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General network production theory for good and bad inputs and outputs

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Abstract

The paper systematically extends a recently developed general production theory for networks of arbitrarily complex structure by utilising multi-criteria analysis to include undesirable objects as inputs and outputs. By embedding Koopmans' linear activity analysis into the broader framework the approach is generic in that it requires rather weak technological properties, thus allowing non-convex and even discrete production possibilities without any disposability property. The modelling approach can often be more appropriate than conventional ones and can be easily extended to dynamic analyses, since its network flow equations are based on balances that can integrate inventories in cases of material inputs and outputs. Theorems are proved that relate important properties of whole networks (e.g. convexity) or of their activities (e.g. efficiency) to those of their individual production units. Common methods of efficiency measurement are integrated in such a way that network data envelopment analysis for bads as inputs or outputs is embedded as the special case focusing specifically on polyhedral technologies. By applying different types of efficiency measures, a systematic procedure for evaluating the performance of network activities is demonstrated for the example of a two-stage production and abatement network and its subsystem with parallel units.

Keywords Network production theory · Undesirable object · Efficiency analysis · Network DEA · Environmental performance assessment

JEL classification C14 · C67 · D24 · Q56

1 Introduction

The question of how to properly model production systems with unintended outputs has proven both controversial and of particular interest to the productivity and efficiency community (Greene et al. 2021). In this context, Rodseth (2014, p. 211) states "that the popular production models that incorporate undesirable outputs may not be applicable to all cases involving pollution production and that more emphasis on appropriate empirical specifications is needed." Murty and Russell (2021, p. 180) assert that "the key to correct modelling of an emission-generating technology lies in a proper formulation of its disposability properties". In past decades, various disposability assumptions were proposed and applied (Dakpo et al. 2016). Dakpo and Ang (2019, p. 690) are in favour of structural representations with multiple equations and remark that the by-production approach "is typical for engineering science and is appealing for economists. [It] opens the black box by making the technical relationships between all inputs and outputs explicit. This increase in accuracy does, however, require appropriate knowledge of the production system …" Therefore, the model designer must have a deep understanding of the realm of reality concerned. This is not only true for *clean technologies* where prevention, abatement, and disposal of bad outputs are integral part of the production technology itself. It is also true for *end-of-pipe technologies*, where the disposal of afore emerged undesirable outputs forms a separate activity of a multi-stage production and abatement network with goods and bads as inputs and outputs (Dyckhoff, 2023, p. 187).

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Networks and undesirable outputs (but rarely bads as inputs) are both topics each of which constitutes a main strand of research regarding efficiency measurement with methods of data envelopment analysis (DEA).[1] However, special reviews of 'network DEA' literature do not emphasise bad inputs or outputs as important topic.[2] Conversely, reviews of environmental efficiency and related literature do not seem to identify networks as essential topic.[3]

To help management "open the black box of production", Avkiran and Parker (2010, S. 4) recommended network DEA as one of four directions of future research. Actually, by combining terms 'DEA' or 'data envelopment analy*' with 'network*' a literature search in the Web of Science results in 3390 papers published until the year 2023 of which 983 explicitly use the term 'network DEA' (with 290 even in its title). A similar search leads to 1594 entries for terms 'DEA' or 'data envelopment analy*' combined with 'bad output*' or 'pollut*' as topics (and to 233 with last terms even in title). The intersection of both research strands results in 184 entries[4], of which 30 contain 'bad output*' or 'pollut*' and 68 'network*' in its title, eight of them both[5]. Nearly all of the 184 sources that are concerned with bads (usually as outputs, rarely as inputs) in network DEA use a simple network structure (mostly of two stages) for a particular application area.[6] Thus, it appears that fundamentally new methods or theoretical developments for *general networks with undesirable objects as inputs or outputs* were not published in the literature on network DEA. Aim of the present paper is to develop a generic approach for analysing the efficiency of activities in networks of arbitrary structures that include those with bads as inputs or outputs as parts of the network system.

Economic literature and in particular literature on productivity and efficiency measurement commonly refer to Shephard (1970) as their production theoretical foundation. His theory has been further developed and applied mainly in the context of general or e.g. agrarian economics. It has not found much interest in business and engineering sciences, neither in the theory of business economics nor in research and teaching on production and operations management and engineering. This is somewhat surprising because the theoretical, technological, and practical knowledge regarding multi-stage and network processes within these disciplines could be fruitful to analyse realistic models and propose valid assertions for networks of e.g. working stations, plants, companies, or even whole countries.

In stark contrast, numerous mathematical models and methods that are based on Koopmans' (1951) *Activity Analysis of Production and Allocation* have been developed to date to deal with economic planning, scheduling, and accounting problems. The use of such models and methods is common practice in larger companies of industries that are heavily affected by coupled production or characterised by a network of interconnected plants, like the chemical or iron and steel industries (Dyckhoff and Souren, 2023, p. 1043). Activity analysis is furthermore the standard approach to modelling production networks in the literature of sustainable production and supply chain management since the 1990s (Thies et al. 2021).

Against this background, Dyckhoff and Souren (2024) chose activity analysis to form the starting point for a *general network production theory* that allows for efficiency analyses and can serve as a building block for network modelling. The theory is characterised by following features which distinguish it from previous network production technologies for good and bad outputs of e.g. Bostian et al. (2025) that are based on Shephard's (1970) production theory:

- This network production theory generalises Koopmans' linear activity analysis by using similar underlying modelling features and fundamental assumptions.
- Analysed networks may possess arbitrary structures that are formed by production units whose technologies can be non-convex and even discrete. Disposability is not presupposed.
- The modelling approach is not only theoretically founded, but also suitable for analysing complex networks of e.g. supply chains or closed-loop systems with recycling. Like linear activity analysis it can easily be extended to dynamic analyses by taking account of inventories.
- Common methods of efficiency measurement known from network DEA can be integrated into the theory so that improvements are facilitated. For example, it can be shown that calculating an overall efficiency score for a network as average of individual scores of its units is inappropriate.

[1] Cf. e.g. Liu et al. (2013), (2016), Lampe & Hilgers (2015), Emrouznejad & Yang (2018, p. 7), Panwar et al. (2022, p. 5415).

[2] E.g. Cook, Liang, & Zhu (2010), Chen et al. (2013), Kao (2014), Alves & Meza (2023). In contrast, the review of Ratner et al. (2023) implicitly illustrates the growing importance of bad inputs and outputs ("eco-efficiency") in the most recent network DEA literature.

[3] E.g. Liu et al. (2010), Song et al. (2012), Dakpo, Jeanneaux, & Latruffe (2016), Halkos & Petrou (2019), Emrouznejad et al. (2023).

[4] Four of these 184 papers are reviews, but exclusively focus a specific application area.

[5] These are Lozano (2015) & (2016), Cui (2017), Cui et al., (2017), Zhang et al. (2018), Li et al. (2020), Michali et al. (2021), Zhao et al. (2022). Although further articles on network DEA with bads exist that are not found by this specific search, e.g. Lotfi et al. (2023), they also seem to deal with simple network structures only.

[6] This also holds for the review of Bostian et al. (2025) on "Network production technologies for good and bad outputs". An exception is Lozano (2016) who developed a slacks-based DEA approach for general networks.

Since this general network production theory supposes desirable objects (goods) as inputs and outputs, following open research questions remain to be answered:

- How can undesirable objects (bads) be integrated into the theory?
- Do all general findings derived from this theory also hold in case of bads?
- What are the advantages and limits of this theory for including bads?

In order to answer these questions for including undesirable objects the present paper systematically extends the general network production theory of Dyckhoff and Souren (2024) by utilising a framework that integrates multi-criteria analysis and production theory (Dyckhoff and Souren, 2022).

The structure of the paper is as follows: Section 2 extends the activity-analytic modelling approach for networks of Dyckhoff and Souren (2024), thus providing fundamental definitions, assumptions, and first results that allow for a systematic and theoretically founded inclusion of bad inputs or outputs into network efficiency analysis. In Section 3, general and more specific theorems are derived that relate important properties of networks (e.g. convexity) or of their activities (e.g. efficiency) to corresponding ones of production units which constitute the network. While the considered networks may feature arbitrary structures and rather general production technologies of their units in principle, Section 4 applies the developed theory to network DEA by focusing on polyhedral technologies. A systematic procedure for measuring efficiency of network activities with goods and bads is demonstrated by the example of a two-stage production and abatement network, using an illustrative data set of decision-making units. In Section 5, the example of a hierarchical network with a subsystem of parallel units illustrates a stepwise procedure how the developed theory allows to construct general, complex networks of e.g. supply chains and closed-loop systems. Section 6 summarises key findings. Contributions and limitations of the new approach are discussed. An appendix contains the proofs of all propositions.

To develop the new theory it is necessary to introduce terms and notations that lack parallels in common literature. Consistency of the general network production theory for goods and bads developed in Sections 2 and 3 with its application to network DEA requires some deviations from conventional DEA models and established notations in Sections 4 and 5. "In order to formulate a DEA model for general networks of processes the notation used can be of a great help if it is chosen appropriately" (Lozano, 2016, p. 74). In this paper, terms and notations of Dyckhoff and Souren (2022; 2024) are used. They are partly adapted from Koopmans' (1951) pioneering contribution on the "Analysis of production as an efficient combination of activities". Nevertheless, to prevent confusion it is attempted to avoid departures from standard conventions and established literature as much as it makes sense.

In the following, $\mathbb{R}^\kappa$ denotes the κ-dimensional Euclidean space, $\mathbb{R}^\kappa_+$ and $\mathbb{R}^\kappa_-$ its nonnegative or nonpositive orthant. Let $\mathbf{0}$ be the vector with all components equal to 0. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^\kappa$, the inequality $\mathbf{a} \geq \mathbf{b}$ ($\mathbf{a} \gg \mathbf{b}$) means that $a_i \geq b_i$ (a_i b_i) for all $i = 1, \ldots, \kappa$, whereas $\mathbf{a} > \mathbf{b}$ denotes $\mathbf{a} \geq \mathbf{b}$, $\mathbf{a} \neq \mathbf{b}$.

2 Basics of a general network production theory with bads

This section extends the network production theory of Dyckhoff and Souren (2024) from networks with solely desirable objects to those which additionally include undesirable objects as inputs or outputs. Subsection 2.1 summarises their general algebraic model of production networks. It does not change by including bads. In contrast, the second and third subsection postulate fundamental properties of networks and their units and deal with basics of production efficiency that both have to be substantially (but not necessarily formally) modified to include bads.

2.1 Modelling production possibilities of networks

By applying an enhanced systematic mode of graphically modelling production systems (Dyckhoff, 1992), Fig. 1 shows the input/output-graph of an example used by Fukuyama and Weber (2014) to review different DEA model approaches for networks with bad outputs. It is a two-stage network of production units A and B with nine relevant object types of which three primary factors #1 to #3 produce two intermediate products #4 and #5 on the first stage that in turn are exclusively used to produce three final goods #6 to #8 and bad output #9 on the second stage. Intermediate products may leave the system if their output from stage A exceeds the respective demand from stage B.

Production units are depicted by squares, object types by circles. White circles indicate desirable and dark grey circles undesirable objects. The outer ('white') box surrounding squares and circles determines the production system formed by a network of arrows connecting them. Each arrow that connects a circle with a square describes a *process input or output* within the interior of the box. Arrows connecting circles with the environment of the box are known as (*system*) *inputs or outputs* from "black box" models that ignore the interior network. Each arrow represents a flow of quantities of objects of that type it is connected with the respective circle, which may be restricted by

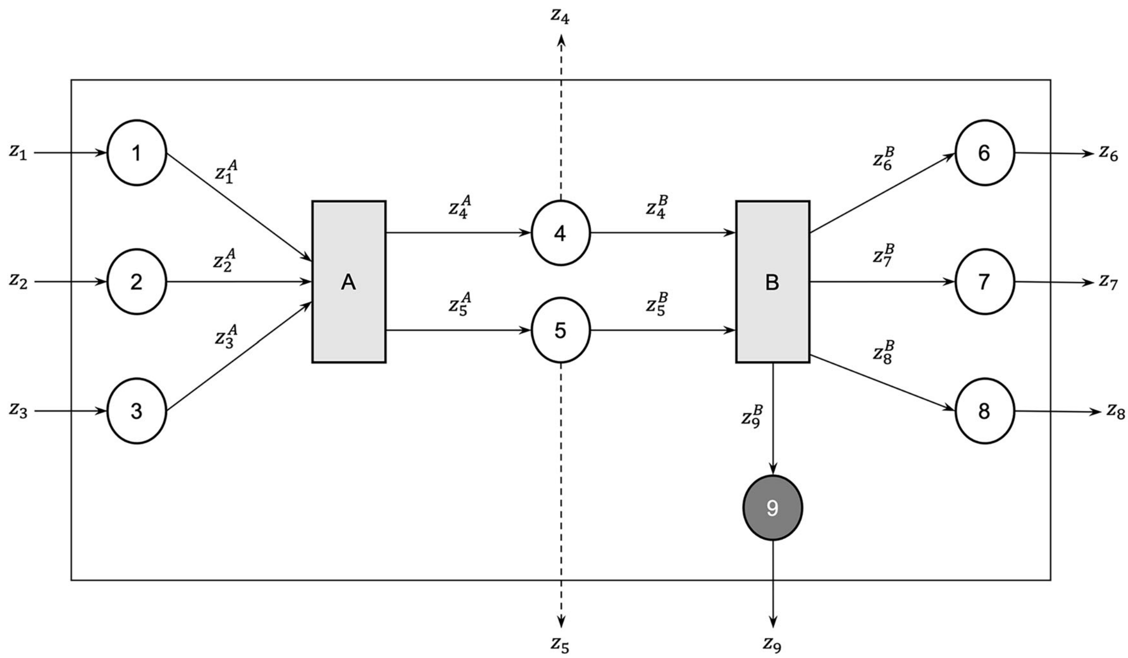

Fig. 1 Two-stage network with bad output adapted from Fukuyama and Weber (2014, p. 459)

exogeneous constraints in case of a dashed arrow. In contrast, qualitative changes of object types exclusively take place by the transformation processes of squares.

Instead of $(\boldsymbol{x};\boldsymbol{y})$ with nonnegative inputs $\boldsymbol{x} \in \mathbb{R}_+^\kappa$ and outputs $\boldsymbol{y} \in \mathbb{R}_+^\kappa$, we mainly use $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$ to describe production activities when developing the general theory, but not necessarily when applying it to specific cases. Positive elements of vector $\boldsymbol{z}$ depict net outputs, negative elements net inputs. Activity $\boldsymbol{z}$ comprises all object types $k \in \mathbb{K} = \{1, \ldots, \kappa\}$ that are relevant for the description of each single unit as well as the whole network as production system. However, for the activities $\boldsymbol{z}^\rho \in \mathcal{P}^\rho$ of a specific production unit, most of their elements may be zero (and thus neglected). With $\rho \in \mathbb{P}$ denoting the individual units of a network, the *production possibility set* (PPS) of unit ρ is determined by its feasible activities:

$$\mathcal{P}^\rho = \{\boldsymbol{z} \in \mathbb{R}^\kappa | \text{Activity}\, \boldsymbol{z}\, \text{can be realised by unit}\, \rho\} \tag{1}$$

The PPS is determined through its technology and may be restricted by individual constraints (that are not caused by the network itself). It may be known as ‘blue-print technology’ from its construction by engineers or derived from observed data, e.g. by DEA methods.

Assumption 2.1: Each PPS $\mathcal{P}^\rho (\rho \in \mathbb{P})$ is non-empty and closed.

Further fundamental properties will be assumed later on. In any case, to derive corresponding properties of the whole production system it is necessary to model the connections between the individual units existing in the network. Vector $\boldsymbol{z}^\mathbb{S} \in \mathbb{R}^\kappa$ (or simply $\boldsymbol{z}$) denotes total input and output quantities of the whole system determined by the network. To derive properties of the whole production system it is necessary to model the connections between the individual units forming the network.

Assumption 2.2: It is supposed that no leakages or losses of object flows within the network occur, so that the activity of the whole system completely results from the activities of its single units. For each object type $k \in \mathbb{K}$ the flows of process inputs and outputs $\boldsymbol{z}^\rho$ of all individual production units $\rho \in \mathbb{P}$ are balanced with the total system input and output $\boldsymbol{z}^\mathbb{S}$ within the considered production period such that differences result in a change Δs_k of stock of this object type:

$$\Delta s_k = \sum_{\rho \in \mathbb{P}} z_k^\rho - z_k^\mathbb{S} \tag{2}$$

That is, the total quantity of any object type that occurs in the system must on the one hand stem from parts that are either procured from outside or produced inside or taken from stock and must on the other hand be used as parts that are consumed inside or delivered outwards or stored, alternatively. Immaterial objects like services cannot be stored so that $\triangle s_k = 0$. Regarding our static analysis, material objects taken from stock $(\Delta s_k < 0)$ can be subsumed under system inputs, whereas those stored $(\Delta s_k > 0)$ can be added to system outputs. The PPS of the whole system is thus determined by:

$$\mathcal{P}^\mathbb{S} = \left\{ \boldsymbol{z} \in \mathbb{R}^\kappa | \boldsymbol{z} = \sum_{\rho \in \mathbb{P}} \boldsymbol{z}^\rho, \boldsymbol{z}^\rho \in \mathcal{P}^\rho, \rho \in \mathbb{P} \right\} \tag{3}$$

Production possibilities of units are not only connected but moreover restricted because object flows between them may become binding. Objects of a type that can only be supplied to the system but not delivered from it are restricted by $z_k \le 0$; or by $z_k \ge 0$ in the opposite case. A pure intermediate product is characterised by $z_k = 0$. They are special cases of lower and upper bounds $\underline{z}_k \le z_k \le \overline{z}_k$ for system inputs or outputs that may exist in general but need often not be binding.

Definition 2.1: The network of a production system is *loose* when each unit of the system is free to choose its process inputs and outputs without any possibly binding exogeneous constraints regarding the resulting input and output of the whole system. Otherwise it is called *tied*.

In case (3), the PPS is a loose network that is uniquely determined by the combined object flows resulting from the unrestricted flows of its individual units. If it is possibly tied by lower or upper bounds one obtains in general:

$$\mathcal{P}^{\mathbb{S}} = \left\{ z \in \mathbb{R}^{\kappa} | z = \sum_{\rho \in \mathbb{P}} z^{\rho}, z^{\rho} \in \mathcal{P}^{\rho}, \rho \in \mathbb{P}, \underline{z}_k \le z_k \le \overline{z}_k, k \in \mathbb{K} \right\} \tag{4}$$

Example 2.1: Regarding the network of Fig. 1, each PPS of units A and B is generally described by its feasible process inputs and outputs:

$$\mathcal{P}^{A} = \{ (z_1^A, z_2^A, z_3^A, z_4^A, z_5^A, 0, 0, 0, 0) | (z_1^A, z_2^A, z_3^A) \in \mathbb{R}^3_- \text{can produce} (z_4^A, z_5^A) \in \mathbb{R}^2_+ \}$$

$$\mathcal{P}^{B} = \{ (0, 0, 0, z_4^B, z_5^B, z_6^B, z_7^B, z_8^B, z_9^B) | (z_4^B, z_5^B) \in \mathbb{R}^2_- \text{can produce} (z_6^B, z_7^B, z_8^B, z_9^B) \in \mathbb{R}^4_+ \}$$

According to (2) and (3) network balances of system inputs and outputs in Fig. 1 are determined by:

$$z_i = z_i^A (i \in \{1, 2, 3\}), z_k = z_k^A + z_k^B (k \in \{4, 5\}), z_r = z_r^B (r \in \{6, \dots, 9\})$$

System inputs and outputs of both intermediate products in Fig. 1 may be restricted by $0 \le z_k \le \overline{z}_k (k \in \{4, 5\})$, allowing for excess production in case of free disposability. Thus, together with above balances and both units' PPS, network PPS (4) of Fig. 1 is determined.

Two particular instances of the general network framework are well-known. They are characterised by specific technologies of the units' PPS according to following remarks.

Remark 2.1: Let the PPS of each production unit be determined by a single *basic activity* $\boldsymbol{a}^{\rho} = (a_1^{\rho}, \dots, a_{\kappa}^{\rho}) \in \mathbb{R}^{\kappa}$ that can be arbitrarily multiplied such that:

$$\mathcal{P}^{\rho} = \{ z \in \mathbb{R}^{\kappa} | z = \boldsymbol{a}^{\rho} \cdot \lambda, \lambda \ge 0 \} \tag{5}$$

Then, the PPS of a corresponding loose network is described by:

$$\mathcal{P}^{\mathbb{S}} = \left\{ z \in \mathbb{R}^{\kappa} | z = \sum_{\rho \in \mathbb{P}} \boldsymbol{a}^{\rho} \lambda^{\rho}, \lambda^{\rho} \ge 0, \rho \in \mathbb{P} \right\} \tag{6}$$

Such *polyhedric cone technologies* are the subject of Koopmans' (1951) original paper, which thus can be interpreted as an analysis of networks of the specific type of ray technologies (5).

Remark 2.2: *Network DEA* considers those special cases where n decision making units (DMUs) $j \in \mathbb{J} = \{1, \dots, n\}$ with an identical structure of production units (or 'divisions') $\rho \in \mathbb{P}$ are observed such that their input and output quantities $\boldsymbol{a}_j^{\rho} = (a_{1j}^{\rho}, \dots, a_{\kappa j}^{\rho})$ are known. Enveloping this data allows to construct an empirically determined PPS for each unit if certain properties of a minimum enveloping hull are assumed. For example, in case of a *linear hull* we obtain:

$$\mathcal{P}^{\rho} = \left\{ z \in \mathbb{R}^{\kappa} | z = \sum_{j \in \mathbb{J}} \boldsymbol{a}_j^{\rho} \lambda_j^{\rho}, \lambda_j^{\rho} \ge 0, j \in \mathbb{J} \right\}, \rho \in \mathbb{P} \tag{7}$$

Then, corresponding PPS of a tied network becomes:

$$\mathcal{P}^{\mathbb{S}} = \left\{ z \in \mathbb{R}^{\kappa} | z = \sum_{j \in \mathbb{J}} \sum_{\rho \in \mathbb{P}} \boldsymbol{a}_j^{\rho} \lambda_j^{\rho}, \lambda_j^{\rho} \ge 0, j \in \mathbb{J}, \rho \in \mathbb{P}, \underline{z}_k \le z_k \le \overline{z}_k, k \in \mathbb{K} \right\} \tag{8}$$

In case of a convex hull, constraints $\underline{z}_k \le z_k \le \overline{z}_k$ are always satisfied if the aggregated observed system data $\boldsymbol{a}_j = \sum_{\rho \in \mathbb{P}} \boldsymbol{a}_j^{\rho}$ of every DMU $j \in \mathbb{J}$ already comply with them. For linear data envelopment (8), however, finite bounds for system imports or exports must not exist so that $\underline{z}_k \in \{-\infty, 0\}$ and $\overline{z}_k \in \{0, +\infty\}$. If the activity levels (or intensity variables) of each DMU are identical for all production units, i.e. $\lambda_j^{\rho} = \lambda_j (\rho \in \mathbb{P})$, one obtains PPS $\mathcal{P}^{\mathbb{O}} \subset \mathcal{P}^{\mathbb{S}}$ as *black box* of the network:

$$\mathcal{P}^{\mathbb{O}} = \left\{ z \in \mathbb{R}^{\kappa} | z = \sum_{j \in \mathbb{J}} \boldsymbol{a}_j \lambda_j, \lambda_j \ge 0, j \in \mathbb{J} \right\} \tag{9}$$

2.2 Inputs and outputs as unsophisticated multiple performance criteria

Dyckhoff and Allen (2001, pp. 320-322) proposed a systematic approach for deriving (ecologically) generalised DEA models by integrating multi-criteria analysis and production theory for performance assessment. For demonstration, they introduced a general additive DEA model with good and bad inputs and outputs (p. 315). It represents the unsophisticated case of the general approach where the input and output quantities themselves are used as simple proxy performance measures. This case will be utilised here for including undesirable objects into general network production theory.

Definition 2.2: Suppose all relevant object types $\mathbb{K} = \{1, \ldots, \kappa\}$ can be subdivided into two categories $\mathbb{G}$ and $\mathbb{B}$ such that $\mathbb{G} \cup \mathbb{B} = \mathbb{K}$ and $\mathbb{G} \cap \mathbb{B} = \varnothing$. Let each activity vector $\boldsymbol{z} = (\boldsymbol{z}_{\mathbb{G}}, \boldsymbol{z}_{\mathbb{B}}) \in \mathbb{R}^{\kappa}$ be differentiated accordingly. Define $\boldsymbol{d} = \boldsymbol{v}(\boldsymbol{z})$ for the specific *multiple value function* $\boldsymbol{v}(\boldsymbol{z}) := (\boldsymbol{z}_{\mathbb{G}}, -\boldsymbol{z}_{\mathbb{B}})$. If positive elements of vector $\boldsymbol{d}$ represent advantages and negative elements disadvantages – called *benefits* and *costs* generated by the considered activity $\boldsymbol{z}$ – objects of category $\mathbb{G}$ are desirable (*goods*) and those of category $\mathbb{B}$ undesirable (*bads*). That is to say, $\boldsymbol{z}_1 = (\boldsymbol{z}_{\mathbb{G}1}, \boldsymbol{z}_{\mathbb{B}1})$ *dominates* $\boldsymbol{z}_2 = (\boldsymbol{z}_{\mathbb{G}2}, \boldsymbol{z}_{\mathbb{B}2})$ if and only if (=when) $\boldsymbol{d}_1 = \boldsymbol{v}(\boldsymbol{z}_1) > \boldsymbol{v}(\boldsymbol{z}_2) = \boldsymbol{d}_2$.

Thus, each cost or benefit is measured in natural, often physical units of the respective object type. Goods as inputs may be called factors and as outputs products, bads as outputs pollutants and as inputs reducts (Dyckhoff and Allen, 2001, p. 315). That is, costs result from consuming factors and emitting pollutants, benefits from generating products and destroying reducts, e.g. in case of a waste incineration power plant: water vaporisation (cubic metres) and carbon dioxide emission (tons) lead to costs, power generation (megawatt-hours) and waste reduction (tons) to benefits. Dominance means less of good inputs or bad outputs or more of good outputs or bad inputs. It presumes that preferences for production activities are compatible with following partial preference order.

Assumption 2.3: If $\boldsymbol{z}_1$ dominates $\boldsymbol{z}_2$, production activity $\boldsymbol{z}_1$ is preferred to $\boldsymbol{z}_2$, i.e. the first is better and the second worse than the other one.

A bad constitutes the opposite of a good from the perspective of the preferences of the decision maker or evaluator. Whereas a good is an object that people would like to have access to and possession of, people seek to rid themselves of a bad and remove it from their sphere of responsibility and disposition, whether due to environmental regulation or individual motivation. A bad is characterised by the fact that it cannot be disposed of easily, but its removal is not costless because it requires the use of additional goods. Otherwise, one would adopt an indifferent (*neutral*) attitude towards this object and simply ignore it (Dyckhoff and Allen, 2001). A bad is thus an undesirable, negatively valued object that is not freely disposable, as a rule, and whose production is also undesirable. While its generation incurs economic, social, or ecological costs, its targeted utilisation or disposal through elimination or conversion as input in 'reduction' processes represents a benefit. In this sense, bads as input into a transformation process represent undesirable factors, of which more process input is preferred to less, ceteris paribus (Dyckhoff, 2023, p. 186).[7]

Hence, good inputs (outputs) and bad outputs (inputs) are treated in the same way. However, it must be stressed that this is only meant syntactically (mathematically), and in no way semantically (interpretively). Choosing a proper modelling is primarily a matter of better handling and would *not* imply that bad outputs *are* inputs or bad inputs *are* outputs. They are only treated *as if they were so*. Nonetheless, it improves clarity to use symbols for the technologically modelled inputs and outputs, but different ones for their preferentially modelled desirability. Therefore, it is appropriate to systematically distinguish the purely technological perspective of activities $\boldsymbol{z}$ on the one hand and the preference perspective of their multiple values $\boldsymbol{v}(\boldsymbol{z})$ according to Definition 2.2 on the other hand (Dyckhoff and Souren, 2022).[8]

Definition 2.3: Any PPS $\mathcal{P}$ with goods and bads corresponds to a *value possibility set* (VPS) $\mathcal{V} = \boldsymbol{v}(\mathcal{P})$ that is defined by

$$\mathcal{V} = \{\boldsymbol{d} \in \mathbb{R}^{\kappa} | \boldsymbol{d} = \boldsymbol{v}(\boldsymbol{z}) = (\boldsymbol{z}_{\mathbb{G}}, -\boldsymbol{z}_{\mathbb{B}}), (\boldsymbol{z}_{\mathbb{G}}, \boldsymbol{z}_{\mathbb{B}}) = \boldsymbol{z} \in \mathcal{P}\}$$

Without undesirable objects ($\mathbb{B} = \varnothing$) this extension reduces to $\boldsymbol{d} = \boldsymbol{z}$ and $\mathcal{V} = \mathcal{P}$, i.e. production theory for goods only ($\mathbb{K} = \mathbb{G}$), so that technological and preferential perspectives coincide formally (but not in terms of content).

With bads, however, the *dominating set* of activity $\boldsymbol{z}_o$ in input-output space and that of its net benefits $\boldsymbol{d}_o$ in value space differ:

$$\mathcal{D}(\boldsymbol{z}_o|\mathcal{P}) = \{\boldsymbol{z} = (\boldsymbol{z}_{\mathbb{G}}, \boldsymbol{z}_{\mathbb{B}}) \in \mathcal{P} | (\boldsymbol{z}_{\mathbb{G}}, -\boldsymbol{z}_{\mathbb{B}}) \geq (\boldsymbol{z}_{\mathbb{G}o}, -\boldsymbol{z}_{\mathbb{B}o})\}$$

$$\mathcal{D}(\boldsymbol{d}_o|\mathcal{V}) = \{\boldsymbol{d} \in \mathcal{V} | \boldsymbol{d} \geq \boldsymbol{d}_o\}$$

[7] The utilisation and disposal of bads is of eminent practical importance in a 'full world' where the ecological impact of economic activities is reaching the physical limits of planet Earth. Contrary to production, which provides society with goods for consumption, these reverse processes – that may be called *reduction* as opposite to production (like assembly versus disassembly) – serve to rid society of undesirable residues of production and consumption through recycling, recovery, and disposal processes, forming the third phase of a circular economy. Bad objects thus generally initially emerge as unintended output of production and consumption, but later form intended input of reduction processes.

[8] The multi-dimensional notions 'value function' and 'value possibility set', based on Dyckhoff and Allen (2001) and termed here in accordance with Dyckhoff and Souren (2022), must be distinguished from the one-dimensional value notion introduced by Halme et al. (1999) with a pseudo-concave preference function. Latter authors developed a theory and procedures for complementing efficiency measurement with preference information, which they call "value efficiency analysis". Both different value concepts are reviewed and categorised by Dyckhoff and Souren (2022, p. 806).

Lemma 2.1: Following assertions are true:

(a) *With* $\boldsymbol{d}_{\mathbb{G}} = z_{\mathbb{G}}$ *and* $\boldsymbol{d}_{\mathbb{B}} = -z_{\mathbb{B}}$, *value function* $\boldsymbol{v}(z)$ *determines a bijective relation between* $\mathcal{P}$ *and* $\mathcal{V}$ *such that:* $z = (z_{\mathbb{G}}; z_{\mathbb{B}}) \in \mathcal{P} \Longleftrightarrow \boldsymbol{d} = (\boldsymbol{d}_{\mathbb{G}}; \boldsymbol{d}_{\mathbb{B}}) \in \mathcal{V}$.

(b) $\mathcal{V}$ *is non-empty, closed, or bounded if and only if* $\mathcal{P}$ *has the respective property, too.*

(c) $\mathcal{D}(z_o|\mathcal{P})$ *is compact if and only if* $\mathcal{D}(\boldsymbol{d}_o|\mathcal{V})$ *is compact.*

With these preparations a further basic assumption about the PPS of individual units, in addition to Assumption 2.1, can be made (implying the same for the whole network according to Theorem 3.1).

Assumption 2.4:[9] The dominating set $\mathcal{D}(z_o|\mathcal{P}^\rho)$ of each feasible activity $z_o \in \mathcal{P}^\rho (\rho \in \mathbb{P})$ is compact.

As a rule, every PPS known from linear activity analysis and network DEA, such as (5) and (7), complies with this assumption. Koopmans (1951, p. 47ff) explicitly postulated four fundamental properties for his linear technologies (6). Dyckhoff and Souren (2024) formulate the first two of these properties plus a third – that is a weak, integrated version of Koopmans' last two – in terms of networks of general technologies, though still without bads. While the first postulate is purely technological – and thus needs not to be changed – the other two must be extended to cover undesirable objects (cf. Dyckhoff (1992), pp. 73-79).

Postulates: Following properties should hold for any (loose or tied) production network and analogously for all its subsystems:

I. *Irreversibility of production*: $\mathcal{P}^{\mathbb{S}} \cap -\mathcal{P}^{\mathbb{S}} \subseteq \{\boldsymbol{0}\}$, i.e., if $z_1, z_2 \in \mathcal{P}^{\mathbb{S}}$ such that $z_1 + z_2 = \boldsymbol{0}$, then $z_1 = z_2 = \boldsymbol{0}$.

II. *Impossibility of the Land of Cockaigne* (*No free lunch*): $\mathcal{V}^{\mathbb{S}} \cap \mathbb{R}_+^\kappa \subseteq \{\boldsymbol{0}\}$, i.e., there exists no activity $z \in \mathcal{P}^{\mathbb{S}}$ with $\boldsymbol{d} = \boldsymbol{v}(z) > \boldsymbol{0}$.

III. *Possibility of production with benefit*: $\mathcal{V}^{\mathbb{S}} \setminus \mathbb{R}_-^\kappa \neq \varnothing$, i.e., there exists an activity $z \in \mathcal{P}^{\mathbb{S}}$ with positive benefit $d_k > 0$, i.e. $z_k > 0, k \in \mathbb{G}$ or $z_k < 0$, $k \in \mathbb{B}$ for at least one $k \in \mathbb{K}$.

With Koopmans (1951, p. 47) it is not claimed "that in all uses of models of production these properties should be present. Rather, it is believed that in a broad class of cases it will be useful to employ models having these properties." The first two postulates reflect the Second Law of thermodynamics which implies that a *perpetuum mobile* is impossible (cf. Remark 2.8 of Dyckhoff and Souren (2024)).

2.3 Efficiency of production with goods and bads

Before analysing properties of general networks with bad inputs and outputs, some fundamental aspects of production efficiency are considered next, of individual units or of whole systems. Activity $z_* \in \mathcal{P}$ is *efficient* when it is not dominated by any other possible production activity. The set $\partial_E \mathcal{P}$ of all efficient activities of PPS $\mathcal{P}$ is called its *efficient frontier*. With Definition 2.2 of multiple values it is determined by

$$\partial_E \mathcal{P} = \{z_* \in \mathcal{P} | \boldsymbol{v}(z) \geq \boldsymbol{v}(z_*) \wedge z \in \mathcal{P} \Longrightarrow z = z_*\} \qquad (10)$$

Since a free lunch is assumed to be impossible (Postulate *II*) and because of $\boldsymbol{v}(\boldsymbol{0}) = \boldsymbol{0}$, 'doing nothing' $z = \boldsymbol{0}$ is always efficient if it is feasible at all, i.e., $\boldsymbol{0} \in \mathcal{P} \Longrightarrow \boldsymbol{0} \in \partial_E \mathcal{P}$.

A different way to characterise efficiency is established by dominating set $\mathcal{D}(z_o|\mathcal{P})$:

$$z_o \in \partial_E \mathcal{P} \Longleftrightarrow \mathcal{D}(z_o|\mathcal{P}) = \{z_o\}$$

Assumption 2.4 of compact dominating sets has following proposition as consequence:

Lemma 2.2: *Let be* $z_o \in \mathcal{P}$ *with compact dominating set* $\mathcal{D}(z_o|\mathcal{P})$. *If* $\tilde{z} \in \mathcal{D}(z_o|\mathcal{P})$ *is inefficient there exists an efficient* $z_* \in \partial_E \mathcal{D}(z_o|\mathcal{P}) = \mathcal{D}(z_o|\mathcal{P}) \cap \partial_E \mathcal{P}$ *dominating it:* $\boldsymbol{v}(z_*) > \boldsymbol{v}(\tilde{z})$.

Since most possible production activities are inefficient it is of interest to measure their degree of inefficiency.

Definition 2.4: A real-valued function $e : \mathcal{P} \rightarrow \mathbb{R}_+$ is an (at least weak) *efficiency measure* for $z \in \mathcal{P}$ regarding PPS $\mathcal{P}$ and value function $\boldsymbol{v}(z)$ when:

(i) $e(z_1) \geq e(z_2)$ if $\boldsymbol{v}(z_1) \geq \boldsymbol{v}(z_2)$
(ii) $e(z_1) > e(z_2)$ if $\boldsymbol{v}(z_1) \gg \boldsymbol{v}(z_2)$
(iii) $e(z) = 1$ if $z \in \partial_E \mathcal{P}$

The efficiency measure is *strong* when:

(iv) $e(z_1) > e(z_2)$ if $\boldsymbol{v}(z_1) > \boldsymbol{v}(z_2)$

With compact dominating sets, any such efficiency measure is well-defined, bounded, and normalised, so that $0 \leq e(z) \leq 1$ for all $z \in \mathcal{P}$. Moreover, it is obvious that any strong efficiency measure identifies efficient activities unambiguously: $e(z) = 1 \Longleftrightarrow z \in \partial_E \mathcal{P}$.

To determine an appropriate efficiency score $e(z_o|\mathcal{P})$ for activity z_o, it is common practice to optimise a certain objective function $\triangle(z; z_o)$ over dominating set $\mathcal{D}(z_o|\mathcal{P})$ with respect to the relevant PPS $\mathcal{P}$. With bads this procedure can be generalised by considering net benefits

[9] Mehdiloo and Podinovski (2021, p. 297) showed by counterexample that the conventional axioms of closeness of the PPS and boundedness of its output sets – without additional assumptions – are not sufficient for the existence of efficient activities. Instead, they prove as sufficient and necessary condition for the existence of at least one efficient activity that there exists a feasible activity whose dominating set is compact, i.e. closed and bounded. Therefore, before assuming general properties of the PPS of any unit or network with goods and bads, dominance of activities had to be introduced.

$\boldsymbol{d} = \boldsymbol{v}(\boldsymbol{z})$, VPS $\mathcal{V} = \boldsymbol{v}(\mathcal{P})$, and $\mathcal{D}(\boldsymbol{d}_o|\mathcal{V}) = \{\boldsymbol{d} \in \mathcal{V}|\boldsymbol{d} \geq \boldsymbol{d}_o\}$ instead:

$$(e(\boldsymbol{z}_o|\mathcal{P}))^{-1} = \triangle(\boldsymbol{d}_*;\boldsymbol{d}_o) = \max\{\triangle(\boldsymbol{d};\boldsymbol{d}_o)|\boldsymbol{d} \in \mathcal{D}(\boldsymbol{d}_o|\mathcal{V}), \boldsymbol{d}_o = \boldsymbol{v}(\boldsymbol{z}_o)\} \quad (11)$$

Vector $\boldsymbol{d} - \boldsymbol{d}_o \in \mathbb{R}^\kappa_+$ describes the *improvements* of realising activity z instead of z_o with respect to each object type $k \in \mathbb{K} = \{1, \ldots, \kappa\}$. Objective function $\Delta:\mathbb{R}^\kappa \times \mathbb{R}^\kappa \to [1, \infty)$ is supposed to be non-decreasing in aggregating all individual improvements into a single *value of advancement* which is larger or equal to one with $\triangle(\boldsymbol{d}_o;\boldsymbol{d}_o) = 1$. Equation (11) states that the efficiency score is defined as inverse of the maximum value of advancement, so that $0 \leq e(\boldsymbol{z}_o|\mathcal{P}) \leq 1$.

Theorem 2.1: *Any efficiency measure of type (11) is monotonously non-increasing when enlarging the PPS, so that* $e(\boldsymbol{z}_o|\mathcal{P}_1) \geq e(\boldsymbol{z}_o|\mathcal{P}_2)$ *for* $\boldsymbol{z}_o \in \mathcal{P}_1 \subset \mathcal{P}_2$.

This monotonicity is well-known from DEA where the efficiency score of a convex envelopment is not smaller than that of the respective linear envelopment. To discuss particular well-known efficiency measures, the notation of nonnegative inputs and outputs, and now also of costs and benefits, that distinguishes between goods and bads is often more appropriate.

Remark 2.3: Regarding the common notation of nonnegative inputs and outputs, activities with goods and bads are described by $(\boldsymbol{x};\boldsymbol{y}) \equiv (\boldsymbol{x}_\mathbb{G}, \boldsymbol{x}_\mathbb{B}; \boldsymbol{y}_\mathbb{G}, \boldsymbol{y}_\mathbb{B}) \in \mathbb{R}^{2\kappa}_+$, costs by $\boldsymbol{c} = (\boldsymbol{x}_\mathbb{G}, \boldsymbol{y}_\mathbb{B}) \in \mathbb{R}^\kappa_+$ and benefits by $\boldsymbol{b} = (\boldsymbol{y}_\mathbb{G}, \boldsymbol{x}_\mathbb{B}) \in \mathbb{R}^\kappa_+$. Thus, $\boldsymbol{d} = \boldsymbol{b} - \boldsymbol{c} = (\boldsymbol{y}_\mathbb{G} - \boldsymbol{x}_\mathbb{G}, \boldsymbol{x}_\mathbb{B} - \boldsymbol{y}_\mathbb{B}) = (\boldsymbol{z}_\mathbb{G}; -\boldsymbol{z}_\mathbb{B})$. Without bads one obtains: $(\boldsymbol{c};\boldsymbol{b}) = (\boldsymbol{x}_\mathbb{G};\boldsymbol{y}_\mathbb{G}) = (\boldsymbol{x};\boldsymbol{y})$.

In this way, with $\boldsymbol{d} = \boldsymbol{b} - \boldsymbol{c}$, improvements can be differentiated into *cost reductions* $\boldsymbol{s} = \boldsymbol{c}_o - \boldsymbol{c} = (\boldsymbol{x}_{\mathbb{G}o} - \boldsymbol{x}_\mathbb{G}, \boldsymbol{y}_{\mathbb{B}o} - \boldsymbol{y}_\mathbb{B}) \in \mathbb{R}^\kappa_+$ and *benefit augmentations* $\boldsymbol{t} = \boldsymbol{b} - \boldsymbol{b}_o = (\boldsymbol{y}_\mathbb{G} - \boldsymbol{y}_{\mathbb{G}o}, \boldsymbol{x}_\mathbb{B} - \boldsymbol{x}_{\mathbb{B}o}) \in \mathbb{R}^\kappa_+$. Without bads they are called (input and output) "slacks" $\boldsymbol{s} = \boldsymbol{x}_o - \boldsymbol{x}$ and $\boldsymbol{t} = \boldsymbol{y} - \boldsymbol{y}_o$.

Extending the unoriented "slack-based efficiency measure" of Tone (2001) by including bads leads to following function, whereby $\mathbb{K}^b_o := \{k \in \mathbb{K}|b_{ko} > 0\}$ and $\mathbb{K}^c_o := \{k \in \mathbb{K}|c_{ko} > 0\}$:

$$\begin{aligned} &\Delta(\boldsymbol{d};\boldsymbol{d}_o) = \tfrac{\eta}{\theta} \text{ with} \\ &\quad \eta = 1 + \tfrac{1}{|\mathbb{K}^b_o|}\sum_{k \in \mathbb{K}^b_o} \tfrac{t_k}{b_{ko}} = \tfrac{1}{|\mathbb{K}^b_o|}\sum_{k \in \mathbb{K}^b_o} \tfrac{b_k}{b_{ko}} \text{ and} \\ &\quad \theta = 1 - \tfrac{1}{|\mathbb{K}^c_o|}\sum_{k \in \mathbb{K}^c_o} \tfrac{s_k}{c_{ko}} = \tfrac{1}{|\mathbb{K}^c_o|}\sum_{k \in \mathbb{K}^c_o} \tfrac{c_k}{c_{ko}} \end{aligned} \quad (12)$$

Analogously, we get a benefit-oriented measure by setting:

$$\Delta(\boldsymbol{d};\boldsymbol{d}_o) = \eta \text{ with } \eta = \min\left\{\frac{b_k}{b_{ko}}|k \in \mathbb{K}^b_o\right\} = 1 + \min\left\{\frac{t_k}{b_{ko}}|k \in \mathbb{K}^b_o\right\} \quad (13)$$

And a cost-oriented measure by:

$$\Delta(\boldsymbol{d};\boldsymbol{d}_o) = \frac{1}{\theta} \text{ with } \theta = \max\left\{\frac{c_k}{c_{ko}}|k \in \mathbb{K}^c_o\right\} = 1 - \min\left\{\frac{s_k}{c_{ko}}|k \in \mathbb{K}^c_o\right\} \quad (14)$$

With $b_k \geq b_{ko}$ and $c_k \leq c_{ko}$ for $\boldsymbol{d} \in \mathcal{D}(\boldsymbol{d}_o|\mathcal{V})$ it follows from (12) to (14) for all kinds of networks with goods and bads, i.e. independent of the technology of the underlying PPS:

$$\frac{\frac{1}{|\mathbb{K}^b_o|}\sum_{k \in \mathbb{K}^b_o} \frac{b_k}{b_{ko}}}{\frac{1}{|\mathbb{K}^c_o|}\sum_{k \in \mathbb{K}^c_o} \frac{c_k}{c_{ko}}} \geq \frac{1}{|\mathbb{K}^b_o|}\sum_{k \in \mathbb{K}^b_o} \frac{b_k}{b_{ko}} \geq \min\left\{\frac{b_k}{b_{ko}}|k \in \mathbb{K}^b_o\right\} \quad (15)$$

$$\frac{\frac{1}{|\mathbb{K}^c_o|}\sum_{k \in \mathbb{K}^c_o} \frac{c_k}{c_{ko}}}{\frac{1}{|\mathbb{K}^b_o|}\sum_{k \in \mathbb{K}^b_o} \frac{b_k}{b_{ko}}} \leq \frac{1}{|\mathbb{K}^c_o|}\sum_{k \in \mathbb{K}^c_o} \frac{c_k}{c_{ko}} \leq \max\left\{\frac{c_k}{c_{ko}}|k \in \mathbb{K}^c_o\right\} \quad (16)$$

Theorem 2.2: *For any PPS $\mathcal{P}$ fulfilling the general requirements of Section 2, an efficiency score (11) based on function (12) is not greater than a respective score that is based on function (13) or (14).*

Remark 2.4: Oriented efficiency measures determined by (11) with functions (13) and (14) are weak so that their optimisation may provide solutions that are merely weakly efficient, i.e. inefficient. This may also be true for function (12), except for activities z_o with $z_k \neq 0$ for all $k \in \mathbb{K}$. In general, strictly increasing functions $\triangle(\boldsymbol{d};\boldsymbol{d}_o)$ are needed to measure strong efficiency in any case.

3 Properties of production networks with goods and bads

Now, networks with goods and bads are analysed, provided they fulfil the assumptions of Section 2. First, a general theorem for loose networks is proven. Second, networks with convex technologies are analysed. Third, production networks with polyhedral technologies are considered, the efficiency of which will be assessed in Section 4.

3.1 General networks of arbitrary technologies

The previous assumptions are rather weak and allow for a non-convex or even discrete PPS of individual network units. Assumptions 2.1 and 2.4 and Lemma 2.2 imply the same properties of the whole network.

Theorem 3.1: *Following properties hold for any network of production units with goods and bads:*

(a) *PPS $\mathcal{P}^{\mathbb{S}}$ is non-empty and closed.*
(b) *Dominating set $\mathcal{D}(z|\mathcal{P}^{\mathbb{S}})$ of each activity $z \in \mathcal{P}^{\mathbb{S}}$ of the whole network is compact.*
(c) *For each production unit as well as for the whole network there exists at least one efficient activity, i.e. $\partial_E \mathcal{P}^{\rho} \neq \emptyset, \rho \in \mathbb{P}$, and $\partial_E \mathcal{P}^{\mathbb{S}} \neq \emptyset$.*

How does the efficiency of individual production units relate to that of the whole system formed by their network? In generalising as well as sharpening a respective proposition of Dyckhoff and Souren (2024) for networks without bads, a fundamental distinction between loose and tied networks must be made.

Theorem 3.2: *Let $z^{\mathbb{S}} = \sum_{\rho=1}^{\pi} z^{\rho} \in \mathcal{P}^{\mathbb{S}}$ be a feasible activity of a network of production units. This system activity is inefficient if one of its unit activities $z^{\rho} \in \mathcal{P}^{\rho}$ is inefficient and can be replaced by a dominating activity such that the so defined new system activity is also feasible. Hence, if $e(z^{\mathbb{S}}|\mathcal{P}^{\mathbb{S}}) = 1$ for strong efficiency measure $e:\mathcal{P}^{\rho} \to \mathbb{R}_{+}$ then $e(z^{\rho}|\mathcal{P}^{\rho}) = 1$ for all production units $\rho \in \mathbb{P}$ in case of a loose network.*

Corollary 3.1 (Efficiency theorem for loose networks with goods and bads): *If the activity of a single production unit is inefficient regarding its own production possibilities, then all system activities of a loose production network that it is part of are inefficient, too.*

Remark 3.1: Using counterexamples without bads, Dyckhoff and Souren (2024) show that an analogous proposition for tied networks as well as a converse assertion for loose networks are not true in general:

(a) If the network is tied the combination of inefficient unit activities may add up to an efficient system activity.
(b) A system activity may be inefficient although it is a combination of efficient unit activities in a loose network.

Remark 3.2: With respect to Theorem 3.2 and Corollary 3.1, it is of crucial importance that any excess output of an intermediate product is indeed desirable if it is classified as "good" according to Definition 2.2 and Assumption 2.3. In contrast, the common assumption in (network) DEA of free disposability of such overproduction states that excess output has no value, signifying a certain kind of inconsistency in valuing (intermediate) products. Free disposability implies that quantities of objects of a certain type are desirable (good) whereas other quantities of the same object type are of no value (free or neutral), i.e. that their desirability is not fixed but depends on their produced or consumed quantity (cf. Dyckhoff (2023) in this regard).

3.2 Networks of production units with convex technologies

By assuming specific properties of the PPS of all units, especially convexity or linearity, one may obtain corresponding ones for the whole network. As long as dominance, efficiency, or other valuation aspects do not play an essential role, respective propositions for networks with solely goods are not changed by extending them to cover undesirable objects, too. This is e.g. true for

Theorem 3.3 (Dyckhoff and Souren, 2024): *Following properties hold for any network of production units:*

(a) *If 'doing nothing' is possible for each individual unit, so for the whole network, too ($\mathbf{0} \in \mathcal{P}^{\rho}$, $\rho \in \mathbb{P} \Longrightarrow \mathbf{0} \in \mathcal{P}^{\mathbb{S}}$), provided existing ties satisfy $\underline{z}_k \leq 0 \leq \bar{z}_k$ for all $k \in \mathbb{K}$.*
(b) *If each unit's PPS is convex, so also the PPS of the whole network.*
(c) *If PPS $\mathcal{P}^{\mathbb{S}}$ is convex and 'doing nothing' is feasible ($\mathbf{0} \in \mathcal{P}^{\mathbb{S}}$), the network exhibits non-increasing returns to scale, i.e., $z \in \mathcal{P}^{\mathbb{S}}, 0 \leq \lambda \leq 1 \Longrightarrow \lambda z \in \mathcal{P}^{\mathbb{S}}$.*

In view of Definition 2.3, following proposition is important if dominance or efficiency are relevant.

Lemma 3.1: *Let $\mathcal{P}$ be any PPS of an individual unit or of a whole network. Its VPS $\mathcal{V} = \boldsymbol{v}(\mathcal{P})$ is linear or convex if and only if $\mathcal{P}$ has the same property, too.*

The next proposition is true for any convex PPS of a unit or network with goods and bads.

Theorem 3.4: *Let $\mathcal{P}$ be any convex PPS and $z_* \in \partial_E \mathcal{P}$ an efficient activity that is a convex combination of two other feasible activities $z_1, z_2 \in \mathcal{P}$ such that $z_* = \alpha z_1 + (1-\alpha) z_2$ for a specific α with $0 < \alpha < 1$. Then $z = \beta z_1 + (1-\beta) z_2 \in \partial_E \mathcal{P}$ for all β with $0 \leq \beta \leq 1$, i.e., activities z_1* and *z_2 as well as all their convex combinations are efficient, too.*

3.3 Network of production units with polyhedral technologies

Network DEA usually considers networks of production units with polyhedral technologies, as defined by (8) in Remark 2.2. They are closed, non-empty with $\mathbf{0} \in \mathcal{P}^{\rho}$, and convex (or even linear) with compact dominating sets. Thus, all general propositions of Sections 2 and 3 are valid for these types of networks with goods and bads, though, solely the respective propositions in case of networks with binding ties. For example, Theorem 2.1 directly implies following general relationship.

Corollary 3.2: *Let be $\mathcal{P}^{\mathbb{S}}$ any PPS of network type (8) of polyhedral technologies and $\mathcal{P}^{\mathbb{O}}$ its corresponding black-box PPS (9) such that $\mathcal{P}^{\mathbb{O}} \subset \mathcal{P}^{\mathbb{S}}$.* Then *$e(z_o|\mathcal{P}^{\mathbb{O}}) \geq e(z_o|\mathcal{P}^{\mathbb{S}})$ for all $z_o \in \mathcal{P}^{\mathbb{O}}$ and each efficiency measure of type (11).*

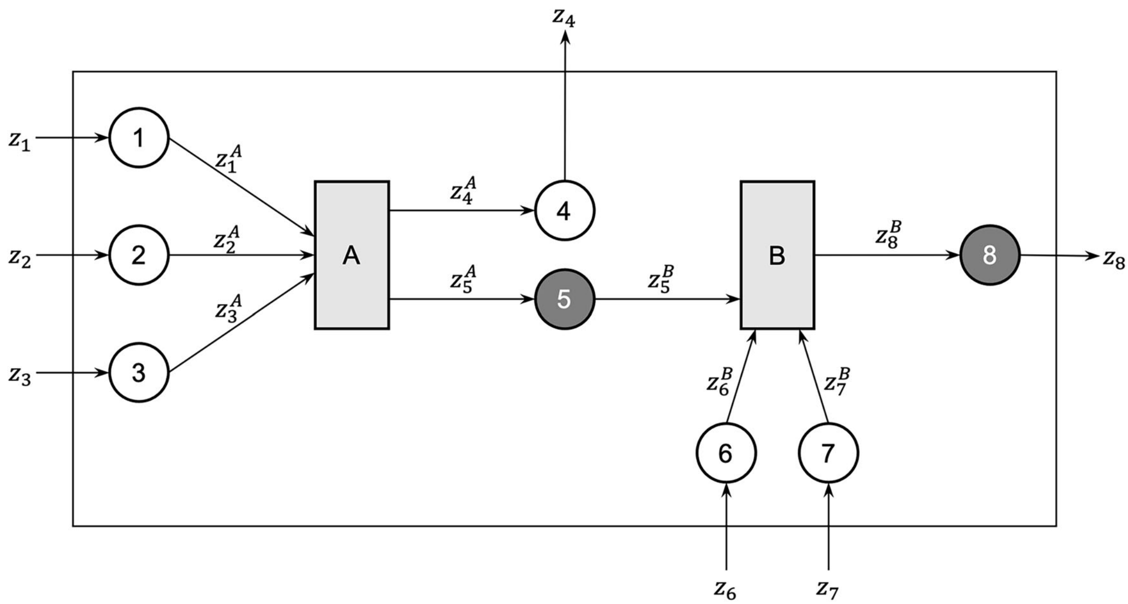

Fig. 2 Network with production and abatement stage adapted from Chen et al. (2018, p. 398)

That is, the black-box efficiency score of any DMU cannot be smaller than its network score. The next section shows that it usually is even definitely larger. Section 4 also illustrates other previous findings and furthermore demonstrates a systematic procedure for measuring efficiency of network activities with goods and bads based on the general network production theory developed here.

4 Application of the general theory to network DEA

The network in Fig. 2 is used as example. It again shows a two-stage network (namely of the industrial water system of 30 regions in China that is adapted from Chen et al. (2018)). Stage A consumes three primary factors #1 to #3 from which final product #4 plus pollutant #5 as pure intermediate product emerge as process outputs. Stage B ("environmental protection") treats pollutant #5 by using two additional primary factors #6 and #7, resulting in a final pollutant #8.

Object types #5 and #8 are assumed to be bads, the other goods. Pollutant #5 is the sole intermediate type of objects, emerging from stage A and completely treated as waste in stage B, where a new bad #8 originates as final output of the network (which should be less harmful than #5 in order to make sense). This implies preferences with following dominating set of any activity z_o regarding a PPS $\mathcal{P}$ in input-output space:

$$\mathcal{D}(z_o|\mathcal{P}) = \{z = (z_1, , z_8) \in \mathcal{P}|z_k \geq z_{ko}, k \in \{1, \ldots, 4, 6, 7\}, z_k \leq z_{ko}, k \in \{5, 8\}\} \quad (17)$$

Now, typical DEA production technologies for the network in Fig. 2 are considered. They represent different envelopments of observed input-output data of DMUs whose performance is to be evaluated by methods of network DEA. With notation defined in Remark 2.2, linear hull (7) of a given data set of DMUs $\mathbb{J}$ as PPS of stage A as well as of stage B leads to:

$$\mathcal{P}^\rho = \left\{ z \in \mathbb{R}^8 | z = \sum_{j \in \mathbb{J}} \boldsymbol{a}_j^\rho \lambda_j^\rho, \lambda_j^\rho \geq 0, j \in \mathbb{J} \right\}, \rho \in \mathbb{P} = \{A, B\} \quad (18)$$

Then, PPS (8) of the *whole network* in Fig. 2 becomes:

$$\mathcal{P}^{\mathbb{S}} = \left\{ z \in \mathbb{R}^8 | z = \sum_{j \in \mathbb{J}} \sum_{\rho \in \mathbb{P}} \boldsymbol{a}_j^\rho \lambda_j^\rho, \lambda_j^\rho \geq 0, j \in \mathbb{J}, \rho \in \mathbb{P}, z_5 = 0 \right\} \quad (19)$$

With identical activity levels $\lambda_j^\rho = \lambda_j$ of all production units $\rho \in \mathbb{P}$ for each DMU $j \in \mathbb{J}$, and by setting $\boldsymbol{a}_j = \sum_{\rho \in \mathbb{P}} \boldsymbol{a}_j^\rho$, network PPS (19) is reduced to that of a *black box* DEA model:

$$\mathcal{P}^{\mathbb{O}} = \left\{ z \in \mathbb{R}^8 | z_k = \sum_{j \in \mathbb{J}} a_{kj} \lambda_j, \lambda_j \geq 0, j \in \mathbb{J}, k \in \mathbb{K} \right\} \quad (20)$$

Regarding Fig. 2, we have $a_{kj} = a_{kj}^A, k \in \{1, \ldots, 4\}$, $a_{5j} = 0, a_{kj} = a_{kj}^B, k \in \{6, 7, 8\}$. Note that $a_{kj}^A = 0$ for $k \in \{6, 7, 8\}$ and $a_{kj}^B = 0$ for $k \in \{1, \ldots, 4\}$.

4.1 Unoriented efficiency measure

To avoid negative numbers, as common practice in DEA literature, define $x_{kj}^{\rho} = -a_{kj}^{\rho}$ if $a_{kj}^{\rho} < 0$, $y_{kj}^{\rho} = a_{kj}^{\rho}$ if $a_{kj}^{\rho} > 0$, and $x_{kj}^{\rho} = y_{kj}^{\rho} = 0$ otherwise. Then, according to PPS (18), feasible process inputs x_k^{ρ} and outputs y_k^{ρ} of each production unit $\rho \in \mathbb{P}$ regarding object type $k \in \mathbb{K}$ are determined by the observed DMUs $j \in \mathbb{J}$ as follows:

$$x_k^{\rho} = \sum_{j \in \mathbb{J}} x_{kj}^{\rho} \lambda_j^{\rho}, y_k^{\rho} = \sum_{j \in \mathbb{J}} y_{kj}^{\rho} \lambda_j^{\rho}, \lambda_j^{\rho} \geq 0 \tag{21}$$

Now, efficiency measure (11) for unoriented function (12) is applied to the network of Fig. 2. Using above defined nonnegative notation x for inputs and y for outputs (instead of z and a) leads to following DEA model for DMU $o \in \mathbb{J}$ with respect to the production unit on stage A:

$$e_o^A = e\left(\boldsymbol{a}_o^A \middle| \mathcal{P}^A\right) = \min \frac{\frac{1}{4}\left(\sum_{i=1}^{3} \frac{x_i^A}{x_{io}^A} + \frac{y_5^A}{y_{5o}^A}\right)}{\frac{y_4^A}{y_{4o}^A}} \, subject\ to (21) \, for\ \rho = A\ with$$
$$x_i^A \leq x_{io}^A (i = 1, 2, 3),\ y_4^A \geq y_{4o}^A, y_5^A \leq y_{5o}^A \tag{22}$$

And with respect to stage B:

$$e_o^B = e\left(\boldsymbol{a}_o^B \middle| \mathcal{P}^B\right) = \min \frac{\frac{1}{3}\left(\sum_{i=6}^{7} \frac{x_i^B}{x_{io}^B} + \frac{y_8^B}{y_{8o}^B}\right)}{\frac{x_5^B}{x_{5o}^B}} \, subject\ to (21) \, for\ \rho = B\ with$$
$$x_5^B \geq x_{50}^B, x_i^B \leq x_{io}^B (i = 6, 7), y_8^B \leq y_{8o}^B \tag{23}$$

Regarding PPS (19) of the whole network, the DEA model corresponding to function (12) is determined by:

$$e_o^{\mathbb{S}} := e\left(\boldsymbol{a}_o \middle| \mathcal{P}^{\mathbb{S}}\right) = \min \frac{\frac{1}{6}\left(\sum_{i=1}^{3} \frac{x_i}{x_{io}} + \sum_{i=6}^{7} \frac{x_i}{x_{io}} + \frac{y_8}{y_{8o}}\right)}{\frac{y_4}{y_{4o}}} \, subject\ to\ (21)\ with$$
$$\begin{aligned} x_i &= x_i^A \leq x_{io}^A = x_{io} (i = 1, 2, 3), y_4 = y_4^A \geq y_{4o}^A = y_{4o} \\ z_5 &= y_5^A - x_5^B = 0 \\ x_i &= x_i^B \leq x_{io}^B = x_{io} (i = 6, 7), y_8 = y_8^B \leq y_{8o}^B = y_{8o} \end{aligned} \tag{24}$$

Regarding PPS (20), the corresponding black box DEA model is given by:

$$e_o^{\mathbb{O}} := e\left(\boldsymbol{a}_o \middle| \mathcal{P}^{\mathbb{O}}\right) = \min \frac{\frac{1}{6}\left(\sum_{i=1}^{3} \frac{x_i}{x_{io}} + \sum_{i=6}^{7} \frac{x_i}{x_{io}} + \frac{y_8}{y_{8o}}\right)}{\frac{y_4}{y_{4o}}} \, subject\ to\ (21)\ with\ \lambda_j^A = \lambda_j^B\ and$$
$$\begin{aligned} x_i &= x_i^A \leq x_{io}^A = x_{io} (i = 1, 2, 3), y_4 = y_4^A \geq y_{4o}^A = y_{4o} \\ x_i &= x_i^B \leq x_{io}^B = x_{io} (i = 6, 7), y_8 = y_8^B \leq y_{8o}^B = y_{8o} \end{aligned} \tag{25}$$

Results of DEA models (22) – (25) for an illustrative numerical data set of DMUs are presented and discussed in Subsection 4.3.

4.2 Oriented efficiency measure

In order to suitably apply an oriented efficiency measure (11), e.g. determined by (13) or (14), let us suppose for the example of Fig. 2 that DMUs cannot influence demand for their final product #4 (that is produced already on the first stage). Hence, any overproduction of it would be of no benefit. (Note that this premise contradicts our prior Definition 2.2 and preferential Assumption 2.3 that the classification of an object type as "good" – in particular type #4 – does not depend on its produced or consumed quantity.) Thus, an efficiency measure for the whole network as well as for stage A seems reasonable which minimises their respective (multi-dimensional) costs for given demand of product #4. Relevant costs of stage A result from inputs #1 to #3 and output of bad #5, those of the whole network from all five inputs #1, #2, #3, #6, and #7 and furthermore from bad output #8 (but not from bad #5 because it is a pure intermediate). Then, to coordinate production and consumption of intermediate #5, stage B should minimise its inputs #6 and #7 as well as its bad output #8, given its input of pollutant #5 emerging from stage A.

When applying cost-oriented function (14) to PPS (18) – (20) we obtain DEA models that are analogous to (22) – (25) and differ from them only with respect to the particular objective function while the constraints do not change in principle. For example, DEA model (22) for unit A changes to:

$$e_o^A = \min \theta, with\ \theta = \max\left\{\frac{x_1^A}{x_{1o}^A}, \frac{x_2^A}{x_{2o}^A}, \frac{x_3^A}{x_{3o}^A}, \frac{y_5^A}{y_{5o}^A}\right\} subject\ to\ (21)\ for\ \rho = A\ with$$
$$x_i^A \leq x_{io}^A (i = 1, 2, 3), y_4^A \geq y_{4o}^A, y_5^A \leq y_{5o}^A \tag{26}$$

This optimisation program is equivalent to the following one that is of a type which are usually called *radial* DEA models:

$$\begin{aligned} e_o^A &= \min \theta\ subject\ to \\ x_i^A &= \sum_{j \in \mathbb{J}} x_{ij}^A \lambda_j^A \leq \theta x_{io}^A\ (i = 1, 2, 3) \\ y_4^A &= \sum_{j \in \mathbb{J}} y_{4j}^A \lambda_j^A \geq y_{4o}^A \\ y_5^A &= \sum_{j \in \mathbb{J}} y_{5j}^A \lambda_j^A \leq \theta y_{5o}^A \\ \lambda_j^A &\geq 0\ (j \in \mathbb{J}) \end{aligned} \tag{27}$$

Cost-oriented DEA models that are analogous to the unoriented ones (23) – (25) can be formulated in the same manner. Thus, the model for production unit B is determined by:

$$\begin{aligned} &e_o^B = \min \theta\ subject\ to\ (21)\ for\ \rho\ =\ B\ with \\ &x_5^B\ \geq\ x_{5o}^B, x_i^B \leq \theta x_i 0^B (i\ =\ 6, 7), y_8^B\ \leq\ \theta y_{8o}^B \end{aligned} \tag{28}$$

Regarding PPS (19) of the whole network, a model that is analogous to (24) is given by:

$$\begin{aligned} &e_o^{\mathbb{S}} = \min \theta \text{ subject to } (21) \text{ with} \\ &x_i = x_i^A, x_{io} = x_{io}^A, x_i \le \theta x_{io} (i = 1,2,3), y_4 = y_4^A, y_{4o} = y_{4o}^A, y_4 \ge y_{4o} \\ &z_5 = y_5^A - x_5^B = 0 \\ &x_i = x_i^B, x_{io} = x_{io}^B, x_i \le \theta x_{io} (i = 6,7), y_8 = y_8^B, y_{8o} = y_{8o}^B, y_8 \le \theta y_{8o} \end{aligned} \quad (29)$$

Then, regarding PPS (20), the cost-oriented black box DEA model is:

$$\begin{aligned} &e_o^{\mathbb{O}} = \min \theta \text{ subject to } (21) \text{ with } \lambda_j^A = \lambda_j^B \text{ and} \\ &x_i = x_i^A, x_{io} = x_{io}^A, x_i \le \theta x_{io} (i = 1,2,3), y_4 = y_4^A, y_{4o} = y_{4o}^A, y_4 \ge y_{4o} \\ &x_i = x_i^B, x_{io} = x_{io}^B, x_i \le \theta x_{io} (i = 6,7), y_8 = y_8^B, y_{8o} = y_{8o}^B, y_8 \le \theta y_{8o} \end{aligned} \quad (30)$$

By inserting Eq. (21), black box model (30) can be simplified, thus taking a well-known form:

$$\begin{aligned} &e_o^{\mathbb{O}} = \min \theta \text{ subject to} \\ &\sum_{j \in \mathbb{J}} x_{ij}^A \lambda_j \le \theta x_{io}^A (i = 1,2,3) \\ &\sum_{j \in \mathbb{J}} y_{4j}^A \lambda_j \ge y_{4o}^A \\ &\sum_{j \in \mathbb{J}} x_{ij}^B \lambda_j \le \theta x_{io}^B (i = 6,7) \\ &\sum_{j \in \mathbb{J}} y_{8j}^B \lambda_j \le \theta y_{8o}^B \\ &\lambda_j \ge 0 (j \in \mathbb{J}) \end{aligned} \quad (31)$$

4.3 Efficiency results for a data set of DMUs

Table 1 reports an illustrative data set for four DMUs $\mathbb{J} = \{1, \ldots, 4\}$ of two-stage network in Fig. 2. Table 2 displays the efficiency scores of these DMUs resulting for each of the four PPS (18) – (20), on the right for models (22) – (25) with respect to unoriented measure defined by (11) and (12), on the left for models (26) – (31) with cost-oriented measure defined by (11) and (14).

In comparison of both measures, Table 2 demonstrates a general proposition which is well known for common DEA models (cf. e.g. Cooper, Seiford, & Tone (2007), p. 103) and generalised by Theorem 2.2. It states that the efficiency scores of the unoriented Tone measure are not greater than the corresponding oriented ones. Actually, they are smaller except for cases where the Tone score is already equal to one. While the cost-oriented score measures a type of efficiency that may be merely weak, the unoriented measure clearly identifies efficiency in the strong sense of Koopmans (1951), as explained by Remark 2.4. For example, oriented black box model (30) and (31) identifies DMU #3 as (at least) weakly cost efficient, though being in fact inefficient with a Tone score of 0.85 according to black box model (25).

Table 1 Data of four DMUs with network structure of Fig. 2

DMU j	x_{1j}^A	x_{2j}^A	x_{3j}^A	y_{4j}^A	$y_{5j}^A = x_{5j}^B$	x_{6j}^B	x_{7j}^B	y_{8j}^B
1	1	1	1	1	2	6	6	6
2	3	3	3	1	4	3	3	3
3	4	4	5	1	6	3	3	3
4	6	6	6	1	4	2	2	1

Table 2 Scores of the DMUs of Table 1 for different efficiency measures

	Cost-oriented measure				Unoriented measure			
DMU j	e_j^A	e_j^B	$e_j^{\mathbb{S}}$	$e_j^{\mathbb{O}}$	e_j^A	e_j^B	$e_j^{\mathbb{S}}$	$e_j^{\mathbb{O}}$
1	1	0.167	1	1	1	0.139	0.569	1
2	0.5	0.667	0.333	1	0.375	0.556	0.306	1
3	0.333	1	0.333	1	0.258	0.833	0.256	0.85
4	0.5	1	0.5	1	0.25	1	0.333	1

Remark 4.1: As the next-to-last column of Table 2 displays, none DMU is network efficient. This result coincides with Corollary 3.2. It contradicts and clarifies common (black box) DEA knowledge which states that at least one DMU must be efficient. It is crucial to notice that such a proposition is no longer valid if each network unit (or division) can *freely* choose from *all* activities that are possible for *each* DMU. This demonstrates that it is fruitful to look into the black box of a DMU, especially if its interior network may consist of single production units whose productivity or efficiency is strongly worse than that of other DMUs of the same network.

Remark 4.2: As DMU #2 shows – and already demonstrated by Dyckhoff and Souren (2024) – the network efficiency score of a DMU can be smaller than all of its divisions. Hence, for measuring the efficiency of a whole system as integrated entity, the proposal by e.g. Tone and Tsutsui (2009, p. 247) to define the "overall efficiency score" of a network DMU as arithmetic or harmonic mean of individual unit scores is inadequate. Any mean is simply what it is: a numerical aggregate of efficiencies of possibly totally unconnected individual units, not necessarily belonging to the same or to any network.

In fact, unoriented network efficiency scores $e_j^{\mathbb{S}}$ of all DMUs are rather small, especially compared to the black box ones $e_j^{\mathbb{O}}$. There is a simple reason for this in view of the specific data of Table 1. Because of the presumed linear PPS (18), stage A of DMU #1 and stage B of DMU #4 clearly dominate the corresponding stages of the other three DMUs. Hence, efficient stage frontiers are determined by

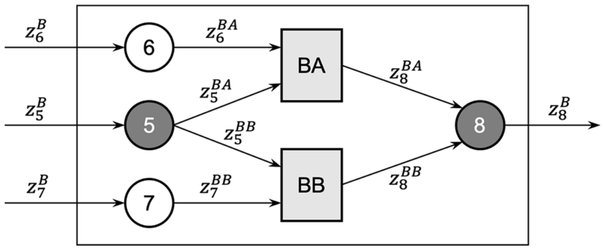

Fig. 3 Unit B of Fig. 2 as subsystem with two parallel subunits BA and BB

them:

$$\partial_E\mathcal{P}^A = \{z \in \mathbb{R}^8 | z = \boldsymbol{a}_1^A\lambda_1^A, \boldsymbol{a}_1^A = (-1,-1,-1,1,2,0,0,0), \lambda_1^A \geq 0\} \tag{32}$$

$$\partial_E\mathcal{P}^B = \{z \in \mathbb{R}^8 | z = \boldsymbol{a}_4^B\lambda_4^B, \boldsymbol{a}_4^B = (0,0,0,0,-4,-2,-2,1), \lambda_4^B \geq 0\} \tag{33}$$

In a similar manner, the combination $\boldsymbol{a}_*^{\mathbb{S}} = \boldsymbol{a}_1^A\lambda_1^A + \boldsymbol{a}_4^B\lambda_4^B$ of these two stage activities by choosing $\lambda_1^A = 1$ and $\lambda_4^B = 0.5$ for the tied network PPS (19) leads again to a corresponding ray as efficient network frontier:

$$\partial_E\mathcal{P}^{\mathbb{S}} = \{z \in \mathbb{R}^8 | z = \boldsymbol{a}_*^{\mathbb{S}}\lambda^{\mathbb{S}}, \boldsymbol{a}_*^{\mathbb{S}} = (-1,-1,-1,1,0,-1,-1,0.5), \lambda^{\mathbb{S}} \geq 0\} \tag{34}$$

5 Complex and hierarchical network structures

All general propositions of Sections 2 and 3 are valid for arbitrary structures of networks. They may thus substantially differ from the two-stage networks of Figs. 1 and 2. For example, Fig. 3 shows a network with two parallel production units BA and BB, both transforming parts of the same bad input #5 into the same bad output #8, thereby using different factors #6 and #7. It may represent production unit B as subsystem of the network of Fig. 2, so that Figs. 2 and 3 together illustrate a *hierarchical network structure*.

In extending the example for stage B of Fig. 2, activities of the interior network PPS $\mathcal{P}^{B\mathbb{S}}$ for units BA and BB of Fig. 3 as well of its a black box $\mathcal{P}^{B\mathbb{O}}$ are characterised by

$$z^\rho = \left(0,0,0,0,-x_5^\rho,-x_6^\rho,-x_7^\rho,y_8^\rho\right) \in \mathcal{P}^\rho \subset \mathbb{R}^8, \rho \in \{BA, BB, B\mathbb{S}, B\mathbb{O}\} \tag{35}$$

By forming the linear hull (7) of an arbitrary data set of DMUs $\mathbb{J}$ for each unit BA and BB analogously to (21) and then composing both to the network of Fig. 3, feasible activities z^ρ are uniquely described by following equations:

$$x_5^{BA} = \sum_{j\in\mathbb{J}} x_{5j}^{BA}\lambda_j^{BA}, x_6^{BA} = \sum_{j\in\mathbb{J}} x_{6j}^{BA}\lambda_j^{BA}, x_7^{BA} = 0, y_8^{BA} = \sum_{j\in\mathbb{J}} y_{8j}^{BA}\lambda_j^{BA}, \lambda_j^{BA} \geq 0 \tag{36}$$

$$x_5^{BB} = \sum_{j\in\mathbb{J}} x_{5j}^{BB}\lambda_j^{BB}, x_6^{BB} = 0, x_7^{BB} = \sum_{j\in\mathbb{J}} x_{7j}^{BB}\lambda_j^{BB}, y_8^{BB} = \sum_{j\in\mathbb{J}} y_{8j}^{BB}\lambda_j^{BB}, \lambda_j^{BB} \geq 0 \tag{37}$$

$$x_5^B = x_5^{BA} + x_5^{BB}, x_6^B = x_6^{BA}, x_7^B = x_7^{BB}, y_8^B = y_8^{BA} + y_8^{BB} \tag{38}$$

Equations (36) and (37) define PPS $\mathcal{P}^{BA}$ and $\mathcal{P}^{BB}$, respectively, all three sets (36) to (38) together network PPS $\mathcal{P}^{B\mathbb{S}}$. With identical activity levels $\lambda_j^\rho = \lambda_j$ of both production units $\rho \in \{BA, BB\}$ for each DMU $j \in \mathbb{J}$, network PPS $\mathcal{P}^{B\mathbb{S}}$ is reduced to black box PPS $\mathcal{P}^{B\mathbb{O}}$:

$$x_k^B = \sum_{j\in\mathbb{J}} x_{kj}^B\lambda_j, \rho \in \{5,6,7\}, y_8^B = \sum_{j\in\mathbb{J}} y_{8j}^B\lambda_j, \lambda_j \geq 0 \tag{39}$$

whereby $x_{5j}^B = x_{5j}^{BA} + x_{5j}^{BB}$, $x_{6j}^B = x_{6j}^{BA}$, $x_{7j}^B = x_{7j}^{BB}$, $y_{8j}^B = y_{8j}^{BA} + y_{8j}^{BB}$. Black box PPS (39) of subsystem B in Fig. 3 is identical with PPS (18) of unit B in Fig. 2: $\mathcal{P}^{B\mathbb{O}} = \mathcal{P}^B$, and also black box DEA model (23) for stage B with unoriented Tone measure to:

$$e_o^{B\mathbb{O}} = e\left(\boldsymbol{a}_o^{B\mathbb{O}} \middle| \mathcal{P}^{B\mathbb{O}}\right) = \min \frac{\frac{1}{3}\left(\sum_{i=6}^{7}\frac{x_i^B}{x_{io}^B} + \frac{y_8^B}{y_{8o}^B}\right)}{\frac{x_5^B}{x_{5o}^B}} \textit{ subject to } (39) \textit{ and}$$
$$x_5^B \geq x_{50}^B, x_i^B \leq x_{io}^B (i = 6,7), y_8^B \leq y_{8o}^B$$

The corresponding network DEA model for subsystem B simply is:

$$e_o^{B\mathbb{S}} = e\left(\boldsymbol{a}_o^{B\mathbb{S}} \middle| \mathcal{P}^{B\mathbb{S}}\right) = \min \frac{\frac{1}{3}\left(\sum_{i=6}^{7}\frac{x_i^B}{x_{io}^B} + \frac{y_8^B}{y_{8o}^B}\right)}{\frac{x_5^B}{x_{5o}^B}} \textit{ subject to } (36) - (38) \textit{ and}$$
$$x_5^B \geq x_{50}^B, x_i^B \leq x_{io}^B (i = 6,7), y_8^B \leq y_{8o}^B$$

This way, the combination of Figs. 2 and 3 demonstrates how complex networks can be hierarchically designed in a stepwise procedure by simply complementing balance equations and production possibilities of subunits and how models for measuring efficiency of complex networks with bads may be hierarchically composed (cf. Dyckhoff and Souren (2024) for a network with recycling and additional types of goods).

6 Conclusions

Search in scientific databases suggests a research gap in terms of fundamentally new methods and theoretical developments for general networks with undesirable objects as inputs or outputs. By systematically extending a recently presented general network production theory for goods, the current paper develops a generic approach for analysing the efficiency of activities in networks of arbitrary structure that include those with bads as inputs or outputs as part of the network system. Following features characterise this new theory and distinguish it from previous network production theories and technologies for goods and bads:

- By utilising a framework that integrates multi-criteria decision analysis and production theory, undesirable objects are included such that the generalised network production theory is structurally and formally analogous to the previous theory without bads. In cases where undesirable objects are actually irrelevant all concepts and findings remain valid for the thus restricted domain and are identical to those of the previous theory with solely goods.
- Bads can be treated in the same way as goods except for their opposite preferences. However, it improves clarity to use different symbols for the technologically modelled inputs and outputs on the one hand, and symbols for their preferentially modelled desirability in terms of costs and benefits on the other. There is a bijective relationship between the physical quantity of an input or output and its so-defined cost or benefit. This allows for a simple generalisation of the pure network production theory for goods to include bads, so that costs and benefits can be treated analogously to inputs and outputs before.
- The theory thus systematically distinguishes the purely technological perspective of activities from the preference perspective of the multiple values added or destroyed by the activity. Primary subject of efficiency analysis are costs and benefits; inputs and outputs matter only as (secondary) unsophisticated performance measures. The quantity of a bad output or the quantity of a good input are both used as proxy performance indicators for costs, which measure disadvantages of the production process, while the quantity of a good output or of a bad input represents benefits as advantages resulting from the process.

The theory presented is based on Dyckhoff and Allen's (2001) multi-criteria approach to measuring environmental efficiency. It is often criticised that input of goods and output of bads are treated in the same way. Emitted pollutants are undeniably outputs; but they are undesirable because their impact on the natural or man-made environment (external effects) generates social (or external) costs. It is a category mistake not to distinguish clearly between (objective) technological information about the process of transforming inputs into outputs, on the one hand, and (subjective) preference information of people valuing these inputs and outputs (perhaps via markets), on the other. Different empirical issues may then be erroneously identified in view of their identical mathematical representation.

Because of the analogy of costs and benefits in the new theory with bads to inputs and outputs in the pure theory for goods, both the previous and the generalised network production theory are symmetric such that corresponding notions, models, and theorems can be formulated that include the previous theory as special case. In particular, following findings have been established in this paper:

- By using similar underlying modelling features and fundamental assumptions, Koopmans' linear activity analysis is generalised to undesirable objects and nearly arbitrary technologies. Resulting networks may possess general structures that are formed by production units whose technologies must satisfy merely very mild conditions so that they can be non-convex and even discrete.
- The modelling approach for goods and bads is not only founded by multi-criteria production theory but may also often be more suitable for analysing complex networks of e.g. supply chains and closed-loop systems with recycling than conventional approaches of economics and network DEA. Furthermore, the approach can easily be extended to dynamic analyses, since its network flow equations are based on balances that can integrate inventories in case of material inputs and outputs.
- Disposability is not presupposed, but the theory is in principle compatible with any kind of such assumption in the literature. It also allows for analysing separate disposal activities such that waste is explicitly treated as intermediate object in a production and abatement network.
- Common methods of efficiency measurement can be integrated such that network DEA for bads as inputs or outputs is embedded in a much more general approach.
- In principle, the presented concepts, models, and theorems for analysing the efficiency of a network and its production units are analogous to those without bads and contain them as special cases. For the relation between system and units' efficiencies it is crucial whether the system is tied by active constraints, especially regarding intermediate products or pollutants. Efficiency scores of network activities are usually definitely smaller than those of their respective black-box evaluation, so that often none DMU is network efficient although many DMUs are efficient if assessed as black box. Moreover, the network efficiency score of a DMU may be smaller than all efficiency scores of their units if the units are assessed without system constraints.

Hence, it is inappropriate to generally define network efficiency scores as a mean of the units' scores as proposed in the literature.

The above listed features (and perhaps others more) of the generalised network production theory for goods and bads may facilitate advancements in future applied research on assessing the productivity and efficiency of complex networks with undesirable objects, especially multi-stage production, abatement, and recycling networks in regard to environmental pollution. However, (at least) following important topics should be dealt with by further theoretical research in future (Dyckhoff and Souren, 2024):

- The first topic also applies to networks without bads: How can proper efficiency measures for arbitrary network structures and technologies be defined that allow for a meaningful combined efficiency assessment of any network and its production units? Complex multi-stage networks consist not only of primary factors, intermediate and final products on middle stages, and parallel production units at the same stage, and may also include recycling. Thus, in general, the production units of a network use different types of inputs and outputs each, so that outputs of one unit are inputs of others. It is not immediately obvious how to design a general methodology for assessing network performance when comparing reductions of different good inputs and bad outputs with increases of different good outputs and bad inputs. Although the presented theory allows for a simple and constructive modelling of general networks, the efficiency analysis of the illustrative examples in Sections 4 and 5 cannot easily be generalised to networks of arbitrary complexity.
- Free disposability for excess output of intermediates is often postulated, rarely with any deeper reflection of its economic meaning and consequences in applications. This may be problematic because the efficiency score of a DMU can be distinctly smaller in case of free disposability than in case of pure intermediate products. Alves and Meza (2023, p. 2746) draw as conclusion from their literature review that the "inclusion of intermediate variables in the efficiency measure" should be a main line for future research. That is the more important the more undesirable objects have to be taken account of.
- Overproduction of unintended joint products is often unavoidable to fulfil certain demands for intermediate or final products that cannot be procured externally. Assuming free disposability of an intermediate product implies that its overproduction is of no value for the system as a whole whereas other quantities of the same intermediate product are assigned positive values if they are maximised as process output of a production stage or minimised as process input of the subsequent stage in order to calculate efficiency scores of both stages. Thus, depending on its (produced or consumed) quantity, one and the same type of object may change its value from positive (*good*) to zero (*neutral* or *free*) – or even negative (*bad*) if its disposal is costly (Dyckhoff, 2023). Such facts are widely ignored in common literature on network productivity and efficiency and thus form another important research gap.

Data Availability

No datasets were generated or analysed during the current study.

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Compliance with ethical standards

Conflict of interest The authors declare no competing interests.

7 Appendix: Proofs[10]

Proof of Lemma 2.1: (a) Value function $(\boldsymbol{d}_{\mathbb{G}}, \boldsymbol{d}_{\mathbb{B}}) = \boldsymbol{v}(\boldsymbol{z}_{\mathbb{G}}, \boldsymbol{z}_{\mathbb{B}}) = (\boldsymbol{z}_{\mathbb{G}}, -\boldsymbol{z}_{\mathbb{B}})$ is invertible such that $\boldsymbol{v}^{-1}(\boldsymbol{d}_{\mathbb{G}}, \boldsymbol{d}_{\mathbb{B}}) = (\boldsymbol{z}_{\mathbb{G}}, -\boldsymbol{z}_{\mathbb{B}})$, which proves bijectivity. (b) This trivial value function implies that $\mathcal{V}$ and $\mathcal{P}$ contain an equal number of elements such that both must be non-empty together. Since the value function and its inverse are moreover continuous, properties such as closeness or

[10] Proofs of Lemma 2.2 and Theorems 3.1, 3.2, 3.4 are generalisations of proofs given by Dyckhoff and Souren (2024) for corresponding lemmata and theorems in the special case with solely goods, i.e. without bads.

boundedness must hold either for both sets or for none of them. (c) The same is also true for subsets of $\mathcal{V}$ and $\mathcal{P}$ such as dominating sets.■

Proof of Lemma 2.2: Consider following optimisation program:

$$\max \sum_{k=1}^{\kappa} v_k(z) = \max\left(\sum_{k\in\mathbb{G}} z_k - \sum_{k\in\mathbb{B}} z_k\right) \text{such that}$$
$$z_{\mathbb{G}} \geq \tilde{z}_{\mathbb{G}}, z_{\mathbb{B}} \leq \tilde{z}_{\mathbb{B}}, z \in \mathcal{D}(z_o|\mathcal{P})$$

Because $\mathcal{D}(z_o|\mathcal{P})$ is compact and $\tilde{z} \in \mathcal{D}(z_o|\mathcal{P})$, the feasible set of this program is compact, too. Because the objective function is linear and hence continuous, an optimal solution $z_* \in \mathcal{D}(z_o|\mathcal{P})$ exists. Since the objective function is strictly increasing with an increasing (positive) output quantity or (negative) input quantity of goods, but strictly decreasing with those of bads, z_* must be efficient. Hence $\boldsymbol{v}(z_*) > \boldsymbol{v}(\tilde{z})$ for inefficient $\tilde{z}$.■

Proof of Theorem 2.1: Because of Definition 2.3 (VPS) and Lemma 2.1 (bijectivity), $\mathcal{P}_1 \subset \mathcal{P}_2$ is equivalent to $\mathcal{V}_1 = \boldsymbol{v}(\mathcal{P}_1) \subset \boldsymbol{v}(\mathcal{P}_2) = \mathcal{V}_2$, which implies $\mathcal{D}(\boldsymbol{d}_o|\mathcal{V}_1) \subset \mathcal{D}(\boldsymbol{d}_o|\mathcal{V}_2)$ for the dominating sets. Hence, maximum of $\triangle(\boldsymbol{d};\boldsymbol{d}_o)$ for $\boldsymbol{d} \in \mathcal{D}(\boldsymbol{d}_o|\mathcal{V}_1)$ cannot be larger than for $\boldsymbol{d} \in \mathcal{D}(\boldsymbol{d}_o|\mathcal{V}_2)$. As inverse of the maximum in (11), it follows for the efficiency score $e(z_o|\mathcal{P}_1) \geq e(z_o|\mathcal{P}_2)$.■

Proof of Theorem 2.2: With $b_k \geq b_{ko}$ and $c_k \leq c_{ko}$ for $\boldsymbol{d} \in \mathcal{D}(\boldsymbol{d}_o|\mathcal{V})$ the assertions follow from (15) and (16) by inserting functions (12) to (14) into the definition of efficiency measure (11).■

Proof of Theorem 3.1: (a) $\mathcal{P}^{\mathbb{S}}$ is non-empty because of Postulate *III*. With definition (3) for $\mathcal{P}^{\mathbb{S}}$, closeness of $\mathcal{P}^{\rho}$ according to Assumption 2.2 implies that $\mathcal{P}^{\mathbb{S}}$ is closed, too. This also holds for networks that are tied by restrictions of type $\underline{z}_k \leq z_k \leq \bar{z}_k$.

(b) Since PPS $\mathcal{P}^{\mathbb{S}}$ of the whole network is closed according to Theorem 3.1a, this is also true for such subsets as defined by $\mathcal{D}(z_0|\mathcal{P}^{\mathbb{S}})$. It remains to show that $\mathcal{D}(z_0|\mathcal{P}^{\mathbb{S}})$ is bounded and thus compact. Let be $\tilde{z} = \sum_{\rho\in\mathbb{P}} \tilde{z}^{\rho} \in \mathcal{D}(z_0|\mathcal{P}^{\mathbb{S}})$ arbitrary. Due to Assumption 2.4, dominating sets $\mathcal{D}(\tilde{z}^{\rho}|\mathcal{P}^{\rho})$ are bounded with respect to each individual production unit $\rho \in \mathbb{P}$ and any $\tilde{z}^{\rho} \in \mathcal{P}^{\rho}$. Then, in view of Lemma 2.1c, $\mathcal{D}\left(\tilde{\boldsymbol{d}}^{\rho}|\mathcal{V}^{\rho}\right)$ is bounded for each $\rho \in \mathbb{P}$ and $\tilde{\boldsymbol{d}}^{\rho} = \boldsymbol{v}(\tilde{z}^{\rho}) \in \mathcal{V}^{\rho}$, too. Bounded from above means that there exists some $\bar{\boldsymbol{d}}^{\rho}$ such that $\boldsymbol{d} \leq \bar{\boldsymbol{d}}^{\rho}$ for all $\boldsymbol{d}\in\mathcal{D}\left(\tilde{\boldsymbol{d}}^{\rho}|\mathcal{V}^{\rho}\right)$. Hence, $\tilde{\boldsymbol{d}}=\boldsymbol{v}(\tilde{z})=\boldsymbol{v}\left(\sum_{\rho\in\mathbb{P}}\tilde{z}^{\rho}\right)=\sum_{\rho\in\mathbb{P}}\boldsymbol{v}(\tilde{z}^{\rho})=\sum_{\rho\in\mathbb{P}}\tilde{\boldsymbol{d}}^{\rho}\leq\sum_{\rho\in\mathbb{P}}\bar{\boldsymbol{d}}^{\rho}=:\bar{\boldsymbol{d}}$ for arbitrary $\tilde{\boldsymbol{d}} \in \mathcal{D}(\boldsymbol{d}_0|\mathcal{V}^{\mathbb{S}})$, so that $\mathcal{D}(\boldsymbol{d}_0|\mathcal{V}^{\mathbb{S}}) = \{\boldsymbol{d} \in \mathcal{V}^{\mathbb{S}}|\boldsymbol{d} \geq \boldsymbol{d}_0\}$ is bounded from above, and also bounded from below per definition, namely by $\boldsymbol{d}_0$ itself. That is, set $\mathcal{D}(\boldsymbol{d}_0|\mathcal{V}^{\mathbb{S}})$ is bounded. Thus, due to Lemma 2.1c, $\mathcal{D}(z_0|\mathcal{P}^{\mathbb{S}})$ is bounded, too. Summing up, $\mathcal{D}(z|\mathcal{P}^{\mathbb{S}})$ is compact for all $z \in \mathcal{P}^{\mathbb{S}}$, be the network tied or not (i.e. irrespective whether constraints of types $z_k^{\mathbb{S}} = 0$, $z_k^{\mathbb{S}} \leq 0$ or $z_k^{\mathbb{S}} \geq 0$ for some object types $k = 1, \ldots, \kappa$ exist).

(c) Because of Assumptions 2.1 and 2.4 and Postulate *III*, each PPS $\mathcal{P}^{\rho}(\rho \in \mathbb{P} \cup \{\mathbb{S}\})$ is non-empty, i.e. $\mathcal{P}^{\rho} \neq \emptyset$. For each $z^{\rho} \in \mathcal{P}^{\rho}, \rho \in \mathbb{P} \cup \{\mathbb{S}\}$, its dominating set $\mathcal{D}(z^{\rho}|\mathcal{P}^{\rho})$ is compact according to Assumption 2.4 and Theorem 3.1b, so that there exists at least one efficient $z^* \in \partial_E\mathcal{D}(z^{\rho}|\mathcal{P}^{\rho}) \subseteq \partial_E\mathcal{P}^{\rho}$ due to Lemma 2.2.■

Proof of Theorem 3.2: If $z \in \mathcal{P}^{\mathbb{S}}$ with $z = \sum_{\rho\in\mathbb{P}} z^{\rho}, z^{\rho} \in \mathcal{P}^{\rho}$ is any feasible production of the whole system, let – without limiting generality – activity $z^A \in \mathcal{P}^A, A \in \mathbb{P}$ be inefficient regarding PPS $\mathcal{P}^A$, i.e. $z^A \notin \partial_E\mathcal{P}^A$. Then, there exists $\tilde{z}^A \in \mathcal{P}^A$ such that $\boldsymbol{v}(\tilde{z}^A) > \boldsymbol{v}(z^A)$. Define $\tilde{z} := \tilde{z}^A + \sum_{\rho\in\mathbb{P}\setminus\{A\}} z^{\rho}$. If this system activity $\tilde{z}$ is also feasible, i.e. $\tilde{z} \in \mathcal{P}^{\mathbb{S}}$, the previous activity z must be inefficient because of the linearity of value function $\boldsymbol{v}(z)$ according to Definition 2.2:

$$\boldsymbol{v}(\tilde{z}) = \boldsymbol{v}(\tilde{z}^A) + \boldsymbol{v}\left(\sum_{\rho\in\mathbb{P}\setminus\{A\}} z^{\rho}\right) > \boldsymbol{v}(z^A) + \boldsymbol{v}\left(\sum_{\rho\in\mathbb{P}\setminus\{A\}} z^{\rho}\right) = \boldsymbol{v}(z) \quad ■$$

Proof of Theorem 3.3 (Dyckhoff & Souren, 2024): (a) With (3) follows: $\sum_{\rho\in\mathbb{P}} \boldsymbol{0} = \boldsymbol{0}\mathcal{P}^{\mathbb{S}}$ if $\boldsymbol{0} \in \mathcal{P}^{\rho}$. This is not affected if the network is tied as long as $\underline{z}_k \leq 0 \leq \bar{z}_k$ for potential lower and upper bounds.

(b) Let $z \in \mathcal{P}^{\mathbb{S}}$ and $\tilde{z}\mathcal{P}^{\mathbb{S}}$ be two activities of the whole system such that $z = \sum_{\rho\in\mathbb{P}} z^{\rho}$ and $\tilde{z} = \sum_{\rho\in\mathbb{P}} \tilde{z}^{\rho}$ for $z^{\rho} \in \mathcal{P}^{\rho}$ and $\tilde{z}^{\rho} \in \mathcal{P}^{\rho}$. Then $\alpha z^{\rho} + \beta\tilde{z}^{\rho} \in \mathcal{P}^{\rho}$ for $\alpha, \beta > 0, \alpha + \beta = 1$ because $\mathcal{P}^{\rho}$ is a convex set. Hence $\alpha z + \beta\tilde{z} = \sum_{\rho\in\mathbb{P}}(\alpha z^{\rho} + \beta\tilde{z}^{\rho}) \in \mathcal{P}^{\mathbb{S}}$.

If the network is tied by restrictions of types $\underline{z}_k \leq z_k \leq \bar{z}_k$, convex combinations $\alpha z + \beta\tilde{z}$ of two system activities fulfil such restrictions, too.

(c) Any convex set $\mathcal{P}$ with $\boldsymbol{0} \in \mathscr{P}$ fulfils $\lambda z \in \mathcal{P}$ for all $z \in \mathcal{P}$, $0 \leq \lambda \leq 1$.■

Proof of Lemma 3.1: Value function $\boldsymbol{d} = \boldsymbol{v}(z) = (z_{\mathbb{G}}, -z_{\mathbb{B}})$ for $z = (z_{\mathbb{G}}, z_{\mathbb{B}}) \in \mathcal{P}$ is invertible due to Lemma 2.1a. Both multi-dimensional functions, $\boldsymbol{v}(z)$ and its inverse $\boldsymbol{v}^{-1}(\boldsymbol{d})$, are linear. That is why it suffices to prove that the image $\mathcal{V} = \boldsymbol{v}(\mathcal{P})$ of a linear or convex set $\mathcal{P}$ derived from any linear function $\boldsymbol{v}$ has the same property. Then, vice versa, image $\mathcal{P}=\boldsymbol{v}^{-1}(\mathcal{V})$ must also be linear or convex if $\mathcal{V}$ is linear or convex, respectively, because $\boldsymbol{v}^{-1}$ is linear, too.

Thus, suppose $\mathcal{P}$ is a linear (convex) set, i.e. $z = \lambda_1 z_1 + \lambda_2 z_2 \in \mathcal{P}$ if $z_j \in \mathcal{P}, \lambda_j \geq 0, j \in \{1, 2\}$ (and $\lambda_1 + \lambda_2 = 1$ for convexity). Let be $\boldsymbol{v}(z)$ any multi-dimensional linear function, so that $\mathcal{V}=\boldsymbol{v}(\mathcal{P}) = \{\boldsymbol{d}|\boldsymbol{d} = \boldsymbol{v}(z), z \in \mathcal{P}\}$. Hence, with $\boldsymbol{d}_j \in \mathcal{V}, j \in \{1, 2\}$, one obtains from linearity (or convexity) of $\boldsymbol{v}(z)$:

$$\boldsymbol{d} = \lambda_1 \boldsymbol{d}_1 + \lambda_2 \boldsymbol{d}_2 = \lambda_1 \boldsymbol{v}(z_1) + \lambda_2 \boldsymbol{v}(z_2) = \boldsymbol{v}(\lambda_1 z_1 + \lambda_2 z_2) = \boldsymbol{v}(z) \in \mathcal{V}$$

■

Proof of Theorem 3.4: Let be $z_* = \alpha z_1 + (1-\alpha)z_2 \in \partial_E \mathcal{P}$ for specific α with $0 < \alpha < 1$. Define $z = \beta z_1 + (1-\beta)z_2$ for some β with $0 \le \beta \le 1$. Thus, $z \in \mathcal{P}$ because $\mathcal{P}$ is convex. Assume $z \notin \partial_E \mathcal{P}$, i.e. there would exist dominating $\tilde{z} \in \mathcal{P}$ so that $\boldsymbol{v}(\tilde{z}) > \boldsymbol{v}(z) \in \mathcal{V}$. Without limiting generality let be $\alpha < \beta$. Then, with $\bar{z} := \gamma\tilde{z} + (1-\gamma)z_2 \in \mathcal{P}$, from linearity of $\boldsymbol{v}(z)$ one obtains:

$$\begin{aligned} \boldsymbol{v}(\bar{z}) &= \gamma \boldsymbol{v}(\tilde{z}) + (1-\gamma)\boldsymbol{v}(z_2) > \gamma \boldsymbol{v}(z) + (1-\gamma)\boldsymbol{v}(z_2) \\ &= \gamma[\beta \boldsymbol{v}(z_1) + (1-\beta)\boldsymbol{v}(z_2)] + (1-\gamma)\boldsymbol{v}(z_2) \\ &= \beta\gamma \boldsymbol{v}(z_1) + (1-\beta\gamma)\boldsymbol{v}(z_2) \text{ for all } 0 < \gamma \le 1 \end{aligned}$$

Choosing $\gamma = \alpha/\beta < 1$ implies $\boldsymbol{v}(\bar{z}) > \boldsymbol{v}(z_*)$ which contradicts efficiency of z_*.■

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