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Quiver Grassmannians for the Bott-Samelson Resolution of Type A Schubert Varieties

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Abstract

We realise the Bott-Samelson resolutions of type A Schubert varieties as quiver Grassmannians. In order to explicitly describe this isomorphism, we introduce the notion of a *geometrically compatible* decomposition for any permutation in the symmetric group S_n. For smooth type A Schubert varieties, we identify a suitable dimension vector such that the corresponding quiver Grassmannian is isomorphic to the Schubert variety. To obtain these isomorphisms, we construct a special quiver with relations and investigate two classes of quiver Grassmannians for this quiver.

1 Introduction

Schubert varieties first appeared at the end of the 19th century in the context of Schubert calculus, whose purpose is to determine the number of solutions of certain intersection problems, and have become some of the best understood examples of complex projective varieties. They have recently been linked to degenerate flag varieties and quiver Grassmannians. Two examples of such connections are in [11], where the authors show that any type A or C degenerate flag variety is isomorphic to a Schubert variety in an appropriate partial flag manifold, and later in [10], which proves that some Schubert varieties arise as irreducible components of certain quiver Grassmannians. The Bott-Samelson(-Demazure-Hansen) varieties provide natural resolutions of Schubert varieties. They were introduced independently by Demazure and Hansen, and named Bott-Samelson by Demazure [14].

Given a quiver Q and a Q-representation M, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is the projective variety parametrising subrepresentations $N \subseteq M$ of dimension vector $\mathbf{e}$. They first appeared in [12, 29] and have since been extensively studied, for instance as a tool in

cluster algebra theory [4] or for studying linear degenerations of the flag variety [7, 8, 17, 18]. Notably, every projective variety arises as a quiver Grassmannian [26], or, more generally, as a quiver Grassmannian of every wild quiver [27].

In this paper, we construct a special quiver with relations and a rigid representation for this quiver, to then consider the quiver Grassmannian that corresponds to opportune choices of a dimension vector for the quiver. We show how this quiver Grassmannian can realise the Bott-Samelson resolution for Schubert varieties and how, for a different dimension vector, it is isomorphic to a chosen smooth Schubert variety. Firstly, in Definition (4.2), we define the special class of quivers with relations (Γ, I) and a (Γ, I)-representation M. Given the ambient dimension $n + 1 = 4$, then (Γ, I) and M are

$$
\begin{array}{ccccc}
\overset{\mathbb{C}}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}}{\bullet} \\
\downarrow^{2,1} \circlearrowright & & \downarrow^{2,1} \circlearrowright & & \downarrow^{2,1} \\
\overset{\mathbb{C}^2}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^2}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^2}{\bullet} \\
\downarrow^{3,2} \circlearrowright & & \downarrow^{3,2} \circlearrowright & & \downarrow^{3,2} \\
\overset{\mathbb{C}^3}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^3}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^3}{\bullet} \\
\downarrow^{4,3} \circlearrowright & & \downarrow^{4,3} \circlearrowright & & \downarrow^{4,3} \\
\overset{\mathbb{C}^4}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^4}{\bullet} & \overset{\text{id}}{\longrightarrow} & \overset{\mathbb{C}^4}{\bullet}
\end{array} \; \cdot
$$

Here, each map $\iota_{i+1,i}$ denotes the inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$ represented (with respect to a chosen basis) by the matrix

$$
\iota_{i+1,i} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Because of the shape of this particular quiver, it is convenient to visualise it as a grid, or matrix, and denote its vertices by double indices (i, j). Consequently, given M as above (and analogously for any other (Γ, I)-representation), we denote by $M_{i,j}$ the vector space associated to vertex (i, j).

Our first result is the following:

Theorem 1 (Proposition 4.4) *M is a rigid representation of (Γ, I).*

In Corollary 4.6, we deduce from Theorem 1 that any quiver Grassmannian associated to the (Γ, I)-representation M is a smooth and irreducible variety.

We then consider a Schubert variety X_w in the flag variety Fl_{n+1}, for any fixed permutation $w \in S_{n+1}$, and define the dimension vector $\mathbf{r}^w$ for (Γ, I) according to w.

Theorem 2 (Theorem 5.19) *The quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ is isomorphic to any Bott-Samelson resolution of X_w associated to a geometrically compatible decomposition of w.*

As part of the proof of Theorem 2 (see Theorem 5.19) we give an explicit description of the isomorphism between the Bott-Samelson resolution of X_w and our quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$. We remark that the quiver, the representation and the corresponding quiver Grassmannian we use are different from the ones that are obtained from the realisation of projective varieties as quiver Grassmannians given in [26]. This allows us not only to exploit an algebraic property of the representation M (i.e. its rigidity) to deduce geometrical properties of the associated variety $\mathrm{Gr}_{\mathbf{r}^w}(M)$, but also to give a straightforward correspondence between the points of $\mathrm{Gr}_{\mathbf{r}^w}(M)$ and those of the Bott-Samelson resolution of X_w which relies only on the combinatorial definition of the latter (see Definition 5.5).

Finally, we make use of a combinatorial characterisation of smooth Schubert varieties: it was first proved in [24] that a Schubert variety X_w is smooth if and only if w avoids the patterns [4231] and [3412], and this criterion was later characterised in [21] in terms of the conditions that define the flags in X_w. We exploit this characterisation and provide an explicit isomorphism between a fixed smooth Schubert variety and the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ of (Γ, I), for a special dimension vector $\mathbf{e}^w$:

Theorem 3 (Theorem 6.4) *If $w \in S_{n+1}$ avoids the patterns* [4231] *and* [3412]*, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ is isomorphic to the Schubert variety X_w.*

Future perspectives The topics and results of this paper raise a few natural questions. Firstly, can we describe linear degenerations of Schubert varieties (or of their Bott-Samelson resolutions) in the sense of [6]? A discussion on this will appear in a follow-up paper in preparation, based on the PhD thesis by the author of this paper (see [23]). Then, we might ask if it is possible to find a map between the desingularisations constructed in [9] and the realisation of the Bott-Samelson resolutions presented in this paper.

Further, it is reasonable to consider the action of the one-dimensional torus $\mathbb{C}^*$ on the representation M in the sense of [8] and prove that it extends to an action on the quiver Grassmannian. We can then investigate whether this action induces a cellular decomposition of the quiver Grassmannian in terms of attracting sets.

The paper is organised as follows: Sections 2 and 3 are dedicated, respectively, to basic facts about quiver Grassmannians and about Schubert varieties in the flag variety of type A. In Section 4, we define the quiver (Γ, I), its representation M and prove Theorem 1. Section 5 is mainly concerned with proving that all permutations admit a certain reduced decomposition, called *geometrically compatible* decomposition, in order to prove Theorem 2. In Section 6, we define a special dimension vector for (Γ, I) and prove Theorem 3.

2 Background on quiver Grassmannians

We first collect some facts about quiver representations and quiver Grassmannians. Standard references are [13, 28].

Definition 2.1 A finite **quiver** $Q = (Q_0, Q_1, s, t)$ is given by a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps $s, t : Q_1 \to Q_0$ assigning to each arrow its source, resp. target.

Definition 2.2 A **relation** on a quiver Q is a subspace of the path algebra of Q spanned by linear combinations of paths with common source and target, of length at least 2. Given a two-sided ideal I of $\mathbb{K}Q$ generated by relations, the pair (Q, I) is a **quiver with relations** and the quotient algebra $\mathbb{K}Q/I$ is the path algebra of (Q, I).

A system of relations for I is defined as a subset R of $\cup_{i,j \in Q_0} iIj$, where i denotes the trivial path on vertex i, such that R, but no proper subset of R, generates I as a two-sided ideal. For any two vertices i and j, we denote by $r(i, j, R)$ the cardinality of the set $R \cap iIj$, which contains those elements in R that are linear combinations of paths starting in i and ending in j. If Q contains no oriented cycle, then the numbers $r(i, j, R)$ are independent of the chosen system of relations (see for instance [3]), and can therefore be denoted by $r(i, j)$.

Definition 2.3 Given a quiver Q, the finite-dimensional Q-**representation** M over an algebraically closed field $\mathbb{K}$ is the ordered pair $((M_i)_{i \in Q_0}, (M^\alpha)_{\alpha \in Q_1})$, where M_i is a finite-dimensional $\mathbb{K}$-vector space attached to vertex $i \in Q_0$ and $M^\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $\mathbb{K}$-linear map for any $\alpha \in Q_1$. The **dimension vector** of M is $\textbf{dim}\, M := (\dim_{\mathbb{K}} M_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{|Q_0|}$.

A **subrepresentation** of M, denoted by $N = ((N_i)_{i \in Q_0}, (M^\alpha \restriction_{N_{s(\alpha)}})_{\alpha \in Q_1})$, is a Q-representation such that $N_i \subseteq M_i$ for any $i \in Q_0$ and $M^\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)}$ for any $\alpha \in Q_1$.

From now on, we consider only bound quivers in the sense of Schiffler (see [28, Definition 5.1]).

The finite-dimensional Q-representations over $\mathbb{K}$ form a category, denoted by $\text{rep}_{\mathbb{K}}(Q)$, where a morphism ϕ between M and M' in $\text{rep}_{\mathbb{K}}(Q)$ is given by linear maps $\phi_i : M_i \to M_i'$ $\forall i \in Q_0$ such that $\phi_{t(\alpha)} \circ M^\alpha = M'^\alpha \circ \phi_{s(\alpha)}$. Similarly, the category $\text{rep}_{\mathbb{K}}(Q, I)$ consists of the finite-dimensional representations of Q that satisfy the relations in I. It is known (see [28, Theorem 5.4] for a proof) that $\text{rep}_{\mathbb{K}}(Q)$ is equivalent to the category A-mod of finite-dimensional modules over the path algebra $A = \mathbb{K}Q$ of Q. Furthermore, $\text{rep}_{\mathbb{K}}(Q)$ is Krull-Schmidt ([28, Theorem 1.2]) and hereditary ([28, Theorem 2.24]).

Definition 2.4 A representation $P \in \text{rep}_{\mathbb{K}}(Q)$ is called **projective** if the functor $\text{Hom}(P, -)$ maps surjective morphisms to surjective morphisms. Dually, $I \in \text{rep}_{\mathbb{K}}(Q)$ is called **injective** if the functor $\text{Hom}(-, I)$ maps injective morphisms to injective morphisms.

If Q a quiver without oriented cycles, then to each vertex $i \in Q_0$ corresponds exactly one indecomposable projective representation, denoted by $P(i)$. Such projective representations are easy to describe: the basis of the vector space $P(i)_k$ at vertex k is given by the set of all possible paths from vertex i to vertex k, and the actions of the maps between the vector spaces are induced by the concatenation of paths. Dually, for every vertex $i \in Q_0$ there is exactly one indecomposable injective representation $I(i)$, whose basis for each vector space $I(i)_k$ is given by the set of all possible paths from vertex k to vertex i and whose maps act by concatenation of paths.

The following result holds in any additive category.

Proposition 2.5 *[28, Proposition 2.7] Let P, P', I and I' be representations of Q. Then:*

(1) $P \oplus P'$ is projective $\Longleftrightarrow$ P and P' are projective;
(2) $I \oplus I'$ is injective $\Longleftrightarrow$ I and I' are injective.

Definition 2.6 Given a Q-representation M, a **projective resolution** of M is an exact sequence

$$\ldots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a projective Q-representation. An **injective resolution** of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \ldots,$$

where each I_i is an injective Q-representation.

Definition 2.7 The **projective dimension** of M is the smallest integer d such that there exists a projective resolution of the form

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

The **injective dimension** of M is the smallest integer d such that there exists an injective resolution of the form

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{d-1} \to I_d \to 0.$$

For $M, N \in \mathrm{rep}_{\mathbb{K}}(Q)$ we use the standard notation

$$[M, N] := \dim_{\mathbb{K}} \mathrm{Hom}_Q(M, N), \ [M, N]^1 := \dim_{\mathbb{K}} \mathrm{Ext}^1_Q(M, N).$$

We call a representation M **rigid** if it has no self-extensions, which means $[M, M]^1 = 0$. Finally, we denote by

$$\langle M, N \rangle = [M, N] - [M, N]^1$$

the Euler-Ringel form of Q, which can be computed (see for instance [3] for details) via the bilinear form $\langle -, - \rangle : \mathbb{Z}^{|Q_0|} \times \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$ defined as

$$\langle \mathbf{d}^M, \mathbf{d}^N \rangle = \langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle := \sum_{i \in Q_0} d_i^M d_i^N - \sum_{\alpha \in Q_1} d_{s(\alpha)}^M d_{t(\alpha)}^N. \tag{2.8}$$

For a quiver with relations (Q, I) and no oriented cycles, the Euler-Ringel form is given by

$$\langle \mathbf{d}^M, \mathbf{d}^N \rangle = \sum_{i \in Q_0} d_i^M d_i^N - \sum_{\alpha \in Q_1} d_{s(\alpha)}^M d_{t(\alpha)}^N + \sum_{i,j \in Q_0} r(i, j) d_i^M d_j^N. \tag{2.9}$$

More background and details can be found for instance in [3, Section 2.2] or [15]. To simplify notation, we will sometimes denote a Q-representation M by its tuple of vector spaces $(M_i)_{i \in Q_0}$ when the assignment of the linear maps is clear from context.

Definition 2.10 Consider a quiver Q, a Q-representation M and a dimension vector $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$ such that $e_i \leq \dim M_i \ \forall i \in Q_0$. The **quiver Grassmannian** $\mathrm{Gr}_{\mathbf{e}}(M)$ parametrises the subrepresentations N of M with $\dim N_i = e_i$ for all $i \in Q_0$.

Analogous to Grassmannians and flag varieties, non-empty quiver Grassmannians can be realised as closed subvarieties of products of Grassmannians, via the closed embedding

$$\iota : \mathrm{Gr}_{\mathbf{e}}(M) \to \prod_{i \in Q_0} \mathrm{Gr}(e_i, M_i)$$

which sends a subrepresentation N of M to the collection of e_i-dimensional subspaces N_i of M_i.

Example 2.11 (The flag variety of type A) In [8, Proposition 2.7], the authors realise the (linear degenerate) flag variety of type A as quiver Grassmannians associated to certain representations of the equioriented quiver of type $\mathbb{A}_n$. In particular, the complete flag variety Fl_{n+1} in $\mathbb{C}^{n+1}$ can be realised as follows.

Consider the quiver with n vertices, labelled from 1 to n, and $n - 1$ arrows of the form $i \to i + 1$. We fix the dimension vector $\mathbf{e} = (1, 2, \ldots, n)$ and the representation M with $M_i = \mathbb{C}^{n+1}$ for $i = 1, \ldots, n$ and $M^\alpha = \mathrm{id}$ for all arrows α:

$$\overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\mathrm{id}} \overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\mathrm{id}} \ldots \xrightarrow{\mathrm{id}} \overset{\mathbb{C}^{n+1}}{\bullet}.$$

The quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ consists precisely of the subrepresentations N of M with $\dim(N_i) = i$, i.e. full flags of vector subspaces.

3 Background on Schubert Varieties

Given any $v_1, \ldots, v_r$ in $\mathbb{C}^{n+1}$, we denote by $\langle v_1, \ldots, v_r \rangle$ their $\mathbb{C}$-linear span. To define Schubert varieties in Fl_{n+1}, we first fix a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_{n+1}\}$ of $\mathbb{C}^{n+1}$ and denote by

$F_\bullet$ the standard flag $\langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \cdots \subseteq \langle b_1, b_2, \ldots, b_{n+1} \rangle$ and by S_{n+1} the symmetric group on $n + 1$ elements. More facts and details about Schubert varieties can be found for instance in [20, Part III].

Definition 3.1 For $w \in S_{n+1}$, the **Schubert cell** X_w° is

$$X_w^\circ = \{V_\bullet \in Fl_{n+1} : \dim(F_p \cap V_q) = \#\{k \leq q : w(k) \leq p\}, \ 1 \leq p, q \leq n + 1\}.$$

Definition 3.2 The **Schubert variety** X_w is defined as the closure in Fl_{n+1} of the cell X_w°, that is

$$X_w = \{V_\bullet \in Fl_{n+1} : \dim(F_p \cap V_q) \geq \#\{k \leq q : w(k) \leq p\}, \ 1 \leq p, q \leq n + 1\}.$$

For the purposes of this paper, we consider the intersections $F_p \cap V_q$ instead of the standard $V_p \cap F_q$ in the definition of Schubert varieties, and to simplify the notation we write $r_{p,q}^w$ for the numbers $\#\{k \leq q : w(k) \leq p\}$.

We observe that the conditions on the intersections between the F_p and the V_q imply, for each pair p, q, one of the following : $F_p \subset V_q$, $F_p \supset V_q$, $F_p = V_q$ or $F_p \cap V_q = U$ with $0 \leq \dim(U) < \min\{p, q\}$. A minimal set of conditions that imply all the conditions defining a Schubert variety X_w has been described in terms of essential sets of the permutation w (see [19, Section 3] or [21, Section 4]). Each Schubert variety X_w is an irreducible subvariety of Fl_{n+1}, and its dimension is given by the number of inversions in w, called length:

$$\ell(w) = \#\{i < j : w(i) > w(j)\}.$$

The length of a permutation w is also the minimal number of simple transpositions needed to form a decomposition of w, called reduced decomposition: $w = s_{\ell(w)} \cdots s_1$, where s_i denotes the swap of i and $i + 1$. We recall that, in general, a permutation admits more than one reduced decomposition.

The length of w is closely related to the numbers $r_{p,q}^w$ appearing in Definition 3.1. In fact, it is easily recovered by representing each flag $V_\bullet \in X_w^\circ$ via a certain matrix, as explained in [20, Section 10.2], and counting how many entries of this matrix are arbitrary complex numbers.

Given $u, w \in S_{n+1}$, we say that the relation $u \leq w$ holds under the Chevalley-Bruhat (partial) order if a reduced expression of w contains a subexpression which is a reduced expression for u. The Schubert variety X_w consists of the cell X_w°, which is open and dense in X_w, and of the cells corresponding to permutations that are smaller than w with respect to the Chevalley-Bruhat order $\leq$ on S_{n+1}: $X_w = \sqcup_{u \leq w} X_u^\circ$.

We represent a permutation w in S_n by listing its (naturally) ordered images, that is, its one-line notation $w = [w(1)w(2) \ldots w(n)]$.

Example 3.3 For $e = [1\,2 \ldots n + 1]$ and $w_0 = [n + 1\,n \ldots 1]$ in S_{n+1}, it is easy to compute from Definition 3.2 the Schubert varieties of minimal and maximal dimension, respectively $X_e = \{F_\bullet\}$ and $X_{w_0} = Fl_{n+1}$.

Smooth Schubert varieties were characterised combinatorially in [24]: a Schubert variety X_w is smooth if and only if w avoids the patterns [4231] and [3412]. We recall that a permutation $w = [w(1)w(2) \ldots w(n)]$ avoids a pattern π if no subsequence of w has the same relative order as the entries of π. In [21, Theorem 1.1], the authors prove that this pattern-avoiding condition is equivalent to X_w being defined by non-crossing inclusions:

Definition 3.4 ([21, Section 1]) A Schubert variety X_w is **defined by inclusions** if the defining conditions on each V_q (see Definition 3.2) are a conjunction of conditions of the form $V_q \subseteq F_p$ and $V_q \supseteq F_s$, for some p and s. A pair of conditions $V_q \subset F_p$ and $F_{p'} \subset V_{q'}$ is **crossing** if $q < q'$ and $p > p'$.

If X_w is defined by inclusions and its conditions do not contain any crossing pair, then X_w is **defined by non-crossing inclusions**.

Example 3.5 All permutations in S_3 are defined by non-crossing inclusions.

In S_5, the permutation $w = [31542]$ avoids both patterns $[4231]$ and $[3412]$, which means that X_w is defined by non-crossing inclusions. We can compute these inclusions using Definition 3.2: a flag $V_\bullet$ is in X_w if and only if

$$V_1 \subseteq F_3, \ F_1 \subseteq V_2 \subseteq F_3, \ F_1 \subseteq V_3, \ F_1 \subseteq V_4.$$

The same conditions can be described without redundancy as $F_1 \subseteq V_2 \subseteq F_3$, which is a pair of non-crossing inclusions.

A permutation in S_5 that yields crossing inclusions is $\tau = [45312]$, which contains the pattern $[3412]$. A flag $V_\bullet$ is in X_τ if and only if $V_1 \subseteq F_4$ and $F_1 \subseteq V_4$. Finally, the permutation $\pi = [53421]$ in S_5 contains the pattern $[4231]$ and defines a non-trivial condition on X_π that is not an inclusion: a flag $V_\bullet$ is in X_π if and only if $\dim(F_3 \cap V_2) \geq 1$.

4 A Special Quiver With Relations and a Special Representation

We now define a quiver with relations and construct a rigid representation for this quiver. This will then be exploited in Sections 5 and 6, together with two different, appropriate choices of dimension vectors, to recover the Bott-Samelson resolution for Schubert varieties and to realise smooth Schubert varieties as the corresponding quiver Grassmannian, respectively.

The following construction of the quiver and its special representation depends on the fixed ambient dimension $n + 1$ but not on the chosen Schubert variety (that is, it's independent of the specific permutation w in S_{n+1}).

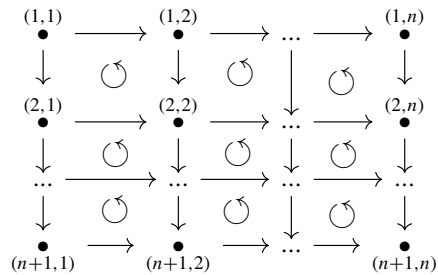Given $n \in \mathbb{N}_{\geq 2}$, we consider the following quiver $\Gamma = (\Gamma_0, \Gamma_1)$:
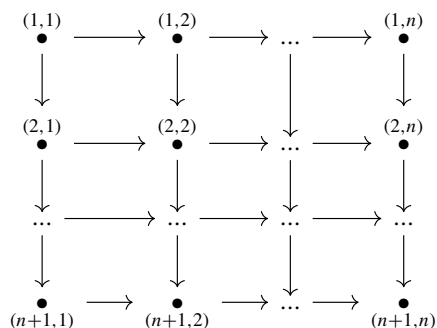
where each vertex in Γ_0 is labelled by a pair (i, j), for $i = 1, \dots, n+1$ and $j = 1, \dots, n$. We denote by $\alpha_{(i,j)}^{(k,l)}$ the arrow going from vertex (i, j) to vertex (k, l).

Then, we consider the following relations on Γ:

$$\alpha_{(i,j+1)}^{(i+1,j+1)}\alpha_{(i,j)}^{(i,j+1)} = \alpha_{(i+1,j)}^{(i+1,j+1)}\alpha_{(i,j)}^{(i+1,j)} \tag{4.1}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, n - 1$, and denote by I the ideal of $\mathbb{C}\Gamma$ generated by these relations. We write (Γ, I) for the quiver with relations:
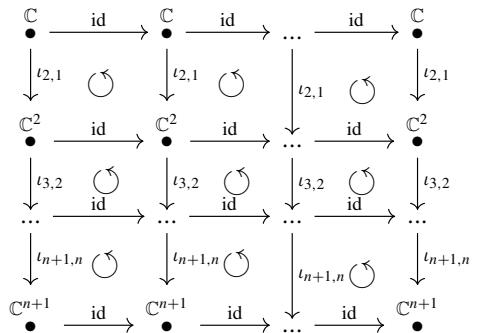

Now, we define the (Γ, I)-representation $M = ((M_{i,j})_{(i,j)\in\Gamma_0}, (M^\alpha)_{\alpha\in\Gamma_1})$ as

$$
M_{i,j} = \mathbb{C}^i, \quad M^\alpha = \begin{cases} \iota_{i+1,i} & \text{if } s(\alpha) = (i, j), t(\alpha) = (i + 1, j) \\ \text{id} & \text{if } s(\alpha) = (i, j), t(\alpha) = (i, j + 1) \end{cases} \tag{4.2}
$$

where $\iota_{i+1,i}$ denotes the inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$, represented with respect to the chosen basis $\mathcal{B} = \{b_1, b_2, \ldots, b_{n+1}\}$ by the matrix $\iota_{i+1,i} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The relations imposed on Γ are trivially satisfied by the representation M:

We now want to show that M is a rigid representation of (Γ, I). To do so, we first consider the following subquiver Γ' of Γ

i.e. the equioriented Dynkin quiver $\mathbb{A}_{n+1}$, and call M' the restriction of the representation M to Γ':

Lemma 4.3 M' *is a rigid representation of* Γ'.

Proof The representation M' is the direct sum of the indecomposable projective representations $P(i)$ of $\mathbb{A}_{n+1}$, which are of the form:

$$P(i): \quad \underset{1}{\overset{0}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \underset{n+1}{\overset{\mathbb{C}}{\bullet}}$$

for $i = 1, \ldots, n+1$. More precisely, $M' = \bigoplus_{i=1}^{n+1} P(i)$. By Proposition 2.5, M' is projective, which implies that $\mathrm{Ext}^1_{\Gamma'}(M', M') = 0$. $\qquad\square$

Proposition 4.4 M *is a rigid representation of* (Γ, I).

Proof Consider Γ' and M' as in Lemma 4.3 and the functor

$$\Phi : \mathrm{rep}_{\mathbb{C}}(\Gamma') \to \mathrm{rep}_{\mathbb{C}}(\Gamma, I)$$

defined on $R \in \mathrm{rep}_{\mathbb{C}}(\Gamma')$ as follows. For all $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$, we set $\Phi(R)_{i,j} = R_i$. For each arrow $i \to i+1$ in Γ' and $j = 1, \ldots, n$, the map $\Phi(R)_{i,j} \to \Phi(R)_{i+1,j}$ is defined as the map $R_i \to R_{i+1}$. Finally, for each $i = 1, \ldots, n+1$ and $j = 1, \ldots, n-1$, the map $\Phi(R)_{i,j} \to \Phi(R)_{i,j+1}$ is id_{R_i}. From the definition of Φ, it follows that $\Phi(M') = M$. As shown in [25, Lemma 2.3, Lemma 2.5], Φ is an exact, fully faithful functor that takes projective objects to projective objects. This implies ([25, Corollary 2.6]) that $\mathrm{Ext}^i_{(\Gamma, I)}(\Phi(V), \Phi(W)) \cong \mathrm{Ext}^i_{\Gamma'}(V, W)$ for every $V, W \in \mathrm{rep}_{\mathbb{C}}(\Gamma')$ and $i \geq 0$. In particular, we have

$$\mathrm{Ext}^1_{(\Gamma, I)}(M, M) \cong \mathrm{Ext}^1_{\Gamma'}(M', M') = 0.$$

$\qquad\square$

We will prove in Corollary 4.6 that all quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}(M)$ associated to M, independently of the dimension vector $\mathbf{e}$, are smooth, irreducible projective varieties of dimension $\langle \mathbf{e}, \dim M - \mathbf{e} \rangle$. The smoothness and irreducibility of quiver flag varieties were already investigated in [30, Theorem 4.12] using a scheme theoretic approach. Here, we provide an alternative proof exploiting instead the rigidity of the representation M together with a few homological properties. For this purpose, we prove the following facts (see Definition 2.7 for projective and injective dimension of a representation):

Proposition 4.5 *Given* (Γ, I) *and* M *as above,*

(1) the projective dimension of M *is 0;*
(2) the injective dimension of M *is 1.*

Proof (1) The representation M is a projective representation of (Γ, I), and therefore has projective dimension equal to zero, because it is a direct sum of projective (Γ, I)-representations (by Proposition 2.5). Let us denote by $P(i, j)$ the indecomposable projective representation of (Γ, I) at vertex (i, j); then, $M = P(1, 1) \oplus P(2, 1) \oplus \cdots \oplus P(n, 1) \oplus P(n+1, 1)$.

(2) In order to show that the injective dimension of M is one, we construct an injective resolution of M (see Definition 2.6). We define the injective representations I_0, I_1 of (Γ, I) as the following sums of indecomposable injective representations:

$$I_0 = \bigoplus_{i=1}^{n+2} I(n+1, n), \quad I_1 = \bigoplus_{i=1}^{n+1} I(i, n).$$

Then, the sequence

$$\varepsilon : 0 \longrightarrow M \xrightarrow{f} I_0 \xrightarrow{g} I_1 \longrightarrow 0$$

where f is the injective map embedding M into I_0 and g is the surjective map projecting I_0 onto I_1 (which implies $\mathrm{Im}(f) = M = \ker(g)$) is a short exact sequence.

□

Corollary 4.6 *Given (Γ, I) and M as above, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$, if not empty, is a smooth and irreducible projective variety for any dimension vector $\mathbf{e}$. Its dimension is $\langle \mathbf{e}, \mathbf{dim}\ M - \mathbf{e} \rangle$.*

Proof As shown in Proposition 4.4, M is a rigid representation of (Γ, I), therefore the irreducibility of $\mathrm{Gr}_{\mathbf{e}}(M)$ follows directly from [5, Proposition 38]. In order to prove the remaining claims, we show that all hypotheses of [9, Proposition 7.1] hold. The representation M has projective dimension zero, since it is a projective representation of (Γ, I), and its injective dimension is one (see Proposition 4.5). It is straightforward to verify that the quotient algebra $\mathbb{C}\Gamma/I$ has global dimension two, since it can be realised as the tensor product of two well-known path algebras. Namely, we consider the path algebra of the cartesian product of an equioriented $\mathbb{A}_n$ quiver and an equioriented $\mathbb{A}_{n+1}$ quiver and take the quotient over the commutativity relations on all resulting squares. It is known that the global dimension of the path algebra of any type $\mathbb{A}_n$ quiver (for $n \geq 2$) is one (see, for instance, [28, Section 2.2]). Then, we apply [2, Theorem 16] and obtain that the global dimension of $\mathbb{C}\Gamma/I$ is the sum of the global dimensions of the path algebras of the two quivers of type $\mathbb{A}_n$. □

5 Recovering the Bott-Samelson Resolution for Schubert Varieties

We consider the quiver (Γ, I) and its representation M constructed in the previous section, and fix a permutation w in S_{n+1}. The conditions that define the elements $V_\bullet$ in X_w are of the form $\dim(F_p \cap V_q) \geq \#\{k \leq q : w(k) \leq p\}$, for $1 \leq p, q \leq n + 1$ (see Definition 3.2). Notice that for $q = n + 1$ and any p these conditions are trivial, since $n + 1$ is the dimension of the ambient space $\mathbb{C}^{n+1}$, and therefore it is enough to consider $q = 1, \ldots, n$.

Now we define the dimension vector $\mathbf{r}^w = (r_{i,j}^w)$ for the quiver (Γ, I) as

$$r_{i,j}^w := \#\{k \leq j : w(k) \leq i\}, \quad i = 1, \ldots, n+1, \quad j = 1, \ldots, n. \tag{5.1}$$

Before introducing the Bott-Samelson resolution for Schubert varieties, let us make a few remarks about this definition for the dimension vector $\mathbf{r}^w$, in particular about how its entries change as we move from w to permutations that are bigger than w with respect to the Bruhat order in S_{n+1}. The following lemma describes which (unique) row and which columns of the dimension vector are affected, and how they change, when we left-multiply by a simple transposition which increases by one the length of the permutation we are considering.

Lemma 5.2 *Consider $r_{p,q}^{\hat{w}} = \#\{j \leq q : \hat{w}(j) \leq p\}$ for $1 \leq p, q \leq n + 1$ and a fixed $\hat{w} \in S_{n+1}$ (see Definition 3.2). Then, for a simple transposition s_i such that $\ell(s_i \hat{w}) = \ell(\hat{w}) + 1$, the numbers $r_{p,q}^{s_i \hat{w}} = \#\{j \leq q : s_i \hat{w}(j) \leq p\}$ are given by*

$$\begin{cases} r_{p,q}^{s_i \hat{w}} = r_{p,q}^{\hat{w}} - 1 & \text{if } p = i \text{ and } q_i \leq q < q_{i+1} \\ r_{p,q}^{s_i \hat{w}} = r_{p,q}^{\hat{w}} & \text{otherwise} \end{cases}$$

where $q_i = \hat{w}^{-1}(i)$ and $q_{i+1} = \hat{w}^{-1}(i + 1)$.

Proof It is straightforward to verify that, since s_i only swaps i and $i + 1$, the count is not affected when $p \neq i$ or when $p = i$ and $q < q_i$ or $q \geq q_{i+1}$.

If $p = i$ and $q_i \leq q < q_{i+1}$, there is exactly one j that satisfies $j \leq q \wedge \hat{w}(j) \leq p$ but not $j \leq q \wedge s_i \hat{w}(j) \leq p$, that is $j = q_i$, and so in this case the count decreases by one. □

Example 5.3 We fix $\hat{w} = [34251] \in S_5$ and compute the corresponding dimension vector $\mathbf{r}^{\hat{w}}$ according to the definition given in (5.1):

$$\mathbf{r}^{\hat{w}} = \begin{pmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 1 \\ 1\ 1\ 2\ 2 \\ 1\ 2\ 3\ 3 \\ 1\ 2\ 3\ 4 \end{pmatrix}.$$

Then we apply s_3, obtaining $w := s_3 \hat{w} = [43251]$, and we know from Lemma 5.2 that $\mathbf{r}^w$ differs from $\mathbf{r}^{\hat{w}}$ only at entry $r_{3,1}^w$:

$$\mathbf{r}^w = \begin{pmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 1 \\ 0\ 1\ 2\ 2 \\ 1\ 2\ 3\ 3 \\ 1\ 2\ 3\ 4 \end{pmatrix}.$$

Remark 5.4 An important consequence of Lemma 5.2 is that some information about the reduced decompositions of w can be read directly off the corresponding dimension vector. In Example 5.3, we can compare $\mathbf{r}^w$ to the dimension vector corresponding to the identity in S_5:

$$\mathbf{r}^{\mathrm{id}} = \begin{pmatrix} 1\ 1\ 1\ 1 \\ 1\ 2\ 2\ 2 \\ 1\ 2\ 3\ 3 \\ 1\ 2\ 3\ 4 \\ 1\ 2\ 3\ 4 \end{pmatrix}$$

and observe that $r_{2,2}^w = r_{2,2}^{\mathrm{id}} - 2$. By Lemma 5.2, this can only happen if the simple transposition s_2 appears at least two times in any reduced decomposition of w. Similarly, we deduce that s_1, s_3 and s_4 appear at least one time in any reduced decomposition of w. The converse is also true: if a simple transposition s_i appears k times in all reduced decompositions of a given permutation w (that is, there are k instances of s_i that are not part of any braid $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$), then there exists an entry in the i-th row of the dimension vector $\mathbf{r}^w$ that has decreased by k from its value in $\mathbf{r}^{\mathrm{id}}$. We do not include a proof of this statement as it is not relevant to the purpose of this section, but an idea of the strategy can be found in the proof of Theorem 5.18, since knowing that these k instances of s_i are not part of any braid allows us to describe which simple transpositions can appear between them.

In order to show that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ is isomorphic to certain Bott-Samelson resolutions of X_w, we recall the following definition of Bott-Samelson varieties:

Definition 5.5 ([22, Definition 3.1]) Given a permutation $w \in S_{n+1}$ of length N and a reduced decomposition $w = s_{i_N} \cdots s_{i_1}$, the **Bott-Samelson variety** $\mathrm{BS}(s_{i_N} \cdots s_{i_1})$ is a subvariety of $(Fl_{n+1})^N$ defined as follows:

$$\mathrm{BS}(s_{i_N} \cdots s_{i_1}) = \{(V_\bullet^0, V_\bullet^1, \ldots, V_\bullet^N) \in (Fl_{n+1})^N : V_i^{k-1} = V_i^k, \forall k = 1, \ldots, N,$$
$$\forall i = 1, \ldots, n, i \neq i_k\}$$

where $V_\bullet^0 = F_\bullet = \langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \cdots \subseteq \langle b_1, b_2, \ldots, b_{n+1} \rangle$, the standard flag in Fl_{n+1}.

Example 5.6 We fix again the permutation $w = [43251] \in S_5$ from Example 5.3 and its reduced decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$. The elements $V_\bullet^\bullet$ of $BS(s_1 s_2 s_3 s_1 s_2 s_1 s_4)$ are given by tuples of seven complete flags, each living in Fl_5, of the following form:

$$\langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, \qquad V_1^4 \subseteq V_2^3 \subseteq V_3^5 \subseteq V_4^1,$$
$$V_1^2 \subseteq \langle b_1, b_2 \rangle \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, \qquad V_1^4 \subseteq V_2^6 \subseteq V_3^5 \subseteq V_4^1,$$
$$V_1^2 \subseteq V_2^3 \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, \qquad V_1^7 \subseteq V_2^6 \subseteq V_3^5 \subseteq V_4^1.$$
$$V_1^4 \subseteq V_2^3 \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1,$$

Remark 5.7 Bott-Samelson varieties provide an explicit desingularisation for type A Schubert varieties: the desingularisation map $BS(s_{i_N} \cdots s_{i_1}) \to X_w$ sends the tuple $(V_\bullet^0, V_\bullet^1, \ldots, V_\bullet^N)$ to its last entry $V_\bullet^N$.

As shown in [1, Chapter 18, Lemma 2.1, Lemma 2.3], the Bott-Samelson varieties corresponding to different reduced decompositions of the same permutation w are birational, since they are all birational to the Schubert variety X_w. It is known that Bott-Samelson varieties corresponding to reduced decompositions that differ only by a commutation move are isomorphic (see [16, Section 2]). If two reduced decompositions of a permutation differ by braid moves, it is not true in general that the corresponding Bott-Samelson varieties are isomorphic.

In order to show that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ is birational to any Bott-Samelson resolution of X_w, it is enough to pick an opportune reduced decomposition of w.

Remark 5.8 To recover certain Bott-Samelson resolutions of Schubert varieties via quiver Grassmannians, we started by defining a quiver and a representation for this quiver (in Section 4) which only depend on the ambient dimension $n + 1$. Then, we fixed a permutation $w \in S_{n+1}$ and computed the dimension vector $\mathbf{r}^w$ according to Definition 3.2. We remark that, given a Bott-Samelson variety, we can find an immediate realisation as a quiver Grassmannian for the grid. Here, we show this through an example and discuss how it differs from our construction.

For instance, consider the permutation $w = [43251] \in S_5$ from Example 5.3, its reduced decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$ and the corresponding Bott-Samelson variety described in Example 5.6. We construct the following quiver Q and Q-representation:

and consider the following dimension vector:

$$\mathbf{b}^{s_1 s_2 s_3 s_1 s_2 s_1 s_4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}.$$

Then, each flag $V_\bullet^j$ in $\mathrm{BS}(s_1 s_2 s_3 s_1 s_2 s_1 s_4)$ is realised in the j-th column of Q. The identity maps corresponding to horizontal arrows impose the inclusion conditions between the appropriate subspaces of the flags $V_\bullet^j$, while the zero maps point to the free subspaces $V_4^1, V_1^2, V_2^3, V_1^4, V_3^5, V_2^6$ and V_1^7, in order.

In this construction, the entries of the considered dimension vector do not depend on w. The number of columns of Q and of the dimension vector, on the other hand, depends on the length of w, and the representation of Q can only be constructed after choosing a specific reduced decomposition of w. Different choices of reduced decompositions yield different Q-representations.

More significantly, we cannot apply to this Q-representation the same arguments as in Section 4. There, we proved the rigidity of the representation M for the quiver with relations (Γ, I). By Corollary 4.6, every quiver Grassmannian for the representation M is smooth and irreducible, and its dimension is given by the Euler-Ringel form of (Γ, I). This means that, using our construction, we realise the chosen Bott-Samelson variety as a quiver Grassmannian that is part of a larger family of varieties which share some common, desirable properties.

Definition 5.9 Let $w \in S_{n+1}$ and denote by $R = (R_{p,q})$, for $p = 1, \ldots, n+1$ and $q = 1, \ldots, n$, an element of $\mathrm{Gr}_{\mathbf{r}^w}(M)$.

- We denote by $\mathcal{M}^w$ the nonordered multiset $[r_{p,q}^w]$ for all the p, q such that

$$
\begin{cases}
r_{p,q}^w < p \\
r_{p,q}^w > r_{p-1,q}^w \\
r_{p,q}^w > r_{p,q-1}^w
\end{cases}.
\tag{5.10}
$$

- We call a reduced decomposition $w = s_{i_N} \cdots s_2 s_1$ **geometrically compatible** if $[i_N, \ldots, i_2, i_1]$ and $\mathcal{M}^w$ coincide as nonordered multisets.

Remark 5.11 The numbers contained in $\mathcal{M}^w$ are the dimensions of exactly those subspaces $R_{p,q}$ of $\mathbb{C}^p$ that are not trivial and do not coincide with a subspace to their left or above them. Fixing an element R in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ means precisely to make a choice for all such subspaces $R_{p,q}$. Hence the name "geometrically compatible" decomposition: it is a reduced decomposition of w from which we can read the dimensions of all the subspaces that are relevant to determine R.

Because of the geometrical significance of the conditions given in (5.10), we will interchangeably refer to the $r_{p,q}^w$ and to the $R_{p,q}$ that satisfy these conditions.

Example 5.12 We consider $w = [43251]$ and the corresponding dimension vector $\mathbf{r}^w$ as in 5.3. Given any subrepresentation R in $\mathrm{Gr}_{\mathbf{r}^w}(M)$, the subspaces whose dimensions satisfy all conditions in (5.10) are $R_{2,3}, R_{3,2}, R_{4,1}, R_{3,3}, R_{4,2}, R_{4,3}$ and $R_{5,4}$. Their dimensions are, in order, 1,1,1,2,2,3, and 4, so a geometrically compatible decomposition of w contains three s_1, two s_2, one s_3 and one s_4. The decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$ considered in 5.6 is geometrically compatible, while, for instance, $w = s_3 s_1 s_2 s_1 s_3 s_2 s_4$ is not.

Now we want to show that all permutations admit a geometrically compatible decomposition. To do so, firstly we characterise the multiset $\mathcal{M}^w$, that is, what follows from the fact that a certain $R_{p,q}$ is not a trivial subspace of $\mathbb{C}^p$ in terms of the reduced decompositions of w.

Recall that the length of a permutation w can be equivalently defined as the number of inversions appearing in w or as the number of simple transpositions that form any reduced decomposition of w.

Lemma 5.13 *Given $w \in S_{n+1}$ and a subrepresentation $R = (R_{p,q})$ in $\mathrm{Gr}_{\mathbf{r}^w}(M)$, with $p = 1, \ldots, n+1$ and $q = 1, \ldots, n$, the cardinality of $\mathcal{M}^w$ is exactly the length of w.*

Proof We denote by N the length of w and write $w = s_{i_N} \cdots s_2 s_1$. By Lemma 5.2, the left-multiplication of each s_{i_k} results in a new subspace (namely $R_{i_k+1, q_{i_k}}$, for q_{i_k} as in the notation of Lemma 5.2) satisfying the conditions in (5.10).

On the other hand, applying one simple transposition cannot cause two new subspaces to satisfy the conditions in (5.10): all the affected entries of the dimension vector decrease by the same amount (by one), and the absolute difference between neighbouring entries of $\mathbf{r}^w$ is always zero or one (this follows from Definition 3.2).

The claim follows by iterating this argument on each s_{i_k}, for $k = 1, \ldots, N$. □

Lemma 5.14 *Let $w \in S_{n+1}$ and R any subrepresentation in $\mathrm{Gr}_{\mathbf{r}^w}(M)$. If the numbers in $\mathcal{M}^w$ are all distinct, then w admits a geometrically compatible decomposition.*

Proof In order for a subspace of dimension d to satisfy the conditions in (5.10), the simple transposition s_d must appear at least once in any reduced decomposition of w. If this is not the case, by Lemma 5.2 the d-th row of $\mathbf{r}^w$ is equal to the d-th row of $\mathbf{r}^{\mathrm{id}}$, which implies that all subspaces appearing in R of dimension d have to coincide with $\mathbb{C}^d$ - and therefore do not satisfy the conditions in (5.10). The result then follows immediately from Lemma 5.13. □

Remark 5.15 A straightforward consequence of Lemma 5.2 is that if the reduced decompositions of $w \in S_{n+1}$ consist of all distinct simple transpositions, then they are geometrically compatible. As shown in the lemma, each of these transpositions s_{i_k} affects the corresponding row of the dimension vector, resulting in the subspace R_{i_k+1, q_i} (which has dimension i_k) satisfying the conditions in (5.10).

Remark 5.15 and Lemma 5.14 describe the same situation, but from two different perspectives: in the first case we assume the numbers in $\mathcal{M}^w$ to be all distinct, and this allows us to write down a reduced decomposition of w, in the second case we already have a reduced decomposition of w and deduce that it is also geometrically compatible.

We recall the following notation from Lemma 5.2: if we left-multiply a permutation w by a simple transposition s_j, we denote by q_j the pre-image $w^{-1}(j)$ of j via w.

Lemma 5.16 *Let $w \in S_{n+1}$ with reduced decomposition $w = s_{i_N} \ldots s_{i_1}$, s_j be a simple transposition such that $\ell(s_j w) = \ell(w) + 1$. Let R_{j+1, q_j} be the subspace that satisfies the conditions in (5.10) if R is any subrepresentation in $\mathrm{Gr}_{\mathbf{r}^{s_j w}}(M)$ (but does not satisfy them if R is in $\mathrm{Gr}_{\mathbf{r}^w}(M)$). Then, the dimension of R_{j+1, q_j} is $\hat{j}$ for some $\hat{j} \leq j$. In particular, $\hat{j} < j$ can only happen if all reduced decompositions of $s_j w$ are of the form $s_j w = s_j s_{i_N} \ldots s_{i_k} \ldots s_{i_1}$, where $i_k = j$ for some k such that $i_t \neq j + 1$ for all $k < t < N$.*

Proof A subspace $R_{p,q}$ can satisfy the conditions in (5.10) only if $p \geq \dim(R_{p,q}) + 1$, and we know from Lemma 5.2 that the only effect of s_j on the corresponding dimension vector is to decrease certain entries in row j by one. By Definition 5.1 of the dimension vector, all entries are bounded by their corresponding numbers of row and column. This implies that a dimension j' can only appear from row j' downwards, and so the dimension of R_{j+1, q_j} cannot be greater than j.

For the second statement, we know that an index k such that $i_k = j$ exists: otherwise, as stated in Remark 5.15, if the simple transpositions appearing in the reduced decomposition of $s_j w$ are all distinct, then the dimension of R_{j+1, q_j} is j. Then, we suppose that s_{j+1} occurs between these two instances of s_j and look at which entries of the dimension vector decrease

when w is left-multiplied by s_j. According to Lemma 5.2, the entries in columns q_j and $q_j + 1$ (and possibly more) decrease by one. This meanins that the subspace R_{j+1,q_j}, for $R \in \mathrm{Gr}_{\mathbf{r}^{s_j w}}(M)$, cannot satisfy the conditions in (5.10), which contradicts the assumption.□

Remark 5.17 Lemma 5.16 characterises when it is possible to perform braid moves in a decomposition of $w \in S_{n+1}$ in terms of the numbers contained in $\mathcal{M}^w$. The second statement in Lemma 5.16 implies that if a transposition s_i appears k times in all reduced decompositions of w (i.e. these k instances of s_i are not part of any braid move) then there are (at least) k subspaces $R_{p,q}$ of dimension i that satisfy the conditions in (5.10). On the other hand, if we apply s_j after w and obtain a reduced decomposition of $s_j w$ that is not geometrically compatible, we know that it is possible to perform a braid move on $s_j s_{j-1} s_j$. This follows from the fact that we can move s_j to the right via commutation until we find an instance of s_{j-1}, and similarly move the second instance of s_j to the left until s_{j-1} (s_{j+1} cannot occur in between by Lemma 5.16).

For instance, we saw in Example 5.12 a reduced decomposition for $w = [43251]$ that is not geometrically compatible: $w = s_3 s_1 s_2 s_1 s_3 s_2 s_4$. We obtain $w = s_1 s_3 s_2 s_3 s_1 s_2 s_4$ by commutation on the two occurrences of s_3, then perform a braid move as described above and get $w = s_1 s_2 s_3 s_2 s_1 s_2 s_4$. Finally, we perform a braid move on $s_2 s_1 s_2$ and obtain the geometrically compatible decomposition of w shown in Example 5.12: $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$.

Theorem 5.18 *All permutations admit a geometrically compatible decomposition.*

Proof We consider a permutation $w \in S_{n+1}$ and denote by $m_i(w)$ the multiplicity of s_i in a given reduced decomposition of w (i.e. how many times s_i appears in this given reduced decomposition). Then, we consider the total number of repetitions $m = \sum_i m_i(w)$ and define $t = \ell(w) - m$. We prove the statement by double induction on t and m.

The base case of the induction ($t = 1$ and $m = 0$) and the induction step on t ($t \geq 1$ and $m = 0$) follow directly from Remark 5.15: a reduced decomposition of w without repetitions consists of distinct simple transpositions, and is therefore geometrically compatible.

For the induction step on m we show that, if a permutation with $m \geq 0$ repetitions admits a geometrically compatible decomposition, then a permutation with $m + 1$ repetitions admits a geometrically compatible decomposition (for any $t \geq 1$). Let $w = s_{i_N} \dots s_{i_1}$ with m repetitions be a geometrically compatible decomposition of w. For $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$, let d_i denote the number of subspaces $R_{p,q}$ of dimension i that satisfy the conditions in (5.10). Since the fixed decomposition of w is geometrically compatible, we have $m_i(w) = d_i$ for all i.

Let then $w' := s_j w = s_j s_{i_N} \dots s_{i_1}$ such that w' has $m + 1$ repetitions, which means $j = i_k$ for some k, and such that $\ell(w') = \ell(w) + 1$. If the corresponding new free subspace appearing in R has dimension j, then this reduced decomposition of w' is already geometrically compatible. If not, then by Lemma 5.16 the dimension of the new free subspace must be $\hat{j} < j$, and therefore $d_{\hat{j}}$ has increased by one. As described in Remark 5.17, we move s_j via commutation and perform a braid move: $w' = s_j s_{i_N} \dots s_{i_1} = s_{i_N} \dots s_j s_{j-1} s_j \dots s_{i_1} = s_{i_N} \dots s_{j-1} s_j s_{j-1} \dots s_{i_1}$, which implies $m_j(w') = m_j(w) - 1$ and $m_{j-1}(w') = m_{j-1}(w) + 1$. Now, if $\hat{j} = j - 1$ we have again $m_i(w') = d_i$ for each i, meaning that this reduced decomposition of w' is geometrically compatible. Otherwise, we denote by $\hat{w}$ the subword of w starting from the second instance of s_{j-1}: $\hat{w} = s_{j-1} \dots s_{i_1}$ and observe that $\hat{w}$ has m repetitions. Therefore, by the induction hypothesis, $\hat{w}$ admits a geometrically compatible decomposition.

We know that the current decomposition of $\hat{w}$ is not geometrically compatible, because the number of $s_{\hat{j}}$ appearing in $\hat{w}$ is $d_{\hat{j}} - 1$. The geometrically compatible decomposition of $\hat{w}$ must then be obtained by performing a sequence of braid moves until the braid $s_{\hat{j}+1} s_{\hat{j}} s_{\hat{j}+1} =$

$s_{\hat{j}}s_{\hat{j}+1}s_{\hat{j}}$. Each braid move decreases by one the number of s_{l+1} and increases by one the number of s_l, for $j - 2 \leq l \leq \hat{j}$. Since the number of all other transpositions appearing in $\hat{w}$ (and in w') is not changed during this process, in the end we get $m_i(w') = d_i$ for all i, which means that we obtained a geometrically compatible decomposition of w'. $\qquad\square$

Theorem 5.19 *Given a permutation $w \in S_{n+1}$ and a geometrically compatible decomposition $w = s_{i_N} \cdots s_1$, the Bott-Samelson resolution $\mathrm{BS}(s_{i_N} \cdots s_{i_1})$ of the Schubert variety X_w is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$.*

Before proving Theorem 5.19, we introduce the main tool for constructing the explicit isomorphism between the Bott-Samelson resolution and the quiver Grassmannian. Given a geometrically compatible decomposition $w = s_{i_N} \cdots s_1$, we define the map φ_w according to the correspondence between the ordered set of indices of the transpositions appearing in w and the vector space $R_{p,q}$ for any $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$:

$$\varphi_w : \{i_N, \ldots, i_1\} \to \{n + 1\} \times \{n\}$$
$$i_k \mapsto (p(k), q(k)) := (i_k + 1 + n_k, i_k + m_k) \tag{5.20}$$

with $n_k := \#\{j : j < k, i_j = i_k\}$ and $m_k := \#\{j : j > k, q_{i_j} \leq q(k) < q_{i_j+1}\}$, where q_{i_j} and q_{i_j+1} are defined as in Lemma 5.2. The map φ_w provides the explicit correspondence between the points of the Bott-Samelson resolution and the subrepresentations in the quiver Grassmannian. We illustrate this in the following example.

Example 5.21 Given a permutation w', the map φ_w defined in Equation (5.20) describes explicitly which subspace $R_{p,q}$, for $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$, becomes a nontrivial subspace of $\mathbb{C}^i$ when s_i is applied to w', with $\ell(w) = \ell(w') + 1$. Consider the geometrically compatible decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$ of Example 5.12, where s_1 appears three times, as s_{i_2}, s_{i_4} and s_{i_7}. The images of i_2, i_4 and i_7 via the map φ_w defined in (5.20) are $(i_2 + 1 + n_2, i_2 + m_2) = (2, 3)$, $(i_4 + 1 + n_4, i_4 + m_4) = (3, 2)$ and $(i_7 + 1 + n_7, i_7 + m_7) = (4, 1)$. The one-dimensional subspaces $R_{2,3}$, $R_{3,2}$ and $R_{4,1}$, which correspond respectively to the subspaces V_1^2, V_1^4 and V_1^7 considered in Example 5.6, are placed at the red vertices of (Γ, I):
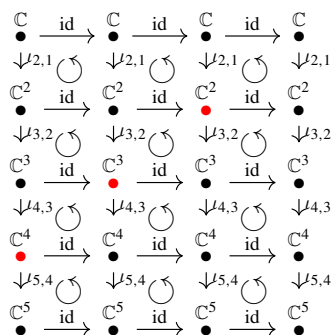
Proof of Theorem 5.19 We prove the statement by induction on the length of $w \in S_{n+1}$. For $w = \mathrm{id}$, the corresponding Bott-Samelson resolution and quiver Grassmannian coincide since they consist of a single point.

We then denote $w' = s_{i_{N-1}} \cdots s_{i_1}$ and assume $\mathrm{BS}(s_{i_{N-1}} \cdots s_{i_1}) \cong \mathrm{Gr}_{\mathbf{r}^{w'}}(M)$, where the isomorphism is given by $\varphi_{w'}$, defined in Eq. 5.20. This means that the explicit correspondence between an element $V_\bullet^\bullet \in \mathrm{BS}(s_{i_{N-1}} \cdots s_{i_1})$ and a subrepresentation $R' \in \mathrm{Gr}_{\mathbf{r}^{w'}}(M)$ is $V_a^b =$

$R'_{p(b),q(b)}$, therefore they are defined by the same inclusion conditions. We now consider $w = s_{i_N} w'$ such that $\ell(w) = \ell(w') + 1$. The image of i_N via φ_w is $(p(N), q(N)) = (i_N + 1 + n_N, i_N + m_N)$: we need to show that the subspace $R_{p(N),q(N)}$ is isomorphic to the subspace $V^N_{i_N}$, whose defining conditions are $V^b_a \subseteq V^N_{i_N} \subseteq V^c_d$ for $b, c < N$ and $a < i_N < d$. By the induction hypothesis, the subspaces $R_{p(b),q(b)}$ and $R_{p(c),q(c)}$ realise respectively V^b_a and V^c_d for all such a, b, c, d. We observe that the dimension of $R_{p(N),q(N)}$ is i_N due to the choice of a geometrically compatible decomposition of w. The statement then follows from the fact that R is a subrepresentation of M, which implies $R_{p,q} \subseteq R_{p(N),q(N)} \subseteq R_{p',q'}$ for all $p \leq p(N), q \leq q(N), p' \geq p(N), q' \geq q(N)$ and so in particular for $p = p(b), q = q(b)$, $p' = p(c), q' = q(c)$. $\qquad\square$

Corollary 5.22 *The Bott-Samelson resolutions corresponding to different reduced decompositions of the same permutation are birational (see Remark 5.7), therefore they are all birational to* $\mathrm{Gr}_{\mathbf{r}^w}(M)$.

6 Realisation of Smooth Schubert Varieties

In Section 5, we recovered the Bott-Samelson resolution for Schubert varieties by defining the dimension vector $\mathbf{r}^w$ for the quiver (Γ, I) as

$$r^w_{i,j} = \#\{k \leq j : w(k) \leq i\}$$

for all i, j. In this section, we give a construction for a different dimension vector for the quiver (Γ, I), denoted by $\mathbf{e}^w$, and show how the corresponding quiver Grassmannian realises the Schubert variety X_w if it is smooth, i.e. if w is pattern-avoiding. We recall from Section 3 that a permutation $w \in S_{n+1}$ corresponds to a smooth Schubert variety if and only if it avoids the patterns [4231] and [3412], and that this is equivalent to X_w being defined by non-crossing inclusions (see Definition 3.4).

Consider again the quiver (Γ, I) and its representation M constructed in Section 4, and fix a permutation w in S_{n+1} that avoids the patterns [4231] and [3412]. For $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$, we now define the dimension vector $\mathbf{e}^w = (e^w_{i,j})$ for the quiver (Γ, I) as:

$$\begin{cases} e^w_{i,j} := r^w_{i,j} & \text{if } r^w_{i,j} = \min\{i, j\} \\ & \text{or } r^w_{i,j} = 0 \\ e^w_{i,j} := \max\{e^w_{i-1,j}, e^w_{i,j-1}\} & \text{if } 0 < r^w_{i,j} < \min\{i, j\} \end{cases} \qquad (6.1)$$

Notice that the value of $r^w_{1,1}$ is either 0 or 1 (according to w) and falls therefore under the first case of Definition (6.1), meaning that $e^w_{1,1}$ is well-defined.

Example 6.2 We compute the conditions defining the flags $V_\bullet$ in X_w for $w = [65124837] \in S_8$ according to Definition 3.2. These conditions are of the form $\dim(F_p \cap V_q) \geq r_{p,q}$, for $1 \leq p, q \leq n+1$, where $r_{p,q} := \#\{k \leq q : w(k) \leq p\}$. We write the value of each $r_{p,q}$ as

the entry of a matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Since w avoids the patterns [4231] and [3412], X_w is smooth and defined by non-crossing inclusions. These inclusions, which follow from the inequalities above, are:

$$V_1 \subset F_6, V_2 \subset F_6, F_1 \subset V_3 \subset F_6, F_2 \subset V_4 \subset F_6, \\ F_2 \subset V_5 \subset F_6, F_2 \subset V_6, F_6 \subset V_7. \tag{6.3}$$

The corresponding dimension vector $\mathbf{e}^w$ obtained from Eq. 6.1 is

$$\mathbf{e}^w = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 & 2 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

By reading each entry $e^w_{i,j}$ as the dimension of the intersection $F_p \cap V_q$ and comparing $\mathbf{e}^w$ with the defining conditions in (6.3), we see how $\mathbf{e}^w$ encodes the same information on $V_\bullet$.

Theorem 6.4 *If $w \in S_{n+1}$ avoids the patterns [4231] and [3412], the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ is isomorphic to the Schubert variety X_w. The isomorphism is given by*

$$\psi : \mathrm{Gr}_{\mathbf{e}^w}(M) \to X_w \\ N \mapsto N_\bullet \tag{6.5}$$

where $N_\bullet = N_{n+1,1} \subseteq N_{n+1,2} \subseteq \cdots \subseteq N_{n+1,n}$.

Proof By the definition of M and $\mathbf{e}^w$, we have

$$N_{n+1,j} \subseteq N_{n+1,j+1}, \; \dim(N_{n+1,j}) = j$$

for all j, implying $N_\bullet \in Fl_{n+1}$. Since w avoids the patterns [4231] and [3412], all flags $V_\bullet = V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$ in X_w are defined by conditions of the following form: for each $q \in \{1, \ldots, n\}$, V_q is defined by $F_{p'_q} \subseteq V_q \subseteq F_{p_q}$ for some p_q, p'_q. These conditions are equivalent, respectively, to $\dim(F_{p'_q} \cap V_q) = \min(p'_q, q) = p'_q$ and $\dim(F_{p_q} \cap V_q) = \min(p_q, q) = q$. The definition of the dimension vector $\mathbf{e}^w$ (in the first line of (6.1)) imposes on $N_\bullet$ exactly these conditions, meaning that $F_{p'_q} \subseteq N_{n+1,q} \subseteq F_{p_q}$ for all q and the corresponding p'_q, p_q. The statement follows from the fact that, whenever the condition $\dim(F_i \cap V_j) \geq \#\{k \leq j : w(k) \leq i\}$ is not defining for $V_\bullet$ (i.e. it is redundant), the corresponding subspace $N_{i,j}$ in $N_\bullet$ is set to either $N_{i-1,j}$ or $N_{i,j-1}$ (second line of (6.1)).□

Remark 6.6 In Theorem 5.19, we proved that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ is isomorphic to the Bott-Samelson resolutions of X_w corresponding to geometrically compatible decompositions of w (see Definition 5.9). In general, the Bott-Samelson resolution of a Schubert variety (see Definition 5.5) is not a strict resolution, meaning that it is not an isomorphism over the smooth locus of the Schubert variety.

This implies that the quiver Grassmannians $\mathrm{Gr}_{\mathbf{r}^w}(M)$ and $\mathrm{Gr}_{\mathbf{e}^w}(M)$ are generally not isomorphic, even if the fixed Schubert variety is smooth.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval not applicable to this work.

Competing Interests The authors declare no competing interests.

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