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Bipartite q-Kneser graphs and two-generated irreducible linear groups

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Let $V := (\mathbb{F}_q)^d$ be a d-dimensional vector space over the field $\mathbb{F}_q$ of order q. Fix positive integers e_1, e_2 satisfying $e_1 + e_2 = d$. Motivated by analysing a fundamental algorithm in computational group theory for recognising classical groups, we consider a certain quantity $P(e_1, e_2)$ which arises in both graph theory and group representation theory: $P(e_1, e_2)$ is the proportion of 3-walks in the 'bipartite q-Kneser graph' Γ_{e_1,e_2} that are closed 3-arcs. We prove that, for a group G satisfying $\mathrm{SL}_d(q) \trianglelefteq G \leqslant \mathrm{GL}_d(q)$, the proportion of certain element-pairs in G called '(e_1, e_2)-stingray duos' which generate an irreducible subgroup is also equal to $P(e_1, e_2)$. We give an exact formula for $P(e_1, e_2)$, and prove that

$$1 - q^{-1} - q^{-2} < P(e_1, e_2) < 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-5}$$

for $2 \leqslant e_2 \leqslant e_1$ and $q \geqslant 2$. These bounds have implications for the complexity analysis of the state-of-the-art algorithms to recognise classical groups, which we discuss in the final section.

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1. Introduction

Let $V := (\mathbb{F}_q)^d$ be a d-dimensional vector space over the field $\mathbb{F}_q$ of order q. Given positive integers e_1, e_2 satisfying $e_1 + e_2 = d$, let X_i denote the set of e_i-dimensional subspaces of V for $i \in \{1, 2\}$. The *bipartite q-Kneser graph* $\Gamma = \Gamma_{e_1,e_2}$ has vertex set $V\Gamma = X_1 \dot{\cup} X_2$ (disjoint union) and edge set $E\Gamma = \{\{S_1, S_2\} \mid S_1 \in X_1, S_2 \in X_2, S_1 \cap S_2 = 0\}$. We emphasise that the subsets X_1, X_2 are regarded as disjoint vertex-subsets, even in the case $e_1 = e_2$. A 4-tuple $(S_0, S_1, S_2, S_3) \in X_2 \times X_1 \times X_2 \times X_1$ is called a 3-*walk* if $\{S_0, S_1\}$, $\{S_1, S_2\}$, $\{S_2, S_3\}$ are edges of Γ; a 3-walk is called *closed* if $\{S_3, S_0\}$ is also an edge, and it is called a 3-*arc* if $S_0 \neq S_2$ and $S_1 \neq S_3$. Define $P(e_1, e_2)$ to be the proportion of 3-walks of Γ_{e_1,e_2} which are closed 3-arcs. Our first result determines an exact formula for $P(e_1, e_2)$ in terms of the following function:

$$\omega(e) := \prod_{i=1}^{e} (1 - q^{-i}), \qquad \text{for } e \in \mathbb{Z} \text{ with } e \geqslant 1, \text{ and } \omega(0) = 1. \tag{1.1}$$

Theorem 1.1. *Let $d = e_1 + e_2$ with $1 \leqslant e_2 \leqslant e_1$. Let $q > 1$ be a prime power, and let $P(e_1, e_2)$ denote the proportion of 3-walks of Γ_{e_1,e_2} which are closed 3-arcs. Then*

$$\begin{aligned} P(e_1, e_2) &= -(1 - q^{-e_1 e_2})q^{-e_1 e_2} + \sum_{\ell=0}^{e_2-1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1 - e_2 + \ell)\omega(\ell)} \\ &= 1 - O(1/q). \end{aligned} \tag{1.2}$$

Although the exact formula (1.2) for the proportion $P(e_1, e_2)$ is intricate, it allows us to prove that $P(e_1, e_2)$ is close to $1 - q^{-1} - q^{-2}$. This requires very delicate calculations, especially for 'small' q. The lower bound given below was our primary objective to help understand the complexity of a probabilistic generation algorithm, especially for 'small' q.

Theorem 1.2. *If $e_2 = 1$, then*

$$\begin{aligned} &1 - q^{-1} - q^{-2} < P(e_1, 1) < 1 - q^{-1} \quad \text{for } e_1 \geqslant 3, \text{ and} \\ &1 - q^{-1} - q^{-2} < P(e_1, e_2) < 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-5} \quad \text{for } 2 \leqslant e_2 \leqslant e_1. \end{aligned}$$

The proportion $P(e_1, e_2)$ also arises when considering certain 2-generated irreducible subgroups of $\mathrm{GL}(V)$. To describe this key connection, we need additional terminology.

An element $g \neq 1$ of the general linear group $\mathrm{GL}(V) = \mathrm{GL}_d(q)$ is called an *e-stingray* element if g acts irreducibly on the image $U = V(g - 1) = \mathrm{im}(g - 1)$ of $g - 1$, and $\dim(U) = e$. A pair (g_1, g_2) of elements in $\mathrm{GL}(V)$ is called an *(e_1, e_2)-stingray duo* if g_i is an e_i-stingray element for $i \in \{1, 2\}$, and $U_1 \cap U_2 = \{0\}$ where $U_i = \mathrm{im}(g_i - 1)$. A pair $(g_1, g_2) \in G \times G$ is called *irreducible* if the 2-generated subgroup $\langle g_1, g_2 \rangle$ of $\mathrm{GL}(V)$ acts irreducibly, that is, the only subspaces of V invariant under $\langle g_1, g_2 \rangle$ are V

and $\{0\}$. Clearly, for an (e_1, e_2)-stingray duo (g_1, g_2), the subgroup $\langle g_1, g_2\rangle$ is reducible whenever $e_1 + e_2 < d$ as it fixes the proper subspace $U_1 + U_2$. The graph-theoretic invariant $P(e_1, e_2)$ described in Theorem 1.1 has a representation-theoretic interpretation as described below.

Theorem 1.3. *If $d = e_1 + e_2$ with $1 \leqslant e_2 \leqslant e_1$, and if $\mathrm{SL}_d(q) \trianglelefteqslant G \leqslant \mathrm{GL}_d(q)$, then with $P(e_1, e_2)$ as in (1.2),*

$$P(e_1, e_2) = \frac{\textit{Number of irreducible } (e_1, e_2)\textit{-stingray duos in } G \times G}{\textit{Number of } (e_1, e_2)\textit{-stingray duos in } G \times G}.$$

An (e_1, e_2)-stingray duo (g_1, g_2) is called *generating* if $\langle g_1, g_2\rangle$ contains the special linear group $\mathrm{SL}_d(q)$. For algorithmic purposes we are interested in the proportion of (e_1, e_2)-stingray duos that are generating. This will be a smaller proportion than the proportion $P(e_1, e_2)$ of stingray duos that are irreducible. We believe that this smaller proportion is also $1 - O(q^{-1})$, but a proof of this fact requires a very careful analysis of when $\langle g_1, g_2\rangle \cap \mathrm{SL}_d(q)$ is a proper subgroup of $\mathrm{SL}_d(q)$. When non-generation occurs, it is almost always because $\langle g_1, g_2\rangle$ is a reducible subgroup. By the previous two theorems, the proportion of (e_1, e_2)-stingray duos in $G \times G$ that are reducible is $1 - P(e_1, e_2) < q^{-1} + q^{-2}$. By contrast, we believe that the proportion of (e_1, e_2)-stingray duos that are non-generating and irreducible is substantially smaller: at most $O(q^{-ce_1e_2})$ for some constant c. This is known to be the case when $e_1 = e_2 = d/2$, see [14, Theorems 5 and 6] and Section 7, which is the case underpinning the algorithm presented in [4]. Theorem 1.2 and Corollary 7.4 lead to an improved complexity analysis for the algorithm in [4], for further details see the forthcoming paper [8].

The estimates obtained in this paper will be applied in [8] to prove a key theorem underpinning the complexity analysis of a new generation of recognition algorithms for classical groups. These new algorithms are described in Rademacher's PhD thesis [15] where, in addition, information is given on implementation details and comparative timings for the new algorithm against the current-best algorithm in [4] for various values of the dimension and field size. The latter indicate considerably improved running times for the new algorithm, especially as the dimension increases.

The bipartite q-Kneser graph Γ_{e_1,e_2} was also used to solve another problem that arises from computational group theory. Suppose that V is a $(e_1 + e_2)$-dimensional classical space (symplectic or orthogonal) over a finite field $\mathbb{F}_q$, and let Y_i be the set of *non-degenerate* e_i-subspaces of V (of a particular type in the orthogonal case). A pair $(U_1, U_2) \in Y_1 \times Y_2$ is called *spanning* if $V = U_1 + U_2$, that is to say, if $\{U_1, U_2\}$ is an edge of Γ_{e_1,e_2}. Thus the proportion of pairs $(U_1, U_2) \in Y_1 \times Y_2$ that are spanning equals the proportion of pairs $(U_1, U_2) \in Y_1 \times Y_2$ that are edges of the induced subgraph $[Y_1 \dot\cup Y_2]$ of Γ_{e_1,e_2} with vertex set $Y_1 \dot\cup Y_2$. It was shown in [6, Theorem 1.1] that this proportion is $1 - O(q^{-1})$, and a stronger estimate was obtained in [5, Theorem 1.1]: namely the proportion is at most $1 - \frac{3}{2q}$ unless $(e_1, e_2, q) = (1, 1, 2)$. (The bound also applies for a unitary space $(\mathbb{F}_{q^2})^d$ if we replace q with q^2.) The proof in [5] uses the

Bipartite Expander Mixing Lemma, and knowledge of the eigenvalues of the induced subgraph $[Y_1 \dot\cup Y_2]$ which are determined via a deep geometric algorithm [3] dating back to Brouwer [2], and the representation theory of the symmetric group. By contrast, the present paper uses elementary combinatorics, linear algebra and group theory to prove the above theorems.

The research in this paper sheds light on a very difficult part of a larger problem of recognising classical groups computationally. Much has been written on the computational complexity of classical recognition algorithms including [4,6,7,11,10,14], and with the benefit of hindsight, the 'right' proportions were not always studied. An overview of this larger picture, and some of the nuances, is discussed more fully in Section 7. In particular, by noting that some of these algorithms work with stingray duos instead of 'stingray pairs' we obtain probability estimates which allow the hypothesis $q > 4$ to be removed from [4, Theorem 1.2], and the hypothesis $q > 2$ to be removed from [14, Theorem 2]. Application of our Theorem 1.2 also improves the analogous upper bound for duos in the linear case in [14, Theorem 5].

Although a major motivation for this paper is analysing a classical recognition algorithm, including the case of special linear groups, we stress that this paper is *motivated by the linear case*. The hard problem for the analysis is showing that the subgroup $\langle g_1, g_2\rangle$, generated by a stingray duo (g_1, g_2), lies inside a proper subgroup of the quasisimple classical group, with low probability. It is shown in [8] that in the symplectic, unitary, and orthogonal cases the probability is at most $c_1 q^{-c_2 d}$ for positive constants c_1, c_2. By contrast, in the linear case, Theorems 1.2 and 1.3 of this paper imply that even the probability that $\langle g_1, g_2\rangle$ lies in a reducible subgroup of $\mathrm{GL}_d(q)$ is much higher, namely just less than $q^{-1} + q^{-2}$. Obtaining this highly accurate estimate has proved critical for giving complexity estimates for the recognition algorithm for $\mathrm{SL}_d(q)$ that are effective even for small values of q (which previous results were unable to handle).

In Section 2, we give a more general definition of a bipartite q-Kneser graph where $e_1 + e_2 \leqslant d$, but because of our applications, we focus on the case where $e_1 + e_2 = d$. We count the number of 3-walks and 3-arcs in Γ_{e_1,e_2}, see Remark 2.1 and Lemma 2.2. In Section 3, we analyse stingray pairs and duos. Lemma 3.5 gives criteria for a stingray duo (g_1, g_2) to be irreducible in terms of subspaces $U_1 = \mathrm{im}(g_1-1)$ and $U_2 = \mathrm{im}(g_2-1)$, and these are shown in Lemma 3.6 to hold precisely when we have a closed 3-arc in Γ_{e_1,e_2}. Section 4 counts the number of closed 3-walks and closed 3-arcs of Γ_{e_1,e_2}. These counts are used in Section 5 to compute the proportion $P(e_1, e_2)$ of 3-walks that are closed 3-arcs. The q-identity in Corollary 4.2 allows us to highlight the dominant terms of $P(e_1, e_2)$. This is used in Section 6 to prove the upper and lower bounds for $P(e_1, e_2)$ in Theorem 1.2. Finally, the computational context for this work is described in Section 7.

2. The q-Kneser graph

2.1. Walks and arcs in graphs

A *k-walk* of a graph Γ is a sequence $(v_0, v_1, \ldots, v_k)$ of $k+1$ vertices of Γ such that $\{v_{i-1}, v_i\}$ is an edge for $1 \leqslant i \leqslant k$. We call a k-walk $(v_0, v_1, \ldots, v_k)$ of Γ a *k-arc* if $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i < k$, see [1, p. 130]. For a k-arc, imagine 'walking' from v_0 to v_k via edges of Γ, where we cannot walk from v_i directly back to v_{i-1}, but we can walk to any other adjacent vertex. A 1-arc (v_0, v_1) is commonly called an *arc*, and is viewed as the edge $\{v_0, v_1\}$ directed from v_0 to v_1. A k-walk or a k-arc $(v_0, v_1, \ldots, v_k)$ is called *closed*[1] if $\{v_k, v_0\}$ is an edge of Γ. If Γ has n vertices and its adjacency matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicities), then it follows from [1, Additional Results 2h] that the number of closed k-walks of Γ equals $\sum_{i=1}^{n} \lambda_i^{k+1}$. We define the bipartite q-Kneser graph below, and show that the number of closed 3-walks can be counted *without* using eigenvalues.

If Γ is a bipartite graph, then a 3-arc has four *distinct* vertices, as the only edges join the 'left' vertices to the 'right' vertices of Γ.

2.2. Bipartite q-Kneser graphs

Let $V := (\mathbb{F}_q)^d$ be a d-dimensional vector space over the field $\mathbb{F}_q$ of order q. Given positive integers e_1, e_2 satisfying $e_1 + e_2 \leqslant d$, let

$$X_i \text{ denote the set of } e_i\text{-subspaces of } (\mathbb{F}_q)^d \text{ for } i \in \{1, 2\}. \tag{2.1}$$

We follow [5] and define the *bipartite q-Kneser graph* $\Gamma := \Gamma_{d,e_1,e_2}$. The vertex set $V\Gamma$ is the disjoint union $X_1 \dot\cup X_2$, and the edge set $E\Gamma$ comprises all 2-subsets $\{S_1, S_2\}$ with $S_1 \in X_1$, $S_2 \in X_2$ and $S_1 \cap S_2 = \{0\}$. If $e_1 + e_2 > d$, then Γ_{d,e_1,e_2} has no edges, and if $d = e_1 + e_2$, then we write Γ_{e_1,e_2} instead of Γ_{d,e_1,e_2}.

When $e_1 = e_2$ there is a *non-bipartite q-Kneser graph* $\widetilde{\Gamma}$ with vertex set $X_1 = X_2$ and $\{S_1, S_2\}$ is an edge precisely when $S_1 \cap S_2 = \{0\}$. The bipartite graph Γ is the standard bipartite double-cover of the non-bipartite graph $\widetilde{\Gamma}$, so that λ is an eigenvalue of $\widetilde{\Gamma}$ if and only if $\pm\sqrt{\lambda}$ are eigenvalues of Γ as explained in [5]. We henceforth consider Γ and not $\widetilde{\Gamma}$. We warn the reader that we sometimes refer to Γ simply as the *q-Kneser graph*, omitting the important adjective 'bipartite'.

Because of a link which we uncover in Section 3 between irreducible 2-generated linear groups and closed 3-arcs in Γ_{e_1,e_2} (see Lemma 3.6) we shall assume after Section 4 that $e_1 + e_2 = d$ holds. We refer to the quantity $\omega(e, q) = |\mathrm{GL}_e(q)|/q^{e^2}$ simply as $\omega(e)$, suppressing q, as per the definition given in (1.1). We count the number of 3-walks of

[1] Our definition of 'closed k-walk' differs from [1]. The $(k+2)$-tuple $(S_0, \ldots, S_k, S_0)$ is a closed $(k+1)$-*walk* according to Biggs [1, p. 12]. For us, the $(k+1)$-tuple $(S_0, \ldots, S_k)$ a closed k-*walk* when $\{S_k, S_0\}$ is an edge.

$\Gamma_{e_1+e_2,e_1,e_2}$, and the number of 3-arcs, in Lemma 2.2. The different types of 3-walks, with possible repeated vertices, are illustrated in Fig. 1; only the leftmost represents a 3-arc.

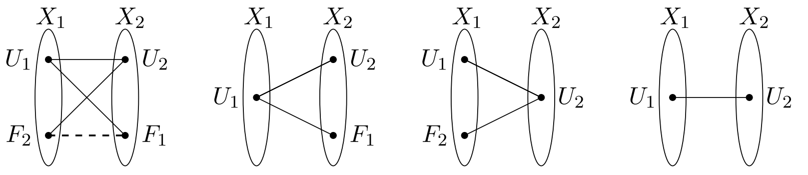


Fig. 1. 3-walks (F_1, U_1, U_2, F_2) with possible repeated vertices; in the second, third and fourth diagrams we have $U_1 = F_2$, $U_2 = F_1$ and $(U_1, U_2) = (F_2, F_1)$, respectively.

Remark 2.1. As Γ_{d,e_1,e_2} is a bipartite graph with vertex set $X_1 \dot\cup X_2$, there are two types of 3-walks: those in $W_1 := X_1 \times X_2 \times X_1 \times X_2$ and those in $W_2 := X_2 \times X_1 \times X_2 \times X_1$. The reversal map $(S_0, S_1, S_2, S_3) \mapsto (S_3, S_2, S_1, S_0)$ is a bijection $W_1 \to W_2$ which preserves 3-walks, 3-arcs, closed 3-walks, and closed 3-arcs. Hence the *proportion* of 3-walks (resp. 3-arcs) in Γ_{d,e_1,e_2} that are closed equals the proportion of 3-walks (resp. 3-arcs) in W_2 that are closed. This explains why we restrict to W_2 in Lemma 2.2 and Theorems 4.3, 4.4.

Lemma 2.2. *If $d = e_1 + e_2$, then the number of 3-walks in $X_2 \times X_1 \times X_2 \times X_1$ is $q^{4e_1e_2}\xi$, and the number of 3-arcs is $q^{4e_1e_2}\xi(1 - q^{-e_1e_2})^2$, where $\xi = \frac{\omega(e_1+e_2)}{\omega(e_1)\omega(e_2)}$ with $\omega(e)$ as in (1.1).*

Proof. We first count the number of 3-walks $(F_1, U_1, U_2, F_2) \in X_2 \times X_1 \times X_2 \times X_1$. The number of choices of F_1 is

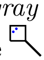
$$|X_2| = |\{\, e_2\text{-subspaces of} (\mathbb{F}_q)^{e_1+e_2}\,\}| = q^{e_1e_2}\xi \quad \text{where} \quad \xi := \frac{\omega(e_1 + e_2)}{\omega(e_1)\omega(e_2)}. \tag{2.2}$$

There are $q^{e_1e_2}$ choices for each of the complements U_1 of F_1, U_2 of U_1 and F_2 of U_2, so the number of 3-walks of Γ is $q^{4e_1e_2}\xi$ with ξ as in the statement. Similarly, the number of 3-arcs of Γ is $q^{e_1e_2}\xi q^{e_1e_2}(q^{e_1e_2} - 1)^2 = q^{4e_1e_2}\xi(1 - q^{-e_1e_2})^2$ as $U_2 \neq F_1$ and $F_2 \neq U_1$. □

Remark 2.3. For $d = e_1+e_2$ and $e_2 \leqslant e_1$ it is proved in [5] that Γ_{e_1,e_2} has $2(e_2+1)$ distinct eigenvalues and their values are $\pm\mu_0, \ldots, \pm\mu_{e_2}$ where $\mu_j = q^{e_1e_2-j(e_1+e_2-j)/2}$ for $0 \leqslant j \leqslant e_2$. Let M_j denote the *multiplicity* of the eigenvalue μ_j. This is also the multiplicity of $-\mu_j$. By [1, Additional Results 2h] the number of closed k-walks is $2\sum_{j=0}^{e_2} M_j\mu_j^{k+1}$ and hence the number of closed 3-walks in Γ_{e_1,e_2} is $2\sum_{j=0}^{e_2} M_j q^{4e_1e_2-2j(e_1+e_2-j)}$. Computing formulas for the μ_j is not elementary. Indeed, the proof in [5] uses the representation theory of the symmetric group, and a sophisticated geometric algorithm due to [2] and [3]. We shall compute both the number of closed 3-walks in Γ_{e_1,e_2}, and the number of closed 3-arcs, by using linear algebra and group theory only. This bypasses the need to calculate the multiplicities M_j, $0 \leqslant j \leqslant e_2$, which were not described in [5] for $j \neq 0$. Our proof involves elementary arguments only. $\square$

3. Stingray duos and the q-Kneser graph

We view $V := (\mathbb{F}_q)^d$ as the natural module for the general linear group $G_d := \mathrm{GL}_d(q)$. Let $e_1 + e_2 \leqslant d$ with each $e_i \geqslant 1$. Fix $g_1, g_2 \in \mathrm{GL}_d(q)$ and set $U_i := \mathrm{im}(g_i - 1)$, $F_i := \ker(g_i - 1)$ and $e_i = \dim(U_i)$ for $i \in \{1, 2\}$ where $\ker(x)$ denotes the kernel of a linear transformation x, and as in Section 1, $Vx = \mathrm{im}(x)$ denotes its image.

Definition 3.1. An element $g \in \mathrm{GL}(V)$ is called a *stingray element* if g acts irreducibly and non-trivially on $U := \mathrm{im}(g-1)$. A stingray element g is called an *e-stingray element* if $\dim(U) = e$. (The first author coined this term as the matrix of g looks like  relative to a suitable basis, where the body has $\dim(U)$ rows, and the tail has $\dim(V/U)$ rows.)

Definition 3.2. Given stingray elements $g_1, g_2 \in \mathrm{GL}(V)$ set $U_i := \mathrm{im}(g_i - 1)$ and $F_i := \ker(g_i - 1)$ for $i \in \{1, 2\}$. We call a pair (g_1, g_2) a *stingray pair*, or an (e_1, e_2)-*stingray pair*, if $\dim(U_1) = e_1$ and $\dim(U_2) = e_2$. We call a stingray pair a *stingray duo* if $U_1 \cap U_2 = \{0\}$.

Remark 3.3. If the minimal polynomial of $g \in \mathrm{GL}(V)$ is a product $a(t)b(t)$ of coprime polynomials, then $V = \ker(a(g)) \oplus \ker(b(g))$ and $\ker(a(g)) = \mathrm{im}(b(g))$. If g is an e-stingray element, then its minimal polynomial has this form with $a(t)$ irreducible and $b(t) = t - 1$, and we have $V = U \oplus F$ where $U = \ker(a(g))$ and $F = \ker(g - 1)$. If $\dim(V) = e_1 + e_2$, then an (e_1, e_2)-stingray duo has $V = U_1 \oplus U_2 = U_1 \oplus F_1 = U_2 \oplus F_2$, and hence $\{U_1, U_2\}$, $\{U_1, F_1\}$ and $\{U_2, F_2\}$ are edges of Γ_{e_1,e_2} and (F_1, U_1, U_2, F_2) is a 3-walk of Γ_{e_1,e_2}. $\square$

Definition 3.4. A stingray duo (g_1, g_2) is called an *irreducible stingray duo* if $\langle g_1, g_2 \rangle$ is an irreducible subgroup of $\mathrm{GL}(V)$, otherwise it is called a *reducible stingray duo*.

If (g_1, g_2) is an (e_1, e_2)-stingray duo in $\mathrm{GL}(V)$, then $e_1 + e_2 \leqslant d := \dim(V)$ since $U_1 \oplus U_2 \leqslant V$. The following lemma is reminiscent of [7, Lemma 3.7]. It characterises an irreducible stingray duo (g_1, g_2) in terms of the subspaces U_i, F_i. As $\langle g_1, g_2 \rangle$ fixes $U_1 + U_2$, an irreducible (e_1, e_2)-stingray duo must have $d = e_1 + e_2$ and $V = U_1 \oplus U_2$.

Lemma 3.5. *Suppose that (g_1, g_2) is an (e_1, e_2)-stingray duo in $\mathrm{GL}_d(q)$. Then the 2-generated subgroup $\langle g_1, g_2 \rangle$ of $\mathrm{GL}_d(q)$ acts irreducibly on $V = (\mathbb{F}_q)^d$ if and only if*

(a) $V = U_1 \oplus U_2$ (so $d = e_1 + e_2$),
(b) $F_1 \cap F_2 = \{0\}$, and
(c) $U_1 \neq F_2$ and $U_2 \neq F_1$.

Proof. Certainly g_i preserves the decomposition $V = U_i \oplus F_i$, and fixes F_i elementwise. Hence if $U_i \leqslant W \leqslant V$, then g_i fixes W. Thus U_1+U_2 is invariant under $G := \langle g_1, g_2\rangle$, and also G fixes $F_1 \cap F_2$ elementwise. Moreover, if $U_1 = F_2$, then G fixes F_2, and similarly if $U_2 = F_1$, then G fixes F_1. Hence if G is irreducible, then the conditions (a)-(c) all hold.

Conversely, suppose that (a)-(c) hold. We argue first that a g_i-invariant subspace Z of V satisfies $Z \leqslant F_i$ or $U_i \leqslant Z$. If some $z \in Z$ satisfies $z \notin F_i$, then $z(g_i - 1)$ is a non-zero element of U_i. However, g_i acts irreducibly on U_i and so $U_i \leqslant Z$ as claimed. Suppose now that W is a non-zero proper subspace invariant under $G := \langle g_1, g_2\rangle$. Then $W \leqslant F_1$ or $U_1 \leqslant W$, and $W \leqslant F_2$ or $U_2 \leqslant W$. It is not possible that $U_1 \leqslant W$ and $U_2 \leqslant W$ as then $V = U_1 + U_2 \leqslant W$ by (a). Also, it is not possible that $W \leqslant F_1$ and $W \leqslant F_2$ as then $W \leqslant F_1 \cap F_2 = \{0\}$ by (b). Hence either $W \leqslant F_1$ and $U_2 \leqslant W$, or $W \leqslant F_2$ and $U_1 \leqslant W$. In the former case, $U_2 \leqslant F_1$ and $d = \dim(V) = e_1 + e_2$ by (a), so that $e_2 = \dim(U_2) \leqslant \dim(F_1) = d - e_1 = e_2$ and hence $U_2 = F_1$. In the latter case, similar reasoning shows that $U_1 = F_2$. In either case, this contradicts (c). Therefore G preserves no proper non-zero subspace, and so G acts irreducibly on V as claimed. □

For the subgroup $\langle g_1, g_2\rangle$ to be irreducible, we require $d = e_1 + e_2$ by Lemma 3.5(a). Under this assumption we have multiple important links with the q-Kneser graph Γ_{e_1,e_2}.

Lemma 3.6. *Let $d = e_1 + e_2$. Let g_1, g_2 be stingray elements of $G_d = \mathrm{GL}_d(q)$. Write $e_i = \dim(\mathrm{im}(g_i - 1))$ and $\mathscr{C}_i = \{x^{-1}g_ix \mid x \in G_d\}$ for $i \in \{1, 2\}$.*

(a) For each i, the map $\phi_i : \mathscr{C}_i \to E\Gamma_{e_1,e_2} : g \mapsto \{\mathrm{im}(g-1), \ker(g-1)\}$ defines a surjection from $\mathscr{C}_i$ to the set $E\Gamma_{e_1,e_2}$ of edges of Γ_{e_1,e_2}.
(b) Consider the map $\psi : \mathscr{C}_1 \times \mathscr{C}_2 \to X_2 \times X_1 \times X_2 \times X_1 : (g_1', g_2') \mapsto (F_1', U_1', U_2', F_2')$, where $U_i' = \mathrm{im}(g_i' - 1)$ and $F_i' = \ker(g_i' - 1)$, for $i = 1, 2$.

  (i) The pair (g_1', g_2') is a stingray duo if and only if $\psi((g_1', g_2'))$ is a 3-walk in Γ_{e_1,e_2}.
  (ii) Restricting ψ to stingray duos yields a surjection onto the set of 3-walks in Γ_{e_1,e_2}.
  (iii) For an (e_1, e_2)-stingray duo (g_1', g_2'), the subgroup $\langle g_1', g_2'\rangle \leqslant \mathrm{GL}_d(q)$ acts irreducibly on $(\mathbb{F}_q)^d$ if and only if the image $\psi((g_1', g_2'))$ is a closed 3-arc in Γ_{e_1,e_2}.

Proof. (a) For $g \in \mathscr{C}_i$ we see that $\{\mathrm{im}(g - 1), \ker(g - 1)\}$ forms a direct decomposition of the vector space $V = (\mathbb{F}_q)^d$, and hence is an edge of Γ_{e_1,e_2} by Remark 3.3. Each edge arises as an image of some element of $\mathscr{C}_i$ under ϕ_i since $G_d = G_{e_1+e_2}$ acts transitively on the set of decompositions $V = U \oplus F$ with $\dim(U) = e_i$.

(b) Note first that each conjugate of an e-stingray element is also an e-stingray element, and a conjugate in $G_d \times G_d$ of an (e_1, e_2)-stingray duo (g_1, g_2) is also an (e_1, e_2)-stingray duo. Fix $(g_1', g_2') \in \mathscr{C}_1 \times \mathscr{C}_2$. Then, for each $i \in \{1, 2\}$, we have $U_i' \cap F_i' = \{0\}$ so $\{F_1', U_1'\}, \{U_2', F_2'\} \in E\Gamma_{e_1,e_2}$. Thus (F_1', U_1', U_2', F_2') is a 3-walk in Γ_{e_1,e_2} if and only if

$\{U_1', U_2'\}$ is also an edge, that is, if and only if $U_1' \cap U_2' = \{0\}$. It follows from Definition 3.2 that (F_1', U_1', U_2', F_2') is a 3-walk if and only if (g_1', g_2') is a stingray duo, proving part (i).

Suppose that the 3-walk (F_1', U_1', U_2', F_2') is the image of the stingray duo (g_1', g_2') in $\mathscr{C}_1 \times \mathscr{C}_2$ under ψ, and let (F_1, U_1, U_2, F_2) be an arbitrary 3-walk in Γ_{e_1,e_2}. Since G_d acts transitively on the decompositions $V = U_i' \oplus F_i'$ with $\dim(U_i') = e_i$, for each $i = 1, 2$, there exist $x_1, x_2 \in G_d$ such that $(F_1', U_1')^{x_1} = (F_1, U_1)$ and $(U_2', F_2')^{x_2} = (U_2, F_2)$. Then ψ maps the pair $((g_1')^{x_1}, (g_2')^{x_2}) \in \mathscr{C}_1 \times \mathscr{C}_2$ to (F_1, U_1, U_2, F_2), and by part (i), $((g_1')^{x_1}, (g_2')^{x_2})$ is a stingray duo. This proves part (ii).

Finally, we prove part (iii). Let (g_1', g_2') be a stingray duo, so $U_i' \cap F_i' = \{0\}$ and $\dim(F_i') = d - e_i$ for each $i \in \{1, 2\}$, and also $U_1' \cap U_2' = \{0\}$; and by part (i), (F_1', U_1', U_2', F_2') is a 3-walk. Suppose first that $\langle g_1', g_2' \rangle$ acts irreducibly on V. Then by Lemma 3.5(b), we have $F_1' \cap F_2' = \{0\}$ so $\{F_1', F_2'\}$ is also an edge of Γ_{e_1,e_2}, so the 3-walk (F_1', U_1', U_2', F_2') is closed; also $U_1' \neq F_2'$ and $U_2' \neq F_1'$ by Lemma 3.5(c), and therefore (F_1', U_1', U_2', F_2') is a 3-arc. Suppose conversely that (F_1', U_1', U_2', F_2') is a closed 3-arc in Γ_{e_1,e_2}. Then the vertices F_1', U_1', U_2', F_2' are pairwise distinct so Lemma 3.5(c) holds; $\{F_1', F_2'\}$ is an edge so Lemma 3.5(b) holds; and also the condition $V = U_1' \oplus U_2'$ of Lemma 3.5(a) holds since $\{U_1', U_2'\}$ is an edge. Thus $\langle g_1', g_2' \rangle$ acts irreducibly by Lemma 3.5. $\square$

We next prove that each of the maps ϕ_1, ϕ_2, ψ in Lemma 3.6 has fibres of constant size, and moreover that the G_d-conjugacy classes of stingray elements are also conjugacy classes for any group between $\mathrm{SL}_d(q)$ and G_d.

For an e-stingray element $g \in \mathrm{GL}(V)$ let $U(g) := \mathrm{im}(g - 1)$ and $F(g) := \ker(g - 1)$.

Lemma 3.7. *Let g be an e-stingray element of $G_d = \mathrm{GL}_d(q)$. For any subgroup G of $\mathrm{GL}_d(q)$ containing $\mathrm{SL}_d(q)$ let $\mathscr{C}_g(G) = \{x^{-1}gx \mid x \in G\}$ be the G-conjugacy class of $g \in G_d$. We allow $g \notin G$.*

(a) The G-conjugacy class $\mathscr{C}_g(G)$ is independent of the choice of G, and has size

$$|\mathscr{C}_g(G)| = \frac{|G_d|}{(q^e - 1) \cdot |G_{d-e}|}.$$

(b) The number of $g' \in \mathscr{C}_g(G)$ such that $(U(g), F(g)) = (U(g'), F(g'))$ is $|G_e|/(q^e - 1)$.

(c) For $i = 1, 2$, let g_i be an e_i-stingray element in G_d and let $\mathscr{C}_i$ be the G_d-conjugacy class containing g_i. Then the number of pairs $(g_1', g_2') \in \mathscr{C}_1 \times \mathscr{C}_2$ such that the 4-tuple $(F(g_1'), U(g_1'), U(g_2'), F(g_2'))$ equals $(F(g_1), U(g_1), U(g_2), F(g_2))$, is

$$\frac{|G_{e_1}| \cdot |G_{e_2}|}{(q^{e_1} - 1)(q^{e_2} - 1)}. \tag{3.1}$$

Proof. (a) Let h be the restriction of g to $U(g)$. Then h is irreducible on $U(g)$ by Definition 3.1, and $C_{\mathrm{GL}(U(g))}(h) = Z_{q^e-1}$ is cyclic of order $q^e - 1$ by [9, Satz II.7.3, p.187]. Therefore $X := C_{G_d}(g) = Z_{q^e-1}\times \mathrm{GL}(F(G))$. In particular, X contains a diagonal matrix of arbitrary non-zero determinant so $|\det(X)| = q-1$. Since G contains $\mathrm{SL}_d(q)$, this implies that $G_d = GX$, so $|\det(G)|$ divides $q-1$, so

$$|\mathscr{C}_g(G)| = \frac{|G|}{|C_G(g)|} = \frac{|G|}{|G\cap X|} = \frac{|GX|}{|X|} = \frac{|G_d|}{|X|} = |\mathscr{C}_g(G_d)|.$$

As $\mathscr{C}_g(G) \subseteq \mathscr{C}_g(G_d)$, equality holds and $|\mathscr{C}_g(G)| = \frac{|G_d|}{(q^e-1)|G_{d-e}|}$.

(b) By part (a), the number of $g' \in \mathscr{C}_g(G)$ such that $U(g') = U(g)$ and $F(g') = F(g)$ is the number of $g' \in \mathscr{C}_g(G_d)$ with this property. Thus we may, and shall, assume that $G = G_d$. Since g is an e-stingray element, we have $V = U(g) \oplus F(g)$ with $\dim(U(g)) = e$. If $g' = g^x$ where $x \in G$, we have $U(g') = U(g)x$ as

$$U(g') = V(g'-1) = V(x^{-1}gx-1) = Vx^{-1}(g-1)x = V(g-1)x = U(g)x.$$

Similarly $F(g') = F(g)x$. Thus the number of $g' \in \mathscr{C}_g(G)$ such that $U(g') = U(g)$ and $F(g') = F(g)$ is equal to the number of choices for $x \in G$ with $U(g)x = U(g)$ and $F(g)x = F(g)$ divided by $|C_{G_d}(g)|$. As $G = G_d$, this equals $(|\mathrm{GL}(U(g))\times \mathrm{GL}(F(g))|)/|X|$ with $X = C_{G_d}(g) = Z_{q^e-1}\times \mathrm{GL}(F(G))$ as in part (a). Therefore, the number we seek is $|\mathrm{GL}(U(g))|/(q^e-1) = |G_e|/(q^e-1)$, and part (b) is proved.

(c) By part (b) there are $|\mathrm{GL}_{e_1}(q)|/(q^{e_1}-1)$ choices for $g_1' \in \mathscr{C}_1$ and $|\mathrm{GL}_{e_2}(q)|/(q^{e_2}-1)$ choices for $g_2' \in \mathscr{C}_2$. Hence the number of pairs $(g_1', g_2') \in \mathscr{C}_1 \times \mathscr{C}_2$ is as claimed. □

4. Closed 3-walks and closed 3-arcs of Γ_{e_1,e_2}

In the light of Lemma 3.5 we shall assume henceforth that $e_1 + e_2 = d$ and therefore $V = \mathbb{F}_q^{e_1+e_2}$. In this section we count the number of closed 3-walks and closed 3-arcs of Γ_{e_1,e_2}. This is twice the number that occur in $X_2 \times X_1 \times X_2 \times X_1$ by Remark 2.1. Recall from (2.1) that X_i denotes the set of e_i-subspaces of $V = (\mathbb{F}_q)^{e_1+e_2}$ for $i \in \{1, 2\}$. It will be convenient to represent a subspace of V as the row space of a block matrix. Let $M_{e\times d}$ denote the vector space of $e \times d$ matrices over $\mathbb{F}_q$. As the general linear group G_d is transitive on the set of decompositions $V = U_1 \oplus U_2$ where $U_1 \in X_1$ and $U_2 \in X_2$, we write U_1 as the row space of $(I \mid 0) \in M_{e_1\times d}$, and U_2 as the row space of $(0 \mid I) \in M_{e_2\times d}$. We use the shorthand $U_1 = \mathrm{RS}(I \mid 0)$ and $U_2 = \mathrm{RS}(0 \mid I)$ where the number of rows of I and 0 can be inferred from $\dim(U_i) = e_i$. The number of columns of 0 can be inferred, as I is always a square matrix, and there are d columns in total. Therefore $U_1 = \mathrm{RS}(I \mid 0)$ has $0 \in M_{e_1\times e_2}$ and $U_2 = \mathrm{RS}(0 \mid I)$ has $0 \in M_{e_2\times e_1}$.

We define the action of the group $G_{e_2} \times G_{e_1}$ on a matrix $A \in M_{e_2\times e_1}$, and on a pair $(A, B) \in M_{e_2\times e_1} \times M_{e_1\times e_2}$: for $(X, Y) \in G_{e_2} \times G_{e_1}$ write

$$A^{(X,Y)} = X^{-1}AY \qquad \text{and} \qquad (A, B)^{(X,Y)} = (X^{-1}AY, Y^{-1}BX). \tag{4.1}$$

(These are actions as $(A^{(X_1,Y_1)})^{(X_2,Y_2)} = A^{(X_1X_2,Y_1Y_2)}$ and $((A,B)^{(X_1,Y_1)})^{(X_2,Y_2)} = (A,B)^{(X_1X_2,Y_1Y_2)}$.)

Since the rank of A equals the dimension of the row space of A, and the dimension of the column space of A, it follows that A and $X^{-1}AY$ have the same rank. Indeed, the set of matrices of a given rank forms an orbit under the group $G_{e_2} \times G_{e_1}$. Since $e_2 \leqslant e_1$, it follows that $G_{e_2} \times G_{e_1}$ has $e_2 + 1$ orbits on $M_{e_2 \times e_1}$, and $G_{e_1} \times G_{e_2}$ has $e_2 + 1$ orbits on $M_{e_1 \times e_2}$. Denote the set of $e_2 \times e_1$ matrices rank k over $\mathbb{F}_q$ by $M^{(k)}_{e_2 \times e_1}$. Let $R^{(k)}_{e_2 \times e_1}$ be $(G_{e_2} \times G_{e_1})$-orbit representatives for $M^{(k)}_{e_2 \times e_1}$ where

$$R^{(k)}_{e_2 \times e_1} := \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M^{(k)}_{e_2 \times e_1} \subseteq M_{e_2 \times e_1} \qquad \text{for } 0 \leqslant k \leqslant e_2. \tag{4.2}$$

The matrices in the second row of $R^{(k)}_{e_2 \times e_1}$ have $e_2 - k$ rows, and those in the second column have $e_1 - k$ columns. In particular, $R^{(0)}_{e_2 \times e_1}$ is the $e_2 \times e_1$ zero matrix.

The cardinality $|M^{(k)}_{e_2 \times e_1}|$ is known, see Morrison [12, §1.7]. We give a short proof to describe the structure of a stabiliser and to introduce our notation in the next lemma.

Lemma 4.1. *If $0 \leqslant k \leqslant e_2 \leqslant e_1$, then the stabiliser of $R^{(k)}_{e_2 \times e_1}$ in $G_{e_2} \times G_{e_1}$ equals*

$$H_k = \left\{ (X,Y) \in G_{e_2} \times G_{e_1} \mid X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix} \text{ and } Y = \begin{pmatrix} X_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix} \right\} \text{ where}$$

$X_{11} = Y_{11} \in G_k, X_{22} \in G_{e_2-k}, Y_{22} \in G_{e_1-k}, X_{12} \in M_{k \times (e_2-k)}$ and $Y_{21} \in M_{(e_1-k) \times k}$. Hence the number of rank-k matrices in $M_{e_2 \times e_1}$ is $|M^{(k)}_{e_2 \times e_1}| = |G_{e_1} \times G_{e_2}|/|H_k|$ where $|H_k|$ equals $|G_k||G_{e_1-k}||G_{e_2-k}|q^{k(e_1+e_2-2k)}$.

Proof. The stabiliser H_k of $R^{(k)}_{e_2 \times e_1}$ in $G_{e_2} \times G_{e_1}$ is easy to compute because the identity $R^{(k)}_{e_2 \times e_1} = (R^{(k)}_{e_2 \times e_1})^{(X,Y)}$ says that $XR^{(k)}_{e_2 \times e_1} = R^{(k)}_{e_2 \times e_1}Y$. In terms of matrices, this says

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right) \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \quad \text{that is} \quad \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ 0 & 0 \end{pmatrix}.$$

Hence $X_{21} = 0$, $Y_{12} = 0$ and $X_{11} = Y_{11} \in G_k$. Thus $X_{22} \in G_{e_2-k}$ and $Y_{22} \in G_{e_1-k}$. Also $X_{12} \in M_{k \times (e_2-k)}$ and $Y_{21} \in M_{(e_1-k) \times k}$ are arbitrary. It follows that the stabiliser H_k of $R^{(k)}_{e_2 \times e_1}$ is as claimed, and its order is $|G_k||G_{e_1-k}||G_{e_2-k}|q^{k(e_1+e_2-2k)}$. By the orbit-stabiliser lemma the set of rank-k matrices in $M_{e_2 \times e_1}$ has cardinality $|G_{e_2} \times G_{e_1}|/|H_k|$. $\square$

When counting algebraic objects over $\mathbb{F}_q$, factoring out the dominant power of q determines the asymptotic behaviour as $q \to \infty$. For example, $|\mathrm{GL}_e(q)| = q^{e^2}\omega(e)$ using (1.1), and $\omega(e) = \prod_{i=0}^{e-1}(1-q^{-i}) = 1-O(q^{-1})$ so $|\mathrm{GL}_e(q)| \sim q^{e^2}$ as $q \to \infty$. By Lemma 4.1, $|M^{(k)}_{e_2 \times e_1}| \sim q^x$ where $x = e_2^2+e_1^2-k^2-(e_1-k)^2-(e_2-k)^2-k(e_1+e_2-2k) = k(e_1+e_2-k)$. A precise formula for $|M^{(k)}_{e_2 \times e_1}|$ follows from Lemma 4.1:

$$|M^{(k)}_{e_2 \times e_1}| = \frac{\omega(e_2)\omega(e_1)q^{k(e_1+e_2-k)}}{\omega(k)\omega(e_1-k)\omega(e_2-k)} \qquad \text{where } 0 \leqslant k \leqslant e_2. \tag{4.3}$$

Corollary 4.2. *If $1 \leqslant e_2 \leqslant e_1$ and $q > 1$ is a prime power, then using (1.1),*

$$\sum_{k=0}^{e_2} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-k)(e_2-k)}}{\omega(k)\omega(e_1-k)\omega(e_2-k)} = \sum_{\ell=0}^{e_2} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_2-\ell)\omega(e_1-e_2+\ell)\omega(\ell)} = 1.$$

Proof. Since $\sum_{k=0}^{e_2} |M_{e_2\times e_1}^{(k)}| = |M_{e_2\times e_1}|$, we have $\sum_{k=0}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{k(e_1+e_2-k)}}{\omega(k)\omega(e_1-k)\omega(e_2-k)} = q^{e_1e_2}$ by (4.3). Setting $\ell = e_2 - k$, the result follows from

$$-e_1e_2 + k(e_1+e_2-k) = -(e_1-k)(e_2-k) = -(e_1-e_2+\ell)\ell. \quad \square$$

We are now ready to count the closed 3-walks.

Theorem 4.3. *Suppose that $d = e_1 + e_2$ and $q > 1$. Then the number of closed 3-walks $(F_1, U_1, U_2, F_2) \in X_2 \times X_1 \times X_2 \times X_1$ in the bipartite q-Kneser graph Γ_{e_1,e_2} equals*

$$q^{4e_1e_2}\omega(e_1+e_2)\sum_{\ell=0}^{e_2} \frac{q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)}.$$

Proof. As G_d is transitive on decompositions $V = U_1 \oplus U_2$, we may assume that $U_1 = \mathrm{RS}(I \mid 0) \in X_1$ and $U_2 = \mathrm{RS}(0 \mid I) \in X_2$. There are $q^{e_1e_2}$ complements $F_1 \in X_2$ to U_1 in V and these can be written uniquely as $F_1 = \mathrm{RS}(A \mid I)$ where $A \in M_{e_2\times e_1}$ and $I = I_{e_2}$ has e_2 rows (and columns). Similarly, the $q^{e_1e_2}$ complements of U_2 can be written uniquely as $F_2 = \mathrm{RS}(I \mid B) \in X_1$ where $B \in M_{e_1\times e_2}$. The number of pairs (U_1, U_2) is

$$\frac{|G_{e_1+e_2}|}{|G_{e_1} \times G_{e_2}|} = q^{2e_1e_2}\xi \qquad \text{where } \xi := \frac{\omega(e_1+e_2)}{\omega(e_1)\omega(e_2)}.$$

Hence the number of 3-walks $(F_1, U_1, U_2, F_2) \in X_2 \times X_1 \times X_2 \times X_1$ is $q^{4e_1e_2}\xi$ (as we saw in Lemma 2.2). The number of closed 3-walks in $X_2 \times X_1 \times X_2 \times X_1$ is less than this, namely $N \cdot q^{2e_1e_2}\xi$ where N is the number of choices of pairs (F_1, F_2), given (U_1, U_2), such that $V = U_1 \oplus F_1 = U_2 \oplus F_2 = F_1 \oplus F_2$. We now find N.

The following are equivalent: $\{F_1, F_2\}$ is an edge of Γ_{e_1,e_2}; $F_1 \cap F_2$ equals $\{0\}$; and $F_1 + F_2$ equals V. In terms of matrices, each of these conditions is equivalent to the constraint that the $d\times d$ matrix $\begin{pmatrix} A & I \\ I & B \end{pmatrix}$ is invertible. However, $\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}\begin{pmatrix} A & I \\ I & B \end{pmatrix} = \begin{pmatrix} 0 & I-AB \\ I & B \end{pmatrix}$. Therefore the condition that $\{F_1, F_2\}$ lies in $E\Gamma_{e_1,e_2}$ is equivalent to the condition that $I_{e_2} - AB$ is invertible. It is possible for $I_{e_2} - AB$ to have full rank e_2, and lie in G_{e_2}, due to our assumption that $e_2 \leqslant e_1$.

Thus we must determine the number N of pairs $(A, B) \in M_{e_2\times e_1} \times M_{e_1\times e_2}$ for which $I - AB$ is an invertible $e_2 \times e_2$ matrix. Recall the action (4.1) of $(X, Y) \in G_{e_2} \times G_{e_1}$ on a pair (A, B) in $M_{e_2\times e_1} \times M_{e_1\times e_2}$. This action preserves pairs (A, B) with $I - AB$ invertible since $(A', B') = (A, B)^{(X,Y)}$ implies

$$I - A'B' = I - (X^{-1}AY)(Y^{-1}BX) = I - X^{-1}ABX = X^{-1}(I - AB)X,$$

and hence $\det(I - A'B') = \det(I - AB)$. Therefore $I - AB$ is invertible precisely when $I - A'B'$ is invertible, and

$$N = \sum_{k=0}^{e_2} \#\{A \mid A \in M^{(k)}_{e_2\times e_1}\} \cdot \#\{B \in M_{e_1\times e_2} \mid I - AB \in G_{e_2}\}.$$

For $A \in M^{(k)}_{e_2\times e_1}$, we may choose $(X, Y) \in G_{e_2} \times G_{e_1}$ so that $X^{-1}AY = R^{(k)}_{e_2\times e_1}$ is the rank-k representative defined by (4.2). Therefore,

$$N = \sum_{k=0}^{e_2} |M^{(k)}_{e_2\times e_1}| \cdot \#\{B \in M_{e_1\times e_2} \mid I - R^{(k)}_{e_2\times e_1}B \in G_{e_2}\}.$$

Write $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ where $B_{11} \in M_{k\times k}$. Since $I - R^{(k)}_{e_2\times e_1}B = \begin{pmatrix} I-B_{11} & -B_{12} \\ 0 & I \end{pmatrix}$, a necessary and sufficient condition for $I - R^{(k)}_{e_2\times e_1}B$ to be invertible is that $I - B_{11} \in G_k$. The matrices B_{12}, B_{21}, B_{22} may be chosen arbitrarily. However, the number of matrices B_{11} with $I - B_{11} \in G_k$ is the number of $k \times k$ matrices not having 1 as an eigenvalue. This equals the number of $k\times k$ matrices not having 0 as an eigenvalue, that is $|G_k| = q^{k^2}\omega(k)$. In summary, the number of choices for $B_{11}, B_{12}, B_{21}, B_{22}$ is $q^{k^2}\omega(k)$, $q^{k(e_2-k)}$, $q^{(e_1-k)k}$, $q^{(e_1-k)(e_2-k)}$, respectively. Consequently, there are $q^{e_1e_2}\omega(k)$ choices for B.

The formula (4.3) for $|M^{(k)}_{e_2\times e_1}|$ shows that

$$N = \sum_{k=0}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{k(e_1+e_2-k)}}{\omega(k)\omega(e_1-k)\omega(e_2-k)} \cdot q^{e_1e_2}\omega(k) = \sum_{k=0}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{e_1e_2+k(e_1+e_2-k)}}{\omega(e_1-k)\omega(e_2-k)}.$$

Multiplying by the number $q^{2e_1e_2}\xi$ of pairs (U_1, U_2), the number of closed 3-walks is

$$\frac{q^{2e_1e_2}\omega(e_1+e_2)}{\omega(e_1)\omega(e_2)} \sum_{k=0}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{e_1e_2+k(e_1+e_2-k)}}{\omega(e_1-k)\omega(e_2-k)}.$$

Observing that $e_1e_2 + k(e_1 + e_2 - k) = 2e_1e_2 - (e_1 - k)(e_2 - k)$, this equals

$$q^{4e_1e_2}\omega(e_1+e_2) \sum_{k=0}^{e_2} \frac{q^{-(e_1-k)(e_2-k)}}{\omega(e_1-k)\omega(e_2-k)} = q^{4e_1e_2}\omega(e_1+e_2) \sum_{\ell=0}^{e_2} \frac{q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)}$$

where in the last step $\ell := e_2 - k$ ranges from e_2 down to 0. □

Now we determine the number of closed 3-arcs.

Theorem 4.4. *Suppose that $1 \leqslant e_2 \leqslant e_1$ and $q > 1$ is a prime power. Then the number of closed 3-arcs $(F_1, U_1, U_2, F_2) \in X_2 \times X_1 \times X_2 \times X_1$ in the q-Kneser graph Γ_{e_1,e_2} equals*

$$q^{4e_1e_2}\omega(e_1+e_2) \sum_{\ell=0}^{e_2-1} \frac{q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)} \left(1 - \frac{q^{-e_1e_2}}{\omega(e_2-\ell)}\right).$$

Proof. Our proof uses the arguments and the notation of Theorem 4.3. We count the closed 3-arcs (F_1, U_1, U_2, F_2) with $V = U_1 \oplus U_2$. There are $q^{2e_1e_2}\xi$ choices for the pair (U_1, U_2) in $X_1 \times X_2$. Each choice of (U_1, U_2) gives rise to the same number of choices of (F_1, F_2) in $X_2 \times X_1$ such that (F_1, U_1, U_2, F_2) is a closed 3-arc. Hence we take $U_1 = \mathrm{RS}(I \mid 0)$ and $U_2 = \mathrm{RS}(0 \mid I)$. Following the proof of Theorem 4.3, we write $F_1 = \mathrm{RS}(A \mid I)$ and $F_2 = \mathrm{RS}(I \mid B)$ where $A \in M_{e_2\times e_1}$ and $B \in M_{e_1\times e_2}$. We require that $F_1 \neq U_2$, $F_2 \neq U_1$, and that $I - AB \in M_{e_2\times e_2}$ is invertible. That is, $A \neq 0$, $B \neq 0$ and $I - AB \in G_{e_2}$.

Suppose that A has rank k. As $A \neq 0$ and $e_2 \leqslant e_1$, we have $|M^{(k)}_{e_2\times e_1}|$ choices for A where $k \in \{1, \dots, e_2\}$. Each $A \in M^{(k)}_{e_2\times e_1}$ gives the same number of choices for B, so we may assume that A equals $R^{(k)}_{e_2\times e_1}$ as per (4.2). Writing $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $B_{11} \in M_{k\times k}$, we require that $B \neq 0$ and $I - B_{11} \in G_k$. If $B_{11} = 0$, there are $q^{e_1e_2-k^2} - 1$ choices for B, namely any non-zero matrix $B = \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$. If $B_{11} \neq 0$, then $B_{11} = I - g$ where $g \in G_k$ is non-identity and B_{12}, B_{21}, B_{22} are arbitrary, so the number of choices for B is

$$\left(q^{k^2}\omega(k) - 1\right) \cdot q^{k(e_2-k)} \cdot q^{(e_1-k)k} \cdot q^{(e_1-k)(e_2-k)} = q^{e_1e_2}\left(\omega(k) - q^{-k^2}\right).$$

Hence for each $A \in M^{(k)}_{e_2\times e_1}$ the total number of choices for $B \in M_{e_1\times e_2}$ is

$$q^{e_1e_2}\omega(k)x_k \qquad \text{where} \quad x_k := 1 - \frac{q^{-e_1e_2}}{\omega(k)}.$$

Therefore, the number of pairs (F_1, F_2) is the number of pairs (A, B) namely (using the formula (4.3))

$$\sum_{k=1}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{k(e_1+e_2-k)}}{\omega(k)\omega(e_1-k)\omega(e_2-k)} \cdot q^{e_1e_2}\omega(k)x_k = \sum_{k=1}^{e_2} \frac{\omega(e_2)\omega(e_1)q^{e_1e_2+k(e_1+e_2-k)}x_k}{\omega(e_1-k)\omega(e_2-k)}.$$

Multiplying by the number $q^{2e_1e_2}\xi$ of pairs (U_1, U_2), the number of closed 3-arcs is

$$q^{2e_1e_2}\omega(e_1+e_2)\sum_{k=1}^{e_2} \frac{q^{e_1e_2+k(e_1+e_2-k)}x_k}{\omega(e_1-k)\omega(e_2-k)}.$$

Observing that $e_1e_2 + k(e_1 + e_2 - k) = 2e_1e_2 - (e_1 - k)(e_2 - k)$, this equals

$$q^{4e_1e_2}\omega(e_1+e_2)\sum_{k=1}^{e_2} \frac{q^{-(e_1-k)(e_2-k)}x_k}{\omega(e_1-k)\omega(e_2-k)} = q^{4e_1e_2}\omega(e_1+e_2)\sum_{\ell=0}^{e_2-1} \frac{q^{-(e_1-e_2+\ell)\ell}x_{e_2-\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)}$$

where in the last step $\ell := e_2 - k$ ranges from $e_2 - 1$ down to 0. □

4.1. Proof of Theorem 1.1: an explicit formula for $P(e_1, e_2)$

We finish this section by drawing together the results on 3-walks and 3-arcs in Γ_{e_1,e_2} to prove Theorem 1.1. Thus we assume that $q > 1$ is a prime power and $d = e_1 + e_2$

with $1 \leqslant e_2 \leqslant e_1$, and we consider the proportion $P(e_1, e_2)$ of 3-walks of Γ_{e_1,e_2} which are closed 3-arcs. To prove Theorem 1.1 we must prove that (1.2) holds for $P(e_1, e_2)$, that is,

$$P(e_1, e_2) = -(1 - q^{-e_1e_2})q^{-e_1e_2} + \sum_{\ell=0}^{e_2-1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1 - e_2 + \ell)\omega(\ell)} = 1 - O(1/q).$$

In addition, it suffices to consider 3-walks and 3-arcs in $X_2 \times X_1 \times X_2 \times X_1$ by Remark 2.1. Let $V = (\mathbb{F}_q)^d$ and note that we may choose decompositions $V = (\mathbb{F}_q)^d = U_1 \oplus F_1 = U_2 \oplus F_2$ with $U_1, F_2 \in X_1$ of dimension e_1 and $U_2, F_1 \in X_2$ of dimension e_2.

The number n of closed 3-arcs in $W_2 := X_2 \times X_1 \times X_2 \times X_1$ is then given by Theorem 4.4, and the number of 3-walks in W_2 is $q^{4e_1e_2}\xi$ by Lemma 2.2 where $\xi = \frac{\omega(e_1+e_2)}{\omega(e_1)\omega(e_2)}$. Thus by Remark 2.1, $P(e_1, e_2) = \frac{n}{q^{4e_1e_2}\xi}$ and we have

$$\begin{aligned} P(e_1, e_2) &= \frac{q^{4e_1e_2}\omega(e_1+e_2)}{q^{4e_1e_2}\xi} \sum_{\ell=0}^{e_2-1} \frac{q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)} \left(1 - \frac{q^{-e_1e_2}}{\omega(e_2-\ell)}\right) \\ &= \omega(e_1)\omega(e_2) \sum_{\ell=0}^{e_2-1} \frac{q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)} \left(1 - \frac{q^{-e_1e_2}}{\omega(e_2-\ell)}\right). \end{aligned}$$

Corollary 4.2 implies that $\sum_{\ell=0}^{e_2-1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_2-\ell)\omega(e_1-e_2+\ell)\omega(\ell)} = 1 - q^{-e_1e_2}$. Hence

$$P(e_1, e_2) = \left(\sum_{\ell=0}^{e_2-1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)}\right) - (1 - q^{-e_1e_2})q^{-e_1e_2}$$

as in (1.2). When $\ell = 0$ the summand is $\frac{\omega(e_1)\omega(e_2)}{\omega(e_1-e_2)} = 1 - O(1/q)$. Finally, this implies that $P(e_1, e_2) = 1 - O(1/q)$, as claimed. □

5. Proportions: closed 3-arcs and stingray duos

The aim of this section is to prove Theorem 1.3. We do this in two steps. Our first result considers stingray duos in a fixed pair of conjugacy classes.

Theorem 5.1. *Let $d = e_1 + e_2$ with $1 \leqslant e_2 \leqslant e_1$, let $\mathrm{SL}_d(q) \trianglelefteq G \leqslant \mathrm{GL}_d(q)$, and for $i = 1, 2$, let $\mathscr{C}_i$ be a G-conjugacy class of e_i-stingray elements. Then the proportion*

$$\frac{\textit{Number of irreducible stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2}{\textit{Number of stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2}$$

equals the proportion $P(e_1, e_2)$ given by (1.2) of 3-walks of Γ_{e_1,e_2} which are closed 3-arcs.

Proof. By Lemma 3.7(a), $\mathscr{C}_1$ and $\mathscr{C}_2$ are conjugacy classes of $G_d = \mathrm{GL}_d(q)$, and so we may assume that $G = G_d$. It follows from Lemma 3.7(c) that, for each 3-walk (F_1, U_1, U_2, F_2) in $X_2 \times X_1 \times X_2 \times X_1$ of Γ_{e_1,e_2}, the equalities

$$U_i = \mathrm{im}(g_i - 1) \text{ and } F_i = \ker(g_i - 1), \text{ for } i = 1, 2, \tag{5.1}$$

all hold for the same number (3.1) of stingray duos $(g_1, g_2) \in \mathscr{C}_1 \times \mathscr{C}_2$. This is true, in particular, for closed 3-arcs (F_1, U_1, U_2, F_2). Further, by Lemma 3.6(b), either all or none of the pairs (g_1, g_2) satisfying the equalities in (5.1) have $\langle g_1, g_2 \rangle$ irreducible, and irreducibility occurs precisely when (F_1, U_1, U_2, F_2) is a closed 3-arc. Hence the proportion of (e_1, e_2)-stingray duos (g_1, g_2) in $\mathscr{C}_1 \times \mathscr{C}_2$ for which $\langle g_1, g_2 \rangle$ is irreducible, equals the proportion of 3-walks $(F_1, U_1, U_2, F_2) \in X_2 \times X_1 \times X_2 \times X_1$ which are closed 3-arcs. Moreover, this proportion is the quantity $P(e_1, e_2)$ in (1.2) by Remark 2.1. $\square$

Theorem 5.1 is an important component in the proof of Theorem 1.3 below.

Proof of Theorem 1.3. Recall that $d = e_1 + e_2$, $1 \leqslant e_2 \leqslant e_1$, and the subgroup G satisfies $\mathrm{SL}_d(q) \trianglelefteq G \leqslant G_d = \mathrm{GL}_d(q)$. Let $\mathscr{E}_i$ be the set of all e_i-stingray elements in G, for $i = 1, 2$. Clearly G acts on $\mathscr{E}_i$. Let $\mathcal{Y}_i$ be the set of G-conjugacy classes that partition $\mathscr{E}_i$. Then $\mathcal{Y}_1 \times \mathcal{Y}_2 := \{\mathscr{C}_1 \times \mathscr{C}_2 \mid \mathscr{C}_1 \in \mathcal{Y}_1, \mathscr{C}_2 \in \mathcal{Y}_2\}$ is the set of $(G \times G)$-conjugacy classes that partition $\mathscr{E}_1 \times \mathscr{E}_2$. For a subset Z of $\mathscr{E}_1 \times \mathscr{E}_2$ let $D(Z)$ denote the set of (e_1, e_2)-stingray duos in Z, and let $I(Z)$ denote the (sub)set of *irreducible* (e_1, e_2)-stingray duos in Z. Paraphrasing Theorem 5.1 gives $|I(\mathscr{C}_1 \times \mathscr{C}_2)| = P(e_1, e_2) \cdot |D(\mathscr{C}_1 \times \mathscr{C}_2)|$. We must prove that $|I(\mathscr{E}_1 \times \mathscr{E}_2)| = P(e_1, e_2) \cdot |D(\mathscr{E}_1 \times \mathscr{E}_2)|$. Since $I(\mathscr{E}_1 \times \mathscr{E}_2)$ is a disjoint union of $I(\mathscr{C}_1 \times \mathscr{C}_2)$ where $\mathscr{C}_1 \times \mathscr{C}_2$ ranges over $\mathcal{Y}_1 \times \mathcal{Y}_2$, and similarly for $D(\mathscr{E}_1 \times \mathscr{E}_2)$ and $D(\mathscr{C}_1 \times \mathscr{C}_2)$, we have

$$\begin{aligned} |I(\mathscr{E}_1 \times \mathscr{E}_2)| &= \sum_{\mathscr{C}_1 \times \mathscr{C}_2} |I(\mathscr{C}_1 \times \mathscr{C}_2)| = \sum_{\mathscr{C}_1 \times \mathscr{C}_2} P(e_1, e_2) \cdot |D(\mathscr{C}_1 \times \mathscr{C}_2)| \\ &= P(e_1, e_2) \cdot |D(\mathscr{E}_1 \times \mathscr{E}_2)|. \quad \square \end{aligned}$$

6. Explicit upper and lower bounds for $P(e_1, e_2)$

Recall that $P(e_1, e_2)$ is the proportion of (e_1, e_2)-stingray duos in $\mathrm{GL}_d(q)$ that are irreducible by Theorem 1.3. In this section we prove Theorem 1.2 giving precise upper and lower bounds for $P(e_1, e_2)$ where $1 \leqslant e_2 \leqslant e_1$, and $q > 1$ is an arbitrary prime power.

The ordering of two real numbers written in base-q is determined by the largest power of q where the digits differ. For example, if $a(q) = \sum_{i \leqslant s} a_i q^i$ and $b(q) = \sum_{j \leqslant s} b_j q^j$ are (necessarily convergent) Laurent series in q, with $a_i, b_j \in \{0, 1, \ldots, q-1\}$, $b_s \neq 0$ and $a_s < b_s$, then $a(q) \leqslant b(q)$ holds. Further, if $a_s < b_s$, then $a(q) = b(q)$ holds precisely when $b_s = a_s + 1$ and, for each $i < s$, $a_i = q - 1$ and $b_i = 0$. An example with $s = 0$ is $\sum_{i<0} (q-1) q^i = q^0$. Note that we can only compare Laurent series in q if the coefficients are not too large (in absolute value). For example,

$$1 - 2q^{-2} - 2q^{-3} + q^{-4} + 4q^{-5} + q^{-6} - 2q^{-7} - 2q^{-8} + q^{10} \leqslant 1 - 2q^{-2}$$

holds for all $q \geqslant 5$. However, direct evaluation shows that it is also true for $q \in \{2, 3, 4\}$.

Proof of Theorem 1.2. Upper and lower bounds for $P(e_1, 1)$ are best handled separately. Substituting $e_2 = 1$ into the formula (1.2) for $P(e_1, e_2)$ in Theorem 1.1 gives

$$P(e_1, 1) = -(1 - q^{-e_1})q^{-e_1} + (1 - q^{-1})(1 - q^{-e_1}) = (1 - q^{-e_1})(1 - q^{-1} - q^{-e_1}).$$

Therefore $P(e_1, 1) = 1 - q^{-1} - 2q^{-e_1} + q^{-e_1-1} + q^{-2e_1}$. For $e_1 \geqslant 3$ and for all $q \geqslant 2$ we have $1 - q^{-1} - q^{-2} < P(e_1, 1) < 1 - q^{-1}$. (The upper bound even holds for $e_1 \geqslant 1$.)

Henceforth suppose that $2 \leqslant e_2 \leqslant e_1$. The case when $e_1 = e_2$ requires delicate estimations. We first prove the upper bound when $e_1 = e_2 = e \geqslant 2$. Theorem 1.1 gives

$$P(e, e) = -(1 - q^{-e^2})q^{-e^2} + \sum_{\ell=0}^{e-1} \frac{\omega(e)^2 q^{-\ell^2}}{\omega(\ell)^2}.$$

Setting $e = 2$ gives

$$P(2, 2) = 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-4} - q^{-5} + q^{-6} + q^{-8}.$$

It follows that $1 - q^{-1} - q^{-2} < P(2, 2) < 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-5}$. Suppose now that $e_1 = e_2 = e \geqslant 3$. Since $\omega(e) \leqslant \omega(3)$ and $\omega(\infty) < \omega(\ell)$, we have

$$\begin{aligned} P(e, e) &< \sum_{\ell=0}^{\infty} \frac{\omega(e)^2 q^{-\ell^2}}{\omega(\ell)^2} < \omega(3)^2 \left(1 + \frac{q^{-1}}{\omega(1)^2} + \frac{q^{-4}}{\omega(2)^2} + \sum_{\ell=3}^{\infty} \frac{q^{-\ell^2}}{\omega(\infty)^2}\right) \\ &< \omega(3)^2 \left(1 + \frac{q^{-1}}{\omega(1)^2} + \frac{q^{-4}}{\omega(2)^2} + \sum_{\ell=3}^{\infty} \frac{q^{-8(\ell-2)}}{\omega(\infty)^2}\right); \end{aligned}$$

where the last line uses the fact that $8(\ell - 2) \leqslant l^2$ for all $\ell \geqslant 3$. Adding the geometric series $\sum_{\ell=3}^{\infty} q^{-8(\ell-2)}$ gives $\frac{q^{-8}}{1-q^{-8}}$. We use the inequality $1 - q^{-1} - q^{-2} + q^{-5} < \omega(\infty)$, which follows from the proof of [13, Lemma 3.5], so that

$$P(e, e) < \omega(3)^2 \left(1 + \frac{q^{-1}}{\omega(1)^2} + \frac{q^{-4}}{\omega(2)^2} + \frac{q^{-8}}{(1 - q^{-1} - q^{-2} + q^{-5})^2(1 - q^{-8})}\right). \tag{6.1}$$

Since $(1 - q^{-1} - q^{-2} + q^{-5})^2(1 - q^{-8}) > \frac{1}{16} \geqslant q^{-4}$ for all q, we have by (6.1) that

$$\begin{aligned} P(e, e) &< \omega(3)^2 \left((1 + q^{-4}) + \frac{q^{-1}}{\omega(1)^2} + \frac{q^{-4}}{\omega(2)^2}\right) \\ &= \omega(3)^2(1 + q^{-4}) + (1 - q^{-2})^2(1 - q^{-3})^2 q^{-1} + (1 - q^{-3})^2 q^{-4}. \end{aligned}$$

The inequality below is true for $q \geqslant 5$ by comparing Laurent series in q:

$$(1-q^{-2})^2(1-q^{-3})^2 = 1-2q^{-2}-2q^{-3}+q^{-4}+4q^{-5}+q^{-6}-2q^{-7}-2q^{-8}+q^{10}$$
$$\leqslant 1-2q^{-2}.$$

However, the inequality $(1-q^{-2})^2(1-q^{-3})^2 \leqslant 1-2q^{-2}$ is also true for $q \in \{2,3,4\}$. Hence $\omega(3)^2 < (1-q^{-1})^2(1-2q^{-2})$, and so

$$P(e,e) < (1-q^{-1})^2(1-2q^{-2})(1+q^{-4}) + (1-2q^{-2})q^{-1} + (1-q^{-3})^2q^{-4}$$
$$= 1-q^{-1}-q^{-2}+2q^{-3}-2q^{-5}-q^{-6}+2q^{-7}-2q^{-8}+q^{-10}.$$

The above reasoning proves that the upper bound $P(e,e) < 1-q^{-1}-q^{-2}+2q^{-3}-2q^{-5}$ holds for all $e \geqslant 2$ and all prime powers $q \geqslant 2$.

We next prove that the lower bound $1-q^{-1}-q^{-2} < P(e,e)$ holds for $e \geqslant 3$ and $q \geqslant 2$. Setting $e_1 = e_2 = e$ in Theorem 1.1 gives

$$P(e,e) = -(1-q^{-e^2})q^{-e^2} + \sum_{\ell=0}^{e-1} \frac{\omega(e)^2 q^{-\ell^2}}{\omega(\ell)^2} > -q^{-9} + \omega(e)^2 \sum_{\ell=0}^{2} \frac{q^{-\ell^2}}{\omega(\ell)^2}.$$

Now

$$\omega(2)^2 \sum_{\ell=0}^{2} \frac{q^{-\ell^2}}{\omega(\ell)^2} = 1-q^{-1}-q^{-2}+2q^{-3}-q^{-5}+q^{-6}$$
$$> 1-q^{-1}-q^{-2}+2q^{-3}-q^{-5} = (1-q^{-2})(1-q^{-1}+q^{-3}).$$

Using $\omega(e) > \omega(\infty) > 1-q^{-1}-q^{-2}+q^{-5}$ and the above inequality gives

$$P(e,e) > -q^{-9} + \frac{\omega(e)^2}{\omega(2)^2}(1-q^{-2})(1-q^{-1}+q^{-3}) = -q^{-9} + \frac{\omega(e)^2(1-q^{-1}+q^{-3})}{(1-q^{-1})^2(1-q^{-2})}$$
$$> -q^{-9} + \frac{(1-q^{-1}-q^{-2}+q^{-5})^2(1-q^{-1}+q^{-3})}{(1-q^{-1})^2(1-q^{-2})}.$$

We approximate the denominator using: $(1-q^{-1})^{-2} = \sum_{i=0}^{\infty}(i+1)q^{-i} > \sum_{i=0}^{6}(i+1)q^{-i}$ and $(1-q^{-2})^{-1} = \sum_{i=0}^{\infty} q^{-2i} > \sum_{i=0}^{2} q^{-2i}$. Hence

$$P(e,e) > -q^{-9} + (1-q^{-1}-q^{-2}+q^{-5})^2(1-q^{-1}+q^{-3}) \left(\sum_{i=0}^{6}(i+1)q^{-i}\right)\left(\sum_{i=0}^{2} q^{-2i}\right)$$
$$= 1-q^{-1}-q^{-2}+q^{-5}-2q^{-6}-8q^{-7}+16q^{-8}+q^{-9}-5q^{-10}+q^{-11}-5q^{-12}$$
$$+26q^{-13}-12q^{-14}-7q^{-15}-q^{-16}-9q^{-17}+6q^{-18}-17q^{-19}-3q^{-20}$$
$$+5q^{-21}+6q^{-22}+7q^{-23}.$$

The above Laurent series is therefore greater than $1-q^{-1}-q^{-2}$ for $q \geqslant 27$, and a computer calculation shows it is greater than $1-q^{-1}-q^{-2}$ for $3 \leqslant q < 26$. The same lower bound for $P(e,e)$ also holds for $q=2$ if we replace the bound $\omega(\infty) > 1-q^{-1}-q^{-2}+q^{-5} = 0.28125$ with the sharper inequality $\omega(\infty) > 0.288$. Thus the bound $P(e,e) > 1-q^{-1}-q^{-2}$ holds for $e \geqslant 3$ and all prime powers $q \geqslant 2$. This establishes the lower bound in Theorem 1.2 for the case $e_1 = e_2$. Henceforth assume that $e_1 \neq e_2$.

We next prove that $P(e_1,e_2) < 1-q^{-1}-q^{-2}+2q^{-3}-2q^{-5}$ when $2 \leqslant e_2 < e_1$. It is convenient to write $P(e_1,e_2) = -(1-q^{-e_1e_2})q^{-e_1e_2} + L(e_1,e_2) + M(e_1,e_2)$ where

$$L(e_1,e_2) := \sum_{\ell=0}^{1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)} \quad \text{and} \quad M(e_1,e_2) := \sum_{\ell=2}^{e_2-1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)}.$$

When e_2 or q is large, the term $L(e_1,e_2)$ dominates $P(e_1,e_2)$. Rearranging gives

$$\begin{aligned} L(e_1,e_2) &:= \sum_{\ell=0}^{1} \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+\ell)\ell}}{\omega(e_1-e_2+\ell)\omega(\ell)} = \frac{\omega(e_1)\omega(e_2)}{\omega(e_1-e_2)} + \frac{\omega(e_1)\omega(e_2)q^{-(e_1-e_2+1)}}{\omega(e_1-e_2+1)\omega(1)} \\ &= \frac{\omega(e_1)\omega(e_2)}{\omega(e_1-e_2+1)\omega(1)} \left((1-q^{-(e_1-e_2+1)})(1-q^{-1}) + q^{-(e_1-e_2+1)}\right) \\ &= \frac{\omega(e_1)\omega(e_2)}{\omega(e_1-e_2+1)\omega(1)} \left(1-q^{-1}+q^{-(e_1-e_2+2)}\right). \end{aligned} \tag{6.2}$$

The inequalities $\omega(e_1) < \omega(e_1-e_2+\ell)$, $\omega(e_2) < \omega(\ell)$ and $e_1-e_2 \geqslant 1$ imply that

$$M(e_1,e_2) < \sum_{\ell=2}^{e_2-1} q^{-(\ell+1)\ell} < q^{-6}+\sum_{\ell=3}^{\infty} q^{-(\ell+1)\ell} < q^{-6}+\sum_{\ell=3}^{\infty} q^{-4\ell} \leqslant q^{-6}+2q^{-12} < q^{-6}+q^{-7}.$$

If $e_2 = 2$, then $M(e_1,e_2) = 0$. Using (6.2) and the inequality $\omega(e_1) < \omega(e_1-e_2+1)$ gives

$$\begin{aligned} P(e_1,2) < L(e_1,2) &= (1-q^{-e_1})(1-q^{-2})(1-q^{-1}+q^{-e_1}) \\ &= (1-q^{-2e_1}-q^{-1}+q^{-(e_1+1)})(1-q^{-2}) \\ &< (1-q^{-1}+q^{-4})(1-q^{-2}) = 1-q^{-1}-q^{-2}+q^{-3}+q^{-4}-q^{-6} \\ &< 1-q^{-1}-q^{-2}+2q^{-3}-2q^{-5}. \end{aligned}$$

Suppose now that $3 \leqslant e_2 < e_1$. In this case $\omega(e_2) \leqslant \omega(3)$, so we have the upper bounds $L(e_1,e_2) < \frac{\omega(3)}{\omega(1)}(1-q^{-1}+q^{-3})$ and $M(e_1,e_2) < q^{-6}+q^{-7}$. Hence

$$\begin{aligned} P(e_1,e_2) < L(e_1,e_2)+M(e_1,e_2) &< \frac{\omega(3)}{\omega(1)}(1-q^{-1}+q^{-3}) + q^{-6}+q^{-7} \\ &= (1-q^{-2})(1-q^{-3})(1-q^{-1}+q^{-3}) + q^{-6}+q^{-7} \\ &= 1-q^{-1}-q^{-2}+q^{-3}+q^{-4}-q^{-6}+q^{-7}+q^{-8} \end{aligned}$$

$$< 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-5}.$$

Thus the bound $P(e_1, e_2) < 1 - q^{-1} - q^{-2} + 2q^{-3} - 2q^{-5}$ holds for $2 \leqslant e_2 \leqslant e_1$ and $q \geqslant 2$.

We now prove that $1 - q^{-1} - q^{-2} < P(e_1, e_2)$ for $2 \leqslant e_2 < e_1$. By Theorem 1.1, $P(e_1, e_2) = -(1 - q^{-e_1 e_2})q^{-e_1 e_2} + L(e_1, e_2) + M(e_1, e_2)$. If $e_2 = 2$, then $M(e_1, 2) = 0$ and

$$\begin{aligned}
P(e_1, 2) &= -(1 - q^{-2e_1})q^{-2e_1} + L(e_1, 2) \\
&= -(1 - q^{-2e_1})q^{-2e_1} + (1 - q^{-e_1})(1 - q^{-2})(1 - q^{-1} + q^{-e_1}) \qquad \text{by (6.2)} \\
&> -q^{-2e_1} + (1 - q^{-2})(1 - q^{-1} + q^{-e_1} - q^{-e_1} + q^{-e_1-1} - q^{-2e_1}) \\
&> -q^{-4} + (1 - q^{-2})(1 - q^{-1}) \\
&= -q^{-4} + 1 - q^{-1} - q^{-2} + q^{-3} > 1 - q^{-1} - q^{-2}.
\end{aligned}$$

Suppose now that $3 \leqslant e_2 < e_1$. Since $-(1 - q^{-e_1 e_2})q^{-e_1 e_2} \geqslant -(1 - q^{-9})q^{-9} > -q^{-9}$ and $M(e_1, e_2) \geqslant 0$, we have $P(e_1, e_2) > -q^{-9} + L(e_1, e_2)$. We next find a sharp lower bound for $L(e_1, e_2)$ for $3 \leqslant e_2 < e_1$. Set $\ell_0 := e_1 - e_2 + 2$. Then $3 \leqslant \ell_0 < e_1$ and

$$\frac{\omega(e_1)}{\omega(\ell_0 - 1)} = \prod_{i=\ell_0}^{e_1} (1 - q^{-i}) > 1 - \sum_{i=\ell_0}^{e_1} q^{-i} > 1 - \sum_{i=\ell_0}^{\infty} q^{-i} = 1 - \frac{q^{-\ell_0}}{1 - q^{-1}}.$$

In addition,

$$\frac{\omega(e_2)}{\omega(1)} > \frac{\omega(\infty)}{1 - q^{-1}}.$$

Multiplying the previous two inequalities shows:

$$\frac{\omega(\infty)(1 - q^{-1} - q^{-\ell_0})}{(1 - q^{-1})^2} < \frac{\omega(e_1)\omega(e_2)}{\omega(e_1 - e_2 + 1)\omega(1)}. \tag{6.3}$$

Recall that $\ell_0 := e_1 - e_2 + 2$ and $3 \leqslant e_2 < e_1$. Using the formula (6.2) for $L(e_1, e_2)$ and equation (6.3) gives

$$\begin{aligned}
P(e_1, e_2) &> -q^{-9} + \frac{\omega(\infty)(1 - q^{-1} - q^{-\ell_0})(1 - q^{-1} + q^{-\ell_0})}{(1 - q^{-1})^2} \\
&= -q^{-9} + \frac{\omega(\infty)(1 - 2q^{-1} + q^{-2} - q^{-2\ell_0})}{(1 - q^{-1})^2}.
\end{aligned}$$

We next use $\omega(\infty) > 1 - q^{-1} - q^{-2} + q^{-5}$ (see [13, Lemma 3.5]) and

$$\frac{1}{(1 - q^{-1})^2} = \sum_{i=0}^{\infty} (i + 1)q^{-i} > \sum_{i=0}^{6} (i + 1)q^{-i}.$$

Since $\ell_0 \geqslant 3$, we have

$$\begin{aligned} P(e_1,e_2) &> -q^{-9} + (1-q^{-1}-q^{-2}+q^{-5})(1-2q^{-1}+q^{-2}-q^{-6}) \left(\sum_{i=0}^{6}(i+1)q^{-i}\right) \\ &= 1-q^{-1}-q^{-2}+q^{-5}-q^{-6}-9q^{-7}+15q^{-8}+q^{-9}-5q^{-10} \\ &\quad + 2q^{-11}-6q^{-12}+17q^{-13}+3q^{-14}-5q^{-15}-6q^{-16}-7q^{-17} \\ &\geqslant 1-q^{-1}-q^{-2}; \end{aligned}$$

the last inequality holds since $q^{-5}-q^{-6}-9q^{-7}+15q^{-8}+\cdots-7q^{-17}>0$ for $q>17$, and direct calculation shows that it also holds for $2 \leqslant q \leqslant 17$. This establishes the lower bound of Theorem 1.2 for $2 \leqslant e_2 < e_1$ and $q \geqslant 2$, and completes the proof. □

7. Motivation from computational group theory

We noted in Section 1 that the results of this paper relate to a larger problem of recognising finite classical groups, see [4,6,7,11,14]. This section describes the context of this problem, and why the lower bound $1-q^{-1}-q^{-2}$ for the proportion $P(e_1,e_2)$ given in this paper is helpful, while the lower bound $1-2q^{-1}+O(q^{-2})$ given in [14], for a related proportion, is problematic.

Let $\mathrm{G}\mathbf{X}_d(q)$ denote a classical group over the field $\mathbb{F}_{q^\delta}^d$ of type $\mathbf{X}$, where $\mathbf{X}$ is $\mathbf{L}$ (linear), $\mathbf{U}$ (unitary), $\mathbf{S}$ (symplectic), or $\mathbf{O}$ (orthogonal) and $\delta=1$ unless $\mathbf{X}=\mathbf{U}$ when $\delta=2$. Then $\mathrm{G}\mathbf{X}_d(q)$ acts naturally on a vector space $V=\mathbb{F}_{q^\delta}^d$. If $\mathrm{G}\mathbf{X}_d(q)$ is not solvable, let $\Omega\mathbf{X}_d(q)$ be the smallest normal subgroup of $\mathrm{G}\mathbf{X}_d(q)$ for which $\mathrm{G}\mathbf{X}_d(q)/\Omega\mathbf{X}_d(q)$ is solvable and let $\mathrm{S}\mathbf{X}_d(q)$ denote the subgroup of $\mathrm{G}\mathbf{X}_d(q)$ comprising matrices with determinant 1. If $\mathbf{X}=\mathbf{L},\mathbf{S},\mathbf{U}$ then $\mathrm{S}\mathbf{X}_d(q)=\Omega\mathbf{X}_d(q)$, whereas $|\mathrm{SO}_d(q):\Omega_d(q)|=2$ when $\mathbf{X}=\mathbf{O}$. We present the key strategy for constructive recognition algorithms of classical groups, first introduced in the state-of-the-art algorithm [4], a strategy which is also adopted in a new algorithm currently being developed.

Conceptually we start with a group $G=\langle A\rangle$ which is (either known to be, or believed to be) a classical group $G=\mathrm{S}\mathbf{Y}_n(q)$ of type $\mathbf{Y}$ acting on a vector space $V=\mathbb{F}_{q^\delta}^n$ as above. The aim of a constructive recognition algorithm is to write the elements of a pre-defined generating set for $\mathrm{S}\mathbf{Y}_n(q)$ as words in A. Algorithms following the strategy in [4] are recursive. In the first step, see for example [4, p. 232 or Section 5], they construct a 'first subgroup H' which, with respect to an appropriate basis, has the form

$$H = \begin{pmatrix} \mathrm{S}\mathbf{X}_d(q) & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

where $d<n$ and either $\mathbf{X}=\mathbf{Y}$ or $(\mathbf{Y},\mathbf{X},q)=(\mathbf{S},\mathbf{O},\text{even})$. To construct H, the algorithms seek a *generating* (e_1,e_2)-stingray duo, namely a (e_1,e_2)-stingray duo (g_1,g_2) in $\mathrm{G}\mathbf{X}_d(q)$ for which $\langle g_1,g_2\rangle$ contains $\Omega\mathbf{X}_d(q)$. The correctness and the complexity analysis, as well as the practical runtime, of a given constructive recognition algorithm,

is underpinned by tight estimates for the proportion of (e_1, e_2)-stingray duos (g_1, g_2) in $\Omega\mathbf{X}_d(q)$ (or in some specified pair of $\Omega\mathbf{X}_d(q)$-conjugacy classes) that are generating and for which each g_i has prime order. For example, the algorithm in [4] seeks pairs of (e_1, e_2)-stingray duos of the form (g, g^x) with $g \in G$ and g^x a random G-conjugate, where $e = e_1 = e_2 = d/2$ for some $d \in [n/3, 2n/3]$. With high probability (g, g^x) is a generating (e, e)-stingray duo, see [4, Section 5.1]. This is implied by [4, Lemma 5.8] in case $\mathbf{L}$, and in [7, Lemma 3.4] for cases $\mathbf{X} = \mathbf{S}, \mathbf{U}$, and $\mathbf{O}$. (There was an unfortunate gap in extending the proof of [4, Lemma 5.8] to cases other than $\mathbf{L}$ which was repaired in [7, Lemma 3.4], see [7, Remark 3.5].) In forthcoming algorithms, we choose d to be $O(\log(n))$ and construct (e_1, e_2)-stingray duos with $d = e_1 + e_2$.

The results in the present paper yield substantial improvements for the analyses of constructive recognition algorithms in three ways.

Firstly, they are required for the analysis of the above mentioned forthcoming algorithm, which hinges on proving that the proportion of generating (e_1, e_2)-stingray duos among all (e_1, e_2)-stingray duos in $\Omega\mathbf{X}_{e_1+e_2}(q)$ is large, which will be proved in [8].

Secondly, they improve existing estimates for the proportion of (e, e)-stingray duos of the form (g, g^x) for $g \in \Omega\mathbf{X}_d(q)$ that are generating, where difficulties in handling small values of q have resulted in restrictions in the applicability of complexity estimates for these algorithms. For example, the main theorem for the constructive recognition of classical groups in even characteristic [4, Theorem 1.2] requires $q > 4$ because the estimation result about stingray elements it relied on (from a preprint version of [14]) had this restriction. In the published version [14, Theorem 2], the only restriction is '$q > 2$ in the orthogonal case', see our discussion in [7, Remark 3.5(b)].

Finally, our results improve the bounds for the complexity analysis of existing algorithms, by noting that the proportions used to estimate the performance of algorithms underestimated the proportion of elements actually constructed in the algorithm.

We address these aspects in the following two subsections.

Estimating the proportion of (e_1, e_2)*-stingray duos*

Let G be a group satisfying $\Omega\mathbf{X}_d(q) \trianglelefteqslant G \leqslant \mathrm{G}\mathbf{X}_d(q)$. The proportion of (e_1, e_2)-stingray duos (g_1, g_2) in G (or in some specified pair of G-conjugacy classes) that are generating can be estimated by bounding the proportion of (e_1, e_2)-stingray duos (g_1, g_2) which do not generate, and therefore lie in a proper maximal subgroup of $\Omega\mathbf{X}_d(q)$. This approach was pioneered in [14, Theorem 6] for a classical group $\mathrm{S}\mathbf{X}_d(q)$ of type $\mathbf{X}$ with $d = 2e$ even, and an $\mathrm{S}\mathbf{X}_d(q)$-conjugacy class $\mathscr{C}$ of e-stingray elements of prime order: the results [14, Theorem 2, 5, 6] show that with probability $1 - O(q^{-\delta})$ a pair in $\mathscr{C} \times \mathscr{C}$ will generate a subgroup $\mathrm{S}\mathbf{X}'_d(q)$ of type $\mathbf{X}'$ (provided $q > 2$ if $\mathbf{X} = \mathbf{O}$), where as noted above, $\mathbf{X}' = \mathbf{X}$ or $(\mathbf{X}, \mathbf{X}', q) = (\mathbf{S}, \mathbf{O}, \text{even})$. Equivalently, the probability that such a pair (g, g^x) fails to generate is $O(q^{-\delta})$. The probability that $\langle g, g^x \rangle$ is irreducible and lies in a proper maximal subgroup of $\mathrm{S}\mathbf{X}_d(q)$, i.e. it fails to generate, is shown to be very small, namely $O(q^{-cd^2})$ where c is a constant depending on the type $\mathbf{X}$, [14, Theorem 6]. Therefore, the leading terms in the estimates correspond to pairs where $\langle g, g^x\rangle$ is reducible on $\mathbb{F}_{q^\delta}^d$. In forthcoming work [8] we are finding that a similar dichotomy holds for general (e_1, e_2)-stingray duos. Also here, the most difficult problem arises when $\mathbf{X} = \mathbf{L}$, where the probability that an (e_1, e_2)-stingray duo is not generating is dominated by the probability that an (e_1, e_2)-stingray duo is reducible. Using the lower bound for $P(e_1, e_2)$ obtained in Theorem 1.2 we arrive at such a bound.

Corollary 7.1. *Suppose that $d = e_1 + e_2$ with $2 \leqslant e_2 \leqslant e_1$, and $\mathrm{SL}_d(q) \trianglelefteqslant G \leqslant \mathrm{GL}_d(q)$. Then*

$$\frac{\textit{Number of reducible } (e_1, e_2)\textit{-stingray duos in } G \times G}{\textit{Number of } (e_1, e_2)\textit{-stingray duos in } G \times G} \leq q^{-1} + q^{-2}.$$

Similarly, if $\mathscr{C}_i$ is a $\mathrm{GL}_d(q)$-conjugacy class of e_i-stingray elements for $i = 1, 2$, then

$$\frac{\textit{Number of reducible } (e_1, e_2)\textit{-stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2}{\textit{Number of } (e_1, e_2)\textit{-stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2} \leq q^{-1} + q^{-2}.$$

Proof. The stated proportion is $1 - P(e_1, e_2)$ by Theorem 5.1. Then by Theorem 1.2, $1 - q^{-1} - q^{-2} < P(e_1, e_2)$ holds, so that $1 - P(e_1, e_2) < q^{-1} + q^{-2}$ as claimed. The bound when restricted to stingray duos in $\mathscr{C}_1 \times \mathscr{C}_2$ follows by the same argument applying Theorem 1.3 rather than Theorem 5.1. □

Stingray pairs versus stingray duos

We now discuss how our results improve the analysis of the algorithm given in [4] originally based on the results in [14]. The bounds did not accurately reflect the practical performance in the linear case $\mathbf{L}$ where the classical group is $\mathrm{SL}_d(q)$ with natural module $\mathbb{F}_q^d$, where $d = 2e$ is even. Let $\mathscr{C}$ be an $\mathrm{SL}_d(q)$-conjugacy class of e-stingray elements. By [14, Lemma 5.3], the proportion of pairs from $\mathscr{C} \times \mathscr{C}$ that generate a reducible subgroup is at most

$$2q^{-1} + q^{-2} - 2q^{-3} - q^{-4} + 2q^{-d^2/4}. \tag{7.1}$$

By [14, Theorem 6],[2] the proportion of pairs from $\mathscr{C} \times \mathscr{C}$ that generate an irreducible proper subgroup of $\mathrm{SL}_d(q)$ is $O(q^{-d^2/4+d/2+2})$. Thus, for sufficiently large d, the probability that a random pair $\mathscr{C} \times \mathscr{C}$ generates $\mathrm{SL}_d(q)$ is at least $1 - 2q^{-1} + O(q^{-2})$, which is not a very useful bound if $q = 2$!

Our suggested solution is to exploit the fact that the algorithm in [4], and other algorithms under development, do not choose random pairs in $\mathscr{C} \times \mathscr{C}$, rather they construct stingray duos. Thus the appropriate proportions to estimate are of the form:

[2] In the course of our work for [8] we discovered a problem with the proof of [14, Lemma 10.1] which is corrected and generalised in [8], so [14, Theorem 6] is valid.

$$\frac{\text{Number of stingray duos in } \mathscr{C} \times \mathscr{C} \text{ with a desired property}}{\text{Number of stingray duos in } \mathscr{C} \times \mathscr{C}},$$

or in the more general setting where we are considering (e_1, e_2)-stingray duos coming from $\mathscr{C}_1 \times \mathscr{C}_2$, where $\mathscr{C}_i$ is a conjugacy class of e_i-stingray elements, estimates of the form:

$$\frac{\text{Number of stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2 \text{ with a desired property}}{\text{Number of stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2}.$$

In other words, the proportions required for the complexity analysis of the recognition algorithms mentioned above require a *different denominator*. For the case where the desired property we wish to estimate is reducibility, Corollary 7.1 yields an upper bound that is independent of e_1, e_2 and, in particular, is an improvement on the bound obtained by applying Equation (7.1) to the case relevant to [4,14]. However, to derive estimates for the proportion of stingray duos which are irreducible but non-generating using the estimates from [14, Theorem 6], we need to know the proportion of pairs in $\mathscr{C}_1 \times \mathscr{C}_2$ which are stingray duos. This proportion is given in Theorem 7.2 below.

Theorem 7.2. *Let $d = e_1 + e_2$ and let $\mathscr{C}_i$ be a $\mathrm{GL}_d(q)$-conjugacy class of e_i-stingray elements, for $i = 1, 2$. Then*

$$\frac{\textit{Number of stingray duos in } \mathscr{C}_1 \times \mathscr{C}_2}{|\mathscr{C}_1 \times \mathscr{C}_2|} = \frac{1}{\xi}, \qquad \text{where } \xi = \frac{\omega(d)}{\omega(e_1)\omega(e_2)}.$$

Moreover, if $2 \leqslant e_2 \leqslant e_1$, then $\frac{(1-q^{-d})(1-q^{-(d-1)})}{(1-q^{-1})(1-q^{-2})} \leqslant \xi < \frac{1}{1-q^{-1}-q^{-2}+q^{-5}}$.

Proof. Let $G_d = \mathrm{GL}_d(q)$. By Lemma 3.7(a),

$$|\mathscr{C}_1 \times \mathscr{C}_2| = \frac{|G_d|}{|G_{e_2}| \cdot (q^{e_1} - 1)} \cdot \frac{|G_d|}{|G_{e_1}| \cdot (q^{e_2} - 1)} = \frac{|G_d|^2}{|G_{e_1}| \cdot |G_{e_2}| \cdot (q^{e_1} - 1)(q^{e_2} - 1)}$$

and we note that

$$\frac{|G_d|}{|G_{e_1}| \cdot |G_{e_2}|} = q^{d^2 - e_1^2 - e_2^2}\xi = q^{2e_2e_2}\xi.$$

Next, by Lemmas 3.6(b)(ii) and 3.7(c), the number of stingray duos in $\mathscr{C}_1 \times \mathscr{C}_2$ is equal to the number of 3-walks in Γ_{e_1,e_2} times $\frac{|G_{e_1}| \cdot |G_{e_2}|}{(q^{e_1}-1)(q^{e_2}-1)}$. Moreover, by Lemma 2.2, the number of 3-walks in Γ_{e_1,e_2} is $q^{4e_1e_2}\xi$. Thus the number N of stingrays duos in $\mathscr{C}_1 \times \mathscr{C}_2$ is equal to

$$N = q^{4e_1e_2}\xi \cdot \frac{|G_{e_1}| \cdot |G_{e_2}|}{(q^{e_1} - 1)(q^{e_2} - 1)}.$$

The proportion of stingray duos in $\mathscr{C}_1 \times \mathscr{C}_2$ is therefore equal to

$$\frac{N}{|\mathscr{C}_1 \times \mathscr{C}_2|} = q^{4e_1e_2}\xi \cdot \frac{|G_{e_1}| \cdot |G_{e_2}|}{(q^{e_1}-1)(q^{e_2}-1)} \cdot \frac{|G_{e_1}| \cdot |G_{e_2}| \cdot (q^{e_1}-1)(q^{e_2}-1)}{|G_d|^2} = \frac{1}{\xi}$$

as claimed. In [7] the q-binomial notation $\binom{d}{e}_q = \frac{\omega(d)}{\omega(e)\omega(d-e)}$ is used. Using this notation $\xi = \binom{d}{e_2}_q$, and $\frac{(1-q^{-d})(1-q^{-(d-1)})}{(1-q^{-1})(1-q^{-2})} \leqslant \xi < \omega(\infty)^{-1}$ by [7, Lemma 5.2], with $\omega(\infty)$ as in (1.1). Moreover $\omega(\infty) > 1 - q^{-1} - q^{-2} + q^{-5}$ by [7, Lemma 5.1]. □

Note that Theorem 7.2 yields a new upper bound on the proportion of pairs from $\mathscr{C}_1 \times \mathscr{C}_2$ that generate a reducible subgroup.

Lemma 7.3. *Let $d = e_1 + e_2$ with $2 \leqslant e_2 \leqslant e_1$, and let $\mathscr{C}_i$ be a $\mathrm{GL}_d(q)$-conjugacy class of e_i-stingray elements, for $i = 1, 2$. Then the proportion of pairs from $\mathscr{C}_1 \times \mathscr{C}_2$ that generate a reducible subgroup is $1 - \frac{P(e_1,e_2)}{\xi}$ with ξ as in Theorem 7.2, and*

$$1 - \frac{P(e_1, e_2)}{\xi} < 2q^{-1} + q^{-2} - 2q^{-3} - q^{-4}.$$

Proof. Let $\mathcal{I}$ be the set of pairs $(g_1, g_2) \in \mathscr{C}_1 \times \mathscr{C}_2$ such that $\langle g_1, g_2 \rangle$ is irreducible, so the proportion we need to bound is $1 - |\mathcal{I}|/|\mathscr{C}_1 \times \mathscr{C}_2|$. If $(g_1, g_2) \in \mathscr{C}_1 \times \mathscr{C}_2$ and $\langle g_1, g_2 \rangle$ is irreducible, then the corresponding subspaces U_1, U_2 in Definition 3.2 must be disjoint, as otherwise $\langle g_1, g_2 \rangle$ would leave invariant the proper subspace $U_1 + U_2$. Hence (g_1, g_2) is a stingray duo. Thus $\mathcal{I}$ is the set of irreducible stingray duos in $\mathscr{C}_1 \times \mathscr{C}_2$. Therefore, if N = Number of stingray duos in $\mathscr{C}_1 \times \mathscr{C}_2$, then Theorem 5.1 gives

$$\frac{|\mathcal{I}|}{|\mathscr{C}_1 \times \mathscr{C}_2|} = \frac{|\mathcal{I}|}{N} \cdot \frac{N}{|\mathscr{C}_1 \times \mathscr{C}_2|} = P(e_1, e_2) \cdot \frac{N}{|\mathscr{C}_1 \times \mathscr{C}_2|}.$$

Theorem 7.2 gives $\frac{N}{|\mathscr{C}_1 \times \mathscr{C}_2|} = \frac{1}{\xi} > 1 - q^{-1} - q^{-2} + q^{-5}$ and, in addition, Theorem 1.2 gives $P(e_1, e_2) > 1 - q^{-1} - q^{-2}$. The following calculation completes the proof as $q^{-6} + q^{-7} < q^{-5}$

$$\begin{aligned} 1 - \frac{|\mathcal{I}|}{|\mathscr{C}_1 \times \mathscr{C}_2|} &= 1 - \frac{P(e_1, e_2)}{\xi} < 1 - (1 - q^{-1} - q^{-2}) \cdot (1 - q^{-1} - q^{-2} + q^{-5}) \\ &= 2q^{-1} + q^{-2} - 2q^{-3} - q^{-4} - q^{-5} + q^{-6} + q^{-7}. \quad \square \end{aligned}$$

The upper bound in Lemma 7.3 with $e_1 = e_2$ always improves the bound in (7.1). When $q = 2$, this yields the upper bound 15/16 for all d. Finally, the following lower bound for the proportion of pairs sought in the algorithm in [4] improves its analysis, and better reflects the fact that the algorithm produces stingray duos.

Corollary 7.4. *Let $\mathscr{C}$ be an $\mathrm{SX}_d(q)$-conjugacy class of e-stingray elements with $e = \frac{d}{2}$. Then, for some positive constant c,*

$$\frac{\textit{Number of generating stingray duos in } \mathscr{C} \times \mathscr{C}}{\textit{Number of stingray duos in } \mathscr{C} \times \mathscr{C}} > 1 - q^{-1} - q^{-2} - c \cdot q^{-d^2/4 + d/2 + 2}.$$

Proof. Applying the bound in Theorem 7.2 together with [14, Theorem 6] to the linear case **L** where $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}$ is an $\mathrm{SL}_d(q)$-conjugacy class of e-stingray elements for $e = d/2$ of order equal to a primitive prime divisor of $q^{d/2} - 1$ we find:

$$\frac{\text{Number of irreducible but non-generating stingray duos in } \mathscr{C} \times \mathscr{C}}{\text{Number of stingray duos in } \mathscr{C} \times \mathscr{C}} =$$
$$\frac{\text{Number of irreducible but non-generating stingray duos in } \mathscr{C} \times \mathscr{C}}{|\mathscr{C} \times \mathscr{C}|/\xi} < c \cdot q^{-d^2/4+d/2+2}$$

for some positive constant c. By Theorem 1.3, the proportion of irreducible (e, e)-stingray duos is $P(e, e)$, and by Theorem 1.2, $P(e, e) > 1 - q^{-1} - q^{-2}$, and hence the proportion of stingray duos in $\mathscr{C} \times \mathscr{C}$ which generate $\mathrm{SL}_d(q)$ is at least

$$P(e, e) - c \cdot q^{-d^2/4+d/2+2} > 1 - q^{-1} - q^{-2} - c \cdot q^{-d^2/4+d/2+2}. \quad \square$$

Declaration of competing interest

The authors declare that there are no competing interests.

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Data availability

No data was used for the research described in the article.

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