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On the Analysis of Suboptimal Nonlinear Model Predictive Control Schemes Without Terminal Constraints

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***Abstract*—Model predictive control (MPC) – one of the most important control methodologies, with many real-world applications and being the focus of intense research efforts – is based on repeatedly solving open-loop optimal control problems. Since this requires in general online numerical optimisation, in practice the optimal control problems are solved only up to some suboptimality, though many analyses do not take this into account. Motivated by this gap, we prove performance bounds and stability results under a given level of suboptimality for nonlinear model predictive control schemes without terminal constraints or costs.**

***Index Terms*—Predictive control for nonlinear systems, optimal control.**

I. Introduction

MODEL predictive control (MPC) is one of the most important modern control methodologies, with many real-world applications, especially in industrial contexts, and being supported by an extensive theory [1]. MPC is based on solving (open loop) optimal control problems online, and applying the first portion of the optimal trajectory in a receding horizon fashion. Since the optimal control problem is solved by numerical methods, in most cases only suboptimal input trajectories are used, whereas most of the existing theory assumes optimal inputs. If terminal constraints are utilized, one can formulate MPC schemes that provably work even with suboptimal input trajectories, cf. [2]. In many industrial applications, nonlinear MPC (NMPC) schemes without terminal constraints are used, both due to practical as well as theoretical considerations, and stability guarantees in this setting are closely linked to optimality [3, Chapters 6, 7]. However, in practice the input sequences resulting from the optimal control problems are usually suboptimal. On the one hand, these optimisation problems are highly nonlinear and nonconvex, and global optimisation methods are infeasible in an online context. On the other hand, since the optimisations are performed online and hence subject to real-time constraints, the optimisation algorithms are often not run until (even local) optimality.

Analysing MPC schemes without terminal constraints has received considerable attention [4], [5], [6], [7]. Assuming that optimal open-loop input trajectories are used in the MPC law, precise non-asymptotic performance and stability results are available [3, Chapter 6]. Most analyses of nonlinear MPC schemes do not take the suboptimality of the open-loop input sequences into account. Motivated by this situation, in the present work, we investigate nonlinear MPC schemes without terminal constraints or costs under suboptimality, in particular, how the closed-loop performance and stability guarantees behave with respect to the level of suboptimality. Essentially, we analyse NMPC schemes with a *suboptimal* optimisation oracle, assuming an a priori bound on the suboptimality gap, but making no assumptions regarding the specifics of the optimisation method.

Closest to our setting and goal is the work [8]. However, it considers suboptimality in terms of a performance measure (corresponding to our $\alpha_{N,\bar{\gamma}}$ below), whereas we start with a concrete (additive) suboptimality level of the underlying open-loop optimal control problems. There are several works that consider suboptimality in the context of a specific numerical optimisation method, like interior point algorithms or projected gradient methods. Typical examples of such investigations are [9], [10], [11]. In contrast, we consider general MPC schemes without any specific numerical optimisation algorithm, and describe suboptimality in terms of the total cost optimality gap. Our work is therefore complementary to the aforementioned references. Finally, the present setting is related, but distinct from the one of approximate model predictive control (AMPC). Since solving an optimisation problem online at each step can be computationally expensive, in AMPC an MPC law is approximated by an efficient explicit representation like a piecewise-linear feedback law [12], a neural network [13], or a kernel machine [14]. This is different from our problem setting, where we consider suboptimality in the optimal control problem inside the MPC scheme. Furthermore, suboptimality of the open-loop trajectories used in the receding horizon control can be interpreted as an input disturbance, and there are MPC schemes that are robust to such perturbations, which in turn can be used for AMPC

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with guarantees [13], [15]. However, in the present context, we consider nominal MPC schemes (not taking any potential disturbances into account) and quantify the suboptimality in terms of total cost, not by the difference in the input space.

Outline: In Section II, we introduce our notation, formalize the setting, and present some preliminary results. In Section III, we formulate and prove a performance bound for the closed-loop system under suboptimal input sequences, and in Section IV, we investigate asymptotic and practical asymptotic stability using Lyapunov techniques. Finally, a summary and discussion of our results in Section V close the paper.

II. Preliminaries

Consider a discrete-time nonlinear control system

$$x_+ = f(x, u), \tag{1}$$

where $f : X \times U \to X$ is the transition function, and the state space X and the input set U are arbitrary non-empty sets. Furthermore, let $\mathbb{Z} \subseteq X \times U$ be a constraint set, inducing state constraints

$$\mathbb{X} = \{x \in X \mid \exists u \in U : (x, u) \in \mathbb{Z}\} \tag{2}$$

and input constraints

$$\mathbb{U}(x) = \{u \in U \mid (x, u) \in \mathbb{Z}\} \tag{3}$$

for $x \in X$ (of course, $\mathbb{U}(x) = \emptyset$ for $x \notin \mathbb{X}$). Furthermore, for $N \in \mathbb{N}_+$ and $x \in X$, define $\mathbb{U}^N(x)$ as the set of control inputs $u \in U^N$ such that $(x(n; x, u), u(n)) \in \mathbb{Z},\ : x(N; x, u) \in \mathbb{X}$ for all $n = 0, \ldots, N-1$, and define also

$$\mathbb{U}^\infty(x) = \{u \in U^\infty \mid \forall N \in \mathbb{N}_+ : u|_{\{0,\ldots,N-1\}} \in \mathbb{U}^N(x)\} \tag{4}$$

as well as

$$\mathbb{X}_N = \{x \in \mathbb{X} \mid \mathbb{U}^N(x) \neq \emptyset\}. \tag{5}$$

Additionally, given some $\tilde{X} \subseteq X$, define

$$\mathbb{U}^N_{\tilde{X}}(x) = \{u \in \mathbb{U}^N(x) \mid x(N; x, u) \in \tilde{X}\}. \tag{6}$$

Assumption 1: $\mathbb{X}$ is viable, i.e., for all $x \in \mathbb{X}$ there exists $u_x \in \mathbb{U}(x)$ such that $f(x, u_x) \in \mathbb{X}$.

Note that in general Assumption 1 is rather strong, but standard in the nonlinear MPC literature, cf. [3, Chapter 6]. It is possible to relax this assumption, cf. [16], and we suspect that our analysis in Sections III and IV can be adapted to this setting. However, due to space constraints, such a generalization is beyond the scope of the present work.

Using a feedback map $\kappa : X \to U$, system (1) can be turned into a closed loop system

$$x_+ = f(x, \kappa(x)),$$

leading to the recursively defined state trajectory

$$x_\kappa(0; x) = x$$
$$x_\kappa(n+1; x) = f(x_\kappa(n; x), \kappa(x))$$

and input trajectory $u_\kappa(n; x) = \kappa(x_\kappa(n; x))$. We call $\kappa : \mathbb{X} \to U$ admissible if for all $x \in \mathbb{X}$ we have $\kappa(x) \in \mathbb{U}(x)$ and $f(x, \kappa(x)) \in \mathbb{X}$.

Similarly, a time-varying feedback map $\kappa : \mathbb{N}_0 \times X \to U$ induces a time-varying closed loop system, for which a state trajectory is defined by

$$x_\kappa(n; n, x) = x$$
$$x_\kappa(n+k+1; n, x) = f(x_\kappa(n+k; n, x), \kappa(n+k; x_\kappa(n+k; n, x)))$$

and the corresponding input trajectory by $u_\kappa(n+k; n, x) = \kappa(n+k; x_\kappa(n+k; n, x))$. For brevity, in this time-varying context define $x_\kappa(n; x) = x_\kappa(n; 0, x)$ and $u_\kappa(n; x) = u_\kappa(n; 0, x)$.

In the following, we work with a stage cost $\ell : X \times U \to \mathbb{R}$, and define for $N \in \mathbb{N}_+ \cup \{\infty\}$ the total cost functional

$$J_N(x, u) = \sum_{n=0}^{N-1} \ell(x(n; x, u), u(n)) \tag{7}$$

and set by convention $J_0 \equiv 0$. Define in addition

$$\ell_*(x) = \inf_{u \in \mathbb{U}(x)} \ell(x, u). \tag{8}$$

If we have a feedback map $\kappa : X \to U$, define the closed loop total cost for horizon $N \in \mathbb{N}_+ \cup \{\infty\}$ and initial state $x \in X$ by

$$J^\kappa_N(x) = J_N(x, u_\kappa(\cdot; x)) = \sum_{n=0}^{N-1} \ell(x_\kappa(n; x), u_\kappa(n; x))$$

and for a time-varying feedback map $\kappa : \mathbb{N}_0 \times X \to U$ by

$$J^\kappa_N(n, x) = \sum_{k=0}^{N-1} \ell(x_\kappa(n+k; n, x), u_\kappa(n+k; n, x)).$$

For $n = 0$ define for brevity $J^\kappa_N(x) = J^\kappa_N(0, x)$. The total cost (7) has a corresponding value function

$$V_N(x) = \inf_{u \in \mathbb{U}^N(x)} J_N(x, u). \tag{9}$$

In this work we are interested in ϵ-suboptimal controls, i.e., instead of optimisers we only get $u_\epsilon \in \mathbb{U}^N(x)$ with

$$J_N(x, u_\epsilon) \leq V_N(x) + \epsilon \tag{10}$$

for a given suboptimality level $\epsilon \in \mathbb{R}_{>0}$, and we denote any such u_ϵ by $u^\epsilon_N(\cdot \mid x)$. In this situation, the following result demonstrates how an approximate Relaxed Dynamic Programming (RDP) inequality can be used for performance estimates. It is a direct adaption of known RDP results, cf. [3, Chapter 4].

Proposition 1: Consider $\tilde{V} : \mathbb{N}_0 \times \mathbb{X} \to \mathbb{R}_{\geq 0}$ and an admissible feedback $\kappa : \mathbb{N}_0 \times \mathbb{X} \to U$. Furthermore, assume that there exist $\alpha \in (0, 1]$ and $(\epsilon_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{R}_{\geq 0}$ such that for all $n \in \mathbb{N}_0$, $x \in \mathbb{X}$, we have

$$\tilde{V}(n+1, f(x, \kappa(n, x))) \leq -\alpha \ell(x, \kappa(n, x)) + \tilde{V}(n, x) + \epsilon_n. \tag{11}$$

Then for all $M \in \mathbb{N}_+$ and $x \in \mathbb{X}$ it holds that

$$J^\kappa_M(x) \leq \frac{1}{\alpha}\tilde{V}(0, x) + \frac{1}{\alpha}\sum_{m=0}^{M-1} \epsilon_m.$$

If $(\epsilon_n)_n$ is summable, we get

$$J^\kappa_\infty(x) \leq \frac{1}{\alpha}\tilde{V}(0, x) + \frac{1}{\alpha}\sum_{m=0}^{\infty} \epsilon_m. \tag{12}$$

Remark 1: If ℓ is nonnegative, then for all $x \in \mathbb{X}$, $\lim_{M\to\infty} J_M^\kappa(x)$ and $\sum_{m=0}^\infty \epsilon_m$ exist (in $\mathbb{R}_{\geq 0} \cup \{\infty\}$), and

$$J_\infty^\kappa(x) \leq \frac{1}{\alpha}\tilde{V}(0,x) + \frac{1}{\alpha}\sum_{m=0}^{\infty}\epsilon_m, \tag{13}$$

even if $(\epsilon_n)_n$ is not summable.

Proof: For all $m \in \mathbb{N}_0$, $x \in \mathbb{X}$, we get from (11)

$$\begin{aligned}&\alpha\ell(x_\kappa(m;x), u_\kappa(m;x))\\ &\quad\leq \tilde{V}(m, x_\kappa(m;x)) - \tilde{V}(m+1, f(x, u_\kappa(m;x))) + \epsilon_m\\ &\quad= \tilde{V}(m, x_\kappa(m;x)) - \tilde{V}(m+1, x_\kappa(m+1;x)) + \epsilon_m.\end{aligned}$$

Summing over $m = 0, \ldots, M-1$ results in

$$\begin{aligned}&\alpha\sum_{m=0}^{M-1}\ell(x_\kappa(m;x), u_\kappa(m;x))\\ &\qquad\leq \tilde{V}(0,x) - \tilde{V}(M, x_\kappa(M;x)) + \sum_{m=0}^{M-1}\epsilon_m,\end{aligned}$$

hence (recall that $\tilde{V}(M, x_\kappa(M;x)) \in \mathbb{R}_{\geq 0}$)

$$J_M^\kappa(x) \leq \frac{1}{\alpha}\tilde{V}(0,x) + \frac{1}{\alpha}\sum_{m=0}^{M-1}\epsilon_m.$$

If $(\epsilon_n)_n$ is summable, we furthermore get

$$J_M^\kappa(x) \leq \frac{1}{\alpha}\tilde{V}(0,x) + \frac{1}{\alpha}\sum_{m=0}^{\infty}\epsilon_m$$

and since the right-hand side is now finite and independent of M, and the left-hand side is increasing with M, the limit of the latter exists and we get the desired bound. ■

III. Performance Bound

There are three common approaches to analyse (tracking or stabilizing) NMPC schemes without terminal conditions [17]. We consider a method based on a certain cost controllability condition [5], [6], in the form as presented in [17], where it is called Variant 2. The following assumption formalizes this condition.

Assumption 2: For all $N \geq 2$ there exists $\gamma_N \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{X}$ we have $V_N(x) \leq \gamma_N \ell_*(x)$.

In order to adapt the aforementioned analysis method to approximately optimal sequences, we need an approximate version of the principle of optimality.

Proposition 2: Let $N \geq 2$, $\epsilon \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{X}$, and $u_\epsilon \in \mathbb{U}^N(x)$ such that $J_N(x, u_\epsilon) \leq V_N(x) + \epsilon$. Then for all $K = 1, \ldots, N-1$ we have

$$J_{N-K}(x(K;x,u_\epsilon), u_\epsilon(\cdot + K)) \leq V_{N-K}(x(K;x,u_\epsilon)) + \epsilon. \tag{14}$$

Proof: Let $K = 1, \ldots, N-1$ be arbitrary, then

$$\begin{aligned}&J_K(x,u_\epsilon) + J_{N-K}(x(K;x,u_\epsilon), u_\epsilon(\cdot+K)) = J_N(x,u_\epsilon)\\ &\quad\leq V_N(x) + \epsilon\\ &\quad= \inf_{\tilde{u}\in\mathbb{U}^K_{\mathbb{X}_{N-K}}(x)} J_K(x,\tilde{u}) + V_{N-K}(x(K;x,\tilde{u})) + \epsilon\\ &\quad\leq J_K(x,u_\epsilon) + V_{N-K}(x(K;x,u_\epsilon)) + \epsilon,\end{aligned}$$

where the first equality follows from the definition of the total cost functional (we have no terminal cost), the first inequality follows from the choice of u_ϵ, then we used the Dynamic Programming Principle (for finite horizons) in the second equality, and the last inequality holds because $u_\epsilon \in \mathbb{U}^K_{\mathbb{X}_{N-K}}(x)$ (since $u_\epsilon \in \mathbb{U}^N(x)$). Subtracting $J_K(x, u_\epsilon)$ from both sides establishes the claim. ■

We can now state the corresponding RDP inequality for approximate optimisers. The proof is based on the same arguments as in [6], [17], but taking the suboptimality into account via Proposition 2.

Theorem 1: Under Assumptions 1 and 2, for all $N \geq 2$, $\epsilon \in \mathbb{R}_{\geq 0}$ and any ϵ-optimal control sequence $u_\epsilon \in \mathbb{U}^N(x)$, we have

$$V_N(f(x, u_\epsilon(0))) \leq V_N(x) - \alpha_{N,\vec{\gamma}}\ell(x, u_\epsilon(0)) + C_\epsilon\epsilon, \tag{15}$$

where

$$\alpha_{N,\vec{\gamma}} = 1 - (\gamma_2 - 1)(\gamma_N - 1)\prod_{k=2}^{N-1}\frac{\gamma_k - 1}{\gamma_k} \tag{16}$$

and

$$\begin{aligned}C_\epsilon &= C_\epsilon(N,\vec{\gamma}) = 1 + (\gamma_2 - 1)\\ &\quad\times\left(\prod_{k=2}^{N-1}\frac{\gamma_k-1}{\gamma_k}\right)\left(1 + \sum_{k=2}^{N-1}\frac{1}{\gamma_k - 1}\prod_{\ell=k+1}^{N-1}\frac{\gamma_\ell}{\gamma_\ell - 1}\right)\end{aligned} \tag{17}$$

Proof: Let $N \geq 2$, $\epsilon \in \mathbb{R}_{\geq 0}$ be arbitrary and $u_\epsilon \in \mathbb{U}^N(x)$ any ϵ-optimal control sequence. For brevity, define $\ell(n) = \ell(x(n;x,u_\epsilon), u_\epsilon(n))$ for all $n = 0, \ldots, N-1$.

Step 1 For all $K = 0, \ldots, N-2$ we have

$$\begin{aligned}\sum_{n=K}^{N-1}\ell(n) &= \sum_{n=0}^{N-1-K}\ell(x(n;x(K;x,u_\epsilon), u_\epsilon(\cdot+K))\\ &= J_{N-K}(x(K;x,u_\epsilon), u_\epsilon(\cdot+K))\\ &\leq V_{N-K}(x(K;x,u_\epsilon)) + \epsilon\\ &\leq \gamma_{N-K}\ell_*(x(K;x,u_\epsilon)) + \epsilon\\ &\leq \gamma_{N-K}\ell(K) + \epsilon,\end{aligned}$$

where we used Proposition 2 in the first inequality and Assumption 2 in the second inequality. Rearranging leads to

$$\sum_{n=K+1}^{N-1}\ell(n) \leq (\gamma_{N-K} - 1)\ell(K) + \epsilon \tag{18}$$

and

$$\ell(K) \geq \frac{1}{\gamma_{N-K}-1}\left(\sum_{n=K+1}^{N-1}\ell(n)\right) - \frac{1}{\gamma_{N-K}-1}\epsilon. \tag{19}$$

This implies that

$$\begin{aligned}\sum_{n=K}^{N-1}\ell(n) &= \ell(K) + \sum_{n=K+1}^{N-1}\ell(n)\\ &\geq \frac{1}{\gamma_{N-K}-1}\left(\sum_{n=K+1}^{N-1}\ell(n)\right) - \frac{1}{\gamma_{N-K}-1}\epsilon\end{aligned}$$

$$+\sum_{n=K+1}^{N-1}\ell(n)$$
$$=\frac{\gamma_{N-K}}{\gamma_{N-K}-1}\left(\sum_{n=K+1}^{N-1}\ell(n)\right)-\frac{1}{\gamma_{N-K}-1}\epsilon.$$

Repeating the previous steps from $K=1$ to $K=N-1$ results in

$$\sum_{n=1}^{N-1}\ell(n)\geq\left(\prod_{k=2}^{N-1}\frac{\gamma_k}{\gamma_k-1}\right)\ell(N-1)-\left(\sum_{k=2}^{N-1}\frac{1}{\gamma_k-1}\prod_{\ell=k+1}^{N-1}\frac{\gamma_\ell}{\gamma_\ell-1}\right)\epsilon$$
$$=:\tilde{C}_1\ell(N-1)-\tilde{C}_2\epsilon.$$

Using now (18) for $K=0$ yields

$$(\gamma_N-1)\ell(0)+\epsilon\geq\sum_{n=1}^{N-1}\ell(n),$$

so altogether

$$(\gamma_N-1)\ell(0)+\epsilon\geq\sum_{n=1}^{N-1}\ell(n)\geq\tilde{C}_1\ell(N-1)-\tilde{C}_2\epsilon,$$

which can be rewritten as

$$\ell(N-1)\leq\frac{\gamma_N-1}{\tilde{C}_1}\ell(0)+\frac{1+\tilde{C}_2}{\tilde{C}_1}\epsilon=:\tilde{C}_3\ell(0)+\tilde{C}_4\epsilon \tag{20}$$

(note that $\tilde{C}_1>0$).

Step 2 Let $\epsilon_2\in\mathbb{R}_{>0}$ and $p\in\{1,\dots,N-1\}$ be arbitrary, and choose some ϵ_2-optimal $\tilde{u}\in\mathbb{U}^{N-(p-1)}(x(p;x,u_\epsilon))$. Note that $\mathbb{U}^{N-(p-1)}(x(p;x,u_\epsilon))\neq\emptyset$ since $u_\epsilon\in\mathbb{U}^N(x)$ and $\mathbb{X}$ is viable. Define now

$$\hat{u}=(u_\epsilon(1),\dots,u_\epsilon(p-1),\tilde{u}(0),\dots,\tilde{u}(N-(p-1)-1))$$

and observe that by construction $\hat{u}\in\mathbb{U}^N(x(1;x,u_\epsilon))$. We then get

$$V_N(f(x,u_\epsilon(0)))\leq J_N\big(f(x,u_\epsilon(0)),\hat{u}\big)$$
$$=\sum_{n=0}^{(p-1)-1}\ell(x(n;f(x,u_\epsilon(0)),u_\epsilon(\cdot+1)),u_\epsilon(n+1))$$
$$+\sum_{n=0}^{N-(p-1)-1}\ell(x(n;x(p;x,u_\epsilon),\tilde{u}),\tilde{u}(n))$$
$$=\sum_{n=1}^{p-1}\ell(x(n;x,u_\epsilon),u_\epsilon(n))+J_{N-(p-1)}(x(p;x,u_\epsilon),\tilde{u})$$
$$\leq\left(\sum_{n=0}^{p-1}\ell(x(n;x,u_\epsilon),u_\epsilon(n))\right)-\ell(0)$$
$$+V_{N-(p-1)}(x(p;x,u_\epsilon))+\epsilon_2$$
$$\leq\left(\sum_{n=0}^{p}\ell(x(n;x,u_\epsilon),u_\epsilon(n))\right)-\ell(0)-\ell(p)$$
$$+\gamma_{N-(p-1)}\ell(p)+\epsilon_2$$
$$=\left(\sum_{n=0}^{p}\ell(x(n;x,u_\epsilon),u_\epsilon(n))\right)-\ell(0)$$
$$+\big(\gamma_{N-(p-1)}-1\big)\ell(p)+\epsilon_2.$$

Setting $p=N-1$ finally yields

$$V_N(f(x,u_\epsilon(0)))\leq\left(\sum_{n=0}^{N-1}\ell(x(n;x,u_\epsilon),u_\epsilon(n))\right)-\ell(0)$$
$$+(\gamma_2-1)\ \ell(N-1)+\epsilon_2$$
$$=J_N(x,u_\epsilon)-\ell(0)+(\gamma_2-1)\ell(N-1)+\epsilon_2.$$

Since the right-hand side is independent of $\tilde{u}$, and $\epsilon_2\in\mathbb{R}_{>0}$ was arbitrary, we get that

$$V_N(f(x,u_\epsilon(0)))\leq J_N(x,u_\epsilon)-\ell(0)+(\gamma_2-1)\ell(N-1).$$

Step 3 We can now continue with

$$V_N(f(x,u_\epsilon(0)))\leq J_N(x,u_\epsilon)-\ell(0)+(\gamma_2-1)\ell(N-1)$$
$$\leq V_N(x)+\epsilon-\ell(0)+(\gamma_2-1)\tilde{C}_3\ell(0)+(\gamma_2-1)\tilde{C}_4\epsilon$$
$$=V_N(x)-\Big(1-(\gamma_2-1)\tilde{C}_3\Big)\ell(0)+\Big(1+(\gamma_2-1)\tilde{C}_4\Big)\epsilon,$$

where we used (20) and the choice of u_ϵ in the second inequality. Using the definitions of $\tilde{C}_3,\tilde{C}_4$ establishes the claim. ■

As expected, the performance of the closed-loop system deteriorates linearly in the suboptimality gap, and if no suboptimality occurs ($\epsilon=0$), then we recover the existing performance bounds from the literature, cf. [17]. If $\gamma_N=\gamma$ for all $N\geq 2$ in Assumption 2, then the constants simplify considerably.

Proposition 3: Under Assumptions 1 and 2 with $\gamma_N=\gamma$ for all $N\geq 2$, we have for all $N\geq 2$, $\epsilon\in\mathbb{R}_{\geq 0}$, and any ϵ-optimal control sequence $u_\epsilon\in\mathbb{U}^N(x)$ that

$$V_N(f(x,u_\epsilon(0)))\leq V_N(x)-\alpha_{N,\gamma}\ell(x,u_\epsilon(0))+C_\epsilon\epsilon, \tag{21}$$

where

$$\alpha_{N,\gamma}=1-\gamma(\gamma-1)\left(\frac{\gamma-1}{\gamma}\right)^{N-1}\text{ and }C_\epsilon=\gamma \tag{22}$$

Proof: Follows directly from Theorem 1 with an elementary, though slightly lengthy calculation. ■

IV. Stability and Practical Stability

We now use the preceding results to investigate asymptotic stability and practical asymptotic stability of the closed loop system. For greater generality, we formalize this using a *state measure* $\sigma:X\to\mathbb{R}_{\geq 0}$, as is common in the MPC literature [4]. For example, if (X,d_X) is a metric space and $(x_*,u_*)\in X\times U$ an equilibrium pair (so $x_*=f(x_*,u_*)$), then stability of x_* can be investigated by choosing $\sigma(x)=d_X(x,x_*)$. First, we need to link the stage cost to the state measure using the following standard assumption.[1] For a detailed discussion of related controllability assumptions and possible choices of stage costs, we refer to [18].

Assumption 3: There exist $\underline{\alpha}_\ell,\bar{\alpha}_\ell\in\mathcal{K}_\infty$ such that for all $x\in\mathbb{X}$ we have $\underline{\alpha}_\ell(\sigma(x))\leq\ell_*(x)\leq\bar{\alpha}_\ell(\sigma(x))$.

[1]We use the common comparison function formalism, cf. e.g. [3].

As is well-known, the preceding bound on the stage cost implies corresponding bounds on the value function, which we recall for convenience.

Lemma 1: Under Assumptions 2 and 3, for all $N \in \mathbb{N}_+$ there exists $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}$, we have

$$\underline{\alpha}(\sigma(x)) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)). \tag{23}$$

We start with the case where an arbitrary small suboptimality level cannot necessarily be reached. The following result shows that in this case one can at least ensure that the finite-horizon value function V_N behaves like a Lyapunov function outside a given set.

Proposition 4: Let Assumptions 1, 2, and 3 hold, and assume that $\alpha_{N,\vec{\gamma}} \in (0, 1]$. Let $\epsilon > 0$, and for all $x \in \mathbb{X}$ define $\kappa(x) = u_N^\epsilon(0 \mid x)$. Choose $0 < p, q < 1$ such that $p + q = 1$, and define $\underline{\sigma} = \underline{\alpha}^{-1}(C_\epsilon \epsilon/(q\alpha_{N,\vec{\gamma}}))$ and $\mathbb{X}_V = \{x \in \mathbb{X} \mid \sigma(x) > \underline{\sigma}\}$. Then there exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}$ we have

$$\underline{\alpha}(\sigma(x)) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)) \tag{24}$$

and for all $x \in \mathbb{X}_V$ we have

$$V_N(f(x, \kappa(x))) \leq -p\underline{\alpha}(\sigma(x)) + V_N(x). \tag{25}$$

Remark 2: There is a tradeoff in the choice of p, q. A large p leads to fast convergence to the set $\mathbb{X}_V$, but this set is then also small, since $\underline{\alpha}^{-1}(\frac{C_\epsilon \epsilon}{q\alpha_{N,\vec{\gamma}}})$ will be large. Conversely, a small p allows a larger $\mathbb{X}_V$, but slows down the guaranteed convergence.

Proof: The existence of $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ and the first two inequalities follow from Lemma 1. Let now $x \in \mathbb{X}_V$ and set $\kappa(x) = u_N^\epsilon(0 \mid x)$, then we have

$$\begin{aligned}
V_N(f(x, \kappa(x))) &\leq V_N(x) - \alpha_{N,\vec{\gamma}}\ell(x, u_N^\epsilon(0 \mid x)) + C_\epsilon \epsilon \\
&\leq V_N(x) - \alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)) + C_\epsilon \epsilon \\
&= V_N(x) - p\alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)) + C_\epsilon \epsilon - q\alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)) \\
&< V_N(x) - p\alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)) + C_\epsilon \epsilon \\
&\quad - q\alpha_{N,\vec{\gamma}}\underline{\alpha}\left(\underline{\alpha}^{-1}\left(\frac{C_\epsilon \epsilon}{q\alpha_{N,\vec{\gamma}}}\right)\right) \\
&= V_N(x) - p\alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)),
\end{aligned}$$

where we used Theorem 1 in the first step, Assumption 3 in the second step, and the definition of $\mathbb{X}_V$ in the third inequality. ■

The preceding result does not claim that V_N is a practical Lyapunov function in sense of [3, Th. 2.20], since the set $\mathbb{X}_V$ is not necessarily forward invariant. The next result provides such a practical Lyapunov function. It is based on a standard argument from the MPC literature, cf. [19, Lemma 7] for an example, though our setting is slightly more general than usual.

Proposition 5: Consider the situation of Proposition 4, and define in addition

$$P = \left\{x \in \mathbb{X} \mid V_N(x) \leq \max\left\{\bar{\alpha}(\underline{\sigma}), \bar{\alpha}(\underline{\alpha}^{-1}(\bar{\alpha}(\underline{\sigma}) + C_\epsilon \epsilon))\right\}\right\}.$$

The set P is forward-invariant under the admissible feedback κ, and there exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}$ we have that

$$\underline{\alpha}(\sigma(x)) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)) \tag{26}$$

and for all $x \in \mathbb{X} \setminus P$ that

$$V_N(f(x, \kappa(x))) \leq -p\underline{\alpha}(\sigma(x)) + V_N(x). \tag{27}$$

Proof: The existence of $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ and the bounds (26) follow as before from Lemma 1. Define now $\tilde{\mathbb{X}} = \{x \in \mathbb{X} \mid \sigma(x) \leq \underline{\sigma}\}$ and $\bar{\mathbb{X}} = \{x \in \mathbb{X} \mid \sigma(x) \leq \underline{\alpha}^{-1}(\bar{\alpha}(\underline{\sigma}) + C_\epsilon \epsilon)\}$. If $x \in \tilde{\mathbb{X}}$, then $V_N(x) \leq \bar{\alpha}(\sigma(x)) \leq \bar{\alpha}(\underline{\sigma})$, so $x \in P$. If $x \in \bar{\mathbb{X}}$, then $V_N(x) \leq \bar{\alpha}(\sigma(x)) \leq \bar{\alpha}(\underline{\alpha}^{-1}(\bar{\alpha}(\underline{\sigma}) + C_\epsilon \epsilon))$, so also $x \in P$. This shows that $\tilde{\mathbb{X}}, \bar{\mathbb{X}} \subseteq P$. That κ is an admissible feedback is clear, and for brevity define now $x_+ = f(x, \kappa(x))$.

If $x \in \tilde{\mathbb{X}}$, so $\sigma(x) \leq \underline{\sigma}$, then

$$\begin{aligned}
\sigma(x_+) &= \underline{\alpha}^{-1}(\underline{\alpha}(\sigma(x_+))) \leq \underline{\alpha}^{-1}(V_N(x_+))) \\
&\leq \underline{\alpha}^{-1}\big(V_N(x) - \alpha_{N,\vec{\gamma}}\underline{\alpha}(\sigma(x)) + C_\epsilon \epsilon\big) \\
&\leq \underline{\alpha}^{-1}(V_N(x) + C_\epsilon \epsilon) \leq \underline{\alpha}^{-1}(\bar{\alpha}(\sigma(x)) + C_\epsilon \epsilon) \\
&\leq \underline{\alpha}^{-1}\big(\bar{\alpha}(\underline{\sigma}) + C_\epsilon \epsilon\big),
\end{aligned}$$

where we used (26) in the first inequality, Theorem 1 together with Assumption 3 in the second inequality, (26) again in the fourth inequality, and finally the definition of $\bar{\mathbb{X}}$. The preceding inequalities show that $x_+ \in \bar{\mathbb{X}} \subseteq P$.

If $x \in P \setminus \tilde{\mathbb{X}}$, then by construction $x \in \mathbb{X}_V$ (since $\mathbb{X}_V = \mathbb{X} \setminus \tilde{\mathbb{X}}$), and hence Proposition 5 shows that

$$\begin{aligned}
V_N(x_+) &\leq V_N(x) - p\underline{\alpha}(\sigma(x)) \leq V_N(x) \\
&\leq \max\left\{\bar{\alpha}(\underline{\sigma}), \bar{\alpha}\left(\underline{\alpha}^{-1}(\bar{\alpha}(\underline{\sigma}) + C_\epsilon \epsilon)\right)\right\}
\end{aligned}$$

so $x_+ \in P$. Altogether, this establishes that P is forward invariant.

Finally, as observed above, $\mathbb{X} \setminus P \subseteq \mathbb{X} \setminus \tilde{\mathbb{X}} = \mathbb{X}_V$, so (27) follows from Proposition 5. ■

We now turn to the case where the suboptimality gap can be made arbitrary small, but not zero. In this setting, we can show asymptotic stability of the closed-loop system, at the expense of a time-varying controller. First, define the set $\mathbb{X}_\sigma = \{x \in \mathbb{X} \mid \sigma(x) = 0\}$.

Assumption 4: The set $\mathbb{X}_\sigma$ is controlled positive invariant, i.e., for all $x \in \mathbb{X}_\sigma$ there exists $u_x \in \mathbb{U}(x)$ such that $f(x, u_x) \in \mathbb{X}_\sigma$.

The following result shows that the finite-horizon function V_N behaves like a Lyapunov function under an appropriately chosen receding horizon controller. The control approach used in the result is essentially a discrete-time analog of the one in [11].

Proposition 6: Let Assumptions 1, 2, 3, and 4 hold, and assume that $\alpha_{N,\vec{\gamma}} \in (0, 1]$. For $\epsilon \in \mathbb{R}_{>0}$ and $\alpha_\epsilon \in (0, \alpha_{N,\vec{\gamma}})$, define $\epsilon : \mathbb{N}_0 \times \mathbb{X} \to \mathbb{R}_{>0}$ by

$$\epsilon(n, x) = \min\left\{2^{-(n+1)}\epsilon, \alpha_\epsilon \underline{\alpha}_\ell(\sigma(x))\right\} \tag{28}$$

and $\kappa : \mathbb{N}_0 \times \mathbb{X} \to U$ by

$$\kappa(n, x) = \begin{cases} u_N^{\epsilon(n,x)}(0 \mid x) & \text{if } \sigma(x) > 0 \\ u_x & \text{if } x \in \mathbb{X}_\sigma \end{cases} \tag{29}$$

where $u_N^{\epsilon(n,x)}(0 \mid x) \in \mathbb{U}^N(x)$ is any $\epsilon(n, x)$-approximate optimiser of $J_N(x, \cdot)$ (exists) and $u_x \in \mathbb{U}(x)$ is such that $f(x, u_x) \in \mathbb{X}_\sigma$ if $\sigma(x) = 0$ (also exists in this case).

Then κ is an admissible feedback and we have for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_0$ that

$$\underline{\alpha}(\sigma(x)) \leq V_N(x) \leq \bar{\alpha}(\sigma(x)) \tag{30}$$

$$V_N(f(x, \kappa(n, x))) \leq -\alpha_V(\sigma(x)) + V_N(x), \tag{31}$$

where $\underline{\alpha} = \underline{\alpha}_\ell$, $\bar{\alpha} = \gamma_N \bar{\alpha}_\ell$, $\alpha_V = (\alpha_{N,\vec{\gamma}} - \alpha_\epsilon)\underline{\alpha}_\ell$, and

$$J_\infty^\kappa(x) \leq \frac{1}{\alpha_{N,\vec{\gamma}}} V_N(x) + \frac{\epsilon}{\alpha_{N,\vec{\gamma}}}. \tag{32}$$

Proof: That κ is an admissible feedback is clear from the construction. The bounds (30) follow as before. Let now $n \in \mathbb{N}_0$ and $x \in \mathbb{X}$ be arbitrary. If $\sigma(x) > 0$, then we have

$$\begin{aligned} V_N(f(x, \kappa(n, x))) &= V_N\Big(f(x, u_N^{\epsilon(n,x)}(0 \mid x))\Big) \\ &\leq -\alpha_{N,\vec{\gamma}} \ell\Big(x, u_N^{\epsilon(n,x)}(0 \mid x)\Big) + V_N(x) + \epsilon(n, x) \\ &\leq -\alpha_{N,\vec{\gamma}} \underline{\alpha}_\ell(\sigma(x)) + V_N(x) + \alpha_\epsilon \underline{\alpha}_\ell(\sigma(x)) \\ &= -\big(\alpha_{N,\vec{\gamma}} - \alpha_\epsilon\big)\underline{\alpha}_\ell(\sigma(x)) + V_N(x), \end{aligned}$$

where the first inequality follows from Theorem 1 and the second inequality from the definition of $\epsilon(n, x)$. If $\sigma(x) = 0$, then $V_N(x) \leq \bar{\alpha}(\sigma(x)) = 0$ implies that $V_N(x) = V_N(f(x, \kappa(n, x))) = 0$ (the latter follows from $f(x, \kappa(n, x)) = f(x, u_x) \in \mathbb{X}_\sigma$ and (23)) and hence

$$V_N(f(x, \kappa(n, x))) = 0 \leq -\alpha_V(\sigma(x)) + V_N(x),$$

where we used additionally that $\alpha_V(\sigma(x)) = \alpha_V(0) = 0$. Combining the two cases establishes (31).

Furthermore, if $\sigma(x) > 0$, then

$$\begin{aligned} V_N(f(x, \kappa(n, x))) &= V_N\Big(f(x, u_N^{\epsilon(n,x)}(0 \mid x))\Big) \\ &\leq -\alpha_{N,\vec{\gamma}} \ell\Big(x, u_N^{\epsilon(n,x)}(0 \mid x)\Big) + V_N(x) + \epsilon(n, x) \\ &\leq -\alpha_{N,\vec{\gamma}} \ell(x, \kappa(n, x)) + V_N(x) + \frac{1}{2^{n+1}}\epsilon, \end{aligned}$$

and if $\sigma(x) = 0$, then as before

$$\begin{aligned} V_N(f(x, \kappa(n, x))) = 0 &\leq -\alpha_{N,\vec{\gamma}} \ell(x, u_x) + V_N(x) + \epsilon(n, x) \\ &\leq -\alpha_{N,\vec{\gamma}} \ell(x, \kappa(n, x)) + V_N(x) + \frac{1}{2^{n+1}}\epsilon. \end{aligned}$$

Since $\sum_{m=0}^\infty \frac{1}{2^{m+1}}\epsilon = \epsilon$, Proposition 1 establishes (32). ■

V. Conclusion

We considered nonlinear MPC schemes without terminal constraints subject to suboptimal solutions of the underlying open-loop optimal control problems. We showed that under standard assumptions, existing performance bounds continue to hold up to a deterioriation that grows linearly in the suboptimality gap, as is intuitively expected. Furthermore, we investigated practical asymptotic stability of the closed loop system under a fixed level of suboptimality using Lyapunov techniques, and if the MPC controller can request arbitrary small, but positive suboptimality gaps, we even established asymptotic stability. Our results hold for very general MPC schemes, independent of the specific numerical optimisation algorithm used. An important limitation of our analysis is the assumption of a *given* additive suboptimality gap, which is difficult to verify a priori. Relaxing this assumption, for example by considering local suboptimality gaps, is an interesting direction for future work. Furthermore, as common in NMPC without terminal constraints, we assumed viability of the state constraint set, and relaxing this assumption is an interesting question. Furthermore, it is known that the analysis technique from [3, Chapter 6] can be sharper than [6]. Using the former technique in our setting is an interesting avenue for future work.

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