Abundance of periodic orbits
in Asymptotically Linear Hamiltonian Systems

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Introduction

Celestial origins At the end of the nineteenth century, the field of celestial mechanics was revolutionized by Poincaré's ideas. Straying away from the quest for exact solutions, which had characterized the previous mathematical research on celestial mechanics, Poincaré instead proposed a qualitative study, where the long term behaviour of the motion is to be investigated without the need of an exact formula to express its solutions. He introduced a global, geometric approach to the study of mechanics. These ideas have been so successful that the qualitative study of dynamical systems is still arguably the main branch of dynamical systems theory.

A deceptively simple model, intensely studied by Poincaré and whose dynamical richness escapes a complete understanding to this day, is the circular planar restricted three body problem (CPR3BP). This is a dynamical system approximating the motion of a small satellite moving under the influence of the gravitational field produced by two large celestial bodies, called the *primaries*, which are assumed to be unperturbed by the satellite. The motion of the satellite is assumed to take place in the plane defined by the Keplerian motion of the two primaries. A concrete example would be the study of the motion of a satellite in the gravitational field of the Earth and the Sun. If we introduce a non-autonomous change of coordinates which rotates with the Keplerian motion of the primaries, so that they remain fixed in time after the change of coordinates, surprisingly, the dynamical system remains described by an autonomous Hamiltonian on a four-dimensional phase space, and the treatment can thus be restricted to the study of a flow on a three-dimensional energy level.

Global surfaces of section One of the deepest insights of Poincaré is the idea of finding a *global surface of section* for the flow on the energy level. This is a surface embedded in the energy level, whose boundary consists of closed orbits of the flow, whose interior is transverse to the vector field generating the flow, and moreover for which every orbit other than the ones on the boundary must eventually collide with the interior. The existence of a global surface of section hence reduces a complicated 3-dimensional flow to a discrete dynamical system on a surface, as the first return map associated to this section captures all of the dynamics of the higher dimensional flow. In particular, the periodic points of the first return map represent closed orbits of the flow.

Since the flow in question happens on an energy level of a Hamiltonian in a 4-dimensional phase space, any global surface of section is endowed with a natural area form induced by the ambient symplectic form. The first return map associated to the global surface of section can be shown to be area-preserving.

Poincaré proved the existence of a global surface of section in certain regimes of the CPR3BP. Namely, assuming that the mass ratio of the primaries approaches zero, and that the satellite moves near the heavy primary, Poincaré was able to show through perturbative methods the existence of two closed orbits, called *direct* and *retrograde* orbits, and to show that these orbits spanned an annulus-shaped global surface of section. The two spanning orbits always twist in opposite directions, so the problem of finding further closed orbits is reduced to the study of an area-preserving map on the annulus which twists the boundaries in opposite directions.

The Poincaré-Birkhoff theorem Area preserving maps of the annulus are definitely not simple objects, and already exhibit the full spectrum of dynamical behaviours, from integrable to chaotic. We give an example in figure 1. With the quest for periodic orbits in the CPR3BP in mind, Poincaré, in one of his last papers [35], conjectured the following theorem:

Theorem *An area preserving map of the annulus which twists the boundary in opposite directions must possess at least two fixed points in its interior.*

Unfortunately Poincaré died shortly thereafter, and was unable to give a complete proof of this result. Birkhoff, one year later, proved this theorem [9], which in the dynamical systems community is now known as the *Poincaré-Birkhoff theorem*. If one only assumes that the map has different rotation numbers on the two boundaries, then one may evince the existence of at least two periodic points. By an iteration argument one may prove that such a map must additionally have infinitely many periodic points with growing primitive periods. Applied to the conjectural annulus shaped global surface of section in the CPR3BP, this result would imply the existence of infinitely many closed orbits.

For a more complete account of the theory of global surfaces of section in celestial mechanics, the reader may consult the wonderful book by U. Frauenfelder and O. Van Koert [18].

Legacy of the Poincaré-Birkhoff theorem The Poincaré-Birkhoff theorem is regarded as a landmark, foreshadowing the development of symplectic topology and dynamics. It is sufficient to mention that an interpretation of this theorem by Arnol'd led him to his famous conjecture on the number of fixed points of a Hamiltonian diffeomorphism on a closed symplectic manifold, as explained in Appendix 9 of his celebrated book on mechanics [6]. The effort to prove the Arnol'd conjecture led to the development of Floer homology, which is without a doubt one of the most powerful tools in the study of symplectic topology and dynamics. Floer theory combines two revolutionary developments which happened in the eighties: the holomorphic curve techniques of Gromov with the ideas of Conley concerning homotopical invariants of gradient-like flows. It makes explicit the fundamental role that periodic orbits of Hamiltonian dynamical systems play in the surprising rigidity phenomena found throughout symplectic geometry. Zehnder's account of this story [43] is a must-read.

The introduction of Floer theoretical techniques led to the understanding that the interplay between topology and dynamics in the symplectic world is embodied, in its most fundamental form, by conditions which force the existence of periodic orbits in Hamiltonian systems. A compelling representation of this link between topology and dynamics is given by a conjecture on Hamiltonian dynamical systems, first formulated by Hofer and Zehnder, which can be stated loosely as follows [29]: a Hamiltonian diffeomorphism on a closed symplectic manifold possessing more than the minimal required amount of fixed points, imposed by Arnol'd-type bounds, must have infinitely many periodic points. In other words, the existence of an "unnecessary" fixed point implies infinitely many periodic points. This conjecture has been verified in many cases, e.g. [21, 27, 39].

On non-closed symplectic manifolds, like the standard linear phase space or the annulus, it is less clear what an "unnecessary" fixed point should be. The example of a rigid rotation of the plane by an angle which is not a rational multiple of 2π shows that there can be Hamiltonian diffeomorphisms all whose orbits are bounded, but which possess very few fixed points, and no periodic points at all. The main theme of this thesis is to define and explore one concept of unnecessary fixed point, namely a fixed point with a twisting condition similar to the one found in the Poincaré-Birkhoff theorem, which applies to arbitrary-dimensional asymptotically linear Hamiltonian dynamical systems on the standard linear phase space.

Asymptotically linear Hamiltonian systems

From the annulus to the plane The main theorem of this thesis is inspired by the Poincaré-Birkhoff theorem for area preserving maps of the annulus. In order to explain the connection between the two, it is necessary to first discuss an application of the Poincaré-Birkhoff theorem to a simple class of Hamiltonian diffeomorphisms of the standard symplectic plane.

Let $\varphi \in \mathrm{Ham}(\mathbb{R}^2)$ be a Hamiltonian diffeomorphism such that there exists a compact set $K \subset \mathbb{R}^2$ out of which φ coincides with a rotation of the plane of angle $\theta_\infty \in \mathbb{R}$. Assume that φ admits a fixed point $z_0 \in K$, and that the rotation number of the linearized flow around this fixed point is $\theta_0 \neq \theta_\infty$. Let $D \supset K$ be a large disc whose boundary is an invariant circle and consider the (degenerate) annulus $D^\times = D \setminus z_0$. The Poincaré-Birkhoff theorem applied to $\varphi|_{D^\times}$ implies that φ admits infinitely many periodic points.

Asymptotically linear Hamiltonian systems and their periodic orbits In order to generalize this simple result to higher dimensions and to a wider class of Hamiltonian diffeomorphisms, we give the following definition. Here $S^1 = \mathbb{R}/\mathbb{Z}$.

Definition A 1-periodic non-autonomous Hamiltonian $H \in C^\infty\left(S^1 \times \mathbb{R}^{2n}\right)$ is said to be *asymptotically quadratic* if there exists a 1-periodic smooth path of symmetric $2n \times 2n$-matrices $A\colon S^1 \to \mathrm{Sym}(2n)$ such that

$$H(t,z) = \frac{1}{2}\langle A(t)z, z\rangle + h(t,z)$$

with $h \in C^\infty(S^1 \times \mathbb{R}^{2n})$ a bounded function such that $\nabla h(t,z) = o(|z|)$ as $|z| \to \infty$ for all $t \in S^1$.

The path A defines a non-autonomous quadratic form $Q(t,z) = \frac{1}{2}\langle A(t)z, z\rangle$ which we call the *quadratic Hamiltonian at infinity*. The time-1 map of the flow of an asymptotically quadratic Hamiltonian is called an *asymptotically linear Hamiltonian diffeomorphism*. We call the time-1 map of the quadratic Hamiltonian at infinity the *linear map at infinity*.

Since the flow of the quadratic Hamiltonian at infinity Q is a flow of linear symplectomorphisms, we can associate to it a mean Conley-Zehnder index, which we call *mean index at infinity*, denoted $\overline{\mathrm{ind}}_\infty H$. Similarly we can define the *index at infinity* of an asymptotically quadratic Hamiltonian to be the Conley-Zehnder index of the path generated by the quadratic Hamiltonian at infinity (see Definitions 1.13 and 1.14). Equivalently, it is the Conley-Zehnder index of the origin as a 1-periodic orbit of the quadratic Hamiltonian at infinity. We interpret the mean index at infinity as a generalization of the rotation number at the "outer" boundary component in the Poincaré-Birkhoff theorem. For the plane maps discussed above, the mean index at infinity is precisely twice the rotation angle of the rotation which the map coincides with outside a compact set.

We assume further that the quadratic Hamiltonian at infinity Q is a non-degenerate Hamiltonian. Equivalently, the linear map at infinity is represented by a symplectic matrix whose spectrum does not contain 1. This means that the time-1 map of the linear flow generated by the quadratic Hamiltonian at infinity admits only the origin as a fixed point; no other point returns to itself in time 1. In this case, we say that the corresponding asymptotically linear Hamiltonian diffeomorphism is non-degenerate at infinity.

To generalize the condition of being a rotation outside a compact set, we assume that the linear map at infinity φ_Q^1 is an unitary map, $\varphi_Q^1 \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$.

The twisting condition in the Poincaré-Birkhoff theorem is generalized in the following way. A 1-periodic orbit whose mean Conley-Zehnder index is different from the mean index at infinity will be called a *twist orbit*, and the corresponding fixed point of φ will be called a *twist fixed point*.

Finally, a fixed point of a Hamiltonian diffeomorphism is said to be *homologicaly visible* if it is isolated and its local Floer homology is non-vanishing. In terms of generating functions, this is equivalent to the non-triviality of the local Morse homology of the corresponding critical point of a generating function for the diffeomorphism in a neighborhood of the fixed point in question. For example, if the Lefschetz index of the isolated fixed point is non-zero, then it is homologically visible (but not the viceversa). Another example is a non-degenerate fixed point, namely, such that the differential of the diffeomorphism at the fixed point does not have the eigenvalue one.

Homological visibility is the minimal required hypothesis which allows Floer theoretical techniques to be applied to the problem. It guarantees that the fixed point in question has a mild persistence property, that is, it cannot disappear under small perturbations of the dynamical system.

We are ready to state the main theorem of the thesis.

Theorem *Let φ be an asymptotically linear Hamiltonian diffeomorphism, non-degenerate and unitary at infinity. Assume that φ admits a homologically visible twist fixed point. Then φ has infinitely many fixed points, or φ has infinitely many periodic points with growing primitive period.*

This theorem generalizes the planar version of the Poincaré-Birkhoff theorem, since the class of Hamiltonians in analysis is much wider, and there is no restriction on the dimension of the phase space.

Short history of the problem Around 1980 Amann and Zehnder [4, 5] combine the celebrated variational approach of Rabinowitz with their finite-dimensional saddle-point reduction to attack the existence problem for asymptotically linear Hamiltonian systems with non-degenerate linear system at infinity. They assume that the linear system at infinity is autonomous, and that the asymptotically quadratic Hamiltonian has bounded Hessian. They find the existence of at least one non-trivial periodic orbit for this class of systems.

Shortly after, Conley and Zehnder [12] extend these results to non-autonomous behaviour at infinity. They prove the existence of one periodic orbit for systems non-degenerate at infinity and find a Morse inequality involving the number of non-degenerate periodic orbits and their indices. An interesting consequence of the Morse inequalities is that if the periodic orbit is assumed to be non-degenerate with index different than the index at infinity, then another non-trivial periodic orbit must exist. In general the Morse inequalities are shown to imply the existence of an odd number of periodic orbits, one of which must have index equal to the index at infinity, and the remaining having index difference one. The techniques used to prove these results are still based on a finite-dimensional reduction in the style of Amann-Zehnder, combined with ideas from Conley's index theory.

It is important to note that the discovery of Morse inequalities involving periodic orbits and their indices foreshadowed the existence of some kind of homology theory, generated by periodic orbits and graded by the index, whose Poincaré-Hilbert series would recover these inequalities. Such homology theory appeared shortly thereafter: Floer homology.

All the results mentioned above assume boundedness of the Hessian of the Hamiltonian function, in order to apply the finite-dimensional reduction scheme of Amann. This hypothesis is removed in the work of Abbondandolo [1–3], where ideas from infinite dimensional relative Morse theory are introduced which allow to bypass the finite-dimensional reduction scheme, and additionally the existence of "subharmonics" (non-trivial higher period periodic orbits) is proven. It is interesting to note that the case of systems *degenerate* at infinity is also covered.

A more complete account of this story can be found in [3, Chapter 5] or [10].

Floer homology for asymptotically linear Hamiltonian systems One of the most successful non-perturbative tools used to probe the existence of periodic orbits in Hamiltoinan dynamical systems is Floer homology. This is a kind of infinite dimensional analogue of Morse homology, which is suitable to analyze the critical point theory of strongly indefinite functionals like the Hamiltonian action functional of classical mechanics. A construction of Floer homology for asymptotically linear Hamiltonian systems is explained in this thesis. Considerable focus is pointed at the invariance properties of Floer homology. In fact, such properties permit an immediate recovery of the seminal result obtained by Conley and Zehnder [12], which was already mentioned above:

Theorem *An asymptotically linear Hamiltonian diffeomorphism which is non-degenerate at infinity has at least one fixed point. If this fixed point is non-degenerate and its index is different from the index at infinity, there is a second fixed point. If every fixed point of the asymptotically linear Hamiltonian diffeomorphism is non-degenerate, then there are an odd number of fixed points. One* *of the fixed points must have index equal to the index at infinity, while the remaining form index difference 1 pairs.*

It is worth to note that the techniques which Conley and Zehnder developed to show the above theorem led them to solve the Arnol'd conjecture on the torus [11]. This was the first arbitrary-dimensional instance where the Arnol'd conjecture could be solved.

Technical remarks

Techniques of proof The proof of the Poincaré-Birkhoff type theorem is related in spirit to Salamon and Zehnder's approach [38], and inspired by the treatment of Gürel [25] for the case of compact perturbations of a linear, hyperbolic and autonomous Hamiltonian system. The original contribution of the thesis lies in the techniques developed to make such arguments applicable to the context.

The proof strategy involves the analysis of the continuation morphisms between the filtered Floer homologies of different iterations of the same Hamiltonian. In the case of a closed symplectically aspherical symplectic manifold, continuation morphisms always exist and are always isomorphisms. This fact, combined with estimates on the action shift of a continuation morphism and on the growth of the index of the iteration of a fixed point, is the fundamental underlying reason which makes the Salamon-Zehnder approach work. On non-closed symplectic manifolds instead continuation morphisms are not always defined, and when they are they might not be isomorphisms.

In the case of asymptotically quadratic Hamiltonians, it is shown in this thesis that if two Hamiltonians have the same index at infinity, then the continuation morphisms exist and are isomorphisms. These continuation morphisms are found to exist even without requiring that the non-quadratic part of the asymptotically quadratic Hamiltonian is bounded. Therefore the Floer homology does indeed depend on the Hamiltonian, but only through its index at infinity. It can be argued that this is the philosophical reason why a Salamon-Zehnder style argument works for the case of asymptotically *hyperbolic* Hamiltonian systems. More precisely, since the mean index of a hyperbolic linear flow is *zero*, as we iterate the Hamiltonian system the index at infinity does not grow. It is therefore reasonable to expect that continuations between different iterations of the same Hamiltonian have properties similar to the closed case: the Floer homologies of different iterations of the same Hamiltonian remain always "in reach" of each other.

By contrast, when the dynamics at infinity is unitary, the mean index is *never* zero, and the index at infinity grows linearly with the iteration. This allows the Floer homologies of different iterations of the same Hamiltonian to "walk away" from each other and not be related by continuations with suitable properties. Now, since the problem is only "at infinity", to surmount it we compose the Hamiltonian system with loops of linear symplectomorphisms which "unwind" the dynamics at infinity, changing the index at infinity, but which do not have any effect on the action value of periodic orbits. The effect on Floer homology is to shift the grading in a controlled manner while maintaining the action filtration. We call this procedure *re-indexing* at infinity. It allows us to have well defined continuations between Hamiltonians generating different iterates of the same map.

However, even at this point the continuations found are not suitable. Namely, they do not seem to have a useful action shift estimate: the estimate found grows too fast with the iteration. So to gain a better control on the action shift, an interpolation technique is introduced. The end result is two Hamiltonians whose Floer homologies are computable in terms of the Floer homologies of the iterates, but whose quadratic forms at infinity coincide, making it possible to execute simpler continuations, with action shift estimate suitable for the proof.

The combination of these techniques with some asymptotic properties of prime numbers allow an argument resembling the proof of the main theorem in [25] to be applied.

Abbondandolo's conjecture There is a conjecture of Abbondandolo [3, pg. 130] stating that an asymptotically linear Hamiltonian system should have one or infinitely many periodic orbits. In particular, if the system has an *unnecessary* orbit different from the one continuing from the linear system at infinity, then it should have infinitely many periodic orbits. As far as the author knows, this conjecture in this generality is wide open.

The results in this thesis can be interpreted as a solution of a *homological* version of this conjecture, under additional restrictions on the type of system at infinity and the index of the unnecessary orbit. Some of these restrictions might be avoidable. Perhaps using a better interpolation argument, it should be possible to remove the assumptions of unitarity at infinity. It also seems absolutely natural and desirable to remove the hypothesis of boundedness of the non-quadratic part of the Hamiltonian, but to do so one must produce energy estimates on Floer cylinders whose behaviour under iteration is suitable for the proof, or change the proof strategy entirely.

The presence of twist, instead, is much more substantial, as shown by the example of a linear symplectic diffeomorphism all whose iterates are non-degenerate. In fact, conservative flows with few periodic orbits seem to have strong restrictions, especially on the index spectrum, see e.g. [23, 24] and [13]. Inspired by the theory of pseudo-rotations, the author wonders whether one can find an example of an asymptotically linear Hamiltonian diffeomorphism with more than one fixed point, all fixed points with the same mean index, and no primitive periodic points with period higher than one.

Comparisons and contrasts Other kinds of Poincaré-Birkhoff type theorems have appeared recently in the literature. We would like to compare and contrast the present work with some of these results.

In the paper by Gürel [25], Hamiltonian diffeomorphism which are equal to an autonomous hyperbolic linear symplectic diffeomorphism outside a compact set are studied. There it is proven that if the Hamiltonian diffeomorphism admits a fixed point with non-vanishing local Floer homology and whose mean Conley-Zehnder index is not zero, then there are infinitely many periodic orbits. Here we are interpreting the request of non-vanishing mean Conley-Zehnder index as a twist condition, since an asymptotically hyperbolic Hamiltonian system always has zero mean index at infinity. The proof schema of the main theorem of this thesis is inspired by the proof found there. The difference is that we admit a larger class of Hamiltonians and that the behaviour at infinity is, in some sense, the opposite of hyperbolic.

Another interesting development is found in Moreno and Van Koert's paper [34], where a kind of Poincaré-Birkhoff theorem is proven for certain "twist" Hamiltoinan diffeomorphisms in the completion of Liouville domains which have infinite dimensional symplectic homology. The present work compares to it because one could interpret our class of Hamiltonian systems as such kind of "twist" Hamiltonian diffeomorphims in the completion of an ellipsoid in $\mathbb{R}^{2n}$. The difference is that the symplectic homology of the ball is vanishing, so our techniques cover an orthogonal case.

Shelukhin's paper [39] is also related to the present work, in fact, a very special case of the main theorem in the thesis follows from it. In this paper, the homological Hofer-Zehnder conjecture is proven on closed symplectic manifolds similar to $\mathbb{C}P^n$. If in the main theorem of the thesis one assumes that the Hamiltonian diffeomorphism is *equal* to a unitary linear map outside of a compact set, then it is possible to extend it to $\mathbb{C}P^n$. Such extension has n non-degenerate fixed points induced by the linear part of the map, and one additional fixed point which is guaranteed to exist by continuation from infinity. This additional fixed point could be twist or not. In case it is twist, a higher iteration of the map has a fixed point with Conley-Zehnder index different from the one at infinity, implying the existence of yet another fixed point continuing from infinity. In case it is not, in the main theorem of the thesis the existence of an additional one which is twist and homologically visible is assumed. All in all, possibly after iterating, we have $n+2$ fixed points, all homologically visible, therefore the local Floer homologies of these fixed points all contribute non-trivially to the homological count. Arnold's homological lower bound would then be exceeded, and Shelukhin's theorem gives infinitely many periodic points. If instead we only require the Hamiltonian diffeomorphism to be generated by an asymptotically quadratic Hamiltonian, then the diffeomorphism extends, but only as a homeomorphism of $\mathbb{C}P^n$. Therefore, the main theorem of this thesis does not follow from the proof of the Hofer-Zehnder conjecture. It is also important to remark that the techniques in the proof of the Hofer-Zehnder conjecture are very different than the ones developed here, which are in some sense simpler. Shelukhin's techniques rely on a detailed analysis of the behaviour of the Floer barcode and the equivariant pants product of a Hamiltonian diffeomorphism under iteration.

Structure of the thesis In chapter 1 we revise some linear symplectic geometry and give a definition of the Conley-Zehnder index. In chapter 2 we introduce asymptotically linear Hamiltonian systems and show that their 1-periodic orbits are contained in a compact subset of $\mathbb{R}^{2n}$. In chapter 3, we gather some aspects of the analysis of the Floer equation, and study the spaces of solutions of the Floer equation, especially their compactifications. In chapter 4 we explain the construction of the filtered Floer homology for asymptotically linear Hamiltonian systems, and give a somewhat detailed account of the invariance properties of filtered Floer homology. This will lead to a definition of Floer homology for degenerate Hamiltoinans, and to the study of the local Floer homology of isolated fixed points. In chapter 5 we introduce two techniques, which are new as far as the author knows, to relate the Floer homologies of different iterates of the same asymptotically linear Hamiltonian system. One is a procedure which changes the index of the linear system at infinity without changing the action filtration on the Floer homology, the other is an interpolation procedure to change the quadratic Hamiltonian at infinity. These two techniques are then applied in the proof of the Poincaré-Birkhoff type theorem. There are three appendices: Appendix A, Appendix B and C. In the first, we explain some elementary concepts of Fredholm theory. In the second, we gather some elementary facts in number theory which are necessary for the proof of the main theorem are collected. In the third we lay out the conventions used in this thesis, in an effort to avoid the headaches typical of students of symplectic geometry, of which the author has suffered plenty.

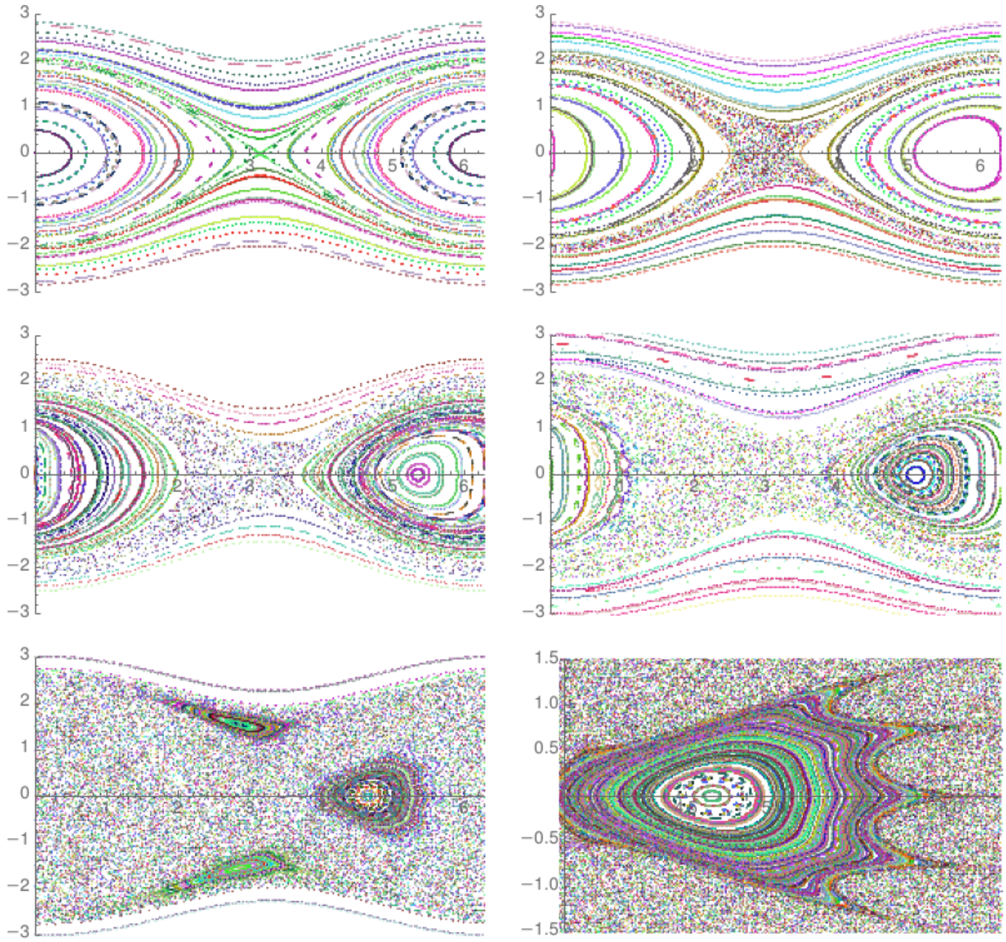

Figure 1. Phase plots of area preserving maps of the annulus, from simple to complicated. All these maps twist the boundary of the annulus in opposite directions. Points of the same color belong to the same orbit (unfortunately there is a finite number of colors available for the plot, so sometimes they repeat for different orbits). The first is the pendulum, regarded as an area preserving map of the annulus. The successive maps are obtained by perturbing the pendulum with a periodic forcing term. The transition from integrable, to nearly integrable, to chaotic is shown by the growth of the "hyperbolic sea" which develops from a homoclinic intersection at the hyperbolic equilibrium point. It is interesting that even very far from the integrable model, the map still seems to admit invariant tori which survive the perturbation, implying that many periodic orbits survive the perturbation. Some evidence for this fact is given by the last image, which is a zoom into the interesting invariant set of the map at its left. Perhaps with an even larger perturbation these tori might disappear.

1. Linear Symplectic Geometry and the Conley-Zehnder Index

In this chapter we recall some elementary facts about linear symplectic geometry and construct a homotopy invariant of paths into the symplectic group starting at the identity: the Conley-Zehnder index.

1.1. Elementary properties of the Symplectic Group

In this section we give the preliminary facts that we need for the definition of the Conley-Zehnder index. Nothing here is new and almost everything can be found in [33].

A symplectic vector space is a real vector space V equipped with a non-degenerate skew bilinear form $\omega\colon V\times V\to\mathbb{R}$. A simple induction argument shows that in the finite-dimensional case, V must have even dimension, $\dim V=2n$. Given two symplectic vector spaces (V,ω), (V',ω') we say that a linear map $\varphi\colon V\to V'$ is a (linear) symplectomorphism if $\varphi^*\omega'=\omega$, i.e. for any $v,w\in V$ we have $\omega'(\varphi(v),\varphi(w))=\omega(v,w)$. In particular, $\varphi^*\omega'$ is a symplectic form, hence φ must be an isomorphism. Even more is true: every symplectic vector space V of dimension $2n$ has a basis $\{v_1,\dots,v_n,w_1,\dots,w_n\}$ such that $\omega(v_i,w_i)=1=-\omega(w_i,v_i)$, $\omega(v_i,w_j)=\omega(v_i,v_j)=\omega(w_i,w_j)=0$ for $i\neq j$. This means that it is symplectically isomorphic to the symplectic vector space $(\mathbb{R}^{2n},\omega_0)$ where we equip the real vector space with the symplectic form ω_0 whose Gram matrix in the canonical basis is

$$\Omega_0=\begin{pmatrix}\mathbb{O}_n & \mathbb{I}_n\\ -\mathbb{I}_n & \mathbb{O}_n\end{pmatrix}$$

Such a basis is usually called a symplectic basis for the symplectic vector space. To study the group of linear symplectomorphisms $\varphi\colon(V,\omega)\to(V,\omega)$, one may just as well choose a symplectic basis and study the linear symplectomorphisms $\varphi\colon(\mathbb{R}^{2n},\omega_0)\to(\mathbb{R}^{2n},\omega_0)$. We denote this group by $\mathrm{Sp}(2n)$.

1.1.1. Algebraic and differential properties

Having chosen the canonical symplectic basis for our symplectic $2n$-space, the group of symplectomorphisms is canonically isomorphic to a matrix subgroup of the general linear group.

Definition 1.1 A matrix $M \in \mathrm{GL}(2n,\mathbb{R})$ is said to be a symplectic matrix if $M^T \Omega_0 M = \Omega_0$. The set of symplectic matrices is denoted by $\mathrm{Sp}(2n)$.

Remark In most treatments of linear symplectic geometry, one starts with the matrix $J_0 = -\Omega_0$ and defines a symplectic matrix as a matrix satisfying $M^T J_0 M = J_0$. I decided to use Ω_0 instead because J_0 is thought of as the complex structure of $\mathbb{R}^{2n}$, while Ω_0 is the matrix representing the symplectic structure, and in the conventions of this thesis they are not the same matrix.

The following Lemma follows from a computation.

Lemma 1.1.1 *The set* $\mathrm{Sp}(2n)$ *is a group under matrix multiplication.*

A more interesting property is that $\mathrm{Sp}(2n)$ sits inside $\mathrm{SL}(2n)$.

Proposition 1.1.1 (Linear Liouville Theorem) *Symplectic matrices have determinant 1.*

Proof. Clearly symplectic matrices have determinant of absolute value 1: if $M \in \mathrm{Sp}(2n)$ then $1 = \det \Omega_0 = \det M^T \Omega_0 M = (\det M)^2$. To see that they have determinant exactly 1, it is more convenient to think of the associated linear symplectomorphism $\varphi\colon (\mathbb{R}^{2n},\omega_0) \to (\mathbb{R}^{2n},\omega_0)$. Such a linear map has matrix whose determinant is 1 if and only if preserves the orientation of a chosen basis. In terms of exterior algebra, this is equivalent to saying that the linear map preserves the canonical volume form on $\mathbb{R}^{2n}$: $\varphi^*(e_1 \wedge \ldots \wedge e_{2n}) = e_1 \wedge \ldots \wedge e_{2n}$. Now, since ω_0 is a skew symmetric bilinear form, we can express it as a 2-form: by inspecting the Gram matrix it is easy to see that $\omega_0 = \sum_{i=1}^n e_i \wedge e_{n+i}$. But now

$$\omega_0^{\wedge n} = n!\, e_1 \wedge e_2 \wedge \ldots \wedge e_{2n} \implies \frac{\omega_0^{\wedge n}}{n!} = e_1 \wedge \ldots \wedge e_{2n}$$

Hence if φ is a linear symplectomorphism

$$\varphi^*(e_1 \wedge \ldots \wedge e_{2n}) = \varphi^* \frac{\omega_0^{\wedge n}}{n!} = \frac{(\varphi^*\omega)^{\wedge n}}{n!} = \frac{\omega_0^{\wedge n}}{n!} = e_1 \wedge \ldots \wedge e_{2n}$$

□

Like most groups underlying geometric structures, the symplectic group is a Lie group.

Proposition 1.1.2 $\mathrm{Sp}(2n)$ *is a Lie subgroup of the Lie group* $\mathrm{GL}(2n,\mathbb{R})$.

Proof. All we have to show is that it's a submanifold. Define the map

$$\begin{aligned} f : M_{2n\times 2n}(\mathbb{R}) &\to M_{2n\times 2n}(\mathbb{R}) \\ M &\mapsto f(M) = M^T \Omega_0 M \end{aligned}$$

First of all notice that

$$f(M)^T = \left(M^T \Omega_0 M\right)^T = -M^T \Omega_0 M = -f(M)$$

meaning that the image of f is contained in the Lie algebra of SO($2n$), the matrix algebra $\mathfrak{so}(2n) = \{B \in M_{2n\times 2n}(\mathbb{R}) : B^T = -B\}$. Therefore we can restrict the range of the map and think of it as a function

$$f : M_{2n\times 2n}(\mathbb{R}) \to \mathfrak{so}(2n)$$

Since $\mathrm{Sp}(2n) = f^{-1}(\Omega_0)$, if we show that Ω_0 is a regular value, then by the submersion theorem we've shown that Sp($2n$) is a submanifold. This can be done with a simple computation. □

Corollary 1.1.1 *The Lie algebra is* $T_{\mathbb{I}_{2n}} \mathrm{Sp}(2n) = \ker df(\mathbb{I}_{2n})$ *hence*

$$\mathfrak{sp}(2n) = \ker df(\mathbb{I}_{2n}) = \left\{R \in M_{2n\times 2n}(\mathbb{R}) : R^T \Omega_0 + \Omega_0 R = \mathbb{O}_{2n}\right\}$$

Lemma 1.1.2 *The Lie algebra of the symplectic group* Sp($2n$) *is equivalently described as*

$$\mathfrak{sp}(2n) = \{\Omega_0 A : A \in \mathrm{Sym}(2n)\}$$

Proof. Indeed if A is a symmetric matrix,

$$(\Omega_0 A)^T \Omega_0 + \Omega_0 \Omega_0 A = A^T - A = \mathbb{O}_{2n}$$

while if $R \in \mathfrak{sp}(2n)$ then the matrix $A = -\Omega_0 R$ is clearly symmetric and $R = \Omega_0 A$. □

Remark The exponential map $\exp\colon \mathfrak{sp}(2n) \to \mathrm{Sp}(2n)$ is not surjective, reflecting the fact that Sp($2n$) is not a compact group. For example, the matrix $\mathrm{diag}(-2, -1/2) \in \mathrm{Sp}(2)$ is not the exponential of any matrix in $\mathfrak{sp}(2)$. Indeed, if it were, it would have a matrix logarithm in $\mathfrak{sp}(2)$. But clearly any logarithm of this matrix will have complex coefficients.

1.1.2. Complex aspects

Real symplectic space $(\mathbb{R}^{2n}, \omega_0)$ is an even-dimensional real vector space. As such, it has also a natural complex structure.

Definition 1.2 A *complex structure* on a $2n$-dimensional real vector space V is an endomorphism $J\colon V \to V$ such that $J^2 = -\mathrm{id}_V$. A *complex linear* map between vector spaces V, V' with complex structures resp. J, J' is a linear map $\varphi\colon V \to V'$ such that $\varphi \circ J = J' \circ \varphi$.

Clearly $\mathbb{C}^n$ seen as a real $2n$-vector space has the complex structure given by multiplication by i. Namely, we have the real vector space isomorphism

$$\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{C}^n$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto z = x + iy$$

Then multiplication by i acts as the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy \mapsto ix - y \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$$

Hence there is a canonical complex structure on $\mathbb{R}^{2n}$ given in block matrix form by

$$J_0 = \begin{pmatrix} \mathbb{O}_n & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}$$

Remark Notice that as matrices $J_0 = \Omega_0^T = \Omega_0^{-1} = -\Omega_0$, hence when we endow $\mathbb{R}^{2n}$ with the standard symplectic form, the bilinear form $\omega_0(\cdot, J_0 \cdot)$ is nothing but the standard Euclidean inner product of $\mathbb{R}^{2n}$. In this sense ω_0 and J_0 are *compatible*, and $\mathbb{R}^{2n}$ is Kähler. Obviously given two out of three between standard inner product $\langle \cdot, \cdot \rangle$, complex structure J_0 and symplectic structure ω_0, we can recover the third. Moreover when seen on $\mathbb{C}^n$, we have that the standard hermitian form h is $h = \langle \cdot, \cdot \rangle + i\omega_0$.

Lemma 1.1.3 1. *The group homomorphism*

$$\iota \colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$$
$$Z = F + iB \mapsto \iota(Z) = \begin{pmatrix} F & -B \\ B & F \end{pmatrix}$$

is injective. Moreover $\iota(\mathrm{U}(n)) \subset \mathrm{Sp}(2n)$.

2. *Identify* $\mathrm{GL}(n, \mathbb{C}), \mathrm{U}(n)$ *with their image inside* $\mathrm{GL}(2n, \mathbb{R}), \mathrm{Sp}(2n)$. *Then*

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n)$$
$$\mathfrak{u}(n) = \mathfrak{so}(2n) \cap \mathfrak{sp}(2n) = \mathfrak{gl}(n, \mathbb{C}) \cap \mathfrak{sp}(2n)$$

Proof. 1. That ι is a morphism can be easily seen by a computation. Clearly $\iota(X + iY) = \mathbb{I}_{2n}$ if and only if $X = \mathbb{I}_n$ and $Y = \mathbb{O}_n$, so if and only if $X + iY = \mathbb{I}_n$. Notice that $\iota(i\mathbb{I}_n) = J_0$, which is just another way of saying that J_0 is multiplication by i. $U \in \mathrm{U}(n)$ means $U^\dagger U = \mathbb{I}_n$. Now notice that $\iota\left(U^\dagger\right) = \iota(U)^T$, so

$$\iota(U)^T J_0 \iota(U) = \iota\left(U^\dagger i U\right) = \iota(i\mathbb{I}) = J_0 \iff \iota(U) \in \mathrm{Sp}(2n).$$

2. Continuing the reasoning from the previous line,

$$\mathbb{I}_{2n} = \iota(\mathbb{I}_n) = \iota\left(U^\dagger U\right) = \iota(U)^T \iota(U) \iff \iota(U) \in \mathrm{O}(2n).$$

so $U \in \mathrm{U}(n) \iff \iota(U) \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$. Now if we have a $Z \in \mathrm{GL}(n, \mathbb{C})$ such that $\iota(Z) \in \mathrm{Sp}(2n)$, then

$$\iota\left(Z^\dagger Z\right) = \iota(Z)^T \iota(Z) = -J_0 \iota(Z)^{-1} J_0 \iota(Z) = -J_0 J_0 = \mathbb{I}_{2n} = \iota(\mathbb{I}_n)$$

Therefore we conclude that $Z^\dagger Z = \mathbb{I}_n$, i.e. $Z \in \mathrm{U}(n)$. The other inclusion is trivial.

□

From now on, we slightly abuse notation by forgetting the homomorphism in the distinction between the complex matrix groups and their image inside the larger real groups. To avoid confusion, we give the following definition.

Definition 1.3 Let $M \in M_{n\times n}(\mathbb{C}) \subset M_{2n\times 2n}(\mathbb{R})$. The *complex determinant* of M is the determinant of M seen as a complex linear map:

$$M = \begin{pmatrix} F & -B \\ B & F \end{pmatrix} \implies \det_{\mathbb{C}}(M) := \det(F + iB) \in \mathbb{C}$$

1.1.3. Topology of the symplectic group

In this section we'll show that the inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n)$ is a homotopy equivalence.

Lemma 1.1.4 *Let $P \in \mathrm{Sp}(2n) \cap \mathrm{Sym}(2n)$ be a symmetric positive definite symplectic matrix. Then $P^\alpha \in \mathrm{Sp}(2n)$ $\forall \alpha \geq 0$.*

Proof. Since P is symmetric and symplectic, it is symplectically diagonalizable, so without loss of generality we may assume that in the canonical basis $\{e_1, \ldots, e_{2n}\}$ the matrix is diagonal, $P = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})$ for not necessarily distinct $\lambda_k > 0$. In this basis, clearly

$$\omega(e_i, e_j) = \omega(Pe_i, Pe_j) = \lambda_i \lambda_j \omega(e_i, e_j)$$

Since P is symplectic, $\omega(e_i, e_j) = 0$ or $\lambda_i \lambda_j = 1$. In any case

$$\omega(e_i, e_j) = (\lambda_i \lambda_j)^\alpha \omega(e_i, e_j) = \omega(\lambda_i^\alpha e_i, \lambda_j^\alpha e_j) = \omega(P^\alpha e_i, P^\alpha e_j)$$

which means that P^α is symplectic. □

The following lemma is crucial and easily proven by a calculation.

Lemma 1.1.5 *Every symplectic matrix* $M \in \mathrm{Sp}(2n)$ *has a* symplectic polar decomposition

$$M = UP, \quad U = M\left(M^T M\right)^{-\frac{1}{2}} \in \mathrm{U}(n), \quad P = \left(M^T M\right)^{\frac{1}{2}} \in \mathrm{Sp}(2n)$$

where P *is* positive definite and symmetric.

Remark Notice that this decomposition realizes a symplectic matrix as the *product* of exponentials of matrices in $\mathfrak{sp}(2n)$. Indeed the unitary part is the image of the exponential of a matrix in $\mathfrak{u}(n)$, since $\mathrm{U}(n)$ is a compact and connected Lie group, and the positive definite part is also an exponential, since real symmetric positive definite matrices have a real and symmetric logarithm.

Lemma 1.1.6 *The set* $S_+\,\mathrm{Sp}(2n)$ *of positive definite, symmetric symplectic matrices is contractible.*

Proof. As remarked just above, real symmetric positive definite matrices have a real and symmetric logarithm, which is also unique. Therefore the logarithm gives a homeomorphism $S_+\,\mathrm{Sp}(2n) \simeq \mathrm{Sym}(2n) \cap \mathfrak{sp}(2n)$ which is a vector space. □

Proposition 1.1.3 *The inclusion* $\mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n)$ *is a homotopy equivalence.*

Proof. Consider the map

$$\begin{aligned} f : [0,1] \times \mathrm{Sp}(2n) &\to \mathrm{Sp}(2n) \\ (t, M) &\mapsto M\left(M^T M\right)^{-t/2} \end{aligned}$$

First of all, this is a well defined continuous map. Clearly, $f(0, M) = M \implies f(0, -) = \mathrm{id}_{\mathrm{Sp}(2n)}$. Moreover, $f(1, M) = M(M^T M)^{-1/2} \in \mathrm{U}(n)$. Finally if $M \in \mathrm{U}(n)$, then $M \in O(2n)$ and $M(M^T M)^{-t/2} = M\mathbb{I}^{-t/2} = M$, meaning that

$$f|_{[0,1]\times \mathrm{U}(n)} = \mathrm{id}_{\mathrm{U}(n)}$$

so f is a strong deformation retraction of $\mathrm{Sp}(2n)$ on $\mathrm{U}(n)$. □

1.1.3.1. Topology of the Unitary Group

Here we work with complex coefficients, i.e. we don't think of $M_{n\times n}(\mathbb{C})$ as a subset of real matrices. To underline the difference and avoid confusion we continue to write $\det_{\mathbb{C}}$ instead of det.

Lemma 1.1.7 1. $\mathrm{U}(n)$ *is homeomorphic to* $\mathrm{U}(1) \times \mathrm{SU}(n)$.

2. $\mathrm{SU}(n)$ *is simply connected.*

Proof. 1. Define the maps

$$\begin{aligned} h\colon \mathrm{U}(1)\times \mathrm{SU}(n) &\to \mathrm{U}(n)\\ (e^{i\theta},V) &\mapsto \mathrm{diag}(e^{i\theta},1,\ldots,1)V\\ g\colon \mathrm{U}(n) &\to \mathrm{U}(1)\times \mathrm{SU}(n)\\ U &\mapsto \left(\det_{\mathbb{C}} U, \mathrm{diag}\left((\det_{\mathbb{C}} U)^{-1},1,\ldots,1\right)U\right) \end{aligned}$$

These maps are obviously well defined and continuous. A direct calculation shows they are inverse to each other.

2. The proof is by induction. For $n=1$, $\mathrm{SU}(1)=\{1\}$ which is simply connected. Next, for the induction step, let's think of $\mathrm{SU}(n)$ as the matrices of change of coordinates between oriented orthonormal bases of $\mathbb{C}^n$. This means that in the canonical basis of $\mathbb{C}^n$, the columns of a matrix $V\in\mathrm{SU}(n)$ are just another orthonormal basis of $\mathbb{C}^n$. Hence they are vectors belonging to the unit sphere in $\mathbb{C}^n$. This way we have the map

$$\begin{aligned} \mathrm{SU}(n) &\to S^{2n-1}\\ V=\left(v_{ij}\right)_{i,j=1}^{n} &\mapsto \left(v_{1j}\right)_{j=1}^{n} \end{aligned}$$

This is a surjective map. The fiber over one fixed vector of S^{2n-1} is the orthonormal basis of $\mathbb{C}^n$ with one vector fixed, hence equivalent to the set of orthonormal bases of $\mathbb{C}^{n-1}$. This can be identified with $\mathrm{SU}(n-1)$, which by inductive hypothesis is simply connected. We constructed a fiber bundle $\mathrm{SU}(n)\to S^{2n-1}$ with fiber $\mathrm{SU}(n-1)$. Now we use the long exact sequence of homotopy groups to obtain the exact sequence

$$\pi_1(\mathrm{SU}(n-1))\to \pi_1(\mathrm{SU}(n))\to \pi_1\left(S^{2n-1}\right)$$

Since $n>1$, S^{2n-1} is simply connected. Hence $\pi_1(\mathrm{SU}(n))=0$.

□

Corollary 1.1.2 *The fundamental group of* $\mathrm{U}(n)$*, hence of* $\mathrm{Sp}(2n)$*, is* $\mathbb{Z}$*.*

Remark The isomorphism is realized by the (complex) determinant:

$$\begin{aligned} \pi_1(\mathrm{U}(n)) &\to \pi_1(\mathrm{U}(1)) \xrightarrow{\cong} \mathbb{Z}\\ [U_t] &\mapsto [\det_{\mathbb{C}} U_t] \to \deg \det_{\mathbb{C}} U_t \end{aligned}$$

1.1.4. The Maslov Index

We can do something analogous for $\mathrm{Sp}(2n)$ by using this map and the symplectic polar decomposition.

Definition 1.4 Let $\Psi\colon S^1 \to \mathrm{Sp}(2n)$, $\Psi(t) = M_t$ be a continuous loop of symplectic matrices. The symplectic polar decomposition $M_t = U_t P_t$ gives an unique loop of unitary matrices $U_t \in \mathrm{U}(n)$, so we have a well defined map

$$M_t \mapsto \deg \det_{\mathbb{C}} U_t = \mathrm{Mas}(\Psi)$$

called the *Maslov index* of the loop Ψ.

The following proposition gathers the defining properties of the Maslov index, all which can be easily shown by calculations.

Proposition 1.1.4 *The Maslov index has the following properties.*

1. *The Maslov index is a complete homotopy invariant of based loops of symplectic matrices, i.e. two loops starting at the identity are homotopic if and only if they have the same Maslov index. Hence it induces the isomorphism* $\mathrm{Mas}\colon \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$.
2. *For any two continuous loops* $\Psi_1, \Psi_2\colon S^1 \to \mathrm{Sp}(2n)$ *one has*

$$\mathrm{Mas}(\Psi_1\Psi_2) = \mathrm{Mas}(\Psi_1) + \mathrm{Mas}(\Psi_2)$$

   In particular the constant loop $\Psi(t) = \mathbb{I}$ *has* $\mathrm{Mas}(\Psi) = 0$.
3. *Let* $n' + n'' = n$ *and identify* $\mathrm{Sp}(2n') \times \mathrm{Sp}(2n'')$ *as a subgroup of* $\mathrm{Sp}(2n)$ *by sending matrices* $(M', M'') \in \mathrm{Sp}(2n') \times \mathrm{Sp}(2n'')$ *to the block matrix*

$$M' \oplus M'' = \begin{pmatrix} M' & \mathbb{O}_{n'\times n''} \\ \mathbb{O}_{n''\times n'} & M'' \end{pmatrix}$$

   Given two loops $\Psi'\colon S^1 \to \mathrm{Sp}(2n')$, $\Psi''\colon S^1 \to \mathrm{Sp}(2n'')$, *we have*

$$\mathrm{Mas}(\Psi' \oplus \Psi'') = \mathrm{Mas}(\Psi') + \mathrm{Mas}(\Psi'')$$

4. *The loop* $\Psi\colon S^1 \to \mathrm{Sp}(2)$ *given by*

$$\Psi(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

   has $\mathrm{Mas}(\Psi) = 1$.

Remark One may show that these properties characterize the Maslov index uniquely, i.e. any map with such properties is equal to the Maslov index.

1.1.5. The spectrum of a Symplectic Matrix

We study of the spectrum of a symplectic matrix, but from the point of view of a natural extension of the concept to $\mathbb{C}^{2n}$. The standard extension of a matrix $P \in M_{2n\times 2n}(\mathbb{R})$ to $\mathbb{C}^{2n}$ is by setting

$$P\zeta = P(x+iy) = Px + iPy, \quad \forall \zeta = x+iy \in \mathbb{C}^{2n}$$

Let h be the standard Hermitian product on $\mathbb{C}^{2n}$, $h(u,v) = u^\dagger v = \bar{u}^T v$.

Definition 1.5 Introduce the inner product on $\mathbb{C}^{2n}$

$$g(\zeta_0, \zeta_1) = h(\zeta_0, -iJ_0\zeta_1)$$

A g-unitary matrix is a matrix $M \in M_{2n\times 2n}(\mathbb{C})$ such that $g(Mu, Mv) = g(u,v)$. Equivalently, M is g-unitary if and only if $M^\dagger J_0 M = J_0$.

Notice that a g-unitary matrix M with real coefficients is just a symplectic matrix. Notice further that a real matrix which is both h-unitary and g-unitary is a matrix in $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$, and viceversa.

Denote the spectrum of a matrix M as $\sigma(M)$. Being g-unitary has the following consequences on the spectrum, which follow immediately from $M = J_0^{-1} M^{-\dagger} J_0$:

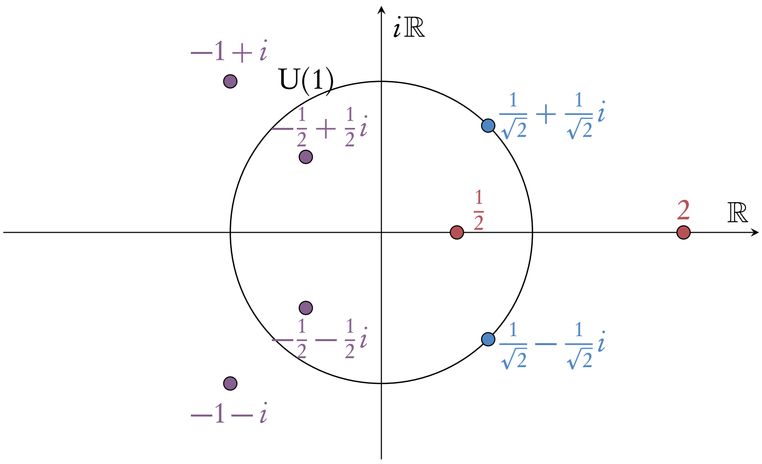

Figure 1.1. Existence of an eigenvalue of one color implies the existence of all eigenvalues of the same color.

Lemma 1.1.8 *The spectrum of a g-unitary matrix M has the following symmetry: $\lambda \in \sigma(M) \implies \overline{\lambda^{-1}} \in \sigma(M)$. Moreover if M is also real, i.e. $M \in \mathrm{Sp}(2n)$, then $\lambda \in \sigma(M) \implies \bar{\lambda} \in \sigma(M)$ and the eigenvalues ± 1 always have even algebraic multiplicity.*

In other words, the eigenvalues of a symplectic matrix come in the following tuples (see figure 1.1):

- If $\pm 1 \in \sigma(M)$ then it has even multiplicity.
- If $\lambda \in \sigma(M) \cap \mathbb{R}$ then $\lambda^{-1} \in \sigma(M)$.
- If $\lambda \in \sigma(M) \cap \mathrm{U}(1)$ then $\bar{\lambda} \in \sigma(M)$.
- If $\lambda \in \sigma(M) \setminus (\mathbb{R} \cup \mathrm{U}(1))$ then $\lambda^{-1}, \bar{\lambda}, \overline{\lambda^{-1}} \in \sigma(M)$

1.1.5.1. Uniformly non-resonant iterations of a symplectic matrix

Here we state the first original result of the thesis. It is going to be crucial for the proof of the Poincaré-Birkhoff type theorem, but it only concerns the spectrum of a symplectic matrix.

Proposition 1.1.5 *Let $M \in \mathrm{Sp}(2n)$ be a symplectic matrix with $1 \notin \sigma(M)$. Then there exists a $c > 0$ and an increasing sequence $(p_j)_{j\in\mathbb{N}}$ of prime numbers such that:*

1. *The spectrum of the p_jth iterate of M remains uniformly away from 1:*

$$d_{\mathrm{U}(1)}\left(\sigma\left(M^{p_j}\right),1\right) > c$$

2. *The gaps in this sequence of primes are distributed like the gaps in the sequence of all primes:*

$$p_{j+m} - p_j = o\left(p_j\right) \quad \textit{as } j \to \infty \quad \forall m \in \mathbb{N}$$

Proof. We aim to apply Vinogradov's equidistribution theorem on prime multiples of irrational numbers [30, 41] (see also Appendix B).

Notice that the only part of the spectrum that may possibly approach 1 as we iterate is given by the eigenvalues on U(1). So let $\{e^{\pm i\alpha_1},\dots,e^{\pm i\alpha_l}\}$ be the eigenvalues on U(1), listed repeating the multiple eigenvalues, when necessary. Let $\alpha_1,\dots,\alpha_l$ be choices of arguments of these eigenvalues, again repeated according to their multiplicity, and $a_j = \alpha_j/2\pi$. These are well defined numbers mod 1. There are two cases: the set $\{1,a_1,\dots,a_l\}$ spans a $\mathbb{Q}$-subspace of $\mathbb{R}$ of rank 1, or it spans a $\mathbb{Q}$-subspace of $\mathbb{R}$ of rank at least 2. In the first case, each a_j is a rational number, or equivalently we have that every eigenvalue of M on the unit circle is a root of unity. In this case it suffices to take a prime number $p_0 >> 2$ larger than the largest prime factor of any order of these roots of unity, and $p_i, i \geq 1$ will be all the prime numbers larger than p_0 listed in increasing order. By the prime number theorem, this sequence satisfies the second point. For $c > 0$ we can now choose

$$c = \min\left\{\frac{2\pi}{j} : j \text{ order of root of unity in } \sigma(M)\right\} > 0$$

Now, assume that $\{1,a_1,\dots,a_l\}$ spans a $\mathbb{Q}$-subspace of rank $q \geq 2$. This is equivalent to saying that there are exactly q rationally independent irrational numbers in $\{a_1,\dots,a_l\}$. Without loss of generality we may assume that $a_1,\dots,a_q$ are such numbers, and that $a_i = \sum_{j=1}^{q} r_{ij}a_j$ for all $i \geq q+1$ where $r_{ij} \in \mathbb{Q}$ for all $i \geq q+1$ and $j \leq q$. Set $r_{ij} = \delta_{ij}$ when $i,j \leq q$. Now, since $\{a_1,\dots,a_q\}$ is a rationally independent set of irrational numbers, we can use Vinogradov's equidistribution theorem [41] combined with [30, Theorem 6.3] to conclude that the sequence $(P\vec{a})_{P \text{ prime}} \subset \mathbb{R}^q$ is equidistributed mod 1, where $\vec{a} = (a_1,\dots,a_q)$. This means that for every measurable $C \subset [0,1]^q$ of measure $|C|$ it holds that

$$\lim_{N\to\infty} \frac{\#\{P \text{ prime } : P \leq N,\ P\vec{a} \bmod 1 \in C\}}{\pi(N)} = |C| \tag{1.1}$$

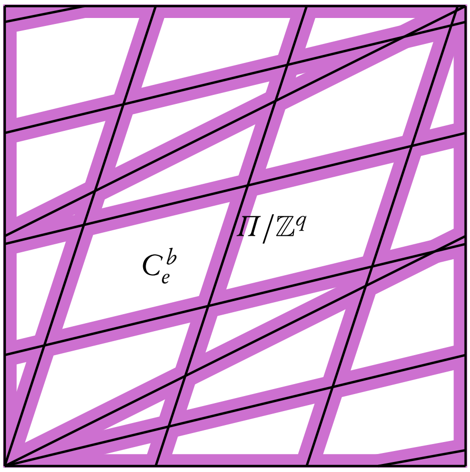

Figure 1.2. The black lines are $\Pi/\mathbb{Z}^q$. Notice that the "boundary box" is included, since $r_{ij} = \delta_{ij}$ when $i, j \le q$. The magenta area represents their e-thickening. The white area represents C_e^b, which by construction has measure $\ge 1 - b$. As long as this is positive, the prime iterates which stay uniformly away from resonances are equidistributed mod 1.

Here $\pi\colon \mathbb{N} \to \mathbb{N}$ is the prime counting function, $\pi(N) = \#\{P \text{ prime} : P \le N\}$. We are therefore finished if we can show that there exists a set $C \subset \mathbb{R}^q/\mathbb{Z}^q$ of positive measure and a $c > 0$ such that $P\vec{a} \bmod 1 \in C \implies d_{S^1}\big(\sigma(M^P), 1\big) > c$. Notice that $1 \in \sigma(M^P)$ if and only if $Pa_i = 0 \bmod 1$ for at least one $i \in \{1, \dots, l\}$. Consider the following collection of hyperplanes in $\mathbb{R}^q$:

$$\Pi_i = \left\{ x \in \mathbb{R}^q : \sum_{j=1}^{q} r_{ij} x_j = 0 \right\}, \qquad \Pi = \bigcup_{i=1}^{n} \Pi_i$$

The set Π is the union of finitely many hyperplanes defined by equations with *rational* coefficients. Therefore, its projection $\Pi/\mathbb{Z}^q \subset \mathbb{R}^q/\mathbb{Z}^q$ is a *finite* collection (see Figure 1.2) of hyperplanes such that $P\vec{a} \bmod 1 \in \Pi/\mathbb{Z}^q \iff 1 \in \sigma(U^P)$. Moreover, since $\Pi/\mathbb{Z}^q$ is a proper closed subset of zero measure, for any $0 < b < 1$ there exists an $e > 0$ such that

$$C_e^b = (\mathbb{R}^q/\mathbb{Z}^q) \setminus \bigcup_{x \in \Pi/\mathbb{Z}^q} B_e(x)$$

has measure $\left|C_e^b\right| \ge 1 - b$. Clearly $e \to 0$ as $b \to 0$, so we can choose b small enough so that $e < 1/2$. By construction

$$P\vec{a} \bmod 1 \in C_e^b \implies d_{\mathrm{U}(1)}\big(\sigma(U^P), 1\big) > 2\pi e = c$$

This finishes the proof: by equidistribution, since $|C_e^b| \ge 1 - b > 0$, there is an increasing sequence $(p_j)_{j\geq 1}$ of prime numbers such that $p_j\vec{a} \bmod 1 \in C_e^b$ for every j, and moreover from (1.1) their cumulative distribution function satisfies (see Lemma B.1):

$$\#\left\{j \in \mathbb{N} : p_j \leq N\right\} = \left|C_e^b\right| \pi(N) + o(\pi)$$

Therefore our sequence $(p_j)_{j\in\mathbb{N}}$ also satisfies the prime number theorem, since it is distributed like the primes up to an irrelevant multiplicative constant (see Lemma B.2). This implies the estimate on the gaps. □

Remark Fast-forwarding one moment, call a symplectic matrix *non-degenerate* when it does not have 1 as an eigenvalue (see Definition 1.11 for the relevance of this concept). What is shown here is that given a non-degenerate symplectic matrix, there exists a wealth of iterates which not only are non-degenerate themselves, but stay uniformly far from having the 1 in the spectrum. We call these iterates the *uniformly non-resonant iterations* of the matrix.

1.2. The Conley-Zehnder index

In this section we study the spectrum of a symplectic matrix and use its properties to construct a homotopy invariant of paths $M : [0,1] \to \mathrm{Sp}(2n)$ with $M(0) = \mathbb{I}$, called the Conley-Zehnder index. The rough idea is to track how many times the eigenvalues on the unit circle wind around as $t \in [0,1]$. Complications arise because the spectrum of a symplectic matrix is quite symmetric, and because conjugate eigenvalues might collide and leave the unit circle. To surmount them, one labels the eigenvalues with their *Krein signature*.

For the Krein theory, we follow the beautiful book of Ekeland [14]. For the definition of the Conley-Zehnder index and its properties we follow sometimes Abbondandolo [3] and sometimes other sources, like [12, 26].

The origins of the Conley-Zehnder index can be traced to before the times of Conley and Zehnder themselves, and lie in the work of Gel'fand and Lidskiĭ, [19], on the theory of strong stability in linear Hamiltonian systems. From this point of view, the Conley-Zehnder index counts the instants at which the linear system leaves the region of strong stability. This should be compared to the interpretation of the Morse index of a geodesic in terms of the count of conjugate points along it. A good source for this theory is the book by Yakubovich and Starzhinskiĭ, [42], and also the aforementioned book by Ekeland.

1.2.1. Rudiments of Krein theory

Recall from Definition 1.5 the inner product g on $\mathbb{C}^{2n}$ defined by $g = h \circ [\mathbb{I} \times (-iJ_0)]$ where $h(u,v) = \bar{u}^T v$ is the standard Hermitian product. Let M be a g-unitary matrix. For $\lambda \in \sigma(M)$, denote by E_λ the *generalized* eigenspace of λ, i.e. the space of vectors in $\mathbb{C}^{2n}$ annihilated by some power of $M - \lambda\mathbb{I}$.

Lemma 1.2.1 *Let M be a g-unitary matrix. If $\lambda, \mu \in \sigma(M)$ are such that $\overline{\lambda}\mu \neq 1$, then E_λ and E_μ are g-orthogonal.*

Proof. Let $u \in E_\lambda$ and $v \in E_\mu$ be non-zero vectors. Then there exist a, b such that

$$(M-\lambda\mathbb{I})^a\, u = 0, \quad (M-\mu\mathbb{I})^b\, v = 0.$$

Set $m = a+b$ and argue by induction on m. For $m=2$, u and v are eigenvectors and we can directly calculate

$$0 = g\,(Mu-\lambda u, Mv-\mu v) = (1-\overline{\lambda}\mu)g(u,v)$$

so since $\overline{\lambda}\mu \neq 1$ we must conclude $g(u,v)=0$.

Now assume the result holds for all $2 \le a+b \le N-1$. Set $m=N$ and define $u' = (M-\lambda\mathbb{I})u$, $v' = (M-\mu\mathbb{I})v$. Notice that

$$(M-\lambda\mathbb{I})^{a-1}\, u' = 0, \quad (M-\mu\mathbb{I})^{b-1}\, v' = 0.$$

By the induction hypothesis, we conclude that

$$g\left(u', v\right) = g\left(u, v'\right) = g\left(u', v'\right) = 0.$$

Substituting the definition of u' and v' in these three equations, we get the three equations

$$g(Mu,v) = \overline{\lambda}g(u,v), \quad g(u,Mv) = \mu g(u,v)$$
$$g(Mu,Mv) - \overline{\lambda}g(u,Mv) - \mu g(Mu,v) + \overline{\lambda}\mu g(u,v) = 0$$

Combining them together we obtain $(1-\overline{\lambda}\mu)g(u,v)=0$, which leads to the conclusion. □

By taking $\lambda=\mu$ in this lemma, we obtain

Corollary 1.2.1 *If $\lambda \in \sigma(M)$ and $|\lambda| \neq 1$, then E_λ is g-isotropic.*

These results motivate the following notation.

Definition 1.6 Let M be a g-unitary matrix. Set:

$$\begin{cases} F_\lambda = E_\lambda \oplus E_{\overline{\lambda^{-1}}}, & \text{if } |\lambda| \neq 1 \\ F_\lambda = E_\lambda, & \text{if } |\lambda| = 1 \end{cases}$$

Then we have $\lambda \neq \mu \implies F_\lambda \perp_g F_\mu$ and hence a g-orthogonal splitting

$$\mathbb{C}^{2n} = \bigoplus_{\lambda\in\sigma(M),\ |\lambda|\le 1} F_\lambda$$

Definition 1.7 Let $\lambda \in \sigma(M)$ be an eigenvalue of a g-unitary matrix M, and $F_\lambda \subset \mathbb{C}^{2n}$ the spaces entering the g-orthogonal splitting as above. The *Krein signature* $(\varkappa_\lambda, \bar{\varkappa}_\lambda)$ of λ is the signature of $g|_{F_\lambda}$. If $g|_{F_\lambda}$ is positive, resp. negative definite, then we say that λ is positive, resp. negative *Krein-definite*. Otherwise we say λ is Krein-indefinite.

Recall that an eigenvalue is said to be *semi-simple* when its algebraic multiplicity coincides with its geometric multiplicity. This means that every irreducible invariant subspace of its generalized eigenspace is 1-dimensional. In other words, the generalized eigenspace splits into genuine eigenspaces.

Lemma 1.2.2 *Let $\lambda \in \sigma(M)$ be an eigenvalue of a g-unitary matrix M.*

1. *If $\lambda \in \mathrm{U}(1)$ is not semi-simple, then there is a g-isotropic vector in E_λ.*
2. *If $|\lambda| \neq 1$ then $\varkappa_\lambda = \bar{\varkappa}_\lambda$.*

In particular a Krein-definite eigenvalue is always unitary and semi-simple, and all other eigenvalues are Krein-indefinite.

Proof. 1. Since the eigenvalue λ is not semi-simple, there is an irreducible invariant subspace in E_λ which has dimension at least 2. Therefore there exists a vector $u \in E_\lambda$ and an eigenvector v of eigenvalue λ such that $Mu = \lambda u + v$. Now

$$g(u,v) = g(Mu, Mv) = |\lambda|^2 g(u,v) + \lambda g(v,v) \implies g(v,v) = 0.$$

2. g is non-degenerate on F_λ, $F_\lambda = E_\lambda \oplus E_{\overline{\lambda^{-1}}}$ and both factors are isotropic subspaces of (complex) dimension equal to the geometric multiplicity of λ.

The last statement is the contrapositive of the first. □

Remark A good intuitive picture to have is that a non-semi-simple eigenvalue λ counts as $\varkappa_\lambda + \bar{\varkappa}_\lambda$ eigenvalues, $\varkappa_\lambda$ of which are positive Krein-definite, and $\bar{\varkappa}_\lambda$ of which are negative Krein-definite.

Let's focus now on the relevant case of a symplectic matrix, i.e. g-unitary and real. Then we can say a little bit more about the Krein signature of eigenvalues, and we can say something about the eigenvalues ± 1.

Lemma 1.2.3 *Let $M \in \mathrm{Sp}(2n)$ be a symplectic matrix. If $\{\lambda, \bar{\lambda}\} \subset \sigma(M)$, then the Krein signature of $\bar{\lambda}$ is opposite to the Krein signature of λ: $\left(\varkappa_{\bar{\lambda}}, \bar{\varkappa}_{\bar{\lambda}}\right) = \left(\bar{\varkappa}_\lambda, \varkappa_\lambda\right)$. In particular, if $\pm 1 \in \sigma(M)$, then they are always Krein-indefinite eigenvalues, with $\varkappa_{\pm 1} = \bar{\varkappa}_{\pm 1}$.*

Proof. Indeed $E_{\bar{\lambda}} = \overline{E_\lambda}$ implies that every $w \in E_{\bar{\lambda}}$ is of the form $w = \bar{v}$ for an unique $v \in E_\lambda$. Hence

$$g(w,w) = h\left(\bar{v}, \overline{iJ_0}\bar{v}\right) = \overline{h(v, iJ_0 v)} = \overline{-g(v,v)} = -g(v,v)$$

for all $w = \bar{v} \in E_\lambda$. □

Remark We now have a complete picture of the Krein signature of the eigenvalues of a symplectic matrix. The only Krein-definite eigenvalues are semi-simple eigenvalues on $\mathrm{U}(1) \setminus \{\pm 1\}$, all the other eigenvalues are Krein-indefinite. Moreover, if $\lambda \in \sigma(M) \cap \mathrm{U}(1)$ is positive Krein-definite, then $\bar{\lambda} \in \sigma(M) \cap \mathrm{U}(1)$ is negative Krein-definite.

1.2.2. Eigenvalues of the first kind

Definition 1.8 Let $M \in \mathrm{Sp}(2n)$ and $\lambda \in \sigma(M)$. We say that λ is an *eigenvalue of the first kind* if $|\lambda| < 1$, or $|\lambda| = 1$ and λ is positive Krein-definite.

Remark The eigenvalues of the "second kind" are the unitary, negative Krein-definite eigenvalues. We shall not need this terminology in the following, but kept it to conform to the existing literature.

Recall that if $\lambda \in \sigma(M)$ is positive Krein-definite, then it must be semi-simple and unitary. Moreover $\lambda^{-1} = \bar{\lambda}$ is then negative Krein-definite. Therefore the spectrum of a symplectic matrix $M \in \mathrm{Sp}(2n)$ can always written as

$$\sigma(M) = \left\{\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}\right\}$$

where $\lambda_1, \ldots, \lambda_n$ are of the first kind and perhaps repeated according to their (algebraic) multiplicity.

A very important property of the spectrum of a symplectic matrix is that the collection of eigenvalues of the first kind depends continuously on the matrix. The proof of this proposition can be found in [42, p. 191].

Proposition 1.2.1 *Let the permutation group $\mathfrak{S}_n$ act on $\mathbb{C}^n$ by permutation of coordinates. The map $\mathrm{Sp}(2n) \to \mathbb{C}^n/\mathfrak{S}_n$ associating to any $M \in \mathrm{Sp}(2n)$ the unordered n-tuple of its eigenvalues of the first kind, repeated with multiplicity, is continuous.*

Corollary 1.2.2 *Let $M\colon [0,1] \to \mathrm{Sp}(2n)$ be a continuous, resp. smooth path of symplectic matrices. There exist continuous, resp. smooth paths $\lambda_1, \ldots, \lambda_n\colon [0,1] \to \mathbb{C}$ such that $\lambda_1(t), \ldots, \lambda_n(t)$ are the eigenvalues of the first kind of $M(t)$, repeated with multiplicity, and the map $\Lambda\colon [0,1] \to \mathbb{C}^n/\mathfrak{S}_n$ given by $t \mapsto [\lambda_1(t), \ldots, \lambda_n(t)]$ is continuous.*

Remark Unless we give a definition of what smoothness means for maps into a singular space, we can't really say that Λ is smooth.

1.2.3. Weighted complex determinant

Definition 1.9 The *weighted complex determinant* $\mathrm{wdet}_{\mathbb{C}}\colon \mathrm{Sp}(2n) \to \mathrm{U}(1)$ of a symplectic matrix is defined as

$$\mathrm{wdet}_{\mathbb{C}}(M) = \prod_{\substack{\lambda \in \sigma(M), \\ \lambda\ 1^{\text{st}}\ \text{kind}}} \frac{\lambda}{|\lambda|} = \prod_{\lambda \in \sigma(M),\ |\lambda| \le 1} \left(\frac{\lambda}{|\lambda|}\right)^{\varkappa_\lambda}$$

Remark The function $\mathrm{wdet}_{\mathbb{C}}$ was introduced by Gel'fand and Lidskiĭ in [19]. In the literature $\mathrm{wdet}_{\mathbb{C}}$ is sometimes called *rotation function*, and sometimes not named at all. We propose the name "weighted complex determinant" since $\mathrm{wdet}_{\mathbb{C}}$ coincides with $\det_{\mathbb{C}}$ on $\mathrm{U}(n)$, as we shall see below.

Lemma 1.2.4 *We can rewrite* $\mathrm{wdet}_{\mathbb{C}}$ *as*

$$\mathrm{wdet}_{\mathbb{C}}(M) = (-1)^m \prod_{\lambda \in \sigma(M) \cap \mathrm{U}(1) \setminus \{\pm 1\}} \lambda^{\varkappa_\lambda}$$

where $2m$ *is the total geometric multiplicity of the negative real eigenvalues of* M.

Proof. Indeed, if $|\lambda| < 1$, $\lambda \notin \mathbb{R}$, then $\bar{\lambda} \neq \lambda$. By Lemma 1.2.2.2 and Lemma 1.2.3 the Krein signature of $\bar{\lambda}$ is the same of λ, so we see the following two terms in the product:

$$\left(\frac{\lambda}{|\lambda|}\right)^{\varkappa_\lambda} \left(\frac{\bar{\lambda}}{|\lambda|}\right)^{\varkappa_{\bar{\lambda}}} = \left(\frac{\lambda\bar{\lambda}}{|\lambda|^2}\right)^{\varkappa_\lambda} = 1.$$

If $\lambda \in \mathbb{R}$, $0 < \lambda \leq 1$, then its corresponding term is equal to 1. Finally if instead $-1 \leq \lambda < 0$, its term contributes with a $(-1)^{\varkappa_\lambda}$. Now again by Lemma 1.2.2.2 and Lemma 1.2.3.2, its Krein signature is of the form $(\varkappa_\lambda, \varkappa_\lambda)$, and so the geometric multiplicity of λ is even. Therefore the total geometric multiplicity of the negative real eigenvalues is always even. To multiply out these contributions one has to calculate

$$\sum_{\lambda \in \sigma(M),\ -1 \leq \lambda < 0} \varkappa_\lambda = \frac{1}{2} \sum_{\lambda \in \sigma(M),\ -1 \leq \lambda < 0} \dim F_\lambda = m$$

which is our claim. □

The following proposition collects the properties of the weighted complex determinant.

Proposition 1.2.2 *The weighted complex determinant has the following properties:*

1. $\mathrm{wdet}_{\mathbb{C}}$ *is continuous.*
2. $\mathrm{wdet}_{\mathbb{C}}$ *is a linear symplectic invariant:* $\mathrm{wdet}_{\mathbb{C}}(L^{-1}ML) = \mathrm{wdet}_{\mathbb{C}}(M)$ *for all* $L, M \in \mathrm{Sp}(2n)$.
3. $\mathrm{wdet}_{\mathbb{C}} U = \det_{\mathbb{C}} U$ *for all* $U \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$.
4. *If* M *has no eigenvalues on* $\mathrm{U}(1)$, *then* $\mathrm{wdet}_{\mathbb{C}}(M) = \pm 1$.
5. $\mathrm{wdet}_{\mathbb{C}}(M' \oplus M'') = \mathrm{wdet}_{\mathbb{C}}(M')\mathrm{wdet}_{\mathbb{C}}(M)$.
6. $\mathrm{wdet}_{\mathbb{C}}(M^k) = \mathrm{wdet}_{\mathbb{C}}(M)^k$ *for all* $M \in \mathrm{Sp}(2n)$ *and all* $k \in \mathbb{Z}$.

Proof.
1. This follows from continuity of the unordered set of eigenvalues of the first kind.
2. The spectrum and the Krein signatures are g-unitary invariants.
3. The spectrum of a matrix in $U \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ is always on $\mathrm{U}(1)$ and each eigenvalue is semi-simple. Therefore each eigenvalue is Krein-definite, and

$$\mathrm{wdet}_{\mathbb{C}}(U) = \prod_{\lambda \in \sigma(U)} \lambda^{\varkappa_\lambda}$$

Let $V^{\pm} \subset \mathbb{C}^{2n}$ be the eigenspaces of J_0 corresponding to the eigenvalues $\pm i$. Notice that $g|_{V^+}$ is positive definite. Notice further that if $U^{\mathbb{C}} \in \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ denotes the unitary matrix corresponding to U thought of as acting on $\mathbb{C}^n$, then $U|_{V^+} = U^{\mathbb{C}}$. Since $J_0 U = U J_0$, all the E_λ for $\lambda \in \sigma(U)$ are J_0-invariant, and

$$V^+ = \bigoplus_{\lambda \in \sigma(U)} E_\lambda \cap V^+$$

Since g is positive definite on V^+, $\varkappa_\lambda = \dim E_\lambda \cap V^+$. Hence

$$\det_{\mathbb{C}} U = \det U^{\mathbb{C}} = \prod_{\lambda \in \sigma(U)} \lambda^{\dim E_\lambda \cap V^+} = \prod_{\lambda \in \sigma(U)} \lambda^{\varkappa_\lambda} = \operatorname{wdet}_{\mathbb{C}} U$$

4. M has no Krein definite eigenvalue, so $\operatorname{wdet}_{\mathbb{C}} M = (-1)^m$ where $2m$ is the total multiplicity of the negative real eigenvalues of M.

The remaining points are found by direct calculation. □

Remark To define the Conley-Zehnder index, one may work directly with the complex determinant. But it does not have the nice property 6 found above, which is very useful when thinking about the index of iterations of periodic orbits.

An interesting property that is implied by the point 3 above is the following

Lemma 1.2.5 *Let $L\colon [0,1] \to \mathrm{Sp}(2n)$ be a continuous loop of symplectic matrices based at $\mathbb{I}$. Then*

$$\deg \operatorname{wdet}_{\mathbb{C}} \circ L = \operatorname{Mas}(L)$$

1.2.4. Non-degenerate symplectic matrices

Definition 1.10 The *Maslov cycle* is the subset

$$\mathrm{Sp}^0(2n) = \{M \in \mathrm{Sp}(2n) : \det(M - \mathbb{I}) = 0\}$$

Its complement is called the set of *non-degenerate symplectic matrices*, and is denoted by

$$\mathrm{Sp}^*(2n) = \{M \in \mathrm{Sp}(2n) : \det(M - \mathbb{I}) \neq 0\}$$

Notice that

$$\mathrm{Sp}^*(2n) = \mathrm{Sp}^+(2n) \sqcup \mathrm{Sp}^-(2n), \quad \mathrm{Sp}^\pm(2n) = \{M \in \mathrm{Sp}^*(2n) : \pm\det(M - \mathbb{I}) > 0\}.$$

Lemma 1.2.6 *A matrix $M \in \mathrm{Sp}^*(2n)$ lies in $\mathrm{Sp}^+(2n)$ if and only if the total multiplicity of real, positive eigenvalues smaller than 1 is even.*

Lemma 1.2.7 *The connected components of $\mathrm{Sp}^*(2n)$ are precisely $\mathrm{Sp}^\pm(2n)$.*

The proof of the two lemmata above is done via the study of *normal forms* of symplectic matrices, which would make us stray away from the aim of this chapter, namely the definition of the Conley-Zehnder index. A complete proof may be found in any of these references: [3, 26, 31].

Sketch of proof of 1.2.7. It is convenient to fix a distinguished element in each $\mathrm{Sp}^{\pm}(2n)$. We take:

$$W^- = \begin{pmatrix} 2 & 0 & \\ 0 & \frac{1}{2} & \\ & & -\mathbb{I} \end{pmatrix} \in \mathrm{Sp}^-(2n), \quad W^+ = -\mathbb{I} \in \mathrm{Sp}^+(2n).$$

Any symplectic matrix can be perturbed in an arbitrarily small way to a symplectic matrix all whose eigenvalues are distinct, complex and non-unitary. If the resulting matrix lies in $\mathrm{Sp}^+(2n)$, it can be connected to a matrix all whose eigenvalues are equal to -1. The resulting matrix always has a real logarithm, so can be connected to the matrix W^+. For $\mathrm{Sp}^-(2n)$ the argument is similar, but involves more intricate perturbation. □

We chose the two matrices $W^{\pm}$ because it is very easy to calculate their weighted complex determinant. Indeed, from Lemma 1.2.4 it's immediate to calculate

$$\mathrm{wdet}_{\mathbb{C}} W^- = (-1)^{n-1}, \quad \mathrm{wdet}_{\mathbb{C}} W^+ = (-1)^n$$

This fact will be important in order to define the Conley-Zehnder index by tracking the degree of $\mathrm{wdet}_{\mathbb{C}}$ along a path of symplectic matrices.

Non-degeneracy is a crucial property for the definition of the Conley-Zehnder index. The reason lies in the following lemma, which can be interpreted as a rule to define the "argument" of a symplectic matrix.

Lemma 1.2.8 *There exist continuous functions* $\theta_1, \ldots, \theta_n \colon \mathrm{Sp}^*(2n) \to [0, 2\pi]$ *with the following property. If* $\lambda_1, \ldots, \lambda_n$ *are the eigenvalues of the first kind of* $M \in \mathrm{Sp}^*(2n)$*, repeated according to their multiplicity, then*

$$\frac{\lambda_j}{|\lambda_j|} = e^{i\theta_j(M)}$$

In particular

$$\mathrm{wdet}_{\mathbb{C}} M = e^{i\sum_j \theta_j(M)}$$

Remark By re-ordering the eigenvalues of the first kind of M, we can always assume that $0 \le \theta_1(M) \le \theta_2(M) \le \cdots \le \theta_n(M) \le 2\pi$.

Proof. Let $M \in \mathrm{Sp}^*(2n)$ and $\lambda \in \sigma(M)$ be an eigenvalue of the first kind. If λ is unitary or λ is real and negative, the formula

$$\frac{\lambda}{|\lambda|} = e^{i\theta}$$

defines $\theta \in (0, 2\pi)$ uniquely (when $\lambda < 0$ obviously we're setting $\theta = \pi$). The problem is only when $\lambda > 0$, since we could set $\theta = 0$ or $\theta = 2\pi$. If $M \in \mathrm{Sp}^+(2n)$, then it has an *even* number of positive eigenvalues of the first kind, counted with multiplicity. We choose the θ_j so that the number of them which equal 0 is always the same as the number of them which equal 2π. If $M \in \mathrm{Sp}^-(M)$, we know that it has an *odd* number $2l+1$ of positive eigenvalues of the first kind, counted with multiplicity. In this case choose θ to be 0 for l of these eigenvalues, and 2π for the remaining $l+1$. With this choice, $\theta_1|_{\mathrm{Sp}^-(2n)} = 0$. This choice for the values of the θ_j in the case of positive real eigenvalues of the first kind guarantees that the θ_j are continuous functions. □

Remark
1. Notice that by construction no matrix $M \in \mathrm{Sp}^*(2n)$ can have $\theta_j(M) = 0$ for every j or $\theta_j(M) = 2\pi$ for every j.
2. The proof of the lemma shows that we can choose the θ_j such that $\theta_j(W^+) = \pi$ for all j and $\theta_1(W^-) = 0, \theta_2(W^-) = \theta_3(W^+) = \cdots = \pi$.

Lemma 1.2.9 *Any loop in* $\mathrm{Sp}^*(2n)$ *is contractible in* $\mathrm{Sp}(2n)$.

Proof. It suffices to show that for any loop $t \mapsto M_t$ in $\mathrm{Sp}^*(2n)$ we have that $t \mapsto \mathrm{wdet}_{\mathbb{C}} M_t \in \mathrm{U}(1)$ is a contractible loop. But this is clear from the existence and the first property of the functions θ_j remarked above. □

Remark The fact that every loop in $\mathrm{Sp}^*(2n)$ is contractible in $\mathrm{Sp}(2n)$ is an indicator of the fact that the Maslov index can be interpreted with some care as an intersection number of a loop with the Maslov cycle. The difficulty lies in the fact that the Maslov cycle is a singular set. Moreover, we consider almost only loops based at $\mathbb{I}$, which lies in the singular part of the Maslov cycle.

An illustration of the situation for $n = 1$ can be found in Figure 1.3.

1.2.5. Definition of the Conley-Zehnder index

In this section we use the constructions developed above to give a homotopy invariant of paths of linear symplectomorphisms.

Definition 1.11 Define

$$\mathrm{SP}(2n) = \left\{ M \in C^0([0,1], \mathrm{Sp}(2n)) : M(0) = \mathbb{I} \right\}$$

We call this the space of based paths in $\mathrm{Sp}(2n)$. Define

$$\mathrm{SP}^*(2n) = \left\{ M \in C^0([0,1], \mathrm{Sp}(2n)) : M(0) = \mathbb{I},\ M(1) \in \mathrm{Sp}^*(2n) \right\}$$

We call this the space of non-degenerate based paths in $\mathrm{Sp}(2n)$.

Remark We commit a small abuse of notation by denoting matrices and paths of matrices with the same symbols. Usually we denote time-dependence as so: $M_t = M(t)$.

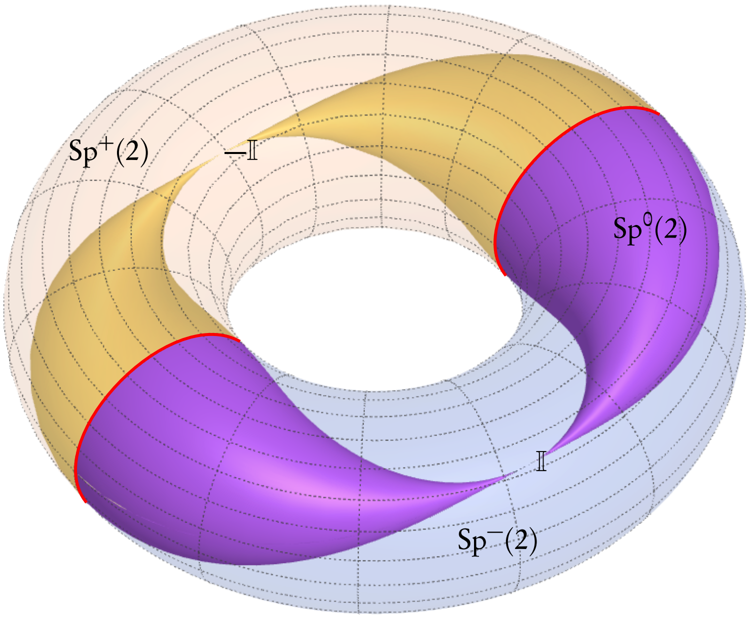

Figure 1.3. The group Sp(2) is diffeomorphic to a solid torus without its border, parametrized by coordinates $(\theta, r, \tau) \in [0, 2\pi) \times [0, 1) \times [0, 2\pi)$. Here θ is the angle which goes around the "horizontal" circle, where $(0, 0, 0) = \mathbb{I}$, while τ parametrizes the "vertical" circle. The coordinate r gives the distance of the point from the central horizontal circle, i.e. the radius at which the point lies on its corresponding slice $\theta = \text{const}$. The explicit parametrization can be found in [3]. The subgroup $\mathrm{U}(1) \subset \mathrm{Sp}(2)$ is the set $\{(\theta, 0, 0)\}$, the circle at the core of the torus passing through $\pm\mathbb{I}$. The singular surface, drawn inside with solid colors, is the set of matrices with double eigenvalue ± 1, which can be seen to be parametrized as $\{(\theta, r, \tau) : r = \sin^2 \theta\}$. This surface has two connected components, one drawn in yellow and one in purple. The yellow part is the connected component corresponding to the matrices with double eigenvalue -1, containing in particular $-\mathbb{I}$. This component lies inside $\mathrm{Sp}^+(2)$. The purple surface is the Maslov cycle $\mathrm{Sp}^0(2)$, which contains the matrix $\mathbb{I}$. The circles drawn in red are to denote where the surface touches the boundary of the solid torus. The Maslov cycle breaks $\mathrm{Sp}^*(2)$ into the two connected components $\mathrm{Sp}^*(2) = \mathrm{Sp}^+(2) \sqcup \mathrm{Sp}^-(2)$. These are depicted by the shaded areas, the yellow shaded area being $\mathrm{Sp}^+(2)$ and the blue shaded area being $\mathrm{Sp}^-(2)$. One thing that cannot be seen well in this drawing is that the Maslov cycle bounds within it a piece of $\mathrm{Sp}^+(2)$, not $\mathrm{Sp}^-(2)$. In fact $\mathrm{U}(1) \setminus \{\mathbb{I}\} \subset \mathrm{Sp}^+(2)$, and the same is true in higher dimensions. Notice that every loop in $\mathrm{Sp}^-(2)$ is contractible in $\mathrm{Sp}(2)$ but there are loops in $\mathrm{Sp}^-(2)$ which are not contractible in $\mathrm{Sp}^-(2)$. This does not happen for $\mathrm{Sp}^+(2)$.

We endow the spaces $\mathrm{SP}(2n)$, $\mathrm{SP}^*(2n)$ with the compact open topology. The connected components of $\mathrm{SP}^*(2n)$ are the homotopy classes of paths in $\mathrm{Sp}(2n)$ which are based at $\mathbb{I}$ and whose unfixed endpoint never touches $\mathrm{Sp}^0(2n)$.

Definition 1.12 For a path $M\colon[0,1]\to\mathrm{Sp}(2n)$ choose any function $\theta\colon[0,1]\to\mathbb{R}$ such that

$$\mathrm{wdet}_{\mathbb{C}} M(t)=e^{i\theta(t)}$$

Define, for $\tau\in[0,1]$,

$$\Delta(M,\tau)=\frac{\theta(\tau)-\theta(0)}{\pi} \tag{1.2}$$

This number represents *twice* the total angle swept by $t\mapsto \mathrm{wdet}_{\mathbb{C}}(M(t))$ up to time τ in units of 2π. It does not depend on the choice of θ.

Remark Since $\mathrm{wdet}_{\mathbb{C}}$ is multiplicative under direct sums, we see that $\Delta(\cdot,\tau)$ is additive under direct sums for all τ.

Now let $M\in\mathrm{Sp}^*(2n)$. Then either $M\in\mathrm{Sp}^+(2n)$ or $M\in\mathrm{Sp}^-(2n)$. Choose a continuous path $\gamma_M\colon[0,1]\to\mathrm{Sp}^*(2n)$ such that $\gamma_M(0)=M$ and $\gamma_M(1)=W^\pm$ according to whether $M\in\mathrm{Sp}^\pm(2n)$. Notice that $\Delta(\gamma_M,1)$ does not depend on the choice of γ_M, since all loops in $\mathrm{Sp}^*(2n)$ are contractible in $\mathrm{Sp}(2n)$. Therefore we can define a function

$$\mathscr{R}\colon \mathrm{Sp}^*(2n)\to\mathbb{R},\quad \mathscr{R}(M)=\Delta(\gamma_M,1)$$

Lemma 1.2.10 $|\mathscr{R}|<n$.

Proof. Indeed since $\gamma_M(t)\in\mathrm{Sp}^*(2n)$ for all t, we can set $\theta(t)=\sum_j\theta_j(\gamma_M(t))$, where the θ_j are the functions constructed in Lemma 1.2.8. Recall that we chose to order the eigenvalues of the first kind of a matrix $M\in\mathrm{Sp}^*(2n)$ in such a way that $0\le\theta_1(M)\le\theta_2(M)\le\cdots\le\theta_n(M)\le 2\pi$. We can calculate

$$\mathscr{R}(M)=\Delta(\gamma_M,1)=\frac{1}{\pi}\sum_{j=1}^{n}\theta_j(W^\pm)-\theta_j(M)$$

where as before we have chosen $W^\pm$ according to where M is. If $M\in\mathrm{Sp}^+(2n)$, then we have chosen θ_j such that $\theta_j(W^+)=\pi$ for all j. We see that

$$M\in\mathrm{Sp}^+(2n)\implies|\mathscr{R}(M)|\le\frac{1}{\pi}\sum_{j=1}^{n}\left|\pi-\theta_j(M)\right|<n$$

the strictness of the last inequality following from the fact that the numbers $\theta_j(M)$ are never all equal to 0 or all equal to 2π. If instead $M\in\mathrm{Sp}^-(2n)$, then we know that $\theta_1(W^-)=\theta_1(M)=0$, and the rest are equal to π. Therefore analogously as above,

$$M\in\mathrm{Sp}^-(2n)\implies|\mathscr{R}(M)|=\frac{1}{\pi}\left|\sum_{j=2}^{n}\pi-\theta_j(M)\right|\le n-1 \qquad\square$$

Definition 1.13 The Conley-Zehnder index of a path $M \in \mathrm{SP}^*(2n)$ is defined as

$$\mathrm{CZ}(M_t : t \in [0,1]) = \Delta(M,1) + \mathscr{R}(M(1))$$

Remark We chose to use the notation above because it is very convenient when dealing with iterates and periodic orbits. When the time domain is clear from the context, for brevity we write $\mathrm{CZ}(M_t) = \mathrm{CZ}(M_t : t \in [0,1])$.

Theorem 1 *The Conley-Zehnder index is an integer valued function. In fact,*

$$\mathrm{CZ}(M_t : t \in [0,1]) = \deg \mathrm{wdet}_{\mathbb{C}}^2 \circ M$$

Two paths $M, M' \in \mathrm{SP}^*(2n)$ *belong to the same path component of* $\mathrm{SP}^*(2n)$ *if and only if their Conley-Zehnder indices coincide.*

Proof. For notational convenience, if two paths $\gamma_0, \gamma_1 : [0,1] \to \mathrm{Sp}(2n)$ are such that $\gamma_0(1) = \gamma_1(0)$, denote by $\gamma_1 \wedge \gamma_0$ the concatenation of γ_1 after γ_0, i.e.

$$\gamma_1 \wedge \gamma_0(t) = \begin{cases} \gamma_0(2t), & t \in [0,1/2] \\ \gamma_1(2t-1), & t \in [1/2,1] \end{cases}$$

Let $M \in \mathrm{SP}^*(2n)$ and fix a $\gamma_M : [0,1] \to \mathrm{Sp}^*(2n)$ such that $\gamma_M(0) = M(1)$ and $\gamma_M(1) = W^{\pm}$ according to whether $M(1) \in \mathrm{Sp}^{\pm}(2n)$. Notice that by definition the Conley-Zehnder index is twice the total angle, in units of 2π, swept by $\mathrm{wdet}_{\mathbb{C}}$ on the concatenation of M with γ_M, or equivalently the total angle in units of 2π swept by the path

$$[0,1] \ni t \mapsto (\mathrm{wdet}_{\mathbb{C}}(\gamma_M \wedge M(t)))^2 \in \mathrm{U}(1)$$

Since $\gamma_M(1) = W^{\pm}$ and $\mathrm{wdet}_{\mathbb{C}} W^{\pm} = \pm 1$, we see that this path is actually a loop based at $1 \in \mathrm{U}(1)$, so the total angle swept by it in units of 2π is its degree.

To show that two paths in $\mathrm{SP}^*(2n)$ are in the same path component if and only if their indices coincide, notice that two paths $M, M' \in \mathrm{SP}^*(2n)$ are in the same path component if and only if they end both in $\mathrm{Sp}^+(2n)$ or both in $\mathrm{Sp}^-(2n)$ and the loop given by the concatenation

$$\xi = \overline{M'} \wedge \overline{\gamma_{M'}} \wedge \gamma_M \wedge M$$

where $\overline{M'}$ denotes the path M' traversed in reverse, is contractible. But this is equivalent to saying that the two loops $\gamma_M \wedge M, \gamma_{M'} \wedge M'$ are homotopic, which is equivalent to them having the same degree. □

Remark One could just as well take the formula for the Conley-Zehnder index in terms of the degree as a definition, and derive the formula in terms of Δ and $\mathscr{R}$ as a theorem. We chose to take this point of view because $\Delta(M,1)$ and $\mathscr{R}(M(1))$ are useful concepts also for the rest of this chapter.

Proposition 1.2.3 *The Conley-Zehnder has the following properties:*

1. *If* $M \in \mathrm{SP}^*(2n)$ *and* $N \in \mathrm{Sp}(2n)$*, then*
$$\mathrm{CZ}\left(N^{-1} M_t N\right) = \mathrm{CZ}(M_t) \tag{1.3}$$
2. *If* $M \in \mathrm{SP}^*(2n)$ *is such that* $\sigma(M_t) \cap \mathrm{U}(1) = \varnothing$ *for all* t*, then* $\mathrm{CZ}(M_t) = 0$.
3. *If* $M \in \mathrm{SP}^*(2n_0)$ *and* $M' \in \mathrm{SP}^*(2n_1)$*, then*
$$\mathrm{CZ}\left(M_t \oplus M'_t\right) = \mathrm{CZ}(M_t) + \mathrm{CZ}\left(M'_t\right) \tag{1.4}$$
4. *If* $M \in \mathrm{SP}^*(2n)$ *then*
$$\mathrm{CZ}\left(M_t^{-1}\right) = \mathrm{CZ}\left(M_t^T\right) = -\mathrm{CZ}(M)$$
5. *If* $M \in \mathrm{SP}^*(2n)$ *and* $L\colon [0,1] \to \mathrm{Sp}(2n)$ *is a loop based at* $\mathbb{I}$*, then*
$$\mathrm{CZ}(L_t M_t) = \mathrm{CZ}(M_t) + 2\,\mathrm{Mas}(L) \tag{1.5}$$
where Mas *denotes the Maslov index, see Definition 1.4.*
6. *The parity of the Conley-Zehnder index depends only on the endpoint of the path, and is determined as follows:*
$$(-1)^{n-\mathrm{CZ}(M_t)} = \operatorname{sign}\det(\mathbb{I} - M_1)$$
7. *If* $M_t = \exp(-J_0 A t)$ *with* $A \in \mathrm{Sym}(2n)$ *such that* $|A| < 2\pi$*, then*
$$\mathrm{CZ}(M_t) = \frac{1}{2}\operatorname{sign} A$$
where $\operatorname{sign} A$ *denotes the signature of* A*, i.e. the number of positive eigenvalues minus the number of negative eigenvalues of* A*.*

Proof.

1. This follows from symplectic invariance of $\mathrm{wdet}_{\mathbb{C}}$.
2. For such a path, $\mathrm{wdet}_{\mathbb{C}}(M_t) = \pm 1$ for all t.
3. This follows from the fact that $\mathrm{wdet}_{\mathbb{C}}(M_t \oplus M'_t) = \mathrm{wdet}_{\mathbb{C}}(M_t)\,\mathrm{wdet}_{\mathbb{C}}(M'_t)$ (see Proposition 1.2.2.5).
4. The first equality follows from the first point, because $M^{-1} = J_0^{-1} M^T J_0$. The last equality follows from Proposition 1.2.2.6.
5. The product $t \mapsto L_t M_t$ is homotopic to the concatenation $L \wedge M$.
$$\mathrm{CZ}(L_t M_t) = \deg \mathrm{wdet}_{\mathbb{C}}^2 \circ (L \wedge M) = 2 \deg \mathrm{wdet}_{\mathbb{C}} L + \deg \mathrm{wdet}_{\mathbb{C}}^2 \circ M$$
The last equality follows from the fact that $t \mapsto \mathrm{wdet}_{\mathbb{C}} L_t$ is already a loop. The conclusion follows from Lemma 1.2.5.
6. Let $\gamma\colon [0,1] \to \mathrm{Sp}^*(2n)$ be a path such that $\gamma(1) = W^{\pm}$. Notice that $\deg \mathrm{wdet}_{\mathbb{C}}^2 \circ \gamma$ is even whenever $\mathrm{wdet}_{\mathbb{C}}\,\gamma(1) = 1$, and odd whenever $\mathrm{wdet}_{\mathbb{C}}\,\gamma(1) = -1$. Recall that $\mathrm{wdet}_{\mathbb{C}}\, W^+ =$

$(-1)^n$ and $\operatorname{wdet}_{\mathbb{C}} W^- = (-1)^{n-1} = -\operatorname{wdet}_{\mathbb{C}} W^+$. For $\gamma = \gamma_M \wedge M$, $\gamma(1) = W^\pm \iff \operatorname{sign}\det(\mathbb{I} - M_1) = \pm 1$. Putting everything together we obtain the claim.

7. We can assume without loss of generality that $A = \operatorname{diag}(a_1, b_1, \dots, a_n, b_n)$ in a symplectic splitting $(\mathbb{R}^{2n}, \omega_0) = (\mathbb{R}^2, \omega_0) \oplus \cdots \oplus (\mathbb{R}^2, \omega_0)$. Since $|A| < 2\pi$, $\exp(-J_0 A) \in \mathrm{Sp}^*(2n)$. From points 2 and 3 above, the only blocks of $M_t = \exp(-J_0 A t)$ that contribute to the index are the blocks with eigenvalues on the unit circle. Therefore we can assume without loss of generality that $a_j b_j > 0$ for all j, i.e. they always have the same sign. This is because blocks with a_j of different sign than b_j generate hyperbolic blocks, which have eigenvalues off the unit circle at all times. Now, in the jth factor of the splitting, the linear flow is the flow of a harmonic oscillator, which can be deformed, without changing the homotopy class of the flow, to a rotation in the plane of angle strictly less than 2π. This rotation is counter-clockwise when $a_j > 0$ and clockwise when $a_j < 0$. To finish the proof, it suffices to look at Section 1.2.7 for an explicit calculation of the index of a non-degenerate rotation depending only on its definition.

□

Remark These properties characterize the Conley-Zehnder index uniquely, as shown in e.g. [26].

1.2.5.1. Extension to degenerate paths

In applications, it is useful to define a Conley-Zehnder index also for paths which end up on the Maslov cycle. This extension is not canonical, and there are a few different choices one can make, depending on the kind of properties one needs. Following [3], we think of the Conley-Zehnder index as a kind of Morse index for the critical points of the action functional. The Morse index is lower-semicontinuous (LSC), so we take the maximal LSC-extension of the Conley-Zehnder index.

Specifically, let $M \in \mathrm{SP}(2n)$ be a based path, perhaps degenerate. Notice that $\mathrm{SP}^*(2n) \subset \mathrm{SP}(2n)$ is an open dense subset. Set

$$\mathrm{CZ}(M_t : t \in [0,1]) = \liminf_{\substack{M' \to M, \\ M' \in \mathrm{SP}^*(2n)}} \mathrm{CZ}\left(M'_t : t \in [0,1]\right)$$

Since $\Delta(\cdot, 1)\colon \mathrm{SP}(2n) \to \mathbb{R}$ is already continuous,

Lemma 1.2.11 *It holds that*

$$\mathrm{CZ}(M_t : t \in [0,1]) = \Delta(M, 1) + \underline{\mathscr{R}}(M_1)$$

where $\underline{\mathscr{R}}\colon \mathrm{Sp}(2n) \to \mathbb{R}$ *is the maximal LSC-extension of* $\mathscr{R}$*:*

$$\underline{\mathscr{R}}(N) = \liminf_{\substack{N' \to N \\ N' \in \mathrm{Sp}^*(2n)}} \mathscr{R}(N')$$

The function $\underline{\mathscr{R}}$ has a similar bound as $\mathscr{R}$, which is proven in a completely analogous way:

Lemma 1.2.12 $|\underline{\mathscr{R}}| \leq n$.

Remark The possible equality is because we cannot guarantee that not all θ_j are 0 or 2π.

The index of a degenerate path has the same properties of the non-degenerate counterpart except continuity. We list the ones we need:

Proposition 1.2.4 *The lower semicontinuous extension* $\mathrm{CZ}\colon \mathrm{SP}(2n) \to \mathbb{Z}$ *of the Conley-Zehnder index satisfies the following properties:*

1. *If* $M \in \mathrm{SP}(2n)$ *and* $N \in \mathrm{Sp}(2n)$, *then*

$$\mathrm{CZ}\left(N^{-1} M_t N\right) = \mathrm{CZ}\left(M_t\right)$$

2. *If* $M \in \mathrm{SP}(2n_0)$ *and* $M' \in \mathrm{SP}(2n_1)$, *then*

$$\mathrm{CZ}\left(M_t \oplus M_t'\right) = \mathrm{CZ}\left(M_t\right) + \mathrm{CZ}\left(M_t'\right)$$

3. *If* $M \in \mathrm{SP}(M)$ *then*

$$\mathrm{CZ}\left(M_t^{-1}\right) = \mathrm{CZ}\left(M_t^T\right) = -\mathrm{CZ}\left(M_t\right)$$

4. *If* $M \in \mathrm{SP}(2n)$ *and* $L\colon [0,1] \to \mathrm{Sp}(2n)$ *is a loop based at* 1, *then*

$$\mathrm{CZ}\left(L_t M_t\right) = \mathrm{CZ}(M_t) + 2\,\mathrm{Mas}(L_t)$$

   In particular, if L is a loop based at $\mathbb{I}$, $\mathrm{CZ}(L_t) = 2\,\mathrm{Mas}(L_t)$.

1.2.6. Iterations and the mean Conley-Zehnder index

In the study of the Conley-Zehnder index of periodic orbits, it is important to understand the index of an iterated orbit. Here we study the linear theory.

Let $M \in \mathrm{SP}^*(2n)$. We extend it periodically to a map $M\colon \mathbb{R} \to \mathrm{Sp}(2n)$ by setting $M_{t+1} = M_t M_1$. In particular, $M_k = M_1^k$. The reason for this choice is that this rule is fulfilled by the matrizants of linear Hamiltonian equations with 1-periodic coefficients.

We need some preliminaries. Let $M \in \mathrm{SP}^*(2n)$ be extended to $\mathbb{R}$ as just explained, and pick a function $\theta\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{wdet}_{\mathbb{C}} M(t) = e^{i\theta(t)}$$

For $\tau \in \mathbb{R}$, define, analogously as in (1.2),

$$\Delta(M, \tau) = \frac{\theta(\tau) - \theta(0)}{\pi}$$

Lemma 1.2.13 *For $M \in \mathrm{SP}^*(2n)$ extended periodically and $k \in \mathbb{Z}$,*

$$\Delta(M,k) = k\Delta(M,1)$$

Proof. Consider the following homotopy

$$\Gamma : [0,1] \times [0,1] \to \mathrm{Sp}(2n), \quad \Gamma(s,t) = M((1-s)t)$$

Notice that $\Gamma(0,t) = M_t$ for all t and $\Gamma(1,t) = \mathbb{I}$ for all t. Therefore Γ is a homotopy between M and the constant path at $\mathbb{I}$. Extend Γ periodically to a function

$$\Gamma : [0,1] \times \mathbb{R} \to \mathrm{Sp}(2n), \quad \Gamma(s,t+1) = \Gamma(s,t)\Gamma(s,1)$$

In particular, whenever $k \in \mathbb{Z}$ we have $\Gamma(s,k) = \Gamma(s,1)^k$. Let $\theta : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a choice of a continuous function such that

$$\mathrm{wdet}_{\mathbb{C}}\, \Gamma(s,t) = e^{i\theta(s,t)} \quad \forall (s,t), \quad \theta(s,0) = 0 \quad \forall s$$

Notice that

$$\Delta(M,k) - k\Delta(M,1) = \frac{1}{\pi}[\theta(0,k) - \theta(0,0) - k\theta(0,1) + k\theta(0,0)] = \frac{1}{\pi}[\theta(0,k) - k\theta(0,1)]$$

Now,

$$\mathrm{wdet}_{\mathbb{C}}\, \Gamma(s,k) = \mathrm{wdet}_{\mathbb{C}}\left(\Gamma(s,1)^k\right) = (\mathrm{wdet}_{\mathbb{C}}\, \Gamma(s,1))^k \quad \forall k \in \mathbb{Z} \quad \forall s \in [0,1]$$

This implies that for each $s \in [0,1]$, $\theta(s,k) - k\theta(s,1)$ is an integer multiple of 2π. Since θ is continuous in s, $\theta(s,k) - k\theta(s,1)$ must therefore be constant in s. Since $\theta(0,t) = 0$ for all t the claim is proven. □

Definition 1.14 Let $k \in \mathbb{Z}$. Define

$$\mathrm{CZ}(M_t : t \in [0,k]) = \mathrm{CZ}(M_{kt} : t \in [0,1]), \quad \overline{\mathrm{CZ}}(M_t : t \in [0,1]) = \lim_{k \to \infty} \frac{1}{k} \mathrm{CZ}(M_t : t \in [0,k])$$

The quantity $\overline{\mathrm{CZ}}(M_t : t \in [0,1])$ is called *mean Conley-Zehnder index*.

Remark Notice that if $\sigma(M_1)$ contains a kth root of unity, then M_k is degenerate, so $\mathrm{CZ}(M_t : t \in [0,k])$ has to be interpreted with the LSC extension explained above.

Lemma 1.2.14 *The mean Conley-Zehnder index is well defined and*

$$\overline{\mathrm{CZ}}(M_t : t \in [0,1]) = \Delta(M,1)$$

Proof. Lemma 1.2.13 together with Lemma 1.2.12 implies that $\mathrm{CZ}(M_t : t \in [0,k])$ has linear growth in k. Indeed

$$\lim_{k\to\infty} \frac{\mathrm{CZ}(M_t : t\in[0,k])}{k} = \lim_{k\to\infty} \frac{\Delta(M,k)+\underline{\mathscr{R}}\left(M_1^k\right)}{k} = \lim_{k\to\infty} \frac{k\Delta(M,1)}{k} + \underbrace{\frac{\underline{\mathscr{R}}\left(M_1^k\right)}{k}}_{|\cdot|\leq\frac{n}{k}\to 0} = \Delta(M,1)$$

□

Proposition 1.2.5 *For every $k\in\mathbb{Z}$ we have the estimate*

$$k\overline{\mathrm{CZ}}(M_t)-n \leq \mathrm{CZ}(M_t : t\in[0,k]) \leq k\overline{\mathrm{CZ}}(M_t)+n$$

The inequalities are strict whenever $M_k = M_1^k \in \mathrm{Sp}^*(2n)$.

Proof. Notice first of all that Lemma 1.2.13 together with Lemma 1.2.14 implies that

$$\overline{\mathrm{CZ}}(M_t : t\in[0,k]) = k\overline{\mathrm{CZ}}(M_t : t\in[0,1])$$

Now, notice that

$$\mathrm{CZ}(M_t : t\in[0,k]) - k\overline{\mathrm{CZ}}(M_t : t\in[0,1]) = \Delta(M,k)+\underline{\mathscr{R}}(M_k) - k\Delta(M,1) = \underline{\mathscr{R}}(M_k)$$

The claim follows from Lemma 1.2.12. □

Remark Looking at the proof of Lemma 1.2.10, we see that the inequality is strict for iterates with at least one eigenvalue different from 1. This kind of paths are called *weakly non-degenerate* after the work of Salamon and Zehnder [38].

The following proposition will be crucial in the proof of the main theorem of the thesis.

Proposition 1.2.6 *Assume that $k,l\in\mathbb{Z}$ are such that M_1^k and M_1^l are non-degenerate. If k and l have the same parity, then* $\mathrm{CZ}(M_t : t\in[0,k])$ *and* $\mathrm{CZ}(M_t : t\in[0,l])$ *have the same parity.*

Proof. By Proposition 1.2.3, point (6), we see that it suffices to show that M_1^l and M_1^k belong to the same connected component of $\mathrm{Sp}^*(2n)$, that is,

$$M_1^l \in \mathrm{Sp}^{\pm}(2n) \iff M_1^k \in \mathrm{Sp}^{\pm}(2n) \quad \forall k,l \text{ odd}$$

By induction it is sufficient to prove this for $k=l+2$, i.e. $M_1^k = M_1^l M_1^2$. Recall further from Lemma 1.2.6 that $N\in\mathrm{Sp}^+(2n)$ if and only if the total multiplicity of real positive eigenvalues of N smaller than 1 is even, and $N\in\mathrm{Sp}^-(2n)$ if and only if the total multiplicity of real positive eigenvalues of N smaller than 1 is odd. The claim is thus reduced to showing that the total multiplicity of real positive eigenvalues of M_1^l smaller than 1 cannot change when multiplying it on the right by M_1^2. But this is obvious, because it will change the total multiplicity of the eigenvalues by an even number. □

Remark This result is also true for iterations k, l which are degenerate, but we will not really need it in the proof of the main theorem of the thesis.

1.2.7. The Conley-Zehnder index of a non-degenerate unitary path

The proof of the main theorem of the thesis involves some index calculations in the special case where the paths in analysis are autonomous, unitary and non-degenerate. Therefore, to conclude the chapter, we give a general formula for the index of an autonomous unitary and non-degenerate path.

Let $U\colon [0,1] \to \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ be an autonomous non-degenerate unitary path. Then by definition $U_t = \exp(\Theta t)$ for some matrix $\Theta \in \mathfrak{u}(n) \subset \mathfrak{sp}(2n)$. Recall that $\Theta \in \mathfrak{u}(n) \subset \mathfrak{sp}(2n)$ if and only if $J_0\Theta = \Theta J_0$ and moreover $\Theta^T = -\Theta$. These properties imply that there exists a symplectic splitting $(\mathbb{R}^{2n}, \omega_0) = (\mathbb{R}^2, \omega_0) \oplus \cdots \oplus (\mathbb{R}^2, \omega_0)$ and real numbers $\alpha_1, \ldots, \alpha_n$ such that in this splitting

$$\Theta = \begin{pmatrix} 0 & -\alpha_1 & & & \\ \alpha_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\alpha_n \\ & & & \alpha_n & 0 \end{pmatrix} \tag{1.6}$$

This is easily seen by unitarily diagonalizing the corresponding Hermitian matrix.

Remark It is worth to stress that $\alpha_j \in \mathbb{R}$, so when reaching time 1 the flow generated by Θ may wind around many times and also in both directions.

We found that an autonomous path of unitary matrices can be unitarily diagonalized for all times and split into rotations in symplectic planes. Therefore by symplectic invariance (1.3) the sum formula (1.4) it suffices to calculate the Conley-Zehnder index of a rotation on $\mathbb{R}^2$. The same is true for the mean Conley-Zehnder index, since $\Delta(\cdot, 1)$ is also additive under direct sums.

For a rotation of the plane, non-degeneracy means that the rotation angle is not equal to an integer multiple of 2π. So let $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and consider

$$U_t = \exp\begin{pmatrix} 0 & -\alpha t \\ \alpha t & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{pmatrix} \tag{1.7}$$

The spectrum of U_t is

$$\sigma(U_t) = \left\{ e^{i\alpha t}, e^{-i\alpha t} \right\}$$

We have to determine which one is the positive Krein-definite eigenvalue. The eigenspaces in $\mathbb{C}^2$ of the eigenvalues $\lambda_t = e^{i\alpha t}$, $\bar{\lambda}_t = e^{-i\alpha t}$ are respectively

$$E_{\lambda_t} = \left\langle \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle, \quad E_{\bar{\lambda}_t} = \left\langle \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\rangle$$

which don't depend on t. It's immediate to see that λ_t is always positive Krein-definite eigenvalue, while $\bar{\lambda}_t$ is always negative Krein-definite. This does not depend on the sign of α, and confirms that $\mathrm{wdet}_{\mathbb{C}}\, U_t = \det_{\mathbb{C}} U_t = e^{i\alpha t}$.

We can immediately calculate the mean Conley-Zehnder index:

Lemma 1.2.15 *Let $U_t = \exp \Theta t$ with $\Theta = J_0 \operatorname{diag}(\alpha_1,\dots,\alpha_n,\alpha_1,\dots,\alpha_n)$. Then*

$$\overline{\mathrm{CZ}}(U_t : t \in [0,1]) = \frac{1}{\pi}\sum_{j=1}^{n} \alpha_j$$

Proof. As just explained, the situation is reduced to a rotation of the plane by additivity. For U_t as in (1.7), we can choose $\theta(t) = \alpha t$ in the definition of the function $\Delta(U,\cdot)$. In particular

$$\Delta(U,1) = \frac{\theta(1) - \theta(0)}{\pi} = \frac{\alpha}{\pi} = \overline{\mathrm{CZ}}(U_t : t \in [0,1])$$

the last equality being Lemma 1.2.14. □

Remark This calculation is valid even if some $\alpha_j \in 2\pi\mathbb{Z}$.

The fact that $U \in \mathrm{SP}^*(2)$ implies that U_1 has no real positive eigenvalues. In fact it has no real eigenvalues at all, unless $\alpha = (2k+1)\pi$ for some $k \in \mathbb{Z}$, which implies $U_1 = -\mathbb{I}$. Therefore $U_1 \in \mathrm{Sp}^+(2)$ always. Compare this with figure 1.3.

Define $[\![\cdot]\!] : \mathbb{R} \to \mathbb{Z}$, $r \mapsto [\![r]\!]$ to be the identity on $\mathbb{Z}$ and the nearest odd integer on $\mathbb{R}\setminus\mathbb{Z}$.

Lemma 1.2.16 *For $U = U_t$ a non-degenerate rotation of the plane of angle $\alpha \in \mathbb{R}\setminus 2\pi\mathbb{Z}$,*

$$\mathrm{CZ}(U_t : t \in [0,1]) = \left[\!\!\left[\frac{\alpha}{\pi} \right]\!\!\right] \tag{1.8}$$

Therefore if $U_t = \exp\Theta t$ for Θ as in (1.6), *we have*

$$\mathrm{CZ}(U_t : t \in [0,k]) = \sum_{j=1}^{n} \left[\!\!\left[\frac{k\alpha_j}{\pi} \right]\!\!\right] \quad \forall k \in \mathbb{Z}$$

Proof. The only thing to prove is the formula (1.8). Since $U_1 \in \mathrm{Sp}^+(2)$ for any α, to calculate CZ we have to connect U_1 to $W^+ = -\mathbb{I}$ without passing through $\mathbb{I}$. If $\alpha = (2k+1)\pi$ then $U_1 = -\mathbb{I}$ and $\mathrm{CZ}(U_t) = \Delta(U,1) = 2k+1 = [\![\alpha/\pi]\!]$. If $\alpha \notin \pi\mathbb{Z}$ then to connect U_1 to $-\mathbb{I}$ without passing through $\mathbb{I}$ we proceed as follows. There is always some $k \in \mathbb{Z}$ such that $\alpha \in ((2k-1)\pi,(2k+1)\pi)$. If $\alpha \in ((2k-1)\pi, 2k\pi)$ then γ_U is the paths of rotations in the opposite direction until we reach $-\mathbb{I}$, and so $\mathrm{CZ}(U_t) = 2k-1 = [\![\alpha/\pi]\!]$. If $\alpha \in (2k\pi,(2k+1)\pi)$ then γ_U is the path of rotations in the same direction until we reach $-\mathbb{I}$, and so $\mathrm{CZ}(U_t) = 2k+1 = [\![\alpha/\pi]\!]$. The claim is proven. □

Remark Notice that we've also just shown that in dimension 2, the Conley-Zehnder index of a non-degenerate path of rotations is always odd, and clearly every odd integer is reached as the Conley-Zehnder index of a non-degenerate path of rotations. This is not true in higher dimensions: we can obtain any integer as the Conley-Zehnder index of an autonomous unitary non-degenerate path.

2. Asymptotically linear Hamiltonian systems

In this chapter we introduce the class of asymptotically linear Hamiltonian systems. We show that each Hamiltonian system in such class has the property that for any fixed period T there exists a compact set which contains all the T-periodic orbits of the system. On the way, we analyze linear Hamiltonian systems and obtain a crucial inversion formula for the linear operator representing a linear Hamiltonian system.

2.1. Hamiltonian systems on linear phase space

Consider $\mathbb{R}^{2n}$ with coordinates $z = (q, p)$, $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n)$. We equip $\mathbb{R}^{2n}$ with the standard euclidean product $\langle \cdot, \cdot \rangle \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by $\langle z_0, z_1 \rangle = z_0^T z_1$ and an identification with $\mathbb{C}^n$ by sending $z = (q, p)$ to $q + ip \in \mathbb{C}^n$. Notice that multiplication by $i \in \mathbb{C}$ on $\mathbb{C}^n$ is represented by the linear map $J_0 \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by the matrix

$$J_0 = \begin{pmatrix} \mathbb{O}_n & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}$$

in the splitting $\mathbb{R}^{2n} = \mathbb{R}^n_q \oplus \mathbb{R}^n_p$.

The standard symplectic form on $\mathbb{R}^{2n}$ is the 2-form $\omega_0 \in \Omega^2\left(\mathbb{R}^{2n}\right)$,

$$\omega_0 = \sum_i dq_i \wedge dp_i$$

This is a symplectic form in the sense that $d\omega_0 = 0$ and $\omega^{\wedge n} \neq 0$. As a bilinear form, it is represented by the matrix $-J_0 = \Omega_0$:

$$\omega_0(u, v) = \langle u, -J_0 v \rangle = -u^T J_0 v \quad \forall u, v \in \mathbb{R}^{2n} \tag{2.1}$$

Any smooth function $H \in C^\infty(\mathbb{R}^{2n})$ defines a vector field $X_H \in \mathscr{X}(\mathbb{R}^{2n})$, called its *Hamiltonian vector field*, by the equation

$$dH = i_{X_H} \omega_0$$

where $i_{X_H}\omega_0 \in \Omega^1(\mathbb{R}^{2n})$ is the 1-form $v \mapsto i_{X_H}\omega_0(v) = \omega_0(X_H, v)$. A *Hamiltonian system* is the ODE defined by a Hamiltonian vector field, and the flow of a Hamiltonian vector field is called a *Hamiltonian flow*. Usually we denote the flow of a Hamiltonian vector field φ_{X_H} by φ_H since X_H is uniquely defined by H up to addition of a constant to H. The flow at any fixed time defines a diffeomorphism which we call an (autonomous) *Hamiltonian diffeomorphism*.

We can enlarge the class of systems under study by considering smooth, *non-autonomous* Hamiltonians $H \in C^\infty\left([0,T]\times\mathbb{R}^{2n}\right)$ and defining the non-autonomous Hamiltonian vector field by the same equations:

$$dH_t = i_{X_{H_t}}\omega_0$$

where $H_t(z) = H(t,z)$. Using the representation (2.1) we find that

$$X_H(t,z) = -J_0\nabla H_t(z)$$

where ∇ is the gradient with respect to z only.

2.1.1. On non-autonomous differential equations and subharmonics

Recall that a non-autonomous vector field $X\colon [0,T]\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$, $X = X_t(z)$, defines a *non-autonomous* flow, that is a map $(s,t,z)\mapsto \varphi_X^{s,t}(z)$ defined by the ODE

$$\begin{cases} \frac{d}{dt}\varphi_X^{s,t}(z) = X_t\left(\varphi_X^{s,t}(z)\right), & t \geq s \\ \varphi_X^{s,s}(z) = z \end{cases}$$

In general, the non-autonomous flow might not be defined everywhere. Using uniqueness of solutions of Cauchy problems, the non-autonomous flow is seen to satisfy the following properties:

$$\varphi_X^{t,t}(z) = z, \quad \varphi_X^{s,t+t'} = \varphi_X^{t,t'}\circ\varphi_X^{s,t},$$

i.e. the flow depends on both the initial and final times, and not only on the elapsed time.

There is no reason to expect that a general non-autonomous flow admits periodic orbits. We therefore restrict our study to non-autonomous systems with periodic coefficients. To be clear, there is still no reason to expect that a non-autonomous but periodic differential equation admits periodic solutions in general, but such an assumption helps. This is because if a non-autonomous vector field, say $X = X_t(z)$, is periodic in time with period T, i.e. $X_{t+T} = X_t$ for all $t\in\mathbb{R}$, then its non-autonomous flow satisfies

$$\varphi_X^{s,t+T} = \varphi_X^{s,t}$$

by uniqueness of solutions of differential equations. Therefore also the non-autonomous flow is somewhat periodic and might define periodic solutions. In particular, we have a well defined map $\varphi_X^T = \varphi_X^{0,T}$. The T-periodic orbits correspond precisely to the fixed points of φ_X^T, and moreover for every $k\in\mathbb{Z}$

$$\left(\varphi_X^T\right)^k = \varphi_X^{0,T}\circ\cdots\circ\varphi_X^{0,T} = \varphi_X^{(k-1)T,kT}\circ\cdots\circ\varphi_X^{T,2T}\circ\varphi_X^{0,T} = \varphi_X^{0,kT}$$

so even the kT-periodic orbits are fixed points of the k-th iteration of φ_X^T. This is the reason that in older literature, higher-period periodic orbits are also sometimes called "subharmonics".

We will only consider non-autonomous Hamiltonian systems with periodic coefficients, namely, the Hamiltonians will always have a periodic dependence on time. We will see them as smooth functions $H \in C^\infty(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^{2n})$. Notice that if H is such a Hamiltonian, then $TH(Tt, z)$ is a 1-periodic Hamiltonian whose flow is just a re-parametrized version of the flow of H. So without loss of generality we can always take $T = 1$ and consider smooth Hamiltonians $H \in C^\infty(S^1 \times \mathbb{R}^{2n})$, where here and below $S^1 = \mathbb{R}/\mathbb{Z}$. We write $\varphi_H^1 = \varphi_{X_H}^{0,1}$.

Remark We make a distinction between $S^1 = \mathbb{R}/\mathbb{Z}$, the unit *length* circle, and $\mathrm{U}(1) \subset \mathbb{C}$ the unit *radius* circle.

2.2. Asymptotically quadratic Hamiltonians

The class of all Hamiltonian systems on $\mathbb{R}^{2n}$ is too wild to study, for example, "most" Hamiltonians don't generate a complete flow on $\mathbb{R}^{2n}$. Moreover, it is quite easy to come up with Hamiltonian systems on $\mathbb{R}^{2n}$ without any periodic orbits, like a constant non-zero vector field, which is generated by a linear Hamiltonian.

A simple class of Hamiltonian systems, which we have secretly already studied in the previous chapter and will pick up again in Section 2.3.1, is given by non-autonomous linear Hamiltonian systems. These are the Hamiltonian systems defined by non-autonomous linear Hamiltonian vector fields, and hence non-autonomous quadratic Hamiltonians, which we take to be with 1-periodic coefficients, as explained above. The question of periodic orbits for linear Hamiltonian systems is reduced to the study of the spectrum of the linear map representing the time-1 flow of the linear Hamiltonian vector field.

A more interesting and much less understood class of Hamiltonian systems is obtained by deforming the linear Hamiltonian systems with sub-quadratic terms.

Definition 2.1 Consider a smooth function $H\colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$.

1. We say that H is *weakly asymptotically quadratic* if there exists a smooth loop of symmetric matrices $A \in C^\infty(S^1, \mathrm{Sym}(2n))$ such that

$$\sup_{t\in S^1} |\nabla H_t(z) - A_t z| = o(|z|) \quad \text{as } |z| \to \infty$$

We refer to the quadratic form $Q_t(z) = \frac{1}{2}\langle A_t z, z\rangle$ as the *quadratic Hamiltonian at infinity*. Usually we denote by

$$h_t(z) = H_t(z) - Q_t(z)$$

the non-quadratic part of H.

Let $H = Q + h$ be a weakly asymptotically quadratic Hamiltonian.

2. We say that the quadratic Hamiltonian at infinity Q is *non-degenerate* if the spectrum of φ_Q^1 does not contain the eigenvalue 1.

3. We say that H is *asymptotically quadratic* if h is bounded:

$$||h||_{L^\infty(S^1\times\mathbb{R}^{2n})} < \infty$$

It is convenient to fix the following notation.

$$\mathfrak{wH} = \left\{ H \in C^\infty\left(S^1 \times \mathbb{R}^{2n}\right) : \begin{array}{c}\text{weakly asympototically quadratic Hamiltonian}\\ \text{with non-degenerate quadratic Hamiltonian at infinity.}\end{array}\right\}$$
$$\mathfrak{H} = \left\{ H \in C^\infty\left(S^1 \times \mathbb{R}^{2n}\right) : \begin{array}{c}\text{asympototically quadratic Hamiltonian}\\ \text{with non-degenerate quadratic Hamiltonian at infinity.}\end{array}\right\}$$

Remark The condition of being weakly asymptotically quadratic is well defined, since if $H = Q + h = Q' + h'$ with both ∇h, $\nabla h'$ of sublinear growth, then $h' = Q - Q' + h$, so

$$\left|\nabla h'_t(z)\right| \geq \left|\left(A_t - A'_t\right) z\right| - |\nabla h_t(z)|$$

If $A \neq A'$ then the first always grows linearly as $|z| \to \infty$. So for $\nabla h'$ to grow sublinearly, it must hold $A' = A$, and therefore also $h = h'$. A fortiori also asymptotically quadratic Hamiltonians are well defined.

Definition 2.2 If $\varphi \in \mathrm{Ham}(\mathbb{R}^{2n})$ is a Hamiltonian diffeomorphism such that $\varphi = \varphi_H^1$ for H some (weakly) asymptotically quadratic Hamiltonian, we say φ is a *(weakly) asymptotically linear Hamiltonian diffeomorphism*. The linear symplectomorphism φ_Q^1 is called the *linear map at infinity*.

Lemma 2.2.1 *The linear map at infinity of a weakly asymptotically linear Hamiltonian diffeomorphism does not depend on the chosen generating weakly asymptotically quadratic Hamiltonian.*

Proof. Assume $\varphi = \varphi_H^1 = \varphi_{H'}^1$ with $H = Q + h$ and $H' = Q' + h'$. Fix $0 \leq \tau \leq 1$ and $z \in \mathbb{R}^{2n}$. Let's estimate:

$$\begin{aligned}\left|\varphi_H^\tau(z) - \varphi_Q^\tau(z)\right| &= \left|\int_0^\tau X_H\left(\varphi_H^t(z)\right) - X_Q\left(\varphi_Q^t(z)\right) dt\right| = \left|\int_0^\tau \nabla H_t\left(\varphi_H^t(z)\right) - A_t \cdot \varphi_Q^t(z) dt\right| \leq \\ &\leq \int_0^\tau \left|\nabla H_t\left(\varphi_H^t(z)\right) - A_t \cdot \varphi_H^t(z)\right| dt + ||A||_{L^\infty} \int_0^\tau \left|\varphi_H^\tau(z) - \varphi_Q^\tau(z)\right| dt = \\ &= \int_0^\tau \left|\nabla h_t\left(\varphi_H^t(z)\right)\right| dt + ||A||_{L^\infty} \int_0^\tau \left|\varphi_H^\tau(z) - \varphi_Q^\tau(z)\right| dt\end{aligned} \tag{2.2}$$

Now, since ∇h_t is $o(|z|)$, for any $\varepsilon > 0$ there exists an $M_\varepsilon > 0$ such that if $|\zeta| > M_\varepsilon$ then $|\nabla h_t(\zeta)| < \varepsilon|\zeta|$. Therefore define

$$I_\varepsilon = \{t \in [0, \tau] : |\varphi_H^t(z)| > M_\varepsilon\}, \quad J_\varepsilon = \{t \in [0, \tau] : |\varphi_H^t(z)| \leq M_\varepsilon\}$$

and we can estimate

$$\int_0^\tau |\nabla h_t(\varphi_H^t(z))|\,dt = \int_{I_\varepsilon} |\nabla h_t(\varphi_H^t(z))|\,dt + \int_{J_\varepsilon} |\nabla h_t(\varphi_H^t(z))|\,dt \leq$$
$$\leq \varepsilon \int_0^\tau |\varphi_H^t(z)|\,dt + \tau \max_{t\in S^1,\ |\zeta|\leq M_\varepsilon} |\nabla h_t(\zeta)| \leq$$
$$\leq \varepsilon \left(\int_0^\tau \left|\varphi_H^t(z) - \varphi_Q^t(z)\right| dt + \int_0^\tau \left|\varphi_Q^t(z)\right| dt \right) + \tau C_\varepsilon \leq$$
$$\leq \varepsilon \int_0^\tau \left|\varphi_H^t(z) - \varphi_Q^t(z)\right| dt + \varepsilon \cdot B^{-1}(e^{B\tau} - 1)|z| + \tau C_\varepsilon$$

where $B = ||A||_{L^\infty}$, $C_\varepsilon = \max_{t\in S^1,\ |\zeta|\leq M_\varepsilon} |\nabla h_t(\zeta)|$. Combining with (2.2) we get

$$\left|\varphi_H^\tau(z) - \varphi_Q^\tau(z)\right| \leq (B+\varepsilon) \int_0^\tau \left|\varphi_H^t(z) - \varphi_Q^t(z)\right| dt + \varepsilon \cdot e^{B\tau}|z| + \tau C_\varepsilon$$

Dividing by $|z|$ both sides and then using Grönwall's lemma, we obtain that for every $\varepsilon > 0$ there exists an $M_\varepsilon > 0$ such that

$$\frac{\left|\varphi_H^1(z) - \varphi_Q^1(z)\right|}{|z|} \leq \left(\varepsilon \frac{e^B - 1}{B} + \frac{C_\varepsilon}{|z|} \right) e^{B+\varepsilon} \xrightarrow[|z|\to\infty]{\varepsilon = |z|^{-1}} 0$$

This is because, taking $\varepsilon = |z|^{-1}$, either M_ε stays bounded as $|z| \to \infty$, or it goes to infinity. In the second case, $C_\varepsilon = \max_{t\in S^1,\ |\zeta|\leq M_\varepsilon} |\nabla h_t(\zeta)|$ either stays bounded, or runs off to infinity, but by definition, slower than $|z|$. In the first case the same conclusion is reached even more easily. Therefore in any case

$$\left|\varphi_H^1(z) - \varphi_Q^1(z)\right| = o(|z|) \quad \text{as} \quad |z| \to \infty$$

The exact same estimates hold for the other flows, so $\left|\varphi_{H'}^1(z) - \varphi_{Q'}^1(z)\right| = o(|z|)$ as $|z| \to \infty$. Therefore

$$\left|\varphi_Q^1(z) - \varphi_{Q'}^1(z)\right| \leq \left|\varphi_H^1(z) - \varphi_Q^1(z)\right| + \left|\varphi_{H'}^1(z) - \varphi_{Q'}^1(z)\right| + \left|\varphi_H^1(z) - \varphi_{H'}^1(z)\right| =$$
$$= \left|\varphi_H^1(z) - \varphi_Q^1(z)\right| + \left|\varphi_{H'}^1(z) - \varphi_{Q'}^1(z)\right| = o(|z|) \quad \text{as} \quad |z| \to \infty$$

But since both φ_Q^1 and $\varphi_{Q'}^1$ are linear maps, this is possible if and only if $\varphi_Q^1 = \varphi_{Q'}^1$. □

2.2.1. Index of a fixed point and index at infinity

Given a fixed point of a Hamiltonian diffeomorphism and a generating Hamiltonian, we can talk about the (mean) index of the fixed point with respect to the given Hamiltonian:

Definition 2.3 Let $H \in C^{\infty}(S^1 \times \mathbb{R}^{2n})$ be any Hamiltonian and $z_0 \in \operatorname{Fix} \varphi_H^1$. We denote

$$\mathrm{CZ}(z_0, H) = \mathrm{CZ}\big(d\varphi_H^t(z_0) : t \in [0,1]\big), \quad \overline{\mathrm{CZ}}(z_0, H) = \lim_{j \to \infty} \frac{\mathrm{CZ}\big(d\varphi_H^t(z_0) : t \in [0,j]\big)}{j}$$

In the specific case of asymptotically linear Hamiltonian diffeomorphism, the choice of generating Hamiltonian also induces a choice of linear isotopy connecting the linear map at infinity with the identity. With this isotopy we can give a notion of index at infinity:

Definition 2.4 Given H weakly asymptotically quadratic, define the *index at infinity* $\operatorname{ind}_\infty(H)$ and the *mean index at infinity* $\overline{\operatorname{ind}_\infty}(H)$ of H as

$$\operatorname{ind}_\infty(H) = \mathrm{CZ}(0, Q) = \mathrm{CZ}\Big(\varphi_Q^t : t \in [0,1]\Big), \quad \overline{\operatorname{ind}_\infty}(H) = \overline{\mathrm{CZ}}(0, Q)$$

Both the index of a fixed point and the index at infinity depend on the chosen generating Hamiltonian. Namely, the change of index formula (1.5) implies immediately the following

Lemma 2.2.2 *Let $F, G \in \mathfrak{wH}$ be weakly asymptotically quadratic Hamiltonians such that*

$$\varphi_F^1 = \varphi_G^1 = \varphi.$$

1. *If $z_0 \in \operatorname{Fix} \varphi$, then*

$$\mathrm{CZ}(z_0, F) = \mathrm{CZ}(z_0, G) + 2\mu$$

where $\mu \in \mathbb{Z}$ is the Maslov index of the loop $\Lambda \colon S^1 \to \mathrm{Sp}(2n)$ given by

$$\Lambda(t) = d\varphi_F^t(z_0)\big[d\varphi_G^t(z_0)\big]^{-1}.$$

2. *Writing $F = P + f$ and $G = Q + g$, we have*

$$\operatorname{ind}_\infty F = \operatorname{ind}_\infty G + 2\mu'$$

where $\mu' \in \mathbb{Z}$ is the Maslov index of the loop $\Lambda' \colon S^1 \to \mathrm{Sp}(2n)$ given by

$$\Lambda'(t) = \varphi_P^t \circ \big[\varphi_Q^t\big]^{-1}.$$

2.3. Dynamical properties of systems non-degenerate at infinity

The non-degeneracy at infinity of a weakly asymptotically linear Hamiltonian system has important consequences on the corresponding dynamics. Namely, for every fixed period, there exists a compact set containing all periodic orbits of that period. This is the minimal required amount of compactness needed for the analysis of the problem of periodic orbits.

2.3.1. (Invertible) linear Hamiltonian systems

We start with the linear theory, where one may describe the periodic orbits completely. In particular, one may show that linear Hamiltonian systems which are non-degenerate have only one 1-periodic orbit, which is the stationary solution at the origin. The consequences of this will be of central importance for the construction of Floer homology.

Let $Q_t(z) = \frac{1}{2}\langle A_t z, z\rangle$ be a quadratic Hamiltonian. Its Hamiltonian vector field

$$X_Q(t,z) = -J_0 A_t z$$

is linear. Therefore, its flow is by linear symplectomorphisms.

Define the *matrizant*, or *fundamental solution*, of the linear ODE associated to Q to be the path of symplectic matrices $M\colon [0,1] \to \mathrm{Sp}(2n)$ defined by the matrix ODE

$$\begin{cases} \dot{M}_t = -J_0 A_t M_t \\ M_0 = \mathbb{I} \end{cases} \tag{2.3}$$

The name "fundamental solution" comes from the fact that a curve $[0,1] \ni t \mapsto z_t$ solves the linear ODE defined by X_Q with initial datum z_0 if and only if $z_t = M_t z_0$.

The proof of the next lemma follows [14, III, Proposition 2].

Lemma 2.3.1 *The operator*

$$\Lambda_A\colon W^{1,2}(S^1,\mathbb{R}^{2n}) \subset L^2(S^1,\mathbb{R}^{2n}) \to L^2(S^1,\mathbb{R}^{2n}), \quad \Lambda_A \xi = J_0\dot{\xi} - A_t\xi$$

is a self-adjoint closed operator. Its image $\operatorname{im}\Lambda_A \subset L^2$ *is closed in* L^2 *and its kernel* $\ker\Lambda_A$ *has dimension at most* $2n$*. Moreover, its kernel is the* L^2*-orthogonal complement to its image:*

$$L^2 = \ker\Lambda_A \oplus \operatorname{im}\Lambda_A$$

Finally, the double restriction

$$\Lambda_A\colon W^{1,2} \cap \operatorname{im}\Lambda_A \to \operatorname{im}\Lambda_A$$

is an isomorphism with compact inverse.

Proof. It is immediate to check that $\langle \Lambda_A\xi, \eta\rangle_{L^2} = \langle \xi, \Lambda_A\eta\rangle_{L^2}$, hence $\Lambda_A^* = \Lambda_A$. From this it follows that Λ_A^* has dense domain in L^2, hence also that Λ_A is closed, as $\Lambda_A^{**} = -\Lambda_A^* = \Lambda_A$.

Invertibility of Λ_A is equivalent to saying that the equation $J_0\dot{\xi} - A_t\xi = \eta$ has an unique solution for any $\eta \in L^2(S^1,\mathbb{R}^{2n})$. The variations of constants formula immediately gives us the proposed solution in terms of the fundamental solution $t \mapsto M_t$ of the linear ODE, i.e. the solution to the equation (2.3):

$$\xi(t) = M_t\xi(0) - \int_0^t M_\tau^{-1} J_0 \eta(\tau)\, d\tau \tag{2.4}$$

Clearly $\xi \in W^{1,2}\big([0,1],\mathbb{R}^{2n}\big)$ and $\Lambda_A\xi = \eta$. Now, we must impose $\xi(0)=\xi(1)$. Calculating

$$\xi(1)=M_1\xi(0)-\int_0^1 M_\tau^{-1}J_0\eta(\tau)d\tau = M_1\xi(0)+\bar{\eta}$$

where $\bar{\eta} = -\int_0^1 M_\tau^{-1}J_0\eta(\tau)d\tau \in \mathbb{R}^{2n}$ is just a vector uniquely determined by A and η. We obtain the relation

$$(M_1-\mathbb{I})\xi(0)=\bar{\eta}$$

If $1\notin\sigma(M_1)$ then there is an unique solution to this equation, and thus Λ_A is invertible. If $1\in\sigma(M_1)$, then we found that

$$\int_0^1 M_\tau^{-1}J_0\eta(\tau)d\tau \in \operatorname{im}(M_1-\mathbb{I})\subset\mathbb{R}^{2n} \tag{2.5}$$

This characterizes the image of Λ_A, and shows that its codimension in L^2 is at most $2n$.

Since Λ_A is self-adjoint and densely defined, we have

$$\ker\Lambda_A = \ker\Lambda_A^* = (\operatorname{im}\Lambda_A)^{\perp_{L^2}}.$$

which shows that $L^2 = \ker\Lambda_A\oplus\operatorname{im}\Lambda_A$. From this and the closed graph theorem we obtain that $\Lambda_A\colon W^{1,2}\cap\operatorname{im}\Lambda_A\to\operatorname{im}\Lambda_A$ is an isomorphism. As $W^{1,2}\big(S^1,\mathbb{R}^{2n}\big)\subset C^0\big(S^1,\mathbb{R}^{2n}\big)$, the map $W^{1,2}\ni\xi\mapsto M_1\xi(0)\in\mathbb{R}^{2n}$ is continuous. We conclude compactness of the inverse from formula (2.4). □

We are interested in the consequences of this Lemma to the linear Hamilton equations.

Corollary 2.3.1 *Let $Q_t(z)=\frac{1}{2}\langle A_t z,z\rangle$ be a quadratic Hamiltonian, and define the linear operator*

$$D_A\colon W^{1,2}\big(S^1,\mathbb{R}^{2n}\big)\to L^2\big(S^1,\mathbb{R}^{2n}\big),\quad D_A\xi=\dot{\xi}+J_0A_t\xi.$$

The operator D_A is a Fredholm index zero operator. D_A is invertible and only if Q is a non-degenerate quadratic Hamiltonian. The inverse $D_A^{-1}\colon L^2\big(S^1,\mathbb{R}^{2n}\big)\to W^{1,2}\big(S^1,\mathbb{R}^{2n}\big)$ is given by

$$D_A^{-1}\eta(t)=M_t\left[(\mathbb{I}-M(1))^{-1}\int_0^1 M(\tau)^{-1}\eta(\tau)d\tau+\int_0^t M(\tau)^{-1}\eta(\tau)d\tau\right] \tag{2.6}$$

Proof. Notice that $D_A=-J_0\Lambda_A$. Since $\ker\Lambda_A$ is L^2-orthogonal to $\operatorname{im}\Lambda_A$, Λ_A is a Fredholm index 0 operator. Hence the same is true for D_A. The invertibility of D_A is equivalent to the invertibility of Λ_A, which by the previous Lemma we found to be equivalent to non-degeneracy of the quadratic Hamiltonian Q, and the formula for the inverse is just an application of the variation of constants formula. □

2.3.2. Periodic orbits of systems non-degenerate at infinity

Recall the classical variational set-up for the periodic orbit problem in Hamiltonian systems on $\mathbb{R}^{2n}$. Let $H \in C^\infty(S^1 \times \mathbb{R}^{2n})$. We consider the Hilbert space $L^2\left(S^1, \mathbb{R}^{2n}\right) = L^2$ and its dense subspace $W^{1,2}\left(S^1, \mathbb{R}^{2n}\right) = W^{1,2}$. Define the action functional on $W^{1,2}$ by

$$\mathscr{A}_H(x) = \int_{S^1} \left[\frac{1}{2}\langle J_0 \dot{x}, x\rangle - H_t \circ x\right] dt$$

Proposition 2.3.1 *The differential of the action functional is given by*

$$d\mathscr{A}_H|_x \xi = \int_{S^1} \omega_0(\dot{x} - X_H \circ x, \xi)$$

Therefore the critical points of the action functional $\mathscr{A}_H$ on $W^{1,2}$ are precisely the 1-periodic orbits of H.

Remark In principle, the critical points of $\mathscr{A}_H$ on $W^{1,2}$ are *weak* solutions of the Hamilton's equations. But it is not difficult to bootstrap from $W^{1,2}$ to smooth (actual) solutions, given the smoothness of the Hamiltonian vector field, see e.g. [29].

Proof. We can compute the (Fréchet) differential in terms of the (Gateaux) directional derivative as follows:

$$\begin{aligned} d\mathscr{A}_H|_x \xi &= \frac{d}{d\sigma} \mathscr{A}_H(x + \sigma\xi)\Big|_{\sigma=0} = \\ &= \frac{d}{d\sigma} \int_0^1 \frac{1}{2}\left\langle J_0\left(\dot{x} + \sigma\dot{\xi}\right), x + \sigma\xi\right\rangle - H_t(x + \sigma\xi)\, dt\Big|_{\sigma=0} = \\ &= \int_0^1 \frac{1}{2}\left[\langle J_0\dot{x}, \xi\rangle + \left\langle J_0\dot{\xi}, x\right\rangle\right] - \langle \nabla H_t(x), \xi\rangle\, dt \end{aligned}$$

Now, notice that

$$\left\langle J_0\dot{\xi}, x\right\rangle = \frac{d}{dt}\langle J_0\xi, x\rangle + \langle J_0\dot{x}, \xi\rangle$$

Moreover,

$$\int_0^1 \frac{d}{dt}\langle J_0\xi, x\rangle\, dt = \langle J_0\xi(1), x(1)\rangle - \langle J_0\xi(0), x(0)\rangle = 0$$

Putting these facts together,

$$\begin{aligned} d\mathscr{A}_H|_x \xi &= \int_0^1 \langle J_0\dot{x}, \xi\rangle - \langle \nabla H_t(x), \xi\rangle\, dt = \int_0^1 \langle J_0\dot{x} - \nabla H_t \circ x, \xi\rangle\, dt = \\ &= \int_{S^1} \omega_0(\dot{x} - X_H(t,x), \xi)\, dt \end{aligned}$$

as claimed. In the last equality we used the identity

$$\omega_0(u,v) = \langle J_0 u, v\rangle$$

i.e. the fact that $-J_0$ is the Gram matrix of ω_0. That the critical points are 1-periodic orbits now follows from non-degeneracy of ω_0 as a bilinear form. □

Borrowing the terminology of Floer [16], we define the "unregularized gradient" of $\mathscr{A}_H$ by the identity

$$\langle \nabla_{L^2}\mathscr{A}_H(x), \xi\rangle_{L^2} = d\mathscr{A}_H|_x \xi \quad \forall \xi \in W^{1,2}$$

Therefore, the previous proposition shows that

$$\nabla_{L^2}\mathscr{A}_H(x) = J_0\dot{x} - \nabla H_t \circ x$$

Remark The specific choice of the complex structure made here, namely the use of J_0, is not essential. Given any compatible almost-complex structure, the same formula holds *mutatis mutandis.*

Lemma 2.3.2 *Let H be a weakly asymptotically quadratic Hamiltonian with A its associated path of symmetric matrices. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|\nabla H \circ x - Ax\|_{L^2} \leq \varepsilon \|x\|_{L^2} + \delta \quad \forall x \in W^{1,2} \tag{2.7}$$

Proof. Since H is weakly asymptotically quadratic, $\nabla H_t(z) - A_t z = o(|z|)$ as $|z| \to \infty$. Namely for every $\varepsilon > 0$ there exists an $M_\varepsilon > 0$ such that if $|z| > M_\varepsilon$ then $|\nabla H_t(z) - A_t z| < \varepsilon |z|$. Therefore

$$\begin{aligned}\|\nabla H \circ x - Ax\|_{L^2}^2 = &\int_{S^1} |\nabla H_t(x_t) - A_t x_t|^2 \, dt = \\ &= \int_{\{t:|x_t|<M_\varepsilon\}} |\nabla H_t(x_t) - A_t x_t|^2 \, dt + \int_{\{t:|x_t|>M_\varepsilon\}} |\nabla H_t(x_t) - A_t x_t|^2 \, dt \leq \\ &\leq \max_{(t,z)\in S^1\times B_{M_\varepsilon}(0)} |\nabla H_t(z) - A_t z|^2 + \varepsilon^2 \|x\|_{L^2}^2\end{aligned}$$

Set $\max_{(t,z)\in S^1\times B_{M_\varepsilon}(0)} |\nabla H_t(z) - A_t z|^2 = \delta^2$. We obtain

$$\|\nabla H \circ x - Ax\|_{L^2} \leq \varepsilon \|x\|_{L^2} + \delta$$

as claimed. □

The non-degeneracy condition at infinity is equivalent to the fact that the Hamiltonian system approaches an invertible linear Hamiltonian system at infinity. This translates into the following crucial property of the action functional, which can be seen as a sort of properness of the unregularized gradient. Here we follow Conley and Zehnder's [12].

Lemma 2.3.3 *Let $H \in \mathfrak{wH}$. There exist constants $\nu, \delta > 0$ such that*

$$||\nabla_{L^2}\mathscr{A}_H(x)||_{L^2} = ||\dot{x} - X_H \circ x||_{L^2} \geq \frac{\nu}{2}||x||_{L^2} - \delta \tag{2.8}$$

Proof. Let $A\colon S^1 \to \mathrm{Sym}\, 2n$ be the corresponding loop of symmetric matrices defining the linear system at infinity. With a small abuse of notation denote by J_0A the operator on L^2 given by multiplication by J_0A: $x \mapsto J_0Ax(t) = J_0A_tx(t)$. Since the quadratic Hamiltonian at infinity of H is non-degenerate, the operator $D_A = \frac{d}{dt} + J_0A\colon W^{1,2} \to L^2$ is invertible. Set $\left\|D_A^{-1}\right\|_{\mathrm{op}} = \nu^{-1}$. We obtain the first estimate

$$||D_Ax||_{L^2} \geq \nu||x||_{L^2} \tag{2.9}$$

Now notice that $\dot{x} - X_H \circ x - J_0D_Ax = -\nabla H_t \circ x + Ax$. Therefore

$$\begin{aligned} ||\dot{x} - X_H \circ x||_{L^2} &= ||J_0D_Ax - (J_0D_Ax - \dot{x} + X_H \circ x)||_{L^2} \geq \\ &\geq ||D_Ax||_{L^2} - ||\nabla H \circ x - Ax||_{L^2} \end{aligned} \tag{2.10}$$

Now we use Lemma 2.3.2 with $\varepsilon = \frac{\nu}{2}$, and combine the estimate (2.7) with (2.9) and (2.10) to get

$$||\dot{x} - X_H \circ x||_{L^2} \geq \frac{\nu}{2}||x||_{L^2} - \delta$$

as claimed. Notice that δ depends only on ν and h, and ν depends only on the behaviour of the linear system at infinity. □

The main consequence of the above result is that the 1-periodic orbits of H are contained in a fixed compact set.

Lemma 2.3.4 *Let $H \in \mathfrak{wH}$. There exists a constant $R > 0$ such that if $x\colon S^1 \to \mathbb{R}^{2n}$ is a 1-periodic orbit of H then*

$$||x||_{L^\infty} \leq R$$

Proof. From the estimate (2.8) applied to a 1-periodic orbit $x \in W^{1,2}$

$$||x||_{L^2} \leq \frac{2\delta}{\nu} = C_0$$

Let $A\colon S^1 \to \mathrm{Sym}\, 2n$ be the loop of symmetric matrices defining the linear system at infinity. Similarly as the previous lemma, notice that

$$D_Ax = J_0(A_tx - \nabla H_t \circ x)$$

From Lemma 2.3.2 with $\varepsilon = 1$ we obtain

$$||D_Ax||_{L^2} = ||\nabla H \circ x - Ax||_{L^2} \leq ||x||_{L^2} + C_1$$

where $C_1 > 0$ depends only on H. Now, recall that non-resonance at infinity is equivalent to $D_A\colon W^{1,2} \to L^2$ being an invertible operator. Moreover any 1-periodic (weak) solution is a bounded function, since $W^{1,2}(S^1, \mathbb{R}^{2n})$ compactly embeds in $L^\infty(S^1, \mathbb{R}^{2n})$ by the Sobolev inequalities. Therefore we get

$$\begin{aligned} \|x\|_{L^\infty} &\le C_2 \|x\|_{W^{1,2}} \le C_2 \left\| D_A^{-1} \right\|_{\text{op}} \|D_A x\|_{L^2} \le \\ &\le C_3 (\|x\|_{L^2} + C_1) \le C_3(C_0 + C_1) =: R \end{aligned}$$

where R now depends only on H. □

Remark Up to this point, we don't know that an asymptotically linear Hamiltonian system non-degenerate at infinity admits at least one periodic orbit. We shall show this using Floer theory, but this was already known to Amann, Conley and Zehnder [4, 5, 12].

2.4. Non-degenerate Hamiltonians

In this last section, we investigate a special class of Hamiltonians, those which have only non-degenerate 1-periodic orbits. They will become important for the Fredholm theory of the Floer equation. The aim of the section is to show that they are dense (in fact residual) in the set of all Hamiltonians.

Definition 2.5 Let H be a smooth Hamiltonian. We say that a 1-periodic orbit of H is *non-degenerate* if the linearization of the flow along it is given by a non-degenerate path of symplectic matrices. If all orbits of H are non-degenerate, we say that H is non-degenerate. It is convenient to fix the following notation:

$$\begin{aligned} \mathfrak{w}\mathfrak{H}_* &= \{H \in \mathfrak{w}\mathfrak{H} : H \text{ non-degenerate } \} \\ \mathfrak{H}_* &= \{H \in \mathfrak{H} : H \text{ non-degenerate}\} \end{aligned}$$

Denote by $\operatorname{Per}^1 H \subset C^\infty(S^1, \mathbb{R}^{2n})$ the set of 1-periodic orbits of X_H. A first, extremely important property of non-degenerate periodic orbits is that they are always isolated.

Proposition 2.4.1 *Let $H \in \mathfrak{w}\mathfrak{H}$ and $\gamma \in \operatorname{Per}^1 H$ be a non-degenerate orbit. Then γ is isolated as a 1-periodic orbit: there exists an open neighborhood $\mathcal{U} \subset S^1 \times \mathbb{R}^{2n}$ of the graph of γ which does not intersect the graph of any other 1-periodic orbit of H.*

Proof. This follows immediately from the fact that $\det\left(\mathbb{I} - D\varphi_H^1(\gamma(0))\right) \neq 0$. □

Remark Non-degenerate 1-periodic orbits correspond to non-degenerate critical points of the action functional $\mathscr{A}_H$. Indeed, from Proposition 2.3.1 we know that the first variation of the action functional is given by

$$d\mathscr{A}_H(\gamma)\xi = \int_{S^1} \omega_0(\dot{\gamma} - X_H(\gamma), \xi)$$

A similar calculation shows that the second variation of the action functional at a critical point γ is

$$\text{Hess}\,\mathscr{A}_H(\gamma)(\xi_0, \xi_1) = \int_{S^1} \omega_0\left(\dot{\xi}_0 - DX_H(\gamma)\xi_0, \xi_1\right)$$

The second variation at the critical point γ thus has a kernel if and only if the linearized Hamilton equations along γ have a solution, which is equivalent to the orbit γ being degenerate.

As a corollary of this together with Lemma 2.3.4 we obtain a first important property of non-degenerate, asymptotically quadratic Hamiltonian systems with non-degenerate linear system at infinity:

Corollary 2.4.1 *If $H \in \mathfrak{wH}_*$ then H has finitely many 1-periodic orbits.*

We equip the set $\mathfrak{wH}$ with the C^∞_{loc}-topology, i.e. a sequence $\left(H^{(k)}\right)_{k\in\mathbb{N}} \subset \mathfrak{wH}$ converges to an $H \in \mathfrak{wH}$ if and only if it converges uniformly on compact sets with all derivatives. With this topology, $\mathfrak{wH}$ is a Fréchet space, so a Baire space, i.e. every residual set in $\mathfrak{wH}$ is dense. Notice that it is *not* a Banach space.

A very useful property of non-degenerate Hamiltonians is that any Hamiltonian has an arbitrarily C^∞-close non-degenerate Hamiltonian. To show this, we follow [15, §3.1].

First we introduce a Banach space of perturbations of Hamiltonians, called *Floer's $C^\infty_{\underline{\varepsilon}}$ space*. Let $\underline{\varepsilon} = (\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$ be a sequence of real numbers converging to zero. Denote by $C^\infty_{\underline{\varepsilon}}(\mathbb{R}^m)$ the set of functions $\mathfrak{h}\colon \mathbb{R}^m \to \mathbb{R}$ such that

$$||\mathfrak{h}||_{\underline{\varepsilon}} = \sum_{k=0}^{\infty} \varepsilon_k \, ||\mathfrak{h}||_{C^k(\mathbb{R}^m)} < \infty$$

The space $C^\infty_{\underline{\varepsilon}}(\mathbb{R}^m)$ equipped with the above norm is a Banach space. Notice that if $\underline{\varepsilon}'$ is a sequence converging to zero faster than $\underline{\varepsilon}$, then there is a *continuous* inclusion $C^\infty_{\underline{\varepsilon}'}(\mathbb{R}^m) \subset C^\infty_{\underline{\varepsilon}}(\mathbb{R}^m)$. We therefore are free to choose the sequence $\underline{\varepsilon}$ to converge as fast as we need.

The following important property of Floer's $C^\infty_{\underline{\varepsilon}}$ space is proven in Floer's original paper [16].

Lemma 2.4.1 *If $\underline{\varepsilon}$ converges to zero fast enough, then $C^\infty_{\underline{\varepsilon}}(\mathbb{R}^m)$ contains functions with arbitrary small compact support near any point, and attaining any value at that point.*

Proof. Let $\rho\colon \mathbb{R} \to [0,1]$ be a smooth function such that $\rho(r) = 0$ for all $r \leq 0$ and $\rho(r) = 1$

for all $r \geq 1$. Let $p \in \mathbb{R}^{2n}$, $\delta \in (0,\infty)^m$ and $\alpha \in \mathbb{R}$ be fixed. Define

$$\mathfrak{h}_{p,\delta,\alpha}(z) = \alpha \prod_{j=1}^{m} \left[1 - \rho\left(\frac{|z_j - p_j|}{\delta_j} \right) \right]$$

This function is a smooth function with support contained in a rectangle around p, attaining the value α at p. The rectangle is as small as we want by choosing δ suitably. Define

$$\varepsilon_k = \frac{1}{k^k \, \|\rho\|_{C^k}}$$

Then obviously $\mathfrak{h}_{p,\delta,\alpha} \in C^\infty_{\underline{\varepsilon}}(\mathbb{R}^m)$. □

When we write $C^\infty_{\underline{\varepsilon}}\left(S^1 \times \mathbb{R}^{2n}\right)$ we mean the closed subspace of $C^\infty_{\underline{\varepsilon}}\left(\mathbb{R}^{1+2n}\right)$ of functions 1-periodic in the first variable. See Appendix A for the necessary concepts and theorems in Fredholm theory.

Proposition 2.4.2 $\mathfrak{wH}_* \subset \mathfrak{wH}$ *is dense.*

Proof. Denote $C^\infty_{\underline{\varepsilon}}\left(S^1 \times \mathbb{R}^{2n}\right) = C^\infty_{\underline{\varepsilon}}$. We show that for every $H^{(0)} \in \mathfrak{wH}$ there exists an $\underline{\varepsilon}$ and a residual set $\mathfrak{P} \subset C^\infty_{\underline{\varepsilon}}$ such that $\mathfrak{h} \in \mathfrak{P} \implies H^{(0)} + \mathfrak{h} \in \mathfrak{wH}_*$.

First of all, Lemma 2.4.1 shows that there exists an $\underline{\varepsilon}$ such that if $\mathfrak{h} \in C^\infty_{\underline{\varepsilon}}$, then $H^{(0)} + \mathfrak{h} \in \mathfrak{wH}$ and $\mathrm{ind}_\infty H^{(0)} + \mathfrak{h} = \mathrm{ind}_\infty H^{(0)}$.

Notice that $H \in \mathfrak{wH}_*$ is equivalent to the set of its 1-periodic orbits $\mathrm{Per}^1 H \subset W^{1,2}\left(S^1, \mathbb{R}^{2n}\right)$ being transversely cut out. Consider $f : \mathscr{E}_0 = W^{1,2}\left(S^1, \mathbb{R}^{2n}\right) \times C^\infty_{\underline{\varepsilon}} \to L^2\left(S^1, \mathbb{R}^{2n}\right) = \mathscr{E}_1$ defined by

$$f(x,\mathfrak{h}) = \dot{x} - X_{H^{(0)}+\mathfrak{h}} \circ x = \dot{x} - X_{H^{(0)}} \circ x - X_{\mathfrak{h}} \circ x$$

Define $\Sigma = f^{-1}(0)$. This is the solution set $\Sigma = \left\{(x,\mathfrak{h}) : \dot{x} = X_{H^{(0)}+\mathfrak{h}} \circ x\right\}$. Now, for fixed $\mathfrak{h}$, the map $f(\cdot,\mathfrak{h})\colon W^{1,2} \to L^2$ is an index zero Fredholm map. Indeed, for any $\xi \in W^{1,2}$,

$$Df(x,\mathfrak{h})(\xi,0) = \dot{\xi} - DX_{H^{(0)}+\mathfrak{h}}(x)\xi = \dot{\xi} + J_0 A_t \xi$$

where we set $A_t = \mathrm{Hess}\left(H^{(0)} + \mathfrak{h}\right)(t, x(t)) \in \mathrm{Sym}(2n)$. All the following facts follow from Lemma 2.3.1 and its Corollary 2.3.1: first, we see that $Df(x,\mathfrak{h})(\xi,0) = D_A\xi$ and we know that D_A has Fredholm index zero. Further, it's immediate to check that

$$Df(x,\mathfrak{h})(\xi,\mathfrak{g}) = \dot{\xi} - DX_{H^{(0)}+\mathfrak{h}}(x)\xi - X_{\mathfrak{g}} \circ x$$

for every $(\xi,\mathfrak{g}) \in \mathscr{E}_0$. We use this formula to show that $Df(x,\mathfrak{h})\colon \mathscr{E}_0 \to \mathscr{E}_1$ is onto for every $(x,\mathfrak{h}) \in \Sigma$. Indeed, for fixed $\theta \in \mathscr{E}_1$, we have to find suitable $(\xi,\mathfrak{g}) \in \mathscr{E}_0$ for which

$$\dot{\xi} - DX_{H^{(0)}+\mathfrak{h}}(x)\xi - X_{\mathfrak{g}}(x) = \theta \iff \dot{\xi} + J_0 A_t \xi = \eta \iff D_A\xi = \eta$$

where $\eta(t) = \theta(t) + X_{\mathfrak{g}}(t, x(t)) \in L^2$. That the equation admits a solution for a suitable choice of $\mathfrak{g}$ follows from the characterization of the image of Λ_A given in equation (2.5), which implies that the image of D_A is characterized by

$$\eta \in \operatorname{im} D_A \iff \int_0^1 M_t M_\tau^{-1} \eta(\tau) d\tau \in \operatorname{im}(M_1 - \mathbb{I}) \subset \mathbb{R}^{2n}.$$

Finally, $\ker Df(x, \mathfrak{h})$ splits, because it is finite dimensional.

We conclude by Theorem A.1 that $\Sigma \subset \mathscr{E}_0$ is a smooth submanifold. Now, let $p_2 \colon \mathscr{E}_0 \to C^\infty_{\underline{\varepsilon}}$ be the projection. The restriction $p = p_2|_\Sigma \colon \Sigma \to C^\infty_{\underline{\varepsilon}}$ is a Fredholm map of index zero, by Lemma A.4. By the Sard-Smale theorem (Theorem A.3), there is a residual set $\mathfrak{P} \subset C^\infty_{\underline{\varepsilon}}$ of regular values for p. But the regular values of p are exactly the functions $\mathfrak{h} \in C^\infty_{\underline{\varepsilon}}$ such that $H^{(0)} + \mathfrak{h} \in \mathfrak{w}\mathfrak{H}_*$.

To show that $\mathfrak{w}\mathfrak{H}_*$ is dense in $\mathfrak{w}\mathfrak{H}$ in the C^∞_{loc}-topology, it suffices to observe that for any fixed $H^{(0)} \in \mathfrak{w}\mathfrak{H}$ we can choose some sequence $\mathfrak{h}^k \in \mathfrak{P}$ with $\left\|\mathfrak{h}^k\right\|_{\underline{\varepsilon}} \to 0$ as $k \to \infty$ and

$$H^{(k)} = H^{(0)} + \mathfrak{h}^k \in \mathfrak{w}\mathfrak{H}_*$$

Then $H^{(k)} \to H^{(0)}$ even in the strong C^∞-topology of uniform convergence over all $S^1 \times \mathbb{R}^{2n}$ together with all derivatives, so *a fortiori* in the C^∞_{loc}-topology too. □

Remark 1. With a bit more effort, one may show that the set $\mathfrak{w}\mathfrak{H}_* \subset \mathfrak{w}\mathfrak{H}$ is even *residual*, and not just dense.

2. The proof shows that in order to perturb a degenerate Hamiltonian to a non-degenerate one, it suffices to add a $C^\infty_{\underline{\varepsilon}}$-perturbation with compact support containing the (compact) set of 1-periodic orbits of the original Hamiltonian.

3. Elements of Analysis of the Floer equation

In this chapter we study some aspects of the analysis of the Floer equation in order to discuss the moduli spaces of its solutions. In particular, we will argue that the solutions of various versions of the Floer equation come in smooth manifolds which admit a compactification by adding boundary and corners.

3.1. The Floer equation: two points of view

Unregularized gradient flow Recall from the previous chapter that the unregularized gradient of the action functional $\nabla_{L^2}\mathscr{A}_H\colon W^{1,2}\to L^2$ is given by

$$\nabla_{L^2}\mathscr{A}_H(x)=J_0\dot{x}-\nabla H_t(x)=J_0[\dot{x}-X_H(x)],\quad \forall x\in W^{1,2}$$

Floer's idea is to build a Morse-Smale-Witten complex where the generators are critical points of the action functional – so 1-periodic orbits of the Hamiltonian sytstem in study – and the differential counts "unregularized anti-gradient flow trajectories". These anti-gradient flow trajectories are maps from the cylinder $u\colon\mathbb{R}\times S^1\to\mathbb{R}^{2n}$ seen as paths of loops $s\mapsto u(s,\cdot)\colon S^1\to\mathbb{R}^{2n}$ which solve the gradient descent equation obtained using the unregularized gradient $\nabla_{L^2}\mathscr{A}_H$. Namely, for u to be an "anti-gradient trajectory" of the action functional, it must solve the equation

$$\partial_s u=-\nabla_{L^2}\mathscr{A}_H(u(s,\cdot))$$

Substituting the formula for the unregularized gradient and putting everything on one side, we obtain

$$\partial_s u+J_0[\partial_t u-X_H(u)]=0$$

This is Floer's equation for the couple (H,J_0).

Notice that we have implicitly chosen J_0 as complex structure on $\mathbb{R}^{2n}$. This is a natural choice, but is not the only one. In fact, below we will develop the analysis of the Floer equation for a general almost-complex structure depending on all variables $(s,t,z)\in\mathbb{R}\times S^1\times\mathbb{R}^{2n}$.

Warning Clearly the unregularized gradient is not a vector field on $W^{1,2}$ so it is not correct to claim that the Floer equation is *really* giving an anti-gradient flow, since it is not giving a flow at all. The miracle is that it acts as a surrogate for an anti-gradient flow equation in all the ways that matter for building a Morse complex.

Perturbation of Cauchy-Riemann equations $(\mathbb{R}^{2n}, J_0)$ can be seen as the complex manifold $(\mathbb{C}^n, i)$, and $\mathbb{R}\times S^1$ is also a complex curve when equipped with the complex structure j_0 defined by $j_0\partial_s = \partial_t$, $j_0\partial_t = -\partial_s$ where $(s,t)\in\mathbb{R}\times S^1$ are the coordinates on the cylinder. From this point of view the Floer equation can be seen as a perturbation of the Cauchy-Riemann equations for maps $u\colon(\mathbb{R}\times S^1, j_0)\to(\mathbb{R}^{2n}, J_0)$. Namely, the Cauchy-Riemann equations for such a map are the equations

$$du\circ j_0 = J_0\circ du$$

which simply say that the differential of u is a complex-linear map. Writing in coordinates:

$$du = \partial_s u\, ds + \partial_t u\, dt, \quad J_0\circ du\circ j_0 = J_0\circ[\partial_t u\, ds - \partial_s u\, dt] = (J_0\partial_t u)\, ds + (-J_0\partial_s u)\, dt$$

Equating this last expression with $-du$ we obtain

$$\begin{cases} J_0\partial_t u = -\partial_s u \\ -J_0\partial_s u = -\partial_t u \end{cases}$$

The second equation is just $-J_0$ multiplied the first, so we are left with

$$\partial_s u + J_0\partial_t u = 0$$

We define $\overline{\partial}_{j_0 J_0} = \partial_s + J_0\partial_t$ the Cauchy-Riemann operator associated to (j_0, J_0). The Floer equation may be seen as a perturbation of this operator, by adding the non-linear term $\nabla H_t(u)$. The asymptotic linearity implies that the operator $u\mapsto\nabla H_t(u)$ is compact (the image of a bounded sequence has a converging subsequence) on suitable Sobolev spaces, hence the operator defining the Floer equation is a compact perturbation of the Cauchy-Riemann operator. This will be an important point of view to obtain elliptic estimates on its solutions.

3.1.1. Almost complex structures

As mentioned above, to guarantee that the solutions of the Floer equation come in smooth families, in general we must perturb the complex structure J_0. We introduce the class of objects within which the perturbation occurs.

Definition 3.1 An *almost-complex* structure on $\mathbb{R}^{2n}$ is a smooth section of the endomorphism bundle $J\colon\mathbb{R}^{2n}\to\operatorname{End}\mathbb{R}^{2n}$ such that $J^2 = -\mathbb{I}$. An almost-complex structure J is *compatible* with the symplectic structure ω_0 if the family of bilinear forms

$$g_J(z)(v_0, v_1) = \omega_0(v_0, J(z)v_1)$$

defines a Riemannian metric on $\mathbb{R}^{2n}$. Similarly if Σ is some smooth manifold, a Σ-family of almost complex structures is a smooth map $J\colon\Sigma\times\mathbb{R}^{2n}\to\operatorname{End}\mathbb{R}^{2n}$ which gives a section for every fixed point in Σ, and such that $J(\sigma,z)^2 = -\mathbb{I}$ for all $(\sigma,z)\in\Sigma\times\mathbb{R}^{2n}$. Such a family is ω_0-compatible if

$$g_J(\sigma,z) = \omega_0\circ(\mathbb{I}\times J(\sigma,z))$$

defines a Σ-parameter family of Riemannian metrics on $\mathbb{R}^{2n}$.

The proof of the following lemma can be found in [32].

Lemma 3.1.1 *The set of ω_0-compatible almost-complex structures is non-empty and contractible in the C^∞_{loc}-topology.*

3.1.2. The full-fledged Floer equation

We are ready to introduce the equation we will actually work with. We also discuss the most general form of the Floer equation needed, that is, the equation needed to define the so-called *continuation morphisms* on Floer homology. These will be of utmost importance for the whole treatment.

Definition 3.2 Consider a smooth function $\mathscr{H}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$, $\mathscr{H}(s,t,z) = \mathscr{H}^s_t(z)$, with the following properties.

1. There exists a bounded closed interval $\mathscr{S} \subset \mathbb{R}$ such that $\mathscr{H}$ depends on s only in $\mathscr{S}$. We set $H^0_t(z) = \mathscr{H}^s_t(z)$ for some (and then all) $s < \min \mathscr{S}$ and $H^1_t(z) = \mathscr{H}^s_t(z)$ for some (and then all) $s > \max \mathscr{S}$.
2. There exists a smooth $\mathbb{A} = \mathbb{A}^s_t : \mathbb{R} \times S^1 \to \mathrm{Sym}(2n)$ such that $\mathbb{A}$ depends on s only in $\mathscr{S}$, and
$$\sup_{(s,t)\in\mathbb{R}\times S^1} |\nabla \mathscr{H}^s_t(z) - \mathbb{A}^s_t z| = o(|z|) \quad \text{as } |z| \to \infty$$
We set $A^0_t = \mathbb{A}^s_t$ for some $s < \min \mathscr{S}$ and $A^1_t = \mathbb{A}^s_t$ for some $s > \max \mathscr{S}$.
3. $A^i : S^1 \to \mathrm{Sym}(2n)$ both generate non-degenerate linear systems. In other words, H^0 and H^1 belong to $\mathfrak{w}\mathfrak{H}$.

We call such an $\mathscr{H}$ an *asymptoticaly quadratic continuation* between the Hamiltonians H^0 and H^1.

Remark We do not assume that $\mathbb{A}^s \in C^\infty\big(S^1, \mathrm{Sym}(2n)\big)$ generates a non-degenerate linear system for all $s \in \mathscr{S}$.

We also fix the behaviour of the families of almost-complex structures required to define the Floer equation. A family of almost-complex structures $\mathscr{J}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \mathrm{End}\,\mathbb{R}^{2n}$, $\mathscr{J}(s,t,z) = \mathscr{J}_{s,t}(z)$ is said to be *adequate* if

1. There exists a bounded closed interval $\mathscr{S} \subset \mathbb{R}$ such that $\mathscr{J}$ depends on s only in $\mathscr{S}$.
2. $\mathscr{J}_{s,t}$ is ω_0-compatible for all $(s,t) \in \mathbb{R} \times S^1$.
3. $\|\mathscr{J}\|_{L^\infty(\mathbb{R}^{2n}, \mathrm{End}\,\mathbb{R}^{2n})} < \infty$.

We say that a path of almost-complex structures $J: S^1 \times \mathbb{R}^{2n} \to \mathrm{End}\,\mathbb{R}^{2n}$ is adequate if the corresponding s-constant family $\mathscr{J}_{s,t} \equiv J_t$ is adequate.

Finally, if $\mathscr{H}$ and $\mathscr{J}$ are as above, we call the couple $(\mathscr{H}, \mathscr{J})$ an *adequate pair*.

Given an adequate family of almost-complex structures $\mathcal{J}$, we define a family of Riemannian metrics on $\mathbb{R}^{2n}$ by

$$g_{\mathcal{J}}(s,t,z)(u,v) = \omega_0\big(u, \mathcal{J}_{s,t}(z)v\big)$$

Often we suppress the dependence on (s,t) and simply write $g_{\mathcal{J}}$. The associated family of norms is denoted by $|\cdot|_{g_{\mathcal{J}}}$. Notice that since $||\mathcal{J}||_{L^\infty} < \infty$, all of these norms can be compared with the standard euclidean metric via an uniform constant: setting $C = \sqrt{||\mathcal{J}||_{L^\infty}}$ we see that

$$\frac{1}{C}|v| \leq |v|_{g_{\mathcal{J}(s,t,z)}} \leq C|v| \quad \forall v \in \mathbb{R}^{2n} \quad \forall (s,t,z) \in \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \tag{3.1}$$

Given a smooth function $\mathcal{F}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$, we define its $\mathcal{J}$-gradient by the identity

$$g_{\mathcal{J}}(s,t,z)\Big(\nabla_{\mathcal{J}}\mathcal{F}(s,t,z), v\Big) = d_z\mathcal{F}(s,t,z)v \quad \forall v \in \mathbb{R}^{2n}$$

where $d_z\mathcal{F}$ is the differential with respect to the $\mathbb{R}^{2n}$-coordinates only. Recall that the Hamiltonian vector field $X_{\mathcal{F}} = X_{\mathcal{F}}(s,t,z)$ of a function $\mathcal{F}$ as above is defined by the identity

$$i_{X_{\mathcal{F}}(s,t,\cdot)}\omega_0 = d_z\mathcal{F}(s,t,\cdot)$$

It follows that since $g_{\mathcal{J}} = \omega_0 \circ (\mathbb{I} \times \mathcal{J})$, we have the identity

$$X_{\mathcal{F}}(s,t,z) = -\mathcal{J}(s,t,z)\nabla_{\mathcal{J}}\mathcal{F}(s,t,z).$$

Similar constructions apply when $J: S^1 \times \mathbb{R}^{2n} \to \operatorname{End}\mathbb{R}^{2n}$ is an adequate almost-complex structure independent of $s \in \mathbb{R}$.

Definition 3.3 Let $H \in \mathfrak{wH}$ and $J: S^1 \times \mathbb{R}^{2n} \to \operatorname{End}\mathbb{R}^{2n}$ an adequate almost-complex structure. Let $\mathcal{H}$ be an asymptotically quadratic continuation and $\mathcal{J}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \operatorname{End}\mathbb{R}^{2n}$ an adequate almost-complex structure.

1. A map $u: \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is said to solve the *(autonomous)* (H,J)*-Floer equation* when

$$\partial_s u + J_t(u)[\partial_t u - X_H(t,u)] = 0$$

The Floer operator associated to the adequate pair (H,J) is the non-linear operator defined by

$$\overline{\partial}_{H,J} u = \partial_s u + J_t(u)\partial_t u + \nabla_J H(t,u)$$

2. A map $u: \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is said to solve the *continuation* $(\mathcal{H}, \mathcal{J})$*-Floer equation* when

$$\partial_s u + \mathcal{J}_{s,t}(u)[\partial_t u - X_{\mathcal{H}^s}(s,t,u)] = 0$$

The *Floer operator*, associated to the adequate pair $(\mathcal{H}, \mathcal{J})$ is the non-linear operator defined by

$$\overline{\partial}_{\mathcal{H},\mathcal{J}} u = \partial_s u + \mathcal{J}(s,t,u)\partial_t u + \nabla_{\mathcal{J}}\mathcal{H}(s,t,u)$$

3.1.2.1. Energy of a Floer trajectory

Definition 3.4 1. Let $H \in \mathfrak{w}\mathfrak{H}$ and J be an adequate almost-complex structure. The (H,J)*-energy* of a map $u : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is defined to be

$$E_{H,J}(u) = \frac{1}{2}\int_{\mathbb{R}\times S^1} |\partial_s u|^2_{g_J} + |\partial_t u - X_H(t,u)|^2_{g_J}\, ds \wedge dt$$

2. Let $\mathscr{H}$ be an asymptotically quadratic continuation and $\mathscr{J}$ an adequate almost-complex structure. The $(\mathscr{H},\mathscr{J})$-energy of a map $u : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is defined to be

$$E_{\mathscr{H},\mathscr{J}}(u) = \frac{1}{2}\int_{\mathbb{R}\times S^1} |\partial_s u(s,t)|^2_{g_{\mathscr{J}}} + |\partial_t u(s,t) - X_{\mathscr{H}^s}(t,u(s,t))|^2_{g_{\mathscr{J}}}\, ds \wedge dt$$

If u solves the continuation Floer equation for $(\mathscr{H},\mathscr{J})$, then

$$E_{\mathscr{H},\mathscr{J}}(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2_{g_{\mathscr{J}}}$$

and analogously for the autonomous case of a solution of the (H,J)-Floer equation. Combined with the comparison between the $g_{\mathscr{J}}$-metrics and the standard euclidean metric of (3.1), we see that

$$||\partial_s u||^2_{L^2(\mathbb{R}\times S^1)} \le ||\mathscr{J}||^2_{L^\infty} \cdot E_{\mathscr{H},\mathscr{J}}(u) \tag{3.2}$$

Therefore a bound on the energy corresponds to a bound on the L^2-norm of $\partial_s u$.

A useful and simple observation is that zero-energy Floer trajectories must be constant in the s-coordinate, and therefore coincide with some 1-periodic orbit of our Hamiltonian system. This follows immediately from the estimate (3.2). This simple fact justifies calling 1-periodic orbits of H the "constant solutions" of the Floer equation, by abusing the analogy between the Floer equation and the gradient flow of a function. Viceversa, cylinders with positive energy will be sometimes called "non-constant solutions".

Remark The definition of energy for a Floer trajectory already gives us a glimpse of the kind functional space we will be working in. For the energy to be finite, one might want to require our maps u to be $W^{1,2}$. Notice that $p = 2$ is precisely the critical exponent in the Sobolev embedding theorem for maps from 2-dimensional manifolds, meaning in particular that there are $W^{1,2}$ maps from the cylinder which are not continuous. Thus we will have to ask our solutions to be $W^{1,p}$ with $p > 2$. This strays us from the Hilbert world and forces us to work in a Banach setting.

3.2. Analysis of the Floer equation: an overview

We will not explain in detail the complete analytical apparatus which is involved in the construction of Floer homology, because the treatment found in [7] can be easily modified to work also for the class of asymptotically linear Hamiltonian systems. Instead, we explain the main ideas needed to construct Floer homology and carry out a few proofs which, according to the author's taste, capture the peculiar techniques the analytical study of the Floer equation necessitates.

Strategy We want to use Floer's equation as a surrogate equation, replacing the ill behaved gradient descent equation for the action functional. Therefore we want to study the space of its finite-energy solutions and show that it has all the properties necessary to define a kind of Morse-Smale-Witten complex for the action functional.

1. The first analytical task is to show that the Floer equation has the regularizing properties typical of elliptic operators. This is usually called elliptic bootstrapping: the task is to show that a weak solution of possibly low regularity is actually smooth and thus a strong solution. In particular, the moduli spaces of solutions are contained in spaces of *smooth* maps.
2. The second analytical task is to show that finite-energy solutions are asymptotic to critical points. This is the minimal required property to interpret them as gradient trajectories between critical points.
3. The third analytical task is to show that the Floer operator can be seen as a Fredholm section of a Banach bundle of maps, and that, up to choosing the Hamiltonian and almost-complex structure adequately, it intersects the zero section transversely. These two facts together combine to imply that the zero locus, which is nothing but the set of solutions of Floer's equation, is a finite-dimensional smooth manifold with (local) dimension the Fredholm index of the linearization. Such index can be computed in terms of asymptotic data at the ends, namely, the Conley-Zehnder indices of the asymptotic periodic orbits.
4. The fourth analytical task is to study the compactification of the moduli spaces of trajectories with fixed ends. This is divided in roughly two sub-tasks. The first is to show compactness of the moduli space of finite-energy trajectories in the C^∞_{loc} topology. Here our treatment must deviate slightly from [7], because their approach works only for *compact* target symplectic manifolds. Our target is $\mathbb{R}^{2n}$ which is not compact. Therefore, in Section 3.3.1 we explain in detail the missing estimates. The second sub-task is to describe the boundary in the compactification in terms of lower-dimensional moduli spaces, via the process of *gluing*.

3.3. Elliptic regularity and its consequences

An in-depth treatment of the elliptic regularity theory we sketch here can be found in McDuff and Salamon's [32, Appendix B] for the case of the Cauchy-Riemann operator (which is morally the only case), and for the Floer operator in [7, Chapter 12]

Elliptic regularity is the fundamental *primum movens* over which the rest of the analysis of the Floer equation rests. It can be loosely described as a bootstrapping process which imbues low regularity weak solutions with the same regularity as the coefficients of the equation. This is to be contrasted, for example, with the case of weak solutions of *hyperbolic* partial differential equations, which are known to develop shocks, or discontinuities, in finite time.

Elliptic regularity for the linear Cauchy-Riemann equation Let $p, q, r \in \mathbb{R}$ be such that $2 < p \leq \infty$, $1 \leq r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $\Omega \subset \mathbb{C}$ be an open subset, with coordinates $(s,t) = s + it \in \Omega \subset \mathbb{C}$. Let $J \in W^{1,p}_{\text{loc}}\left(\Omega, \operatorname{End}\mathbb{R}^{2n}\right)$ be such that $J^2 = -\mathbb{I}$. Denote by $\overline{\partial}_J = \partial_s + J(s,t)\partial_t$ the (domain-dependent, but linear) Cauchy-Riemann operator. Finally let $\eta \in L^r_{\text{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$.

Definition 3.5 We say that a function $u \in L^q_{\text{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$ is a *weak solution* of the Cauchy-Riemann equation

$$\overline{\partial}_J u = \eta$$

if it satisfies the integral equation

$$\int_\Omega \left\langle \partial_s \varphi + J^T \partial_t \varphi, u \right\rangle = -\int_\Omega \langle \varphi, \eta + (\partial_t J) u \rangle \quad \forall \varphi \in C^\infty_c\left(\Omega, \mathbb{R}^{2n}\right)$$

Any *actual* solution with the necessary regularity satisfies this integral equation, as one can see by a few integrations by parts. A weak solution instead doesn't have *a priori* enough regularity to be a true solution.

The following proposition gives the fundamental elliptic estimates we need to prove elliptic regularity.

Proposition 3.3.1 *Let $p, q, r \in \mathbb{R}$ be such that $2 < p \leq \infty$, $1 \leq r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $\Omega \subset \mathbb{C}$ be an open set. Let $J \in W^{1,p}_{\text{loc}}\left(\Omega, \operatorname{End}\mathbb{R}^{2n}\right)$ be a domain-dependent almost complex structure. For every pre-compact open subsets $U \subset V \subset \Omega$ with $\overline{U} \subset V$, there exists a $c > 0$ such that*

$$\|u\|_{W^{1,r}(U)} \leq c\left(\left\|\overline{\partial}_J u\right\|_{L^r(V)} + \|u\|_{L^q(V)}\right) \quad \forall u \in W^{1,r}_{\text{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$$

In principle, a weak solution isn't even $W^{1,r}_{\text{loc}}$. This missing first jump in regularity is

Proposition 3.3.2 *Let p, q, r and J be as above, and consider an $\eta \in L^r_{\text{loc}}(\Omega, \mathbb{R}^{2n})$. Any weak solution $u \in L^q_{\text{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$ of $\overline{\partial}_J u = \eta$ is actually of regularity $W^{1,r}_{\text{loc}}$ and satisfies the equation $\overline{\partial}_J u = \eta$ almost everywhere.*

A proof of both these results can be found in [32, Lemma B.4.6]. They are a consequence of the corresponding regularizing properties of the Laplacian. These propositions, together with a smart choice of Sobolev exponents and induction, show the following important result.

Corollary 3.3.1 *Let $\Omega \subset \mathbb{C}$ be an open set, $k \geq 1$ a fixed integer and $p > 2$. Let $J \in W^{k,p}_{\text{loc}}\left(\Omega, \operatorname{End}\mathbb{R}^{2n}\right)$ be an almost-complex structure and $\eta \in W^{k,p}_{\text{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$. For every pre-compact open $U \subset V \subset \Omega$ with $\overline{U} \subset V$ there exists a $c > 0$ such that*

$$\|u\|_{W^{k+1,p}(U)} \leq c\left(\left\|\overline{\partial}_J u\right\|_{W^{k,p}(V)} + \|u\|_{W^{k,p}(V)}\right)$$

In particular, if J and η are smooth, and $u \in L^p_{\mathrm{loc}}\left(\Omega, \mathbb{R}^{2n}\right)$ is a weak solution of the Cauchy-Riemann equation $\overline{\partial}_J u = \eta$, then u is a smooth function.

Roughly, the proof of smoothness goes as follows. Any weak solution is actually $W^{1,p}_{\mathrm{loc}}$, so has one weak derivative in L^p_{loc}. Now if we look at $v = \partial_s u$, we get that $\partial_s v + J\partial_t v = \eta'$ weakly for some new $\eta' \in L^p_{\mathrm{loc}}$. So we are again in the hypotheses of the Proposition 3.3.2, and we have that $v \in W^{1,p}_{\mathrm{loc}}$ itself! Moreover $\partial_t u = -J\partial_s u$, and hence $v \in W^{1,p}_{\mathrm{loc}} \implies u \in W^{2,p}_{\mathrm{loc}}$. We pulled ourselves from the bootstraps jumping from zero derivatives to one, then to two... inductively we reach infinity. The full proof can be found in [32, Prop. B.4.9].

Elliptic regularity for the Floer equation Similar regularizing properties for the *non-linear* Cauchy-Riemann operator can actually be derived from this linear result. These can be extended without substantial troubles to the case of the Floer equation. Indeed fix $p > 2$. If we have an adequate couple $(\mathcal{H}, \mathcal{J})$ and a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that

$$\partial_t u - \mathcal{J}^s_t(u)[\partial_t u - X_{\mathcal{H}}(s,t,u)] = 0$$

weakly, then we can set $J^s_t = \mathcal{J}^s_t(u(s,t))$, $\eta(s,t) = \nabla_{\mathcal{J}} \mathcal{H}(s,t,u(s,t))$. Since $p > 2$, u is a continuous function. This implies, together with the smoothness of $\mathcal{H}$ and $\mathcal{J}$, that $J \in W^{1,p}_{\mathrm{loc}}$ and $\eta \in W^{1,p}_{\mathrm{loc}}$. Therefore u solves the equation $\overline{\partial}_J u = \eta$ in the weak sense.

Remark Notice that one must require a bit more regularity at the start here, since if u is only L^q_{loc} like in the linear theory, we cannot expect that $\mathcal{J} \circ u$ and $\nabla_{\mathcal{J}} \mathcal{H} \circ u$ will result to be $W^{1,p}_{\mathrm{loc}}$, as u might not even be continuous.

This sketch of an argument is the fundamental reason why the next proposition holds. A complete proof can be found in [7, §12.4].

Proposition 3.3.3 *Let $p > 2$ and $(\mathcal{H}, \mathcal{J})$ an adequate pair. If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ solves the Floer equation for $(\mathcal{H}, \mathcal{J})$, then $u \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.*

Let $\mathcal{M}(\mathcal{H}, \mathcal{J}) \subset W^{1,p}_{\mathrm{loc}}\left(S^1 \times \mathbb{R}, \mathbb{R}^{2n}\right)$ be the set of solutions of the Floer equation associated to the adequate pair $(\mathcal{H}, \mathcal{J})$. The previous proposition can be restated by saying that $\mathcal{M}(\mathcal{H}, \mathcal{J}) \subset C^\infty\left(S^1 \times \mathbb{R}, \mathbb{R}^{2n}\right)$. In principle, the *space* $\mathcal{M}(\mathcal{H}, \mathcal{J})$ is equipped with the $W^{1,p}_{\mathrm{loc}}$ topology, since it is defined as a subspace of $W^{1,p}_{\mathrm{loc}}\left(S^1 \times \mathbb{R}, \mathbb{R}^{2n}\right)$, while $C^\infty\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$ has a natural topology, the C^∞_{loc} topology, which is much finer than the $W^{1,p}_{\mathrm{loc}}$ topology. But elliptic regularity implies that on the space of solutions, these two topologies coincide:

Proposition 3.3.4 *Let $u_j \in \mathcal{M}(\mathcal{H}, \mathcal{J})$ be a sequence of solutions of the Floer equation for the adequate pair $(\mathcal{H}, \mathcal{J})$. If $u_j \to u$ in $W^{1,p}_{\mathrm{loc}}$ for some $p > 2$, then $u \in \mathcal{M}(\mathcal{H}, \mathcal{J})$ and $u_j \to u$ in C^∞_{loc}. In other words the C^∞_{loc} and the $W^{1,p}_{\mathrm{loc}}$ topologies coincide on $\mathcal{M}(\mathcal{H}, \mathcal{J})$.*

Proof sketch. That the limit is a solution follows immediately from $W^{1,p}_{\mathrm{loc}}$ convergence alone. By Proposition 3.3.3 the limit u is smooth.

To show that the convergence holds also in the C^∞_{loc} topology, we show that the limit holds in the $W^{k,p}_{\mathrm{loc}}$ topology for all $k \geq 1$. We identify $\mathbb{R} \times S^1 \simeq \mathbb{C} \setminus \{0\} = \Omega \subset \mathbb{C}$ using coordinates $(s,t) \mapsto e^{s+it}$. By induction, assume that the convergence is in $W^{k,p}_{\mathrm{loc}}(\Omega)$, for some $k \geq 1$. Define $J_j(s,t) = \mathscr{J}^s_t(u_j(s,t))$ and $\eta_j(s,t) = \nabla_{\mathscr{J}}\mathscr{H}\left(s,t,u_j(s,t)\right)$. Since $u_j \to u$ in $W^{k,p}_{\mathrm{loc}}$ with $p > 2$, then $J_j \to J$ and $\eta_j \to \eta$ in $W^{k,p}_{\mathrm{loc}}$, where $J(s,t) = \mathscr{J}^s_t(u(s,t))$ and $\eta(s,t) = \nabla_{\mathscr{J}}\mathscr{H}(s,t,u(s,t))$. Notice that the hypotheses of $p > 2$ and $k \geq 1$ are crucial for this to hold. Now, clearly u_j solves $\overline{\partial}_{J_j} u_j = \eta_j$ for all j and consequently $\overline{\partial}_J u = \eta$. Now look at $v_j = u - u_j$. Since $\overline{\partial}_{J_j} u_j = \eta_j$ and $\overline{\partial}_J u = \eta$,

$$\begin{aligned}\overline{\partial}_{J_j} v_j &= \partial_s\left(u-u_j\right) + J_j\partial_t\left(u-u_j\right) = \partial_s u + J\partial_t u - J\partial_t u + J_j\partial_t u - \partial_s u_j - J_j\partial_t u_j = \\ &= \eta - \eta_j + (J_j - J)\partial_t u = \Theta_j\end{aligned}$$

Notice that since u is smooth and by our inductive assumption, $\Theta_j \to 0$ and $v_j \to 0$, both in $W^{k,p}_{\mathrm{loc}}$. By Corollary 3.3.1 for any pre-compact open $U \subset V \subset \Omega$ there exists a constant $c > 0$ such that

$$\begin{aligned}\left\|v_j\right\|_{W^{k+1,p}(U)} &\leq c\left[\left\|\overline{\partial}_{J_j} v_j\right\|_{W^{k,p}(V)} + \left\|v_j\right\|_{W^{k,p}(V)}\right] = \\ &= c\left[\left\|\Theta_j\right\|_{W^{k,p}(V)} + \left\|v_j\right\|_{W^{k,p}(V)}\right] \to 0 \text{ as } j \to \infty\end{aligned}$$

This implies that $u_j \to u$ in $W^{k+1,p}_{\mathrm{loc}}$. □

3.3.1. A priori uniform estimates

Since $\mathbb{R}^{2n}$ is a non-compact target manifold, nothing guarantees that families of solutions of the Floer equation must be uniformly bounded. This is the minimal requirement needed to obtain compactness properties of families of solutions. Therefore, we must argue in some way that an uniform L^∞ bound may be reached for solutions of the Floer equation.

We derive the uniform L^∞ bounds in the most general case we need, that of a Floer cylinder solving the Floer equation for an asymptotically quadratic continuation.

Proposition 3.3.5 *Let $\mathscr{H}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ be an asymptotically quadratic continuation between Hamiltonians $H^0, H^1 \in \mathfrak{wH}$, and $\mathscr{J}: \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \operatorname{End}\mathbb{R}^{2n}$ an adequate family of almost complex structures. For every $E > 0$ there exists an $R > 0$ with the following significance.*

If $u : \mathbb{R}\times S^1 \to \mathbb{R}^{2n}$ solves the continuation Floer equation for the pair $(\mathcal{H},\mathcal{J})$ and

$$E_{\mathcal{H},\mathcal{J}}(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2_{\mathcal{J}} < E$$

then we have

$$||u||_{L^\infty} < R$$

Proof. We can assume that the interval $\mathcal{S} \subset \mathbb{R}$ where $\mathcal{H}$ depends on s is the same as the one for $\mathcal{J}$. The uniform energy bound together with (3.2) gives

$$||\partial_s u||_{L^2} = \int_{\mathbb{R}\times S^1} |\partial_s u|^2 < E'$$

where E' depends only on E and $\mathcal{J}$. Since each H^i has non-degenerate quadratic Hamiltonian at infinity, there exist constants $\nu, \delta > 0$ such that

$$||\nabla_{L^2}\mathcal{A}_{\mathcal{H}^s}(x)||_{L^2(S^1)} \geq \frac{\nu}{2}||x||_{L^2} - \delta \quad \forall x \in W^{1,2}(S^1,\mathbb{R}^{2n}) \quad \forall s \in \mathbb{R}\setminus\mathcal{S}$$

Here the weak-L^2 gradient is with respect to the L^2-metric arising from the euclidean inner product. Notice that $u(s,\cdot) \in W^{1,2}(S^1)$ for all $s \in \mathbb{R}$ by the regularity theory of the Floer equation. Since u solves the Floer equation, and using again (3.2),

$$||\partial_s u(s,\cdot)||_{L^2(S^1)} \geq B\,||\nabla_{L^2}\mathcal{A}_{\mathcal{H}^s}(u(s,\cdot))||_{L^2(S^1)} \geq \frac{\nu'}{2}||u(s,\cdot)||_{L^2(S^1)} - \delta' \quad \forall s \in \mathbb{R}\setminus\mathcal{S} \qquad (3.3)$$

Consider an arbitrary $\alpha > 0$ to be determined later. Define

$$S_\alpha = \left\{ s \in \mathbb{R} : ||\partial_s u(s,\cdot)||_{L^2(S^1)} \leq \alpha \right\}$$

Notice that (3.3) implies

$$||u(s,\cdot)||_{L^2(S^1)} \leq \frac{2(\alpha+\delta)}{\nu'} =: R_\alpha \quad \forall s \in S_\alpha \setminus \mathcal{S} \qquad (3.4)$$

The length of the complement of S_α can be estimated as

$$|\mathbb{R}\setminus S_\alpha| \leq \frac{1}{\alpha^2}\int_{\mathbb{R}} ||\partial_s u(s,\cdot)||^2_{L^2(S^1)} ds \leq \frac{E'}{\alpha^2} = L_\alpha$$

Therefore if $I \subset \mathbb{R}$ is an interval of length $|I| = L > L_\alpha$, the previous estimate implies that $S_\alpha \cap I \neq \varnothing$ so there exists some $s_* \in S_\alpha \cap I$. Now, if we fix $\alpha < \sqrt{\frac{E'}{|\mathcal{S}|}}$, then $L_\alpha > |\mathcal{S}|$ so we can assume that $s_* \in (S_\alpha \setminus \mathscr{S}) \cap I$. Using the identity

$$u(s,t) = u(s_*,t) + \int_{s_*}^{s} \partial_s u(\sigma,t) d\sigma$$

and estimate (3.4) we obtain $\forall s \in I$

$$\begin{aligned} \|u(s,\cdot)\|_{L^2(S^1)} &= \int_{S^1} |u(s,t)|^2 dt \leq 2\left(\int_{S^1} |u(s_*,t)|^2 dt + \int_{S^1}\int_{s_*}^{s} |\partial_s u(\sigma,t)|^2 d\sigma dt\right) \leq \\ &\leq 2\left(R_\alpha^2 + LE'\right) = B_0^2 \end{aligned}$$

Integrating over I, we obtain that if $L_\alpha < |I| < \infty$ then

$$\|u\|_{L^2(I\times S^1)} \leq \sqrt{|I|} B_0 \tag{3.5}$$

Now, let $I \subset \mathbb{R}$ be an interval of length $L = L_\alpha + 1$ and $I' \supset I$ an interval of length at most $2L$. Denote $\overline{\partial}_{\mathscr{J}} u = \partial_s u + \mathscr{J}_{s,t}(u)\partial_t u$ the Cauchy-Riemann operator associated to $\mathscr{J}$. By the Calderòn-Zygmund inequality, for every $p \in [2,\infty)$ there exists a constant $C_p > 0$ such that

$$\|\nabla u\|_{L^p(I\times S^1)} \leq C_p \left[\|u\|_{L^p(I'\times S^1)} + \left\|\overline{\partial}_{\mathscr{J}} u\right\|_{L^p(I'\times S^1)}\right] \quad \forall j \in \mathbb{Z} \tag{3.6}$$

The constant C_p depends only on p and the length of I, i.e. L. Moreover $\overline{\partial}_{\mathscr{J}} u = \nabla_{\mathscr{J}} \mathscr{H} \circ u$, where $\nabla_{\mathscr{J}} \mathscr{H}$ is the gradient of $\mathscr{H}$ with respect to $g_{\mathscr{J}}$. Since $\mathscr{H}$ is asymptotically linear, we can use Lemma 2.3.2 to estimate

$$\begin{aligned} \left\|\overline{\partial}_{\mathscr{J}} u\right\|_{L^p(I'\times S^1)} &= \left\|\left(\nabla_{\mathscr{J}} \mathscr{H}\right)\circ u\right\|_{L^p(I'\times S^1)} \leq \left\|\left(\nabla_{\mathscr{J}} \mathscr{H}\right)\circ u - \mathbb{A}u\right\|_{L^p(I'\times S^1)} + \|\mathbb{A}u\|_{L^p(I'\times S^1)} \leq \\ &\leq B_1\left(\|u\|_{L^p(I'\times S^1)} + 1\right) \end{aligned} \tag{3.7}$$

where $B_1 > 0$ depends only on $\mathscr{H}$, $\mathbb{A}$ and $\mathscr{J}$. Now fix $p = 2$ and using the estimates (3.5), (3.6), (3.7) we obtain

$$\|u\|_{W^{1,2}(I\times S^1)} \leq \|u\|_{L^2(I\times S^1)} + \|\nabla u\|_{L^2(I\times S^1)} \leq \sqrt{2L}B_0 + C_p\left[\sqrt{2L}B_0 + B_1\left(\sqrt{2L}B_0 + 1\right)\right] = M_1$$

By the Sobolev embedding theorem for every $2 \leq p < \infty$ there is a constant $R_p > 0$ such that

$$\|u\|_{L^p(I\times S^1)} \leq R_p \|u\|_{W^{1,2}(I\times S^1)} = R_p M_1$$

where R_p depends only on the lenght of I. So applying Calderon-Zygmund again we have

$$\|u\|_{W^{1,p}(I\times S^1)} \leq \|u\|_{L^p(I'\times S^1)} + \|\nabla u\|_{L^p(I'\times S^1)} \leq M_2$$

and again M_2 depends only on p and the length L of I. We are now allowed to take a fixed $p > 2$ and use the Sobolev embedding theorem again to reach our conclusion:

$$||u||_{L^\infty(I\times S^1)} \leq B_p ||u||_{W^{1,p}(I\times S^1)} \leq B_p M_2 = R$$

where we are again the Sobolev constant B_p depends only on the length of I. Covering $\mathbb{R}$ by intervals of length $L \leq I \leq 2L$ therefore supplies us with the wanted estimate. All in all, since L depends only on E and $\mathscr{J}$, we have that R depends only on E, $\mathscr{J}$ and $\mathscr{H}$. The estimate is therefore independent of the particular solution u. □

3.3.2. First compactness result: uniform energy bounds

Combining the elliptic regularity estimates with the uniform L^∞-estimates, we obtain the first compactness result, which states that the space of solutions with uniformly bounded energy is compact. A key step is to show that an uniform energy bound implies an uniform gradient bound. Here the important concept of "bubbling of holomorphic curves" shows up.

Let us introduce some convenient notation.

Definition 3.6 1. Let $H \in \mathfrak{w}\mathfrak{H}$ and $J \in C^\infty\left(S^1 \times \mathbb{R}^{2n}, \operatorname{End}\mathbb{R}^{2n}\right)$ an adequate almost-complex structure. For $E > 0$, define

$$M(E;H,J) = \left\{u \in C^\infty\left(S^1 \times \mathbb{R}, \mathbb{R}^{2n}\right) : \overline{\partial}_{H,J} u = 0, \quad E_{H,J}(u) \leq E\right\}$$

2. Let $\mathscr{H}$ be an asymptotically quadratic continuation and $\mathscr{J} \in C^\infty\left(\mathbb{R} \times S^1 \times \mathbb{R}^{2n}, \operatorname{End}\mathbb{R}^{2n}\right)$ an adequate almost-complex structure. For $E > 0$, define

$$\mathscr{M}(E;\mathscr{H},\mathscr{J}) = \left\{u \in C^\infty\left(S^1 \times \mathbb{R}, \mathbb{R}^{2n}\right) : \overline{\partial}_{\mathscr{H},\mathscr{J}} u = 0, \quad E_{\mathscr{H},\mathscr{J}}(u) \leq E\right\}$$

These sets of solutions of Floer equations are usually called *spaces of bounded trajectories* (as in e.g. [29]), in analogy with finite-dimensional Morse theory.

Preparatory lemmata The first result we need is that there are no non-constant holomorphic planes in $\mathbb{R}^{2n}$. This follows from Gromov's removal of singularities theorem, and will be used in the bubbling argument which produces uniform gradient bounds from energy bounds.

Lemma 3.3.1 *Let $J \in C^\infty(\mathbb{R}^{2n}, \operatorname{End}\mathbb{R}^{2n})$ be an ω_0-compatible almost-complex structure and $v : \mathbb{C} \to \mathbb{R}^{2n}$ a J-holomorphic map with finite energy and bounded image. Then v is constant.*

Proof. By Gromov's removal of singularities theorem (see e.g. [32]), the map v extends to a J-holomorphic sphere. If v is non-constant, it must have positive ω_0-energy, in contradiction to Stokes' theorem. □

The second preparatory lemma we need is an abstract result in metric spaces. It's usually called Hofer's lemma or Half-Maximum lemma.

Lemma 3.3.2 *Let (X, d) be a complete metric space and $f: X \to \mathbb{R}$ a continuous, positive function. For every $x \in X$ and $\delta > 0$ there exists a $y \in X$ and $\varepsilon \in (0, \delta]$ such that:*

1. $d(x, y) \le 2\delta$
2. $\delta f(x) \le \varepsilon f(y)$
3. $\sup_{B_\varepsilon(x)} f \le 2f(y)$.

The proof of this lemma and similar results can be found in [14, Chapter IV].

Gradient bounds from bubbling Here we prove the key step, namely, that energy bounds imply gradient bounds. The proof hinges on the conformal "almost-symmetry" of the Floer equation.

Proposition 3.3.6 *For any $E > 0$ there exists a $B > 0$ such that*

$$u \in \mathcal{M}(E; \mathcal{H}, \mathcal{J}) \implies ||\nabla u||_{L^\infty} < B$$

Proof. Assume by contradiction that there is a sequence (σ_k, τ_k) such that

$$\lim_{k \to \infty} |\nabla u_k(\sigma_k, \tau_k)| = +\infty$$

Choose a sequence $\delta_k \to 0$ such that we still have

$$\lim_{k \to \infty} \delta_k |\nabla u_k(\sigma_k, \tau_k)| = +\infty$$

Using Hofer's lemma we obtain sequences $\varepsilon_k < \delta_k \to 0$ and $(s_k, t_k) \in \mathbb{R} \times S^1$ such that

$$\begin{aligned} |(\sigma_k - s_k, t_k - \tau_k)| &\le 2\delta_k \to 0, \\ \varepsilon_k |\nabla u_k(s_k, t_k)| &\ge \delta_k |\nabla u_k(\sigma_k, \tau_k)| \to +\infty, \\ 2|\nabla u_k(s_k, t_k)| &\ge \max_{(s,t) \in B_{\varepsilon_k}((s_k, t_k))} |\nabla u_k(s, t)| \end{aligned} \tag{3.8}$$

Set $R_k = |\nabla u_k(s_k, t_k)|$, so that $\varepsilon_k R_k \to +\infty$. Define

$$\begin{aligned} v_k(s, t) &= u_k\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k\right) \\ \mathcal{H}_k(s, t, z) &= \frac{1}{R_k} \mathcal{H}\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k, z\right) \\ \mathcal{J}_k(s, t, z) &= \mathcal{J}\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k, z\right) \end{aligned}$$

Notice that since the u_k are solutions of the $(\mathcal{H}, \mathcal{J})$-Floer equation,

$$\partial_s v_k + \mathcal{J}_k(s, t, v_k)\left[\partial_s v_k - X_{\mathcal{H}_k}(s, t, v_k)\right] = 0.$$

First, we claim that v_k is *never* a constant function. This follows from

$$\nabla v_k(s,t) = \frac{1}{R_k}\nabla u_k\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k\right) \implies |\nabla v_k(0,0)| = 1 \quad \forall k$$

Next, we claim that the sequence of functions

$$v_k : B_{\varepsilon_k R_k}(0,0) \to \mathbb{R}^{2n}$$

has a C^0_{loc}-converging subsequence. Indeed, by the last inequality in (3.8),

$$|\nabla v_k(s,t)| = \left|\frac{1}{R_k}\nabla u_k\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k\right)\right| \le \frac{2}{R_k}|\nabla u_k(s_k,t_k)| \le 2 \quad \forall (s,t) \in B_{\varepsilon_k R_k}(0,0).$$

This implies that the sequence v_k is equicontinuous. Moreover, notice that

$$\sup_k E_{\mathscr{H},\mathscr{J}}(u_k) \le E \implies \sup_k ||u_k||_{L^\infty} \le R$$

for an $R > 0$ given by Proposition 3.3.5. We conclude that

$$\sup_k ||v_k||_{L^\infty\left(B_{\varepsilon_k R_k}(0,0)\right)} \le R$$

since v_k is just a shifted and rescaled version of u_k. Hence the Ascoli-Arzelà theorem implies that there exists a subsequence, still denoted by v_k, such that $v_k \to v$ in C^0_{loc} where $v : \mathbb{R}^2 \to \mathbb{R}^{2n}$ is a continuous function. Using the elliptic regularity machine of Section 3.3, we can promote this limit to a C^∞_{loc} limit, so that we obtain a further subsequence, denoted still by v_k, which converges C^∞_{loc} to a *smooth* $v : \mathbb{R}^2 \to \mathbb{R}^{2n}$.

Since the v_k solve equation 3.3.2, we claim that, up to choosing a further subsequence, v solves the holomorphic curve equation $\overline{\partial}_J v = 0$ for an appropriate almost-complex structure J on $\mathbb{R}^{2n}$. Indeed, since all the v_k have image contained in the ball $B_R(0)$, we can estimate, remembering that $\mathscr{H}$ depends on s only on a bounded interval $\mathscr{S} \subset \mathbb{R}$,

$$\left|X_{\mathscr{H}_k}(s,t,v_k(s,t))\right| = \left|\frac{1}{R_k}X_{\mathscr{H}}\left(\frac{s}{R_k} - s_k, \frac{t}{R_k} - t_k, v_k(s,t)\right)\right| \le \frac{1}{R_k}\max_{\mathscr{S}\times S^1\times B_R(0)}|X_{\mathscr{H}}| \xrightarrow[k\to\infty]{} 0$$

This shows that the Hamiltonian term vanishes in the limit. To find the almost-complex structure, there are two possible cases: either s_k is bounded or it is unbounded. In case it is bounded, we extract a converging sequence $s_k \to s_*$. Since $t_k \in S^1$ always, we also can assume $t_k \to t_*$. Hence we can set $J(z) = \mathscr{J}(-s_*, -t_*, z)$. In case it is unbounded, we can extract a subsequence $s_k \to -\infty$ or a subsequence $s_k \to +\infty$. In both cases, since $\mathscr{J}$ depends on s only on a bounded subset $\mathscr{S}$ of $\mathbb{R}$, we are allowed to set $J(z) = \mathscr{J}(s, -t_*, z)$ for $s > \max \mathscr{S}$ or $s < \min \mathscr{S}$ accordingly.

Finally notice that $|\nabla v(0,0)| = 1$, $||\nabla v||_{L^\infty} \le 2$. So we found that v is a non-constant (i,J)-

holomorphic plane, almost a contradiction. If we show that its energy is bounded, then a contradiction is reached. Set $B_k = B_{\varepsilon_k}((s_k, t_k))$.

$$\begin{aligned}\int_{B_{\varepsilon_k R_k}(0,0)} |\nabla v_k|^2_{g_J} = \int_{B_k} |\nabla u_k|^2_{g_J} \leq \int_{B_k} |\nabla u_k|^2_{g_J} \leq \\ \leq \int_{B_k} |\partial_s u_k|^2_{g_J} + |\partial_t u_k - X_H \circ u_k|^2_{g_J} + \int_{B_k} |X_H \circ u_k|^2_{g_J} \leq \\ \leq E_{H,J}(u_k) + \int_{B_k} |X_H \circ u_k|^2_{g_J} \leq E + e_k\end{aligned}$$

Now $e_k \to 0$ because $B_k = B_{\varepsilon_k}((s_k, t_k))$ which shrinks to 0 as $k \to \infty$. We conclude from this that $E_J(v) \leq E$. So by removal of singularities v extends to a *non-constant* holomorphic sphere $v\colon S^2 \to \mathbb{R}^{2n}$ of finite energy, which is a contradiction. □

Remark The holomorphic sphere which was found in the contradiction argument is called a *bubble*, and the rescaling and limiting process which leads to it is called *bubbling*. It is worth to note that bubbling is ruled out only because we are studying the Floer equation in an exact symplectic manifold, in which Lemma 3.3.1 holds. A more general situation where the same result holds would be the case of *symplectically aspherical* manifolds, which are those symplectic manifolds in which the symplectic form evaluates to zero over every spherical homology class. In more general situations, bubbling cannot be avoided, and is a true obstruction to compactness which has to be reckoned with by different means.

The first compactness theorem Proposition 3.3.6 lets us obtain a first compactness theorem.

Theorem 2 *Let $H \in \mathfrak{w}\mathfrak{H}$, J an adequate almost-complex structure, and $E > 0$. The space $M(E;H,J)$ is compact in the C^∞_{loc} topology. The same can be said for the space $\mathscr{M}(E;\mathscr{H},\mathscr{J})$ where $\mathscr{H}$ is an asymptotically quadratic continuation and $\mathscr{J}$ an adequate almost-complex structure.*

Proof. We prove the theorem for $\mathscr{M}(E;\mathscr{H},\mathscr{J})$ as the autonomous case follows from that. Let $(u_k) \subset \mathscr{M}(E;\mathscr{H},\mathscr{J})$ be any sequence. By Proposition 3.3.5 and Proposition 3.3.6 we can apply Ascoli-Arzelà and obtain a C^0_{loc} converging subsequence $u_k \to u$. By elliptic regularity 3.3.4 this limit is promoted to a C^∞_{loc} limit. It follows that $u \in \mathscr{M}(E;\mathscr{H},\mathscr{J})$ which concludes the proof. □

3.4. Asymptotics of trajectories

The Floer equation is *not really* a gradient flow for the action functional. So the asymptotics of a solution don't follow immediately from its definition. We nevertheless can prove that Floer curves are asymptotic to periodic orbits at their ends.

Let $H \in \mathfrak{w}\mathfrak{H}$, i.e. a weakly asymptotically quadratic Hamiltonian with non-degenerate quadratic Hamiltonian at infinity, and J an adequate almost-complex structure. In this section we work with half-cylinder solutions $u\colon [S_0,+\infty)\times S^1 \to \mathbb{R}^{2n}$ of the Floer equation

$$\partial_s u + J_t(u)[\partial_t u - X_H(t,u)] = 0$$

as this permits the result to be applied to both the autonomous and continuation Floer equation. This is because we take continuations to be eventually independent of the s-coordinate. Completely analogous properties hold when one works with a negative half-cylinder.

Theorem 3 *Let $u\colon [S_0,+\infty)\times S^1 \to \mathbb{R}^{2n}$ be a Floer half-cylinder with finite energy and bounded image. For every sequence $\sigma_k^+ \to +\infty$ there exist a subsequence $\sigma_{k_j}^+ \to +\infty$ and a 1-periodic orbit $\gamma_+\colon S^1 \to \mathbb{R}^{2n}$ of X_H such that*

$$\lim_{j\to\infty} u\left(\sigma_{k_j}^+,\cdot\right) = \gamma, \quad \lim_{j\to\infty} \partial_s u\left(\sigma_{k_j}^+,\cdot\right) = 0 \quad \text{in } C^\infty$$

Proof. Let $\left(\sigma_k^+\right)\subset\mathbb{R}$ be a monotone sequence diverging to $+\infty$. Define the maps

$$u_k\colon [S_0,+\infty)\times S^1 \to \mathbb{R}^{2n} \quad u_k(s,t) = u(s+\sigma_k^+,t)$$

These are of course smooth maps, and, since J and X_H are independent of the s-coordinate, they are also solutions of the Floer equation. We claim that the sequence u_k converges (up to subsequences) in C^∞_{loc} to a Floer cylinder $u_{+\infty}$. We know that the sequence u_k is uniformly bounded, since u has bounded image. With a bubbling argument completely analogous to the one found in Proposition 3.3.6, one shows that u_k must be C^1-bounded. By Ascoli-Arzelà, $u_k \to u_{+\infty}$ in C^0_{loc} up to a subsequence. By the elliptic regularity machine we can extract a (further) subsequence which converges in C^∞_{loc} to a $u_{+\infty} \in C^\infty([S_0,+\infty)\times S^1,\mathbb{R}^{2n})$. This function $u_{+\infty}$ must be a solution of the Floer equation, because it is the C^∞_{loc} limit of solutions of the Floer equation.

Now we have to estimate the energy of $u_{+\infty}$. Since J is adequate, there is a constant depending only on J such that

$$E_{HJ}(u_{+\infty}) \le C \lim_{S\to\infty} \int_{-S}^{S} \|\partial_s u_{+\infty}(s,\cdot)\|^2_{L^2(S^1)}$$

But now, since $|\partial_s u| \in L^2\left(\mathbb{R}\times S^1\right)$, and using Fatou's lemma, for any $S>0$

$$\int_{-S}^{S} \|\partial_s u_{+\infty}(s,\cdot)\|^2_{L^2(S^1)} = \int_{-S}^{S} \lim_{k\to\infty} \left\|\partial_s u(s+\sigma_k^+,\cdot)\right\|^2_{L^2(S^1)} ds \le$$

$$\le \liminf_{k\to\infty} \int_{-S+\sigma_k^+}^{S+\sigma_k^+} \|\partial_s u(s,\cdot)\|^2_{L^2(S^1)} ds = 0$$

Hence our Floer cylinder has zero energy $E_{H,J}(u) = 0$. But this implies that $\partial_s u_{+\infty} = 0$, so setting $\gamma_+(t) = u_{+\infty}(t)$, the Floer equation reads

$$J_t(\gamma_+)\left[\dot{\gamma}_+ - X_H(t,\gamma_+)\right] = 0 \iff \dot{\gamma}_+ = X_H(t,\gamma_+)$$

To obtain the limit of the s-derivative, one simply uses the Floer equation and the C^∞ convergence found above. □

By Proposition 3.3.5, autonomous and continuation Floer cylinders have an *a priori* uniform L^∞-bound. Hence, a half-cylinder obtained by chopping off one end from a Floer cylinder has bounded image. From this the next corollary follows.

Corollary 3.4.1 *1. Let $H \in \mathfrak{wH}$ and J be an adequate almost complex structure. Let $u \in M(E;H,J)$ for some $E > 0$. For every $\sigma_k^\pm \to \pm\infty$ there exist 1-periodic orbits $\gamma_\pm \in \mathrm{Per}^1 H$ such that, up to a subsequence,*

$$\lim_{k\to\infty} u\left(\sigma_k^\pm, t\right) = \gamma_\pm(t), \quad \lim_{k\to\infty} \partial_s u\left(\sigma_k^\pm, t\right) = 0 \quad \text{in } C^\infty$$

2. Let $\mathcal{H}$ be an asymptotically quadratic continuation between Hamiltonians $H^\pm \in \mathfrak{wH}$ and $\mathcal{J}$ an adequate almost-complex structure. Let $u \in \mathcal{M}(E;\mathcal{H},\mathcal{J})$ for some $E > 0$. For every $\sigma_k^\pm \to \pm\infty$ there exist 1-periodic orbits $\gamma_\pm \in \mathrm{Per}^1 H^\pm$ such that, up to a subsequence,

$$\lim_{k\to\infty} u\left(\sigma_k^\pm, t\right) = \gamma_\pm(t), \quad \lim_{k\to\infty} \partial_s u\left(\sigma_k^\pm, t\right) = 0 \quad \text{in } C^\infty$$

Remark We would like to stress that the limit orbit obtained is not unique, and depends on the sequence $\sigma_k \to +\infty$ chosen. Different choices of sequences can lead to different asymptotic orbits. Examples of this phenomenon are known in other contexts, see e.g. the paper of Siefring [40] for non-uniqueness of the limiting orbit of a holomorphic plane in the symplectization of a degenerate contact manifold. It is therefore natural to expect that such phenomenon arises also in Floer homology. Uniqueness is guaranteed *only* when orbits are isolated, for example in the non-degenerate case, as we shall discuss in the next section.

3.4.1. Uniqueness of the asymptotics

The goal here is to show that if the 1-periodic orbits of a Hamiltonian are isolated, then the asymptotics of its Floer trajectories are unique. From the uniqueness of the asymptotics we will obtain an energy calculation for Floer cylinders with non-degenerate ends.

We will also derive a kind of quantitative refinement of this phenomenon, which states that when a piece of Floer cylinder has small enough energy, its image must be contained in a neighborhood of a fixed 1-periodic orbit. This is known in the trade as the "long cylinders with small energy" lemma.

3.4.1.1. Uniqueness of the asymptotics and an energy calculation

Definition 3.7 A 1-periodic orbit $\gamma\colon S^1 \to \mathbb{R}^{2n}$ is said to be isolated when there exists an open neighborhood $\mathscr{U} \subset S^1 \times \mathbb{R}^{2n}$ of the graph of γ which does not contain the graph of any other 1-periodic orbit.

From isolation and Theorem 3 it follows immediately that

Proposition 3.4.1 *Let $H \in \mathfrak{wH}$ and J be an adequate almost complex structure. Assume that all the 1-periodic orbits of H are isolated. If $u\colon [S_0, +\infty) \times S^1 \to \mathbb{R}^{2n}$ is a finite-energy Floer half-cylinder with bounded image, then there exists an unique $\gamma_+ \in \mathrm{Per}^1 H$ such that*

$$\lim_{s\to+\infty} u(s,t) = \gamma_+(t), \quad \lim_{s\to+\infty} \partial_s u(s,t) = 0 \quad \textit{in } C^\infty$$

Having unique asymptotics is useful in order to estimate the energy of a Floer trajectory. This is contained in the following calculation.

Lemma 3.4.1 *Let $\mathscr{H} = \mathscr{H}^s$ be a asymptotically quadratic continuation between weakly asymptotically quadratic Hamiltonians $H^\pm$ all whose 1-periodic orbits are isolated. Let $\mathscr{J}$ be an adequate almost-complex structure. Take $\gamma^\pm \in \mathrm{Per}^1 H^\pm$. If $u \in C^\infty\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$ solves the continuation Floer equation for $(\mathscr{H}, \mathscr{J})$ with asymptotic orbits $\gamma^\pm$ as $s \to \pm\infty$, then*

$$E_{\mathscr{H},\mathscr{J}}(u) = \mathscr{A}_{H^-}\left(\gamma^-\right) - \mathscr{A}_{H^+}\left(\gamma^+\right) - \int_{\mathbb{R}\times S^1} (\partial_s \mathscr{H})(s,t,u(s,t))\, ds\, dt$$

Proof. Recall that if $u \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ solves the continuation Floer equation

$$\partial_s u + \mathscr{J}_t^s(u)[\partial_t u - X_{\mathscr{H}_t^s}(u)] = 0$$

then its energy satisfies

$$E_{\mathscr{H},\mathscr{J}}(u) = \|\partial_s u\|^2_{L^2(g_{\mathscr{J}})} = \int_{\mathbb{R}\times S^1} (\omega_0)_{u(s,t)} (\mathscr{J}_t^s(u(s,t))\partial_s u(s,t), \partial_s u(s,t))\, ds\, dt$$

Here we've used the formula for the associated family of Riemannian structures:

$$g_{\mathscr{J}}(s,t,z)(u,v) = \omega_0(u, \mathscr{J}_t^s(z)v)$$

Recall further that the differential of the action functional $\mathscr{A}_H\colon W^{1,2}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ is given by

$$d\mathscr{A}_H|_\gamma \xi = \int_{S^1} (\omega_0)_{\gamma(t)} (\dot\gamma(t) - X_H \circ \gamma(t), \xi(t))\, dt \quad \forall \xi \in W^{1,2}\left(\gamma^* T\mathbb{R}^{2n}\right)$$

Thinking of u as an s-family of loops from γ^- to γ^+, and using the continuation Floer equation, we can compute:

$$
\begin{aligned}
E_{\mathcal{H},\mathcal{J}}(u) &= \int_{\mathbb{R}\times S^1} (\omega_0)_{u(s,t)}(\partial_s u(s,t), \mathcal{J}_t^s(u(s,t))\partial_s u(s,t))\, ds dt = \\
&= \int_{\mathbb{R}\times S^1} (\omega_0)_u(-\mathcal{J}_t^s(u)[\partial_t u - X_{\mathcal{H}}(u)], \mathcal{J}_t^s(u)\partial_s u)\, ds dt = \\
&= -\int_{\mathbb{R}}\int_{S^1} (\omega_0)_u(\partial_t u - X_{\mathcal{H}}(u), \partial_s u)\, ds dt = -\int_{\mathbb{R}} d\mathcal{A}_{\mathcal{H}^s}|_{u(s,\cdot)}\partial_s u(s,\cdot) ds =
\end{aligned}
$$

Now, we use the chain rule and compute

$$
\begin{aligned}
-\int_{\mathbb{R}} d\mathcal{A}_{\mathcal{H}^s}|_{u(s,\cdot)}\partial_s u(s,\cdot) ds &= -\int_{\mathbb{R}} \left\{ \frac{d}{ds}[\mathcal{A}_{\mathcal{H}^s}(u(s,\cdot))] - \frac{\partial \mathcal{A}_{\mathcal{H}^s}}{\partial s}(u(s,\cdot)) \right\} ds = \\
&= -\mathcal{A}_{\mathcal{H}^s}(u(s,\cdot))|_{s\to-\infty}^{s\to+\infty} + \int_{\mathbb{R}} \frac{\partial \mathcal{A}_{\mathcal{H}^s}}{\partial s}(u(s,\cdot)) ds = \\
&= \mathcal{A}_{H^-}(\gamma^-) - \mathcal{A}_{H^+}(\gamma^+) - \int_{\mathbb{R}\times S^1} (\partial_s \mathcal{H})(s,t,u(s,t))\, ds dt
\end{aligned}
$$

which was our claim. □

Remark In particular, in the autonomous case, we obtain the well known energy formula: if $H \in \mathfrak{wH}$ has only isolated 1-periodic orbits, J adequate and $u \in M(E;H,J)$ for some $E > 0$, then

$$
E_{H,J}(u) = \mathcal{A}_H(\gamma_-) - \mathcal{A}_H(\gamma_+) \tag{3.9}
$$

where $u(s,\cdot) \to \gamma_\pm$ as $s \to \pm\infty$.

3.4.1.2. Long cylinders with small energy

Now we give a refinement of Theorem 3 in the case that the orbits are non-degenerate. It shows that cylinders with small energy must approach 1-periodic orbits and stay within a neighborhood of them for long times. It is completely analogous to one of the main theorem in [28].

Let $F \in \mathfrak{wH}$ have only isolated 1-periodic orbits. Since by Lemma 2.3.4 these are contained in a compact set, they must come in a finite number. Therefore, the set of 1-periodic orbits

$$
\mathrm{Per}^1 F = \{\gamma \in C^\infty(S^1, \mathbb{R}^{2n}) : \dot\gamma = X_F \circ \gamma\} \subset C^\infty(S^1, \mathbb{R}^{2n})
$$

admits a neighborhood $\mathcal{W}$ in the C^∞-topology with the property that every connected component of $\mathcal{W}$ contains only one periodic orbit.

Define the “quantum of energy” associated to F by

$$
\hbar_F = \min\{|\mathcal{A}_F(\gamma_1) - \mathcal{A}_F(\gamma_0)| : \gamma_0, \gamma_1 \in \mathrm{Per}^1 F,\ \mathcal{A}_F(\gamma_0) \neq \mathcal{A}_F(\gamma_1)\}
$$

By finiteness of the set of orbits we conclude that $\hbar_F > 0$.

Proposition 3.4.2 *Let $F \in \mathfrak{wH}$ have only isolated 1-periodic orbits and $\mathscr{W} \subset C^\infty(S^1, \mathbb{R}^{2n})$ as above. Choose any adequate almost complex structure J. For every neighborhood $\mathscr{W}' \subset \mathscr{W}$ there exists a constant $\overline{S} > 0$ with the following property. If $S > \overline{S}$, $u \in C^\infty([-S,S] \times S^1, \mathbb{R}^{2n})$ is a non-constant solution of the (F,J)-Floer equation with*

$$E_{F,J}(u) < \hbar_F$$

and bounded image, then

$$u(s,\cdot) \in \mathscr{W}' \quad \forall s \in \left[-S+\overline{S}, S-\overline{S}\right]$$

Proof. We argue by contradiction. Assume there exists a non-decreasing sequence $S_k \to \infty$ with $S_0 > \overline{S}$, a sequence $u_k \in C^\infty\left([-S_k, S_k] \times S^1, \mathbb{R}^{2n}\right)$ of non-constant solutions of the (F,J)-Floer equation with $E_{F,J}(u_k) < \hbar_F$ for all k but

$$u_k(s_k,\cdot) \notin \mathscr{W}' \text{ for some sequence } s_k \in \left[-S_k+\overline{S}, S_k-\overline{S}\right]$$

Shifting everything by s_k we can actually assume that $u_k(0,\cdot) \notin \mathscr{W}'$. By a bubbling off analysis completely analogous to the one in Proposition 3.3.6, we conclude that the gradients of u_k are uniformly bounded. Bootstrapping gives us C^∞_{loc} convergence, up to subsequences, to a solution $u: \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ of the (F,J)-Floer equation. The solution u must be non-constant because for $s = 0$ it leaves the neighborhood $\mathscr{W}'$, so it is not close to any periodic orbit of F. The uniform energy bound along the sequence gives $E_{F,J}(u) < \hbar_F$. But since u is non-constant, by (3.9) we know that $E_{F,J}(u) = \mathscr{A}_F(\gamma_1) - \mathscr{A}_F(\gamma_0) < \hbar_F$ for some $\gamma_1 \neq \gamma_0$. But this is possible only if $\mathscr{A}_F(\gamma_0) = \mathscr{A}_F(\gamma_1)$, by definition of $\hbar_F$. In that case, $E_{F,J}(u) = 0$, which implies that u must be a constant solution. This is a contradiction. □

Remark Notice that by the way we have chosen the neighborhood $\mathscr{W}$, it follows that there exists a 1-periodic orbit $\gamma \in \text{Per}^1 F$ and a neighborhood $\mathscr{U}$ of $\gamma(S^1)$ where u enters and stays within for a long enough time.

Remark We gain the following picture of the set of Floer trajectories with bounded energy, at least in the non-degenerate case (see Figure 3.1): a sequence of Floer trajectories with bounded energy has a converging subsequence, whose limit also has bounded energy. The limit therefore also is asymptotic to some 1-periodic orbits, but they don't necessarily coincide with the asymptotics of cylinders in the initial sequence. We will see how to refine this picture in the next section.

3.5. Transversality and moduli spaces

Here we delve in the description of the moduli spaces with fixed asymptotics. The main goal is to show that families of Floer cylinders with fixed asymptotics are smooth manifolds. We will refer to the relevant chapters in [7] for the proofs of the statements which we present. There it

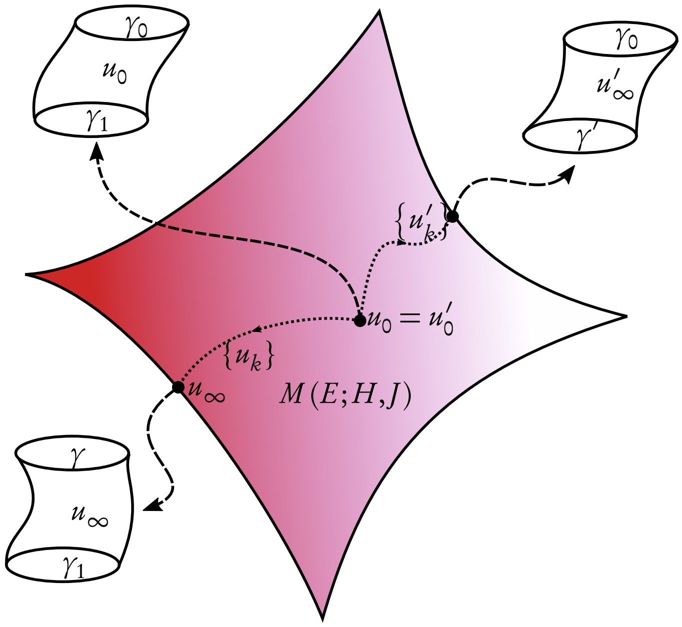

Figure 3.1. We know that the set of trajectories with bounded energy is C^∞_{loc}-compact, and that in the non-degenerate case trajectories with bounded energy have unique asymptotics. But the limit of a sequence with fixed asymptotics is not guaranteed to have the same asymptotics of the sequence converging to it. This phenomenon is the starting observation behind the theory of *broken convergence*.

is assumed that the target symplectic manifold is compact. We substitute this assumption with the uniform L^∞-estimates found above. Once that is done, the proofs all follow almost word by word, the only difference being that regarding the continuation Floer equation, one must *provide* the uniform energy bounds needed to guarantee the uniform L^∞-bounds.

In the definition of Floer homology, we need three types of moduli spaces of Floer cylinders with fixed asymptotics: the autonomous, non-autonomous, and parametrized moduli spaces. The first is necessary for the definition of the differential of the Floer chain complex, the second for the definition of continuation morphisms on Floer homology, and the third for the definition of chain homotopies of continuation morphisms. All three have mostly similar differential topological properties. To show that they are smooth finite-dimensional manifolds, one shows that the Floer operator is a Fredholm section of some Banach bundle, and that, up to a generic choice, it intersects the zero section of this bundle transversely. Moreover their dimension, which is the Fredholm index of the operator defining the linearization of the Floer equation, is computed in terms of the algebraic invariants attached to the ends of the cylinders, i.e. the Conley-Zehnder indexes of the asymptotic orbits.

3.5.1. Transversality theory for the Floer equation

The solution set of the Floer equation can be interpreted geometrically as a subset of a suitable Banach bundle, given by the intersection of the zero section with the section defined by the Floer operator. The linearization of the Floer equation gives rise to a Fredholm operator, which means that the Floer operator is a Fredholm section of the Banach bundle in question. These two facts combined with the Sard-Smale theorem give us a way to show that the moduli spaces of Floer trajectories are smooth manifolds of finite dimension, and a way to compute their dimension. We treat the case of the "autonomous" Floer equation, i.e.

$$\partial_s u + J_t(u)[\partial_t u - X_H(t,u)] = 0, \quad \lim_{s\to\pm\infty} u(s,\cdot) = \gamma^\pm \in \mathrm{Per}^1 H$$

for an adequate pair (H,J) with $H \in \mathfrak{w}\mathfrak{H}_*$. The case of the continuation Floer equation is, in some sense, easier: in the non-autonomous case of the continuation Floer equation, there is a "larger" space of perturbations which makes transversality more likely to be reached. We follow again [15, §3.2].

3.5.1.1. Setting up the Fredholm theory

Fix once and for all a $p > 2$. Let $\gamma^\pm \in \mathrm{Per}^1 H$. Let $u_0 \in C^\infty\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$ be a smooth map such that there exists an $s_0 > 0$ for which

$$u(s,t) = \gamma^-(t) \quad \forall s \le -s_0, \quad u(s,t) = \gamma^+(t) \quad \forall s \ge s_0.$$

Define $\mathscr{B}_0 = u_0 + W^{1,p}\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$ and $\mathscr{E}_1 = L^p\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$. $\mathscr{B}_0$ is to be thought of as an affine space modelled over $W^{1,p}\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$. In particular, if $u \in \mathscr{B}_0$, then $T_u\mathscr{B}_0 = W^{1,p}\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$ is the set of $W^{1,p}$-vector fields along u.

The Floer operator is a section

$$\overline{\partial}_{H,J}\colon \mathscr{B}_0 \to \mathscr{E}_1, \quad \overline{\partial}_{H,J}(u) = \partial_s u + J_t(u)[\partial_t u - X_H(u)]$$

We need to show that if $u \in C^\infty\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$ solves the Floer equation and has finite energy, then it belongs to $\mathscr{B}_0$ for suitably chosen $\gamma^\pm$. We know that for a given finite-energy solution u, since the Hamiltonian is non-degenerate, there exist unique $\gamma^\pm \in \mathrm{Per}^1 H$ to which u tends as $s \to \pm\infty$. Now, non-degeneracy of the asymptotic orbits is a crucial hypothesis to obtain estimates on the rate of convergence of this limit, which are necessary to show that $u \in \mathscr{B}_0$. In order to do that we need the the linearization of the Floer operator, i.e. its derivative $D\overline{\partial}_{H,J}(u)\colon W^{1,p}\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right) \to L^p\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right)$. A simple computation shows that the linearization of $\overline{\partial}_{H,J}$ at an $u \in \mathscr{B}_0$ applied to a $Y \in W^{1,p}$ is given by

$$\begin{aligned} D\overline{\partial}_{H,J}(u)Y &= \partial_s Y + DJ_t(u)Y\cdot[\partial_t u - X_H(u)] + J_t(u)[\partial_t Y + J_0 \operatorname{Hess} H(u) Y] = \\ &= \partial_s Y + \tilde{J}_{s,t}\partial_t Y + \tilde{J}_{s,t}J_0 \operatorname{Hess} H_t(u)Y + D\left\{\tilde{J}_{s,t}V_{s,t}\right\}Y - \tilde{J}_{s,t}DV_{s,t}\cdot Y \end{aligned} \tag{3.10}$$

We abbreviated $\tilde{J}_{s,t} = J_t(u(s,t))$ and $V = \partial_t u - X_H(u)$. Notice that this is a linear operator of the form

$$D\overline{\partial}_{H,J}(u)Y = \partial_s Y + \tilde{J}_{s,t}\left[\partial_t Y + J_0 A_t^s Y\right] + \eta_{s,t} Y$$

where

$$\tilde{J}_{s,t} = J_t(u(s,t)), \quad A_t^s = \operatorname{Hess} H(t, u(s,t)), \quad \eta_{s,t} \xrightarrow[s\to\pm\infty]{} \mathbb{O}$$

the limit of $\eta_{s,t}$ being zero since $\partial_t u - X_H(u) \to 0$ as $s \to \pm\infty$. Moreover, notice that the operator

$$D_{A_t^s} = \partial_t + J_0 A_t^s$$

tends to invertible operators as $s \to \pm\infty$, by non-degeneracy of the asymptotic orbits. These invertible operators are called *asymptotic operators*.

We now have the following estimates for functions solving the linear Floer equation with non-degenerate asymptotic operator.

Proposition 3.5.1 *Let $\tilde{J} \in C^\infty\left(\mathbb{R} \times S^1, \operatorname{End}\mathbb{R}^{2n}\right)$ be a family of ω_0-compatible domain dependent complex structures, $A \in C^\infty\left(\mathbb{R} \times S^1, \operatorname{Sym}(2n)\right)$, $\eta \in C^\infty\left(\mathbb{R} \times S^1, \operatorname{End}\mathbb{R}^{2n}\right)$. Assume that $\tilde{J}_{s,t} \to J_t^\pm$ some ω_0-compatible t-dependent complex structures as $s \to \pm\infty$, $A_t^s \to A_t^\pm$ some loops of symmetric matrices generating* non-degenerate *paths of symplectic matrices as $s \to \pm\infty$ and $\eta_{s,t} \to \mathbb{O}$ as $s \to \pm\infty$, all limits being C^∞ limits. If $Y : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is a function such that* $\sup_{s\in\mathbb{R}} \|Y(s,\cdot)\|_{L^2(S^1)} < \infty$ *and which weakly solves the equation*

$$\partial_s Y + \tilde{J}_{s,t}\left[\partial_t Y + J_0 A_t^s Y\right] + \eta_{s,t} Y = 0$$

then $Y \in C^\infty\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$ and there exists constants $b, c > 0$ such that

$$|Y(s,t)| \le b e^{-c|s|} \quad \forall (s,t) \in \mathbb{R} \times S^1$$

The proof can be found essentially in [7, §8.9]. The idea is that using invertibility of the asymptotic operators, one may show that the function $f(s) = \frac{1}{2}\|Y(s)\|_{L^2(S^1)}$ solves a differential inequality of the form $f'' \ge -cf$ for a $c > 0$ and $|s|$ large, and that $g(s,t) = |Y(s,t)|^2$ solves a differential inequality of the form $\Delta g \ge -\alpha g$ for an $\alpha > 1$ and $|s|$ large. The estimates follow from standard estimates of decay for solutions of these differential inequalities.

This proposition can be used to show that a finite-energy solution u of the (H,J)-Floer equation belongs to $\mathscr{B}_0$ for $\gamma^\pm$ its non-degenerate asymptotic orbits. Indeed, set $Y = \partial_s u$. The finite energy condition implies that $\sup_{s\in\mathbb{R}} \|\partial_s u(s,\cdot)\|_{L^2(S^1)} < \infty$. Moreover,

$$\begin{aligned}\partial_s Y &= -DJ_t(u)Y\left[\partial_t u - X_H(u)\right] - J_t(u)\left[\partial_t Y - DX_H(u)Y\right] = \\ &= -\tilde{J}_{s,t}\left[\partial_t Y + J_0 A_t^s Y\right] - \eta_{s,t} Y\end{aligned}$$

with $\tilde{J}_{s,t}$, A_t^s and $\eta_{s,t}$ satisfying the hypotheses of the Proposition 3.5.1. Therefore we obtain an estimate

$$|\partial_s u(s,t)| \le b e^{-c|s|} \quad \forall (s,t) \in \mathbb{R} \times S^1$$

Iterating this reasoning on higher-order derivatives in s we conclude that for every $\alpha \in \mathbb{N}$, $\alpha \geq 1$ there exist $b_\alpha, c_\alpha > 0$ such that

$$|\partial_s^\alpha u(s,t)| \leq b_\alpha e^{-c_\alpha |s|} \quad \forall (s,t) \in \mathbb{R} \times S^1$$

From these estimates it follows immediately that $u \in \mathscr{B}_0$.

Remark We stress the fact that these estimates were reached under two crucial hypotheses, which, as far as the author knows, *cannot* be removed: the solution u has finite energy and its asymptotic orbits are non-degenerate.

3.5.1.2. The transversality theorem

In this section we explain the following properties of the Floer equation. First of all, that $\overline{\partial}_{HJ}$ is a (smooth) Fredholm map. Then the Fredholm index of its linearization at a solution u is given by the difference of the Conley-Zehnder indices of the asymptotic orbits $\gamma^\pm$ of u. Finally, for every fixed $H \in \mathfrak{w}\mathfrak{H}_*$ there exists a residual set of adequate almost-complex structures J such that 0 is a regular value for $\overline{\partial}_{HJ}$ for all choices of asymptotic orbits $\gamma^\pm$.

Proposition 3.5.2 $\overline{\partial}_{HJ} \colon \mathscr{B}_0 \to \mathscr{E}_1$ *is a Fredholm map.*

Proof sketch. The equation (3.10) shows that the operator $D\overline{\partial}_{HJ}(u) \colon W^{1,p} \to L^p$ is of the form

$$D\overline{\partial}_{HJ}(u)Y = \partial_s Y + \tilde{J}_{s,t}\partial_t Y + \Theta_{s,t} Y = \left[\overline{\partial}_{\tilde{J}} + \Theta\right] Y$$

where we have set

$$\Theta(s,t) = D\left[J_t(u)(\partial_t u - X_H(t,u))\right] - J_t(u) D\left[\partial_t u - X_H(t,u)\right]$$

The local elliptic estimates for the linear Cauchy-Riemann operator found in Proposition 3.3.1 and the characterization of pre-Fredholm operators found in Lemma A.1 can be used to show that $D\overline{\partial}_{HJ}(u)$ has finite dimensional kernel and closed range, i.e. it is a pre-Fredholm operator. To conclude that it is a Fredholm operator, we apply a similar treatment to its L^2-adjoint operator, concluding that it is pre-Fredholm. If an operator and its adjoint are pre-Fredholm, then they are obviously both Fredholm. The argument is standard and can be found e.g. in Salamon's lecture notes [37] or in [7, §8.7]. □

The computation of the Fredholm index of the linearized Floer operator is the following proposition. Its proof hinges on the invariance of the Fredholm index under perturbations by compact operators, and invariance under homotopy. This reduces the problem to a very simple, explicit model where the kernel and cokernel may be exhibited explicitly via a computation. This strategy of proof is the one of [7, §8.8], and we do not repeat it here.

Proposition 3.5.3 *If $u \in \mathscr{B}_0$ is a solution of the Floer equation, then $D\overline{\partial}_{H,J}(u)$ has Fredholm index*

$$\operatorname{ind} D\overline{\partial}_{H,J}(u) = \mathrm{CZ}(\gamma^-, H) - \mathrm{CZ}(\gamma^+, H)$$

Remark Another interesting approach is via the spectral flow of a path of self-adjoint operators, which measures the net change in the number of negative eigenvalues along the path. This is the approach chosen by Robbin and Salamon [36]. The appeal of this approach is that philosophically, it justifies the idea that the difference of Conley-Zehnder indices is a kind of relative Morse index, where the Morse index itself is not well defined. In the work of Abbondandolo [2, 3] one may find a definition of the relative Morse index and its relation with the Conley-Zehnder index.

Denote by $\mathfrak{J}$ the space of S^1-families of adequate almost-complex structures, i.e. smooth ω_0-compatible $t \in S^1$-dependent almost-complex structures which are bounded, in the sense that $||J||_{L^\infty(S^1 \times \mathbb{R}^{2n}, \operatorname{End}\mathbb{R}^{2n})} < \infty$ for all $J \in \mathfrak{J}$ (see Definition 3.2). For a fixed $J \in \mathfrak{J}$, the tangent space of $\mathfrak{J}$ at J is given by the set of smooth maps $\mathrm{j}: S^1 \times \mathbb{R}^{2n} \to \operatorname{End}\mathbb{R}^{2n}$ such that

$$\omega_0(\mathrm{j}_t(z)v_0, v_1) + \omega_0(v_0, \mathrm{j}_t(z)v_1) = 0, \quad \mathrm{j}_t(z)J_t(z) + J_t(z)\mathrm{j}_t(z) = 0$$

Similarly as we did for Hamiltonians, we need our almost-complex structures to belong to a Banach space, in order to use the inverse function theorem for Banach spaces. Therefore we must define a Banach space of rapidly vanishing perturbations and a corresponding Banach manifold of almost complex structures. Choose a sequence $\underline{\varepsilon} = (\varepsilon_k)_{k \in \mathbb{N}}$ with $\varepsilon_k \to 0$, and define

$$||\mathrm{j}||_{\underline{\varepsilon}} = \sum_{k=0}^{\infty} \varepsilon_k \, ||\mathrm{j}||_{C^k(S^1 \times \mathbb{R}^{2n}, \operatorname{End}\mathbb{R}^{2n})}$$

For a fixed $J \in \mathfrak{J}$, denote by $C^\infty_{\underline{\varepsilon}}(J)$ the Banach space of maps j as above such that $||\mathrm{j}||_{\underline{\varepsilon}}$ is finite.

Notice that if $\mathrm{j} \in C^\infty_{\underline{\varepsilon}}(J)$, then $\tilde{J}_t(z) = J_t(z)\exp(-J_t(z)\mathrm{j}_t(z))$ defines another ω_0-compatible almost-complex structure with $||\tilde{J}||_{L^\infty} < \infty$. Therefore $C^\infty_{\underline{\varepsilon}}(J)$ provides a suitable space of perturbations for the almost-complex structures. We will use it to define a Banach chart of nearby almost-complex structures as follows.

Definition 3.8 Fix a reference $J_* \in \mathfrak{J}$, a sequence $\underline{\varepsilon} = (\varepsilon_k)_{k \in \mathbb{N}}$ converging to zero and a $\delta > 0$. Define

$$U(J_*, \underline{\varepsilon}, \delta) = \left\{ J = J_* \exp(-J_* \mathrm{j}) : ||\mathrm{j}||_{\underline{\varepsilon}} < \delta \right\}$$

Clearly $U(J_*, \underline{\varepsilon}, \delta)$ is a Banach manifold for δ small enough, as it has a global parametrization in terms of $B_\delta(0) \subset C^\infty_{\underline{\varepsilon}}(J)$, an open ball in a Banach space.

Lemma 3.5.1 *For every $J_* \in \mathfrak{J}$ there exists a $\delta > 0$ such that $U(J_*, \underline{\varepsilon}, \delta)$ is a Banach manifold, irregardless of $\underline{\varepsilon}$, and $U(J_*, \underline{\varepsilon}, \delta) \subset \mathfrak{J}$.*

Remark It is not true that $U(J_*, \underline{\varepsilon}, \delta)$ is a submanifold of $\mathfrak{J}$, where $\mathfrak{J}$ is considered with its natural Fréchet manifold structure in the C^∞_{loc} topology. It is just a subset, and with the subspace topology it would *not* be a submanifold.

Theorem 4 *For every $H \in \mathfrak{wH}_*$ there exists a neighborhood $\mathfrak{U} \subset \mathfrak{J}$ of J_0 and a residual set $\mathfrak{R}_{J_0} \subset \mathfrak{U}$ such that for every $J \in \mathfrak{R}$, the operator $\overline{\partial}_{H,J} : \mathscr{B}_0 \to \mathscr{E}_1$ has 0 as a regular value.*

Proof. We give a sketch of the proof following in part the seminal paper of Floer, Hofer and Salamon [17].

Fix $H \in \mathfrak{wH}_*$ and the reference almost-complex structure $J_* \equiv J_0$. Define the universal ambient space $\mathscr{E}_0 = \mathscr{B}_0 \times U(J_0, \underline{\varepsilon}, \delta)$, where $\underline{\varepsilon}$ and δ are chosen as in Lemma 3.5.1. This is a Banach manifold.

The Floer operator gives a map

$$\mathscr{F} : \mathscr{E}_0 \to \mathscr{E}_1, \quad \mathscr{F}(u,J) = \overline{\partial}_{H,J}(u) = \partial_s u - J_t(u)[\partial_t u - X_H(u)]$$

The *universal moduli space* is by definition the preimage of zero via this function:

$$\mathscr{U}\mathscr{M} = \mathscr{F}^{-1}(0) = \left\{(u,J) : \overline{\partial}_{H,J}(u) = 0\right\}$$

We first want to show that this is a smooth manifold, by showing that 0 is a regular value, i.e. $D\mathscr{F}$ is surjective at all points of $\mathscr{U}\mathscr{M}$. A simple calculation shows that for every $(u,J) \in \mathscr{U}\mathscr{M}$,

$$D\mathscr{F}(u,J)(\xi, \mathrm{j}) = D\overline{\partial}_{H,J}(u)\xi + \mathrm{j}_t(u)[\partial_t u - X_H(u)]$$

By Proposition 3.5.2, $D\overline{\partial}_{H,J}(u)$ is Fredholm. So by Corollary A.1, the image of $D\mathscr{F}(u,J)$ is closed. To show surjectivity we therefore only have to show that the image of $D\mathscr{F}(u,J)$ is dense. The proof of this is by contradiction. Let $q > 1$ be chosen such that $\frac{1}{p} + \frac{1}{q} = 1$. Fix $(u,J) \in \mathscr{U}\mathscr{M}$. Assume that $0 \neq \eta \in L^q\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$ is such that

$$\int_{\mathbb{R}\times S^1} \langle \eta(s,t), \theta(s,t) \rangle = 0 \quad \forall \theta \in \mathrm{im} D\mathscr{F}(u,J) \subset L^p\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$$

i.e. η as a functional on L^p annihilates every element in the image of $D\mathscr{F}(u,J)$. Since $\theta \in \mathrm{im} D\mathscr{F}(u,J)$, we conclude that η is a weak solution of a Cauchy-Riemann type equation. By boostrapping (Propositions 3.3.1, 3.3.2) we conclude that η is actually smooth. Since η is smooth and by assumption non-zero, there exists some open subset $\Omega \subset \mathbb{R} \times S^1$ such that $\eta(s,t) \neq 0$ for all $(s,t) \in \Omega$. Now, one uses the fact that the set of *regular points* of Floer trajectories (the analog of *injective points* of J-holomorphic curves) are dense in $\mathbb{R} \times S^1$, in particular, in Ω, to construct a $\xi \in W^{1,p}$ such that

$$\int_{\mathbb{R}\times S^1} \langle \eta(s,t), [D\mathscr{F}(u,J)\xi](s,t) \rangle \neq 0$$

in contradiction to the hypothesis on η. For the details of the construction of ξ starting from the existence of (at least) one injective point in Ω we refer to [32, pg. 51]. See also [17, pp. 267–269]. By Lemma A.3 we see that $D\mathscr{F}(u,J)$ has a right inverse. Using Theorem A.1, we conclude that the universal moduli space $\mathcal{U}\mathcal{M} \subset \mathcal{E}_0$ is a smooth Banach submanifold.

Set $\mathfrak{U} = U(J_0, \underline{\varepsilon}, \delta)$. This is an open neighborhood of J_0 in the $C^\infty_{\underline{\varepsilon}}$ topology, which is finer than the C^∞_{loc} topology. The projection $p_2 \colon \mathcal{E}_0 \to \mathfrak{U}$ restricts to $p = p_2|_{\mathcal{U}\mathcal{M}} \colon \mathcal{U}\mathcal{M} \to \mathfrak{U}$, which is a Fredholm map, as shown by Lemma A.4, of the same index of $D\overline{\partial}_{H,J}$, which is computed by Propostion 3.5.3. By the Sard-Smale theorem A.3, there is a residual set $\mathfrak{R}_{J_0} \subset \mathfrak{U}$ of regular values of the projection. The almost-complex structures in this set are precisely the ones for which 0 is a regular value of $\overline{\partial}_{H,J}$. □

Remark One may obtain a residual set $\mathfrak{R} \subset \mathfrak{J}$ in the C^∞_{loc}-topology on $\mathfrak{J}$ by following Taubes' argument as in [32, Theorem 3.1.6 (ii), pg. 54-56]. See also Remark 3.2.7 ibid.

In the following subsections, we give the consequences of the above theorem and its variant for continuation Floer cylinders and a parametrized version for the definition of chain homotopies.

3.5.2. Autonomous moduli spaces

Autonomous moduli spaces are moduli spaces of cylinders which solve the "autonomous" Floer equation, i.e. the Floer equation where (H,J) do not depend on the evolution variable $s \in \mathbb{R}$ but only on the "internal" variable $t \in S^1$.

Definition 3.9 Let $H \in \mathfrak{w}\mathfrak{H}_*$, J an adequate almost complex structure, and $\gamma_0, \gamma_1 \in \mathrm{Per}^1(H)$. Denote

$$M(\gamma_0,\gamma_1;H,J) = \left\{ u \in C^\infty\left(\mathbb{R}\times S^1, \mathbb{R}^{2n}\right) : \begin{array}{c} \partial_s u + J_t(u)[\partial_t u - X_H(u)] = 0, \\ u \to \gamma_0 \text{ as } s \to -\infty, \\ u \to \gamma_1 \text{ as } s \to +\infty \end{array} \right\}$$

This set is well defined by uniqueness of the limits. We call this the moduli space of Floer cylinders between γ_0 and γ_1. In principle, the space $M(\gamma_0,\gamma_1;H,J)$ is equipped with the $W^{1,p}_{\text{loc}}$ topology, which is suitable for the Fredholm theory of the Floer equation. By Proposition 3.3.4, we know that on $M(\gamma_0,\gamma_1;H,J)$ the C^∞_{loc} and $W^{1,p}_{\text{loc}}$ topology coincide. Therefore we state our theorems for the C^∞_{loc} topology.

3.5.2.1. Transversality and dimension, autonomous case

From Theorem 4 we may conclude the following proposition. For a complete proof one may consult [7, Chapter 8] or the seminal paper of Floer, Hofer and Salamon [17].

Proposition 3.5.4 *There exists a residual set of adequate* 1*-parameter family of almost complex structures* $J \colon S^1 \times \mathbb{R}^{2n} \to \mathrm{End}\,\mathbb{R}^{2n}$ *near* J_0 *such that for any* γ_0, γ_1 1*-periodic orbits of* H*, the space* $M(\gamma_0,\gamma_1;H,J)$ *in the* C^∞_{loc} *topology is a smooth finite-dimensional manifold of dimension*

$$\dim M(\gamma_0,\gamma_1;H,J) = \mathrm{CZ}(\gamma_0,H) - \mathrm{CZ}(\gamma_1,H)$$

Definition 3.10 A pair (H,J) for which Proposition 3.5.4 holds is called a *regular pair*.

Let (H,J) be a regular pair. Notice that there is a natural $\mathbb{R}$-action on $M(\gamma,\gamma';H,J)$ given by translations in the s-direction, i.e. $(\sigma,u)\mapsto\sigma\cdot u$ where $\sigma\cdot u(s,t)=u(s+\sigma,t)$. This is because H and J do not depend on the s-coordinate, i.e. the Floer equation in analysis is autonomous. It is easy to see that this action is smooth, proper and free (unless $\gamma=\gamma'$) so the quotient $\underline{M}(\gamma,\gamma';H,J)=M(\gamma,\gamma';H,J)/\mathbb{R}$ is again a smooth manifold. The dimension formula tells us that

$$\underline{M}(\gamma,\gamma';H,J)=\mathrm{CZ}(\gamma,H)-\mathrm{CZ}(\gamma',H)-1$$

So if $\mathrm{CZ}(\gamma,H)=\mathrm{CZ}(\gamma',H)$ then we conclude that the moduli space of cylinders $M(\gamma,\gamma';H,J)$ must be empty, unless $\gamma=\gamma'$ and then it can contain only the trivial cylinder $u=\gamma$.

3.5.3. Non-autonomous moduli spaces

Non-autonomous moduli spaces are moduli spaces of cylinders which solve the continuation Floer equation for continuation datum $(\mathscr{H},\mathscr{J})$. Since the continuation datum depends on the evolution variable s, we think of this equation as non-autonomous.

Let $H^\pm\in\mathfrak{wH}_*$ and $\mathscr{H}$ an asymptotically quadratic continuation between them. Let $\mathscr{J}$ be an adequate family of almost complex structures. Fix $\gamma^\pm\in\mathrm{Per}^1(H^\pm)$. Denote

$$\mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right)=\left\{u\in C^\infty\left(\mathbb{R}\times S^1,\mathbb{R}^{2n}\right):\begin{matrix}\partial_s u+\mathscr{J}_{s,t}(u)[\partial_t u-X_{\mathscr{H}}(s,u)]=0,\\ u\to\gamma^\pm \text{ as } s\to\pm\infty\end{matrix}\right\}$$

We call this the moduli space of continuation Floer cylinders between γ^- and γ^+, and refer to the collection of moduli spaces of continuation Floer cylinders as the non-autonomous moduli spaces.

Similar remarks on the topology which this moduli space carries hold as for the autonomous ones. Namely, we are thinking of $\mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$ as equipped with the C^∞_{loc} topology, which coincides with the $W^{1,p}_{\mathrm{loc}}$ topology on it.

3.5.3.1. Transversality and dimension, non-autonomous case

The proof of this proposition is the content of [7, §11.1.b].

Proposition 3.5.5 *There exists a residual set of adequate 2-parameter family of almost complex structures $\mathscr{J}:\mathbb{R}\times S^1\to\mathrm{End}\,\mathbb{R}^{2n}$ near J_0 such that for any $\gamma^\pm$ 1-periodic orbits of $H^\pm$, the space $\mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$ with the C^∞_{loc} topology is a smooth finite-dimensional manifold of dimension*

$$\mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right)=\mathrm{CZ}\left(\gamma^-,H^-\right)-\mathrm{CZ}\left(\gamma^+,H^+\right)$$

Definition 3.11 A pair $(\mathscr{H},\mathscr{J})$ as above for which Proposition 3.5.5 holds, is called a *regular pair*.

We cannot conclude that zero-dimensional non-autonomous moduli spaces are necessarily trivial, as there is no natural $\mathbb{R}$-action when $\mathscr{H}$ and $\mathscr{J}$ depend on s.

3.5.4. Parametrized moduli spaces

We need one last technical ingredient, which is used to show that continuation morphisms on Floer homology do not depend on the chosen continuation Hamiltonian. The proofs of the statements in this subsection can be found in [7, §11.3-4].

Let $H^{\pm} \in \mathfrak{wH}_{*}$, $J^{\pm}$ adequate almost-complex structures such that $(H^{\pm},J^{\pm})$ are two regular pairs, and $(\mathscr{H},\mathscr{J})$, $(\mathscr{G},\mathscr{I})$ be two regular pairs of asymptotically quadratic continuations and adequate almost-complex structures between them. Let $\mathbb{H}\colon [0,1]\times\mathbb{R}\times S^1\times\mathbb{R}^{2n}\to\mathbb{R}$, $\mathbb{H}=\mathbb{H}(\rho,s,t,z)$ be a homotopy of asymptotically quadratic continuations which is constantly equal to $\mathscr{H}$ when ρ is near 0, and constantly equal to $\mathscr{G}$ when ρ is near 1. Let $\mathbb{J}$ be a homotopy of adequate almost-complex structures with similar boundary conditions. We denote by $\mathbb{H}^{\rho},\mathbb{J}^{\rho}$ the asymptotically quadratic continuations obtained by fixing ρ in $\mathbb{H},\mathbb{J}$.

We are interested in the Floer equation

$$\partial_s u^{\rho}+\mathbb{J}^{\rho}(s,t,u^{\rho})[\partial_t u^{\rho}-X_{\mathbb{H}^{\rho}}(s,t,u^{\rho})]=0 \tag{3.11}$$

which depends on the parameter ρ. We form the fiber product, called *parametrized* moduli space

$$\mathbb{M}\big(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\big)=\big\{(\rho,u^{\rho}):\rho\in[0,1],\ u^{\rho}\text{ sol. of (3.11)},\ u^{\rho}\to\gamma^{\pm}\text{ as }s\to\pm\infty\big\}$$

Notice that the ends of the cylinders are fixed for every parameter ρ, and that we have a projection $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})\to[0,1]$ whose fiber over $\rho\in[0,1]$ is $\mathscr{M}(\gamma^-,\gamma^+;\mathbb{H}^{\rho},\mathbb{J}^{\rho})$. We equip this space with the natural fiber product topology, which is the subspace topology given by the inclusion $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})\subset[0,1]\times W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$. As in the previous cases concerning the topologies on moduli spaces, first of all $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$ actually lies in $[0,1]\times C^{\infty}\big(\mathbb{R}\times S^1,\mathbb{R}^{2n}\big)$, and by Proposition 3.3.4 the moduli space can be equipped with the subspace topology given by this inclusion, where $[0,1]\times C^{\infty}\big(\mathbb{R}\times S^1,\mathbb{R}^{2n}\big)$ is equipped with the product of the standard topology with the $C^{\infty}_{\mathrm{loc}}$ topology.

3.5.4.1. Transversality and dimension, parametrized case

This parametrized moduli space has more or less similar transversality and compactness properties to the ones explained above for the moduli spaces of Floer cylinders explained above. This is shown in [7, §11.3.b].

Proposition 3.5.6 *There exists a residual set of adequate 3-parameter families of almost complex structures $\mathbb{J}$ near J_0 such that for any $\gamma^{\pm}$ 1-periodic orbits of $H^{\pm}$, the space $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$ with the fiber product $C^{\infty}_{\mathrm{loc}}$ topology is a smooth finite-dimensional manifold of dimension*

$$\dim\mathbb{M}\big(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\big)=\mathrm{CZ}\big(\gamma^-,H^1\big)-\mathrm{CZ}\big(\gamma^+,H^+\big)+1$$

Definition 3.12 A pair $(\mathbb{H},\mathbb{J})$ for which Proposition 3.5.6 holds is called *regular pair*.

Remark It is *not* true in general that every fiber $\mathcal{M}(\gamma^-,\gamma^+;\mathbb{H}^\rho,\mathbb{J}^\rho)$ is transversely cut out for each fixed $\rho\in(0,1)$. For example, when studying the case of $\dim\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})=0$, it is inevitable that for some $\rho\in(0,1)$ the fiber $\mathcal{M}(\gamma^-,\gamma^+;\mathbb{H}^\rho,\mathbb{J}^\rho)$ contains cylinders of index -1. In fact, these cylinders of index -1 are precisely the ones we want to count to define a chain homotopy between the continuation morphisms induced by the regular pairs $(\mathscr{H},\mathscr{J})$ and $(\mathscr{G},\mathscr{I})$, as we shall see below.

Notice that differently from the previous two cases, the parametrized moduli spaces are manifolds with boundary. The boundary is easy to describe: since we chose the homotopy to be stationary near $\rho=0,1$, it is easily proven that

$$\partial\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})=\{0\}\times\mathcal{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})\cup\{1\}\times\mathcal{M}(\gamma^-,\gamma^+;\mathscr{G},\mathscr{I}).$$

We call this the *regular boundary* of the parametrized moduli space. This does not mean that the moduli space is compact; in fact, we will have to compactify the moduli space by adding an *exceptional part* of the boundary.

3.6. Broken convergence and gluing

In the previous section we've seen that families of Floer cylinders of various flavors form smooth manifolds. In general, these are open manifolds, possibly with boundary, like in the parametrized case. In this section we first provide a suitable notion of compactification of the moduli spaces. To describe the compactification of the moduli spaces, one must introduce the notion of broken convergence of Floer cylinders. This leads to defining the boundary of the compactification of the moduli spaces in terms of broken configurations of Floer cylinders.

The cylinders entering the broken configurations are points in a lower-dimensional moduli space. This means that the boundary of a moduli space of Floer cylinders can be described in terms of lower-dimension moduli spaces. A theory of gluing of Floer trajectories permits us to equip the one dimensional compactified moduli spaces with the structure of a smooth manifold with boundary.

3.6.1. Broken convergence and compactification

The aim of this section is to investigate how one can exploit the compactness property of the set of Floer cylinders with bounded energy, that is to say Theorem 2, and the (eventual) translational symmetry of the Floer equation, to compactify the space of cylinders with fixed asymptotics. We will treat the autonomous case somewhat in detail. The non-autonomous and parametrized cases are quite analogous to it.

3.6.1.1. The autonomous case

Let $H \in \mathfrak{w}\mathfrak{H}_*$ and J be an adequate almost-complex structure. We are interested in describing the limit of a sequence of solutions to the Floer equation defined by (H,J) *with fixed asymptotics*. Given that the asymptotics are fixed, we have an uniform energy bound on the sequence, so by Theorem 2 the sequence has a limit. The problem is that we cannot guarantee that the limit cylinder has the same asymptotics as the cylinders in the sequence. This means that the moduli space with fixed asymptotics is *not compact*. But, inspired by Morse theory, by choosing a suitable translation of the sequence, we might "access" different cylinders which, as we will show, connect at intermediate 1-periodic orbits of H. This is the phenomenon of broken convergence, and we will use it to *define* the compactification of the moduli space with fixed asymptotics.

Proposition 3.6.1 *Let γ, γ' be any 1-periodic orbits of $H \in \mathfrak{w}\mathfrak{H}_*$. Let $(u_k)_k \subset M\left(\gamma,\gamma';H,J\right)$ be a sequence of Floer cylinders where J is an adequate almost-complex structure.*

There exists a subsequence $(u_{k_j})_j$ of $(u_k)_k$, 1-periodic orbits $\gamma_0 = \gamma, \gamma_1, \dots, \gamma_{l+1} = \gamma'$ of H, sequences $(\sigma_j^r)_j$ for $r \in \{0,\dots,l\}$ and Floer cylinders $u^r \in M\left(\gamma_r,\gamma_{r+1};H,J\right)$ for $r \in \{0,\dots,l\}$ such that, denoting by $(\sigma \cdot u)(s,t) = u(\sigma + s, t)$,

$$\lim_{j\to\infty} \sigma_j^r \cdot u_{k_j} = u^r \text{ in } C^\infty_{\text{loc}} \quad \forall r \in \{0,\dots,l\}.$$

Moreover, if (H,J) is a regular pair, then the dimension formula for the moduli spaces gives a bound on the maximum number of cylinders in a broken Floer cylinder:

$$l < \mathrm{CZ}(\gamma,H) - \mathrm{CZ}(\gamma',H)$$

The collection of Floer cylinders which appear as limits of the shifted sequence found above is called a *broken Floer cylinder*, and the phenomenon of convergence to a broken cylinder is referred to as *breaking*.

Proof. The following proof was inspired by a conversation I had with Dr. Urs Fuchs.

Let $\mathscr{W} \subset C^\infty\left(S^1,\mathbb{R}^{2n}\right)$ be a neighborhood of $\mathrm{Per}^1 H$ such that every connected component of $\mathscr{W}$ contains an unique orbit of H. This exists and has finitely many connected components because $H \in \mathfrak{w}\mathfrak{H}_*$ is non-degenerate up to infinity. Recall that $H \in \mathfrak{w}\mathfrak{H}_*$ implies that there is a quantum of energy $\hbar_H > 0$ below which any long enough piece of (H,J)-Floer trajectory must remain within one connected component of $\mathscr{W}$ (Proposition 3.4.2).

Take $u \in M\left(\gamma,\gamma';H,J\right) \subset M(E;H,J)$ where we set $E = \mathscr{A}_H(\gamma) - \mathscr{A}_H(\gamma')$. We chop up the cylinder into pieces with small energy, as follows. Fix an arbitrary $0 < e < \hbar_H$. Define $-\infty = \sigma^{-1}(u) < \sigma^0(u) \le \dots \le \sigma^L(u) < \sigma^{L+1}(u) = +\infty \in \mathbb{R}$ by

$$E_{H,J}\left(u|_{(\sigma^r(u),\sigma^{r+1}(u))}\right) = e \quad \forall -1 \le r \le L$$

Notice that L is a finite number because $E < \infty$, so we can estimate $L \le E/\hbar_H$, and that

$-\infty < \sigma^0(u),\dots,\sigma^L(u) < +\infty$ are all finite because u must enter the fixed neighborhood of γ as $s \to -\infty$ and the fixed neighborhood of γ' as $s \to +\infty$.

Now take a sequence $(u_k) \subset M(\gamma,\gamma') \subset M(E;H,J)$. Define $\sigma_k^r = \sigma^r(u_k)$ for $r = 0,\dots,L_k$. *A priori* L_k depends on k. But clearly there is a subsequence of the u_k for which it is eventually constant in k, since $E_{HJ}(u_k) = E$ for every k. Therefore without loss of generality we assume $L_k = L$ for all k.

The tuple of sequences $\left(\sigma_k^r\right)_{r,k}$ will give us the shifts needed to capture the cylinders breaking in the limit. By Theorem 2 each sequence

$$\sigma_k^r \cdot u_k(s,t) = u_k\left(\sigma_k^r + s, t\right)$$

converges up to a subsequence to a limit $u^r \in M(E;H,J)$ in C^∞_{loc}. We will still index the subsequence by k in order to keep the notation light. Define a pre-order on sequences $(\sigma_k) \subset \mathbb{R}$ by setting

$$(\sigma_k) \preccurlyeq \left(\sigma_k'\right) \iff \limsup_{k\to\infty}\left(\sigma_k - \sigma_k'\right) < +\infty.$$

We denote by $\sim$ the equivalence relation defined by the pre-order, that is,

$$(\sigma_k) \sim \left(\sigma_k'\right) \iff (\sigma_k) \preccurlyeq \left(\sigma_k'\right) \text{ and } \left(\sigma_k'\right) \preccurlyeq (\sigma_k) \tag{3.12}$$

We apply the equivalence relation (3.12) on the finite set of sequences $\left(\sigma_k^0\right),\dots,\left(\sigma_k^L\right)$. By construction it always holds that

$$q < r \implies \sigma_k^q < \sigma_k^r \ \forall k \implies \left(\sigma_k^q\right) \preccurlyeq \left(\sigma_k^r\right).$$

Lemma 3.6.1 *If $\left(\sigma_k^q\right) \sim \left(\sigma_k^r\right)$ then there exists a $\sigma_* \in \mathbb{R}$ such that $u^r = \sigma_* \cdot u^q$.*

Proof. Assume without loss of generality that $q < r$. Set

$$0 < \sigma_* = \limsup_{k\to\infty}\left(\sigma_k^r - \sigma_k^q\right) < +\infty$$

Let $k_j \to \infty$ be such that $\sigma_{k_j}^r - \sigma_{k_j}^q \to \sigma_*$ as $j \to \infty$. Then, up to a further subsequence,

$$u^r \xleftarrow[\infty\leftarrow j]{} \sigma_{k_j}^r \cdot u_{k_j} = \left(\sigma_{k_j}^r - \sigma_{k_j}^q + \sigma_{k_j}^q\right)\cdot u_{k_j} = \left(\sigma_{k_j}^r - \sigma_{k_j}^q\right)\cdot \sigma_{k_j}^q \cdot u_{k_j} \xrightarrow[j\to\infty]{} \sigma_* \cdot u^q$$

We conclude by uniqueness of the C^∞_{loc} limit. □

This last claim means that $u^q = u^r$ in $\underline{M}(\gamma,\gamma')$ for some $\gamma,\gamma' \in \operatorname{Per}^1 H$. Notice that

$$\left(\sigma_k^q\right) \sim \left(\sigma_k^r\right), \quad q < k \implies \left(\sigma_k^q\right) \sim \left(\sigma_k^p\right) \quad \forall q < p < r$$

Thus we can discard all the sequences in the same equivalence class but one, and re-naming them incrementally according to the pre-order $\preccurlyeq$ we end up with a possibly smaller tuple $(\sigma_k^0), \dots, (\sigma_k^l)$, $l \le L$. As before, up to a subsequence,

$$\sigma_k^r \cdot u_k \xrightarrow[k\to\infty]{C^\infty_{\text{loc}}} u^r, \quad 0 \le r \le l.$$

with the difference that now all the cylinders u^r are distinct, even up to shifts.

It is clear that $u^0 \to \gamma$ as $s \to -\infty$ and $u^l \to \gamma'$ as $s \to +\infty$.

Lemma 3.6.2 *If $u^r \to \gamma_{r+1} \in \mathrm{Per}^1 H$ as $s \to +\infty$, then $u^{r+1} \to \gamma_{r+1}$ as $s \to -\infty$.*

Proof. Since we discarded equivalent sequences, we have that

$$I_k = \left[\sigma_k^r, \sigma_k^{r+1}\right]$$

has unbounded length in k. Moreover, by definition

$$E_{H,J}\left(u_k|_{I_k}\right) = e < \hbar_H \quad \forall k$$

Hence by (a minor modification of) Proposition 3.4.2 for every k large enough there exists an $\overline{S}_k > 0$ such that

$$\varnothing \neq I_k' = \left[\sigma_k^r + \overline{S}_k, \sigma_k^{r+1} - \overline{S}_k\right] \subset I_k, \quad \text{and } u_k(s,\cdot) \in \mathscr{W} \quad \forall s \in I_k' \tag{3.13}$$

Equivalently

$$\sigma_k^r \cdot u_k(s,\cdot) \in \mathscr{W} \quad \forall s \in \left[\overline{S}_k, (\sigma_k^{r+1} - \sigma_k^r) - \overline{S}_k\right] \subset \left[0, \sigma_k^{r+1}\right]$$

There are two cases. Either $\left(\sigma_k^{r+1} - \sigma_k^r\right) - \overline{S}_k < C$ for some $C > 0$ independent of k, or it is unbounded. In the first case, combining with (3.13) we see that

$$\sigma_k^{r+1} - \sigma_k^r - C \le \overline{S}_k \le \sigma_k^{r+1} - \sigma_k^r \quad \forall k$$

But this means that $\overline{S}_k$ is unbounded, which implies that eventually

$$\left[\overline{S}_k, (\sigma_k^{r+1} - \sigma_k^r) - \overline{S}_k\right] = \varnothing$$

which is impossible, as it implies that $I_k' = \varnothing$. Therefore $\left(\sigma_k^{r+1} - \sigma_k^r\right) - \overline{S}_k$ is always unbounded. Passing to the limit on a subsequence which diverges, we conclude that $u_k(s,\cdot)$ is in the component $\mathscr{W}\left(\gamma_{r+1}\right)$ of $\mathscr{W}$ around γ_{r+1} for all $s \in I_k'$, since by hypothesis $u^r \to \gamma_{r+1}$ as $s \to +\infty$.

On the other hand,

$$\sigma_k^{r+1} \cdot u_k(s,\cdot) \in \mathscr{W}\left(\gamma_{r+1}\right) \quad \forall s \in \left[\sigma_k^r - \sigma_k^{r+1} + \overline{S}_k, -\overline{S}_k\right] \subset \left[\sigma_k^r - \sigma_k^{r+1}, 0\right]$$

Arguing identically as before we can show that $\sigma^r - \sigma^{r+1} + \overline{S}_k$ must diverge to $-\infty$, so we can pass to the limit and conclude that $u^{r+1} \to \gamma_{r+1}$ as $s \to -\infty$. □

Finally, when (H,J) is a regular pair, the bound on l follows immediately from the transversality theory (Proposition 3.5.4). This concludes the proof of the proposition. □

This proposition tells us that to compactify the moduli space, one should add to it configurations of Floer cylinders with pair-wise matching ends, and the notion of convergence should be C^∞_{loc} up to shifts. Moreover, the proof shows that to really capture all the breaking that happens along the sequence, the shifts should be chosen appropriately. This prompts the following definition.

Definition 3.13 Let $(\sigma_k^r) \subset \mathbb{R}$, $(u_k) \subset M(\gamma,\gamma')$ and $(u^0,\dots,u^l) \in M(\gamma,\gamma_1) \times \cdots \times M(\gamma_l,\gamma')$ be such that $\sigma_k^r \cdot u_k \to u^r$ in C^∞_{loc} for all $1 \le r \le l$.

1. We say that the tuple of shifts $(\sigma_k^r)_{r,k}$ is *ordered* if $(\sigma_k^r) \preccurlyeq (\sigma_k^{r+1})$ for every r. We say that it is *totally ordered* if $\preccurlyeq$ is a total order on the tuple of sequences.
2. We say that an ordered tuple of shifts $(\sigma_k^r)_{r,k}$ is *exhaustive* when it fulfills the following properties. Set for convenience $\sigma_k^{-1} = -\infty$ and $\sigma_k^{l+1} = +\infty$. Let $(\sigma_k) \subset \mathbb{R}$ be any sequence.
   a) If there exists an $1 \le r \le l$ and a $C > 0$ such that $\left|\sigma_k - \sigma_k^r\right| < C$, then there exists a $\sigma_* \in \mathbb{R}$ such that $\sigma_k \cdot u_k \to \sigma_* \cdot u^r$ in C^∞_{loc} up to a subsequence.
   b) If $\left|\sigma_k - \sigma_k^r\right| \to \infty$ as $k \to \infty$ for every r, then $\sigma_k \cdot u_k \to \gamma_r$ in C^∞_{loc} up to sub-sequences, where $-1 \le r \le l+1$ is such that $\sigma_k^r \le \sigma_k \le \sigma_k^{r+1}$ for the relevant sub-sequence and we set $\gamma_{-1} = \gamma$, $\gamma_{l+1} = \gamma'$.

Remark Given a tuple of shifts, we can always assume it is ordered by re-naming its components. If an ordered tuple is exhaustive, we can identify the components which are at a bounded distance to obtain a totally ordered exhaustive tuple of shifts.

We are now ready to compactify the space $\underline{M}(\gamma,\gamma';H,J)$. Set, for (H,J) a regular pair,

$$d = \mathrm{CZ}(\gamma';H) - \mathrm{CZ}(\gamma,H) - 1 = \dim \underline{M}(\gamma,\gamma';H,J)$$

The proposition tells us that we should add to it the set of *broken configurations of Floer cylinders*

$$\underline{M}^d(\gamma,\gamma') = \bigcup \left\{ \underline{M}(\gamma,\gamma_1) \times \cdots \times \underline{M}(\gamma_l,\gamma') : l \le d,\ \gamma_r \in \mathrm{Per}^1 H,\ \mathrm{CZ}(\gamma_r) < \mathrm{CZ}(\gamma_{r+1}) \right\} \tag{3.14}$$

Important With a substantial abuse of notation, we denote the compactification of $\underline{M}(\gamma,\gamma')$ with the same symbol, that is, from now on $\underline{M}(\gamma,\gamma')$ means the quotiented moduli space united with $\underline{M}^d(\gamma,\gamma')$.

The topology on $\underline{M}(\gamma,\gamma')$ is defined as so:

Definition 3.14 A sequence $(u_k) \subset \underline{M}(\gamma, \gamma')$ converges to a broken configuration $(u^0, \ldots, u^l)$ in $\underline{M}^d(\gamma, \gamma')$ when, for any choice of lifts of u_k to $M(\gamma, \gamma')$, there exist an *exhaustive, totally ordered* l-tuple of sequences $(\sigma_k^r) \subset \mathbb{R}$, such that $\sigma_k^r \cdot u_k \to u^r$ in C^∞_{loc} for every $r \in \{1, \ldots, l\}$. The integer $l+1$ will be called the *number of levels* of the broken configuration, and the index r will be called the *level*.

This topology coincides with the quotient by the $\mathbb{R}$-action of the C^∞_{loc} topology on the interior. The same notion of convergence applies to points in $\underline{M}^d(\gamma, \gamma')$, as we can work level by level. This shows that

Theorem 5 *The space* $\underline{M}(\gamma, \gamma')$ *with the broken convergence topology is compact.*

Remark For now, the set of broken Floer cylinders $\underline{M}^d(\gamma, \gamma')$ only provides us with the boundary of $\underline{M}(\gamma, \gamma')$ in the sense of topology. But $\underline{M}(\gamma, \gamma')$ has the structure of a smooth manifold. Later we shall see some hints pointing towards the fact that the set of broken configurations gives the boundary and corners of the moduli space as a smooth manifold.

Global convergence à la Gromov We conclude the section with a proposition which elucidates the "global nature" of the notion of convergence by breaking. It justifies the intuitive picture that most working Floer theorists have of broken convergence.

Fix a smooth strictly increasing diffeomorphism $S\colon (-1,1) \to \mathbb{R}$ such that $S(\tau) \to \pm\infty$ as $\tau \to \pm 1$, for example $S(\tau) = \tan \frac{\pi\tau}{2}$. If $v \in M(\gamma, \gamma')$, define a map $V\colon [-1,1] \times S^1 \to \mathbb{R}^{2n}$ by setting

$$V(\tau, t) = \begin{cases} \gamma(t), & \tau = -1 \\ v(S(\tau), t), & \tau \in (-1,1) \\ \gamma'(t), & \tau = 1 \end{cases}$$

Since v converges to γ and γ' at its ends with exponential decay of all its s-derivatives, V is a continuous function.

Let $(u^0, \ldots, u^l) \in M(\gamma, \gamma_1) \times \cdots \times M(\gamma_l, \gamma')$. Similarly as just explained, extend these to maps $U^0, \ldots, U^l\colon [-1,1] \times S^1 \to \mathbb{R}^{2n}$. Define a map $U\colon [-1,1] \times S^1 \to \mathbb{R}^{2n}$ by setting

$$U(\tau, t) = U^r\left((l+1)\tau + l - 2r, t\right) \quad \forall \tau \in \left[-1 + \frac{2r}{l+1}, -1 + \frac{2(r+1)}{l+1}\right], \quad \forall 0 \le r \le l$$

It's easy to see that U is a continuous map, because each u^r converges exponentially fast to its ends, and the positive end of each cylinder matches with the negative end of the next.

Remark The specific formula for U is not important. The idea is that the map U gives the stacked-up broken configuration defined by our tuple $(u^0, \ldots, u^l)$. The intuitive picture which most working Floer theorists have in mind is that broken convergence happens as in figure 3.2.

Finally, let $(u_k) \subset M(\gamma, \gamma')$ be some sequence and $U_k\colon [-1,1] \times \mathbb{R}^{2n}$ the extensions of the maps.

Proposition 3.6.2 *If $u_k \to (u^0, \dots, u^l)$ in $\underline{M}(\gamma, \gamma')$ then there exist homeomorphisms*

$$\varphi_k : [-1,1] \times S^1 \to [-1,1] \times S^1$$

such that $U_k \circ \varphi_k \to U$ uniformly.

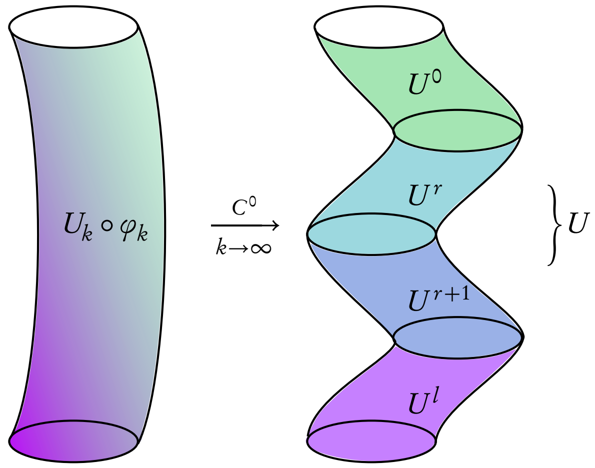

Figure 3.2.

Proof. Assume that $u_k \to (u^0, \dots, u^l)$ in $\underline{M}(\gamma, \gamma')$. Choose arbitrary lifts of u_k to $M(\gamma, \gamma')$ and obtain the totally ordered exhaustive tuple of shifts $(\sigma_k^r)_{r,k}$, such that $\sigma_k^r \cdot u_k \to u^r$ in C^∞_{loc}. Since $(\sigma_k^r)_{r,k}$ is exhaustive and totally ordered, we know that

$$u_k\left(\frac{\sigma_k^{r+1} - \sigma_k^r}{2}, \cdot\right) \to \gamma_r \text{ in } C^\infty(S^1, \mathbb{R}^{2n})$$

Set for notational convenience $s_k^r = \frac{\sigma_k^{r+1} - \sigma_k^r}{2}$. Define

$$\tau_k^r = S^{-1}(s_k^r), \quad \overline{\tau_r} = -1 + \frac{2r}{l+1}, \quad 1 \le r \le l$$

Let $\varphi_k : [-1,1] \times S^1 \to [-1,1] \times S^1$, $\varphi_k(\tau, t) = (f_k(\tau), t)$ where $f_k : [-1,1] \to [-1,1]$ is a continuous strictly increasing function chosen so that $f_k(\pm 1) = \pm 1$ and $f_k(\tau_k^r) = \overline{\tau_r}$. In order to avoid pathological situations, we choose the piece-wise linear function fulfilling these requirements. Since $\sigma_k^r \cdot u_k \to u^r$ in C^∞_{loc}, we know already that

$$U_k \circ \varphi_k \to U \text{ in } C^0_{\text{loc}}\left([-1,1] \setminus \bigcup_{r=1}^{l} \overline{\tau_r}\right)$$

Let's focus on a breaking point $\overline{\tau_r}$. Recall the quantum of energy $\hbar_H > 0$ from the theory of long cylinders with small energy. Let $s_* > 0$ be chosen such that

$$E_{HJ}\left(u^r|_{[s_*,+\infty)}\right) < \hbar_H, \quad E_{HJ}\left(u^{r+1}\Big|_{(-\infty,-s_*]}\right) < \hbar_H.$$

Define $\varepsilon = 1 - S^{-1}(s_*) > 0$. From the C^∞_{loc} convergence it follows that

$$U_k \circ \varphi_k(\overline{\tau_r} \pm \varepsilon, \,\cdot\,) \to U(\overline{\tau_r} \pm \varepsilon, \,\cdot\,) \text{ in } C^\infty\left(S^1\right)$$

In particular, the actions of the corresponding loops must also converge. Unpacking what this means in terms of the u_k and u^r, u^{r+1} and using the fact that both S and f_k are non-decreasing functions, we see that given our $\varepsilon > 0$ there exists a $\delta > 0$ and a sequence s'_k with $s'_k \leq S'\left(\tau^r_k\right)\delta$ such that

$$\mathscr{A}_H\left(u_k\left(s^r_k - s'_k, \cdot\right)\right) \to \mathscr{A}_H\left(u^r\left(s_* + S\left(-2 + \frac{2r}{l+1}\right), \cdot\right)\right)$$

Given the piece-wise linearity of f_k one could even express δ explicitly in terms of ε, τ^k_r and $\overline{\tau_r}$. Since both $s^r_k \cdot u_k$ and u^r are asymptotic to γ_r as $s \to +\infty$,

$$E_{HJ}\left(u_k|_{[s^r_k - s'_k, +\infty)}\right) \to E_{HJ}\left(u^r|_{[s_* + S(-2+2r/(l+1)), +\infty)}\right) < \hbar_H$$

Thus eventually the left-hand side is less than $\hbar_H$. Applying the long cylinders with small energy Proposition 3.4.2, for any isolating C^∞-neighborhood $\mathscr{W}(\gamma_r)$ of γ_r we find an $s''_k < s^r_k - s'_k$ such that

$$u_k|_{[s^r_k - s'_k + s''_k, +\infty)} \subset \mathscr{W}(\gamma_r) \quad \forall k \text{ large enough.}$$

Let $\delta' > 0$ be such that $\tau^r_k - \delta' = S^{-1}\left(s^r_k - s'_k + s''_k\right)$ and $\varepsilon' \leq \varepsilon$ such that

$$f_k\left(\overline{\tau_r} + \varepsilon'\right) = \tau^r_k - \delta'.$$

What we gain is that $U_k \circ \varphi_k(\tau, \cdot) \in \mathscr{W}(\gamma_r)$ for all $\tau \in \left[\overline{\tau_r} - \varepsilon', \overline{\tau_r}\right]$ for every k large enough. Arguing similarly for the other side using the smallness of the energy of u^{r+1}, we can show that the same holds on $\left[\overline{\tau_r} - \varepsilon', \overline{\tau_r} + \varepsilon'\right]$ perhaps with a smaller ε'. What we've shown is that for any small enough neighborhood $\mathscr{W}(\gamma_r)$ of γ_r in the loop space, there exists an $\varepsilon' > 0$ such that $U_k \circ \varphi_k(\tau, \cdot)$ and $U(\tau, \cdot)$ are both in $\mathscr{W}(\gamma_r)$ for every $\tau \in [\overline{\tau_r} - \varepsilon', \overline{\tau_r} + \varepsilon']$ and every k large enough. This implies that $U_k \circ \varphi_k \to U$ in C^0 on a neighborhood of $\overline{\tau_r}$, and that concludes the proof. □

Remark It's not so hard to see that actually also the converse is true. For the other direction, the uniform convergence of the maps $U_k \circ \varphi_k$ to U implies the C^0_{loc} convergence up to shifts of the maps u_k. Bootstrapping then gives C^∞_{loc} convergence.

Remark If we have a sequence of cylinders with a tuple of shifts which is *not* exhaustive, then there could be some energy concentration which we do not keep track of, implying that we are forgetting some cylinder that breaks off. In this case, the proposition cannot be true.

3.6.1.2. The non-autonomous case

Let $\mathscr{H}$ be an asymptotically quadratic continuation between $H^{\pm} \in \mathfrak{w}\mathfrak{H}_*$ and $\mathscr{J}$ an adequate almost-complex structure. Similarly as in the previous section, we have the following notion of broken convergence of sequences of $(\mathscr{H}, \mathscr{J})$-Floer cylinders with fixed asymptotics.

Proposition 3.6.3 *Let $\gamma^{\pm}$ be any 1-periodic orbits of $H^{\pm}$. Assume that $\mathscr{H}$ is chosen such that there exists an $E = E(\gamma^-, \gamma^+, \mathscr{H}, \mathscr{J})$ for which*

$$u \in \mathscr{M}\left(\gamma^-, \gamma^+; \mathscr{H}, \mathscr{J}\right) \implies E_{\mathscr{H},\mathscr{J}}(u) < E$$

Let $(u_k)_k \subset \mathscr{M}(\gamma^-, \gamma^+; \mathscr{H}, \mathscr{J})$ be a sequence of Floer cylinders. There exists a subsequence $(u_{k_j})_j$ of $(u_k)_k$, 1-periodic orbits $\gamma_0^- = \gamma^-, \dots, \gamma_{l^-}^-$ of H^-, 1-periodic orbits $\gamma_1^+, \dots, \gamma_{l^++1}^+ = \gamma^+$ of H^+, sequences $(\sigma_j^{r,-})_j$ for $r \in \{0, \dots, l^-\}$ tending to $-\infty$, sequences $(\sigma_j^{r,+})_j$ for $r \in \{1, \dots, l^+ + 1\}$ tending to $+\infty$, Floer cylinders $u^{r,-} \in M\left(\gamma_r^-, \gamma_{r+1}^-; H^-, J^-\right)$ for $r \in \{0, \dots, l^-\}$, Floer cylinders $u^{r,+} \in M\left(\gamma_r^+, \gamma_{r+1}^+; H^-, J^-\right)$ for $r \in \{1, \dots, l^+\}$, and a Floer cylinder $u^* \in \mathscr{M}\left(\gamma_{l^-}^-, \gamma_1^+; \mathscr{H}, \mathscr{J}\right)$ such that

$$\lim_{j\to\infty} \sigma_j^{r,-} \cdot u_{k_j} = u^{r,-}, \quad \lim_{j\to\infty} \sigma_j^{r,+} \cdot u_{k_j} = u^{r,+}, \quad \lim_{j\to\infty} u_{k_j} = u^*.$$

All these limits are intended in $C^{\infty}_{\mathrm{loc}}$. Moreover, if $(\mathscr{H}, \mathscr{J})$ is a regular pair, then the dimension formula for the moduli spaces gives a bound on the maximum number of cylinders in a broken Floer cylinder:

$$l^- + l^+ \leq \mathrm{CZ}(\gamma^-, H^-) - \mathrm{CZ}(\gamma^+, H^+)$$

The proof of this proposition is completely analogous to its autonomous counterpart, since braking happens in the "autonomous regions" where $\mathscr{H}$ is s-constant.

Remark The presence of an uniform energy bound in the case of continuation Floer cylinders is not automatic, and it is crucial, because it is necessary to prove the compactness of the space of finite-energy trajectories. Below we will present some cases in which this uniform energy bound is achieved.

In the same way as in the autonomous case, we can define the set of broken configurations of Floer cylinders, and this proposition defines the topology of the compactification of the moduli space of Floer cylinders between two fixed orbits. This time the set of broken configurations is as follows: we set $d = \mathrm{CZ}(\gamma^-) - \mathrm{CZ}(\gamma^+)$ and define

$$\mathscr{M}^d(\gamma_0, \gamma_1) = \bigcup \left\{ \begin{array}{c} \underline{M}\left(\gamma^-, \gamma_1^-\right) \times \dots \\ \dots \times \mathscr{M}\left(\gamma_{l^-}^-, \gamma_1^+\right) \times \dots \\ \dots \times \underline{M}\left(\gamma_{l^+}^+, \gamma^+\right) \end{array} : \begin{array}{c} l^+ + l^- \leq d, \\ \xi_r^{\pm} \in \mathrm{Per}^1 H^{\pm}, \\ \mathrm{CZ}\left(\xi_r^{\pm}\right) < \mathrm{CZ}\left(\xi_{r+1}^{\pm}\right), \\ \mathrm{CZ}\left(\gamma_{l^-}^-\right) \leq \mathrm{CZ}\left(\gamma_1^+\right) \end{array} \right\}$$

Again we denote the compactification of the moduli spaces with the same symbols as their non-compactified counterparts. Similar considerations on the compactness and global convergence properties hold also in this non-autonomous case, as the breaking always happens in the autonomous regions.

3.6.1.3. Parametrized case

Let $\mathscr{H},\mathscr{G}$ be two asymptotically quadratic continuations between $H^\pm \in \mathfrak{w}\mathfrak{H}_*$ and $\mathscr{J},\mathscr{I}$ two adequate almost-complex structures. Let $(\mathbb{H},\mathbb{J})$ be a homotopy between $(\mathscr{H},\mathscr{J})$ and $(\mathscr{G},\mathscr{I})$.

Here we describe the suitable notion of broken convergence.

Proposition 3.6.4 *Fix 1-periodic orbits $\gamma^\pm$ of $H^\pm$. Assume that $\mathbb{H}$ is chosen in such a way that there exists an $e>0$ for which the following holds:*

$$(\rho,u^\rho)\in\mathbb{M}\left(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\right) \implies E_{\mathbb{H}^\rho,\mathbb{J}^\rho}(u^\rho)<e \tag{3.15}$$

For every sequence $(\rho_n,u_n^{\rho_n})\in\mathbb{M}(\gamma^-,\gamma^+)$ there exists a subsequence $(\rho_{n_j},u_{n_j}^{\rho_{n_j}})$, 1-periodic orbits $\gamma_0^-=\gamma^-,\gamma_1^-,\dots,\gamma_{l^-}^-$ of H^-, 1-periodic orbits $\gamma_1^+,\dots,\gamma_{l^++1}^+=\gamma^+$ of H^+, real sequences $(\sigma_j^{r,-})\subset\mathbb{R}$ for $r\in\{0,\dots,l^-\}$ tending to $-\infty$, real sequences $(\sigma_j^{r,+})\subset\mathbb{R}$ for $r\in\{1,l^++1\}$ tending to $+\infty$, Floer cylinders $u^{r,-}\in\mathscr{M}\left(\gamma_r^-,\gamma_{r+1}^-\right)$ for $r\in\{0,\dots,l^--1\}$, Floer cylinders $u^{r,+}\in\mathscr{M}\left(\gamma_r^+,\gamma_{r+1}^+\right)$ for $r\in\{1,\dots,l^+\}$ and a pair $(\rho_*,u^{\rho_*})\in\mathbb{M}\left(\gamma_{l^-}^-,\gamma_1^+\right)$ such that

$$\lim_{j\to\infty}\sigma_j^{r,-}\cdot u_{n_j}^{\rho_{n_j}}=u^{r,-},\quad \lim_{j\to\infty}\sigma_j^{r,+}\cdot u_{n_j}^{\rho_{n_j}}=u^{r,+},\quad \lim_{j\to\infty}u_{n_j}^{\rho_{n_j}}=u^{\rho_*},\quad \lim_{j\to\infty}\rho_{n_j}=\rho_*$$

where all limits except the last are in C^∞. Moreover, when all pairs $(\mathscr{H},\mathscr{J})$, $(\mathscr{G},\mathscr{I})$ and $(\mathbb{H},\mathbb{J})$ are regular, we can estimate $l^-+l^+\le \mathrm{CZ}(\gamma^-,H^-)-\mathrm{CZ}(\gamma^+,H^+)+1$.

This proposition prompts us with the correct notion of broken Floer trajectory and defines the convergence in the compactification of this moduli space. The compactification will again be denoted by the same symbol of the non-compactified moduli space.

Remark Notice that the energy bound (3.15) can also be weakened to a ρ-pointwise uniform energy bound, by compactness of $[0,1]$. Therefore there is basically no additional requirement here other than the ones already present in the compactness theory of the non-autonomous moduli spaces.

3.6.2. Gluing theory

In the previous three sections we've seen how a sequence of Floer cylinders converges to a broken configuration of Floer cylinders coming from lower-dimensional moduli spaces. In this section we show a sort of converse to this phenomenon, where a broken configuration with two levels is shown to have a 1-parameter family of Floer cylinders converging to it. A more elaborate treatment would lead us to conclude that the compactified moduli spaces can be endowed with the structure of compact smooth manifolds with boundaries and corners.

We will give an overview of gluing in the autonomous case, which already presents all the analytical difficulties of the theory, following the analogous treatment for holomorphic spheres in the book by McDuff and Salamon [32, Chap. 10]. The non-autonomous and parametrized cases are similar in spirit, always because of the fact that the continuation Hamiltonians and homotopy Hamiltonians considered are stationary outside compact sets of the parameters.

3.6.2.1. Gluing up two Floer cylinders

Let (H,J) be a regular pair and $\gamma_0,\gamma_1,\gamma_2 \in \mathrm{Per}^1 H$ be orbits with indices satisfying

$$\mathrm{CZ}(\gamma_0,H) > \mathrm{CZ}(\gamma_1,H) > \mathrm{CZ}(\gamma_2,H).$$

Consider two (H,J)-Floer cylinders u^0, u^1 and assume that $u^0 \in M(\gamma_0,\gamma_1)$ and $u^1 \in M(\gamma_1,\gamma_2)$. Therefore, the images $u^0(\mathbb{R}\times S^1)$ and $u^1(\mathbb{R}\times S^1)$ intersect at the orbit γ_1. We can thus "pre-glue" the two cylinders as follows. Fix a smooth non-decreasing function $\chi:\mathbb{R}\to[0,1]$ such that $\chi(s)=0$ for all $s\le 0$, $\chi(s)=1$ for all $s\ge 1$, $\chi'(s)\le 2$ for all s and $|\chi''(s)|\le 8$ for all s. For every $R>0$ large, define the "pre-glued cylinders" by the formula

$$u^0\widehat{\#_R}u^1(s,t)=\begin{cases} u^0(s+R,t) & s\le -\frac{R}{2}-1\\ \chi\left(-s-\frac{R}{2}\right)u^0(s+R,t)+\left[1-\chi\left(-s-\frac{R}{2}\right)\right]\gamma_1(t), & -\frac{R}{2}-1\le s\le -\frac{R}{2}\\ \gamma_1(t), & -\frac{R}{2}\le s\le \frac{R}{2}\\ \left[1-\chi\left(s-\frac{R}{2}\right)\right]\gamma_1(t)+\chi\left(s-\frac{R}{2}\right)u^1(s-R,t), & \frac{R}{2}\le s\le \frac{R}{2}+1\\ u^1(s-R,t) & s\ge \frac{R}{2}+1\end{cases} \tag{3.16}$$

This formula clearly gives us a smooth map $u^0\widehat{\#_R}u^1:\mathbb{R}\times S^1\to\mathbb{R}^{2n}$ which is asymptotic to γ_0 as $s\to-\infty$ and to γ_2 as $s\to+\infty$. Moreover, $u^0\widehat{\#_R}u^1(s,t)=\gamma_1(t)$ for all $s\in[-\varepsilon,\varepsilon]$. Recall the affine space $\mathscr{B}_0$ from Section 3.5.1.1 with asymptotic orbits γ_0,γ_2. It is immediate to see that $u^0\widehat{\#_R}u^1\in\mathscr{B}_0$, because of the exponential decay of the s-derivatives of both u^0 and u^1. From the formula it also follows that

Lemma 3.6.3 *$u^0\widehat{\#_R}u^1$ converges in C^∞_{loc} to γ_1 as $R\to+\infty$, which is a (constant) solution of the (H,J)-Floer equation. Moreover $(-R)\cdot\left(u^0\widehat{\#_R}u^1\right)\to u^0$ and $R\cdot\left(u^0\widehat{\#_R}u^1\right)\to u^1$ both in C^∞_{loc} as $R\to+\infty$.*

Hence, it is plausible to expect that for every R large enough the implicit function theorem will give us an actual solution which is near to the pre-glued cylinder. The aim of this section is to prove this claim. First, let's formulate a function-analytical model. According to the author's taste, the clearest account of this is McDuff and Salamon's [32, Prop. A.3.4].

Lemma 3.6.4 (Newton-Picard iteration) *Let E,F be Banach spaces, $U\subset E$ an open set and $f:U\to F$ a C^1 map. Let $u\in U$ be such that $Df(u):E\to F$ is surjective and has a bounded right* *inverse $G\colon F\to E$. Choose $\delta, c>0$ such that $\|G\|_{\mathrm{op}}\le c$, $B^E_\delta(u)\subset U$ and*

$$\left\|Df\left(u'\right)-Df(u)\right\|_{\mathrm{op}}\le \frac{1}{2c}\quad \forall u'\in B^E_\delta(u).$$

Assume that $w\in E$ satisfies

$$\|f(w)\|_F<\frac{\delta}{4c},\quad \|w-u\|_E<\frac{\delta}{8}$$

Then there exists an unique $v\in E$ such that

$$f(v)=0,\quad v-w\in \operatorname{im} G,\quad v\in B^E_\delta(u).$$

Moreover we have the estimate

$$\|v-w\|_E\le \frac{3c}{2}\|f(w)\|_F.$$

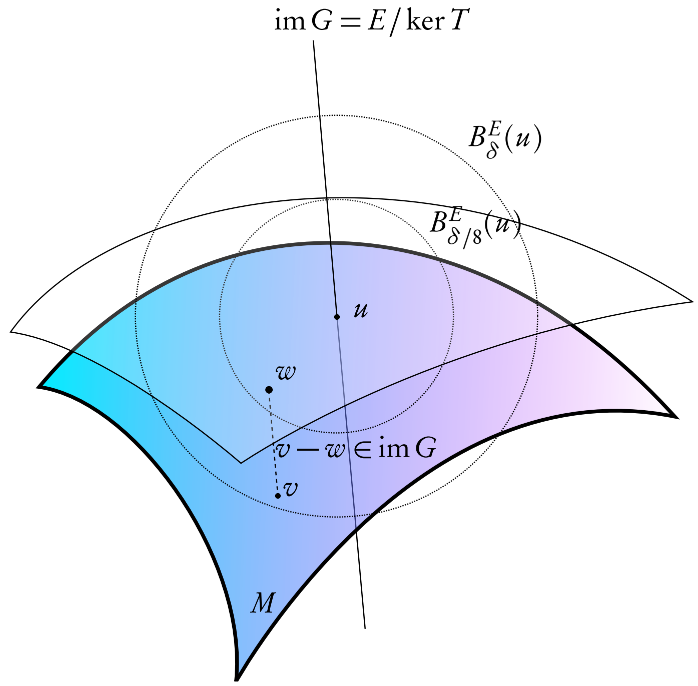

Figure 3.3. Schematic picture of the Newton-Picard iteration scheme in the Banach setting.

The proof can be found in [32, Prop. A.3.4]. We wish to apply this lemma to

$$E=W^{1,p}\left(\mathbb{R}\times S^1,\mathbb{R}^{2n}\right),\quad F=L^p\left(\mathbb{R}\times S^1,\mathbb{R}^{2n}\right),\quad f=\overline{\partial}_{H,J},\quad w_R=u^0\widehat{\#_R}u^1$$

Since (H,J) is a regular pair, $Df(u)$ is a Fredholm operator with bounded right inverse for every $u \in f^{-1}(0)$ (see Theorem 4). The problem is that we don't know this for $w_R = u^0 \widehat{\#_R} u^1$. So first one must construct an approximate inverse for $Df(w_R)$.

Lemma 3.6.5 *Let (H,J) be a regular pair and $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Per}^1 H$ with $\mathrm{CZ}(\gamma_0) > \mathrm{CZ}(\gamma_1) > \mathrm{CZ}(\gamma_2)$. There exist constants $R_0, c > 0$ and a smooth map assigning to every $u^0 \in M(\gamma_0, \gamma_1)$, $u^1 \in M(\gamma_1, \gamma_2)$ and $R > R_0$ a bounded linear operator*

$$Q_R : L^p\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right) \to W^{1,p}\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$$

with the following property. Denote by $w_R = u^0 \widehat{\#_R} u^1$ and

$$T_R = D\overline{\partial}_{HJ}(w_R) : W^{1,p}\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right) \to L^p\left(\mathbb{R} \times S^1, \mathbb{R}^{2n}\right)$$

Then for any $\eta \in L^p$ and $R > R_0$ we have the estimates

$$\|T_R Q_R \eta - \eta\|_{L^p} \le \frac{1}{2} \|\eta\|_{L^p}, \quad \|Q_R \eta\|_{W^{1,p}} \le c \|\eta\|_{L^p}$$

Here are some comments on the construction of this operator-valued map. Recalling the definition of the pre-glued cylinder (3.16), set

$$u_R^0(s,t) = \begin{cases} u^0 \widehat{\#_R} u^1(s,t), & s \le 0 \\ \gamma_1(t), & s \ge 0 \end{cases}, \quad u_R^1(s,t) = \begin{cases} \gamma_1(t), & s \le 0 \\ u^0 \widehat{\#_R} u^1(s,t), & s \ge 0. \end{cases}$$

From the formula, it is clear that

$$w_R(s,t) = \begin{cases} u_R^0(s,t), & s \le 0 \\ u_R^1(s,t), & s \ge 0. \end{cases}$$

Notice further that u_R^0 is a $W^{1,p}$-small perturbation of $(-R) \cdot u^0$ as $R \to +\infty$. The same can be said for u_R^1 and $R \cdot u^1$. Therefore, the operator $D\overline{\partial}_{HJ}\left(u_R^0\right)$ gets arbitrarily close in the strong topology to $D\overline{\partial}_{HJ}\left((-R) \cdot u^0\right)$ as $R \to +\infty$, which is onto because (H,J) is a regular pair, and the same can be said about the corresponding operators for u_R^1 and $R \cdot u^1$. This can be seen from scrutinizing the formula (3.10). We conclude that there exists an $R_0 > 0$ such that the operator

$$\widetilde{T_R} = D\overline{\partial}_{HJ}\left(u_R^0\right) \oplus D\overline{\partial}_{HJ}\left(u_R^1\right) : W^{1,p} \times W^{1,p} \to L^p \times L^p$$

is onto for every $R > R_0$. Therefore, we can define

$$\widetilde{Q_R} : L^p \times L^p \to W^{1,p} \times W^{1,p}$$

to be the unique right inverse of $\widetilde{T_R}$ whose range is L^2-orthogonal to the kernel of $\widetilde{T_R}$. It can be shown that (see [32, Lemma 10.6.1]) there exists a $c > 0$ and a (new) $R_0 > 0$ (possibly the same as before, but likely larger), such that

$$\left\|\widetilde{Q_R}\left(\eta^0,\eta^1\right)\right\|_{W^{1,p}\times W^{1,p}} \leq c\left\|\left(\eta^0,\eta^1\right)\right\|_{L^p\times L^p} \quad \forall R>R_0,\ \forall\left(\eta^0,\eta^1\right)\in L^p\times L^p.$$

The construction now proceeds exactly as in [32, Prop. 10.5.1].

Given the approximate right inverse Q_R to T_R, we can easily define a true right inverse

$$G_R\colon L^p\to W^{1,p},\quad G_R=Q_R\left(T_RQ_R\right)^{-1}$$

which has the same image as our approximate right inverse T_R. It is now a simple matter to check that this family of right inverses has an uniform bound on their norm, obtaining

Lemma 3.6.6 *There exists a $c>0$ and an $R_0>0$ such that*

$$\|G_R\|_{\mathrm{op}}\leq c\quad \forall R>R_0$$

Combining Lemmata 3.6.3, 3.6.4 and 3.6.6 we conclude that there exists an $R_0>0$ such that for every $R>R_0$, there exists an unique solution

$$u^0\#_Ru^1\in M\left(\gamma_0,\gamma_2\right)$$

such that

$$\left\|u^0\#_Ru^1-u^0\widehat{\#_R}u^1\right\|_{W^{1,p}}\leq\frac{3\|G_R\|_{\mathrm{op}}}{2}\left\|\overline{\partial}_{HJ}\left(u^0\widehat{\#_R}u^1\right)\right\|_{L^p}\to 0 \text{ as } R\to\infty \tag{3.17}$$

3.6.2.2. Gluing a 2-level broken configuration to the interior

We can arrange the glued solutions into a smooth map

$$\beta\colon \underline{M}(\gamma_0,\gamma_1)\times\underline{M}(\gamma_1,\gamma_2)\times(R_0,+\infty)\to\underline{M}(\gamma_0,\gamma_2),\quad \beta\left(u^0,u^1,R\right)=u^0\#_Ru^1$$

The smoothness of this map follows from the smoothness of all the operations involved in defining it. We implicitly chose an s-parametrization for each cylinder. The result does not depend on the chosen parametrizations, since it is a solution of the autonomous Floer equation.

Lemma 3.6.7 *$\beta\left(u^0,u^1,R\right)$ converges to the broken configuration $(u^0,u^1)\in\underline{M}(\gamma_0,\gamma_1)\times\underline{M}(\gamma_1,\gamma_2)$ as $R\to+\infty$. Viceversa, let $(u_k)\subset\underline{M}(\gamma_0,\gamma_2)$ be a sequence. If u_k converges to the broken configuration $\left(u^0,u^1\right)\in\underline{M}(\gamma_0,\gamma_1)\times\underline{M}(\gamma_1,\gamma_2)$, then for every k large enough there exist $R_k\in[R_0,+\infty)$ such that $u_k=\beta\left(u^0,u^1,R_k\right)$.*

Proof sketch. Recall that $(-R)\cdot u^0\widehat{\#_R}u^1\to u^0$ in C^0_{loc} as $R\to+\infty$ and $R\cdot u^0\widehat{\#_R}u^1\to u^1$ in C^0_{loc} as $R\to+\infty$. This combined with (3.17) gives $W^{1,p}_{\mathrm{loc}}$ convergence of the corresponding shifts of $u^0\#_Ru^1$. Elliptic regularity gives C^∞_{loc} convergence. Viceversa, recall that broken convergence is equivalent to “global convergence”, in the sense of Proposition 3.6.2. Therefore, for k large enough, u_k belongs to a neighborhood of the pre-glued trajectory. But then by the uniqueness in Lemma 3.6.4 it must be that u_k is of the claimed form. □

Proposition 3.6.5 *The map β is a smooth embedding.*

Proof sketch. The details of this proof can be found in [32, Theorem 10.1.2]. Lemma 3.6.7 implies that β is proper. Next one shows that it is an immersion. To do this, it suffices to prove that

$$D(\#_R)(u^0,u^1): T_{u^0}M(\gamma_0,\gamma_1)\oplus T_{u^1}M(\gamma_1,\gamma_2)\to T_{u^0\#_R u^1}M(\gamma_0,\gamma_2)$$

is injective. Consider a smooth path

$$[-\varepsilon,\varepsilon]\ni\tau\mapsto\left(u^{0,\tau},u^{1,\tau}\right)\in M(\gamma_0,\gamma_1)\times M(\gamma_1,\gamma_2).$$

Denote by $w_R^\tau = u^{0,\tau}\widehat{\#_R}u^{1,\tau}$, $v_R^\tau = u^{0,\tau}\#_R u^{1,\tau}$ and $w_R^\tau - v_R^\tau = \eta_R^\tau$. Denote further $T_R^\tau = D\overline{\partial}_{H,J}(w_R^\tau)$ and G_R^τ its right inverse. From the Newton-Picard Lemma 3.6.4 we know that

$$||\eta_R^\tau||_{W^{1,p}}\le\frac{3||G_R^\tau||_{\text{op}}}{2}\left\|\overline{\partial}_{H,J}w_R^\tau\right\|_{L^p}\le C_0\left\|\overline{\partial}_{H,J}w_R^\tau\right\|_{L^p}$$

The constant $C_0>0$ doesn't depend on R as long as it is large enough, by Lemma 3.6.6. In [32, Proposition 10.5.4] it is shown that:

$$\left\|\frac{d\eta_R^\tau}{d\tau}\right\|_{W^{1,p}}\le\frac{C_0}{R^{2/p}}\left(\left\|\frac{du^{0,\tau}}{d\tau}\right\|_{L^2}+\left\|\frac{du^{1,\tau}}{d\tau}\right\|_{L^2}\right)$$

This is the crucial estimate of the proof. Notice that since $\frac{du^{i,\tau}}{d\tau}$ is a tangent vector to a transversely cut out moduli space, it belongs to a finite-dimensional space, so the choice of the L^2 norm here is somewhat arbitrary.

Next, by the chain rule

$$\frac{dw_R^\tau}{d\tau}=D\left(\widehat{\#_R}\right)\left(u^{0,\tau},u^{1,\tau}\right)\left(\frac{du^{0,\tau}}{d\tau},\frac{du^{1,\tau}}{d\tau}\right)$$

Obviously by definition of the pre-gluing map, whenever $\eta^0\in T_{u^0}M(\gamma_0,\gamma_1)=\ker D\overline{\partial}_{H,J}(u^0)$ and $\eta^1\in T_{u^1}M(\gamma_1,\gamma_2)=\ker D\overline{\partial}_{H,J}(u^1)$, we have

$$\begin{aligned}\left\|D\left(\widehat{\#_R}\right)\left(u^0,u^1\right)\left(\eta^0,\eta^1\right)\right\|_{L^2}^2 &\ge\left\|\eta^0\right\|_{L^2((-\infty,R-1]\times S^1)}^2+\left\|\eta^1\right\|_{L^2([R+1,+\infty)\times S^1)}^2\ge\\ &\ge C_1\left(\left\|\eta^0\right\|_{L^2}^2+\left\|\eta^1\right\|_{L^2}^2\right)\end{aligned}$$

The last inequality follows because each η^i is in the kernel of some linear Cauchy-Riemann type operator, so one may use the unique continuation properties of solutions of linear Cauchy-Riemann type equations (see e.g. [17]) to show that the L^2 norm on a half-infinite cylinder controls the L^2 norm of the whole cylinder. Also here the choice of L^2 norm is somewhat arbitrary.

Now, we combine the previous estimates:

$$\begin{aligned}\left\|\frac{dv_R^\tau}{d\tau}\right\|_{W^{1,p}} &\geq \left\|\frac{dw_R^\tau}{d\tau}\right\|_{W^{1,p}} - \left\|\frac{d\eta_R^\tau}{d\tau}\right\|_{W^{1,p}} \geq \\ &\geq \left(C_1 - \frac{C_0}{R^{2/p}}\right)\left[\left\|\frac{du^{0,\tau}}{d\tau}\right\|_{L^2} + \left\|\frac{du^{1,\tau}}{d\tau}\right\|_{L^2}\right]\end{aligned}$$

which is positive as long as R is large enough. This shows that when R is large enough, $D\#_R$ is injective.

Finally one shows that β is injective, which concludes the proof. This is done as follows. The uniform estimates on the inverse of Lemma 3.6.6 can be used to show that for R large enough, the gluing map is injective on sufficiently small $W^{1,p}$-balls. Then, since $(-R)\cdot u^0\#_R u^1$ is arbitrarily close to u^0 and $R \cdot u^0\#_R u^1$ is arbitrarily close to u^1 as $R \to +\infty$, two couples with the same image under the gluing map must be $W^{1,p}$-close as much as we want by taking R large. But then they lie within a sufficiently small $W^{1,p}$-ball, where we know the gluing map is injective. □

3.6.2.3. A sketch of the general case

The gluing construction can be generalized to glue up a broken configuration of Floer cylinders. Namely, let $\gamma, \gamma' \in \mathrm{Per}^1 H$ and recall the space $\underline{M}^d(\gamma,\gamma')$ defined in (3.14), with $d = \mathrm{CZ}(\gamma) - \mathrm{CZ}(\gamma') - 1$. The space $\underline{M}^d(\gamma,\gamma')$ can be stratified into the subspaces of configurations with number of levels exactly $l \leq d$. For a fixed stratum, define a map

$$\beta \colon \underline{M}(\gamma,\gamma_1) \times \cdots \times \underline{M}(\gamma_l,\gamma') \times (R_0,+\infty)^l \to \underline{M}(\gamma_0,\gamma_1)$$

by setting

$$\beta(u^0,\ldots,u^l,R_1,\ldots,R_l) = u^0\#_{R_1}u^1\#_{R_2}\ldots\#_{R_{l-1}}u^{l-1}\#_{R_l}u^l$$

Here the right hand side is obtained in an analogous way as for two cylinders, by defining a pre-glued cylinder and running a similar analysis as the two cylinder case. One must take care that the pre-gluing is defined in such a way that the single flattened-out cylinders don't interact for any $R > 0$.

Remark One might think that one could obtain a gluing map on a general broken configuration by gluing up the cylinders iteratively, i.e. gluing the first two, then gluing the result to the third, and so forth. This procedure does not allow us to obtain a smooth embedding of the interior of a corner into the interior of the moduli space, so it is not the correct strategy.

This map will turn out to have similar properties as the two level case, in particular, it is a smooth embedding of the stratum as a corner of the moduli space $\underline{M}(\gamma,\gamma')$.

Remark To show that the compactified moduli space $\underline{M}(\gamma,\gamma')$ really is a manifold with boundary and corners, the map β is not sufficient. In fact, we haven't shown that the map gives an embedding including the boundary, which would correspond to the case that $R_1,\ldots,R_l=+\infty$. Also, the set up sketched above is not suitable to discuss coordinate changes, as there is no "ambient space" containing all the moduli spaces entering the corner structure. Hence the gluing theory sketched here cannot be used to give an atlas consisting of corner charts in the usual sense. The author was not able to find a full discussion of such issues in the "traditional" literature on Floer homology. It is reasonable to expect that these issues could be resolved via polyfold technology, but the author could not find a written down account of this as of the writing of the thesis.

3.6.3. Parametrizing the boundaries via gluing

Finally, we conclude our study of the moduli spaces of Floer trajectories with fixed asymptotics by gathering together the broken convergence and gluing theories to describe their boundaries in terms of lower dimensional moduli spaces. We do this only in the cases relevant for the definition of Floer homology, namely, only when the moduli space involved has dimension 1.

3.6.3.1. Boundary of autonomous moduli spaces

Let (H,J) be a regular pair. The proof of the following proposition can be also found in detail in [7, §9.2-6].

Proposition 3.6.6 *Let $\gamma,\gamma'\in\mathrm{Per}^1(H)$ be such that $\mathrm{CZ}(\gamma,H)-\mathrm{CZ}(\gamma',H)=2$. Then the boundary of the compactified moduli space $\underline{M}(\gamma,\gamma';H,J)$ is given by*

$$\partial\underline{M}(\gamma,\gamma';H,J)\cong\bigcup_{\substack{\gamma_1\in\mathrm{Per}^1(H):\\ \mathrm{CZ}(\gamma,H)=\mathrm{CZ}(\gamma_0,H)-1}}\underline{M}(\gamma,\gamma_1;H,J)\times\underline{M}(\gamma_1,\gamma';H,J)\tag{3.18}$$

3.6.3.2. Boundary of non-autonomous moduli spaces

Let $H^\pm\in\mathfrak{wH}_*$. Let $(\mathscr{H},\mathscr{J})$ be a regular pair of continuation data, with $\mathscr{H}$ an asymptotically quadratic continuation. Similarly as before, one may consult [7, §11.2] for a complete proof of the following proposition.

Proposition 3.6.7 *Let $\gamma^\pm\in\mathrm{Per}^1(H^\pm)$ be such that $\mathrm{CZ}(\gamma^-,H^-)-\mathrm{CZ}(\gamma^+,H^+)=1$. Then the*

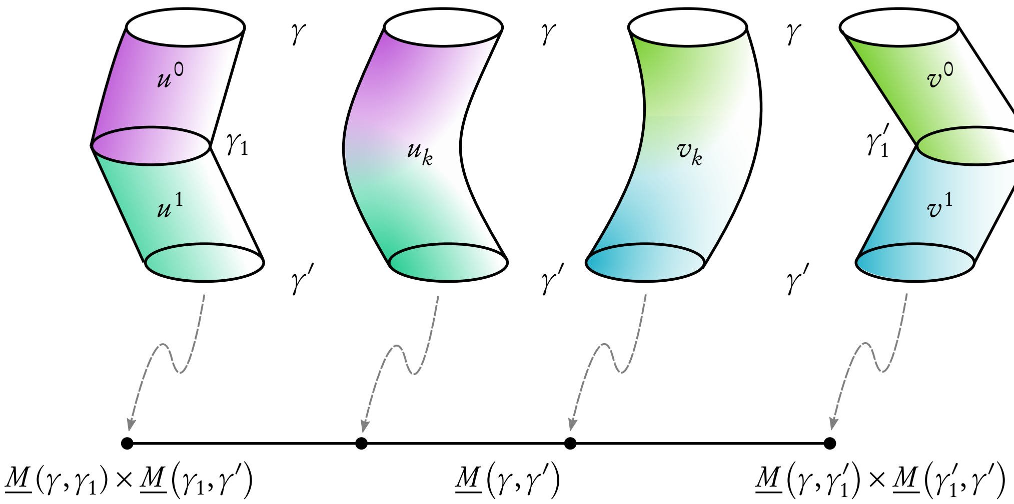

Figure 3.4. Schematic drawing of the compactification of one connected component of a one-dimensional quotiented moduli space.

boundary of the compactified moduli space $\mathcal{M}(\gamma^-,\gamma^+;\mathcal{H},\mathcal{J})$ *is given by*

$$\partial\mathcal{M}(\gamma^-,\gamma^+;\mathcal{H},\mathcal{J})\cong \bigcup_{\substack{\gamma_1^-\in\mathrm{Per}^1(H^-):\\ \mathrm{CZ}(\gamma_1^-,H^-)=\mathrm{CZ}(\gamma^-,H^-)-1}} \underline{M}(\gamma^-,\gamma_1^-;H^-,J^-)\times\mathcal{M}(\gamma_1^-,\gamma^+;\mathcal{H},\mathcal{J})\sqcup \sqcup \bigcup_{\substack{\gamma_1^+\in\mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma_1^+,H^+)=\mathrm{CZ}(\gamma^-,H^-)}} \mathcal{M}(\gamma^-,\gamma_1^+;\mathcal{H},\mathcal{J})\times\underline{M}(\gamma_1^+,\gamma^+;H^+,J^+) \tag{3.19}$$

3.6.3.3. Boundary of parametrized moduli spaces

Let $H^\pm$ be as in the previous section, $(\mathcal{H},\mathcal{J})$ and $(\mathcal{G},\mathcal{I})$ two regular pairs of continuation data between H^- and H^+, and $(\mathbb{H},\mathbb{J})$ a regular homotopy between the continuations.

Proposition 3.6.8 *Let* $\gamma^\pm\in\mathrm{Per}^1(H^\pm)$ *be such that* $\mathrm{CZ}(\gamma^+,H^+)=\mathrm{CZ}(\gamma^-,H^-)$. *Then the*

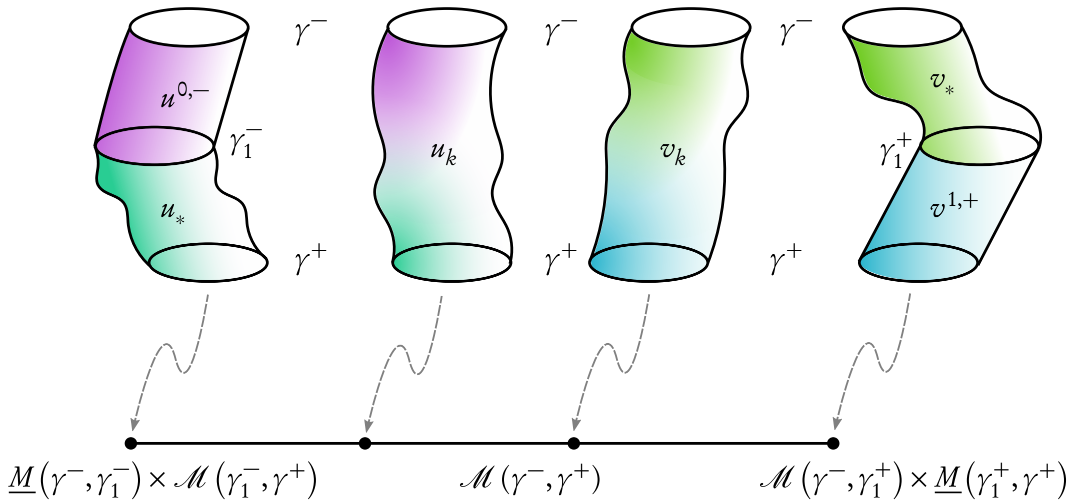

Figure 3.5. Schematic drawing of the compactification of a component of a one-dimensional non-autonomous moduli space. The wavy cylinders are meant to represent solutions of the non-autonomous Floer equation. The sequence u_k has $l^- = 1, l^+ = 0$, while the sequence v_k has $l^- = 0, l^+ = 1$.

boundary of the compactified moduli space $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$ *is given by*

$$\begin{aligned}
\partial\mathbb{M}\left(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\right) \cong \{0\} \times \mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right) \sqcup \{1\} \times \mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{G},\mathscr{I}\right) \sqcup \\
\sqcup \bigcup_{\substack{\gamma_1^- \in \mathrm{Per}^1(H^-): \\ \mathrm{CZ}(\gamma_1^-,H^-)=\mathrm{CZ}(\gamma^-,H^-)-1}} \underline{M}\left(\gamma^-,\gamma_1^-;H^-,J^-\right) \times \mathbb{M}\left(\gamma_1^-,\gamma^+;\mathbb{H},\mathbb{J}\right) \sqcup \\
\sqcup \bigcup_{\substack{\gamma_1^+ \in \mathrm{Per}^1(H^+): \\ \mathrm{CZ}(\gamma_1^+,H^+)=\mathrm{CZ}(\gamma^+,H^+)+1}} \mathbb{M}\left(\gamma^-,\gamma_1^+;\mathbb{H},\mathbb{J}\right) \times \underline{M}\left(\gamma_1^+,\gamma^+;H^+,J^+\right)
\end{aligned} \tag{3.20}$$

Remark It can be useful to make a small distinction between the two main components of the boundary, namely the first line in (3.20) and the bottom two. The first component, given in terms of non-autonomous moduli spaces, is already present before the compactification given by broken convergence. We call this first component the *regular part* of the boundary, and the second component the *exceptional part*. This is because of the following property: if the couple $(\mathbb{H}^\rho,\mathbb{J}^\rho)$ is a regular pair of continuation data for every fixed ρ, the exceptional part of the boundary is always empty. Indeed, notice that all the moduli spaces entering the exceptional part of the boundary have dimension zero. Therefore, if each pair $(\mathbb{H}^\rho,\mathbb{J}^\rho)$ is regular for every fixed $\rho \in (0,1)$, then these moduli spaces must be empty, otherwise we are in contradiction with the standing assumption that both $(\mathscr{H},\mathscr{J})$ and $(\mathscr{G},\mathscr{I})$ are regular pairs.

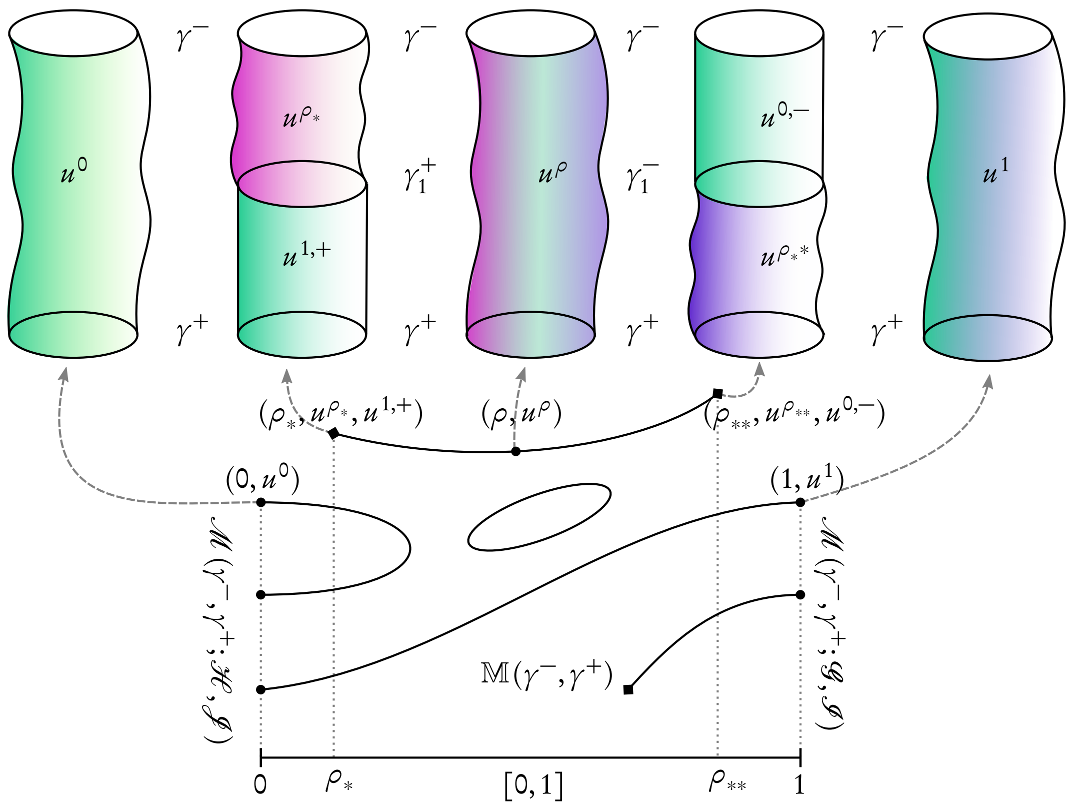

Figure 3.6. Schematic drawing of the compactification of a parametrized moduli space of dimension 1. The moduli space fibers over $[0,1]$, which is depicted at the bottom. The exceptional part of the boundary is denoted by diamond shaped points. In order not to clutter the picture too much, the cylinder corresponding to the compactification of the half-open branch issuing back from $\mathscr{M}(\gamma^-, \gamma^+; \mathscr{G}, \mathscr{I})$ was not represented. Such half-open branches may also issue out of $\mathscr{M}(\gamma^-, \gamma^+; \mathscr{H}, \mathscr{J})$. Notice the closed boundary-less component. If $(\mathbb{H}^\rho, \mathbb{J}^\rho)$ is a regular pair for all $\rho \in [0,1]$, then half-open and open branches cannot exist, and $\mathbb{M}(\gamma^-, \gamma^+)$ provides us with a cobordism between $\mathscr{M}(\gamma^-, \gamma^+; \mathscr{H}, \mathscr{J})$ and $\mathscr{M}(\gamma^-, \gamma^+; \mathscr{G}, \mathscr{I})$.

4. Floer homology for asymptotically linear Hamiltonian systems

Now that we have a rough understanding of the moduli spaces of Floer trajectories and their topological properties, we are ready to use the combinatorics of the low-dimensional moduli spaces to define Floer homology. This will be the homology of a chain complex generated by 1-periodic orbits, graded by their Conley-Zehnder index, and whose differential is given in terms of a count of Floer trajectories connecting two fixed orbits. Since we are only interested in existence of periodic orbits, it suffices to work with chain complexes with $\mathbb{Z}/2$-coefficients. We will also explain how to treat the case of degenerate Hamiltonians, leading to local Floer homology and its interactions with filtered Floer homology.

4.1. Floer homology

Let (H,J) be a regular pair, which we recall consists in an asymptotically quadratic, non-degenerate Hamiltonian, non-degenerate at infinity and a 1-periodic family of adequate almost complex structures such that all moduli spaces of Floer trajectories are transversally cut out. Recall that the 1-periodic orbits of such a Hamiltonian come in a finite number.

We define a chain complex, for each $k \in \mathbb{Z}$,

$$\mathrm{CF}_k(H,J) = \mathrm{span}_{\mathbb{Z}/2}\left\{\gamma \in C^\infty(S^1,\mathbb{R}^{2n}) : \dot{\gamma} = X_H \circ \gamma,\ \mathrm{CZ}(\gamma,H) = k\right\}$$

i.e. the free $\mathbb{Z}/2$-vector space with generators the 1-periodic orbits of H of Conley-Zehnder index k. By the remark above, this space is finite-dimensional for any k.

Often we identify a 1-periodic orbit γ of H with a fixed point $z = \gamma(0)$ of φ_H^1, and we think of $\mathrm{CF}_k(H,J)$ as generated by fixed points.

We define a boundary operator on the generators as follows

$$\begin{aligned} d_{H,J} : \mathrm{CF}_k(H,J) &\to \mathrm{CF}_{k-1}(H,J) \\ \gamma &\mapsto \sum_{\gamma'} \#_{\mathbb{Z}/2} \underline{M}\left(\gamma,\gamma';H,J\right)\gamma' \end{aligned}$$

where $\#_{\mathbb{Z}/2}$ denotes the parity of the cardinality. Since $\mathrm{CZ}(\gamma) = \mathrm{CZ}(\gamma') + 1$, the quotient of the moduli space is zero-dimensional and compact, and its cardinality is finite.

Remark One may take coefficients in a more general abelian group by discussing suitable $\mathbb{Z}$-orientations of the moduli spaces, and replacing the parity of the cardinality with a signed count.

Lemma 4.1.1 $d_{H,J} \circ d_{H,J} = 0$.

Proof. Abbreviate $d = d_{H,J}$. It suffices to show the claim on a generator $\gamma \in \mathrm{Per}^1(H)$ of index $\mathrm{CZ}(\gamma, H) = k$.

$$dd\gamma = \sum_{\substack{\gamma' \in \mathrm{Per}^1(H):\\ \mathrm{CZ}(\gamma,H)=k-2}} \sum_{\substack{\gamma_1 \in \mathrm{Per}^1(H):\\ \mathrm{CZ}(\gamma,H)=k-1}} \#_{\mathbb{Z}/2}\underline{M}(\gamma,\gamma_1;H,J) \cdot \#_{\mathbb{Z}/2}\underline{M}\left(\gamma_1,\gamma';H,J\right)\gamma'$$

Comparing this with the description of the boundary (3.18), we see that

$$\sum_{\substack{\gamma_1 \in \mathrm{Per}^1(H):\\ \mathrm{CZ}(\gamma,H)=k-1}} \#_{\mathbb{Z}/2}\underline{M}(\gamma,\gamma_1;H,J) \cdot \#_{\mathbb{Z}/2}\underline{M}\left(\gamma_1,\gamma';H,J\right) = \#_{\mathbb{Z}/2}\partial\,\underline{M}\left(\gamma,\gamma';H,J\right)$$

But since $\underline{M}(\gamma,\gamma';H,J)$ is a smooth 1-dimensional manifold, the cardinality of its boundary is even. This concludes the proof. □

Define the Floer homology of the Hamiltonian H to be the homology of the Floer chain complex:

$$\mathrm{HF}_k(H) = H_k\left(\mathrm{CF}_*(H,J), d_{H,J}\right)$$

From the definition only, it is not clear whether $\mathrm{HF}(H)$ depends on J or H, although the chosen notation suggests the answer. To show that it is independent on J and only "mildly dependent" on H, we construct continuation isomorphisms.

4.1.1. Continuation morphisms

We aim to explore the dependence of Floer homology on H and J. Let $(H^\pm, J^\pm)$ and $(\mathscr{H}, \mathscr{J})$ be regular pairs, where $\mathscr{H}$ is an asymptotically quadratic continuation between H^- and H^+ in the sense of Definition 3.2. The idea is to use $(\mathscr{H}, \mathscr{J})$ as a "homotopy of data", and hope that the moduli spaces of the corresponding continuation Floer equation have sufficient compactness to define a morphism, in the following way:

$$\begin{aligned}\mathscr{C}(\mathscr{H},\mathscr{J}) : \mathrm{CF}_*\left(H^-,J^-\right) &\to \mathrm{CF}_*\left(H^+,J^+\right)\\ \gamma^- &\mapsto \sum_{\substack{\gamma^+ \in \mathrm{Per}^1(H^+)\\ \mathrm{CZ}(\gamma^+,H^+)=\mathrm{CZ}(\gamma^-,H^-)}} \#_{\mathbb{Z}/2}\mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right)\gamma^+\end{aligned} \tag{4.1}$$

Of course, here the count makes sense only if the moduli space is compact. Proposition 3.6.3 tells us that this is possible when the Hamiltonian $\mathscr{H}$ leads to an uniform energy bound across all relevant moduli spaces. Namely, if $u \in \mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$, then we know that

$$E_{\mathscr{H},\mathscr{J}}(u) \leq \mathscr{A}_{H^-}(\gamma^-) - \mathscr{A}_{H^+}(\gamma^+) + \int_{\mathbb{R}\times S^1} \partial_s \mathscr{H}(s,t,u)\,ds\,dt.$$

In order to obtain the necessary compactness of the moduli spaces, one must guarantee that there exists a $B > 0$ such that

$$\int_{\mathbb{R}\times S^1} \partial_s \mathscr{H}(s,t,u)\,ds\,dt < B \quad \forall u \in \mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right).$$

Assuming that such a B exists for our chosen $\mathscr{H}$, one may show

Lemma 4.1.2 $\mathscr{C}(\mathscr{H},\mathscr{J}) : \mathrm{CF}_*(H^-,J^-) \to \mathrm{CF}_*(H^+,J^+)$ *is a morphism of chain complexes.*

Proof. Set for simplicity $d^{\pm} = d_{H^{\pm}J^{\pm}}$ and $\mathscr{C}(\mathscr{H},\mathscr{J}) = \mathscr{C}$. We have to show

$$d^+ \circ \mathscr{C} = \mathscr{C} \circ d^-$$

It suffices to show this equality on a generator $\gamma^- \in \mathrm{Per}^1(H^-)$ of index $\mathrm{CZ}(\gamma^-,H^-) = k$. The left hand side reads:

$$d^+\mathscr{C}\gamma^- = \sum_{\substack{\gamma^+\in\mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k-1}} \sum_{\substack{\gamma_1^+\in\mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma_1^+,H^+)=k}} \#_{\mathbb{Z}/2}\mathscr{M}\left(\gamma^-,\gamma_1^+;\mathscr{H},\mathscr{J}\right)\cdot \#_{\mathbb{Z}/2}\underline{M}\left(\gamma_1^+,\gamma^+;H^+,J^+\right)\gamma^+ \tag{4.2}$$

The right hand side reads:

$$\mathscr{C}d^-\gamma^- = \sum_{\substack{\gamma^+\in\mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k-1}} \sum_{\substack{\gamma_1^-\in\mathrm{Per}^1(H^-):\\ \mathrm{CZ}(\gamma_1^-,H^-)=k-1}} \#_{\mathbb{Z}/2}\underline{M}\left(\gamma^-,\gamma_1^-;H^-,J^-\right)\cdot \#_{\mathbb{Z}/2}\mathscr{M}\left(\gamma_1^-,\gamma^+;\mathscr{H},\mathscr{J}\right)\gamma^+ \tag{4.3}$$

Now, we subtract the right hand sides of equations (4.2) and (4.3) and compare the result with the description of the boundary of the moduli space $\mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$ given in (3.19). What we find is

$$\left[d^+\circ\mathscr{C} - \mathscr{C}\circ d^-\right]\gamma^- = \sum_{\substack{\gamma^+\in\mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k-1}} \#_{\mathbb{Z}/2}\partial\mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\right)\gamma^+$$

Since $\mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$ is a compact 1-dimensional smooth manifold, the cardinality of its boundary is even. This concludes the proof. □

Therefore, assuming for a moment that the continuation Hamiltonian allows us to reach the uniform energy bounds, we obtain a well defined morphism on the Floer homologies

$$\mathscr{C}(\mathscr{H},\mathscr{J})\colon \mathrm{HF}_*(H^-)\to \mathrm{HF}_*(H^+)$$

defined by equation (4.1). The first question to settle is dependence on the chosen homotopy.

Lemma 4.1.3 *If $(\mathscr{H},\mathscr{J})$ and $(\mathscr{G},\mathscr{I})$ are regular pairs of continuation data between (H^-,J^-) and (H^+,J^+) which lead to an uniform energy bound for their Floer cylinders, then $\mathscr{C}(\mathscr{H},\mathscr{J})$ is chain homotopic to $\mathscr{C}(\mathscr{G},\mathscr{I})$.*

Proof. This will follow from defining a chain homotopy using the parametrized moduli space $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$ of dimension zero, and then basically from understanding the description of the boundary via Proposition 3.6.8. Therefore take $\gamma^\pm\in \mathrm{Per}^1(H^\pm)$ such that

$$\mathrm{CZ}\left(\gamma^-,H^1\right)+1=\mathrm{CZ}\left(\gamma^+,H^+\right)$$

We use the zero-dimensional parametrized moduli spaces to define a putative chain homotopy

$$\mathfrak{X}(\mathbb{H},\mathbb{J}):\mathrm{CF}_*\left(H^-,J^-\right)\to \mathrm{CF}_{*+1}\left(H^+,J^+\right)$$

given on generators by the count

$$\mathfrak{X}(\mathbb{H},\mathbb{J})\gamma^-=\sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+)\\ \mathrm{CZ}(\gamma^+,H^+)=\mathrm{CZ}(\gamma^-,H^-)+1}} \#_{\mathbb{Z}/2}\mathbb{M}\left(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\right)\gamma^+$$

By compactness of the zero-dimensional parametrized moduli spaces $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$, this morphism is well defined.

Set for simplicity $\mathfrak{X}=\mathfrak{X}(\mathbb{H},\mathbb{J})$ and $d^\pm=d_{H^\pm,J^\pm}$. We have to show that it is truly a chain homotopy between $\mathscr{C}(\mathscr{H},\mathscr{J})$ and $\mathscr{C}(\mathscr{G},\mathscr{I})$, i.e.

$$\mathscr{C}(\mathscr{H},\mathscr{J})-\mathscr{C}(\mathscr{G},\mathscr{I})=\mathfrak{X}\circ d^-+d^+\circ\mathfrak{X}$$

We have to show this equality mod 2, so signs are not important. We write this equation applied to one generator γ^- of degree $\mathrm{CZ}(\gamma^-,H^-)=k$. The right hand side reads:

$$\begin{aligned}
&\mathfrak{X}d^-\gamma^-+d^+\mathfrak{X}\gamma^-=\\
&=\sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k}}\ \sum_{\substack{\gamma_1^-\in \mathrm{Per}^1(H^-):\\ \mathrm{CZ}(\gamma_1^-,H^-)=k-1}} \#_{\mathbb{Z}/2}\underline{M}(\gamma^-,\gamma_1^-;H^-,J^-)\cdot \#_{\mathbb{Z}/2}\mathbb{M}\left(\gamma_1^-,\gamma^+;\mathbb{H},\mathbb{J}\right)\gamma^+ +\\
&+\sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k}}\ \sum_{\substack{\gamma_1^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma_1^+,H^+)=k+1}} \#_{\mathbb{Z}/2}\mathbb{M}(\gamma^-,\gamma_1^+;\mathbb{H},\mathbb{J})\cdot \#_{\mathbb{Z}/2}\underline{M}\left(\gamma_1^+,\gamma^+;H^+,J^+\right)\gamma^+
\end{aligned}\tag{4.4}$$

The left hand side reads:

$$\mathscr{C}(\mathscr{H},\mathscr{J})\gamma^- - \mathscr{C}(\mathscr{G},\mathscr{I})\gamma^- = \sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+)=k}} \Big[\#_{\mathbb{Z}/2}\mathscr{M}\big(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\big) - \#_{\mathbb{Z}/2}\mathscr{M}\big(\gamma^-,\gamma^+;\mathscr{G},\mathscr{I}\big)\Big]\gamma^+ \tag{4.5}$$

Remember that the coefficients we take are $\mathbb{Z}/2$, so signs are somewhat inconsequential in all the above equations. Subtracting the right hand sides of the equalities (4.4) and (4.5), we obtain the following sum:

$$\sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k}} \Bigg[\#_{\mathbb{Z}/2}\mathscr{M}\big(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J}\big) - \#_{\mathbb{Z}/2}\mathscr{M}\big(\gamma^-,\gamma^+;\mathscr{G},\mathscr{I}\big) + \\ - \sum_{\substack{\gamma_1^-\in \mathrm{Per}^1(H^-):\\ \mathrm{CZ}(\gamma_1^-,H^-)=k-1}} \#_{\mathbb{Z}/2}\underline{M}(\gamma^-,\gamma_1^-;H^-,J^-)\cdot \#_{\mathbb{Z}/2}\mathbb{M}\big(\gamma_1^-,\gamma^+;\mathbb{H},\mathbb{J}\big) + \\ - \sum_{\substack{\gamma_1^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma_1^+,H^+)=k+1}} \#_{\mathbb{Z}/2}\mathbb{M}(\gamma^-,\gamma_1^+;\mathbb{H},\mathbb{J})\cdot \#_{\mathbb{Z}/2}\underline{M}\big(\gamma_1^+,\gamma^+;H^+,J^+\big)\Bigg]\gamma^+$$

Comparing the above sum with the description of the boundary of the parametrized moduli space (3.20), we see that

$$\big[\mathscr{C}(\mathscr{H},\mathscr{J}) - \mathscr{C}(\mathscr{G},\mathscr{I}) - \mathfrak{X}\circ d^- - d^+\circ \mathfrak{X}\big]\gamma^- = \\ = \sum_{\substack{\gamma^+\in \mathrm{Per}^1(H^+):\\ \mathrm{CZ}(\gamma^+,H^+)=k}} \#_{\mathbb{Z}/2}\partial\mathbb{M}\big(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J}\big)\gamma^+$$

But since $\mathbb{M}(\gamma^-,\gamma^+;\mathbb{H},\mathbb{J})$ is a compact 1-manifold, its boundary must have even cardinality. This concludes the proof. □

Remark Notice that if one can arrange the homotopy of continuations datum $(\mathbb{H},\mathbb{J})$ to give regular pairs $(\mathbb{H}^\rho,\mathbb{J}^\rho)$ for every fixed ρ, then the morphisms $\mathscr{C}(\mathscr{H},\mathscr{J})$ and $\mathscr{C}(\mathscr{G},\mathscr{I})$ can be shown to be *equal* on the level of chain complexes, and not just chain homotopic (so equal on the level of homology). This is because the "error" given by the non-emptiness of the exceptional part of the boundary of the parametrized moduli spaces is zero.

4.1.1.1. Functoriality of continuations

Continuation morphisms enjoy a certain kind of functoriality under concatenation of continuation Hamiltonians. To state it, fix three regular pairs (H^-,J^-), (H^*,J^*), (H^+,J^+). Assume we can define continuations between all the three Hamiltonians, via continuation data $(\mathscr{H},\mathscr{J})$, resp. $(\mathscr{G},\mathscr{I})$, resp. $(\mathscr{F},\mathscr{L})$ between (H^-,J^-) and (H^*,J^*), resp. (H^-,J^-) and (H^+,J^+), resp. (H^*,J^*) and (H^+,J^+).

Lemma 4.1.4 1. *Let $H \in \mathfrak{w}\mathfrak{H}_*$. The continuation morphism induced by the constant continuation $\mathscr{H}^s = H$ is the identity.*

2. *$\mathscr{C}(\mathscr{G},\mathscr{I})$ is chain homotopic to $\mathscr{C}(\mathscr{F},\mathscr{L}) \circ \mathscr{C}(\mathscr{H},\mathscr{I})$.*

The proof of the first point is obvious from the definition of the continuation morphisms. The proof of the second point is as follows: one translates $\mathscr{F}$ and $\mathscr{H}$ in the s-direction until they can be concatenated in the region where they equal H^*, and then shows that the induced morphism is exactly $\mathscr{C}(\mathscr{F},\mathscr{L}) \circ \mathscr{C}(\mathscr{H},\mathscr{I})$. This step follows fundamentally from the gluing theorem, and can be found in [7, §11.5]. Finally since we found a asymptotically quadratic continuation between H^- and H^+ inducing $\mathscr{C}(\mathscr{F},\mathscr{L}) \circ \mathscr{C}(\mathscr{H},\mathscr{I})$, by the previous Lemma it must be chain homotopic to $\mathscr{C}(\mathscr{G},\mathscr{I})$.

As an immediate corollary of the above two lemmata, we obtain

Lemma 4.1.5 $\mathrm{HF}_*(H)$ *doesn't depend on J.*

Proof. Let J^- and J^+ be almost-complex structures such that (H,J^-) and (H,J^+) are both regular pairs. Consider a homotopy of almost-complex structures $\mathscr{J}$ such that $(H,\mathscr{J})$ is a regular pair, where H is considered as the constant homotopy. Then clearly $\partial_s H = 0$ so there is no issue of compactness in defining the continuation morphism as in (4.1). Flipping the direction of $\mathscr{J}$ gives the homotopy inverse continuation morphism. □

4.1.2. Uniform energy bounds along asymptotically quadratic continuations

It remains to show that one may indeed find a asymptotically quadratic continuation producing the adequate compactness, and the same for homotopies of continuations. By our L^∞-estimate found in Proposition 3.3.5, we have to provide Hamiltonians which reach uniform energy bounds across their moduli spaces. The brunt of the argument will be showing this for asymptotically quadratic continuations, because for homotopies of continuations we can work point-wise in the homotopy parameter.

4.1.2.1. Hamiltonians with the same quadratic form at infinity

The simplest case, which is also relevant for the proof of the Poincaré-Birkhoff theorem, is when H^-, H^+ are such that $H^\pm = Q + h^\pm$, i.e. they have the *same quadratic form at infinity*. For this kind of Hamiltonians, we can define

$$\mathscr{H}_t^s(z) = Q_t(z) + \chi(s)h_t^+(z) + (1-\chi(s))h_t^-(z)$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth non-decreasing function such that $\chi(s) = 0$ for all $s \leq 0$ and $\chi(s) = 1$ for all $s \geq 1$. Pick an adequate family $\mathscr{J}$ of almost complex structures and fix asymptotics $\gamma^{\pm} \in \operatorname{Fix} \varphi^1_{H^{\pm}}$. Clearly

$$\begin{aligned} E_{\mathscr{H},\mathscr{J}}(u) &\leq \mathscr{A}_{H^-}\left(\gamma^-\right) - \mathscr{A}_{H^+}(\gamma^+) - \int_{\mathbb{R}\times S^1} \partial_s \mathscr{H}^s_t(u(s,t))\,ds\,dt = \\ &= \mathscr{A}_{H^-}\left(\gamma^-\right) - \mathscr{A}_{H^+}(\gamma^+) - \int_0^1\int_0^1 \chi'(s)\left[h^+_t(u(s,t)) - h^-_t(u(s,t))\right] ds\,dt \leq \\ &\leq \mathscr{A}_{H^-}\left(\gamma^-\right) - \mathscr{A}_{H^+}(\gamma^+) - \min_{(t,\zeta)\in S^1\times\mathbb{R}^{2n}} \left(h^+_t(\zeta) - h^-_t(\zeta)\right) \leq \\ &\leq \mathscr{A}_{H^-}\left(\gamma^-\right) - \mathscr{A}_{H^+}(\gamma^+) + \left\|h^+ - h^-\right\|_{L^\infty} \end{aligned}$$

Notice that here it is crucial that $H^{\pm}$ are asymptotically quadratic, and not just weakly asymptotically quadratic.

4.1.2.2. Weakly asymptotically quadratic Hamiltonians with the same index at infinity

Here we show that an uniform energy estimate is reachable in a much wider context, namely, that two Hamiltonians H^- and $H^+ \in \mathfrak{wH}$ which have the same index at infinity may be connected by an asymptotically quadratic continuation reaching the uniform energy bound, and moreover such that the linear system at infinity of this asymptotically quadratic continuation is always non-degenerate.

Let $(H^{\pm}, J^{\pm})$ be two regular pairs, where $H^{\pm}_t = Q^{\pm}_t + h^{\pm}_t \in \mathfrak{wH}$ and $Q^{\pm}_t(z) = \frac{1}{2}\left\langle A^{\pm}_t z, z\right\rangle$ are non-degenerate quadratic Hamiltonians. Assume that $H^{\pm}$ have the same index at infinity. Then there is a path of loops $\mathbb{A}\colon [0,1]\times S^1 \to \mathrm{Sym}(2n)$, such that $\mathbb{A}^0 = A^-$, $\mathbb{A}^1 = A^+$, and $\mathbb{A}^s\colon S^1 \to \mathrm{Sym}(2n)$ defines a non-degenerate linear Hamiltonian system for all $s\in[0,1]$. We extend $\mathbb{A}$ constantly outside $s\in[0,1]$ to a smooth $\mathbb{A}\colon \mathbb{R}\times S^1 \to \mathrm{Sym}(2n)$.

Take a non-decreasing smooth $\chi\colon \mathbb{R}\to[0,1]$ such that $\chi(s)=0\ \forall s\leq 0$, $\chi(s)=1\ \forall s\geq 1$ and $\chi'\leq 2$. Fix a $\beta\in(0,1)$. We define a family of asymptotically quadratic continuations depending on β:

$$\mathscr{H}^{\beta,s}_t(z) = \frac{1}{2}\left\langle \mathbb{A}^{\beta s}_t z, z\right\rangle + (1-\chi(\beta s))h^0(t,z) + \chi(\beta s)h^1(t,z) =: \mathscr{Q}^s_t(z) + \mathscr{K}^s_t(z)$$

Notice that $\mathscr{H}^{\beta,s} \in \mathfrak{wH}$ for all $s\in\mathbb{R}$. Moreover notice that it depends on s only for $s\in[0,\beta^{-1}] = \mathscr{S}$. Therefore we know from Lemma 2.3.3 that there exist constants $\nu_s, \delta_s > 0$ such that

$$\|\dot{x} - X_{\mathscr{H}^{\beta,s}}(x)\|_{L^2} \geq \frac{\nu_s}{2}\|x\|_{L^2} - \delta_s \quad \forall x\in W^{1,2}\left(S^1,\mathbb{R}^{2n}\right)$$

Recall from the proof of Lemma 2.3.3 that $\nu_s = \left\|D^{-1}_{\mathbb{A}^s}\right\|^{-1}_{\mathrm{op}}$, and that δ_s is the L^∞-norm of $\nabla\mathscr{K}^s_t$ over a ball whose radius depends only on ν_s. We set

$$\bar{\nu} = \min_{s\in[0,1]} \nu_s, \quad \overline{\delta} = \max_{s\in[0,1]} \delta_s$$

Finally, pick an adequate family of almost-complex structures $\mathscr{J}$. The following proposition shows that an uniform energy bound may be reached adiabatically.

Proposition 4.1.1 *Let $\gamma^{\pm}$ be 1-periodic orbits of $H^{\pm}$. Set $C_{\mathbb{A}} = \|\partial_s \mathbb{A}\|_{L^\infty([0,1]\times S^1, \mathrm{Sym}\, 2n)}$. If $\beta \le \frac{\bar{\nu}^2}{4C_{\mathbb{A}}}$, then there exists a $C = C(H^{\pm})$ such that*

$$E_{\mathscr{H}^\beta,\mathscr{J}}(u) \le \mathscr{A}_{H^-}(\gamma^-) - \mathscr{A}_{H^+}(\gamma^+) + C \tag{4.6}$$

for all $u \in \mathscr{M}\left(\gamma^-, \gamma^+; \mathscr{H}^\beta, \mathscr{J}\right)$.

Proof. Set $\Delta\mathscr{A} = \mathscr{A}_{H^-}(\gamma^-) - \mathscr{A}_{H^+}(\gamma^+)$. We start from the usual estimate:

$$\begin{aligned} E_{\mathscr{H}^\beta,\mathscr{J}}(u) &\le \Delta\mathscr{A} + \int_{\mathbb{R}\times S^1} \left|\left(\partial_s \mathscr{H}_t^{\beta,s}\right)(u(s,t))\right| ds\,dt \le \\ &\le \Delta\mathscr{A} + \int_{\mathscr{S}\times S^1} \frac{1}{2}\left|\left\langle \partial_s\left(\mathbb{A}_t^{\beta s}\right) u(s,t), u(s,t)\right\rangle\right| + \\ &\quad + \int_{\mathscr{S}\times S^1} \beta\chi'(\beta s)\left|h_t^+(u(s,t)) - h_t^-(u(s,t))\right| ds\,dt \le \\ &\le \Delta\mathscr{A} + \frac{\bar{\nu}^2}{8}\|u\|^2_{L^2(\mathscr{S}\times S^1)} + 2\left[\left\|h^-\circ u\right\|_{L^1(\mathscr{S}\times S^1)} + \left\|h^+\circ u\right\|_{L^1(\mathscr{S}\times S^1)}\right] \end{aligned} \tag{4.7}$$

Now, recall that $H^{\pm}$ are weakly asymptotically quadratic. The sublinearity of $\nabla h^{\pm}$ as $|z| \to \infty$ implies that $h^{\pm}$ is sub-quadratic as $|z| \to \infty$. Therefore, analogously as in Lemma 2.3.2, we can show that for every $\varepsilon > 0$ there exist constants $D_\varepsilon^{\pm} > 0$ such that

$$\left\|h^{\pm}\circ x\right\|_{L^1(S^1)} \le \frac{\varepsilon}{4}\|x\|^2_{L^2(S^1)} + D_\varepsilon^{\pm} \quad \forall x \in W^{1,2}\left(S^1, \mathbb{R}^{2n}\right)$$

Setting $D_\varepsilon = \max\{2D_\varepsilon^-, 2D_\varepsilon^+\}$ we can estimate the last term in (4.7) and obtain, for any fixed $\varepsilon > 0$,

$$E_{\mathscr{H}^\beta,\mathscr{J}}(u) \le \Delta\mathscr{A} + D_\varepsilon + \left[\frac{\bar{\nu}^2}{8} + \varepsilon\right]\|u\|^2_{L^2(\mathscr{S}\times S^1)} \tag{4.8}$$

On the other hand, since the quadratic Hamiltonian defined by $\mathbb{A}^s$ is non-degenerate for all $s \in \mathbb{R}$,

$$\begin{aligned} E_{\mathscr{H}^\beta,\mathscr{J}}(u) &= \|\partial_s u\|^2_{L^2} = \|\partial_t u - X_{\mathscr{H}^\beta}\circ u\|^2_{L^2} \ge \\ &\ge \int_{\mathscr{S}} \|\partial_t u(s,\cdot) - X_{\mathscr{H}^\beta}(u(s,\cdot))\|^2_{L^2(S^1)} ds \ge \int_{\mathscr{S}} \left[\frac{\bar{\nu}}{2}\|u(s,\cdot)\|_{L^2(S^1)} - \bar{\delta}\right]^2 ds\,dt = \\ &= \frac{\bar{\nu}^2}{4}\|u\|^2_{L^2(\mathscr{S}\times S^1)} - \bar{\delta}\bar{\nu}\|u\|_{L^2(\mathscr{S}\times S^1)} + |\mathscr{S}|\,\bar{\delta}^2 \end{aligned} \tag{4.9}$$

Comparing the estimates (4.8) and (4.9), we obtain that for any $\varepsilon > 0$ there exists a $D_\varepsilon > 0$ such that

$$\frac{\overline{\nu}^2}{4}||u||^2_{L^2(\mathscr{S}\times S^1)} - \overline{\delta}\overline{\nu}||u||_{L^2(\mathscr{S}\times S^1)} + |\mathscr{S}|\overline{\delta}^2 \leq \Delta\mathscr{A} + D_\varepsilon + \left[\frac{\overline{\nu}^2}{8} + \varepsilon\right]||u||^2_{L^2(\mathscr{S}\times S^1)} \iff$$
$$\left[\frac{\overline{\nu}^2}{4} - \varepsilon\right]||u||^2_{L^2(\mathscr{S}\times S^1)} - \overline{\delta}\overline{\nu}||u||_{L^2(\mathscr{S}\times S^1)} \leq -|\mathscr{S}|\overline{\delta}^2 + \Delta\mathscr{A} + D_\varepsilon$$

As long as $\varepsilon < \frac{\overline{\nu}^2}{4}$, the coefficient of the leading quadratic term is positive. But then, $||u||_{L^2(\mathscr{S}\times S^1)}$ must be uniformly bounded. Re-inserting such fact in (4.8) and fixing, for example, $\varepsilon = \frac{\overline{\nu}^2}{8}$, gives the wanted uniform bound of the energy. □

Remark The specific form of $\mathscr{H}^\beta$ constructed here is not crucial to the proof. It has only been chosen this way to make the calculations simpler and the constants a bit more explicit.

Having obtained an uniform energy bound over the moduli space, one may construct continuation morphisms as explained above. For the purposes of filtered Floer homology, we state the following

Corollary 4.1.1 *Let $H^\pm \in \mathfrak{w}\mathfrak{H}$ and β be as above. There exits a compact set $K = K(\mathscr{H}^\beta)$ and a constant $e = e(\mathscr{H}^\beta)$ such that*

$$E_{\mathscr{H}^\beta,\mathscr{J}}(u) \leq \mathscr{A}_{H^-}(\gamma^-) - \mathscr{A}_{H^+}(\gamma^+) + \|h^- - h^+\|_{L^\infty(S^1\times K)} + e$$

for all $u \in \mathscr{M}\left(\gamma^-,\gamma^+;\mathscr{H}^\beta,\mathscr{J}\right)$.

Proof. By Propositions 3.3.5 and 4.1.1 there exists a compact set K where all solutions of the continuation Floer equation $u \in \mathscr{M}(\gamma^-,\gamma^+;\mathscr{H},\mathscr{J})$ are contained. Now look at (4.7). The conclusion is reached by setting

$$e = \frac{1}{2}||\partial_s\mathbb{A}||_{L^\infty} \cdot (\mathrm{diam}\, K)^2$$

and estimating the term $|h_t^- \circ u - h_t^+ \circ u|$ over K. □

On chain homotopies From the above calculation, we see that the necessary uniform energy bound on the parametrized moduli spaces arising in the definition of a chain homotopy as in Lemma 4.1.3 is reached when all the asymptotically quadratic continuations involved are between Hamiltonians with the same index at infinity.

4.1.2.3. Intermediate case: close quadratic Hamiltonians at infinity

We need a last case, which is somewhat intermediate between the easy case of Hamiltoinans in $\mathfrak{H}$ with the same quadratic Hamiltonian at infinity, and Hamiltonians in $\mathfrak{w}\mathfrak{H}$ with the same index at infinity. It will be important for the proof of invariance of filtered Floer homology.

Let $H^i \in \mathfrak{wH}$, $H = Q^i + h^i$, and $Q^i_t(z) = \frac{1}{2}\left\langle A^i_t z, z\right\rangle$, $i = 0, 1$. If A^0 and A^1 are sufficiently C^0 close, then clearly $\operatorname{ind}_\infty H^0 = \operatorname{ind}_\infty H^1$, and we can obtain an uniform energy bound. We would like to obtain a manageable action shift estimate which is small with the distance of the quadratic Hamiltonians at infinity. The task is to find an explicit path of symmetric matrices which generate a family of non-degenerate quadratic Hamiltonians.

Lemma 4.1.6 *For every $A^0 \in C^\infty\left(S^1, \mathrm{Sym}(2n)\right)$ there exists a constant $\alpha > 0$ with the following property. Let $A^1 \in C^\infty\left(S^1, \mathrm{Sym}(2n)\right)$ be such that $||A^0 - A^1||_{L^\infty} < \alpha$. Denote by $M_i \colon [0,1] \to \mathrm{Sp}(2n)$, $t \mapsto M^t_i$ the paths of symplectic matrices representing the flow of the linear symplectic vector fields $-J_0 A^i_t$. If M^1_0 does not have the eigenvalue 1, then the path $\mathbb{A} \colon [0,1] \times S^1 \to \mathrm{Sym}(2n)$ defined by*

$$\mathbb{A}^s_t = (1-s)A^0_t + sA^1_t$$

generates a homotopy of paths $\mathbb{M} \colon [0,1] \times S^1 \to \mathrm{Sp}(2n)$, $(s,t) \mapsto \mathbb{M}^t_s$ between M_0 and M_+ such that $\mathbb{M}^1_s$ never has the eigenvalue 1. In particular $\mathrm{CZ}\left(M^t_0\right) = \mathrm{CZ}\left(M^t_+\right)$.

Proof. Let $A^1 \in C^\infty\left(S^1, \mathrm{Sym}(2n)\right)$ be such that $||A^0 - A^1||_{L^\infty} < 1$. For fixed $s \in [0,1]$, denote the flow of the linear vector field $-J_0\mathbb{A}^s$ by

$$\mathbb{M}_s \colon [0,1] \to \mathrm{Sp}(2n), \quad \mathbb{M}^t_s = \varphi^t_{-J_0\mathbb{A}^s}$$

We estimate

$$\begin{aligned}\left|\mathbb{M}^\tau_s - M^\tau_-\right| &= \left|\int_0^\tau s\left(A^1_t - A^0_t\right)\mathbb{M}^t_s + A^0_t\mathbb{M}^t_s - A^0_t M^t_- \, dt\right| \leq \\ &\leq s\left\|A^0 - A^1\right\|_{L^\infty} ||\mathbb{M}_s||_{L^1([0,1],\mathrm{Sp}(2n))} + \left\|A^0\right\|_{L^\infty}\int_0^\tau \left|\mathbb{M}^t_s - M^t_-\right| dt\end{aligned}$$

Now,

$$\begin{aligned}|\mathbb{M}^\tau_s| &\leq \int_0^\tau \left|s\left(A^1_t - A^0_t\right) + A^0_t\right| \cdot |\mathbb{M}^t_s| \, dt \leq \left(\left\|A^0\right\|_{L^\infty} + 1\right)\int_0^\tau |\mathbb{M}^t_s| \, dt \implies \\ ||\mathbb{M}_s||_{L^1([0,1],\mathrm{Sym}(2n))} &\leq \int_0^1 e^{(||A^0||_{L^\infty}+1)\tau} d\tau = \frac{e^{||A^0||_{L^\infty}+1} - 1}{||A^0||_{L^\infty} + 1} = c_0\end{aligned}$$

Hence we obtain that

$$\left|\mathbb{M}^\tau_s - M^\tau_-\right| \leq c_0\left\|A^0 - A^1\right\|_{L^\infty} + \left\|A^0\right\|_{L^\infty}\int_0^\tau \left|\mathbb{M}^t_s - M^t_-\right| dt$$

Using Grönwall lemma again, we obtain the estimate

$$\left|\mathbb{M}^1_s - M^1_-\right| \leq C \cdot \left\|A^0 - A^1\right\|_{L^\infty}, \quad C = c_0 e^{||A^0||_{L^\infty}}$$

Now, since the matrix M^1_- does not have the eigenvalue 1, there exists a constant $C_- > 0$ such that

$$\left|M_-^1 z - z\right| \geq C_-|z| \quad \forall z \in \mathbb{R}^{2n}$$

Hence we can estimate

$$\begin{aligned}\left|\mathbb{M}_s^1 z - z\right| = \left|\mathbb{M}_s^1 z - M_-^1 z + M_-^1 z - z\right| \geq \\ \geq \left|M_-^1 z - z\right| - \left|\left(\mathbb{M}_s^1 - M_-^1\right) z\right| \geq \left[C_- - C\left\|A^0 - A^1\right\|_{L^\infty}\right]|z|\end{aligned}$$

Now, if we take

$$\left\|A^0 - A^1\right\|_{L^\infty} < \min\left\{\frac{C_-}{C}, 1\right\} = \alpha$$

then the path $t \mapsto \mathbb{M}_s^t$ never has the eigenvalue 1. The constant α clearly depends only on A^0. □

This lemma justifies the following definition.

Definition 4.1 Let Q^i be quadratic Hamiltonians, with $Q_t^i(z) = \frac{1}{2}\left\langle A_t^i z, z\right\rangle$. Define the *distance* of the quadratic Hamiltonians by

$$d\left(Q^0, Q^1\right) = \left\|A^0 - A^1\right\|_{L^\infty}$$

Now let $\mathscr{H}$ be the following asymptotically quadratic continuation

$$\mathscr{H}_t^s(z) = (1 - \chi(s)) H_t^0(z) + \chi(s) H_t^1(z) \tag{4.10}$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth non-decreasing function with $\chi(s) = 0$ for all $s \leq 0$, $\chi(s) = 1$ for all $s \geq 1$ and $\chi' \leq 2$ and $H^i = Q^i + h^i$. By the Lemma above there exists an $\alpha > 0$ depending only on Q^0 such that if $d\left(Q^0, Q^1\right) < \alpha$, this asymptotically quadratic continuation $\mathscr{H}$ has non-degenerate quadratic Hamiltonians at infinity for every $s \in \mathbb{R}$. Moreover, since $|\partial_s \mathbb{A}_t^s| \leq 2|A_t^1 - A_t^0|$, up to choosing A^0 closer to A^1 we are in the hypotheses of Proposition 4.1.1. By Corollary 4.1.1 we therefore have proven the following

Proposition 4.1.2 *For every $H^0 = Q^0 + h^0 \in \mathfrak{wH}$ there exists a constant $\alpha > 0$ depending only on Q^0 with the following property. If $H^1 = Q^1 + h^1 \in \mathfrak{wH}$ is such that $d\left(Q^0, Q^1\right) < \alpha$, then there exists a compact set $K \subset \mathbb{R}^{2n}$ for which*

$$E_{\mathscr{H},\mathscr{J}}(u) \leq \mathscr{A}_{H^0}\left(\gamma^0\right) - \mathscr{A}_{H^1}\left(\gamma^1\right) + \left\|h^0 - h^1\right\|_{L^\infty(S^1 \times K)} + d\left(Q^0, Q^1\right) \cdot (\operatorname{diam} K)^2 \tag{4.11}$$

where $\mathscr{H}$ is defined in (4.10) *and $\mathscr{J}$ is an adequate almost-complex structure.*

4.1.3. Global calculation of Floer homology

The uniform energy estimate for a continuation between Hamiltonians with the same index at infinity implies already a surprising fact. Indeed, if $H \in \mathfrak{w}\mathfrak{H}_*$ with quadratic form at infinity Q, then Q as a Hamiltonian has the same index at infinity of H. Therefore the Floer homology of H can be computed directly from the quadratic form Q:

$$\mathrm{HF}_*(H) \cong \mathrm{HF}_*(Q) = \begin{cases} \mathbb{Z}/2, & * = \mathrm{ind}_\infty(H) \\ 0, & \text{otherwise} \end{cases}$$

It is worth to use this global calculation to show an existence theorem for 1-periodic orbits in asymptotically linear Hamiltonian systems, recovering the results of Conley and Zehnder in [12].

Theorem 6 *Let $H \in \mathfrak{w}\mathfrak{H}_*$. Then X_H has at least one 1-periodic orbit γ_0. If $\mathrm{CZ}(\gamma_0) \neq \mathrm{ind}_\infty H$, then X_H has another 1-periodic orbit γ_1 with $|\mathrm{CZ}(\gamma_0) - \mathrm{CZ}(\gamma_1)| = 1$. Finally, X_H always has an odd number of 1-periodic orbits, one of which always has index equal to $\mathrm{ind}_\infty H$, and all the others having index difference 1. In particular, if it has two, it has three, and one of them has index equal to $\mathrm{ind}_\infty H$.*

Proof. By the global calculation above, we see immediately that X_H must have at least one 1-periodic orbit γ_0. If its index is $\mathrm{ind}_\infty H$, we cannot say anything more. If its index is not $\mathrm{ind}_\infty H$, then first of all there must be a 1-periodic orbit γ_∞ whose index is $\mathrm{CZ}(\gamma_\infty) = \mathrm{ind}_\infty H$. There must also be a third 1-periodic orbit γ_1 with index $\mathrm{CZ}(\gamma_1) = \mathrm{CZ}(\gamma_0) \pm 1$ otherwise the homology of $\mathrm{CF}(H,J)$ would not be the one computed. Similarly, if there are m orbits, then one must have index $\mathrm{ind}_\infty H$ and all the others must pair up in couples with index difference 1 for them to kill each other off and not show up in the homology. □

Remark Notice that with these formal algebraic arguments we cannot go beyond the existence of one or three 1-periodic orbits.

Another approach to the proof of the theorem above is the following. Define the Hilbert-Poincaré series

$$p_C(t) = \sum_k \dim \mathrm{CF}_k(H,J)\, t^k, \quad p_H(t) = \sum_k \dim \mathrm{HF}_k(H)\, t^k$$

Then defining $d = \min\{\mathrm{CZ}(\gamma) : \gamma \in \mathrm{Per}^1 H\}$, we have the relation

$$p_C(t) = p_H(t) + t^{-d}(1+t)Q(t)$$

where Q is some polynomial in t with non-negative integer coefficients. The global calculation implies that

$$p_H(t) = t^{\mathrm{ind}_\infty H}$$

If the 1-periodic orbits are $\gamma_0, \dots, \gamma_m$ with indices $k_0, \dots, k_m$, then

$$p_C(t) = \sum_{l=0}^{m} t^{k_l} \implies \sum_{l=0}^{m} t^{k_l} = t^{\mathrm{ind}_\infty H} + t^{-d}(1+t)Q(t)$$

From this equality, all the claims in the theorem can be proven by comparison of values of the polynomials.

4.2. Filtered Floer homology

The global calculation of Floer homology found above does not allow us to gain access to finer dynamical information, as it is sensitive only to the index at infinity of the Hamiltonian system. As the Floer chain complex formally encodes the Morse theory of the action functional, we can equip it with a filtration given by the action value of the periodic orbits. We will show that continuation morphisms are filtered morphisms only up to a shift in the filtration, so that we can use this shift to gain further information which gets lost in the continuation at the global level.

4.2.1. Action filtration on the Floer chain complex

Let $H \in \mathfrak{wH}_*$ and J a generic adequate almost-complex structure such that (H,J) is a regular pair. Since $\mathrm{CF}_r(H,J)$ is generated by 1-periodic orbits of H of index r, for any $a \in \mathbb{R}$ we can consider the subspace $\mathrm{CF}_r^{(-\infty,a]}(H,J)$ generated by the 1-periodic orbits γ of action $\mathscr{A}_H(\gamma) \leq a$. The energy calculation of a Floer trajectory implies that the differential d_{HJ} decreases the action, so the chain complex

$$\left(\mathrm{CF}_*^{(-\infty,a]}(H,J), d_{H,J}\right)$$

is a sub-complex of the Floer chain complex. Set, for $b > a$,

$$\mathrm{CF}_*^{(a,b]}(H,J) = \mathrm{CF}_*^{(-\infty,b]}(H,J) \Big/ \mathrm{CF}_*^{(-\infty,a]}(H,J)$$

which is equivalently the space generated by the orbits with action in $(a,b]$. $\mathrm{CF}_*^{(a,b]}(H,J)$ is a chain complex when endowed with the quotient differential. Its homology is filtered Floer homology, denoted by $\mathrm{HF}_*^{(a,b]}(H)$.

If $a < b < c$, then there is an obvious exact sequence of chain complexes

$$0 \to \mathrm{CF}_*^{(a,b]}(H,J) \to \mathrm{CF}_*^{(a,c]}(H,J) \to \mathrm{CF}_*^{(b,c]}(H,J) \to 0$$

where the first non-trivial map is an inclusion and the second a quotient, which induces a long exact sequence in homology

$$\dots \to \mathrm{HF}_*^{(a,b]}(H) \to \mathrm{HF}_*^{(a,c]}(H) \to \mathrm{HF}_*^{(b,c]}(H) \to \mathrm{HF}_{*-1}^{(a,b]}(H) \to \dots \tag{4.12}$$

Following [20], we call the first arrow $i : \mathrm{HF}_*^{(a,b]}(H) \to \mathrm{HF}_*^{(a,c]}(H)$ the *inclusion* morphism, and the middle arrow $q : \mathrm{HF}_*^{(a,c]}(H) \to \mathrm{HF}_*^{(b,c]}(H)$ the *quotient* morphism.

As an important particular case, consider $b > a$ and $C > 0$. Consider first the long exact sequence (4.12) with $a < b < b + C$, and then with $a < a + C < b + C$. We obtain

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{HF}_*^{(a,b]}(H) & \xrightarrow{i} & \mathrm{HF}_*^{(a,b+C]}(H) & \xrightarrow{q} & \mathrm{HF}_*^{(b,b+C]}(H) & \longrightarrow & \cdots \\
& & & & \| & & & & \\
\cdots & \longrightarrow & \mathrm{HF}_*^{(a,a+C]}(H) & \xrightarrow{i} & \mathrm{HF}_*^{(a,b+C]}(H) & \xrightarrow{q} & \mathrm{HF}_*^{(a+C,b+C]}(H) & \longrightarrow & \cdots
\end{array}$$

The composition of the upper inclusion with the lower quotient is called *inclusion-quotient morphism*:

$$\Phi_{(a,b]}(C) \colon \mathrm{HF}_*^{(a,b]}(H) \to \mathrm{HF}_*^{(a+C,b+C]}(H) \tag{4.13}$$

The window $(a, b]$ and the shift C are almost always clear from the context, so we usually denote this only by Φ. Also, when $I = (a, b]$, we denote $I + C = (a + C, b + C]$.

4.2.2. Action shift of continuation morphisms

We now explain the effect of continuations on the filtered Floer homology, which is without a doubt one of the most crucial elements in the proof of the Poincaré-Birkhoff theorem.

Let $H^0, H^1 \in \mathfrak{wH}_*$ have the same index at infinity. Let $\mathscr{H}$ be the asymptotically quadratic continuation defined as in Proposition 4.1.1, with a $\beta > 0$ fixed as explained there, and $\mathscr{J}$ an adequate generic almost-complex structure. The uniform energy estimate given in (4.6) implies that a necessary condition for the moduli space of continuation Floer trajectories between fixed 1-periodic orbits γ^0, γ^1 to be non-empty is that

$$0 < E_{\mathscr{H},\mathscr{J}}(u) < \mathscr{A}_{H^0}(\gamma^0) - \mathscr{A}_{H^1}(\gamma^1) + C \implies \mathscr{A}_{H^1}(\gamma^1) < \mathscr{A}_{H^0}(\gamma^0) + C$$

We see that the continuation morphism $\mathscr{C}(\mathscr{H}, \mathscr{J}) \colon \mathrm{CF}_*(H^0, J^0) \to \mathrm{CF}_*(H^1, J^1)$ is a filtered chain complex morphism only up to a shift:

$$\mathscr{C}(\mathscr{H}, \mathscr{J}) \colon \mathrm{CF}_*^{(-\infty,a]}(H^0, J^0) \to \mathrm{CF}_*^{(-\infty,a+C]}(H^1, J^1)$$

Therefore the continuation descends to a morphism of filtered complexes

$$\mathscr{C} : \mathrm{CF}_*^{(a,b]}\left(H^0, J^0\right) \to \mathrm{CF}_*^{(a,b]+C}\left(H^1, J^1\right)$$

and induces a morphism on the filtered homologies

$$\mathscr{C} : \mathrm{HF}_*^{(a,b]}\left(H^0\right) \to \mathrm{HF}_*^{(a,b]+C}\left(H^1\right)$$

The naturality of the long exact sequence in homology implies that the long exact sequence

(4.12) is functorial with respect to continuations:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{HF}_*^{(a,b]}\big(H^0\big) & \xrightarrow{\ i\ } & \mathrm{HF}_*^{(a,c]}\big(H^0\big) & \xrightarrow{\ q\ } & \mathrm{HF}_*^{(b,c]}\big(H^0\big) & \longrightarrow \cdots \\
& \big\downarrow \mathscr{C} & & \big\downarrow \mathscr{C} & & \big\downarrow \mathscr{C} & \\
\cdots \longrightarrow & \mathrm{HF}_*^{(a+C,b+C]}\big(H^1\big) & \xrightarrow{\ i\ } & \mathrm{HF}_*^{(a+C,c+C]}\big(H^1\big) & \xrightarrow{\ q\ } & \mathrm{HF}_*^{(b+C,c+C]}\big(H^1\big) & \longrightarrow \cdots
\end{array}$$

This fact implies the following lemma:

Lemma 4.2.1 *The inclusion-quotient commutes with continuations, i.e. the following square commutes:*

$$\begin{array}{ccc}
\mathrm{HF}_*^{(a,b]}\big(H^0\big) & \xrightarrow{\ \mathscr{C}\ } & \mathrm{HF}_*^{(a,b]+C}\big(H^1\big) \\
\big\downarrow \Phi_{(a,b]}(D) & & \big\downarrow \Phi_{(a,b]+C}(D) \\
\mathrm{HF}_*^{(a,b]+D}\big(H^0\big) & \xrightarrow{\ \mathscr{C}\ } & \mathrm{HF}_*^{(a,b]+C+D}\big(H^1\big)
\end{array}$$

In the proof of the Poincaré-Birkhoff theorem, at the crucial diagram (5.5), we used the following particular form of functoriality "up to a shift":

Lemma 4.2.2 *Let $H^i = Q^i + h^i \in \mathfrak{w}\mathfrak{H}_*$, $i \in \{0,1\}$, $a < b \in \mathbb{R}$ and $I = (a,b]$. Let $\mathscr{C}$ be the continuation morphism from H^0 to H^1 and $\overline{\mathscr{C}}$ be the continuation morphism from H^1 back to H^0. Then $\overline{\mathscr{C}} \circ \mathscr{C}$ factors the inclusion-quotient morphism $\Phi_{(a,b]}(2C)$:*

$$\begin{array}{ccccc}
& \overset{\mathscr{C}}{\nearrow} & \mathrm{HF}_*^{I+C}\big(H^1\big) & \overset{\overline{\mathscr{C}}}{\searrow} & \\
\mathrm{HF}_*^{I}\big(H^0\big) & & \xrightarrow{\qquad \Phi_I(2C) \qquad} & & \mathrm{HF}_*^{I+2C}\big(H^0\big)
\end{array}$$

Proof sketch. Let $\mathscr{C}$ be induced on homology by a choice of asymptotically quadratic continuation and generic adequate almost-complex structure $(\mathscr{H}, \mathscr{J})$. Then we can choose the reversed continuation $\mathscr{G}^s = \mathscr{H}^{1-s}$ with a generic adequate almost-complex structure $\mathscr{I}$ to induce $\overline{\mathscr{C}}$. Functoriality implies that the morphisms

$$\begin{cases} \mathscr{C}(\mathscr{H},\mathscr{J}) : \mathrm{CF}_*\big(H^0,J^0\big) \to \mathrm{CF}_*\big(H^1,J^1\big), \\ \mathscr{C}(\mathscr{G},\mathscr{I}) : \mathrm{CF}_*\big(H^1,J^1\big) \to \mathrm{CF}_*\big(H^0,J^0\big) \end{cases}$$

are homotopy inverses. Hence $\mathscr{C}$ and $\overline{\mathscr{C}}$ are inverses on the total homologies. At the filtered level, they are inverse to each other only up to shifts. Namely, $\overline{\mathscr{C}} \circ \mathscr{C} : \mathrm{HF}_*^I(H^0) \to \mathrm{HF}_*^{I+2C}(H^0)$ maps all classes with action in $I \cap (I+2C)$ to themselves, while all the others are mapped to zero. This is precisely the inclusion-quotient morphism. □

We end this section with a very simple but important result which shows that the filtered Floer homology of a Hamiltonian H does not change unless the filtration crosses a critical value. This property justifies the intuition that Floer homology is a kind of Morse homology for the Hamiltonian action functional. The proof follows immediately from the existence of the inclusion-quotient morphism.

Definition 4.2 Denote by $\mathbb{S}(H)$ the set of critical values of $\mathscr{A}_H$, i.e. the set of actions of 1-periodic orbits of H. This is called the *action spectrum* of H.

Proposition 4.2.1 *Let $H \in \mathfrak{w}\mathfrak{H}_*$ and $a, b \in \mathbb{R}$. For every $\lambda \in \mathbb{R}$ such that*

$$(a,b] \cap \mathbb{S}(H) = (a+\lambda, b+\lambda] \cap \mathbb{S}(H)$$

we have that

$$\mathrm{HF}_*^{(a,b]}(H) \cong \mathrm{HF}_*^{(a,b]+\lambda}(H)$$

4.3. Floer homology for degenerate Hamiltonians and local Floer homology

In this section we start by investigating to which extent the filtered Floer homology depends on the Hamiltonian chosen to define it. It turns out that the filtered Floer homology is, in some sense, locally constant in the Hamiltonian. This makes it possible to define filtered Floer homology groups also for degenerate Hamiltonians, by a small C^∞-perturbation to a non-degenerate Hamiltonian.

We start with an important property of the action spectrum, which holds for any Hamiltonian in $\mathfrak{w}\mathfrak{H}$. We follow [29].

Theorem 7 *Let $H \in \mathfrak{w}\mathfrak{H}$. The set $\mathbb{S}(H)$ is compact and nowhere dense in $\mathbb{R}$.*

Proof. Let's show that the set of critical points of $\mathscr{A}_H$ is compact in the C^∞-topology on $C^\infty(S^1, \mathbb{R}^{2n})$. Let $\gamma_k \in \mathrm{Per}^1 H$ be a sequence of 1-periodic orbits. Recall that since $H \in \mathfrak{w}\mathfrak{H}$ has non-degenerate quadratic form at infinity, there exists an $R > 0$ such that $||\gamma_k||_{L^\infty} < R$ for all k. The sequence is therefore uniformly bounded. Moreover,

$$|\dot{\gamma}_k| = |X_H \circ \gamma_k| \leq \max_{(t,z)\in S^1 \times B_R(0)} |\nabla H_t(z)|$$

hence the sequence is equicontinuous. By the Ascoli-Arzelà theorem, the sequence admits a subsequence, which abusing notation we denote again by γ_k, which converges C^0 to some $\gamma \in C^0(S^1, \mathbb{R}^{2n})$. We can iterate this reasoning on the derivatives to conclude that γ_k converges uniformly with all derivatives to γ, which is thus smooth and a solution of the Hamilton equations. This means that the set of critical points of $\mathscr{A}_H$ is C^∞-compact.

To show that $\mathbb{S}(H)$ is nowhere dense, we show that $\mathbb{S}(H)$ is contained in the set of critical values of a smooth function from $\mathbb{R}^{2n}$ to $\mathbb{R}$. After that, the claim follows from Sard's theorem.

Let $\chi : [0,1] \to [0,1]$ be a smooth function, constantly equal to 1 near 0 and to 0 near 1. Define $\psi : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\psi(t,z) = \chi(t)\varphi_H^t(z) + (1-\chi(t))\varphi_H^t\left(\left(\varphi_H^1\right)^{-1}(z)\right)$$

Notice that $\psi(1,z) = \psi(0,z) = z$ for all $z \in \mathbb{R}^{2n}$. Moreover, if $z_0 \in \operatorname{Fix}\varphi_H^1$, then $\psi(t,z_0) = \varphi_H^t(z_0)$. Define the function

$$f : \mathbb{R}^{2n} \to \mathbb{R}, \quad f(z) = \mathscr{A}_H(\psi(\cdot,z))$$

Notice that the loop $t \mapsto \psi(t,z)$ is smooth for all z, so the function $z \mapsto \psi(\cdot,z)$ maps $\mathbb{R}^{2n}$ into $W^{1,2}(S^1,\mathbb{R}^{2n})$. The action functional $\mathscr{A}_H$ is smooth on $W^{1,2}(S^1,\mathbb{R}^{2n})$ (see e.g. [29, Appendix 3]). Moreover, if $z_0 \in \operatorname{Fix}\varphi_H^1$ then $z_0 \in \operatorname{Crit} f$. Since f is smooth, its set of critical values is nowhere dense. This concludes the proof. □

4.3.1. Non-degenerate perturbations

The next lemma is a perturbation result for non-degenerate periodic orbits bifurcating out of a possibly degenerate periodic orbit. Since the C^∞-topology on $C^\infty(S^1,\mathbb{R}^{2n})$ is metrizable, we can choose a metric d_{C^∞} which induces it. Recall from Section 3.5.1 that $\mathfrak{wH}$ is equipped with the C^∞_{loc}-topology.

Lemma 4.3.1 *For every $H \in \mathfrak{wH}$ and every $\delta > 0$ there exists a neighborhood $\mathfrak{U} \subset \mathfrak{wH}$ of H with the following property. If $\tilde{H} \in \mathfrak{U} \cap \mathfrak{wH}_*$ and $\xi \in \operatorname{Per}^1 \tilde{H}$ is a 1-periodic orbit, then there exists a $\gamma \in \operatorname{Per}^1 H$ such that*

$$d_{C^\infty}(\gamma,\xi) < \delta$$

Proof. We argue by contradiction: assume that there is a $\delta > 0$ such that for any neighborhood $\mathfrak{U} \subset \mathfrak{wH}$ of H in the C^∞_{loc} topology, the 1-periodic orbits of any $\tilde{H} \in \mathfrak{U} \cap \mathfrak{wH}_*$ stay at distance at least δ from all 1-periodic orbits of H. As the C^∞-strong topology is finer than the C^∞_{loc}-topology, we can further assume that there exists a $\delta > 0$ and a sequence $(H^{(k)})_{k\in\mathbb{N}} \subset \mathfrak{wH}_*$ with $H^{(k)} = H + \mathfrak{h}^{(k)}$ such that $\mathfrak{h}^{(k)} \to 0$ uniformly on the whole $\mathbb{R}^{2n}$ together with all their derivatives, and a sequence $\xi_k \in \operatorname{Per}^1 H^{(k)}$ such that

$$d_{C^\infty}(\xi_k,\gamma) \geq \delta \quad \forall \gamma \in \operatorname{Per}^1 H \tag{4.14}$$

Since $H^{(k)} \to H$ in the strong C^∞-topology, the same can be said about the paths of symmetric matrices defining their quadratic Hamiltonians at infinity. Hence, inspecting the proofs of Lemmata 2.3.2, 2.3.3 and 2.3.4 one may conclude that there is a compact set $K \subset \mathbb{R}^{2n}$ such that $\xi_k(S^1) \subset K$ for all k. Therefore $(\xi_k)_k$ is an uniformly bounded sequence of smooth maps. Since

$$\left|\dot{\xi}_k\right| = |X_{H^{(k)}} \circ \xi_k| \leq \varepsilon_k + \max_{S^1\times K} |\nabla H|, \quad \text{for some } \varepsilon_k \to 0$$

by Ascoli-Arzelà we find a subsequence $(\xi_{k_l})_{l\in\mathbb{N}}$ which C^0-converges to some continuous loop $\gamma \in C^0\big(S^1,\mathbb{R}^{2n}\big)$. Since the Hamiltonians $H^{(k)}$ strongly C^∞-converge to H as $k \to \infty$, we can iterate the reasoning and diagonalize to obtain a C^∞-converging subsequence, therefore $\gamma \in C^\infty(S^1,\mathbb{R}^{2n})$. Now, the limit γ is a 1-periodic orbit of X_H, since the Hamiltonians C^∞-converge. Therefore we found a subsequence ξ_{k_l} C^∞-converging to a 1-periodic orbit of H, in contradiction with (4.14). □

Remark This result implies that for every $H \in \mathfrak{wH}$ and $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U} \subset \mathfrak{wH}$ of H with the following property. If $\tilde{H} \in \mathfrak{U} \cap \mathfrak{wH}_*$ and $\xi \in \mathrm{Per}^1 \tilde{H}$ is a 1-periodic orbit, then there exists an $a_0 \in \mathbb{S}(H)$ such that

$$\left|\mathscr{A}_{\tilde{H}}(\xi) - a_0\right| < \varepsilon$$

In fact, the action spectrum $\mathbb{S}(H)$ as a function of H is a lower semicontinuous multivalued function, as observed in [8].

4.3.2. Homotopy invariance of filtered Floer homology

In this section we investigate in which way *filtered* Floer homology depends on the Hamiltonian chosen to define it. We follow the line of thought of [8, 15].

Let $a < b \in \mathbb{R}$ and define

$$\mathfrak{wH}^{a,b}_{(*)} = \left\{H \in \mathfrak{wH}_{(*)} : a, b \notin \mathbb{S}(H)\right\}$$

Notice that $\mathfrak{wH}^{a,b} \subset \mathfrak{wH}$ is open. Therefore, $\mathfrak{wH}^{a,b}_* \subset \mathfrak{wH}^{a,b}$ is dense.

Two Hamiltonians $H^0, H^1 \in \mathfrak{wH}^{a,b}$ are in the same path component of $\mathfrak{wH}^{a,b}$ if and only if they can be connected by an asymptotically quadratic continuation $\mathscr{H} = \mathscr{H}^s$ such that $a, b \notin \mathbb{S}(\mathscr{H}^s)$ for all s.

Proposition 4.3.1 *Any two Hamiltonians $H^0, H^1 \in \mathfrak{wH}^{a,b}_*$ in the same path component of $\mathfrak{wH}^{a,b}$ have isomorphic homologies $\mathrm{HF}^{(a,b]}_*(H^0) \cong \mathrm{HF}^{(a,b]}_*(H^1)$.*

Proof. Let $H = Q + h \in \mathfrak{wH}^{a,b}$ be fixed. First we show that there exists a small enough strong C^∞-neighborhood $\mathfrak{U}$ of H such that any $H^-, H^+ \in \mathfrak{U} \cap \mathfrak{wH}^{a,b}_*$ have isomorphic filtered Floer homologies. Choose an $\varepsilon > 0$ so small that

$$\mathbb{S}(H) \cap [a - 4\varepsilon, a + 4\varepsilon] = \varnothing = \mathbb{S}(H) \cap [b - 4\varepsilon, b + 4\varepsilon]$$

This is possible because $\mathbb{S}(H) \subset \mathbb{R}$ is compact and nowhere dense, and because $a, b \notin \mathbb{S}(H)$. The proof of Lemma 4.3.1 shows that there exists a strong C^∞-neighborhood $\mathfrak{U} \subset \mathfrak{wH}^{a,b}$ of H such that every $G \in \mathfrak{U}$ satisfies

$$\mathbb{S}(G) \cap [a - 4\varepsilon, a + 4\varepsilon] = \varnothing = \mathbb{S}(G) \cap [b - 4\varepsilon, b + 4\varepsilon]$$

By taking $\mathfrak{U}$ smaller, we can assume that every $G = P + g \in \mathfrak{U}$ is such that $d(P, Q) < \alpha$, where d denotes the distance of the quadratic forms introduced in Definition 4.1, and $\alpha > 0$ is the constant, depending only on Q, of Proposition 4.1.2, where $H^0 = H$ and $H^1 = G$. Therefore, for any $H^\pm = Q^\pm + h^\pm \in \mathfrak{U}$ there exists a compact set $K \subset \mathbb{R}^{2n}$ for which the energy estimate (4.11) in Proposition 4.1.2 holds. By taking $\mathfrak{U}$ even smaller, we can also assume that $H^\pm = Q^\pm + h^\pm \in \mathfrak{U}$ are such that

$$\|h^- - h^+\|_{L^\infty(S^1\times K)} < \frac{\varepsilon}{2}, \quad d\left(Q^-, Q^+\right) < \min\left\{\frac{\varepsilon}{2(\operatorname{diam} K)^2}, \alpha\right\}$$

In particular, , we can use the continuation Hamiltonian as in (4.10) to define a continuation from H^- to H^+ (or viceversa). By the energy estimate (4.11) such continuation will shift the action at most by

$$\|h^- - h^+\|_{L^\infty(S^1\times K)} + d\left(Q^-, Q^+\right) \cdot (\operatorname{diam} K)^2 < \varepsilon$$

Set $\mathfrak{U}_* = \mathfrak{U} \cap \mathfrak{wH}_*$. Looking at the proof of Proposition 2.4.2, we see that $\mathfrak{wH}_*$ is residual in $\mathfrak{wH}$ even in the strong C^∞-topology, so $\mathfrak{U}_*$ is dense in $\mathfrak{U}$. Take $H^\pm \in \mathfrak{U}_*$. A continuation morphism from H^- to H^+ will give

$$\mathscr{C} : \mathrm{HF}_*^{(a,b]}\left(H^-\right) \to \mathrm{HF}_*^{(a,b]+\varepsilon}\left(H^+\right) \cong \mathrm{HF}_*^{(a,b]}\left(H^+\right)$$

A continuation morphism from H^+ to H^- will give

$$\overline{\mathscr{C}} : \mathrm{HF}_*^{(a,b]}\left(H^+\right) \to \mathrm{HF}_*^{(a,b]+\varepsilon}\left(H^-\right) \cong \mathrm{HF}_*^{(a,b]}\left(H^-\right)$$

The isomorphisms written at the end are from Proposition 4.2.1. The composition of these continuations will give a morphism

$$\overline{\mathscr{C}} \circ \mathscr{C} : \mathrm{HF}_*^{(a,b]}\left(H^-\right) \to \mathrm{HF}^{(a,b]+2\varepsilon}\left(H^+\right) \cong \mathrm{HF}_*^{(a,b]}\left(H^+\right)$$

By Lemma 4.2.2, this composition gives the inclusion-quotient morphism, which is an isomorphism by Proposition 4.2.1 and the choice of action window and ε. This proves the claim. Now, if $H^0, H^1 \in \mathfrak{wH}_*^{a,b}$ are in the same path component, we can find a continuous path $[0,1] \ni s \mapsto H^s \in \mathfrak{wH}$ connecting them such that $H^s \in \mathfrak{wH}^{a,b}$ for all s. We can cover its image by finitely many strong C^∞-neighborhoods having the property that any two non-degenerate Hamiltonians within these neighborhoods have isomorphic Floer homologies. Composing these finitely many isomorphisms gives the claimed isomorphism between the filtered Floer homologies of H^0 and H^1. □

4.3.3. Construction of filtered Floer homology for a degenerate Hamiltonian

Proposition 4.3.1 implies that two close enough non-degenerate perturbations of any Hamiltonian $H \in \mathfrak{wH}^{a,b}$ have isomorphic Floer homologies. This means that we can take the Floer homology of any such non-degenerate Hamiltonian as the Floer homology of the degenerate one.

Definition 4.3 Let $H \in \mathfrak{wH}^{a,b}$. Let $\tilde{H}^{(k)} \in \mathfrak{wH}^{a,b}_*$ be such that $H^{(k)} = H + \mathfrak{h}^{(k)}$ and $\mathfrak{h}^{(k)} \to 0$ uniformly with all their derivatives on the whole $\mathbb{R}^{2n}$. Define

$$\mathrm{HF}^{(a,b]}_*(H) = \mathrm{HF}^{(a,b]}_*\left(\tilde{H}^{(k)}\right)$$

where $k > 0$ is so large that $\mathrm{HF}^{(a,b]}_*\left(\tilde{H}^{(k_0)}\right) \cong \mathrm{HF}^{(a,b]}_*\left(\tilde{H}^{(k_1)}\right)$ for all $k_0, k_1 > k$. That such a k exists follows from the proof of Proposition 4.3.1.

This notion is well defined because any two sequences of non-degenerate Hamiltonians converging to H will eventually lie in a small enough strong C^∞-neighborhood, so by Proposition 4.3.1 their filtered Floer homologies will be eventually all isomorphic.

Remark Notice moreover that the proof of Proposition 2.4.2 implies that we can even take the $\mathfrak{h}^{(k)}$ to have compact support, as long as it is always containing the compact set where all the 1-periodic orbits of H lie.

Remark It is also possible to define a genuine filtered chain complex whose filtered homology computes the same filtered Floer homology for a degenerate Hamiltonian. This is done by taking the colimit of the Floer chain complexes of non-degenerate Hamiltonians converging C^∞-strong to H. The algebraic details though are a bit more involved, so this shorter route was chosen.

Important observation All the lemmata and propositions regarding filtered Floer homology explained in Sections 4.2 and 4.2.2 continue to hold for degenerate Hamiltonians, since they hold for non-degenerate ones. The only thing to be careful of is that *the end-points of the action windows must be guaranteed not to lie in the action spectrum of the degenerate Hamiltonian*, since in that case there is no canonical definition of filtered Floer homology. Therefore, versions of those lemmata and propositions for degenerate Hamiltonians must include in their hypotheses that the extrema of the action windows in consideration do not touch the action spectrum.

4.3.3.1. Global calculation, degenerate case

Let $H \in \mathfrak{wH}$. Since $\mathbb{S}(H)$ is compact, there exists $a_\infty, b_\infty \in \mathbb{R}$ such that $a_\infty, b_\infty \notin \mathbb{S}(H)$ and $\mathbb{S}(H) \subset (a_\infty, b_\infty]$. We set by definition

$$\mathrm{HF}_*(H) = \mathrm{HF}^{(a_\infty, b_\infty]}_*(H)$$

Let $H = Q + h$. Since Q is a non-degenerate quadratic Hamiltonian,

$$\mathrm{HF}_*(H) \cong \mathrm{HF}_*(Q) = \begin{cases} \mathbb{Z}/2, & * = \mathrm{ind}_\infty(H) \\ 0, & \text{otherwise} \end{cases}$$

As a corollary we again recover one of the main results in [12].

Theorem 8 *Let $H \in \mathfrak{wH}$. Then H has at least one 1-periodic orbit γ_0. If this 1-periodic orbit is non-degenerate and has index different from* $\operatorname{ind}_\infty H$*, then there is at least one other periodic orbit γ_1. If this second periodic orbit is also non-degenerate, there is a third 1-periodic orbit. One of the three 1-periodic orbits must have index equal to* $\operatorname{ind}_\infty H$

Remark

1. We say that the 1-periodic orbit with index $\operatorname{ind}_\infty H$ is *continuing from infinity*. Notice that it's not clear whether the corresponding fixed point is twist or not. Indeed, it has the same CZ-index as the index at infinity, but the indices of its iterates don't necessarily have to match up with the index at infinity of the iterates.
2. The global calculation above shows that any fixed point, other than the one continuing from infinity, is "unnecessary" in the sense of the Hofer-Zehnder conjecture ([29, pg. 263]). Abbondandolo [3, pg. 129-130] conjectures that the existence of two 1-periodic orbits should imply infinitely many periodic orbits. The main theorem in this thesis gives a positive answer to this conjecture in the case that the additional 1-periodic orbit, not continuing from infinity, is twist, and the linear map at infinity is unitary.

4.3.4. Local Floer homology

In the filtered Floer homology of a degenerate Hamiltonian H, the 1-periodic orbits of the original Hamiltonian H cannot be interpreted as generators, unless they are non-degenerate orbits. The generators are instead the non-degenerate infinitesimal bifurcations of the degenerate orbits. Therefore to understand the Floer homology of a degenerate Hamiltonian, it might be helpful to localize the perturbation process around the 1-periodic orbits. This is meaningful only in the case of isolated 1-periodic orbits.

Let $H \in \mathfrak{wH}$. Recall that $\gamma \in \operatorname{Per}^1 H$ is said to be isolated when there exists a neighborhood $\mathcal{U} \subset S^1 \times \mathbb{R}^{2n}$ of the graph of γ such that no graph of any other $\xi \neq \gamma \in \operatorname{Per}^1 H$ may intersect $\mathcal{U}$. We call such an $\mathcal{U}$ an *isolating neighborhood* of the periodic orbit.

Since $\gamma(S^1)$ is compact in $\mathbb{R}^{2n}$, we can assume that $\overline{\mathcal{U}}$ is compact in $S^1 \times \mathbb{R}^{2n}$, i.e. $\mathcal{U}$ is a pre-compact neighborhood of the graph of γ. We need a notation for non-degenerate perturbations of H within $\mathcal{U}$. Denote

$$\mathfrak{wH}_{(*)}(H, \mathcal{U}) = \left\{ G \in \mathfrak{wH} : G|_{S^1 \times \mathbb{R}^{2n} \setminus \overline{\mathcal{U}}} = H|_{S^1 \times \mathbb{R}^{2n} \setminus \overline{\mathcal{U}}}, \quad (G|_{\mathcal{U}} \text{ non-degenerate}) \right\}$$

The set $\mathfrak{wH}(H, \mathcal{U}) \subset \mathfrak{wH}$ is a C^∞-open neighborhood of H, because $\overline{\mathcal{U}}$ is compact. Therefore $\mathfrak{wH}_*(H, \mathcal{U}) \subset \mathfrak{wH}(H, \mathcal{U})$ is dense. The following Lemma is an immediate consequence of Lemma 4.3.1.

Lemma 4.3.2 *Let $\mathcal{U}$ be a pre-compact isolating neighborhood of γ. For every $\mathcal{V} \subset \mathcal{U}$ isolating neighborhood of γ there exists a neighborhood $\mathfrak{V} \subset \mathfrak{wH}(H, \mathcal{V})$ of H such that whenever $G \in \mathfrak{V} \cap \mathfrak{wH}_*(H, \mathcal{V})$ the graphs of all 1-periodic orbits of G are either contained in $\mathcal{V}$ or outside $\mathcal{U}$.*

In other words, all non-degenerate 1-periodic orbits bifurcating out of γ are contained in its isolating neighborhood when the perturbation is small enough.

Let $\gamma_0 \in \mathrm{Per}^1 H$ be an isolated 1-periodic orbit with isolating neighborhood $\mathscr{U}$. We can always take $\mathscr{U}$ small enough such that its closure is compact. For $G \in \mathfrak{wH}(H, \mathscr{U})$, denote by $\mathrm{Per}^1 G \cap \mathscr{U}$ the set of 1-periodic orbits of G whose graph is contained in $\mathscr{U}$. Define the following $\mathbb{Z}/2$-vector space:

$$\mathrm{CF}_r^{\mathrm{loc}}(G, \mathscr{U}) = \bigoplus \left\{ \mathbb{Z}/2 : \xi \in \mathrm{Per}^1 G \cap \mathscr{U}, \quad \mathrm{CZ}(\xi, G) = r \right\}$$

This is simply the free vector space generated by the 1-periodic orbits of G with graph in $\mathscr{U}$. We would like to define a differential $d_{G,J} : \mathrm{CF}_r^{\mathrm{loc}}(G, \mathscr{U}) \to \mathrm{CF}_{r-1}^{\mathrm{loc}}(G, \mathscr{U})$ by counting Floer cylinders which connect orbits of G in $\mathscr{U}$.

First of all, we need a notion of adequate almost complex structure which is tuned to the problem at hand. We may assume that the almost complex structures are fixed outside $\mathscr{U}$, for example are equal to J_0 outside $\mathscr{U}$. A pair (G, J) where $G \in \mathfrak{wH}_*(H, \mathscr{U})$ will be called regular if transversality is achieved for the Floer cylinders asymptotic to orbits within $\mathscr{U}$. For this to be well defined, we must guarantee that these Floer cylinders remain in $\mathscr{U}$ whenever G is a small enough perturbation of H.

Proposition 4.3.2 *For every $\mathscr{V} \subset \mathscr{U}$ there exists an open neighborhood $\mathfrak{W} \subset \mathfrak{wH}(H, \mathscr{V})$ of H with the following property. For every $G \in \mathfrak{W} \cap \mathfrak{wH}_*(H, \mathscr{V})$, every adequate almost-complex structure J and $\xi^\pm \in \mathrm{Per}^1 G \cap \mathscr{V}$, any $u \in M(\xi^-, \xi^+; G, J)$ is such that $u\left(\mathbb{R} \times S^1\right) \subset \mathscr{V}$.*

Proof. We argue by contradiction. Assume that there exists a sequence $G^{(k)} \in \mathfrak{wH}_*(H, \mathscr{V})$ with $G^{(k)} \to H$ in C^∞, sequences $\xi_\pm^{(k)} \in \mathrm{Per}^1 G^{(k)} \cap \mathscr{V}$ with $\xi_\pm^{(k)} \to \gamma$ in C^∞, a sequence of adequate almost complex structures $J^{(k)}$ and a sequence $u^{(k)} \in M\left(\xi_-^{(k)}, \xi_+^{(k)}; G^{(k)}, J^{(k)}\right)$ such that graph $u^{(k)}(s_k, \cdot) \not\subseteq \mathscr{V}$ for some $s_k \in \mathbb{R}$. Notice that under these hypotheses,

$$E_{G^{(k)}, J^{(k)}}\left(u^{(k)}\right) \to 0 \text{ as } k \to \infty$$

Up to shifting each $u^{(k)}$ by s_k in the s-direction, we may assume that graph $u^{(k)}(0, \cdot) \not\subseteq \mathscr{V}$. But by Proposition 3.4.2, we see that as $k \to \infty$ $u^{(k)}(0, \cdot)$ must get arbitrarily close to some 1-periodic orbit of $G^{(k)}$ within $\mathscr{U}$. But the isolation hypothesis together with Lemma 4.3.2 implies that no such orbit may exist. □

This last result implies that one may define a Floer chain complex which is generated by the bifurcating 1-periodic orbits and whose differential counts the Floer cylinders between them. For the sake of transversality, we can restrict our attention to almost-complex structures J which coincide with J_0 outside a compact set. All results in Section 3.5.1 hold for this class, namely, transversality can be achieved by choosing a generic almost-complex structure in this class (see e.g. [15], where this is proven explicitly).

An argument completely analogous to the proof of Proposition 4.3.2 shows that continuation Floer cylinders between 1-periodic orbits of two small enough perturbations also stay within a pre-compact neighborhood of the degenerate orbit, and similarly for the cylinders entering the definition of a homotopy between continuations. Combined with an argument analogous to the proof of Proposition 4.2.1, this implies the following proposition.

Proposition 4.3.3 *There exists an open neighborhood $\mathfrak{U} \subset \mathfrak{w}\mathfrak{H}(H,\mathcal{U})$ such that if $G^0, G^1 \in \mathfrak{U} \cap \mathfrak{w}\mathfrak{H}_*(H,\mathcal{U})$ and J^0, J^1 are generic adequate almost-complex structures, then*

$$\left(\mathrm{CF}_*^{\mathrm{loc}}(G^0,J^0), d_{G^0 J^0}\right) \text{ is quasi-isomorphic to } \left(\mathrm{CF}_*^{\mathrm{loc}}(G^1,J^1), d_{G^1 J^1}\right)$$

namely, there exist morphisms between them which are homotopy inverses to each other.

The following definition is thusly justified.

Definition 4.4 Let $H \in \mathfrak{w}\mathfrak{H}$ and $\gamma \in \mathrm{Per}^1 H$ be an isolated 1-periodic orbit with a fixed, small, pre-compact isolating neighborhood $\mathcal{U}$. Let $z_0 \in \mathrm{Fix}\,\varphi_H^1$ be the corresponding fixed point. The *local Floer homology* of z_0 is

$$\mathrm{HF}_*^{\mathrm{loc}}(H,z_0) = H_*\left(\mathrm{CF}^{\mathrm{loc}}(G,\mathcal{U}), d_{GJ}\right)$$

where $G \in \mathfrak{w}\mathfrak{H}_*(H,\mathcal{U})$ is within a neighborhood of H so small that all the results above hold, and J is a generic adequate almost complex structure for which transversality is achieved.

We do not make much difference between 1-periodic orbits and fixed points, since we carry the generating Hamiltonian in the notation.

Remark Recall that non-degenerate 1-periodic orbits are always isolated. Therefore their local Floer homologies are computed immediately without need of perturbation. They will have one generator in the degree corresponding to the Conley-Zehnder index of the orbit, and will be zero in all other degrees. From this point of view, the local Floer homology of a degenerate 1-periodic orbit can be regarded as a kind of derived object, in the spirit of the famous derived intersection numbers of Serre for non-transverse intersections of algebraic varieties. Here the non-transverse intersection would be between the diagonal and the graph of the Hamiltonian diffeomorphism in the product, locally around the fixed point. This is an intersection of Lagrangian submanifolds.

4.3.4.1. Properties of local Floer homology

We start with identifying in which degrees the local Floer homology can be non-trivial.

Lemma 4.3.3 *Let $z_0 \in \mathrm{Fix}\,\varphi_H^1$ be an isolated fixed point. Define*

$$\deg\operatorname{supp}\mathrm{HF}_*^{\mathrm{loc}}(H,z_0) = \left\{s \in \mathbb{Z} : \mathrm{HF}_s^{\mathrm{loc}}(H,z_0) \neq \{0\}\right\} \tag{4.15}$$

Then we have

$$\deg\operatorname{supp}\mathrm{HF}_*^{\mathrm{loc}}(H,z_0) \subseteq \left[\overline{\mathrm{CZ}}(z_0,H) - n, \overline{\mathrm{CZ}}(z_0,H) + n\right]$$

The proof of this lemma is an immediate consequence of the fact that we are taking the lower semicontinous extension of the Conley-Zehnder index when dealing with degenerate orbits, namely, it follows from the estimate in Lemma 1.2.12. In fact, a tighter bound can be proven in terms of the nullity of the periodic orbit, but we don't need it.

Another important property of local Floer homology, which in some sense justifies its name, is that it depends on the Hamiltonian only in a neighborhood of the fixed point in analysis. The proof of this fact follows immediately from its definition.

Lemma 4.3.4 *Let $\mathscr{H} = \mathscr{H}^s$ be a 1-parameter family of Hamiltonians such that there exists a $z_0 \in \mathbb{R}^{2n}$ with the following property: the orbit $t \mapsto \varphi^t_{\mathscr{H}^s}(z_0) = \gamma^s(t)$ is a 1-periodic orbit of $\mathscr{H}^s$ for every s, and there exists a neighborhood U_0 of the orbit γ^s,* independent of s, *such that γ^s is the* only *1-periodic orbit in U_0. Then*

$$\mathrm{HF}^{\mathrm{loc}}_*(\mathscr{H}^{s_0}, \gamma^{s_0}) \cong \mathrm{HF}^{\mathrm{loc}}_*(\mathscr{H}^{s_1}, \gamma^{s_1}) \quad \forall s_0, s_1$$

In particular, if F, G are Hamiltonians such that there exists a $z_0 \in \mathrm{Fix}\,\varphi^1_F \cap \mathrm{Fix}\,\varphi^1_G$ and an $\mathscr{U}_0$ isolating the 1-periodic orbits of both F and G stemming from z_0, and such that $F|_{\mathscr{U}_0} = G|_{\mathscr{U}_0}$, then $\mathrm{HF}^{\mathrm{loc}}_*(F, z_0) \cong \mathrm{HF}^{\mathrm{loc}}_*(G, z_0)$.

Using the isolation hypothesis, it is not too hard to prove the following lemma.

Proposition 4.3.4 *Let $z_0 \in \mathrm{Fix}\,\varphi^1_H$ be an isolated fixed point such that its critical value $\mathscr{A}_H(z_0) = a \in \mathbb{R}$ is isolated: there is an $\varepsilon > 0$ is such that $I = (a-\varepsilon, a+\varepsilon]$ contains no critical values other than a. Then there exists an* injective *morphism*

$$\mathrm{HF}^{\mathrm{loc}}_*(H, z_0) \hookrightarrow \mathrm{HF}^{(a-\varepsilon, a+\varepsilon]}_*(H)$$

Proof. Recall that the Floer homology of H in action window $(a-\varepsilon, a+\varepsilon]$ is by definition the Floer homology of any small enough non-degenerate perturbation $\tilde{H}$ in the same action window. It is clear that choosing a $G \in \mathfrak{w}\mathfrak{H}_*(H, \mathscr{U})$ there always exists a $\tilde{H} \in \mathfrak{w}\mathfrak{H}_*$ which coincides with G in $\mathscr{U}$. In particular, there is an inclusion of chain complexes $\mathrm{CF}^{\mathrm{loc}}_*(G, \mathscr{U}) \hookrightarrow \mathrm{CF}^{(a-\varepsilon, a+\varepsilon]}_*\left(\tilde{H}, J\right)$ whenever J is used to define the differential of the local chain complex. What we have to show is that this morphism remains an injection at the level of homologies, when G and $\tilde{H}$ are chosen close enough to H.

We argue by contradiction. Set $\gamma(t) = \varphi^t_H(z_0)$ the 1-periodic orbit corresponding to z_0. Assume there exist sequences $\tilde{H}^{(k)} \in \mathfrak{w}\mathfrak{H}_*$, $G^{(k)} \in \mathfrak{w}\mathfrak{H}_*(H, \mathscr{U})$ converging to H in C^∞, such that $G^{(k)}|_{\mathscr{U}} = \tilde{H}^{(k)}|_{\mathscr{U}}, J^{(k)} \to J$ adequate generic almost complex structures, $\varepsilon_k \to 0$ and chains $0 \neq c_k \in \mathrm{CF}^{\mathrm{loc}}_*\left(G^{(k)}, \mathscr{U}\right)$ for which $c_k \in \mathrm{Im}\, d_{\tilde{H}^{(k)}, J^{(k)}} \subset \mathrm{CF}^{(a-\varepsilon_k, a+\varepsilon_k]}\left(\tilde{H}^{(k)}, J\right)$. In particular there are sequences of 1-periodic orbits $\xi_k, \chi_k \in \mathrm{Per}^1\,\tilde{H}^{(k)}$ with

$$a - \varepsilon_k < \mathscr{A}_{H^{(k)}}(\chi_k), \mathscr{A}_{H^{(k)}}(\xi_k) \leq a + \varepsilon_k \tag{4.16}$$

and $(\tilde{H}^{(k)}, J^{(k)})$-Floer cylinders u_k with $u_k(s, \cdot) \to \chi_k$ as $s \to -\infty$ and $u_k(s, \cdot) \to \xi_k$ as $s \to +\infty$ for all k. We can even assume that χ_k lie outside $\mathscr{U}$ because the case of χ_k inside $\mathscr{U}$ does not contradict the claim. From (4.16) we see that

$$E_{H^{(k)},J^{(k)}}(u_k) \to 0$$

Up to shifting u_k we can assume that graph $u_k(0,\cdot) \cap \partial\mathcal{U} \neq \varnothing$ for all k. By compactness we can assume that $u_k \to u$ in C^∞_{loc} where u is a non-constant (H,J)-Floer cylinder of zero energy, i.e. a 1-periodic orbit of H. Since u_k touches $\partial\mathcal{U}$ for all k, also this 1-periodic orbit must intersect $\mathcal{U}$. But this contradicts isolation of $\gamma = \varphi^t_H(z_0)$. □

As a corollary, we state the following

Lemma 4.3.5 *Assume that* Fix φ^1_H *is a discrete set. Then since it is contained in a compact set, it is finite. In particular, if $a \in \mathbb{R}$ is a critical value of $\mathcal{A}_H$, then it is isolated as a critical value and*

$$\mathrm{HF}_*^{(a-\varepsilon,a+\varepsilon]}(H) \cong \bigoplus \left\{ \mathrm{HF}^{\text{loc}}\left(H,z'\right) : z' \in \mathrm{Fix}\,\varphi^1_H,\ \mathcal{A}_H\left(z'\right) = a \right\}$$

for any $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon]$ contains only a as a critical value.

Remark When defining the filtered Floer homology with a chain complex model, the relationship between local and filtered Floer homology arises in the guise of the spectral sequence associated to the action filtration.

Finally, it is important to know how local Floer homology behaves under iteration.

Lemma 4.3.6 *Let $H \in \mathfrak{wH}$ and $z_0 \in$ Fix φ^1_H. Assume that $\mathrm{HF}^{\text{loc}}_*(H,z_0) \neq \{0\}$. Then for every $k \in \mathbb{Z}$ which is not a multiple of any order of root of unity in the spectrum of $d\varphi^1_H(z_0)$, we have that $\mathrm{HF}^{\text{loc}}_*(H^{\times k}, z_0) \neq \{0\}$.*

A proof of this lemma which applies also to our situation can be found in [22]. In fact, much more can be proven: these local homologies are isomorphic up to a shift in degree, depending on the iteration. Moreover the growth rate of this shift in degree as one iterates can be shown to be exactly the mean Conley-Zehnder index of the 1-periodic orbit in analysis.

Remark Local Floer homology does not have to be restricted to isolated 1-periodic orbits. It can in fact be defined for any isolated invariant set, since an isolated invariant set bifurcates into a finite collection of non-degenerate 1-periodic orbits when perturbing to a non-degenerate Hamiltonian. It would be interesting to understand how this invariant compares with the Conley index.

5. A Poincaré-Birkhoff theorem for ALHDs

In this chapter, in order to prove our main application to dynamical systems, we develop two techniques whose aim is to make it possible to relate the Floer homologies of different iterates of the same asymptotically quadratic Hamiltonian. The first is a procedure, called re-indexing at infinity, which can be used to change the index at infinity of an asymptotically quadratic Hamiltonian without changing its periodic points nor their action value. The second is an interpolation at infinity, which starts with two Hamiltonians with the same index at infinity and produces two Hamiltonians with the same quadratic form at infinity, without creating new 1-periodic orbits. The Floer homology of the new Hamiltonian can be related to the old one. These techniques are employed to prove the Poincaré-Birkhoff theorem through a contradiction argument.

5.1. Proof of the Poincaré-Birkhoff theorem

We start the chapter by proving the main application to dynamical systems, a Poincaré-Birkhoff theorem for ALHDs. In order to streamline the argument, we would like to first state a proposition that incapsulates the constructions which are developed below, and show how to conclude the Poincaré-Birkhoff theorem from it.

Proposition 5.1.1 *Let φ be an ALHD whose linear map at infinity $\varphi_\infty \in \mathrm{Sp}(2n)$ is non-degenerate and unitary. Assume that φ has finitely many fixed points and that the set of primitive periods of periodic points of φ is also finite. Then there exist:*

- *(a) an increasing sequence of prime numbers $(p_j)_{j\in\mathbb{N}}$, with p_0 larger than the largest primitive period of any periodic point of φ,*
- *(b) an asymptotically quadratic Hamiltonian H, non-degenerate at infinity, generating φ,*
- *(c) a sequence of asymptotically quadratic Hamiltonians G_j such that $\varphi^{p_j} = \varphi^1_{G_j}$*

such that for any fixed $m \in \mathbb{N}$, we have the following properties:

1. *The spectrum of $\varphi_\infty^{p_j}$ never contains 1.*
2. *$p_{j+m} - p_j = o(p_j)$, i.e. the gaps in the sequence $(p_j)_{j\in\mathbb{N}}$ are distributed like the gaps in the sequence of all primes.*

3. *For any* $z \in \operatorname{Fix}\varphi^1_{G_j} = \operatorname{Fix}\varphi^{p_j} = \operatorname{Fix}\varphi$ *we have*

$$\begin{cases} \mathscr{A}_{G_j}(z) = \mathscr{A}_{H^{\times p_j}}(z) = p_j \mathscr{A}_H(z), \\ \mathrm{CZ}\left(z, G_j\right) = \mathrm{CZ}(z, H^{\times p_j}), \quad \overline{\mathrm{CZ}}\left(z, G_j\right) = \overline{\mathrm{CZ}}(z, H^{\times p_j}) = p_j \overline{\mathrm{CZ}}(z, H) \\ \mathrm{HF}^{\mathrm{loc}}\left(G_j, z\right) \cong \mathrm{HF}^{\mathrm{loc}}\left(H^{\times p_j}, z\right) \end{cases} \tag{5.1}$$

Moreover, for any fixed $m \in \mathbb{N}$, *there exists a sequence* $\left(\sigma_{j,m}\right)_{j \in \mathbb{N}}$ *of integers and a sequence of asymptotically quadratic Hamiltonians* $F_{j,m}$ *such that:*

4. $\left(p_{j+m} - p_j\right)\overline{\mathrm{ind}}_\infty H - n \leq \sigma_{j,m} \leq \left(p_{j+m} - p_j\right)\overline{\mathrm{ind}}_\infty H + n.$
5. G_j *and* $F_{j,m}$ *have the same quadratic Hamiltonian at infinity. Moreover*

$$\left\| G_j - F_{j,m} \right\|_{L^\infty} = O\left(p_{j+m} - p_j\right) \tag{5.2}$$

6. *Every* p_{j+m}*-periodic orbit of* H *is a 1-periodic orbit of* $F_{j,m}$, *and viceversa. Moreover, for every* $z \in \operatorname{Fix}\varphi^1_{F_{j,m}} = \operatorname{Fix}\varphi^{p_{j+m}} = \operatorname{Fix}\varphi$, *we have*

$$\begin{cases} \mathscr{A}_{F_{j,m}}(z) = \mathscr{A}_{H^{\times p_{j+m}}}(z) = p_{j+m} \mathscr{A}_H(z), \\ \overline{\mathrm{CZ}}\left(z, F_{j,m}\right) = \overline{\mathrm{CZ}}\left(z, H^{\times p_{j+m}}\right) = p_{j+m} \overline{\mathrm{CZ}}(z, H) \end{cases} \tag{5.3}$$

and finally, for every $z \in \operatorname{Fix}\varphi^1_{F_{j,m}} = \operatorname{Fix}\varphi^{p_{j+m}} = \operatorname{Fix}\varphi$, *we have*

$$\begin{cases} \mathrm{CZ}\left(z, F_{j,m}\right) = \mathrm{CZ}\left(z, H^{p_{j+m}}\right) - \sigma_{j,m}, \\ \mathrm{HF}^{\mathrm{loc}}_*\left(F_{j,m}, z\right) \cong \mathrm{HF}^{\mathrm{loc}}_{*+\sigma_{j,m}}\left(H^{\times p_{j+m}}, z\right) \end{cases} \tag{5.4}$$

The proof of this proposition will be shown in Section 5.2.3. Now, we are ready to state and prove the main theorem of the paper. Recall that a fixed point $z_0 \in \operatorname{Fix}\varphi$ is said to be *twist*, when $\overline{\mathrm{CZ}}(z_0, H) \neq \overline{\mathrm{ind}}_\infty H$ for some generating asymptotically quadratic Hamiltonian H, and that it is said to be homologically visible when $\mathrm{HF}^{\mathrm{loc}}(H, z_0) \neq \{0\}$ for some generating asymptotically quadratic Hamiltonian H. Both these conditions do not depend on the choice of generating Hamiltonian.

Theorem 9 *Let* φ *be an ALHD with non-degenerate and unitary linear map at infinity* $\varphi_\infty \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$. *Assume that* φ *admits a homologically visible twist fixed point* $z_0 \in \operatorname{Fix}\varphi$. *Then* φ *has infinitely many fixed points or infinitely many periodic points with increasing primitive period.*

The proof is inspired by the proof of Gürel [25].

Proof. Assume by contradiction that φ has finitely many fixed points and finitely many integers appear as primitive periods of periodic points of φ. Therefore, Proposition 5.1.1 holds.

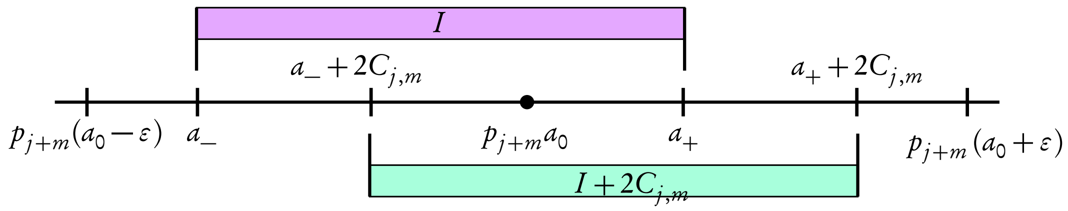

Figure 5.1. Special action windows. For clarity, set $a_\pm = p_{j+m}\left(a_0 \pm \frac{\varepsilon}{3}\right)$

Since φ has finitely many fixed points, the action value $a_0 = \mathscr{A}_H(z_0)$ of the 1-periodic orbit corresponding to the fixed point z_0 is isolated. Therefore there exists an $\varepsilon > 0$ such that $(a_0 - \varepsilon, a_0 + \varepsilon]$ contains only a_0 as critical value of the action functional.

Notice that by construction, since p_0 is larger than the largest primitive period of any periodic point of φ, all p_j-periodic points are iterates of fixed points for all j, which come in a finite number. Therefore, also $p_j a_0$, resp. $p_{j+m}a_0$ is an isolated critical value of $\mathscr{A}_{G_j}$, resp. $\mathscr{A}_{F_{j,m}}$. In fact, these two functionals have no critical value other than $p_j a_0$, resp. $p_{j+m}a_0$, in $\left(p_j(a_0 - \varepsilon), p_j(a_0 + \varepsilon)\right]$, resp. $\left(p_{j+m}(a_0 - \varepsilon), p_{j+m}(a_0 + \varepsilon)\right]$.

First, we analyze the filtered Floer homology of $F_{j,m}$ in action windows around the critical value $p_{j+m}a_0$ and identify action windows for which a certain inclusion-quotient morphism is non-vanishing (see Section 4.2.2 and specifically (4.13)).

Set $C_{j,m} = \left\|F_{j,m} - G_j\right\|_{L^\infty}$. This is the action shift given by a continuation between $F_{j,m}$ and G_j, see Section 4.2.2. Since $C_{j,m} = O(p_{j+m} - p_j) = o(p_j)$ by point (2) in Proposition 5.1.1, we know that for any $\delta > 0$ there exists a $j_0 > 0$ such that $\delta p_{j+m} > \delta p_j > 6C_{j,m}$ for every $j > j_0$. So take $\delta = \varepsilon$ as above and set

$$I = \left(p_{j+m}\left(a_0 - \frac{\varepsilon}{3}\right), p_{j+m}\left(a_0 + \frac{\varepsilon}{3}\right)\right]$$

Then we have (see Figure 5.1)

$$\left.\begin{aligned} I \cup \left(I + C_{j,m}\right) \cup \left(I + 2C_{j,m}\right) \subset \left(p_{j+m}(a_0 - \varepsilon), p_{j+m}(a_0 + \varepsilon)\right] \\ p_{j+m}a_0 \in I \cap \left(I + C_{j,m}\right) \cap \left(I + 2C_{j,m}\right) \end{aligned}\right\} \forall j > j_0$$

Since $p_{j+m}a_0$ is an isolated critical value, we know that

$$\mathrm{HF}_*^J\left(F_{j,m}\right) \cong \bigoplus \left\{\mathrm{HF}_*^{\mathrm{loc}}\left(F_{j,m}, z\right) : z \in \mathrm{Fix}\,\varphi^1_{F_{j,m}} = \mathrm{Fix}\,\varphi,\ \mathscr{A}_{F_{j,m}}(z) = p_{j+m}a_0\right\}$$
$$\forall J \in \left\{I, I + 2C_{j,m}\right\}$$

see Lemma 4.3.5. Therefore by Lemma 4.3.6 the local Floer homology of z_0 contributes non-trivially to this sum, since p_{j+m} is by construction never a multiple of the order of the roots of unity in the spectrum of $d\varphi(z_0)$. Moreover, as we take $j > j_0$ large enough, we see that the supports in degree (see (4.15)) of the local Floer homology summands corresponding to fixed points with different mean Conley-Zehnder index become disjoint. This lets us conclude that

$$\mathrm{HF}_s^J\left(F_{j,m}\right) \cong \bigoplus \left\{ \mathrm{HF}_s^{\mathrm{loc}}\left(F_{j,m}, z\right) : \begin{array}{c} z \in \mathrm{Fix}\,\varphi, \\ \mathscr{A}_{F_{j,m}}(z) = p_{j+m} a_0, \\ \overline{\mathrm{CZ}}(z,H) = \overline{\mathrm{CZ}}(z_0,H) \end{array} \right\}$$
$$\forall s \in \deg\operatorname{supp} \mathrm{HF}^{\mathrm{loc}}\left(F_{j,m}, z_0\right), \quad \forall J \in \left\{I, I+2C_{j,m}\right\}$$

Recall from (5.4) that

$$\mathrm{HF}_*^{\mathrm{loc}}\left(F_{j,m}, z_0\right) \cong \mathrm{HF}_{*+\sigma_{j,m}}^{\mathrm{loc}}\left(H^{\times p_{j+m}}, z_0\right).$$

Hence (Lemma 4.3.3)

$$\begin{aligned} \deg\operatorname{supp} \mathrm{HF}^{\mathrm{loc}}\left(F_{j,m}, z_0\right) \subseteq \Delta(z_0, j, m) = & \\ = \Big[p_{j+m}\overline{\mathrm{CZ}}(z_0,H) - \sigma_{j,m} - n, \; & p_{j+m}\overline{\mathrm{CZ}}(z_0,H) - \sigma_{j,m} + n \Big] \end{aligned}$$

As remarked above, since z_0 is homologically visible and p_{j+m} is a large prime, we can apply Lemma 4.3.6 together with (5.4) and conclude that

$$\{0\} \neq \mathrm{HF}_*^{\mathrm{loc}}\left(H^{\times p_{j+m}}, z_0\right) \cong \mathrm{HF}_*^{\mathrm{loc}}\left(F_{j,m}, z_0\right)$$

In particular

$$\{0\} \neq \mathrm{HF}_*^J\left(F_{j,m}\right) \cong \mathrm{HF}_*^{\mathrm{loc}}\left(F_{j,m}, z_0\right) \oplus \cdots \quad \forall J \in \{I, I+2C_{j,m}\}$$

This implies that the inclusion-quotient morphism (see (4.13)) from action window I to action window $I+2C_{j,m}$ is non-zero:

$$0 \neq \Phi : \mathrm{HF}_*^I\left(F_{j,m}\right) \to \mathrm{HF}_*^{I+2C_{j,m}}\left(F_{j,m}\right)$$

Moreover, by definition, the classes corresponding to $\mathrm{HF}_*^{\mathrm{loc}}\left(F_{j,m}, z_0\right)$ are sent to themselves under Φ.

Now, since $F_{j,m}$ and G_j have the same quadratic Hamiltonian at infinity, we can perform a continuation between their filtered Floer homologies with action shift $C_{j,m} = \left\|F_{j,m} - G_j\right\|_{L^\infty}$ (Section 4.2.2):

$$\begin{cases} \mathscr{C} & : \mathrm{HF}_*^J\left(F_{j,m}\right) \to \mathrm{HF}_*^{J+C_{j,m}}\left(G_j\right), \\ \overline{\mathscr{C}} & : \mathrm{HF}_*^J\left(G_j\right) \to \mathrm{HF}_*^{J+C_{j,m}}\left(F_{j,m}\right) \end{cases} \quad \forall J \subset \mathbb{R}$$

The inclusion quotient morphism is factored by the composition of these morphisms, see

Lemma 4.2.2:

$$\begin{array}{ccc} & \mathrm{HF}_*^{I+C_{j,m}}\left(G_j\right) & \\ \nearrow^{\mathscr{C}} & & \searrow^{\overline{\mathscr{C}}} \\ \mathrm{HF}_*^{I}\left(F_{j,m}\right) & \xrightarrow{\quad\Phi\quad} & \mathrm{HF}_*^{I+2C_{j,m}}\left(F_{j,m}\right) \end{array} \tag{5.5}$$

We derive a contradiction from this factorization, by comparing the degrees where Φ is certainly non-zero with the degrees where $\mathscr{C}, \overline{\mathscr{C}}$ are both certainly zero.

As observed above, Φ is certainly non-zero for some degrees $s \in \Delta(z_0, j, m)$ for all j large enough and all m. On the other hand, $\mathscr{C}$ and $\overline{\mathscr{C}}$ are non-zero only in the degrees corresponding to the supports in degree of local Floer homologies of fixed points of G_j. These in turn are:

$$\deg\operatorname{supp}\mathrm{HF}_*^{\mathrm{loc}}\left(G_j, z\right) \subseteq \Gamma(z, j) = \left[p_j\overline{\mathrm{CZ}}(z, H) - n, p_j\overline{\mathrm{CZ}}(z, H) + n\right]$$

There are two cases: either $z \in \operatorname{Fix}\varphi$ is such that $\overline{\mathrm{CZ}}(z, H) = \overline{\mathrm{CZ}}(z_0, H)$, or not. In the first case, we use the twist condition. Recalling that by Proposition 5.1.1 we have

$$\left(p_{j+m} - p_j\right)\overline{\mathrm{ind}}_\infty H - n \leq \sigma_{j,m} \leq \left(p_{j+m} - p_j\right)\overline{\mathrm{ind}}_\infty H + n$$

it follows that

$$\begin{aligned} \Delta(z_0, j, m) \cap \Gamma(z, j) \neq \varnothing &\iff \left|\left(p_{j+m} - p_j\right)\overline{\mathrm{CZ}}(z_0, H) - \sigma_{j,m}\right| < 2n \\ &\iff p_{j+m} - p_j < \frac{2n}{\left|\overline{\mathrm{CZ}}(z_0) - \overline{\mathrm{ind}}_\infty H\right|} \end{aligned}$$

But $p_{j+m} - p_j > 2m$, so we can choose m large enough to make $\Delta(z_0, j, m)$ and $\Gamma(z, j)$ disjoint. If instead $\overline{\mathrm{CZ}}(z, H) \neq \overline{\mathrm{CZ}}(z_0, H)$, then

$$\Delta(z_0, j, m) \cap \Gamma(z, j) \neq \varnothing \iff \left|p_{j+m}\overline{\mathrm{CZ}}(z_0, H) - \sigma_{j,m} - p_j\overline{\mathrm{CZ}}(z, H)\right| < 2n$$

Notice that point (1) in Proposition 5.1.1 implies that $p_{j+m}/p_j \to 1$ as $j \to \infty$ and point (3) that $\sigma_{j,m}/p_j \to 0$ as $j \to \infty$. Therefore dividing everything by p_j we see

$$\frac{1}{p_j}\left|p_{j+m}\overline{\mathrm{CZ}}(z_0, H) - \sigma_{j,m} - p_j\overline{\mathrm{CZ}}(z, H)\right| \xrightarrow[j \to \infty]{} \left|\overline{\mathrm{CZ}}(z_0, H) - \overline{\mathrm{CZ}}(z, H)\right| > 0$$

But since p_j is unbounded in j, also $\left|p_{j+m}\overline{\mathrm{CZ}}(z_0, H) - \sigma_{j,m} - p_j\overline{\mathrm{CZ}}(z, H)\right|$ must be, and hence eventually larger than $2n$. So for every j large enough and with no condition on m, $\Delta(z_0, j, m)$ and $\Gamma(z, j)$ are disjoint.

This concludes the proof, because we found that for every j large enough there always exists an m such that Φ is non-zero for some degrees in $\Delta(z_0, j, m)$ while $\mathscr{C}$ and $\overline{\mathscr{C}}$ are both zero for all degrees in $\Delta(z_0, j, m)$. □

5.2. Re-indexing and interpolation at infinity

In this section we first explain the constructions involving the proof of Proposition 5.1.1, and then tie everything together to prove Proposition 5.1.1.

We start with some notation.

Definition 5.1 Choose a smooth non-decreasing step function $\rho\colon [0,1] \to [0,1]$ such that $\rho'(t) < 2$. If F, G are two Hamiltonians, define

$$(F\#G)_t(z) = F_t(z) + G_t\left(\left(\varphi_F^t\right)^{-1}(z)\right),$$
$$(F \wedge G)_t(z) = \begin{cases} 2\rho'(2t)G_{\rho(2t)}(z), & t \in \left[0, \frac{1}{2}\right] \\ 2\rho'(2t-1)F_{\rho(2t-1)}(z), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Denote by $\overline{F}_t(z) = -F_t\left(\varphi_F^t(z)\right)$.

The proof of the following lemma is a simple calculation.

Lemma 5.2.1 *These Hamiltonians generate the following flows:*

$$\varphi_{\overline{F}}^t = \left(\varphi_F^t\right)^{-1}, \quad \varphi_{F\#G}^t = \varphi_F^t \circ \varphi_G^t, \quad \varphi_{F\wedge G}^t = \begin{cases} \varphi_G^{\rho(2t)}, & t \in \left[0, \frac{1}{2}\right] \\ \varphi_F^{\rho(2t-1)} \circ \varphi_G^1, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

In particular $\varphi_{F\#G}^1 = \varphi_F^1 \circ \varphi_G^1 = \varphi_{F\wedge G}^1$ *and* $\varphi_{\overline{F}}^1 = \left(\varphi_F^1\right)^{-1}$.

5.2.1. Re-indexing at infinity

In general, the index at infinity of an asymptotically quadratic Hamiltonian grows linearly under iteration. This implies that as we iterate a Hamiltonian system, the different iterates will in general lie in different homotopy classes of linear systems at infinity. Since we can expect the continuation morphisms to be isomorphisms only within a fixed homotopy class of linear systems at infinity, we must find a way to modify the index at infinity without producing or destroying periodic orbits. To do this, we try our best to forget the flow and focus on the time-k maps only. The k-periodic orbits of H are in 1-1 correspondence with $\operatorname{Fix}\varphi^k$. Now, for $k \in \mathbb{N}$, φ^k can be generated by the Hamiltonian $H^{\times k}$, but this is not the only possible choice. In fact, if we are only interested in $\operatorname{Fix}\varphi^k$, we have the freedom to compose any Hamiltonian generating φ^k with a Hamiltonian generating a *loop* of Hamiltonian diffeomorphisms. Since all we have to do is change the index at infinity, linear loops suffice.

We start with a useful lemma. Recall from Section 2.2 that $\mathfrak{H}$ is the set of asymptotically quadratic Hamiltonians with non-degenerate quadratic Hamiltonian at infinity. The difference with $\mathfrak{wH}$ is that the non-quadratic part of the Hamiltonian is assumed to be bounded.

Lemma 5.2.2 *Let $G \in \mathfrak{wH}$, $G_t = R_t + g_t$, and $P_t(z) = \frac{1}{2}\langle B_t z, z\rangle$ a quadratic Hamiltonian generating a* loop *in* $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$, $\varphi_P^t =: N^t \in \mathrm{U}(n)$, $N^0 = N^1 = \mathbb{I}$. *Then:*

1. *$P\#G \in \mathfrak{wH}$. If $G \in \mathfrak{H}$ then also $P\#G \in \mathfrak{H}$.*
2. *For every $z \in \mathrm{Fix}\,\varphi^1_{P\#G} = \mathrm{Fix}\,\varphi^1_G$, we have*

$$\mathscr{A}_{P\#G}(z) = \mathscr{A}_G(z)$$

Proof. 1. By definition $P\#G_t = P\#(R_t + g_t) = P_t + R_t \circ (N^t)^{-1} + g_t \circ (N^t)^{-1}$. Clearly

$$\left|\nabla\left(g_t \circ (N^t)^{-1}\right)(z)\right| = \left|\nabla g_t\left((N^t)^{-1} z\right)\right| = o(|z|) \quad \text{as } |z| \to \infty$$

since $N^t \in \mathrm{U}(n)$. If g is bounded, then also $g \circ (N^t)^{-1}$ is bounded. Since the time-1 map is untouched by composition with φ_P^t, it is still non-degenerate at infinity.

2. We calculate

$$\begin{aligned}\mathscr{A}_{P\#G}(z) &= \int_0^1 \frac{1}{2}\left\langle N^t\varphi_G^t(z), J_0 \frac{d}{dt}\left(N^t\varphi_G^t(z)\right)\right\rangle - P\#G_t\left(N^t\varphi_G^t(z)\right)dt = \\ &= \int_0^1 \frac{1}{2}\left\langle N^t\varphi_G^t(z), A_t N^t\varphi_G^t(z)\right\rangle - P_t\left(N^t\varphi_G^t(z)\right)dt + \\ &\quad + \int_0^1 \left\langle N^t\varphi_G^t(z), J_0 N^t\dot\varphi_G^t(z)\right\rangle - G_t\left(\varphi_G^t(z)\right)dt = \\ &= \int_0^1 \left\langle \varphi_G^t(z), J_0\dot\varphi_G^t(z)\right\rangle - G_t\left(\varphi_G^t(z)\right)dt = \mathscr{A}_G(z)\end{aligned}$$

since $\langle N^t v, J_0 N^t w\rangle = \left\langle N^t v, (N^t)^{-T} J_0 w\right\rangle = \left\langle (N^t)^{-1} N^t v, J_0 w\right\rangle = \langle v, J_0 w\rangle$. Notice that the assumption on P to generate a loop is only needed to conclude that the fixed point sets coincide, $\mathrm{Fix}\,\varphi^1_G = \mathrm{Fix}\,\varphi^1_{P\#G}$.

□

Remark The calculation above also shows that the action of a fixed point of a linear flow is always zero in these conventions. In fact, we are tacitly imposing a *normalization condition* on linear symplectomorphisms: we are asking our linear flows to be generated by genuine quadratic forms, so the freedom of adding a constant to the quadratic Hamiltonian is ruled out.

This lemma tells us that if we compose our Hamiltonian H with a quadratic Hamiltonian generating a loop, we might be able to change the index at infinity without changing the fixed points nor their action.

Let $H_t = Q_t + h_t \in \mathfrak{w}\mathfrak{H}$, $\varphi = \varphi_H^1$ and $\varphi_\infty = \varphi_Q^1$. Recall that

$$\begin{aligned}\operatorname{ind}_\infty\left(H^{\times k}\right) &= \operatorname{CZ}\left(\varphi^t_{Q^{\times k}} : t\in[0,1]\right) = \operatorname{CZ}\left(\varphi^t_Q : t\in[0,k]\right),\\ \overline{\operatorname{ind}}_\infty(H) &= \lim_{k\to\infty}\frac{1}{k}\operatorname{ind}_\infty\left(H^{\times k}\right)\end{aligned}$$

If k is a multiple of an order of a root of unity in the spectrum of φ^1_Q, then the index at time k is defined by lower-semicontinuous extension as in Section 1.2.5.1.

Fix two odd integers $k > l \geq 1$ which are not multiples of orders of roots of unity in $\sigma(\varphi_\infty)$. Then by Proposition 1.2.6 the parity of the indices at infinity of the iterates $H^{\times k}$ and $H^{\times l}$ are the same, so that

$$\operatorname{ind}_\infty\left(H^{\times k}\right) - \operatorname{ind}_\infty\left(H^{\times l}\right) = 2\mu$$

for some $\mu \in \mathbb{Z}$. Notice that Proposition 1.2.5 implies an automatic bound on μ in terms of the mean index at infinity, namely

$$(k-l)\overline{\operatorname{ind}}_\infty H - n \leq 2\mu \leq (k-l)\overline{\operatorname{ind}}_\infty H + n$$

Let $P^\mu_t(z) = \frac{1}{2}\left\langle B^\mu_t z, z\right\rangle$ be a quadratic Hamiltonian generating a *loop* $N^t = \varphi^t_{P^\mu} \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ such that the Maslov index of $[0,1] \ni t \mapsto N^t$ is precisely μ. By the elementary theory of the Maslov index (Propositions 1.1.2 and 1.1.4) such a path of unitary matrices always exists, and correspondingly a generating quadratic Hamiltonian P^μ. Define the following Hamiltonian

$$H^{k\ominus l} = \left(\overline{P^\mu}\# H^{\times(k-l)}\right)\wedge H^{\times l}$$

Lemma 5.2.3 *The Hamiltonian $H^{k\ominus l}$ has the following properties.*

0. $H^{k\ominus l} \in \mathfrak{w}\mathfrak{H}$.
1. *It generates φ^k_H in time 1.*
2. $\operatorname{ind}_\infty\left(H^{k\ominus l}\right) = \operatorname{ind}_\infty\left(H^{\times l}\right)$. *Moreover if $\bar z \in \operatorname{Fix}\varphi^k_H = \operatorname{Fix}\varphi^1_{H^{k\ominus l}}$ then*

$$\operatorname{CZ}\left(\bar z, H^{k\ominus l}\right) = \operatorname{CZ}\left(\bar z, H^{\times k}\right) - 2\mu$$

3. *If $z_0 \in \operatorname{Fix}\varphi^1_H$ is seen as a k-periodic point, then $\mathscr{A}_{H^{k\ominus l}}(z_0) = \mathscr{A}_{H^{\times k}}(z_0) = k\mathscr{A}_H(z_0)$.*

Proof. 0. This is just Lemma 5.2.2.

1. P^μ generates a loop based at the identity in time 1, and $H^{\times(k-l)} \wedge H^{\times l}$ generates φ^k in time 1.
2. The path $t \mapsto \varphi^t_{H^{k\ominus l}}$ is Hamiltonian-isotopic to the path $t \mapsto \left(\varphi^t_{P^\mu}\right)^{-1} \circ \varphi^{kt}_H$, which is generated by the Hamiltonian $\overline{P}^\mu \# H^{\times k}$. The quadratic form at infinity of this Hamiltonian

is $\overline{P}^{\mu}\#Q^{\times k}$. Therefore

$$\begin{aligned}\operatorname{ind}_\infty\left(H^{k\ominus l}\right) &= \mathrm{CZ}\left(\left(\varphi^t_{P^\mu}\right)^{-1}\circ\varphi^t_{Q^{\times k}} : t\in[0,1]\right) = \\ &= \mathrm{CZ}\left(\varphi^t_{Q^{\times k}} : t\in[0,1]\right) + 2\,\mathrm{Mas}\left(\left(\varphi^t_{P^\mu}\right)^{-1} : t\in[0,1]\right) = \\ &= \operatorname{ind}_\infty\left(H^{\times k}\right) - 2\mu = \operatorname{ind}_\infty\left(H^{\times k}\right)\end{aligned}$$

The calculation of the index of periodic orbits is completely analogous.

3. Notice that $z_0\in\operatorname{Fix}\varphi^1_H \implies z_0\in\operatorname{Fix}\varphi^l_H\cap\operatorname{Fix}\varphi^{k-l}_H$. So we are reduced to the situation of two Hamiltoinans F,G and a point $z_0\in\operatorname{Fix}\varphi^1_F\cap\operatorname{Fix}\varphi^1_G$. Then $z_0\in\operatorname{Fix}\varphi^1_F\circ\varphi^1_G$ and

$$\begin{aligned}\mathscr{A}_{F\wedge G}(z_0) &= \int_0^{\frac{1}{2}}\left(\varphi_G^{\rho(2\cdot)}(z_0)\right)^*\lambda_0 - 2\rho'(2t)G_{\rho(2t)}\circ\varphi_G^{\rho(2t)}(z_0)dt + \\ &+ \int_{\frac{1}{2}}^1\left(\varphi_F^{\rho(2\cdot-1)}\circ\varphi^1_G(z_0)\right)^*\lambda_0 - 2\rho'(2t)F_{\rho(2t-1)}\circ\varphi_F^{\rho(2t-1)}\circ\varphi^1_G(z_0)dt \\ &= \int_0^1\left(\varphi^{\cdot}_G(z_0)\right)^*\lambda_0 - G_t\circ\varphi^t_G(z_0)dt + \int_0^1\left(\varphi^{\cdot}(z_0)\right)^*\lambda_0 - F_t\circ\varphi^t_F(z_0)dt = \\ &= \mathscr{A}_G(z_0) + \mathscr{A}_F(z_0)\end{aligned}$$

Using Lemma 5.2.2, we calculate

$$\mathscr{A}_{H^{k\ominus l}}(z_0) = \mathscr{A}_{H^{\times l}}(z_0) + \mathscr{A}_{\overline{P}^\mu\#H^{k-l}}(z_0) = l\,\mathscr{A}_H(z_0) + (k-l)\mathscr{A}_H(z_0) = k\,\mathscr{A}_H(z_0)$$

□

5.2.1.1. Action of linear loops on Floer homology

Re-indexing at infinity has a simple effect on the Floer homology: it shifts the grading of the Floer homology by an explicit integer and does not shift the action filtration at all.

Lemma 5.2.4 *Let $P_t(z)=\frac{1}{2}\langle B_t z, z\rangle$ be a quadratic Hamiltonian generating a* loop *of linear symplectomorphisms $N^t=\varphi^t_P$, $N^0=N^1=\mathbb{I}$ of Maslov index $\mu\in\mathbb{Z}$. Consider $H\in\mathfrak{w}\mathfrak{H}_*$. Let $I\subset\mathbb{R}$ be an interval. Let J be an adequate family of almost-complex structures on $\mathbb{R}^{2n}$ with (H,J) a regular pair. We have*

$$\mathrm{CF}^I_*(P\#H, N^*J)\cong\mathrm{CF}^I_{*-2\mu}(H,J)$$

Proof. To exhibit the isomorphism, it suffices to give a bijection of sets of 1-periodic orbits and a bijection of moduli spaces of Floer cylinders.

The bijection on the orbits is given by sending a 1-periodic orbit γ of H to $\xi=N\cdot\gamma$ which is a 1-periodic orbit of $P\#H$, since N is a loop. By Lemma 5.2.2 we already know that $\mathscr{A}_H(\gamma)=\mathscr{A}_{P\#H}(\xi)$, so the bijection respects the filtration on the chain complexes. Moreover by the change of index formula (Equation (1.5)) for the Conley-Zehnder index of the composition of a loop in Sp($2n$) with a non-degenerate path in Sp($2n$), we get

$$\mathrm{CZ}(N\cdot\gamma)=\mathrm{CZ}(\gamma)+2\mu$$

therefore this bijection sends the generators of $\mathrm{CF}^I_*(P\#H,J)$ to the generators of $\mathrm{CF}^I_{*-2\mu}\left(H,\tilde{J}\right)$ where $\tilde{J}$ is to be determined by studying the corresponding bijection on the moduli spaces of Floer cylinders.

Namely, if we take $u\colon\mathbb{R}\times S^1\to\mathbb{R}^{2n}$ solving the Floer equation for (H,J), then $v(s,t)=N^t u(s,t)$ solves the Floer equation for $(P\#H,N^*J)$ where $(N^*J)_{(t,z)}=[(N^t)^*J_t]_z$. Notice that since N is a *loop* of linear symplectomorphisms, the family N^*J is still an adequate family of almost-complex structures.

All in all, the bijection extends by linearity to a chain complex isomorphism as claimed. □

Remark Since filtered Floer homology groups are independent of the almost-complex structures used to define the chain complex, we see that this chain complex isomorphism descends to an isomorphism between $\mathrm{HF}^I_*(P\#H)$ and $\mathrm{HF}^I_{*-2\mu}(H)$.

Corollary 5.2.1 *Let $H\in\mathfrak{w}\mathfrak{H}$, $k>l$ two odd numbers, and $2\mu=\mathrm{ind}_\infty H^{\times k}-\mathrm{ind}_\infty H^{\times l}$. Then for any fixed point $z_0\in\mathrm{Fix}\,\varphi^1_H$ we have*

$$\mathrm{HF}^{\mathrm{loc}}_*\left(H^{k\ominus l},z_0\right)\cong\mathrm{HF}^{\mathrm{loc}}_{*+2\mu}\left(H^{\times k},z_0\right)$$

5.2.2. Interpolations at infinity

Consider two asymptotically quadratic Hamiltonians H^0,H^1, both non-degenerate at infinity. Assume that the two systems have the same index at infinity. In this situation, it is possible to define continuation maps between the Floer homologies of H^0 and H^1, as we've seen in Section 4.1.2.2. But a problem arises in trying to find a good estimate for the action shift of the continuation Floer morphism. In this section we explain how to circumvent this problem in the case that the quadratic forms generate linear *unitary* flows, by interpolating one of the two quadratic forms to the other. The result is a new Hamiltonian which has the same fixed points as the starting one, and equal to it in a large ball. This will permit us to obtain a better control of the action shift constant.

We start with a preliminary lemma on dynamical systems.

Lemma 5.2.5 *Let $X_t=E_t+F_t$ be a vector field on $\mathbb{R}^m$ such that $E_t(0)=0$, $|DE_t(z)|<C$ and $|F_t(z)|=o(|z|)$ as $|z|\to\infty$ for all t. If there exist $M_0,C_0>0$ such that*

$$\left|\varphi^1_E(z)-z\right|>C_0|z|\quad\forall|z|>M_0$$

then there exist $M_1,C_1>0$ such that

$$\left|\varphi^1_X(z)-z\right|>C_1\quad\forall|z|>M_1$$

Proof. The argument is just a routine application of the Grönwall lemma. Fix a $z \in \mathbb{R}^m$ and a $\tau \in [0,1]$. First of all let's estimate the distance between $\varphi_E^\tau(z)$ and $\varphi_X^\tau(z)$.

$$
\begin{aligned}
|\varphi_X^\tau(z) - \varphi_E^\tau(z)| &\leq \int_0^\tau \left|X_t \circ \varphi_X^t(z) - E_t \circ \varphi_E^t(z)\right| dt \leq \\
&\leq \int_0^\tau |X_t \circ \varphi_X^t(z) - E_t \circ \varphi_X^t(z)| + \left|E_t \circ \varphi_X^t(z) - E_t \circ \varphi_E^t(z)\right| dt \leq \\
&\leq \int_0^\tau |F_t \circ \varphi_X^t(z)|\, dt + C \int_0^\tau \left|\varphi_X^t(z) - \varphi_E^t(z)\right| dt
\end{aligned}
\tag{5.6}
$$

We have to estimate the first integral. To do this, fix an arbitrary $\varepsilon \in (0,1)$. Then $\exists M_\varepsilon > 0$ such that $|x| > M_\varepsilon \implies |F_t(x)| < \varepsilon |x|$. Set

$$
I_\varepsilon = \{t \in [0,\tau] : |\varphi_X^t(z)| \leq M_\varepsilon\}, \quad J_\varepsilon = \{t \in [0,\tau] : |\varphi_X^t(z)| > M_\varepsilon\}
$$

We obtain

$$
\begin{aligned}
&\int_0^\tau |F_t \circ \varphi_X^t(z)|\, dt = \int_{I_\varepsilon} |F_t \circ \varphi_X^t(z)|\, dt + \int_{J_\varepsilon} |F_t \circ \varphi_X^t(z)|\, dt \leq \\
&\leq \max_{(t,x) \in [0,1] \times B_{M_\varepsilon}(0)} |F_t(x)| + \int_{J_\varepsilon} \varepsilon\, |\varphi_X^t(z)|\, dt \leq D_\varepsilon + \varepsilon \int_0^\tau |\varphi_X^t(z)|\, dt
\end{aligned}
$$

Notice that D_ε becomes possibly very big as $\varepsilon \to 0$. Now

$$
\int_0^\tau |\varphi_X^t(z)|\, dt \leq \int_0^\tau \left|\varphi_X^t(z) - \varphi_E^t(z)\right| dt + \int_0^\tau \left|\varphi_E^t(z)\right| dt
$$

but since $E_t(0) = 0$ for all t and $|DE_t(z)| < C$, we can estimate using the Grönwall lemma

$$
\left|\varphi_E^t(z)\right| = \left|\varphi_E^t(z) - \varphi_E^t(0)\right| \leq e^{Ct} |z|
$$

Plugging in these estimates in (5.6), we obtain

$$
|\varphi_X^\tau(z) - \varphi_E^\tau(z)| \leq D_\varepsilon + \varepsilon \cdot \frac{\left(e^{C\tau} - 1\right)}{C} |z| + (C + \varepsilon) \int_0^\tau \left|\varphi_X^t(z) - \varphi_E^t(z)\right| dt
$$

We can apply the Grönwall lemma to obtain the estimate, for $\tau = 1$,

$$
\left|\varphi_X^1(z) - \varphi_E^1(z)\right| \leq e^{C+\varepsilon} \left(D_\varepsilon + \frac{\varepsilon \left(e^C - 1\right)}{C} |z| \right) \leq S_\varepsilon + \varepsilon T |z|
$$

where $S_\varepsilon = e^{C+1}D_\varepsilon$, $T = e^{C+1}\frac{e^C-1}{C}$. Now, take $|z| > M_0$. Then

$$\begin{aligned}\left|\varphi_X^1(z) - z\right| \geq \left|\varphi_E^1(z) - z\right| - \left|\varphi_X^1(z) - \varphi_E^1(z)\right| \geq \\ \geq C_0|z| - S_\varepsilon - \varepsilon T|z| = (C_0 - \varepsilon T)|z| - S_\varepsilon\end{aligned}$$

To conclude, we take $\varepsilon < \frac{C_0}{T}$ so that $C_0 - \varepsilon T > 0$. Then $M_1 = \frac{S_\varepsilon}{C_0 - \varepsilon T} + 1$, $C_1 = C_0 - \varepsilon T$. □

Next, we work at the linear level: we interpolate the quadratic forms without creating fixed points.

Proposition 5.2.1 *Let Q^0 and Q^1 be two non-degenerate quadratic Hamiltonians of the same index, i.e. $\varphi^1_{Q^i}$ does not contain 1 in its spectrum for both $i = 0,1$ and*

$$\mathrm{CZ}\left(\varphi^t_{Q^0} : t \in [0,1]\right) = \mathrm{CZ}\left(\varphi^t_{Q^1} : t \in [0,1]\right).$$

Assume that $\varphi^t_{Q^i} \in \mathrm{U}(n)$ for all t and all $i = 0,1$. Then for every $R_0 > 0$ there exists an $R_1 > R_0$ and a Hamiltonian $K \in C^\infty(S^1 \times \mathbb{R}^{2n})$ with the following properties:

1. *K interpolates between Q^0 and Q^1:*

$$K_t|_{B_{R_0}} = Q^0_t, \quad K_t|_{\mathbb{R}^{2n}\setminus B_{R_1}} = Q^1_t$$

2. *K has no non-trivial 1-periodic orbits:*

$$\mathrm{Fix}\,\varphi^1_K = \{0\}$$

Proof. Write $Q^i_t(z) = \frac{1}{2}\left\langle A^i_t z, z\right\rangle$ for $A^i : S^1 \to \mathrm{Sym}(2n)$. Let $\mathbb{A} : [0,1] \times S^1 \to \mathrm{Sym}(2n)$ be a homotopy of symmetric matrices such that $\mathbb{A}^0 = A^0$, $\mathbb{A}^1 = A^1$, $\mathbb{A}^s : S^1 \to \mathrm{Sym}(2n)$ is non-degenerate for all s, and $-J_0\mathbb{A}^s_t \in \mathfrak{u}(n)$ for all s,t. This exists because Q^0 and Q^1 have the same index at infinity, and their flows are unitary at all times.

Fix an arbitrary $R_0 > 0$ and a $R_1 > R_0$ to be determined later. Consider a non-decreasing smooth function $\chi : [0,\infty) \to [0,1]$ such that $\chi(r) = 0$ for all $r < R_0$ and $\chi(r) = 1$ for all $r > R_1$. We want to determine R_1 so that the Hamiltonian system defined by

$$K(t,z) = \frac{1}{2}\left\langle \mathbb{A}_t^{\chi(|z|^2)} z, z\right\rangle$$

has only $0 \in \mathbb{R}^{2n}$ as 1-periodic orbit. We calculate the Hamiltonian vector field

$$X_K(t,z) = -J_0\mathbb{A}_t^{\chi(|z|^2)} z - \chi'\left(|z|^2\right)\left\langle \left.\frac{\partial \mathbb{A}^s_t}{\partial s}\right|_{s=\chi(|z|^2)} z, z\right\rangle J_0 z$$

The Hamilton equations can thus be written as

$$\dot{x}+J_0\mathbb{A}_t^{\chi(|x|^2)}x=-\chi'\left(|x|^2\right)\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi(|x|^2)} x,x\right\rangle J_0x$$

Let's show that the norm of an integral curve of K is constant. Indeed, if $x:[0,T]\to\mathbb{R}^{2n}$ is an integral curve of K,

$$\frac{1}{2}\frac{d}{dt}|x|^2=\langle x,\dot{x}\rangle=-\left\langle x,J_0\mathbb{A}_t^{\chi(|x|^2)}x\right\rangle-\chi'\left(|x|^2\right)\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi(|x|^2)} x,x\right\rangle\langle x,J_0x\rangle=0$$

because $J_0\mathbb{A}_t^s\in\mathfrak{u}(n)\subset\mathfrak{o}(2n)$ which is the set of skew symmetric matrices. Therefore, if we set $r_0=|x(0)|^2$, $\chi_0=\chi(r_0)$, $\chi_0'=\chi'(r_0)$, we can use the fact that $\mathbb{A}^s$ generates a non-degenerate system for all s to invert the operator $\frac{d}{dt}+J_0\mathbb{A}_t^{\chi_0}$ using the variation of constants method, as in Lemma 2.3.1 and its Corollary 2.3.1. We obtain the integral expression (see equation (2.6))

$$x_t=\mathbb{M}_t^{\chi_0}\left[x_0-\int_0^t\chi_0'\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi_0} x_\tau,x_\tau\right\rangle\left(\mathbb{M}_\tau^{\chi_0}\right)^{-1}J_0x_\tau d\tau\right]$$

where $\mathbb{M}^s:[0,1]\to U(n)\subset Sp(2n)$ is the path of symplectic matrices generated by $\mathbb{A}^s$. We use this formula to estimate

$$\begin{aligned}|x_1-x_0|&=\left|\mathbb{M}_1^{\chi_0}x_0-x_0-\int_0^1\chi_0'\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi_0} x_\tau,x_\tau\right\rangle\mathbb{M}_{1-\tau}^{\chi_0}x_\tau d\tau\right|\geq\\ &\geq\left|\left|\mathbb{M}_1^{\chi_0}x_0-x_0\right|-\left|\int_0^1\chi_0'\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi_0} x_\tau,x_\tau\right\rangle\mathbb{M}_{1-\tau}^{\chi_0}x_\tau d\tau\right|\right|\end{aligned}\tag{5.7}$$

Since the systems defined by $\mathbb{A}$ are all non-degenerate, there exists a $C_0>0$ such that

$$|\mathbb{M}_1^s z-z|>C_0|z|\quad\forall z\in\mathbb{R}^{2n}\quad\forall s\in[0,1]$$

Therefore to reach our conclusion we have to bound from above the integral in the second part of equation (5.7) by $C_0|x_0|$. We start with

$$\left|\int_0^1\chi_0'\left\langle \left.\frac{\partial \mathbb{A}_t^s}{\partial s}\right|_{s=\chi_0} x_\tau,x_\tau\right\rangle\mathbb{M}_{1-\tau}^{\chi_0}x_\tau d\tau\right|\leq\int_0^1 C_1\chi'\left(|x_0|^2\right)|x_0|^3d\tau=C_1\chi'(r_0)r_0^{3/2}$$

where $C_1=||\partial_s\mathbb{A}||_{L^\infty}$. We are led to impose the point-wise constraint

$$\chi'(r)\leq\frac{C_0}{C_1 r}$$

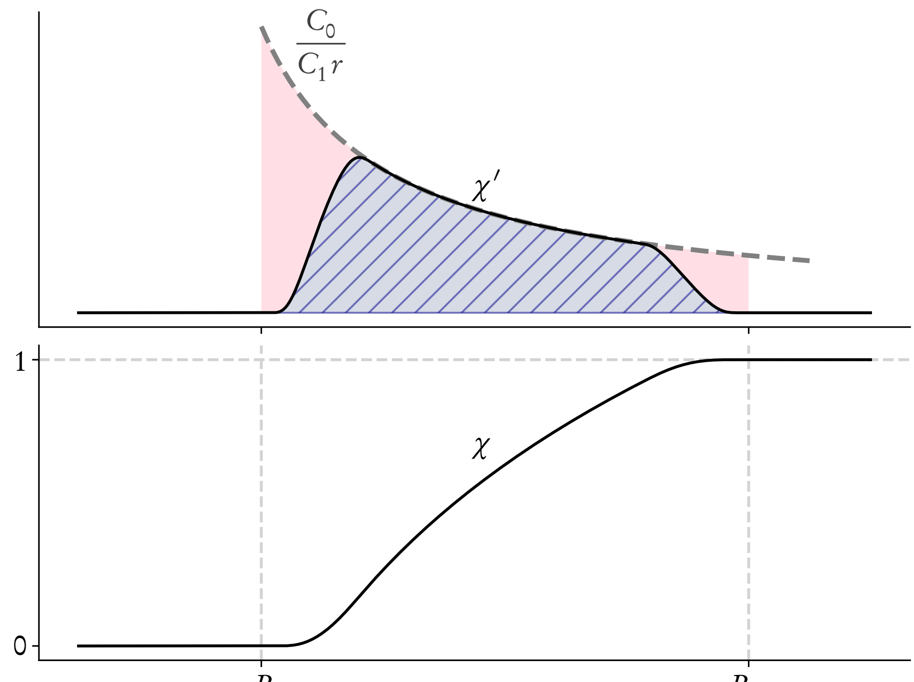

Figure 5.2. Construction of the step function χ. The smoothing is done in such a way that the pink area contributes e.g. $\frac{1}{4}$ to the total area and the blue striped area contributes 1 to the total area. The resulting primitive, which is our χ, is drawn below.

Observe that $\int_{\mathbb{R}} \chi' = 1$, while r^{-1} has diverging integral. Therefore given any fixed $R_0 > 0$, let $R_1 > R_0$ be such that

$$\int_{R_0}^{R_1} \frac{C_0}{C_1 r} dr = 1 + \frac{1}{4} \iff R_1 = e^{\frac{5}{4}\frac{C_1}{C_0}} R_0$$

Define χ' to be a smoothed out version of $\mathbb{I}_{[R_0,R_1]}\frac{C_0}{C_1 r}$, where $\mathbb{I}_{[R_0,R_1]}$ denotes the indicator function of $[R_0, R_1]$ (see Figure 5.2). We can assume that the smoothing is done in such a way that $\int_{\mathbb{R}} \chi' = 1$. Imposing the boundary condition that $\chi(R_0) = 0$ gives an unique primitive χ of our χ' which will have the sought-for properties. ☐

Remark We gather the formulas for the two constants C_0 and C_1 which enter the definition of R_1, since their behaviour under iteration is crucial for the proof of the main proposition:

$$C_0 = \min_{s\in[0,1],\, z\in\mathbb{R}^{2n}} \frac{\left|\mathbb{M}_1^s z - z\right|}{|z|} > 0, \quad C_1 = ||\partial_s \mathbb{A}||_{L^\infty([0,1]\times S^1, \mathrm{Sym}\, 2n)} \tag{5.8}$$

The fact that $C_0 > 0$ is implied by the fact that $1 \notin \sigma(\mathbb{M}_1^s)$ for all $s \in [0,1]$.

Proposition 5.2.2 *Let $H_t^i = Q_t^i + h_t^i$, $i \in \{0,1\}$ be two (weakly) asymptotically quadratic Hamil-*

tonians whose quadratic forms at infinity $Q^i_t(z) = \frac{1}{2}\left\langle A^i_t z, z\right\rangle$ *are non-degenerate and generate flows of linear unitary maps. Assume that* $\operatorname{ind}_\infty H^0 = \operatorname{ind}_\infty H^1$. *Then there exists an* $R_0 > 0$ *and a Hamiltonian* $F \in C^\infty(S^1 \times \mathbb{R}^{2n})$ *such that:*

1. F *is (weakly) asymptotically quadratic, has* Q^1 *as quadratic Hamiltonian at infinity, and*

$$F_t|_{B_{R_0}} = H^0_t\Big|_{B_{R_0}}$$

2. $\operatorname{Fix}\varphi^1_F = \operatorname{Fix}\varphi^1_{H^0}$.

Proof. By Proposition 5.2.1, for any $R_0 > 0$ we find an $R_1 > R_0$ and a $K \in C^\infty(S^1 \times \mathbb{R}^{2n})$ interpolating between Q^0 and Q^1 without creating 1-periodic orbits. So let $R_0 > 0$ be so large that $B_{R_0}(0)$ contains all 1-periodic orbits of H^0. This always exist because of Lemma 2.3.4. So set

$$F = K + h^0 = Q^1 + \left(K - Q^1 + h^0\right) = Q^1 + \widetilde{h^0}$$

The first point follow immediately from this definition. For the second point, we apply Lemma 5.2.5 to $X = X_F$, $E = X_K$, $F = X_{h^0}$. The hypotheses on E and F are clearly met in this case. All in all, the resulting F will have the same fixed points as H^0, and no other fixed points. □

5.2.3. Proof of the main proposition

First of all, we show that any ALHD φ with $\varphi_\infty \in \mathrm{U}(n)$ can be generated by an asymptotically quadratic Hamiltonian $H = Q + h$ such that $\varphi^t_Q \in \mathrm{U}(n)$ for all t.

Lemma 5.2.6 *Let* $H'_t = Q'_t + h'_t \in \mathfrak{H}$ *where* $\varphi^1_{Q'} \in \mathrm{U}(n)$. *There exists a (possibly non-autonomous) quadratic Hamiltonian* P *generating a* loop *of unitary maps such that* $H_t = P\#H'_t = Q + h_t$ *is an asymptotically quadratic Hamiltoninan with* autonomous *quadratic form at infinity such that* $\varphi^t_Q \in \mathrm{U}(n)$ *for all* t *and* $\varphi^1_H = \varphi^1_{H'}$.

Proof. The proof is standard Krein theory. Let $\varphi^1_{Q'} = U \in \mathrm{U}(n)$. Since U is unitary, it has a logarithm $b = \log U \in \mathfrak{u}(n)$. Notice that $J_0 b = b J_0$ and $b^t = -b$, since $\mathfrak{u}(n) = \mathfrak{o}(2n) \cap \mathfrak{gl}(n, \mathbb{C})$. Set $B = J_0 b$. Then B is symmetric: $B^T = -b^T J_0 = B$. Define

$$Q(z) = \frac{1}{2}\langle Bz, z\rangle, \quad P = Q\#\overline{Q'}$$

Notice $\varphi^t_Q \in \mathrm{U}(n)$ and $\varphi^1_Q = U$ by construction. Then $\varphi^1_P = \varphi^1_Q \circ \left(\varphi^1_{Q'}\right) = \mathrm{id}$ so φ^t_P is a loop of unitary matrices. Finally

$$\begin{aligned} P\#H_t &= P_t + H_t \circ \left(\varphi^t_P\right)^{-1} = Q - Q'_t \circ \varphi^t_{Q'} \circ \varphi^{-t}_Q + Q'_t \circ \varphi^t_{Q'} \circ \varphi^{-t}_Q + h'_t \circ \varphi^t_{Q'} \circ \varphi^{-t}_Q = \\ &= Q + h_t \end{aligned}$$

where $h_t = h'_t \circ \left(\varphi_P^t\right)^{-1}$. □

Moreover, since every Hermitian matrix can be unitarily diagonalized, we can find a basis of $\mathbb{C}^n$ for which Q is the following diagonal quadratic form:

$$Q(z) = \frac{1}{2}\sum_{r=1}^{n} \alpha_r |z_r|^2, \quad z = (z_1, \ldots, z_n) \in \mathbb{C} \oplus \cdots \oplus \mathbb{C} = \mathbb{C}^n \tag{5.9}$$

Notice that we are not imposing that $\alpha_r > 0$ for any r.

The Hamiltonian H of point (c) in Proposition 5.1.1 is any H which is autonomous at infinity and whose flow at infinity is unitary and diagonal. We have $\varphi = \varphi_H^1$ and $\varphi_\infty = \varphi_Q^1$. Recall that we are assuming that φ has finitely many fixed points and finitely many integers are attained as primitive periods of periodic orbits of φ.

Next we find the sequence $(p_j)_{j\in\mathbb{N}}$. To do this, we simply apply Proposition 1.1.5 to the map φ_∞ and we obtain an increasing sequence of prime numbers $(p_j)_j$ which satisfies properties (1) and (2) in Proposition 5.1.1, and such that

$$d_{\mathrm{U}(1)}\left(\sigma(\varphi_\infty^{p_j}), 1\right) > c$$

for some $c > 0$ independent of j. Without loss of generality we can assume that the first element p_0 of the sequence is larger than the largest primitive period of a periodic point of φ.

The sequence $\sigma_{j,m}$ is defined as follows:

$$\sigma_{j,m} = \frac{1}{2}\left[\mathrm{ind}_\infty H^{\times p_{j+m}} - \mathrm{ind}_\infty H^{\times p_j}\right]$$

That this is an integer follows from the fact that p_j and p_{j+m} are odd, so the indices have the same parity. The estimate on the growth of $\sigma_{j,m}$ follows immediately from the estimate of the iterated Conley-Zehnder index in terms of the mean Conley-Zehnder index.

The Hamiltonians G_j are defined as $G_j = 0 \wedge H^{\times p_j}$. The properties of equation (5.1) in Proposition 5.1.1 are clear.

Next we define $F_{j,m}$. Notice that G_j and $H^{p_{j+m}\ominus p_j}$ have the same index at infinity by construction of the re-indexed Hamiltonian $H^{p_{j+m}\ominus p_j}$. So we aim to apply Proposition 5.2.2 to $H^0 = H^{p_{j+m}\ominus p_j}$ and $H^1 = G_j$. But to gain control over the iteration process, we must construct the non-resonant path $\mathbb{A}$ in the proof of Proposition 5.2.2 explicitly.

For convenience set $k = p_{j+m}$ and $l = p_j$. Let's construct $H^{k\ominus l}$ explicitly. Here the diagonalization (5.9) helps to write explicit formulae. In fact we can compute explicitly the indices at infinity and the Maslov index of the unwinding loop, and obtain

$$P^\mu(z) = \frac{1}{2}\sum_r 2\pi \left\lfloor \frac{(k-l)\alpha_r}{2\pi} \right\rfloor |z_r|^2$$

where $\lfloor \alpha \rfloor = \max\{j \in \mathbb{Z} : j \leq \alpha\}$. We calculate

$$\overline{P^\mu} \# H_t^{\times(k-l)}(z) = \frac{1}{2}\sum_r 2\pi \left[\frac{(k-l)\alpha_r}{2\pi} - \left\lfloor \frac{(k-l)\alpha_r}{2\pi} \right\rfloor \right] |z_r|^2 + (k-l) h_{(k-l)t} \circ \varphi^t_{P^\mu}(z) =$$
$$= R(z) + h_t^{\times(k-l)} \circ \varphi^t_{P^\mu}(z)$$

This gives us

$$H^{k\ominus l} = \left(\overline{P^\mu} \# H^{\times(k-l)}\right) \wedge H^{\times l} = R \wedge Q^{\times l} + \left(h^{\times(k-l)} \circ \varphi^{\cdot}_{P^\mu}\right) \wedge h^{\times l}$$

By construction $\mathrm{ind}_\infty(H^{k\ominus l}) = \mathrm{ind}_\infty(H^{\times l}) = \mathrm{ind}_\infty(0 \wedge H^{\times l})$, which is the index at infinity of G_j. Therefore we next construct an explicit non-degenerate homotopy to perform the interpolation at infinity of section 5.2.1. Set for notational convenience $R(z) = \frac{1}{2}\sum_j \beta_r |z_r|^2$. Notice that $\beta_r \in (0, 2\pi)$. We define

$$R^s(z) = \frac{1}{2}\sum_r \beta_r^s |z_r|^2$$

where the angular velocities β_r^s are determined in the following way: fix an r and notice that the flow of $Q^{\times l}$ fixes the r-th complex line in the decomposition $\mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ and restricts to it to the map

$$z_r \mapsto e^{-l\alpha_r i t} z_r$$

Set $\zeta_r = e^{-l\alpha_r i}$ i.e. the image of the vector 1 under the map above. Now consider the arc

$$\gamma : [0,1] \to S^1 \subset \mathbb{C}, \quad \gamma(s) = e^{-i\beta_r(1-s)}\zeta_r$$

We know by hypothesis that $\gamma(1) \neq 1$. If there is no $s \in (0,1)$ such that $\gamma(s) = 1$, then we set $\beta_r^s = (1-s)\beta_r$. Otherwise $\beta_r^s = (1-s)(2\pi - \beta_r)$, i.e. we trace the complementary arc in the unit circle. By construction, $R^0 = R$ and $R^1 = 0$. Therefore $s \mapsto R^s \wedge Q^{\times l}$ is a homotopy between $R \wedge Q^{\times l}$ and $0 \wedge Q^{\times l}$. A moment of thought shows that for every s, the linear system defined by $R^s \wedge Q^{\times l}$ is non-degenerate by construction, exactly because we avoid the possible resonance which might happen if the arc traced by γ at some s hits 1 for some r. Notice that by construction

$$0 \leq |\beta_r^s| < 2\pi \implies \max_{(s,z)\in[0,1]\times B_\rho(0)} |R^s(z)| < 2\pi\rho^2$$

We thus define $F_{j,m}$ to be the interpolation at infinity of $H^0 = H^{p_{j+m}\ominus p_j}$ to $H^1 = G_j$ with the non-resonant path $s \mapsto R^s \wedge Q^{\times l}$. Notice that the initial radius of interpolation R_0 can be chosen independent of j and m, because we are working under the hypothesis that all p_{j+m}-periodic orbits of H are iterations of fixed points, so they are all contained in the same compact set. For this choice, the properties in equation (5.3) and equation 5.4 are clear.

The last thing to show is the estimate on the uniform distance of $F_{j,m}$ and G_j in equation (5.2). Again for notational convenience we are denoting $k = p_{j+m}$ and $l = p_j$. Recall the interpolation Hamiltonian $K_t(z) = R^{\chi(|z|^2)} \wedge Q^{\times l}(z)$. We write $F = 0 \wedge Q^{\times l} + 0 \wedge h^{\times l}$, $G = 0 \wedge Q^{\times l} + g$ where

$$g = K - 0 \wedge Q^{\times l} + \left(h^{\times(k-l)} \circ \varphi_{p^\mu}^{\cdot}\right) \wedge h^{\times l} = R^{\chi} \wedge 0 + \left(h^{\times(k-l)} \circ \varphi_{p^\mu}^{\cdot}\right) \wedge h^{\times l}$$

Notice that $z \mapsto \left(R^{\chi(|z|^2)} \wedge 0\right)(z)$ has support precisely $B_{R_1}(0)$. We can therefore estimate

$$\left\|F_{j,m} - G_j\right\|_{L^\infty} \leq \left\|0 \wedge h^{\times l} - g\right\|_{L^\infty} \leq 4(k-l)\|h\|_{L^\infty} + 2 \max_{(s,z)\in[0,1]\times B_{R_1}(0)} |R^s(z)| \leq$$
$$\leq 4(k-l)\|h\|_{L^\infty} + 4\pi R_1^2$$

We are finished if we can control R_1. Recall that $R_1 = e^{\frac{5C_1}{4C_0}} R_0$ where C_0, C_1 are defined in (5.8). It suffices to show that we can take C_0 and C_1 independent of j and m, because, as remarked above, R_0 clearly can be taken independent of j and m. This is clear for C_0, because $C_0 = \|\partial_s \mathbb{A}\|_{L^\infty} < 2\pi$ for our path $s \mapsto R^s \wedge Q^{\times l}$. The last is C_1. Our non-resonant path never touches the eigenvalue 1 for $s \in (0,1)$, but a priori the end-points might be approaching a resonance as we iterate. This is guaranteed not to happen, because we took the uniformly non-resonant iterations (p_j), for which the spectrum of $\varphi_\infty^{p_j}$ always remains at a fixed distance from 1. Therefore R_1 is also independent of j and m. This concludes the proof of the main Proposition 5.1.1.

Appendix A.

Rudimental Fredholm theory

A.1. Fredholm operators

In this section, when we say “Banach space”, we mean a real, *separable*, normed vector space which is complete. When X,Y are Banach spaces, we denote by $\mathscr{B}(X,Y)$ the space of continuous linear operators from X to Y, equipped with the operator norm, i.e. the strong topology.

Definition A.1 Let X,Y be Banach spaces. A bounded linear operator $D\colon X \to Y$ is said to be *Fredholm* if it has finite-dimensional kernel and cokernel. Its index is defined as the number

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D$$

The set of Fredholm operators is open in $\mathscr{B}(X,Y)$. The index is a locally constant function on it. The adjoint $D'\colon Y' \to X'$ of a Fredholm operator is always Fredholm, with index $\operatorname{ind} D' = -\operatorname{ind} D$.

Definition A.2 A linear operator $K\colon X \to Y$ is said to be *compact* when for any bounded subset $W \subset X$ the image KW has compact closure in Y. Compact operators are automatically bounded. The set of compact operators is closed in $\mathscr{B}(X,Y)$. The adjoint of a compact operator is always compact. The useful property of compact operators is that the image of any bounded sequence in X has a converging subsequence in Y.

Lemma A.1 *Let X,Y,Z be Banach spaces, $D\colon X \to Y$ a bounded linear operator, $K\colon X \to Z$ a compact operator. Assume that $\exists c > 0$ such that for any $x \in X$, we have an estimate*

$$||x||_X \le c\,(||Dx||_Y + ||Kx||_Z)$$

Then D has finite-dimensional kernel and closed image i.e. D is a pre-Fredholm *operator.*

Proof. $\ker D$ is finite dimensional. To see this topologically, we can show that the unit ball $B_1^{\ker D}(0) = \ker D \cap B_1^X(0)$ in $\ker D$ is compact. Let $(x_k)_k \subset B_1^{\ker D}(0)$ be a sequence. The estimate says that

$$||x_k||_X \le c\,||Kx_k||_Z \quad \forall k \in \mathbb{N}$$

Since $KB_1^{\ker D}(0)$ is pre-compact, the sequence $(Kx_k)_k$ has a subsequence $(Kx_{n_k})_k$ converging in the closure. Passing to x_{n_k}, the estimate tells us that it is Cauchy, hence it converges. So every sequence admits a converging subsequence, meaning that $B_1^{\ker D}(0)$ is compact.

To see that $\operatorname{im} D$ is closed, it suffices to show that if $(x_k)_k$ is a sequence in X such that $Dx_k = y_k \to y \in Y$, then $y \in \operatorname{im} D$. Now, (x_k) is either bounded or unbounded. If (x_k) is bounded, then (Kx_k) has a converging subsequence (Kx_{n_k}). Passing to this subsequence, it still holds that $y_{n_k} \to y$. The estimate tells us

$$\left\|x_{n_k}\right\|_X \le c\left(\left\|Dx_{n_k}\right\|_Y + \left\|Kx_{n_k}\right\|_Z\right)$$

Hence (x_{n_k}) is bounded by converging sequences, implying it is Cauchy. So it must converge to an $x \in X$ by completeness. Finally, since D is continuous, $Dx = y$. If (x_k) is not bounded, then it must have a norm-diverging subsequence, which by the sake of the argument can be taken as the full sequence. Since $\ker D$ is finite-dimensional, it has a complement in X, say $X = \ker D \oplus X_1$. If $x_k \in \ker D$ for all k, then there is nothing to prove, so we might as well take $x_k \in X_1$ for all $k \in \mathbb{N}$. Now set $u_k = x_k / \|x_k\| \in B_1^{X_1}(0)$. Clearly $\|Du_k\| \to 0$, and since (Ku_k) is pre-compact, it has a Cauchy subsequence. Hence the estimate again tells us that (u_k) has a converging subsequence. Now, the only possible limit is $u_k \to 0$ while $\|u_k\| = 1$ for all k. This contradiction shows that (x_k) cannot be an unbounded sequence. □

If X is a Banach space, we denote by X' its topological dual. If $D \in \mathscr{B}(X, Y)$, then an adjoint $D^*\colon Y' \to X'$ is defined naturally by the formula $(D^*\psi)x = \psi(Dx)$. This is clearly a linear operator. It is continuous in the *strong* topology on the duals. The following lemma is obvious.

Lemma A.2 *If $D\colon X \to Y$ and the adjoint $D^*\colon Y' \to X'$ are* both *pre-Fredholm operators, then they are both Fredholm operators.*

Towards regular values The following lemmata are used in the proof of the transversality theorem for the projection of the universal moduli space of Floer trajectories.

Lemma A.3 *Let X, Y, Z be Banach spaces, $D\colon X \to Y$ a Fredholm operator, $L\colon Z \to Y$ a bounded linear operator such that $D \oplus L\colon X \oplus Z \to Y$ is surjective. Then $D \oplus L$ has a right inverse.*

Proof. First, $\ker D$ is finite dimensional, so it has a complement X_1 in X – that is, there exists a closed Banach subspace X_1 such that $X_1 \oplus \ker D = X$. Then, $\operatorname{coker} D = Y / \operatorname{im} D$ is finite dimensional, so $\operatorname{im} D$ has a finite dimensional complement. Hence we may pick finitely many vectors $z_\nu \in Z$, $\nu = 1, \ldots, N$ such that $\operatorname{span}\{Lz_\nu : 1 \le \nu \le N\}$ is the complement in Y to $\operatorname{im} D$. Now, $D \oplus L$ is surjective so for each $y \in Y$ we can always choose $(x, z) \in X \oplus Z$ such that $y = Dx + Lz$. We may always pick $x = x_1 \in X_1$, because otherwise $Dx = 0$, and we may always pick $z = \sum_\nu \lambda_\nu z_\nu$ because otherwise $z \in \operatorname{im} D$. Hence we get the operator

$$\begin{aligned} Y &\to X \oplus Z \\ y &\mapsto \left(x_1, \sum_\nu \lambda_\nu z_\nu\right) \end{aligned}$$

which by our specific choices for each $y \in Y$ is obviously a right inverse to $D \oplus L$. □

Corollary A.1 *Let X, Y, Z be Banach spaces, $D: X \to Y$ a Fredholm operator, $L: Z \to Y$ a bounded linear operator. Then the operator $D \oplus L: X \oplus Z \to Y$ has closed image.*

Lemma A.4 *Let X, Y, Z be Banach spaces, $D: X \to Y$ a Fredholm operator and $L: Z \to Y$ a bounded linear operator, such that $D \oplus L: X \oplus Z \to Y$ is surjective. The operator*

$$\begin{aligned} P\colon \ker(D \oplus L) &\to Z \\ (x, z) &\mapsto z \end{aligned}$$

given by the projection $X \oplus Z \to Z$ restricted to $\ker(D \oplus L)$ *is Fredholm with kernel* $\ker D$ *and cokernel* $\operatorname{coker} D$. *In particular* $\operatorname{ind} D = \operatorname{ind} P$.

Proof. Obviously $\ker P \cong \ker D \oplus \{0\}$. So $\dim\ker P = \dim\ker D < \infty$. Moreover, its image is

$$\begin{aligned} \operatorname{im} P = &\{z \in Z : \exists x \in X \text{ s.t. } (x, z) \in \ker(D \oplus L)\} = \\ &= \{z \in Z : \exists x \in X \text{ s.t. } Lz + Dx = 0\} \cong L^{-1}(\operatorname{im} D) \end{aligned}$$

so by elementary linear algebra we get the chain of isomorphisms

$$\begin{aligned} \operatorname{coker} P = Z/\operatorname{im} P \cong Z/L^{-1}(\operatorname{im} D) \cong \operatorname{im} L/\operatorname{im} L \cap \operatorname{im} D \cong \\ \cong (\operatorname{im} L + \operatorname{im} D)/\operatorname{im} D = Y/\operatorname{im} D = \operatorname{coker} D \end{aligned}$$

Hence the kernel and cokernel of P are finite dimensional, P is Fredholm and with the same index of D. □

A.2. Non-linear Fredholm theory

Let $\mathscr{E}_0, \mathscr{E}_1$ be separable Banach manifolds and $f: \mathscr{E}_0 \to \mathscr{E}_1$ be a C^1 map. We say that f is a *Fredholm map* at $u \in \mathscr{E}_0$ if the tangent map $Tf|_u : T_u\mathscr{E}_0 \to T_{f(u)}\mathscr{E}_1$ is a Fredholm operator. We say that f is a *Fredholm map* if it is at every point $u \in \mathscr{E}_0$. The index of a Fredholm map is the Fredholm index of its tangent map, which is thus locally constant, since Tf is a continuous map of Banach bundles.

Fredholm maps transverse to submanifolds in their target cut out finite-dimensional submanifolds in their domain. As for its finite-dimensional counterpart, this property hinges on the inverse function theorem, which holds on (separable) Banach manifolds. Moreover, always in the Banach setting, there is a Sard-type theorem, called the Sard-Smale theorem, which guarantees that Fredholm maps can be generically put in a transverse situation. This is the main mechanism that is used to show that the moduli spaces of Floer trajectories are smooth finite-dimensional manifolds.

Remark In the Fréchet setting of spaces of *smooth* maps, the main theorems we cited above do not hold, specifically, there is no Fréchet inverse function theorem. This fact makes the Proposition 3.3.4 absolutely crucial for the analysis of the Floer equation. Indeed, we can work with a Banach ambient space but still obtain results which hold for the C^∞_{loc} topology on the space of solutions.

We state the two main theorems used in the transversality theory for the Floer equation. The first is a direct consequence of the implicit function theorem, and mimics the corresponding result in the finite-dimensional case.

Recall that a regular value of a smooth function $f : \mathscr{E}_0 \to \mathscr{E}_1$ between separable Banach manifolds is a point $v \in \mathscr{E}_1$ such that for all points $u \in f^{-1}(v)$, it holds that $Tf|_u : T_u\mathscr{E}_0 \to T_v\mathscr{E}_1$ is surjective and has a continuous right inverse, i.e. its kernel has a closed complementary subspace.

Theorem A.1 *Let $\mathscr{E}_0, \mathscr{E}_1$ be separable Banach manifolds and $f : \mathscr{E}_0 \to \mathscr{E}_1$ be a smooth map between them. If $v \in \mathscr{E}_1$ is a regular value of f, then $f^{-1}(v) \subset \mathscr{E}_0$ is a smooth submanifold.*

Notice that if f is Fredholm, the kernel of its linearization is always finite-dimensional, therefore always admits a closed complementary subspace. We conclude

Theorem A.2 *Let $\mathscr{E}_0, \mathscr{E}_1$ be separable Banach manifolds and $f : \mathscr{E}_0 \to \mathscr{E}_1$ be a smooth Fredholm map between them. If $v \in \mathscr{E}_1$ is a regular value of f, then $f^{-1}(v) \subset \mathscr{E}_0$ is a smooth finite-dimensional manifold, of dimension*

$$\dim f^{-1}(v) = \operatorname{ind} Tf|_u = \dim \ker Tf|_u$$

where u is any point in the fiber.

Finally, one may show that the set of regular values of a smooth Fredholm map is "large" in the target. This is the content of the famous Sard-Smale theorem, which we recall below.

Theorem A.3 (Sard-Smale theorem) *The set of regular values of a Fredholm map between separable Banach manifolds is residual in the target.*

Appendix B.

Equidistribution and prime iterates

Here we gather some lesser known facts about prime multiples of irrational numbers, and some simple calculations regarding the distribution of gaps of primes. These facts are used in a crucial manner for the construction of the uniformly non-resonant sequence of prime iterations, which is in turn crucial for the proof of the Poincaré-Birkhoff theorem.

We start with a number-theoretical definition from [30]. There it is referred to as *uniform distribution mod 1*, but we preferred to retain the nomenclature from dynamical systems.

Definition B.1 Let $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ be a sequence of points in $\mathbb{R}^n$. We say that (x_j) is *equidistributed mod* 1 when for any measurable subset $C \subset [0,1]^n$ of volume $|C|$, one has

$$\lim_{N\to\infty} \frac{\#\left\{x_j \bmod 1 : j \le N, x_j \in C\right\}}{N} = |C|$$

We are interested in the following criterion for equidistribution [30, Theorem 6.3]:

Proposition B.1 *The sequence $(x_j)_{j\in\mathbb{N}}$ is equidistributed mod 1 if and only if for every non-zero lattice point $h \in \mathbb{Z}^n \setminus \{0\}$ the real sequence $\left(\left\langle x_j, h\right\rangle\right)_j \subset \mathbb{R}$ is equidistributed mod 1.*

This result is combined with the following theorem of Vinogradov [41]:

Proposition B.2 *Let a be an irrational number. Enumerate the prime numbers in increasing order, $2 = P_1 < P_2 < \cdots < P_l < \cdots$. Then the sequence $(P_l a)_{l\in\mathbb{N}}$ is equidistributed mod 1.*

From these two results, we immediately obtain the following

Corollary B.1 *Let $\vec{a} = (a_1, \ldots, a_q)$, $q \ge 2$, be a vector of rationally independent irrational numbers. The sequence $(P_l\vec{a})_{l\in\mathbb{N}} \subset \mathbb{R}^m$ is equidistributed mod 1.*

Remark The condition for $\vec{a} = (a_1, \ldots, a_q)$ to be a vector of rationally independent irrational numbers is equivalent to saying that the set $\{1, a_1, \ldots, a_q\} \subset \mathbb{R}$ is rationally independent, so spanning a $\mathbb{Q}$-subspace of $\mathbb{R}$ of dimension $q+1$.

This corollary is the starting point for the construction of the uniformly non-resonant prime iteration sequence.

Lemma B.1 *Let $\vec{a} = (a_1, \ldots, a_q)$ be a vector of rationally independent irrational numbers. Let $C \subset [0,1]^q$ be a measurable subset of measure $|C|$. Denote by π the cumulative distribution of the prime numbers, i.e. $\pi(M) = \#\{P \text{ prime} : P \leq M\}$. Define*

$$\mathcal{V}_C(M) = \#\{P \text{ prime} : P \leq M,\ P\vec{a} \bmod 1 \in C\}$$

Then

$$\mathcal{V}_C(M) = |C|\pi(M) + o(\pi)$$

Proof. Let P_k be an enumeration of the prime numbers P such that $P\vec{a} \bmod 1 \in C$. By definition of equidistribution mod 1, we have that P_k is an infinite sequence of primes, and that

$$\lim_{N\to\infty} \frac{\#\{P_k : k \leq N,\ P_k\vec{a} \bmod 1 \in C\}}{N} = |C|$$

Define

$$L_N = \frac{\#\{P_k : k \leq N,\ P_k\vec{a} \bmod 1 \in C\}}{N}$$

Notice that the function $\pi\colon \mathbb{N} \to \mathbb{N}$ is a monotone increasing function. Therefore we can consider the subsequence $(L_{\pi(M)})_{M\in\mathbb{N}}$. We obtain thusly that

$$|C| = \lim_{M\to\infty} L_{\pi(M)} = \lim_{M\to\infty} \frac{\#\{P_k : k \leq \pi(M),\ P_k\vec{a} \bmod 1 \in C\}}{\pi(M)}$$

Now $k \leq \pi(M)$ if and only if $P_k \leq M$. Therefore we can forget k and obtain

$$|C| = \lim_{M\to\infty} \frac{\#\{P \text{ prime} : P \leq M,\ P\vec{a} \bmod 1 \in C\}}{\pi(M)}$$

which is our claim. □

Lemma B.2 *Let $\vec{a}$ and C be as above. Let $\left(p_j\right)_{j\in\mathbb{N}}$ be an increasing enumeration of the set $\{P \text{ prime} : P\vec{a} \bmod 1 \in C\}$. Then for any fixed $m \geq 1$ we have*

$$p_{j+m} - p_j = o\left(p_j\right) \text{ as } j \to \infty$$

Proof. Notice that

$$p_{j+m} - p_j = \sum_{k=0}^{m-1} p_{j+k+1} - p_{j+k}$$

therefore the claim is reduced to showing that

$$p_{j+1} - p_j = o\left(p_j\right) \text{ as } j \to \infty.$$

In order to do that, we use the prime number theorem:

$$\pi(N) = \frac{N}{\log N} + o\left(\frac{N}{\log N}\right)$$

where π is the prime counting function. Since $\mathscr{V}_C(N) = |C|\pi(N) + o(\pi)$, we also have that

$$\#\left\{j : p_j \leq N\right\} = \mathscr{V}_C(N) = |C|\frac{N}{\log N} + o\left(\frac{N}{\log N}\right)$$

In particular, we have that

$$\lim_{j\to\infty} \frac{p_j}{j \log j} = |C|$$

Hence we may compute

$$\lim_{j\to\infty} \frac{p_{j+1}}{p_j} = \lim_{j\to\infty} \frac{p_{j+1}}{(j+1)\log(j+1)} \frac{j \log j}{p_j} \frac{(j+1)\log(j+1)}{j \log j} = \frac{|C|}{|C|} \lim_{j\to\infty} \frac{(j+1)\log(j+1)}{j \log j} = 1$$

whence it follows that

$$\lim_{j\to\infty} \frac{p_{j+1} - p_j}{p_j} = \lim_{j\to\infty} \frac{p_{j+1}}{p_j} - 1 = 0$$

which was our claim. □

Appendix C.

Conventions

In the following, the split forms refer to the splitting of $\mathbb{R}^{2n}$ into symplectic 2-spaces $\mathbb{R}^{2n} \cong \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$ with corresponding coordinates $z = (z_1, \ldots, z_{2n}) = (q_1, p_1, \ldots, q_n, p_n)$.

1. $\omega_0 = \sum_i dq_i \wedge dp_i = \sum_{i=1}^{n-1} dz_i \wedge dz_{i+1}$.
2. $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus n}$.

Therefore we have the identity

$$\omega_0(v, w) = \langle J_0 v, w\rangle \tag{C.1}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product, $\langle v, w\rangle = v^T w$. Hence, an almost-complex structure J is compatible with ω_0 when the following is a Riemannian metric:

$$g_J = \omega_0 \circ (\mathbb{I} \times J)$$

The Hamilton equations for a Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$ are

$$i_{X_H}\omega_0 = dH$$

Combining this with the identity (C.1) we obtain the formula for the Hamiltonian vector field in these conventions:

$$X_H = -J_0 \nabla H$$

We often use the following radial anti-primitive of ω_0,

$$\lambda_0 = \frac{1}{2}\sum_i q_i dp_i - p_i dq_i, \quad \lambda_0 |_z (v) = \frac{1}{2}\langle z, J_0 v\rangle, \quad -d\lambda_0 = \omega_0$$

This gives us that the action of a curve $\gamma : [0,1] \to \mathbb{R}^{2n}$ is

$$\mathscr{A}_H(\gamma) = \int_0^1 \gamma^* \lambda_0 - H_t \circ \gamma(t) dt = \int_0^1 \frac{1}{2}\langle \gamma(t), J_0 \dot{\gamma}(t)\rangle - H(t, \gamma(t)) dt$$

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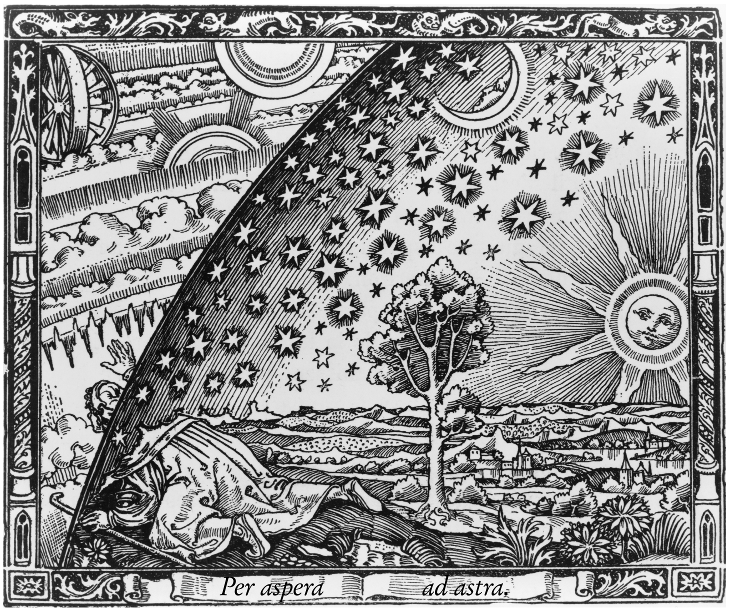
Unquantifiable amounts of gratitude are due to my family, my partner and my friends.

I dedicate this thesis to my grandmothers.



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Bibliography

[1] A. Abbondandolo. *Sub-harmonics for two-dimensional Hamiltonian systems*. *NoDEA, Nonlinear Differ. Equ. Appl.* 6.(**4**) (1999). DOI: 10.1007/s000300050007.

[2] A. Abbondandolo. *Morse theory for asymptotically linear Hamiltonian systems*. *Nonlinear Anal.* 39.(**8**) (2000). DOI: 10.1016/S0362-546X(98)00265-X.

[3] A. Abbondandolo. *Morse theory for Hamiltonian systems*. Vol. 425. Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[4] H. Amann and E. Zehnder. *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 7.(**4**) (1980).

[5] H. Amann and E. Zehnder. *Periodic solutions of asymptotically linear Hamiltonian systems*. *Manuscripta Math.* 32.(**1-2**) (1980). DOI: 10.1007/BF01298187.

[6] V. I. Arnol'd. *Mathematical methods of classical mechanics*. Vol. 60. Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.

[7] M. Audin and M. Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. DOI: 10.1007/978-1-4471-5496-9.

[8] P. Biran, L. Polterovich, and D. Salamon. *Propagation in Hamiltonian dynamics and relative symplectic homology*. English. *Duke Math. J.* 119.(**1**) (2003). DOI: 10.1215/S0012-7094-03-11913-4.

[9] G. D. Birkhoff. *Proof of Poincaré's geometric theorem*. *Trans. Am. Math. Soc.* 14 (1913). DOI: 10.2307/1988766.

[10] K. C. Chang. *Infinite-dimensional Morse theory and multiple solution problems*. Vol. 6. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, MA, 1993. DOI: 10.1007/978-1-4612-0385-8.

[11] C. C. Conley and E. Zehnder. *The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold*. *Invent. Math.* 73.(**1**) (1983). DOI: 10.1007/BF01393824.

[12] C. Conley and E. Zehnder. *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*. *Comm. Pure Appl. Math.* 37.(**2**) (1984). DOI: 10.1002/cpa.3160370204.

[13] D. Cristofaro-Gardiner et al. *Contact three-manifolds with exactly two simple Reeb orbits*. 2022. arXiv: 2102.04970 [math.SG].

[14] I. Ekeland. *Convexity methods in Hamiltonian mechanics*. Vol. 19. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1990. DOI: 10.1007/978-3-642-74331-3.

[15] A. Floer and H. Hofer. *Symplectic homology. I: Open sets in $\mathbb{C}^n$*. English. *Math. Z.* 215.(**1**) (1994). DOI: 10.1007/BF02571699.

[16] A. Floer. *The unregularized gradient flow of the symplectic action*. *Communications on Pure and Applied Mathematics* 41.(**6**) (1988). DOI: https://doi.org/10.1002/cpa.3160410603. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160410603.

[17] A. Floer, H. Hofer, and D. A. Salamon. *Transversality in elliptic Morse theory for the symplectic action*. *Duke Mathematical Journal* 80 (1995).

[18] U. Frauenfelder and O. van Koert. *The restricted three-body problem and holomorphic curves*. Pathways in Mathematics. Birkhäuser/Springer, Cham, 2018. DOI: 10.1007/978-3-319-72278-8.

[19] I. M. Gel'fand and V. B. Lidskiĭ. *On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients*. *Uspehi Mat. Nauk (N.S.)* 10.(**1(63)**) (1955).

[20] V. L. Ginzburg. *Coisotropic intersections*. *Duke Math. J.* 140.(**1**) (2007). DOI: 10.1215/S0012-7094-07-14014-6.

[21] V. L. Ginzburg. *The Conley conjecture*. *Ann. of Math. (2)* 172.(**2**) (2010). DOI: 10.4007/annals.2010.172.1129.

[22] V. L. Ginzburg and B. Z. Gürel. *Local Floer homology and the action gap*. *J. Symplectic Geom.* 8.(**3**) (2010). DOI: 10.4310/JSG.2010.v8.n3.a4.

[23] V. L. Ginzburg and B. Z. Gürel. *Pseudo-rotations versus rotations*. *J. Lond. Math. Soc. (2)* 106.(**4**) (2022). DOI: 10.1112/jlms.12665.

[24] V. L. Ginzburg and E. Kerman. *Homological resonances for Hamiltonian diffeomorphisms and Reeb flows*. *Int. Math. Res. Not. IMRN* (**1**) (2010). DOI: 10.1093/imrn/rnp120.

[25] B. Z. Gürel. *Periodic orbits of Hamiltonian systems linear and hyperbolic at infinity*. *Pacific J. Math.* 271.(**1**) (2014). DOI: 10.2140/pjm.2014.271.159.

[26] J. Gutt. *The Conley-Zehnder index for a path of symplectic matrices* (2012). arXiv: 1201.3728 [math.DG].

[27] N. Hingston. *Subharmonic solutions of Hamiltonian equations on tori*. *Ann. of Math. (2)* 170.(**2**) (2009). DOI: 10.4007/annals.2009.170.529.

[28] H. Hofer, K. Wysocki, and E. Zehnder. *Finite energy cylinders of small area*. *Ergodic Theory and Dynamical Systems* 22.(**5**) (2002). DOI: 10.1017/S0143385702001013.

[29] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2011. DOI: 10.1007/978-3-0348-0104-1.

[30] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[31] Y. Long. *A Maslov-type index theory for symplectic paths*. *Topol. Methods Nonlinear Anal.* 10.(**1**) (1997). DOI: 10.12775/TMNA.1997.021.

[32] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*. Second. Vol. 52. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2012.

[33] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Third Edition. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017. DOI: 10.1093/oso/9780198794899.001.0001.

[34] A. Moreno and O. van Koert. *A generalized Poincaré-Birkhoff theorem*. *J. Fixed Point Theory Appl.* 24.(**2**) (2022). DOI: 10.1007/s11784-022-00957-6.

[35] H. Poincaré. *Sur un théorème de géométrie*. *Rend. Circ. Mat. Palermo* 33 (1912). DOI: 10.1007/BF03015314.

[36] J. Robbin and D. Salamon. *The spectral flow and the Maslov index*. *Bull. London Math. Soc.* 27.(**1**) (1995). DOI: 10.1112/blms/27.1.1.

[37] D. Salamon. *Lectures on Floer homology*. *Symplectic Geometry and Topology*. Ed. by Y. Eliashberg and L. Traynor. IAS/Park City Mathematics Series. AMS, 1999. DOI: https://doi.org/10.1090/pcms/007.

[38] D. Salamon and E. Zehnder. *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*. *Comm. Pure Appl. Math.* 45.(**10**) (1992). DOI: 10.1002/cpa.3160451004.

[39] E. Shelukhin. *On the Hofer-Zehnder conjecture*. *Ann. of Math. (2)* 195.(**3**) (2022). DOI: 10.4007/annals.2022.195.3.1.

[40] R. Siefring. *Finite-energy pseudoholomorphic planes with multiple asymptotic limits*. *Math. Ann.* 368.(**1-2**) (2017). DOI: 10.1007/s00208-016-1478-y.

[41] I. M. Vinogradov. *The method of trigonometrical sums in the theory of numbers*. Dover Publications, Inc., Mineola, NY, 2004.

[42] V. A. Yakubovich and V. M. Starzhinskiĭ. *Linear Differential Equations with Periodic Coefficients 1*. John Wiley and Sons, New York-Toronto, 1975.

[43] E. Zehnder. *The beginnings of symplectic topology in Bochum in the early eighties*. English. *Jahresber. Dtsch. Math.-Ver.* 121.(**2**) (2019). DOI: 10.1365/s13291-019-00201-0.