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Quotienting: A new perspective on the control of n-D systems

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Abstract: In this work, we investigate the process of quotienting the module of observables for a given n-D system by a suitable pre-defined submodule (obtained from a chosen ideal). We show that the functional meaning of this algebraic process is restricting the given n-D system, called the "plant," to a special type of a smaller subsystem, called the "controlled system." We further show that this controlled system can be independently characterized via a Malgrange-like duality result involving the above-mentioned process of quotienting. Finally, by proving an injectivity result, we show that this duality possesses a Galois connection, much akin to the conventional module-behaviour duality.

Keywords: Distributed parameter systems, linear control systems, algebraic methods.

1. INTRODUCTION

One of the most fundamental tools in algebra is the process of quotienting a ring (or a module) by an ideal (or a submodule). In this paper, we investigate the effect of quotienting on a given n-D system by a suitably chosen ideal.

By n-D systems, in this paper, we mean a system of linear partial differential equations (PDEs) with constant real coefficients. Such systems are known to arise when modelling various engineering phenomena: distributed parameter models of electrical systems, civil engineering structures, convolutional neural networks, etc. This paper can be categorized as a work in the area of algebraic analysis, where algebraic methods/techniques are utilized to deduce useful information about the behaviour of such n-D systems.

An important issue concerning n-D systems is their controllability and, subsequently, their control by introducing new laws coming from a controller (see Pillai and Shankar (1999); Pommaret and Quadrat (1999); Wood et al. (1999) for more on controllability of n-D systems, see also Rocha and Willems (1991) for the special case of $n = 2$). We show in this paper that quotienting the module of equations by a suitably chosen ideal results in a new "controlled" behaviour, whose trajectories satisfy both the given laws of the concerned n-D system and those imposed by the ideal. One of the main results of this paper is that the above-mentioned controlled behaviour admits a Malgrange-like duality (see Oberst (1990) for Malgrange's Theorem, see also Wood (2000)) with a suitably constructed module over the quotient ring.

We then consider two special cases, the first of which is when the ideal is assumed to be prime. The ideal being prime results in the quotient ring being an integral domain. This allows us to shift the analysis of the controlled behaviour to be over the field of fractions of the quotient integral domain. By utilizing this trick, we then formulate necessary and sufficient conditions for the controlled behaviour to admit an image representation. This is of crucial importance, because image representations are known to provide a parameterization of n-D systems that are conducive to the notions of controllability and control.

In the second case, we consider ideals whose varieties are straight lines. After proving a few facts about the corresponding function space, we show that the controllability of the quotiented system with respect to such an ideal is directly related to the trajectory level notion of line-controllability.

A special case of the results presented in this paper, namely when $n = 2$, has appeared in the authors' recent paper Pal and Zerz (2024). Some of the proofs for the general case follows immediately from the corresponding proofs in Pal and Zerz (2024) since these proofs are not specific to $n = 2$. We skip such proofs here, and request the interested reader to kindly go through Pal and Zerz (2024). All other results for the general n are proved here.

2. NOTATION AND PRELIMINARIES

We use standard notation in this paper. The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real numbers and complex numbers, respectively. For a given positive integer n, the symbols $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional Euclidean spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. We often use ∂_i to denote the i^{th} partial differential operator $\frac{\partial}{\partial x_i}$. Collectively, we use ∂ to denote the n-tuple $(\partial_1, \partial_2, \ldots, \partial_n)$. The n-variable polynomial ring in the variables $\{\partial_1, \ldots, \partial_n\}$ with coefficients from $\mathbb{R}$ is denoted by $\mathbb{R}[\partial]$; throughout the

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paper we use the symbol $\mathcal{A}$ to denote the ring $\mathbb{R}[\partial]$. We use $\mathcal{F}$ to denote the space of infinitely often differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$, i.e., $\mathcal{F} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$

By an n-D system we mean a system of linear partial differential equations (PDEs) over $\mathbb{R}^n$ with constant real coefficients. We use the succinct matrix-vector equation format to describe such systems: $R(\partial)w = 0$, where $R \in \mathcal{A}^{g\times q}$ and $w \in \mathcal{F}^q$. The solution set of a given system of PDEs is of crucial importance: we call this set the *behaviour* of the n-D system and denote it by $\mathfrak{B}$. Thus

$$\mathfrak{B} := \{w \in \mathcal{F}^q \mid R(\partial)w = 0\} = \ker R(\partial).$$

The representation of $\mathfrak{B}$, as described above, is called a *kernel representation* of the behaviour/n-D system.

The $\mathcal{A}$-module generated by the rows of the matrix $R(\partial)$ is called the *equation module* of the given n-D system, and is denoted by $\mathcal{R}$. Clearly, $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. The $\mathcal{A}$-module of equivalence classes corresponding to the equivalence relation defined in $\mathcal{A}^{1\times q}$ by $r_1 :\sim r_2$ if and only if $r_1 - r_2 \in \mathcal{R}$ is going to be of crucial importance. This module is called the *module of observables* and is denoted by $\mathcal{M}$. Malgrange's Theorem states that the behaviour $\mathfrak{B} = \ker R(\partial)$ is isomorphic as an $\mathcal{A}$-module to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}))$.

Like the kernel representation, we also need *image representation* of given n-D systems. An image representation for an n-D system is given in the following manner:

$$\mathfrak{B} = \{w = M(\partial)\ell \mid \ell \in \mathcal{F}^m\},$$

where $M \in \mathcal{A}^{q\times m}$.

Controllability is one of the cornerstone notions in systems theory. An n-D system, with behaviour $\mathfrak{B}$, is said to be *controllable* if for every pair of $w_1, w_2 \in \mathfrak{B}$ and every pair of sets $U_1, U_2 \subseteq \mathbb{R}^n$ such that $\overline{U_1} \cap \overline{U_2} = \emptyset$ there exists $w \in \mathfrak{B}$ such that $w|_{U_1} = w_1|_{U_1}$ and $w|_{U_2} = w_2|_{U_2}$.

The following result is well-known. See Pillai and Shankar (1999).

Proposition 1. An n-D system with behaviour $\mathfrak{B}$ is controllable if and only if it admits an image representation.

A special case of controllability holds a significant place in the literature (see Rocha and Wood (1997); Pillai and Shankar (1999); Shankar (2014)). This is the case, when the image representation turns out to be *observable*, that is, the image representation matrix $M \in \mathcal{A}^{q\times m}$ admits a left-inverse. This is equivalent to the module of observables, $\mathcal{M}$, being a free module over $\mathcal{A}$, which, in turn, is equivalent to the behaviour $\mathfrak{B}$ admitting a kernel representation matrix $R \in \mathcal{A}^{g\times q}$ that satisfies $\mathrm{rank}\,(R(\lambda)) = g$ for all $\lambda \in \mathbb{C}^n$. The equation module $\mathcal{R}$ in this case turns out to be a direct-summand of $\mathcal{A}^{1\times q}$ (see Eisenbud (2013)). This special case of controllability is known in the literature as *strong controllability*.

3. THE SIGNAL SPACE

Given an ideal $\mathfrak{a} \subseteq \mathcal{A}$, the signal space of relevance is defined as

$$\mathcal{F}_{\mathfrak{a}} := \{f \in \mathcal{F} \mid a(\partial)f = 0 \text{ for all } a \in \mathfrak{a}\}.$$

For the ideal $\mathfrak{a}$, we define the corresponding quotient ring of equivalence classes modulo $\mathfrak{a}$ by $\mathcal{A}/\mathfrak{a}$ and the corresponding canonical surjection map is denoted by $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$.

Definition 2. The *action of $\mathcal{A}/\mathfrak{a}$ on $\mathcal{F}_{\mathfrak{a}}$* is defined as follows: For any $f \in \mathcal{F}_{\mathfrak{a}}$ and any $a \in \mathcal{A}/\mathfrak{a}$, let $\widehat{a}(\partial) \in \mathcal{A}$ be such that $\pi(\widehat{a}(\partial)) = a$, define

$$a \cdot f := \widehat{a}(\partial)f.$$

Since $f \in \mathcal{F}_{\mathfrak{a}}$, this action is well-defined.

The following fundamental properties of $\mathcal{F}_{\mathfrak{a}}$ will be important in the sequel.

Proposition 3. The following are true for $\mathcal{F}_{\mathfrak{a}}$.

(1) $\mathcal{F}_{\mathfrak{a}}$ has the structure of an $\mathcal{A}/\mathfrak{a}$-module.
(2) $\mathcal{F}_{\mathfrak{a}}$ is an injective cogenerator as an $\mathcal{A}/\mathfrak{a}$-module.

Proof. Please see (Pal and Zerz, 2024, Proposition 3) for the proof of Statement 1. Please see (Pal and Zerz, 2024, Theorem 3) for the proof of Statement 2. □

4. THE QUOTIENTED BEHAVIOUR

Let the ideal $\mathfrak{a} \subseteq \mathcal{A}$ be given. Recall the canonical projection map $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$. We extend the action of π to row vectors (in $\mathcal{A}^{1\times q}$) and matrices (in $\mathcal{A}^{g\times q}$), naturally, by making π act element-wise. Thus, given a kernel representation of a behaviour $\mathfrak{B} = \ker R(\partial)$, with $R \in \mathcal{A}^{g\times q}$, the object $\pi(R(\partial))$ is understood to be an element of $(\mathcal{A}/\mathfrak{a})^{g\times q}$. Since the signal space $\mathcal{F}_{\mathfrak{a}}$ has the structure of an $\mathcal{A}/\mathfrak{a}$-module, the action of $\pi(R(\partial))$ is well-defined on a q-tuple $w \in \mathcal{F}_{\mathfrak{a}}^q$. This allows us to define the following.

Definition 4. Given an ideal $\mathfrak{a} \subseteq \mathcal{A}$, and a behaviour $\mathfrak{B} = \ker R(\partial)$, the $\mathfrak{a}$-*quotiented behaviour*, denoted by $\mathfrak{B}_{\mathfrak{a}}$ is defined as

$$\mathfrak{B}_{\mathfrak{a}} := \{w \in \mathcal{F}_{\mathfrak{a}}^q \mid \pi(R(\partial)) \cdot w = 0\}.$$

The following is a simple but crucial property of $\mathfrak{B}_{\mathfrak{a}}$.

Proposition 5. Let $\mathfrak{B}$ be the behaviour of an n-D system, and $\mathfrak{a} \subseteq \mathcal{A}$ an ideal. Then the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_{\mathfrak{a}}$ satisfies the following

$$\mathfrak{B}_{\mathfrak{a}} = \mathfrak{B} \cap \mathcal{F}_{\mathfrak{a}}^q.$$

Proof. Please see (Pal and Zerz, 2024, Proposition 4). □

Note that Proposition 5 brings out the fact that the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_{\mathfrak{a}}$ is the "controlled behaviour" obtained by attaching a "controller" – with behaviour given by $\mathcal{F}_{\mathfrak{a}}^q$ – to a "plant" – with behaviour given by $\mathfrak{B}$. Interestingly, this controlled behaviour can be directly associated with its corresponding (dual) algebraic object. This is proved in Theorem 7 below. Theorem 7 is analogous to the classical Malgrange's Theorem; the proof follows essentially the same line of argument as in the proof of the classical version. We need the following crucial definition in order to proceed.

Definition 6. Let $\mathfrak{B}$ be the behaviour of an n-D system with equation module $\mathcal{R}$. Further, let $\mathfrak{a} \subseteq \mathcal{A}$ be an ideal. Then the $\mathfrak{a}$-*quotiented module of observables*, denoted by $\overline{\mathcal{M}}$, is defined as

$$\overline{\mathcal{M}} := (\mathcal{A}/\mathfrak{a})^{1\times q}/\mathrm{rowspan}\,\pi(R(\partial)).$$

Theorem 7. Let $\mathfrak{B}$ be the behaviour of an n-D system with equation module $\mathcal{R}$, and $\mathfrak{a} \subseteq \mathcal{A}$ an ideal. Let the $\mathfrak{a}$-quotiented module of observables, $\overline{\mathcal{M}}$ be as defined above. Then the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_{\mathfrak{a}}$ is isomorphic to $\mathrm{Hom}_{\mathcal{A}/\mathfrak{a}}\left(\overline{\mathcal{M}}, \mathcal{F}_{\mathfrak{a}}\right)$ as $\mathcal{A}/\mathfrak{a}$-modules.

Proof. In order to show the isomorphism, we first explicitly construct a mapping from $\mathfrak{B}_{\mathfrak{a}}$ to $\mathrm{Hom}_{\mathcal{A}/\mathfrak{a}}\left(\overline{\mathcal{M}}, \mathcal{F}_{\mathfrak{a}}\right)$. Then we show that this mapping is bijective $\mathcal{A}/\mathfrak{a}$-module homomorphism.

Let $w \in \mathfrak{B}_{\mathfrak{a}}$ be arbitrary, define $\Psi(w) \in \mathrm{Hom}_{\overline{\mathcal{A}}}\left(\overline{\mathcal{M}}, \mathcal{F}_{\mathfrak{a}}\right)$ to be equal to the $\mathcal{A}/\mathfrak{a}$-module map $\varphi_w : \overline{\mathcal{M}} \to \mathcal{F}_{\mathfrak{a}}$ given by $\varphi_w(m) := m \cdot w$.

We spare a few sentences to clarify the meaning of the action $m \cdot w$ for its crucial importance in the proof. First note that $\overline{\mathcal{M}} \simeq \frac{\mathcal{A}^{1\times q}}{\mathcal{R}+\mathfrak{a}\mathcal{A}^{1\times q}}$, where $\mathfrak{a}\mathcal{A}^{1\times q}$ denotes the submodule of $\mathcal{A}^{1\times q}$ consisting of vectors whose every entry belongs to $\mathfrak{a}$. Now, for any $m \in \overline{\mathcal{M}}$, let $\widehat{m}(\partial) \in \mathcal{A}^{1\times q}$ be its preimage under the canonical projection $\mathcal{A}^{1\times q} \twoheadrightarrow \frac{\mathcal{A}^{1\times q}}{\mathcal{R}+\mathfrak{a}\mathcal{A}^{1\times q}}$. Then we define

$$m \cdot w := \widehat{m}(\partial) w. \tag{1}$$

Since $w \in \mathfrak{B}_{\mathfrak{a}}$ and $\mathfrak{B}_{\mathfrak{a}} = \mathfrak{B} \cap \mathcal{F}_{\mathfrak{a}}^q$, the action defined by equation (1) is well-defined.

That φ_w is an $\mathcal{A}$-module map can be checked easily. Further, since $w \in \mathcal{F}_{\mathfrak{a}}^q$, for all $a(\partial) \in \mathfrak{a}$ we have $\varphi_w\left(a(\partial)m\right) = a(\partial)m \cdot w = m \cdot (a(\partial)w) = 0$. Thus, φ_w is naturally an $\mathcal{A}/\mathfrak{a}$-module map. Hence, Ψ, too, is an $\mathcal{A}/\mathfrak{a}$-module map. It remains to show that Ψ is a bijection.

(Injectivity) Let $w_1, w_2 \in \mathfrak{B}_{\mathfrak{a}}$ be such that $\Psi(w_1) = \Psi(w_2)$. This means, for all $m \in \overline{\mathcal{M}}$, $m \cdot w_1 = m \cdot w_2$. That is, $m \cdot (w_1 - w_2) = 0$. Taking one-by-one m to be the images of the standard basis row-vectors e_is, we get that $w_1 = w_2$. This proves injectivity.

(Surjectivity) Let $\varphi \in \mathrm{Hom}_{\overline{\mathcal{A}}}\left(\overline{\mathcal{M}}, \mathcal{F}_{\mathfrak{a}}\right)$ be arbitrary. Define $w \in \mathcal{F}$ such that $w_i := \varphi(\overline{e_i})$, where $e_i \in \mathcal{A}^{1\times q}$ is the ith basis row-vector. Since $\overline{\mathcal{M}} \simeq \frac{\mathcal{A}^{1\times q}}{\mathcal{R}+\mathfrak{a}\mathcal{A}^{1\times q}}$, for every $r(\partial) \in \mathcal{R}$, $r(\partial)w = \varphi\left(\overline{r(\partial)}\right) = 0$, and for every $a(\partial) \in \mathfrak{a}$, $a(\partial)w_i = \varphi\left(\overline{a(\partial)e_i}\right) = 0$. Thus $w \in \mathfrak{B} \cap \mathcal{F}_{\mathfrak{a}}^q = \mathfrak{B}_{\mathfrak{a}}$. From the definition of this w it can be easily verified that $\Psi(w) = \varphi$. This proves surjectivity. □

A consequence of Proposition 3 and Theorem 7 is the following result, which is analogous to the well-known *fundamental principle* of algebraic analysis.

Proposition 8. For $M \in (\mathcal{A}/\mathfrak{a})^{q\times k}$ and $y \in \mathcal{F}_{\mathfrak{a}}^q$, there exists $x \in \mathcal{F}_{\mathfrak{a}}^k$ satisfying $M \cdot x = y$ if and only if $r \cdot y = 0 \in \mathcal{F}_{\mathfrak{a}}$ for all $r \in (\mathcal{A}/\mathfrak{a})^{1\times q}$ such that $rM = 0 \in (\mathcal{A}/\mathfrak{a})^{1\times k}$.

Proof. See (Pal and Zerz, 2024, Theorem 4). □

5. THE GALOIS CONNECTION

Let $\mathscr{P}\left((\mathcal{A}/\mathfrak{a})^{1\times q}\right)$ denote the set of all submodules of $(\mathcal{A}/\mathfrak{a})^{1\times q}$. Likewise, let $\mathscr{P}\left(\mathcal{F}_{\mathfrak{a}}^q\right)$ be the set of all $\mathfrak{a}$-quotiented behaviours with q number of manifest variables. We now define the following mapping between the two sets $\mathscr{P}\left((\mathcal{A}/\mathfrak{a})^{1\times q}\right)$ and $\mathscr{P}\left(\mathcal{F}_{\mathfrak{a}}^q\right)$:

$$\begin{aligned} \mathbb{G} : \mathscr{P}\left((\mathcal{A}/\mathfrak{a})^{1\times q}\right) &\to \mathscr{P}\left(\mathcal{F}_{\mathfrak{a}}^q\right) \\ \overline{\mathcal{R}} &\mapsto \mathrm{Hom}_{\mathcal{A}/\mathfrak{a}}\left(\overline{\mathcal{M}}, \mathcal{F}_{\mathfrak{a}}\right), \end{aligned} \tag{2}$$

where $\overline{\mathcal{M}} := \frac{(\mathcal{A}/\mathfrak{a})^{1\times q}}{\overline{\mathcal{R}}}$ as defined above. It is obvious that $\mathbb{G}$ is surjective. That it is also injective is a consequence of $\mathcal{F}_{\mathfrak{a}}$ being an injective cogenerator as an $\mathcal{A}/\mathfrak{a}$-module.

Theorem 9. Let $\mathbb{G}$ be as defined in Eq. (2) above. Then the following hold:

(1) $\mathbb{G}$ is inclusion reversing.
(2) $\mathbb{G}$ is bijective.

In other words, $\mathbb{G}$ is a bijective Galois connection.

Proof. The result follows immediately from (Oberst, 1990, Corollary 48). See also (Wood, 2000, Corollary 1 in Section 2.4).

6. SPECIAL CASE I: $\mathfrak{a}$ IS A PRIME IDEAL

The analysis of controllability, in conjunction with the notion of quotient behaviours, becomes more tractable when the ideal $\mathfrak{a}$ is assumed to be prime. When $\mathfrak{a}$ is prime, the quotient ring $\mathcal{A}/\mathfrak{a}$ turns out to be an integral domain. We then define the following two $\mathcal{A}/\mathfrak{a}$-modules and establish a crucial relationship between them. Please see (Pillai and Shankar, 1999, Remark on pp 396) for a more elaborate discussion on a similar development.

Let $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ be a given equation module. Define $\overline{\mathcal{R}} := \langle \pi(\mathcal{R}) \rangle \subseteq (\mathcal{A}/\mathfrak{a})^{1\times q}$. For this $\overline{\mathcal{R}}$, define the following $\mathcal{A}/\mathfrak{a}$-module

$$\overline{\mathcal{R}}^{\perp} := \left\{ m \in (\mathcal{A}/\mathfrak{a})^{q\times 1} \mid rm = 0 \in \mathcal{A}/\mathfrak{a} \ \forall r \in \overline{\mathcal{R}} \right\}.$$

Likewise, it is possible to define

$$\left(\overline{\mathcal{R}}^{\perp}\right)^{\perp} := \left\{ r \in (\mathcal{A}/\mathfrak{a})^{1\times q} \mid rm = 0 \in \mathcal{A}/\mathfrak{a} \ \forall m \in \overline{\mathcal{R}}^{\perp} \right\}.$$

It is easy to see that $\overline{\mathcal{R}} \subseteq \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$, in general. As mentioned earlier, $\mathcal{A}/\mathfrak{a}$ is an integral domain because $\mathfrak{a}$ has been assumed to be a prime ideal. Tensoring the two submodules with the field of fractions of $\mathcal{A}/\mathfrak{a}$, i.e., $\mathrm{qt}\,(\mathcal{A}/\mathfrak{a})$, over $\mathcal{A}/\mathfrak{a}$, we get that the two vector-spaces (over $\mathrm{qt}\,(\mathcal{A}/\mathfrak{a})$) are equal:

$$\overline{\mathcal{R}} \otimes_{\mathcal{A}/\mathfrak{a}} \mathrm{qt}\,(\mathcal{A}/\mathfrak{a}) = \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp} \otimes_{\mathcal{A}/\mathfrak{a}} \mathrm{qt}\,(\mathcal{A}/\mathfrak{a}). \tag{3}$$

As a consequence, every row-vector in $\left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$ can be written as a linear combination of the generating row-vectors of $\overline{\mathcal{R}}$ with coefficients from the field $\mathrm{qt}\,(\mathcal{A}/\mathfrak{a})$. Since $\mathcal{A}/\mathfrak{a}$ is Noetherian (so, both the modules $\overline{\mathcal{R}}$ and $\left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$ are finitely generated), it follows that there exist matrices $R \in (\mathcal{A}/\mathfrak{a})^{g\times q}$ and $R_{\mathrm{c}} \in (\mathcal{A}/\mathfrak{a})^{g'\times q}$ such that

$$\mathrm{rowspan}_{\mathcal{A}/\mathfrak{a}} R = \overline{\mathcal{R}} \text{ and } \mathrm{rowspan}_{\mathcal{A}/\mathfrak{a}} R_{\mathrm{c}} = \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}.$$

It then follows from equation (3) that there exists a matrix $\widetilde{F} \in \mathrm{qt}\,(\mathcal{A}/\mathfrak{a})^{g'\times g}$ such that

$$R_{\mathrm{c}} = \widetilde{F} R.$$

Multiplying both sides of the equation above by the product (or, equivalently, by the LCM) of the denominators in the entries of the matrix $\widetilde{F}$ we get that there exist $f \in \mathcal{A}/\mathfrak{a}$ and $F \in (\mathcal{A}/\mathfrak{a})^{g'\times g}$ that

$$f R_{\mathrm{c}} = F R. \tag{4}$$

This leads to the following proposition.

Proposition 10. In the notation introduced above, for a given submodule $\overline{\mathcal{R}} \subseteq (\mathcal{A}/\mathfrak{a})^{1\times q}$, the module $\left(\overline{\mathcal{R}}^{\perp}\right)^{\perp} / \overline{\mathcal{R}}$

contains all the torsion elements of the $\mathfrak{a}$-quotiented module of observables, $\overline{\mathcal{M}}$.

Proof. It is clear from Eq. (4) that every element of $\left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}/\overline{\mathcal{R}}$ is a torsion. It remains to show the converse. Suppose $h \in (\mathcal{A}/\mathfrak{a})^{1\times q}$ be such that its image in $\overline{\mathcal{M}}$ is a torsion. It then follows that there exists $f \in \mathcal{A}/\mathfrak{a}$ such that $fh \in \overline{\mathcal{R}}$. Note that this means, for every $m \in \overline{\mathcal{R}}^{\perp}$ we have $fhm = 0$. Since $\mathcal{A}/\mathfrak{a}$ is an integral domain (because $\mathfrak{a}$ has been assumed to be prime), it follows that $hm = 0$. Therefore, $h \in \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$ and the claim follows. □

With this observation, we are now in a place to prove one of the main results of this paper.

Theorem 11. Let $\mathfrak{B}$ be the behaviour of an n-D system and $\mathfrak{a} \subseteq \mathcal{A}$ a prime ideal. Then the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_{\mathfrak{a}}$ admits an image representation if and only if the $\mathfrak{a}$-quotiented module of observables, $\overline{\mathcal{M}}$, is torsion-free.

Proof. (If) Assuming $\overline{\mathcal{M}}$ to be torsion-free, we want to show that $\mathfrak{B}_{\mathfrak{a}}$ admits an image representation. Since $\overline{\mathcal{M}}$ is torsion-free, by Proposition 10, we must have $\overline{\mathcal{R}} = \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$. Let $M \in (\mathcal{A}/\mathfrak{a})^{q\times k}$ be such that $\mathrm{colspan}_{\mathcal{A}/\mathfrak{a}} M = \overline{\mathcal{R}}^{\perp}$. (Such a matrix M exists because $\mathcal{A}/\mathfrak{a}$ is Noetherian, and, therefore, $\overline{\mathcal{R}}^{\perp}$ is a finitely generated $\mathcal{A}/\mathfrak{a}$-module.) Let $\mathfrak{B} = \ker R(\partial)$. Clearly, $\pi(R(\partial)) \in (\mathcal{A}/\mathfrak{a})^{g\times q}$ is such that $\mathrm{rowspan}_{\mathcal{A}/\mathfrak{a}}(\pi(R(\partial))) = \overline{\mathcal{R}}$. Since $\overline{\mathcal{R}} = \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$, it follows from Proposition 8 that $w = M\ell$ for some $\ell \in \mathcal{F}_{\mathfrak{a}}^{k}$ if and only if $w \in \ker(\pi(R(\partial)))$. In other words, $\mathfrak{B}_{\mathfrak{a}}$ admits an image representation given by $\mathfrak{B}_{\mathfrak{a}} = \mathrm{im}\, M$.

(Only if) Assuming $\mathfrak{B}_{\mathfrak{a}}$ admits an image representation, we want to prove that $\mathcal{M}$ is torsion-free. Suppose, on the contrary, that $\mathcal{M}$ is not torsion-free. By Proposition 10 it follows that $\mathcal{R} \subsetneq \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$. Let an image representation of $\mathfrak{B}_{\mathfrak{a}}$ be $\mathfrak{B}_{\mathfrak{a}} = \mathrm{im}\, M$, where $M \in (\mathcal{A}/\mathfrak{a})^{q\times k}$. Thus, $\mathfrak{B}_{\mathfrak{a}}$ is parameterized by $w = M\ell$, with $\ell \in \mathcal{F}_{\mathfrak{a}}^{k}$. Let $R(\partial)$ be such that $\mathfrak{B} = \ker R(\partial)$. Therefore, $\pi(R(\partial))\, w = \pi(R(\partial))\, M\ell = 0$. Since $\ell \in \mathcal{F}_{\mathfrak{a}}^{k}$ is arbitrary, it must follow that $\pi(R(\partial))\, M = 0$. Note that $\mathrm{rowspan}_{\mathcal{A}/\mathfrak{a}}(\pi(R(\partial))) = \overline{\mathcal{R}}$. Therefore, it follows that $\mathrm{colspan}_{\mathcal{A}/\mathfrak{a}}(M) \subseteq \overline{\mathcal{R}}^{\perp}$. Consequently, every row-vector $r \in \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$ must satisfy $rM = 0 \in (\mathcal{A}/\mathfrak{a})^{1\times k}$. Therefore, every trajectory $w = M\ell$, for some $\ell \in \mathcal{F}_{\mathfrak{a}}^{k}$, must satisfy $rw = 0 \in \mathcal{F}_{\mathfrak{a}}$ for all $r \in \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$, by Proposition 8. By Theorem 7 and Proposition 3 it follows that, since $\overline{\mathcal{R}} \subsetneq \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$, there exists $w \in \mathfrak{B}_{\mathfrak{a}}$ that does not satisfy $rw = 0 \in \mathcal{F}_{\mathfrak{a}}$ for all $r \in \left(\overline{\mathcal{R}}^{\perp}\right)^{\perp}$. That is, $\mathfrak{B}_{\mathfrak{a}} \supsetneq \mathrm{im}\, M$. This is a contradiction. □

7. SPECIAL CASE II: QUOTIENTING BY AN IDEAL WHOSE VARIETY IS A STRAIGHT LINE

Consider a line $\mathcal{L} \subseteq \mathbb{R}^n$ given by

$$\mathcal{L} := \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} t \mid t \in \mathbb{R} \right\}, \tag{5}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $(a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0)$. We may assume without loss of generality that $\sum_{i=1}^{n} a_i^2 = 1$ and $a_n \neq 0$.

For such a line $\mathcal{L}$ as defined above, define the following ideal

$$\begin{aligned}\mathfrak{a} := \langle a_n\partial_1 - a_1\partial_n - b_1, a_n\partial_2 - a_2\partial_n - b_2, \cdots, \\ a_n\partial_{n-1} - a_{n-1}\partial_n - b_{n-1}\rangle,\end{aligned} \tag{6}$$

where $b_1, \ldots, b_{n-1} \in \mathbb{R}$.

Proposition 12. For the ideal $\mathfrak{a} \subseteq \mathcal{A}$ defined above, the following hold.

(1) $\mathfrak{a}$ is a prime ideal.
(2) The signal space $\mathcal{F}_{\mathfrak{a}}$ corresponding to $\mathfrak{a}$ satisfies that $\mathcal{F}_{\mathfrak{a}}|_{\mathcal{L}} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$.

Proof. (1) Under the lexicographic term ordering $\partial_1 \succ \partial_2 \succ \cdots \succ \partial_n$, the given generators of the ideal form a Gröbner basis. It then follows that the set of standard monomials for this case is $\{1, \partial_n, \partial_n^2, \ldots\}$. Therefore, the quotient ring $\mathcal{A}/\mathfrak{a}$ is isomorphic – as an $\mathbb{R}$-algebra – to the 1-variable polynomial ring $\mathbb{R}[\partial_n]$ (see Sturmfels (2002)). Since $\mathbb{R}[\partial_n]$ is an integral domain, it follows from the First Isomorphism Theorem that $\mathfrak{a}$ must be prime.

(2) Note that $\mathcal{F}_{\mathfrak{a}}$ contains all the functions of the following form

$$f(x_1, \ldots, x_n) = \phi\left(\sum_{i=1}^{n} a_i x_i\right) \exp\left(\frac{1}{a_n}\sum_{i=1}^{n-1} b_i x_i\right),$$

where $\phi \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ is arbitrary. Such an f, when restricted to the straight line $\mathcal{L}$, takes the following form:

$$\begin{aligned} f|_{\mathcal{L}}(t) := f(a_1 t, \ldots, a_n t) &= \phi\left(t\sum_{i=1} a_i^2\right) \exp\left(\frac{t}{a_n}\sum_{i=1}^{n-1} b_i a_i\right) \\ &= \phi(t) \exp\left(\frac{t}{a_n}\sum_{i=1}^{n-1} b_i a_i\right). \end{aligned}$$

Since $\exp\left(\frac{t}{a_n}\sum_{i=1}^{n-1} b_i a_i\right)$ admits a smooth inverse, it follows that every function in $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ can be obtained by restricting an $f \in \mathcal{F}_{\mathfrak{a}}$ to $\mathcal{L}$. □

Lemma 13. For the ideal $\mathfrak{a} \subseteq \mathcal{A}$ defined above and an $m(\partial) \in \mathcal{A}$, let $m(\partial) \notin \mathfrak{a}$. Then $\ker_{\mathcal{F}_{\mathfrak{a}}} m(\partial)$ is a strongly autonomous system.

Proof. Note that

$$\ker_{\mathcal{F}_{\mathfrak{a}}}(m(\partial)) = \ker_{\mathcal{F}} \begin{bmatrix} m(\partial) \\ a_n\partial_1 - a_1\partial_n - b_1 \\ a_n\partial_2 - a_2\partial_n - b_2 \\ \vdots \\ a_n\partial_{n-1} - a_{n-1}\partial_n - b_{n-1} \end{bmatrix}.$$

The equation ideal for this behaviour is $\mathfrak{a} + \langle m(\partial)\rangle$. Recall that, under the lexicographic term ordering $\partial_1 \succ \partial_2 \succ \cdots \succ \partial_n$, the given generators of the ideal form a Gröbner basis. It then follows that the set of standard monomials for this case is $\{1, \partial_n, \partial_n^2, \ldots\}$. Since $m(\partial) \notin \mathfrak{a}$, the remainder after division of $m(\partial)$ by this Gröbner basis is a non-zero polynomial $\rho(\partial_n) \in \mathbb{R}[\partial_n]$. It then follows that the quotient ring $\mathcal{A}/(\mathfrak{a} + \langle m(\partial)\rangle)$ is isomorphic to the quotient ring $\mathbb{R}[\partial_n]/\langle\rho(\partial_n)\rangle$. Since $\rho(\partial) \neq 0$, we must have $\mathbb{R}[\partial_n]/\langle\rho(\partial_n)\rangle$ to be zero-dimensional. Therefore, $\mathcal{A}/(\mathfrak{a} + \langle m(\partial)\rangle)$ must also be zero-dimensional, and hence, $\ker_{\mathcal{F}_\mathfrak{a}} m(\partial)$ must be strongly autonomous. □

7.1 Quotient controllability

Definition 14. Given an n-D behaviour $\mathfrak{B}$ and an ideal $\mathfrak{a} \subseteq \mathcal{A}$, the behaviour $\mathfrak{B}$ is said to be $\mathfrak{a}$-quotient controllable if the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_\mathfrak{a}$ admits an image representation.

Proposition 15. If $\mathfrak{a} \subseteq \mathcal{A}$ is a prime ideal then a behaviour $\mathfrak{B}$ is $\mathfrak{a}$-quotient controllable if and only if the $\mathfrak{a}$-quotiented module of observables $\overline{\mathcal{M}}$ is torsion-free.

Proof. Follows trivially from Theorem 11. □

Definition 16. Let $\mathcal{L} \subseteq \mathbb{R}^n$ be a line as defined in equation (5). An n-D behaviour $\mathfrak{B}$ is said to be *line controllable* with respect to $\mathcal{L}$ if for every pair $w_1, w_2 \in \mathfrak{B}$ and $\tau > 0$ there exists $w_3 \in \mathfrak{B}$ such that

$$w_3(a_1t, a_2t, \ldots, a_nt) = \begin{cases} w_1(a_1t, a_2t, \ldots, a_nt) & \text{for } t \leqslant 0 \\ w_2(a_1t, a_2t, \ldots, a_nt) & \text{for } t \geqslant \tau \end{cases}$$

Theorem 17. Let $\mathcal{L}$ be as defined in equation (5) and $\mathfrak{a}$ be the corresponding ideal as defined in equation (6). Then an n-D behaviour $\mathfrak{B}$ is $\mathfrak{a}$-quotient controllable if and only if the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_\mathfrak{a}$ is line controllable with respect to $\mathcal{L}$.

Proof. (Only if) Assuming $\mathfrak{B}$ is $\mathfrak{a}$-quotient controllable we want to show that $\mathfrak{B}_\mathfrak{a}$ is line controllable with respect to $\mathcal{L}$. Since $\mathfrak{B}$ is $\mathfrak{a}$-quotient controllable, it follows that the $\mathfrak{a}$-quotiented module of observables, $\overline{\mathcal{M}}$, is torsion-free. Therefore, there exists $M \in (\mathcal{A}/\mathfrak{a})^{q\times k}$ such that

$$\mathfrak{B}_\mathfrak{a} = \left\{w \in \mathcal{F}_\mathfrak{a}^q \mid \exists \ell \in \mathcal{F}_\mathfrak{a}^k \text{ such that } w = M \cdot \ell\right\}.$$

That is, $\mathfrak{B}_\mathfrak{a} = M \cdot \mathcal{F}_\mathfrak{a}^k$. Therefore, when restricted to the line $\mathcal{L}$, we have $\mathfrak{B}_\mathfrak{a}|_\mathcal{L} = M \cdot (\mathcal{F}_\mathfrak{a}|_\mathcal{L})^k$. But, it has been shown in Proposition 12 that $\mathcal{F}_\mathfrak{a}|_\mathcal{L} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. Therefore, $\mathfrak{B}_\mathfrak{a}|_\mathcal{L} = M \cdot \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^k)$, which means $\mathfrak{B}_\mathfrak{a}$ is line controllable with respect to $\mathcal{L}$.

(If) We take the contrapositive route to prove this implication. Suppose $\mathfrak{B}$ is not $\mathfrak{a}$-quotient controllable. So, the $\mathfrak{a}$-quotiented module of observables, $\mathcal{M}$, must then contain torsion elements. This means, there exist $r \in (\mathcal{A}/\mathfrak{a})^{1\times q}$ and $a \in \mathcal{A}/\mathfrak{a}$ such that $r \notin \langle\pi(\mathcal{R})\rangle$, but $ar \in \langle\pi(\mathcal{R})\rangle$. It then follows that there exists a trajectory $w \in \mathfrak{B}_\mathfrak{a}$ such that $v := r \cdot w \neq 0 \in \mathcal{F}_\mathfrak{a}$, but $a \cdot v = 0 \in \mathcal{F}_\mathfrak{a}$. Let $\widehat{a}(\partial) \in \mathcal{A}$ be such that $\pi(\widehat{a}(\partial)) = a \in \mathcal{A}/\mathfrak{a}$. Clearly $v \in \ker_{\mathcal{F}_\mathfrak{a}} \widehat{a}(\partial)$. By Lemma 13 this scalar behaviour is strongly autonomous, and hence, the restricted trajectory $v|_\mathcal{L}$ is not patchable with the zero trajectory. This means $w|_\mathcal{L}$ is not patchable with the zero trajectory. Therefore, $\mathfrak{B}_\mathfrak{a}$ cannot be line controllable with respect to $\mathcal{L}$. □

7.2 A theorem of alternatives for controllability

We now utilize the above-mentioned notion of line-controllability to provide a theorem of alternatives for the class of controllable n-D systems that admit a full row-rank (FRR) kernel representation. Since the result is derived from the trajectory-level notion of line-controllability, it provides a classification of controllable n-D systems – that are kernels of FRR matrices – based completely on trajectory-level properties. Note that the assumption on FRR kernel representation is not restrictive. In fact, for 2-D systems, every controllable system must necessarily admit such an FRR kernel representation. On the other hand, for $n \geqslant 3$, the set of all n-D systems that admit kernel representations by FRR matrices is Zariski closed. And hence, the property of admitting an FRR kernel representation matrix is generic. See Shankar (2014) for details.

Before proceeding, we introduce the following notation: Let $\mathfrak{L}$ denote the set of all 1-D subspaces of $\mathbb{R}^n$ passing through the origin. Thus a line $\mathcal{L}$ given by Eq. (5) satisfies $\mathcal{L} \in \mathfrak{L}$. To get a corresponding ideal, $\mathfrak{a}$, as given in Eq. (6) we need a vector $b \in \mathbb{R}^{n-1}$. Together, $(\mathcal{L}, b) \in \mathfrak{L} \times \mathbb{R}^{n-1}$ defines an affine straight-line in $\mathbb{R}^n$ and it corresponds uniquely to an ideal $\mathfrak{a}$ as given in Eq. (6).

The notion of *cancellation ideal* plays a crucial role in the proof of Theorem 20 below.

Definition 18. Let $R \in \mathcal{A}^{g\times q}$ be of full row-rank. The *cancellation ideal* corresponding to R, denoted by $\mathfrak{i}_{\text{can}}$, is the ideal generated by the $(g \times g)$ minors of R.

Proposition 19. Let $R_1, R_2 \in \mathcal{A}^{g\times q}$, both full row-rank, be such that the row-span of R_1 equals that of R_2 (both over $\mathcal{A}$). Let $\mathfrak{i}_{\text{can},1}$ and $\mathfrak{i}_{\text{can},2}$ be the cancellation ideals of R_1 and R_2, respectively. Then $\mathfrak{i}_{\text{can},1} = \mathfrak{i}_{\text{can},2}$.

Proof. Follows from (Eisenbud, 2013, Corollary 20.4) by noting that the cancellation ideal is the 0^{th} Fitting ideal of the module of observable $\mathcal{M}$ corresponding to the row module $\mathcal{R}$ that is the spanned by the rows of R_1 or R_2.

Note that from Proposition 19 it follows that, assuming $\mathfrak{B}$ admits a kernel representation with a full row-rank matrix, we can uniquely attach a cancellation ideal with a given $\mathfrak{B}$. Also, the affine variety

$$\mathbb{V}(\mathfrak{i}_{\text{can}}) := \{\lambda \in \mathbb{C}^n \mid f(\lambda) = 0\ \forall\ f \in \mathfrak{i}_{\text{can}}\}$$

of the cancellation ideal is called the *cancellation variety* and is denoted by $\mathbb{V}_{\text{can}}$.

Theorem 20. Let $\mathfrak{B}$ be a controllable n-D system having a kernel representation with a kernel representation matrix having full row-rank. Then exactly one of the following two statements is true:

(1) $\mathfrak{B}$ is strongly controllable, i.e., $\mathfrak{B}$ admits an observable image representation.
(2) There exists $(\mathcal{L}, b) \in \mathfrak{L} \times \mathbb{R}^{n-1}$ such that $\mathfrak{B}_\mathfrak{a}$ is not line-controllable with respect to $\mathcal{L}$, where $\mathfrak{a}$ is the corresponding ideal given by Eq. (6).

Proof. It is enough to prove that if $\mathfrak{B}$ is not strongly controllable then Statement 2 holds. Hence, we assume $\mathfrak{B}$ is not strongly controllable. It then follows from (Shankar, 2014, Proposition 2.2) that the cancellation variety $\mathbb{V}_{\text{can}} \neq \emptyset$. Hence we can choose an arbitrary $\lambda \in \mathbb{V}_{\text{can}}$. Let the i^{th} component of λ be denoted by $\lambda_i = \alpha_i + \iota\beta_i$, where $\alpha_i, \beta_i \in \mathbb{R}$, $\iota := \sqrt{-1}$, and $i = 1, \ldots, n$. Consider the linear equation

$$\begin{bmatrix} -\beta_n & 0 & \cdots & 0 & \beta_1 \\ 0 & -\beta_n & \cdots & 0 & \beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta_n & \beta_{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = 0. \tag{7}$$

Note that, Eq. (7) always admits non-zero real solutions. Choose such a nonzero solution $(a_1, \dots, a_n)$. We may assume that $a_n \neq 0$. If it is not so, then a permutation of the independent variables $x_1, \dots, x_n$ is needed to enforce it. With this choice of $(a_1, \dots, a_n)$ let $(b_1, \dots, b_{n-1}) \in \mathbb{R}^{n-1}$ be such that

$$\begin{bmatrix} a_n & 0 & \cdots & 0 & a_1 \\ 0 & a_n & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n & a_{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}. \tag{8}$$

With this $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^{n-1}$ define the corresponding ideal $\mathfrak{a}$ as given in Eq. (6). It can be easily verified that $\lambda \in \mathbb{V}(\mathfrak{a})$. Thus $\lambda \in \mathbb{V}(\mathfrak{i}_{\text{can}}) \cap \mathbb{V}(\mathfrak{a})$. By Hilbert's strong Nullstellensatz, this means there exists a maximal ideal $\mathfrak{m} \subseteq \mathcal{A}$ such that $\mathfrak{m} \supseteq \mathfrak{i}_{\text{can}} + \mathfrak{a}$. Recalling the definition of the quotient ring $\mathcal{A}/\mathfrak{a}$ and the corresponding natural surjection $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$ we can rewrite the above containment as follows: there exists a maximal ideal $\overline{\mathfrak{m}} \subseteq \mathcal{A}/\mathfrak{a}$ such that $\overline{\mathfrak{m}} \supseteq \langle \pi(\mathfrak{i}_{\text{can}}) \rangle$. Noting that $\langle \pi(\mathfrak{i}_{\text{can}}) \rangle$ is the cancellation ideal of $\pi(R(\partial))$, it follows that the $\pi(R(\partial))$ is a full row-rank matrix that loses rank upon further quotienting by the maximal ideal $\overline{\mathfrak{m}} \subseteq \mathcal{A}/\mathfrak{a}$. Since $\mathcal{A}/\mathfrak{a}$ is isomorphic to the 1-variable polynomial ring, this implies the $\mathfrak{a}$-quotiented module of observables $\overline{\mathcal{M}}$ contains torsion elements. And thus, by Theorem 17, the $\mathfrak{a}$-quotiented behaviour $\mathfrak{B}_{\mathfrak{a}}$ is not line controllable with respect to the line corresponding to parameter $a \in \mathbb{R}^n$ as chosen above.

8. CONCLUDING REMARKS

In this paper, we investigated the effect of quotienting the equation module, corresponding to a given n-D system, by a chosen ideal. We first showed that the trajectory level meaning of quotienting is restricting the trajectories to be from a special function space characterized by the chosen ideal. We argued that this restriction is nothing but creating a controlled system by attaching a controller whose equations are component-wise given by the chosen ideal.

With this notion of control by quotienting, we proceeded to show that the controlled behaviour – or, the quotiented behaviour, as they are called in this paper – can be characterized using a module over the quotient ring. More specifically, we proved a Malgrange-like duality result for quotiented behaviours. We then utilized the fact that the special function space, corresponding to the quotienting ideal, is an injective module over the quotient ring to show that there is a bijective Galois correspondence between quotiented behaviours and submodules (over the quotient ring) of the free module (of rank equal to the number of dependent variables in the n-D system at hand) over the quotient ring.

Next, we considered two special cases: first, when the chosen ideal is a prime ideal, and second, when the chosen ideal's variety is a line. We exploited the prime-ness of the ideal, and its consequence that the quotient ring is an integral domain, to characterize the property of the quotiented behaviour to be representable as an image with the property of the dual module being torsion-free. For the second special case, we showed how the trajectory level idea of line-controllability turns out to be equivalent to the algebraic idea of quotient controllability.

In Shankar (2014), the idea of a Hautus test for large class of n-D systems was proposed. As per this test, controllability is equivalent to the cancellation variety of an equation module having dimension strictly less than $n - 1$. This naturally raises the question that: is the dimension of this variety related to a gradation of controllability? We strongly believe that the idea of line-controllability/quotient-controllability holds the key to resolving this issue of gradation in controllability. We shall pursue this as a part of our future work.

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REFERENCES

Eisenbud, D. (2013). *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer.

Oberst, U. (1990). Multidimensional constant linear systems. *Acta Applicandae Mathematicae*, 20, 1–175.

Pal, D. and Zerz, E. (2024). The graded Hautus test of controllability for 2-d systems. *IEEE Control Systems Letters*, 8, 490–495.

Pillai, H.K. and Shankar, S. (1999). A behavioral approach to control of distributed systems. *SIAM Journal on Control and Optimization*, 37(2), 388–408.

Pommaret, J.F. and Quadrat, A. (1999). Algebraic analysis of linear multidimensional control systems. *IMA Journal of Mathematical Control and Information*, 16, 275–297.

Rocha, P. and Wood, J. (1997). A new perspective on controllability properties for dynamical systems. *Applied Mathematics and Computer Science*, 7(9), 869–879.

Rocha, P. and Willems, J.C. (1991). Controllability of 2-D systems. *IEEE Transactions on Automatic Control*, 36(4), 413–423.

Shankar, S. (2014). The Hautus test and genericity results for controllable and uncontrollable behaviors. *SIAM Journal on Control and Optimization*, 52(1), 32–51.

Sturmfels, B. (2002). *Solving Systems of Polynomial Equations.* American Mathematical Society with support from the National Science Foundation.

Wood, J., Rogers, E., and Owens, D.H. (1999). Controllable and autonomous nD systems. *Multidimensional Systems and Signal Processing*, 10, 33–69.

Wood, J. (2000). Modules and behaviours in n-d systems theory. *Multidimensional Systems and Signal Processing*, 11, 11–48.