Quaternionic Modular Forms of Degree two over $\mathbb{Q}(-3, -1)$

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

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Tag der mündlichen Prüfung: 30.11.2012

Diese Dissertation ist auf den Internetseiten der Hochschulbibliothek online verfügbar
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Introduction

Until now, the maximal order $\mathcal{O} = \mathbb{Z} + \mathbb{Z} \frac{1+i\sqrt{3}}{2} + \mathbb{Z} i_2 + \mathbb{Z} \frac{1+i\sqrt{3}}{2} i_2$, where $\mathbb{H} = \mathbb{R} + i_1 \mathbb{R} + i_2 \mathbb{R} + i_1 i_2 \mathbb{R}$ is the skew field of real Hamilton quaternions, and the quaternionic modular forms attached to them have not been studied in detail, yet. In order to be able to analyze the associated graded rings, it is essential to develop advanced insights into the theory of quaternionic modular forms over $\mathcal{O}$.

So as to define modular forms we require the notion of modular groups and half-spaces. Thus, let $\text{Sp}_2(\mathcal{O}) = \{ M \in \mathcal{O}^{4 \times 4} ; \overline{M} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \}$ be the quaternionic modular group of degree two with respect to $\mathcal{O}$, $\text{Sp}_2(\mathcal{O})$ acts on the quaternionic half-space $\mathcal{H}(\mathbb{H}) = \{ Z = X + iY \in \mathbb{H}^{2 \times 2} \otimes_{\mathbb{R}} \mathbb{C} ; Z = Z' : = \overline{X} + iY' , Y > 0 \}$ of degree two as a group of biholomorphic automorphisms via $Z \mapsto M(Z) = (AZ + B)(CZ + D)^{-1}$, $M = \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \text{Sp}_2(\mathcal{O})$. The main objects of this thesis are quaternionic modular forms of degree two with respect to subgroups of $\text{Sp}_2(\mathcal{O})$, i.e. holomorphic functions $f : \mathcal{H}(\mathbb{H}) \to \mathbb{C}$ with the special transformation behavior

$$f(M(Z)) = \nu(M) \cdot (\det(\hat{C}Z + \hat{D}))^{k/2} \cdot f(Z)$$

for all $M = \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ belonging to a subgroup $\Gamma$ of $\text{Sp}_2(\mathcal{O})$ and all $Z \in \mathcal{H}(\mathbb{H})$. Here $\nu$ denotes the embedding of $\mathbb{H}$ in $\mathbb{C}^{2 \times 2}$ and $\nu$ is a multiplier system for $\Gamma$ of weight $k \in \mathbb{Z}$. In particular for $\Gamma = \text{Sp}_2(\mathcal{O})$ the $\mathbb{C}$-vector space $[\text{Sp}_2(\mathcal{O}), k, \nu]$ of these functions is finite dimensional. So the main objective is to determine these spaces in terms of a basis – or at least in terms of generators. Moreover, noting $[\text{Sp}_2(\mathcal{O}), k_1, \nu_1] \cdot [\text{Sp}_2(\mathcal{O}), k_2, \nu_2] \subset [\text{Sp}_2(\mathcal{O}), k_1 + k_2, \nu_1 \nu_2]$ one can also analyze the graded ring $\bigoplus_{k, \nu} [\text{Sp}_2(\mathcal{O}), k, \nu]$, which initially motivated the work on this thesis.

The systematic theory of modular forms in higher dimensions began with C. F. Siegel’s work “Einführung in die Theorie der Modulfunktionen n-ten Grades” in 1939, analyzing Siegel modular forms with respect to the symplectic groups $\text{Sp}_n(\mathbb{Z})$. They are a generalization of one-dimensional elliptic modular forms. These elliptic modular forms had been discovered in the 19-th century. Topics related to them – in particular all kinds of automorphic forms – have been a major field of study since then, influencing vast domains of mathematics and also having applications in physics. In the 1960’s Igusa used theta-constants to construct generators for the graded ring of Siegel modular forms with respect to $\text{Sp}_2(\mathbb{Z})$ (cf. [Ig62] and [Ig64]). The graded ring which only considers even weights and the trivial character turns out to be isomorphic to the polynomial ring over $\mathbb{C}$ in four variables. One generalization of Siegel modular forms are Hermitian modular forms, replacing the Siegel half-space by the Hermitian half-space and $\text{Sp}_2(\mathbb{Z})$ by $\text{Sp}_2(o)$. Here, $o$ is the integral closure of some imaginary quadratic number field. Using the same approach like Igusa did, in 1967 Freitag determined generators for the graded ring of symmetric Hermitian modular forms of degree two over the Gaussian number field (cf. [Fr67]). Again, the graded ring is isomorphic to a polynomial ring over $\mathbb{C}$, this time in five variables, in case one only considers the character $\det^{k/2}$ (cf. [DK03]). Choosing a different method, Dern and Krieg also determined the graded rings of (not
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obligatorily symmetric) Hermitian modular forms of degree two over \( Q(i) \), \( Q(i\sqrt{2}) \) and \( Q(i\sqrt{3}) \) (cf. [De01], [DK03], and [DK04]). Dern used so-called Borcherds products (cf. [Bo98]) to reduce the problem of analyzing Hermitian modular forms to the consideration of Siegel modular forms and paramodular forms. A further generalization of Siegel modular forms are quaternionic modular forms. They were introduced by Krieg in 1985, who considered quaternionic modular forms with respect to the Hurwitz order (cf. [Kr85]). Furthermore, all these types of modular forms correspond to orthogonal modular forms for certain orthogonal groups \( O(2, 2 + l) \). Here, \( l = 1, 2, 4 \) regarding Siegel, Hermitian and quaternionic modular forms, respectively. Klöcker determined the associated graded rings with respect to two special orthogonal modular groups for \( l = 3 \) using similar methods like Dern and making use of his results (cf. [Kl06]). Moreover, Freitag, Hermann and Salvati-Manni investigated orthogonal modular forms in a general sense, and also with particular emphasis on some further special orthogonal modular groups in [FH00] and [FS07] – especially those orthogonal modular forms corresponding to quaternionic modular forms over the Hurwitz order. Again using Borcherds products and building up upon the results of Klöcker, Freitag and Hermann, in 2005 Krieg was able to describe the graded ring of quaternionic modular forms over the Hurwitz order (cf. [Kr05]). Once more it turned out that, when considering the extended quaternionic modular group and sticking to the trivial character, the graded ring is isomorphic to a polynomial ring over \( C \), more precisely the polynomial ring in seven variables. This final result is based on an upward process using Borcherds products: Knowing the structure of Siegel modular forms and certain paramodular forms, Dern was able to investigate Hermitian modular forms, and hence certain orthogonal modular forms for \( O(2, 4) \). Based on these results, Klöcker could progress upwards to investigate some cases for \( O(2, 5) \). Finally, these achievements lead to the conclusions of Krieg concerning the Hurwitz order, or in other words concerning one important case for \( O(2, 6) \).

In this thesis, we investigate another order which has not been studied in full detail yet, namely the above mentioned \( O \). Therefore, we have to begin by working out the first basic facts on quaternionic modular forms over this order. Note that there already exists some research in this field (cf. [K98]). Based on this we will develop advanced insights into the theory of quaternionic modular forms over \( O \). In order to initialize a reduction process similar to the proceeding concerning the Hurwitz order, first important quaternionic modular forms have to be constructed. This includes theta-constants, which helped to construct generators for graded rings of modular forms in other settings (cf. [Ig62], [Ig64], [Fr67], [Kr10] and [GK10]). Furthermore, we need so-called Maaß lifts (or arithmetical liftings in the sense of [Gr95]) in analogy to the Maaß lifts with respect to the Hurwitz order (cf. [Kr87], [Kr90] and [Kr96]). The advantage of Maaß lifts are their easily computable Fourier-expansions which are essential to the verification of certain results. Moreover, quaternionic Eisenstein-series and their Fourier-expansions are investigated. Their importance lies in the fact that apart from some special Borcherds products (which are mainly needed to cover non-trivial abelian characters) Eisenstein-series sufficed to generate the graded rings of modular forms in all cases mentioned above. And finally, we will present Borcherds products, and in that context a possible reduction process in order to analyze the structure of the graded ring of quaternionic modular forms of degree two over \( O \). Unlike the circumstances given for the Hurwitz order, the graded rings of modular forms being involved in the reduction process regarding \( O \) have not been determined, yet. They also turn out to be much more complicated in their nature. This is mainly due to the obstruction spaces regarding
Borcherds products having dimensions greater or equal to two, with the Fourier-expansions of their generators yielding too many and too complicated constraints for possible combinations of divisors.

After we constructed and analyzed all quaternionic modular forms which should – in the light of the results concerning the previous settings described above and the numerous similarities between the Hurwitz order and \( O \) – suffice to describe all spaces of quaternionic modular forms of degree two over \( O \), we will switch from the symplectic setting to the orthogonal world. Quaternionic modular forms over \( O \) correspond to orthogonal modular forms with respect to another orthogonal modular group in \( O(2,6) \). Using Borcherds products we will reduce the investigation regarding \( O(2,6) \) to lower dimensional settings. Finally, we will end up with a special orthogonal modular group in \( O(2,3) \). This yields orthogonal modular forms that correspond to paramodular forms of degree two and level 7. Just like Marschner, who examined level 5 (cf. [Mar04]), we will not be able to completely determine the graded ring of paramodular forms, which would be crucial to initiate the described upward process. Nevertheless, we will point out where the critical points lie in order to further investigate this issue in possible future work. Furthermore, as said before we will develop the necessary proceedings, requirements and propositions to be able to finally work out the details of a successful upward process once the issue concerning the paramodular forms has been solved.

We will finish this thesis by constructing a set of seven algebraically independent quaternionic modular forms. These should be minimal regarding the occurring weights. In order to do so we will have to combine several parts of the afore developed theories of this thesis. It is not without reason that the graded ring of quaternionic modular forms of degree two over \( O \) with respect to the extended quaternionic modular group and the trivial character might again be isomorphic to the polynomial ring over \( \mathbb{C} \) in seven variables. Moreover, the seven algebraically independent forms are reasonable candidates as generators of this graded ring.

We give a short description of each chapter, now:

In chapter one, we begin by introducing the quaternionic and symplectic setting, and in particular quaternionic modular forms and their basic properties. We summarize important facts previously investigated in [Kr85] and also work out special and new results regarding the order \( O \). The groups of abelian characters of \( \text{Sp}_2(O) \) and of the yet to introduce extended quaternionic modular group \( \Gamma(O) \) are completely characterized, and as well all possible multiplier systems for \( \text{Sp}_2(O) \). It turns out that in contrast to the case of the Hurwitz order there exist multiplier systems of odd weight. Furthermore, a fundamental domain concerning the action of \( \text{Sp}_2(O) \) on \( \mathcal{H}(\mathbb{H}) \) is constructed. This in turn helps to obtain first bounds on the dimensions of the spaces of quaternionic modular forms. Furthermore, the theory of cusp forms and Fourier-expansions is described.

The second chapter deals with quaternionic theta-series. In particular, theta-constants and quaternionic theta-series of the second kind are analyzed in full detail, including their transformation behavior under the extended modular group. A formula for the Fourier-coefficients of the theta-constants is developed. Moreover, we verify that they are quaternionic modular forms of weight two with respect to the principal congruence subgroup \( \text{Sp}_2(O)[p] \) of level \( p \), where \( p = i_1 \sqrt{3}O \) is an important two-sided ideal. Invariants in these 21 theta-constants can be constructed to obtain quaternionic modular forms with respect to the full quaternionic modular
group with easily computable Fourier-expansions.

In the third chapter the theory of quaternionic Maaß lifts is developed. In [Kl98] Maaß lifts of even weight and trivial character were considered. In this thesis we construct Maaß lifts of odd weight with respect to the two multiplier systems \( \nu_i \) and \( \nu_{-i} \), which are the only multiplier systems of odd weight for \( \text{Sp}_2(\mathcal{O}) \). Both are not abelian characters, which makes the corresponding theory more complicated. Constructing these quaternionic Maaß lifts of odd weight and showing that they are non-identically vanishing also verifies that there actually exist quaternionic modular forms of odd weight with respect to \( \text{Sp}_2(\mathcal{O}) \). This is not the case for the Hurwitz order. The proceeding to obtain these liftings is to decompose quaternionic modular forms using Fourier-Jacobi-decompositions and theta-decompositions to get elliptic modular forms serving as input functions. These are then lifted back to quaternionic modular forms using certain Hecke-operators for quaternionic Jacobi-forms. Furthermore, to explicitly determine the spaces of elliptic modular forms which are the input of these liftings the theory of elliptic newforms and the theory of Hecke-operators are presented. The final result is the exact shape of the input functions and the dimensions of the spaces of Maaß lifts of odd weight being \( \left[ \frac{k}{6} \right] \). An application of the Maaß lifts is that we are able to completely specify the spaces of quaternionic modular forms of weights \( k \leq 7 \) for the trivial character and the multiplier systems \( \nu_i \) and \( \nu_{-i} \).

The fourth chapter deals with quaternionic Eisenstein-series (with input function \( f \equiv 1 \), so no Klingens Eisenstein-series are considered). First, the basic facts regarding Eisenstein-series are presented. The main achievement of this chapter is to explicitly determine the Fourier-expansion of the Eisenstein-series. In particular, this result is needed to verify that the aforementioned seven algebraically independent forms are independent, indeed. The analysis concerning this Fourier-expansion is quite involving. Apart from the introductory section the whole chapter deals with this issue. Ultimately, it turns out that the Eisenstein-series are special Maaß lifts for the trivial character. Therefore, we begin by investigating the spaces of elliptic modular forms serving as the input to the lifting in much greater detail than it was initially done in [Kl98]. We have to do so because for our special purpose we need the explicit shape of the Fourier-coefficients and not just the theoretical background on how to construct Maaß lifts. Afterwards, we introduce the theory of Hecke-operators in the setting of quaternionic modular forms. In particular, we investigate how the special Hecke-operators \( T_2(p) \) act on the spaces of Maaß lifts. This is the most involving part of the approach. It turns out that the action of \( T_2(p) \) is given by the action of another Hecke-operator (in degree one) on the input function of the lifting. This also means that \( T_2(p) \) preserves the spaces of Maaß lifts for the trivial character. The final step is to show that the Eisenstein-series are eigenforms of the Hecke-operators \( T_2(p) \), and that every other quaternionic modular form with non-vanishing constant Fourier-coefficient also being such an eigenform only differs from the Eisenstein-series of the same weight by a constant factor. Therefore, the lifting of an appropriately normalized eigenfunction with respect to the corresponding Hecke-operator for the input functions has to coincide with the Eisenstein-series of the according weight.

The fifth chapter finally describes the link between quaternionic modular forms over \( \mathcal{O} \) and orthogonal modular forms for \( O(2,6) \). We develop a dictionary in order to be able to switch between the symplectic and the orthogonal world. So we can make use of the advantages of both. The orthogonal setting and also particular lattices which are needed for a reduction process
are introduced. This includes the structures of the attached orthogonal modular groups and their groups of abelian characters. We also investigate how orthogonal modular forms restrict to half-spaces attached to sublattices, which yields important information on the subsequent reduction process. Lastly, we also present the theory of the metaplectic covering of \( \text{SL}_2(\mathbb{Z}) \) and (weakly holomorphic) vector-valued modular forms of half-integral weight, since this will be needed to discuss and construct Borcherds products.

The sixth and final chapter presents the reduction process. We begin by demonstrating the fascinating construction of Borcherds specialized to our setting. It lifts certain weakly holomorphic vector-valued modular forms with respect to the dual of the Weil representation to orthogonal modular forms with prescribed divisors. These divisors are completely determined by the principal part of the lifted vector-valued form. They turn out to be rational quadratic divisors corresponding to embedded orthogonal half-spaces of codimension one. Even without knowing the complete vector-valued modular form it is possible to construct principal parts (and corresponding constant terms) belonging to some appropriate vector-valued modular form by making use of the so-called obstruction space. Afterwards, we actually produce Borcherds products for all orthogonal settings we are interested in. We show how a reduction process can be achieved similarly to the respective progress arising over the Hurwitz order. In particular, the zeros of the Borcherds products which we construct for the spaces of quaternionic modular forms over \( \mathcal{O} \) lie on a single rational quadratic divisor (and those divisors congruent to it modulo the orthogonal modular group). They are of first order, meaning that if we knew the structure of the graded rings of orthogonal modular forms corresponding to these embedded orthogonal half-spaces, then it should be easy to determine the graded rings of quaternionic modular forms, as well. But in order to actually analyze these graded rings with the help of Borcherds products, we have to reduce the dimension even further. Ultimately, we end up with the investigation of paramodular forms of degree two and level 7. Hence we also introduce the theory of paramodular forms and examine the special case of level 7. Unfortunately, we will not be able to completely specify the attached graded ring. But we point out where the difficulties lie and how the upward process should work once this issue has been solved. We finish this last chapter by constructing a set of seven algebraically independent quaternionic modular forms with respect to the extended quaternionic modular group \( \Gamma(\mathcal{O}) \) and the trivial character. It is given by \( \{ E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14} \} \), where the \( E_k \) are quaternionic Eisenstein-series, \( F_{12} \) is the square of a Borcherds product of weight six and \( F_{14} \) is the product of the two unique Maass lifts for \( \nu \) and \( \nu_- \) of weight seven. This set should be minimal regarding the weights. Moreover, it is a reasonable and possible candidate for a set of generators for the graded ring of quaternionic modular forms with respect to the extended quaternionic modular group and the trivial character.

This thesis was developed and written at the Lehrstuhl A für Mathematik, RWTH Aachen University. The work on this topic was supervised by Prof. Dr. A. Krieg. I would like to express my deepest gratitude to him for the suggestion of this topic, his encouragement and his valuable and continuous advice. Without his support this research on quaternionic modular forms would not have been possible.

Furthermore, I would like to thank Dr. habil. S. Kraußhar for accepting to act as second referee. Moreover, I would like to express my thanks to all my present and former colleagues at
the Lehrstuhl A für Mathematik for all the helpful and inspiring discussions, especially Dr. M. Raum for his frequent advice.

And finally, I thank my parents and my girlfriend for their continued support, help, encouragement and love during all my time of studying and developing this thesis.
0 Basic Notation

We use the following notation: \( \mathbb{N} \) is the set of positive integers, \( \mathbb{N}_0 \) is the set of non-negative integers, \( \mathbb{Z} \) is the ring of integers, \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) are the fields of rational, real and complex numbers, respectively. \( i \) will always, without exception, denote the imaginary unit. \( \mathcal{H} = \{ z \in \mathbb{C} ; \text{Im}(z) > 0 \} \) is the upper complex half-plane.

\( \mathcal{H} \) denotes the skew field of real Hamilton quaternions with standard basis \( \mathcal{H} = \mathbb{R} + i_1 \mathbb{R} + i_2 \mathbb{R} + i_3 \mathbb{R} \), where \( i_1^2 = i_2^2 = -1 \) and \( i_3 = i_1 i_2 = -i_2 i_1 \). For \( a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \in \mathcal{H}, \) where \( a_j \in \mathbb{R}, \) the real part is given by \( \text{Re}(a) = a_0, \) the conjugate by \( \bar{a} = a_0 - a_1 i_1 - a_2 i_2 - a_3 i_3 \) and the norm by \( N(a) = a \bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2. \) If not noted any differently, \( a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \in \mathcal{H} \) will always imply \( a_0, \ldots, a_3 \in \mathbb{R} \). Calculation rules can be found in [Kr85, ch.I].

Let \( R \) be a ring with unity. As usual, \( R^{n \times m} \) is the (additive) group of \( n \times m \) matrices over \( R, \) \( \text{GL}_n(R) \) and \( \text{SL}_n(R) \) are the general linear group and the special linear group in \( R^{n \times n}, \) respectively. \( R^n = R^{n \times 1} \) always stands for column vectors. For \( A \in R^{n \times n} \) and \( B \in R^{n \times m}, \) the transpose of \( B \) is denoted by \( B', \) and \( \text{tr}(A) \) stands for the trace of \( A. \) If there exists some conjugation in \( R, \) we denote the conjugate transpose of \( B \) by \( \overline{B}. \)\) and define \( A[B] := \overline{B} AB. \) Note that for \( R \subset \mathcal{H}, (AB)' = \overline{B}' \overline{A}' \) holds for all \( A,B \in R^{n \times n} \) according to (1.35). The subset of symmetric matrices of \( R^{n \times n} \) is denoted by \( \text{Sym}_n(R), \) while \( \text{Her}_n(R) \) denotes the subset of Hermitian matrices, i.e. \( \text{Her}_n(R) = \{ A \in R^{n \times n} ; A = \overline{A}' \}. \) If such a definition is possible for \( R, \) then for \( A \in \text{Her}_n(R) \) we write \( A > 0 \) if \( A \) is positive definite and we write \( A \geq 0 \) if \( A \) is positive semi-definite. \( \text{Pos}_n(R) \) then denotes the set of all Hermitian, positive definite matrices of \( R^{n \times n}. \) \( I_n \) will always denote the identity matrix in \( R^{n \times n}, \) or simply \( I \) if the dimension is obvious. \( I_{jk} \) is the \( n \times n \) matrix which has only 0 as entries except the \((j,k)-\)entry, which is 1. The \( j \)-th unit vector in \( R^n \) is denoted by \( e_j. \) Note that the dimension \( n \) will always be clear from the context.

If not explicitly noted any different, \( M = ( A_{ij} B_{ij} ) \) is always supposed to be the decomposition of \( M \in R^{2n \times 2n} \) into \( n \times n \) blocks. For \( a_1, \ldots, a_n \in R \) the diagonal matrix with diagonal entries \( a_1, \ldots, a_n \) is denoted by \( \text{diag}(a_1, \ldots, a_n). \) And by abuse of notation we also write \( \text{diag}(A_1, \ldots, A_n) \) for matrices \( A_j \in R^{m_j \times m_j}. \) This denotes the quadratic block-diagonal matrix of degree \( m_1 + \ldots + m_n \) whose “diagonal entries” (which means its diagonal blocks) are given by the matrices \( A_j. \)

Let \( G \) be a group. For \( g,h \in G \) the commutator of \( g \) and \( h \) is defined by \( [g,h] := ghg^{-1}h^{-1}. \) The commutator subgroup of \( G \) is denoted by \( G' = \{ [g,h] ; g,h \in G \}, \) and the commutator factor group by \( G^{ab} := G/G'. \) We also denote the group of abelian characters \( G \to \mathbb{C}^* \) by \( G^{ab}, \) as it is isomorphic to the commutator factor group.

If for example a theorem is denoted by “(1.1) Theorem.”, then we will refer to it by writing “(1.1)”. On the other hand, if for example an equation is numerated by (1.1) on the right margin of a page, then we refer to it by writing “1.1”, so that it is clear if are referring to a theorem (and the like) or to such a numeration of an equation (and the like).
1 Quaternionic Modular Forms

This first chapter serves as an introduction to quaternionic modular forms, in particular of degree two, which are the main objects of interest in this thesis. We will define the quaternionic half-space, which can be seen as a convex cone in $\mathbb{C}^m$ (for an appropriate $m \in \mathbb{N}$) the quaternionic modular forms live on – where $m = 6$ when considering quaternionic modular forms of degree two. We will introduce the symplectic group over the real Hamilton quaternions and a special discrete subgroup of it, the quaternionic modular group over the particular maximal order of $\mathbb{H}$ this thesis deals with. The first section is mainly about these objects and some of their important properties. We will cite many facts and results from [Kr85], where the symplectic setting concerning modular forms for $\mathbb{H}$ is discussed and analyzed – although for another order, namely the Hurwitz order. Nevertheless, deriving from the many similarities between these two orders a lot of results can be adapted to our setting. And since a basic introduction concerning this setting was already given in [Kl98], we will also cite several results from that work. Moreover, we will even present some new achievements on this topic in the first introductory section, mainly concerning the abelian characters for the orthogonal modular group and the extended quaternionic modular group.

In the second section a fundamental domain regarding the action of the quaternionic modular group on the quaternionic half-space will be constructed. We will restrict ourselves to the case of degree two there, since we are mainly interested in quaternionic modular forms of degree two in this thesis. In that context we will also get to know a reduction process concerning positive definite matrices over $\mathbb{H}$ with respect to $\text{GL}_2(\mathcal{O})$, which will be useful later on when dealing with Fourier-expansions and bounds on the dimensions of spaces of quaternionic modular forms.

The third section finally introduces quaternionic modular forms and first basic facts about these objects, including Fourier-expansions. To define quaternionic modular forms we will need so-called multiplier systems, which possess an attribute called the weight. In the case of even weights multiplier systems simply are abelian characters, but in general this no longer holds true for arbitrary weight. Moreover, we will show that multiplier systems for the whole quaternionic modular group can only exist for integral weights, and we will investigate which multiplier systems are possible for odd weights. It will turn out that there are exactly two, denoted by $\nu_i$ and $\nu_{-i}$.

In the fourth section we will examine spaces of quaternionic modular forms in greater detail. The theory of cusp forms will be presented and we will see that under certain conditions (and these conditions will be fulfilled for all the cases we are interested in) the spaces of quaternionic modular forms with respect to fixed subgroups, weights and multiplier systems are finite dimensional, and we will obtain first bounds for these dimensions.

The last section is about the restriction of quaternionic modular forms to certain submanifolds. If we choose the correct ones, then one obtains Hermitian and Siegel modular forms of the same weight. This gives the link in what way quaternionic modular forms can be seen as one possible
generalization of Siegel modular forms.

1.1 The quaternionic half-space and the quaternionic modular group

To define symplectic modular forms, we first need to take a look at the so-called half-spaces, symplectic groups and modular groups. As we will work with several quaternionic, Hermitian and Siegel modular forms, the definitions will be very general so that it is not necessary to define them individually. Note that \( \mathbb{R} \) and \( \mathbb{C} \) are subrings of \( \mathbb{H} \) by simply putting \( a_1 = a_2 = a_3 = 0 \), or \( a_2 = a_3 = 0 \) respectively for \( a_0 + a_1i_1 + a_2i_2 + a_3i_3 \in \mathbb{H} \). The conjugate in \( \mathbb{H} \) is compatible with the complex conjugate in \( \mathbb{C} \), while it is the identity for \( \mathbb{R} \).

Note that \( \mathbb{H} \) is an \( \mathbb{R} \)-vector space with basis \( i_0 = 1, i_1, i_2, i_3 \). In the forthcoming definition we will speak of subspaces \( R \) of \( \mathbb{H} \). Precisely, we mean \( \mathbb{R} \)-subvector spaces of \( \mathbb{H} \), e.g. \( R = \mathbb{H}, \mathbb{C}, \mathbb{R}, \{a_0 + a_1i_1 + a_2i_2 \in \mathbb{H}\} \), such that the notation \( R^{\times n} \otimes_\mathbb{R} \mathbb{C} \) makes sense.

(1.1) Definition. Let \( n \in \mathbb{N} \) and \( R \) a subspace of \( \mathbb{H} \) satisfying \( \pi \in R \) for all \( a \in R \). The half-space of degree \( n \) with respect to \( R \) is given by

\[
\mathcal{H}_n(R) := \{ Z = X + iY \in R^{\times n} \otimes_\mathbb{R} \mathbb{C} ; Z = Z' := \bar{X} + iY', Y > 0 \}.
\]

Since we are mainly interested in degree two, we will simply write \( \mathcal{H}(R) := \mathcal{H}_2(R) \). Writing \( Z = X + iY \in \mathcal{H}_n(R) \) will always imply \( X, Y \in R^{\times n} \).

Remark. a) \( Z = Z' \) is equivalent to \( X, Y \in \text{Her}_n(R) \). For \( Y \in \text{Her}_n(\mathbb{H}) \), \( Y[x] \in \mathbb{R} \) holds for all \( x \in \mathbb{H}^n \) according to [Kr85, p.21], so that \( Y > 0 \) makes sense. Furthermore, because according to [Kr85, ch.I, prop.1.3] \( (a, b) \mapsto \text{Re}(ab) \) is a scalar product on the \( \mathbb{R} \)-vector space \( \mathbb{H} \) and because of [Kr85, p.23, (3)], \( \mathcal{H}_n(R_1) \subset \mathcal{H}_n(R_2) \) holds for subspaces \( R_1 \leq R_2 \leq \mathbb{H} \).

b) Note that also for \( R = \mathbb{C} \) (or something similar) you have to differ between \( i_1 \) and \( i \). Only the entries of \( X \) and \( Y \), i.e. the \( i_1 \)'s, are being conjugated in \( Z' \), not \( i \). Usually, the Hermitian half-space, i.e. \( R = \mathbb{C} \), is defined differently, namely \( \mathcal{H}_n(\mathbb{C}) = \{ Z \in \mathbb{C}^{n \times n} ; \frac{1}{2}(Z + \bar{Z}) > 0 \} \), where no \( i_1 \) occurs. But an easy calculation shows that those two half-spaces are isomorphic by mapping \( \mathcal{H}_n(\mathbb{C}) \rightarrow \mathcal{H}_n(R), Z \mapsto \mathcal{R}(Z) + i\mathcal{I}(Z) \), where \( \mathcal{R}(Z) := \frac{1}{2}(Z + \bar{Z}), \mathcal{I}(Z) := \frac{1}{2i}(Z - \bar{Z}) \in \mathbb{H}^{n \times n} \) are identified in \( \mathbb{H} \). Further easy calculations show that this isomorphism commutes with symplectic transformations, the operations needed to define modular forms, so that we will indeed speak about the same Hermitian modular forms commonly known. The same holds true for other two-dimensional subalgebras, \( R = \{a_0 + a_1(i_1 + i_2) \in \mathbb{H}\} \) for example would result in the Hermitian half-space, too.

c) For \( R = \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathcal{H}_n(R) \) is called the quaternionic, Hermitian or Siegel half-space of degree \( n \), respectively. As explained in b) \( \mathcal{H}_n(R) \) also leads to the Hermitian half-space (and Hermitian modular forms) if \( R \) is a two-dimensional subalgebra with \( \mathbb{R} \subset R \).

d) Clearly one has \( \mathcal{H}_1(\mathbb{H}) = \mathcal{H}_1(\mathbb{C}) = \mathcal{H}_1(\mathbb{R}) = \{ z \in \mathbb{C} ; \text{Im}(z) > 0 \} = \mathcal{H} \), which is the upper complex half-plane.
1.1 The quaternionic half-space and the quaternionic modular group

(1.2) Definition. Let \( n \in \mathbb{N}, J_n := \begin{pmatrix} 0 & \bar{I}_n \\ \bar{I}_n & 0 \end{pmatrix} \) and \( R \) a subring of \( \mathbb{H} \) such that \( 1 \in R \) and \( \bar{a} \in R \) for all \( a \in R \). The symplectic group of degree \( n \) with respect to \( R \) is defined as

\[
\text{Sp}_n(R) := \{ M \in R^{2n \times 2n} ; J_n[M] = J_n \}.
\]

Again, as we will mainly have to deal with degree two, we simply write \( J := J_2 \).

Clearly, \( \text{Sp}_n(R) \) is a subgroup of \( \text{GL}_n(R) \), since \( J_n \) is invertible. From [Kr85, ch.I, le.1.1] we cite the following lemma as the proof would be the same for subrings:

(1.3) Lemma. Given \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(R) \) with \( R \) as in (1.2), the following assertions are equivalent:

(i) \( M \in \text{Sp}_n(R) \).

(ii) \( M' \in \text{Sp}_n(R) \).

(iii) \( \bar{A}'C - \bar{C}'A = \bar{B}'D - \bar{D}'B = 0, \bar{A}'D - \bar{C}'B = 1 \).

(iv) \( A\bar{B}' - B\bar{A}' = C\bar{D}' - D\bar{C}' = 0, AD' - BC' = 1 \).

In this case one has

\[
M^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}
\]

The identities in the lemma above are called fundamental relations. With those fundamental relations one can easily show with some calculations that the following holds true for \( R \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \):

\[
\text{Sp}_1(R) = \{ \epsilon M ; M \in \text{SL}_2(R), \epsilon \in R, N(\epsilon) = 1 \}.
\]

Apart from \( J_n \), there are two special types of matrices belonging to \( \text{Sp}_n(R) \) which we will need frequently:

\[
\text{Rot}(U) := \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix} \text{ for } U \in \text{GL}_n(R)
\]

\[
\text{Trans}(S) := \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix} \text{ for } S \in \text{Her}_n(R)
\]

Furthermore, we have the following embedding:

\[
\text{Sp}_m(R) \times \text{Sp}_n(R) \hookrightarrow \text{Sp}_{m+n}(R)
\]

\[
(M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}) \mapsto M_1 \times M_2 := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}
\]

(1.1)

Now, before looking at orders, we cite another lemma about generators for some special cases. According to [Kr85, ch.I, le.1.4], we have:
(1.4) Lemma. For $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $\text{Sp}_n(R)$ is generated by the matrices

$$J_n, \quad \text{Rot}(U), \quad U \in \text{GL}_n(R), \quad \text{Trans}(S), \quad S \in \text{Her}_n(R).$$

The symplectic groups operate on the half-spaces. For $R$, we will only need the special cases in (1.4). Again, note that if $R$ is a two-dimensional subalgebra of $\mathbb{H}$ with $\mathbb{R} \subseteq R$, i.e. $R = \mathbb{R} + R\omega$ with $\omega \in \mathbb{H} \setminus \mathbb{R}$, then $R$ is isomorphic to $\mathbb{C}$, and thus $\mathcal{H}_n(R)$ is (isomorphic to) the Hermitian half-space. So the following special cases suffice for all the symplectic groups we are going to look at, cited from [Kr85, ch.II, thm.1.7, thm.1.8]:

(1.5) Theorem. Let $n \in \mathbb{N}$ and $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $Z = X + iY \in \mathcal{H}_n(R)$ define $\widetilde{Z}' = X' + iY'$, $\mathbf{Z} = X - iY$, $\mathfrak{Re}(Z) = X$ and $\mathfrak{Im}(Z) = Y$. Furthermore, define a transformation $\tau : R^{n \times n} \otimes_\mathbb{R} \mathbb{C} \to R^{2 \times 2} \otimes_\mathbb{R} \mathbb{C}$, $Z \mapsto Z'$.

Let $M = (\begin{smallmatrix} A & B \\ \mathbf{C} & \mathbf{D} \end{smallmatrix})$, $M_1 \in \text{Sp}_n(R)$ and $Z = X + iY \in \mathcal{H}_n(R)$. Then one has:

- $\text{a) } M\{Z\} := CZ + D$ and $\widetilde{M}\{Z\} = \widetilde{CZ} + \widetilde{D}'$ are invertible in $R^{n \times n} \otimes_\mathbb{R} \mathbb{C}$.

- $\text{b) } M\{Z\} := (AZ + B)(CZ + D)^{-1} \in \mathcal{H}_n(R)$.

- $\text{c) } \mathfrak{Im}(M\{Z\}) = (\widetilde{M}\{Z\})^{-1}Y(M\{Z\})^{-1}, \quad \mathfrak{Im}(M\{Z\})^{-1} = Y[C] + Y^{-1}[XC' + D']$.

- $\text{d) } M\{M_1\{Z\}\} = (MM_1)\{Z\}$.

- $\text{e) } (MM_1)\{Z\} = M\{M_1\{Z\}\}M_1\{Z\}$.

- $\text{f) } \mathcal{H}_n(R) \to \mathcal{H}_n(R), \quad Z \mapsto M\{Z\}$ is a biholomorphic function of $\mathcal{H}_n(R)$, and so is $\tau$ on $\mathcal{H}_2(\mathbb{H})$ and on $\mathcal{H}_n(\mathbb{C})$ for $n \geq 2$. (Cf. [Kr85, ch.I, pp.47] for details about holomorphic and biholomorphic quaternionic functions.)

- $\text{g) }$ If $n \geq 2$, $Z \mapsto M\{Z\}$ and $Z \mapsto M_1\{Z\}$ coincide if and only if there exists $\epsilon \in R$, such that $\epsilon$ commutes with all elements of $R$ and $N(\epsilon) = 1$, with $M_1 = \epsilon M$.

- $\text{h) }$ For $R = \mathbb{R}$, $n \geq 1$ or $R = \mathbb{H}$, $n \geq 3$, the group of all biholomorphic functions of $\mathcal{H}_n(R)$ is given by $\{Z \mapsto M\{Z\} ; \ M \in \text{Sp}_n(R)\}$.

- $\text{i) }$ For $R = \mathbb{C}$, $n \geq 2$ or $R = \mathbb{H}$, $n = 2$, the group of all biholomorphic functions of $\mathcal{H}_n(R)$ is given by $\{Z \mapsto \tau^\epsilon(Z) ; \ M \in \text{Sp}_n(R), \ \epsilon = 0, 1\}$.

The maps $Z \mapsto M\{Z\}, \ M \in \text{Sp}_n(R)$ are called symplectic transformations.

Now if we want to define Hermitian modular forms, we have to fix an imaginary quadratic number field and its integers, first. In the case of quaternionic modular forms, we need an order in analogy with the integers of a number field. In this work, we are mainly interested in one specific order, which is, so to say, the Eisenstein integers and a copy of them:
(1.6) **Definition.** Let \( \mathcal{O} := \mathbb{Z} + \mathbb{Z} \frac{1 + i_1 \sqrt{3}}{2} + \mathbb{Z} i_2 + \mathbb{Z} \frac{1 + i_2 \sqrt{3}}{2} \). If not noted differently, writing \( a = a_0 + a_1 \frac{1 + i_1 \sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1 + i_2 \sqrt{3}}{2} \in \mathcal{O} \) will always imply \( a_0, \ldots, a_3 \in \mathbb{Z} \), respectively \( a_0, \ldots, a_3 \in \mathbb{R} \) if \( a \in \mathbb{H} \) in general. Speaking of the coefficients of \( a \) in the standard basis of \( \mathcal{O} \) will always refer to \( a_0, \ldots, a_3 \).

Some important properties of this order are collected in the following proposition. They can all be found in [Kl98, ch.1] and are the analog of those described by Hurwitz about the Hurwitz order \( \mathbb{Z} + \mathbb{Z} i_1 + \mathbb{Z} i_2 + \mathbb{Z} \frac{1 + i_1 + i_2}{2} \) (cf. [Hu19]).

(1.7) **Proposition.**

a) \( \mathcal{O} \) is a maximal order of \( \mathbb{H} \). (Cf. [Kl98, p.4] for a definition.)

b) \( N(a) = a_0^2 + a_0 a_1 + a_1^2 + a_2^2 + a_2 a_3 + a_3^2 \)

for \( a = a_0 + a_1 \frac{1 + i_1 \sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1 + i_2 \sqrt{3}}{2} \in \mathcal{O} \).

c) The unit group of \( \mathcal{O} \) is given by

\[
\mathcal{E} = \{ \pm 1, \frac{1}{2}(\pm 1 \pm i_1 \sqrt{3}), \pm i_2, \frac{1}{2}(\pm 1 \pm i_1 \sqrt{3})i_2 \}.
\]

It has order 12 and is generated by \( \frac{1}{2}(-1 + i_1 \sqrt{3}) \) of order 3 and \( i_2 \) of order 4.

d) \( \mathcal{I}(\mathcal{O}) := \{ 0 \neq a \in \mathcal{O} ; a\mathcal{O} = \mathcal{O}a \} = \{ n, n i_1 \sqrt{3} e ; n \in \mathbb{N}, e \in \mathcal{E} \} \). \( \mathcal{I}(\mathcal{O}) \) is called the set of invariant elements.

e) For all \( a \in \mathbb{H} \) there exists \( g \in \mathcal{O} \) such that \( N(a - g) \leq \frac{7}{8} \), i.e. \( \mathcal{O} \) is euclidean and for all \( a, b \in \mathcal{O} \), \( b \neq 0 \) there exist \( x, y, z, w \in \mathcal{O} \) with \( a = bx + y \) and \( N(y) < N(b) \) and as well \( a = zb + w \) and \( N(w) < N(z) \). (The factor \( \frac{7}{8} \) can even be improved to \( \frac{2}{3} \), but we will get to that in detail later on.)

f) \( \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} \), \( (a, b) \mapsto \text{Re}(\overline{ab}) \) is a positive definite bilinear form. The dual lattice \( \mathcal{O}^\ast \) of \( \mathcal{O} \) with respect to this bilinear form is given by

\[
\mathcal{O}^\ast := \{ a \in \mathbb{H} ; \text{Re}(\overline{ab}) \in \mathbb{Z} \vee b \in \mathcal{O} \} = \frac{2}{3} i_1 \sqrt{3} \mathcal{O} = \mathcal{O} \frac{2}{3} i_1 \sqrt{3}
\]

\[
= \mathbb{Z}(1 + \frac{1}{3} i_1 \sqrt{3}) + \mathbb{Z} \frac{2}{3} i_1 \sqrt{3} + \mathbb{Z}(1 + \frac{1}{3} i_2 \sqrt{3})i_2 + \mathbb{Z} \frac{2}{3} i_1 \sqrt{3}i_2.
\]

Furthermore \( \text{vol} (\mathcal{O}^\ast) = \text{vol} (\mathcal{O})^{-1} = \frac{4}{3} \).

g) \( \# \{ a \in \mathcal{O} ; N(a) = m \} = 12 \sum_{d|m, 3|d} d \). In particular, for every \( m \in \mathbb{N}_0 \) there exists \( a \in \mathcal{O} \) such that \( N(a) = m \).

h) For \( a \in \mathcal{O} \) with \( N(a) = \prod_{j=1}^{n} p_j \) where the \( p_j \) are primes, there exist \( a_j \in \mathcal{O} \) with \( N(a_j) = p_j \) and \( a = a_1 \cdot \ldots \cdot a_n \).

For later purposes, we need a special two-sided ideal in \( \mathcal{O} \). According to the proposition above, it is indeed a two-sided ideal.
(1.8) Definition. \( p := i_1 \sqrt{3} \mathcal{O} = \mathcal{O} i_1 \sqrt{3} \subseteq \mathcal{O}. \)

Note that \( p \) being a two-sided ideal also implies \( -i_1 \mathcal{O} i_1 = \mathcal{O}. \)

(1.9) Proposition. \( p = \{ a \in \mathcal{O} \mid N(a) \in 3 \mathbb{Z} \}. \)

Proof: Suppose \( a \in p \), then \( a = i_1 \sqrt{3} b \) for some \( b \in \mathcal{O} \), hence \( N(a) = 3 N(b) \in 3 \mathbb{Z} \). The other way around, suppose \( a \in \mathcal{O} \), \( N(a) = 3m \), \( m \in \mathbb{N}_0 \). Of course, the assertion immediately follows in view of (1.7): There are \( q, b \in \mathcal{O} \) with \( N(q) = 3 \) and \( a = qb \). According to (1.7) again, there are exactly 12 elements in \( \mathcal{O} \) of norm three. Of course, these are given by \( i_1 \sqrt{3} \epsilon, \epsilon \in \mathcal{E} \) as \( \# \mathcal{E} = 12 \). So \( a = i_1 \sqrt{3} b \in p \). But since this assertion is so important later on, let us give a complete proof, since there is no actual proof in [Kl98] that (1.7) holds, and the reader is only referred to [Hu19], where the assertion is verified for the Hurwitz order, only. (Of course, the assertion could be verified in an analogous way, but it simply was not done so far. Furthermore, in chapter 4, once we have analyzed some further number theoretical background of \( \mathcal{O} \), we will give a full proof of (1.7)h), since it is so important.)

So suppose there is no \( b \in \mathcal{O} \) satisfying \( a = i_1 \sqrt{3} b \). Let \( I \) denote the right-sided ideal generated by \( i_1 \sqrt{3} \) and \( a \), i.e.

\[
I = \{ i_1 \sqrt{3} u + av \mid u, v \in \mathcal{O} \}.
\]

Obviously, \( I \) is a right-sided ideal, indeed. Let \( 0 \neq c \in I \) denote an element with smallest possible norm in \( I \), which means \( N(c) = \min \{ N(b) \mid 0 \neq b \in I \} \). Of course we have \( c \mathcal{O} \subseteq I \). Now, let \( 0 \neq b \in I \). According to (1.7), there exist \( x, y \in \mathcal{O} \) with \( b - cx = y \in I \) and \( N(y) < N(c) \). Since the norm of \( c \) is minimal in \( I \), \( y = 0 \) follows. This proves \( I = c \mathcal{O} \). Since \( i_1 \sqrt{3} \in \mathcal{O} \) with \( N(i_1 \sqrt{3}) = 3 \), \( c \) has to fulfill \( N(c) \in \{ 1, 2, 3 \} \). \( N(c) = 2 \) would be a contradiction, because otherwise every element in \( I \) would have a norm divisible by 2, but \( N(i_1 \sqrt{3}) = 3 \). But also \( N(c) = 3 \) would be a contradiction: According to (1.7) (whereas one could even check it manually, since there can only exist finitely many elements in \( \mathcal{O} \) of a given norm), all elements of norm 3 are given by \( i_1 \sqrt{3} \epsilon, \epsilon \in \mathcal{E} \), because there are exactly 12 elements of norm 3 and \( \# \mathcal{E} = 12 \). But then \( I = i_1 \sqrt{3} \mathcal{O} \) follows, which contradicts \( a \in I \) and \( a \notin i_1 \sqrt{3} \mathcal{O} \). Hence we obtain \( I = \mathcal{O} \).

And thus, by definition, there exist \( x, y \in \mathcal{O} \) such that

\[
1 = i_1 \sqrt{3} x + ay,
\]

or in other words \( ay = 1 - i_1 \sqrt{3} x \). This gives

\[
N(a) N(y) = N(1 - i_1 \sqrt{3} x) = 1 + 3 N(x) - i_1 \sqrt{3} x - (-\bar{x} i_1 \sqrt{3}) = 1 + 3 N(x) + i_1 \sqrt{3} (-x + y)
\]

for some \( y \in \mathcal{O} \), since \( i_1 \sqrt{3} \in \mathcal{I}(\mathcal{O}) \) in virtue of (1.7). Note that this is already a contradiction. According to the assumption, \( N(a) = 3m = (i_1 \sqrt{3})(-i_1 \sqrt{3})m \) holds, and thus

\[
1 = (i_1 \sqrt{3})(-i_1 \sqrt{3}) N(y) - (i_1 \sqrt{3})(-i_1 \sqrt{3}) N(x) - i_1 \sqrt{3} (-x + y) \in i_1 \sqrt{3} \mathcal{O},
\]

but \( 1 \notin i_1 \sqrt{3} \mathcal{O} \), since we have already seen that every element in \( p \) has a norm divisible by 3. Hence the assertion follows. \( \square \)
(1.10) Lemma. $\mathcal{O}/p \simeq \mathbb{F}_9$, where $\mathcal{O}/p$ is the factor ring and $\mathbb{F}_9$ is the field with nine elements.

Proof: To keep it clearly arranged, for now we will simply write $a \equiv b$ for $a \equiv b \mod p$, i.e. $a + p = b + p$. For $a_0 + a_1 \frac{1 + i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1 + i\sqrt{3}}{2} i_2 \in \mathcal{O}$, one calculates

$$i_1 \sqrt{3}(a_0 + a_1 \frac{1 + i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1 + i\sqrt{3}}{2} i_2) = (-a_0 - 2a_1) + (2a_0 + a_1) \frac{1 + i\sqrt{3}}{2} + (-a_2 - 2a_3) i_2 + (2a_2 + a_3) \frac{1 + i\sqrt{3}}{2} i_2.$$ 

So we have $3, -2 + \frac{1 + i\sqrt{3}}{2}, 3i_2, -2i_2 + \frac{1 + i\sqrt{3}}{2} i_2 \in p$. By simply adding multiples of these, we see that for $a \in \mathcal{O}$

$$a \equiv n + mi_2 \quad \text{where } n, m \in \{0, 1, -1\}.$$ 

As $p$ is a two-sided ideal, $\mathcal{O}/p$ is a (possibly non-commutative) unitary ring, which is a simple fact from linear algebra. Because of the observation above, $ab \equiv (n_a + m_ai_2)(n_b + m_bi_2) = (n_b + m_b i_2)(n_a + m_ai_2) \equiv ba$ for $a, b \in \mathcal{O}$ (with appropriate $n_a, m_a, n_b, m_b \in \mathbb{Z}$), so that $\mathcal{O}/p$ is also commutative. Furthermore, let $a \in \mathcal{O}$ with $a \equiv n + mi_2 \neq 0$, $n, m \in \{0, 1, -1\}$. If $n = 0$ or $m = 0$, then $a(n - mi_2) \equiv (n + mi_2)(n - mi_2) = 1$. If $n, m \neq 0$, then $a(-n + mi_2) \equiv (n + mi_2)(-n + mi_2) = -2 \equiv 1$. So every $a \in \mathcal{O}$, $a \neq 0$, has a multiplicative inverse mod $p$. Putting it all together, $\mathcal{O}/p$ is a field. So we only have to show $|\mathcal{O}/p| = 9$, i.e. $M = \{n + mi_2 ; n, m \in \{0, 1, -1\}\}$ is a transversal, as there exists (up to isomorphy) only one field of order 9.

For $i_1 \sqrt{3}a \in p$ (with $a \in \mathcal{O}$), $N(i_1 \sqrt{3}a) = 3 N(a) \in 3\mathbb{Z}$. As one can easily check, $N(n + mi_2) = n^2 + m^2 \in 3\mathbb{Z}$ for $n, m \in \mathbb{Z}$ if and only if $n, m \in 3\mathbb{Z}$. So $a - b \not\equiv p$, i.e. $a \neq b$, for all $a, b \in M$, $a \neq b$, and $M$ is a transversal of $\mathcal{O}/p$. 

\( \square \)

(1.11) Remark. As we will need them later on, let us have a closer look at the congruences mod $p$. We have

$$\begin{align*}
\pm 1 & \equiv \pm 1 \\
\frac{1 \pm i_1 \sqrt{3}}{2} & \equiv -1 \\
\frac{-1 \pm i_1 \sqrt{3}}{2} & \equiv 1 \\
\pm i_2 & \equiv \pm i_2 \\
\frac{1 \pm i_1 \sqrt{3}}{2} i_2 & \equiv -i_2 \\
\frac{-1 \pm i_1 \sqrt{3}}{2} i_2 & \equiv i_2
\end{align*}$$

which directly leads to

$$a_0 + a_1 \frac{1 + i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1 + i\sqrt{3}}{2} i_2 \equiv (a_0 - a_1 \mod 3) + (a_2 - a_3 \mod 3)i_2.$$ 

Furthermore, note that $\mathbb{F}_9$ can be realized as $\mathbb{F}_3[X]/(X^2 + 1)$, so $\mathbb{F}_9 = \mathbb{F}_3 + \omega \mathbb{F}_3$ with $\omega^2 = -1$. The Frobenius automorphism is the unique, non-trivial Galois automorphism of $\mathbb{F}_9/\mathbb{F}_3$. It is an involution given by $\phi_3 : a + b\omega \mapsto a - b\omega$. Because of (1.10) and the congruences above, one can
easily check that
\[ \pi_p : \mathcal{O} / \mathfrak{p} \to \mathbb{F}_9, \quad (n + mi_2) + \mathfrak{p} \mapsto n + m\omega \]
is a canonical isomorphism (where \( n, m \) are identified accordingly in \( \mathbb{F}_9 \)), as well as
\[ \pi_p(\bar{a} + \mathfrak{p}) = \phi_3(\pi_p(a + \mathfrak{p})). \]
By abuse of notation, we will also write \( \pi_p : \mathcal{O} \to \mathbb{F}_9, \ a \mapsto \pi_p(a + \mathfrak{p}) \). This, of course, is a ring homomorphism with \( \pi_p(\bar{a}) = \phi_3(\pi_p(a)) \).

Finally, we can define the quaternionic modular group, the extended quaternionic modular group, and some special subgroups.

(1.12) Definition.  
a) \( \text{Sp}_n(\mathcal{O}) = \text{Sp}_n(\mathbb{H}) \cap \mathcal{O}^{2n \times 2n} \) is called the quaternionic modular group of degree \( n \) (with respect to \( \mathcal{O} \)). Note that \( \text{Sp}_n(\mathcal{O}) \subset \text{GL}_{2n}(\mathcal{O}) \) because of (1.3).
b) \( \Gamma(\mathcal{O}) := \langle Z \mapsto M(Z), \ \tau ; \ M \in \text{Sp}_2(\mathcal{O}) \ or \ M = i_1I_4 \rangle \) is called the extended quaternionic modular group (of degree 2 with respect to \( \mathcal{O} \)).
c) Let \( P \triangleleft \mathcal{O} \) be a two-sided ideal of \( \mathcal{O} \). The principal congruence subgroup of level \( P \) is defined as \( \text{Sp}_n(\mathcal{O})[P] := \{ M \in \text{Sp}_n(\mathcal{O}) ; \ M \equiv I \ mod \ P \} \). A subgroup of \( \text{Sp}_n(\mathcal{O}) \) is called a congruence subgroup of level \( P \) if it contains \( \text{Sp}_n(\mathcal{O})[P] \).
d) A special congruence subgroup of level \( P \) is given by the so-called theta-group \( \text{Sp}_n(\mathcal{O})[P]_0 := \{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathcal{O}) ; \ C \equiv 0 \ mod \ P \} \) of level \( P \).

As we will see in (1.20), \( \Gamma(\mathcal{O}) \) it is an extension of \( \text{Sp}_2(\mathcal{O}) \) of order four; or to be more precise an extension of \( \{ Z \mapsto \langle Z \rangle ; \ M \in \text{Sp}_2(\mathcal{O}) \} \). From now, we will also write \( \text{Sp}_2(\mathcal{O}) \) for this group of automorphisms, as it will always be clear from the context whether the modular group or its associated group of symplectic transformations is meant.

Another maximal, euclidean order was analyzed by A. Krieg in [Kr85] (and of course the quaternionic modular group and the quaternionic modular forms attached to it), namely the Hurwitz order \( \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}^{1+i+h+n} \). Many of his results on this order as well as results concerning the quaternionic modular group with respect to this order are only based on the euclidean property, like the determination of generators, so they can be carried over to our setting without the need to prove them again, since the proofs would be completely analogous. So we will just cite the original results. Some of them can already be found in [Kl98].

(1.13) Proposition. [Kr85, ch.I, prop.1.9]. Every left respectively right ideal \( P \) of \( \mathcal{O} \) is principal, i.e. it is generated by a single element, i.e. \( P = \mathcal{O}a \) respectively \( P = a\mathcal{O} \). Each two-sided ideal of \( \mathcal{O} \) is generated by an invariant element (see (1.7)) or 0.

(1.14) Theorem. [Kr85, ch.I, thm.2.2]. Given \( n > 1 \) the unimodular group \( \text{GL}_n(\mathcal{O}) \) is generated by the matrices \( I_n + I_n, \ \text{diag}(\varepsilon, 1, \ldots, 1) \), where \( \varepsilon \in \mathcal{E} \) and the permutation matrices \( P_\pi := (\varepsilon_{\pi(1)}, \ldots, \varepsilon_{\pi(n)}) \), where \( \pi \in S_n \) is a permutation of \( n \) elements.
1.1 The quaternionic half-space and the quaternionic modular group

(1.15) Theorem. [Kr85, ch.II, thm.2.3] The modular group $\text{Sp}_n(O)$ is generated by the matrices

$$J_n, \quad \text{Trans}(S), \quad S \in \text{Her}_n(O), \quad \text{Rot}(U), \quad U \in \text{GL}_n(O).$$

Thus we also have $\text{Sp}_1(O) = \{\varepsilon M ; \varepsilon \in E, M \in \text{SL}_2(Z)\}$. Next we remind of the so-called Hua’s identity which can be verified by a simple calculation.

(1.16) Lemma. Given $n \in \mathbb{N}$ and $S \in \text{Her}_n(H) \cap \text{GL}_n(H)$, the following identity holds true:

$$\text{Trans}(S) \cdot J_n \cdot \text{Trans}(S^{-1}) \cdot J_n \cdot \text{Trans}(S) \cdot J_n^{-1} = \text{Rot}(S).$$

This identity is called Hua’s identity.

Because of Hua’s identity we also get a finite set of generators for $\text{Sp}_n(O)$. $P_{\pi_1}P_{\pi_2} = P_{\pi_1 \circ \pi_2}$ is well known. $S_n$ is generated by transpositions. The associated permutation matrices are then symmetric (hence also hermitian). Because of $\left(\frac{5}{3} \frac{3}{2} \frac{2}{1}\right)\left(\frac{1}{3} \frac{1}{2} \frac{1}{1}\right) = \left(\frac{1}{1} \frac{1}{1} \frac{1}{1}\right)$, $I_n + I_{1,n}$ is a product of symmetric (hermitian) matrices. Thus, because of (1.14), $\text{GL}_n(O)$ is generated by hermitian matrices and $\text{diag}(\varepsilon, 1, \ldots, 1)$, where $\varepsilon \in E$. Clearly one has $\text{Trans}(S_1) \text{Trans}(S_2) = \text{Trans}(S_1 + S_2)$ and $\text{Trans}(S)^{-1} = \text{Trans}(-S)$. Then Hua’s identity, (1.15), (1.7) and the special structure of hermitian matrices particularly lead to

(1.17) Corollary. The modular group $\text{Sp}_2(O)$ is generated by the matrices

$$J, \quad \text{Trans}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad \text{Trans}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad \text{Trans}\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right), \quad \text{Rot}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

with $a \in \left\{1, \frac{1+i\sqrt{3}}{2}, i_2, \frac{1+i\sqrt{3}}{2}, i_2\right\}$ and $\varepsilon \in \left\{-\frac{1+i\sqrt{3}}{2}, i_2\right\}$.

Furthermore, according to [Kl98, Kor.1.22] (or just as an easy consequence of (1.17)), we also have the following

(1.18) Corollary. $\text{Sp}_1(O) = \{\varepsilon M ; \varepsilon \in E, M \in \text{SL}_2(Z)\}$.

Now that we have all the necessary definitions and already known properties, we can come to the first new result concerning the abelian characters of the quaternionic modular and extended modular group. Abelian characters are group homomorphisms $\nu : \Lambda \to \mathbb{C}^*$ for subgroups $\Lambda \leq \text{Sp}_p(O)$ (or $\Lambda \leq \Gamma(O)$). This leads to $\nu(\Lambda) \subset S^1 := \{z \in \mathbb{C} ; |z| = 1\}$ if $\#\nu(\Lambda) < \infty$, which is fulfilled if the commutator factor group $\Lambda/\Lambda'$ is finite, in particular. Again, we are only interested in the case of degree two. First, we need to define a special abelian character. It is the only non-trivial one for $\text{Sp}_2(O)$, which we will see in (1.20).

As we described in (1.10), $O/p \simeq \mathbb{F}_9$. Thus, again by abuse of notation, with $\pi_p$ from (1.11) we get a canonical homomorphism $\pi_p : \text{Sp}_2(O) \to \mathbb{F}_9^{4 \times 4}$, $A = (a_{ij,k}) \mapsto \pi_p(A) := (\pi_p(a_{ij,k}))$. In $\mathbb{F}_9^{4 \times 4}$, the determinant is defined (which in general is not the case in $\text{Sp}_2(O)$, as $H$ is not commutative). So $\det \circ \pi_p$ clearly is a group homomorphism $\text{Sp}_2(O) \to \mathbb{F}_9$. Now, with (1.17) and the explicit identification in (1.11), one easily verifies $\det \circ \pi_p(M) = \pm 1 \in \mathbb{F}_9$ for all $M \in \text{Sp}_2(O)$. By simply identifying $\pm 1$ in $S^1$, we can define the following abelian character:
(1.19) Definition. 

\[ \nu_{\text{det}} : \text{Sp}_2(\mathcal{O}) \to \{ \pm 1 \}, \ M \mapsto \det \circ \pi_p(M) \]

is an abelian character of \( \text{Sp}_2(\mathcal{O}) \) of order 2.

We will now determine the group of abelian characters for \( \text{Sp}_2(\mathcal{O}) \) and \( \Gamma(\mathcal{O}) \). Here \( C_n \) denotes the cyclic group of order \( n \).

(1.20) Theorem. 

a) \( \text{Sp}_2(\mathcal{O}) \leq \Gamma(\mathcal{O}) \) is an extension of order 4.

b) \( \text{Sp}_2(\mathcal{O})^{\text{ab}} = \langle \nu_{\text{det}} \rangle \cong C_2 \).

c) \( \Gamma(\mathcal{O})^{\text{ab}} = \langle \nu_{\text{det}}, \nu_i, \nu_\tau \rangle \cong C_2 \times C_2 \times C_2 \) where

\[
\begin{align*}
\nu_{\text{det}}(i_1 I) &:= \nu_{\text{det}}(\tau) := 1, \\
\nu_i|_{\text{Sp}_2(\mathcal{O})} &:= 1, \nu_i(i_1 I) := -1, \nu_i(\tau) := 1, \\
\nu_\tau|_{\text{Sp}_2(\mathcal{O})} &:= 1, \nu_\tau(i_1 I) := 1, \nu_\tau(\tau) := -1.
\end{align*}
\]

Proof: 
a) Of course, \( i_1 I \not\in \text{Sp}_2(\mathcal{O}) \), but there might exist \( M \in \text{Sp}_2(\mathcal{O}) \) such that \( M(\cdot) = i_1 I(\cdot) \). So let us assume that this is the case. Then, due to (1.5) \( M = \varepsilon(i_1 I) \) with \( N(\varepsilon) = 1 \) and \( \varepsilon \) lies in the center of \( \mathbb{H} \), so we get \( M = \pm i_1 I \), which is a contradiction to \( \pm i_1 I \not\in \text{Sp}_2(\mathcal{O}) \). So \( \Lambda := \langle \text{Sp}_2(\mathcal{O}), i_1 I \rangle \) is a non-trivial expansion of the associated group of symplectic transformations of \( \text{Sp}_2(\mathcal{O}) \).

Next, note that \( \phi_i : \mathcal{O} \to \mathcal{O}, a \mapsto -i_1 a_i \) is a bijection with \( \phi_i(a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2) = a_0 + a_1 \frac{1+i\sqrt{3}}{2} - a_2 i_2 - a_3 \frac{1+i\sqrt{3}}{2} i_2 \) and \( \phi_i(\bar{a}) = \phi_i(a) \). For the generators in (1.17) we have

\[
\begin{align*}
(-i_1 I)J(i_1 I) &= J \\
(-i_1 I)\text{Trans} \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) (i_1 I) &= \text{Trans} \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) \\
(-i_1 I)\text{Trans} \left( \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) (i_1 I) &= \text{Trans} \left( \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) \\
(-i_1 I)\text{Rot} \left( \left( \begin{array}{c} \varepsilon \\ 0 \end{array} \right) \right) (i_1 I) &= \text{Rot} \left( \left( \begin{array}{c} \phi_i(\varepsilon) \\ 0 \end{array} \right) \right) \\
\end{align*}
\]

(1.2)

Note that \( i_1 I \in \text{Sp}_2(\mathbb{H}) \), and thus we clearly have

\[ i_1 I \text{Sp}_2(\mathcal{O}) = \text{Sp}_2(\mathcal{O})i_1 I. \]

(To be exact, we have, by abuse of notation, \((i_1 I) M = \phi_i(M)(i_1 I) \).) By definition, every element of \( \Lambda \) can be written as \((i_1 I)^{n_0} M_1 (i_1 I)^{n_1} \ldots M_m (i_1 I)^{n_m} \) for some appropriate \( m \in \mathbb{N} \), \( n_0, \ldots, n_m \in \mathbb{Z} \), \( M_1, \ldots, M_m \in \text{Sp}_2(\mathcal{O}) \), and is, according to the observation above, equal to \((i_1 I)^{n_0 + \ldots + n_m} M \) for some \( M \in \text{Sp}_2(\mathcal{O}) \). Moreover, as \((i_1 I)^2 = -I \), this is equal to \((i_1 I)\varepsilon(\pm M) \) with \( \pm M \in \text{Sp}_2(\mathcal{O}) \) and \( \varepsilon \in \{0,1\} \). This shows that \( \Lambda / \text{Sp}_2(\mathcal{O}) = \{ i_1 I \text{Sp}_2(\mathcal{O}), \text{Sp}_2(\mathcal{O}) \} \) and \( \Lambda \) is an extension of \( \text{Sp}_2(\mathcal{O}) \) of order two (seen both as subgroups of \( \text{Sp}_2(\mathbb{H}) \) or as the
associated groups of symplectic transformations).

If we see $\Lambda$ as a subgroup of $\text{Sp}_2(\mathbb{H})$, then according to [Kr85, ch.II, thm.1.8] there is no $M \in \Lambda$ with $M \cdot \cdot \cdot \equiv \tau$ on $\mathcal{H}(\mathbb{H})$. So, seen as a group of automorphisms, $\Gamma(\mathcal{O})$ is a non-trivial extension of $\Lambda$. Some calculations show that for $Z \in \mathcal{H}(\mathbb{H})$, the generators in (1.17) and $i_1I$ we have

$$
\tau(I(Z)) = J(\tau(Z))
$$

$$
\tau(\text{Trans}((1 \ 0) (Z)) = \text{Trans}((1 \ 0))(\tau(Z))
$$

$$
\tau(\text{Trans}((0 \ 0) (Z)) = \text{Trans}((0 \ 0)))(\tau(Z))
$$

$$
\tau(\text{Rot}((\epsilon \ 0) (Z)) = \text{Rot}((1 \ 0))(\tau(Z))
$$

$$
\tau(i_1I(Z)) = i_1I(\tau(Z))
$$

and according to (1.14), (1.16), (1.17), the matrices occuring on the right-hand side generate $\Lambda$, too. So, again seen as a group of automorphisms, we have

$$
\tau \circ \Lambda = \Lambda \circ \tau.
$$

Since $\tau \circ \tau$ is the identity, the same argument as above leads to $|\Gamma(\mathcal{O})/\Lambda| = 2$, hence $\Gamma(\mathcal{O})$ is an extension of $\text{Sp}_2(\mathcal{O})$ of order 4.

By making no difference in the notation of elements in $M \in \text{Sp}_2(\mathbb{H})$ and their associated symplectic transformation $M \cdot \cdot \cdot$, again, we summarize

$$
\Gamma(\mathcal{O}) = \text{Sp}_2(\mathcal{O}) \cup i_1I\text{Sp}_2(\mathcal{O}) \cup \tau \circ \text{Sp}_2(\mathcal{O}) \cup \tau \circ i_1I\text{Sp}_2(\mathcal{O})
$$

Furthermore, note that due to the same arguments as above, $\tilde{\Lambda} = \langle \tau, \text{Sp}_2(\mathcal{O}) \rangle$ is an extension of $\text{Sp}_2(\mathcal{O})$ of order two, and $\Gamma(\mathcal{O})$ is an extension of $\tilde{\Lambda}$ of order two, as well.

b) We will now take a closer look at the commutator subgroup $\text{Sp}_2(\mathcal{O})'$ of $\text{Sp}_2(\mathcal{O})$. We will show that $|\text{Sp}_2(\mathcal{O})'\text{ab}| = |\text{Sp}_2(\mathcal{O})/\text{Sp}_2(\mathcal{O})'| \leq 2$ holds true. Now $\nu_{\text{det}}$ is a nontrivial character of order 2, and this finally leads to $\text{Sp}_2(\mathcal{O})'\text{ab} = \{\text{id}, \nu_{\text{det}}\} = \langle \nu_{\text{det}} \rangle$.

The ideas to prove the claim $|\text{Sp}_2(\mathcal{O})/\text{Sp}_2(\mathcal{O})'| \leq 2$ can already be found in [Ma64, Le.1], and as well in [De98], where the Siegel respectively Hermitian cases are considered.

First, we construct some basic elements of $\text{Sp}_2(\mathcal{O})'$. For $U \in \text{GL}_2(\mathcal{O})$ and $S \in \text{Her}_2(\mathcal{O})$ one easily calculates

$$
[\text{Rot}(U), \text{Trans}(S)] = \text{Trans}(T) \quad \text{with} \quad T = S[U] - S
$$

and we get

$$
T = \begin{pmatrix}
N(a) & a \\
\bar{a} & 0
\end{pmatrix}
$$

for $U = \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$$
T = \begin{pmatrix}
n & 0 \\
0 & -n
\end{pmatrix}
$$

for $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}$,
with \( n \in \mathbb{Z}, a \in \mathcal{O} \). Since \( 2 \cdot \text{Re}(\frac{1+i\sqrt{3}}{2}) = 1 \), we obtain \( \text{Trans}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}), \text{Trans}(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \), \( \text{Trans}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) \) \( \in \text{Sp}_2(\mathcal{O})' \) \( (a \in \mathcal{O}) \). The arguments of \( \text{Trans}(-) \) are generators of \( \text{Her}_2(\mathcal{O}) \), hence \( \text{Trans}(S) \in \text{Sp}_2(\mathcal{O})' \) for all \( S \in \text{Her}_2(\mathcal{O}) \).

Because of Hua’s identity (1.16) and the fact that \( \text{Sp}_2(\mathcal{O})' / \text{Sp}_2(\mathcal{O})' \) is abelian,

\[
-J\text{Sp}_2(\mathcal{O})' = -\text{Trans}(I)J\text{Trans}(I)\text{Trans}(I)\text{Sp}_2(\mathcal{O})'
= -J^2\text{Sp}_2(\mathcal{O})' = -(I_4)\text{Sp}_2(\mathcal{O})' = \text{Sp}_2(\mathcal{O})'
\]

and thus \(-J \in \text{Sp}_2(\mathcal{O})' \) follows, as well as \( J \in \text{Sp}_2(\mathcal{O})' \) because of \(-I_4 = (-I)(-J) \in \text{Sp}_2(\mathcal{O})' \). So again, deriving from Hua’s identity and because of the same arguments that led to (1.17), we also get \( \text{Rot}(U) \in \text{Sp}_2(\mathcal{O})' \) for all \( U \in \text{GL}_2(\mathbb{Z}) \) as well as all \( U \in \text{Her}_2(\mathcal{O}) \cap \text{GL}_2(\mathcal{O}) \).

Another easy calculation yields

\[
\left[ \text{Rot} \left( \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \right), \text{Rot} \left( \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \right) \right] = \text{Rot} \left( \begin{pmatrix} 1 & \varepsilon - 1 \\ 0 & 1 \end{pmatrix} \right)
\]

for all \( \varepsilon \in \mathcal{E} \). Multiplying with \( \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \) \( \in \text{Sp}_2(\mathcal{O})' \) leads to \( \text{Rot}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \) \( \in \text{Sp}_2(\mathcal{O})' \) for all \( \varepsilon \in \mathcal{E} \). As \( \mathcal{O} \) is (additively) generated by \( \mathcal{E} \) and by again multiplying with \( \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \) \( \in \text{Sp}_2(\mathcal{O})' \) from both sides leads to

\[
\text{Rot} \left( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right), \text{Rot} \left( \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right) \in \text{Sp}_2(\mathcal{O})' \quad \text{for all} \ a \in \mathcal{O}.
\]

\( \mathcal{O} \) is euclidean in virtue of (1.7). So by applying the Euclidean Algorithm and multiplying with appropriate matrices from above, every \( \text{Rot}(U), U \in \text{GL}_2(\mathcal{O}) \) is congruent to \( \text{Rot}(\begin{pmatrix} \varepsilon & 0 \\ 0 & \delta \end{pmatrix}) \) modulo \( \text{Sp}_2(\mathcal{O})' \) for some \( \varepsilon, \delta \in \mathcal{E} \). As we have seen above, \( \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \) \( \in \text{Sp}_2(\mathcal{O})' \) and therefore

\[
\text{Rot} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \cdot \text{Rot} \left( \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \right) = \text{Rot} \left( \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \right) \in \text{Sp}_2(\mathcal{O})' \quad \text{for all} \ \delta \in \mathcal{E}.
\]

So we conclude that every \( \text{Rot}(U), U \in \text{GL}_2(\mathcal{O}) \) is congruent to \( \text{Rot}(\begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}) \) modulo \( \text{Sp}_2(\mathcal{O})' \) for some \( \varepsilon \in \mathcal{E} \). By multiplying this with \( \text{Rot}(\begin{pmatrix} \varepsilon & 0 \\ 0 & \delta \end{pmatrix}) \) from both sides (remember that \( \text{Sp}_2(\mathcal{O})' / \text{Sp}_2(\mathcal{O})' \) is abelian) \( \varepsilon \) can be chosen modulo the action \( \varepsilon \mapsto \delta \varepsilon \varepsilon, \delta \in \mathcal{E} \).

One easily checks \( (\frac{1+i\sqrt{3}}{2})^3 = -1, (\frac{1-i\sqrt{3}}{2})^3 = 1, (\frac{1+i\sqrt{3}}{2})(\frac{1+i\sqrt{3}}{2})(\frac{1+i\sqrt{3}}{2}) = i_2, (\frac{1+i\sqrt{3}}{2})(\frac{1+i\sqrt{3}}{2})(\frac{1+i\sqrt{3}}{2}) = -i_2 \). As \( \text{Rot}(\begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}) \in \text{Sp}_2(\mathcal{O})' \), we can finally conclude that we can choose \( \varepsilon \in \{1, i_2\} \) and we have shown

\[
\text{Sp}_2(\mathcal{O}) = \text{Sp}_2(\mathcal{O})' \cup \text{Rot}(\begin{pmatrix} i_2 & 0 \\ 0 & 1 \end{pmatrix}) \text{Sp}_2(\mathcal{O})'.
\]
(where initially the union does not have to be disjoint). So $|\text{Sp}_2(\mathcal{O})/\text{Sp}_2(\mathcal{O})'| \leq 2$, and even $|\text{Sp}_2(\mathcal{O})/\text{Sp}_2(\mathcal{O})'| = 2$ (and the union actually is disjoint), when taking the statement from the beginning into account.

c) Again, for simplicity, we will write $M$ instead of $M\langle \cdot \rangle$ if we mean the associated automorphism of $M \in \text{Sp}_2(\mathbb{H})$. In a) we saw that

$$\Gamma(\mathcal{O}) = \text{Sp}_2(\mathcal{O}) \cup i_1 I \text{Sp}_2(\mathcal{O}) \cup \tau \circ \text{Sp}_2(\mathcal{O}) \cup \tau \circ i_1 I \text{Sp}_2(\mathcal{O})$$

and $\Lambda = \langle \text{Sp}_2(\mathcal{O}), i_1 I \rangle$ as well as $\tilde{\Lambda} = \langle \tau, \text{Sp}_2(\mathcal{O}) \rangle$ are subgroups of $\Gamma(\mathcal{O})$ of index two. It is a well known fact that $\Lambda$ and $\tilde{\Lambda}$ hence are normal subgroups and $\Gamma(\mathcal{O})/\Lambda \simeq \langle i_1 I \rangle \simeq C_2$, $\Gamma(\mathcal{O})/\tilde{\Lambda} \simeq \langle \tau \rangle \simeq C_2$. $C_2$ has exactly one non-trivial abelian character, mapping its single generator to $-1$. So by definition and a) we have to show

$$\tau \circ (i_1 I)^{\varepsilon_1} \cdot M = (-1)^{\varepsilon_2} \quad \text{and} \quad \tau \circ (i_1 I)^{\varepsilon_2} = (-1)^{\varepsilon_1}.$$

As both $i_1 I$ and $\tau$ are of order two, we already have $|\Gamma(\mathcal{O})^{ab}| \geq 4$. Furthermore, as $\text{Sp}_2(\mathcal{O})' \leq \Gamma(\mathcal{O})'$, $|\text{Sp}_2(\mathcal{O})/\text{Sp}_2(\mathcal{O})'| = 2$ and $|\Gamma(\mathcal{O})/\text{Sp}_2(\mathcal{O})| = 4$, $|\Gamma(\mathcal{O})^{ab}| \leq 8$ follows. Now $v_{\det}$ is an abelian character of $\text{Sp}_2(\mathcal{O})$ (with $\text{Sp}_2(\mathcal{O})$ a matrix group). $v_{\det}(-I) = 1$ and according to (1.5) the associated group of automorphisms $\{M\langle \cdot \rangle : M \in \text{Sp}_2(\mathcal{O})\}$ is isomorphic to $\text{Sp}_2(\mathcal{O})/\{\pm I\}$. As $\{\pm I\} \subset \ker(v_{\det})$, $v_{\det}$ factors through $\text{Sp}_2(\mathcal{O})/\{\pm I\}$ and is therefore a well defined homomorphism of the associated group of automorphisms.

Again, for simplicity, we will just write $v_{\det}(M)$ for an element $M\langle \cdot \rangle$.

According to a) every element of $\Gamma(\mathcal{O})$ can be written uniquely as $\tau^{\varepsilon_1} \circ (i_1 I)^{\varepsilon_2} \cdot M$ with $\varepsilon_1, \varepsilon_2 \in \{0,1\}, M \in \text{Sp}_2(\mathcal{O})$. (That is, if we see $M$ as an automorphism. If seen as a matrix, $M$ is only unique up to multiplication with $-I$.) So the following map is well defined (and again denoted by $v_{\det}$ to keep it simple):

$$v_{\det} : \Gamma(\mathcal{O}) \rightarrow \{\pm 1\}, \quad \tau^{\varepsilon_1} \circ (i_1 I)^{\varepsilon_2} \cdot M \mapsto v_{\det}(M)$$

Of course, $v_{\det}$ is not necessarily a homomorphism. We still have to prove

$$v_{\det}(\tau^{\varepsilon_1} \circ (i_1 I)^{\varepsilon_1} \cdot M_1) \cdot v_{\det}(\tau^{\varepsilon_2} \circ (i_1 I)^{\varepsilon_2} \cdot M_2) = v_{\det}((\tau^{\varepsilon_1} \circ (i_1 I)^{\varepsilon_1} \cdot M_1) \circ (\tau^{\varepsilon_2} \circ (i_1 I)^{\varepsilon_2} \cdot M_2)).$$

So by definition and a) we have to show

$$v_{\det}(M_1) \cdot v_{\det}(M_2) = v_{\det}((\tau^{\varepsilon_1+\varepsilon_2} \circ (i_1 I)^{\varepsilon_1+\varepsilon_2} \cdot (((i_1 I)^{-\varepsilon_2} \circ \tau^{\varepsilon_2}) \circ M_1 \circ (\tau^{\varepsilon_2} \circ (i_1 I)^{\varepsilon_2}) \cdot M_2)))$$

$$= v_{\det}(((i_1 I)^{-\varepsilon_2} \circ \tau^{\varepsilon_2}) \circ M_1 \circ (\tau^{\varepsilon_2} \circ (i_1 I)^{\varepsilon_2}) \cdot M_2)$$

$$= v_{\det}(((i_1 I)^{-\varepsilon_2} \circ \tau^{\varepsilon_2}) \circ M_1 \circ (\tau^{\varepsilon_2} \circ (i_1 I)^{\varepsilon_2}) \cdot v_{\det}(M_2))$$
which is equivalent to proving
\[ v_{\det}(-i_1 I \cdot M \cdot i_1 I) = v_{\det}(M), \quad v_{\det}(\tau \circ M \circ \tau) = v_{\det}(M) \]
for all \( M \in \text{Sp}_2(O) \). Since \( v_{\det} \) is a homomorphism and \((-i_1 I \cdot M_1 M_2 \cdot i_1 I) = (-i_1 I \cdot M_1 \cdot i_1 I) \cdot (-i_1 I \cdot M_2 \cdot i_1 I) \) (the same for \( \tau \)), it suffices to prove the above for generators, only.
But for generators this can easily be checked by looking at the transformation laws 1.2 and 1.3 in a). Note that we also have \( \pi_p \circ \phi_3 = \phi_3 \circ \pi_p \) as well as \( \pi_p \circ \tau = \phi_3 \circ \pi_p \) by those transformation laws. But as \( \det(\pi_p(M)) \in \{ \pm 1 \} \subset \mathbb{F}_3 \), the determinant is not changed. With all this in mind, it is an easy calculation to prove that the identities from above hold true for the generators of \( \text{Sp}_2(O) \). So \( v_{\det} \) turns out to be an abelian character of \( \Gamma(O) \).
It is clear that \( v_{\det} \notin \langle v_{i_1}, v_{\tau} \rangle \) and that \( v_{\det} \) is of order two. \( \Gamma(O)^{ab} \simeq C_2 \times C_2 \times C_2 \) immediately follows.

There is one last thing that remains to be done in this section. For later purposes we need generators for the theta-groups of level \( P \leq O \), and even an explicit, finite set of generators for the special theta-group \( \text{Sp}_2(O)|p|_0 \). We will use the same methods already found in [Kr85, ch.I, thm.4.2], but since that work defines congruence subgroups and hence theta-groups for \( P = qO, q \in \mathbb{N} \), only, we cannot simply adopt the theorem. Nonetheless, the proof turns out to be the same. But note again that we can adopt other facts about the Hurwitz order and its attached modular and linear groups to our maximal order, since no other structure aside from the euclidean property is needed. So, in the subsequent proof we will cite those facts by substituting the occurring maximal order. Note that due to (1.13) every two-sided ideal of \( O \) is principal and generated by an invariant element.

(1.21) Theorem. Given a two-sided ideal \( P = qO \leq O, q \in \mathcal{I}(O) \) the theta-group \( \text{Sp}_n(O)[P]_0 \) of level \( P \) is generated by the matrices
\[
\text{Rot}(U), \quad \text{Trans}(S), \quad -J_n \text{Trans}(-T)J_n = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix}, \quad I \times K,
\]
where \( U \in \text{GL}_n(O), S \in \text{Her}_n(O), T \in \text{Her}_n(O) \cap P^{n \times n} \) and \( K \in \text{Sp}_1(O)|P|_0 \).

Proof: For \( q \in \mathcal{E} \), which is equivalent to \( P = O, \text{Sp}_n(O)[P]_0 = \text{Sp}_n(O) \) holds by definition. So
\[
\begin{pmatrix} I & 1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \begin{pmatrix} I & 1 \\ -I & I \end{pmatrix} \begin{pmatrix} I & 1 \\ 0 & I \end{pmatrix} = J_n
\]
and (1.17) yield the assertion in the case \( q \in \mathcal{E} \).
Now suppose \( q \notin \mathcal{E} \), i.e. \( P \neq O \). By (1.7), \( P = mO \) or \( P = mi_1 \sqrt{3}O \) for some \( m \in \mathbb{N} \), so without loss of generality, \( q = m_i, m > 1 \), or \( q = mi_1 \sqrt{3}, m \in \mathbb{N} \). Then of course \( m \in P \) or \( 3m \in P \) respectively, as \( i \sqrt{3}(-i \sqrt{3}) = 3 \). Now suppose \( c \in P \cap \mathbb{Z} \). So \( c \in m\mathbb{Z} \) or \( c \in mi_1 \sqrt{3}O \cap \mathbb{Z} \). In the latter case, \( c = mi_1 \sqrt{3}a \) for some \( a \in O \), so \( c^2 = N(mi_1 \sqrt{3}a) = 3m^2 N(a), c^2 \in 3m^2 \mathbb{Z} \). As 3 is a prime, a simple prime-factorization argument leads to \( 3m|c \). So we have shown \( P \cap \mathbb{Z} = m\mathbb{Z} \) or \( P \cap \mathbb{Z} = 3m\mathbb{Z} \) respectively.
1.1 The quaternionic half-space and the quaternionic modular group

We will now use induction on \( n \). The case \( n = 1 \) is trivial. (In this case \( I \) in \( I \times K \) is of degree 0, so \( I \times K \) is defined to be \( K \).) But nonetheless, let us make a remark: According to (1.18), \( \text{Sp}_1(O) = \{ \epsilon M ; \epsilon \in \mathcal{E}, M \in \text{SL}_2(Z) \} \). Clearly, for \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(Z) \) and \( \epsilon \in \mathcal{E} \), we have \( \epsilon c \in P \) if and only if \( c \in P \cap Z \). So if defined accordingly, \( \epsilon M \in \text{Sp}_1(O)[P]_0 \) is equivalent to \( M \in \text{SL}_2(Z)[m][0] \) or \( M \in \text{SL}_2(Z)[3m][0] \) respectively.

So let \( n > 1 \) and define \( \Delta_n \) to be the subgroup of \( \text{Sp}_n(O)[P]_0 \) generated by the matrices above. Suppose \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_n(O)[P]_0 \). Multiplying with \( \text{Rot}(U) \in \Delta_n, U \in \text{GL}_n(O) \) gives

\[
M_1 = \text{Rot}(U)M = \left( \begin{array}{cc} U^T A U^{-1} & U^T B U^{-1} \\ U^T C U^{-1} & U^T D U^{-1} \end{array} \right) = \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right).
\]

By applying [Kr85, ch.I, cor.2.4] to our order \( O \), we can choose \( U \in \text{GL}_n(O) \) such that the first column of \( U^T A \) possesses the form \((a_0,0,\ldots,0)\). If \( a \) and \( q \) are not relatively right-prime, then by definition such a potential non-unital right divisor would also be a right divisor of the first column of \( M_1 \in \text{Sp}_n(O)[P]_0 \). This contradicts [Kr85, ch.I, le.2.8] as \( M_1 \in \text{GL}_2n(O) \) by (1.12). So \( a \) and \( q \) are relatively right-prime, and in particular \( a \neq 0 \). Now let \( (e_1,\ldots,e_n) \) denote the first column of \( c_1 \) and suppose that \( u \) is a greatest common right divisor of \( a \) and \( c_1 \) (cf. [Kr85, ch.I, p.10] for details), which implies that there exist \( a, \gamma \in \mathcal{O} \) such that \( a = au, c_1 = \gamma u, (1.3) \) yields

\[
\left( \begin{array}{c} \bar{a}c_1 \\ * \\ * \end{array} \right) = \bar{\alpha}_1 c_1 = \gamma_1 \left( \begin{array}{c} a \\ * \\ * \end{array} \right),
\]

hence \( \bar{a} c_1 = \bar{\alpha}_1 a = \bar{\alpha}_1 a \in \mathcal{O} \cap R = Z \). And since \( u \neq 0 \), this implies

\[
\bar{a} \gamma = u^{-1} \bar{a} c_1 u = N(u)^{-1} \bar{a} c_1 = N(u)^{-1} \bar{\alpha}_1 a = \bar{\gamma} a.
\]

Furthermore, \( a \) and \( \gamma \) have to be relatively right-prime by definition, because \( u \) is defined to be a greatest common right divisor. So by adopting [Kr85, ch.II, cor.2.4] for our case, there is \( K \in \text{Sp}_1(Z) \) having \((a,\gamma)\) as its first column. Deriving from (1.18), there are \( \tilde{a}, \tilde{\gamma} \in Z \) and \( \tilde{e} \in \mathcal{E} \) with \( a = \tilde{a} \tilde{e}, \gamma = \tilde{\gamma} \tilde{e} \). Therefore, without restriction we may assume \( a, \gamma \in Z \), hence \( K \in \text{Sp}_1(Z) = \text{SL}_2(Z) \). Now, since \( a \) and \( q \) are relatively right-prime, \( u \) and \( q \) must have the same property as \( a = au \). As we have seen above, \( q = m \) or \( q = \gamma_1 \sqrt{3} m \) for some \( m \in Z \). In the first case, \( c_1 = \gamma u \in m \mathcal{O} \). So for \( u = u_0 + u_1 \frac{1+i \sqrt{3}}{2} + u_2 i_2 + u_3 \frac{1+i \sqrt{3}}{2} i_2, \gamma u_0, \ldots, \gamma u_3 \in m \mathcal{Z} \). As \( u \) and \( q = m \) are relatively right-prime, gcd\((u_0,\ldots,u_3,m)\) = 1 follows. Combining this yields \( \gamma \in m \mathcal{Z} \). If \( q = \gamma_1 \sqrt{3} m \), then by the same argument \( \gamma \in m \mathcal{Z} \), so \( \frac{\gamma}{m} \in Z \), \( \frac{\gamma}{m} u \in \sqrt{3} \mathcal{O} = p \) and \( \frac{\gamma^2}{m^2} N(u) \in 3 \mathcal{Z} \). \( N(u) \in 3 \mathcal{Z} \) would imply \( u \in p \) according to (1.9), but then \( \sqrt{3} \) would be a right divisor of \( q \) and \( u \), which is a contradiction. So \( \frac{\gamma^2}{m^2} \in 3 \mathcal{Z} \), which also implies \( \frac{\gamma}{m} \in 3 \mathcal{Z} \), hence \( \gamma \in 3 m \mathcal{Z} \). So according to what we have already seen at the beginning of the proof, \( \gamma \in P \cap Z \) holds in both cases, and thus \( K \in \text{Sp}_1(O)[P]_0 \) as well as \( K^{-1} \in \text{Sp}_1(O)[P]_0 \).

Now let \( \pi \) denote the permutation \((1,n)\), i.e. \( \pi \) permutes 1 and \( n \) and fixes all \( 1 < j < n \), and define \( U \) to be the corresponding permutation matrix. One computes

\[
K^{-1} \times I = \text{Rot}(U)(I \times K^{-1}) \text{Rot}(U) \in \Delta_n.
\]
Now if we define

\[ M_2 = \text{Trans}(I)(K^{-1} \times I)M_1 \]

another calculation shows that \((u, c_2, \ldots, c_n, 0, c_2, \ldots, c_n)\) turns out to be the first column of \(M_2\). Once again, according to [Kr85, ch.II, cor.2.4] \(u, c_2, \ldots, c_n\) have to be relatively right-prime, hence \((u, c_2, \ldots, c_n)'\) is the first column of some \(U \in \text{GL}_n(O)\) in virtue of [Kr85, ch.I, le.2.8]. By definition \(U^{-1}(u, c_2, \ldots, c_n)' = (1, 0, \ldots, 0)'\) holds and thus the first column of

\[ M_3 = \text{Rot}(U)M_2 \]

is equal to \((1, 0, \ldots, 0, \bar{c}_1, \ldots, \bar{c}_n)\) for some \(\bar{c}_1, \ldots, \bar{c}_n \in P\). As we have seen above, \(\bar{c}_1 = \bar{\epsilon}_1 \in \mathbb{Z}\) has to hold (this time \(a = 1\)). So we can choose \(T \in \text{Her}_n(O) \cap P^{n \times n}\) having \(-(\bar{c}_1, \ldots, \bar{c}_n)'\) as its first column, and one computes that \((1, 0, \ldots, 0)'\) is the first column of

\[ M_4 = \begin{pmatrix} 1 & * & * & * \\ 0 & A_0 & B_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & C_0 & D_0 & 0 \end{pmatrix}, \quad \text{where } M_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \text{Sp}_{n-1}(O)[P]_0. \]

The fundamental relations in (1.3) and an easy calculation show that \(M_4\) has to be of the shape

\[ M_4 = \begin{pmatrix} 1 & * & * & * \\ 0 & A_0 & B_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & C_0 & D_0 & 0 \end{pmatrix}, \quad \text{where } M_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \text{Sp}_{n-1}(O)[P]_0. \]

The induction hypothesis yields \(M_0 \in \Delta_{n-1}\) as well as \(M^{-1} \in \Delta_{n-1}\), and therefore \(I_2 \times M_0^{-1} \in \Delta_n\), since obviously \(I_2 \times \Delta_{n-1} \subset \Delta_n\) holds true. Another short computation yields

\[ M_5 = (I_2 \times M_0^{-1})M_4 = \begin{pmatrix} A_5 & B_5 \\ 0 & D_5 \end{pmatrix}, \]

where 0 is the \(n \times n\) zero-matrix. From the fundamental relations in (1.3) we obtain \(A_5 \bar{D}_5 = I\) as well as \(A_5 \bar{B}_5 = B_5 \bar{A}_5\). Hence \(A_5, D_5 \in \text{GL}_n(O)\), and \(A_5 \bar{B}_5 = B_5 \bar{A}_5\) if and only if \(A_5^{-1}B_5 = \bar{B}_5(\bar{A}_5)^{-1} = (A_5^{-1}B_5)\). Therefore, \(A_5^{-1}B_5 \in \text{Her}_n(O)\) and another short calculation finally yields

\[ M_5 = \text{Rot}(\bar{A}_5) \text{Trans}(A_5^{-1}B_5) \in \Delta_n, \]

and \(M \in \Delta_n\) follows. □

We will now give an explicit, finite set of generators for \(\text{Sp}_2(O)[p]_0\) as an application of (1.21).

**Corollary.** The theta-group \(\text{Sp}_2(O)[p]_0\) of level \(p\) is generated by the matrices

\[
\begin{align*}
\text{Rot} \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right), & \quad \text{Rot} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), & \quad \text{Rot} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \\
\text{Trans} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), & \quad \text{Trans} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), & \quad \text{Trans} \left( \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right), \\
-I \text{Trans} \left( \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \right) I, & \quad -I \text{Trans} \left( \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} \right) I, & \quad -I \text{Trans} \left( \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right) I,
\end{align*}
\]
where \( \varepsilon \in \{ -\frac{1+i\sqrt{3}}{2}, i_2 \} \), \( a \in \{ 1, \frac{1+i\sqrt{3}}{2}, i_2, \frac{1+i\sqrt{3}}{2} \} \) and \( b \in \{ 3, \frac{3+i\sqrt{3}}{2}, 3i_2, \frac{3+i\sqrt{3}}{2} i_2 \} \).

**Proof:** According to (1.21), \( \text{Sp}_2(O)[p]_0 \) is generated by the matrices

\[
\text{Rot}(U), \quad \text{Trans}(S), \quad -J\text{Trans}(-T)J, \quad I \times K,
\]

where \( U \in \text{GL}_2(O), S \in \text{Her}_2(O), T \in \text{Her}_2(O) \cap p^{2 \times 2} \) and \( K \in \text{Sp}_1(O)[p]_0 \).

As we have seen in the proof of (1.21), \( \text{Sp}_1(O)[p]_0 = \mathcal{E} \cdot \text{SL}_2(Z)[3]_0 \). For \( \varepsilon \in \mathcal{E} \) and \( M \in \text{SL}_2(Z)[3]_0 \), \( I \times \varepsilon M \) can be written as

\[
I \times \varepsilon M = \text{Rot} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) (I \times M).
\]

From [De96] we obtain that \( \text{SL}_2(Z)[3]_0 \) is generated by

\[
-I, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.
\]

Now

\[
I \times (-I) = \text{Rot} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \quad I \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \text{Trans} \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad I \times \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = -J\text{Trans} \left( \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right) J,
\]

and thus the matrices \( I \times K \) are not needed as generators. From (1.7) and (1.14) we obtain that \( \text{GL}_2(O) \) is generated by

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 5 & 1 \end{pmatrix}
\]

with \( \varepsilon \in \{ -\frac{1+i\sqrt{3}}{2}, i_2 \} \). And of course the group of all \( \text{Trans}(S), S \in \text{Her}_2(O) \) is generated by the matrices \( \text{Trans}(\cdot) \) quoted above.

It is easy to check that \( (-J\text{Trans}(-T_1))(-J\text{Trans}(-T_2)) = -J\text{Trans}(-(T_1 + T_2)) J \) holds for all \( T_1, T_2 \in \text{Her}_2(O) \cap p^{2 \times 2} \). By definition

\[
p = i_1\sqrt{3}O = i_1\sqrt{3}Z + \frac{3+i\sqrt{3}}{2}i_2Z + i_1i_2\sqrt{3}Z + \frac{3+i\sqrt{3}}{2}i_2Z = 3Z + \frac{3+i\sqrt{3}}{2}i_2Z + 3i_2Z + \frac{3+i\sqrt{3}}{2}i_2Z,
\]

where the last equality is easy to check. Hence the claim follows. \( \Box \)

### 1.2 A fundamental domain

We are now going to determine a fundamental domain of \( \mathcal{H}(\mathbb{H}) \) with respect to \( \text{Sp}_2(O) \), which is more or less a transversal of the action of \( \text{Sp}_2(O) \) on the quaternionic half space, or to be more precise, the closure of a special connected one. We will need this fundamental domain for later purposes, like Eisenstein-series or rough estimates on the dimension of spaces of modular forms.

**Definition.** Let \( T \) be a topological space and \( G \) a subgroup of the group of automorphisms of \( T \), which is denoted by \( \text{Aut} T := \{ f : T \to T ; f \text{ is a homeomorphism} \} \). A subset \( \mathcal{F} \) of \( T \) is called a fundamental domain of \( T \) with respect to \( G \) if it fulfills:

\[ (1.23) \]
(F.1) \( \mathcal{F} \) is closed in \( \mathcal{T} \) and possesses interior points.

(F.2) For all \( x \in \mathcal{T} \) there exists \( g \in \mathcal{G} \) such that \( g(x) \in \mathcal{F} \).

(F.3) If \( x \) and \( g(x) \) are interior points of \( \mathcal{F} \) for some \( g \in \mathcal{G} \), then \( g = \text{id}_{\mathcal{T}} \).

(F.4) For all compact subsets \( \mathcal{C} \) of \( \mathcal{T} \) there exist only finitely many \( g \in \mathcal{G} \) such that \( g(\mathcal{F}) \cap \mathcal{C} \neq \emptyset \).

(F.5) The set \( \{ g \in \mathcal{G} : g(\mathcal{F}) \cap \mathcal{F} \neq \emptyset \} \) is finite.

We are now going to determine a fundamental for \( \mathcal{T} = \mathcal{H}(\mathbb{H}) \) and \( \mathcal{G} = \text{Sp}_2(\mathcal{O}) \). It will be denoted by \( \mathcal{F}_2(\mathcal{O}) \). This is going to need a lot of preparation. We begin with the announced proof of the remark in (1.7) concerning the fact that \( \mathcal{O} \) is a euclidean ring.

(1.24) Proposition. For all \( a \in \mathbb{H} \) there exists \( g \in \mathcal{O} \) such that \( N(a-g) \leq \frac{2}{3} \).

Proof: Let \( a = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 b + a_3 \frac{1+i\sqrt{3}}{2} i \in \mathbb{H} \). We will now choose \( g = g_0 + g_1 \frac{1+i\sqrt{3}}{2} + g_2 b + g_3 \frac{1+i\sqrt{3}}{2} i \in \mathcal{O} \) adequately and denote \( b = a - g = b_0 + b_1 \frac{1+i\sqrt{3}}{2} + b_2 b + b_3 \frac{1+i\sqrt{3}}{2} i \).

As the interval \( \left[ -\frac{1}{3}, \frac{2}{3} \right] \) is of length 1, we can choose \( g_0 \) such that \( b_0 \in \left[ -\frac{1}{3}, \frac{2}{3} \right] \). If \( b_0 \in \left[ -\frac{1}{3}, \frac{1}{3} \right] \), we choose \( g_1 \) such that \( b_1 \in \left[ -\frac{1}{2} - \frac{1}{2} b_0, \frac{1}{2} - \frac{1}{2} b_0 \right] \) (this interval is again of length 1). If \( b_0 \in \left( \frac{1}{3}, \frac{2}{3} \right] \), choose \( g_1 \) such that \( b_1 \in \left[ -1 + b_0, b_0 \right] \). And if in the last case \( b_0 > \frac{1}{2} \), replace \( g_0 \) by \( g_0 + 1 \) (and thus the original \( b_0 \) by \( b_0 - 1 \)). This yields \( b_0 \in \left[ -\frac{3}{2}, -\frac{1}{2} \right] \), \( b_1 \in \left[ b_0, b_0 + 1 \right] \) and as well \( b_0 \geq -\frac{1}{2} - \frac{1}{2} b_1 \). If \( b_0 \in \left( \frac{1}{3}, \frac{2}{3} \right] \) and \( b_0 \leq \frac{1}{2} - \frac{1}{2} b_1, b_1 \leq b_0 \) always holds true. And if \( b_0 \in \left( -\frac{3}{2}, -\frac{1}{2} \right] \) and \( b_0 \geq -\frac{2}{3} - \frac{1}{2} b_1, b_1 \geq b_0 \) always holds true. So, to summarize, we can choose \( g_0, g_1 \) such that one of the following cases holds true:

- \( b_0 \in \left[ -\frac{1}{3}, \frac{1}{3} \right], \quad b_1 \in \left[ -\frac{1}{2} - \frac{1}{2} b_0, \frac{1}{2} - \frac{1}{2} b_0 \right] \)
- \( b_0 \in \left( \frac{1}{3}, \frac{2}{3} \right], \quad b_1 \in \left[ -1 + b_0, 1 - 2 b_0 \right] \)
- \( b_0 \in \left[ -\frac{2}{3}, -\frac{1}{3} \right], \quad b_1 \in \left[ -1 - 2 b_0, 1 + b_0 \right] \)

Note that this is equivalent to say

\[
|b_0 - b_1| \leq 1, \quad |2b_0 + b_1| \leq 1, \quad |b_0 + 2b_1| \leq 1.
\]

The proof of this statement is just an easy, yet quite long calculation. The first implication is to simply go through the three possible cases and check each condition by a calculation. The other implication is just as straightforward. Start with checking that the conditions yield \( -\frac{2}{3} \leq b_0 \leq \frac{2}{3} \), then check the three cases for \( b_0 \) from above. The full proof is omitted here, as the long calculation would not yield anything noteworthy. But for an easier understanding figure 1.1 shows the set of \((b_0, b_1)\) in \(\mathbb{R}^2\) fulfilling the conditions above.

Of course the same can be done with \(b_2 \) and \(b_3\), i.e. we choose \( g \) such that

\[
\begin{align*}
|b_0 - b_1| & \leq 1, & |2b_0 + b_1| & \leq 1, & |b_0 + 2b_1| & \leq 1, \\
|b_2 - b_3| & \leq 1, & |2b_2 + b_3| & \leq 1, & |b_2 + 2b_3| & \leq 1.
\end{align*}
\]
Next we will show that every \((b_0, b_1)\) fulfilling the conditions above also satisfies \(N(b_0 + b_1 \frac{\sqrt{3}}{2}) \leq \frac{1}{3}\), which immediately leads to \(N(b) \leq \frac{2}{3}\) as \(N(b) = N(b_0 + b_1 \frac{\sqrt{3}}{2}) + N(b_2 + b_3 \frac{\sqrt{3}}{2})\).

So let \(b_0\) be fixed and define \(f(b_1) = N(b_0 + b_1 \frac{\sqrt{3}}{2}) = b_1^2 + b_0 b_1 + b_0^2\). As a normalized polynomial of degree two, \(f\) will always take its maximum on the boundary of an interval. So for \(b_0\) fixed, we only have to evaluate \(f\) at the minimal and maximal possible \(b_1\).

- \(b_0 \in [-\frac{1}{3}, \frac{1}{3}]\), \(b_1 \in [-\frac{1}{2} - \frac{1}{2} b_0, \frac{1}{2} - \frac{1}{2} b_0]\). Then \(f(-\frac{1}{2} - \frac{1}{2} b_0) = f(\frac{1}{2} - \frac{1}{2} b_0) = \frac{1}{4} + \frac{3}{2} b_0^2 \leq \frac{1}{3}\).

- \(b_0 \in [\frac{1}{3}, \frac{2}{3}]\), \(b_1 \in [-1 + b_0, 1 - 2 b_0]\). Then \(f(-1 + b_0) = f(1 - 2 b_0) = 3 b_0^2 - 3 b_0 + 1\). Again, this polynomial takes its maximum at the boundary. \(3(\frac{1}{3})^2 - 3 \frac{1}{3} + 1 = 3(\frac{2}{3})^2 - 3 \frac{2}{3} + 1 = \frac{1}{3}\) yields the assertion.

- \(b_0 \in [-\frac{2}{3}, -\frac{1}{3}]\), \(b_1 \in [1 - 2 b_0, 1 + b_0]\). This is handled in the same way as the second case. □

We will now follow the idea of constructing a fundamental domain for \(T = \mathcal{H}(\mathbb{H})\) and \(G = \text{Sp}_2(\mathcal{O})\) which can already be found in [Kr85, ch.II.3] for the case of \(\mathcal{O}\) standing for the Hurwitz order. Of course, the idea has to be modified to fit our case. So, following this idea, we first need the concept of reduced marices. To state it again, we will only consider the case of degree two. The basics for higher degrees will already be given by looking at degree two, but only sticking to degree two will yield a much clearer representation.

First, we cite a lemma from [Kr98]:

(1.25) **Lemma.** Define

\[
\mathcal{F}(\mathbb{H}) := \left\{ a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \mathbb{H} \mid a_0 \geq 0, |a_1| \leq \frac{1}{\sqrt{3}} a_0, |a_2| \leq a_0, |a_3| \leq \frac{2}{\sqrt{3}} a_0 - \frac{1}{\sqrt{3}} |a_2| \right\}
\]
Then \( F(H) \) is a fundamental domain of \( H \) with respect to \( G = \{ \mathbb{H} \to \mathbb{H}, a \mapsto ea; e \in \mathcal{E} \} \) as well as \( G = \{ \mathbb{H} \to \mathbb{H}, a \mapsto ae; e \in \mathcal{E} \} \), and \( F^*(H) \) is a fundamental domain of \( H \) with respect to \( G = \{ \mathbb{H} \to \mathbb{H}, a \mapsto ead; e, d \in \mathcal{E} \} \).

Now, note that \( \mathcal{O} \) is a euclidean ring in view of (1.24). As said before, in [Kr85] the Hurwitz order was investigated, which is euclidean, too. (The upper bound \( \frac{2}{3} \) in (1.24) is \( \frac{1}{2} \) in this case.) All proofs in [Kr85] concerning \( \mathcal{O} \) which are needed for the following proposition only use an analog of (1.24) (with \( \frac{1}{2} \) instead of \( \frac{2}{3} \)), so all proofs would be the same for the order \( \mathcal{O} \) which we are interested in, this time. So we simply cite the following proposition, a composition of [Kr85, ch.I, prop.4.1, cor.4.2, thm.4.3]. Note, that the constant \( d = 3 \) in c) arises from the bound \( \frac{2}{3} \) in (1.24) in contrast to \( \frac{1}{2} \) by taking a closer look at the proof of [Kr85, ch.I, thm.4.3]. For the definition of \( \det(S) \) for \( S \in \text{Her}_n(\mathbb{H}) \), confer [Kr85, ch.I, p.21].

(1.26) Proposition. Given \( S \in \text{Pos}_n(\mathbb{H}) \), we define the minimum \( \mu(S) \) of \( S \) by

\[
\mu(S) := \inf\{\|g\|; g \in \mathcal{O}^n \setminus \{0\}\}.
\]

We then have

a) There exists \( 0 \neq g \in \mathcal{O}^n \) such that \( \mu(S) = \|g\| > 0 \). Furthermore, \( \mu(S) = \mu(S[U]) \) holds for all \( U \in \text{GL}_n(\mathcal{O}) \) as well as \( \mu(S) > \mu(T) \) if \( T \in \text{Pos}_n(\mathbb{H}) \) with \( S > T \).

b) There exists \( U \in \text{GL}_n(\mathcal{O}) \) such that \( S[U] = T = (t_{k,l}) \), where \( t_{1,1} = \mu(S) = \mu(T) \).

c) There is \( d > 0 \) such that

\[
\mu(S)^n \leq d^{n(n-1)/2} \det(S)
\]

holds for all \( S \in \text{Pos}_n(\mathbb{H}) \). One can choose \( d = 3 \). In particular

\[
\mu(S)^2 \leq 3 \det(S) = 3(s_{1,1}s_{2,2} - N(s_{1,2}))
\]

holds for all \( S \in \text{Pos}_2(\mathbb{H}) \).

Now, we can define reduced matrices. We will stick to the case of degree two to keep it simple, as we will not need higher dimensions.
\section*{1.2 A fundamental domain}

\setcounter{equation}{27}

\textbf{(1.27) Definition.} \( \mathcal{R}(O) \) is the set of all reduced matrices, which means that it is the set of all \( S = (s_{kj}) \in \text{Pos}_2(H) \) satisfying the following conditions:

\begin{enumerate}
  \item[(R.1)] \( s_{1,2} \in \mathcal{F}^+(H) \),
  \item[(R.2)] \( S[g] \geq s_{1,1} = \mu(S) \) for all \( g = (g_1, g_2) \in O^2 \) where \( g_1 \) and \( g_2 \) are relatively right-prime, as well as \( S[g] \geq s_{2,2} \) for all \( g = (g_1, g_2) \in O^2 \) where \( g_2 \in E \).
\end{enumerate}

\textbf{(1.28) Proposition.} \quad a) Given \( S \in \text{Pos}_2(H) \) there exists \( U \in \text{GL}_2(O) \) such that \( S[U] \in \mathcal{R}(O) \).

b) \( \mathcal{R}(O) \) is closed in \( \text{Pos}_2(H) \) and contains interior points.

c) Suppose that \( S, S[U] \in \mathcal{R}(O) \) for some \( U \in \text{GL}_2(O) \), \( U \neq \pm I \), then \( S \) and \( S[U] \) belong to the boundary of \( \mathcal{R}(O) \).

d) If \( S = (s_{kj}) \in \mathcal{R}(O) \), then the following holds true:

\begin{enumerate}
  \item[i)] \( s_{1,1} = \mu(S) \).
  \item[ii)] \( s_{1,1} \leq s_{2,2} \).
  \item[iii)] For \( s_{1,2} = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \), one has
    \[
    |a_0 - a_1| \leq s_{1,1} , \quad |2a_0 + a_1| \leq s_{1,1} , \quad |a_0 + 2a_1| \leq s_{1,1} ,
    \]
    \[
    |a_2 - a_3| \leq s_{1,1} , \quad |2a_2 + a_3| \leq s_{1,1} , \quad |a_2 + 2a_3| \leq s_{1,1} ,
    \]
    which also leads to
    \[ N(s_{1,2}) \leq \frac{3}{2} s_{1,1} . \]
  \item[iv)] \( s_{1,1} \cdot s_{2,2} \leq 3 \det(S) \).
\end{enumerate}

\textbf{Proof:} a) – d)ii) Because of the reasons discussed above, we can cite already known facts from [Kr85] again, as the proofs would be exactly the same because no special structure for the maximal order we are looking at is required, except from its property to be euclidean. To be precise, we cite propositions 4.4 to 4.7 in chapter I from [Kr85].

\textbf{d) Choose} \( \epsilon \in E \) and \( g = (1, -\epsilon) \). One calculates
\[
S[g] = s_{1,1} + s_{2,2} - 2 \text{Re}(s_{1,2}\epsilon) .
\]
\( S[g] \geq s_{2,2} \) holds because of \( S \in \mathcal{R}(O) \) and (R.2), so we have
\[
2 \text{Re}(s_{1,2}\epsilon) \leq s_{1,1} \quad \text{for all} \ \epsilon \in E .
\]

Now, we simply insert all twelve \( \epsilon \in E \) and calculate that this is equivalent to claiming
\[
|a_0 - a_1| \leq s_{1,1} , \quad |2a_0 + a_1| \leq s_{1,1} , \quad |a_0 + 2a_1| \leq s_{1,1} ,
\]
\[
|a_2 - a_3| \leq s_{1,1} , \quad |2a_2 + a_3| \leq s_{1,1} , \quad |a_2 + 2a_3| \leq s_{1,1} .
\]
This also means that $s_{1,2}/s_{1,1}$ fulfills the conditions 1.4 in (1.24). Thus we immediately obtain $N(s_{1,2}/s_{1,1}) = N(s_{1,2})/s_{1,1}^2 \leq \frac{3}{4}$.

Finally, this leads to

$$\det(S) = s_{1,1} \cdot s_{2,2} - N(s_{1,2}) \geq s_{1,1} \cdot s_{2,2} - \frac{2}{3}s_{1,1}^2 = s_{1,1}(s_{2,2} - \frac{2}{3}s_{1,1}) \geq s_{1,1} \cdot \frac{1}{3}s_{2,2}. \quad \square$$

According to [Kr85, ch.I, cor.3.10] every $S \in \text{Pos}_n(\mathbb{H})$ possesses a unique so-called Jacobian representation, i.e. $S = D[B]$, where $D > 0$ is a real, positive definite, diagonal matrix and $B \in \mathbb{H}^{n \times n}$ is an upper triangular matrix whose diagonal entries are all equal to 1. With this representation we can analyze reduced matrices even further. The following proposition is an analog to [Kr85, ch.I, prop.4.9].

(1.29) Proposition. For $\alpha > 0$ define the elementary set $\mathcal{E}_n(\mathbb{H})[\alpha]$ consisting of all matrices $S = D[B]$ in Jacobian representation (with $D = (d_{k,l})$, $B = (b_{k,l})$), such that $0 < d_{i,j} < \alpha d_{i+1,j+1}$ for all $1 \leq j < n$ and $N(b_{j,k}) < \alpha^2$ for all $1 \leq j < k \leq n$. Then for all $\alpha > 3$

$$\mathcal{R}(O) \subset E_2(\mathbb{H})[\alpha]$$

holds.

Proof: Let $S \in \mathcal{R}(O)$ with Jacobian decomposition

$$S = \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{pmatrix} = \begin{pmatrix} d_{1,1} & 0 \\ 0 & d_{2,2} \end{pmatrix} \begin{pmatrix} 1 & b_{1,2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{1,1} & d_{1,1}b_{1,2} \\ d_{1,1}b_{1,2} & d_{1,1}N(b_{1,2}) + d_{2,2} \end{pmatrix}.$$  

As $S \in \mathcal{R}(O)$, (1.28) yields

$$d_{1,1}d_{2,2} = \det(S) \geq \frac{1}{3}s_{1,1}s_{2,2} = \frac{1}{3}d_{1,1}s_{2,2}$$

$$\Leftrightarrow \quad \frac{d_{1,1}}{d_{2,2}} \geq \frac{1}{3}s_{2,2}$$

$$\Rightarrow \quad \frac{d_{1,1}}{d_{2,2}} \leq \frac{3}{3}d_{1,1} = \frac{3}{3}s_{1,1} \leq 3 < \alpha$$

Furthermore, (1.28) yields

$$d_{1,1}^2N(b_{1,2}) = N(s_{1,2}) \leq \frac{2}{3}s_{1,1}^2 = \frac{2}{3}d_{1,1}^2 \quad \Rightarrow \quad N(b_{1,2}) \leq \frac{2}{3} < \alpha^2$$

Thus, $S \in \mathcal{E}_2(\mathbb{H})[\alpha]$ by definition. \quad \square

Because of the preceding proposition, [Kr85, ch.I, prop.4.10] and [Kr85, ch.I, prop.4.11] can be adopted literally for our case (this is only necessary for (F.4) and (F.5)). So we summarize the mentioned results in

(1.30) Theorem. $\mathcal{R}(O)$ is a fundamental domain of $\text{Pos}_2(\mathbb{H})$ with respect to the action $S \mapsto S[U]$, $U \in \text{GL}_2(O)$, and a convex cone. All $S = (s_{k,l}) \in \mathcal{R}(O)$ satisfy:

a) $s_{1,1} \leq s_{2,2}$,
1.2 A fundamental domain

b) \( N(s_{1,2}) \leq \frac{2}{3}s_{1,1}^2 \),

c) \( s_{1,1} \cdot s_{2,2} \leq 3 \text{det}(S) \),

d) \( 2 \left( \begin{array} {cc} s_{1,1} & 0 \\ 0 & s_{2,2} \end{array} \right) \geq S \geq (1 - \alpha) \left( \begin{array} {cc} s_{1,1} & 0 \\ 0 & s_{2,2} \end{array} \right) \) for all \( \alpha \geq \sqrt{\frac{2}{3}} \).

Proof: Only d) remains to be shown. Note that a matrix is positive (semi) definite if and only if all its principal minors are positive (non-negative) – cf. [Kr85, ch.I, thm.3.11, prop.3.12]. The first principal minor of

\[
T := 2 \begin{pmatrix} s_{1,1} & 0 \\ 0 & s_{2,2} \end{pmatrix} - S = \begin{pmatrix} \frac{s_{1,1}}{s_{2,2}} & -s_{1,2} \\ -s_{1,2} & \frac{s_{2,2}}{s_{1,1}} \end{pmatrix}
\]

is \( s_{1,1} \), the second is \( \text{det}(T) = s_{1,1}s_{2,2} - N(-s_{1,2}) = \text{det}(S) > 0 \), so \( T \) is positive definite. Now let

\[
T := S - (1 - \alpha) \begin{pmatrix} s_{1,1} & 0 \\ 0 & s_{2,2} \end{pmatrix} = \begin{pmatrix} \alpha s_{1,1} & -s_{1,2} \\ -s_{1,2} & \alpha s_{2,2} \end{pmatrix},
\]

Again, the first principal minor is positive since \( \alpha > 0 \) and \( s_{1,1} > 0 \). Because of a) and b), for the second principal minor we get

\[
\text{det}(T) = \alpha^2 s_{1,1}s_{2,2} - N(s_{1,2}) \geq s_{1,1}(\alpha^2 s_{2,2} - \frac{2}{3}s_{1,1}) \geq (\alpha^2 - \frac{2}{3})s_{1,1}s_{2,2} \geq 0,
\]

since \( \alpha \geq \sqrt{\frac{2}{3}} \).

Because we are only investigating the case of degree two, it is even possible to specify \( \mathcal{R}(\mathcal{O}) \) explicitly.

(1.31) Proposition. Define

\[
M := \left\{ S = \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{pmatrix} \in \text{Her}_2(\mathbb{H}) \mid s_{1,2} \in \mathcal{F}^+(\mathbb{H}), 2 \text{Re}(s_{1,2}) \leq s_{1,1} \leq s_{2,2}, s_{1,1} > 0 \right\}
\]

\[
= \left\{ S = \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{pmatrix} \in \text{Her}_2(\mathbb{H}) \mid s_{1,2} = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2, 
\quad 0 \leq a_1 \leq a_0, 3|a_3| \leq 2a_2 + 3a_0 + a_1 \leq s_{1,1} \leq s_{2,2}, s_{1,1} > 0 \right\}.
\]

Then \( \mathcal{R}(\mathcal{O}) = M \) holds true.

Proof: Due to (1.27) and (1.28), \( \mathcal{R}(\mathcal{O}) \subset M \) is clear, already. So let \( \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{pmatrix} \in M \) and \( g = (g_1, g_2)' \in \mathcal{O}^2, \ g \neq 0 \). A short calculation yields

\[
S[g] = s_{1,1} \cdot N(g_1) + 2 \text{Re}(s_{1,2}g_2 g_1^*) + s_{2,2} \cdot N(g_2).
\]
Set \( s_{1,2} = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \) and \( g_{2G_1} = b_0 + b_1 \frac{1+i\sqrt{3}}{2} + b_2 i_2 + b_3 \frac{1+i\sqrt{3}}{2} i_2 \). As \( s_{1,2} \in \mathcal{F}^*(\mathcal{H}) \), (1.25) yields \( 2a_0 + a_1 \geq 2a_2 + a_3 \geq 3|a_3|, 2a_0 + a_1 \geq 3a_1 \). So another calculation gives

\[
2 \text{Re}(s_{1,2}g_{2G_1}) = 2((a_0 + a_1)(b_0 + \frac{b_1}{2}) - 3a_1 \frac{b_1}{2} - (a_2 + a_3)(b_2 + \frac{b_3}{2}) - 3a_3 \frac{b_3}{2}) \\
\geq -(2a_0 + a_1)(|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + \frac{|b_3}{2}|) \\
\geq -s_{1,1}(|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|)
\]

and

\[
S[g] \geq s_{1,1} \cdot \left[ N(g_1) + N(g_2) - (|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|) \right]
\]

follows. Now choose \( c_0, c_1, c_2, c_3 \in \mathbb{Q} \) such that

\[
2c_0 + c_1, 3c_1, 2c_2 + c_3, 3c_3 \in \{\pm 1\}
\]

with

\[
(2c_0 + c_1)(b_0 + \frac{b_1}{2}) = -|b_0 + \frac{b_1}{2}|, \quad -3c_1(\frac{b_1}{2}) = -|\frac{b_1}{2}|, \\\n-(2c_2 + c_3)(b_2 + \frac{b_3}{2}) = -|b_2 + \frac{b_3}{2}|, \quad -3c_3(\frac{b_3}{2}) = -|\frac{b_3}{2}|.
\]

If we define \( c = c_0 + c_1 \frac{1+i\sqrt{3}}{2} + c_2 i_2 + c_3 \frac{1+i\sqrt{3}}{2} i_2 \), then by construction \( N(\frac{1}{2}c) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} \) and

\[
T = \left( \begin{array}{cc} 1 & \frac{1}{2}c \\ \frac{1}{2}c & 1 \end{array} \right)
\]

is positive definite. So the same calculation as above yields

\[
N(g_1) + N(g_2) - (|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|) = T[g] > 0.
\]

Because of \( g_1, g_2 \in \mathcal{O} \), the left-hand side is an integer, and this gives rise to

\[
N(g_1) + N(g_2) - (|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|) \geq 1 \quad (1.5)
\]

as well as

\[
S[g] \geq s_{1,1}.
\]

In particular \( S \in \text{Pos}_2(\mathcal{H}) \) follows.

Now we still have to prove \( S[g] \geq s_{2,2} \) for all \( g = (g_1, g_2)' \in \mathcal{O}^2 \) where \( g_2 \in \mathcal{E} \). In this case, as \( N(g_2) = 1, 1.5 \) immediately yields

\[
N(g_1) + (|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|) \geq 0.
\]

Finally, the same calculation as above leads to

\[
S[g] \geq s_{1,1} \cdot \left[ N(g_1) - (|b_0 + \frac{b_1}{2}| + |\frac{b_1}{2}| + |b_2 + \frac{b_3}{2}| + |\frac{b_3}{2}|) \right] + s_{2,2} \cdot N(g_2) \geq s_{2,2}.
\]

So the condition (R.2) is fulfilled, and because (R.1) holds true by definition, \( S \in \mathcal{R}(\mathcal{O}) \) ensues. \( \square \)
1.2 A fundamental domain

To finish studying reduced matrices, we cite some last propositions that can be applied to our case without the need to prove them again as this would only result in citing the proof literally. Note that for \( S = (s_{kl}) \in \text{Her}_2(O) \) we get \( \det(S) = s_{11}s_{22} - N(s_{12}) \in \mathbb{Z} \). (Again, cf. [Kr85, ch.I, pp.21] for details about the determinant of Hermitian quaternionic matrices.)

(1.32) Proposition. \( a \) \( S \geq (1 - \sqrt{\frac{2}{3}}) \mu(S)I \) for every \( S \in \mathcal{R}(O) \)

\( b \) Given \( \gamma, \delta > 0 \) the set 
\[ \{ S \in \mathcal{R}(O) ; \det(S) \leq \gamma, \mu(S) \geq \delta \} \]

is bounded.

\( c \) Given \( n \in \mathbb{N} \), let \( c(n, H) \) denote the number of equivalence classes of matrices \( S \in \text{Pos}_2(O) \) satisfying \( \det(S) = n \). Then the so-called class numbers \( c(n, H) \), \( n \geq 1 \) are well defined, i.e. \( c(n, H) \in \mathbb{N} \) is finite.

Proof: \( b, c \) Cf. [Kr85, ch.I, prop.5.1, thm.5.2].

\( a \) \( S \geq (1 - \sqrt{\frac{2}{3}}) \left( \begin{smallmatrix} 1 & 0 \\ 0 & s_{22} \end{smallmatrix} \right) \) according to (1.30), while \( \left( \begin{smallmatrix} 1 & 0 \\ 0 & s_{22} \end{smallmatrix} \right) \geq \mu(S)I \) follows from (1.28). \( \square \)

Next we need to identify quaternionic matrices with complex and real matrices of higher degree to be able to define the so-called determinant-forms in \( \mathbb{H}^{2 \times 2} \otimes \mathbb{R} \mathbb{C} \). These determinant-forms will also be necessary to define modular forms. For details confer [Kr85, ch.I, pp.14] as well as [Kr85, ch.II, pp. 46].

(1.33) Definition. Define the maps

\[ \wedge : \mathbb{H} \rightarrow \mathbb{R}^{4 \times 4}, \quad a = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 \mapsto \hat{a} = \left( \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{array} \right), \]

\[ \vee : \mathbb{H} \rightarrow \mathbb{C}^{2 \times 2}, \quad a = a_0 + a_1i_1 + a_2i_2 + a_3i_1i_2 \mapsto \tilde{a} = \left( \begin{array}{cc} a_0 & a_1i \\ -a_2 & a_3i \\ a_2 & a_3 \end{array} \right). \]

We extend these maps to matrices over \( \mathbb{H} \), i.e.

\[ \wedge : \mathbb{H}^{n \times n} \rightarrow \mathbb{R}^{4n \times 4n}, \quad A = (a_{kl}) \mapsto \hat{A} = (\hat{a}_{kl}), \]

\[ \vee : \mathbb{H}^{n \times n} \rightarrow \mathbb{C}^{2n \times 2n}, \quad A = (a_{kl}) \mapsto \tilde{A} = (\tilde{a}_{kl}), \]

as well as \( \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C} \) by

\[ \wedge : \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C} \rightarrow \mathbb{C}^{4n \times 4n}, \quad Z = X + iY \mapsto \hat{Z} = \hat{X} + i\hat{Y}, \]

\[ \vee : \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C} \rightarrow \mathbb{C}^{2n \times 2n}, \quad Z = X + iY \mapsto \tilde{Z} = \tilde{X} + i\tilde{Y}. \]

This also gives us the so-called determinant-forms \( \det(\hat{Z}) \) and \( \det(\tilde{Z}) \) for \( Z \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C} \).
We will now state some further facts about the embeddings \(^\wedge\) and \(^\vee\) cited from [Kr85], again.

(1.34) Proposition. a) For \(a \in \mathbb{H}\), \(\det(\hat{a}) = N(a)\) and \(\det(\bar{a}) = N(a)^2\) hold.

b) \(^\wedge\) and \(^\vee\) are (whether in the extended versions or not) injective \(\mathbb{R}\)-algebra homomorphisms with \(\hat{1}_n = I_{4n}, \hat{I}_n = I_{2n}\), and \(A \in \text{GL}_n(\mathbb{H})\) (or simply \(A\) invertible if \(A \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C}\) if and only if \(\hat{A} \in \text{GL}_{4n}(\mathbb{R})\) (or \(\hat{A} \in \text{GL}_{4n}(\mathbb{C})\) or \(\hat{A} \in \text{GL}_{2n}(\mathbb{C})\), respectively. Furthermore, \(\det(\hat{S}) = \det(S)^4\) and \(\det(\hat{S'}) = \det(S)^2\) for all \(S \in \text{Pos}_n(\mathbb{H})\) (or \(S = S' \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C}\)). Also, \(\hat{A} = (\hat{A})'\) and \(\hat{A}' = (\hat{A})\) for all \(A \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C}\), but in this case \((\hat{A}) = (\hat{A})'\) holds.

c) \(\det(\hat{A}) \in \mathbb{Z}\) for all \(A \in \mathcal{O}^{n \times n}\) and \(A \in \text{GL}_n(\mathcal{O})\) \(\iff\) \(\det(\hat{A}) = 1\). (cf. [Ki98, Kor.1.16, Le.1.17])

d) \(\det(\bar{Z}) = \det(Z)^4\) and \(\det(\bar{Z}) = \det(Z)^2\) for all \(Z \in \mathcal{H}_n(\mathbb{H})\).

e) \(\mathcal{H}_n(\mathbb{H}) \to \mathcal{H}_{4n}(\mathbb{R}), Z \mapsto \hat{Z}\) and \(\mathcal{H}_n(\mathbb{H}) \to \mathcal{H}_{2n}(\mathbb{C}), Z \mapsto \bar{Z}\) yield embeddings of the quaternionic half-space into Siegel and Hermann half-spaces of higher degrees.

(1.35) Remark. Note that because of (1.34) it is easy to see that
\[
(\widehat{AB})' = \overline{B'}A'
\]
holds for all \(A, B \in \mathbb{H}^{n \times n}\) (or even for all \(A, B \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \mathbb{C}\):
\[
(\widehat{AB})' = (\hat{A}\hat{B})' = \overline{B'}A' = \overline{(B'A')}.
\]
The injectivity of \(^\wedge\) completes the proof.

Finally, we can work out a fundamental domain for \(\mathcal{H}(\mathbb{H})\) and \(\text{Sp}_2(\mathcal{O})\) with respect to modular transformations.

(1.36) Definition. Define the set \(\mathcal{F}_2(\mathcal{O})\) to consist of all matrices \(Z = X + iY \in \mathcal{H}(\mathbb{H})\) satisfying

- \(X \in \mathcal{C}(\mathcal{O}) := \{X = (x_{i,j}) \in \text{Her}_2(\mathbb{H}) ; \ -\frac{1}{2} \leq x_{1,1}, x_{2,2} \leq \frac{1}{2}, x_{1,2} = a_0 + a_1 x_{1,2} + a_2 i \leq \frac{1}{2}, j = 0, \ldots, 3\},\)
- \(Y \in \mathcal{R}(\mathcal{O})\),
- \(|\det(\hat{M}(Z))| \geq 1\) for all \(M \in \text{Sp}_2(\mathcal{O})\).
To prove that $\mathcal{F}_2(\mathcal{O})$ actually is a fundamental domain we will use the same proceeding already found in [Kr85, ch.II.3].

**(1.37) Proposition.** For all $Z = X + iY \in \mathcal{F}_2(\mathcal{O})$

$$Y \geq \frac{\sqrt{3}}{2}(1 - \sqrt{\frac{3}{2}})I$$

holds true.

**Proof:** Let $Z = X + iY = (z_{k,l}) = (x_{k,l}) + i(y_{k,l}) \in \mathcal{F}_2(\mathcal{O})$. Define $M := J_1 \times I_2 \in \text{Sp}_2(\mathcal{O})$. A short calculation and the prerequisites yield

$$1 \leq |\det(M\{Z\})| = |z_{1,1}|^4 = (x_{1,1}^2 + y_{1,1}^2)^2.$$  

Now $C \in C(\mathcal{O})$ implies $x_{1,1}^2 \leq \frac{1}{4}$, hence $y_{1,1} \geq \frac{\sqrt{3}}{2}$. So $\mu(Y) = y_{1,1} \geq \frac{\sqrt{3}}{2}$ by (1.28) and

$$Y \geq (1 - \sqrt{\frac{3}{2}})\mu(Y)I \geq \frac{\sqrt{3}}{2}(1 - \sqrt{\frac{3}{2}})I$$

follows from (1.32). \[\square\]

The next results can be adopted from [Kr85] without the need to prove them here again, since no special results about the maximal order $\mathcal{O}$ are needed.

**(1.38) Lemma.**

a) $ii \in \mathcal{F}_2(\mathcal{O})$

b) Given $Z = X + iY \in \mathcal{F}_2(\mathcal{O})$ then $Z_\lambda := X + i\lambda Y \in \mathcal{F}_2(\mathcal{O})$ holds for all $\lambda \geq 1$.

c) $\mathcal{F}_2(\mathcal{O})$ is connected.

**Proof:** [Kr85, ch.II, le.3.2]. \[\square\]

**(1.39) Proposition.** Given $Z \in \mathcal{H}(\mathcal{H})$ there is an $M_0 \in \text{Sp}_2(\mathcal{O})$ such that

$$\det(\Im(M(Z))) \leq \det(\Im(M_0(Z)))$$

holds for all $M \in \text{Sp}_2(\mathcal{O})$.

**Proof:** [Kr85, ch.II, prop.3.3]. \[\square\]

**(1.40) Lemma.** Given $Z \in \mathcal{H}(\mathcal{H})$ there exists an $M \in \text{Sp}_2(\mathcal{O})$ such that

$$M(Z) \in \mathcal{F}_2(\mathcal{O}).$$

**Proof:** [Kr85, ch.II, le.3.4]. \[\square\]
(1.41) Proposition. If \( Z, M(Z) \in \mathcal{F}_2(\mathcal{O}) \) for some \( M \in \text{Sp}_2(\mathcal{O}), M \neq \pm I \), then \( Z \) and \( M(Z) \) are contained in the boundary of \( \mathcal{F}_2(\mathcal{O}) \).

Proof: [Kr85, ch.II, prop.3.5] □

For \( \alpha > 0 \) we define the so-called Siegel's elementary set:

\[
S_n(\mathbb{H})[\alpha] := \{ Z = (x_{k,l}) + i(y_{k,l}) \in \mathcal{H}_n(\mathbb{H}); N(x_{k,l}) < \alpha^2, Y \in \mathcal{E}_n(\mathbb{H})[\alpha], 1 < \alpha y_{1,1} \}
\]

(1.42) Proposition. 

a) For all \( \alpha > 3 \) the following holds true:

\[
\mathcal{F}_2(\mathcal{O}) \subset S_2(\mathbb{H})[\alpha].
\]

b) Given a compact subset \( C \subset \mathcal{H}_n(\mathbb{H}) \) there exists \( \beta > 0 \) such that

\[
C \subset S_n(\mathbb{H})[\beta].
\]

Proof: a) Let \( Z = (x_{k,l}) + i(y_{k,l}) \in \mathcal{F}_2(\mathcal{O}) \). According to (1.29), \( \mathcal{R}(\mathcal{O}) \subset \mathcal{E}_2(\mathbb{H})[\alpha] \) for all \( \alpha > 3 \). In the proof of (1.37) we have seen that \( y_{1,1} \geq \frac{\sqrt{3}}{2} > \frac{1}{8} \) holds true. Finally, \( (x_{k,l}) \in \mathcal{C}(\mathcal{O}) \) yields \( N(x_{k,l}) \leq 6 \cdot \frac{1}{8} < \alpha^2 \) for all \( 1 \leq k \leq l \leq 2 \), and the claim follows.

b) Identical proof as for [Kr85, ch.II, prop.3.6] □

Again, [Kr85, ch.II, prop.3.7, cor.3.8, prop.3.9] can be adopted literally for our case (with [Kr85, ch.II, cor.3.8] only being necessary for (F.4) and (F.5)). Furthermore, the co-called symplectic volume of \( \mathcal{F}_2(\mathcal{O}) \) is given by

\[
\text{vol}(\mathcal{F}_2(\mathcal{O})) := \int_{\mathcal{F}_2(\mathcal{O})} dv := \int_{\mathcal{F}_2(\mathcal{O})} (\det(Y))^{-6} dY dX.
\]

It is finite because of the same arguments found in [Kr85, ch.II, prop.3.9]. Here, \( dv := (\det(Y))^{-6} dY dX \) is a volume element in \( \mathcal{H}(\mathbb{H}) \) which is invariant under all biholomorphic functions of \( \mathcal{H}(\mathbb{H}) \) (cf. [Kr85, ch.II, thm.1.10]). Now we summarize the preceding results in

(1.43) Theorem. \( \mathcal{F}_2(\mathcal{O}) \) is a fundamental domain of the quaternionic half-space \( \mathcal{H}(\mathbb{H}) \) with respect to the action of the quaternionic modular group \( \text{Sp}_2(\mathcal{O}) \) and \( \mathcal{F}_2(\mathcal{O}) \) possesses finite symplectic volume. \( \mathcal{F}_2(\mathcal{O}) \) is connected and closed in \( \text{Her}_2(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \).

For later purposes, we need an upper bound for a certain quantity:

(1.44) Proposition. The quantity

\[
s_2(\mathcal{O}) := \sup \{ \text{tr}(Y^{-1}); Z = X + iY \in \mathcal{F}_2(\mathcal{O}) \}
\]

is finite and satisfies

\[
s_2(\mathcal{O}) \leq 4\sqrt{3}.
\]

Proof: According to (1.37), \( Y \geq \frac{\sqrt{3}}{2} (1 - \sqrt{3}) I \) holds for all \( Z = X + iY = (x_{k,l}) + i(y_{k,l}) \in \mathcal{F}_2(\mathcal{O}) \).
So of course \( Y \) is invertible and for \( \det(Y) = y_{1,1}y_{2,2} - N(y_{1,2}) \) a short calculation yields

\[
\text{tr}(Y^{-1}) = \text{tr}\left( \frac{1}{\det(Y)} \begin{pmatrix} y_{2,2} & -y_{1,2} \\ -y_{1,2} & y_{1,1} \end{pmatrix} \right) = \frac{y_{1,1} + y_{2,2}}{y_{1,1}y_{2,2} - N(y_{1,2})},
\]

which is the usual result one would obtain for 2 × 2 matrices over fields (which \( \mathbb{H} \) is not). Since \( Y \in \mathcal{R}(O) \), (1.28) yields

\[
\det(Y) = y_{1,1}y_{2,2} - N(y_{1,2}) \geq \frac{1}{3}y_{1,1}y_{2,2} - \frac{2}{3}y_{1,2}y_{2,2} = \frac{1}{3}y_{1,1}y_{2,2},
\]

and thus

\[
\text{tr}(Y^{-1}) \leq \frac{y_{1,1} + y_{2,2}}{\frac{1}{3}y_{1,1}y_{2,2}} = 3(y_{1,1}^{-1} + y_{2,2}^{-1}).
\]

Furthermore, we have \( y_{2,2} \geq y_{1,1} \geq \sqrt{\frac{3}{2}} \) (as seen in the proof of (1.42)), and finally we get

\[
\text{tr}(Y^{-1}) \leq 3 \cdot 2 \cdot \frac{2}{\sqrt{3}} = 4\sqrt{3}.
\]

\[\square\]

1.3 Introduction to quaternionic modular forms

Now that we have defined and analyzed all necessary requirements, we can finally define quaternionic modular forms and give some first important facts. We begin by defining a positive definite bilinear form on \( \text{Her}_n(\mathbb{H}) \). The fact that it is a positive definite bilinear form on \( \text{Her}_n(\mathbb{H}) \) indeed, can be found in [Kr85, ch.I, prop.3.1].

\[\begin{align*}
(1.45) \text{ Definition.} & \quad \text{For matrices } A, B \in \mathbb{H}^{m \times n} \text{ (or also for } A, B \in \mathbb{H}^{m \times n} \otimes_{\mathbb{R}} \mathbb{C}) \text{ we put } \\
& \quad \tau(A, B) := \frac{1}{2} \text{tr}(AB' + BA'). \\
& \quad \text{Note that } \tau \text{ turns out to be the canonical scalar product on } \mathbb{H}^{m \times n} \text{ according to [Kr85, ch.I, p.20].}
\end{align*}\]

From [Kr85, ch.IV, prop.1.1] we cite a few facts about \( \tau \) which we will need quite often. The last fact (which means c)) can be obtained by an easy computation.

\[\begin{align*}
(1.46) \text{ Proposition.} & \quad a) \quad \text{Suppose that } S = S' \in \mathbb{H}^{m \times m} \otimes_{\mathbb{R}} \mathbb{C}, T = T' \in \mathbb{H}^{n \times n} \otimes_{\mathbb{R}} \mathbb{C} \text{ and } A, B \in \mathbb{H}^{m \times n} \otimes_{\mathbb{R}} \mathbb{C}. \text{ Then } \\
& \quad \tau(A, B) = \tau(B, A) = \tau(A', B'), \\
& \quad \tau(S[A + B], T) = \tau(S[A] + S[B], T) + 2\tau(SA, BT).
\end{align*}\]

\[\begin{align*}
& \quad b) \quad \text{Given } A \in \mathbb{H}^{m \times n} \otimes_{\mathbb{R}} \mathbb{C}, B \in \mathbb{H}^{n \times p} \otimes_{\mathbb{R}} \mathbb{C}, \text{ and } C \in \mathbb{H}^{m \times p} \otimes_{\mathbb{R}} \mathbb{C} \text{ one has } \\
& \quad \tau(AB, C) = \tau(A, CB') = \tau(B, A'C).
\end{align*}\]
c) Given $A = (a_{k,l})$, $B = (b_{k,l}) \in H^{n \times n} \otimes_R C$ then
\[
\tau(A, B) = \sum_{1 \leq k \leq n, 1 \leq l \leq n} \text{Re}(a_{k,l} \overline{b}_{k,l})
\]
holds if we define $\overline{z} = \overline{x} + i\overline{y}$ and $\text{Re}(z) = \frac{1}{2}(z + \overline{z}) = \text{Re}(x) + i\text{Re}(y) \in C$ for $z = x + iy \in H \otimes_R C$. Furthermore, if in addition $A = \overline{A}$, $B = \overline{B}$, the formula is reduced to
\[
\tau(A, B) = \sum_{j=1}^{n} a_{jj} b_{jj} + \sum_{1 \leq k < l \leq n} 2 \text{Re}(\overline{a}_{k,l} b_{k,l})
\]
because $\text{Re}(\overline{ab}) = \text{Re}(\overline{a}b)$ holds for all $a, b \in H \otimes_R C$.

**Remark.** Note that one should not become confused by the abuse of notation, here. For $Z = X + iY \in H_n(H)$ we defined $\Re(Z) = X$, while for $z = x + iy \in H \otimes_R C$ we define $\Re(z) = \frac{1}{2}(z + \overline{z}) = \Re(x) + i\Re(y)$. Furthermore, if we are obviously talking about $\tau = x + iy \in C$ (for example $\tau \in H$) in the one-dimensional setting, then $\Re(\tau) = x$ as usual. This notation was chosen to keep the formulas that will occur as simple as possible. But note that it should always be clear what kind of “real part” is meant: The notation $\Re(z) = \Re(x) + i\Re(y)$ will only be used in the context of Fourier-expansions of quaternionic modular forms of degree greater or equal to two, since $\tau(\cdot, \cdot)$ will play a role there (see (1.54) further below). And note that for $z_1, z_2 \in H \otimes_R C$, $\Re(z_1 \overline{z}_2)$ can also be written (or defined) as
\[
\Re(z_1 \overline{z}_2) := \tau(z_1, z_2) \quad (1.7)
\]
So for $a, b \in H$ one obtains $\tau(a, b) = \frac{1}{2}(ab + \overline{a} \overline{b}) = \Re(ab)$, indeed (where $\Re(\cdot)$ is the normal real part of a quaternion). So we just “continued” this fact and notation to $H \otimes_R C$. Further below we will give an explicit expression in the standard basis of $O$.

Next, we can define the dual lattice of the lattice $\text{Her}_n(O)$ with respect to $\tau$. We will need it for the Fourier-expansions of quaternionic modular forms. The occurring equalities can already be found in [Kl98, p.18].

**Definition.** The dual lattice of the lattice $\text{Her}_n(O)$ with respect to $\tau$ is given by
\[
\text{Her}_n^\tau(O) := \{ T \in \text{Her}_n(H) \mid \tau(S, T) \in \mathbb{Z} \text{ for all } S \in \text{Her}_n(O) \}
\]
\[
= \{ T = (t_{k,l}) \in \text{Her}_n(H) \mid t_{jj} \in \mathbb{Z}, 2t_{k,l} \in \mathcal{O}^2, 1 \leq j \leq n, 1 \leq k < l \leq n \}
\]
\[
= \{ T \in \text{Her}_n(H) \mid T[g] \in \mathbb{Z} \text{ for all } g \in \mathcal{O}^n \}.
\]
Moreover, we then have $\text{vol}(\text{Her}_n(O)) = (\text{vol}(\mathcal{O}))^{n(n-1)/2} = \left( \frac{3}{4} \right)^{n(n-1)/2}$, in particular.

To define modular forms, we first need the concept of an automorphy factor. Note, that $H_n(H)$ is a convex cone according to [Kr85, ch.I, prop.3.12], and that $Z \mapsto \det(\mathcal{M}(Z))$, $M \in \text{Sp}_n(H)$ is holomorphic on $H_n(H)$ without zeros, because it is just a polynomial in the complex entries of $Z$ and because $\mathcal{M}(Z)$ is always invertible according to (1.5) and (1.34). So a holomorphic branch
As both sides are holomorphic functions, in fact $k$ the ordinary power $(\det(M\{\tilde{Z}\}))'$. Furthermore, according to (1.5)

$$\det((\tilde{M}_1\tilde{M}_2}\{\tilde{Z}\}) = \det(\tilde{M}_1\{\tilde{M}_2\{\tilde{Z}\})\cdot \det(\tilde{M}_2\{\tilde{Z}\})$$ \hspace{1cm} (1.9)

holds for $M_1, M_2 \in \text{Sp}_n(H)$ and $Z \in \mathcal{H}_n(H)$. For $r \in Z$ this leads to

$$(\det((\tilde{M}_1\tilde{M}_2}\{\tilde{Z}\}))' = (\det(\tilde{M}_1\{\tilde{M}_2\{\tilde{Z}\}))' \cdot (\det(\tilde{M}_2\{\tilde{Z}\}))' \cdot (\det(M_1\{M_2\{Z\})').$$ \hspace{1cm} (1.10)

But note that for arbitrary $r \in C$ we only get

$$(\det((\tilde{M}_1\tilde{M}_2}\{\tilde{Z}\}))' = \varepsilon_r(M_1, M_2) \cdot (\det(\tilde{M}_1\{\tilde{M}_2\{\tilde{Z}\}))' \cdot (\det(\tilde{M}_2\{\tilde{Z}\}))' \cdot (\det(M_1\{M_2\{Z\})').$$ \hspace{1cm} (1.11)

As both sides are holomorphic functions, in fact $\varepsilon_r(M_1, M_2)$ only depends on $M_1$ and $M_2$ (and $r$ of course) as indicated, and not on $Z$. So we get

$$\varepsilon_r(M_1, M_2) = \frac{(\det((\tilde{M}_1\tilde{M}_2}\{\tilde{Z}\}))' \cdot (\det(M_1\{M_2\{Z\})')}{(\det(\tilde{M}_1\{\tilde{M}_2\{\tilde{Z}\}))' \cdot (\det(\tilde{M}_2\{\tilde{Z}\})'}. \hspace{1cm} (1.12)

The relations 1.9 and 1.10 are called cocycle relations.

Now, we can introduce the so-called (quaternionic) slash-operator. Given a holomorphic function $f : \mathcal{H}_n(H) \to C$, $M \in \text{Sp}_n(H)$ and $k \in C$, we define

$$f|_k M : \mathcal{H}_n(H) \to C, \quad Z \mapsto (\det(M\{Z\}))^{-k/2} f(M(Z)).$$ \hspace{1cm} (1.13)

If $k \in 2\mathbb{Z}$, then 1.10 and (1.5) lead to

$$f|_k (M_1M_2) = f|_k M_1|_k M_2$$ \hspace{1cm} (1.14)

for all $M_1, M_2 \in \text{Sp}_n(H)$. Again, for arbitrary $k \in C$, 1.11 only gives us

$$f|_k (M_1M_2) = \varepsilon_{k/2}(M_1, M_2)^{-1} \cdot f|_k M_1|_k M_2.\hspace{1cm} (1.15)$$

The reason why we choose the exponent to be $\frac{k}{2}$ and not $k$ (as “$|_k$” might indicate) will become clear in section 1.5. Anyways, this slash-operator gives rise to define so-called automorphy factors and multiplier systems.

**1.48 Definition.** Let $\Gamma$ be a subgroup of $\text{Sp}_n(O)$. A map $j : \Gamma \times \mathcal{H}_n(H) \to C^*$ is called an automorphy factor with regard to $\Gamma$ if

$$\mathcal{H}_n(H) \to C, \quad Z \mapsto j(M, Z)$$
is holomorphic for every $M \in \Gamma$ and the so-called cocycle relation

$$j(M_1 M_2, Z) = j(M_1, M_2/Z) \cdot j(M_2, Z)$$

holds for all $M_1, M_2 \in \Gamma$. A map $\nu : \Gamma \to \mathbb{C}^*$ is called a multiplier system for $\Gamma$ of weight $k \in \mathbb{C}$ if

$$j(M, Z) := \nu(M) \cdot (\det(\tilde{M} \{ \tilde{Z} \}))^{k/2}$$

is an automorphy factor.

1.49 Remark. a) According to 1.10, $j(M, Z) := (\det(\tilde{M} \{ \tilde{Z} \}))^{k/2}$ is an automorphy factor if $k \in 2\mathbb{Z}$. So suppose $\nu$ is a multiplier system of weight $k \in 2\mathbb{Z}$. Then

$$\nu(M_1 M_2) \cdot (\det(\tilde{M}_1 \tilde{M}_2 \{ \tilde{Z} \}))^{k/2}$$

$$= \left[ \nu(M_1) \cdot (\det(\tilde{M}_1 \{ \tilde{Z} \}))^{k/2} \right] \cdot \left[ \nu(M_2) \cdot (\det(\tilde{M}_2 \{ \tilde{Z} \}))^{k/2} \right]$$

if and only if $\nu(M_1 M_2) = \nu(M_1)\nu(M_2)$ holds for all $M_1, M_2 \in \Gamma$, i.e. if and only if $\nu$ is an abelian character of $\Gamma$.

b) For arbitrary $k \in \mathbb{C}$, we can only refer to 1.11 and the relation from above no longer holds true in general. But again both sides are holomorphic functions, so in this case $\nu$ is a multiplier system of weight $k$ if and only if

$$\nu(M_1 M_2) = \nu(M_1)\nu(M_2) \cdot \varepsilon_{k/2}(M_1, M_2)^{-1}$$

$$= \nu(M_1)\nu(M_2) \cdot \frac{(\det(\tilde{M}_1 \{ \tilde{Z} \}))^{k/2} \cdot (\det(\tilde{M}_2 \{ \tilde{Z} \}))^{k/2}}{(\det((\tilde{M}_1 \tilde{M}_2 \{ \tilde{Z} \}))^{k/2})}$$

(1.16)

holds for all $M_1, M_2 \in \Gamma$. This is a condition not easy to check and in general $\nu$ will not be an abelian character.

c) Suppose $\Gamma = \text{Sp}_2(\mathcal{O})$. By means of (1.20), we already determined all multiplier systems of even weight $k \in 2\mathbb{Z}$, i.e. all abelian characters. Later on, we will see that a multiplier system of weight $k \in \mathbb{C}$ exists if and only if $k \in \mathbb{Z}$, and that in the case of an odd $k$ exactly two multiplier systems exist. At the end of this section, we will show that $k \in \mathbb{Z}$ is a necessity. The existence will have to wait until we introduce so-called Maaß forms of odd weight.

d) Suppose $\nu$ is a multiplier system for $\Gamma$ of weight $k \in \mathbb{C}$ and $f : \mathcal{H}_n(\mathbb{H}) \to \mathbb{C}$, then by definition

$$\nu(M_1 M_2)^{-1} \cdot f|_k(M_1 M_2) = \nu(M_1)^{-1} \cdot f|_k M_1 |_k M_2$$

(1.17)

holds for all $M_1, M_2 \in \Gamma$.

Finally, we can get to the definition of quaternionic modular forms.
(1.50) Definition. Let $k \in \mathbb{C}$, $\Gamma \leq \text{Sp}_{n}(\mathcal{O})$ of finite index and $\nu$ a multiplier system for $\Gamma$ of weight $k$. A function $f : \mathcal{H}_{n}(\mathbb{H}) \to \mathbb{C}$ is called a quaternionic modular form of weight $k$ and degree $n$ with respect to $\Gamma$ and $\nu$ if the following conditions are fulfilled:

(M.1) $f$ is holomorphic.

(M.2) $f|_{k}M = v(M) \cdot f$ for all $M \in \Gamma$.

(M.3) In the case of $n = 1$ the functions $f|_{k}M$, $M \in \text{Sp}_{1}(\mathcal{O})$ are bounded in the domain $\{z \in \mathbb{C} : \text{Im}(z) \geq \beta\}$ for any $\beta > 0$.

The $\mathbb{C}$-vector space of all quaternionic modular forms of weight $k$ with respect to $\Gamma$ and $\nu$ is denoted by $[\Gamma, k, \nu]$.

If $n = 2$ and $k \in 2\mathbb{Z}$ we also allow $\Gamma$ to be a subgroup of $\Gamma(\mathcal{O})$ with finite index and $\nu$ an abelian character of $\Gamma$. Again, in this case $f : \mathcal{H}(\mathbb{H}) \to \mathbb{C}$ is called a quaternionic modular form of weight $k$ and degree 2 with respect to $\Gamma$ and $\nu$ if (M.1) and (M.2) are fulfilled, where for $\tau$ (M.2) translates to

$$f(Z') = v(\tau) \cdot f(Z).$$

Let $\epsilon = v(\tau)$. $f$ is then called $\epsilon$-symmetric, or simply symmetric if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$.

If again $\Gamma \leq \text{Sp}_{2}(\mathcal{O})$ and $\epsilon \in \mathbb{C}$, then the $\mathbb{C}$-vector space of all $\epsilon$-symmetric quaternionic modular forms of weight $k$ with respect to $\Gamma$ and $\nu$ is denoted by $[\Gamma, k, \nu]_{\epsilon}$.

Of course $f \equiv 0$ is always a quaternionic modular form of any weight, and if $\nu \equiv 1$ every constant function is a quaternionic modular form of weight 0. If $\nu_{1}$ and $\nu_{2}$ are multiplier systems for $\Gamma$ with weights $k_{1}, k_{2} \in \mathbb{C}$, then by definition $\nu_{1}\nu_{2}$ is a multiplier system of weight $k_{1} + k_{2}$. It is easy to see that

$$f \cdot g \in [\Gamma, k_{1} + k_{2}, \nu_{1}\nu_{2}] \quad (1.18)$$

holds for $f \in [\Gamma, k_{1}, \nu_{1}]$ and $g \in [\Gamma, k_{2}, \nu_{2}]$. Now, note that by definition and due to (1.34), $(\det(\text{Trans}(S)\{Z\}))^{k/2} = (\det(\text{Rot}(U)\{Z\}))^{k/2} = 1$ holds for all $Z \in \mathcal{H}_{n}(\mathbb{H})$, $S \in \text{Her}_{n}(\mathbb{H})$ and $U \in \text{GL}_{n}(\mathcal{O})$. $\text{Sp}_{n}(\mathcal{O})$ is generated by $J_{n}$, Trans$(S)$, and Rot$(U)$ according to (1.15). Together with 1.17 this leads to the following

(1.51) Lemma. A holomorphic function $f : \mathcal{H}_{n}(\mathbb{H}) \to \mathbb{C}$ (satisfying (M.3) if $n = 1$) is a modular form of weight $k$ for $\Gamma = \text{Sp}_{n}(\mathcal{O})$ with multiplier system $\nu$ if and only if the following properties are satisfied for all $Z \in \mathcal{H}_{n}(\mathbb{H})$:

(i) $f(Z + S) = v(\text{Trans}(S)) \cdot f(Z)$ for all $S \in \text{Her}_{n}(\mathcal{O})$,

(ii) $f(Z[U]) = v(\text{Rot}(U)) \cdot f(Z)$ for all $U \in \text{GL}_{n}(\mathcal{O})$,

(iii) $f(-Z^{-1}) = v(J_{n}) \cdot (\det(\hat{Z}))^{k/2}f(Z)$.

Because of (1.34), (iii) can also be written as $f(-Z^{-1}) = v(J_{n}) \cdot (\det(Z))^{k}f(Z)$ if $k \in 2\mathbb{Z}$. 

1.3 Introduction to quaternionic modular forms
Remark. Note that in the case of \( n = 1 \) we are speaking about ordinary elliptic modular forms. As we have seen, \( \mathcal{H}_1(\mathbb{H}) = \mathcal{H}_1(\mathbb{R}) = \mathcal{H} \) is the upper complex half-plane and \( \text{Sp}_1(\mathcal{O}) = \{ \epsilon M ; M \in \text{SL}_2(\mathbb{Z}), \epsilon \in \mathcal{E} \} \). Thus (M.2) translates to

\[
\nu(\epsilon M) \cdot f(\tau) = f|_k(\epsilon M)(\tau) = (\det((\tilde{e}I \cdot \tilde{M})\{\tilde{\tau}\}))^{-k/2}f((\epsilon M)\langle\tau\rangle)
= (\det(eI) \cdot \det(M\{\tau\}))^{-k/2}f((a\tau + b)(c\tau + d)^{-1})
= (N(\epsilon)^2 \cdot (M\{\tau\})^2)^{-k/2}f(M\langle\tau\rangle) = \delta_M \cdot (M\{\tau\})^{-k}f(M\langle\tau\rangle)
\]

for \( \tau \in \mathcal{H}_1(\mathbb{R}) \), \( M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \) and some \( \delta_M \in \mathbb{C} \). This is just the ordinary definition of an elliptic modular form, where the multiplier system would be \( \nu(M) \cdot \delta_M^{-1} \). We just have the extra condition \( \nu(\epsilon M) = \nu(M) \) or otherwise \( f \) vanishes identically. So we restrict ourselves to \( \epsilon = 1 \) and multiplier systems that fulfill the required condition. Confer [KK07] for further details about elliptic modular forms.

For further analysis, we need to regard quaternionic modular forms and multiplier systems from a reversed perspective.

(1.52) Lemma. Let \( \Gamma \leq \text{Sp}_n(\mathcal{O}), k \in \mathbb{C} \). Suppose that \( f : \mathcal{H}_n(\mathbb{H}) \to \mathbb{C} \) is holomorphic, \( f \not\equiv 0 \), and that \( f \) fulfills

\[
f|_k M = \nu(M) \cdot f \quad \text{for all } M \in \Gamma
\]

for some map \( \nu : \Gamma \to \mathbb{C}^* \). Then \( \nu \) turns out to be a multiplier system for \( \Gamma \) of weight \( k \). The same holds true if a priori the functional equation from above only holds for a generating set of \( \Gamma \) by extending \( \nu \) to \( \Gamma \) by setting

\[
\nu(M_1M_2) = \nu(M_1)\nu(M_2) \cdot \frac{(\det(M_1\{M_1^{\langle\tilde{I}\rangle}\}))^{k/2} \cdot (\det(M_2\{\tilde{I}\}))^{k/2}}{(\det((M_1M_2)\{\tilde{I}\}))^{k/2}}
\]

for \( M_1, M_2 \in \Gamma \), while \( \nu(I) := 1 \). In particular, \( \nu \) is well defined by the equation above and then the functional equation holds true for all \( M \in \Gamma \).

Proof: Since the first assertion follows from the second one, we will look at the case that for some generating set \( G \) of \( \Gamma \),

\[
f|_k M = \nu(M) \cdot f
\]

holds for all \( M \in G \). Now, let \( M_1, M_2 \in G \), then the assumption, 1.12 and 1.15 lead to

\[
f|_k(M_1M_2) = \frac{(\det(M_1\{M_1^{\langle\tilde{I}\rangle}\}))^{k/2} \cdot (\det(M_2\{\tilde{I}\}))^{k/2}}{(\det((M_1M_2)\{\tilde{I}\}))^{k/2}} \cdot \nu(M_1)\nu(M_2) \cdot f.
\]

Since \( f \not\equiv 0 \), the prefactor of \( f \) on the right side of the equation cannot depend on the special choice of \( M_1 \) and \( M_2 \), only on the product \( M_1M_2 \), and \( \nu(M_1M_2) \) becomes well defined, as asserted. So an extension of \( \nu \) to \( \Gamma \) has to exist and is given by both the functional equations of \( f \) and also the defining equation seen above. Now, we only have to prove that \( \nu \) is a multiplier system for \( \Gamma \) of weight \( k \), indeed. But this is given immediately by looking at the “defining equation” of \( \nu \) and using 1.16 in (1.49).
With help of the preceding lemma, we can show now that the vector spaces of modular forms for congruent subgroups and certain multiplier systems are isomorphic.

**Lemma.** Suppose that \( \Gamma \leq \text{Sp}_n(\mathcal{O}) \) is a subgroup of finite index, \( k \in \mathbb{C} \) and \( \nu \) is a multiplier system for \( \Gamma \) of weight \( k \). Then for every \( f \in [\Gamma, k, \nu] \) and \( M \in \text{Sp}_n(\mathcal{O}) \)

\[
\nu_M(M^{-1}\Gamma M, k, \nu_M) = \varphi_k(M, M^{-1}\Gamma M) \cdot f|_k(M^{-1}\Gamma M)
\]

holds, where \( \nu_M \) is a certain multiplier system for \( M^{-1}\Gamma M \) of weight \( k \). If \( k \in 2\mathbb{Z} \), \( \nu_M \) is given by \( \nu_M(M^{-1}\Gamma M) = \nu(K) \) for all \( K \in \Gamma \).

**Proof:** So let \( f \in [\Gamma, k, \nu], f \neq 0, M \in \text{Sp}_n(\mathcal{O}) \) and \( K \in \Gamma \). According to 1.15, we get

\[
\nu_M(M^{-1}\Gamma M) := \frac{\varphi_k(M, M^{-1}\Gamma M)}{\varphi_k(K, M)} \cdot f|_k(KM) = \frac{\varphi_k(M, M^{-1}\Gamma M)}{\varphi_k(K, M)} \cdot f|_k\nu(K) \cdot f|_kM.
\]

Thus, according to (1.52),

\[
\nu_M(M^{-1}\Gamma M) := \frac{\varphi_k(M, M^{-1}\Gamma M)}{\varphi_k(K, M)} \cdot f|_k\nu(K), \quad K \in \Gamma
\]

is a multiplier system for \( M^{-1}\Gamma M \) of weight \( k \), and \( f|_k\nu_M(M^{-1}\Gamma M, k, \nu_M) = \nu(K) \cdot f|_kM \). If \( k \in 2\mathbb{Z} \), then \( \varphi_k(\cdot, \cdot) \equiv 1 \) holds according to 1.10. \( \square \)

We are going to have a look at the Fourier-expansion of a quaternionic modular form, now. Since \( \Gamma \) is of finite index in \( \text{Sp}_n(\mathcal{O}) \), there is also a subgroup \( G \leq \text{Her}_n(\mathcal{O}) \) (in an additive sense, so “submodule” would be more precise) of finite index such that \( \text{Trans}(S) \in \Gamma \) for all \( S \in G \). Now, suppose there is also a subgroup \( H \leq G \) (again rather a submodule) of finite index with \( \nu(\text{Trans}(S)) = 1 \) for all \( S \in H \). Then \( f(Z + S) = f(Z) \) for all \( S \in H \) and \( Z \in \text{Her}_n(\mathcal{H}) \), and by the same methods used in [Kr85, ch.III, thm.1.2] any quaternionic modular form \( f \in [\Gamma, k, \nu] \) possesses some Fourier-expansion

\[
f(Z) = \sum_T a_f(T)e^{2\pi i \nu(T, Z)}, \quad Z \in \text{Her}_n(\mathcal{H}),
\]

where \( T \) runs through some discrete subset of \( \text{Her}_n(\mathcal{H}) \). This subset is rational in the sense that the entries of \( T \) have rational coefficients with respect to the basis of \( \mathcal{O} \). By means of the Koecher’s principle in [Kr85, ch.III, le.1.5], even \( T \geq 0 \) holds. We can get more concrete if actually \( f(Z + S) = f(Z) \) holds for all \( S \in \text{Her}_n(\mathcal{O}) \). Then, we can cite [Kr85, Satz 2.2.2]. By adopting the Koecher’s principle found in [Kr85, ch.III, le.1.5] to our case, and also [Kr85, ch.III, thm.1.6], we summarize those results in the following theorem, which is crucial for the upcoming considerations.
(1.54) Theorem. Suppose \( f : \mathcal{H}_n(\mathbb{H}) \to \mathbb{C} \) is holomorphic and satisfies \( f(Z + S) = f(Z) \) for all \( Z \in \mathcal{H}_n(\mathbb{H}) \) and all \( S \in \text{Her}_n(\mathcal{O}) \) (so \( f \) is said to be periodic), then \( f \) possesses a Fourier-expansion

\[
f(Z) = \sum_{T \in \text{Her}_n(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}_n(\mathbb{H}).
\]

It converges absolutely and uniformly on \( \{ Z \in \mathcal{H}_n(\mathbb{H}) \mid \text{Im}(Z) \geq \beta \} \) for all \( \beta > 0 \). The Fourier-coefficients \( \alpha_f(T) \) do not depend on \( Z \), are uniquely determined by \( f \) and given by

\[
\alpha_f(T) = \text{vol}(\text{Her}_n(\mathcal{O})) \cdot e^{2\pi i \tau(T,Y)} \cdot \int_{\mathcal{C}_n(\mathcal{O})} f(X + iY)e^{-2\pi i \tau(T,X)} \, dX,
\]

where \( Y \in \text{Pos}_n(\mathbb{H}) \) is fixed, \( dX \) is the euclidean volume element of \( \text{Her}_n(\mathbb{H}) \) and \( \mathcal{C}_n(\mathcal{O}) \) is defined accordingly to \( \mathcal{C}(\mathcal{O}) \) in (1.36) for higher dimensions.

Now, the shape of the Fourier-expansion 1.19 in the general case follows, since for \( f \in [\Gamma, k, \nu] \) with the appropriate preconditions there is some \( m \in \mathbb{Z} \) such that \( g(Z) := f(mZ) \) is periodic.

Because of (1.51) there is a certain relation between the Fourier-coefficients lying in the same orbit under the operation \( T \mapsto T[U] \):

(1.55) Corollary. Suppose \( k \in \mathbb{C} \) and \( \nu \) is a multiplier system of \( \text{Sp}_n(\mathcal{O}) \) with \( \nu(\text{Trans}(S)) = 1 \) for all \( S \in \text{Her}_n(\mathcal{O}) \). Let \( f \in [\text{Sp}_n(\mathcal{O}), k, \nu] \) with Fourier-expansion according to (1.54). Then

\[
\alpha_f(T[U]) = \nu(\text{Rot}((\overline{U})^{-1})) \cdot \alpha_f(T)
\]

holds for all \( U \in \text{GL}_n(\mathcal{O}) \) and all \( T \in \text{Her}_n^T(\mathcal{O}) \) with \( T \geq 0 \).

Proof: Of course \( (\overline{U})^{-1} \in \text{GL}_n(\mathcal{O}) \) holds true for all \( U \in \text{GL}_n(\mathcal{O}) \). Because of the third description of \( \text{Her}_n^T(\mathcal{O}) \) in (1.47) and because \( S[M] \geq 0 \) holds for all \( S \in \text{Her}_n(\mathbb{H}) \), \( S \geq 0 \) and \( M \in \mathbb{H}^{n \times n} \) (cf. [Kr85, ch.I, cor.3.7]), \( T \mapsto T[U] \) is a bijection of the set \( \{ T \in \text{Her}_n^T(\mathcal{O}) \mid T \geq 0 \} \) for all \( U \in \text{GL}_n(\mathcal{O}) \). Now (1.46), (1.51) and (1.54) yield

\[
\nu(\text{Rot}((\overline{U})^{-1})) \cdot f(Z) = f(Z[(\overline{U})^{-1}]) = \sum_{T \in \text{Her}_n(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,Z[(\overline{U})^{-1}]})
\]

\[
= \sum_{T \in \text{Her}_n(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,U^{-1}Z)}
\]

\[
= \sum_{T \in \text{Her}_n(\mathcal{O}), T \geq 0} \alpha_f(T[U])e^{2\pi i \tau(T,Z)}.
\]

By (1.54), the uniqueness of the Fourier-coefficients yields the assertion. \( \square \)

As announced, to finish this introduction to quaternionic modular froms we will now show that a multiplier system for the whole modular group \( \text{Sp}_n(\mathcal{O}) \) can only exist if \( k \in \mathbb{Z} \). In the case of degree two, we can even say which multiplier systems of odd weight are hypothetically
possible at all. Their existence will be proved later on. Remember that for \( k \) even, the multiplier systems are given exactly by the abelian characters.

Recall the definition of \( \pi_p \) in (1.19). Again, \( \det \circ \pi_p \) clearly is a group homomorphism \( \text{GL}_2(\mathcal{O}) \to \mathbb{F}_9 \). And thus, using calculation rules for the determinant (see 1.8). As both sides are holomorphic in \( \mathbb{F}_9 \), \( \phi \) is holomorphic in \( \mathcal{O} \). Now \( \omega^2 = -1 \) holds by definition, thus the group \( \langle \omega \rangle \leq \mathbb{F}_9 \) can easily be identified with \( \langle i \rangle \leq \mathbb{C}^* \) by mapping \( \phi_1 : \omega \mapsto i \). So \( \phi_1 \circ \det \circ \pi_p : \text{GL}_n(\mathcal{O}) \to \mathbb{C}^* \) is an abelian character of \( \text{GL}_n(\mathcal{O}) \) with \( \phi_1 \circ \det \circ \pi_p(\text{GL}_n(\mathcal{O})) = \{ \pm 1, \pm i \} \). The same holds true for the second identification \( \phi_2 : \omega \mapsto -i \). Note that \( \phi_1 \circ \det \circ \pi_p \) is of order 4 with \( (\phi_1 \circ \det \circ \pi_p)^3 = \phi_2 \circ \det \circ \pi_p \).

**Theorem (1.56).** Let \( n \geq 2 \) and \( \nu \) a multiplier system for a congruence subgroup \( \Gamma \leq \text{Sp}_n(\mathcal{O}) \) of weight \( k \in \mathbb{C} \). Then \( k \in \mathbb{Q} \) holds. If \( \Gamma = \text{Sp}_n(\mathcal{O}) \), even \( k \in \mathbb{Z} \) holds. Furthermore, if \( n = 2 \) and \( k \in \mathbb{Z} \) is odd, \( \nu \) has to fulfill

\[
\nu(\text{Trans}(S)) = 1 \quad \text{for all } S \in \text{Her}_2(\mathcal{O}),
\nu(\text{Rot}(U)) = \phi_1/2 \circ \det \circ \pi_p(U) \quad \text{for all } U \in \text{GL}_2(\mathcal{O}),
\nu(J) = -1.
\]

**Proof:** To prove the first claim, we will restrict ourselves to the Siegel case, i.e. \( Z \in \mathcal{H}_n(\mathbb{R}) \subset \mathcal{H}_n(\mathbb{H}) \) and \( M \in \text{Sp}_n(\mathbb{H}) \). By definition,

\[
j : \text{Sp}_n(\mathcal{O}) \times \mathcal{H}_n(\mathbb{H}), \quad j(M, Z) := \nu(M) \cdot (\det(\tilde{M}(\tilde{Z})))^{1/2}
\]

is an automorphy factor. Now note that for \( A \in \mathbb{R}^{n \times n} \otimes_{\mathbb{R}} \mathbb{C} \), we have \( \det(\tilde{A}) = \det(A)^2 \). This is an easy calculation by taking into account that \( \tilde{A} \) derives from \( A \) by duplicating \( A \) in the following way (see the definition in (1.33)): Every entry of \( A \) becomes a \( 2 \times 2 \) diagonal matrix with exactly that entry on the diagonal. And thus, using calculation rules for the determinant yields the asserted result. So for \( Z \in \mathcal{H}_n(\mathbb{R}) \) and \( M \in \text{Sp}_n(\mathbb{Z}) \) we get

\[
j(M, Z) = \nu(M) \cdot \exp\left(\frac{\nu}{2} \log(\det(\tilde{M}(\tilde{Z})))\right) = e^{\pi\text{im}k} \cdot \nu(M) \cdot \exp\left(k \log(\det(M\{Z\}))\right).
\]

for some \( m \in \mathbb{Z} \) by calculation rules for logarithms, where log is some branch of the logarithm (see 1.8). As both sides are holomorphic in \( Z \in \mathcal{H}_n(\mathbb{R}) \) without zeros and since \( \mathcal{H}_n(\mathbb{R}) \) is connected (cf. [Kr85, ch.I, prop.3.12]), \( m \) cannot depend on \( Z \) and is therefore fixed for fixed \( M \). If we denote it by \( m_M \), we can define

\[
\tilde{j} : \text{Sp}_n(\mathbb{Z}) \times \mathcal{H}_n(\mathbb{R}), \quad \tilde{j}(M, Z) := e^{\pi\text{im}Mk} \cdot \nu(M) \cdot (\det(M\{Z\}))^k = e^{\pi\text{im}Mk} \cdot \nu(M) \cdot \exp\left(k \log(\det(M\{Z\}))\right)
\]

and

\[
\tilde{\nu} : \text{Sp}_n(\mathbb{Z}) \to \mathbb{C}^*, \quad M \mapsto e^{\pi\text{im}Mk} \cdot \nu(M).
\]

By definition we have \( \tilde{j} = j|_{\text{Sp}_n(\mathbb{Z}) \times \mathcal{H}_n(\mathbb{R})} \), so \( \tilde{j} \) is holomorphic in \( Z \) and fulfills the cocycle relations for all \( Z \in \mathcal{H}_n(\mathbb{R}) \) and \( M \in \text{Sp}_n(\mathbb{Z}) \). Furthermore, \( \text{Sp}_n(\mathbb{Z}) \cap \Gamma \) is a Siegel congruence subgroup as we have seen in the proof of (1.21). Thus, \( \tilde{j} \) is an ordinary automorphy factor by standard definition for the Siegel case (see for example [Ch62]) and \( \tilde{\nu} \) is a multiplier system.
for $\Gamma \cap \text{Sp}_n(Z)$ of weight $k$. It is a well known fact (proved by Christian in [Ch62]), that in the Siegel case multiplier systems for congruence subgroups of degree $n \geq 2$ only exist for rational weight $k$. If $\Gamma = \text{Sp}_n(O)$, then $\text{Sp}_n(Z) \cap \Gamma = \text{Sp}_n(Z)$. In case of the whole Siegel modular group of degree $n \geq 2$ multiplier systems only exist for integral weight $k$. So we have shown $k \in \mathbb{Q}$ and $k \in \mathbb{Z}$ respectively.

For the second claim, let $n = 2$ and $k \in \mathbb{Z}$ be odd. We will look at the automorphy factor $\tilde{j}$, again. As $k \in \mathbb{Z}$, no branch of the logarithm is needed, $\tilde{\nu}$ is an abelian character of $\text{Sp}_2(Z)$ and $\tilde{j}$ is given by the ordinary power

$$\tilde{j}(M,Z) = e^{\pi i \text{Im}Mk} \cdot \nu(M) \cdot (\det(M\{Z\}))^k$$

$$= \pm \nu(M) \cdot (\det(M\{Z\}))^k$$

$$= \tilde{\nu}(M) \cdot (\det(M\{Z\}))^k$$

$$= \nu(M) \cdot \exp\left(\frac{k}{2} \log(\det(M\{\tilde{Z}\}))\right).$$

It was shown in [Ma64] and in [Kli74], as well, that for $n = 2$ exactly two abelian characters exist for the full Siegel modular group. The non-trivial one, denoted by $\chi$, is given by $\chi(j) = 1$ and $\chi(S) = (-1)^{s_{11} + s_{12}}$ for $S = (s_{ij}) \in \text{Sym}_2(Z)$. (Note that it is well known that $\text{Sp}_2(Z)$ is generated by those matrices.) Again, the sign in the second row in the equations above cannot depend on $Z$, so if we knew $\tilde{\nu}(M)$ for some $M \in \text{Sp}_2(Z)$, we can easily compute $\nu(M)$ by comparing the last two rows for $Z = iI$. For $M = J$, $\tilde{\nu}(J) = 1$ holds. As $k$ is odd, one easily computes $(\det(f\{jI\}))^k = (\det(iI)) = -1$, while $\exp\left(\frac{k}{2} \log(\det(iI))\right) = \exp\left(\frac{k}{2} \log(1)\right) = 1$, so

$$\nu(f) = -1$$

has to hold.

As we have seen in the proof of (1.21),

$$\Delta := \{M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \text{Sp}_2(O)\}$$

$$= \{\text{Rot}(U) \cdot \text{Trans}(S) \ ; \ U \in \text{GL}_2(O), S \in \text{Her}_2(O)\} \leq \text{Sp}_2(O)$$

holds, and according to (1.34) we have

$$(\det(M\{\tilde{Z}\}))^{k/2} = \exp\left(\frac{k}{2} \log(\det(D))\right) = 1$$

for all $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta$ since $D \in \text{GL}_2(O)$.

$$\nu(M_1M_2) = \nu(M_1)\nu(M_2) \quad \text{for all } M_1, M_2 \in \Delta$$

immediately follows by 1.16 in (1.49), thus $\nu$ is an abelian character of $\Delta$. So next we have to determine all abelian characters of $\Delta$. Therefore, as usual, we have to look at the commutator subgroup $\Delta'$. In the proof of (1.20) we have seen that

$$[\text{Rot}(U),\text{Trans}(S)] = \text{Trans}(T) \quad \text{with} \quad T = S[U] - S$$

holds for all $U \in \text{GL}_2(O)$ and $S \in \text{Her}_2(O)$. Furthermore, we could show that by this relation alone, $\text{Trans}(S) \in \text{Sp}_2(O)'$ holds for all $S \in \text{Her}_2(O)$, and so again we conclude $\text{Trans}(S) \in \Delta'$
for all $S \in \text{Her}_2(O)$, hence
\[
\nu(\text{Trans}(S)) = 1 \quad \text{for all } S \in \text{Her}_2(O)
\]
follows. But this also means $\tilde{\nu}(\text{Trans}(S)) = 1$ for all $S \in \text{Sym}_2(Z)$, thus $\tilde{\nu}$ has to be the trivial character due to what we have seen above. So according to what we discussed above, $\nu$ is already completely determined for all $M \in \text{Sp}_2(R)$ and given by
\[
\nu(M) = \frac{(\det(M\{iI\}))^k}{(\det(M\{i\})))^{k/2}},
\]
while
\[
\nu(M) \cdot (\det(M\{i\}))^{k/2} = (\det(M\{Z\}))^k
\]
holds for all $M \in \text{Sp}_2(R)$ and $Z \in \mathcal{H}(R)$.

Now, for $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(Z)$ we have $\tilde{\nu}(\text{Rot}(U)) = 1$ and, since $k$ is odd, we calculate $(\det(\text{Rot}(U\{i\}))^k = (\det(U^{-1}))^k = -1$, while $\exp(\frac{i}{2} \log(\det(U^{-1}))) = \exp(\frac{i}{2} \log(1)) = 1$.

Again, we conclude
\[
\nu \left( \text{Rot} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right) = -1.
\]
Of course, $\nu$ is also an abelian character on the subgroup $\{ \text{Rot}(U); U \in \text{GL}_2(O) \} \subseteq \Delta$. As clearly $\text{Rot}(U_1) \cdot \text{Rot}(U_2) = \text{Rot}(U_1U_2)$ holds, looking at the commutator subgroup of this group is equivalent to looking at $\text{GL}_2(O)'$, and every abelian character is also an abelian character of this group. Thus the abelian characters coincide. So let us determine the abelian characters of $\text{GL}_2(O)$.

First, we compute some elements of the commutator subgroup. For $a \in O$ and $\varepsilon \in \mathcal{E}$ one computes
\[
\left[ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & a \varepsilon - a \varepsilon \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)',
\]
so in particular, if we choose $\varepsilon = -1$, $\begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)'$ holds for all $a \in O$. Choosing $a = -\frac{1}{2}i\sqrt{3}$ and $\varepsilon = \frac{1}{2}i\sqrt{3}$, one calculates $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)'$ and therefore $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)'$.

Choosing $a = \varepsilon$ yields $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)'$ as well as $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)'$ for all $\varepsilon \in \mathcal{E}$ by multiplying with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So
\[
\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(O)' \quad \text{for all } \varepsilon \in \mathcal{E}
\]
immediately follows. Repeating the same calculation with $\left[ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \right]$ yields
\[
\begin{pmatrix} 1 & \varepsilon \\ a & 1 \end{pmatrix} \in \text{GL}_2(O)' \quad \text{for all } a \in O.
\]
A further computation gives
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left[ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right] \in \text{GL}_2(O)'.
\]
Now, by multiplying an arbitrary $U \in \text{GL}_2(O)$ with the matrices from above from the left and the right, the Euclidean Algorithm yields that $U$ can be reduced to $\begin{pmatrix} \varepsilon & 0 \\ 0 & \delta \end{pmatrix}$ modulo $\text{GL}_2(O)'$ for some $\varepsilon, \delta \in \mathcal{E}$, i.e.
\[
U \cdot \text{GL}_2(O)' = \begin{pmatrix} \varepsilon & 0 \\ 0 & \delta \end{pmatrix} \cdot \text{GL}_2(O)'.
\]
Another calculation leads to
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{O})'.
\]
So we can reduce $U$ even further and get
\[
U \cdot \text{GL}_2(\mathcal{O})' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \text{GL}_2(\mathcal{O})'
\]
for some $\epsilon \in \mathcal{E}$. By multiplying with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ from the right and its conjugate from the left for some $\delta \in \mathcal{E}$, we can even choose $\epsilon$ as an element of a transversal for the operation $\epsilon \mapsto \delta \epsilon \delta$. A computation using the computer algebra system [SAGE] yields that such a transversal is given by $\{ \pm 1, \pm i \}$. Thus we have shown
\[
| \text{GL}_2(\mathcal{O}) / \text{GL}_2(\mathcal{O})' | \leq 4,
\]
and by what we have already seen,
\[
| \Delta / \Delta' | \leq 4
\]
follows. Now, for \((A_1, \ast), (A_2, \ast) \in \Delta, (A_1, \ast)(A_2, \ast) = (A_1 A_2, \ast)\) holds. So
\[
\varphi : \Delta \rightarrow \text{GL}_2(\mathcal{O}), \quad (A, \ast) \mapsto A
\]
is a surjective homomorphism. Therefore
\[
\varphi_{\text{GL}} := \varphi_1 \circ \det \circ \pi_p \circ \varphi : \Delta \rightarrow \{ \pm 1, \pm i \} \subset \mathbb{C}^*
\]
is an abelian character of $\Delta$. As we have seen, it is of order 4. So $\Delta^\text{ab}$ is of order 4 and generated by $\varphi_{\text{GL}}$. By what we have discussed above, there is $m \in \{0, \ldots, 3\}$ such that
\[
v|_\Delta = \varphi_{\text{GL}}^m
\]
holds. Let $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As we have seen, $v(\text{Rot}(U)) = -1$. One immediately checks that $\varphi_{\text{GL}}^m(\text{Rot}(U)) = (-1)^m$, and therefore $m \in \{1, 3\}$ follows. Note that
\[
\varphi_{\text{GL}}(\text{Rot}(U)) = \varphi_1 \circ \det \circ \pi_p (U) = \varphi_2 \circ \det \circ \pi_p (U) = \varphi_{\text{GL}}^3(\text{Rot}(U))
\]
holds for all $U \in \text{GL}_2(\mathcal{O})$, hence the assertion follows.

Until now, we have not shown that the two hypothetically possible multiplier systems of odd weight constructed in the preceding theorem are indeed multiplier systems. So far, we only know that if such multiplier systems exist, they have to take the values on the generators of $\text{Sp}_2(\mathcal{O})$ as described in the theorem. Later on, we will construct non-identically vanishing quaternionic modular forms that behave accordingly for those generators, and hence by what we have seen those multiplier systems have to exist in the way we described them. But for now, let us just fix some notation for the two hypothetically possible multiplier systems:
1.4 The vector space of quaternionic modular forms

(1.57) Definition. The two hypothetically possible multiplier systems for $\text{Sp}_2(\mathcal{O})$ of odd weight constructed in (1.56) shall be denoted by $\nu_i$ and $\nu_{-i}$. Their values on the generators described in (1.17) are as follows:

\[
\begin{align*}
\nu_i(J) &= \nu_{-i}(J) = -1 \\
\nu_i(\text{Trans}(S)) &= \nu_{-i}(\text{Trans}(S)) = 1 \\
\nu_i \left( \text{Rot} \left( \begin{pmatrix} -1+i\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \right) \right) &= \nu_{-i} \left( \text{Rot} \left( \begin{pmatrix} -1+i\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \right) \right) = 1 \\
\nu_i \left( \text{Rot} \left( \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right) \right) &= i, \quad \nu_{-i} \left( \text{Rot} \left( \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right) \right) = -i
\end{align*}
\]

where $S \in \text{Her}_2(\mathcal{O})$.

Note that in the case of the Hurwitz order, there exists a multiplier system for the whole modular group of weight $k$ if and only if $k \in 2\mathbb{Z}$ is even (cf. [KW98]). So the possibility that in our case a multiplier system for the whole modular group of odd weight – and thus maybe also quaternionic modular forms of odd weight – might exist (and both actually do, as we will see in chapter 3), is quite special, indeed.

We finish this section with a final remark concerning the possibility to extend $\nu_i$ and $\nu_{-i}$ to the extended modular group.

(1.58) Remark. The two hypothetically possible multiplier systems $\nu_i$ and $\nu_{-i}$ for $\text{Sp}_2(\mathcal{O})$ of odd weight cannot be extended to $\langle \text{Sp}_2(\mathcal{O}), i_1 I \rangle$: Suppose that this is indeed possible. Then just like in the proof of (1.56), $\nu_i$ and $\nu_{-i}$ have to be abelian characters on the subgroup $\langle \text{Rot}(U) \rangle; U \in \text{GL}_2(\mathcal{O})$ or $U = i_1 I_2$. Since $(i_1 I_1)^2 = -I$, $\nu_i(i_1 I_1)^2 = \nu_{-i}(i_1 I_1)^2 = \nu_i(-I) = 1$ would have follow, and thus $\nu_i(i_1 I_1), \nu_{-i}(i_1 I_1) \in \{ \pm 1 \}$. But then $i = (-1)^2 i = \nu_i((-i_1 I) \text{Rot} \left( \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right) (i_1 I)) = \nu_i \left( \text{Rot} \left( \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right) \right) = -i$ is a contradiction (and the same follows for $\nu_{-i}$). Therefore, $\nu_i$ and $\nu_{-i}$ cannot be extended to $i_1 I$.

But note that they can actually be extended to $\tau: \mathbb{Z} \mapsto \mathbb{Z}'$. As we will see in chapter 3, there do exist non-identically vanishing quaternionic modular forms of odd weight with respect to the multiplier systems $\nu_i$ and $\nu_{-i}$ — so-called quaternionic Maaß lifts of odd weight. They will turn out to be skew-symmetric. Hence, $\nu_i$ and $\nu_{-i}$ can be extended to $\tau$ with $\nu_i(\tau) = \nu_{-i}(\tau) = -1$. And moreover, we will see that there also exist non-identically vanishing skew-symmetric modular forms of even weight (i.e. with character $\nu_i$). According to 1.18, the product of such a quaternionic modular form with a quaternionic Maaß lift of odd weight will again yield a non-identically vanishing quaternionic modular form for the multiplier system $\nu_i$ or $\nu_{-i}$ — but this time a symmetric one. Thus $\nu_i$ and $\nu_{-i}$ can also be extended to $\tau$ with $\nu_i(\tau) = \nu_{-i}(\tau) = 1$.

1.4 The vector space of quaternionic modular forms

In this section we will have a closer look at quaternionic modular forms and their vector spaces. It will turn out that these vector spaces are indeed finite-dimensional if $\Gamma$ is a congruence subgroup and the multiplier system fulfills a certain condition. We will start with a subvector space by introducing so-called cusp forms.
The following lemma can already be found in [Kl98, Le.2.9] for $\Gamma = \text{Sp}_n(\mathcal{O})$. But as we are going to look at cusp forms in a more general sense, we will fit the proof for this more general case of congruence subgroups. Suppose $\Gamma \leq \text{Sp}_n(\mathcal{O})$ is a congruence subgroup, so there is some two-sided ideal $P \leq \mathcal{O}$ with $\text{Sp}_n(\mathcal{O})|P| \leq \Gamma$. Now consider $\Delta$ from the proof of (1.56). Again, a multiplier system $\nu$ of $\Gamma$ is an abelian character of $\Delta \cap \Gamma$, and in particular of $\Delta \cap \text{Sp}_n(\mathcal{O})|P|$. Just as in the proof of (1.20), for $U \in \text{GL}_2(\mathcal{O})$ and $S \in \text{Her}_2(\mathcal{O})$ one calculates

\[
[\text{Rot}(U), \text{Trans}(S)] = \text{Trans}(T) \quad \text{with} \quad T = S[U] - S
\]

and we get

\[
T = \begin{pmatrix} n \cdot N(a) & n \cdot a \\ n \cdot a & 0 \end{pmatrix} \quad \text{for} \quad U = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}.
\]

with $n \in \mathbb{Z} \cap P, a \in P$. Choosing $-a$ instead of $a$ and multiplying the resulting matrices leads to $\text{Trans} \left( \begin{pmatrix} 2n \cdot N(a) & 0 \\ 0 & 0 \end{pmatrix} \right), \text{Trans} \left( \begin{pmatrix} 0 & 0 \\ 0 & 2n \cdot N(a) \end{pmatrix} \right) \in (\Delta \cap \text{Sp}_n(\mathcal{O})|P|)'$, hence also $\text{Trans} \left( \begin{pmatrix} 0 & 2n \cdot a \\ 2n \cdot a & 0 \end{pmatrix} \right) \in (\Delta \cap \text{Sp}_n(\mathcal{O})|P|)'$. Thus $\nu(\text{Trans}(S)) = 1$ holds for all $S \in H$ for some subgroup $H \subset \text{Her}_n(\mathcal{O})$ of finite index. So as we have seen in the remark before (1.54), every $f \in [\Gamma, k, \nu]$ possesses an absolutely and locally uniformly convergent Fourier-expansion

\[
f(Z) = \sum_{T \geq 0} \alpha_f(T) e^{2\pi i T(Z)}, \quad Z \in \mathcal{H}_n(\mathbb{H}),
\]

where $T \geq 0$ runs through some discrete subset of $\text{Her}_n(\mathbb{H})$. As we will not need the more general case, from now on we will always assume that $\Gamma$ is a congruence subgroup, and hence $f$ will always possess a Fourier-expansion.

(1.59) Lemma. Let $n \in \mathbb{N}, k \in \mathbb{Q}, \Gamma_n \leq \text{Sp}_n(\mathcal{O})$ a congruence subgroup and $\nu_n$ a multiplier system for $\Gamma_n$ of weight $k$. The map

\[
\Phi: [\Gamma_n, k, \nu_n] \to [\Gamma_{n-1}, k, \nu_{n-1}], \quad f \mapsto f|\Phi
\]

with

\[
f|\Phi(Z) := \lim_{y \to \infty} f \left( \begin{pmatrix} Z & 0 \\ 0 & iy \end{pmatrix} \right), \quad \text{if} \ n \geq 2 \ \text{and} \ Z \in \mathcal{H}_{n-1}(\mathbb{H}),
\]

\[
f|\Phi := \lim_{y \to \infty} f(iy), \quad \text{if} \ n = 1
\]

is well-defined and linear, where $\Gamma_{n-1} := \{ M \in \text{Sp}_{n-1}(\mathcal{O}) ; M \times I \in \Gamma_n \}$, $\nu_{n-1}$ is some multiplier system of $\Gamma_{n-1}$ and $[\Gamma_0, k, \nu_0] := \mathbb{C}$.

Proof: As $\Gamma_n$ is a congruence subgroup, there is some two-sided ideal $P \leq \mathcal{O}$ with $\text{Sp}_n(\mathcal{O})|P| \leq \Gamma_n$. By definition, $M \times I \in \text{Sp}_n(\mathcal{O})|P|$ for all $M \in \text{Sp}_{n-1}(\mathcal{O})|P|$, and therefore $\text{Sp}_{n-1}(\mathcal{O})|P| \leq \Gamma_{n-1}$, i.e. $\Gamma_{n-1}$ is again a congruence subgroup of level $P$.
Now, let \( f \in [\Gamma_n, k, \nu_n] \). According to what we have seen before, the Fourier-expansion of \( f \) is absolutely and locally uniformly convergent. So the limit can be interchanged in the following calculations: For \( n = 1 \) we get

\[
\lim_{y \to \infty} f(iy) = \sum_{m \geq 0} \lim_{y \to \infty} \alpha_f(m)e^{-2\pi my} = \alpha_f(0) \in \mathbb{C}
\]

where the sum ranges over some discrete subset of \( \mathbb{R} \). For \( n > 1 \) we compute

\[
f|\Phi(Z) = \lim_{y \to \infty} f\left(\begin{array}{cc} Z & 0 \\ 0 & iy \end{array}\right) = \sum_{T = (T_1 T_0) \geq 0} \alpha_f(T) e^{2\pi iT(Z_i Z)} \lim_{y \to \infty} e^{-2\pi my} = \sum_{T = (T_1 T_0) \geq 0} \alpha_f(T_1 0) e^{2\pi iT(Z_i Z)}
\]

where the sums range over some discrete subsets of \( \text{Her}_n(\mathbb{H}) \) and \( \text{Her}_{n-1}(\mathbb{H}) \), respectively. Of course, this Fourier-expansion converges absolutely and locally uniformly on \( \mathcal{H}_{n-1}(\mathbb{H}) \) as it is a partial series of the original Fourier-expansion. Thus \( f|\Phi \) is holomorphic. The required bounding conditions for the case \( n = 2 \) immediatley follow, since the Fourier-expansions only range over non-negative exponents \( T_1 \). The transformation behavior of \( f|\Phi \) results from the transformation behavior of \( f \): Let \( M \in \Gamma_{n-1} \), which means \( M \times I \in \Gamma_n \). It is easy to verify that

\[
f|_k(M \times I)\left(\begin{array}{cc} Z & 0 \\ 0 & iy \end{array}\right) = (\det(M\{\mathcal{Z}\}))^{-k/2} f\left(\begin{array}{cc} M(Z) & 0 \\ 0 & iy \end{array}\right)
\]

holds. Now \( f|_k(M \times I) = \nu_n(M \times I) \cdot f \) and passing this to the limit \( y \to \infty \) yields

\[
(f|\Phi)|_k M = \nu_n(M \times I) \cdot f|\Phi \quad \text{for all } M \in \Gamma_{n-1}.
\]

So \( f|\Phi \) is again a quaternionic modular form of weight \( k \), and according to (1.52),\( \nu_{n-1}(M) = \nu_n(M \times I) \) for \( M \in \Gamma_{n-1} \) turns out to be a multiplier system for \( \Gamma_{n-1} \) of weight \( k \). Hence the map \( \Phi \) is well-defined. The linearity is obvious. \( \square \)

The homomorphism \( \Phi \) defined in the preceding lemma is called Siegel’s \( \Phi \)-operator.

Note that \( \text{Sp}_n(\mathcal{O})|P \leq \text{Sp}_n(\mathcal{O}) \) is normal, hence for a congruence subgroup \( \Gamma \), \( M^{-1}\Gamma M \) is a congruence subgroup again for all \( M \in \text{Sp}_n(\mathcal{O}) \). For that reason and because of (1.53) the following definition makes sense:

**1.60 Definition.** Suppose \( \Gamma \leq \text{Sp}_n(\mathcal{O}) \) is a congruence subgroup with a multiplier system \( \nu \) of weight \( k \) and \( f \in [\Gamma, k, \nu] \). If

\[
(f|_k M)\Phi \equiv 0
\]

holds for all \( M \in \text{Sp}_n(\mathcal{O}) \), then \( f \) is said to be a (quaternionic) cusp form. The subvector space of all cusp forms (for \([\Gamma, k, \nu]\)) is denoted by \([\Gamma, k, \nu]_0\).
Note that like for the Hurwitz order (see [Kr85, ch.III, pp.83]) we only have to consider 
\[ M \in \text{Sp}_n(\mathcal{O}) \] (and not \( M \in \text{Sp}_n(\mathbb{Q} : \mathcal{O}) \)) for the definition of cusp forms due \( \mathcal{O} \) being euclidean.
The reason for this are the following considerations:

Once more, we can adopt a proposition for the Hurwitz order from [Kr85] as the proof would
be completely the same, taking only the euclidean property into account.

(1.61) **Proposition.** Let \( T \in \text{Her}_n(\mathbb{H}) \) be rational, i.e. the entries of \( T \) have rational coefficients with
respect to the basis of \( \mathcal{O} \). If \( T \geq 0 \) and \( 0 < \text{rank}(T) = m < n \), there exists \( U \in \text{GL}_n(\mathcal{O}) \) and
\( T_1 \in \text{Her}_m(\mathbb{H}) \) satisfying \( T_1 > 0 \) and
\[
T[U] = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

**Proof:** [Kr85, ch.III, prop.2.2], and as well [Kl98, Prop.2.10]. \(\square\)

We will now characterize cusp forms via the Fourier-expansions, which also gives the answer
to the remark above.

(1.62) **Lemma.** Suppose \( \Gamma \leq \text{Sp}_n(\mathcal{O}) \) is a congruence subgroup with a multiplier system \( \nu \) of weight \( k \)
and \( f \in [\Gamma, k, \nu] \). Then \( f \) is a cusp form if and only if \( f|_k M \) possesses a Fourier-expansion of the shape
\[
f|_k M(Z) = \sum_{T > 0} \alpha_{f,M}(T)e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}_n(\mathbb{H})
\]
for all \( M \in \text{Sp}_n(\mathcal{O}) \), i.e. all Fourier-coefficients \( \alpha_{f,M}(T) \) vanish, for which \( T \) is not positive definite.
Again, the sum ranges over some rational, discrete subset of \( \text{Her}_n(\mathcal{O}) \).

**Proof:** By the same calculation as in the proof of (1.59), the Fourier-expansion of \((f|_k M)|\Phi\) is given by
\[
(f|_k M)|\Phi(Z) = \sum_{T_0 \geq 0} \alpha_{f,M}(T_0) e^{2\pi i \tau(T_0,Z)}.
\]
So if \( \alpha_{f,M}(T) \) vanishes for all \( T \) which are not positive definite, \((f|_k M)|\Phi \equiv 0\) immediately
follows.
Conversely, let \( f \) be a cusp form. Due to the shape of the Fourier-expansion above, \( \alpha_{f,M}(T_0) = 0 \) follows for all \( T_0 \) over which the Fourier-expansion of \((f|_k M)|\Phi\) ranges. Now, let \( T_0 \geq 0 \) be
arbitrary such that \( T_0 \) occurs in the Fourier-expansion of \( f|_k M \) and such that \( T_0 \) is not positive
definite. Thus \( 0 < \text{rank}(T_0) < n \) follows (or \( T_0 = 0 \), which is the trivial case). According to
(1.61), there is \( U \in \text{GL}_n(\mathcal{O}) \) such that \( T_0[U] = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \) with \( T_1 \in \text{Her}_{n-1}(\mathbb{H}), T_1 \geq 0 \). Analogous
to the proof of (1.55), the Fourier-expansion of \( f|_k M|k \text{Rot}(\overline{U}') \) is given by
\[
f|_k M|k \text{Rot}(\overline{U}') (Z) = \sum_{T \geq 0} \alpha_{f,M}(T)e^{2\pi i \tau(T[U],Z)}
= \sum_{S \geq 0} \alpha_{f,M}(S[U^{-1}])e^{2\pi i \tau(S,Z)}
= \sum_{S \geq 0} \frac{\epsilon_k/2}{(M, \text{Rot}(\overline{U}'))(S)} \cdot \alpha_{f,M,\text{Rot}(\overline{U}')} (S) e^{2\pi i \tau(S,Z)}
\]
where the sum ranges over all $T$ occuring in the Fourier-expansion of $f|_k M$ and defining $S = T[U]$. Due to the uniqueness of the Fourier-coefficients, $\epsilon_{k/2}(M, \text{Rot}(U')) \cdot \alpha_{f, M \text{Rot}(U')}(T[U]) = \alpha_{f, M}(T)$ follows, in particular for $T = T_0$. But $\alpha_{f, M \text{Rot}(U')}(T_0[U]) = 0$ has to hold regarding the shape of the Fourier-expansion of $(f|_k (M \text{Rot}(U')))|\Phi$, since $(f|_k (M \text{Rot}(U')))|\Phi$ vanishes. This completes the proof. \hfill \qedsymbol

We will now restrict ourselves to the case of degree two, again. We will show that for congruence subgroups (of genus two) and for the multiplier system fulfilling a certain condition, the vector spaces of modular forms are finite dimensional. We will only use methods already found in [Kr85]. The detailed examination of the case of degree two will show that the methods are exactly the same, so the results will coincide completely for arbitrary degree considering [Kr85, ch.III, pp.84]. We would have to construct fundamental domains like we did for the case of degree two. The reason why we worked out the fundamental domain for degree two was to get explicit and satisfying bounds. The fundamental domains for higher degrees can be constructed using a combination of what we did for degree two and what can be found in [Kr85].

We begin with an easy method to construct fundamental domains for congruence subgroups $\Gamma \leq \text{Sp}_2(\mathcal{O})$. It can be found in [KK07], for example, and we will adjust it to our matters. Note that congruence subgroups are always of finite index in $\text{Sp}_2(\mathcal{O})$.

1.63 Proposition. Let $\Gamma \leq \text{Sp}_2(\mathcal{O})$ a congruence subgroup and $\Gamma' = \langle \Gamma, -I \rangle$. Suppose $l = |\text{Sp}_2(\mathcal{O})/\Gamma'|$ is the index and $M_1, \ldots, M_l$ is a right transversal of $\Gamma' \setminus \text{Sp}_2(\mathcal{O})$, i.e.

$$\text{Sp}_2(\mathcal{O}) = \bigcup_{1 \leq m \leq l} \Gamma'M_m$$

is a disjunct union. Then

$$\mathcal{F}_2(\mathcal{O}, \Gamma) := \bigcup_{1 \leq m \leq l} M_m \langle \mathcal{F}_2(\mathcal{O}) \rangle$$

is a fundamental domain of $\mathcal{H}(\mathbb{H})$ and $\Gamma$ with respect to modular transformations.

Proof: Of course the requirements (F.1), (F.4) and (F.5) are met since $\mathcal{F}_2(\mathcal{O}, \Gamma)$ is a finite union and $\mathcal{F}_2(\mathcal{O})$ meets these. So let $Z \in \mathcal{H}(\mathbb{H})$ and choose $L \in \text{Sp}_2(\mathcal{O})$ such that $L[Z] \in \mathcal{F}_2(\mathcal{O})$ according to (1.43). As the $M_m$ are a right transversal we can choose $M_m$ and $M \in \Gamma$ such that $L^{-1} = \pm M^{-1} M_m$ which is equivalent to $\pm M = M_m L$. So we get

$$M[Z] = M_m \langle L[Z] \rangle \in M_m \langle \mathcal{F}_2(\mathcal{O}) \rangle \subset \mathcal{F}_2(\mathcal{O}, \Gamma).$$

Now, suppose $M \in \Gamma$ such that $Z$ and $M[Z]$ are interior points of $\mathcal{F}_2(\mathcal{O}, \Gamma)$. Choose an open neighborhood $U \subset \mathcal{F}_2(\mathcal{O}, \Gamma)$ of $Z$ in $\mathcal{F}_2(\mathcal{O}, \Gamma)$ sufficiently small such that $M[U] \subset \mathcal{F}_2(\mathcal{O}, \Gamma)$.
(recall that \( M(\cdot) \) is continuous). We have
\[
U = \bigcup_{1 \leq m \leq l} U_m, \quad \text{for } U_m := U \cap M_m(U)
\]
and therefore at least one of the \( U_m \) has to contain interior points. Without restriction suppose that \( U_1 \) contains an interior point \( V \in U_1 \). Since \( U_1 \subset M_1(F_2(O)) \), \( W := M_1^{-1}(V) \) is an interior point of \( F_2(O) \). By assumption and definition there is an \( M_m \) satisfying \( M(V) \in M_m(F_2(O)) \). So finally, \( W := M_m^{-1}MM_1(W) \in F_2(O) \) holds, and as we have seen, \( W \) is an interior point of \( F_2(O) \). Therefore, deriving from (1.41), \( M_m^{-1}MM_1 = \pm I \) and \( W = \tilde{W} \) follow, and thus \( M_m = \pm MM_1 \). Since the \( M_m \) are a transversal, \( m = 1 \) and \( M = \pm I \) follow, which finally proves the claim. \( \Box \)

Next, we need to examine some auxiliary function. The proof is more or less the same as in [Kr85, ch.III, le.2.4], but some more details have to be worked out to fit our case.

(1.64) Lemma. Let \( f \in [\Gamma, k,v] \) for some congruence subgroup \( \Gamma \leq \text{Sp}_2(O) \) and a multiplier system \( v \) of weight \( k \) such that \( |v(M)| = 1 \) for all \( M \in \Gamma \). Then the function
\[
\tilde{f}(Z) := (\det(Y))^{k/2} f(Z), \quad Z = X + iY \in \mathcal{H}(\mathbb{I})
\]
is invariant under modular transformations with respect to \( \Gamma \). If \( f \) is a cusp form, then \( \tilde{f} \) is bounded and there exists \( Z_0 \in F_2(O,\Gamma) \) satisfying
\[
\tilde{f}(Z_0) = \max \{ \tilde{f}(Z) \ ; \ Z \in \mathcal{H}(\mathbb{I}) \}.
\]

Proof: Let \( Z = X + iY \in \mathcal{H}(\mathbb{I}) \) and \( M \in \Gamma \). Note that \( \det(Y) > 0 \) holds since \( Y > 0 \), and also \( \det(\tilde{Y}) = (\det(Y))^2 > 0 \) (see (1.34)) as well as \( \det(\tilde{M}(\tilde{Z})) > 0 \). The behavior of \( f \) under modular transformations, (1.5) and (1.34) lead to
\[
\tilde{f}(M(Z)) = (\det(\tilde{M}(\tilde{Z})))^{k/2} |f(M(Z))|
\]
\[
= |(\det((\tilde{M}\{\tilde{Z}\})^{-1}\tilde{Y}(\tilde{M}\{\tilde{Z}\})^{-1}))|^{k/4} \cdot |v(M) \det(\tilde{M}\{\tilde{Z}\})|^{k/2} |f(Z)|
\]
\[
= |v(M)| \cdot |\det(\tilde{M}\{\tilde{Z}\})|^{-k/2} \cdot |\det(\tilde{M}\{\tilde{Z}\})|^{k/2} \cdot (\det(Y))^{k/2} |f(Z)| = \tilde{f}(Z),
\]
hence this results in the invariance with respect to \( \Gamma \). The same calculation also yields
\[
\tilde{f}(M(Z)) = \tilde{f}|_k M(Z)
\]
for all \( M \in \text{Sp}_2(O) \) and \( Z \in \mathcal{H}(\mathbb{I}) \). Now, let \( f \) be a cusp form. Because of the first equation it suffices to examine \( \tilde{f} \) in \( F_2(O,\Gamma) \). So suppose \( M_1, \ldots, M_l \) is a right transversal of \( \Gamma' \setminus \text{Sp}_2(O) \) as in (1.63), hence
\[
F_2(O,\Gamma) := \bigcup_{1 \leq m \leq l} M_m(F_2(O)).
\]
So for every \( Z \in F_2(O,\Gamma) \) there is some \( M_m \) with \( M_m^{-1}(Z) =: W \in F_2(O) \) and according to the
second equation from above

\[ \tilde{f}(Z) = f_{kM_n}(W) \]

holds. Therefore it suffices to examine \( f_{kM_1, \ldots, f_{kM_l}} \) in \( F_2(\mathcal{O}) \). The procedure would be the same for all the \( f_{kM_n} \) as \( f \) is a cusp form and only one specific property is needed which is fulfilled for all \( f_{kM} \) according to (1.62), namely that \( f \) (and all the \( f_{kM} \)) have a Fourier-expansion of the shape

\[ f(Z) = \sum_{T > 0} \alpha_f(T)e^{2\pi i T Z}, \quad Z \in \mathcal{H}_n(\mathbb{H}). \]

Now (1.37) and the properties of reduced matrices yield that the set

\[ \{ Z = X + iY = (x_{j,k}) + i(y_{j,k}) \in F_2(\mathcal{O}) ; y_{2.2} \leq c \}, \]

where \( c > 0 \), is compact and therefore \( \tilde{f} \) is bounded on such sets. Hence it suffices to show that

\[ \lim_{y_{2.2} \to \infty} \tilde{f}(Z) = 0 \]

holds, where \( Z = X + iY = (x_{j,k}) + i(y_{j,k}) \in F_2(\mathcal{O}) \). So let \( Z = X + iY = (x_{j,k}) + i(y_{j,k}) \in F_2(\mathcal{O}) \) and \( T = (t_{j,k}) \geq 0 \) such that it appears in the Fourier-expansion of \( f \). Then according to [Kr85, ch.1, thm.3.6] there is \( U \in \text{GL}_2(\mathbb{H}) \) such that \( T = \overline{U} U \). Moreover, \( Y[U] > 0 \) holds. So according to (1.46) we have

\[ \tau(T, Y) = \tau(I, Y[\overline{U}]) = \text{tr}(Y[\overline{U}]) > 0 \]

since the trace of positive definite matrices is positive. Deriving from (1.30), there exists \( \beta > 0 \) such that \( Y \geq \beta \begin{pmatrix} y_{1.1} & 0 \\ 0 & y_{2.2} \end{pmatrix} \) holds for all \( Z \in F_2(\mathcal{O}) \), since \( Z \in F_2(\mathcal{O}) \) implies \( Y \in \mathcal{R}(\mathcal{O}) \). So due to the inequality above and the linearity of the trace form we compute

\[ \tau(T, Y) \geq \beta \tau \left( T, \begin{pmatrix} y_{1.1} & 0 \\ 0 & y_{2.2} \end{pmatrix} \right) = \beta(t_{1,1}y_{1,1} + t_{2,2}y_{2,2}). \]

Because \( \text{det}(Y) = y_{1,1}y_{2,2} - N(y_{1,2}) \leq y_{1,1}y_{2,2} \),

\[ (\text{det}(Y))^{k/2} \leq (\text{det} \begin{pmatrix} y_{1.1} & 0 \\ 0 & y_{2.2} \end{pmatrix})^{k/2} \]

holds for \( k \geq 0 \). According to (1.30) we have \( y_{1,1}y_{2,2} \leq 3 \text{det}(Y) \) which leads to

\[ (\text{det}(Y))^{k/2} \leq 3^{-k/2}(\text{det} \begin{pmatrix} y_{1.1} & 0 \\ 0 & y_{2.2} \end{pmatrix})^{k/2} \]

for \( k < 0 \). Now given \( \rho > 0 \), the function \( y \mapsto y^{k/2}e^{-\rho y} \) is bounded in \( [\frac{1}{2}\sqrt{3}, \infty] \). So regarding (1.30) as well as the proof of (1.37) and due to what we have seen so far, there exists \( C > 0 \) such that

\[ (\text{det}(Y))^{k/2}e^{-2\pi \tau(T, Y)} \leq C \cdot e^{-\pi \beta(t_{1,1}y_{1,1} + t_{2,2}y_{2,2})} \]

holds for all \( T > 0 \) as there is a global lower bound on the diagonal entries of \( T \). Considering
the Fourier-expansion of $f$ (or the $f|_k M$) from above, we observe that
\[ \tilde{f}(Z) \leq C \sum_{T \geq 0} |\alpha_f(T)| e^{-\pi|t_{1,1}y_{1,1} + t_{2,2}y_{2,2}|} \]
holds for all $Z = X + iY = (x_{j,k}) + i(y_{j,k}) \in \mathcal{F}_2(\mathcal{O})$, where $T = (t_{j,k})$ and the sum ranges over some discrete, rational subset of $\text{Her}_2(\mathbb{H})$. In virtue of (1.54) (or more its analog for congruence subgroups) we may distribute the limit $y_{2,2} \to \infty$ through the infinite sum. Now $T > 0$ (and $Y > 0$) implies $t_{2,2} > 0$ (as well as $t_{1,1} > 0, y_{1,1} > 0$) and therefore
\[ \lim_{y_{2,2} \to \infty} e^{-\pi|t_{1,1}y_{1,1} + t_{2,2}y_{2,2}|} = 0 \]
which completes the proof. \( \square \)

Recall that for $\Gamma \leq \text{Sp}_2(\mathcal{O})$ a congruence subgroup and $f \in [\Gamma, k, \nu]$ there exists $m \in \mathbb{N}$ such that $f(Z + S) = f(Z)$ holds for all $S \in \text{Her}_2(m\mathcal{O})$. Thus $g(Z) = f(mZ)$ is holomorphic and periodic and that this is how we obtain a Fourier-expansion of $f$. According to (1.54), it is thus given by
\[
f(Z) = g\left(\frac{1}{m}Z\right) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_g(T) e^{2\pi i \tau(T, \frac{1}{m}Z)} = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_g(T) e^{2\pi i \tau(T, Z)} = \sum_{mT \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_g(mT) e^{2\pi i \tau(T, Z)} = \sum_{T \in \frac{1}{m} \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(T, Z)}, \tag{1.20}
\]
where $\alpha_f(T) := \alpha_g(mT)$. It converges absolutely and uniformly on \{ $Z \in \mathcal{H}_m(\mathbb{H})$ ; $3\text{Im}(Z) \geq \beta I$ \} for all $\beta > 0$. The Fourier-coefficients $\alpha_g(T)$ (and hence $\alpha_f(T)$) do not depend on $Z$, are uniquely determined by $f$ and given by
\[
\alpha_f(T) = \alpha_g(mT) = \text{vol}(\text{Her}_2(\mathcal{O})) \cdot e^{2\pi i \tau(mT, Y)} \cdot \int_{C(\mathcal{O})} g(X + iY) e^{-2\pi i \tau(mT, X)} \, dX = \text{vol}(\text{Her}_2(\mathcal{O})) \cdot e^{2\pi i \tau(mT, Y)} \cdot \int_{C(\mathcal{O})} f(mX + imY) e^{-2\pi i \tau(mT, X)} \, dX
\]
where $Y \in \text{Pos}_2(\mathbb{H})$ is fixed. As they do not depend on $Z$, we can substitute $Z$ by $\frac{1}{m}Z$ and obtain
\[
\alpha_f(T) = \text{vol}(\text{Her}_2(\mathcal{O})) \cdot e^{2\pi i \tau(T, Y)} \cdot \int_{C(\mathcal{O})} f(mX + iY) e^{-2\pi i \tau(T, mX)} \, dX.
\]
This is the reason why we get the following analog of [Kr85, ch.III, le.2.6], saying that henceforth we only have to consider non-negative weights.
(1.65) Lemma. Suppose $\Gamma \leq \text{Sp}_2(O)$ is a congruence subgroup with some multiplier system $\nu$ of weight $k (\in \mathbb{Q}), k < 0$ such that $|\nu(M)| = 1$ for all $M \in \Gamma$. Then $[\Gamma, k, \nu] = \{0\}$.

Proof: It is well known that if we would look at degree one instead of degree two, the assertion would hold true since we would consider ordinary elliptic modular forms, and (holomorphic) elliptic modular forms of negative weight are always identically zero (cf. [Se73, p.88, thm.4]). Hence $(f|_k M)|\Phi \equiv 0$ holds for all $f \in [\Gamma, k, \nu]$ and all $M \in \text{Sp}_2(O)$, i.e. all $f$ are cusp forms. So in virtue of (1.64), $\tilde{f}$ is bounded on $\mathcal{H}(\mathbb{H})$. So there exists $C > 0$ such that

$$|f(Z)| \leq C \cdot (\det(Y))^{-k/2}$$

holds for all $Z = X + iY \in \mathcal{H}(\mathbb{H})$. By means of how the Fourier-Coefficients $a_f(T)$ of $f$ are given (see above), we get

$$|a_f(T)| \leq \text{vol}(\text{Her}_2(O)) \cdot e^{2\pi T(Y)} \cdot \int_{C(O)} |f(mX + iY)e^{-2\pi iT,|mX|}| \, dX$$

$$\leq C \cdot \text{vol}(\text{Her}_2(O)) \cdot e^{2\pi T(Y)} \cdot (\det(Y))^{-k/2} \cdot \int_{C(O)} dX$$

$$=: \tilde{C} \cdot e^{2\pi T(Y)} \cdot (\det(Y))^{-k/2},$$

where $Z = X + iY > 0$ is fixed, but arbitrary. By construction, $\tilde{C}$ does not depend on $Z$. Thus

$$|a_f(T)| \cdot (\det(Y))^{k/2} \cdot e^{-2\pi T(Y)} ,$$

where $T \in \frac{1}{m} \text{Her}_2^\perp(O), T > 0$ (for some $m \in \mathbb{N}$ such that the Fourier-expansion of $f$ ranges over $\frac{1}{m} \text{Her}_2^\perp(O)$ like we have seen above), is bounded by $\tilde{C}$, whereas $\tilde{C}$ does not depend on $Y \in \text{Pos}_2(\mathbb{H})$. Hence, since $Y = \left( \begin{smallmatrix} 0 & 0 \\ \epsilon & \epsilon \end{smallmatrix} \right) \in \text{Pos}_2(\mathbb{H})$ holds for all $\epsilon > 0$,

$$|a_f(T)| \epsilon^k \cdot e^{-2\pi \text{tr}(T)}$$

remains bounded for $\epsilon > 0$. And thus $a(T) = 0$ follows by passing on to $\epsilon \downarrow 0$ and noting $k < 0$. Therefore $f$ vanishes identically.

Note that if we choose $Y = T^{-1} > 0$ in the equation above we have also shown that

$$|a_f(T)| \leq \tilde{C} \cdot e^{2\pi \text{tr}(T)} \cdot (\det(T))^{k/2} =: C \cdot (\det(T))^{k/2}$$

holds for all $T > 0$ for a cusp form $f$.

Suppose that $k > 0$ and $f \in [\Gamma, k, \nu]$ had no zero, i.e. $f(Z) \neq 0$ for all $Z \in \mathcal{H}(\mathbb{H})$. Then, an easy consideration shows that $\frac{1}{T} \in [\Gamma, -k, \nu^{-1}]$, which is a contradiction to the preceding lemma (1.65). Thus we have proved

(1.66) Corollary. Suppose that $\Gamma \leq \text{Sp}_2(O)$ is a congruence subgroup with some multiplier system $\nu$ of weight $k > 0$ such that $|\nu(M)| = 1$ for all $M \in \Gamma$. Then for every $f \in [\Gamma, k, \nu]$ there exists $Z \in \mathcal{H}(\mathbb{H})$ such that $f(Z) = 0$. 

Before we will estimate the dimension of the spaces of quaternionic modular forms for congruence subgroups in the case of degree two, we have to examine the special case of quaternionic modular forms for the whole quaternionic modular group \( \text{Sp}_2(\mathcal{O}) \). Note that in virtue of (1.65) and (1.56) the weight \( k \) has to be a non-negative integer. Also note that every possible multiplier system \( \nu \) fulfills \( |\nu(M)| = 1 \) for all \( M \in \text{Sp}_2(\mathcal{O}) \) and \( \nu(\text{Trans}(S)) = 1 \) for all \( S \in \text{Her}_2(\mathcal{O}) \) according to (1.20) and (1.57). Furthermore, we have \( \nu|\Phi| \in [\text{SL}_2(Z), k, 1] \) (because \( k \) being odd would imply \( f|\Phi| \equiv 0 \) as \( f\left( \frac{2k+1}{0 \ 0 \ z_{22}^2} \right) = -f\left( \frac{2k+1}{0 \ 0 \ z_{22}^2} \right) \) by (1.57), since then \( \nu(\text{Rot}(\text{diag}(-1,1))) = -1 \), while \( \text{Rot}(\text{diag}(-1,1)) \) leaves \( \left( \frac{2k+1}{0 \ 0 \ z_{22}^2} \right) \) invariant – and the same holds for \( k \) even and \( \nu = \nu_{\text{det}} \) regarding \( \nu_{\text{det}}(\text{Rot}(\text{diag}(i_2,1))) = -1 \). So in all cases \( f \) would achieve its maximum at some point in \( \mathcal{F}_2(\mathcal{O}) \) according to (1.64). Based on this analysis we can reformulate [Kr85, ch.III, thm.2.8] for our case without the need to prove it again. In the notation of this theorem, set \( s(2; H) := s_2(\mathcal{O}) \leq 4\sqrt{3} \) (see (1.44), and note that \( 4\sqrt{3} \geq \frac{2}{3}\sqrt{3} \) which is a fact needed in this theorem). All other properties of quaternionic modular forms for \( \text{Sp}_2(\mathcal{O}) \) (rather than for the Hurwitz quaternions) needed in this theorem have been proved here. So it would be just literally copying the proof. Therefore we formulate the following theorem without giving the proof, again:

(1.67) Theorem. Suppose that \( \nu \) is some multiplier system of \( \text{Sp}_2(\mathcal{O}) \) of non-negative weight \( k \) (hence \( k \in \mathbb{N}_0 \)). Let \( f \in [\text{Sp}_2(\mathcal{O}), k, \nu] \) with Fourier-expansion
\[
f(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}(H).
\]
If \( \alpha_f(T) = 0 \) holds for all \( T \in \text{Her}_2(\mathcal{O}), T \geq 0 \), such that
\[
\text{tr}(T) \leq \frac{k}{4\pi} \cdot s_2(\mathcal{O}) \leq \frac{\sqrt{3}}{\pi} \cdot k,
\]
then \( f \) vanishes identically.

This directly implies

(1.68) Corollary. \( [\text{Sp}_2(\mathcal{O}), 0, 1] = \mathbb{C} \) and \( [\text{Sp}_2(\mathcal{O}), 0, \nu_{\text{det}}] = \{0\} \).

Proof: Of course \( \mathbb{C} \subset [\text{Sp}_2(\mathcal{O}), 0, 1] \) holds. Suppose \( f \in [\text{Sp}_2(\mathcal{O}), 0, 1] \). Then, the constant term in the Fourier-expansion of \( f - \alpha_f(0) \in [\text{Sp}_2(\mathcal{O}), 0, 1] \) vanishes and thus \( f - \alpha_f(0) \equiv 0 \) according to (1.67).

Now, let \( f \in [\text{Sp}_2(\mathcal{O}), 0, \nu_{\text{det}}] \). Then (1.20) and (1.55) imply \( \alpha_f(0) = \nu(\text{Rot}(\bar{T})) \cdot \alpha_f(0|U) = -\alpha_f(0) \) for \( U = \left( \begin{array}{cc} i & 0 \\ 0 & 1 \end{array} \right) \), and thus \( f \equiv 0 \) follows. \( \square \)

Note that there are only finitely many \( T \in \text{Her}_2(\mathcal{O}), T \geq 0 \) satisfying \( \text{tr}(T) \leq c \) for \( c > 0 \). So (1.67) immediately yields
(1.69) **Corollary.** Suppose that \( \nu \) is some multiplier system of \( \text{Sp}_2(\mathcal{O}) \) of weight \( k \). Then
\[
\dim[\text{Sp}_2(\mathcal{O}), k, \nu] < \infty
\]
holds.

Once we introduced Maass lifts we can even state the explicit dimension for small weights (see (3.36)). We will also introduce a more concrete bound in (4.9), taking into account that Fourier-coefficients for modulo \( \text{GL}_2(\mathcal{O}) \) congruent matrices \( T \in \text{Her}_{2}^{0} \), \( T \geq 0 \) only differ by some constant factor (see (1.55)) – hence it suffices to consider a certain subset of \( \mathcal{R}(\mathcal{O}) \). But note that this bound will not be very satisfying, since it only takes into account the number of incongruent matrices in a bounded set. The dimensions themselves turn out to much smaller, since there might also exist certain correlations for Fourier-coefficients belonging to incongruent matrices (for example, see the yet to introduce Maass lifts). But at least the bound deriving from (4.9) will give an estimate on how many Fourier-coefficients have to be compared to verify that two quaternionic modular forms coincide. This number is far less than the one deriving from (1.67), only.

Now, suppose \( f \in [\Gamma, k, 1] \) for some congruence subgroup \( \Gamma \leq \text{Sp}_2(\mathcal{O}) \) and even \( k \in \mathbb{N}_0 \). Let \( M_1, \ldots, M_l \) be a transversal of \( \Gamma \setminus \text{Sp}_2(\mathcal{O}) \). Then, an easy consideration shows that
\[
\Pi(f) := \prod_{1 \leq m \leq l} f|_{kM_m} \in [\text{Sp}_2(\mathcal{O}), kl, 1]
\]
and the product is independent of the special choice of the transversal. This leads to

(1.70) **Corollary.** Suppose \( \Gamma \leq \text{Sp}_2(\mathcal{O}) \) is a congruence subgroup, \( k \in 2\mathbb{N}_0 \) and \( \nu \) is an abelian character such that there exists \( m \in \mathbb{N} \) with \( \text{Sp}_2(\mathcal{O})|_{m\mathcal{O}} \subset \Gamma \) and \( \nu|_{\text{Sp}_2(\mathcal{O})|_{m\mathcal{O}}} \equiv 1 \) (hence \( \nu \) is said to be an abelian character of \( \Gamma \) mod \( m \)). Define \( \Gamma^* = \{ M \in \Gamma ; \nu(M) = 1 \} \leq \text{Sp}_2(\mathcal{O}) \) (then \( \text{Sp}_2(\mathcal{O})|_{m\mathcal{O}} \subset \Gamma^* \) and \( \Gamma^* \) is of finite index) and \( l = |\text{Sp}_2(\mathcal{O})/\Gamma^*| \). If \( f \in [\Gamma, k, \nu] \) and \( M \in \text{Sp}_2(\mathcal{O}) \) such that
\[
\alpha_{(f, M)}(T) = 0 \quad \text{for all } T \in \frac{1}{m} \text{Her}_2^0(\mathcal{O}), \ T \geq 0, \ \text{tr}(T) \leq \sqrt{\frac{3}{\pi}} \cdot kl,
\]
then \( f \equiv 0 \) follows.

**Proof:** By definition, \( f \in [\Gamma^*, k, 1] \). So if \( M_1, \ldots, M_l \) is a transversal of \( \Gamma^* \setminus \text{Sp}_2(\mathcal{O}) \), then
\[
g := \Pi(f) = \prod_{1 \leq m \leq l} f|_{kM_m} \in [\text{Sp}_2(\mathcal{O}), kl, 1].
\]

Considering the shape of the Fourier-expansion of the \( f|_{kM_m} \) we discussed above and multiplying them out yields \( \alpha_k(T) = 0 \) for all \( T \in \text{Her}_2^0(\mathcal{O}), \ T \geq 0, \ \text{tr}(T) \leq \sqrt{\frac{3}{\pi}} \cdot kl \) and therefore \( g \equiv 0 \) according to (1.67). Thus the assertion follows as the ring of holomorphic functions on \( \mathcal{H}(\mathbb{H}) \) is free of zero divisors, which can be found in standard literature for holomorphic functions in several variables, for example confer [Ra86].

Finally we can receive the result that the spaces of quaternionic modular forms for congruence
subgroups are finite dimensional, at least if the multiplier system fulfills a certain condition, specified below.

(1.71) **Proposition.** Suppose \( \Gamma \leq \text{Sp}_2(\mathcal{O}) \) is a congruence subgroup with some multiplier system \( v \) of weight \( k \) (\( \in \mathbb{Q}, k \geq 0 \)), such that there is \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \) with \( \text{Sp}_2(\mathcal{O})[m\mathcal{O}] \subset \Gamma \) and \( v^n|_{\text{Sp}_2(\mathcal{O})[m\mathcal{O}]} \equiv 1 \). Then

\[
\dim[\Gamma, k, v] < \infty.
\]

**Proof:** Suppose \( \dim[\Gamma, k, v] > 0 \), hence there is \( f \in [\Gamma, k, v] \), \( f \neq 0 \). As \( k \in \mathbb{Q}, k \geq 0 \), there is \( l \in \mathbb{N} \) such that \( lk \in 2\mathbb{N}_0 \). Now, let \( g \in [\Gamma, k, v] \), then

\[
f^{nl-1}g \in [\Gamma, nlk, v^n]
\]

and therefore

\[
f^{nl-1}[\Gamma, k, v] \leq [\Gamma, nlk, v^n]
\]

holds, which leads to

\[
\dim[\Gamma, k, v] = \dim(f^{nl-1}[\Gamma, k, v]) \leq \dim([\Gamma, nlk, v^n]).
\]

Because \( nlk \in 2\mathbb{N}_0 \) and \( v^n|_{\text{Sp}_2(\mathcal{O})[m\mathcal{O}]} \equiv 1 \), the prerequisites of (1.70) are met. As there are only finitely many \( T \in \frac{1}{m} \text{Her}^2(\mathcal{O}), T \geq 0 \), with \( \text{tr}(T) \leq c \) for \( c > 0 \), \( \dim[\Gamma, nlk, v^n] < \infty \) follows, and hence the assertion is derived as well. 

\[ \square \]

### 1.5 Hermitian and Siegel modular forms

In this last section of the first introductory chapter we will introduce and discuss Hermitian and Siegel modular forms. As we will see, there is one main difference between Hermitian resp. Siegel modular forms and quaternionic modular forms if the weight is not an even integer, but apart from that their definition and properties are completely the same, just in another setting. Therefore we will just give a short overview. The properties are the same as those we have seen for quaternionic modular forms so far, with the only difference that in cases where the weight is required to be an even integer, the weight can also be odd.

We already introduced the Siegel half-space \( \mathcal{H}_n(\mathbb{R}) \) and the Hermitian half-space \( \mathcal{H}_n(\mathbb{C}) \) in (1.1) and the subsequent remark. Usually, the Hermitian half-space is given by \( \mathcal{H}_n(\mathbb{C}) = \{ Z \in \mathbb{C}^{n \times n} ; \frac{1}{2i}(Z - Z^\dagger) > 0 \} \). But as we have seen those half-spaces are isomorphic and the same holds true if we look at \( \mathcal{H}_n(\mathbb{R}) \) for two-dimensional subalgebras \( R \) of the shape \( R = \{ a_0 + a_1\omega \in \mathbb{H} ; a_0, a_1 \in \mathbb{R} \}, \) where \( \omega \in \mathbb{H} \setminus \mathbb{R} \) (and \( \omega \) is algebraic over \( \mathbb{R} \) – see below). It is easy to check that those isomorphisms commute with the needed symplectic transformations, hence for reasons of clarification, we will call \( \mathcal{H}_n(\mathbb{C}) \) the Hermitian half-space, but keep in mind that, when dealing with Hermitian modular forms, no matter how we actually define the Hermitian half-space (that is \( \mathcal{H}_n(\mathbb{C}), \mathcal{H}_n(\mathbb{R}) \) or \( \mathcal{H}_n(\mathbb{C}) \)) we are speaking about the same objects as in standard literature because of this isomorphism.

Now, instead of taking modular transformations with respect to \( \text{Sp}_n(\mathbb{H}) \) into account, we
need the Siegel and the Hermitian symplectic groups, i.e. $\text{Sp}_n(\mathbb{R})$ and $\text{Sp}_n(\mathbb{C})$. As $\mathbb{R}$ and $\mathbb{C}$ are commutative, in contrast to $\mathbb{H}$, we can use the normal determinant instead of the determinant form $\det(\mathcal{M}\{\hat{Z}\})^{-k/2n}$. So the slash-operator for a holomorphic function $f : \mathcal{H}_n(\mathbb{R}) \to \mathbb{C}$ or $f : \mathcal{H}_n(\mathbb{C}) \to \mathbb{C}$ is defined as

$$f|_k M(Z) := (\det(\mathcal{M}\{Z\}))^{-k}f(M(Z)) := e^{-k\log(\det(\mathcal{M}|_k Z))}f(M(Z))$$

where we have to fix a branch of the logarithm, again.

The quaternionic modular group is substituted by the Siegel modular group $\text{Sp}_n(\mathbb{Z})$ and by Hermitian modular groups $\text{Sp}_n(q)$. Here $q$ is the integral closure of some imaginary quadratic number field $\mathbb{Q}(\sqrt{-\Delta})$. Details about that can be found in [De01] (while details about Siegel modular forms can be found in [Fr82]). The Siegel modular group and the Hermitian modular groups are generated by the respective matrices found in (1.17), where in the Siegel case the generators $\text{Rot}(U)$ are not needed and in the Hermitian case the $e$'s are to be chosen from a set of generators of $q^*$. Again, one defines multiplier systems for subgroups of finite index just as in (1.48), with the only difference that the normal determinant is used instead of $\det(\mathcal{M}\{\hat{Z}\})^{-k/2n}$. Siegel and Hermitian modular forms are then defined just the same way as quaternionic modular forms are defined in (1.50), where in (M.2) the slash-operator defined above is used instead of the quaternionic slash-operator. The spaces of modular forms are denoted by $[\Gamma, k, \nu]$, again, where $\Gamma$ indicates if we are talking about Siegel or Hermitian modular forms, of course.

It is important that if we restrict quaternionic modular forms to certain submanifolds we will obtain Siegel and Hermitian modular forms. We get Siegel modular forms by restricting to $\mathcal{H}_n(\mathbb{R})$, and we obtain Hermitian modular forms by restricting to $\mathcal{H}_n(\mathbb{C})$, where $R = \{a_0 + a_1\omega \in \mathbb{H} \mid a_0, a_1 \in \mathbb{R}\}$, with $\omega \in \mathbb{H} \setminus \mathbb{R}$ is a two-dimensional subalgebra of $\mathbb{H}$, but this is a bit more involved (also confer [Kl98, pp.25] for more details on that). First of course, $R = \{a_0 + a_1\omega \in \mathbb{H} \mid a_0, a_1 \in \mathbb{R}\}$ is always a subvector space of $\mathbb{H}$, but we require it to be a subalgebra, indeed. And as we need a complex structure, $\overline{\omega} \in R$ has to hold (but both conditions are always fulfilled – see below). If these conditions are fulfilled, $R$ turns out to be a field: Let $0 \neq a = a_0 + a_1\omega \in R$, then $a^{-1} = N(a)^{-1} \cdot (a_0 + a_1\overline{\omega}) \in R$ holds. Of course, $R$ is commutative, and thus $R$ turns out to be a field and $R = \mathbb{R}(\omega)$ is a non-trivial field extension of $\mathbb{R}$. So if $\omega$ was algebraic over $\mathbb{R}$, then $R$ had to be isomorphic to $\mathbb{C}$. So the final property we require is that $Q(\omega)$ is indeed an algebraic field extension of $Q$ of degree two, and thus $Q(\omega)$ turns out to be an imaginary quadratic number field (or at least isomorphic to one).

We are now going to restrict the choice of $\omega$ such that the conditions above are guaranteed. It suffices to do this since we will not have to consider the general case. So let $\omega \in O \setminus \mathbb{R}$. As $\overline{\omega} = 2\text{Re}(\omega) - \omega$ we compute $\omega^2 = 2\text{Re}(\omega) \cdot \omega - N(\omega)$. Thus $R = \{a_0 + a_1\omega \in \mathbb{H} \mid a_0, a_1 \in \mathbb{R}\}$ is a subalgebra of $\mathbb{H}$, $\overline{\omega} \in R$ holds and $\omega$ is algebraic over $\mathbb{R}$, indeed. (And so $R = \mathbb{R}(\omega)$ is always isomorphic to $\mathbb{C}$, no matter how $\omega \in \mathbb{H} \setminus \mathbb{R}$ is chosen.) Furthermore, $\omega$ is algebraic over $Q$ of degree two and also integral in $Q(\omega)$ because $2\text{Re}(\omega) \in \mathbb{Z}$ and $N(\omega) \in \mathbb{N}_0$ hold for all $\omega \in O$. Thus $Q(\omega)$ is an imaginary quadratic number field (and we identify it as $Q(\omega) \subset \mathbb{C}$) and

$$O(\omega) := \mathbb{Z} + \omega\mathbb{Z} \subset O$$

is a subring of the integral closure of $Q(\omega)$. So the following theorem makes sense, indeed:
(1.72) Theorem. Suppose $\Gamma \leq \text{Sp}_n(\mathcal{O})$ is a congruence subgroup with a multiplier system $\nu$ of weight $k (\in \mathbb{Q}, k \geq 0)$, $\omega \in \mathcal{O} \setminus \mathbb{R}$ and $R := \mathbb{R}(\omega) = \{ a_0 + a_1 \omega \in \mathbb{H} ; a_0, a_1 \in \mathbb{R} \}$. Then for every $f \in [\Gamma, k, \nu]$

$$f |_{H_n(R)} \in [\Gamma_R, k, \nu_R] \quad \text{and} \quad f |_{H_n(R)} \in [\Gamma_R, k, \nu_R]$$

hold, where

$$\Gamma_R := \Gamma \cap \mathbb{R}^{2n \times 2n} \subset \text{Sp}_n(Z) \quad \text{and} \quad \Gamma_R := \Gamma \cap \mathbb{R}^{2n \times 2n} \subset \text{Sp}_n(\mathcal{O}(\omega)) \subset \text{Sp}_n(\mathbb{C})$$

and $\nu_R$ and $\nu_R$ are some multiplier systems of weight $k$ for $\Gamma_R$ and $\Gamma_R$, respectively. If $\Gamma = \text{Sp}_n(\mathcal{O})$, then they are given by the following values on the generators:

$$\nu_R(\text{Trans}(S)) = \nu_R(\text{Trans}(S)) = \nu(\text{Trans}(S)),$$

$$\nu_R(\text{Rot}(U)) = \nu_R(\text{Rot}(U)) = (\det(U))^k \cdot \nu(\text{Rot}(U)),$$

$$\nu_R(J_n) = \nu_R(J_n) = \frac{(i^n)^k}{((-1)^n)^{k/2}} \cdot (\nu(R))^k.$$ 

Proof: Of course, the restrictions are again holomorphic. Furthermore, the case $n = 1$ is trivial as $\mathcal{H}_1(\mathbb{H}) = \mathcal{H}_1(\mathbb{R})$. So we just have to check the transformation behavior of the restrictions under modular transformations with respect to $\Gamma_R$ and $\Gamma_R$. As there is no difference in the calculations (and after all it is well known that Hermitian modular forms restricted to the Siegel half-space are Siegel modular forms), we will only consider the Hermitian case. So let $Z \in \mathcal{H}_n(R) (\simeq \mathcal{H}_n(\mathbb{C}))$ and $M \in \Gamma_R(\subset \text{Sp}_n(\mathbb{C}))$. Just as in [Kr85] one can check that $|\det(M\{Z\})| = |\det(M\{Z\})|^2$ holds. It can be done as follows: Both $\det(\cdot)^*$ and $(\det(\cdot))^2$ are automorphy factors that fulfill the cocycle relations 1.10. Hence we only have to check if $|\det(\cdot)|^2 = |\det(\cdot)|^2$ holds for generators of the modular groups. For $\text{Trans}(S)$ and $\text{Rot}(U)$ the assertion holds true in virtue of (1.34) and noting $|\det(U)| = 1$ for all $U \in \text{GL}_n(\mathcal{O}(\omega))$ (cf. [De96]). For $J_n$ the claim holds because of (1.34) as $\det(\hat{Z}) = \det(Z)^2$ for all $Z \in \mathcal{H}_n(\mathbb{H})$. And so there is some $\delta_M$ independent on $Z$ such that $(\det(M\{\hat{Z}\}))^{-k/2} = \delta_M \cdot (\det(M\{Z\}))^{-k}$ as both sides are holomorphic. This is also the reason why we have to use the exponent $\frac{1}{2}$ instead of $k$ in the quaternionic case, or otherwise the weight would change when restricting to the Hermitian or Siegel half-space. Thus we compute

$$\nu(M)f(Z) = f|_{H_n(M)}(Z) = (\det(M\{\hat{Z}\}))^{-k/2}f(M\{Z\}) = \delta_M \cdot (\det(M\{Z\}))^{-k}f(M\{Z\})$$

for $Z \in \mathcal{H}_n(R)$ and $M \in \Gamma_R$, which means

$$(f |_{H_n(R)}) |_{H_n(M)} = \delta_M^{-1} \nu(M) \cdot f |_{H_n(R)}(Z)$$

holds if $|_{H_n(M)}$ now stands for the “new” slash-operator for Hermitian (or Siegel) modular forms. So just analogous to (1.52), $\nu_R(M) := \delta_M^{-1} \nu(M)$ turns out to be a multiplier system for $\Gamma_R$ of weight $k$ (as long as there is some $f$ with $f |_{H_n(R)} \neq 0$, but if not, then this is the trivial case,
For the second claim we simply have to compute \( \delta_M \) for special matrices, namely the generators of the modular group. For \( M = \text{Trans}(S) \), \( S \in \text{Her}_n(O(\omega)) \), \( \delta_M = 1 \) is trivial as \( (\det(M\{\tilde{Z}\}))^{-k/2} = (\det(\tilde{I}))^{-k/2} = 1 = (\det(M\{Z\}))^{-k} \). So let \( M = \text{Rot}(U) \), \( U \in \text{GL}_n(O(\omega)) \subset \text{GL}_n(O) \). According to (1.34), \( (\det(M\{\tilde{Z}\}))^{-k/2} = (\tilde{U}^{-1})^{-k/2} = 1 \) holds, whereas \( (\det(M\{Z\}))^{-k} = (\det(U))^{k} \), and thus by definition \( \delta^{-1}_M = (\det(U))^{k} \). The remaining case is \( M = I_n \). Of course, \( \delta_M \) can be determined by inserting one single point, and by how the branch of the logarithm is defined, the easiest one is \( Z = iI \). By definition we get

\[
(\det(I_n\{\tilde{I}\}))^{-k/2} = (i^{n})^{-k/2} = (-1)^{n}^{-k/2}
\]

and

\[
(\det(I_n\{iI\}))^{-k} = (i^{n})^{-k},
\]

hence the assertion follows.

Another interesting issue is the shape of the Fourier-expansion of the restricted modular forms. Of course, Siegel and Hermitian modular forms possess some Fourier-expansion like 1.19 as long as the necessary requirements are met. Now, because the Fourier-expansions are absolutely and locally uniformly convergent, we can rearrange the series to our desire.

So let \( T = (t_{j,k}) \in \text{Her}_n(H) \), \( T \geq 0 \), \( Z = (z_{j,k}) \in \mathcal{H}_n(H) \). Then according to (1.46)

\[
\tau(T,Z) = \sum_{j=1}^{n} t_{j,j} z_{j,j} + \sum_{1 \leq k < l \leq n} 2 \text{Re}(\tilde{t}_{k,l} z_{k,l})
\]

holds if we define \( \text{Re}(z) = \text{Re}(x) + i \text{Re}(y) = \frac{1}{2} (z + \overline{z}) \) for \( z = x + iy \in H \otimes_{R} C \) like in 1.7. Furthermore, if \( t = t_0 + t_1 \frac{1+i\sqrt{3}}{2} + t_2 i_2 + \frac{1+i\sqrt{7}}{2} z_2 i_2 + z_3 \frac{1+i\sqrt{3}}{2} i_2 \in H \otimes_{R} C \) (where \( z_0, \ldots, z_3 \in C \)), one computes

\[
2 \text{Re}(\tilde{t}z) = 2t_0 z_0 + 2t_1 z_1 + 2t_2 z_2 + 2t_3 z_3 + 2t_3 z_3.
\]

So if \( z \in R \otimes_{R} C \) (i.e. \( z_1 = z_2 = z_3 = 0 \)), then \( 2 \text{Re}(\tilde{t}z) = (2t_0 + t_1) z_0 \) and thus \( 2 \text{Re}(\tilde{t}z) \) only depends on the real part of \( t \) if \( z \) is arbitrary but fixed. Next, if we are looking at the case \( \omega \in O \setminus R \) and \( R := R(\omega) \) and thus \( z = z_0 + z_1 \omega \) (where \( z_0, z_1 \in C \)) we get

\[
2 \text{Re}(\tilde{t}z) = 2 \text{Re}(\tilde{t}z_0) + 2 \text{Re}(\tilde{t}z_1 \omega) = 2 \text{Re}(t) \cdot z_0 + 2 \text{Re}(\tilde{t} \omega) \cdot z_1,
\]

which means that \( 2 \text{Re}(\tilde{t}z) \) only depends on the real part of \( t \) and the real part of \( \tilde{t} \omega \). According to (1.7), \( H \times H \rightarrow R \), \( (a,b) \rightarrow \text{Re}(\overline{ab}) \) is a positive definite (and reflexive) bilinear form. This means we can find a basis \( H = R + \omega R + b_2 R + b_3 R \) such that \( b_2 \) and \( b_3 \) are orthogonal to 1 and \( \omega \), i.e. \( \text{Re}(b_2) = \text{Re}(b_3) = \text{Re}(\tilde{t} \omega) = \text{Re}(\tilde{b}_3 \omega) = 0 \). So if we write \( t \) in the form \( t = t_0 + t_1 \omega + t_2 b_2 + t_3 b_3 \), the equation above gives us

\[
2 \text{Re}(\tilde{t}z) = 2 \text{Re}(\tilde{t}_0 + \tilde{t}_1 \omega) \cdot z_0 + 2 \text{Re}(\tilde{t}_0 + \tilde{t}_1 \omega) \cdot \omega \cdot z_1.
\]

This might seem to be an odd way to write down the trace form, but it will provide us the right
shape to write down the Fourier-expansions: Every Matrix $M \in H^{n \times n}$ possesses a representation of the form

$$M = M_0 + i_1M_1 + i_2M_2 + i_1i_2M_3$$

as well as

$$M = \tilde{M}_0 + \omega \tilde{M}_1 + b_2\tilde{M}_2 + b_3\tilde{M}_3$$

with $M_0, \ldots, M_3, \tilde{M}_0, \ldots, \tilde{M}_3 \in \mathbb{R}^{n \times n}$. We consider the maps

$$\text{Re} : H^{n \times n} \to \mathbb{R}^{n \times n}, \quad \text{Re}(M) := M_0,$$

$$\text{Co}_\omega : H^{n \times n} \to \mathbb{R}^{n \times n}, \quad \text{Co}_\omega(M) := \tilde{M}_0 + \omega \tilde{M}_1.$$ 

Note that for $M \in \text{Her}_n(\mathbb{H})$ the diagonal entries of $M_1, M_2, M_3, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ are always zero. Furthermore, $\tilde{b}_2 = -b_2$ and $\tilde{b}_3 = -b_3$ hold by construction, and therefore $M \in \text{Her}_n(\mathbb{H})$ leads to $\text{Co}_\omega(M) \in \text{Her}_n(\mathbb{R})$ (and of course $\text{Re}(M) \in \text{Sym}_n(\mathbb{R})$). Now let $T \in \text{Her}_n(\mathbb{H})$, $T \geq 0$. Due to what we have seen above,

$$\tau(T, Z) = \tau(\text{Re}(T), Z) = \text{tr}(TZ)$$

holds for all $Z \in \mathcal{H}_n(\mathbb{R})$, and

$$\tau(T, Z) = \tau(\text{Co}_\omega(T), Z)$$

for all $Z \in \mathcal{H}_n(\mathbb{R}(\omega))$, as well. Therefore, we obtain the following theorem:

**Theorem.** Suppose $\Gamma \leq \text{Sp}_n(\mathcal{O})$ is a congruence subgroup with a multiplier system $\nu$ of weight $k (\in \mathbb{Q}, k \geq 0)$, $\omega \in \mathcal{O} \setminus \mathbb{R}$ and $R := \mathbb{R}(\omega)$. If $f \in [\Gamma, k, \nu]$ has the Fourier-expansion

$$f(Z) = \sum_{T \geq 0} a_f(T)e^{2\pi i \tau(T, Z)}, \quad Z \in \mathcal{H}_n(\mathbb{H}),$$

(where the sum ranges over $\frac{1}{m} \text{Her}_n^m(\mathcal{O})$ for some $m \in \mathbb{N}$,) then $f|_{\mathcal{H}_n(\mathbb{R})}$ and $f|_{\mathcal{H}_n(\mathbb{R})}$ possess the Fourier-expansions

$$f|_{\mathcal{H}_n(\mathbb{R})}(Z) = \sum_{T \geq 0} \beta_f(T)e^{2\pi i \text{tr}(TZ)}, \quad Z \in \mathcal{H}_n(\mathbb{R}),$$

$$f|_{\mathcal{H}_n(\mathbb{R})}(Z) = \sum_{T \geq 0} \gamma_f(T)e^{2\pi i \tau(T, Z)}, \quad Z \in \mathcal{H}_n(\mathbb{R}),$$

where the first sum ranges over all $T \in \text{Sym}_n(\mathbb{R})$ such that there is some $\tilde{T} \in \text{Her}_n(\mathbb{H})$ which occurs in the Fourier-expansion of $f$ with $\text{Re}(\tilde{T}) = T$ and the second sum ranges over all $T \in \text{Her}_n(\mathbb{R})$ such that there is some $\tilde{T} \in \text{Her}_n(\mathbb{H})$ which occurs in the Fourier-expansion of $f$ with $\text{Co}_\omega(\tilde{T}) = T$. The Fourier-coefficients for those $T$ are given by

$$\beta_f(T) = \sum_{\tilde{T} \geq 0, \text{Re}(\tilde{T}) = T} a_f(\tilde{T}),$$

$$\gamma_f(T) = \sum_{\tilde{T} \geq 0, \text{Co}_\omega(\tilde{T}) = T} a_f(\tilde{T}).$$
**Proof:** We have seen that
\[ \tau(T, Z) = \tau(\text{Re}(T), Z) = \text{tr}(TZ) \]
holds for all \( Z \in \mathcal{H}_n(\mathbb{R}) \) and that
\[ \tau(T, Z) = \tau(\text{Co}_\omega(T), Z) \]
holds for all \( Z \in \mathcal{H}_n(\mathbb{R}) \). Furthermore, as \( b_2 \) and \( b_3 \) are orthogonal to \( 1 \) and \( \omega \) and the diagonal entries are real, an easy calculation shows that \( T[g] = \text{Co}_\omega(T)[g] \) holds for all \( g \in \mathbb{R}^n \) (and of course \( T[g] = \text{Re}(T)[g] \) for all \( g \in \mathbb{R}^n \)). This immediately leads to \( \text{Co}_\omega(T) \geq 0 \) as well as \( \text{Re}(T) \geq 0 \) if \( T \geq 0 \). (Note that we have already seen that \( \text{Co}_\omega(M) \in \text{Her}_n(\mathbb{R}) \) and \( \text{Re}(M) \in \text{Sym}_n(\mathbb{R}) \) hold for \( T \in \text{Her}_n(\mathbb{H}) \).) So because of the absolute and local uniform convergence the Fourier-series can be rearranged to the asserted shape. \( \square \)

Note that the sums for the computation of \( \beta_f(T) \) and \( \gamma_f(T) \) are finite: The diagonal entries of \( T, \text{Re}(T) \) and \( \text{Co}_\omega(T) \) coincide and there are only finitely many \( T \in \frac{1}{m} \text{Her}_n^O(\mathcal{O}) \) with fixed diagonal entries.

A much more involved question is which Hermitian and Siegel modular forms arise from restricting quaternionic modular forms. And whether the restricting map \( [\Gamma, k, \nu] \mapsto [\Gamma_R, k, \nu_R] \) is injective or surjective. In general, this is an unsolved problem, but indeed essential if we want to determine the exact structure of spaces of quaternionic modular forms. For example, confer [Kr05] for the case of \( \mathcal{O} \) being the Hurwitz order. In the next chapter we will consider theta-series. Here, we can give some answers to the questions mentioned above.
2 Quaternionic Theta-Series

In this chapter we will give first important examples of quaternionic modular forms by means of theta-series and theta-constants. Theta-constants often play a role in constructing generators for graded rings of many types of modular forms. For example, Igusa constructed generators for the graded ring of Siegel modular forms of degree two using theta constants (cf. [Ig62], [Ig64]). And so did Freitag regarding symmetric Hermitian modular forms over \( \mathbb{Q}(i) \) (cf. [Fr67]). Moreover, quaternionic theta-constants (over the Hurwitz order) can be used to generate the graded ring of quaternionic modular forms over the Hurwitz order with respect to the extended modular group and the trivial character and as well concerning a certain principal subgroup – although in both cases one additionally needs a further quaternionic modular form which is a Maass lift (cf. [Kr05], [Kr10], [FH00], [FS07], and also [GK10] regarding some further setting). Similarly, theta-constants were needed in [Kl06] to construct generators for the graded ring of certain orthogonal modular forms for \( \mathcal{O}(2,5) \). So in other words, theta-constants often play an important role regarding generators for spaces of modular forms.

So we will also investigate quaternionic theta-constants with respect to \( \mathcal{O} \) in this thesis. And since quaternionic theta-constants are just a special case of general quaternionic theta-series, we start by introducing these more general objects. So the first introductory section will be about quaternionic theta-series which were already investigated in great detail in [Kr85]. We will mainly cite results from that work, but will also give some new ones regarding the order of interest in this thesis. In the second section we will analyze quaternionic theta-constants in full detail (which turn out to be 21 in number), including their transformation behavior even with respect to \( \Gamma(\mathcal{O}) \), their Fourier-expansions, the space of quaternionic modular forms they belong to and their restrictions to the Hermitian and Siegel half-spaces. The final section is about another important type of quaternionic theta-series, namely quaternionic theta-series of the second kind. We will examine these in the same way like the quaternionic theta-constants, again including their transformation behavior with respect to \( Z \mapsto Z' \) and as well \( \text{Rot}(i_1 I) \).

2.1 Quaternionic theta-series

In the next section we will define and have a closer look at quaternionic theta-constants. But first, we will start by defining general theta-series on \( \mathcal{H}_n(\mathbb{H}) \) and presenting some of their important properties. All further theta-series and theta-constants that will be introduced in this chapter are special cases of the general theta-series. All of the following definitions, propositions and theorems derive from the work of A. Krieg [Kr85, pp.100]. We will cite them in order to be able to recourse to them when we will finally talk about quaternionic theta-constants.
(2.1) Definition. Suppose that \( Z \in \mathcal{H}_n(H) \), \( S \in \text{Pos}_m(H) \), \( P, Q \in H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \) and that \( \Lambda \) is a lattice in \( H^{m \times n} \). Then
\[
\Theta_{P,Q}(Z;S;\Lambda) := \sum_{G \in \Lambda} e^{\pi i \tau(S[G+P],Z)+2\pi i \tau(Q,G+P)}
\]
is called the theta-series on \( \mathcal{H}_n(H) \) (or quaternionic theta-series) in \( Z \) and \( S \) with respect to the characteristic \( (P,Q) \) and the lattice \( \Lambda \).

Of course, one can replace \( H \) by \( \mathbb{C} \) or \( \mathbb{R} \) in order to obtain theta-series on the Hermitian and Siegel half-space. The subsequent properties also hold true for those cases.

Every \( A \in H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \) possesses a representation of the form \( A = (a_{k,l}) \) with \( a_{k,l} = a_{k,l}^{(0)} + a_{k,l}^{(1)}/i + a_{k,l}^{(2)}/2 + a_{k,l}^{(3)}/i2 \), where \( a_{k,l}^{(j)} \in \mathbb{C} \). Again, we call a function \( f : H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \), \( A \mapsto f(A) \) holomorphic if \( f \) is a holomorphic function of the coefficients \( a_{k,l}^{(j)} \) in the usual sense. With that in mind, we have the following theorem:

(2.2) Theorem. a) \( \Theta_{P,Q}(Z;S;\Lambda) \) converges absolutely and locally uniformly in dependence on \( Z \), \( S \), \( P \) and \( Q \).

b) Given \( \rho > 0 \) then \( \Theta_{P,Q}(Z;S;\Lambda) \) converges absolutely and uniformly in the domain \( \Im(Z) \geq \rho \), \( S \geq \rho I \), \( P \in H^{m \times n} \), \( \tau(P,P) \leq \rho^{-2} \), \( Q \in H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \), \( \tau(\Im(Q),\Im(Q)) \leq \rho^{-2} \).

c) \( \Theta_{P,Q}(Z;S;\Lambda) \) considered as a function of \( Z \) is holomorphic on the quaternionic half-space \( \mathcal{H}_n(H) \), and considered as a function of \( P \) resp. \( Q \) the theta-series turns out to be a holomorphic function on \( H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \).

In analogy to \( \text{Her}_n^+(\mathbb{O}) \) (see (1.47)) we define the dual lattice \( \Lambda^* \) for a lattice \( \Lambda \) as
\[
\Lambda^* := \{ K \in H^{m \times n} ; \tau(H,K) \in \mathbb{Z} \text{ for all } H \in \Lambda \}.
\]

(2.3) Proposition. Let \( H \in \Lambda \), \( K \in \Lambda^* \) and \( U \in H^{m \times n} \otimes_{\mathbb{R}} \mathbb{C} \).

a) \( \Theta_{P+H,Q}(Z;S;\Lambda) = \Theta_{P,Q}(Z;S;\Lambda) \).

b) \( \Theta_{P,Q+K}(Z;S;\Lambda) = e^{2\pi i \tau(P,K)} \Theta_{P,Q}(Z;S;\Lambda) \).

c) \( \Theta_{P,Q+S(U,Z;S;\Lambda)} = e^{-2\pi i \tau(S[U],Z)-2\pi i \tau(Q,U)} \Theta_{P+U,Q}(Z;S;\Lambda) \).

(2.4) Proposition. Let \( \Lambda \) and \( \Lambda^* \) be two lattices in \( H^{m \times n} \) and suppose that \( U \in \text{GL}_n(H) \), \( V \in \text{GL}_m(H) \) such that the map
\[
\Lambda \to \Lambda^*, \quad H \mapsto VHU
\]
is bijective. Then, one has
\[
\Theta_{P,Q}(Z;S;\Lambda^*) = \Theta_{V^{-1}P \pi^{-1}V',Q}(Z[U],S[V];\Lambda) \,.
\]
2.1 Quaternionic theta-series

In virtue of 1.6 and (1.7), one easily checks that

\[(O^{m \times n})^T = ((O^T)^{m \times n})^T = (2i_1 \sqrt{3}O)^{m \times n} = 2i_1 \sqrt{3}(O^{m \times n})\]

holds. Thus we get the following analogy of [Kr85, ch.IV, cor.1.7]:

(2.5) Corollary. a) Given \( U \in \text{GL}_n(O) \) and \( V \in \text{GL}_m(O) \) then

\[\Theta_{P,Q}(Z, S; O^{m \times n}) = \Theta_{V^{-1}PT^{-1}VT, U}(Z[U], S[V]; O^{m \times n}).\]

b) Put \( \alpha = 2i_1 \sqrt{3}, \) then

\[\Theta_{P,Q}(Z, S; (O^{m \times n})^T) = \Theta_{\alpha^{-1}P, -aQ}(Z, S[a1]; O^{m \times n}) = \Theta_{P, -aQ}(Z[a1], S; O^{m \times n}).\]

(2.6) Definition. A matrix \( S \in \text{Her}_n(H) \) is called \( O \)-even if \( \frac{1}{2}S \in \text{Her}_n^+(O) \). Furthermore, an even matrix \( S \in \text{Pos}_n(H) \) is called \( O \)-stable if in addition \( \det(S) = \text{vol}(O)^{-n/2} = \left(\frac{4}{3}\right)^{n/2} \) holds.

(2.7) Remark. According to [Kl98, prop.3.22] and [Kr85, ch.IV, cor.2.5] it holds that if \( S \in \text{Her}_n(H) \) is \( O \)-stable, then \( n \) has to be even.

\[S_\Theta := \begin{pmatrix} 2 & 2i_1 \sqrt{3} + i_2 \sqrt{3} \\ -2(i_1 \sqrt{3} + i_2 \sqrt{3}) & 2 \end{pmatrix}\]

turns out to be \( O \)-stable (cf. [Kl98, p.47]). Note that if \( S_1 \) and \( S_2 \) are \( O \)-stable, then

\[\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}\]

is \( O \)-stable, too. Hence there exist \( O \)-stable matrices of degree \( n \) as long as \( n \) is even.

Once more, the proof of the following proposition would be exactly the same as in [Kr85, ch.IV, prop.1.11] for the Hurwitz order. Therefore we simply cite

(2.8) Proposition. Let \( S \in \text{Pos}_m(H) \) be \( O \)-even. Then

\[\Theta_{P,Q}(Z + T, S; O^{m \times n}) = e^{-\pi i \tau(S[P], T)} \Theta_{P,Q,SPT}(Z, S; O^{m \times n})\]

holds for all \( Z \in \mathcal{H}_n(H), T \in \text{Her}_n(O) \) and \( P, Q \in H^{m \times n} \otimes R C.\)

(2.9) Theorem. (Theta-transformation-formula)

Let \( Z \in \mathcal{H}_n(H), S \in \text{Pos}_m(H), P, Q \in H^{m \times n} \otimes R C \) and \( \Lambda \) be a lattice in \( H^{m \times n} \). Then, one has

\[\Theta_{-Q,P}(-Z^{-1}, S^{-1}; \Lambda^\tau) = (\text{vol}(\Lambda)) \cdot (\text{det}(\frac{1}{2}Z))^{2m} \cdot (\text{det}(S))^{2n} \cdot e^{-2\pi i \tau(P,Q)} \cdot \Theta_{P,Q}(Z, S; \Lambda).\]
Combining the theta-transformation-formula and (2.5) leads to

\[(2.10) \textbf{Corollary.} \text{ If } \alpha = \frac{2}{3}i \sqrt{3} \text{ one has}
\]
\[(\det(\frac{i}{2}Z))^2m \cdot (\det(S))^2n \cdot \Theta_{P,Q}(Z,S;O^{m \times n})
= (\frac{4}{3})^{m\alpha} \cdot e^{2\pi i (P,Q)} \cdot \Theta_{-\alpha^{-1}Q,-\alpha P}(-Z^{-1},S^{-1}[\alpha I];O^{m \times n}) .
\]

We will now introduce the first non-identically vanishing quaternionic modular forms for the whole quaternionic modular group with trivial character, namely the so-called \textit{Theta-Nullwerte}
\[
\Theta(Z,S;O) := \Theta^{(n)}(Z,S;O) := \Theta_{0,0}(Z,S;O^{m \times n})
\]
(2.1) for \(S \in \text{Pos}_m(H), Z \in \mathcal{H}_n(H)\). Furthermore, for \(T \in \text{Her}_n(H)\) and \(S \in \text{Her}_m(H)\) we need to define the so-called set of representations of \(S\) by \(T\), that is
\[
\Lambda(S,T;O) := \{G \in O^{m \times n} ; S[G] = T \} .
\]
(2.2)

We call
\[
\#(S,T;O) := \#(\Lambda(S,T;O))
\]
(2.3) the number of representations of \(T\) by \(S\). Completely analogous to [Kr85, ch.I, prop.5.3] we have the following

\[(2.11) \textbf{Proposition.} \text{ Suppose that } S \in \text{Her}_m(H) \text{ and } T \in \text{Her}_n(H). \text{ Given } U \in \text{GL}_m(O) \text{ and } V \in \text{GL}_n(O) \text{ it holds that}
\]
\[
\#(S[U],T[V];O) = #(S,T;O) .
\]
If \(S \in \text{Pos}_m(H)\), then \(#(S,T;O)\) becomes finite.

The following theorem can be found in [Kr85, ch.IV, thm.2.6] for the Hurwitz order. It can be applied to our case, but this time we will give some more details on the proof as we have to take a closer look at some of the used considerations.

\[(2.12) \textbf{Theorem.} \text{ Let } S \in \text{Pos}_m(H) \text{ be } O\text{-stable. Then}
\]
\[
\Theta^{(n)}(\cdot,S;O) \in [\text{Sp}_n(O),2m,1]
\]
with Fourier-expansion
\[
\Theta(Z,S;O) = \sum_{T \in \text{Her}_m(O), T \geq 0} #(S,2T;O) \cdot e^{2\pi i (T,Z)} , \quad Z \in \mathcal{H}_n(H) .
\]

One has
\[
\Theta^{(n)}(\cdot,S;O)|\Phi = \Theta^{(n-1)}(\cdot,S;O)
\]
where \(\Theta^{(0)}(\cdot,S;O) := 1\).
2.1 Quaternionic theta-series

**Proof:** By definition and (2.2),

$$\Theta(Z, S; \mathcal{O}) = \sum_{G \in \mathcal{O}^{m \times n}} e^{\pi i r(S[G], Z)}$$

is absolutely and locally uniformly convergent in $\mathcal{H}_n(\mathbb{H})$ and holomorphic (both with respect to $Z$). So we can rearrange the series and collect all $e^{\pi i r(S[G], Z)}$ such that $S[G] = T$ for all possibly occurring $T \in \text{Her}_n(\mathbb{H})$, $T \geq 0$. Now $S$ is $\mathcal{O}$-even, i.e. $\frac{1}{2}S \in \text{Her}_n^+(\mathcal{O})$ and thus $\frac{1}{2}S[G] \in \text{Her}_n^+(\mathcal{O})$ holds for all $G \in \mathcal{O}^{m \times n}$ by definition and (1.46). Hence only such $T = S[G]$ occur such that $T \in \text{Her}_n^+(\mathcal{O})$ which yields the asserted shape of the Fourier-expansion. The statement about the $\Phi$-operator can be verified in exactly the same way as it is already done in [Kr85, ch.IV, thm.2.6].

If $n = 1$, the bounding condition (M.3) is obvious regarding the shape of the Fourier-expansion. So only (M.2) remains to be shown, which means that the theta-series behave accordingly under modular transformations. Of course, it suffices to look at the generators of $\text{Sp}_n(\mathcal{O})$. According to (2.5),

$$\Theta(Z, S; \mathcal{O})|_{2m} \text{Rot}(U) = \Theta(Z[U], S; \mathcal{O}) = \Theta(Z, S; \mathcal{O})$$

holds for all $U \in \text{GL}_n(\mathcal{O})$, as well as

$$\Theta(Z, S; \mathcal{O})|_{2m} \text{Trans}(T) = \Theta(Z + T, S; \mathcal{O}) = \Theta(Z, S; \mathcal{O})$$

for all $T \in \text{Her}_n(\mathcal{O})$ in virtue of (2.8). So we only have to check the transformation behavior with respect to $J_n$. Let $\alpha = \frac{2}{3}i\sqrt{3}$, then according to (2.10)

$$\Theta(-Z^{-1}, S^{-1}[\alpha I]; \mathcal{O}) = (\det(Z))^{2m} \cdot \Theta(Z, S; \mathcal{O}) = (\det(Z))^{m} \cdot \Theta(Z, S; \mathcal{O})$$

holds because $(\det(S))^{2n} = \left(\frac{3}{4}\right)^{mn}$ by assumption and $(\det(\frac{1}{2}Z))^{2n} = (\det(Z))^{2n} = (\det(Z))^{m}$ as $m$ is even (see (2.7), while the second equality is correct due to (1.34)). Now, note that $S = (s_{ij}) \in 2 \text{Her}_n^+(\mathcal{O})$, hence $s_{ij} \in 2\mathbb{Z}$ and $s_{ij} \in \mathcal{O} = \alpha \mathcal{O}$ holds for $1 \leq j \leq m, 1 \leq k, l \leq m, k \neq l$ (see (1.47)). So because $\alpha^{-1} = -\frac{1}{2}i\sqrt{3}$, $\alpha^{-1}s_{ij} \in i\sqrt{3}\mathbb{Z} \subset \mathcal{O}$ as well as $\alpha^{-1}s_{kj} \in \mathcal{O}$ follow, which means $W := \alpha^{-1}S \in \mathcal{O}^{m \times m}$. By means of (1.34) one computes

$$\det(W) = N(a)^{-m} \cdot (\det(S))^2 = \left(\frac{3}{4}\right)^{-m} \cdot \left(\left(\frac{3}{4}\right)^{m/2}\right)^2 = 1.$$ 

Thus $W \in \text{GL}_m(\mathcal{O})$ follows in virtue of (1.34). As $S^{-1}[\alpha I][W] = S^{-1}[\alpha I] = S' = S$ holds, (2.5) leads to

$$\Theta(-Z^{-1}, S^{-1}[\alpha I]; \mathcal{O}) = \Theta(-Z^{-1}, S^{-1}[\alpha I][W]; \mathcal{O}) = \Theta(-Z^{-1}, S; \mathcal{O}) .$$

So we have shown

$$\Theta(Z, S; \mathcal{O})|_{2m} J_n = (\det(Z))^{-m} \cdot \Theta(-Z^{-1}, S; \mathcal{O}) = \Theta(Z, S; \mathcal{O}) ,$$

which completes the proof. □

So with $\Theta(Z, S; \mathcal{O}) \in [\text{Sp}_n(\mathcal{O}), 4, 1]$ we have constructed a first example of a non-identically vanishing quaternionic modular form for the whole modular group, and thus for all weights $k \in 4\mathbb{N}_0$ in terms of powers of this theta-series.
2.2 Quaternionic theta-constants

In the previous section we defined general quaternionic theta-series and exposed some of their important properties. We also looked at the special case of a Theta-Nullwert. In the current section we will have a closer look at another special case of theta-series, namely the quaternionic theta-constants. They are another important type of quaternionic modular forms, constructed in analogy the quaternionic theta-constants found in [Kr10]. The great importance of (Hurwitz) theta-constants. They are another important type of quaternionic modular forms, constructed in analogy to [Kr10] and [DK03]. Essentially, quaternionic theta constants are just a special case of quaternionic theta-series for $\mathbb{O}$. They are defined in analogy to [Kr10] and [DK03].

(2.13) Definition. Let $p \in \mathbb{O}^2$. The quaternionic theta-constant with characteristic $p$ is defined as

$$\vartheta_p(Z) := \sum_{g \in \frac{1}{\sqrt{3}}p + \mathbb{O}^2} e^{2\pi i Z |g|} = \sum_{g \in \frac{1}{\sqrt{3}}p + \mathbb{O}^2} e^{2\pi i \tau(Z, g)}$$

$Z \in \mathcal{H} (\mathbb{H})$.

Note that we will see in (2.27) that quaternionic theta-constants also arise from ordinary Siegel theta-constants living on $\mathcal{H}_3 (\mathbb{R})$ and Hermitian theta-constants living on $\mathcal{H}_4 (\mathbb{C})$ by restricting these to $\mathcal{H} (\mathbb{H})$.

(2.14) Remark. a) The chosen notation and description is only for comparability to [Kr10] and [DK03]. Essentially, quaternionic theta constants are just a special case of quaternionic theta-series of degree two. Let $p = (p_1, p_2)' \in \mathbb{O}^2$. Define $\Lambda = \mathbb{O}^{1 \times 2}, P = \left( \frac{i}{\sqrt{3}} p_1, \frac{i}{\sqrt{3}} p_2 \right) = \left( \frac{i}{\sqrt{3}} p \right) \in \frac{i}{\sqrt{3}} \Lambda$ and $Q = 0 \in \Lambda$. Using (1.46), we compute

$$\vartheta_p(Z) = \sum_{g \in \frac{1}{\sqrt{3}}p + \mathbb{O}^2} e^{2\pi i Z |g|} = \sum_{g \in \frac{1}{\sqrt{3}}p + \mathbb{O}^2} e^{2\pi i \tau(Z, g)} = \sum_{g \in \mathbb{O}^2} e^{\pi i \tau(2|p + g|, Z)}$$

$$= \sum_{G \in \Lambda} e^{\pi i \tau(2|p + g|, Z)} = \Theta_{p, Q}(Z, 2; \Lambda) .$$

Note that $2 \in \text{Pos}_1 (\mathbb{H})$ is $\mathbb{O}$-even. Thus we can apply all properties about quaternionic theta-series for $\mathbb{O}$-even matrices from the preceding section.

b) Let $h \in p^2 = i \sqrt{3} \mathbb{O}^2$, then $H = \left( \frac{i}{\sqrt{3}} h \right)' \in \Lambda$. In virtue of a) and (2.3),

$$\vartheta_{p+h}(Z) = \Theta_{p+H, 0}(Z, 2; \Lambda) = \Theta_{p, 0}(Z, 2; \Lambda) = \vartheta_p(Z)$$

holds. Thus we can choose $p$ modulo $p^2$. According to (1.10), $\mathbb{O}^2/p^2 \simeq \mathbb{F}_q^2$, hence there are at most 81 theta-constants. Now let $\varepsilon \in \mathcal{E}$. According to (2.5),

$$\vartheta_{p\varepsilon}(Z) = \Theta_{p, 0}(Z, 2; \Lambda) = \Theta_{p, 0}(Z, 2; \varepsilon) = \Theta_{p, 0}(Z, 2; \Lambda) = \vartheta_p(Z)$$

holds. Note that we can choose $p$ modulo $p^2$, while $\mathbb{O}^2/p^2 \simeq \mathbb{F}_q^2$ is commutative, and thus
2.2 Quaternionic theta-constants

also

\[ \theta_{p'}(Z) = \theta_p(Z) \]

holds. Moreover, this means that we can reduce the choice of \( p \) even further, namely modulo \( E \). Considering \( p \) to be an element of \( \mathbb{F}_2^{12} \), this means we can reduce modulo \{\pm 1, \pm i_2\} (or \{\pm 1, \pm \omega\} with \( F_9 = F_3(\omega) \), see 1.11). Doing so yields 21 remaining choices for \( p \):

\[
\begin{align*}
p_1 &= (0 \quad 0) && \quad p_2 = (0 \quad 1) && \quad p_3 = (0 \quad 1 + i_2) \\
p_4 &= (1 \quad 0) && \quad p_5 = (1 \quad 1) && \quad p_6 = (1 \quad -1) \\
p_7 &= (1 \quad i_2) && \quad p_8 = (1 \quad -i_2) && \quad p_9 = (1 \quad 1 + i_2) \\
p_{10} &= (1 \quad 1 - i_2) && \quad p_{11} = (1 \quad -1 + i_2) && \quad p_{12} = (1 \quad -1 - i_2) \\
p_{13} &= (1 + i_1 \quad 0) && \quad p_{14} = (1 + i_2 \quad 1) && \quad p_{15} = (1 + i_2 \quad -1) \\
p_{16} &= (1 + i_2 \quad i_2) && \quad p_{17} = (1 + i_2 \quad -i_2) && \quad p_{18} = (1 + i_2 \quad 1 + i_2) \\
p_{19} &= (1 + i_2 \quad 1 - i_2) && \quad p_{20} = (1 + i_2 \quad -1 + i_2) && \quad p_{21} = (1 + i_2 \quad -1 - i_2)
\end{align*}
\]

To keep it concise, we define

\[ \theta_j := \theta_{p_j}, \quad 1 \leq j \leq 21. \]

Further below, we will see that these 21 quaternionic theta-constants are indeed linearly independent.

c) \( g = (g_1, g_2)' \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} p + O^2 \) yields \( g_1, g_2 \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} p = O \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} \), and therefore \( g_3 g_k' \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} O \) for \( j, k \in \{1, 2\} \). Thus \( g g' \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} \text{Her}_2(O) \) holds. Note that \( g g' \geq 0 \) according to [Kr85, ch.I, cor.3.7]. Just as in (2.12) we can rearrange the theta-series as they are absolutely and locally uniformly convergent:

\[
\theta_j(Z) = \sum_{g \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} p + O^2} e^{2\pi i \tau(g g', Z)} = \sum_{T \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} \text{Her}_2(O), T \geq 0} \alpha_j(T) \cdot e^{2\pi i \tau(T, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]

where

\[ \alpha_j(T) = \#\{g \in \begin{pmatrix} i_3 \sqrt{3} \end{pmatrix} p + O^2 \mid g g' = T\}. \]

We will determine the Fourier-coefficients \( \alpha_j(T) \) later. But note that \( \alpha_j(T) \neq 0 \) implies that \( T \) is not invertible, of course, i.e. \( \det(T) = 0 \), since for \( T = g g' \) and \( g = (g_1, g_2)' \) one easily computes \( \det(T) = N(g_1) N(g_2) - N(g_1 g_2') = 0 \)

d) Let \( U \in \text{GL}_2(O) \). As we have seen before, \( \phi_1 : O \to O, a \mapsto -i_1 a i_1 \) is a bijection, and \( -i_1 U i_1 \in \text{GL}_2(O) \) holds because of (1.34). Note that \( \frac{i_3}{\sqrt{3}} U p = (-i_1 U i_1) \frac{i_3}{\sqrt{3}} p \). So in virtue
of (2.5) we compute
\[
\vartheta_{U_p}(Z) = \Theta_{\frac{1}{2} i U_1}^{-\frac{1}{2} \lambda_p} (Z; 2; \Lambda) = \Theta_{\frac{1}{2} i U_1}^{-\frac{1}{2} \lambda_p} (Z; 2; \Lambda) \\
= \Theta_{\pi, 0} (Z[-i_1 U_1], 2; \Lambda) = \vartheta_p(Z[-i_1 U_1]).
\]
Again, \( p \) can be chosen modulo \( p^2 \). As we have seen in the proof of (1.20), \( \pi_p \circ \phi_1 = \phi_3 \circ \pi_p \) holds, while \( \pi_p(a + p) = \phi_3(\pi_p(a + p)) \) according to (1.11). In other words, \( -i_1 U_1 \equiv \mathfrak{U} \) modulo \( p \). So we can reformulate the result from above as
\[
\vartheta_p(Z[U]) = \vartheta_{\mathfrak{U} p}(Z)
\]
for all \( Z \in \mathcal{H}(\mathbb{H}) \) and \( U \in \text{GL}_2(\mathcal{O}) \).
Now let \( S \in \text{Her}_2(\mathcal{O}) \). We get
\[
\vartheta_p(Z + S) = \sum_{g \in \frac{i}{\sqrt{3}} p + \mathcal{O}^2} e^{2\pi i(Z + S)[g]} = \sum_{g \in \mathcal{O}^2} e^{2\pi iZ[\frac{i}{\sqrt{3}} p + g] + 2\pi is[\frac{i}{\sqrt{3}} p + g]}
\]
with
\[
S[\frac{i}{\sqrt{3}} p + g] = \tau(S[\frac{i}{\sqrt{3}} p + g], 1) = \tau(S, (\frac{i}{\sqrt{3}} p + g)(\frac{i}{\sqrt{3}} p + g)^{\prime}) \\
= S[\frac{i}{\sqrt{3}} p] + S[g] + \tau(S, \frac{i}{\sqrt{3}} p g^{\prime} + \tau(S, g p^{\prime} \frac{i}{\sqrt{3}}) \\
= S[\frac{i}{\sqrt{3}} p] + S[g] + \tau(S, \frac{2}{3} i_1 \sqrt{3} p g^{\prime})
\]
which derives from (1.46), since \( S = \mathfrak{S} \). Now, \( S[g] \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z} \). Moreover, \( \frac{2}{3} i_1 \sqrt{3} p g^{\prime} \in (\mathcal{O}^2)^{2 \times 2} = (\mathcal{O}^2)^{2 \times 2} \) and thus \( \tau(S, \frac{2}{3} i_1 \sqrt{3} p g^{\prime}) \in \mathbb{Z} \) holds, too. We conclude
\[
\vartheta_p(Z + S) = e^{2\pi iS[\frac{i}{\sqrt{3}} p]} \cdot \vartheta_p(Z).
\]
Note that \( S[\frac{i}{\sqrt{3}} p] = \frac{1}{3} (-i_1 S i_1)[p] \in \frac{1}{3} \mathbb{Z} \). So the prefactor \( e^{2\pi iS[\frac{i}{\sqrt{3}} p]} \) is always a third root of unity.

So in terms of quaternionic modular forms and generators of \( \text{Sp}_2(\mathcal{O}) \), the \( \text{Rot}(U) \) permute the 21 theta-constants, while the \( \text{Trans}(S) \) provide some possible prefactors. What remains to be analyzed is the transformation behavior under \( J \). This problem can be solved using the theta-transformation-formula. The transformation with \( J \) yields a sum of theta-constants:

\textbf{(2.15) Theorem.} Let \( p = (p_1, p_2)^{\prime} \in \mathcal{O}^2 \), then
\[
\vartheta_p(-Z^{-1}) = \frac{1}{2} (\det(Z))^2 \sum_{q = (q_1, q_2)^{\prime} \in \mathcal{O}^2 / p^2} e^{-\frac{4}{3} \pi i \text{Re}(\varpi_1 q_1 + \varpi_2 q_2)} \cdot \vartheta_q(Z)
\]
holds for all \( Z \in \mathcal{H}(\mathbb{H}) \).
Proof: Let \( Z \in \mathcal{H}(\mathbb{H}) \) and define \( \Lambda = O^{1 \times 2} \), \( P = \left( \frac{1}{\sqrt{3}} p_1, \frac{1}{\sqrt{3}} p_2 \right) = \left( \frac{1}{\sqrt{3}} p \right) \in \frac{1}{\sqrt{3}} \Lambda \) and \( \alpha = \frac{2}{3} i_1 \sqrt{3} \). In virtue of (1.46), (2.10) and (2.14), we calculate

\[
\vartheta_p(-Z^{-1}) = \Theta_{p,0}(-Z^{-1}; 2; \Lambda) = (\frac{4}{3})^2 \cdot 2^{-4} \cdot e^{2\pi i \tau(P,0)} \cdot (\det(-1/2Z))^2 \cdot \Theta_{0,-\alpha}(Z, 1/2 [\alpha I_1]; \Lambda) = \frac{1}{\alpha} (\det(Z))^2 \sum_{g \in O} \mu_{\tau(Z)}(g) e^{2\pi i \tau(P, -\pi g')} = \frac{1}{\alpha} (\det(Z))^2 \sum_{g \in O} e^{\pi i Z|g| + 2\pi i (P, -\pi g')}.
\]

As the quaternionic theta-constants are absolutely and locally uniformly convergent, we may rearrange the infinite sum. Every \( g \in O^2 \) can be uniquely written as \( g = q + i_1 \sqrt{3} a \), where \( q \in O^2/p^2 \) and \( a \in O^2 \), and hence (because \( Z|g| \in \mathbb{R} \otimes \mathbb{R} \mathbb{C} \) for \( g \in \mathbb{H}^2 \))

\[
\vartheta_p(-Z^{-1}) = \frac{1}{\alpha} (\det(Z))^2 \sum_{q \in O^2/p^2} \sum_{g \in O} e^{\pi i Z|q + i_1 \sqrt{3} a| + 2\pi i (P, -\pi (g' - g' i_1 \sqrt{3}))} = \frac{1}{\alpha} (\det(Z))^2 \sum_{-i_1 q_1} \sum_{g' \in O} e^{\pi i Z|q + i_1 \sqrt{3} a| - i_1 i_1 i_1 \sqrt{3}) + 2\pi i (P, -\pi (g' - g' i_1 \sqrt{3}))} = \frac{1}{\alpha} (\det(Z))^2 \sum_{q \in O^2/p^1} \sum_{i_1 q_1} e^{\pi i Z|q + g' + i_1 \sqrt{3} a| + 2\pi i (P, -\pi (g' - g' i_1 \sqrt{3}))}
\]

follows. Furthermore,

\[
\tau(P, a(-i_1 q' i_1 + i_1 \sqrt{3} g')) = \tau(P, \frac{2}{3} \sqrt{3} q' i_1 - \frac{2}{3} \sqrt{3} \sqrt{3} g') = \tau(P, q' i_1^2 + g' i_1 \sqrt{3} a) = \tau(P, q' i_1^2 + \sqrt{3} a)
\]

holds, and therefore

\[
2\pi i \tau(P, a(-i_1 q' i_1 + i_1 \sqrt{3} g')) = -\frac{4}{3} \pi i \left( \text{Re}(q_1 q_1 + q_2 q_2) - \text{Re}(q_1 q_1 + q_2 q_2) \right)
\]

for \( g = (g_1, g_2)' \) and \( q = (q_1, q_2) \), because \( P, \bar{a} = -\frac{2}{3} (\bar{p}_1, \bar{p}_2) \). Noting \( \bar{p}_1 i_1 \sqrt{3} g_1 + \bar{p}_2 i_1 \sqrt{3} g_2 \in p \), we obtain \( \text{Re}(q_1 q_1 + q_2 q_2) \in \frac{1}{3} Z \), because \( p = i_1 \sqrt{3} O = i_1 \sqrt{3} Z + i_1^3 \sqrt{3} Z + i_1 i_1^3 \sqrt{3} Z + i_1^3 \sqrt{3} Z \). Thus \( \frac{4}{3} \pi i \text{Re}(q_1 q_1 + q_2 q_2) \) holds, and

\[
\vartheta_p(-Z^{-1}) = \frac{1}{\alpha} (\det(Z))^2 \sum_{q = (q_1, q_2)' \in O^2/p^2} \sum_{g \in O} e^{2\pi i Z|q + g|} - \frac{4}{3} \pi i \text{Re}(q_1 q_1 + q_2 q_2) \cdot \vartheta_q(Z)
\]

follows.

\[\square\]

We have determined the transformation behavior of \( \vartheta_p | L M \) for all \( M \) in the list of generators.
of $\text{Sp}_2(\mathcal{O})$ in (1.15). Hence we are able to determine $\vartheta_p|_2M$ for all $M \in \text{Sp}_2(\mathcal{O})$ (which is always a certain sum of quaternionic theta-constants) in view of 1.14. This can be described in the following corollary. The proof is obvious by what we have seen so far.

(2.16) Corollary. Let $\vartheta = (\vartheta_1, \ldots, \vartheta_{21})'$. There is a homomorphism $\Psi: \text{Sp}_2(\mathcal{O}) \rightarrow \text{GL}_{21}(\mathbb{C})$ such that

$$\vartheta|_2M := (\vartheta_1|_2M, \ldots, \vartheta_{21}|_2M)' = \Psi(M) \cdot \vartheta$$

holds for all $M \in \text{Sp}_2(\mathcal{O})$.

The explicit images of the generators of $\text{Sp}_2(\mathcal{O})$ under $\Psi$ can be found in the appendix. They can be deduced from (2.14) and (2.15), easily. Using [MAGMA] one can check that the order of $\Psi(\text{Sp}_2(\mathcal{O}))$ is 6,531,840. So because this order as well as the degree 21 are too large, it seems impossible to determine the ring of invariants $\langle \vartheta_1, \ldots, \vartheta_{21}\rangle_{\text{Sp}_2(\mathcal{O})}$ using a computer algebra system like it was done in [Krt05], for example.

Next, we want to show that the quaternionic theta-constants are indeed quaternionic modular forms for a certain congruence subgroup. To be more precise, we will show that $\vartheta_p$ can be specified explicitly. So as $\text{Sp}_2(\mathcal{O})|_p \subset \text{Sp}_2(\mathcal{O})|_0$, we will be able to prove that $\vartheta_p|_2M = \vartheta_p$ holds for all $M \in \text{Sp}_2(\mathcal{O})|_0$ without the need to determine generators.

(2.17) Theorem. Let $p \in \mathcal{O}^2$ and $M = (A \; B)
\begin{array}{c|c}
C & D
\end{array} \in \text{Sp}_2(\mathcal{O})|_0$. Then

$$\vartheta_p|_2M(Z) = M \ast \vartheta_p(Z) := e^{2\pi i(A\overline{B})|_{\mathbb{R}}|\frac{1}{p}|} \ast \vartheta_p(Z)$$

holds for all $Z \in \mathcal{H}(\mathbb{H})$.

Proof: Note that $AB^t \in \text{Her}_2(\mathcal{O})$ holds according to the fundamental relations in (1.3), hence $(A\overline{B})|_{\mathbb{R}}|\frac{1}{p}| \in \mathbb{R}$ follows. So the definition of $M \ast \vartheta_p$ makes sense.

First, we will show that the assertion holds true for all generators of $\text{Sp}_2(\mathcal{O})|_0$. Afterwards, we will prove that if the assertion holds for two matrices $M_1$ and $M_2$, then it holds for the product $M_1M_2$, too (which is equivalent to verify $(M_1M_2) \ast \vartheta_p = M_2 \ast (M_1 \ast \vartheta_p)$ in virtue of 1.14), so that it holds for the whole group, consequently.

According to (1.22), $\text{Sp}_2(\mathcal{O})|_0$ is generated by

$$\text{Rot}(U), \; U \in \text{GL}_2(\mathcal{O}), \quad \text{Trans}(S), \; S \in \text{Her}_2(\mathcal{O}), \quad -J \text{Trans}(-T)J, \; T \in \text{Her}_2(p).$$
Let us start with $M = \text{Rot}(U)$, $U \in \text{GL}_2(\mathcal{O})$ (thus $A = \overline{U}$, $B = 0$). (2.14) yields
\[
\vartheta_p|_2 M(Z) = \vartheta_p(Z[U]) = \vartheta_{\overline{U}p}(Z) = \vartheta_{A'p}(Z)
\]
and hence the assertion for $\text{Rot}(U)$.

Next, let $M = \text{Trans}(S)$, $S \in \text{Her}_2(\mathcal{O})$ (and thus $A = I$, $B = S = \overline{S} = \overline{B}$). Again, (2.14) yields
\[
\vartheta_p|_2 M(Z) = \vartheta_p(Z + S) = e^{2\pi i S[\frac{1}{\sqrt{p}}]} \cdot \vartheta_p(Z) = e^{2\pi i \overline{B}[\frac{1}{\sqrt{p}}]} \cdot \vartheta_p(Z).
\]

Last, let $M = -j \text{Trans}(-T)J = (\frac{1}{T} I T)$, $T \in \text{Her}_2(p)$ (which means $A = I$, $B = 0$). According to (2.15), there is a homogeneous multivariate polynomial $r_p \in \mathbb{C}[X_1, \ldots, X_{21}]$ of degree 1 such that
\[
\vartheta_p|_2(-J) = \vartheta_p|_2 J = r_p(\vartheta_1, \ldots, \vartheta_{21}).
\]

Furthermore, because $T \in \text{Her}_2(p) \subset i_1 \sqrt{3} \mathbb{O}^{2 \times 2} = \mathbb{O}^{2 \times 2} i_1 \sqrt{3}$,
\[
T[\frac{1}{\sqrt{3}}pJ] \in \mathbb{R} \cap \frac{i}{\sqrt{3}} \mathcal{O} = \mathbb{Z}
\]
holds (as $\frac{i}{\sqrt{3}} \mathcal{O} = \frac{i}{\sqrt{3}} \mathbb{Z} + (\frac{i}{2\sqrt{3}} - \frac{1}{2}) \mathbb{Z} + (\frac{i}{2\sqrt{3}} - \frac{1}{2})i_2 \mathbb{Z}$) and therefore
\[
\vartheta_j|_2 \text{Trans}(-T)(Z) = \vartheta_j(Z - T) = e^{-2\pi i T[\frac{1}{\sqrt{3}}p]} \cdot \vartheta_j(Z) = \vartheta_j(Z)
\]
follows for all $j \in \{1, \ldots, 21\}$. So 1.14 yields
\[
\vartheta_p|_2 M = \vartheta_p|_2(-J)|_2 \text{Trans}(-T)|_2 J
\]
\[
= r_p(\vartheta_1|_2 \text{Trans}(-T), \ldots, \vartheta_{21}|_2 \text{Trans}(-T))|_2 J
\]
\[
= r_p(\vartheta_1, \ldots, \vartheta_{21})|_2 J = \vartheta_p|_2(-J) = \vartheta_p|_2 J = \vartheta_p.
\]

Hence the assertion holds for all generators of $\text{Sp}_2(\mathcal{O})|p|_0$. As we have seen above, only
\[(M_1 M_2) \ast \vartheta_p = M_2 \ast (M_1 \ast \vartheta_p) \quad \text{for all } M_1, M_2 \in \text{Sp}_2(\mathcal{O})|p|_0
\]
remains to be verified. So let $M_j = (\begin{array}{c|c} A_j & B_j \\ \hline C_j & D_j \end{array}) \in \text{Sp}_2(\mathcal{O})|p|_0$, $j \in \{1, 2\}$. First, note that
\[
M_1 M_2 = (\begin{array}{c|c} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ \hline A_2 C_1 + B_2 D_1 & A_2 D_1 + B_2 C_1 \end{array})
\]
holds. $\vartheta_p$ only depends on $p \mod p^2$ as we have seen in (2.14), where $\mathcal{O}^2/p^2 = \mathbb{F}_p^2$ is commutative according to (1.10). In general, $(AB)' = B'A'$ does not hold (only $\overline{AB} = B'\overline{A}$) for $A, B \in \mathbb{H}^{n \times n}$ as $\mathbb{H}$ is not commutative, but because of the before mentioned facts
\[
\vartheta_{(AB)'p} = \vartheta_{B'A'p}
\]
holds for all $A, B \in \mathcal{O}^{2 \times 2}$. And thus by definition
\[
(M_1 M_2) \ast \vartheta_p(Z) = e^{2\pi i ((A_1 A_2 + B_1 C_2)(\overline{B_2} \overline{A_1} + \overline{D_2} \overline{D_1}))[\frac{1}{\sqrt{p}}]} \cdot \vartheta_p((A_2' A_1' + C_2' D_1')p)(Z)
\]
while
\[ M_2 * (M_1 * \theta_p)(Z) = e^{2\pi i (A_1 \overline{B}_1)[\frac{i}{\sqrt{3}} p]} . e^{2\pi i (A_2 \overline{B}_2)[\frac{i}{\sqrt{3}} A'[p]} . \theta_{A_2 A'[p]}(Z). \]

Hence we have to show the equality of those two expressions. First, note that \( C'_2 \in \mathbb{p}^{2 \times 2} \). Thus \( C'_2 B'_1 p \in \mathbb{p}^2 \) holds as \( p \leq O \) is a two-sided ideal, and on that account
\[ \theta(A'_2 A'[p] + C'_2 B'_1)p = \theta_{A'_2 A'[p]} \]
follows according to (2.14). So only
\[ e^{2\pi i ((A_1 A_2 + B_1 C_2)(B'_2 \overline{A}_1' + D'_2 B'_1))[\frac{i}{\sqrt{3}} p]} = e^{2\pi i (A_1 \overline{B}_1)[\frac{i}{\sqrt{3}} p]} . e^{2\pi i (A_2 \overline{B}_2)[\frac{i}{\sqrt{3}} A'[p]} \]
remains to be shown. We calculate
\[ ((A_1 A_2 + B_1 C_2)(B'_2 \overline{A}_1' + D'_2 B'_1))[\frac{i}{\sqrt{3}} p] = (A_1 A_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p] + (B_1 C_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p] + (A_1 A_2 D'_2 B'_1)[\frac{i}{\sqrt{3}} p] + (B_1 C_2 D'_2 B'_1)[\frac{i}{\sqrt{3}} p]. \]

In view of the fundamental relations in (1.3), \( A_2 D'_2 = I + B_2 \overline{C}_2 \). Moreover, note that for any \( M \in \mathbb{H}^{2 \times 2} \) and \( a \in \mathbb{H}^2 \), \( M[a] \in \mathbb{H} \) and therefore \( \overline{M} = M[a] = \overline{a} M \). So we get
\[ (B_1 C_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p] + (A_1 A_2 D'_2 B'_1)[\frac{i}{\sqrt{3}} p] = 2 \Re((B_1 C_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p]) + (A_1 \overline{B}_1)[\frac{i}{\sqrt{3}} p]. \]

Now, \( C_2 \in \mathbb{p}^{2 \times 2} \) implies \( (B_1 C_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p] \in \frac{i}{\sqrt{3}} O \), and
\[ 2 \Re((B_1 C_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p]) \in \mathbb{Z} \]
follows because \( \frac{i}{\sqrt{3}} O = \frac{i}{\sqrt{3}} \mathbb{Z} + (\frac{i}{\sqrt{3}} - \frac{1}{2}) \mathbb{Z} + \frac{i}{\sqrt{3}} \mathbb{Z} + (\frac{i}{\sqrt{3}} - \frac{1}{2}) \mathbb{Z} \). Next, \( C_2 D'_2 \in \text{Her}_2(O) \) according to the fundamental relations, and thus \( B_1 C_2 D'_2 B'_1 = C_2 D'_2 B'_1 \in \text{Her}_2(p) \) (recall that \( C_2 \in \mathbb{p}^{2 \times 2} \)). So as we have seen before,
\[ (B_1 C_2 D'_2 B'_1)[\frac{i}{\sqrt{3}} p] \in \mathbb{R} \cap \frac{i}{\sqrt{3}} O = \mathbb{Z} \]
holds. All this and the equality \( (A_1 A_2 B'_2 \overline{A}_1)[\frac{i}{\sqrt{3}} p] = (A_2 B'_2)[\overline{A}_1][\frac{i}{\sqrt{3}} p] \) finally lead to
\[ e^{2\pi i ((A_1 A_2 + B_1 C_2)(B'_2 \overline{A}_1' + D'_2 B'_1))[\frac{i}{\sqrt{3}} p]} = e^{2\pi i (A_2 \overline{B}_2'[\overline{A}_1'][\frac{i}{\sqrt{3}} p]} . e^{2\pi i (A_1 \overline{B}_1)[\frac{i}{\sqrt{3}} p]\}\]

Finally, what remains to be done is verifying the identity
\[ e^{2\pi i (A_2 \overline{B}_2)[\frac{i}{\sqrt{3}} A'[p]]} = e^{2\pi i (A_2 \overline{B}_2)[\overline{A}_1'][\frac{i}{\sqrt{3}} p]} \]
Just as we have seen in (2.14), \( A'_1 \equiv -i_1 \overline{A}'_1 i_1 \) modulo \( p \), i.e. there is some \( M \in \mathbb{p}^{2 \times 2} \) such that
2.2 Quaternionic theta-constants

\[ A_1' = -i_1 \overline{A}_1 i_1 + M. \]  
This leads to

\[ e^{2\pi i (A_2 B_2') \frac{i}{\sqrt{3}} A_1' p} = e^{2\pi i (A_2 B_2') \frac{i}{\sqrt{3}} A_1' p} \cdot e^{2\pi i (A_2 B_2') \frac{i}{\sqrt{3}} M p} . \]

\[ \frac{i}{\sqrt{3}} M \in O^{2 \times 2} \] and \( A_2 B_2' \in \text{Her}_2(O) \) (according to the fundamental relations) yield

\[ (A_2 B_2') \frac{i}{\sqrt{3}} M p \in R \cap O = Z \]

and thus the assertion follows. This completes the proof. \( \square \)

The next theorem merely is a corollary of the preceding theorem, but because of its significance we formulate it as a theorem which finally states that the quaternionic theta-constants are quaternionic modular forms with respect to the principal congruence subgroup \( \text{Sp}_2(O)[p] \), indeed.

(2.18) Theorem. \( \vartheta_p \in [\text{Sp}_2(O)[p], 2, 1] \) holds for all \( p \in O^2 \).

Proof: In virtue of (2.17) and (2.14) we only have to show that

\[ p \equiv A' p \quad \text{modulo } p \]

as well as

\[ (A B') \frac{i}{\sqrt{3}} p \in Z \]

hold for all \( p \in O^2 \) and all \( M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \text{Sp}_2(O)[p] \). By definition, \( A \equiv I \) modulo \( p \), i.e. there exists \( P \in p^{2 \times 2} \) such that \( A' = I + P \). Therefore,

\[ A' p = p + Pp \equiv p \quad \text{modulo } p \]

is obvious. \( B \in p^{2 \times 2} \) and the fundamental relations imply \( A B' \in \text{Her}_2(p) \). As we have seen several times before, this leads to

\[ (A B') \frac{i}{\sqrt{3}} p \in R \cap \frac{i}{\sqrt{3}} O = Z . \quad \square \]

Note that \( p_1 = (0, 0)' \). Thus (2.17) even immediately implies

\[ \vartheta_1 \in [\text{Sp}_2(O)[p], 2, 1] . \quad (2.4) \]

We will now determine the explicit Fourier-expansions of the quaternionic theta-constants. Note that \( 3O \subset p \), so by 1.20 we already know that the Fourier-expansions will have the shape

\[ \vartheta_j(Z) = \sum_{T \in \frac{i}{3} \text{Her}_2(O), T \geq 0} \alpha_j(T) e^{2\pi i T(Z)} , \quad Z \in \mathcal{H}(H) \]

for all \( j \in \{1, \ldots, 21\} \). We saw that before in (2.14) by rearranging the infinite sums. According
to this rearrangement, the Fourier-expansion is given by
\[ \vartheta_j(Z) = \sum_{T \in \frac{1}{2} \text{Her}_2(\mathcal{O}) \cap \mathbb{H}^2, T \geq 0, \det(T) = 0} \alpha_j(T) e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}) \]
with
\[ \alpha_j(T) = \#\{g \in \frac{j}{\sqrt{3}} p + \mathcal{O}^2 ; \ g\mathcal{O}' = T\} . \]
We will now determine the \( \alpha_j(T) \) explicitly and start with the easiest case, namely \( \vartheta_1 \).

\[ \text{(2.19) Proposition.} \]
\[ \vartheta_1(Z) = 1 + \sum_{0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0, \det(T) = 0} \left( 12 \sum_{l | e(T), 3|l} l \right) e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}) \]
where
\[ \epsilon(T) := \max\{l \in \mathbb{N} ; \frac{1}{3} T \in \text{Her}_2(\mathcal{O})\} . \]

\[ \text{Proof:} \]
\( p_1 = (0,0)' \) implies that we have to consider \( g \in \mathcal{O}^2 \), which means that \( g\mathcal{O}' \in \text{Her}_2(\mathcal{O}) \subset \text{Her}_2(\mathcal{O}) \). So according to what we mentioned above,
\[ \vartheta_1(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0, \det(T) = 0} \alpha_1(T) e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}) \]
with
\[ \alpha_1(T) = \#\{g \in \mathcal{O}^2 ; g\mathcal{O}' = T\} \]
holds. A short calculation yields
\[ g\mathcal{O}' = \left( \begin{array}{c} N(g_1) \\ g_1 \mathcal{O} \\ g_2 \end{array} \right) \]
\[ \text{(2.5)} \]
for \( g = (g_1, g_2)' \) (even for all \( g \in \mathbb{H}^2 \)). So of course \( \alpha_1(0) = 1 \). Now, let \( 0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0, \det(T) = 0 \). Then \( T \) is of rank one. Thus according to (1.61) there exists \( U \in \text{GL}_2(\mathcal{O}) \) such that
\[ T[U] = \left( \begin{array}{cc} d & 0 \\ 0 & 0 \end{array} \right) , \]
where \( d \in \mathbb{N} \). As \( \vartheta_1 \) is invariant under Rot(\( U \)) (see 2.4), one calculates – like we have already done before –
\[ \sum_{S \in \text{Her}_2(\mathcal{O}), S \geq 0, \det(S) = 0} \alpha_1(S) e^{2\pi i \tau(S,Z)} = \vartheta_1(Z) = \vartheta_1(Z[\overline{(U)}^{-1}]) \]
\[ = \sum_{S \in \text{Her}_2(\mathcal{O}), S \geq 0, \det(S) = 0} \alpha_1(S) e^{2\pi i \tau(S[[U^{-1}]],Z)} \]
\[ = \sum_{S \in \text{Her}_2(\mathcal{O}), S \geq 0, \det(S) = 0} \alpha_1(S[U]) e^{2\pi i \tau(S,Z)} . \]
The uniqueness of the Fourier-coefficients yields \( a_1(S) = a_1(S|U) \) for all \( S \in \text{Her}_2(O) \), in particular \( a_1(T) = a_1(T|U) \). In view of 2.5 (which implies \( g_2 = 0 \) and (1.7) we obtain

\[
a_1(T) = a_1(T|U) = \# \{ g_1 \in O : N(g_1) = d \} = 12 \sum_{l|d, 3|l} l.
\]

Finally, \( \varepsilon(T) = d \) is obvious because \( \frac{1}{T}(T|U) = (\frac{1}{T}|U) \), and this completes the proof. \( \square \)

With the help of the explicit Fourier-expansion of \( \vartheta_1 \) we are now going to determine the Fourier-expansion of the other quaternionic theta-constants.

**Theorem.** Let \( p = (p_1, p_2)' \in O^2 \setminus p^2 \). Then

\[
\vartheta_p(Z) = \sum_{0 \neq T \in \text{Her}_2(O)} \left( 3 \sum_{l|\varepsilon(T), 3|l} l \right) \cdot e^{2\pi i \frac{1}{T}Z}, \quad Z \in \mathcal{H}(H)
\]

holds, where

\[
\varepsilon(T) := \max \{ l \in \mathbb{N} ; \frac{1}{T} \in \text{Her}_2(O) \}.
\]

**Proof:** First, we need to do some preliminary considerations. Because \( Z[g] \in \mathbb{C} \) for all \( Z \in \mathcal{H}(H) \) and all \( g \in O^2 \), \( Z[g] \mid \frac{i}{\sqrt{3}} \mid = Z[g] \mid \frac{1}{\sqrt{3}} g \mid = (\frac{1}{T}Z)|g| \) holds. \( O^2 \rightarrow O^2 \), \( g \mapsto -iv_1gi_1 \) is a bijection (see the remark after (1.8)). According to (2.14), \( \vartheta_p = \vartheta_{ep+h} = \vartheta_{pe+h} \) holds for all \( p \in O^2 \), \( h \in p^2 \) and all \( e \in E \). Furthermore, it is easy to check that under multiplication with \( \pi_p(E) = \{ \pm 1, \pm \omega \} \) (see (1.11)) every orbit of \( (O^2/p^2) \setminus \{ 0 \} = F_2^2 \setminus \{ 0 \} \) consists of four elements (namely \( \{ \pm \pi_p(p), \pm \omega \pi_p(p) \} \)). By abuse of notation the transversal \( \{ p_1, \ldots, p_{21} \} \) of these orbits (plus \( p_1 \)) shall be denoted by \( (O^2/p^2)/E \). Every \( g \in O^2 \) can be uniquely written as \( g = p + i1 \sqrt{3}a \) where \( p \in O^2/p^2 \) and \( a \in O^2 \), and thus also every \( g \in \frac{i}{\sqrt{3}} O^2 \) can be uniquely written as \( g = \frac{i}{\sqrt{3}} p + a \) where \( p \in O^2/p^2 \) and \( a \in O^2 \). The theta-series are absolutely and locally uniformly convergent, hence the infinite sums can be rearranged. Keeping all that in mind and using (2.19) as well as the shape Fourier-expansions of the \( \vartheta_p \) from above we calculate for \( Z \in \mathcal{H}(H) \):

\[
1 + \sum_{0 \neq T \in \text{Her}_2(O)} \left( 12 \sum_{l|\varepsilon(T), 3|l} l \right) \cdot e^{2\pi i \frac{1}{T}Z} = \vartheta_1(Z) = \sum_{g \in O^2} e^{2\pi i Z[g] \mid \frac{1}{\sqrt{3}} g \mid} = \sum_{g \in \frac{i}{\sqrt{3}} O^2} e^{2\pi i Z[g]}
\]

\[
= \sum_{p \in O^2/p^2} e^{2\pi i Z[\frac{i}{\sqrt{3}} p + a]} = \sum_{p \in O^2/p^2} \vartheta_p(Z).
\]

\[
= \vartheta_1(Z) + \sum_{0 \neq p \in (O^2/p^2)/E} 4 \vartheta_p(Z) = \vartheta_1(Z) + \sum_{j=2}^{21} 4 \vartheta_j(Z)
\]
An easy consideration yields
\[ \pi \left( \sum_{T \in \frac{1}{4} \text{Her}_2(O), T \geq 0, \det(T) = 0} a_1(T) e^{2\pi i T Z} + \sum_{j=2}^{21} \left( \sum_{T \in \frac{1}{4} \text{Her}_2(O), T \geq 0, \det(T) = 0} 4a_j(T) e^{2\pi i T Z} \right) \right) \]

where
\[ a_j(T) = \# \{ g \in \frac{1}{\sqrt{3}} p_j + O^2 : g g^T = T \} \]

The first and the last line (if one would re-sort the infinite sums by the occurring exponents \( \tau(T, Z) \)) of this equation are both some Fourier-expansion, and thus the Fourier-coefficients have to coincide. Note that 2.5 implies that if \( a_j(T) \neq 0 \), then \( T \in \frac{1}{4} \text{Her}_2(O) \) holds, i.e. \( a_j(T) = 0 \) for all \( T \in \frac{1}{4} \text{Her}_2(O) \) \( \setminus \frac{1}{4} \text{Her}_2(O) \) and \( j \in \{ 1, \ldots, 21 \} \).

Next, we are going to prove the following assertion: If \( p, q \in O^2 \) such that \( p \) and \( q \) are not congruent modulo \( p^2 \) and modulo \( E \), i.e. \( p \neq q \) in \( (O^2/p^2)/E \), or in other words \( p \equiv q \) modulo \( p, j, k \in \{ 1, \ldots, 21 \}, j \neq k \), then there exists no \( T \in \frac{1}{4} \text{Her}_2(O) \) such that \( a_p(T) \neq 0 \) and \( a_q(T) \neq 0 \) (where \( a_p \) and \( a_q \) denote the Fourier-coefficients of \( \theta_p \) and \( \theta_q \)).

For \( p = (p_1, p_2)' \), \( q = (q_1, q_2)' \in O^2 \) and \( a = \frac{i}{\sqrt{3}} p + g \) we compute
\[ a a' = -\frac{1}{3}i \left( N(p_1 - i_1 \sqrt{3} g_1) + N(p_2 - i_1 \sqrt{3} g_2) \right) \]

Note that \( N(p_1 - i_1 \sqrt{3} g_1) = N(p_1) + 3N(g_1) + 2 \text{Re}(-i_1 \sqrt{3} g_1 \bar{p}_1) \). \( i_1 \sqrt{3} O = i_1 \sqrt{3} Z + \frac{i_1 \sqrt{3} - 3}{2} i_1 Z + i_1 i_2 \sqrt{3} Z + \frac{i_1 \sqrt{3} - 3}{2} i_2 Z \) implies \( 2 \text{Re}(-i_1 \sqrt{3} g_1 \bar{p}_1) \in 3Z \). Furthermore, note that \( 3 \in p \) and \( -i_1 O i_1 = O \) hold. Thus
\[ N(p_1 - i_1 \sqrt{3} g_1) \equiv N(p_1) \mod 3 \]
\[ N(p_2 - i_1 \sqrt{3} g_2) \equiv N(p_2) \mod 3 \]
\[ p_1 \bar{p}_2 + p_1 g_2 i_1 \sqrt{3} - i_1 \sqrt{3} g_1 \bar{p}_2 - 3 i_1 g_1 g_2 i_1 \equiv p_1 \bar{p}_2 \mod p \]

follows. The same holds true if we replace \( p \) with \( q \). Now, suppose the assertion is false. Because of the shape of the Fourier-coefficients, there have to exist \( a, b \in O^2 \) such that \( T = c c' = d d' \) for \( c = \frac{i}{\sqrt{3}} p + a \) and \( d = \frac{i}{\sqrt{3}} q + b \). Hence
\[ N(p_1) \equiv N(q_1) \mod 3 \]
\[ N(p_2) \equiv N(q_2) \mod 3 \]
\[ p_1 \bar{p}_2 \equiv q_1 \bar{q}_2 \mod p \]

has to hold. Now, consider the norm in \( F_9 = F_3 + \omega F_3 \) given by \( N(a + \omega b) = a^2 + b^2 = (a + \omega b) \cdot \phi_3(a + \omega b) \). So in view of (1.11), \( \pi_p(N(a)) = N(\pi_p(a)) \) holds for all \( a \in O \), and thus \( N(\pi_p(p_1)) = N(\pi_p(q_1)), N(\pi_p(p_2)) = N(\pi_p(q_2)) \). As \( F_9 \) is a field, an easy consequence of this is that there exist \( f_1, f_2 \in F_9 \), \( N(f_1) = N(f_2) = 1 \) such that
\[ \pi_p(p_1) = \pi_p(q_1) \cdot f_1, \quad \pi_p(p_2) = \pi_p(q_2) \cdot f_2. \]

An easy consideration yields \( \pi_p(E) = \{ f \in F_9 ; N(f) = 1 \} \). So in other words there exist
\[ \epsilon_1, \epsilon_2 \in \mathcal{E} \text{ (with } \pi_p(\epsilon_1) = f_1, \pi_p(\epsilon_2) = f_2 \text{) such that} \]
\[ p_1 \equiv q_1 \epsilon_1, \quad p_2 \equiv q_2 \epsilon_2 \mod p. \]

Next, \( p_1 \mathbf{\overline{q}}_2 \equiv p_1 \mathbf{\overline{q}}_2 \) is equivalent to saying \( \pi_p(p_1 \mathbf{\overline{q}}_2) = \pi_p(q_1 \mathbf{\overline{q}}_2) \), and in virtue of (1.11) we obtain

\[ \pi_p(q_1 \mathbf{\overline{q}}_2) = \pi_p(p_1 \mathbf{\overline{q}}_2) = \pi_p(q_1) f_1 \phi_3(f_2) \pi_p(q_2) = f_1 \phi_3(f_2) \pi_p(q_1 \mathbf{\overline{q}}_2). \]

There are several cases. If \( \pi_p(q_1 \mathbf{\overline{q}}_2) \neq 0 \) (i.e. \( q_1, q_2 \not\in p \)), \( f_1 \phi_3(f_2) = 1 \), or equivalently \( f_1 = f_2 \) has to hold, which means \( \epsilon_1 \equiv \epsilon_2 \mod p \). On the other hand, \( q_1 \in p \) or \( q_2 \in p \) implies \( p_1 \in p \) or \( p_2 \in p \) resp. by what we have seen so far. Thus in all possible cases there exists \( f \in \mathbb{F}_9 \) \( (f \in \{f_1, f_2\}) \) such that

\[ \pi_p(p_1) = \pi_p(q_1) \cdot f \quad \text{and} \quad \pi_p(p_2) = \pi_p(q_2) \cdot f \]

hold, or in other words there exists \( \epsilon \in \mathcal{E} \) \( (\epsilon \in \{\epsilon_1, \epsilon_2\}) \) such that

\[ p_1 \equiv q_1 \epsilon, \quad p_2 \equiv q_2 \epsilon \mod p \]

holds. So we have shown that there exist \( \epsilon \in \mathcal{E} \) and \( h \in p^2 \) such that \( p = qe + h \), which contradicts the assumption. Hence the assertion follows.

Now, let us get back to the beginning of the proof where we had to compare the Fourier-coefficients of the two Fourier-expansions. On the most left hand side of that equation, every \( T \in \text{Her}_2(O) \), \( T \geq 0 \), \( \det(T) = 0 \) occurs as an “exponent” in \( \tau(T, Z) \) with non-vanishing Fourier-coefficient \( \alpha(T) \). Therefore, it also has to occur on the most right hand side, again with non-vanishing Fourier-coefficient. We have seen that the set of occurring “exponents” \( T \) with non-vanishing Fourier-coefficients in the Fourier-expansions of \( \vartheta_1, \ldots, \vartheta_21 \) are pairwise disjoint. So every \( T \in \text{Her}_2(O) \), \( T \geq 0 \), \( \det(T) = 0 \) can “occur” in exactly one of the Fourier-expansions on the most right hand side of the equation above. So we can simply read off the Fourier-coefficients on the left-hand side (while taking the prefactor 4 into account). Note that if \( p = (p_1, p_2) \in \mathcal{O}^2 \setminus p^2 \), then \( 3 \nmid N(p_1) \) or \( 3 \nmid N(p_2) \) has to hold in view of (1.9). So we can ignore the condition \( 3 \nmid l \) in “\( l | \epsilon(T), 3 | l' \)”.

There is one last thing to mention to complete the proof. Let \( p = (p_1, p_2)' \in \mathcal{O}. \) As we have seen, every \( T \in \text{Her}_2(O) \), \( T \geq 0 \), \( \det(T) = 0 \) occurring in the Fourier-expansion of \( \vartheta_p \) as an exponent in \( \tau(T, Z) \) with non-vanishing Fourier-coefficient is of the shape

\[ T = -\frac{1}{3} i_1 \begin{pmatrix} n & t \\ \overline{t} & m \end{pmatrix} i_1, \quad n \equiv N(p_1) \mod 3, \quad m \equiv N(p_2) \mod 3, \quad t \equiv p_1 \overline{p}_2 \mod p \]

(where \( n, m \in \mathbb{N}_0 \), \( t \in \mathcal{O} \) – and every such \( T \) occurs). Note that \( -i_1 n i_1 = n \) and \( -i_1 m i_1 = m \). And as we have seen in the proof of (1.20), \( \pi_p \circ \phi_{i_1} = \phi_3 \circ \pi_p \) holds, while \( \pi_p(\mathbf{\overline{a}} + p) = \phi_3(\pi_p(a + p)) \) according to (1.11). In other words,

\[ -i_1 t i_1 \equiv \mathbf{\overline{t}} \equiv p_2 \overline{p}_1 \equiv \overline{p}_1 p_2 \mod p \]

holds. This finally completes the proof. \( \square \)
(2.21) Remark. As we have seen in the preceding theorem, the sets of \( T \in \frac{1}{2} \text{Her}_2(O) \) occuring in the Fourier-expansions of the 21 quaternionic theta-constants with non-vanishing Fourier-coefficients are pairwise disjoint. So in view of the uniqueness of Fourier-coefficients, the 21 quaternionic theta-constants are indeed linearly independent.

Now that we have determined the explicit Fourier-expansions of the quaternionic theta-constants, we can even describe their behavior under the extended modular group.

(2.22) Theorem. Let \( p \in O^2 \), then

\[
\vartheta_p(Z') = \vartheta_p|_{2}(i1I)(Z) = \vartheta_p(Z)
\]

holds for all \( Z \in \mathcal{H}(\mathbb{H}) \).

Proof: As we have seen several times before, \(-i1pi_1 \equiv \overline{p} \mod p\) and \(-i1O^2i_1 = O^2\) hold. So in virtue of (2.14) (and \( Z|g \in \mathbb{R} \otimes \mathbb{R} \mathbb{C} \) for all \( Z \in \mathcal{H}(\mathbb{H}), g \in \mathbb{H}^2 \)) we calculate

\[
\vartheta_p|_{2}(i1I)(Z) = (\text{det}(i1I))^{2/2} \vartheta_p(i1I(Z)) = \vartheta_p(Z[i1I]) = \sum_{g \in O^2} e^{2\pi iZ[i1I] \left[\frac{1}{2}p+g\right]}
\]

\[
= \sum_{g \in O^2} e^{2\pi iZ[i1I] \left[\frac{1}{2}p+g\right] - i1} = \sum_{g \in O^2} e^{2\pi iZ \left[\frac{1}{2}(-i1p1) + (-i1g1)\right]}
\]

\[
= \vartheta_{-i1p1}(Z) = \vartheta_p(Z).
\]

Furthermore, an easy consequence of the formula in (1.46) is

\[
\tau(T, Z') = \tau(T', Z) = \tau(T, Z)
\]

for all \( Z \in \mathcal{H}(\mathbb{H}) \) and \( T \in \text{Her}_2(\mathbb{H}) \), since \( T' \equiv T \). If \( p \in p^2 \), then

\[
\vartheta_p(Z') = \vartheta_1(Z') = \vartheta_1(Z) = \vartheta_p(Z)
\]

is an easy consequence of the Fourier-expansion in (2.19), since obviously \( \varepsilon(T) = \varepsilon(T) \) holds. So let \( p \in O^2 \setminus p^2 \). As \( O/p \) is commutative and with the help of the Fourier-expansion in (2.20) we conclude

\[
\vartheta_p(Z') = \sum_{0 \neq T = \left(\begin{array}{c} n \\
T \\
m \end{array}\right) \in \text{Her}_2(O) \atop T \geq 0, \det(T) = 0} \left(\sum_{l | (T)} 3 \right) \cdot e^{2\pi i \tau \left(\frac{1}{2}T, Z'\right)}
\]

\[
= \sum_{0 \neq T = \left(\begin{array}{c} n \\
T \\
m \end{array}\right) \in \text{Her}_2(O) \atop T \geq 0, \det(T) = 0} \left(\sum_{l | (T)} 3 \right) \cdot e^{2\pi i \tau \left(\frac{1}{2}T, Z\right)}
\]

\[n \equiv N(p_1)(3), m \equiv N(p_2)(3), t \equiv p_1 p_2 (p)\]
According to (1.72) there are several possibilities to do so and each one yields Hermitian theta-constants that can be written as a homogeneous polynomial of degree 2

\[ j \in \mathbb{Z} \]

holds for \( j \in \{0, \ldots, 4\} \) and every Hermitian modular form \( f \in \text{Sp}_2(\mathbb{Z}[\omega]) \), \( k \), \( \det^k \), \( k \in \mathbb{N}_0 \) can be written as a homogeneous polynomial of degree \( 2k \) in \( \tilde{\vartheta}_0, \ldots, \tilde{\vartheta}_4 \), which are algebraically independent. And thus of course also every Hermitian modular form \( f \in \text{Sp}_2(\mathbb{Z}[\omega]) \), \( 2k \), \( \det^{2k} \), \( k \in \mathbb{N}_0 \) can be written as a homogeneous polynomial of degree \( k \) in the two-products \( \tilde{\vartheta}_j \), \( j, l \in \{0, \ldots, 4\} \).

As announced, we are now going to restrict the quaternionic theta-constants to the Hermitian half-space. According to (1.72) there are several possibilities to do so and each one yields Hermitian modular forms with respect to a certain Hermitian congruence subgroup that is a subgroup of \( \text{Sp}_2(\mathbb{Q}(\omega)) \), where we restrict to \( \mathcal{H}_2(\mathbb{R}) \) with \( R = \mathbb{R}(\omega), \omega \in \mathcal{O} \setminus \mathbb{R} \). Here, we are interested in the principal congruence subgroups of \( \mathcal{H}_2(\mathbb{C}) \) for \( \omega \in \mathcal{O} \setminus \mathbb{R} \), which are algebraically independent. And thus of course also every Hermitian modular form \( f \in \text{Sp}_2(\mathbb{Z}[\omega]) \), \( 2k \), \( \det^{2k} \), \( k \in \mathbb{N}_0 \) can be written as a homogeneous polynomial of degree \( k \) in the two-products \( \tilde{\vartheta}_j \), \( j, l \in \{0, \ldots, 4\} \).

As we are now going to have a closer look at the restrictions to the Hermitian half-space. According to (1.72) there are several possibilities to do so and each one yields Hermitian modular forms with respect to a certain Hermitian congruence subgroup that is a subgroup of \( \text{Sp}_2(\mathbb{Q}(\omega)) \), where we restrict to \( \mathcal{H}_2(\mathbb{R}) \) with \( R = \mathbb{R}(\omega), \omega \in \mathcal{O} \setminus \mathbb{R} \). Here, we are interested in the principal congruence subgroups of \( \mathcal{H}_2(\mathbb{C}) \) for \( \omega \in \mathcal{O} \setminus \mathbb{R} \), which are algebraically independent. And thus of course also every Hermitian modular form \( f \in \text{Sp}_2(\mathbb{Z}[\omega]) \), \( 2k \), \( \det^{2k} \), \( k \in \mathbb{N}_0 \) can be written as a homogeneous polynomial of degree \( k \) in the two-products \( \tilde{\vartheta}_j \), \( j, l \in \{0, \ldots, 4\} \).

As we are now going to have a closer look at the restrictions to the Hermitian half-space.
Furthermore, \(80 = \varrho \in H_1\). Using the identifications in (1.72),
\[
\vartheta_{|R_1} \in [Sp_2(O_1)|3O_1], 2, \det \quad \text{and} \quad \vartheta_{|R_3} \in [Sp_2(O_3)[i\sqrt{3}O_1], 2, \det^2]^1
\]
hold for all \(j \in \{1, \ldots, 21\}\). The Fourier-expansions are given by
\[
\vartheta_{|R_1}(Z) = 1 + \sum_{0 \neq T = (\frac{n}{m}, \frac{l}{m}) \geq 0} \left( \frac{12}{l} \right) \sum_{0 \neq S = (\frac{n}{m}, \frac{s}{m}) \in \text{Herz}(\mathcal{O})} \sum_{S \geq 0, \det(S) = 0} \sum_{l \in (3, 3l)} e^{2\pi i(T,Z)}
\]
where \(Z \in H_2(C) \simeq H_2(R_1)\), \(p = (p_1, p_2) \in \mathcal{O}^2 \setminus \mathcal{p}^2\) and \(\pi_\mathcal{p}(\mathcal{F}, 1) = \varepsilon_1 + \varepsilon_\omega \omega \in \mathbb{F}_9 = \mathbb{F}_3 + \omega \mathbb{F}_3\), as well as
\[
\vartheta_{|R_3}(Z) = 1 + \sum_{0 \neq T = (\frac{n}{m}, \frac{l}{m}) \geq 0} \left( \frac{12}{l} \right) \sum_{0 \neq S = (\frac{n}{m}, \frac{s}{m}) \in \text{Herz}(\mathcal{O})} \sum_{S \geq 0, \det(S) = 0} \sum_{l \in (\sqrt{3}, 3l)} e^{2\pi i(T,Z)}
\]
where \(Z \in H_2(C) \simeq H_2(R_3)\), \(p = (p_1, p_2) \in \mathcal{O}^2 \setminus \mathcal{p}^2\) and again \(\pi_\mathcal{p}(\mathcal{F}, 1) = \varepsilon_1 + \varepsilon_\omega \omega \in \mathbb{F}_9\). Furthermore, \(\vartheta_{|R_1}, \ldots, \vartheta_{|R_{21}}\) are linearly independent, while
\[
\vartheta_{1|R_3} = \tilde{\vartheta}_0, \quad \vartheta_{2|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_2, \quad \vartheta_{3|R_3} = \tilde{\vartheta}_2, \quad \vartheta_{4|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_4, \quad \vartheta_{5|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_5, \quad \vartheta_{6|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_3, \quad \vartheta_{7|R_3} = \tilde{\vartheta}_1 \tilde{\vartheta}_2, \quad \vartheta_{8|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_4, \quad \vartheta_{9|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_3, \quad \vartheta_{10|R_3} = \tilde{\vartheta}_1 \tilde{\vartheta}_4, \quad \vartheta_{11|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_5, \quad \vartheta_{12|R_3} = \tilde{\vartheta}_1 \tilde{\vartheta}_5, \quad \vartheta_{13|R_3} = \tilde{\vartheta}_3 \tilde{\vartheta}_4, \quad \vartheta_{14|R_3} = \tilde{\vartheta}_1 \tilde{\vartheta}_3, \quad \vartheta_{15|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_6, \quad \vartheta_{16|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_6, \quad \vartheta_{17|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_7, \quad \vartheta_{18|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_7, \quad \vartheta_{19|R_3} = \tilde{\vartheta}_0 \tilde{\vartheta}_8, \quad \vartheta_{20|R_3} = \tilde{\vartheta}_2 \tilde{\vartheta}_8, \quad \vartheta_{21|R_3} = \tilde{\vartheta}_1 \tilde{\vartheta}_8
\]
2.2 Quaternionic theta-constants

and

\[ \vartheta_7|_{R_3} = \vartheta_8|_{R_3}, \quad \vartheta_9|_{R_3} = \vartheta_{10}|_{R_3}, \quad \vartheta_{11}|_{R_3} = \vartheta_{12}|_{R_3}, \]
\[ \vartheta_{14}|_{R_3} = \vartheta_{16}|_{R_3}, \quad \vartheta_{15}|_{R_3} = \vartheta_{17}|_{R_3}, \quad \vartheta_{19}|_{R_3} = \vartheta_{20}|_{R_3} \]

hold.

**Proof:** Because of (1.72) and (2.18), it is already clear that the restrictions of the quaternionic theta-constants are Hermitian modular forms of weight 2 with respect to some congruence subgroups of \( \text{Sp}_2(O_1) \) and \( \text{Sp}_2(O_3) \) and some abelian character. It is obvious that (with appropriate identifications) \( \text{Sp}_2(O)[p] \cap R_1^{4 \times 4} = \text{Sp}_2(O_1)[3O_1] \) and \( \text{Sp}_2(O)[p] \cap R_3^{4 \times 4} = \text{Sp}_2(O_3)[i \sqrt{3}O_3] \) hold as \( p = i_1 \sqrt{3}Z + \frac{-3+i_1 \sqrt{3}}{2}Z + i_2 \sqrt{3}Z + \frac{-3+i_2 \sqrt{3}}{2}Z. \) Thus the restrictions are Hermitian modular forms with respect to these principal congruence subgroups. So only the corresponding abelian character has to be clarified. As we have seen in (1.72), the abelian characters simply arise from the old ones by comparing the different actions of the slash-operators for quaternionic and Hermitian modular forms. Note that \( \frac{(p^2)}{|(-1)p^2|} = 1 \), and thus making use of (1.72), or rather the proof of it, we obtain

\[
(\vartheta_p|_R|_{2 \text{Trans}(S)})|_R = (\vartheta_p|_R)|_{2 \text{Trans}(S)},
\]
\[
(\vartheta_p|_R|_{\text{Rot}(U)})|_R = (\det(U))^{-\frac{1}{2}} \cdot (\vartheta_p|_R)|_{2 \text{Rot}(U)},
\]
\[
(\vartheta_p|_R|_{2 I})|_R = (\vartheta_p|_R)|_{2 I}
\]

where \( p \in O_2, R \in \{R_1, R_3\}, S \in \text{Her}_2(O_1) \subset \text{Her}_2(O) \) or \( S \in \text{Her}_2(O_3) \subset \text{Her}_2(O), U \in \text{GL}_2(O_1) \subset \text{GL}_2(O) \) or \( U \in \text{GL}_2(O_3) \subset \text{GL}_2(O) \) (in each case with appropriate identifications), while on the left sides the quaternionic slash-operator and on the right sides the Hermitian slash-operator is used. Now every \( M \in \text{Sp}_2(O_1) \) or \( M \in \text{Sp}_2(O_3) \) is a product of these generating matrices, and note that \( \det(\text{Trans}(S)) = \det(I) = 1 \), while \( \det(\text{Rot}(U)) = \det(U) \), \( \det(U)^{-1} = \det(U)^{-2} \). So let \( M \in \text{Sp}_2(O_1) \) or \( M \in \text{Sp}_2(O_3) \), which also implies \( M \in \text{Sp}_2(O)[p] \), and let \( M = M_1 \cdots M_j \) be some representation of \( M \) as a product of the generating matrices from above. Hence we compute

\[
(\vartheta_p|_R)|_{2M} = (\vartheta_p|_R)|_{2M_1} \cdots 2M_j = (\det(M_1))^{-1} \cdots (\det(M_j))^{-1} \cdot (\vartheta_p|_R|_{2M_1} \cdots 2M_j)|_R
\]
\[
= (\det(M))^{-1} \cdot (\vartheta_p|_R|_{2M}) = (\det(M))^{-1} \cdot (\vartheta_p|_R).
\]

Note that \( (\det(M))^2 = 1 \) holds for all \( M \in \text{Sp}_2(Z[i]) \) according to [De01], while \( (\det(M))^3 = 1 \) holds for all \( M \in \text{Sp}_2(Z[\frac{1}{2}(1 + i \sqrt{3})]) \), and thus the assertion about the corresponding abelian characters is verified. We also claimed that the quaternionic theta-constants restricted to \( R_3 \) are symmetric. This follows because of (2.22) and the explicit shape of the Fourier-expansions (which we will deal with further below):

It holds true that \( \vartheta_p|_{R_3}(Z') = \vartheta_p|_{R_3}(Z) \). So let \( p = (p_1, p_2) \notin p^2. \) (Note that \( \vartheta_1|_{R_3}(Z') = \vartheta_1|_{R_3}(Z) \) is clear, already.) As we will see further below, the explicit shape of the Fourier-expansion only depends on \( N(p_1), N(p_2) \) and \( \varepsilon_1 \), where \( \pi_p(p_1, p_2) = \varepsilon_1 + \varepsilon_2 \omega \in F_9. \) But \( N(p_1) = N(p_2) \),
with non-vanishing Fourier-coefficients. As we have seen in (1.11),
\[ p \mapsto \pi_p(p_1 p_2) = \pi_p(p_1 p_2) = \phi_3(\pi_p(p_1 p_2)) = \epsilon_1 - \epsilon_\omega \omega \] hold true, and hence
\[ \vartheta_p|_{R_3}(Z') = \vartheta_p|_{R_3}(Z) = \vartheta_p|_{R_3}(Z) \]
follows. \( \vartheta_p|_{R_3} = \vartheta_p|_{R_3} \) also yields the stated equalities of the six pairs of restricted quaternionic theta-constants.

The explicit shape of the Fourier-expansions also yields the linear independence of the quaternionic theta-constants restricted to \( R_1 \). This was also verified by computing sufficiently many Fourier-coefficients using [SAGE].

The same was done with the restrictions to \( R_3 \). Using [SAGE], both sufficiently many Fourier-coefficients of the restricted quaternionic theta-constants as well as the two-products of the Fourier-coefficients using\([SAGE]\). Hermitian theta-constants were computed and compared. Because of an analogue of (1.70) for coefficients of the restricted quaternionic theta-constants as well as the two-products of the\( \vartheta_p \) follows.

\[ \vartheta_p \] for all \( \{ \}\) to see that such a basis is given by
\[ \vartheta_p \] and \( \vartheta \). Then for \( \vartheta \)
\[ \vartheta \] or in other words \( \vartheta \). Then \( \vartheta \) can be written as
\[ \vartheta \]

So only the explicit shape of the Fourier-expansions of the restricted quaternionic theta-constants remains. But in view of (1.73), we already know this implicitly. According to that theorem, we first need to find a \( \mathbb{R} \)-basis of \( H = \mathbb{R} + \omega_1 \mathbb{R} + b_2 \mathbb{R} + b_3 \mathbb{R}, j \in \{1, 3\} \) such that \( b_2 \) and \( b_3 \) are orthogonal to \( R_1 \) or \( R_3 \) with respect to the bilinear form \( (a, b) \mapsto \Re(\overline{a}b) \). It is easy to see that such a basis is given by \{1, \omega_1, i_1, i_1 i_2\} and \{1, \omega_3, i_2, \frac{1+i\sqrt{3}}{2} i_2\} respectively. Let \( a = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \in \mathbb{H} \). Then \( a \) can be written as
\[ a = (a_0 + \frac{1}{2}a_1) + (a_2 + \frac{1}{2}a_3) \omega_1 + \frac{\sqrt{3}}{2} i_1 + \frac{\sqrt{3}}{2} a_3 i_2 = a_0 + a_1 \omega_3 + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 . \]

So let \( p \in O^2 \). According to (1.73) we get
\[ \vartheta \] for all \( Z \in H_2(\mathbb{C}) \simeq H_2(R_1) \) and
\[ \vartheta \] for all \( Z \in H_2(\mathbb{C}) \simeq H_2(R_3) \). We already determined the Fourier-coefficients in (2.19) and (2.20). So the only remaining matter is to specify which \( T \)'s occur in the Fourier-expansion with non-vanishing Fourier-coefficients. As we have seen in (1.11), \( a = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \equiv (a_0 - a_1 \text{ mod } 3) + (a_2 - a_3 \text{ mod } 3)i_2 \) holds modulo \( p \), or in other words \( \pi_p(a) = (a_0 - a_1 \text{ mod } 3) + (a_2 - a_3 \text{ mod } 3)i_2 \in \mathbb{F}_9 \) for all \( a \in \mathbb{H} \). So suppose \( p = (p_1, p_2)' \in O^2 \), \( \pi_p(p_1 p_2) = \epsilon_1 + \epsilon_\omega \omega \in \mathbb{F}_9 \). Then for \( s = s_0 + s_1 \frac{1+i\sqrt{3}}{2} + s_2 i_2 + s_3 \frac{1+i\sqrt{3}}{2} i_2 \in \mathbb{H} \) we have \( s \equiv \overline{p}_1 p_2 \)
if and only if \( s_0 - s_1 \equiv \epsilon_1 \) and \( s_2 - s_3 \equiv \epsilon_\omega \) mod 3. So in the case of restricting to \( R_1 \) this leads to
\[
t = t_0 + t_2 i = (s_0 + \frac{1}{2}s_1) + (s_2 + \frac{1}{2}s_3)i = \frac{1}{2}(2s_0 + s_1) + \frac{1}{2}(2s_2 + s_3)i
\]
and thus we get the conditions
\[
2t_0 \equiv 2s_0 + s_1 \equiv -s_0 + s_1 \equiv \epsilon_1 , \quad 2t_2 \equiv 2s_2 + s_3 \equiv -s_2 + s_3 \equiv \epsilon_\omega \mod 3 ,
\]
whereas in the case of restricting to \( R_3 \) we analogously get
\[
t = t_0 + t_1 \frac{1+i\sqrt{3}}{2} , \quad t_0 - t_1 = s_0 - s_1 \equiv \epsilon_1 \mod 3 .
\]
Putting it all together obviously results in the asserted shapes of the Fourier-expansion. □

As we have seen in the preceding theorem, restricting the quaternionic theta-constants to \( R_3 \) (hence resulting in Hermitian modular forms for \( Q(i\sqrt{3}) \)) yields all two-products of the Hermitian theta-constants \( \tilde{\vartheta}_j \), [DK03] immediately gives

\[
(2.24) \text{Corollary. Every Hermitian modular form } f \in [\text{Sp}_2(\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]), 2k, \det^{2k}], \quad k \in \mathbb{N}_0 \text{ can be written as a homogeneous polynomial of degree } k \text{ in the restricted quaternionic theta-constants } \tilde{\vartheta}_j|_{R[\frac{1+i\sqrt{3}}{2}]}, \quad j \in \{1, \ldots, 21\}.
\]

\[
(2.25) \text{Remark. Of course, the quaternionic theta-constants can also be restricted to the Siegel half-space. Just analogous to (2.23) we have}
\]
\[
\tilde{\vartheta}_j|_{R} \in [\text{Sp}_2(\mathbb{Z}[3\mathbb{Z}]), 2, 1]
\]
for all \( j \in \{1, \ldots, 21\} \). The corresponding Fourier-expansions can be obtained in the same way. Furthermore, as every Siegel modular form of even weight with respect to the whole Siegel modular group and the trivial character is a restriction of a symmetric Hermitian modular form for \( Q(i\sqrt{3}) \) (cf. [DK03]), (2.24) directly implies: Every Siegel modular form \( f \in [\text{Sp}_2(\mathbb{Z}), 2k, 1], \quad k \in \mathbb{N}_0 \) can be written as a homogeneous polynomial of degree \( k \) in the restricted quaternionic theta-constants \( \tilde{\vartheta}_j|_{R}, \quad j \in \{1, \ldots, 21\} \).

There is one last thing left to do: Above, we considered restrictions of the quaternionic theta-constants. On the other hand, the quaternionic theta-constants themselves are restrictions of ordinary Siegel theta-constants living on \( \mathcal{H}_8(\mathbb{R}) \) and also restrictions of Hermitian theta-constants living on \( \mathcal{H}_4(\mathbb{C}) \):

According to (1.34) \( \mathcal{H}(\mathbb{H}) \) is embedded in \( \mathcal{H}_8(\mathbb{R}) \) and in \( \mathcal{H}_4(\mathbb{C}) \) via \( Z \mapsto \hat{Z} \) and \( Z \mapsto \check{Z} \). Therefore, we can restrict functions living on \( \mathcal{H}_8(\mathbb{R}) \) or \( \mathcal{H}_4(\mathbb{C}) \) to \( \mathcal{H}(\mathbb{H}) \). We are now going to determine such functions whose restrictions to the quaternionic half-space yield our quaternionic theta-constants.
We need two further maps. For \( m, n \in \mathbb{N} \) let
\[
\begin{align*}
\mu_R : \mathbb{H}^{m \times n} \otimes \mathbb{R} \mathbb{C} &\to \mathbb{R}^{4m \times n} \otimes \mathbb{R} \mathbb{C}, & A = (a_1, \ldots, a_n) &\mapsto (b_1, \ldots, b_n), \\
\mu_C : \mathbb{H}^{m \times n} \otimes \mathbb{R} \mathbb{C} &\to \mathbb{C}^{2m \times n} \otimes \mathbb{R} \mathbb{C}, & A = (a_1, \ldots, a_n) &\mapsto (c_1, \ldots, c_n),
\end{align*}
\]
where \( a_1, \ldots, a_n \) are the columns of \( A \), \( b_j \) is the first column of \( \hat{a}_j \in \mathbb{R}^{4m \times 4} \otimes \mathbb{R} \mathbb{C} \) and \( c_j \) is the first column of \( \hat{a}_j \in \mathbb{C}^{2m \times 2} \otimes \mathbb{R} \mathbb{C} \), \( j = 1, \ldots, n \). Obviously (and also confer \([Kl98]\)), \( \mu_R \) and \( \mu_C \) are vector space isomorphisms, and lattices in \( \mathbb{H}^{m \times n} \otimes \mathbb{R} \mathbb{C} \) are mapped to lattices in \( \mathbb{R}^{4m \times n} \otimes \mathbb{R} \mathbb{C} \) and \( \mathbb{C}^{2m \times n} \otimes \mathbb{R} \mathbb{C} \), respectively. From \([Kr85, p. 106]\) we cite:

\((2.26)\) Lemma. Let \( A, B \in \mathbb{H}^{m \times n} \otimes \mathbb{R} \mathbb{C} \). Then we have
\[
\tau(A, B) = \tau(\mu_R(A), \mu_R(B)) = \tau(\mu_C(A), \mu_C(B))
\]

Note that this is just a straightforward calculation. Furthermore, since \( \land \) and \( \lor \) are algebra homomorphisms (see \((1.34)\)), we obviously obtain
\[
\mu_R(Zg) = \hat{Z}\mu_R(g), \quad \mu_C(Zg) = \hat{Z}\mu_C(g)
\]
for \( Z \in \mathcal{H}(\mathbb{H}) \) and \( g \in \mathbb{H}^{2 \times 1} \), since \( \mu_R(Zg) \) is the first column of \( \hat{Z}\hat{g} \) (and the according statement for \( \mu_C(Zg) \)). But note that also a simple calculation verifies these identities. The facts from above suffice to prove

\((2.27)\) Proposition. Let \( p = (p_1, p_2)' \in \mathcal{O}^2 \) with \( p_1 = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \) and \( p_2 = b_0 + b_1 \frac{1+i\sqrt{3}}{2} + b_2 i_2 + b_3 \frac{1+i\sqrt{3}}{2} i_2 \). Regarding the embeddings above, we have:
\[
\theta_p(\cdot) = \Theta_{P_R,0}(\cdot ; 2; \Lambda_R) |_{\mathcal{H}(\mathbb{H})} = \Theta_{P_C,0}(\cdot ; 2; \Lambda_C) |_{\mathcal{H}(\mathbb{H})},
\]
where \( \Theta_{P_R,0}(\cdot ; 2; \Lambda_R) : \mathcal{H}_8(\mathbb{R}) \to \mathbb{C} \) and \( \Theta_{P_C,0}(\cdot ; 2; \Lambda_C) : \mathcal{H}_4(\mathbb{C}) \to \mathbb{C} \) are Siegel and Hermitian theta series defined like in \((2.1)\) with \( \mathbb{R} \) and \( \mathbb{C} \) instead of \( \mathbb{H} \). The parameters are given by
\[
P_R = \left(-\frac{1}{2}a_1, \frac{\sqrt{3}}{2}a_0 + \frac{\sqrt{3}}{6}a_1, -\frac{1}{2}a_3, \frac{\sqrt{3}}{2}a_2 + \frac{\sqrt{3}}{6}a_3, -\frac{1}{2}b_1, \frac{\sqrt{3}}{2}b_0 + \frac{\sqrt{3}}{6}b_1, -\frac{1}{2}b_3, \frac{\sqrt{3}}{6}b_2 + \frac{\sqrt{3}}{6}b_3\right)
\]
and
\[
P_C = \left(-\frac{1}{2}a_1 + i\left(-\frac{\sqrt{3}}{2}a_0 - \frac{\sqrt{3}}{6}a_1\right), -\frac{1}{2}a_3 + i\left(-\frac{\sqrt{3}}{2}a_2 - \frac{\sqrt{3}}{6}a_3\right),
\right.
\left.-\frac{1}{2}b_1 + i\left(-\frac{\sqrt{3}}{2}b_0 - \frac{\sqrt{3}}{6}b_1\right), -\frac{1}{2}b_3 + i\left(-\frac{\sqrt{3}}{2}b_2 - \frac{\sqrt{3}}{6}b_3\right)\right)
\]
\[
= \frac{i}{\sqrt{3}} \left(a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 \frac{1+i\sqrt{3}}{2} a_3 + b_0 + b_1 \frac{1+i\sqrt{3}}{2} b_2 + b_3 \frac{1+i\sqrt{3}}{2} i_2\right),
\]
and the lattices are given by
\[
\Lambda_R = (1,0,0,0,0,0,0,0)Z + (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0,0,0,0,0,0)Z + (0,0,1,0,0,0,0,0)Z + (0,0,0,1,0,0,0,0)Z + (0,0,0,0,1,0,0,0)Z + (0,0,0,0,0,1,0,0)Z + (0,0,0,0,0,0,1,0)Z + (0,0,0,0,0,0,0,1)Z
\]
\[
+ (0,0,0,0,0,0,0,0)Z + (0,0,0,0,1,0,0,0)Z + (0,0,0,0,0,1,0,0)Z + (0,0,0,0,0,0,1,0)Z + (0,0,0,0,0,0,0,1)Z
\]
\[ \Lambda_C = (1, 0, 0, 0)Z + (\frac{1}{2} + i\frac{\sqrt{3}}{2}, 0, 0, 0)Z + (0, 1, 0, 0)Z + (0, \frac{1}{2} + i\frac{\sqrt{3}}{2}, 0, 0)Z + (0, 0, 1, 0)Z + (0, 0, 0, \frac{1}{2} + i\frac{\sqrt{3}}{2})Z \]
\[ = \left(Z^{1 + i\sqrt{3}} \right)^4 \]

**Proof:** Let \( Z \in \mathcal{H}(\mathbb{H}) \). By definition (2.13) and by taking (1.34), (1.46), (2.26) and 2.7 into account we compute
\[
\vartheta_p(Z) = \sum_{g \in \frac{3}{2}p + \mathbb{O}^2} e^{2\pi i (Z, g_g)} = \sum_{g \in \frac{3}{2}p + \mathbb{O}^2} e^{2\pi i (\tilde{Z}, g,g)} = \sum_{g \in \frac{3}{2}p + \mathbb{O}^2} e^{2\pi i (\tilde{Z}, g,g)} = \sum_{g \in \frac{3}{2}p + \mathbb{O}^2} e^{2\pi i (\tilde{Z}, g,g)} = \Theta_{P_R, 0}(\tilde{Z}, \tilde{Z}; (\mu_R(\mathcal{O}^{2 \times 1}))',
\]
where
\[
P_R = \left( \mu_R \left( \frac{3}{2}p \right) \right)',
\]
\[= \left( -\frac{1}{2}a_1, \frac{\sqrt{3}}{2}a_1, -\frac{\sqrt{3}}{2}a_2 + \frac{\sqrt{3}}{6}a_3, -\frac{1}{2}b_1, \frac{\sqrt{3}}{2}b_0 + \frac{\sqrt{3}}{6}b_1, -\frac{1}{2}b_3, \frac{\sqrt{3}}{2}b_2 + \frac{\sqrt{3}}{6}b_3 \right),
\]
with the last step following analogous to the calculations done in (2.14). Exactly the same proceeding also yields
\[
\vartheta_p(Z) = \sum_{g \in \mu_C(\frac{3}{2}p) + \mathbb{O}^2} e^{2\pi i (\tilde{Z}, g,g)} = \Theta_{P_C, 0}(\tilde{Z}, \tilde{Z}; (\mu_C(\mathcal{O}^{2 \times 1}))')
\]
with
\[
P_C = \left( \mu_C \left( \frac{3}{2}p \right) \right)',
\]
\[= \left( -\frac{1}{2}a_1 + i(-\frac{\sqrt{3}}{3}a_0 - \frac{\sqrt{3}}{6}a_1), -\frac{1}{2}a_3 + i(-\frac{\sqrt{3}}{3}a_2 - \frac{\sqrt{3}}{6}a_3), -\frac{1}{2}b_1 + i(-\frac{\sqrt{3}}{3}b_0 - \frac{\sqrt{3}}{6}b_1), -\frac{1}{2}b_3 + i(-\frac{\sqrt{3}}{3}b_2 - \frac{\sqrt{3}}{6}b_3) \right).
\]
And finally, we have
\[
\mu_R(\mathcal{O}^{2 \times 1})' = \Lambda_R, \mu_C(\mathcal{O}^{2 \times 1})' = \Lambda_C
\]
by definition. Hence the assertion follows.

Thus by defining Hermitian theta-constants \( \tilde{\vartheta}_p \) on \( \mathcal{H}_4(\mathbb{C}) \) analogous to (2.13) by
\[
\tilde{\vartheta}_p(Z) := \sum_{g \in \frac{3}{2}p + (Z^{1 + i\sqrt{3}})^4} e^{2\pi i (\tilde{Z}, g,g)}, \quad Z \in \mathcal{H}_4(\mathbb{C})
\]
for $\tilde{p} \in \left(\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]\right)^4$, then in particular one obtains

$$\vartheta_p = \tilde{\vartheta}_{\tilde{p}}|_{\mathcal{H}(\mathbb{H})},$$

where $p$ and $\tilde{p}$ correspond to each other via the bijection

$$\mathcal{O} \rightarrow \left(\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]\right)^2, \quad a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \mapsto \left(a_0 + a_1 \frac{1+i\sqrt{3}}{2}, a_2 + a_3 \frac{1+i\sqrt{3}}{2}\right).$$

Of course, the same can be done for $\mathcal{H}_8(\mathbb{R})$, whereas one does not obtain such a “nice” description.

### 2.3 Quaternionic theta-series of the second kind

Just as in [Kr10] we are now going to study another important special case of quaternionic theta-series, namely the quaternionic theta-series of the second kind. Recall the definition of the $\mathcal{O}$-stable matrix

$$S_{\mathcal{O}} = \begin{pmatrix} 2 & \frac{2}{3}(i_1 \sqrt{3} + i_1 i_2 \sqrt{3}) \\ -\frac{2}{3}(i_1 \sqrt{3} + i_1 i_2 \sqrt{3}) & 2 \end{pmatrix}$$

from section 2.1. In the first section of this chapter we took a closer look at the special theta-series $\Theta_{0,0}(Z, S; O^{m \times n})$ for $\mathcal{O}$-stable matrices $S$ which turned out to be quaternionic modular forms for the whole quaternionic modular group. We will now permit the first characteristic to be non-zero and have a closer look at theta-series of the shape $\Theta_{A,0}(Z, S_{\mathcal{O}}; O^{2 \times 2})$.

**2.28 Definition.** Let $A \in O^{2 \times 2}$ and $Z \in \mathcal{H}(\mathbb{H})$.

$$\Theta_A(Z; S_{\mathcal{O}}) := \sum_{G \in \frac{1}{\mathcal{O}^{2 \times 2}}} e^{\pi i t(S_G[Z])} = \sum_{G \in \mathcal{O}^{2 \times 2}} e^{\pi i t(S_G[-\frac{1}{\sqrt{3}} + G,Z])} = \Theta_{-\frac{1}{\sqrt{3}},0}(Z, S_{\mathcal{O}}; O^{2 \times 2})$$

is called the quaternionic theta-series of the second kind with characteristic $A$ (for $S_{\mathcal{O}}$).

As we will need it at some points, $\Theta_A(Z; S)$ shall denote the according theta-series for arbitrary $S \in \text{Pos}_2(\mathbb{H})$ replacing $S_{\mathcal{O}}$.

Just like for the quernionic theta-constants we get the following first results:

**2.29 Remark.**

a) Again, as we have already seen in the definition, theta-series of the second kind are just a special case of general quaternionic theta-series. But note that $S_{\mathcal{O}}$ is $\mathcal{O}$-stable, which will result in some special properties. In particular, we can apply all properties of quaternionic theta-series for $\mathcal{O}$-even matrices from the first section.

b) Let $B \in \mathcal{P}^{2 \times 2} = i_1 \sqrt{3} \mathcal{O}^{2 \times 2}$, then $H = -\frac{B}{\sqrt{3}} \in \mathcal{O}^{2 \times 2}$. So again, like in the case of quaternionic theta-constants, we obtain

$$\Theta_{A+B}(Z; S_{\mathcal{O}}) = \Theta_{-\frac{1}{\sqrt{3}} + H,0}(Z, S_{\mathcal{O}}; O^{2 \times 2}) = \Theta_{-\frac{1}{\sqrt{3}},0}(Z, S_{\mathcal{O}}; O^{2 \times 2}) = \Theta_A(Z; S_{\mathcal{O}})$$
2.3 Quaternionic theta-series of the second kind

in virtue of (2.3). Hence we can choose $A$ modulo $p^{2 \times 2}$. According to (1.10), $O^{2 \times 2} / p^{2 \times 2} \cong \mathbb{F}_p^{2 \times 2}$, thus there are at most $q^4$ quaternionic theta-series of the second kind. With the help of some further results, we will later see that there are exactly 19 quaternionic theta-series of the second kind. The proof will be based on the following transformation law:

Define

$$\text{Aut}(S_\mathcal{O}) := \{ V \in O^{2 \times 2} ; S_\mathcal{O}[V] = S_\mathcal{O} \}$$

to be the so-called automorphy group of $S_\mathcal{O}$. Of course, $\text{Aut}(S_\mathcal{O}) \subset \text{GL}_2(\mathcal{O})$ holds true (e.g. in virtue of (1.34), while it will even be proved in another way later on). Then for $A \in O^{2 \times 2}$ and $V \in \text{Aut}(S_\mathcal{O})$ we get

$$\Theta_A^V(Z; S_\mathcal{O}) = \Theta_{-N\mathcal{A}^{-\frac{1}{2}}}_{\mathcal{A}, 0}(Z, S_\mathcal{O}; O^{2 \times 2}) = \Theta_{-N\mathcal{A}^{-\frac{1}{2}}}_{\mathcal{A}, 0}(Z, S_\mathcal{O}[V]; O^{2 \times 2}) = \Theta_A(Z; S_\mathcal{O})$$

by means of (2.5).

c) $G \in \frac{i}{\sqrt{3}}A + O^{2 \times 2}$ and $S_\mathcal{O} \in \text{Her}_2(\frac{2i}{3\sqrt{3}} \mathcal{O})$ yield $S_\mathcal{O}[G] \in \text{Her}_2(\frac{2i}{3\sqrt{3}} \mathcal{O}) = \frac{2}{3} \text{Her}_2(\mathcal{O})$ as $\frac{2i}{3\sqrt{3}} \mathcal{O} = \frac{2}{3} \frac{i}{\sqrt{3}} \mathcal{O} = \frac{2}{3} (\frac{i}{\sqrt{3}} \mathcal{O} + \frac{1}{\sqrt{3}} \mathcal{O} + \frac{1}{\sqrt{3}} \mathcal{O} + \frac{1}{\sqrt{3}} \mathcal{O})$. Note that $S_\mathcal{O}[G] \geq 0$ according to [Kr85, ch.I, cor.3.7]. Just like in (2.12) we can rearrange the theta-series as they are absolutely and locally uniformly convergent:

$$\Theta_A(Z; S_\mathcal{O}) = \sum_{G \in \frac{i}{\sqrt{3}}A + O^{2 \times 2}} e^{\pi i (S_\mathcal{O}(G), Z)} = \sum_{T \in \frac{1}{\sqrt{2}} \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_A(T) \cdot e^{2\pi i (T, Z)}$$

for all $Z \in \mathcal{H}(\mathbb{H})$, where

$$\alpha_A(T) = \# \{ G \in -\mathcal{A} \frac{i}{\sqrt{3}} + O^{2 \times 2} ; S_\mathcal{O}[G] = T \} .$$

Unfortunately, it does not seem possible to determine the Fourier-coefficients explicitly. But note the following property: It is easy to check that $S_\mathcal{O} \geq \text{diag}(s_0, s_0)$ holds true for $s_0 = 2 - \frac{2\sqrt{2}}{\sqrt{3}}$. This implies

$$S_\mathcal{O}[G] - \text{diag}(s_0, s_0)[G] \geq 0 \Rightarrow \text{tr}(S_\mathcal{O}[G] - \text{diag}(s_0, s_0)[G]) \geq 0 \Rightarrow \text{tr}(S_\mathcal{O}[G]) \geq \text{tr}(\text{diag}(s_0, s_0)[G]) = s_0 (N(a) + N(b) + N(c) + N(d))$$

for $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. And thus a $G$ with $S_\mathcal{O}[G] = T$ has to meet the condition

$$N(a) + N(b) + N(c) + N(d) \leq \frac{1}{s_0} \text{tr}(T) .$$

So because $\frac{i}{\sqrt{3}}A + O^{2 \times 2}$ is discrete, only finitely many $G$ contribute to the calculation of $\alpha_A(T)$. In particular, the computation of the Fourier-coefficients can be implemented on a computer using a computer algebra system. Moreover, in doing so one can verify that the 19 quaternionic theta-series of the second kind (as announced, we will introduce them further below) are actually linearly independent. It was done using [SAGE].

d) Let $U \in \text{GL}_2(\mathcal{O})$. As we have seen before, $\phi_1 : \mathcal{O} \rightarrow \mathcal{O}, a \mapsto -i_1 a i_1$ is a bijection, and
Again, the characteristic can be chosen modulo $p^{2\times 2}$. And as we have seen in (2.14), 
$-i_1 U_{i_1} \equiv \bar{U}$ modulo $p^{2\times 2}$ holds. So we can reformulate the result from above as

$$\Theta_A(Z[U]; S_O) = \Theta_{\Pi A}(Z; S_O)$$

for all $Z \in H(H)$ and $U \in GL_2(O)$.

Now, let $S \in \text{Herm}(2)$. We get

$$\Theta_A(Z + S; S_O) = \sum_{G \in O^{2\times 2}} e^{\pi i r(S_O[-A' \frac{i_1}{\sqrt{3}} + G], Z)} e^{\pi i r(S', [\frac{i_1}{\sqrt{3}} A + \bar{G}', S)}$$

with

$$\tau(S_O[-A' \frac{i_1}{\sqrt{3}} + G], S) = \tau(S_O, S[\frac{i_1}{\sqrt{3}} A + \bar{G}'])$$

$$= \frac{1}{2} \tau(S_O, (-i_1 S_1)[A]) + \tau(S_O, S[\bar{G}'] + GS_{\frac{i_1}{\sqrt{3}}}A + GS_{\frac{i_1}{\sqrt{3}}}A')$$

in view of (1.46), since $S = S'$. Let $T = S[\bar{G}'] + GS_{\frac{i_1}{\sqrt{3}}}A + GS_{\frac{i_1}{\sqrt{3}}}A'$. Of course, $T \in \text{Herm}_2(H)$ holds, and because $GS_{\frac{i_1}{\sqrt{3}}}A \in O^{2\times 2}$ since $S \in i_1 \sqrt{3} O^{2\times 2}$, even $T \in \text{Herm}_2(O)$ holds true. Now $S_O \in 2 \text{Herm}_2^b(O)$ by the definition of $O$-stable matrices, and thus, again by definiton, $\tau(S_O, T) \in 2Z$ has to hold. Hence

$$\Theta_A(Z + S; S_O) = e^{\frac{1}{2} \pi i r(S_O[\bar{A}]\}_{-i_1 S_1})} \cdot \Theta_A(Z; S_O)$$

follows. Unfortunately, there is no transformation law for all $\text{Trans}(S), S \in \text{Herm}_2(O)$, only for $S \in \text{Herm}_2(i_1 \sqrt{3} O)$, as otherwise $\tau(S_O, T) \in 2Z$ no longer holds true in general. But note that the prefactor $e^{\frac{1}{2} \pi i r(S_O[\bar{A}]\}_{-i_1 S_1})}$ is always a third root of unity as $\tau(S_O, (-i_1 S_1)[A]) \in 2Z$.

It is also possible to specify the behavior under $\text{Rot}(i_1 I)$. According to (2.4) and (2.5), and since $G \mapsto -i_1 Gi_1$ is a bijection of $O^{2\times 2}$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \in GL_2(O)$, we calculate

$$\Theta_A(Z[i_1 I]; S_O) = \Theta_{-A' \frac{i_1}{\sqrt{3}}}Z[i_1 I], S_O; O^{2\times 2})$$

$$= \Theta_{-(i_1)A' \frac{i_1}{\sqrt{3}}}Z[i_1 I], S_O; O^{2\times 2})$$

$$= \Theta_{-(i_1)A' \frac{i_1}{\sqrt{3}}}Z[i_1 I], S_O; O^{2\times 2})$$

$$= \Theta_{(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})^{-1} A' \frac{i_1}{\sqrt{3}}}Z[i_1 I], S_O; O^{2\times 2})$$
As we have seen in (2.29), \( \text{Aut} \), applications of the characteristic with 
\( \Theta \) yields no expression in terms of the original theta-series. Partly, this is due to the fact that 
easy to prove (and was done, indeed) that the gained set of matrices in \( \text{Aut} \). 
Using \([\text{SAGE}]\), it is possible to define and compute matrix groups via generators. Thus it is 
matrices to check and the computation only took a few days). This gives a set of 720 matrices. 
all matrices 
So the easiest method is using a computer algebra system (here, \([\text{SAGE}]\) was used) and testing 
satisfy 
initial proof here, as it is simply unnecessary. 
results themselves can easily be proven using a computer algebra system. We will not give the 
Proof: Although the proof can and has been done manually, there is a much nicer way. The 
manual proof is quite long and tedious, but was needed to get the results from above. But the 
results themselves can easily be proven using a computer algebra system. We will not give the 

\[ \begin{align*} 
\Theta & = \Theta_{-iA_1}(1 \of i) \Theta_0(Z; S \mathcal{O}^2; \mathcal{O}^2) \\
& = \Theta_{-iA_1}(1 \of i) (Z; S \mathcal{O}) = \Theta_{\mathcal{A}}(1 \of i) (Z; S \mathcal{O}), 
\end{align*} \]

because an easy verification shows \( S \mathcal{O} [i_1 I] = S \mathcal{O} [(1 \of 0 \of -i_2)] \) and \(-i_1 A_1 \equiv \mathcal{A} \) holds modulo \( p \) as we have seen several times before.

Unfortunately, there also seems to be no way to describe the behavior of the quaternionic 
theta-series of the second kind under \( J \), apart from the theta-transformation-formula, which 
yields no expression in terms of the original theta-series. Partly, this is due to the fact that 
\( \Theta_0(Z; S \mathcal{O}) \) is already a quaternionic modular form with respect to the full quaternionic modular 
group (confer the first section).

So now we will rather prove the announced proposition that modulo \( p^2 \times 2 \) and modulo multi-
lications of the characteristic with \( V' \) from the right, where \( V \in \text{Aut}(S \mathcal{O}) \), only 19 characteristics, 
and thus 19 quaternionic theta-series of the second kind remain. So first, we have to explicitly 
determine \( \text{Aut}(S \mathcal{O}) = \{ V \in O^2 \times 2; S \mathcal{O}[V] = S \mathcal{O} \} \).

\[ \text{(2.30) Lemma. Aut}(S \mathcal{O}) \text{ is described explicitly in the following way:} \]

- \# \text{Aut}(S \mathcal{O}) = 720.
- \text{Aut}(S \mathcal{O}) \text{ is generated by the following matrices:}
  \[
  \begin{pmatrix}
    0 & 1 \\
    -1 & 0
  \end{pmatrix}, 
  \begin{pmatrix}
    i_2 & 0 \\
    0 & -i_2
  \end{pmatrix}, 
  \begin{pmatrix}
    1 + i_1 \sqrt{3} \over 2 & -1 \\
    0 & 1 - i_1 \sqrt{3} \over 2
  \end{pmatrix}, 
  \begin{pmatrix}
    1 + i_1 \sqrt{3} \over 2 & -1 + i_1 \sqrt{3} \over 2 \\
    0 & -1 + i_1 \sqrt{3} \over 2
  \end{pmatrix}.
\]
- \text{Aut}(S \mathcal{O}) \leq \text{GL}_2(\mathcal{O}).
- \text{Aut}(S \mathcal{O}) \simeq \text{SL}_2(\mathcal{F}_9) \text{ via } \pi_g.

\textbf{Proof:} Although the proof can and has been done manually, there is a much nicer way. The 
manual proof is quite long and tedious, but was needed to get the results from above. But the 
results themselves can easily be proven using a computer algebra system. We will not give the 
initial proof here, as it is simply unnecessary.

As we have seen in (2.29), \( \text{Aut}(S \mathcal{O}) \) is finite as every \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O} \) fulfilling \( S \mathcal{O}[V] = S \mathcal{O} \) has to satisfy

\[ N(a) + N(b) + N(c) + N(d) \leq (2 - 2 \sqrt{3})^{-1} \text{tr}(S \mathcal{O}) = 2(1 - \sqrt{3})^{-1} < 11. \]

So the easiest method is using a computer algebra system (here, \([\text{SAGE}]\) was used) and testing 
all matrices \( V \in O^2 \times 2 \) with this property (which is a finite and not too large set – around \( 10^8 \) 
matrices to check and the computation only took a few days). This gives a set of 720 matrices.

Using \([\text{SAGE}]\), it is possible to define and compute matrix groups via generators. Thus it is 
easy to prove (and was done, indeed) that the gained set of matrices in \( \text{Aut}(S \mathcal{O}) \) coincides with
the matrix group given by the generators from above. And of course, in view of the given
generators, it is easy to see that $\text{Aut}(S_O) \leq \text{GL}_2(O)$ holds. By the same methods and using
[SAGE] again, it is possible to prove that the images of the given generators under $\pi_p$ generate
$\text{SL}_2(F_9)$, whereas $\#\text{SL}_2(F_9) = 720$ holds.

Now that we have explicitly determined $\text{Aut}(S_O)$, we can now prove that there are exactly
19 quaternionic theta-series of the second kind: As we have seen, the index $A$ of $\Theta_A(Z; S_O)$
can be chosen modulo $p^{2 \times 2}$, which yields $9^4 = #F_9^{2 \times 2}$ characteristics. Furthermore, we have
seen that we can further reduce the characteristic by multiplication with $V'$ from the
right, where $V \in \text{Aut}(S_O)$. As we have seen in the preceding lemma, modulo $p^{2 \times 2}$ this means
we can reduce the characteristic (seen as a matrix in $F_9^{2 \times 2}$) by multiplication with $V$ from the
right, where $V \in \text{SL}_2(F_9)$, because $V \in \text{Aut}(S_O)$ implies $\pi_p(V) \in \text{SL}_2(F_9)$ and as well
$\pi_p(V') = \phi_3(\pi_p(V')) \in \text{SL}_2(F_9)$.

Again, the easiest method is using a computer algebra system – once more [SAGE] was used.
$\text{SL}_2(F_9)$ operates on $F_9^{2 \times 2}$ by multiplication from the right, and all we need to do is finding
representatives of the orbits under this operation – and [SAGE] provides such tools. So the
proof of the following proposition is a simple recalculation – theoretically, it could be done
manually: Of course, the length of the orbit of $A = 0$ is 1. And obviously, the length of the orbit of $A \in \text{GL}_2(F_9)$ is $\#\text{SL}_2(F_9) = 720$, whereas $\#\text{GL}_2(F_9) = 80 \cdot 72$. This gives another
$80 \cdot 72/720 = 80 \cdot 72/720 = 8$ orbits. Some further considerations yield that the length of the orbit of some
$A \in F_9^{2 \times 2}$ of rank 1 is $720/9 = 80$. As $9^4 - 1 - 8 \cdot 720 = 800$, there have to be 10 such orbits. But
just as stated before, it can all be done using a computer algebra system. The result is:

(2.31) Proposition. Every quaternionic theta-series of the second kind $\Theta_A(Z; S_O)$, $A \in O^{2 \times 2}$ is equal
to one of the following 19 theta-series:

\[ \Theta_j(Z) := \Theta_{A_j}(Z; S_O), \quad j \in \{1, \ldots, 19\} \]

where

\[
\begin{align*}
A_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & A_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, & A_6 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_7 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & A_8 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
A_9 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{10} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
A_{12} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{13} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{14} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{15} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
A_{16} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{17} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{18} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A_{19} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

Just as we did for the quaternionic theta-constants we will now describe the behavior of the
quaternionic theta-series of the second kind under $Z \mapsto Z'$. This time, the proof will be much
more involved as there seems to be no way to explicitly describe the Fourier-coefficients. But
again, the transformation $Z \mapsto Z'$ only permutes the 19 theta-series.
(2.32) Theorem. Let \( A \in \mathcal{O}^{2\times 2} \). If \( \det(\pi_p(A)) = 0 \in \mathbb{F}_9 \), then
\[
\Theta_A(Z'; S_O) = \Theta_{\tau}(Z; S_O)
\]
holds for all \( Z \in \mathcal{H}(\mathbb{H}) \). If otherwise \( \det(\pi_p(A)) \in \mathbb{F}_9 \setminus \{0\} \), then
\[
\Theta_A(Z'; S_O) = \Theta_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}(Z; S_O)
\]
holds for all \( Z \in \mathcal{H}(\mathbb{H}) \).

**Proof:** Let \( Z \in \mathcal{H}(\mathbb{H}) \). At first, we will only prove the assertion for the 19 special quaternionic theta-series of the second kind in (2.31). Afterwards, we will show the assertion for arbitrary \( A \in \mathcal{O}^{2\times 2} \).

Let us start with \( A = A_j, j \in \{1, 3, 4, \ldots, 11\} \), which means that \( A = (a, 0) \) (where \( a \) and \( 0 \) are the columns of \( A \)) with \( a = (0, 0)' \) or \( a = (1, b)' \), \( b \in \{0, \pm 1, \pm 1 + i_2, \pm 1 - i_2\} \). A simple calculation shows that
\[
S_O = C\mathcal{C}' , \quad C = \left( \begin{array}{cc} \sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}}(i_1 + i_1i_2) & \frac{1}{\sqrt{2}}i_1 \end{array} \right)
\]
holds. By another easy consideration we obtain \( \text{tr}(Z|M) = Z[m_1] + Z[m_2] \) if \( M = (m_1, m_2) \in \mathbb{H}^{2\times 2} \), and furthermore, since \( Z = Z' \), we get \( \tau(I, Z|M) = \text{tr}(Z[M]) \). And as \( Z[g] \in \mathbb{R} \otimes \mathbb{R} C \) for all \( g \in \mathbb{H}^2, \ Z[g] = -i_1(Z[g])i_1 = Z[gi_1] \) holds. Moreover, note that every \( h \in \mathcal{O}^2 \) can be written as \( h = u + i_1\sqrt{3}v \), where \( u \in \mathcal{O}^2/p^2, v \in \mathcal{O}^2 \). Keeping all that in mind and in view of (1.46), we compute
\[
\Theta_A(Z; S_O) = \sum_{G \in \mathcal{O}^{2\times 2}} e^{\pi i \tau(CG, Z(\tfrac{i_1}{\sqrt{2}}, A + G))} = \sum_{G \in \mathcal{O}^{2\times 2}} e^{\pi i \tau(I, Z(\tfrac{i_1}{\sqrt{2}}, A + G)C)}
\]
\[
= \sum_{G \in \mathcal{O}^{2\times 2}} e^{\pi i \tau(Z)(\tfrac{i_1}{\sqrt{2}}, A + G)C)}
\]
\[
= \sum_{g, h \in \mathcal{O}^2} e^{\pi i \tau(Zg \sqrt{2} + h(-\sqrt{2}(i_1 + i_1i_2)) + \sqrt{2}v\sqrt{2} + Z[\sqrt{2}hi_1])}
\]
\[
= \sum_{u \in \mathcal{O}^2/p^2} \sum_{v \in \mathcal{O}^2} e^{2\pi i \tau(Z[\tfrac{i_1}{\sqrt{2}}(u(1 + i_2) + a) + (g + v(1 + i_2))] + Z[\tfrac{i_1}{\sqrt{2}}u + v])}
\]
\[
= \sum_{u \in \mathcal{O}^2/p^2} \left( \sum_{v \in \mathcal{O}^2} e^{2\pi i \tau(Z[\tfrac{i_1}{\sqrt{2}}u + v])} \sum_{g \in \mathcal{O}^2} e^{2\pi i \tau(Z[\tfrac{i_1}{\sqrt{2}}(u(1 + i_2) + a) + g])} \right)
\]
\[
= \sum_{u \in \mathcal{O}^2/p^2} \vartheta_u(Z) \cdot \vartheta_{u(1 + i_2) + a}(Z)
\]
where in the second last step \( g \) was substituted by \( g - v(1 + i_2) \), which of course is a bijection. Fortunately, in (2.22) we already determined the behavior of the quaternionic theta-constants. Recall that we can choose the characteristics modulo \( p \), with \( \mathcal{O}/p \simeq \mathbb{F}_9 \) being commutative.
Moreover, \((\begin{smallmatrix} i_2 & 0 \\ 0 & -i_2 \end{smallmatrix}) \in \text{Aut}(S_O)\) according to (2.30). Thus in view of (2.14) we compute
\[
\Theta_A(Z'; S_O) = \sum_{u \in O^2/\mu^2} \theta_u(Z') \cdot \theta_{u(1+i_2)+a}(Z') = \sum_{u \in O^2/\mu^2} \theta_{u(Z)} \cdot \theta_{u(1+i_2)+\pi_1(Z)}
\]
\[
= \sum_{u \in O^2/\mu^2} \theta_{u(Z)} \cdot \theta_{u(1+i_2)+\pi_1(Z)} = \sum_{u \in O^2/\mu^2} \theta_{u(Z)} \cdot \theta_{u(1+i_2)+\pi_1(Z)}
\]
\[
= \Theta_{A_{i_2}}(Z; S_O) = \Theta_{A_{i_2}}((-i_2)^0) (Z; S_O) = \Theta_{A_{i_2}}((1 0 -1) (Z; S_O)
\]
where the last step is possible because the second column of \(A\) is zero.
Next, we will have a look at the special case \(A = A_2\), so \(A = (0, a), a = (0, 1)'\). Another simple calculation shows that
\[
S_O = C\mathbb{C}', \quad C = \left( \frac{\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{2}}, 0 \right)
\]
holds. Almost exactly repeating the calculation from above we obtain
\[
\Theta_A(Z; S_O) = \sum_{G \in O^{2 \times 2}} e^{\pi i \text{tr}(Z)(\frac{1}{2}A + G)}
\]
\[
= \sum_{g, h \in O^2} e^{\pi i g(i_1^2 + i_1 i_2)} + h\sqrt{2} i_1 + \sqrt{2} g i_1) = \sum_{u \in O^2/\mu^2} \theta_u(Z) \cdot \theta_u(1+i_2) + a(Z)
\]
and once more, because of the special choice of \(A\) we calculate
\[
\Theta_A(Z'; S_O) = \Theta_{A_{i_2}}(Z; S_O) = \Theta_{A_{i_2}}((-i_2)^0) (Z; S_O)
\]
\[
= \Theta_{A_{i_2}}((1 0 -1) (Z; S_O) = \Theta_{A}(Z; S_O)
\]
where again the last step is possible because the first column of \(A\) is zero.
So let us consider the remaining cases, i.e. \(A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\), \(a \in \{\pm 1, \pm i_2, \pm 1 + i_2, \pm 1 - i_2\}\). As we have seen in (2.29), \(\Theta_B(Z; S_O)\) possesses the Fourier-expansion
\[
\Theta_B(Z; S_O) = \sum_{T \in \frac{1}{2} \text{Her}_2(O), T \geq 0} \alpha_B(T) \cdot e^{2\pi i \text{tr}(T, Z)}
\]
for all \(Z \in \mathcal{H}(\mathbb{H})\) and \(B \in O^{2 \times 2}\), where
\[
\alpha_B(T) = \#\{G \in -B^T i_1^2 + O^{2 \times 2}; S_O[G] = T\}.
\]
So if \(\alpha_B(T) \neq 0\), then \(T\) has to be of the shape \(T = S_O[-B^T i_1^2 + G], G \in O^{2 \times 2}\). With the help of a straightforward, but tedious computation one can verify that for \(B = A_j, j \in \{1, \ldots, 11\}\) (and
We can compare the Fourier-expansions of both sides, where the entry \( t \) in \( T = \left( \begin{smallmatrix} * & t \\ * & * \end{smallmatrix} \right) , \quad t \in \frac{1}{3} \mathcal{O} \)

while for \( B = A_j = \text{diag}(1, a_j), \ j \in \{12, \ldots, 19\} \) (and hence for all \( B \) such that \( \det(\pi_p(B)) \neq 0 \)), which implies \( \Theta_{\mathcal{O}}(Z; S_{\mathcal{O}}) = \Theta_{A_j}(Z; S_{\mathcal{O}}) \) for some \( j \in \{12, \ldots, 19\} \) such a \( T \) is always of the shape

\[
T = \left( \begin{smallmatrix} * & t \\ * & * \end{smallmatrix} \right) , \quad t \notin \frac{1}{3} \mathcal{O}, \quad \frac{3}{2} i_1 \sqrt{3} t \in a_j(1 - i_2) + i_1 \sqrt{3} \mathcal{O}.
\]

We omit the details here, as it is truly straightforward but very long without giving any important information. So there is no \( T \in \text{Her}_2(\mathbb{H}) \) such that there exist \( j \in \{12, \ldots, 19\} \) and \( k \in \{1, \ldots, 19\} \), \( j \neq k \) with \( a_{A_j}(T) \neq 0 \) and \( a_{A_k}(T) \neq 0 \). Next, just as we mentioned before in (2.29), \( S_{\mathcal{O}}[i_1 I] = S_{\mathcal{O}}[\left( \begin{smallmatrix} 1 & 0 \\ 0 & -i_2 \end{smallmatrix} \right)] \) holds. So together with (2.5) we obtain

\[
\Theta_0(Z; -i_1 S_{\mathcal{O}} i_1) = \Theta_{0,0}(Z, S_{\mathcal{O}}[\left( \begin{smallmatrix} 1 & 0 \\ 0 & -i_2 \end{smallmatrix} \right)]; \mathcal{O}^{2 \times 2}) = \Theta_{0,0}(Z, S_{\mathcal{O}}; \mathcal{O}^{2 \times 2}) = \Theta_0(Z; S_{\mathcal{O}}).
\]

Moreover, recall that \(-i_1 O_{i_1} = O\), so next we calculate

\[
\Theta_0(Z; S_{\mathcal{O}}) = \Theta_0(Z; -i_1 S_{\mathcal{O}} i_1) = \sum_{G \in \mathcal{O}^{2 \times 2}} e^{\pi i \tau((-i_1 S_{\mathcal{O}} i_1)[G], Z)}
\]

\[
= \sum_{A \in \mathcal{O}^{2 \times 2}/p^{2 \times 2}} \sum_{G \in \mathcal{O}^{2 \times 2}} e^{\pi i \tau((-i_1 S_{\mathcal{O}} i_1)[A + i_1 \sqrt{3} G], Z)}
\]

\[
= \sum_{A \in \mathcal{O}^{2 \times 2}/p^{2 \times 2}} \sum_{G \in \mathcal{O}^{2 \times 2}} e^{\pi i \tau(S_{\mathcal{O}}[\frac{1}{2} A + G], Z)}
\]

\[
= \sum_{A \in \mathcal{O}^{2 \times 2}/p^{2 \times 2}} \Theta_A(3Z; S_{\mathcal{O}}) = \sum_{j=1}^{19} \beta_j \Theta_{A_j}(3Z; S_{\mathcal{O}})
\]

where

\[
\beta_j = \begin{cases} 
1, & \text{for } j = 1, \\
80, & \text{for } j = 2, \ldots, 11, \\
720, & \text{for } j = 12, \ldots, 19.
\end{cases}
\]

We already verified that

\[
\Theta_0(Z'; S_{\mathcal{O}}) = \Theta_0(Z; S_{\mathcal{O}})
\]

holds for all \( Z \in \mathcal{H}(\mathbb{H}) \). This immediately implies

\[
\sum_{j=1}^{19} \beta_j \Theta_{A_j}(Z'; S_{\mathcal{O}}) = \sum_{j=1}^{19} \beta_j \Theta_{A_j}(Z; S_{\mathcal{O}}).
\]

We can compare the Fourier-expansions of both sides, where the entry \( t \) in \( T = \left( \begin{smallmatrix} * & t \\ * & * \end{smallmatrix} \right) \) for \( T \) occurring in these Fourier-expansions can be uniquely correlated to the Fourier-expansion of one of the \( \Theta_{A_j}(Z; S_{\mathcal{O}}), \ j \in \{12, \ldots, 19\} \) in case \( t \notin \frac{1}{3} \mathcal{O} \) holds. In the proof of (2.22) we already showed that

\[
\tau(T, Z') = \tau(T', Z) = \tau(T, Z)
\]
holds true for all $Z \in \mathcal{H}(\mathbb{H})$ and $T \in \text{Her}_2(\mathbb{H})$, since $T' = \overline{T}$. Hence we obtain
\[
\sum_{j=1}^{19} \beta_j \sum_{T \in \frac{1}{2} \text{Her}_2(O), T \geq 0} a_{\lambda_i}(T) \cdot e^{2\pi i \tau(T,Z)} = \sum_{j=1}^{19} \beta_j \sum_{T \in \frac{1}{2} \text{Her}_2(O), T \geq 0} a_{\lambda_i}(T) \cdot e^{2\pi i \tau(T,Z)}.
\]
And as we have just seen, for $j \in \{12, \ldots, 19\}$, this implies
\[
\Theta_{A_j}(Z'; S_O) = \sum_{T \in \frac{1}{2} \text{Her}_2(O), T \geq 0} a_{\lambda_i}(T) \cdot e^{2\pi i \tau(T,Z)} = \sum_{T \in \frac{1}{2} \text{Her}_2(O), T \geq 0} a_{\lambda_i}(T) \cdot e^{2\pi i \tau(T,Z)} = \Theta_{A_i}(Z; S_O)
\]
for some $k \in \{12, \ldots, 19\}$. So in order to compare the two Fourier-expansions and hence investigating the behavior under $Z \mapsto Z'$, we need to map $T = \begin{pmatrix} * & i \\ -i & * \end{pmatrix} \mapsto \overline{T} = \begin{pmatrix} * & -i \\ i & * \end{pmatrix}$. And thus, as we have explained above, it suffices to check $\pi_p(\frac{3}{2}i_1\sqrt{3}t)$ in comparison to the “original” $\pi_p(\frac{3}{2}i_1\sqrt{3})$. So let us assume $\frac{3}{2}i_1\sqrt{3}t \in a_j(1 - i_2) + i_1\sqrt{3}O$, i.e. $\frac{3}{2}i_1\sqrt{3}t = a_j(1 - i_2) + i_1\sqrt{3}x$ for some $x \in O$. This is equivalent to
\[
-\frac{3}{2}i_1\sqrt{3} = (1 + i_2)a_j - \overline{x}1\sqrt{3}
\]
\[
\iff \frac{3}{2}i_1\sqrt{3}t = i_1(1 + i_2)a_j + i_1\sqrt{3}\overline{x}.
\]
Thus we have shown
\[
\frac{3}{2}i_1\sqrt{3}t \in i_1(1 + i_2)a_j + i_1\sqrt{3}O = -a_j(1 - i_2) + i_1\sqrt{3}O
\]
because $O/p$ is commutative and $\pi_p(-i_1a_i) = \pi_p(\overline{a})$ holds as we have seen several times before. Therefore, by simply comparing the Fourier-expansions and knowing that the Fourier-expansions can be distinguished by $\pi_p(\frac{3}{2}i_1\sqrt{3})$, we have verified
\[
\Theta_{A}(Z'; S_O) = \Theta_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_i}(Z; S_O).
\]
Now for the general case. Let $B \in O^{2 \times 2}$. According to (2.31), there exist $j \in \{1, \ldots, 19\}$, $V \in \text{Aut}(S_O)$ and $H \in p^{2 \times 2}$ such that $BV + H = A_j$, or equivalently spoken $\pi_p(BV) = \pi_p(B)\pi_p(V) = \pi_p(A_j)$, where $\pi_p(V) \in \text{SL}_2(\mathbb{F}_9)$. Then $\det(\pi_p(A_j)) = \det(\pi_p(B)\pi_p(V)) = \det(\pi_p(B))$ holds, of course.
Now, assume $\det(\pi_p(B)) = 0$, then by what we have proven so far (and recall $\overline{A} \equiv -i_1Ai_1$ modulo $p^{2 \times 2}$ as well as $-i_1O_1 = O$ and $-i_1p_1 = p$) we obtain
\[
\Theta_{B}(Z'; S_O) = \Theta_{A_j}(Z'; S_O) = \Theta_{\overline{A}_i}(Z; S_O)
\]
\[
= \Theta_{\begin{pmatrix} -i_1B_1i_1(\overline{-i_1V_1}) - i_1H_1i_1 \end{pmatrix}}(Z; S_O) = \Theta_{\begin{pmatrix} -i_1B_1i_1 \end{pmatrix}}(Z; S_O) = \Theta_{\pi}(Z; S_O),
\]
because $\pi_p((-i_1B_1)(\overline{-i_1V_1}) - i_1H_1i_1) = \pi_p(-i_1B_1i_1) \cdot \phi_3(\pi_p(V))$, and $\pi_p(V) \in \text{SL}_2(\mathbb{F}_9)$ implies $\phi_3(\pi_p(V)) \in \text{SL}_2(\mathbb{F}_9)$.
On the other hand, assume $\det(\pi_{p}(B)) \neq 0$. Then an analogous calculation shows
\[
\Theta_{B}(Z'; S_{O}) = \Theta_{A_{j}}(Z'; S_{O}) = \Theta_{\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) . A_{j}}(Z; S_{O})
\]
\[
= \Theta_{\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right). (b\overline{\mathcal{H}})}(Z; S_{O}) = \Theta_{\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right). B}(Z; S_{O})
\]
which completes the proof. 

To finish this section about quaternionic theta-series of the second kind we are now going to show that these special theta-series are indeed quaternionic modular forms. In this case it will turn out that they are quaternionic modular forms for $\text{Sp}_{2}(\mathcal{O})[p]$, just like the quaternionic theta-constants, but of weight four and with a certain (in most cases non-trivial) abelian character. Note that in section 2.1 we already discovered $\Theta$ to be a quaternionic modular form of weight four with respect to the whole quaternionic modular group and for the trivial character.

The idea goes back to [Kr85] for the case of the Hurwitz order. We will adapt and work out this idea for the presently observed case, and start with a definition and a lemma from [Kr85, ch.IV, le.3.1].

(2.33) Lemma. Let $S \in \text{Pos}_{n}(\mathbb{H})$ be fixed. Given $M = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{array}\right) \in \text{Sp}_{n}(\mathbb{H})$ and $P, Q \in \mathbb{H}^{m \times n} \otimes_{\mathbb{R}} \mathbb{C}$ we define
\[
M \ast (P, Q) = (P \overline{D'} - S^{-1}C'Q, Q\overline{A'} - SPB'),
\]
\[
\kappa(M; (P, Q)) = e^{\pi i r(S[P]\overline{\mathcal{H}}) + \pi i r(S^{-1}[Q], \overline{\mathcal{C}}) - 2\pi i r(\mathcal{F}, \overline{\mathcal{W}})}. \]
Suppose that $M_{1}, M_{2} \in \text{Sp}_{n}(\mathbb{H})$ then the following holds:

a) $(M_{1}M_{2}) \ast (P, Q) = M_{1} \ast (M_{2} \ast (P, Q))$

b) $\kappa(M_{1}M_{2}; (P, Q)) = \kappa(M_{1}; M_{2} \ast (P, Q))\kappa(M_{2}; (P, Q))$

The next lemma is an analog to [Kr85, ch.IV, thm.3.2]. Recall that $S_{O}$ is $\mathcal{O}$-stable.

(2.34) Lemma. Given $M \in \text{Sp}_{2}(\mathcal{O}), Z \in \mathcal{H}(\mathbb{H})$ and $P, Q \in \mathbb{H}^{2 \times 2} \otimes_{\mathbb{R}} \mathbb{C}$ one has
\[
\Theta_{P,Q}(Z, S; \mathcal{O}^{2 \times 2}) = (\det(\tilde{M}\{\tilde{Z}\}))^{-2} \kappa(M; (P, Q)) \cdot \Theta_{M^{-1}(P,Q)}(M(Z), S; \mathcal{O}^{2 \times 2}).
\]
Proof: Because of (2.33), (1.5) and 1.10 (as $-2 \in 2\mathbb{Z}$) it suffices to prove the assertion for the generators of $\text{Sp}_{2}(\mathcal{O})$ specified in (1.15), i.e. for $\text{Trans}(T), \text{Rot}(U)$ and $J_{n}$ where $T \in \text{Her}_{2}(\mathcal{O}), U \in \text{GL}_{2}(\mathcal{O})$. According to (2.5), the assertion is true for $M = \text{Rot}(U) =\left(\begin{array}{cc} T & 0 \\ 0 & U^{-1} \end{array}\right)$, since
\[
\Theta_{P,Q}(Z, S; \mathcal{O}^{2 \times 2}) = \Theta_{P'\overline{U}^{-1},Q'U}(Z[U], S; \mathcal{O}^{2 \times 2}),
\]
while on the other hand
\[
\text{Rot}(U) \langle Z \rangle = Z[U] ,
\]
\[
(\det(\text{Rot}(U) \{ \hat{Z} \}))^{-2} = (\det(U))^2 = 1 ,
\]
\[
\text{Rot}(U) \ast (P, Q) = (PU^{-1}, QU) ,
\]
\[
\kappa(\text{Rot}(U); (P, Q)) = e^{2\pi i r[S[P],0] + \pi i r(S^{-1}[Q],0) - 2\pi i r(T^Q,0)} = 1 .
\]
The assertion is also true for \( M = \text{Trans}(T) = \left( \begin{array}{cc} 1 & T \\ 0 & 1 \end{array} \right) \), as in view of (2.8) (because \( S_o \) is \( O \)-even) we have
\[
\Theta_{P,Q}(Z, S; O^{2\times 2}) = e^{2\pi i r(S[P],T)} \Theta_{P,Q-SPT}(Z + T, S; O^{2\times 2}) ,
\]
while
\[
\text{Trans}(T) \langle Z \rangle = Z + T ,
\]
\[
(\det(\text{Trans}(T) \{ \hat{Z} \}))^{-2} = (\det(I))^2 = 1 ,
\]
\[
\text{Trans}(T) \ast (P, Q) = (P, Q - SPT^T) ,
\]
\[
\kappa(\text{Trans}(T); (P, Q)) = e^{2\pi i r(S[P],T) + \pi i r(S^{-1}[Q],0) - 2\pi i r(T^Q,0)} = e^{2\pi i r(S[P],T)} .
\]

Last, defining \( a = \frac{2}{3} i \sqrt{3} \) and abbreviating \( S = S_o \) the Theta-transformation-formula (2.10) yields for \( M = J_n \), as \( S \) is \( O \)-stable and because one easily verifies \( S^{-1} a = (i_1 \sqrt{3} 1 - i_2) \in \text{GL}_2(O) \) (with inverse \( \left( \begin{array}{cc} -i_1 \sqrt{3} & 1 + i_2 \\ 1 - i_2 & -i_1 \sqrt{3} \end{array} \right) \)):
\[
\Theta_{P,Q}(Z, S; O^{2\times 2}) = (\det(Z))^{-4} (\det(S))^{-4} (\frac{1}{3})^4 e^{2\pi i r(P,Q)} \cdot \Theta_{-a^{-1}Q,a^{-1}P}(-Z^{-1}, S^{-1}[aI]; O^{2\times 2})
\]
\[
= (\det(Z))^{-4} (\frac{1}{3})^4 (\frac{1}{3})^4 e^{2\pi i r(P,Q)} \cdot \Theta_{-a^{-1}Q,a^{-1}P}(-Z^{-1}, S^{-1}[aI]; O^{2\times 2})
\]
\[
= (\det(Z))^{-4} e^{2\pi i r(P,Q)} \sum_{G \in O^{2\times 2}} \pi i r(S[aG-S^{-1}Q],-Z^{-1}) + 2\pi i r(aP,G-a^{-1}Q)
\]
\[
= (\det(Z))^{-4} e^{2\pi i r(P,Q)} \sum_{G \in O^{2\times 2}} \pi i r(S[aG-S^{-1}Q],-Z^{-1}) + 2\pi i r(aP,G-a^{-1}Q)
\]
\[
= (\det(Z))^{-4} e^{2\pi i r(P,Q)} \Theta_{-S^{-1}Q,SP}(Z, S; O^{2\times 2}) .
\]

On the other hand we have (remember (1.34))
\[
J_n(Z) = -Z^{-1} ,
\]
\[
(\det(J_n \{ \hat{Z} \}))^{-2} = (\det(\hat{Z}))^{-2} = (\det(Z))^{-4} ,
\]
\[
J_n \ast (P, Q) = (-S^{-1}Q, SP) ,
\]
\[
\kappa(J_n; (P, Q)) = e^{2\pi i r(S[P],0) + \pi i r(S^{-1}[Q],0) - 2\pi i r(T^Q,0)} = e^{2\pi i r(P,Q)} .
\]

This completes the proof. \( \square \)
2.3 Quaternionic theta-series of the second kind

The quaternionic theta-series of the second kind being quaternionic modular forms is just an easy consequence of the preceding lemma.

(2.35) Theorem. Given $L \in \mathcal{O}^{2 \times 2}$ we have

$$\Theta_L(Z; S_\mathcal{O}) \in [\text{Sp}_2(\mathcal{O})[p], 4, \nu_L],$$

where for $M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{Sp}_2(\mathcal{O})[p]$ the abelian character is given by

$$\nu_L(M) = e^{\frac{1}{2}\pi i \tau(S_\mathcal{O} | \mathcal{I} ||i_1,l,\mathcal{I}^B)}.$$

**Proof:** Let $P = -\mathcal{I} \sqrt{3}$, then according to (2.28)

$$\Theta_L(Z; S_\mathcal{O}) = \Theta_{P,0}(Z, S_\mathcal{O}; \mathcal{O}^{2 \times 2})$$

holds. So let $M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{Sp}_2(\mathcal{O})[p]$, then in view of (2.34) we get

$$\Theta_L(Z; S_\mathcal{O}) = (\det(M \{ Z \}))^{-2} \kappa(M; (P, 0)) \cdot \Theta_{M, \mathcal{O}^2}(M \langle Z \rangle, S; \mathcal{O}^{2 \times 2}).$$

We have

$$M \ast (P, 0) = (P^\dagger, -S_\mathcal{O} \cdot P^\dagger)$$

and

$$\kappa(M; (P, 0)) = e^{\pi i \tau(S_\mathcal{O} | P, \mathcal{I}^B) + \pi i \tau(0, \mathcal{I}^B) - 2 \pi i \tau(0, \mathcal{I}^C)} = e^{\pi i \tau(S_\mathcal{O} | P, \mathcal{I}^B)} = e^{\frac{1}{2}\pi i \tau(S_\mathcal{O} | \mathcal{I} ||i_1,l,\mathcal{I}^B)}.$$ 

$D \equiv I$ modulo $p$ means there exists $H \in \mathcal{O}^{2 \times 2} = i_1 \sqrt{3} \mathcal{O}^{2 \times 2}$ with $D = I + H$. $\mathcal{H}^\prime \in i_1 \sqrt{3} \mathcal{O}^{2 \times 2}$ and $P \in \frac{i_1}{\sqrt{3}} \mathcal{O}^{2 \times 2}$ implies $P \mathcal{H}^\prime \in \mathcal{O}^{2 \times 2}$ (because $i_1 \mathcal{O} = \mathcal{O} i_1$), and thus in virtue of (2.3)

$$\Theta_{M, \mathcal{O}^2}(M \langle Z \rangle, S_\mathcal{O}; \mathcal{O}^{2 \times 2}) = \Theta_{P + \mathcal{H}, -S_\mathcal{O} \cdot \mathcal{P}^\dagger}(M \langle Z \rangle, S_\mathcal{O}; \mathcal{O}^{2 \times 2})$$

follows. So all that remains to be done is proving

$$\Theta_{P, -S_\mathcal{O} \cdot \mathcal{P}^\dagger}(M \langle Z \rangle, S_\mathcal{O}; \mathcal{O}^{2 \times 2}) = e^{-\frac{1}{2}\pi i \tau(S_\mathcal{O} | \mathcal{I} ||i_1,l,\mathcal{I}^B)} \cdot \Theta_{P,0}(M \langle Z \rangle, S_\mathcal{O}; \mathcal{O}^{2 \times 2}).$$

We have

$$\Theta_{P, -S_\mathcal{O} \cdot \mathcal{P}^\dagger}(M \langle Z \rangle, S_\mathcal{O}; \mathcal{O}^{2 \times 2}) = \sum_{G \in \Lambda} e^{\pi i \tau(S(G + P, M \langle Z \rangle)) + 2 \pi i \tau(-S_\mathcal{O} \cdot \mathcal{P}^\dagger, G + P)}$$

$$= \sum_{G \in \Lambda} e^{\pi i \tau(S(G + P, M \langle Z \rangle))} e^{2 \pi i \tau(-S_\mathcal{O} \cdot \mathcal{P}^\dagger, G)} e^{2 \pi i \tau(-S_\mathcal{O} \cdot \mathcal{P}^\dagger, P)}.$$ 

$\tau(-S_\mathcal{O} \cdot \mathcal{P}^\dagger, G) = -\tau(S_\mathcal{O}, GB^\dagger) \$ holds in view of (1.46). $B \in i_1 \sqrt{3} \mathcal{O}^{2 \times 2}$ and $P^\dagger \in \frac{i_1}{\sqrt{3}} \mathcal{O}^{2 \times 2}$ imply $GB^\dagger \in \mathcal{O}^{2 \times 2}$, again. But then $S_\mathcal{O} \in 2 \text{Her}_2(\mathcal{O})$ leads to $\tau(S_\mathcal{O}, GB^\dagger) \in 2\mathbb{Z}$ (see (1.47) and (2.6),
as \( S_O \) is \( O \)-stable). Hence

\[
\Theta_{P,-S_OP}(M(Z), S_O; O^{2 \times 2}) = e^{2\pi i\tau(-S_O, B|\bar{T}'|)} \cdot \Theta_{P,0}(M(Z), S_O; O^{2 \times 2})
\]

follows. Recall that \( D = I + H \), and thus \( \bar{T}' B = B + H B \), where \( H B \in 3O^{2 \times 2} \). This implies \((\bar{T}' B)|\bar{T}'| = B|\bar{T}'| + (H B)|\bar{T}'| \) with \((H B)|\bar{T}'| \in O^{2 \times 2} \). Now, by definition \( S_O \in 2\text{Her}_2^+(O) \) leads to \( S_O \in O^{2 \times 2} \) (see (1.47)). But then the formula for the trace form and the definition of \( O^i \) implicate \( \tau(S_O, (H B)|\bar{T}'|) \in \mathbb{Z} \). Thus we have shown

\[
e^{2\pi i\tau(-S_O, B|\bar{T}'|)} = e^{2\pi i\tau(-S_O, B|\bar{T}'|)} = e^{2\pi i\tau(-S_O|I|,|I|, B)} \]

So we have verified

\[
\Theta_L(Z; S_O) = (\det(M\{\tilde{Z}\}))^{-2} \cdot \Theta_{M\ast(P,0)}(M(Z), S; O^{2 \times 2})
\]

\[
e^{-\frac{1}{2} \pi i\tau(S_O|I|,|I|, B)}(\det(M\{\tilde{Z}\}))^{-2} \cdot \Theta_{P,0}(M(Z), S; O^{2 \times 2})
\]

\[
e^{-\frac{1}{2} \pi i\tau(S_O|I|,|I|, B)}(\det(M\{\tilde{Z}\}))^{-2} \cdot \Theta_L(M(Z); S_O)
\]

or in other words

\[
\Theta_L(Z; S_O)|_4 M = e^{\frac{1}{2} \pi i\tau(S_O|I|,|I|, B)} \cdot \Theta_L(Z; S_O)
\]

for all \( M = (A B) \in \text{Sp}_2(O)|p| \). So according to (1.52), \( \nu_L(M) := e^{\frac{1}{2} \pi i\tau(S_O|I|,|I|, B)} \) turns out to be a multiplier system for \( \text{Sp}_2(O)|p| \) of weight four (and hence an abelian character as \( 4 \in 2\mathbb{Z} \)), and \( \Theta_L(Z; S_O) \in [\text{Sp}_2(O)|p|, 4, \nu_L] \). \( \square \)

Unfortunately, looking at quaternionic theta-series of the first kind (with respect to \( S_O \)) like it was done in [Kr10] yields no further insights. It turns out – which will not be proven here – that those theta-series \( \Theta_{S_O^{-1}C}(Z; S; O^{2 \times 2}) \) are quaternionic modular forms of weight four with respect to \( \text{Sp}_2(O)|2O| \), not \( \text{Sp}_2(O)|p| \), with a certain character (given by \( \nu(M) = e^{\pi i\tau(S_O^3|B|)+\pi i\tau(S_O^3|C|)} \)). So an appropriate discussion (including the behavior under \( Z \mapsto Z' \)) seems infeasible and also unnecessary since these theta-series are quaternionic modular forms for the “wrong” principal subgroup.
3 Quaternionic Maaß Lifts

In chapter 2 a first important example of non-trivial quaternionic modular forms was given in terms of quaternionic theta-series. In the current chapter we will define and have a closer look at so-called quaternionic Maaß lifts: Special elliptic modular forms (i.e. modular forms for the complex upper half-plane $\mathcal{H} = \mathcal{H}_1(\mathbb{R})$ and subgroups of the special linear group $\text{SL}_2(\mathbb{Z}) = \text{Sp}_1(\mathbb{Z})$), or in general vector-valued modular forms, are lifted to quaternionic modular forms. The general technique is well known, but determining the exact shape of the lift as well as answering the question which elliptic modular forms are needed and how the corresponding space can be explicitly described is quite involving. The advantage of Maaß lifts is that elliptic modular forms are, to a certain degree, very well known (cf. [KK07] and [Mi89]). So, one can gain a lot of information about the lifts. What this means in detail will be shown in chapter 4 which is about Eisenstein-series.

We will start by presenting the already well-known Maaß spaces for even weight and the trivial character, which were determined in [Kl98]. On the one hand, this will just serve as an introduction to quaternionic Maaß lifts. On the other hand, we will need them later on when investigating Eisenstein-series.

We will then answer a question left open in the first chapter: If there exists a quaternionic modular form for the whole quaternionic modular group of odd weight, then the corresponding multiplier system must be given by $\nu_i$ or $\nu_{-i} = \nu_3^i = \overline{\nu}$ (cf. (1.56) and (1.57)). But it was not clear if these two functions are indeed multiplier systems for the whole quaternionic modular group. They are – a priori – only given on the generators of $\text{Sp}_2(\mathcal{O})$. As we have seen in the first chapter, because of the cocycle relations a multiplier system for a subgroup $\Gamma \leq \text{Sp}_2(\mathcal{O})$ is uniquely determined by its values on a set of generators of $\Gamma$. But of course it is not clear whether a function given on the generators can be expanded to the whole group in a well-defined way (via 1.16) to actually become a multiplier system. Thus the question whether non-trivial quaternionic modular forms for the whole quaternionic modular group of odd weight exist or not is open, too. We will answer both questions in the second section with the help of quaternionic Maaß lifts of odd weight. According to (1.52) it will suffice to prove that these constructed lifts are non-identically vanishing, holomorphic and behave accordingly under modular transformations for some set of generators.

Finally, in the third section we will take a closer look at the elliptic modular forms needed as inputs for the quaternionic Maaß lifts of odd weight. We will show that those (hypothetically existent) elliptic modular forms with special properties determined in the second section actually do exist. Furthermore, we will see how they can be constructed explicitly and we will determine the dimensions of the spaces of these special elliptic modular forms, which also yields the dimensions of the spaces of the lifts. Doing so, we have to introduce the theory of newforms and the theory of Hecke-operators for elliptic modular forms, too. And a final application will be that with the help of Maaß lifts we can specify the spaces of quaternionic modular forms with respect to the $\text{Sp}_2(\mathcal{O})$ for small weights explicitly.
3.1 Maaß spaces for the trivial character

As announced, this first section about quaternionic Maaß lifts is a mere summary of the results from [Kl98] about Maaß spaces for the trivial character (and thus for even weights), plus some basic definitions and notations. It serves as a first introduction to the topic. But we will also need the results, later on. Again, we will only look at the case of degree two, so in the whole chapter speaking of quaternionic modular forms will always imply that they are of degree two.

We start with the definition of the Fourier-Jacobi-decomposition. Recall that a quaternionic modular form with respect to a congruence subgroup always possesses an absolutely and locally uniformly convergent Fourier-expansion, where the sum ranges over all \( T \in \frac{1}{l} \text{Her}_2(O), \ T \geq 0 \) for some \( l \in \mathbb{N} \) (see the beginning of section 1.4 and the observations around (1.54)). The absolute and local uniform convergence allows for a simple rearrangement of the infinite sum:

(3.1) Definition. Let \( \Gamma \leq \text{Sp}_2(O) \) a congruence subgroup, \( k \in \mathbb{Q} \) and \( \nu \) a multiplier system for \( \Gamma \) of weight \( k \). Suppose \( f \in [\Gamma, k, \nu] \) possesses the Fourier-expansion

\[
f(Z) = \sum_{T \in \frac{1}{l} \text{Her}_2(O), \ T \geq 0} \alpha_f(T) e^{2\pi i (T, Z)}
\]

for some \( l \in \mathbb{N} \) (where we defined \( \text{Re}(Tz) = \tau(t, z) \) – see 1.7 and 1.21). Then the Fourier-Jacobi-decomposition is given by

\[
f(Z) = \sum_{m \in \frac{1}{l} \mathbb{N}_0} \varphi_{f, m}(\tau, z) e^{2\pi i m \omega}, \quad Z = \left( \frac{\tau}{z} \omega \right) \in \mathcal{H}(\mathbb{H}),
\]

where

\[
\varphi_{f, m}(\tau, z) = \sum_{n \in \frac{1}{l} \mathbb{N}_0, t \in \frac{1}{l} O^\circ, \ nm \geq N(t)} \alpha_f(n \ t \ m) e^{2\pi i (n \tau + m \omega + 2 \text{Re}(Tz))}, \quad \tau \in \mathcal{H}, \ z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}.
\]

\( \varphi_{f, m} \) is called the \( m \)-th Fourier-Jacobi-coefficient of \( f \), or Fourier-Jacobi-coefficient of index \( m \).

For \( \Gamma = \text{Sp}_2(O) \), the first Fourier-Jacobi-coefficient will play the most important role with regard to quaternionic Maaß lifts. But also the Fourier-Jacobi-coefficient of index zero plays a special role:

(3.2) Remark. By definition and by what we saw in the proof of (1.59),

\[
\varphi_{f, 0}(\tau, z) = \sum_{n \in \frac{1}{l} \mathbb{N}_0} \alpha_f(0) e^{2\pi i n \tau} = f|\Phi(\tau), \quad \tau \in \mathcal{H}, \ z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}.
\]

holds for \( f \in [\Gamma, k, \nu] \). In particular, we have \( \varphi_{f, 0}(\cdot, z) = \varphi_{f, 0}(\cdot, 0) \in [\tilde{\Gamma}, k, \tilde{\nu}] \) for all \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \) for some congruence subgroup \( \tilde{\Gamma} \leq \text{SL}_2(\mathbb{Z}) \) and some multiplier system \( \tilde{\nu} \).
The following definitions, propositions and theorems are a summary of the results in [Kl98].

(3.3) Lemma. A transversal of $O^5/2O$ is given by

$$h_{3k+j} = k(1 + \frac{j}{\sqrt{3}}) + j(1 + \frac{2}{\sqrt{3}})i, \quad 0 \leq j, k \leq 2.$$ 

For $j \in \{0, \ldots, 8\}$ we define theta-series

$$\theta_j(\tau, z) := \sum_{g \in O} e^{2\pi i (N(g + \frac{1}{2}h))\tau + 2Re(z(g + \frac{1}{2}h)))}, \quad \tau \in \mathcal{H}, \ z \in \mathbb{H} \otimes \mathbb{C}.$$ 

(3.4) Proposition. Let $k \in 2\mathbb{Z}$ be even and $f \in [\text{Sp}_2(O), k, 1]$ with Fourier-Jacobi-decomposition

$$f(Z) = \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z)e^{2\pi im\omega}, \ Z = \begin{pmatrix} \frac{1}{2} & z \end{pmatrix} \in \mathcal{H}(\mathbb{H}).$$

Then the first Fourier-Jacobi-coefficient is given by

$$\varphi_{f,1}(\tau, z) = \sum_{j=0}^{8} f_j(\tau)\theta_j(\tau, z),$$

where

$$f_0(\tau) = \sum_{l \in \mathbb{N}_0} \alpha_f(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) e^{2\pi i l\tau}, \quad \tau \in \mathcal{H},$$

$$f_1(\tau) = f_2(\tau) = f_3(\tau) = f_6(\tau) = \sum_{l \in \mathbb{N}_0} \alpha_f(\begin{pmatrix} l+1 & \frac{1}{2}h_l \\\ \frac{1}{2} & 1 \end{pmatrix}) e^{\pi i l\tau}, \quad \tau \in \mathcal{H},$$

$$f_4(\tau) = f_5(\tau) = f_7(\tau) = f_8(\tau) = \sum_{l \in \mathbb{N}_0} \alpha_f(\begin{pmatrix} l+1 & \frac{1}{2}(h_l+h_3) \\\ \frac{1}{2} & 1 \end{pmatrix}) e^{2\pi i l\tau}, \quad \tau \in \mathcal{H}.$$ 

Furthermore, defining $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho = e^{2\pi i}$ the following holds:

- $f_0 \in [\text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0, k-2, 1], f_1, f_4 \in [\text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0, k-2, 1]$ 
- $f_1 = -\frac{1}{4}(f_0|_{k-2} + \rho f_0|_{k-2}T_1 + \rho^2 f_0|_{k-2}T_1^2)$ 
- $f_4 = -\frac{1}{4}(f_0|_{k-2} + \rho^2 f_0|_{k-2}T_1 + \rho f_0|_{k-2}T_1^2)$ 
- $$\sum_{M \in \text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0 \setminus \text{SL}_2(\mathbb{Z})} f_0|_{k-2}M = f_0 + f_0|_{k-2} + f_0|_{k-2}T_1 + f_0|_{k-2}T_1^2 \equiv 0.$$ 

In particular, the first Fourier-Jacobi-coefficient is uniquely determined by $f_0$.

Define

$$\pi_{tr} : [\text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0, k-2, 1] \to [\text{SL}_2(\mathbb{Z}), k-2, 1], \quad g \mapsto \pi_{tr}(g) := \sum_{M \in \text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0 \setminus \text{SL}_2(\mathbb{Z})} g|_{k-2}M.$$ 

Then (3.4) yields

$$f_0 \in [\text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0, k-2, 1]_{tr} := \ker(\pi_{tr}).$$
(3.5) Lemma. $\dim [\text{SL}_2(Z) \, |3Z|_0, k-2, 1]_{\text{tr}} = \left[ \frac{k}{4} \right]$ holds for all even $k \geq 4$.

(3.6) Definition. Let $k \in 2\mathbb{Z}$ be even and $f \in [\text{Sp}_2(\mathcal{O}), k, 1]$ with Fourier-expansion

$$f(Z) = \sum_{T \in \text{Her}_n(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i r(T,Z)}, \quad Z \in \mathcal{H}_n(\mathbb{H}).$$

Then $f$ is said to be a quaternionic Maaß form of weight $k$ with trivial character if and only if the Fourier-coefficients satisfy the so-called Maaß-condition

$$\alpha_f(T) = \sum_{d \in \mathbb{N}, d \mid (T)} d^{k-1} \alpha_f(\frac{1}{d} \frac{t/d}{m_n/d})$$

for all $0 \neq T = (\frac{n}{t}, \frac{l}{m}) \in \text{Her}_2^1(\mathcal{O}), T \geq 0$, where again

$$\epsilon(T) := \max\{d \in \mathbb{N}; \frac{1}{d} T \in \text{Her}_2^1(\mathcal{O})\}.$$ 

This is equivalent so the existence of a so-called Maaß-function $\alpha_f^* : \mathbb{N}_0 \to \mathbb{C}$ such that

$$\alpha_f(T) = \sum_{d \in \mathbb{N}, d \mid (T)} d^{k-1} \alpha_f^*(3 \det(T)/d^2)$$

holds for all $0 \neq T \in \text{Her}_2^1(\mathcal{O}), T \geq 0$.

The vector space of all quaternionic Maaß forms of even weight $k$ with trivial character is denoted by $\mathcal{M}(k; \mathcal{O})$.

According to [Kl98, Kor.4.3] we have

$$\mathcal{M}(k; \mathcal{O}) = [\text{Sp}_2(\mathcal{O}), k, 1] = \{0\} \quad \text{for all even } k < 0$$

as well as

$$\mathcal{M}(0; \mathcal{O}) = [\text{Sp}_2(\mathcal{O}), 0, 1] = \mathbb{C}.$$ 

Now, we present the final result about quaternionic Maaß forms for the trivial character, which means we define the Maaß lift for the case of even weight and trivial character:

(3.7) Theorem. Let $k \in 2\mathbb{Z}, k \geq 4$. Then the mapping

$$\mathcal{M}(k; \mathcal{O}) \to [\text{SL}_2(Z) \, |3Z|_0, k-2, 1]_{\text{tr}} \quad f \mapsto f_0$$

with $f_0$ defined in (3.4) is an isomorphism. In particular $\dim \mathcal{M}(k; \mathcal{O}) = \left[ \frac{k}{4} \right]$. The inverse mapping, i.e. the so-called quaternionic Maaß lift for the case of even weight and trivial character is given by the following:

For $f_0 \in [\text{SL}_2(Z) \, |3Z|_0, k-2, 1]_{\text{tr}}$ define

$$f_1 := -\frac{1}{3} (f_0|_{k-2} f_1 + \rho f_0|_{k-2} f_1 T_1 + \rho^2 f_0|_{k-2} f_1 T_1^2) \in [\text{SL}_2(Z) \, |3Z|_0, k-2, 1],$$

$$f_4 := -\frac{1}{3} (f_0|_{k-2} f_4 + \rho^2 f_0|_{k-2} f_4 T_1 + \rho f_0|_{k-2} f_4 T_1^2) \in [\text{SL}_2(Z) \, |3Z|_0, k-2, 1].$$
The Fourier-expansions are given by

\[ f_0(\tau) = \sum_{n \in \mathbb{N}_0} \beta(3n)e^{2\pi i n\tau}, \]
\[ f_1(\tau) = \sum_{n \in \mathbb{N}_0} \gamma(3n + 2)e^{\pi i (2n + \frac{1}{3})\tau}, \]
\[ f_4(\tau) = \sum_{n \in \mathbb{N}_0} \delta(3n + 1)e^{\pi i (2n + \frac{1}{3})\tau} \]

for \( \tau \in \mathcal{H} \) and appropriate maps \( \beta, \gamma, \delta \). Define \( \alpha^* : \mathbb{N}_0 \rightarrow \mathbb{C} \) by

\[ \alpha^*(n) = \begin{cases} 
\beta(n), & \text{if } n \equiv 0 \pmod{3}, \\
\delta(n), & \text{if } n \equiv 1 \pmod{3}, \\
\gamma(n), & \text{if } n \equiv 2 \pmod{3}.
\end{cases} \]

Then the quaternionic Maaß lift \( M_{f_0} \), i.e. the inverse mapping, is given by

\[ M_{f_0}(Z) = -\frac{B_k}{2k} \alpha^*(0) + \sum_{0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0} \left( \sum_{d \in \mathbb{N}, d \mid \det(T)} d^{k-1} \alpha^*(3 \det(T)/d^2) \right) e^{2\pi i T(Z)} \]

for \( Z \in \mathcal{H}_n(\mathbb{H}) \), where \( B_k \) denotes the \( k \)-th Bernoulli number.

Later on, when studying Eisenstein-series, we will determine the spaces \([\text{SL}_2(\mathbb{Z})[3\mathbb{Z}]_{0, k, 1}]_{tr}\) more explicitly and in greater detail. This plus detailed information about Hecke-operators will yield the Fourier-expansions of the Eisenstein-series.

We will get related results to the ones from above in the next section when examining quaternionic Maaß lifts of odd weight. But although the methods used will be similar, the results will differ a lot, especially concerning the underlying special spaces of elliptic modular forms being lifted.

### 3.2 Quaternionic Maaß lifts of odd weight

We will now construct quaternionic Maaß lifts of odd weight, step by step. We will begin by assuming that the two hypothetically possible multiplier systems for \( \text{Sp}_2(\mathcal{O}) \) of odd weight, i.e. \( \nu_i \) and \( \nu_{-i} = \nu_i^3 = i_\mathcal{U} \) (see (1.57)), actually are multiplier systems and that there exists some (non-trivial) quaternionic modular form for the whole group with respect to one of these multiplier systems. We will then decompose such a (hypothetical) quaternionic modular form in a similar way like it was done in the first section. This will result in quite special elliptic modular forms. That such elliptic modular forms actually do exist non-trivially will be proven in the next section. We will then take these elliptic modular forms and lift them – via another Maaß lift – to non-identically vanishing quaternionic modular forms. Ultimately, this will prove that \( \nu_i \) and \( \nu_{-i} \) are multiplier systems, indeed, and that there actually exist non-trivial quaternionic modular forms with respect to the full modular group of odd weight. Note that this is quite special, indeed, since in the case of the Hurwitz order, there do not exist any multiplier systems for the whole modular group of odd weight, and therewith there exist no quaternionic modular forms.
forms with respect to the full modular group of odd weight (cf. [KW98]).

We start with \( v_i \). The calculations for \( v_{-i} \) are completely analogous. So let us assume that \( v_i \) is a multiplier system, indeed. For generators of \( \text{Sp}_2(\mathcal{O}) \) (see (1.15) and (1.17)), \( v_i \) is given by

\[
\begin{align*}
v_i(f) &= -1, \\
v_i(\text{Trans}(S)) &= 1, \\
v_i(\text{Rot}(U)) &= \varphi_1 \circ \det \circ \pi_p(U),
\end{align*}
\]

where \( S \in \text{Her}_2(\mathcal{O}) \), \( U \in \text{GL}_2(\mathcal{O}) \). Or for the finite set of generators in (1.17):

\[
\begin{align*}
v_i(f) &= -1, \\
v_i(\text{Trans}(S)) &= 1, \\
v_i \left( \text{Rot} \left( \frac{-1+i\sqrt{3}}{2} \right) \right) &= 1, \\
v_i \left( \text{Rot} \left( \frac{i}{2} \right) \right) &= i,
\end{align*}
\]

where \( S \in \text{Her}_2(\mathcal{O}) \). Thus, because of 1.15 (and concerning the last equation because of the proof of (1.56)), \( f \in [\text{Sp}_2(\mathcal{O}), k, v_i] \), where \( k \in \mathbb{N} \) is odd, holds if and only if

\[
\begin{align*}
f(Z + S) &= f(Z), \\
f(Z[U]) &= \varphi_1 \circ \det \circ \pi_p(U) \cdot f(Z), \\
f(-Z^{-1}) &= -(\det(\tilde{Z}))^{k/2} f(Z) = (\det(Z))^k f(Z)
\end{align*}
\]

for all \( Z \in \mathcal{H}(\mathbb{H}) \), \( S \in \text{Her}_2(\mathcal{O}) \), \( U \in \text{GL}_2(\mathcal{O}) \), or again for the finite set of generators:

\[
\begin{align*}
f(Z + S) &= f(Z), \\
f \left( Z \left[ \left( \begin{array}{cc} -1+i\sqrt{3} & 0 \\ 2 & 1 \end{array} \right) \right] \right) &= f(Z), \\
f \left( Z \left[ \left( \begin{array}{cc} i & 0 \\ 0 & 1 \end{array} \right) \right] \right) &= if(Z), \\
f(-Z^{-1}) &= -(\det(\tilde{Z}))^{k/2} f(Z) = (\det(Z))^k f(Z)
\end{align*}
\]

for all \( Z \in \mathcal{H}(\mathbb{H}) \) and \( S \in \text{Her}_2(\mathcal{O}) \).

Since \( f(Z + S) = f(Z) \) holds for all \( S \in \text{Her}_2(\mathcal{O}) \), \( f \) possesses the Fourier-expansion

\[
f(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi iT,Z}, \quad Z \in \mathcal{H}_m(\mathbb{H})
\]

in virtue of (1.54). Therefore, just as in (3.1), \( f \) possesses the Fourier-Jacobi-decomposition

\[
f(Z) = \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z) e^{2\pi i m \omega}, \quad Z = (\frac{\tau}{\omega}, \frac{z}{\omega}) \in \mathcal{H}(\mathbb{H}),
\]

where

\[
\varphi_{f,m}(\tau, z) = \sum_{n \in \mathbb{N}_0, l \in \mathbf{O}^f \cap \mathbb{N}(t)} \alpha_f \left( \begin{array}{cc} n & 1 \\ l & m \end{array} \right) e^{2\pi i (n \tau + 2 \text{Re}(l \tau))}, \quad \tau \in \mathcal{H}, \ z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}.
\]
3.2 Quaternionic Maass lifts of odd weight

Like in (3.4), we have to decompose the first Fourier-Jacobi-coefficient \( \varphi_{f,1} \). But first, we will have a closer look at the transformation behavior of the Fourier-Jacobi-coefficients.

According to 3.1, \( f(Z + (s_1, s_2, 0)) = f(Z) \) holds for all \( Z \in \mathcal{H}(\mathbb{H}) \), \( s_1 \in \mathbb{Z} \) and \( s_2 \in \mathcal{O} \). Thus we calculate

\[
\sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z)e^{2\pi im\omega} = f(Z) = f(Z + (s_1, s_2, 0)) = \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau + s_1, z + s_2)e^{2\pi im\omega}
\]

for all \( Z = \left( \frac{z}{\omega}, \omega \right) \in \mathcal{H}(\mathbb{H}) \). Note that for fixed \( \tau \) and \( z \), \( f \) is also a periodic function in \( \omega \) with its Fourier-expansion given by the Fourier-Jacobi-decomposition, where the Fourier-coefficients are exactly the Fourier-Jacobi-coefficients (which are constant for fixed \( \tau \) and \( z \)). Thus the uniqueness of the Fourier-coefficients of a Fourier-expansion yields

\[
\varphi_{f,m}(\tau, z) = \varphi_{f,m}(\tau + s_1, z + s_2)
\]  

for all \( m \in \mathbb{N}_0, \tau \in \mathcal{H}, z \in \mathbb{H} \oplus \mathbb{R} C \), \( s_1 \in \mathbb{Z}, s_2 \in \mathcal{O} \).

Furthermore, \( f(Z \left[ \left( \frac{1}{0} \right) \right]) = f(Z) \) holds for all \( Z \in \mathcal{H}(\mathbb{H}) \) and \( r \in \mathcal{O} \) in view of 3.1. A short calculation yields

\[
\left( \begin{array}{c} \tau \\ z \\ \omega \end{array} \right) \left[ \left( \begin{array}{c} 1 \\ r \\ 0 \\ 1 \end{array} \right) \right] = \left( \begin{array}{c} \tau \\ \tau r + z \\ \tau r + z + \tau N(r) + 2 \text{Re}(\tau z) + \omega \end{array} \right)
\]

and thus for \( Z = \left( \frac{z}{\omega}, \omega \right) \) we compute

\[
\sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z)e^{2\pi im\omega} = f(Z) = f(Z \left[ \left( \frac{1}{0} \right) \right])
\]

\[
= \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z + \tau r)e^{2\pi im(\tau N(r) + 2 \text{Re}(\tau z) + \omega)}
\]

\[
= \sum_{m \in \mathbb{N}_0} \left( e^{2\pi im(\tau N(r) + 2 \text{Re}(\tau z))} \varphi_{f,m}(\tau, z + \tau r) \right)e^{2\pi im\omega}
\]

Again, the uniqueness of the Fourier-coefficients yields

\[
\varphi_{f,m}(\tau, z) = e^{2\pi im(\tau N(r) + 2 \text{Re}(\tau z))} \varphi_{f,m}(\tau, z + \tau r)
\]  

for all \( m \in \mathbb{N}_0, \tau \in \mathcal{H}, z \in \mathbb{H} \oplus \mathbb{R} C \) and \( r \in \mathcal{O} \). Moreover, as we have seen above, the Fourier-Jacobi-coefficients \( \varphi_{f,m} \) are periodic in both arguments (with periods \( Z \) and \( \mathcal{O} \) resp.). Their absolutely and locally uniformly convergent Fourier-expansions are already uniquely given by

\[
\varphi_{f,m}(\tau, z) = \sum_{\substack{n \in \mathbb{N}_0, t \in \frac{1}{2} \mathcal{O}^\circ \mathcal{O} \setminus \mathbb{N}(t) \mathbb{Z} \subset \mathbb{N}(t) \geq N(t)}} \alpha_f \left( \frac{n}{t} \right) \cdot e^{2\pi in(\tau t + 2 \text{Re}(\tau z))}
\]

\[
= \sum_{\substack{n \in \mathbb{N}_0, t \in \frac{1}{2} \mathcal{O}^\circ \mathcal{O} \setminus \mathbb{N}(t) \mathbb{Z} \subset \mathbb{N}(t) \geq N(t)}} \alpha_f(n, t) \cdot e^{2\pi in(\tau N(t) + \text{Re}(\tau z))}, \quad \tau \in \mathcal{H}, z \in \mathbb{H} \oplus \mathbb{R} C,
\]

where

\[
\alpha_f(n, t) = \alpha_f \left( \frac{n}{t} \right).
\]
Also note that the Fourier-Jacobi-coefficients are thus holomorphic in both arguments. We have

\[ \psi_{f,m}(\tau, z) = \sum_{n \in \mathbb{N}_0 \cup \mathbb{O}^2, \tau \in \mathbb{O}^2} \alpha_{f,m}(n, t) \cdot e^{2\pi i (n \tau + \text{Re}(\tau z))} = e^{2\pi i \text{Im}(\tau N(r) + 2 \text{Re}(\tau z))} \psi_{f,m}(\tau, z + \tau r) \]

\[ = \sum_{n \in \mathbb{N}_0 \cup \mathbb{O}^2, \tau \in \mathbb{O}^2} e^{2\pi i \text{Im}(\tau N(r) + 2 \text{Re}(\tau z))} \alpha_{f,m}(n, t) \cdot e^{2\pi i (n \tau + \text{Re}(\tau (\tau + z)))} \]

\[ = \sum_{n \in \mathbb{N}_0 \cup \mathbb{O}^2, \tau \in \mathbb{O}^2} \alpha_{f,m}(n, t) \cdot e^{2\pi i ((n + \text{Re}(\tau r)) + m N(r)) \tau + \text{Re}(2n\tau + \tau z))} \]

\[ = \sum_{n \in \mathbb{N}_0 \cup \mathbb{O}^2, \tau \in \mathbb{O}^2} \alpha_{f,m}(n - \text{Re}(\tau r) - m N(r), t - 2mr) \cdot e^{2\pi i (n \tau + \text{Re}(\tau z))} = \sum_{n \in \mathbb{N}_0 \cup \mathbb{O}^2, \tau \in \mathbb{O}^2} \alpha_{f,m}(n - \text{Re}(\tau r) - m N(r), t - 2mr) \cdot e^{2\pi i (n \tau + \text{Re}(\tau z))} , \]

where the last step follows because of the uniqueness of Fourier-expansions. And once more because of the uniqueness of the Fourier-expansion we have therefore shown (by substituting \( t \) by \( t + 2mr \))

\[ \alpha_{f,m}(n, t + 2mr) = \alpha_{f,m}(n - \text{Re}(\tau r) - m N(r), t) \quad (3.6) \]

for all \( m \in \mathbb{N}_0, n \in \mathbb{N}_0, t \in \mathbb{O}^2 \) and \( r \in \mathbb{O} \) (s.t. \( 4nm \geq N(t + 2mr) \)). With the help of this identity we are going to decompose the Fourier-Jacobi-coefficients. Defining

\[ \psi_{f,m,t}(\tau) := \sum_{n \in \mathbb{N}_0, 4nm \geq N(t)} \alpha_{f,m}(n, t)e^{2\pi in \tau} , \quad \tau \in \mathcal{H} \]

yields

\[ \psi_{f,m}(\tau, z) = \sum_{\tau \in \mathbb{O}^2} \psi_{f,m,t}(\tau) \cdot e^{2\pi i \text{Re}(\tau z)} , \quad \tau \in \mathcal{H}, z \in \mathbb{H} \otimes \mathbb{R} C \, . \]

Note that because \( \psi_{f,m,t} \) is an absolutely and locally uniformly convergent Fourier-series, ranging only over \( n \in \mathbb{N}_0 \), \( \psi_{f,m,t} \) is holomorphic on \( \mathcal{H} \) and bounded on \( \{ \tau \in \mathbb{C} ; \, \text{Im}(z) \geq \beta \} \) for any \( \beta > 0 \). Furthermore, 3.4 leads to

\[ \sum_{t \in \mathbb{O}^2} \psi_{f,m,t}(\tau) \cdot e^{2\pi i \text{Re}(\tau z)} = \psi_{f,m}(\tau, z) = e^{2\pi i \text{Im}(\tau N(r) + 2 \text{Re}(\tau z))} \psi_{f,m}(\tau, z + \tau r) \]

\[ = \sum_{t \in \mathbb{O}^2} e^{2\pi i \text{Im}(\tau N(r) + 2 \text{Re}(\tau z))} \psi_{f,m,t}(\tau) \cdot e^{2\pi i \text{Re}(\tau (\tau + z)))} \]

\[ = \sum_{t \in \mathbb{O}^2} e^{2\pi i \text{Re}(\tau r) + m N(r)) \tau} \psi_{f,m,t}(\tau) \cdot e^{2\pi i \text{Re}(\tau + 2mr) z)} \]

\[ = \sum_{t \in \mathbb{O}^2} \left( e^{2\pi i (\text{Re}(\tau r) + m N(r)) \tau} \psi_{f,m,t-2mr}(\tau) \right) \cdot e^{2\pi i \text{Re}(\tau z)} \]
for all \( m \in \mathbb{N}_0, n \in \mathbb{N}_0, t \in O^i \) and \( r \in O \). Again, for fixed \( \tau \), the unique coefficients of the Fourier-expansions have to coincide, and thus (by substituting \( t \) with \( t + 2mr \))

\[
\psi_{f,m,t+2rm}(\tau) = e^{2\pi i (\text{Re}(\tau)r + m N(r))} \psi_{f,m,t}(\tau)
\]

(3.7)

follows for all \( m \in \mathbb{N}_0, \tau \in \mathcal{H}, t \in O^i \) and \( r \in O \). So we can choose \( t \in O^i \) modulo \( 2mO \), which means the following: We have \( O^i = \frac{2}{3}i_1 \sqrt{3}O \supset \frac{2}{3}i_1 \sqrt{3}(-i_1 \sqrt{3} O) = 2O \), and hence \( 2mO \) is a two-sided ideal of the (additive) group \( O \) for all \( m \in \mathbb{N} \). So every \( s \in O^i \) can be written as \( s = t + 2mr \) where \( t \in O^i / 2mO, r \in O \), and even uniquely if we fix a transversal of \( O^i / 2mO \). So assume we fixed such a transversal. Summing over this (finite) transversal shall be denoted by \( "t : O^i / 2mO" \). With the help of 3.7 we compute (while substituting \( \tau \) with \( \omega \) for the moment) to avoid confusion with the trace form \( \tau (\cdot, \cdot) \), whereat one should note \( \tau (z_1, z_2) = \text{Re}(z_1 z_2) \) for \( z_1, z_2 \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \) as well as \( \tau (x_1, x_2) = x_1 x_2 \) for \( x_1, x_2 \in \mathbb{R} \otimes \mathbb{R} \mathbb{C} \).

\[
\varphi_{f,m}(\omega, z) = \sum_{t \in O^i} \psi_{f,m,t+2mr}(\omega) \cdot e^{2\pi i \text{Re}(\tau z)}
\]

\[
= \sum_{t \in O^i / 2mO} \sum_{r \in O} \psi_{f,m,t+2mr}(\omega) \cdot e^{2\pi i (\text{Re}(\tau r) + m N(r))z}
\]

\[
= \sum_{t \in O^i / 2mO} \psi_{f,m,t}(\omega) \sum_{r \in O} e^{2\pi i (\text{Re}(\tau r) + m N(r) + N(t)/4m)z + \text{Re}(\tau 2mr)z)}
\]

\[
= \sum_{t \in O^i / 2mO} e^{-\pi i N(t)/2m} \psi_{f,m,t}(\omega) \sum_{r \in O} e^{2\pi i (\text{Re}(\tau r) + m N(r) + N(t)/4m + \text{Re}(\tau 2mr)z)}
\]

\[
= \sum_{t \in O^i / 2mO} e^{-\pi i N(t)/2m} \psi_{f,m,t}(\omega) \cdot \Theta_{1/2m,2m}(\omega, 2m; O)
\]

(3.8)

for all \( m \in \mathbb{N}, \omega \in \mathcal{H}, z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \), where

\[
f_{m,t}(\omega) := e^{-\pi i N(t)/2m} \psi_{f,m,t}(\omega)
\]

(3.9)

and

\[
\vartheta_{f,m,t}(\omega, z) := \Theta_{1/2m,2m}(\omega, 2m; O)
\]

(3.10)

\[
= \sum_{r \in O} e^{2\pi i (2m[t r + t/2m]_\omega) + 2\pi i \tau (2mr + t/2m)}
\]

\[
= \sum_{r \in O} e^{2\pi i (2m[t r + t]_\omega) + 4m + \tau (z 2mr + t)}
\]

\[
= \sum_{s \in 2mO + t} e^{2\pi i (4m + \tau (z 2s + t))}.
\]

The decomposition 3.8 of the Fourier-Jacobi-coefficients is called \textit{theta-decomposition}. Its usefulness will become clear soon. But first, we need two further kinds of transformation behavior of the Fourier-Jacobi-coefficients. Let \( M_0 = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) = \text{Sp}_1(\mathbb{Z}) \), then \( M := M_0 \times \mathbb{I} \in \text{SL}_2(\mathbb{Z}) \).
\(\text{Sp}_2(O) \cap \mathbb{Z}^{4 \times 4} = \text{Sp}_2(\mathbb{Z})\) holds according to 1.1. A straightforward calculation yields

\[
M\{\mathbb{Z}\} = \begin{pmatrix} M_0(\tau) & M_0\{\tau\}^{-1}z \\ M_0\{\tau\}^{-1}z & \omega - M_0\{\tau\}^{-1}czz \end{pmatrix}
\]

(3.11)

and as well

\[
\det(\tilde{M}\{\mathbb{Z}\})^{k/2} = ((c\tau + d)^2)^{k/2} = \pm(c\tau + d)^k
\]

for \(Z = (\frac{i}{z} \omega) \in \mathcal{H}(\mathbb{H})\), as \(k\) is supposed to be odd. Now, note that according to the proof of (1.56), \(v_i\) had to be chosen such that

\[
v_i(N) \det(\tilde{N}\{\mathbb{Z}\})^{k/2} = (\det(N\{\mathbb{Z}\}))^k
\]

holds for all \(N \in \text{Sp}_2(\mathbb{Z})\) and \(Z \in \mathcal{H}_2(\mathbb{R})\). So if we would have chosen \(Z = (\frac{i}{z} \omega) \in \mathcal{H}_2(\mathbb{R})\) (which is equivalent to \(Z \in \mathcal{H}(\mathbb{H})\) and \(z \in \mathbb{R} \otimes \mathbb{R} C\)), we would have gotten

\[
v_i(M) \det(\tilde{M}\{\mathbb{Z}\})^{k/2} = (\det(M\{\mathbb{Z}\}))^k = (c\tau + d)^k.
\]

For fixed \(\tau\) and \(\omega\), \(z \mapsto v_i(M) \det(\tilde{M}\{\mathbb{Z}\})^{k/2}\) is holomorphic with possible values \(\pm(c\tau + d)^k\) \(v_i(M) \neq 0\), whereat its value for \(z \in \mathbb{R} \otimes \mathbb{R} C\) is \((c\tau + d)^k\). And thus, since \(\mathcal{H}(\mathbb{H})\) is convex according to [Kr85, ch.I, prop.3.12], \(z \mapsto v_i(M) \det(\tilde{M}\{\mathbb{Z}\})^{k/2}\) has to be constant with value \((c\tau + d)^k\) and we have proven

\[
v_i(M) \det(\tilde{M}\{\mathbb{Z}\})^{k/2} = (c\tau + d)^k
\]

(3.12)

for all \(Z = (\frac{i}{z} \omega) \in \mathcal{H}(\mathbb{H})\). Hence we compute

\[
f(Z) = \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z) e^{2\pi im\omega}
\]

\[
= (v_i(M))^{-1} f_k M(Z) = (v_i(M))^{-1} (\det(\tilde{M}\{\mathbb{Z}\}))^{-k/2} f(M\{\mathbb{Z}\})
\]

\[
= (c\tau + d)^{-k} f \begin{pmatrix} M_0(\tau) & M_0\{\tau\}^{-1}z \\ M_0\{\tau\}^{-1}z & \omega - M_0\{\tau\}^{-1}czz \end{pmatrix}
\]

\[
= (M_0\{\tau\})^{-k} \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(M_0(\tau), M_0\{\tau\}^{-1}z) e^{2\pi im(\omega - M_0\{\tau\}^{-1}c\omega)}
\]

\[
= \sum_{m \in \mathbb{N}_0} \left( (M_0\{\tau\})^{-k} e^{-2\pi im M_0\{\tau\}^{-1}c\omega} \varphi_{f,m}(M_0(\tau), M_0\{\tau\}^{-1}z) \right) e^{2\pi im\omega}
\]

for \(M = M_0 \times I\). As we have done before, comparing the unique Fourier-coefficients of these Fourier-series (in \(\omega\)) yields

\[
\varphi_{f,m}(M_0(\tau), M_0\{\tau\}^{-1}z) = e^{2\pi im M_0\{\tau\}^{-1}c\omega} \cdot (M_0\{\tau\})^k \cdot \varphi_{f,m}(\tau, z)
\]

(3.13)

for all \(m \in \mathbb{N}_0, \tau \in \mathcal{H}, z \in \mathbb{H} \otimes \mathbb{R} C\) and \(M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\). Or in other words, if we define

\[
\varphi|_{k,m}(M_0)(\tau, z) := e^{-2\pi im M_0\{\tau\}^{-1}c\omega} \cdot (M_0\{\tau\})^{-k} \cdot \varphi(M_0(\tau), M_0\{\tau\}^{-1}z)
\]

(3.14)
for holomorphic functions $\varphi : \mathcal{H} \times \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_0 = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, then

$$\varphi_{f,m}|_{k,m}[M_0](\tau, z) = \varphi_{f,m}(\tau, z)$$

(3.15)

holds. Note that

$$\varphi|_{k,m}[M_1 M_2] = (\varphi|_{k,m}[M_1])|_{k,m}[M_2]$$

(3.16)

holds for all $M_1 = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right)$, $M_2 = \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \in \text{SL}_2(\mathbb{Z})$: Using (1.5) we compute

$$\varphi|_{k,m}[M_1]|_{k,m}[M_2](\tau, z)$$

$$= e^{-2\pi i m M_2 \{ \tau \}^{-1} c_2 z} \cdot (M_2 \{ \tau \})^{-k} \cdot \varphi|_{k,m}[M_1](M_2 \{ \tau \}, M_2 \{ \tau \}^{-1} z)$$

$$= e^{-2\pi i m (M_1 M_2) \{ \tau \}^{-1} c_2 z} \cdot (M_2 \{ \tau \})^{-k} \cdot \varphi((M_1 M_2) \{ \tau \}, (M_1 M_2) \{ \tau \}^{-1} z)$$

because

$$(M_1 M_2) \{ \tau \}^{-1} c_2 + M_2 \{ \tau \}^{-1} c_1 = \frac{((c_1 a_2 + d_1 c_2) \tau + (c_1 b_2 + d_1 d_2))c_2 + c_1}{c_2 \tau + d_2}$$

$$= \frac{((c_1 a_2 + d_1 c_2) c_2 \tau + (d_1 c_2 + c_1 a_2) d_2)}{c_2 \tau + d_2}$$

$$= c_1 a_2 + d_1 c_2$$

holds, as $\det(M_2) = a_2 d_2 - b_2 c_2 = 1$.

Finally, let $\epsilon \in \mathcal{E}$ and $M = \text{Rot}(\begin{array}{cc} \epsilon & 0 \\ 0 & 1 \end{array}) \in \text{Sp}_2(\mathbb{O})$. Using (1.34), one easily verifies

$$M(Z) = \begin{pmatrix} \tau & \epsilon z \\ \overline{z} & \omega \end{pmatrix}$$

and

$$\det(M(Z))^{1/2} = (N(\epsilon^2))^{1/2} = 1$$

for $Z = \left( \begin{array}{cc} \tau & z \\ \overline{z} & \omega \end{array} \right) \in \mathcal{H}(\mathbb{I} \mathbb{H})$. Thus we compute

$$f(Z) = \sum_{m \in \mathbb{N}_0} \varphi_{f,m}(\tau, z) e^{2\pi i m \omega} = (\nu_{(M)})^{-1} f|_M(Z)$$

$$= (\nu_{(M)})^{-1} f \begin{pmatrix} \tau & \epsilon z \\ \overline{z} & \omega \end{pmatrix}$$

$$= \sum_{m \in \mathbb{N}_0} \left( \varphi_{(\pi_p(\overline{z}))^{-1} \varphi_{f,m}(\tau, \epsilon z)} \right) e^{2\pi i m \omega}.$$
Once more, comparing Fourier-coefficients yields

\[ \varphi_{f,m}(\tau, \varepsilon z) = \varphi_1(\pi_p(\varepsilon)) \cdot \varphi_{f,m}(\tau, z) \quad (3.17) \]

for all \( m \in \mathbb{N}_0, \tau \in \mathcal{H}, z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) and \( \varepsilon \in \mathcal{E} \). Special choices of \( \varepsilon \) yield

\[
\begin{align*}
\varphi_{f,m}(\tau, -z) & = -\varphi_{f,m}(\tau, z), \\
\varphi_{f,m}(\tau, iz) & = -i \cdot \varphi_{f,m}(\tau, z), \\
\varphi_{f,m}(\tau, -\frac{1+\sqrt{3}}{2}z) & = \varphi_{f,m}(\tau, z).
\end{align*}
\]

Note that the first equality already follows from (3.3).

Now that we have determined all relevant transformation behaviors for the Fourier-Jacobi-coefficients, we can define a special subspace of holomorphic functions on \( \mathcal{H} \times \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) that the Fourier-Jacobi-coefficients belong to:

**Definition.** Let \( \varphi : \mathcal{H} \times \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) be holomorphic, \( k \in \mathbb{Z} \) be odd and \( m \in \mathbb{N}_0 \). Then \( \varphi \) is said to be a quaternionic Jacobi-form of weight \( k \) and index \( m \) with respect to \( \mathcal{O} \) and \( v_i \), if \( \varphi \) fulfills

- \( \varphi(\tau + s_1, z + s_2) = \varphi(\tau, z) \)
- \( \varphi(\tau, z + \tau r) = e^{-2\pi i m(\tau(N(r) + 2 \Re(\tau z)))} \cdot \varphi(\tau, z) \)
- \( \varphi_{|k,m}[M](\tau, z) := e^{-2\pi i m M(\tau) \cdot \zeta z} \cdot (M\{\tau\})^{-k} \cdot \varphi(M(\tau), M\{\tau\}^{-1}z) = \varphi(\tau, z) \)
- \( \varphi(\tau, \varepsilon z) = \varphi_1(\pi_p(\varepsilon)) \cdot \varphi(\tau, z) \)

for all \( \tau \in \mathcal{H}, z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}, s_1 \in \mathbb{Z}, s_2 \in \mathcal{O}, r \in \mathcal{O}, M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}), \varepsilon \in \mathcal{E} \), and if it possesses an absolutely and locally uniformly convergent Fourier-expansion of the shape

\[
\varphi(\tau, z) = \sum_{n \in \mathbb{N}_0, j \in \mathcal{O}^i} \alpha(n, t) \cdot e^{2\pi i (nt + \Re(\tau z))}, \quad \tau \in \mathcal{H}, z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}.
\]

The space of all quaternionic Jacobi-form of weight \( k \) and index \( m \) with respect to \( \mathcal{O} \) and \( v_i \) is denoted by \( \mathcal{J}(k, m, \mathcal{O}, v_i) \).

Note that according to 3.3, 3.4, 3.5, 3.15 and 3.17, \( \varphi_{f,m} \in \mathcal{J}(k, m, \mathcal{O}, v_i) \) holds for all \( m \in \mathbb{N}_0 \) (assuming that \( f \) actually exists non-trivially). On the other hand, every \( \varphi \in \mathcal{J}(k, m, \mathcal{O}, v_i) \) possesses a theta-decomposition of the shape 3.8 and its Fourier-coefficients fulfill the relation 3.6, as no further information was needed to verify those. Furthermore, note that our definition of quaternionic Jacobi-forms is the common one found in the literature (for example, cf. [Kr87] for the case of Hurwitz quaternions or [De01] for the Hermitian case) apart from the requirement that \( \varphi(\tau, \varepsilon z) = \varphi_1(\pi_p(\varepsilon)) \cdot \varphi(\tau, z) \) has to hold. This simply results from the fact that \( v_i \) is a non-trivial multiplier system (while it is yet to show that it actually exists).

We are now going to have a closer look at the first Fourier-Jacobi-coefficient \( \varphi_{f,1} \). Recall that, according to (3.3), a transversal of \( \mathcal{O}^i / 2\mathcal{O} \) is given by

\[
h_{3k+j} = k(1 + \frac{1}{\sqrt{3}}) + j(1 + \frac{1}{\sqrt{3}})i_2, \quad 0 \leq j, k \leq 2. \quad (3.18)
\]
As we will now only deal with the first Fourier-Jacobi-coefficient \( \varphi_{f,1} \), we will get rid of some indices in order to keep the considerations well-arranged. Therefore, the theta-series from 3.10 shall be denoted by

\[
\vartheta_{h_j}(\tau, z) := \vartheta_{f,1,h_j}(\tau, z) = \Theta_{\frac{1}{2}h_j, 2z}(\tau, 2; \mathcal{O}) , \quad j \in \{0, \ldots, 8\},
\]

(3.19)

while (see 3.9)

\[
f_j(\tau) := f_{1,h_j}(\tau) = e^{-\pi i N(h_j) \tau} \varphi_{f,1,h_j}(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{1}{2}h_j} \alpha_f \left( \frac{n}{\frac{1}{2}h_j} \right) e^{2\pi i (n-N(\frac{1}{2}h_j))\tau}, \quad \tau \in \mathcal{H}.
\]

(3.20)

Then according to 3.8, \( \varphi_{f,1} \) is given by

\[
\varphi_{f,1}(\tau, z) = \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_{h_j}(\tau, z), \quad \tau \in \mathcal{H}, \ z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}.
\]

(3.21)

Noting that all \( h_j, j \in \{0, \ldots 8\} \) are linear combinations of \( h_1 \) and \( h_3 \), we compute

\[
f_0(\tau) = \sum_{n \in \mathbb{N}_0} \alpha_f \left( \frac{n}{0} \right) e^{2\pi i n \tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n}{0} \right) e^{2\pi i n \tau}
\]

\[
f_1(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{1}{2}} \alpha_f \left( \frac{n}{\frac{1}{2}h_1} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{1}{2}h_1} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_2(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{4}{3}} \alpha_f \left( \frac{n}{\frac{4}{3}h_1} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{4}{3}h_1} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_3(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{1}{2}} \alpha_f \left( \frac{n}{\frac{1}{2}h_3} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{1}{2}h_3} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_4(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{3}{4}} \alpha_f \left( \frac{n}{\frac{3}{4}(h_1+h_3)} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{3}{4}(h_1+h_3)} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_5(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{1}{2}} \alpha_f \left( \frac{h_1+h_3}{\frac{1}{2}h_3} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{1}{2}(h_1+h_3)} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_6(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{4}{3}} \alpha_f \left( \frac{h_3}{\frac{1}{2}h_3} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{1}{2}h_3} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_7(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{3}{4}} \alpha_f \left( \frac{h_1+h_3}{\frac{3}{4}(h_1+h_3)} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{3}{4}(h_1+h_3)} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

\[
f_8(\tau) = \sum_{n \in \mathbb{N}_0, n \geq \frac{1}{2}} \alpha_f \left( \frac{h_1+h_3}{\frac{1}{2}h_3} \right) e^{2\pi i (n-\frac{1}{2})\tau} = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{\frac{1}{2}h_1+h_3} \right) e^{2\pi i (n+\frac{1}{2})\tau}
\]

The next step is taken from [Kl98], adapted to our case of a non-trivial multiplier system. In
virtue of (1.55),
\[ \alpha_f(T[\mathcal{U}]) = \nu_i(\text{Rot}((\mathcal{U}'^{-1})) \cdot \alpha_f(T) \]
holds for all \( \mathcal{U} \in \text{GL}_2(\mathcal{O}) \) and all \( T \in \text{Her}_2^\tau(\mathcal{O}) \) with \( T \geq 0 \). According to [Kl98, p.38], we have the following identities:
\[
\begin{pmatrix}
  n + 1 & \frac{1}{2}h_1 \\
  \frac{1}{2}h_1 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  0 & -i_2
\end{pmatrix}
= \begin{pmatrix}
  n + 1 & \frac{1}{2}h_3 \\
  \frac{1}{2}h_3 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
  n + 1 & \frac{1}{2}h_3 \\
  \frac{1}{2}h_3 & 1
\end{pmatrix}
\begin{pmatrix}
  -1 & 0 \\
  1 & 0
\end{pmatrix}
= \begin{pmatrix}
  n + 2 & h_3 \\
  h_3 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
  n + 2 & h_3 \\
  h_3 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  0 & i_2
\end{pmatrix}
= \begin{pmatrix}
  n + 2 & h_1 \\
  h_1 & 1
\end{pmatrix}
\]
as well as
\[
\begin{pmatrix}
  n + 1 & \frac{1}{2}(h_1 + h_3) \\
  \frac{1}{2}(h_1 + h_3) & 1
\end{pmatrix}
\begin{pmatrix}
  -i_2 & 0 \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  n + 2 & h_1 + \frac{1}{2}h_3 \\
  h_1 + \frac{1}{2}h_3 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
  n + 1 & \frac{1}{2}(h_1 + h_3) \\
  \frac{1}{2}(h_1 + h_3) & 1
\end{pmatrix}
\begin{pmatrix}
  i_2 & 0 \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  n + 2 & \frac{1}{2}h_1 + h_3 \\
  \frac{1}{2}h_1 + h_3 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
  n + 1 & \frac{1}{2}(h_1 + h_3) \\
  \frac{1}{2}(h_1 + h_3) & 1
\end{pmatrix}
\begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  n + 3 & h_1 + h_3 \\
  h_1 + h_3 & 1
\end{pmatrix}
\]
Therefore, the transformation law for the Fourier-coefficients and the Fourier-expansions from above yield
\[
f_1 = -f_2 = if_3 = -if_6, \quad (3.22)
f_4 = if_5 = -if_7 = -f_8. \quad (3.23)
\]
Furthermore,
\[
\begin{pmatrix}
  n & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  n & 0 \\
  0 & 1
\end{pmatrix}
\]
holds, which leads to
\[
\alpha_f\left( \begin{pmatrix}
  n & 0 \\
  0 & 1
\end{pmatrix} \right) = -\alpha_f\left( \begin{pmatrix}
  n & 0 \\
  0 & 1
\end{pmatrix} \right).
\]
This implies
\[
f_0 \equiv 0
\]
and we get
\[
\varphi_{f,1}(\tau, z) = f_1(\tau) \cdot (\vartheta_{h_1}(\tau, z) - \vartheta_{h_2}(\tau, z) - i\vartheta_{h_3}(\tau, z) + i\vartheta_{h_6}(\tau, z))
+ f_4(\tau) \cdot (\vartheta_{h_4}(\tau, z) - i\vartheta_{h_5}(\tau, z) + i\vartheta_{h_7}(\tau, z) - \vartheta_{h_8}(\tau, z)). \quad (3.24)
\]
Next, we will demonstrate that \((f_1, f_4)\) is a vector-valued modular form (see [De01] or (5.36) for a definition). But in order to do so, we have to examine the transformation behavior of the
3.2 Quaternionic Maaß lifts of odd weight

theta-series \( \vartheta_{h_j} \). As we will need two special matrices quite often now, let us shortly fix some notation for the rest of the thesis:

(3.9) Definition. Let
\[
T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

It is well known (and already stated in (1.15), and also confer [KK07]), that \( T_1 \) and \( J_1 \) generate \( \text{SL}_2(\mathbb{Z}) = \text{Sp}_1(\mathbb{Z}) \). \( f \mid_k M \), where \( M \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), \( k \in \mathbb{Z} \) and \( f \) is holomorphic on \( \mathcal{H} \), should always stand for the common slash-operator, i.e.
\[
f \mid_k M(\tau) = (c\tau + d)^{-k} f(M(\tau)) = (c\tau + d)^{-k} f\left(\frac{at+b}{ct+d}\right)
\]
for \( \tau \in \mathcal{H} \). In particular, we have
\[
f \mid_k T_1(\tau) = f(\tau + 1), \quad f \mid_k J_1(\tau) = \tau^{-k} f(-\tau^{-1})
\]
as well as
\[
\varphi \mid_{k,m}[T_1](\tau, z) = \varphi(\tau + 1, z), \quad \varphi \mid_{k,m}[J_1](\tau, z) = e^{-2\pi i \tau^{-1}z} \cdot \tau^{-k} \cdot \varphi(-\tau^{-1}, \tau^{-1}z).
\]

And to stay closer to common literature (cf. [KK07]), define
\[
\Gamma[n] := \text{Sp}_1(\mathbb{Z})[n\mathbb{Z}] = \{ M \in \text{SL}_2(\mathbb{Z}) ; M \equiv I \text{ mod } n \},
\]
\[
\Gamma(n) := \{ M \in \text{SL}_2(\mathbb{Z}) ; M \equiv \pm I \text{ mod } n \},
\]
\[
\Gamma_0[n] := \text{Sp}_1(\mathbb{Z})[n\mathbb{Z}]_{0} = \{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) ; c \equiv 0 \text{ mod } n \},
\]

Now, we need to determine the transformation behavior of the \( \vartheta_{h_j} \) under the operator \( \mid_k[M] \)”, \( M \in \text{SL}_2(\mathbb{Z}) \). Although the result can already be found in [KL98, Le.3.15], we will demonstrate it here again for the sake of completeness and in the more general light of chapter 2.

(3.10) Lemma. The theta-series \( \vartheta_{h_j}, j \in \{0, \ldots, 8\} \) are linearly independent. Let \( \vartheta = (\vartheta_{h_0}, \ldots, \vartheta_{h_8})' \) and \( \rho = e^{\frac{2\pi i}{3}} \). There exists a unique homomorphism \( \kappa : \text{SL}_2(\mathbb{Z}) \rightarrow \text{GL}_9(\mathbb{Q}(\rho)) \) such that
\[
\vartheta \mid_{2,1}[M] := (\vartheta_{h_0} \mid_{2,1}[M], \ldots, \vartheta_{h_8} \mid_{2,1}[M])' = \kappa(M) \cdot \vartheta
\]
holds for all \( M \in \text{SL}_2(\mathbb{Z}) \). The kernel of this homomorphism is \( \Gamma[3] \). For the two generators of \( \text{SL}_2(\mathbb{Z}) \), \( \kappa \) is given by
\[
\kappa(T_1) = \text{diag}(1, \rho, \rho, \rho^2, \rho^2, \rho, \rho^2, \rho^2)
\]
and

\[ \kappa(j_1) = -\frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \rho & \rho^2 & 1 & \rho & \rho^2 & 1 & \rho \\
1 & \rho^2 & \rho & 1 & \rho^2 & \rho & 1 & \rho \\
1 & 1 & 1 & \rho & \rho & \rho^2 & \rho & \rho \\
1 & \rho & \rho^2 & \rho & \rho^2 & 1 & \rho^2 & 1 & \rho \\
1 & \rho^2 & \rho & \rho & 1 & \rho^2 & \rho & 1 & \rho \\
1 & 1 & 1 & \rho^2 & \rho^2 & \rho & \rho & \rho & \rho \\
1 & \rho & \rho^2 & \rho^2 & 1 & \rho & \rho & \rho^2 & 1 \\
1 & \rho^2 & \rho & \rho & 1 & \rho & 1 & \rho & \rho^2 \\
\end{pmatrix} \cdot . \]

**Proof:** As the \( \vartheta_{h_j} \) are just special theta-series, their absolutely and locally uniformly convergent Fourier-expansions (with respect to the second variable) are given by

\[ \vartheta_{h_j}(\tau, z) = \Theta_{\frac{1}{2}h_j, z}^{(1)}(\tau, 2; \mathcal{O}) = \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + \frac{1}{2}h_j)\tau + \Re(\pi(2g + h_j)))} \]

\[ = \sum_{t \in \mathcal{O}^t} a_{h_j}(t) \cdot e^{2\pi i \Re(tz)}, \quad \tau \in \mathcal{H}, \ z \in \mathcal{H} \otimes \mathbb{C}, \]

where

\[ a_{h_j}(t) = \begin{cases} 
eq e^{2\pi i \Re(t\frac{1}{2})}, & \text{if } t \in 2\mathcal{O} + h_j, \\
0, & \text{else}. \end{cases} \]

Hence there is no “exponent” \( t \in \mathcal{O}^t \) occurring in the Fourier-expansions of two different \( \vartheta_{h_j} \) with non-vanishing Fourier-coefficients, since the \( h_j \) are a transversal of \( \mathcal{O}^t / 2\mathcal{O} \). The uniqueness of Fourier-coefficients clearly yields the linear independency of the \( \vartheta_{h_j} \). (The result can already be found in [Kl98].)

So let us get to the homomorphism \( \kappa \): Suppose there are matrices \( M_1, M_2 \in \text{SL}_2(\mathbb{Z}) \) such that there exist some matrices in \( \text{GL}_9(\mathbb{Q}(\rho)) \), denoted by \( \kappa(M_1) \) and \( \kappa(M_2) \) such that

\[ \vartheta_{|2,1}[M_j] = \kappa(M_j) \cdot \vartheta \]

holds for \( j \in \{1, 2\} \). Then according to 3.16 also

\[ \vartheta_{|2,1}[M_1 M_2] = (\vartheta_{|2,1}[M_1])_{|2,1}[M_2] = (\kappa(M_1) \cdot \vartheta)_{|2,1}[M_2] \]

\[ = \kappa(M_1) \cdot (\vartheta_{|2,1}[M_2]) = (\kappa(M_1) \cdot \vartheta)_{|2,1}[M_2] = \kappa(M_1) \cdot \vartheta \]

holds as the operator \( "|_{k,1}[M]" \) clearly is linear. So if we would show that there exist matrices \( \kappa(T_1) \) and \( \kappa(J_1) \) in \( \text{GL}_9(\mathbb{Q}(\rho)) \) such that \( \vartheta_{|2,1}[T_1] = \kappa(T_1) \cdot \vartheta \) and \( \vartheta_{|2,1}[J_1] = \kappa(J_1) \cdot \vartheta \) hold, then clearly there exists a matrix \( \kappa(M) \in \text{GL}_9(\mathbb{Q}(\rho)) \) for every \( M \in \text{SL}_2(\mathbb{Z}) \) such that \( \vartheta_{|2,1}[M] = \kappa(M) \cdot \vartheta \) holds. And by the same calculation as above, also

\[ (\kappa(M_1) \cdot \kappa(M_2)) \cdot \vartheta = (\vartheta_{|2,1}[M_1])_{|2,1}[M_2] = \vartheta_{|2,1}[M_1 M_2] = \kappa(M_1 M_2) \cdot \vartheta \]
holds for all \( M_1, M_2 \in \text{SL}_2(\mathbb{Z}) \). Therefore, since the \( \vartheta_{h_j} \) are linearly independent,

\[
\kappa(M_1) \cdot \kappa(M_2) = \kappa(M_1 M_2)
\]

has to hold true for all \( M_1, M_2 \in \text{SL}_2(\mathbb{Z}) \). Hence \( \kappa \) is a homomorphism, and unique for the same reason. So let us have closer look at the two generators of \( \text{SL}_2(\mathbb{Z}) \). In view of (3.9) we compute

\[
\vartheta_{h_j}|_{2,1}[T_1](\tau, z) = \vartheta_{h_j}(\tau + 1, z) = \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + \frac{1}{2} h_j)(\tau + 1) + \text{Re}(\tau(2g + h_j)))} = \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + \frac{1}{2} h_j)) e^{2\pi i (N(g + \frac{1}{2} h_j)(\tau + \text{Re}(\tau(2g + h_j)))}}
\]

for \( j \in \{0, \ldots, 8\} \). As we have seen several times so far,

\[
N(g + \frac{1}{2} h_j) = N(g) + N(\frac{1}{2} h_j) + \text{Re}(\overline{g} h_j)
\]

holds, whereas \( g \in \mathcal{O} \) and \( h_j \in \mathcal{O}^2 \) imply \( N(g) \in \mathbb{Z} \) and \( \text{Re}(\overline{g} h_j) \in \mathbb{Z} \). Thus we have shown

\[
\vartheta_{h_j}|_{2,1}[T_1](\tau, z) = e^{2\pi i N(\frac{1}{2} h_j)} \cdot \vartheta_{h_j}|_{2,1}[T_1](\tau, z),
\]

which leads to

\[
\kappa(T_1) = \text{diag}(1, \rho, \rho, \rho^2, \rho^2, \rho, \rho^2, \rho^2)
\]

by simply computing \( N(\frac{1}{2} h_j), j = 0, \ldots, 8 \). For \( J_1 \) we have to use the Theta-transformation formula (2.9). Note that every \( t \in \mathcal{O}^2 \) can be written uniquely as \( t = h_j + 2g \) for some \( j \in \{0, \ldots, 8\} \) and \( g \in \mathcal{O} \). In virtue of (3.9) and (1.7), given \( j \in \{0, \ldots, 8\} \) we compute

\[
\vartheta_{h_j}|_{2,1}[J_1](\tau, z) = e^{-2\pi i (\tau^{-1} \tau z) \cdot \tau^{-2} \cdot \varphi(-\tau^{-1}, \tau^{-1} z) \\
= e^{-2\pi i (\tau^{-1} \tau z) \cdot \tau^{-2} \cdot \Theta_{\frac{1}{2} h_j, 2\tau^{-1} \tau z}(-\tau^{-1}, 2; \mathcal{O}) \\
= e^{-2\pi i (\tau^{-1} \tau z) \cdot \tau^{-2} \cdot \text{vol}(\mathcal{O}^2) \cdot (\tau^{-1} z)^2 \cdot 2^{-2} \cdot e^{2\pi i \text{Re}(\overline{g} h_j)(2\tau^{-1} z))} \Theta_{2\tau^{-1} z, -\frac{1}{2} h_j}(\tau, \frac{1}{2}, \mathcal{O}^2) \\
= -\frac{1}{3} \cdot e^{-2\pi i (\tau^{-1} \tau z) \cdot \sum_{g \in \mathcal{O}^2} e^{2\pi i (N(g + \frac{1}{2} h_j)(\tau + 2 \text{Re}(\overline{g} \tau)) - 2\pi i \text{Re}(\overline{g} h_j))} \\
= -\frac{1}{3} \cdot \sum_{k=0}^{8} \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + \frac{1}{2} h_j) \tau + 2 \text{Re}(g + \frac{1}{2} h_j) \tau \overline{g} h_j)) - 2\pi i \text{Re}(g + \frac{1}{2} h_j))} \\
= -\frac{1}{3} \cdot \sum_{k=0}^{8} e^{-\pi i \text{Re}(h \overline{g} h_j)} \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + \frac{1}{2} h_j) \tau + 2 \text{Re}(g + \frac{1}{2} h_j) \tau \overline{g} h_j))} \\
= -\frac{1}{3} \cdot \sum_{k=0}^{8} e^{-\pi i \text{Re}(h \overline{g} h_j)} \cdot \vartheta_{h_j}(\tau, z),
\]

as \( \text{Re}(g \overline{g} h_j) = \text{Re}(g \overline{g} h_j) + \frac{1}{2} \text{Re}(h \overline{g} h_j) \) holds and \( g \in \mathcal{O}, h_j \in \mathcal{O}^2 \) implies \( \text{Re}(g \overline{g} h_j) \in \mathbb{Z} \). By simply computing the prefactors \( e^{-\pi i \text{Re}(h \overline{g} h_j)} \) we have thus shown that \( \vartheta|_{2,1}[J_1] = \kappa(J_1) \cdot \vartheta \) holds,
with \(\kappa(J_1)\) given as above.
The kernel of \(\kappa\) has already been determined in [KI98, Le.3.15] and we omit the proof here. \(\square\)

With the help of the preceding lemma we are now able to determine the transformation behavior of the \(f_j\). Let \(F = (f_0, \ldots, f_8)'\) and again \(\vartheta = (\vartheta_{h_1}, \ldots, \vartheta_{h_5})'\). According to 3.21, \(\varphi_{f,1}(\tau,z) = (F(\tau))' \cdot \vartheta(\tau,z) = \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_{h_j}(\tau,z)\) holds. Suppose \(M = (a, b) \in \text{SL}_2(\mathcal{Z})\). Using 3.15 and (3.10) we compute

\[
(F(\tau))' \cdot \vartheta(\tau,z) = \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_{h_j}(\tau,z) = \varphi_{f,1}(\tau,z) = \varphi_{f,1}|_1[M](\tau,z)
\]

\[
eq \sum_{j=0}^{8} ((M\{\tau\})^{-(k-2)}f_j(M(\tau))) \cdot (e^{-2\pi i M(\tau)} - 1) \cdot (M\{\tau\})^{-2} \cdot \vartheta_{h_j}(M(\tau), M\{\tau\}^{-1}z)
\]

\[
= \sum_{j=0}^{8} f_j|_{k-2}M(\tau) \cdot \vartheta_{h_j}|_{2,1}[M](\tau,z) = (F|_{k-2}M(\tau))' \cdot \vartheta|_{2,1}[M](\tau,z)
\]

\[
= (F|_{k-2}M(\tau))' \cdot (\kappa(M) \cdot \vartheta(\tau,z)) = (\kappa(M)')^{-1} \cdot F|_{k-2}M(\tau) \cdot \vartheta(\tau,z)
\]

for all \(\tau \in \mathcal{H}\) and \(z \in \mathcal{H} \otimes \mathbb{R} \mathbb{C}\). All involved functions are holomorphic, while the \(\vartheta_{h_j}\) are linearly independent in virtue of (3.10). So the equation from above implies

\[
F|_{k-2}M := (f_0|_{k-2}M(\tau), \ldots, f_8|_{k-2}M(\tau))' = (\kappa(M)')^{-1} \cdot F
\]

for all \(M \in \text{SL}_2(\mathcal{Z})\). A simple calculation shows

\[
(\kappa(T_1))^{-1} = \text{diag}(1, \rho^2, \rho^2, \rho, \rho, \rho, \rho)
\]

and

\[
(\kappa(J_1))^{-1} = -\frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \rho^2 & \rho & 1 & \rho^2 & \rho & 1 & \rho^2 \\
1 & \rho & \rho^2 & 1 & \rho & \rho^2 & 1 & \rho \\
1 & \rho^2 & \rho & \rho^2 & 1 & \rho & \rho^2 & 1 \\
1 & 1 & 1 & \rho^2 & \rho^2 & \rho & \rho^2 & 1 \\
1 & \rho & \rho^2 & \rho & \rho^2 & \rho & \rho^2 & 1 \\
1 & \rho^2 & \rho & 1 & \rho^2 & \rho & 1 & \rho \\
1 & \rho^2 & \rho & 1 & \rho^2 & \rho & 1 & \rho
\end{pmatrix}.
\]

Note that by definition

\[
f_j|_{k-2}(-I)(\tau) = (-1)^{-k-2}f_j(\tau) = -f_j(\tau)
\]

holds as \(k\) is odd. Furthermore, because of the Fourier-expansions of the \(f_j\) in 3.20, where only non-negative exponents \(q\) occur (in \(e^{2\pi i q\tau}\)) and knowing that \(f_j|_{k-2}M, M \in \text{SL}_2(\mathcal{Z})\) is always a linear combination of the \(f_j\), it is clear that \(f_j|_{k-2}M, M \in \text{SL}_2(\mathcal{Z})\) is always bounded in the domain \(\{z \in \mathbb{C} \mid \text{Im}(z) \geq \beta\}\) for any \(\beta > 0\). Using the transformation behavior from above, 3.22 and 3.23 as well as the fact that the kernel of \(\kappa\) is \(\Gamma[3]\), we have proven the following lemma:
(3.11) Lemma. It holds true that

\[ f_1, f_4 \in [\Gamma[3], k - 2, 1] . \]

Moreover, we have the following transformation behavior:

\[
\begin{align*}
    f_1|_{k-2}(-1) &= -f_1 , \\
    f_4|_{k-2}(-1) &= -f_4 , \\
    f_1|_{k-2}T_1 &= \rho^2 f_1 , \\
    f_4|_{k-2}T_1 &= \rho f_4 , \\
    f_1|_{k-2}f_1 &= -\frac{1}{3}(\rho^2 - \rho - i + i) f_1 - \frac{1}{3}(\rho^2 - i\rho + i\rho^2 - \rho) f_4 \\
    &= \frac{i}{\sqrt{3}} f_1 + \frac{1+i}{\sqrt{3}} f_4 , \\
    f_4|_{k-2}f_1 &= -\frac{1}{3}(\rho^2 - \rho - i\rho^2 + i\rho) f_1 - \frac{1}{3}(\rho - i + i - \rho^2) f_4 \\
    &= \frac{1+i}{\sqrt{3}} f_1 - \frac{i}{\sqrt{3}} f_4 ,
\end{align*}
\]

where \( \rho = e^{\frac{2}{3} \pi i} \).

Note that \( f_1 \) and \( f_4 \) contain all the information needed to determine \( \varphi_{f,1} \). The first Fourier-Jacobi-coefficient can always be decomposed into \( \varphi_{f,1} = \sum_{j=1}^{8} f_j \cdot \vartheta_{h_j} \), where the theta-series are independent of \( \varphi_{f,1} \) and always the same, while the \( f_j \) are completely determined by \( f_1 \) and \( f_4 \). Furthermore, the \( f_j \) are unique for every \( \varphi_{f,1} \), of course, as the first Fourier-Jacobi-coefficient is completely determined by them. (But note that the \( f_j \) are only unique for \( \varphi_{f,1} \), not \( f \) itself, as two different quaternionic modular forms might have coinciding first Fourier-Jacobi-coefficients).

So \( f \mapsto (f_1, f_4) \) is a homomorphism (where \( f \in \text{Sp}_2(O), k, \nu_j) \)). Its kernel coincides with the kernel of the homomorphism \( f \mapsto \varphi_{f,1} \).

We will now come to the last, final step in decomposing and describing \( \varphi_{f,1} \). \( (f_1, f_4) \) being a vector-valued modular form (where \( f_1 \) and \( f_4 \) themselves are elliptic modular forms with respect to \( \Gamma[3] \)) does not seem to be sufficient enough as we have to describe the space \( \varphi_{f,1} \) is mapped to, explicitly. Moreover, given any \( (f_1, f_4) \) having all the properties determined to far, we want to define a function \( \varphi \) via \( \varphi := \sum_{j=1}^{8} f_j \cdot \vartheta_{h_j} \) (where the remaining \( f_j \) are defined using 3.22 and 3.23). Indeed, it will turn out that this function is a quaternionic Jacobi-form then, and in a further step we will lift such a Jacobi-form to a quaternionic modular form (having exactly \( \varphi \) as its first Fourier-Jacobi-coefficient). In any way, we have to determine the space \( \varphi_{f,1} \) is mapped to more explicitly. And the most promising approach seems to be looking for a way to isomorphically map \( (f_1, f_4) \) to some space of elliptic modular forms. The reason why this is the most promising approach is that elliptic modular forms are quite well understood, to a vastly greater degree than for example Siegel, or even more, quaternionic modular forms.

Fortunately, there actually exists a way to isomorphically map \( (f_1, f_4) \) (and thus \( \varphi_{f,1} \)) to a certain, quite special space of elliptic modular forms. Again, assume that \( \nu_j \) is a multiplier system of odd weight \( k \), indeed. And note that according to [De96], \( \Gamma_0[3] \) is generated by \(-I, T_1\) and \( J_1^{-1} T_1^{-3} J_1 = \left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right) \). We used that fact before in the proof of (1.22).
(3.12) Proposition. Suppose \( f \in \text{Sp}_2(O), k, v_i \). If \( f_1 \) and \( f_4 \) denote the functions used to decompose the first Fourier-Jacobi-coefficient \( \varphi_{f,1} \) as in 3.24, define

\[
\hat{f}(\tau) := f_1(3\tau) + (1 + i)f_4(3\tau), \quad \tau \in \mathcal{H}.
\]

Then

\[
\hat{f} \in \mathcal{M}_{k-2}^3
\]

holds, where

\[
\mathcal{M}_{k-2}^3 := \left\{ g \in \Gamma_0[3], k-2, \mu_i \mid g(\tau) = \sum_{n \in \mathbb{N}, \, n \neq 0} \alpha(n)e^{2\pi in\tau}, \tau \in \mathcal{H} \right\} \leq \Gamma_0[3], k-2, \mu_i_0.
\]

Here \( \mu_i \) is an abelian character for \( \Gamma_0[3] \). Its values for the generators of \( \Gamma_0[3] \) are given by

\[
\mu_i(-1) = -1, \quad \mu_i(T_i) = 1, \quad \mu_i(J_1^{-1}T_2^{-3}J_1) = \rho := e^{\frac{3}{4}\pi i}.
\]

If the Fourier-expansion of \( f \) is given by

\[
f(Z) = \sum_{\Gamma \in \text{Her}_2(O), \, \Gamma \geq 0} \alpha_f(T)e^{2\pi i\tau(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]

then the Fourier-expansion of \( \hat{f} \) is as follows:

\[
\hat{f}(\tau) = \sum_{n \in \mathbb{N}, \, n \neq 0} \hat{\alpha}_f(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H},
\]

where

\[
\hat{\alpha}_f(n) = \begin{cases} 
0, & \text{if } n \in 3\mathbb{N}_0, \\
(1 + i) \cdot \alpha_f \left( \frac{r+1}{2} \left( \frac{h_1}{h_2} + \frac{h_3}{h_2} \right) \right), & \text{if } n = 3r + 1, \ r \in \mathbb{N}_0, \\
\alpha_f \left( \frac{r+1}{2} \frac{h_1}{h_2} \right), & \text{if } n = 3r + 2, \ r \in \mathbb{N}_0,
\end{cases}
\]

with \( h_1 \) and \( h_3 \) defined as in 3.18.

Proof: So let \( \hat{f}(\tau) = f_1(3\tau) + (1 + i)f_4(3\tau) \). The statement about the Fourier-expansion is clear already, as we have seen above (on page 111) that \( f_1 \) and \( f_4 \) are given by

\[
f_1(\tau) = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{2} \frac{h_1}{h_2} \right) e^{2\pi i(n+\frac{1}{2})\tau},
\]

\[
f_4(\tau) = \sum_{n=0}^{\infty} \alpha_f \left( \frac{n+1}{2} \left( \frac{h_1}{h_2} + \frac{h_3}{h_2} \right) \right) e^{2\pi i(n+\frac{1}{2})\tau}.
\]

Of course, \( \hat{f}(\tau)|_{k-2}(-1)(\tau) = (-1)^{k-2}\hat{f}(\tau) = -\hat{f}(\tau) \) holds as \( k \) is odd. Now, according to (3.11)
we have
\[ f_1(\tau + 1) = \rho^2 f_1(\tau), \quad \tau^{-k+2} f_1(-\tau^{-1}) = \frac{i}{\sqrt{3}} f_1(\tau) + \frac{1+i}{\sqrt{3}} f_4(\tau), \]
\[ f_4(\tau + 1) = \rho f_4(\tau), \quad \tau^{-k+2} f_4(-\tau^{-1}) = \frac{1+i}{\sqrt{3}} f_1(\tau) - \frac{i}{\sqrt{3}} f_4(\tau). \]

Hence we compute
\[ \hat{f}(\tau + 1) = f_1(3\tau + 3) + (1+i)f_4(3\tau + 3) = \rho^6 f_1(3\tau) + (1+i)\rho^3 f_4(3\tau) = \hat{f}(\tau), \]
or in other words \( \hat{f}|_{k-2}T_1 = \hat{f} \). The last generator is a bit more involved. Making use of the transformation laws from above we compute
\[
\hat{f}((J_1^{-1}T_1^{-3}J_1)(\tau)) = \hat{f}(\frac{\tau}{3\tau+1}) = f_1\left(\frac{3\tau}{3\tau+1}\right) + (1+i)f_4\left(\frac{3\tau}{3\tau+1}\right)
\]
\[ = f_1\left(1 - \frac{1}{3\tau+1}\right) + (1+i)f_4\left(1 - \frac{1}{3\tau+1}\right)
\]
\[ = \rho^2 f_1(-3\tau + 1) + (1+i)\rho f_4(-3\tau + 1) \]
\[ = \frac{(3\tau+1)^{k-2}}{\sqrt{3}} \left( \rho^2 i f_1(3\tau + 1) + \rho^2(-1+i) f_4(3\tau + 1) \right)
\]
\[ + \rho(1+i)^2 f_1(3\tau + 1) - \rho(1+i) i f_4(3\tau + 1) \]
\[ = \frac{(3\tau+1)^{k-2}}{\sqrt{3}} \left( \rho^4 i + \rho^3(1+i)^2 f_1(3\tau) + \rho^3(-1+i) - \rho^2(1+i) f_4(3\tau) \right)
\]
\[ = (3\tau + 1)^{k-2} \left( \rho f_1(3\tau) + \rho(1+i) f_4(3\tau) \right) = (3\tau + 1)^{k-2} \rho \hat{f}(\tau), \]

where the second last step is just a straightforward calculation. So we proved \( \hat{f}|_{k-2}(J_1^{-1}T_1^{-3}J_1) = \rho \hat{f} \). And as \( \Gamma_0[3] \) is generated by the matrices we considered, it is clear that \( \hat{f} \) possesses a certain transformation behavior for all \( M \in \Gamma_0[3] \) in virtue of 1.14. Now if we knew that \( \varphi_{f,1} \)

existed non-trivially (and hence \( \hat{f} \) by definition), we would not have to prove that extending \( \mu_i \)

multiplicatively (if possible at all) actually yields an abelian character for \( \Gamma_0[3] \) (confer (1.52)). But yet we do not even know if \( \nu_i \) is a multiplier system, hence we also do not know if \( f \) exists non-trivially. So for the sake of completeness we will prove that there exists an abelian character such that its values on the generators of \( \Gamma_0[3] \) coincide with the values from above, indeed.

So let \( \chi_3 \) be the non-trivial Dirichlet character \( \text{mod } 3 \), i.e.

\[
\chi_3(n) = \begin{cases} 
0, & \text{if } n \equiv 0 \pmod{3}, \\
1, & \text{if } n \equiv 1 \pmod{3}, \\
-1, & \text{if } n \equiv -1 \pmod{3}.
\end{cases}
\]

Then define
\[
\mu_i(a \ b \ c \ d) := \chi_3(d) \cdot \rho^{\chi_3(d)c/3}
\]

for \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0[3] \). (Note that being an element of \( \Gamma_0[3] \) implies \( c \in 3\mathbb{Z} \).) An easy verification yields \( \mu_i(-1) = -1, \mu_i(T_1) = 1 \) and \( \mu_i(J_1^{-1}T_1^{-3}J_1) = \rho \) as desired. So we only need to prove that \( \mu_i \), defined as above, turns out to be an abelian character. Therefore, let \( M_1 = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) \), \( M_2 = \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \), and \( T \) act on \( f \) as follows: (Note that we are free to choose \( c \) and \( d \) in the \( \mu_i \) definition),

\[
\mu_i((J_1^{-1}T_1^{-3}J_1) M_1) = \mu_i((J_1^{-1}T_1^{-3}J_1) M_2).
\]

This completes the proof that \( \mu_i \) is an abelian character.
We already showed that, apart from prefactors, we have already seen that

\[ \mu_i(M_1)\mu_i(M_2) = \chi_3(d_1) \cdot \rho_{\chi_3(d_1)c_1/3} \chi_3(d_2) \cdot \rho_{\chi_3(d_2)c_2/3} = \chi_3(d_1d_2)\rho_{\chi_3(d_1)c_1 + \chi_3(d_2)c_2/3} \]

and

\[ \mu_i(M_1M_2) = \chi_3(c_1b_2 + d_1d_2)\rho_{\chi_3(c_1b_2 + d_1d_2)(c_1a_2 + d_1c_2)/3} = \chi_3(d_1d_2)\rho_{\chi_3(d_1d_2)(c_1a_2 + d_1c_2)/3}, \]

as \( c_1b_2 \in 3\mathbb{Z} \). Note that \( \rho^3 = 1 \), so all we have to check is whether

\[ (\chi_3(d_1)c_1 + \chi_3(d_2)c_2)/3 \equiv \chi_3(d_1d_2)(c_1a_2 + d_1c_2)/3 \mod 3 \]

holds. By definition \( \chi_3(d) \equiv d \mod 3 \) holds, as well as \( \det(M_2) = a_2d_2 - b_2c_2 = 1 \). \( \det(M_1) = \det(M_2) = 1 \) also implies \( d_1, d_2 \not\equiv 0 \mod 3 \), as \( c_1, c_2 \in 3\mathbb{Z} \). This leads to \( d_1^2 \equiv 1 \mod 3 \). Finally, \( c_1c_2 \in 9\mathbb{Z} \) holds true. Keeping all that in mind, we compute

\[ \chi_3(d_1d_2)(c_1a_2 + d_1c_2)/3 \equiv d_1d_2(c_1a_2 + d_1c_2)/3 \equiv d_1(c_1 + c_2b_2 + d_1d_2c_2)/3 \equiv d_1c_1/3 + d_2^2d_2c_2/3 \equiv d_1c_1/3 + d_2c_2/3 \equiv (\chi_3(d_1)c_1 + \chi_3(d_2)c_2)/3 \mod 3. \]

Thus \( \mu_i \) actually turns out to be an abelian character for \( \Gamma_0[3] \). What remains to be done is to prove that \( \hat{f} \) is a cusp form. As we are considering elliptic modular forms, this means that the constant term in the Fourier-expansion of \( \hat{f}|_{k-2}M \) vanishes for all \( M \in \text{SL}_2(\mathbb{Z}) \), so

\[ \hat{f}|_{k-2}M(\tau) = \sum_{n \in \mathbb{N}} a(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}, \]

where \( l \) is some appropriate integer. (In our case, \( l = 9 \), as \( \hat{f} \) is a modular form for \( \Gamma[9] \) with trivial character – see chapter 1). Note that if all these Fourier-expansions have the shape from above, then (M.3) is automatically fulfilled, of course. So we actually have \( \hat{f} \in [\Gamma_0[3], k-2, \mu_i]_0 \). We already showed that, apart from prefactors, \( \hat{f} \) is invariant under \( \hat{f}|_{k-2}M \) for \( M \in \Gamma_0[3] \). Suppose \( M_1, \ldots, M_n \) is some transversal of \( \Gamma_0[3] \backslash \text{SL}_2(\mathbb{Z}) \). Then, for every \( M \in \text{SL}_2(\mathbb{Z}) \) there exist \( j \in \{1, \ldots, n\} \) and \( N \in \Gamma_0[3] \) such that \( M = NM_j \), and we obtain \( \hat{f}|_{k-2}M = \nu(N)\hat{f}|_{k-2}M_j \). So of course, we only have to verify the assumption from above for such a transversal. According to [Kl98, Le.1.24], one such transversal is given by

\[ L, J_1, J_1T_1, J_1T_1^2. \]

We have already seen that

\[ \hat{f}(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \hat{a}_f(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}, \]
holds, hence we obtain the assumption for \( \hat{f}|_{k-2}I \). For \( f_1 \) we get

\[
\hat{f}|_{k-2}f_1(\tau) = \tau^{-k+2}f(-\tau^{-1}) = \tau^{-k+2} \left( f_1(-3\tau^{-1}) + (1 + i)f_4(-3\tau^{-1}) \right)
\]

\[
= 3^{-k+2} \left( \left( \frac{\tau}{3} \right)^{-k+2} f_1(-\left( \frac{\tau}{3} \right)^{-1}) + (1 + i) \left( \frac{\tau}{3} \right)^{-k-2} f_4(-\left( \frac{\tau}{3} \right)^{-1}) \right)
\]

\[
= 3^{-k+2} \frac{1}{\sqrt{3}} \left( (i f_1 \left( \frac{\tau}{3} \right) + (-1 + i)f_4 \left( \frac{\tau}{3} \right)) + (1 + i)^2 f_1 \left( \frac{\tau}{3} \right) - i(1 + i)f_4 \left( \frac{\tau}{3} \right) \right)
\]

by using (3.11). As we have seen above (on page 111), \( f_1 \) and \( f_4 \) possess Fourier-expansions of the shape

\[
f_{1/4}(\tau) = \sum_{n \in \frac{1}{2} \mathbb{N}} \alpha_{1/4}(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H},
\]

and thus \( \hat{f}|_{k-2}f_1 \) possesses a Fourier-expansion of the shape

\[
\hat{f}|_{k-2}f_1(\tau) = \sum_{n \in \frac{1}{4} \mathbb{N}} \alpha(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H},
\]

for appropriate coefficients \( \alpha(n) \). So the constant term vanishes here, too. Finally, we have

\[
\hat{f}|_{k-2}(f_1T_1)(\tau) = (\hat{f}|_{k-2}f_1)|_{k-2}T_1(\tau) = \sum_{n \in \frac{1}{4} \mathbb{N}} \alpha(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H},
\]

and the appropriate expansion for \( f_1T^2_1 \). Thus \( \hat{f} \) actually turns out to be a cusp form, which completes the proof. \( \square \)

Now that we finally found out which elliptic modular forms we have to lift to quaternionic modular forms, we can go all the way back up. We started with the first Fourier-Jacobi-coefficient \( \varphi_{f,1} \). We were able to map it to \( \mathcal{M}_k^{3+} \) with an injective homomorphism. Now we will see that this homomorphism is even surjective. This means we will start with an elliptic modular form \( g \in \mathcal{M}_k^{3+} \), construct appropriate \( f_1 \) and \( f_4 \) (and the rest of the \( f_i \) in view of 3.22 and 3.23), and afterwards some \( \varphi \) via \( \varphi := \sum_{i=1}^8 f_i \cdot \vartheta_{h_i} \), which will turn out to be a quaternionic Jacobi-form. The image under the injective homomorphism from above will be \( g \), again.

But before we start, let us include a remark, first: So far, we have not shown that there is some odd \( k \) such that \( \mathcal{M}_k^{3+} \) is non-trivial (i.e. it does not only contain the zero-function). We have seen that \( \mu_i \) actually is an abelian character, but yet alone the required condition on the Fourier-expansion is really special. So we might still talk about trivial spaces. Fortunately, it will turn out that \( \mathcal{M}_k^{3+} \) is non-trivial if \( k \geq 7 \). And one can, at least theoretically, construct non-trivial elements of these spaces explicitly. And we will even be able to determine the dimension of \( \mathcal{M}_k^{3+} \), which is \( \left[ \frac{k}{6} \right] \). But this will all have to wait until the next section. So far, we only have to bear in mind that \( \mathcal{M}_k^{3+} \) is non-trivial if \( k \geq 7 \) (and \( k \) is odd). And as a final note: The denotation \( \mathcal{M}_k^{3+} \) has been chosen in reminiscence of the so-called Kohnen plus space (cf. [Ko80]), the space of elliptic modular forms needed for the well known Maaß lift in the Siegel case.

So let \( g \in \mathcal{M}_k^{3+} \). We “reconstruct” \( f_1 \) and \( f_4 \) in the following way:
(3.13) Lemma. Suppose $k \in \mathbb{N}$ is odd and let $g \in \mathcal{M}_{k-2}^{3+}$ and $\rho = e^{\frac{2\pi i}{3}}$. Define functions $f_1, f_4 : \mathcal{H} \to \mathbb{C}$ by

$$f_1(\tau) := \frac{1}{1 - \rho} \left( \rho g \left( \frac{\tau + 1}{3} \right) - g \left( \frac{\tau - 1}{3} \right) \right),$$

$$f_4(\tau) := \frac{1}{\rho - 1} \left( g \left( \frac{\tau + 1}{3} \right) - \rho g \left( \frac{\tau - 1}{3} \right) \right) \cdot \frac{1}{1 + i}.$$ 

Then we obtain the following transformation behavior:

$$f_1|_{k-2}(-1) = -f_1,$$

$$f_4|_{k-2}(-1) = -f_4,$$

$$f_1|_{k-2}T_1 = \rho^2 f_1,$$

$$f_4|_{k-2}T_1 = \rho f_4,$$

$$f_1|_{k-2}J_1 = i \sqrt{3} f_1 + \frac{-1 + i}{\sqrt{3}} f_4,$$

$$f_4|_{k-2}J_1 = \frac{1 + i}{\sqrt{3}} f_1 - \frac{i}{\sqrt{3}} f_4.$$ 

Moreover,

$$g(\tau) = f_1(3\tau) + (1 + i)f_4(3\tau)$$

holds for all $\tau \in \mathcal{H}$. Suppose

$$g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \alpha(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}$$

is the Fourier-expansion of $g$, with appropriate Fourier-coefficients $\alpha(n)$. Then the Fourier-expansions of $f_1$ and $f_4$ are given by

$$f_1(\tau) = \sum_{n \in \mathbb{N}, n \equiv 2\ (3)} \alpha(n)e^{2\pi i \frac{2}{3} \tau}, \quad \tau \in \mathcal{H}$$

and

$$f_4(\tau) = \frac{1}{1 + i} \sum_{n \in \mathbb{N}, n \equiv 1\ (3)} \alpha(n)e^{2\pi i \frac{1}{3} \tau}, \quad \tau \in \mathcal{H}.$$ 

They are absolutely and locally uniformly convergent.

**Proof:** So let $g \in \mathcal{M}_{k-2}^{3+}$. By definition we have

$$g|_{k-2}(-1) = -g, \quad g|_{k-2}T_1 = g, \quad g|_{k-2}(J_1 T_1^{-3} J_1) = \rho g,$$

and the absolutely and locally uniformly convergent Fourier-expansion of $g$ is given by

$$g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \alpha(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}$$

for appropriate Fourier-coefficients $\alpha(n)$. First, we will verify that $g(\tau) = f_1(3\tau) + (1 + i)f_4(3\tau)$ holds true, indeed. We can do this by examining the Fourier-expansions of $f_1$ and $f_4$. Note that if $n \equiv 1\ (3)$, then $\rho n + 1 - \rho^{-n} = 0$, since $n + 1 \equiv 2 \equiv -n$ holds mod 3 and $\rho^3 = 1$. On the
other hand, \( n \equiv 2 \mod 3 \) yields \( \rho^{n+1} - \rho^{-n} = 1 - \rho \), since \( n + 1 \equiv 0 \) and \( -n \equiv 1 \mod 3 \). Hence we compute

\[
f_1(\tau) = \frac{1}{1-\rho} \left( \rho g\left(\frac{\tau+1}{3}\right) - g\left(\frac{\tau-1}{3}\right) \right)
= \frac{1}{1-\rho} \left( \rho \sum_{n \in \mathbb{N}, n \neq 0} (\alpha(n) e^{2\pi i n \tau_2/3} - \sum_{n \in \mathbb{N}, n \neq 0} (\alpha(n) e^{2\pi i n \tau - 2\pi i n \tau_2}) \right)
= \frac{1}{1-\rho} \sum_{n \in \mathbb{N}, n \neq 0} \left( \rho^n - \rho^{-n} \right) \alpha(n) e^{2\pi i n \tau}
= \sum_{n \in \mathbb{N}, n \equiv 2(3)} \alpha(n) e^{2\pi i n \tau} \quad \tau \in \mathcal{H}.
\]

Using the same considerations, one easily verifies \( \rho^n - \rho^{-n+1} = \rho - 1 \) for \( n \equiv 1 \mod 3 \) and \( \rho^n - \rho^{-n+1} = 0 \) for \( n \equiv 2 \mod 3 \). Thus we analogously obtain

\[
f_4(\tau) = \frac{1}{\rho - 1} \left( g\left(\frac{\tau+1}{3}\right) - g\left(\frac{\tau-1}{3}\right) \right) \cdot \frac{1}{1-\rho}
= \frac{1}{\rho - 1} \sum_{n \in \mathbb{N}, n \neq 0} (\alpha(n) e^{2\pi i n \tau_2/3} - \rho \sum_{n \in \mathbb{N}, n \neq 0} (\alpha(n) e^{2\pi i n \tau - 2\pi i n \tau_2}) \right)
= \frac{1}{\rho - 1} \sum_{n \in \mathbb{N}, n \neq 0} \left( \rho^n - \rho^{-n+1} \right) \alpha(n) e^{2\pi i n \tau}
= \sum_{n \in \mathbb{N}, n \equiv 1(3)} \alpha(n) e^{2\pi i n \tau} \quad \tau \in \mathcal{H}.
\]

An easy consequence is that \( g(\tau) = f_1(3\tau) + (1 + i)f_4(3\tau) \) holds by simply making use of these Fourier-expansions. Furthermore, according to [Mi89, cor.2.1.6], the Fourier-coefficients \( \alpha(n) \) fulfill \( \alpha(n) = O(n^{k-2}/2) \), since \( g \) is a cusp form for \( \Gamma[9] \) and the trivial character. Thus it is clear that the Fourier-expansions of \( f_1 \) and \( f_4 \) from above converge absolutely and locally uniformly (or confer [Mi89, le.4.3.3], for example).

Next, we verify the asserted transformation laws. As we have seen several times so far, \( f_1|_{k-2}(-1) = \rho f_1 \) and \( f_4|_{k-2}(-1) = -f_4 \) are simple necessities as \( k \) is odd. Using the determined Fourier-expansions (and taking \( \rho^3 = 1 \) into account) we obtain

\[
f_1(\tau + 1) = \sum_{n \in \mathbb{N}, n \equiv 2(3)} \alpha(n) e^{2\pi i n \tau_2/3} + 2\pi i n \tau_2\cdot \rho^2 \sum_{n \in \mathbb{N}, n \equiv 2(3)} \alpha(n) e^{2\pi i n \tau} = \rho^2 f_1(\tau).
\]

for all \( \tau \in \mathcal{H} \), and analogously

\[
f_4(\tau + 1) = \rho f_4(\tau).
\]

Or in other words \( f_1|_{k-2}T_1 = \rho^2 f_1 \) as well as \( f_4|_{k-2}T_1 = \rho f_4 \). Again, the more involved case is the transformation behavior with respect to \( f_1 \). Note that in the proof of (3.12) we explicitly determined the abelian character \( \mu_i \). So \( g \in M_{k-2}^1 \) implies \( g\left(\frac{\tau}{3\tau+1}\right) = (3\tau + 1)^{k-2} \rho g(\tau) \) as well as \( g\left(-\frac{\tau}{3\tau+1}\right) = (-3\tau + 1)^{k-2} \rho^{-1} g(\tau) \) for all \( \tau \in \mathcal{H} \). Let \( \tau \in \mathcal{H} \) and define \( \tau_1 = \frac{-\tau}{3\tau+1} \in \mathcal{H} \).
Another straightforward calculation shows
\[(1 - \rho)f_1(-\tau^{-1}) = \rho g \left( -\tau^{-1} + \frac{1}{3} \right) - g \left( -\tau^{-1} - \frac{1}{3} \right)\]
\[= \rho g \left( \frac{i}{\sqrt{3}} \right) - g \left( -\tau^{-1} - \frac{1}{3} \right)\]
\[= \rho g \left( \frac{i}{\sqrt{3}} + \frac{1}{3} \right) - g \left( -\tau^{-1} - \frac{1}{3} \right)\]
\[= \rho(3\tau_1 + 1)k^{-2}g(\tau_1) - (-3\tau_1 + 1)^{-2}g(\tau_2)\]
\[= \rho^2 \tau^{-k-2}g(\tau_1) - \rho^2 (-\tau)^{k-2}g(\tau + \frac{1}{3})\]
\[= \rho^2 \tau^{-k-2}g(\tau + \frac{1}{3}) + g(\tau + \frac{1}{3})\]

or in other words
\[f_1|_{k-2}f_1 = \frac{\rho^2}{i \rho} \left( g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau + 1}{3} \right) \right).\]

On the other hand, a straightforward calculation shows
\[\frac{i}{\sqrt{3}}f_1(\tau) + \frac{1+i}{\sqrt{3}}f_4(\tau) = \frac{i}{\sqrt{3}} \frac{1}{1+i} \left( \rho g \left( \frac{\tau + 1}{3} \right) - g \left( \frac{\tau - 1}{3} \right) \right) + \frac{1+i}{\sqrt{3}} \left( \frac{1}{1+i}(\rho - 1) \left( g \left( \frac{\tau + 1}{3} \right) - \rho g \left( \frac{\tau - 1}{3} \right) \right) \right)\]
\[= \left( \frac{i}{\sqrt{3}(1-\rho)} + \frac{1+i}{\sqrt{3}(1+i)(\rho-1)} \right) g \left( \frac{\tau + 1}{3} \right) + \left( -\frac{i}{\sqrt{3}(1-\rho)} - \frac{-1+i}{\sqrt{3}(1+i)(\rho-1)} \right) g \left( \frac{\tau - 1}{3} \right)\]
\[= \frac{i}{\sqrt{3}} \left( g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau - 1}{3} \right) \right) = \frac{\rho^2}{i \rho} \left( g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau - 1}{3} \right) \right),\]

and thus
\[f_1|_{k-2}f_1 = \frac{i}{\sqrt{3}}f_1(\tau) + \frac{1+i}{\sqrt{3}}f_4(\tau).\]

The calculation for \(f_4\) is done analogously:
\[(1 + i)(\rho - 1)f_4(-\tau^{-1}) = g \left( -\tau^{-1} + \frac{1}{3} \right) - \rho g \left( -\tau^{-1} - \frac{1}{3} \right)\]
\[= g \left( \frac{\tau + 1}{3} \right) - \rho g \left( \frac{\tau - 1}{3} \right)\]
\[= g \left( \frac{\tau + 1}{3} \right) - \rho g \left( \frac{\tau - 1}{3} \right)\]
\[= (3\tau_1 + 1)k^{-2}g(\tau_1) - \rho(-3\tau_1 + 1)^{-2}g(\tau_2)\]
\[= \rho \tau^{-k-2}g \left( \frac{\tau + 1}{3} \right) - (-\tau)^{k-2}g \left( \frac{\tau - 1}{3} \right)\]
\[= \tau^{k-2} \left( \rho g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau - 1}{3} \right) \right),\]

or again in other words
\[f_4|_{k-2}f_1 = \frac{1}{(1+i)(\rho-1)} \left( \rho g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau - 1}{3} \right) \right).\]

Another straightforward calculation shows
\[\frac{1+i}{\sqrt{3}}f_1(\tau) - \frac{i}{\sqrt{3}}f_4(\tau) = \frac{1+i}{\sqrt{3}} \frac{1}{1+i} \left( \rho g \left( \frac{\tau + 1}{3} \right) - g \left( \frac{\tau - 1}{3} \right) \right) - \frac{i}{\sqrt{3}} \frac{1}{1+i}(\rho - 1) \left( g \left( \frac{\tau + 1}{3} \right) - \rho g \left( \frac{\tau - 1}{3} \right) \right)\]
\[= \left( \frac{(1+i)(\rho)}{\sqrt{3}(1-\rho)} - \frac{i}{\sqrt{3}(1+i)(\rho-1)} \right) g \left( \frac{\tau + 1}{3} \right) + \left( -\frac{-1+i}{\sqrt{3}(1-\rho)} + \frac{i}{\sqrt{3}(1+i)(\rho-1)} \right) g \left( \frac{\tau - 1}{3} \right)\]
\[= \frac{1}{(1+i)(\rho-1)} \left( \rho g \left( \frac{\tau + 1}{3} \right) + g \left( \frac{\tau - 1}{3} \right) \right),\]
and thus

\[ f_k^1|_{k-2} = \frac{1+i}{\sqrt{3}}f_1(\tau) - \frac{i}{\sqrt{3}}f_4(\tau). \]

The easiest way to verify the equalities concerning the prefactors from above would be using a computer algebra system. Here, [MAPLE] was used. Of course, it could be done manually, but (needless to say) we omit the detailed calculations here.

Having “reconstructed” \( f_1 \) and \( f_4 \) from an arbitrary \( g \in M^+_{k-2} \), we obtain the other \( f_j \) quite easily using 3.22 and 3.23. Moreover, the transformation behavior of \( F = (f_0, \ldots, f_8)' \) is quite obvious when observing how we obtained the transformation laws for \( f_1 \) and \( f_4 \) in (3.11).

(3.14) Lemma. Suppose \( k \in \mathbb{N} \) is odd and let \( g \in M^+_{k-2} \) and \( \rho = e^{\pi i/3} \). Define \( f_1 \) and \( f_4 \) like in (3.13). Moreover, define further functions \( f_j \) via the identities

\[
\begin{align*}
    f_1 &\equiv 0, \\
    f_1 &= -f_2 = if_3 = -if_6, \\
    f_4 &= if_5 = -if_7 = -f_8,
\end{align*}
\]

and \( F = (f_0, \ldots, f_8)' \). There exists a homomorphism \( \kappa : SL_2(\mathbb{Z}) \to GL_9(\mathbb{Q}(\rho)) \) such that

\[
F|_{k-2}M := (f_{0}|_{k-2}M(\tau), \ldots, f_{8}|_{k-2}M(\tau))' = (\kappa(M))^{-1} \cdot F
\]

holds for all \( M \in SL_2(\mathbb{Z}) \). For the generators of \( SL_2(\mathbb{Z}) \), \( (\kappa')^{-1} \) is given by

\[
(\kappa(T_1))^{-1} = \text{diag}(1, \rho^2, \rho^2, \rho, \rho, \rho^2, \rho, \rho, \rho)
\]

and

\[
(\kappa(J_1))^{-1} = -\frac{1}{3} \begin{pmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    1 & \rho^2 & \rho & 1 & \rho^2 & \rho & 1 & \rho & \rho \\
    1 & \rho & \rho^2 & 1 & \rho & \rho^2 & 1 & \rho & \rho^2 \\
    1 & 1 & 1 & \rho^2 & \rho^2 & \rho^2 & \rho & \rho & \rho \\
    1 & \rho^2 & \rho & \rho^2 & 1 & \rho & \rho & \rho^2 & 1 \\
    1 & \rho & \rho^2 & \rho^2 & 1 & \rho & \rho & \rho^2 & 1 \\
    1 & 1 & 1 & \rho & \rho & \rho & \rho & \rho & \rho \\
    1 & \rho^2 & \rho & \rho^2 & 1 & \rho & \rho & \rho^2 & 1 \\
    1 & \rho & \rho^2 & \rho^2 & 1 & \rho & \rho & \rho^2 & 1
\end{pmatrix}.
\]

Proof: Of course, there is not much to prove here. As we have seen several times before, the assertion only has to be verified for the generators of \( SL_2(\mathbb{Z}) \). The transformation law regarding \( T_1 \) is obvious in view of the transformation behavior of \( f_1 \) and \( f_4 \) in (3.13) and the definition of the other \( f_j \). But also the asserted transformation law with regard to \( J_1 \) is a simple recalculation using (3.13) and the definition of the other \( f_j \). But for the sake of completeness, let us give two examples – the remaining ones are as simple. According to (3.13) we have

\[
f_1|_{k-2}f_1 = \frac{i}{\sqrt{3}}f_1 + \frac{1+i}{\sqrt{8}}f_4.
\]
So the assertion is that
\[
\frac{i}{\sqrt{3}} f_1 + \frac{-1+i}{\sqrt{3}} f_4 = -\frac{1}{3}(\rho^2 f_1 + \rho f_2 + f_3 + \rho^2 f_4 + \rho f_5 + f_6 + \rho^2 f_7 + \rho f_8) \\
= -\frac{1}{3}(\rho^2 - \rho - i + i) f_1 - \frac{1}{3}(\rho^2 - i\rho + i\rho^2 - \rho) f_4
\]
holds. Thus we only have to verify \( \frac{i}{\sqrt{3}} = -\frac{1}{3}(\rho^2 - \rho - i + i) \) and \( \frac{-1+i}{\sqrt{3}} = -\frac{1}{3}(\rho^2 - i\rho + i\rho^2 - \rho) \). This is a simple recalculation and can be done using [MAPLE], for example. Another example would be \( f_5 \). According to (3.13) we have
\[
f_5|_{k-2} = -i f_4|_{k-2} = \frac{1-i}{\sqrt{3}} f_1 - \frac{1}{\sqrt{3}} f_4.
\]
So this time verifying \( -\frac{1}{3}(\rho - \rho^2 - i\rho^2 + i\rho) = \frac{1-i}{\sqrt{3}} \) and \( -\frac{1}{3}(1 - i\rho + i\rho^2 - 1) = -\frac{1}{\sqrt{3}} \) yields the assertion. Again, this is a simple recalculation. \( \square \)

Finally, we can reconstruct our desired quaternionic Jacobi-form, which will become the first Fourier-Jacobi-coefficient of the final lift.

(3.15) Proposition. Suppose \( k \in \mathbb{N} \) is odd and let \( g \in \mathcal{M}_k^{+} \). Define the functions \( f_0, \ldots, f_8 \) like in (3.14), as well as the theta-series \( \theta_{h_{0}}, \ldots, \theta_{h_{8}} \) according to 3.19. Furthermore, let \( F = (f_0, \ldots, f_8)' \), \( \vartheta = (\theta_{h_{0}}, \ldots, \theta_{h_{8}})' \) and
\[
\varphi(\tau, z) := (F(\tau))' \cdot \vartheta(\tau, z) = \sum_{j=0}^{8} f_j(\tau) \cdot \theta_{h_{j}}(\tau, z), \quad \tau \in \mathcal{H}, z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}.
\]
Then
\[
\varphi \in \mathcal{J}(k, 1, \mathcal{O}, \nu_{i})
\]
holds. Define
\[
\gamma_i(h_1) = 1, \quad \gamma_i(h_2) = -1, \quad \gamma_i(h_3) = -i, \quad \gamma_i(h_6) = i, \\
\gamma_i(h_4) = \frac{1}{i+1}, \quad \gamma_i(h_5) = \frac{-i}{i+1}, \quad \gamma_i(h_7) = \frac{i}{i+1}, \quad \gamma_i(h_8) = \frac{-1}{i+1},
\]
and for arbitrary \( t \in \mathcal{O} \setminus 2\mathcal{O} \) define \( \gamma_i(t) := \gamma_i(h_i) \) if \( t \equiv h_i \mod 2\mathcal{O} \). If \( g \) has the Fourier-expansion
\[
g(\tau) = \sum_{\substack{n \in \mathbb{N}, n \neq 0 \in (3)}} a(n) e^{2\pi in \tau}, \quad \tau \in \mathcal{H}
\]
then the Fourier-expansion of \( \varphi \) is given by
\[
\varphi(\tau, z) = \sum_{n \in \mathbb{N}} \sum_{t \in \mathcal{O} \setminus 2\mathcal{O}} \gamma_i(t) a(3n - \frac{3}{4} N(t)) e^{2\pi i (n \tau + \text{Re}(\tau z))}, \quad \tau \in \mathcal{H}, z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}.
\]

Proof: First, let us have a look at the proposed Fourier-expansion. Define \( r(h_1) = r(h_2) = r(h_3) = r(h_6) = 2 \) and \( r(h_4) = r(h_5) = r(h_7) = r(h_8) = 1 \), and for later purposes \( r(t) := r(h_i) \) for
3.2 Quaternionic Maaß lifts of odd weight

t ∈ \mathcal{O}^2 \setminus 2\mathcal{O}, if t ≡ h_j \mod 2\mathcal{O}. The Fourier-expansions of the \( f_j \) have been determined in (3.13), while the theta-series \( \tilde{\theta}_{h_j} \), \( j \in \{0, \ldots, 8\} \) is already defined as a Fourier-series in (3.10). Since the Fourier-expansions of the \( f_j \) and the \( \tilde{\theta}_{h_j} \) are absolutely and locally uniformly convergent, we compute

\[
\varphi(\tau, z) = \sum_{j=1}^{8} f_j(z) \tilde{\theta}_{h_j}(\tau, z)
\]

\[
= \sum_{j=1}^{8} \left( \sum_{n \in \mathbb{N}_0} \gamma_i(h_j) a(3n + r(h_j)) e^{2\pi i (n+r(h_j)/3) \tau} \cdot \sum_{g \in \mathcal{O}} e^{2\pi i (N(g+\frac{1}{2}h_j) + \Re(\tilde{h}_j g))} \right)
\]

\[
= \sum_{j=1}^{8} \sum_{g \in \mathcal{O}} \sum_{n \in \mathbb{N}_0} \gamma_i(h_j) a(3n + r(h_j)) \cdot e^{2\pi i ((n+r(h_j)/3 + N(g+\frac{1}{2}h_j)) \tau + \Re(\tilde{h}_j g))}.
\]

Define

\[
\delta(h_j) := r(h_j)/3 + N(\frac{1}{2}h_j).
\]

An easy recalculation shows that

\[
\delta(h_1) = 1, \quad \delta(h_2) = 2, \quad \delta(h_3) = 1, \quad \delta(h_4) = 1, \quad \delta(h_5) = 2, \quad \delta(h_6) = 2, \quad \delta(h_7) = 2, \quad \delta(h_8) = 3
\]

holds. This is why \( n + r(h_j)/3 + N(g+\frac{1}{2}h_j) = n + r(h_j)/3 + N(g) + N(\frac{1}{2}h_j) + \Re(\tilde{h}_j g) \) is always a natural number, since \( h_j \in \mathcal{O}^2 \) and \( g \in \mathcal{O} \). So for \( n \in \mathbb{N}_0, g \in \mathcal{O} \) and \( j \in \{1, \ldots, 8\} \) define

\[
\tilde{n} := n + r(h_j)/3 + N(g+\frac{1}{2}h_j) \in \mathbb{N}.
\]

An easy calculation yields

\[
3n + r(h_j) = 3\tilde{n} - \frac{3}{4} N(2g + h_j).
\]

And of course such a \( \tilde{n} \in \mathbb{N} \) occurs if and only if

\[
\tilde{n} \geq \frac{1}{4} N(2g + h_j),
\]

since \( r(h_j)/3 + N(g+\frac{1}{2}h_j) \) is the smallest natural number greater or equal to \( \frac{1}{4} N(2g + h_j) \). Finally, \( \{2g + h_j ; g \in \mathcal{O}, j \in \{1, \ldots, 8\} \} = \mathcal{O}^2 \setminus 2\mathcal{O} \). So a simple renaming of the variables \( (t = 2g + h_j \text{ and } \tilde{n} = n + r(t)/3 + N(\frac{1}{2}t) \) as above – note that \( \gamma_i(t) = \gamma_i(h_j) \) holds by definition) and a rearrangement of the infinite sum yields

\[
\varphi(\tau, z) = \sum_{t \in \mathcal{O}^2 \setminus 2\mathcal{O}} \sum_{\tilde{n} \in \mathbb{N}} \sum_{n \in \mathbb{N}} \gamma_i(t) a(3\tilde{n} - \frac{3}{4} N(t)) e^{2\pi i (n\tau + \Re(\tilde{t}))},
\]

\[
\tau \in \mathcal{H}, \quad z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}
\]

as asserted. The absolute and local uniform convergence of this Fourier-expansion is clear because of the absolute and local uniform convergence of the involved Fourier-expansions of the \( f_j \) and \( \tilde{\theta}_{h_j} \).

We will now deal with the assertion that \( \varphi \in \mathcal{F}(k, 1, \mathcal{O}, \nu_i) \) holds. We already verified the
condition concerning the Fourier-expansion (see (3.8)). And of course, \( \varphi \) is holomorphic in both variables since the \( f_j \) and \( \varphi_{h_j} \) are holomorphic. So let us have a closer look at the transformation behavior of \( \varphi \). According to (3.14), \( F|_{k-2}M = (\kappa(M)')^{-1} \cdot F \) holds for all \( M \in \text{SL}_2(\mathbb{Z}) \), while for \( \varphi \) we obtained \( \varphi|_{2,1}[M] = \kappa(M) \cdot \varphi \) in virtue of (3.10). Hence we compute

\[
\varphi|_{k,1}[M](\tau, z) = \sum_{j=0}^{8} ((M\{\tau\})^{-(k-2)} f_j(M(\tau))) \cdot (e^{-2\pi i M(\tau)^{-1}c \tau} \cdot (M\{\tau\})^{-2} \cdot \varphi_{h_j}(M(\tau), M\{\tau\}^{-1}z))
\]

\[
= \sum_{j=0}^{8} f_j|_{k-2}M(\tau) \cdot \varphi_{h_j}|_{2,1}[M](\tau, z) = (F|_{k-2}M(\tau))' \cdot \varphi|_{2,1}[M](\tau, z)
\]

\[
= ((\kappa(M)')^{-1} \cdot F(\tau))' \cdot (\kappa(M) \cdot \varphi(\tau, z)) = (F(\tau))' \cdot \varphi(\tau, z) = \varphi(\tau, z)
\]

for all \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), \( \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \). So one of the four transformation conditions for quaternionic Jacobi-forms is already fulfilled. Another condition to check is whether

\[
\varphi(\tau, \varepsilon z) = \varphi_1(\pi_\varphi(\varepsilon)) \cdot \varphi(\tau, z)
\]

holds for all \( \varepsilon \in \mathcal{E} \). Since \( \mathcal{E} \to \mathbb{C}, \varepsilon \mapsto \varphi_1(\pi_\varphi(\varepsilon)) \) is a homomorphism, this only needs to be verified for generators of \( \mathcal{E} \), of course. According to (1.7), \( \mathcal{E} \) is generated by \( \frac{1}{2}(-1 + i \sqrt{3}) \) and \( i2 \). Note that \( \varphi_1(\pi_\varphi(\frac{1}{2}(-1 - i \sqrt{3}))) = 1 \) and \( \varphi_1(\pi_\varphi(-i2)) = -i \). So let us examine the transformation behavior of the \( \varphi_{h_j}(\tau, z) \) under \( z \mapsto \varepsilon z \). Therefore, let \( \varepsilon \in \mathcal{E} \). We obtain (since \( N(\varepsilon a) = N(a) \) for all \( a \in \mathbb{H} \))

\[
\varphi_{h_j}(\tau, \varepsilon z) = \sum_{g \in \mathcal{O}} e^{2\pi i (N(\varepsilon g + \frac{1}{2}h_j)\tau + \text{Re}(\pi(2g + h_j)))}
\]

\[
= \sum_{g \in \mathcal{O}} e^{2\pi i (N(\varepsilon g + \frac{1}{2}h_j)\tau + \text{Re}(\pi(2g + h_j)))} = \varphi_{h_j}(\tau, z)
\]

for all \( \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \), which means \( \varphi_{h_j} \) is mapped to \( \varphi_{h_j} \) and \( z \mapsto \varepsilon z \) simply permutes the theta-series, since the \( \varphi_{h_j} \) are a transversal of \( \mathcal{O}^2 / 2\mathcal{O} \), again, and the index \( h_j \) can be chosen modulo \( 2\mathcal{O} \) according to 3.19 and (2.3). Since

\[
-\frac{1-i\sqrt{3}}{2} \cdot (1 + \frac{i}{\sqrt{3}}) = -\frac{2i}{\sqrt{3}} = 1 + \frac{i}{\sqrt{3}} - (1 + i \sqrt{3}) \equiv 1 + \frac{i}{\sqrt{3}} \mod 2\mathcal{O}
\]

holds, it is obvious that we have \( -\frac{1-i\sqrt{3}}{2} \cdot h_j \equiv h_j \mod 2\mathcal{O} \). Hence

\[
\varphi(\tau, \frac{1}{2}(-1 + i \sqrt{3}) \cdot z) = \sum_{j=0}^{8} f_j(\tau) \cdot \varphi_{h_j}(\tau, \frac{1}{2}(-1 + i \sqrt{3}) \cdot z) = \sum_{j=0}^{8} f_j(\tau) \cdot \varphi_{h_j}(\tau, z) = \varphi(\tau, z)
\]

follows. Another straightforward calculation shows that modulo \( 2\mathcal{O} \)

\[
- i_2 h_1 \equiv h_6 , \quad - i_2 h_2 \equiv h_3 , \quad - i_2 h_3 \equiv h_1 , \quad - i_2 h_6 \equiv h_2 ,
\]

\[
- i_2 h_4 \equiv h_7 , \quad - i_2 h_5 \equiv h_4 , \quad - i_2 h_7 \equiv h_8 , \quad - i_2 h_8 \equiv h_5
\]
holds. So we compute
\[
\varphi(\tau, i_2z) = f_1(\tau) \cdot (\vartheta_{h_1}(\tau, i_2z) - \vartheta_{h_2}(\tau, i_2z) - i\vartheta_{h_1}(\tau, i_2z) + i\vartheta_{h_2}(\tau, i_2z)) \\
+ f_4(\tau) \cdot (\vartheta_{h_1}(\tau, i_2z) - i\vartheta_{h_2}(\tau, i_2z) + i\vartheta_{h_1}(\tau, i_2z) - \vartheta_{h_2}(\tau, i_2z)) \\
= f_1(\tau) \cdot (\vartheta_{h_1}(\tau, z) - \vartheta_{h_2}(\tau, z) - i\vartheta_{h_1}(\tau, z) + i\vartheta_{h_2}(\tau, z)) \\
+ f_4(\tau) \cdot (\vartheta_{h_1}(\tau, z) - i\vartheta_{h_2}(\tau, z) + i\vartheta_{h_1}(\tau, z) - \vartheta_{h_2}(\tau, z)) \\
= (-i f_1(\tau)) \cdot (\vartheta_{h_1}(\tau, z) - \vartheta_{h_2}(\tau, z) - i\vartheta_{h_1}(\tau, z) + i\vartheta_{h_2}(\tau, z)) \\
+ (-i f_4(\tau)) \cdot (\vartheta_{h_1}(\tau, z) - i\vartheta_{h_2}(\tau, z) + i\vartheta_{h_1}(\tau, z) - \vartheta_{h_2}(\tau, z)) \\
= -i\varphi(\tau, z).
\]

Thus already two of the four transformation conditions for quaternionic Jacobi-forms are fulfilled. The next transformation behavior is quite obvious since we already determined the Fourier-expansion of \( \varphi \). Note that by definition \( \text{Re}(\mathcal{F}_2) \in \mathbb{Z} \) holds for \( s_2 \in \mathcal{O} \), since \( t \in \mathcal{O}^2 \). So we get
\[
\varphi(\tau + s_1, z + s_2) = \sum_{n \in \mathbb{N}} \sum_{1 \leq \ell \leq 2} \gamma_1(t) \alpha(3n - \frac{3}{4} N(t)) \cdot e^{2\pi i (ns_1 + \text{Re}(\mathcal{F}_2))} e^{2\pi i (it + \text{Re}(\mathcal{F}_2))} = \varphi(\tau, z)
\]
for all \( s_1 \in \mathbb{Z}, s_2 \in \mathcal{O}, \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \). Thus only
\[
\varphi(\tau, z + \tau r) = e^{-2\pi i (\tau N(r) + 2 \text{Re}(\mathcal{F}_2))} \cdot \varphi(\tau, z)
\]
remains to be verified for \( r \in \mathcal{O} \). So let \( r \in \mathcal{O}, \tau \in \mathcal{H}, z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) and \( j \in \{1, \ldots, 8\} \). We obtain
\[
e^{2\pi i (\tau N(r) + 2 \text{Re}(\mathcal{F}_2))} \cdot \vartheta_{h_j}(\tau, z + \tau r) = \sum_{g \in \mathcal{O}} e^{2\pi i ((N(g + \frac{1}{2} h_j) + N(r)) \tau + \text{Re}(g + h_j)) + 2 \text{Re}(\tau r))} \\
= \sum_{g \in \mathcal{O}} e^{2\pi i ((N(g + \frac{1}{2} h_j) + N(r) + 2 \text{Re}(g + h_j)) \tau + \text{Re}(g + h_j + 2r))} \\
= \sum_{g \in \mathcal{O}} e^{2\pi i (N(g + r + \frac{1}{2} h_j) \tau + \text{Re}(2g + h_j))} \\
= \vartheta_{h_j}(\tau, z).
\]
Hence
\[
\varphi(\tau, z + \tau r) = \sum_{j=1}^{8} f_j(\tau) \vartheta_{h_j}(\tau, z + \tau r) = e^{-2\pi i (\tau N(r) + 2 \text{Re}(\mathcal{F}_2))} \cdot \varphi(\tau, z)
\]
is an easy consequence. \( \square \)

One could say that the steps which follow now are a “standard procedure”, somehow. The basic approach can – just to name a few examples – already be found in [K98] for the quaternionic Maass lift with trivial character or in [De01] for the case of Hermitian modular forms. The idea is to construct quaternionic Jacobi-forms of higher indeces (\( \varphi \) has index 1) with the help of \( \varphi \) from the preceding proposition and so-called Hecke-operators for Jacobi-forms.
Let $m \in \mathbb{N}$. We define

$$T(m) := \{ M \in \mathbb{Z}^{2 \times 2} ; \det(M) = m \}.$$ 

Of course, $\text{SL}_2(\mathbb{Z})$ operates on $T(m)$ by multiplication from the left. We cite the following proposition from [Kl98, Prop.4.5] about a transversal of the orbits of this operation:

(3.16) Proposition. A transversal of the orbits of the operation of $\text{SL}_2(\mathbb{Z})$ on $T(m)$, $m \in \mathbb{N}$, is given by

$$\left(\begin{array}{cc} d & b \\ 0 & \frac{m}{d} \end{array} \right), \quad d \in \mathbb{N}, \; d|m, \; b = 0, \ldots, \frac{m}{d} - 1.$$

This also implies

$$T(m) = \bigcup_{d \in \mathbb{N}, \; d|m, \; b = 0, \ldots, \frac{m}{d} - 1} \text{SL}_2(\mathbb{Z}) \left(\begin{array}{cc} d & b \\ 0 & \frac{m}{d} \end{array} \right).$$

We can now define so-called Hecke-operators for quaternionic Jacobi-forms.

(3.17) Definition. Let $k, m \in \mathbb{N}$ and $\varphi : \mathcal{H} \times \mathbb{H} \otimes \mathbb{R} \to \mathbb{C}$ such that

$$\varphi|_{k,1}[N] = \varphi$$

holds for all $M \in \text{SL}_2(\mathbb{Z})$. Then we define the Hecke-operator $[T(m)]$ by

$$\varphi|_{k,1}[T(m)](\tau, z) := m^\frac{1}{2} \sum_{M : \text{SL}_2(\mathbb{Z}) \backslash T(m)} \varphi|_{k,1}[\frac{1}{\sqrt{m}} M](\tau, \sqrt{m}z)$$

for $\tau \in \mathcal{H}$ and $z \in \mathbb{H} \otimes \mathbb{R}$, where $\varphi|_{k,1}[N], N \in \text{SL}_2(\mathbb{R})$ is defined via 3.14 by expanding the definition to $\text{SL}_2(\mathbb{R})$.

Note that the Hecke-operator $[T(m)]$ is well-defined: The verification of 3.16 did not make use of $M_1$ and $M_2$ having integral entries, only $\det(M_1) = \det(M_2) = 1$ was necessary. So if $M \in \text{SL}_2(\mathbb{Z})$ and $N \in T(m)$ are given, then we obtain

$$\varphi|_{k,1}[\frac{1}{\sqrt{m}} MN](\tau, \sqrt{m}z) = (\varphi|_{k,1}[M]|_{k,1}[\frac{1}{\sqrt{m}} N])(\tau, \sqrt{m}z) = \varphi|_{k,1}[\frac{1}{\sqrt{m}} N](\tau, \sqrt{m}z).$$

Thus the definition of the Hecke-operator $[T(m)]$ is independent of the special choice of a transversal. The importance of this Hecke-operator is emphasized in the following

(3.18) Proposition. Suppose $k \in \mathbb{N}$ is odd and let $g \in \mathcal{M}_k^{3+}$, $\mathcal{M}_k^{3+}$. Define the associated quaternionic Jacobi-form $\varphi$ like in (3.15). Then

$$\varphi|_{k,1}[T(m)] \in \mathcal{J}(k, m, \mathcal{O}, v_i).$$
3.2 Quaternionic Maass lifts of odd weight

holds for all \( m \in \mathbb{N} \). Again, define

\[
\gamma_i(h_1) = 1, \quad \gamma_i(h_2) = -1, \quad \gamma_i(h_3) = -i, \quad \gamma_i(h_6) = i, \\
\gamma_i(h_4) = \frac{1}{i+1}, \quad \gamma_i(h_5) = -i \frac{i}{i+1}, \quad \gamma_i(h_7) = i \frac{i}{i+1}, \quad \gamma_i(h_8) = -\frac{1}{i+1},
\]

and for arbitrary \( t \in O^2 \setminus 2O \) define \( \gamma_i(t) := \gamma_i(h_j) \) if \( t \equiv h_j \mod 2O \). If \( g \) has the Fourier-expansion

\[
g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} a(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]

then the Fourier-expansion of \( \varphi|_{k,1}[T(m)] \) is given by

\[
\varphi|_{k,1}[T(m)](\tau, z) = \sum_{n \in \mathbb{N}} \sum_{t \in O \cap 2O \setminus O(t)} \left( \sum_{d \in \mathbb{N}, \gcd(n,m) \atop d \divides O} d^{-1} \gamma_i(\frac{1}{d}) a\left(3 \frac{mn-N(t)/4}{d^2}\right) \right) e^{2\pi i nt + \text{Re}(\tau z)}
\]

for \( \tau \in \mathcal{H} \) and \( z \in H \otimes \mathbb{R} \).

**Proof:** Again, let us start with the asserted Fourier-expansion. Let \( m \in \mathbb{N}, \tau \in \mathcal{H} \) and \( z \in H \otimes \mathbb{R} \). According to (3.15) the absolutely and locally uniformly convergent Fourier-expansion of \( \varphi \) is given by

\[
\varphi(\tau, z) = \sum_{n \in \mathbb{N}} \sum_{t \in O \cap 2O \setminus O(t)} a_\varphi(n, t) \cdot e^{2\pi i nt + \text{Re}(\tau z)}
\]

where

\[
a_\varphi(n, t) = \gamma_i(t)a\left(3n - \frac{3}{2}N(t)\right).
\]

So by definition and regarding (3.16) we compute

\[
\varphi|_{k,1}[T(m)](\tau, z) = \sum_{d \in \mathbb{N}, d \mid m \atop b=0, \ldots, m/d-1} m^{\frac{k}{d}-1} \varphi|_{k,1}\left(\frac{d}{\sqrt{m}}, \frac{b}{\sqrt{m}}\right)(\tau, \sqrt{mz})
\]

\[
= \sum_{d \in \mathbb{N}, d \mid m \atop b=0, \ldots, m/d-1} m^{\frac{k}{d}-1} (\sqrt{m})^{-k} d^k \varphi\left(\frac{d^2}{m} \tau + \frac{bd}{m}, dz\right)
\]

\[
= \sum_{d \in \mathbb{N}, d \mid m \atop b=0, \ldots, m/d-1} m^{-1} d^k \sum_{n \in \mathbb{N}} \sum_{t \in O \cap 2O \setminus O(t)} a_\varphi(n, t) \cdot e^{2\pi i n (\frac{d^2}{m} \tau + \frac{bd}{m}) + \text{Re}(\tau dz)}
\]

\[
= \sum_{d \in \mathbb{N}, d \mid m} m^{-1} d^k \sum_{n \in \mathbb{N}} \sum_{t \in O \cap 2O \setminus O(t)} a_\varphi(n, t) \cdot e^{2\pi i n (\frac{d^2}{m} \tau + \text{Re}(\tau dz))} \sum_{b=0}^{m-1} (2\pi i \frac{bd}{m})^b.
\]
Note that if \( \frac{\alpha d}{m} \notin \mathbb{Z} \) (and thus \( \exp(2\pi i \frac{\alpha d}{m}) \neq 1 \)) we get

\[
\sum_{b=0}^{\frac{m}{d}-1} (e^{2\pi i \frac{\alpha d}{m}})^b = \frac{1 - (e^{2\pi i \frac{\alpha d}{m}})^{\frac{m}{d}}}{1 - e^{2\pi i \frac{\alpha d}{m}}} = 0.
\]

On the other hand, if \( \frac{\alpha d}{m} \in \mathbb{Z} \) (and thus \( \exp(2\pi i \frac{\alpha d}{m}) = 1 \)), which is equivalent to \( \frac{m}{d} | n \), we get

\[
\sum_{b=0}^{\frac{m}{d}-1} (e^{2\pi i \frac{\alpha d}{m}})^b = \frac{m}{d}.
\]

Hence we obtain

\[
\varphi|_{k,1}[T(m)](\tau, z) = \sum_{n \in \mathbb{N}, t \in \mathcal{O} \setminus 2\mathcal{O}} \sum_{d \in \mathbb{N}, \gcd(n, m) \neq d} \frac{m}{d}m^{-1}d^k \alpha \varphi(n, t) \cdot e^{2\pi i (\frac{\alpha d}{m} \tau + \text{Re}(\bar{m}z))}.
\]

Define

\[
\tilde{n} := \frac{mn^2}{m}, \quad \tilde{t} := dt.
\]

Then

\[
\alpha \varphi(n, t) = \alpha \varphi(\frac{\tilde{n}}{\tilde{t}}, \frac{\tilde{t}}{\tilde{n}}).
\]

Furthermore, \( \frac{m}{d} | n \) is equivalent to the existence of \( x \in \mathbb{N} \) such that \( \frac{\tilde{n}}{\tilde{t}} = n = \frac{d}{m}x \) holds, which is equivalent to the existence of \( x \in \mathbb{N} \) such that \( n = xd \), and this is equivalent to \( d \mid \tilde{n} \).

Next, \( 4\frac{\tilde{n}m}{d^2} = 4n \geq N(t) = N(\tilde{t}) \frac{1}{d^2} \) is equivalent to \( 4\tilde{n}m \geq N(\tilde{t}) \). But note that we have seen that \( t \in \mathcal{O} \setminus 2\mathcal{O} \) implies \( \frac{1}{4} N(t) \notin \mathbb{N} \), and thus the condition \( 4n \geq N(t) \) can be replaced by \( 4\tilde{n}m \geq N(\tilde{t}) \), which is equivalent to \( 4\tilde{n}m \geq N(\tilde{t}) \). Next, we observe that \( \tilde{t} \) has to fulfill \( d^{-1}t \in \mathcal{O} \setminus 2\mathcal{O} \). And finally, suppose \( \tilde{n} \in \mathbb{N}_0 \) and \( \tilde{t} \in \mathcal{O} \) such that there exists \( d \in \mathbb{N} \) satisfying \( d \mid m, d \mid \tilde{n} \) (which means there exist \( x \in \mathbb{N} \) and \( y \in \mathbb{N} \) with \( m = xd \) and \( \tilde{n} = yd \)) and \( d^{-1}t \in \mathcal{O} \setminus 2\mathcal{O} \). Then, choose \( t \equiv d^{-1}t \) and \( n = xy \), and we obtain \( \tilde{t} = dt \) as well as \( \tilde{n} = \frac{xyd}{x} = \frac{yd^2}{m} \). Bringing it all together we have shown that

\[
\{(n, t, d) \in \mathbb{N} \times (\mathcal{O} \setminus 2\mathcal{O}) \times \mathbb{N} \mid 4n > N(t), d \mid m, \frac{m}{d} | n\}
\]

\[
\rightarrow \{(n, t, d) \in \mathbb{N} \times \mathcal{O} \times \mathbb{N} \mid 4nm > N(t), d \mid m, d \mid n, d^{-1}t \in \mathcal{O} \setminus 2\mathcal{O}\}
\]

\[
(n, t, d) \mapsto (\frac{nm^2}{m}, dt, d)
\]

is a bijection. And therefore a simple rearrangement of the absolutely and locally uniformly convergent infinite sum yields

\[
\varphi|_{k,1}[T(m)](\tau, z) = \sum_{n \in \mathbb{N}} \sum_{t \in \mathcal{O}} \left( \sum_{d \mid \gcd(n, m), d^{-1}t \in \mathcal{O} \setminus 2\mathcal{O}} d^{k-1} \alpha \varphi(\frac{nm}{d^2}, \tau + \frac{1}{d^2}z) \right) \cdot e^{2\pi i (nt + \text{Re}(\bar{m}z))}
\]

\[
= \sum_{n \in \mathbb{N}} \sum_{t \in \mathcal{O}} \left( \sum_{d \mid \gcd(n, m), d^{-1}t \in \mathcal{O} \setminus 2\mathcal{O}} d^{k-1} \gamma(t, \frac{1}{d^2}) \alpha(3\frac{nm-N(t)/4}{d^2}) \right) \cdot e^{2\pi i (nt + \text{Re}(\bar{m}z))}.
\]
3.2 Quaternionic Maaß lifts of odd weight

Of course, this Fourier-expansion is again absolutely and locally uniformly convergent since it is just a finite sum of absolutely and locally uniformly convergent series according to the first few transformation-steps from above. Thus the necessary condition concerning the Fourier-expansion for quaternionic modular forms in (3.8) is fulfilled. And note that \( \varphi|_{k,1}[T(1)] = \varphi \) obviously holds true.

So next, we need to consider the transformation behavior of \( \varphi|_{k,1}[T(m)] \), again. First, let us have a look at \( \langle \varphi|_{k,1}[T(m)] \rangle_{k,M}[M] \) for \( M \in \text{SL}_2(\mathbb{Z}) \). The following calculation has already been done in [Kl98], but we do it here again for the sake of completeness. So let \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}), \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \). As we discussed above, 3.16 (which is \( \varphi|_{k,1}[M_1M_2] = \langle \varphi|_{k,1}[M_1] \rangle_{k_1}[M_2] \)) also holds for \( M_1, M_2 \in \text{SL}_2(\mathbb{R}) \). We compute

\[
\langle \varphi|_{k,1}[T(m)] \rangle_{k,M}[M](\tau, z) = e^{-2\pi i M(\tau)^{-1} \epsilon z} \cdot \langle M(\tau) \rangle^{-k} \cdot \langle \varphi|_{k,1}[T(m)] \rangle_{k,M}(\tau, M(\tau)^{-1} \epsilon z)
\]

\[
= m^{\frac{k}{2} - 1} \sum_{K \in \text{SL}_2(\mathbb{Z}) \setminus T(m)} e^{-2\pi i M(\tau)^{-1} \epsilon \sqrt{m} z} \cdot \langle M(\tau) \rangle^{-k} \cdot \langle \varphi|_{k,1}[\frac{1}{\sqrt{m}} K] \rangle_{k,1}[M](\tau, \sqrt{m} \epsilon z)
\]

\[
= m^{\frac{k}{2} - 1} \sum_{K \in \text{SL}_2(\mathbb{Z}) \setminus T(m)} \varphi|_{k,1}[\frac{1}{\sqrt{m}} K](\tau, \sqrt{m} \epsilon z).
\]

Hence all that remains to be shown is that if \( K_1, \ldots, K_n \) is a transversal of \( \text{SL}_2(\mathbb{Z}) \setminus T(m) \), then \( K_1M_1, \ldots, K_nM_n \) is, too. Since the length of this assumed transversal is the same of the original one, we only have to prove that \( K_iM_1, K_iM_2 \) do not lie in the same orbit under \( \text{SL}_2(\mathbb{Z}) \) for \( j \neq l \). But this is obvious since the existence of \( N \in \text{SL}_2(\mathbb{Z}) \) satisfying \( NK_1M = K_1M \) would imply \( NK_1 = K_1 \), since \( M \in \text{GL}_2(\mathbb{Z}) \). But this contradicts the assumption of \( K_1, \ldots, K_n \) being a transversal. Hence we obtain

\[
\langle \varphi|_{k,1}[T(m)] \rangle_{k,M}[M] = \varphi|_{k,1}[T(m)]
\]

for all \( m \in \mathbb{N} \) and \( M \in \text{SL}_2(\mathbb{Z}) \). Next, we will verify

\[
\varphi|_{k,1}[T(m)](\tau, \epsilon z) = \varphi(\pi_{\epsilon}(\tau)) \cdot \varphi|_{k,1}[T(m)](\tau, z)
\]

for all \( \epsilon \in \mathcal{E} \). Using (3.15) (meaning that \( \varphi \in \mathcal{J}(k, 1, O, \nu_l) \)) we get (where \( \frac{1}{\sqrt{m}} M = \left( \begin{smallmatrix} \ast & \ast \\ \ast & \ast \end{smallmatrix} \right) \))

\[
\varphi|_{k,1}[T(m)](\tau, \epsilon z) = m^{\frac{k}{2} - 1} \sum_{M \in \text{SL}_2(\mathbb{Z}) \setminus T(m)} e^{-2\pi i (\frac{1}{\sqrt{m}} M(\tau)^{-1} c_{\epsilon 1} \pi z)} \cdot \langle \varphi(\pi_{\epsilon}(\tau)) \rangle \cdot \varphi|_{k,1}[\frac{1}{\sqrt{m}} M](\tau, \sqrt{m} \epsilon z)
\]

\[
= m^{\frac{k}{2} - 1} \sum_{M \in \text{SL}_2(\mathbb{Z}) \setminus T(m)} \varphi(\pi_{\epsilon}(\tau)) \cdot \varphi|_{k,1}[\frac{1}{\sqrt{m}} M](\tau, \sqrt{m} \epsilon z)
\]

for all \( m \in \mathbb{N}, \epsilon \in \mathcal{E}, \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \). Hence we already verified two of the four needed transformation laws for proving \( \varphi|_{k,1}[T(m)] \in \mathcal{J}(k, m, O, \nu_l) \). And just analogous to the proof
of (3.15) it is clear that
\[ \varphi|_{k,1}[T(m)](\tau + s_1, z + s_2) = \varphi|_{k,1}[T(m)](\tau, z) \]
holds for all \( s_1 \in \mathbb{Z} \) and \( s_2 \in \mathcal{O} \) because of the already determined Fourier-expansion. Thus finally, only
\[ \varphi|_{k,1}[T(m)](\tau, z + \tau r) = e^{-2\pi i m(\tau N(r) + 2 \text{Re}(\tau z))} \cdot \varphi|_{k,1}[T(m)](\tau, z) \]
for \( r \in \mathcal{O} \) remains to be shown. So let \( r \in \mathcal{O}, \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \). Note that for \( b, d \in \mathbb{N} \) satisfying \( d | m \) we get
\[ e^{2\pi i m \tau N(r)} = e^{2\pi i N(\frac{r}{d} \tau + \frac{b}{d})} \]
since \( N(\frac{r}{d} \tau) \frac{b}{d} = \frac{r}{d} b \tau N(r) \in \mathbb{Z} \). And moreover
\[ m \text{Re}(\tau z) = \text{Re}(\frac{m}{d} \tau (dz)) \]
holds. Next, note that
\[ \varphi(\tau, z + \tau \frac{m}{d} r) = e^{-2\pi i (\tau N(\frac{r}{d}) + 2 \text{Re}(\frac{m}{d} \tau z))} \cdot \varphi(\tau, z) \]
holds for all \( \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \), since \( \varphi \in J(k, 1, \mathcal{O}, v_i) \). And last,
\[ \varphi(\tau, z + (\frac{b d}{m}))(\frac{r}{d} \tau) = \varphi(\tau, z) \]
holds for all \( \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes \mathbb{R} \mathbb{C} \), since \( (\frac{b d}{m})(\frac{r}{d} \tau) = b r \in \mathcal{O} \). So putting it all together and in virtue of (3.16) we compute
\[
e^{2\pi i m(\tau N(r) + 2 \text{Re}(\tau z))} \cdot \varphi|_{k,1}[T(m)](\tau, z + \tau r) = e^{2\pi i m(\tau N(r) + 2 \text{Re}(\tau z))} \sum_{d \in \mathbb{N}, d|m} \frac{m^{-1} d^k e^{2\pi i (\frac{r}{d} \tau + \frac{b d}{m})}}{b=0, \ldots, m/d-1} \phi(\frac{d}{m} \tau + \frac{b d}{m}, d(z + \tau r))
\]
\[
= \sum_{d \in \mathbb{N}, d|m} \frac{m^{-1} d^k e^{2\pi i (\frac{r}{d} \tau + \frac{b d}{m})}}{b=0, \ldots, m/d-1} \phi(\frac{d}{m} \tau + \frac{b d}{m}, dz + (\frac{d}{m} \tau + \frac{b d}{m})(\frac{r}{d} \tau))
\]
\[
= \sum_{d \in \mathbb{N}, d|m} m^{-1} d^k \phi(\frac{d}{m} \tau + \frac{b d}{m}, dz) = \varphi|_{k,1}[T(m)](\tau, z) .
\]
This completes the proof. \( \square \)

Finally, we have all the necessary tools and we can define our quaternionic Maass lift of odd weight. Note that in principal it is clear that the \( \varphi|_{k,1}[T(m)] \) will yield the correct transformation behavior for the lift. It is common knowledge that Jacobi-forms together with the Hecke-operator from above yield modular forms by summing them up in a certain way – in case this series converges! So to say, the transformation behavior of the lift is not the problem, although there is a lot to show. But most of it is straightforward and not really new. The important question is whether the lift converges. Our advantage is that we start with elliptic modular
forms, because the special growth of Fourier-coefficients of elliptic modular forms will yield the convergence. This is not the case if one directly starts with Jacobi-forms. So one could say the difficulty in starting with Jacobi-forms is the convergence behavior, while the difficulty in starting with elliptic modular forms is to find the right ones to lift – such that we get the right transformation behavior. Since we started with elliptic modular forms and already determined the transformation behavior of the corresponding Jacobi-forms, the convergence as well as the transformation behavior of the lift will be quite straightforward to show. So the actual problem was finding $\mathcal{M}_k^{3+}$ and of course we still have to prove that there are non-trivial elements in $\mathcal{M}_k^{3+}$.

(3.19) **Theorem.** $v_i$ defined in (1.57) and continued to $\text{Sp}_2(\mathcal{O})$ by means of 1.16 is a multiplier system of weight $k$ for all odd $k \in \mathbb{Z}$. Suppose $k \in \mathbb{N}$, $k \geq 3$ is odd and let $g \in \mathcal{M}_k^{3+}$. Define the associated quaternionic Jacobi-form $\varphi$ like in (3.15). The quaternionic Maaß lift of odd weight for $v_i$ is defined by

$$\mathcal{M}_g^{(i)}(Z) := \sum_{m \in \mathbb{N}} \varphi|_{k_1}[T(m)](\tau, z) \cdot e^{2\pi i m \omega}, \quad Z = \left( \frac{\tau}{\omega}, \frac{z}{\omega} \right) \in \mathcal{H}(\mathbb{H}).$$

One has

$$\mathcal{M}_g^{(i)} \in [\text{Sp}_2(\mathcal{O}), k, v_i].$$

Again, define

$$\gamma_i(h_1) = 1, \quad \gamma_i(h_2) = -1, \quad \gamma_i(h_3) = -i, \quad \gamma_i(h_6) = i,$$

$$\gamma_i(h_4) = \frac{1}{1+i}, \quad \gamma_i(h_5) = \frac{-i}{1+i}, \quad \gamma_i(h_7) = \frac{i}{1+i}, \quad \gamma_i(h_8) = \frac{-1}{1+i},$$

and for arbitrary $t \in \mathcal{O}^\times \setminus 2\mathcal{O}$ define $\gamma_i(t) := \gamma_i(h_j)$ if $t \equiv h_j \mod 2\mathcal{O}$. If $g$ has the Fourier-expansion

$$g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \alpha(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H},$$

then the Fourier-expansion of the Maaß lift is given by

$$\mathcal{M}_g^{(i)}(Z) = \sum_{m, n \in \mathbb{N}} \sum_{t \in \mathcal{O}^\times} \left( \sum_{d \mid \text{gcd}(n, m)} d^{k-1} \gamma_i\left( \frac{1}{d} \right) \alpha\left( 3\frac{nm-N(t)}{d^2} \right) \right) \cdot e^{2\pi i (n \tau + m \omega + \text{Re}(t)z)}$$

for all $Z = \left( \frac{\tau}{\omega}, \frac{z}{\omega} \right) \in \mathcal{H}(\mathbb{H})$. The vector space of all quaternionic Maaß lifts for $v_i$ of odd weight $k$ is denoted by $\mathcal{M}(k, v_i; \mathcal{O})$.

**Proof:** To keep it well-arranged, define $f := \mathcal{M}_g^{(i)}$. According to (3.18) we obviously have

$$f(Z) = \sum_{m \in \mathbb{N}} \varphi|_{k_1}[T(m)](\tau, z) \cdot e^{2\pi i m \omega}$$

$$= \sum_{m, n \in \mathbb{N}} \sum_{t \in \mathcal{O}^\times} \left( \sum_{d \mid \text{gcd}(n, m)} d^{k-1} \gamma_i\left( \frac{1}{d} \right) \alpha\left( 3\frac{nm-N(t)}{d^2} \right) \right) \cdot e^{2\pi i (n \tau + m \omega + \text{Re}(t)z)}$$
\[ = \sum_{T = \left( \begin{array}{cc} \frac{n}{m} & \frac{i}{m} \\ \frac{i}{m} & \frac{n}{m} \end{array} \right) \in \text{Her}_2(O), \, T > 0} \left( \sum_{d \in \mathbb{N}} \frac{d^{k-1} \gamma_{\frac{2i}{m}}(\frac{2}{T})\alpha(3 \frac{\det(T)}{d^2})}{d^{-1} \tau \in \frac{1}{2} O \setminus O} \right) \cdot e^{2\pi i r(T, Z)} \]

\[ =: \sum_{T \in \text{Her}_2(O), \, T > 0} \alpha_f(T) \cdot e^{2\pi i r(T, Z)} \]

for all \( Z = \left( \begin{array}{cc} \frac{r}{Z} & \frac{z}{Z} \\ \frac{z}{Z} & \frac{r}{Z} \end{array} \right) \in \mathcal{H}(\mathbb{H}) \) (where one should not get confused by mixing up the variable \( \tau \in \mathcal{H} \) and the trace form \( \tau(T, Z) \)) – assuming that this Fourier-series actually converges. So let us prove that this series converges absolutely and locally uniformly. Let \( T = \left( \begin{array}{cc} \frac{n}{m} & \frac{i}{m} \\ \frac{i}{m} & \frac{n}{m} \end{array} \right) \in \text{Her}_2(O), \, T > 0. \) Then

\[ 0 < \det(T) = nm - N(t) \leq nm < n^2 + 2nm + m^2 = \text{tr}(T)^2 \]

holds. Furthermore, since \( g \in [\Gamma[9], k - 2, 1]_0, \)

\[ \alpha(n) = O(n^{(k-2)/2}) \]

holds according to [Mi89, cor.2.1.6], which means there is a constant \( c_1 > 0 \) satisfying

\[ |\alpha(n)| \leq c_1 n^{(k-2)/2} \]

for all \( n \in \mathbb{N}. \) Defining the divisor sum

\[ \sigma_r(n) := \sum_{d \in \mathbb{N}, \, d \mid n} d^r \]

for \( r \in \mathbb{C}, \, n \in \mathbb{N} \) as usual, we compute

\[ |\alpha_f(T)| = \left| \sum_{d \mid \text{gcd}(n,m)} \frac{d^{k-1} \gamma_{\frac{2i}{m}}(\frac{2}{T})\alpha(3 \frac{\det(T)}{d^2})}{d^{-1} \tau \in \frac{1}{2} O \setminus O} \right| \leq \sum_{d \in \mathbb{N}} \frac{d^{k-1}c_1(3 \frac{\det(T)}{d^2})^{(k-2)/2}}{d^{-1} \tau \in \frac{1}{2} O \setminus O} \sum_{d \mid \text{gcd}(n,m)} d \leq c_2 \text{tr}(T)^{(k-2)} \sigma_1(\text{tr}(T)) \]

for \( c_2 = c_1 3^{(k-2)/2}. \) Next, we have

\[ \# \{ T \in \text{Her}_2(O) ; \, T > 0, \, \text{tr}(T) = l \} \leq (l + 1) \cdot \# \{ t \in O^\natural ; \, N(t) \leq 4l^2 \} = (l + 1) \cdot \# \{ t \in O ; \, N(t) \leq 3l^2 \} = O(l^7) \]

since \( O^\natural = \frac{2}{3} i_1 \sqrt{3} O \) (with \( N(\frac{2}{3} i_1 \sqrt{3} t) = \frac{4}{3} t \) for \( t \in O \)) and

\[ \# \{ t \in O ; \, N(t) \leq 3l^2 \} = \sum_{r=0}^{3l^2} \sum_{d \mid r, 3 \mid d} d \leq 12(3l^2 + 1) \sigma_1(3l^2) = O(l^8) \]
in virtue of (1.7) and because \( \sigma_1(r) \leq r^2 \) holds. Thus there is some \( c_3 > 0 \) satisfying
\[
\# \{ T \in \text{Her}_2^2(\mathcal{O}) ; \ T > 0, \ \text{tr}(T) = I \} \leq c_3 l^7
\]
So let \( y > 0 \) and
\[
B_y := \{ Z = X + iY \in \mathcal{H}(\mathbb{H}) ; \ Y > y I \}.
\]
Again, suppose \( T \in \text{Her}_2^2(\mathcal{O}), \ T > 0 \). Then according to [Kr85, ch.I, thm.3.6] there exists \( W \in \text{GL}_2(\mathbb{H}) \) with \( T = W^* W \). \( Z = X + iY \in B_y \) and thus \( Y - y I \in \text{Pos}_2(\mathbb{H}) \) implies \( (Y - y I)(W^*) \in \text{Pos}_2(\mathbb{H}) \) by the same theorem, which implies \( \text{tr}((Y - y I)(W^*)) > 0 \). Using (1.46) we compute
\[
\tau(Y, T) - \tau(y I, T) = \tau((Y - y I)(W^*), I) = \text{tr}((Y - y I)(W^*)) > 0,
\]
which leads to (noting \( \tau(Z, T) \in \mathbb{C} \))
\[
\text{Im}(\tau(Z, T)) = \text{Im}(\tau(X, T) + i\tau(Y, T)) = \tau(Y, T) > \tau(y I, T) = y \text{tr}(T).
\]
for all \( Z = X + iY \in B_y \). Putting it all together yields
\[
\sum_{T \in \text{Her}_2^2(\mathcal{O}), \ T > 0} |\alpha_f(T) \cdot e^{2\pi i \tau(T, Z)}| \leq \sum_{l \in \mathbb{N}} \sum_{T \in \text{Her}_2^2(\mathcal{O}), \ T > 0, \ \text{tr}(T) = l} |\alpha_f(T)| \cdot e^{-2\pi \text{Im}(\tau(T, Z))}
\]
\[
\leq \sum_{l \in \mathbb{N}} \sum_{T \in \text{Her}_2^2(\mathcal{O}), \ T > 0, \ \text{tr}(T) = l} c_2 l^{k-2} \sigma_1(l) \cdot e^{-2\pi y l}
\]
\[
\leq \sum_{l \in \mathbb{N}} c_2 c_3 l^7 l^{k-2} l^2 \cdot e^{-2\pi y l} < \infty
\]
for all \( Z = X + iY \in B_y \). Hence the Fourier-series converges absolutely and uniformly on \( B_y \), which is equivalent to the absolute and local uniform convergence of the Fourier-expansion of \( f \). And moreover, the Maaß lift is indeed well-defined. So let us consider the transformation behavior of \( f \), now.

Let us start with \( \varepsilon \in \mathcal{E}, \ U = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \) and \( M = \text{Rot}(U) = \text{diag}(\varepsilon, 1, \varepsilon, 1) = \varepsilon I \times I \). For \( Z = \begin{pmatrix} \varepsilon & z \\ \overline{z} & \omega \end{pmatrix} \in \mathcal{H}(\mathbb{H}) \) we compute
\[
f|_k M(Z) = f(Z[U]) = f(\begin{pmatrix} \varepsilon & z \\ \overline{z} & \omega \end{pmatrix}) = \sum_{m \in \mathbb{N}} \varphi_{|k,1}[T(m)](\varepsilon, \overline{z}) \cdot e^{2\pi i m \omega}
\]
\[
= \sum_{m \in \mathbb{N}} \varphi_1(\pi_p(\varepsilon) \varphi_{|k,1}[T(m)](\varepsilon, z) \cdot e^{2\pi i m \omega} = v_1(M) f(Z)
\]
in virtue of (3.18). Next, let \( M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) and \( M = M_0 \times I \). As we have proven before in 3.12,
\[
v_i(M) \det(M\{Z\})^{k/2} = (c \tau + d)^k
\]
holds for all \( Z = \begin{pmatrix} \varepsilon & z \\ \overline{z} & \omega \end{pmatrix} \in \mathcal{H}(\mathbb{H}) \) because of the definition of \( v_i \) and the proof of (1.56), since
$M \in \text{Sp}_2(\mathbb{R})$. Using 3.11 and (3.18) we obtain

$$f|_k M(Z) = \det(M(Z))^{-k/2} f(M(Z)) = v_i(M)(ct + d)^{-k} \left( \begin{array}{cc} M_0(\tau) & M_0(\tau)^{-1} \omega \\ M_0(\tau)^{-1} \omega & M_0(\tau) \end{array} \right)$$

$$= v_i(M)(ct + d)^{-k} \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](M_0(\tau), M_0(\tau)^{-1} \omega) \cdot e^{2\pi i m(\omega - M_0(\tau)^{-1} \omega)}$$

$$= v_i(M) \sum_{\tau \in \mathbb{N}} e^{-2\pi i m M_0(\tau)^{-1} \omega} \varphi|_{k,1}[T(m)](M_0(\tau), M_0(\tau)^{-1} \omega) \cdot e^{2\pi i m \omega}$$

$$= v_i(M) \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau, z) \cdot e^{2\pi i m \omega} = v_i(M) f(Z)$$

for all $Z = \left( \begin{array}{cc} \tau & z \\ 0 & \omega \end{array} \right) \in \mathcal{H}(\mathbb{H})$. Next, let $r \in \mathcal{O}$, $U = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$, $M = \text{Rot}(U)$ and again $Z = \left( \begin{array}{cc} \tau & z \\ 0 & \omega \end{array} \right) \in \mathcal{H}(\mathbb{H})$. As we have seen before, a short calculation yields

$$Z[U] = \left( \begin{array}{cc} \tau & z \\ 0 & \omega \end{array} \right)$$

Furthermore, $v_i(M) = 1$ holds by the definition of $v_i$. Utilizing the transformation behavior of the $\varphi|_{k,1}[T(m)]$ from (3.18) we compute

$$f|_k M(Z) = f(Z[U]) = f(\tau \tau + \tau N(r)+2 \text{Re}(\tau z) + \omega)$$

$$= \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau, z + \tau r) \cdot e^{2\pi i m(\tau N(r)+2 \text{Re}(\tau z) + \omega)}$$

$$= \sum_{\tau \in \mathbb{N}} e^{2\pi i m(\tau N(r)+2 \text{Re}(\tau z))} \varphi|_{k,1}[T(m)](\tau, z + \tau r) \cdot e^{2\pi i m \omega}$$

$$= \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau, z) \cdot e^{2\pi i m \omega} = v_i(M) f(Z) .$$

And finally, let $s_1, s_3 \in \mathbb{Z}$, $s_2 \in \mathcal{O}$, $S = \left( \begin{array}{cc} s_1 & s_2 \\ s_2 & s_3 \end{array} \right)$ and $M = \text{Trans}(S)$. Then again, $v_i(M) = 1$ holds and (3.18) yields

$$f|_k M(Z) = f(Z + S) = \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau + s_1, z + s_2) \cdot e^{2\pi i m(\omega + s_3)}$$

$$= \sum_{\tau \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau, z) \cdot e^{2\pi i m \omega} = v_i(M) f(Z)$$

for all $Z = \left( \begin{array}{cc} \tau & z \\ 0 & \omega \end{array} \right) \in \mathcal{H}(\mathbb{H})$. And thus, in virtue of 1.17, $f$ already meets the following transformation behavior:

$$f|_k M = v(M) \cdot f$$

for

$$\Gamma := \{ (\varepsilon I \times I) \cdot (M_0 \times I) \cdot \text{Rot} (\left( \begin{array}{cc} 1 & \varepsilon \\ 0 & 1 \end{array} \right)) \cdot \text{Trans} \left( \left( \begin{array}{cc} s_1 & s_2 \\ s_2 & s_3 \end{array} \right) \right) ; \ \varepsilon \in \mathcal{E}, \ M_0 \in \text{SL}_2(\mathbb{Z}), \ r, s_2 \in \mathcal{O}, \ s_1, s_3 \in \mathbb{Z} \}$$

and some map $\nu : \Gamma \to \{ \pm 1, \pm i \}$. Now, we assert that

$$\Gamma = \Gamma_{\infty} := \{ M \in \text{Sp}_2(\mathcal{O}) ; \ M = \left( \begin{array}{cc} * & * \\ 0 & 0 \end{array} \right) \}.$$
3.2 Quaternionic Maass lifts of odd weight

holds (where the \( \ast \) are \((3 \times 1)\)-blocks). The proof works as follows: By simply expanding the product and noting that all four types of matrices that are involved belong to \( \text{Sp}_2(\mathcal{O}) \), one easily verifies \( \Gamma \subset \Gamma_\infty \). Conversely, suppose

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_\infty.
\]

The fundamental relations (1.3) yield

\[
CD' - DC' = (\ast c_2 \ast) = 0,
\]

which implies \( c_2 = 0 \), as well as

\[
AD' - BC' = (\ast a_2 \ast) = I,
\]

which implies \( a_2 = 0 \) and \( a_4 = 1 \). So we have

\[
M = \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & 1 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Applying the fundamental relations once more yields

\[
\overline{\mathcal{A}}C - C'A = (\pi c_1 - \overline{c}_1 a_1 \ast) = 0,
\]

which implies \( \overline{a}_1 c_1 = \overline{c}_1 a_1 \),

\[
B'D - D'B = (\overline{b}_1 d_1 - \overline{d}_1 b_1 \ast) = 0,
\]

which leads to \( \overline{b}_1 d_1 = \overline{d}_1 b_1 \), and finally

\[
\overline{A}D - C'B = (\pi d_1 - \overline{c}_1 b_1 \ast) = I,
\]

which yields \( \overline{a}_1 d_1 - \overline{c}_1 b_1 = 1 \). And thus

\[
M_1 = (a_1 b_1 \overline{b}_1 \overline{c}_1 d_1) \in \text{Sp}_1(\mathcal{O})
\]

follows according to (1.3). A straightforward calculation using the fundamental relations then shows

\[
(M_1^{-1} \times I)M = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & b_1 & b_4 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} := \tilde{M}
\]

for some \( s, r \in \mathcal{O} \). Once more, the fundamental relations yield

\[
\begin{pmatrix} 1 & \pi r \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pi s + r \\ 0 & 1 \end{pmatrix} = I,
\]

and thus \( a_3 = -r \). Therefore, another calculation gives

\[
\text{Rot} \left( \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \right) \tilde{M} = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & b_1 & b_4 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
for some $t \in \mathcal{O}$. Applying the fundamental relations one last time yields $\left( \begin{array}{c} 0 \\ b_i \\ t \end{array} \right) \in \text{Her}_2(\mathcal{O})$, i.e. $t \in \mathbb{Z}$ and $b_3 = s$. And finally we have shown

$$M = (M_1 \times I) \cdot \text{Rot} \left( \left( \begin{array}{c} 1 \\ r \end{array} \right) \right) \cdot \text{Trans} \left( \left( \begin{array}{c} 0 \\ s \end{array} \right) \right).$$

Recalling $\text{Sp}_1(\mathcal{O}) = \mathcal{E} \text{SL}_2(\mathbb{Z})$, this yields $M \in \Gamma$, and thus $\Gamma = \Gamma_\infty$. Note that $\Gamma_\infty$ is a subgroup of $\text{Sp}_2(\mathcal{O})$, whereas all generators of $\text{Sp}_2(\mathcal{O})$ specified in (1.17) are already contained in $\Gamma_\infty$ – except $J$. But a straightforward calculation shows that

$$J = (J_1 \times I) \cdot \text{Rot} \left( \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right) \right) \cdot (J_1 \times I) \cdot \text{Rot} \left( \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \right)$$

holds. Hence if we show that also $f|_k \text{Rot} \left( \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right) \right) = f|_k \text{Rot} \left( \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \right) = f$ holds true (note that $v_i (\text{Rot} (\left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right))) = v_i (\text{Rot} (\left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right))) = 1$), then we have proven that

$$f|_k M = v(M) \cdot f$$

holds for all $M \in \text{Sp}_2(\mathcal{O})$ for some map $v : \text{Sp}_2(\mathcal{O}) \to \{ \pm 1, \pm i \}$. So let $U = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)$, $M = \text{Rot}(U)$ and $Z = \left( \frac{\omega}{\tau}, \zeta \right) \in \mathcal{H}(\mathbb{H})$. An easy calculation gives

$$Z[U] = Z[-U] = \left( \frac{\omega}{\tau} - \zeta \right).$$

As we have seen several times so far, $\text{Re}(t) \in \mathbb{Z}$ holds for all $t \in \mathcal{O}^2$, since we have $\text{Re}(t_\mathcal{O}) \in \mathbb{Z}$ for all $g \in \mathcal{O}$ by definition. Therefore, $-\overline{t} = t - 2 \text{Re}(t) \equiv t$ holds modulo $2\mathcal{O}$ for all $t \in \mathcal{O}^2$. We compute

$$f|_k M(Z) = f|_k (-M)(Z) = f(Z[U]) = f \left( \frac{\omega}{\tau} - \zeta \right)$$

$$= \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2} \sum_{\substack{d \mid \text{gcd}(m,n) \\ d^{-1} t \in \mathcal{O}^2 \setminus 2\mathcal{O}}} \left( d^{k-1} \gamma_i \left( \frac{1}{d^2} \right) \alpha \left( \frac{3mn - N(t)/4}{d^2} \right) \right) \cdot e^{2\pi i (\omega t + n \tau + \text{Re}(\overline{t} \zeta))}$$

$$= \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2} \sum_{\substack{d \mid \text{gcd}(m,n) \\ d^{-1} (-t) \in \mathcal{O}^2 \setminus 2\mathcal{O}}} \left( d^{k-1} \gamma_i \left( \frac{-1}{d^2} \right) \alpha \left( \frac{3mn - N(-t)/4}{d^2} \right) \right) \cdot e^{2\pi i (\omega t + n \tau + \text{Re}(\overline{t} \zeta))}$$

$$= \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2} \sum_{\substack{d \mid \text{gcd}(m,n) \\ d^{-1} t \in \mathcal{O}^2 \setminus 2\mathcal{O}}} \left( d^{k-1} \gamma_i \left( \frac{1}{d} \right) \alpha \left( \frac{3mn - N(t)/4}{d} \right) \right) \cdot e^{2\pi i (\omega t + n \tau + \text{Re}(\overline{t} \zeta))}$$

$$= v_i(M) f(Z).$$

So $f$ possesses an appropriate transformation behavior for all $M \in \text{Sp}_2(\mathcal{O})$. We have not yet shown that $g \in \mathcal{M}_{k+2}^+ \exists$ non-trivially. But for the moment, suppose we did. (Actually, we will come up with this proof in the next section.) Thus the explicit Fourier-expansion of $f$ yields that it is non-identically vanishing. Since

$$f|_k M = v(M) \cdot f$$

holds for all $M \in \text{Sp}_2(\mathcal{O})$ for some not explicitly specified map $v : \text{Sp}_2(\mathcal{O}) \to \{ \pm 1, \pm i \}$, $v$ turns
out to be a multiplier system for $\text{Sp}_2(\mathcal{O})$ of weight $k$ in virtue of (1.52). According to (1.56), there are only two possible multiplier systems of odd weight, namely $\nu_i$ and $\nu_{-i}$. But since we have already seen that

$$f|_k \text{Rot} \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) = \varphi_1(\pi_p(i_2)) \cdot f = i \cdot f$$

holds, $\nu = \nu_i$ follows. Thus $\nu_i$ turns out to be a multiplier system and

$$f \in [\text{Sp}_2(\mathcal{O}), k, \nu_i]$$

holds, which completes the proof. □

The quaternionic Maaß lifts for $\nu_i$ also possess an appropriate transformation behavior under $Z \mapsto Z'$. To be more precise, they are skew-symmetric:

**Theorem.** Suppose $k \in \mathbb{N}$, $k \geq 3$ is odd and let $f \in \mathcal{M}(k, \nu_i; \mathcal{O})$. Then $f$ is skew-symmetric, which means

$$f(Z') = -f(Z)$$

holds for all $Z \in \mathcal{H}(\mathbb{H})$.

**Proof:** So let $g \in \mathcal{M}_{k-2}^3$ with Fourier-expansion

$$g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \alpha(n) e^{2\pi in\tau}, \quad \tau \in \mathcal{H}$$

such that

$$f(Z) = \mathcal{M}_g^{(i)}(Z) = \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^1} \left( \sum_{d \mid \gcd(n,m)} \sum_{\frac{d}{2} \in \mathcal{O}^1 \setminus 2\mathcal{O}} d^{k-1} \gamma_i \left( \frac{1}{d} \right) \alpha \left( \frac{3nm-N(t)/4}{d^2} \right) \right) e^{2\pi i(n\tau+m\omega+\text{Re}(\tau))}$$

holds for all $Z = \left( \begin{pmatrix} t & \bar{z} \\ \bar{z} & \omega \end{pmatrix} \right) \in \mathcal{H}(\mathbb{H})$ (see (3.19)). An easy calculation shows that the $h_j$ from (3.3) fulfill

$$\overline{h}_1 \equiv h_2, \quad \overline{h}_3 \equiv h_6, \quad \overline{h}_4 \equiv h_8, \quad \overline{h}_5 \equiv h_7 \mod 2\mathcal{O},$$

and thus by definition of the prefactors $\gamma_i(t)$,

$$\gamma_i(\bar{t}) = -\gamma_i(t)$$

holds for all $t \in \mathcal{O}^1 \setminus 2\mathcal{O}$. This is why we obtain

$$f(Z') = f\left( \begin{pmatrix} t & \bar{z} \\ \bar{z} & \omega \end{pmatrix} \right) = \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^1} \left( \sum_{d \mid \gcd(n,m)} \sum_{\frac{d}{2} \in \mathcal{O}^1 \setminus 2\mathcal{O}} -d^{k-1} \gamma_i \left( \frac{1}{d} \right) \alpha \left( \frac{3nm-N(t)/4}{d^2} \right) \right) e^{2\pi i(n\tau+m\omega+\text{Re}(\tau))}$$

$$= \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^1} \left( \sum_{d \mid \gcd(n,m)} \sum_{\frac{d}{2} \in \mathcal{O}^1 \setminus 2\mathcal{O}} d^{k-1} \gamma_i \left( \frac{1}{d} \right) \alpha \left( \frac{3nm-N(t)/4}{d^2} \right) \right) e^{2\pi i(n\tau+m\omega+\text{Re}(\tau))}$$
\[
= - \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2_{\mathbb{N}(t)}} \left( \sum_{d \in \mathbb{N}} \frac{d^{k-1} \gamma_i \left( \frac{t}{d} \right) \kappa \left( \frac{3^{nm-N(t)/4}}{d^4} \right) \cdot e^{2\pi i (n\tau + m\omega + Re(tz))}}{d^{k-4} \mathcal{O}_{\mathbb{N}(t)\mathcal{O}}} \right) 
\]

\[
= - f(Z)
\]

for all \(Z = \left( \frac{t}{z}, \omega \right) \in \mathcal{H}(\mathbb{H}).\) \(\square\)

Now, let us do a short excursion on the second (possible) multiplier system \(\nu_{-i}.\) The existence of the quaternionic Maaß lift for \(\nu_{-i}\) is just a consequence of what we discovered so far. So let us just formulate it as a remark:

**Remark.** Note that the multiplier systems coincide on all generators of \(\text{Sp}_2(\mathcal{O})\) specified in (1.17), except for \(M = \text{Rot} \left( \begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix} \right) \), since \(\nu_i(M) = i\), while \(\nu_{-i}(M) = -i\) (see (1.57)). So the process of decomposing \(f \in [\text{Sp}_2(\mathcal{O}), k, \nu_{-i}]\) is almost exactly the same as for \(\nu_i\). While decomposing \(f\), the first situation when \(\nu_i(M) = i\) played a role was when the identities concerning the \(f_j\) had to be found, where

\[
\varphi_{f,1}(\tau, z) = \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_{h_j}(\tau, z), \quad \tau \in \mathcal{H}, \ z \in \mathcal{H} \otimes \mathbb{R} \mathbb{C}.
\]

Using the fact that

\[
\alpha_f(T[U]) = \nu_i(\text{Rot}((\overline{U})^{-1})) \cdot \alpha_f(T)
\]

holds for \(f \in [\text{Sp}_2(\mathcal{O}), k, \nu_i]\) yielded

\[
f_1 = -f_2 = if_3 = -if_6 ,
\]

\[
f_4 = if_5 = -if_7 = -f_8 ,
\]

as well as \(f_0 \equiv 0\). Redoing exactly the same calculations with

\[
\alpha_f(T[U]) = \nu_{-i}(\text{Rot}((\overline{U})^{-1})) \cdot \alpha_f(T)
\]

replacing \(\nu_i\), we obtain: If \(f \in [\text{Sp}_2(\mathcal{O}), k, \nu_{-i}]\), then

\[
\varphi_{f,1}(\tau, z) = \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_{h_j}(\tau, z), \quad \tau \in \mathcal{H}, \ z \in \mathcal{H} \otimes \mathbb{R} \mathbb{C} ,
\]

where

\[
f_1 = -f_2 = -if_3 = if_6 ,
\]

\[
f_4 = -if_5 = if_7 = -f_8 ,
\]

as well as \(f_0 \equiv 0\) hold. The transformation behavior of \(F = (f_0, \ldots, f_8)'\) under \(\text{SL}_2(\mathbb{Z})\) is the same, again, since \(\nu_i\) and \(\nu_{-i}\) coincide on \(\text{Sp}_2(\mathbb{R})\). This means that we have \(F |_{k-2} M = (\kappa(M)' \cdot F \end{equation}
for all $M \in \text{SL}_2(\mathbb{Z})$, with the same $\kappa$ as in 3.25. Hence we get

\[ f_1|_{k-2}(-I) = -f_1, \]
\[ f_4|_{k-2}(-I) = -f_4, \]
\[ f_1|_{k-2}T_1 = \rho^2 f_1, \]
\[ f_4|_{k-2}T_1 = \rho f_4, \]
\[ f_1|_{k-2}J_1 = -\frac{1}{3}(\rho^2 - \rho + i - i)f_1 - \frac{1}{2}(\rho^2 + i\rho^2 - \rho)f_4 \]
\[ = \frac{i\sqrt{3}}{\sqrt{3}} f_1 + \frac{1 + i}{\sqrt{3}} f_4, \]
\[ f_4|_{k-2}J_1 = -\frac{1}{3}(\rho^2 - \rho + i\rho^2 - \rho)f_1 - \frac{1}{3}(\rho + i - i - \rho^2)f_4 \]
\[ = -\frac{1 + i}{\sqrt{3}} f_1 - \frac{i}{\sqrt{3}} f_4, \]

where $\rho = e^{\frac{i}{2}\pi}$. This is only slightly different from (3.11). This time, we have to define

\[ \hat{f}(\tau) := f_1(3\tau) + (1 - i)f_4(\tau), \quad \tau \in \mathcal{H}. \]

(Instead of the prefactor $1 + i$ which we needed for $v_i$.) Of course, $\hat{f}|_{k-2}(-I) = \hat{f}$ and $\hat{f}|_{k-2}(T_1) = \hat{f}$ obviously hold true, again. For $J_1$ we compute

\[ \hat{f}((J_1^{-1}T_1^{-3}J_1)(\langle \rangle)) = \hat{f}\left(\frac{\tau}{3\tau + 1}\right) = f_1\left(\frac{3\tau}{3\tau + 1}\right) + (1 - i)f_4\left(\frac{3\tau}{3\tau + 1}\right) \]
\[ = f_1\left(1 - \frac{1}{3\tau + 1}\right) + (1 - i)f_4\left(1 - \frac{1}{3\tau + 1}\right) \]
\[ = \rho^2 f_1(- (3\tau + 1)^{-1}) + (1 - i)\rho f_4(- (3\tau + 1)^{-1}) \]
\[ = \frac{(3\tau + 1)^{k-2}}{\sqrt{3}} \left(\rho^2 f_1(3\tau + 1) + \rho^2 (1 + i)f_4(3\tau + 1) \right. \]
\[ + \rho (-1 + i)(1 - i)f_1(3\tau + 1) - \rho (1 - i)if_4(3\tau + 1) \bigg) \]
\[ = \frac{(3\tau + 1)^{k-1}}{\sqrt{3}} \left((\rho^4 i + \rho^3 (-1 + i)(1 - i))f_1(3\tau) + (\rho^3 (1 + i) - \rho^2 (1 - i)i) f_4(3\tau) \right) \]
\[ = (3\tau + 1)^{k-2} \left(\rho f_1(3\tau) + \rho (1 - i)f_4(3\tau) \right) = (3\tau + 1)^{k-2} \rho \hat{f}(\tau), \]

where the second last step is again a straightforward calculation. This yields $\hat{f} \in M^{3+}_{k-2}$, again.

So we are lifting exactly the same elliptic modular forms, no matter whether we consider quaternionic Maaß lifts for $v_i$ or $v_{-i}$.

So let us start with $g \in M^{3+}_{k-2}$, again. This time, we define

\[ f_1(\tau) := \frac{1}{1 - \rho} \left(\rho g\left(\frac{\tau + 1}{3}\right) - g\left(\frac{\tau - 1}{3}\right)\right), \]
\[ f_4(\tau) := \frac{1}{\rho - 1} \left(g\left(\frac{\tau + 1}{3}\right) - \rho g\left(\frac{\tau - 1}{3}\right)\right) \cdot \frac{1}{1 - i} \]

for $\tau \in \mathcal{H}$, while the rest of the $f_j$ are given by

\[ f_1 \equiv 0, \quad f_1 = -f_2 = -if_3 = if_6, \quad f_4 = -if_5 = if_7 = -f_8. \]

Note that this is again a slight difference compared to the $v_i$-case and a consequence of the
considerations from above. By exactly the same calculations as in (3.13) and (3.14) one can verify that \( f_1 \) and \( f_4 \) as well as \( F = (f_6, \ldots, f_8) \) possess the “right” transformation behaviors, which means the ones from above. We continue the process by defining

\[
\varphi(\tau, z) := \sum_{j=0}^{8} f_j(\tau) \cdot \vartheta_j(\tau, z), \quad \tau \in \mathcal{H}, \; z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}
\]

and

\[
\varphi|_{k,1}[T(m)](\tau, z) := m^{\frac{1}{2k-1}} \sum_{M \in \text{SL}_2(\mathbb{Z}) \setminus T(m)} \varphi|_{k,1}[\frac{1}{\sqrt{m}}M](\tau, \sqrt{m}z).
\]

This time we obtain

\[
\varphi|_{k,1}[T(m)] \in \mathcal{J}(k, m, \mathcal{O}, \nu_{-i})
\]

for all \( m \in \mathbb{N} \), where quaternionic Jacobi-forms for \( \nu_{-i} \) are defined in exactly the same way as quaternionic Jacobi-forms for \( \nu_i \) in (3.8), with the only difference that this time

\[
\varphi(\tau, ez) = \varphi_2(\pi p(e)) \cdot \varphi(\tau, z) \quad (= \varphi_1(\pi p(e)) \cdot \varphi(\tau, z))
\]

has to hold. The calculations to prove this would be exactly the same as in (3.15) and (3.18), where one simply has to note that – compared with the \( \nu_i \)-case – the roles of \( f_3 \) and \( f_6 \) and of \( f_5 \) and \( f_7 \) are interchanged. This is only relevant for the behavior under \( z \mapsto ez \). Furthermore, according to the considerations from above, we have to define

\[
\gamma_{-i}(h_1) = 1, \quad \gamma_{-i}(h_2) = -1, \quad \gamma_{-i}(h_3) = i, \quad \gamma_{-i}(h_6) = -i,
\]

\[
\gamma_{-i}(h_4) = \frac{1}{1-i}, \quad \gamma_{-i}(h_5) = \frac{i}{1-i}, \quad \gamma_{-i}(h_7) = \frac{-1}{1-i}, \quad \gamma_{-i}(h_8) = \frac{1}{1-i},
\]

and again \( \gamma_{-i}(t) := \gamma_{-i}(h_t) \) if \( t \equiv h_t \mod 2\mathcal{O} \) for arbitrary \( t \in \mathcal{O} \setminus 2\mathcal{O} \). By the same methods used for the \( \nu_i \)-case it is clear that if the Fourier-expansion of \( g \) is given by

\[
g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} a(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]

then the Fourier-expansion of \( \varphi|_{k,1}[T(m)] \) is given by

\[
\varphi|_{k,1}[T(m)](\tau, z) = \sum_{n \in \mathbb{N}} \sum_{t \in \mathcal{O}} \left( \sum_{d \mid \gcd(n,m)} \frac{d^{k-1} \gamma_{-i}(\frac{t}{d}) a\left(3\frac{nm-N(t)/4}{d^2}\right)}{d \cdot \gcd(n, m)} \right) e^{2\pi i (nt + \text{Re}(\tau z))}
\]

for \( \tau \in \mathcal{H} \) and \( z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \). Finally, the quaternionic Maass lift \( \mathcal{M}_g^{(-i)} \) for \( \nu_{-i} \) is defined as

\[
\mathcal{M}_g^{(-i)}(Z) := \sum_{m \in \mathbb{N}} \varphi|_{k,1}[T(m)](\tau, z) \cdot e^{2\pi im\omega}, \quad Z = \left( \frac{\tau + m}{\omega} \right) \in \mathcal{H} (\mathbb{H}).
\]

Redoing exactly the same calculations as in (3.19) (but noting that this time \( \varphi|_{k,1}[T(m)](\tau, ez) = \)
\( \varphi_2(\pi_\nu(\sigma)) \cdot \varphi|_{k,1}[\mathcal{M}(m)] \) holds) one verifies
\[
\mathcal{M}_8^{(-i)} \in [\text{Sp}_2(\mathcal{O}), k, v_{-i}],
\]
and this time the Fourier-expansion is given by
\[
\mathcal{M}_8^{(-i)}(Z) = \sum_{m, n \in \mathbb{N}} \sum_{l \in \mathcal{O}} \left( \sum_{d \in \mathbb{N}} \frac{d^{k-1} \gamma_{-i} \left( \frac{1}{4} \right) \alpha \left( 3^{nm-N(t)} / 4 \right)}{d | \text{gcd}(n, m)} \right) e^{2\pi i (nt + mw + \text{Re}(\imath z))}
\]
for all \( Z = (\frac{\tau}{z}, \omega) \in \mathcal{H}(\mathbb{H}) \). So actually, the only difference between the two Maass lifts is the prefactor \( \gamma_i \) and \( \gamma_{-i} \), respectively. And just like in (3.20), \( \mathcal{M}_8^{(-i)} \) is skew-symmetric. Finally, the vector space of all quaternionic Maass lifts for \( v_{-i} \) of odd weight \( k \) shall be denoted by \( \mathcal{M}(k, v_{-i}; \mathcal{O}) \).

We already saw that both quaternionic Maass lifts are quaternionic modular forms for \( \text{Sp}_2(\mathcal{O}) \) that also transform appropriately under \( Z \mapsto Z' \). But they are not quaternionic modular forms with respect to the extended quaternionic modular group \( \Gamma(\mathcal{O}) \). Instead, the transformation under \( i_1 l \) maps the two quaternionic Maass spaces \( \mathcal{M}(k, v_i; \mathcal{O}) \) and \( \mathcal{M}(k, v_{-i}; \mathcal{O}) \) onto each other.

(3.22) Theorem. Suppose \( k \in \mathbb{N}, k \geq 3 \) is odd and let \( g \in \mathcal{M}_{3,+}^{k-2} \). Then
\[
\mathcal{M}_8^{(i)}|_{k}(i_1 I)(Z) = -\mathcal{M}_8^{(-i)}(Z)
\]
holds for all \( Z \in \mathcal{H}(\mathbb{H}) \).

Proof: Suppose that the Fourier-expansion of \( g \) is given by
\[
g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0} \alpha(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}.
\]
Let \( Z = (\frac{\tau}{z}, \omega) \in \mathcal{H}(\mathbb{H}) \). Once more, note that \(-i_1 \mathcal{O} i_1 = \mathcal{O} \) as well as \(-i_1 \mathcal{O}^2 i_1 = \mathcal{O}^2 \) hold. Moreover, if \( t \in \mathcal{O}^2 \), then \( t \equiv i_1 \) modulo \( 2 \mathcal{O} \) for some \( j \in \{0, \ldots 8\} \) is equivalent to \(-i_1 t i_1 \equiv -i_1 h j i_1 \) modulo \( 2 \mathcal{O} \), and \( t \in 2 \mathcal{O} \) if and only if \(-i_1 t i_1 \in 2 \mathcal{O} \), of course. And finally, an easy consideration shows \( \text{Re}(\imath \tau(-i_1 z i_1)) = \text{Re}((i_1 \imath \tau(-i_1)) z) = \text{Re}((-i_1 \imath \tau i_1 \imath) z) \). Hence we compute
\[
\mathcal{M}_8^{(i)}|_{k}(i_1 I)(Z) = \mathcal{M}_8^{(i)}(-i_1 Z i_1) = \mathcal{M}_8^{(i)}(-i_1 z i_1, \omega) = \sum_{m, n \in \mathbb{N}} \sum_{l \in \mathcal{O}} \left( \sum_{d \in \mathbb{N}} \frac{d^{k-1} \gamma_{i} \left( \frac{1}{4} \right) \alpha \left( 3^{nm-N(t)} / 4 \right)}{d | \text{gcd}(n, m)} \right) e^{2\pi i (nt + mw + \text{Re}(\imath z))}
\]
which leads to
\[\sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2} \left( \sum_{d \mid \gcd(n,m)} \sum_{d^{-1} t \in \mathcal{O}^2 \setminus \mathcal{O}} d^{k-1} \gamma_i \left( \frac{-i t i t}{d} \right) a \left( \frac{3 \text{m} \text{m} - N(t) / 4}{d^2} \right) \right) \cdot e^{2\pi i (\nu t + m \omega + \text{Re}(T z))} \cdot d^{k-1} \gamma_i \left( \frac{-i t i t}{d} \right) a \left( \frac{3 \text{m} \text{m} - N(t) / 4}{d^2} \right) \cdot e^{2\pi i (\nu t + m \omega + \text{Re}(T z))} \cdot \]

A straightforward calculation shows that the \( h_j \) from (3.3) fulfill
\[-i_1 h_1 i_1 \equiv h_2, \quad -i_1 h_3 i_1 \equiv h_3, \quad -i_1 h_4 i_1 \equiv h_5, \quad -i_1 h_6 i_1 \equiv h_6, \quad -i_1 h_7 i_1 \equiv h_8 \pmod{2 \mathcal{O}},\]
and thus we compute
\[
\begin{align*}
\gamma_i (-i_1 h_1 i_1) &= -1 = -\gamma_{-i} (h_1), \\
\gamma_i (-i_1 h_3 i_1) &= -i = -\gamma_{-i} (h_3), \\
\gamma_i (-i_1 h_4 i_1) &= \frac{-i}{1 + i} = -\gamma_{-i} (h_4), \\
\gamma_i (-i_1 h_7 i_1) &= \frac{1}{1 + i} = -\gamma_{-i} (h_7), \\
\gamma_i (-i_1 h_2 i_1) &= 1 = -\gamma_{-i} (h_2), \\
\gamma_i (-i_1 h_6 i_1) &= i = -\gamma_{-i} (h_6), \\
\gamma_i (-i_1 h_5 i_1) &= \frac{1}{1 + i} = -\gamma_{-i} (h_5), \\
\gamma_i (-i_1 h_8 i_1) &= \frac{-i}{1 + i} = -\gamma_{-i} (h_8),
\end{align*}
\]
which leads to
\[
\gamma_i (-i_1 t i_1) = -\gamma_{-i} (t)
\]
for all \( t \in \mathcal{O}^2 \setminus 2 \mathcal{O}. \) And therefore we have
\[
\mathcal{M}_{\mathcal{O}}^{(i)}(i_1 I)(Z) = \sum_{m,n \in \mathbb{N}} \sum_{t \in \mathcal{O}^2 \setminus 2 \mathcal{O}} \left( \sum_{d \mid \gcd(n,m)} \sum_{d^{-1} t \in \mathcal{O}^2 \setminus \mathcal{O}} d^{k-1} \gamma_i \left( \frac{-i t i t}{d} \right) a \left( \frac{3 \text{m} \text{m} - N(t) / 4}{d^2} \right) \right) \cdot e^{2\pi i (\nu t + m \omega + \text{Re}(T z))} \cdot d^{k-1} \gamma_i \left( \frac{-i t i t}{d} \right) a \left( \frac{3 \text{m} \text{m} - N(t) / 4}{d^2} \right) \cdot e^{2\pi i (\nu t + m \omega + \text{Re}(T z))} \cdot \]

There are two final remarks: First, the condition \( 4mn \succ N(t) \) in the Fourier-expansions leads to the quaternionic Maaß lifts of odd weight being cusp forms (see (1.62)). And second, suppose \( f \in \mathcal{M}(k, \nu; \mathcal{O}) \) or \( f \in \mathcal{M}(k, \nu; \mathcal{O}) \) and let \( M = \text{Rot}( \left( \begin{array} {cc} 1 & 0 \\ 0 & i_2 \end{array} \right) ) \). Then by definition, \( \nu_i (M) = \nu_{-i} (M) = -1 \) holds. Thus we have
\[
f_{|k} M(Z) = f(-i_2 Z i_2) = -f(Z)
\]
for all \( Z \in \mathcal{H}(\mathbb{H}) \). Define \( R = \mathbb{R} + i_2 \mathbb{R} \). One easily checks that \( -i_2 Z i_2 = Z \) holds for all \( Z \in \mathcal{H}(R) \), and thus
\[
f_{|k} M \equiv 0 \quad \text{(3.26)}
\]
follows. Note that according to (1.72), the restriction to \( R \) yields Hermitian modular forms with respect to \( \mathbb{Q}(i) \), since the integral closure of \( \mathbb{Q}(i) \) is \( \mathbb{Z}(i) \) (cf. [De01] about Hermitian modular forms). We abbreviate and indicate this fact by writing \( f_{|\mathbb{Q}(i)} \) (for any quaternionic modular form \( f \)). Thus for quaternionic Maaß forms of odd weight we have
\[
f_{|\mathbb{Q}(i)} \equiv 0 \cdot
3.3 The space $\mathcal{M}_k^3$

As announced, we are now going to determine the vector spaces $\mathcal{M}_k^3$ for odd $k \geq 1$. The result will be that they are of dimension $\left\lfloor \frac{k+1}{6} \right\rfloor$. And we will present methods to explicitly determine elements of these spaces – at least by using numerical methods since we have to make use of the Atkin-Lehner-involution. Once we have determined the vector spaces $\mathcal{M}_k^3$, then we will also have proved that there actually exist non-trivial quaternionic Maaß lifts of odd weight and that $v_i$ and $v_{-i}$ are multiplier systems, indeed.

We will need a lot of theoretical background about elliptic modular forms. We will have to deal with Hecke-operators for congruence subgroups and with so-called newforms, as well. We will not present the whole theory here, only the parts needed to finally determine $\mathcal{M}_k^3$. The complete theory can be found in [Mi89], where it is presented quite comprehensible and thoroughly.

First, we need to extend the slash-operator for elliptic modular forms to the space of two by two matrices over $\mathbb{R}$ with positive determinant, since it is only defined for $SL_2(\mathbb{R})$, so far. So let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) := \{A \in GL_2(\mathbb{R}) ; \det(A) > 0\}$ and $k \in \mathbb{Z}$. For a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ we define

$$ f|M(\tau) := \det(M)^{k/2} \cdot M \{\tau\}^{-k} \cdot f(M(\tau)) = \det(M)^{k/2} \cdot (c\tau + d)^{-k} \cdot f \left( \frac{a\tau + b}{c\tau + d} \right). \quad (3.27) $$

According to [Mi89, p.37],

$$ (f|M_1)|_k M_2 = f|M_1 M_2 \quad (3.28) $$

holds for all $M_1, M_2 \in GL_2^+(\mathbb{R})$, again. And by definition we have

$$ f|_k (rM) = f|_k M $$

for all $M \in GL_2^+(\mathbb{R})$ and $r > 0$, of course. Next, let us shortly introduce Hecke-operators for the theta groups $\Gamma_0[N], N \in \mathbb{N}$. For details, confer [Mi89, pp.132]. Define

$$ \Delta_0(N) := \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2} ; \quad c \equiv 0 \bmod N, \quad \gcd(a, N) = 1, \quad \det(M) > 0\}. $$

According to [Mi89, le.4.5.2], the following holds: Given $M \in \Delta_0(N)$, there exist $l, m \in \mathbb{N}$ such that $l|m$, $\gcd(l, N) = 1$ and

$$ \Gamma_0[N]M\Gamma_0[N] = \Gamma_0[N]\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma_0[N]. $$

$l$ and $m$ are uniquely determined by $M$. Let $\chi$ be some Dirichlet character mod $N$ (see [Mi89, p.79] for a definition). Then $\chi$ induces an abelian character for $\Gamma_0[N]$ (cf. [Mi89, p.114]). It is again denoted by $\chi$ and defined as

$$ \chi(M) := \chi(d), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0[N]. $$
\(\chi\) can also be extended to a character of \(\Delta_0(N)\) via
\[
\chi(M) := \overline{\chi(a)}, \quad M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Delta_0(N).
\]

(cf. [Mi89, p.134]) Note that for \(M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0[N]\), \(\det(M) = ad - bc = 1\) has to hold, and thus \(a\) is the inverse of \(d\) in \((\mathbb{Z}/N\mathbb{Z})^*\), since \(c \in \mathbb{N}Z\). Therefore, because \(\chi\) is a Dirichlet character mod \(N\), \(\chi(d) = \chi(a)^{-1} = \overline{\chi(a)}\) has to hold. And thus \(\chi\), defined as above, is indeed an extension of the abelian character \(\chi\) of \(\Gamma_0[N]\) to \(\Delta_0(N)\).

Next, suppose \(M \in \Delta_0(N)\). Of course, \(\Gamma_0[N]\) operates on the set \(\Gamma_0[N]\Delta_0[N] \) by multiplication from the left. Let \(I\) be some index set and suppose \(\{ M_j ; j \in I \} \) is a transversal of the orbits of \(\Gamma_0[N]\Delta_0[N] \) with respect to the operation of \(\Gamma_0[N]\), hence
\[
\Gamma_0[N]\Delta_0[N] = \bigcup_{j \in I} \Gamma_0[N]M_j.
\]

Then, given \(f \in \Gamma_0[N], k, \chi\) we define the Hecke-operator \(\Gamma_0[N]\Delta_0[N] \) by
\[
f|_k \Gamma_0[N]\Delta_0[N] := \det(M)^{k/2-1} \sum_{j \in I} \chi(M_j) f|_k M_j.
\] (3.29)

One easily verifies that the Hecke-operator is well defined, i.e. it is independent of the special choice of a transversal, and that again
\[
f|_k \Gamma_0[N]\Delta_0[N] \in [\Gamma_0[N], k, \chi].
\]

(Or confer [Mi89, p.134].) This definition can be extended to formal finite sums of double cosets \(\Gamma_0[N]M_j\Gamma_0[N]\), where \(M_j \in \Delta_0(N)\) for \(j \in I\) for some finite index set \(I\), by
\[
f|_k (\sum_{j \in I} \alpha_j \Gamma_0[N]M_j\Gamma_0[N]) := \sum_{j \in I} \alpha_j f|_k \Gamma_0[N]M_j\Gamma_0[N].
\]

We need some special double cosets and formal sums of this shape: Let \(l, m, n \in \mathbb{N}\) with \(l|m\), \(\gcd(l, N) = 1\), then
\[
T(l, m) := \Gamma_0[N] \left( \begin{array}{cc} l & 0 \\ 0 & m \end{array} \right) \Gamma_0[N] \quad \text{(3.30)}
\]

and
\[
T(n) := \sum_{l'|m'=n, \, l'|m', \, \gcd(l',N) = 1} T(l', m').
\] (3.31)

In particular, if \(p\) is a prime number, then \(T(p) = T(1, p)\). And although \(N\) is not indicated by the denotations \(T(l, m)\) and \(T(n)\), it will always be clear from the context what \(N\) is. Next, we need a special case of [Mi89, le.4.5.6]:
(3.23) Lemma. Let $p$ be a prime number. As a transversal of $\Gamma_0[N]\backslash \Gamma_0[N](\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix})\Gamma_0[N]$ we may take the following set:

\[
\begin{cases}
\left\{ \begin{pmatrix} p^{1-\delta} & m \\ 0 & p^\delta \end{pmatrix} ; \delta \in \{0,1\}, 0 \leq m < p^\delta \right\}, & \text{if } p \nmid N , \\
\left\{ \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} ; 0 \leq m < p \right\}, & \text{if } p | N .
\end{cases}
\]

In particular, a transversal of $T(3)$ is given by

\[
\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},
\]
as long as $N \in 3\mathbb{N}$.

Note that one can also define a product for the considered double cosets $\Gamma_0[N]\Gamma_0[N]$ (cf. [Mi89, pp.69] for details). An addition is already given by the formal sums. Thus we can define the so-called Hecke algebra

\[
\mathcal{R}(N) := \langle \Gamma_0[N]\Gamma_0[N] ; M \in \Delta_0(N) \rangle .
\]

According to [Mi89, thm.4.5.3], $\mathcal{R}(N)$ is a commutative algebra. And it was proved in [Mi89, thm.4.5.9], that the Hecke algebra is given by

\[
\mathcal{R}(N) = \mathbb{Z}[T(p), T(p, p), T(q) ; p, q \text{ prime numbers}, p \nmid N, q | N] .
\]

Before we present some structure theorems for spaces of elliptic modular forms, let us recall the definition of $\mathcal{M}_k^{3+}$:

(3.24) Remark. Given $k \in \mathbb{N}$, $k$ odd, $\mathcal{M}_k^{3+}$ is defined as

\[
\mathcal{M}_k^{3+} := \left\{ g \in [\Gamma_0[3], k, \mu_i]_0 ; g(\tau) = \sum_{n \in \mathbb{N}, n \neq 0 (3)} a(n)e^{2\pi i n \tau}, \tau \in \mathcal{H} \right\} \leq [\Gamma_0[3], k, \mu_i]_0 .
\]

On the generators of $\Gamma_0[3]$, the abelian character $\mu_i$ is given by

\[
\mu_i(-I) = -1, \quad \mu_i(T_1) = 1, \quad \mu_i(J_1^{-1}T_1^{-3}J_1) = \rho := e^{\frac{2}{3}\pi i} .
\]

In the proof of (3.12), we also determined $\mu_i$ explicitly: Let $\chi_3$ be the non-trivial Dirichlet character mod 3, i.e.

\[
\chi_3(n) = \begin{cases} 
0, & \text{if } n \equiv 0 \pmod{3} , \\
1, & \text{if } n \equiv 1 \pmod{3} , \\
-1, & \text{if } n \equiv -1 \pmod{3} .
\end{cases}
\]

Then $\mu_i$ is given by

\[
\mu_i(M) = \chi_3(d) \cdot \rho^{\chi_3(d)c/3} , \quad M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0[3] .
\]
Note that if $c \in 9\mathbb{Z}$, then $\omega^3(c/d)^c/3 = 1$ for all $d \in \mathbb{N}$. And therefore every $g \in \mathcal{M}_{3+}^k$ also fulfills

$$g \in [\Gamma_0(9), k, \omega_3]_0.$$ 

Now let $g \in [\Gamma_0(9), k, \omega_3]$ be arbitrary and suppose its Fourier-expansion is given by

$$g(\tau) = \sum_{n \in \mathbb{N}_0} a(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}$$

for appropriate Fourier-coefficients $a(n)$. Then in virtue of 3.29, 3.31 and (3.23) we have

$$g|kT(3) = 3^{k/2-1} \sum_{m=0}^{2} 3^{k/2-3} g \left( \frac{\tau + m}{3} \right) = \frac{1}{3} \sum_{n=0}^{2} a(n)(\omega^n)^n e^{\frac{2\pi in\tau}{3}}$$

for all $\tau \in \mathbb{N}$, since one easily verifies

$$1 + \omega^n + \omega^{2n} = \begin{cases} 3, & \text{if } n \equiv 0 \pmod{3} \\ 0, & \text{if } n \not\equiv 0 \pmod{3} \end{cases}.$$ 

Hence the equivalence

$$a(n) = 0 \quad \text{for all } n \in 3\mathbb{N}_0 \quad \Leftrightarrow \quad g|kT(3) \equiv 0$$

follows. This means $\mathcal{M}_{3+}^k$ is given as

$$\mathcal{M}_{3+}^k = \{ g \in [\Gamma_0(3], k, \mu]_0 ; g|kT(3) \equiv 0 \},$$

where the Hecke-operator $T(3)$ is meant to be with respect to $\Gamma_0(9)$, i.e. $T(3) = \Gamma_0(9)(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix})\Gamma_0(9)$.

To get back to structure theorems concerning elliptic modular forms, we need two further operators. The first one is the so-called Atkin-Lehner involution or Fricke involution. For $n \in \mathbb{N}$, define

$$\omega_n := \begin{pmatrix} 0 & -1 \\ n & 0 \end{pmatrix} \in \text{GL}_2^+ (\mathbb{R}). \tag{3.32}$$

Suppose $f \in [\Gamma_0[N], k, \chi]$ and $g \in [\Gamma_0[N], k, \chi]_0$ for some mod $N$ Dirichlet character $\chi$. Then according to [Mi89, le.4.3.2],

$$f|k\omega_N \in [\Gamma_0[N], k, \chi],$$

$$g|k\omega_N \in [\Gamma_0[N], k, \chi]_0 \tag{3.33}$$

hold, and the spaces are isomorphic via this map. Now, suppose again that $f \in [\Gamma_0[N], k, \chi]$.
3.3 The space $\mathcal{M}_k^{3+}$

with Fourier-expansion

$$f(\tau) = \sum_{n \in \mathbb{N}_0} a(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}. $$

Then define

$$f_{\rho}(\tau) := f(-\tau) = \sum_{n \in \mathbb{N}_0} \overline{a(n)}e^{2\pi in\tau}, \quad \tau \in \mathcal{H}. \quad (3.34)$$

Again, according to [Mi89, le.4.3.2],

$$f_{\rho} \in \Gamma_0[N], k, \chi \quad \text{and} \quad g_{\rho} \in \Gamma_0[N], k, \chi_0 \quad (3.35)$$

hold for all $f \in \Gamma_0[N], k, \chi$ and $g \in \Gamma_0[N], k, \chi_0$.

The importance of $\omega_n$ concerning $\mathcal{M}_k^{3+}$ lies in the following

(3.25) Lemma. Let $\chi_3$ be the non-trivial Dirichlet character mod 3 (see (3.24)) and let $k \in \mathbb{N}$ be odd. Suppose

$$g \in \Gamma_0[9], k, \chi_3\, \text{and assume that } g \text{ possesses the Fourier-expansion}$$

$$g(\tau) = \sum_{n \in \mathbb{N}, n \equiv 3 \mod 3} a(n)e^{2\pi in\tau} = \sum_{n \in \mathbb{N}_0} a(3n + 2)e^{2\pi i(3n + 2)\tau}, \quad \tau \in \mathcal{H}. $$

Then

$$g|_{k\omega_9} \in \Gamma_0[3], k, \mu_i|_0 $$

holds.

Proof: According to 3.33,

$$g|_{k\omega_9} \in \Gamma_0[9], k, \chi_3\, \text{and } \gamma = \Gamma_0[9], k, \chi_3|_0 \supset \Gamma_0[3], k, \mu_i|_0 $$

already holds, since $\chi_3$ only takes real values. Of course, $\Gamma_0[9] \subseteq \Gamma_0[3]$ holds. A transversal of $\Gamma_0[9] \setminus \Gamma_0[3]$ is given by

$I, \quad J_1^{-1}T_1^{-3}J_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad J_1^{-1}T_1^{-6}J_1 = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, $ 

which can be verified quite easily: Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0[3], M \notin \Gamma_0[9]$. Since $\det(M) = ad - bc = 1$, whereas $c \in 3\mathbb{Z}, a \equiv 1 \mod 3$ or $a \equiv -1 \mod 3$ has to hold. $c \in 3\mathbb{Z}$ and $c \notin 9\mathbb{Z}$ implies $c \equiv 3 \mod 9$ or $c \equiv -3 \mod 9$. So assume $a \equiv 1 \mod 3$ (which implies $3a \equiv 3 \mod 9$) and $c \equiv 3 \mod 9$ (which implies $c - 3a \equiv 0 \mod 9$). Then

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} M = \begin{pmatrix} a & b \\ c - 3a & d - 3b \end{pmatrix} \in \Gamma_0[9].$$

holds, or equivalently $M \in \Gamma_0[9], \text{ since } \begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$. Now assume $a \equiv -1 \mod 3$ and
We observe 
\[(\frac{1}{6} 0) M = (\frac{a}{c-b} a_6 \cdot b_{11}) \in \Gamma_0[9].\]
The two remaining cases are handled accordingly. And since the three matrices obviously belong to distinct cosets, the claim follows. Moreover, since \((\frac{1}{6} 0) = (\frac{3}{1} 1)^2\), we have thus shown
\[\Gamma_0[3] = \langle \Gamma_0[9], (\frac{1}{3} 1) \rangle.\]

And, in view of 1.14, since \(g|\omega_9\) already transforms under \(\Gamma_0[9]\) like we asserted, we only have to verify that
\[(g|\omega_9)|\mathcal{H}M = \rho \cdot g|\omega_9\]
holds true for \(M = (\frac{1}{3} 1)\). Note that we have
\[\omega_9M = (\frac{-3}{9} -1) = (\frac{1}{0} -1/3) \omega_9.\]

The special shape of the Fourier-expansion of \(g\) yields
\[g|\tau(\frac{1}{0} -1/3) = g(\tau - \frac{1}{3}) = \sum_{n \in \mathbb{N}_0} a(3n+2)e^{2\pi i (3n+2)\tau} e^{2\pi i (-3n-2)} \]
\[= \rho^{-2} \sum_{n \in \mathbb{N}_0} a(3n+2)e^{2\pi i (3n+2)\tau} = \rho \cdot g(\tau)\]
for all \(\tau \in \mathcal{H}\), since \(\rho^3 = 1\). So in virtue of 3.28 we compute
\[(g|\omega_9)|\mathcal{H}M = g|\tau(\frac{1}{0} -1) = (g|\tau(\frac{1}{0} -1/3))|\mathcal{H}M = \rho \cdot g|\omega_9,\]
which completes the proof. \([\Box]\)

Now we need to introduce so-called newforms and oldforms. Again, we will only present those parts of the theory needed to determine \(M_{k}^{+}\). The full theory can be found in [Mi89].

First, we need another operator. For \(l \in \mathbb{N}\), put
\[\delta_l := (\frac{1}{0} 1) .\]

We observe
\[f|\delta_l(\tau) = l^{k/2} f(l\tau)\]
by definition. Note that if \(\chi\) is a Dirichlet character mod \(N\), then \(\chi'\), defined by \(\chi'(n) = \chi(n)\) if \(\gcd(n, lN) = 1\) and \(\chi'(n) = 0\) if \(\gcd(n, lN) \neq 1\), is a Dirichlet character mod \(lN\) (cf. [Mi89, p.79]). We say that \(\chi'\) is induced by \(\chi\). To keep it well arranged, we denote it again by \(\chi\) (and not \(\chi'\)). Suppose \(f \in [\Gamma_0[N], k, \chi]\) and \(g \in [\Gamma_0[N], k, \chi_0]\). Then according to [Mi89, le.4.6.1],
\[f|\delta_l \in [\Gamma_0[lN], k, \chi],\]
\[g|\delta_l \in [\Gamma_0[lN], k, \chi_0]\]
hold. And note that we obviously have \([\Gamma_0[N], k, \chi] \subset [\Gamma_0[lN], k, \chi]\) and \([\Gamma_0[N], k, \chi_0] \subset [\Gamma_0[lN], k, \chi_0]\). Furhtermore, we need the co-called conductor \(m_\chi\) for \(\chi\) a Dirichlet charac-
3.3 The space $\mathcal{M}_k^+$

This is the smallest integer $m$ such that $\chi$ is induced by a Dirichlet character mod $m$. If $N = m\chi$, then we call $\chi$ a primitive character mod $N$. Note that $m\chi$ actually exists for any Dirichlet character (cf. [Mi89, p.80]).

So let $k \in \mathbb{N}$ and $\chi$ a Dirichlet character mod $N$. The space of so-called oldforms of $[\Gamma_0[N], k, \chi]_0$, denoted by $[\Gamma_0[N], k, \chi]_0^{\text{old}}$, is the subspace of $[\Gamma_0[N], k, \chi]_0$ generated by the set

$$\bigcup_{M \in \mathbb{N}, M < N} \bigcup_{l \in \mathbb{N}, l \mid \frac{N}{M}} \{ f(lz) ; f \in [\Gamma_0[M], k, \chi]_0 \} .$$

Furthermore, there exists an inner product on the space $[\Gamma_0[N], k, \chi]_0$ given by

$$(f, g) := \nu(\Gamma_0[N] \setminus \mathcal{H}) \int_{\Gamma_0[N] \setminus \mathcal{H}} f(z) \overline{g(z)} y^k \frac{dx \, dy}{y^2} ,$$

for $f, g \in [\Gamma_0[N], k, \chi]_0$, where $z = x + iy$, $\Gamma_0[N] \setminus \mathcal{H}$ a fundamental domain of $\mathcal{H}$ with respect to $\Gamma_0[N]$ (see (1.23)) and

$$\nu(\Gamma_0[N] \setminus \mathcal{H}) := \int_{\Gamma_0[N] \setminus \mathcal{H}} \frac{dx \, dy}{y^2} .$$

It is called the Peterson inner product and actually induces a Hermitian inner product on $[\Gamma_0[N], k, \chi]_0$ (for example, confer [Mi89, p.44]). Thus there exists an orthogonal complement of $[\Gamma_0[N], k, \chi]_0^{\text{old}}$ in $[\Gamma_0[N], k, \chi]_0$ with respect to this inner product. This orthogonal complement is called the space of newforms and denoted by

$$[\Gamma_0[N], k, \chi]_0^{\text{new}} := ([\Gamma_0[N], k, \chi]_0^{\text{old}})^\perp \leq [\Gamma_0[N], k, \chi]_0 .$$

A simple fact from linear algebra is that

$$[\Gamma_0[N], k, \chi]_0 = [\Gamma_0[N], k, \chi]_0^{\text{new}} \oplus [\Gamma_0[N], k, \chi]_0^{\text{old}}$$

then holds. And according to [Mi89, le.4.6.9], we obviously have:

- If $\chi$ is a primitive character mod $N$, then $[\Gamma_0[N], k, \chi]_0 = [\Gamma_0[N], k, \chi]_0^{\text{new}}$.

- If $m\chi \mid M$, $\nu \mid N$ and $M \neq N$, then $[\Gamma_0[M], k, \chi]_0 \subset [\Gamma_0[N], k, \chi]_0^{\text{old}}$.

- By induction, $[\Gamma_0[N], k, \chi]_0$ is generated by the set

$$\bigcup_{M \in \mathbb{N}, M \mid N} \bigcup_{l \in \mathbb{N}, l \mid \frac{N}{M}} \{ f(lz) ; f \in [\Gamma_0[M], k, \chi]_0^{\text{new}} \} .$$

We are now going to concretize this decomposition for $N = 9$ and $\chi = \chi_3$ the non-trivial Dirichlet character mod 3.
We obtain

\[ f = f_0 \bigoplus e^{\infty \frac{1}{2} \pi i \tau} \in H(\Gamma_0[3], k, \chi_3) \bigoplus \mathcal{H}(\Gamma_0[3], k, \chi_3) \bigoplus \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{new}, \]

where \( \mathcal{H}(\Gamma_0[3], k, \chi_3) \bigoplus \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{old} \) and the Fourier-expansion is given by

\[ f|k \delta_3 = \{ f|k \delta_3 : f \in \mathcal{H}(\Gamma_0[3], k, \chi_3) \}. \]

**Proof:** According to 3.41, we can decompose \( \mathcal{H}(\Gamma_0[3], k, \chi_3) \bigoplus \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{old} \) as

\[ \mathcal{H}(\Gamma_0[3], k, \chi_3) \bigoplus \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{new} = \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{old}, \]

so let us have a closer look at \( \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{old} \). Since \( \chi_3 \) is the non-trivial Dirichlet character mod 3, its conductor is 3. So the only integer \( M \) fulfilling \( m_{\chi_3} = 3|M, M|N = 9, M \leq 9 \) is \( M = 3 \), and the only integers \( l \) fulfilling \( l|\frac{N}{M} = 3 \) are \( l = 1 \) and \( l = 3 \). And thus in virtue of 3.38 we have

\[ \mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{old} = \mathcal{H}(\Gamma_0[3], k, \chi_3) + \mathcal{H}(\Gamma_0[3], k, \chi_3)|k \delta_3. \]

Thus we only have to prove that

\[ \mathcal{H}(\Gamma_0[3], k, \chi_3) \cap \mathcal{H}(\Gamma_0[3], k, \chi_3)|k \delta_3 = \{0\} \]

holds. So let \( f \in \mathcal{H}(\Gamma_0[3], k, \chi_3) \) with Fourier-expansion

\[ f(\tau) = \sum_{n \in \mathbb{N}} a(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}. \]

We obtain

\[ f|k \delta_3(\tau) = 3^{k/2}f(3\tau) = \sum_{n \in \mathbb{N}} 3^{k/2}a(n)e^{2\pi i 3n \tau} = \sum_{n \in \mathbb{N}} 3^{k/2}a\left(\frac{n}{3}\right)e^{2\pi i n \tau} \]

for all \( \tau \in \mathcal{H} \). Now suppose \( g \in \mathcal{H}(\Gamma_0[3], k, \chi_3) \cap \mathcal{H}(\Gamma_0[3], k, \chi_3)|k \delta_3. \) By what we have just seen, this implies \( g \in \mathcal{H}(\Gamma_0[3], k, \chi_3) \) and the Fourier-expansion is given by

\[ g(\tau) = \sum_{n \in \mathbb{N}} \beta(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H} \]

for some appropriate Fourier-coefficients \( \beta(n) \) that fulfill \( \beta(n) = 0 \) for all \( n \in \mathbb{N} \setminus 3\mathbb{N} \). This means: Let \( l = 3 \). The conductor of \( \chi_3 \) is \( m_{\chi_3} = 3 \). \( \beta(n) = 0 \) for all \( n \) prime to \( l \), whereas \( \gcd(l, \frac{3}{m_{\chi_3}}) = 1 \). Thus all preconditions of theorem 4.6.8 in [Mi89] are fulfilled, and this theorem then says that \( g \equiv 0 \). This completes the proof.

In particular, the preceding corollary says

\[ \dim(\mathcal{H}(\Gamma_0[3], k, \chi_3)) = 2 \cdot \dim(\mathcal{H}(\Gamma_0[3], k, \chi_3)) + \dim(\mathcal{H}(\Gamma_0[3], k, \chi_3)^\text{new}) \quad (3.42) \]

Next, we need so-called primitive newforms. Note that the Hecke-operators \( T(n) \) map \( \mathcal{H}(\Gamma_0[N], k, \chi) \) onto itself, so the following definition makes sense:
(3.27) Definition. Let \( k \in \mathbb{N} \), \( \chi \) a Dirichlet character mod \( N \) and \( f \in [\Gamma_0[N], k, \chi]_0^{\text{new}} \) with Fourier-expansion
\[
f(\tau) = \sum_{n \in \mathbb{N}} \alpha(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}.
\]
Then \( f \) is said to be a primitive newform of conductor \( N \) if the following conditions are fulfilled:

- \( f \) is a common eigenfunction of all \( T(n) \), where \( n \in \mathbb{N} \) with \( \gcd(n, N) = 1 \), i.e.

\[
\text{there exist } t(n) \text{ such that } f|_kT(n) = t(n) \cdot f \text{ holds for all } n \in \mathbb{N} \text{ with } \gcd(n, N) = 1.
\]

- \( \alpha(1) = 1 \).

The surprising fact lies in the following theorem, cited from [Mi89, thm.4.6.13]:

(3.28) Theorem. Let \( k \in \mathbb{N} \) and \( \chi \) a Dirichlet character mod \( N \). Then \( [\Gamma_0[N], k, \chi]_0^{\text{new}} \) has a basis consisting of primitive newforms. Furthermore, all primitive newforms of conductor \( N \) are common eigentfunctions for all \( T(n), n \in \mathbb{N} \) (which means also if \( \gcd(n, N) \neq 1 \)).

We need some further important facts about primitive newforms in \( [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \). We collect them in the following

(3.29) Proposition. Let \( k \in \mathbb{N}, k \text{ odd}, \chi_3 \text{ the non-trivial Dirichlet character mod } 3 \) (see (3.24)) and \( f \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \) a primitive newform with Fourier-expansion
\[
f(\tau) = \sum_{n \in \mathbb{N}} \alpha(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}.
\]
Then the following holds:

\( a) \) \( f|_kT(n) = \alpha(n) \cdot f \) for all \( n \in \mathbb{N} \).

\( b) \) \( (f|_{k\omega_9})|_kT(n) = \overline{\alpha(n)} \cdot (f|_{k\omega_9}) \) for all \( n \in \mathbb{N} \).

\( c) \) \( f_\rho \) is also a primitive newforms in \( [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \) and we have
\[
f|_{k\omega_9} = c \cdot f_\rho
\]
for some \( c \in \mathbb{C} \) with \( |c| = 1 \). Furthermore,
\[
f_\rho|_{k\omega_9} = -\tau \cdot f
\]
holds.

\( d) \) We have
\[
\alpha(3n) = 0, \quad \alpha(3n + 1) \in \mathbb{R}, \quad \alpha(3n + 2) \in i\mathbb{R}
\]
for all \( n \in \mathbb{N}_0 \) (and \( \alpha(1) = 1 \) holds by definition).

\( e) \) \( f \neq f_\rho \), or in other words there exists \( n \in \mathbb{N}_0 \) such that \( \alpha(3n + 2) \neq 0 \).
Proof: a) Cf. [Mi89, le.4.5.15] and [Mi89, p.165].

b) Cf. [Mi89, thm.4.5.5] and [Mi89, p.165].

c) According to [Mi89, thm.4.6.15], $f_\rho$ is a primitive newforms in $[\Gamma_0[9], k, \chi_3]^{\text{new}}$ (since $\chi_3 = \overline{\chi}_3$) and there exists $c \in \mathbb{C}$ such that

$$f|_k \omega_q = c \cdot f_\rho$$

holds. We required $k$ to be odd, and thus one easily verifies $(f|_k \omega_q)|_k \omega_q = -f$ as well as $f_\rho|_k \omega_q = -(f|_k \omega_q)_\rho$ (or confer [Li75, p.296]), and of course $(f_\rho)_\rho = f$. Hence we get

$$-f = (f|_k \omega_q)|_k \omega_q = (c \cdot f_\rho)|_k \omega_q = -c \cdot ((f|_k \omega_q)_\rho) = -c \cdot (c \cdot (f_\rho)_\rho) = -c \cdot f,$$

which implies $c \overline{c} = |c|^2 = 1$, since $f \neq 0$. A similar computation yields

$$f_\rho|_k \omega_q = -(f|_k \omega_q)_\rho = -(c \cdot f_\rho)_\rho = -c \cdot f.$$

d) Let $n \in \mathbb{N}_0$. Of course, we have $\gcd(3n + 1, 9) = \gcd(3n + 2, 9) = 1$. Thus, according to [Mi89, thm.4.5.4] and [Mi89, p.165], we obtain

$$\frac{\alpha(3n + 1)}{\alpha(3n + 2)} = \chi_3(3n + 1)\alpha(3n + 1) = \alpha(3n + 1), \quad \frac{\alpha(3n + 2)}{\alpha(3n + 2)} = \chi_3(3n + 2)\alpha(3n + 2) = -\alpha(3n + 2).$$

Furthermore, the conductor of $\chi_3$ is $m_{\chi_3} = 3$. Let $q = 3$. Then $q$ is a divisor of $N = 9$. The $q$-components of $N$ and $m_{\chi_3}$ are $N_q = 9$ and $m_q = 3$, of course. We have $q^2 | N_q$ and $N_q \neq m_q$. So the preconditions of theorem 4.6.17 in [Mi89] are met and this theorem then says $\alpha(q) = \alpha(3) = 0$. Thus a) implies

$$f|_k T(3) = \alpha(3) \cdot f \equiv 0.$$

As we have seen in (3.24), this is equivalent to

$$\alpha(3n) = 0$$

for all $n \in \mathbb{N}_0$.

e) Suppose $f = f_\rho$ (which is equivalent to $\alpha(3n + 2) = 0$ for all $n \in \mathbb{N}_0$ in view of d)). Thus

$$f|_k \omega_q = c \cdot f_\rho = c \cdot f$$

holds for some appropriate $c \in \mathbb{C}$. But then the combination of theorem 3 and 7 in [Li75] yields that $\chi_3$ has to be the principal character mod 9, which obviously is not the case. Thus this contradiction yields $f \neq f_\rho$. \qed

Suppose $f \in [\Gamma_0[9], k, \chi_3]^{\text{new}}$ is a primitive newform. By definition, the first Fourier-coefficient $\alpha(1)$ of $f$ is 1, and according to the preceding proposition there exists $n \in \mathbb{N}_0$ such that the $(3n + 2)$-th Fourier-coefficient satisfies $0 \neq \alpha(3n + 2) \in i\mathbb{R}$. This implies that both $f + f_\rho$ and
$f - f_{\rho}$ are non-trivial elements of $[\Gamma_0[9], k, \chi_3]_0^{\text{new}}$. According to (3.28), $[\Gamma_0[9], k, \chi_3]_0^{\text{new}}$ has a basis consisting of primitive newforms. Let such a basis be given by $f_1, \ldots, f_m$. (Note that $m = \dim([\Gamma_0[9], k, \chi_3]_0) < \infty$ according to [Mi89, thm.2.5.3] – we will determine it explicitly later on.)

Suppose $g \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}}$, then there exist appropriate (and unique) constants $c_1, \ldots, c_m \in \mathbb{C}$ such that

$$g = \sum_{j=1}^{m} c_j f_j = \sum_{j=1}^{m} \frac{1}{2} c_j (f_j + (f_j)_{\rho}) + \sum_{j=1}^{m} \frac{1}{2} c_j (f_j - (f_j)_{\rho})$$

holds. According to (3.29), the Fourier-expansions of the $f_j + (f_j)_{\rho}$ and $f_j - (f_j)_{\rho}$ are given by

$$(f_j + (f_j)_{\rho})(\tau) = \sum_{n \in \mathbb{N}_0} a_j(3n + 1)e^{2\pi i (3n+1)\tau}, \quad \tau \in \mathcal{H}$$

and

$$(f_j - (f_j)_{\rho})(\tau) = \sum_{n \in \mathbb{N}_0} i a_j(3n + 2)e^{2\pi i (3n+2)\tau}, \quad \tau \in \mathcal{H}$$

for some appropriate coefficient-functions $a_j : \mathbb{N}_0 \to \mathbb{R}$. Thus we have shown that every $g \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}}$ can be written as

$$g = c_1 g_1 + c_2 g_2$$

for appropriate constants $c_1, c_2 \in \mathbb{C}$ and $g_j \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}}$, where

$$[\Gamma_0[9], k, \chi_3]_0^{\text{new, (j)}} := \left\{ f \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \mid f(\tau) = \sum_{n \in \mathbb{N}_0} a(3n + j \rho)e^{2\pi i (3n+j)\tau}, \tau \in \mathcal{H} \right\}.$$

(3.43)

Of course, $[\Gamma_0[9], k, \chi_3]_0^{\text{new, (1)}} \cap [\Gamma_0[9], k, \chi_3]_0^{\text{new, (2)}} = \{0\}$, and thus

$$[\Gamma_0[9], k, \chi_3]_0^{\text{new}} = [\Gamma_0[9], k, \chi_3]_0^{\text{new, (1)}} \oplus [\Gamma_0[9], k, \chi_3]_0^{\text{new, (2)}}$$

follows. Now suppose $g \in [\Gamma_0[9], k, \chi_3]_0^{\text{new, (1)}} \leq [\Gamma_0[9], k, \chi_3]_0^{\text{new}}$. Again, we obtain

$$g = \sum_{j=1}^{m} \frac{1}{2} c_j (f_j + (f_j)_{\rho}) + \sum_{j=1}^{m} \frac{1}{2} c_j (f_j - (f_j)_{\rho})$$

for appropriate constants $c_1, \ldots, c_m \in \mathbb{C}$. Considering the Fourier-expansion on both sides yields

$$\sum_{j=1}^{m} \frac{1}{2} c_j (f_j - (f_j)_{\rho}) \equiv 0,$$

and thus

$$g = \sum_{j=1}^{m} \frac{1}{2} c_j (f_j + (f_j)_{\rho}).$$

The analog also holds for $g \in [\Gamma_0[9], k, \chi_3]_0^{\text{new, (2)}}$, of course. So this implies

$$[\Gamma_0[9], k, \chi_3]_0^{\text{new, (1)}} = (f_j + (f_j)_{\rho} ; j = 1, \ldots, m), \quad [\Gamma_0[9], k, \chi_3]_0^{\text{new, (2)}} = (f_j - (f_j)_{\rho} ; j = 1, \ldots, m).$$
Let \( m_1 \) denote the dimension of \( \Gamma_0[9], k, \chi_{30}^{\text{new},(1)} \), and without loss of generality suppose that
\[
f_1 + (f_1)_\rho, \ldots, f_{m_1} + (f_{m_1})_\rho
\]
is a basis of \( \Gamma_0[9], k, \chi_{30}^{\text{new},(1)} \). Thus we can define a homomorphism
\[
\varphi : [\Gamma_0[9], k, \chi_{30}^{\text{new},(1)}] \rightarrow [\Gamma_0[9], k, \chi_{30}^{\text{new},(2)}], \quad \sum_{j=1}^{m_1} c_j (f_j + (f_j)_\rho) \mapsto \sum_{j=1}^{m_1} c_j (f_j - (f_j)_\rho).
\]
We are now going to verify that this homomorphism is bijective: We start by considering an arbitrary primitive newform \( f \in [\Gamma_0[9], k, \chi_{30}^{\text{new}} \) with Fourier-expansion
\[
f(\tau) = \sum_{n \in \mathbb{N}} a(n) e^{2\pi in \tau}, \quad \tau \in \mathcal{H}.
\]
(With \( a(3n) = 0 \) for all \( n \in \mathbb{N} \).) According to (3.29),
\[
f|_kT(n) = a(n) \cdot f
\]
holds for all \( n \in \mathbb{N} \). In virtue of the same proposition, the Fourier-expansion of \( f_\rho \) is given by
\[
f_\rho(\tau) = \sum_{n \in \mathbb{N}} a_\rho(n) e^{2\pi in \tau}, \quad \tau \in \mathcal{H},
\]
where
\[
a_\rho(3n) = 0, \quad a_\rho(3n + 1) = a(3n + 1), \quad a_\rho(3n + 2) = -a(3n + 2)
\]
for all \( n \in \mathbb{N}_0 \), and again
\[
f_\rho|_kT(n) = a_\rho(n) \cdot f_\rho.
\]
This implies
\[
(f - f_\rho)|_kT(3n + 2) = a(3n + 2) \cdot (f + f_\rho)
\]
for all \( n \in \mathbb{N}_0 \). Now let \( t_j(n) \in \mathbb{C} \) for \( j = 1, \ldots, m \) and \( n \in \mathbb{N} \) such that we have
\[
f_j|_kT(n) = t_j(n) \cdot f_j.
\]
So let us prove the injectivity, now. Suppose there are constants \( c_1, \ldots, c_{m_1} \) such that
\[
\varphi\left( \sum_{j=1}^{m_1} c_j (f_j + (f_j)_\rho) \right) = \sum_{j=1}^{m_1} c_j (f_j - (f_j)_\rho) = 0.
\]
Let \( l \in \{1, \ldots, m_1\} \) be arbitrary. In view of (3.29) there exists \( n \in \mathbb{N}_0 \) such that \( t_l(3n + 2) \neq 0 \). According to the considerations above we get
\[
0 = \left( \sum_{j=1}^{m_1} c_j (f_j - (f_j)_\rho) \right)|_kT(3n + 2) = \sum_{j=1}^{m_1} c_j t_j(3n + 2) \cdot (f_j + (f_j)_\rho).
\]
We assumed \( f_1 + (f_1)_\rho, \ldots, f_{m_1} + (f_{m_1})_\rho \) to be linearly independent. Thus \( c_j t_j(3n + 2) = 0 \) follows
for all $j \in \{1, \ldots, m_1\}$, so in particular $c_j t_j (3n + 2) = 0$, which implies $c_l = 0$ since $t_j (3n + 2) \neq 0$. The arbitrariness of $l$ then yields the injectivity.

The proof of the surjectivity makes use of the same idea. By definition, $\varphi(f_j + (f_j)_\rho) = f_j - (f_j)_\rho$ already holds for all $j \in \{1, \ldots, m_1\}$. So let $l \in \{m_1 + 1, \ldots, m\}$. Since $f_l + (f_l)_\rho \in [\Gamma_0[9], k, \chi_3]_0^{new,(1)}$, there exist constants $c_1, \ldots, c_{m_1}$ such that

$$f_l + (f_l)_\rho = \sum_{j=1}^{m_1} c_j (f_j + (f_j)_\rho).$$

Again, choose $n \in \mathbb{N}_0$ so that $t_l (3n + 2) \neq 0$. We obtain

$$t_l (3n + 2) \cdot (f_l - (f_l)_\rho) = (f_l + (f_l)_\rho)_{|T(3n + 2)} = \sum_{j=1}^{m_1} c_j t_j (3n + 2) \cdot (f_j - (f_j)_\rho).$$

$t_l (3n + 2) \neq 0$ then leads to

$$(f_l - (f_l)_\rho) = \varphi \left( \sum_{j=1}^{m_1} c_j t_j (3n + 2) \cdot (f_j + (f_j)_\rho) \right).$$

According to the considerations above, $[\Gamma_0[9], k, \chi_3]_0^{new,(2)}$ is generated by $f_j - (f_j)_\rho$, $j = 1, \ldots, m$. Hence the surjectivity follows. So we proved

**(3.30) Proposition.** Let $k \in \mathbb{N}$, $k$ odd and $\chi_3$ the non-trivial Dirichlet character mod 3 (see (3.24)). We have

$$[\Gamma_0[9], k, \chi_3]_0^{new} = [\Gamma_0[9], k, \chi_3]_0^{new,(1)} \oplus [\Gamma_0[9], k, \chi_3]_0^{new,(2)}$$

(with $[\Gamma_0[9], k, \chi_3]_0^{new,(j)}$ from 3.43). Furthermore,

$$[\Gamma_0[9], k, \chi_3]_0^{new,(j)} = \{ f + (-1)^j f_\rho : f \in [\Gamma_0[9], k, \chi_3]_0^{new} \text{ a primitive newform} \}, \quad j \in \{1,2\}$$

and

$$\dim([\Gamma_0[9], k, \chi_3]_0^{new,(1)}) = \dim([\Gamma_0[9], k, \chi_3]_0^{new,(2)}) = \frac{1}{2} \dim([\Gamma_0[9], k, \chi_3]_0^{new})$$

hold.

The importance of $[\Gamma_0[9], k, \chi_3]_0^{new,(2)}$ lies in the following

**(3.31) Proposition.** Let $k \in \mathbb{N}$, $k$ odd and $\chi_3$ the non-trivial Dirichlet character mod 3. Then

$$\varphi_M : [\Gamma_0[9], k, \chi_3]_0^{new,(2)} \rightarrow \mathcal{M}_k^{3+}, \quad f \mapsto f|_k \omega_9$$

is an injective homomorphism.

**Proof:** Let $f \in [\Gamma_0[9], k, \chi_3]_0^{new}$ be a primitive newforms. Thus $f - f_\rho \in [\Gamma_0[9], k, \chi_3]_0^{new,(2)}$ (see
In particular, this implies that \( f - f_\rho \) possesses a Fourier-expansion of the shape
\[
g(\tau) = \sum_{n \in \mathbb{N}_0} a(3n + 2) e^{2\pi i(3n+2)\tau}, \quad \tau \in \mathcal{H}.
\]

But then
\[
\varphi_M(f - f_\rho) \in [\Gamma_0[3], k, \mu_i]_0
\]
already holds in virtue of (3.25). Furthermore, \( f|kT(3) \equiv 0 \) and \( (f|k\omega_9)|kT(3) \equiv 0 \) hold according to (3.29) and there exists \( c \in \mathbb{C} \) with \( f_\rho|k\omega_9 = c \cdot f \). Hence we obtain
\[
(\varphi_M(f - f_\rho))|kT(3) = (f|k\omega_9)|kT(3) - c \cdot f|kT(3) \equiv 0,
\]
which leads to \( \varphi_M(f - f_\rho) \in \mathcal{M}_k^{3+} \) in view of (3.24). According to (3.30), \( [\Gamma_0[9], k, \chi_3]_0^{\text{new}(2)} \) is generated by the set of all \( f - f_\rho, f \) being primitive newforms. The obvious linearity of \( \varphi_M \) then yields the well-definedness. And clearly, \( \varphi_M \) is injective since \( f|k\omega_9 \equiv 0 \) if and only if \( f \equiv 0. \Box
\]

The final step will be to prove that \( \varphi_M \) actually is an isomorphism. Since \( k \) is odd, we have \( (f|k\omega_9)|k\omega_9 = -f \) for any holomorphic function \( f \). Therefore, proving the surjectivity of \( \varphi_M \) is equivalent to proving that \( g|k\omega_9 \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}(2)} \) holds for all \( g \in \mathcal{M}_k^{3+} \). But before we can do so, we need some further traits of newforms in \( [\Gamma_0[9], k, \chi_3]_0 \). Mainly, we need an appropriate method to determine whether some \( f \in [\Gamma_0[9], k, \chi_3]_0 \) is a newform or not.

We will need an operator that maps elliptic modular form in \( [\Gamma_0[9], k, \chi_3]_0 \) to elliptic modular forms in \( [\Gamma_0[3], k, \chi_3]_0 \). Note that \( \chi_3 \) is an abelian character for both \( [\Gamma_0[3]] \) and \( [\Gamma_0[9]] \). Given \( f \in [\Gamma_0[9], k, \chi_3]_0 \), we define
\[
\text{tr}_{\Gamma_0[3]}^{\Gamma_0[9]}(f) := \sum_{M : \Gamma_0[9]|\Gamma_0[3]} \chi_3(M) \cdot f|kM,
\]
where \( M : \Gamma_0[9]|\Gamma_0[3] \) means that we sum over a transversal of the cosets \( \Gamma_0[9]|\Gamma_0[3] \). Note that we already determined one such transversal in the proof of (3.25): \( I, (\frac{1}{3} \frac{1}{3}), (\frac{1}{9} \frac{1}{9}). \)

The operator \( \text{tr}_{\Gamma_0[3]}^{\Gamma_0[9]} \) is well defined, i.e. it is independent on the special choice of a transversal. Let \( \{M_1, M_2, M_3\} \) and \( \{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3\} \) be transversals of \( \Gamma_0[9]|\Gamma_0[3] \) and let \( N_1, N_2, N_3 \in \Gamma_0[9] \) such that \( M_j = N_j\tilde{M}_j, j = 1, 2, 3 \). Bearing in mind that \( f \in [\Gamma_0[9], k, \chi_3]_0 \) and that \( \chi_3^2 \) is the trivial (or principal) character, we obtain
\[
\begin{align*}
\chi_3(M_1) \cdot f|kM_1 + \chi_3(M_2) \cdot f|kM_2 + \chi_3(M_3) \cdot f|kM_3 \\
= \chi_3(N_1\tilde{M}_1) \cdot f|k(N_1\tilde{M}_1) + \chi_3(N_2\tilde{M}_2) \cdot f|k(N_2\tilde{M}_2) + \chi_3(N_3\tilde{M}_3) \cdot f|k(N_3\tilde{M}_3) \\
= \chi_3(\tilde{M}_1)\chi_3^2(N_1) \cdot f|k\tilde{M}_1 + \chi_3(\tilde{M}_2)\chi_3^2(N_2) \cdot f|k\tilde{M}_2 + \chi_3(\tilde{M}_3)\chi_3^2(N_3) \cdot f|k\tilde{M}_3 \\
= \chi_3(\tilde{M}_1) \cdot f|k\tilde{M}_1 + \chi_3(\tilde{M}_2) \cdot f|k\tilde{M}_2 + \chi_3(\tilde{M}_3) \cdot f|k\tilde{M}_3.
\end{align*}
\]
This proves the well-definedness. And of course, \( \text{tr}_{\Gamma_0[9]}^{\Gamma_0[3]}(f) \in [\Gamma_0[3], k, \chi_3]_0 \) holds: Suppose \( N \in \Gamma_0[3] \). Obviously, if \( M_1, M_2, M_3 \) is a transversal of \( \Gamma_0[9]|\Gamma_0[3] \), then so is \( M_1N, M_2N, M_3N \).
Bearing in mind again that \( \chi_3^2 \) is the trivial character, we compute

\[
tr_{\Gamma_0[3]}^{\Gamma_0[9]}(f) \mid_k N = \sum_{M: \Gamma_0[9] \nmid \Gamma_0[3]} \chi_3(M) \cdot f \mid_k (MN)
\]

\[
= \chi_3(N) \sum_{M: \Gamma_0[9] \nmid \Gamma_0[3]} \chi_3(MN) \cdot f \mid_k (MN) = \chi_3(N) \cdot tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f).
\]

And of course, \( tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f) \) is again a cusp form since it is a sum of cusp forms by definition. The operator \( tr_{\Gamma_0[9]}^{\Gamma_0[3]} \) now turns out to be the desired indicator for newforms:

**Proposition.** Let \( k \in \mathbb{N}, k \) odd, \( \chi_3 \) the non-trivial Dirichlet character mod 3 and \( f \in [\Gamma_0[9], k, \chi_3]_0 \). Then we have the equivalence

\[
f \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \iff (tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f) \equiv 0 \land tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f \mid k \omega_3) \equiv 0).
\]

**Proof:** First, let \( f \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \) be a primitive newform. A simple calculation shows

\[
\omega_3 \cdot \left( \begin{array}{c} 1 \\ 3n \\ 1 \end{array} \right) = \frac{(-3n-1)}{9} = \left( \begin{array}{c} 1-n \\ 0 \\ 3 \end{array} \right) \cdot \omega_3
\]

for all \( n \in \mathbb{N} \). In view of (3.29), there exists \( c \in \mathbb{C} \) such that \( f = c \cdot f_p \mid k \omega_3 \), and \( f_p \) is again a primitive newform. By means of the identities above, (3.23) and (3.29) we compute

\[
tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f) = tr_{\Gamma_0[9]}^{\Gamma_0[3]}(c \cdot f_p \mid k \omega_3) = c \cdot \left( f_p \mid k (\omega_9 I) + f_p \mid (\omega_9 \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right)) + f_p \mid_k (\omega_9 \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)) \right)
\]

\[
= c \cdot \left( f_p \mid (\omega_9 I) + f_p \mid_k (\left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) (\left( \begin{array}{c} 1 \\ 3 \\ 0 \end{array} \right)) + f_p \mid_k (\left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) (\left( \begin{array}{c} 1 \\ 3 \\ 0 \end{array} \right))) \right) \mid_k \omega_3
\]

\[
= c \cdot \left( f_p \mid (\omega_9 I) + f_p \mid_k (\left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) (\left( \begin{array}{c} 1 \\ 3 \\ 0 \end{array} \right)) \right) \mid_k \omega_3
\]

\[
= 3^{-k/2+1} c \cdot (f_p \mid T(3)) \mid_k \omega_3 \equiv 0,
\]

since \( f_p \mid T(3) \equiv 0 \). And because \( f \mid k \omega_3 = -c \cdot f_p \), where \( f_p \) is again a primitive newform,

\[
tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f \mid k \omega_3) \equiv 0
\]

obviously holds true, too. According to (3.28), \( [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \) possesses a basis consisting of primitive newforms. The linearity of the operators \( tr_{\Gamma_0[9]}^{\Gamma_0[3]} \) and \( \mid_k \omega_3 \) then yields

\[
tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f) = tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f \mid k \omega_3) \equiv 0
\]

for all \( f \in [\Gamma_0[9], k, \chi_3]_0^{\text{new}} \). On the other hand, suppose \( f \in [\Gamma_0[9], k, \chi_3]_0 \) with \( tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f) = tr_{\Gamma_0[9]}^{\Gamma_0[3]}(f \mid k \omega_3) \equiv 0 \). According to (3.26) we have

\[
[\Gamma_0[9], k, \chi_3]_0 = [\Gamma_0[3], k, \chi_3]_0 \oplus [\Gamma_0[3], k, \chi_3]_0 | k \Delta_3 \oplus [\Gamma_0[9], k, \chi_3]_0^{\text{new}}.
\]
Hence there exist $f_1, f_2 \in [\Gamma_0[3], k, \chi_3]_0$ and $f_3 \in [\Gamma_0[9], k, \chi_3]_{\text{new}}$ such that
\[ f = f_1 + f_2 |\delta_3| f_3. \]

By what we have seen above, $\text{tr}_{\Gamma_0[3]}^{\Gamma_0[9]}(f_3) = \text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}(f_3|k\omega_9) = 0$. The linearity thus yields
\[ \text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}((f_1 + f_2 |\delta_3)|k\omega_9) = 0. \]

Using the identity from the beginning of the proof we get
\[
\begin{align*}
\text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}((f_1 + f_2 |\delta_3)|k\omega_9) &= (f_1 + f_2 |\delta_3)|k(\omega_9 T) + (f_1 + f_2 |\delta_3)|k(\omega_9(\frac{1}{0} \frac{0}{1})) + (f_1 + f_2 |\delta_3)|k(\omega_9(\frac{1}{0} \frac{0}{1})) \\
&= ((f_1 + f_2 |\delta_3)|k(\frac{1}{0} \frac{0}{1}) + (f_1 + f_2 |\delta_3)|k((\frac{1}{0} \frac{2}{3})) + (f_1 + f_2 |\delta_3)|k((\frac{1}{0} \frac{2}{3}))|k\omega_3 \\
&= (f_1 + f_2 |\delta_3)|k(\frac{3}{0} \frac{2}{3}) + (f_1 + f_2 |\delta_3)|k(\frac{3}{0} \frac{2}{3}) + (f_1 + f_2 |\delta_3)|k(\frac{3}{0} \frac{2}{3})|k\omega_3 \\
&= 3^{-k/2 + 1}. (f_1 |kT(3) + (f_2 |\delta_3)|kT(3)|k\omega_3 = 0
\end{align*}
\]
like above. Of course, this implies $f_1 |kT(3) + (f_2 |\delta_3)|kT(3)|k\omega_3 = 0$. Furthermore, we have
\[
(f_2 |k\delta_3)|kT(3) = 3^{k/2 - 1} \cdot \left( f_2 |k(\delta_3(\frac{0}{0} \frac{2}{3})) + f_2 |k(\delta_3(\frac{0}{0} \frac{2}{3})) \right) \\
= 3^{k/2 - 1} \cdot \left( f_2 |k(\frac{3}{0} \frac{3}{3}) + f_2 |k(\frac{3}{0} \frac{3}{3}) + f_2 |k(\frac{3}{0} \frac{3}{3}) \right) \\
= 3^{k/2 - 1} \cdot \left( f_2 |k(\frac{1}{0} \frac{3}{3}) + f_2 |k(\frac{1}{0} \frac{3}{3}) + f_2 |k(\frac{1}{0} \frac{3}{3}) \right) \\
= 3^{k/2} f_2.
\]

Thus
\[ f_1 |kT(3) = -3^{k/2} f_2 \]
has to hold. But then, by assumption,
\[ \text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}(f_1 + (-3^{k/2} f_1 |kT(3)|k\delta_3)) = 0 \]
must be fulfilled. Noting that $\text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}(f_1) = 3 f_1$ and applying $\omega_3$ yields that this is equivalent to
\[ 3 f_1 |k\omega_3 = \left( \text{tr}_{\Gamma_0[9]}^{\Gamma_0[9]}((3^{k/2} f_1 |kT(3)|k\delta_3)) \right) |k\omega_3. \]

(Note that this is only done to simplify the coming calculations.) Since a simple calculation shows
\[
\delta_3(\frac{1}{0} \frac{0}{1}) \omega_3 = (\frac{0}{3} \frac{-3}{-n}),
\]
we compute
\[
\left( \text{tr}_{\Gamma_0[9]}^\Gamma \left( (3^{-k/2} f_1 | k T(3)) | k \delta_3 \right) \right) | k \omega_3
= (3^{-k/2} f_1 | k T(3)) | k \left( \frac{0}{3} - \frac{3}{0} \right) + (3^{-k/2} f_1 | k T(3)) | k \left( \frac{0}{3} - \frac{3}{0} \right) + (3^{-k/2} f_1 | k T(3)) | k \left( \frac{0}{3} - \frac{3}{0} \right) .
\]
Furthermore, we have
\[
3^{-k/2} f_1 | k T(3) = \frac{1}{3} \left( f_1 | k \left( \frac{1}{0} \frac{0}{3} \right) + f_1 | k \left( \frac{1}{0} \frac{3}{0} \right) + f_1 | k \left( \frac{1}{0} \frac{3}{0} \right) \right) .
\]

We will now have a closer look at each summand:

\[
(3^{-k/2} f_1 | k T(3)) | k \left( \frac{0}{3} - \frac{3}{0} \right) = \frac{1}{3} \left( f_1 | k \left( \frac{0}{0} \frac{0}{3} \right) + f_1 | k \left( \frac{0}{0} \frac{3}{0} \right) + f_1 | k \left( \frac{0}{0} \frac{3}{0} \right) \right) | k \left( \frac{0}{1} - \frac{1}{1} \right)
= \frac{1}{3} \left( f_1 | k \left( \frac{0}{1} - \frac{1}{0} \right) + f_1 | k \left( \frac{1}{3} - \frac{3}{1} \right) + f_1 | k \left( \frac{2}{3} - \frac{3}{1} \right) \right)
= \frac{1}{3} \left( f_1 | k \left( \frac{0}{1} - \frac{1}{0} \right) + f_1 | k \left( \frac{2}{3} - \frac{3}{1} \right) - f_1 | k \left( \frac{1}{3} - \frac{3}{1} \right) \right) .
\]

\[
(3^{-k/2} f_1 | k T(3)) | k \left( \frac{0}{3} - \frac{3}{0} \right) = \frac{1}{3} \left( f_1 | k \left( \frac{0}{0} \frac{0}{3} \right) + f_1 | k \left( \frac{0}{0} \frac{3}{0} \right) + f_1 | k \left( \frac{0}{0} \frac{3}{0} \right) \right) | k \left( \frac{0}{1} - \frac{2}{1} \right)
= \frac{1}{3} \left( f_1 | k \left( \frac{0}{1} - \frac{1}{0} \right) + f_1 | k \left( \frac{1}{3} - \frac{3}{1} \right) + f_1 | k \left( \frac{2}{3} - \frac{3}{1} \right) \right)
= \frac{1}{3} \left( f_1 | k \left( \frac{0}{1} - \frac{1}{0} \right) + f_1 | k \left( \frac{2}{3} - \frac{3}{1} \right) - f_1 | k \left( \frac{1}{3} - \frac{3}{1} \right) \right) .
\]

And thus we finally obtain
\[
3 f_1 | k \omega_3 = \left( \text{tr}_{\Gamma_0[9]}^\Gamma \left( (3^{-k/2} f_1 | k T(3)) | k \delta_3 \right) \right) | k \omega_3 = f_1 | k \omega_3 ,
\]
which is equivalent to \( f_1 \equiv 0 \), and \( f_1 | k T(3) = -3^{k/2} f_2 \) implies \( f_2 \equiv 0 \). This means \( f = f_3 \in [\Gamma_0[9], k \chi_3]_{10}^{\text{new}} \) and we have shown the assertion. \( \square \)

We still had to verify that \( g | k \omega_9 \in [\Gamma_0[9], k \chi_3]^{\text{new}, (2)}_0 \) holds for all \( g \in \mathcal{M}_k^{3+} \) to prove the surjectivity of \( \phi_M \). Considering the preceding proposition, we have the appropriate tools to do so, now.
(3.33) Proposition. Let \( k \in \mathbb{N} \), \( k \) odd, \( \chi_3 \) the non-trivial Dirichlet character mod 3 and \( g \in M_k^{3+} \). Then
\[
g|_{k \omega_9} \in [\Gamma_0[9], k, \chi_3]_{0}^{\text{new,}(2)}
\]
holds.

Proof: First, note that in virtue of (3.24) we already have \( g \in [\Gamma_0[9], k, \chi_3]_{0} \), and thus
\[
g|_{k \omega_9} \in [\Gamma_0[9], k, \chi_3]_{0}
\]
according to 3.33, since \( \chi_3 = \chi_3 \). So let us consider the Fourier-expansion of \( g|_{k \omega_9} \). In view of (3.24), \( g|_{k \omega_9} \), \( T(3) \equiv 0 \) holds true. Regarding the transformation behavior of \( g \) (see (3.24) and define \( \rho = e^{\frac{3}{2} \pi i} \)) and making use of (3.23) we compute
\[
0 \equiv (3^{-k/2+1}g|_{k \omega_9} T(3))|_{k \omega_3} = \left( g|_{k \omega_9} \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) + g|_{k \omega_9} \left( \begin{smallmatrix} 1 & 0 \\ 3 & 0 \end{smallmatrix} \right) + g|_{k \omega_9} \left( \begin{smallmatrix} 0 & 0 \\ 3 & 0 \end{smallmatrix} \right) \right)|_{k \omega_3} = g|_{k \omega_9} \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) + g|_{k \omega_9} \left( \begin{smallmatrix} 3 & -1 \\ 0 & 1 \end{smallmatrix} \right) + g|_{k \omega_9} \left( \begin{smallmatrix} 6 & 0 \\ 0 & 0 \end{smallmatrix} \right) = g|_{k \omega_9} \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) + \rho \cdot g|_{k \omega_9} \left( \begin{smallmatrix} 1 & 0 \\ -3 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 3 & -1 \\ 0 & 0 \end{smallmatrix} \right) - \rho^2 \cdot g|_{k \omega_9} \left( \begin{smallmatrix} -1 & 1 \\ 2 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 3 & 1 \\ 0 & 0 \end{smallmatrix} \right).
\]
Thus we obtain
\[
g|_{k \omega_9} = -\rho \cdot g|_{k \omega_9} \left( \begin{smallmatrix} 3 & -1 \\ 0 & 0 \end{smallmatrix} \right) + \rho^2 \cdot g|_{k \omega_9} \left( \begin{smallmatrix} 3 & 1 \\ 0 & 0 \end{smallmatrix} \right).
\]
Suppose the Fourier-expansion of \( g \) is given by
\[
g(\tau) = \sum_{n \in \mathbb{N}_0} a(3n+1) e^{2\pi i (3n+1)\tau} + \sum_{n \in \mathbb{N}_0} a(3n+2) e^{2\pi i (3n+2)\tau}.
\]
We compute
\[
g|_{k \omega_9}(\tau) = -\rho \cdot g(\tau - \frac{1}{3}) + \rho^2 \cdot g(\tau + \frac{1}{3})
\]
\[
= \sum_{n \in \mathbb{N}_0} \left(-\rho \cdot a(3n+1) e^{-2\pi i (3n+1)\tau} + \rho^2 \cdot a(3n+1) e^{2\pi i (3n+1)\tau}\right) e^{2\pi i (3n+1)\tau}
\]
\[
+ \sum_{n \in \mathbb{N}_0} \left(-\rho \cdot a(3n+2) e^{-2\pi i (3n+2)\tau} + \rho^2 \cdot a(3n+2) e^{2\pi i (3n+2)\tau}\right) e^{2\pi i (3n+2)\tau}
\]
\[
= \sum_{n \in \mathbb{N}_0} \left( -\rho \cdot \rho^2 + \rho^2 \cdot \rho \right) a(3n+1) e^{2\pi i (3n+1)\tau}
\]
\[
+ \sum_{n \in \mathbb{N}_0} \left( -\rho \cdot \rho + \rho^2 \cdot \rho^2 \right) a(3n+2) e^{2\pi i (3n+2)\tau}
\]
\[
= \sum_{n \in \mathbb{N}_0} i\sqrt{3} a(3n+2) e^{2\pi i (3n+2)\tau}.
\]
Hence \( g|_{k \omega_9} \) possesses the desired Fourier-expansion needed for \( [\Gamma_0[9], k, \chi_3]_{0}^{\text{new,}(2)} \). So the only thing that is left to prove is that \( g|_{k \omega_9} \) actually is a newform, i.e. \( g|_{k \omega_9} [\Gamma_0[9], k, \chi_3]_{0}^{\text{new}} \). So in
view of (3.32) we have to verify

\[ \text{tr}_{\Gamma_0[9]_{\omega_9}}^{G | k \omega_9} \equiv 0 \quad \text{and} \quad \text{tr}_{\Gamma_0[9]}^{G | k \omega_9}(g | k \omega_9) = - \text{tr}_{\Gamma_0[9]}^{G | k \omega_9}(g) \equiv 0 . \]

We obtain the second identity quite easily by considering the transformation behavior of \( g \):

\[ \text{tr}_{\Gamma_0[9]}^{G | k \omega_9}(g) = g | k (\frac{1}{0} \frac{0}{1}) + g | k (\frac{1}{3} \frac{1}{1}) + g | k (\frac{1}{6} \frac{1}{1}) = (1 + \rho + \rho^2) \cdot g = 0 \cdot g \equiv 0 . \]

And we saw before that

\[ \omega_9 \cdot (\frac{1}{3n} \frac{0}{1}) = (\frac{-3n}{9} \frac{-1}{0}) = (\frac{1}{0} \frac{-n}{3}) \cdot \omega_3 \]

holds for all \( n \in \mathbb{N} \). Thus we compute

\[
\text{tr}_{\Gamma_0[9]}^{G | k \omega_9}(g | k \omega_9) = g | k (\omega_9 (\frac{1}{0} \frac{0}{1})) + g | k (\omega_9 (\frac{1}{3} \frac{1}{1})) + g | k (\omega_9 (\frac{1}{6} \frac{1}{1})) \\
= (g | k (\frac{1}{0} \frac{0}{3}) + g | k (\frac{1}{0} \frac{-1}{3}) + g | k (\frac{1}{0} \frac{-2}{3})) | k \omega_3 \\
= 3^{\frac{k}{2} + 1} (g | k T(3)) | k \omega_3 \equiv 0 ,
\]

and this completes the proof. \( \square \)

As we already stated before, combining (3.31) and (3.33) immediately leads to

**Theorem.** Let \( k \in \mathbb{N} \), \( k \) odd and \( \chi_3 \) the non-trivial Dirichlet character mod 3. Then \( M_{\chi_3}^{3+} \) and \( [\Gamma_0[9], k, \chi_3]_{0}^{\text{new,(2)}} \) are isomorphic via the isomorphism

\[ \varphi_{\mathcal{M}} : [\Gamma_0[9], k, \chi_3]_{0}^{\text{new,(2)}} \rightarrow M_{\chi_3}^{3+} , \ f \mapsto f | k \omega_9 \]

(with inverse \( g \in M_{\chi_3}^{3+} \mapsto -g | k \omega_9 \)).

Finally, all that is left to do is determining the dimension of \( [\Gamma_0[9], k, \chi_3]_{0}^{\text{new,(2)}} \). According to (3.26) and (3.30) we have

\[ \dim([\Gamma_0[9], k, \chi_3]_{0}^{\text{new,(2)}}) = \frac{1}{2} \dim([\Gamma_0[9], k, \chi_3]_{0}) - \dim([\Gamma_0[3], k, \chi_3]_{0}) . \]

Fortunately, [Mi89, thm.2.5.3] provides an explicit formula to determine the dimensions on the right-hand side. As a result we get the following theorem:

**Theorem.** Let \( k \in \mathbb{N} \), \( k \) odd. Then the dimensions of the spaces of quaternionic Maaß forms of odd weight \( k \) for \( v_i \) and \( v_{-i} \) are given by

\[ \dim(\mathcal{M}(k, v_i ; \mathcal{O})) = \dim(\mathcal{M}(k, v_{-i} ; \mathcal{O})) = \left\lfloor \frac{k}{6} \right\rfloor . \]

In particular, there exist non-trivial Maaß lifts of odd weight \( k \) for all odd \( k \geq 7 \).
Proof: In virtue of (3.19), (3.21), (3.34) and the consideration above we get
\[
\dim(M(k + 2, v_i; O)) = \dim(M(k + 2, v_{-i}; O)) = \dim(M_k^{3+i}) = \dim([\Gamma_0[9], k, \chi_3]_0^{\text{new}(2)})
\]
\[
= \frac{1}{2} \dim([\Gamma_0[9], k, \chi_3]_0) - \dim([\Gamma_0[3], k, \chi_3]_0),
\]
where $\chi_3$ is the non-trivial Dirichlet character mod 3. So we have to determine $\dim([\Gamma_0[9], k, \chi_3]_0)$ and $\dim([\Gamma_0[3], k, \chi_3]_0)$. We will use the explicit formula in [Mi89, thm.2.5.3].

Note that we will not present the whole theory concerning the genus, inequivalent elliptic points and inequivalent cusps of congruence subgroups. It can all be found in [Mi89]. But we simply do not need it here. All involved parameters can be determined explicitly using theorems from [Mi89]. So we will just make use of these explicit values without presenting the theoretical background.

Define
\[
\Gamma^+_0[3] := \{ M = (a\ b\ c\ d) \in \text{SL}_2(\mathbb{Z}) \ ; \ c \equiv 0, \ d \equiv 1 \mod 3 \}
\]
and
\[
\Gamma^+_0[9] := \{ M = (a\ b\ c\ d) \in \text{SL}_2(\mathbb{Z}) \ ; \ c \equiv 0 \mod 9, \ d \equiv 1 \mod 3 \}.\]

Then $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$ obviously are subgroups of $\Gamma_0[3]$ and $\Gamma_0[9]$, resp., both of index 2, since $\Gamma_0[N] = (\Gamma^+_0[N]) \cup (-I \cdot \Gamma^+_0[N])$ for $N \in \{3, 9\}$. And of course we have
\[
[\Gamma_0[3], k, \chi_3]_0 = [\Gamma^+_0[3], k, 1]_0, \quad [\Gamma_0[9], k, \chi_3]_0 = [\Gamma^+_0[9], k, 1]_0,
\]
since the requirement $g |_k (-1) = -g$ is not a proper transformation law for $g$, but simply a consequence of $k$ being odd. Note that $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$ are still congruence subgroups of level 3 and 9, respectively. So the preconditions of theorem 2.5.3 in [Mi89] are met. Again, confer [Mi89] for details about the terms used in the following consequence of this theorem:

Let $g_3$ and $g_9$ be the genera of $\Gamma^+_0[3] \setminus \mathcal{H}^*_{\Gamma^+_0[3]}$ and $\Gamma^+_0[9] \setminus \mathcal{H}^*_{\Gamma^+_0[9]}$ (where $\mathcal{H}^*_{\Gamma^+_0[N]}$ is the union of $\mathcal{H}$ with the set of all cusps of $\Gamma^+_0[N]$, i.e. $\mathcal{H}^*_{\Gamma^+_0[N]} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ – cf. [Mi89, p.25, p.107]), $e_{3,1}, \ldots, e_{3,r_3}$ and $e_{9,1}, \ldots, e_{9,r_9}$ the orders of inequivalent elliptic points of $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$, $u_3$ and $u_9$ the numbers of inequivalent regular cusps of $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$, and $v_3$ and $v_9$ the numbers of inequivalent irregular cusps of $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$ (see [Mi89, pp.18] for a definition of all these terms). Then
\[
\dim([\Gamma_0[N], k, \chi_3]_0) = (k - 1)(g_N - 1) + \sum_{j=1}^{r_N} \left[ \frac{1}{2}(1 - \frac{1}{N_j}) \right] + \frac{k-2}{2} u_N + \frac{k-1}{2} v_N
\]
holds for $N \in \{3, 9\}$, if $k \geq 3$. $[x]$ denotes the largest integer $\leq x$, as usual. Note that the special case $\dim(M(3, v_i; O)) = 0$ follows in virtue of (1.67), since all elements of $M(3, v_i; O)$ are cusp forms, and therefore $a_j(T) = 0$ for all $T \in \text{Hom}_2(O)$ with $\text{tr}(T) \leq 1$, whereas $3\sqrt{3}/\pi < 2$. So it suffices that the dimension formula from above only holds for $k \geq 3$ (and thus $k + 2 \geq 5$).

So let us determine the needed parameters. According to [Mi89, le.4.2.6], the orders of all elliptic points of a congruence subgroup are of order 2 or 3. Furthermore, all cusps of $\Gamma^+_0[3]$ and $\Gamma^+_0[9]$ are regular, which can be verified as follows. Again, note that we will not give details on the definitions here and simply use the propositions from [Mi89]. Let $\text{SL}_2(\mathbb{Z})_{\infty} := \{(\delta \quad g\ 0\ \bar{g}) \ ; \ \delta \in \{\pm 1\}, \ n \in \mathbb{Z}\}$, $\text{SL}_2(\mathbb{Z})_{\infty} := \{((1\ 0\ 1) \ ; \ n \in \mathbb{Z}\}$ and $M = (a\ b\ c\ d) \in \text{SL}_2(\mathbb{Z})$. Of
course, $\Gamma_0^+[N] \cdot \text{SL}_2(\mathbb{Z}) = \Gamma_0^+[N](-M) \cdot \text{SL}_2(\mathbb{Z})$ holds ($N \in \{3, 9\}$). Now, assume that also $\Gamma_0^+[N] \cdot \text{SL}_2(\mathbb{Z}) = \Gamma_0^+[N](-M) \cdot \text{SL}_2(\mathbb{Z})$. Then there exist $\tilde{M} \in \Gamma_0^+[N]$ and $n \in \mathbb{Z}$ such that $\tilde{M} \cdot (1, 0) = -M$ holds. An easy calculation shows that this is equivalent to

$$\tilde{M} = \left( \frac{-1 + nac}{n^2}, \frac{-na^2}{n^2} \right).$$

$\tilde{M} \in \Gamma_0^+[N]$ immediately implies $n \equiv 0$ or $c \equiv 0 \mod 3$. But then we get $-1 - nac \equiv -1 \mod 3$, which contradicts the assumption $\tilde{M} \in \Gamma_0^+[N]$. Thus we have $\Gamma_0^+[N] \cdot \text{SL}_2(\mathbb{Z}) \neq \Gamma_0^+[N](-M) \cdot \text{SL}_2(\mathbb{Z})$. Therefore, according to [Mi89, p.108], all cusps of $\Gamma_0^+[N]$ are regular ($N \in \{3, 9\}$).

So let $v_2(\Gamma_0^+[N])$ and $v_3(\Gamma_0^+[N])$ denote the number of inequivalent elliptic points of $\Gamma_0^+[N]$ of order 2 and 3, while $v_\infty(\Gamma_0^+[N])$ shall denote the number of inequivalent cusps of $\Gamma_0^+[N]$ (for $N \in \{3, 9\}$). (Note that, according to the considerations above, we have $u_N = v_\infty(\Gamma_0^+[N])$ and $v_N = 0$ for $N \in \{3, 9\}$.) Then the dimension formula above can be expressed as

$$\dim([\Gamma_0[N], k, \chi_3]) = (k - 1) \cdot (g_N - 1) + v_2(\Gamma_0^+[N]) \left( \frac{k}{4} \right) + v_3(\Gamma_0^+[N]) \left( \frac{k}{3} \right) + \frac{k^2}{3} - v_\infty(\Gamma_0^+[N])$$

for $N \in \{3, 9\}$. So let us determine the needed parameters. Note that the images of $(\Gamma_0^+[3])$ and $(\Gamma_0^+[9])$ in the group of automorphisms of $\mathcal{H}$ (as modular transformations) coincide with those of $(\Gamma_0[3])$ and $(\Gamma_0[9])$, resp., since $-I(\gamma) = \gamma$ for all $\gamma \in \mathbb{C}$. Therefore we have

$$v_2(\Gamma_0^+[N]) = v_2(\Gamma_0[N]) \quad v_3(\Gamma_0^+[N]) = v_3(\Gamma_0[N]) \quad v_\infty(\Gamma_0^+[N]) = v_\infty(\Gamma_0[N])$$

for $N \in \{3, 9\}$ by definition (see [Mi89, pp.18] for details). Hence we can apply [Mi89, thm.4.2.7] and obtain

$$v_2(\Gamma_0^+[3]) = v_2(\Gamma_0^+[9]) = 1 + \left( \frac{-1}{3} \right) = 0,$$

$$v_3(\Gamma_0^+[3]) = 1 + \left( \frac{-2}{3} \right) = 1,$$

$$v_3(\Gamma_0^+[9]) = 0,$$

$$v_\infty(\Gamma_0^+[3]) = \sum_{d \in \mathbb{N}, d \mid 3} \phi(\gcd(d, \frac{3}{3})) = 2,$$

$$v_\infty(\Gamma_0^+[9]) = \sum_{d \in \mathbb{N}, d \mid 9} \phi(\gcd(d, \frac{9}{3})) = 4,$$

where $\left( \frac{n}{p} \right)$ is the Legendre symbol and $\phi$ is Euler’s phi-function. Furthermore, according to [Mi89, thm.4.2.11], the genera $g_N$ are given by

$$g_N = 1 + \frac{[\text{SL}_2(\mathbb{Z}) : \Gamma_0^+[N]]}{12} - \frac{v_2(\Gamma_0^+[N])}{4} - \frac{v_3(\Gamma_0^+[N])}{3} - \frac{v_\infty(\Gamma_0^+[N])}{2}$$

for $N \in \{3, 9\}$, where $\Gamma$ denotes the image of $\Gamma \leq \text{SL}_2(\mathbb{Z})$ in the group of automorphisms of $\mathcal{H}$ as symplectic transformations. This obviously means $[\text{SL}_2(\mathbb{Z}) : \Gamma_0^+[N]] = [\text{SL}_2(\mathbb{Z}) : \Gamma_0[N]]$. We have $[\text{SL}_2(\mathbb{Z}) : \Gamma_0[3]] = 4$ in view of [Kl98, Le.1.24] and $[\Gamma_0[3] : \Gamma_0[9]] = 3$ by means of the proof.
of (3.25), which implies $|\text{SL}_2(\mathbb{Z}) : \Gamma_0[9]| = 12$. Hence we obtain

$$g_3 = 1 + \frac{1}{3} - 0 - \frac{1}{3} - 1 = 0 \quad \text{and} \quad g_9 = 1 + 1 - 0 - 0 - 2 = 0.$$  

And thus we finally get

$$\dim([\Gamma_0[3], k, \chi_3|_0]) = (k - 1)(0 - 1) + 0 + 1 \cdot \left[\frac{k}{3}\right] + \frac{k^2}{2} \cdot 2 = \left[\frac{k}{3}\right] - 1$$

and

$$\dim([\Gamma_0[9], k, \chi_3|_0]) = (k - 1)(0 - 1) + 0 + 0 + \frac{k^2}{2} \cdot 4 = k - 3.$$  

So

$$\dim(\mathcal{M}(k, \nu; \mathcal{O})) = \frac{1}{2}(k - 2 - 3) - \left(\left[\frac{k - 2}{3}\right] - 1\right) = \frac{k - 3}{2} - \left[\frac{k - 2}{3}\right]$$

follows for $k \geq 5$ odd. So the only thing that remains to be verified is that

$$\frac{k - 3}{2} - \left[\frac{k - 2}{3}\right] = \left[\frac{k}{6}\right]$$

holds for all odd $k \geq 5$. To do so, we simply recalculate the three occurring cases: First, let $k = 6n - 1$, $n \in \mathbb{N}$. We get

$$\frac{k - 3}{2} - \left[\frac{k - 2}{3}\right] = 3n - 2 - [2n - 1] = n - 1 = \left[n - 1 + \frac{5}{6}\right] = \left[\frac{k}{6}\right].$$

For $k = 6n + 1$, $n \in \mathbb{N}$ we compute

$$\frac{k - 3}{2} - \left[\frac{k - 2}{3}\right] = 3n - 1 - \left[2n - \frac{1}{3}\right] = n = \left[n + \frac{1}{6}\right] = \left[\frac{k}{6}\right].$$

And finally, if $k = 6n + 3$, $n \in \mathbb{N}$, then we have

$$\frac{k - 3}{2} - \left[\frac{k - 2}{3}\right] = 3n - 2 - \left[2n + \frac{1}{3}\right] = n = \left[n + \frac{3}{6}\right] = \left[\frac{k}{6}\right].$$

\[\Box\]

There is one thing left that we should briefly discuss: How to explicitly construct Maaß lifts of odd weight, which means in terms of Fourier-expansions. Of course, if the Fourier-expansion of $g \in \mathcal{M}_{\kappa-2}^{3+}$ is given explicitly in terms of the Fourier-coefficients (at least up to a given, but arbitrary bound), then the Fourier-coefficients of the Maaß lift $\mathcal{M}_{\kappa}^{(l)}$ (or $\mathcal{M}_{\kappa}^{(-1)}$) can be calculated explicitly with regard to (3.19) (or (3.21)) – most probably not in terms of a self-contained formula, but in terms of explicit values for the Fourier-coefficients up to certain bounds. So the problem is finding elliptic modular forms $g \in \mathcal{M}_{\kappa-2}^{3+}$ with an explicit Fourier-expansion (maybe also only in terms of explicit values for the Fourier-coefficients up to a chosen bound). And this turns out to be a problem that is quite involved:

In view of (3.34) we have to construct elements in $\mathcal{M}_{\kappa}^{3+}$ by mapping elliptic modular forms in $[\Gamma_0[9], k, \chi_3|^{\text{new},(2)}]$ to $\mathcal{M}_{\kappa}^{3+}$ via $\varphi_\mathcal{M}$ (which means $f \mapsto f|_{\kappa}{}^{\omega_9}$) – or at least we have not found another method so far. Elliptic modular forms in $[\Gamma_0[9], k, \chi_3|^{\text{new},(2)}]$ can be constructed quite explicitly, at least in terms of explicit values for the Fourier-coefficients: [SAGE] provides tools
3.3 The space $\mathcal{M}_k^{3+}$

To calculate the Fourier-coefficients of a basis of $[\Gamma_0[9], k, \chi_3]^{\text{new}}$ consisting of primitive newforms. Mapping these to $f - f_\rho$ yields a generating set of $[\Gamma_0[9], k, \chi_3]^{\text{new,(2)}}$ (see (3.30)) – whereas the Fourier-coefficients of $f$ immediately yield the Fourier-coefficients of $f - f_\rho$. Thus obtaining explicit values of Fourier-coefficients for elements in $[\Gamma_0[9], k, \chi_3]^{\text{new,(2)}}$ is not a problem, at least with the help of a computer algebra system. The problem actually lies in the Atkin-Lehner (or Fricke) involution $\omega_9$. According to (3.29), there exists $c \in \mathbb{C}, |c| = 1$ such that $f|_k \omega_9 = c \cdot f_\rho$ and $(f - f_\rho)|_k \omega_9 = \tau \cdot f + c \cdot f_\rho$. So the actual difficulty lies in determining this constant $c$. And so far, there seems to be no standard method which provides it.

Actually, one can at least determine this constant numerically, up to some desired precision: As said before, it is possible to calculate any number of Fourier-coefficients of some primitive newform $f$ with the help of computer algebra systems – at least as many as the computing power admits. So suppose the Fourier-expansion is given by $\sum_{n \in \mathbb{N}} a(n)e^{2\pi i n \tau}$ (with $a(3n) = 0$, $a(3n + 1) \in \mathbb{R}$, $a(3n + 2) \in i\mathbb{R}$ for all $n \in \mathbb{N}$). We know that there has to exist $c \in \mathbb{C}, |c| = 1$ with

$$(3\tau)^{-k} \sum_{n \in \mathbb{N}} a(n)e^{2\pi i n (-9\tau^{-1})} = f|_k \omega_9(\tau) = c \cdot f_\rho(\tau) = c \sum_{n \in \mathbb{N}} a(n)e^{2\pi i n \tau}$$

for all $\tau \in \mathcal{H}$. And since $f \neq 0$, there exists $y \in \mathbb{R}, y > 0$, such that $f(iy) \neq 0$. This yields

$$(3iy)^{-k} \sum_{n \in \mathbb{N}} a(n)e^{-2\pi n/(9y)} = c \sum_{n \in \mathbb{N}} a(n)e^{-2\pi ny}.$$  

According to [Mi89, cor.2.1.6] we have $a(n) = O(n^{1/2})$. Thus each series converges quite fast. So with the help of some computer algebra system one can approximate $c$ quite well and up to any precision – again up to those precisions that the computing power admits. And moreover, assuming that the transformation behavior under $\omega_9$ is “good-natured” with respect to the algebraic properties of $c$, one can determine the constant even explicitly.

This method was used in the lowest weight case, which means $k = 5$ (which yields weight 7 Maaß lifts). According to the preceding theorem, $[\Gamma_0[9], k, \chi_3]^{\text{new}}$ is two-dimensional and $[\Gamma_0[9], k, \chi_3]^{\text{new,(2)}}$ is one-dimensional. Hence there is a basis $\{f, f_\rho\}$ of $[\Gamma_0[9], k, \chi_3]^{\text{new}}$, where $f$ is the one unique primitive newform (unique up to differing between $f$ and $f_\rho$). As noted before, [SAGE] provides tools to calculate the Fourier-coefficients of $f$. According to that, the Fourier-expansion of $f$ starts with

$$f(\tau) = q + 3i\sqrt{2}q^2 - 2q^4 - 21i\sqrt{2}q^5 - 28q^7 + 42i\sqrt{2}q^8 + 126q^{10} + 12i\sqrt{2}q^{11} + \ldots,$$

where $q = e^{2\pi i \tau}$. And in this case one can determine $c$ (at least in a numerical sense) to be $\frac{\sqrt{2} - i}{\sqrt{3}}$ using the method described above. Normalizing accordingly yields that $\mathcal{M}_k^{3+}$ is generated by a unique modular form $g$, whose Fourier-expansion starts with

$$g(\tau) = q - 3q^2 - 2q^4 + 21q^5 - 28q^7 - 42q^8 + 126q^{10} - 12q^{11} + \ldots,$$

where again $q = e^{2\pi i \tau}$. This in return gives explicit Fourier-coefficients for the unique Maaß lifts of weight 7 for $v_i$ and $v_{-i}$.

To finish this chapter, we explicitly determine some spaces of quaternionic modular forms for
low weights.

(3.36) Proposition. For low weights we have

\[ [\text{Sp}_2(O), 0, 1] = C \]
\[ [\text{Sp}_2(O), 2, 1] = \{0\} \]
\[ [\text{Sp}_2(O), k, 1] = \mathcal{M}(k; O) \text{ for } k = 4, 6 \]

and

\[ [\text{Sp}_2(O), k, \nu_{\text{det}}] = [\text{Sp}_2(O), k, \nu_{\text{det}}]_0 \text{ for all } k \in 2\mathbb{N}_0 \]
\[ [\text{Sp}_2(O), k, \nu_{\text{det}}] = \{0\} \text{ for } k = 0, 2 \]

and as well

\[ [\text{Sp}_2(O), k, \nu_i] = \{0\} \text{ for all odd } k \leq 5 \]
\[ [\text{Sp}_2(O), 7, \nu_i] = \mathcal{M}(7, \nu_i; O) . \]

The same holds for \( \nu_{-i} \).

Proof: The weight 0 cases have already been dealt with in (1.68).

If \( f \) is a quaternionic modular form in \( [\text{Sp}_2(O), k, \nu] \), where \( k \in \mathbb{N} \) and \( \nu \) is some multiplier system for \( \text{Sp}_2(O) \) of weight \( k \) (hence \( \nu = 1 \) or \( \nu = \nu_{\text{det}} \) if \( k \) is even, and \( \nu = \nu_i \) or \( \nu = \nu_{-i} \) if \( k \) is odd), and \( f \) possesses the Fourier-expansion

\[
f(Z) = \sum_{T \in \text{Her}_2(O), T \geq 0} \alpha_f(T) e^{2\pi i \tau(T, Z)}, \quad Z \in \mathcal{H}^{(\mathbb{H})},
\]

then, according to (1.55),

\[
\alpha_f(T[U]) = \nu(\text{Rot}((UT)^{-1})) \cdot \alpha_f(T)
\]

holds for all \( U \in \text{GL}_2(O) \) and all \( T \in \text{Her}_2(O) \) with \( T \geq 0 \). Consider \( U = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \), \( T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) with \( n \in \mathbb{N}_0 \) and \( \nu \in \{ \nu_{\text{det}}, \nu_i, \nu_{-i} \} \). Since \( T[U] = T \) holds, the definitions of the multiplier systems yield

\[
\alpha_f(T) = \delta \cdot \alpha_f(T),
\]

where \( \delta \in \{-1, \pm i\} \), and thus \( \alpha_f(T) = 0 \). And therefore, according to the definition of cusp forms (1.60) and the proof of (1.59) (regarding the Fourier-expansion of \( f|\Phi \)), \( f \) has to be a cusp form. This means that if \( \nu \) is not the trivial multiplier system, then the spaces of quaternionic modular forms with respect to the whole modular group and the multiplier system \( \nu \) only consist of cusp forms.

Next, suppose that \( f \) is a cusp form of weight \( k \leq 3 \). Then in view of (1.62), \( \alpha_f(T) \neq 0 \) for \( T \in \text{Her}_2(O) \) implies \( T > 0 \) and thus \( \text{tr}(T) \geq 2 \), or in other words \( \alpha_f(T) = 0 \) for all \( T \in \text{Her}_2(O) \) with \( \text{tr}(T) \leq \frac{\sqrt{3}}{2} \cdot k < 2 \). But then (1.67) yields \( f \equiv 0 \).

Let us have a closer look at the trivial character \( \nu = 1 \) and \( k \in \{2, 4, 6\} \). Then \( \frac{\sqrt{3}}{2} \cdot k < 4 \) holds. Suppose \( T = \begin{pmatrix} n & 1 \\ m & n \end{pmatrix} \in \text{Her}_2(O) \) with \( \text{tr}(T) < 4 \). Since \( T \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} n & 1 \\ m & n \end{pmatrix} \) and because of the
identity $a_f(T[U]) = a_f(T)$, we only have to consider the cases $T = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$ with $n \leq 3$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $n \leq 2$. Note that the Fourier-coefficients for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{N}_0$ uniquely correspond to the first Fourier-Jacobi-coefficient $q_{f,1}$, and according to (3.4), that uniquely corresponds to $f_0 \in \text{SL}_2(\mathbb{Z})[3\mathbb{Z}][k-2,1]$. In view of (3.7), the first Fourier-Jacobi-coefficients of $M_{f_0}$ and $f$ coincide. This means that if we define $g = f - M_{f_0}$, then $a_g\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 0$ holds for all $n \in \mathbb{N}_0$, and thus also $a_g\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$. Hence, in view of the proof of (1.59) we have $g|\Phi \in \text{SL}_2(\mathbb{Z}), k, 1]$ and

$$g|\Phi(\tau) = a_g\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) + a_g\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right) q^2 + a_g\left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\right) q^3 + \ldots$$

for all $\tau \in \mathcal{H}$, where $q = e^{2\pi i \tau}$. According to [KK07, ch.II], there exists no non-identically vanishing elliptic modular form with such a Fourier-expansion (where the Fourier-coefficient $a(1)$ is zero). Thus $g|\Phi \equiv 0$ follows, which implies $a_g\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$ for all $n \in \mathbb{N}_0$, and we have shown $a_g(T) = 0$ for all $T \in \text{Her}_2(\mathcal{O})$ with $\text{tr}(T) \leq \sqrt{3} \cdot k < 4$. Again, (1.67) yields $g \equiv 0$, or $f = M_{f_0}$.

The same method can be used to verify the assertion for $v = v_i$ or $v = v_{-i}$. Here, if $k \leq 7$, we get $\sqrt{3} \cdot k < 4$, again. The only difference is that $a_g\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$ immediately has to hold since $g$ has to be a cusp form, as we have seen above. $\mathcal{M}(k, v_i; \mathcal{O}) = \{0\}$ for $k < 7$ follows from (3.35). □

Note that $\mathcal{M}(k; \mathcal{O})$ has dimension one for $k = 4, 6$, as well as $\mathcal{M}(7, v_i; \mathcal{O})$ and $\mathcal{M}(7, v_{-i}; \mathcal{O})$ (see (3.5) and (3.35)).

(3.37) Remark. To obtain a similar result for $[\text{Sp}_2(\mathcal{O}), 4, \nu_{\text{det}}]$ and $[\text{Sp}_2(\mathcal{O}), 6, \nu_{\text{det}}]$, one would have to construct Maaß lifts of even weight for the abelian character $\nu_{\text{det}}$. The proceeding would be the same like those for the Maaß lifts of odd weights. The first difference will be

$$f_0 \equiv 0, \quad f_1 = f_2 = -f_3 = -f_6, \quad f_4 = -f_5 = -f_7 = f_8,$$

deriving from the character $\nu_{\text{det}}$. From that initial point one has to investigate $f_1$ and $f_4$ like we did concerning the Maaß lifts of odd weight. Actually, one easily computes (again making use of $\kappa(\cdot)$ – see 3.25)

$$f_1|_{k-2} T_1 = \rho^2 f_1, \quad f_1|_{k-2} h_1 = f_1,$$
$$f_4|_{k-2} T_1 = \rho f_4, \quad f_4|_{k-2} h_1 = f_4,$$

which means that this time we immediately obtain elliptic modular forms with respect to the full modular group $\text{SL}_2(\mathbb{Z})$ and some abelian character, without the need to combine $f_1$ and $f_4$ in some way. Moreover, this also means that $f_1$ and $f_4$ can be chosen independently from each other. Now, recall the definition of the famous Dedekind $\eta$-function:

$$\eta: \mathcal{H} \to \mathbb{C}, \quad \tau \mapsto e^{\frac{1}{12} \pi i \tau} \prod_{n \in \mathbb{N}} \left(1 - e^{2\pi i \tau}\right).$$
It is well known (for example, confer \[KK07, pp.187\]) that \(\eta\) is free of zeros and fulfills
\[
\eta^{16}|_{T_1} = \rho^2 \eta^{16}, \quad \eta^{16}|_{\mathfrak{f}_1} = \eta^{16}, \quad \eta^8|_{T_1} = \rho \eta^8, \quad \eta^8|_{\mathfrak{f}_1} = \eta^8.
\]

Finally, note that the smallest exponent in the Fourier-expansions of \(f_1\) and \(\eta^{16}\) is \(\frac{2}{3}\) in both cases (regarding \(e^{2\pi i n \tau}\) and meaning the smallest \(n\)), while for \(f_4\) and \(\eta^8\) it is \(\frac{1}{3}\). This fact, the transformation behaviors, and \(\eta\) possessing no zeros obviously leads to
\[
\eta^{-16}f_1 \in [\text{SL}_2(\mathbb{Z}), k - 2 - 8, 1], \quad \eta^{-8}f_4 \in [\text{SL}_2(\mathbb{Z}), k - 2 - 4, 1].
\]

So we already know which elliptic modular forms we have to lift to obtain Maaß lifts of even weight and abelian character \(v_{\det}\). Denote the space of these lifts by \(\mathcal{M}(k, v_{\det}; \mathcal{O})\). Then we obviously obtain an isomorphism
\[
[\text{SL}_2(\mathbb{Z}), k - 10, 1] \times [\text{SL}_2(\mathbb{Z}), k - 6, 1] \rightarrow \mathcal{M}(k, v_{\det}; \mathcal{O})
\]
given by the considerations above (and using the same methods like for the Maaß lifts of odd weight to actually lift \((f_1, f_4) = (\eta^{16}g_1, \eta^8g_4)\) to a quaternionic modular form with respect to \(\text{Sp}_2(\mathcal{O})\) and \(v_{\det}\), where \(g_1 \in [\text{SL}_2(\mathbb{Z}), k - 10, 1]\) and \(g_4 \in [\text{SL}_2(\mathbb{Z}), k - 6, 1]\).

So we even obtain \([\text{Sp}_2(\mathcal{O}), k, v_{\det}] = \{0\}\) for \(k \leq 4\) and \([\text{Sp}_2(\mathcal{O}), 6, v_{\det}] = \mathcal{M}(6, v_{\det}; \mathcal{O})\), where \(\dim(\mathcal{M}(6, v_{\det}; \mathcal{O})) = 1\).

Moreover, the same considerations done for the Maaß lifts of odd weights regarding the transformation behavior with respect to \(Z \mapsto Z'\) and \(\text{Rot}(i_1 I)\) yield the following: Firstly, these new Maaß lifts are symmetric. Secondly, there are two possibilities then. If \(f_1 \equiv 0\), then the Maaß lift also possesses a proper transformation behavior with respect to \(\text{Rot}(i_1 I)\), hence it is a quaternionic modular form with respect to the extended modular group and the abelian character \(v_{\det}v_{I_1}\). The same holds for \(f_4 \equiv 0\), but then the abelian character is given by \(v_{\det}\). On the other hand, if \(f_1, f_4 \not\equiv 0\), then the Maaß lift (denoted by \(f\)) possesses no proper transformation behavior concerning \(\text{Rot}(i_1 I)\). Nevertheless, in this case \(f - f|_{k}(\text{Rot}(i_1 I)) \in [\Gamma(\mathcal{O}), k, v_{\det}v_{I_1}]\) follows (and \(f + f|_{k}(\text{Rot}(i_1 I)) \in [\Gamma(\mathcal{O}), k, v_{\det}]\)).

In particular, this implies that the unique Maaß lift \(F_6\) (for \(v_{\det}\)) of weight six satisfies \(F_6 \in [\Gamma(\mathcal{O}), 6, v_{\det}v_{I_1}]\), since \(f_1 \equiv 0\) in this case. Moreover, \(F_6\) therefore has to coincide (up to some constant factor \(c \in \mathbb{C}^*\)) with the Borcherds product \(\psi_{1,6}\) (cf. (6.12), since \(F_6\) has to vanish along \(\mathcal{H}_{\Delta_2}\) due to its abelian character – but more on that in chapter 6).

Of course, note that we did not give any details regarding the issue of Maaß lifts of even weight for \(v_{\det}\). But we also did not give many details concerning \(v_{-i}\). The reason is that the proceeding is completely in all three cases. Once it is clear which elliptic modular forms have to be lifted, then the rest is just a straightforward process. So instead of writing down the procedure three times it suffices to determine which elliptic modular forms are the “right ones” to be lifted. And this we did concerning both \(v_{\det}\) and \(v_{-i}\).
4 Quaternionic Eisenstein-Series

In this chapter we will introduce a third important type of non-identically vanishing quaternionic modular forms. They are called Eisenstein-series. The first section will deal with their definition and first properties. But the main goal will be to determine the explicit Fourier-expansions of the Eisenstein-series. This is separated into several steps and sections. It will turn out that Eisenstein-series are quaternionic Maaß lifts which are eigenforms for certain Hecke-operators. Finally, these properties will yield the explicit Fourier-coefficients – explicit in the sense of an explicit formula analogous to [Kr90], which deals with Eisenstein-series and Maaß lifts over the Hurwitz order.

So in the second section we will analyze the spaces of elliptic modular forms needed as input to the Maaß lift of even weight for the trivial character, since we need more explicit information on their Fourier-expansion. Similar to what we did regarding the Maaß lifts of odd weight, we need to find another and more direct description for \([\text{SL}_2(\mathbb{Z})|3\mathbb{Z}|_0, k, 1]_{tr}\). In the third section we will work out some specific number theoretical background regarding \(O\) which will be needed for the examination of Hecke-operators, including certain divisor theories.

The fourth section is about Hecke-operators for quaternionic modular forms. After a short introduction, we will be mainly interested in the special Hecke-operators \(T_2(p)\). First, we determine this Hecke-operator explicitly in terms of a transversal of the cosets. Having this information, we will investigate how \(T_2(p)\) acts on the space of quaternionic Maaß lifts, which is the most involving part. It will turn out that \(T_2(p)\) acts on Maaß lifts via a transformation of the input function, which itself can be expressed as a Hecke-operator for elliptic modular forms. This will be the key in the final section to develop a formula for the Fourier-coefficients of the quaternionic Eisenstein-series, deriving from the fact that Eisenstein-series are eigenforms for all \(T_2(p)\), while every further quaternionic modular form that is also an eigenform and whose constant term in the Fourier-expansion does not vanish has to coincide with the Eisenstein-series of the same weight up to some constant pre-factor – which will be proved in the final section, too. In the last chapter of this thesis the explicitly given Fourier-expansions of the quaternionic Eisenstein-series will be one of the crucial keys to construct the seven algebraically independent forms already mentioned in the introduction.

4.1 Introduction to quaternionic Eisenstein-series

As announced, this section will deal with the definition and first properties of Eisenstein-series. Note that we will only consider Eisenstein-series of the shape \(E^k_{n,0}(Z, 1)\) for \(n \in \{1, 2\}\) in the sense of [Kr85, ch.V, p.152], since we are mainly interested in the Fourier-expansions.

To define Eisenstein-series of degree 1 and 2, we need a special subgroup of the quaternionic modular group \(\text{Sp}_n(O)\), first.
(4.1) Definition. For \( n \in \mathbb{N} \), define
\[
\text{Sp}_n(\mathcal{O})_0 := \{ M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_n(\mathcal{O}) \, ; \, C = 0 \} .
\]

Obviously, \( \text{Sp}_n(\mathcal{O})_0 \) is a subgroup of \( \text{Sp}_n(\mathcal{O}) \). We state some first facts about this subgroup in the following

(4.2) Lemma.  
\( a) \) For \( n \in \mathbb{N} \) we have
\[
\text{Sp}_n(\mathcal{O})_0 = \{ \text{Rot}(U) \cdot \text{Trans}(S) \, ; \, U \in \text{GL}_n(\mathcal{O}), S \in \text{Her}_n(\mathcal{O}) \}
\]
with \( \text{Trans}(S) = \{ S = \left( \begin{array}{cc} I & a b \\ 0 & I \end{array} \right) \} \), equivalently
\[
\text{Sp}_n(\mathcal{O})_0 = \{ \text{Trans}(S) \cdot \text{Rot}(U) \, ; \, S \in \text{Her}_n(\mathcal{O}), U \in \text{GL}_n(\mathcal{O}) \}
\]

\( b) \) A fundamental domain of \( \mathcal{H} \) and \( \mathcal{H}(\mathbb{H}) \) with respect to \( \text{Sp}_1(\mathcal{O})_0 \) and \( \text{Sp}_2(\mathcal{O})_0 \) is given by
\[
\mathcal{F}_1(\mathcal{O})_0 := \{ \tau = x + iy \in \mathcal{H} \, ; \, x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \}
\]
and
\[
\mathcal{F}_2(\mathcal{O})_0 := \{ Z = X + iY \in \mathcal{H}(\mathbb{H}) \, ; \, X \in \mathcal{C}(\mathcal{O}), Y \in \mathcal{R}(\mathcal{O}) \} ,
\]
respectively.

Proof:  
\( a) \) Given \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_n(\mathcal{O})_0 \), the fundamental relations (1.3) imply \( A' D = I \), or equivalently \( D A = I \), and as well \( D B \in \text{Her}_n(\mathcal{O}) \). We obtain
\[
\text{Rot}(D) M = \left( \begin{array}{cc} \overline{D} A & \overline{D} B \\ 0 & D^{-1} D \end{array} \right) = \left( \begin{array}{cc} I & \overline{D} B \\ 0 & I \end{array} \right) = \text{Trans}(\overline{D} B) ,
\]

hence we showed the first identity. The second one can be verified completely analogously.

\( b) \) Note that we have \( \text{Sp}_1(\mathcal{O})_0 = \mathcal{E} : \text{SL}_2(\mathbb{Z})_0 \), where \( \text{SL}_2(\mathbb{Z})_0 = \text{Sp}_1(\mathcal{O})_0 \cap \text{SL}_2(\mathbb{Z}) \). Given \( M = \varepsilon M_0 \) with \( \varepsilon \in \mathcal{E} \) and \( M_0 = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})_0 \) we have
\[
M(\tau) = \tau + b .
\]

Thus the image of \( \text{Sp}_1(\mathcal{O})_0 \) in the group of automorphisms of \( \mathcal{H} \) consists of all shifts \( \tau \mapsto \tau + b \) with any \( b \in \mathbb{Z} \). So the assertion of \( \mathcal{F}_1(\mathcal{O})_0 \) being a fundamental domain is quite obvious in view of definition (1.23).

So let us consider the case of degree two. (F.1) in (1.23) is obvious. Let \( Z = X + iY \in \mathcal{H}(\mathbb{H}) \). According to (1.30), \( \mathcal{R}(\mathcal{O}) \) is a fundamental domain of \( \text{Pos}_2(\mathbb{H}) \) with respect to the action \( S \mapsto S[U], U \in \text{GL}_2(\mathcal{O}) \). This means there exists \( U \in \text{GL}_2(\mathcal{O}) \) with
\[
\text{Rot}(U)[Z] = X[U] + iY[U], \quad Y[U] \in \mathcal{R}(\mathcal{O}) .
\]

And according to the definition of \( \mathcal{C}(\mathcal{O}) \), there obviously exists \( S \in \text{Her}_2(\mathcal{O}) \) such that
\[
(\text{Trans}(S) \text{Rot}(U))[Z] = X[U] + S + iY[U] , \quad X[U] + S \in \mathcal{C}(\mathcal{O})
\]
holds. Hence (F.2) is met. Next, suppose there are \( Z = X + iY \in \mathcal{F}_2(\mathcal{O})_0, S \in \text{Her}_2(\mathcal{O}) \).
and \( U \in \text{GL}_2(\mathcal{O}) \) such that \( Z \) and \((\text{Trans}(S) \text{Rot}(U))(Z) = X[U] + S + iY[U] \) are interior points of \( F_2(\mathcal{O})_0 \). So in particular, \( Y \) and \( Y[U] \) are interior points of \( \mathcal{R}(\mathcal{O}) \). And since \( \mathcal{R}(\mathcal{O}) \) is a fundamental domain of \( \text{Pos}_2(\mathcal{H}) \), \( (F.3) \) (applied to \( \mathcal{R}(\mathcal{O}) \)) implies that \( T \mapsto T[U] \) is the identity on \( \text{Pos}_2(\mathcal{H}) \), which of course leads to \( U = \pm I \) (or cf. (1.28)). Furthermore, both \( X \) and \( X[U] + S = X + S \) lying in the interior of \( \mathcal{C}(\mathcal{O}) \) yields \( S = 0 \) by the definiton of \( \mathcal{C} \). \( Z \mapsto Z[\pm I] = Z \) being the identity yields \((F.3)\). And like in (1.30), analogs of [Kr85, ch.I, prop.4.10] and [Kr85, ch.I, prop.4.11] for our order \( \mathcal{O} \) would yield \((F.4)\) and \((F.5)\). We omit the details here, since it would simply be re-doing the proofs found in [Kr85].  

Next, let us define the central objects of this chapter, namely Eisenstein-series of degree one and two, where the case of degree one is only needed to determine the Fourier-expansions of the Eisenstein-series of degree two. Like before, for a subgroup \( \Gamma \leq \text{Sp}_n(\mathcal{O}) \), “\( M : \Gamma \backslash \text{Sp}_n(\mathcal{O}) \)” means summing over a transversal of the cosets in \( \Gamma \backslash \text{Sp}_n(\mathcal{O}) \).

(4.3) Definition. Let \( k \in 2\mathbb{N} \). For \( k \geq 4 \), the ordinary (elliptic) Eisenstein-series of weight \( k \) is defined as

\[
G_k(\tau) := \sum_{M: \text{Sp}_1(\mathcal{O})_0 \setminus \text{Sp}_1(\mathcal{O})} \det(\tilde{M}\{\tau\})^{-k/2} \quad \tau \in \mathcal{H}.
\]

And for \( k \geq 8 \), the quaternionic Eisenstein-series of degree two and weight \( k \) (with respect to \( \mathcal{O} \)) is given by

\[
E_k(Z) := \sum_{M: \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} \det(\tilde{M}\{\tilde{Z}\})^{-k/2} \quad Z \in \mathcal{H}(\mathcal{H}).
\]

Before we consider the question concerning the convergence of the Eisenstein-series of degree two, let us briefly look at the question of well-definedness and the case of degree one.

(4.4) Remark. a) Without considering convergence, the Eisenstein-series are well-defined in the sense that they are independent of the special choice of a transversal of the cosets \( \text{Sp}_n(\mathcal{O})_0 \setminus \text{Sp}_n(\mathcal{O}) \): Consider \( N = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \text{Sp}_2(\mathcal{O})_0 \). Again, the fundamental relations (1.3) yield \( D \in \text{GL}_2(\mathcal{O}) \). Thus (1.34) leads to \( \det(\tilde{N}\{\tilde{Z}\}) = \det(\tilde{D}) = 1 \). Now, for every \( M \) in a fixed transversal, let \( N_M \in \text{Sp}_2(\mathcal{O})_0 \). Since \( k \) is even, (1.5) yields

\[
\sum_{M: \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} \det((\tilde{N}_M\tilde{M}\{\tilde{Z}\})^{-k/2} = \sum_{M: \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} \det(\tilde{N}_M\{\tilde{M}\{\tilde{Z}\})^{-k/2} \det(\tilde{M}\{\tilde{Z}\})^{-k/2} = \sum_{M: \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} \det(\tilde{M}\{\tilde{Z}\})^{-k/2},
\]

and hence the independence on the special choice of a transversal. Of course, the same considerations can be done for the case of degree one.

b) Note that the degree one yields ordinary elliptic Eisenstein-series, indeed. In view of the proof of (4.2), a transversal of \( \text{SL}_2(\mathcal{O})_0 \setminus \text{SL}_2(\mathcal{O}) \) is also a transversal of \( \text{Sp}_1(\mathcal{O})_0 \setminus \text{Sp}_1(\mathcal{O}) \). For such a transversal, (1.34) yields \( \det(\tilde{M}\{\tau\})^{-k/2} = \det(\tilde{M}\{\tau\})^{-k} = M\{\tau\}^{-k} \) for all.
\( \tau \in \mathcal{H} \) since \( k \) is even. And thus we also have

\[
G_k(\tau) = \sum_{M: SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} M\{\tau\}^{-k} \quad \tau \in \mathcal{H},
\]

and this is exactly the definition of normalized elliptic Eisenstein-series found in [KKK07, ch.III]. And because of that, we already know that the \( G_k \) converge absolutely and locally uniformly on \( \mathcal{H} \) for all even \( k \geq 4 \). And moreover, \( G_k \in [SL_2(\mathbb{Z}), k, 1] \) holds. Further results concerning the \( G_k \) can be found in [KKK07].

c) Assume that the \( E_k \) converge absolutely and locally uniformly for all even \( k \geq 8 \), indeed. Then the definition immediately yields that

\[
E_k \in [Sp_2(\mathcal{O}), k, 1]
\]

has to hold: Suppose \( N \in Sp_2(\mathcal{O}) \). Of course, if \( (M_j)_j \) is a transversal of \( Sp_2(\mathcal{O}) \backslash Sp_2(\mathcal{O}) \), then so is \( (M_j, N)_j \), which is a simple fact from algebra. So since \( k \) is even, (1.5) yields

\[
E_k|N(Z) = (\det(\tilde{N}\{\tilde{Z}\}))^{-k/2} E_k(N(Z)) \\
= \sum_{M:Sp_2(\mathcal{O}) \backslash Sp_2(\mathcal{O})} \det(\tilde{M}\{\tilde{N}\{\tilde{Z}\}\})^{-k/2} \det(\tilde{N}\{\tilde{Z}\}))^{-k/2} \\
= \sum_{M:Sp_2(\mathcal{O}) \backslash Sp_2(\mathcal{O})} \det((\tilde{M}\tilde{N})\{\tilde{Z}\})^{-k/2} = E_k(Z)
\]

for all \( Z \in \mathcal{H}(\mathbb{H}) \). So the convergence is our next aim.

Before we are going to consider the convergence of the quaternionic Eisenstein-series, we need some special volume element of \( \mathcal{H}(\mathbb{H}) \) that is invariant under symplectic transformations.

Of course, the dimension of \( \text{Her}_n(\mathbb{H}) \) as an \( \mathbb{R} \)-vector space is \( h = n + 4^{(n-1)/2} \). For \( x = x_0 + x_1i_1 + x_2i_2 + x_3i_3 \in \mathbb{H} \), let \( x^{(j)} = x_j, j = 0, \ldots, 3 \). And for \( X = (x_{k,l}) \in \text{Her}_n(\mathbb{H}) \), \( x_{k,l} \) shall always denote the \((k,l)\)-th entry, while we abbreviate \( x_j = x_{j,j} \). With that in mind, denote by

\[
dX = \prod_{1 \leq j \leq n} dx_j \prod_{1 \leq k \leq l \leq n} dx_{k,l}^{(j)}
\]

the Euclidean volume element in \( \text{Her}_n(\mathbb{H}) \). The \textit{symplectic volume element} is then defined as

\[
dv := (\det(Y))^{-2h/n} dX dY, \tag{4.1}
\]

where always \( Z = X + iY \). According to [Kr85, ch.II, thm.1.10] we have the following

\textbf{(4.5) Theorem.} \( dv \) is a volume element in \( \mathcal{H}_n(\mathbb{H}) \), which is invariant under symplectic transformations, i.e.

\[
\int_{\mathcal{H}_n(\mathbb{H})} f(Z) dv = \int_{\mathcal{H}_n(\mathbb{H})} f(M(Z)) dv
\]

holds for all \( M \in Sp_n(\mathbb{H}) \) (and integrable \( f : \mathcal{H}_n(\mathbb{H}) \to \mathbb{C} \)).
We will need so-called vertical strips in $\mathcal{H}(\mathbb{H})$. For $\varepsilon > 0$ a vertical strip is defined as

$$\mathcal{V}_\varepsilon := \{Z = X + iY \in \mathcal{H}(\mathbb{H}) ; \ Y \geq \varepsilon I, \ X^2 \leq \varepsilon^{-2}I\}.$$ We collect some needed facts from [Kr85, ch.V, le.2.5, le.2.7] in the following lemma:

(4.6) Lemma. a) Let $\varepsilon > 0$. There exists $c = c(\varepsilon) > 0$ such that

$$|\det(Z + W)| \geq c|\det(iI + W)|$$

as well as

$$|\det(\tilde{M}\{\tilde{Z}\})| \geq c|\det(\tilde{M}\{\tilde{iI}\})|$$

hold for all $Z \in \mathcal{V}_\varepsilon$, $W = U + iV \in \text{Her}_2(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ with $V \geq 0$ and $M \in \text{Sp}_2(\mathbb{H})$.

b) Given a compact subset $C$ in $\mathcal{H}(\mathbb{H})$ there exists $c = c(C)$ such that

$$\Im(M\{Z\}) \leq c\Im(M\{W\})$$

holds for all $Z, W \in C$ and $M \in \text{Sp}_2(\mathbb{H})$.

Finally, we are going to investigate the convergence behavior of the $E_k$. Note that this proof is very similar to the one found in [Kr85, ch.V, thm.2.8] concerning Eisenstein-series for the Hurwitz order. Nevertheless, we will give a complete proof here, since there are some details to be noticed.

(4.7) Theorem. Let $k \in 2\mathbb{N}$, $k \geq 8$. Then the quaternionic Eisenstein-series $E_k$ converges absolutely and uniformly in every vertical strip $\mathcal{V}_\varepsilon$, $\varepsilon > 0$.

Proof: So let $\varepsilon > 0$. We have to prove the uniform convergence of

$$\sum_{M \in \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} |\det(\tilde{M}\{\tilde{Z}\})|^{-k/2}$$

in $\mathcal{V}_\varepsilon$. Note that we already saw in (4.4) that this series is independent of the special choice of a transversal of the cosets $\text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})$. According to (4.6), there exists a constant $c > 0$ such that $|\det(\tilde{M}\{\tilde{Z}\})|^{-k/2} \leq c|\det(\tilde{M}\{\tilde{iI}\})|^{-k/2}$ holds for all $Z \in \mathcal{V}_\varepsilon$ and all $M \in \text{Sp}_2(\mathcal{O})$. Hence it suffices to prove the convergence of

$$\sum_{M \in \text{Sp}_2(\mathcal{O})_0 \setminus \text{Sp}_2(\mathcal{O})} |\det(\tilde{M}\{\tilde{iI}\})|^{-k/2}.$$ Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathcal{O})$. Note that $M\{iI\} \in \mathcal{H}(\mathbb{H})$ according to (1.5), which implies $\Im(M\{iI\}) \in \text{Pos}_2(\mathbb{H})$, and thus $\det(\Im(M\{iI\})) > 0$ in virtue of [Kr85, ch.I, thm.3.11]. So in view of (1.5) and (1.34) we compute

$$((\det(\Im(M\{iI\})))^{k/2} = (\det(\Im(M\{iI\})))^{k/4} = (\det(\tilde{M}\{i\tilde{i}\}))^{k/4}$$

$$= (\det((\tilde{M}\{i\tilde{i}\})^{-1}\tilde{I}(\tilde{M}\{i\tilde{i}\})^{-1}))^{k/4}.$$
According to (4.5),

\[ |\det(iC + \bar{D})(i\tilde{C} + \bar{D})|^{-k/4} \]

\[ = (\det(iC + \bar{D}) \det(i\tilde{C} + \bar{D}))^{-k/4} \]

\[ = (|\det(i\tilde{C} + \bar{D})|^2)^{-k/4} = |\det(M\{iI\}|^{-k/2} \]

where \( \mathfrak{Im}(X + iY) := \hat{Y} \) for \( X + iY \in \mathbb{H}^{2\times2} \otimes \mathbb{R} \). Thus we have to verify the convergence of

\[ \sum_{M: \text{Sp}_2(\mathbb{O})} (\det(\mathfrak{Im}(M\{iI\})))^{k/2} . \]

To keep it well-arranged, define maps \( x \) and \( y \) by \( M(Z) = x(Z) + iy(Z) \) (where \( M \) is still fixed), which means \( \mathfrak{Im}(M(Z)) = y(Z) \). Next, let \( C \) be some compact subset of the fundamental domain \( \mathcal{F}_2(\mathbb{O}) \) with positive symplectic volume \( \int_C dv \). According to (4.6) there exists a constant \( c_1 \) such that \( y(i) \leq c_1 y(Z) \) holds for all \( Z \in C \) (since \( C \cup \{iI\} \) is compact, too), and this constant is independent of \( M \) (which means we can choose the same \( c_1 \) for all \( M \in \text{Sp}_2(\mathbb{O}) \)). This also implies \( \det(y(i)) \leq \det(c_1 y(Z)) \) (by combining [Kr85, ch.I, cor.3.8] and [Kr85, ch.I, thm.3.11]), and thus \( c^{-k} (\det(y(i)))^{k/2} \leq (\det(y(Z)))^{k/2} \). Furthermore, put \( c_2 = c_1^2 (\int_C dv)^{-1} \). We compute

\[
(\det(y(i)))^{k/2} = c_2(\det(y(i)))^{k/2} c_1^{-k} \int_C dv = c_2 \int_C c_1^{-k} (\det(y(i)))^{k/2} dv \leq c_2 \int_C (\det(y(Z)))^{k/2} dv .
\]

According to (4.5), \( dv \) is invariant under symplectic transformations. So defining \( f : \mathcal{H} \to \mathbb{C} \), \( Z \mapsto \det(\mathfrak{Im}(Z))^{k/2} \) yields

\[
(\det(y(i)))^{k/2} \leq c_2 \int_C (\det(y(Z)))^{k/2} dv = c_2 \int_{\mathbb{M}(C)} f(MZ) dv = c_2 \int_{\mathbb{M}(C)} \det(\mathfrak{Im}(Z))^{k/2} dv .
\]

Since \( C \) is compact, there exists \( t := \max\{\det(\mathfrak{Im}(Z)) \mid Z \in C\} \). And because \( C \subset \mathcal{F}_2(\mathbb{O}) \), \( |\det(M\{Z\})| \geq 1 \) holds for all \( M \in \text{Sp}_2(\mathbb{O}) \) and \( Z \in C \) in view of (1.36), and thus also \( |\det(M\{\hat{Z}\})| \geq 1 \) with regard to (1.34). A similar computation like above therefore yields for \( Z = X + iY \in C \):

\[
|\det(\mathfrak{Im}(M(Z)))| = |(\det(\mathfrak{Im}(M(Z))))^2|^{1/2} = |\det(\mathfrak{Im}(M(Z)))|^{1/2} = |\det((\mathfrak{M}\{\hat{Y}\})^{-1}\hat{Y}(\mathfrak{M}\{\hat{Y}\})^{-1})|^{1/2} \]

\[
= |\det((\mathfrak{X}\tilde{C}' + \bar{D}' - i\tilde{Y}\tilde{C}')^{-1}\tilde{Y}(\mathfrak{C}\tilde{X} + \bar{D} + i\tilde{C}\tilde{Y})^{-1})|^{1/2} \]

\[
= |\det(\mathfrak{C}\tilde{X} + \bar{D} + i\tilde{C}\tilde{Y})^{-1}\det(\mathfrak{C}\tilde{X} + \bar{D} + i\tilde{C}\tilde{Y})^{-1}\det(\tilde{Y})|^{1/2} \]

\[
= |\det(M\{\hat{Z}\})|^{-1} |\det(\mathfrak{Im}(Z))| \leq |\det(\mathfrak{Im}(Z))| .
\]
Hence we have shown
\[
\det(\Im(Z)) \leq t \quad \text{for all } Z \in \bigcup_{M \in \text{Sp}_2(O)} M(C).
\]

In virtue of (1.23) and (1.43), there are only finitely many \( M \in \text{Sp}_2(O) \) satisfying \( M(F_2(O)) \cap F_2(O) \neq \emptyset \). The number of \( M \) doing so shall be denoted by \( l \). Now, let \( M_1 \in \text{Sp}_2(O) \) and suppose \( M_2 \) is given such that \( M_1(C) \cap M_2(C) \neq \emptyset \). This is equivalent to \( (M_2^{-1}M_1)(C) \cap C \neq \emptyset \) and implies \( (M_2^{-1}M_1)(F_2(O)) \cap F_2(O) \neq \emptyset \). Thus, the number of \( M_2 \in \text{Sp}_2(O) \) satisfying \( M_1(C) \cap M_2(C) \neq \emptyset \) is at most \( l \) for every \( M_1 \in \text{Sp}_2(O) \).

Now, let \( (M_j)_j \) be a transversal of \( \text{Sp}_2(O) \setminus \text{Sp}_2(O) \). Then by definition we have: For every \( Z \in M_j(C) \) there exists \( N_{Z,j} \in \text{Sp}_2(O)_0 \) such that \( N_{Z,j}(Z) \in F_2(O)_0 \). So the question which arises how often \( F_2(O)_0 \) (or to be more precise, how often each single point in \( F_2(O)_0 \)) is covered at most by that. So suppose there is \( Z \in F_2(O)_0 \) such that there exist \( Z_j \in C, M_j \) and \( N_{Z,j} \), with \( Z = (N_{Z,j}M_j)(Z)_j \). Then every \( Z_m \in C, M_m \) and \( N_{Z_m,m} \) also fulfilling \( Z = (N_{Z_m,m}M_m)(Z_m) \) would satisfy \( (N_{Z,j}M_j)(C) \cap (N_{Z_m,m}M_m)(C) \neq \emptyset \). Hence only \( l \) further such combinations can exist, and mapping the \( M_j(C) \) via the \( N_{Z,j} \) to \( F_2(O)_0 \) only yields an at most \( l \)-times covering of \( F_2(O)_0 \). Furthermore, noting that \( dv \) is invariant under symplectic transformations and that \( \det(\Im(Z)) \leq t \) holds for all \( Z \in \bigcup_{M \in \text{Sp}_2(O)} M(C) \), we get
\[
\sum_{M \in \text{Sp}_2(O)_0 \setminus \text{Sp}_2(O)} (\det(\Im(M(\langle i \rangle))))^{k/2} \leq \sum_{M \in \text{Sp}_2(O)_0 \setminus \text{Sp}_2(O)} \int_{M(C)} \det(\Im(Z))^{k/2} dv \\
\leq c_2 l \int_{F_2(O)_0, \det(Y) \leq t} \det(Y)^{k/2} dv,
\]
which means we have to verify that
\[
\int_{F_2(O)_0, \det(Y) \leq t} \det(Y)^{k/2} dv = \int dX \int_{R(O), \det(Y) \leq t} (\det(Y))^{k/2-2h/2} dY
\]
converges, where \( h = 2 + 4\frac{(2-1)}{2} \), or equivalently
\[
\int_{R(O), \det(Y) \leq t} (\det(Y))^{k/2-6} dY < \infty
\]
for all \( k \geq 8 \). But this holds due to [Kr85, ch.I, cor.5.10]. Note that this corollary only considers the Hurwitz order, but the proof would be exactly the same here, since the only needed fact is \( \int_{\text{R}(O), \det(Y) \leq 1} dY < \infty \), which can be adopted from [Kr85, ch.I, le.5.8] – whereat it is also a simple consequence of (1.30) (in the notation of this theorem, \( \det(S) \leq 1 \) implies \( cs_{1,2}^{(j)} \leq s_{1,1} \leq 3s_{2,2}^{-1} \) for some constant \( c \), so the integrals except that over \( s_{2,2} \) yield a term of the shape \( s_{2,2}^{-5} \), and the integral over this along \( (1, \infty) \) converges, while \( s_{1,1} \leq s_{2,2} \) yields the convergence along \( (0, 1) \)). This completes the proof.

**Remark.** Like in [Kr85, ch.V, pp.145] one can also define Eisenstein-series of higher degree, or
even with an “input” \( f \in \text{Sp}_j(O, k, 1)_0 \) (our “input” was \( 1 \in \text{Sp}_0(O, k, 1)_0 := C - \text{ if } j > 0 \) then the attached Eisenstein-series are also called Klingen Eisenstein-series). The convergence can be verified similar to (4.7). Confer [Kr85, ch.V, thm.2.9] about the details. These further Eisenstein-series are also quaternionic modular forms with respect to \( \text{Sp}_n(O) \), and their behavior under the \( \Phi \)-operator would be the same like the one described in [Kr85, ch.V, thm.2.11] concerning the Hurwitz order. But in this thesis, we stick to the Eisenstein-series of degrees one and two defined in (4.3).

Because of the absolute and local uniform convergence, we have therefore found another important example of non-identically vanishing quaternionic modular forms (see (4.4)). (Whereas the assertion that they do not vanish identically is not proved until the next theorem.)

We will now state an important property of the quaternionic Eisenstein-series \( E_k \). It is an analog of [Kr85, ch.V, thm.2.11], which is a property for Eisenstein-series with respect to the Hurwitz order. But note that the proof of this theorem uses no special attributes of the Hurwitz order that \( O \) does not possess. More or less, the structures of \( \text{Sp}_2(O)_0 \backslash \text{Sp}_2(O) \) and \( \text{Sp}_1(O)_0 \backslash \text{Sp}_1(O) \) have to be compared and the connection to the \( \Phi \)-operator has to be analyzed. So since the proof would be exactly the same (switching \( O \) and the Hurwitz order) we obtain, without literally re-doing the proof:

(4.8) Theorem. We have

\[ E_k|\Phi = G_k \]

for all even \( k \geq 8 \) as well as

\[ G_k|\Phi = 1 \]

for all even \( k \geq 4 \).

The second part of the preceding theorem is due to [KK07, ch.III]. In particular, the Eisenstein-series are non-identically vanishing, and the constant term in their Fourier-expansion equals 1.

For now, we collected all the information about quaternionic Eisenstein-series of degree one and two we need. To verify further properties, we will need a lot of background information, first. In the end, we will prove that the quaternionic Eisenstein-series \( E_k \) are special quaternionic Maaß lifts (for the trivial character) that are eigenforms for certain Hecke-operators. And moreover, we will determine their Fourier-expansions explicitly.

But before we continue, let us also define \( E_4 \) and \( E_6 \). Note that they cannot be defined like the rest of the Eisenstein-series, since the series do not converge absolutely for \( k = 4 \) and \( k = 6 \). But according to (3.36), \( [\text{Sp}_2(O), 4, 1] \) and \( [\text{Sp}_2(O), 6, 1] \) are one-dimensional and coincide with the Maaß spaces \( M(4; O) \) and \( M(6; O) \), respectively. Hence we can define \( E_4 \) and \( E_6 \) to be the unique generators of these spaces (unique up to a normalizing factor). Let \( g_2 \) and \( g_4 \) be generators of \( [\Gamma_0[3], 2, 1]_{\text{tr}} \) and \( [\Gamma_0[3], 4, 1]_{\text{tr}} \). We will see that the Fourier-expansions of \( g_2 \) and \( g_4 \) actually have a non-vanishing constant term. So we define \( g_2 \) and \( g_4 \) to be the unique generators whose constant terms in the Fourier-expansions are \( -\frac{8}{240} = 240 \) and \( -\frac{12}{504} = 504 \). Then we
define
\[ E_4 := M_{g_2}, \quad E_6 := M_{g_4}. \] (4.2)
Note that once it is clear that the Fourier-expansions of \( g_2 \) and \( g_4 \) possess the constant terms as stated above, the constant term in the Fourier-expansions of \( E_4 \) and \( E_6 \) has to be \( 1 \) (see (3.7)). Hence we obtain
\[ E_4|\Phi = G_4, \quad E_6|\Phi = G_6 \] (4.3)
in virtue of [KK07, ch.III], again, since \([\text{SL}_2(\mathbb{Z}), 4, 1]\) and \([\text{SL}_2(\mathbb{Z}), 6, 1]\) are one-dimensional and the \( G_k \) are normalized, i.e. the constant term in their Fourier-expansions is \( 1 \).

Note that we also could have defined \( E_4 \) to be \( \Theta(2)(\mathbb{Z}, S_0; O) \subseteq [\text{Sp}_2(O), 4, 1] \), since this theta-series is normalized, too (cf. (2.12)).

To finish this section, we will give another upper bound on the dimensions of spaces of quaternionic modular forms with respect to \( \text{Sp}_2(O) \).

(4.9) Proposition. Let \( k \in \mathbb{N} \) and \( \nu \) a multiplier system for \( \text{Sp}_2(O) \) of weight \( k \) (hence \( \nu \in \{1, \nu_{\text{det}}\} \) if \( k \in 2\mathbb{N} \) and \( \nu \in \{\nu, \nu-i\} \) if \( k \in 2\mathbb{N} - 1 \). Define
\[ d_{ck} := \begin{cases} 0, & \text{if } k \notin 2\mathbb{N}, \\ \lfloor k/12 \rfloor, & \text{if } k \in 2\mathbb{N}, k \equiv 2 \text{ mod } 12, \\ \lfloor k/12 \rfloor + 1, & \text{if } k \in 2\mathbb{N}, k \not\equiv 2 \text{ mod } 12. \end{cases} \]
(\text{where } \lfloor x \rfloor \text{ denotes the greatest integer } n \text{ such that } n \leq x) \text{ and } \epsilon = 1 \text{ if } \nu = 1 \text{ and } \epsilon = 0 \text{ in all other cases.}

Furthermore, let \( b_\nu = \left\lfloor \frac{\sqrt{3}}{\pi} \cdot k \right\rfloor \) and
\[ d_{\mathcal{R}(O), k} := \#\{n, m \in \mathbb{N}, a_0, a_1, a_2, a_3 \in \mathbb{Z}; n \leq \frac{1}{2} b_\nu, nm \leq \frac{3}{4} k^2, -a_1 \geq a_0 \geq -\frac{1}{2} a_1 \geq 0, m \geq n \geq -a_1 \geq -a_3 \geq |2a_2 + a_3|, 3nm - a_0^2 - a_0 a_1 - a_1^2 - a_2 a_3 - a_3^2 \leq \frac{3}{4} k^2 \} \].

Then
\[ \dim[\text{Sp}_2(O), k, \nu] \leq d_{\mathcal{R}(O), k} + \epsilon d_{ck} \]
holds.

Proof: First, let \( f \in [\text{Sp}_2(O), k, \nu] \). According to (1.55) \( \alpha_f(T[U]) = \nu(\text{Rot}((UT)^{-1})) \cdot \alpha_f(T) \) holds for the Fourier-coefficients of \( f \), where \( T \in \text{Her}_2^1(O) \) and \( U \in \text{GL}_2(O) \). Then by definition we have \( \alpha_f(T[\text{diag}(i_2, 1)]) = \delta \alpha_f(T) \) with \( \delta \neq 1 \) for \( \nu \neq 1 \). Noting \( \text{diag}(n, 0)[\text{diag}(i_2, 1)] = \text{diag}(n, 0) \) for all \( n \in \mathbb{N}_0 \) (and thus \( \delta \alpha_f(\text{diag}(n, 0)) = 0 \)) and taking the proof of (1.59) into account leads to \( f \) being a cusp form. Hence \([\text{Sp}_2(O), k, \nu] = [\text{Sp}_2(O), k, \nu]|_0 \) if \( \nu \neq 1 \). On the other hand, suppose \( f \in [\text{Sp}_2(O), k, 1] \) (which also implies \( k \in 2\mathbb{N} \)). Then in virtue of (1.59) we obtain \( f|\Phi \in [\text{SL}_2(\mathbb{Z}), k, 1] \) (due to what we already discussed concerning \( G_k \in [\text{SL}_2(\mathbb{Z}), k, 1] \)). Furthermore, every \( g \in [\text{SL}_2(\mathbb{Z}), k, 1] \) can be expressed as a polynomial in \( G_4 \) and \( G_6 \) (cf. [KK07, ch.III, Kor.4.3]), while \( G_4 \) and \( G_6 \) are algebraically independent. This means there exists
a polynomial \( p \in \mathbb{C}[X_1, X_2] \) with \( f|\Phi = p(G_4, G_6) \) – and note that \( p \) is homogeneous with respect to the weights of \( G_4 \) and \( G_6 \). But then the linearity of the \( \Phi \)-operator and (4.8) imply \( (f - p(E_4, E_6)) \equiv 0 \), hence \( f - p(E_4, E_6) \in [\text{Sp}_2(O), k, 1]_0 \). Finally, note that \( d_{s,k} = \# \{ s, t \in \mathbb{N}_0; 4s + 6t = k \} \) (cf. [KK07, pp.174]). And since there cannot exist a polynomial \( p \in \mathbb{C}[X_1, X_2] \) (homogeneous in the weights) which is not the zero polynomial such that \( p(E_4, E_6) \equiv 0 \) (or otherwise \( p(G_4, G_6) = p(E_4, E_6) \Phi \equiv 0 \) would have to follow, but \( G_4 \) and \( G_6 \) are algebraically independent), we obviously obtain

\[
\dim[\text{Sp}_2(O), k, v] = ed_{s,k} + \dim[\text{Sp}_2(O), k, v]_0.
\]

So let us determine an upper bound on the dimension of the spaces of cusp forms. According to (1.67) a quaternionic modular form \( f \in [\text{Sp}_2(O), k, v]_0 \) vanishes identically if \( \alpha_f(T) = 0 \) holds for all \( T \in \text{Her}_2^2(O), T > 0 \) with \( \text{tr}(T) \leq b_T \) (where \( \alpha_f(T) \) denotes the Fourier-coefficients of \( f \), of course).

For every \( T \in \text{Her}_2^2(O), T > 0 \) there exists \( U \in \text{GL}_2(O) \) such that \( T[U] \in \mathcal{R}(O) \) (see (1.30)). So for every \( T \in \text{Her}_2^2(O), T > 0 \) with \( \text{tr}(T) \leq b_T \) let \( U_T \in \text{GL}_2(O) \) with \( T[U_T] \in \mathcal{R}(O) \). In view of \( \alpha_f(T[U_T]) = v(\text{Rot}((T^t)^{-1})) \cdot \alpha_f(T) \) it suffices to consider \( \alpha_f(T[U_T]) = 0 \) for all \( T \in \text{Her}_2^2(O), T > 0 \) with \( \text{tr}(T) \leq b_T \).

Let \( T \) be such a matrix and \( T[U_T] = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathcal{R}(O) \cap \text{Her}_2^2(O) \) – hence in particular \( n, m \in \mathbb{N}_0 \) and \( t \in i_3/3 \mathcal{O} \) (see (1.7) and (1.47)). Let \( a_0, \ldots, a_3 \in \mathbb{Z} \) such that \( t = i_3/3 \left( a_0 + a_1 i_1/3 + a_2 i_2 + a_3 i_3/3 i_2 \right) \). The diagonal entries of \( T \) were supposed to be positive integers, and their sum to be less or equal to \( b_T \). So an easy consideration yields \( \det(T) \leq 1/4 b_T^2 \) (this is just some maximality problem). Deriving from (1.34) and \( \det(T) > 0 \) we have \( \det(T[U_T]) = \det(T) \leq 1/4 b_T^2 \). But this implies

\[
nm \leq 3 \det(T[U_T]) = 3nm - a_0^2 - a_0 a_1 - a_1^2 - a_2^2 - a_2 a_3 - a_3^2 \leq 3b_T^2/4.
\]

in view of (1.30). Furthermore \( n = \mu(T) \) holds according to (1.26) and (1.28). But obviously, \( \mu(T) \) has to be less or equal to the minimum of the diagonal entries of \( T \), and one of these has to be less or equal to \( 1/2 b_T \). Therefore, \( n \leq 1/2 b_T \) follows. In (1.31) we determined \( \mathcal{R}(O) \) explicitly.

A straightforward calculation shows that the conditions on the coefficients of \( T[U_T] \in \mathcal{R}(O) \) deriving from that proposition are equivalent to

\[
-4a_1 \geq a_0 \geq -2a_2 \geq 0 \quad \text{and} \quad m \geq n \geq -a_1 \geq -4a_3 \geq |2a_2 + a_3|.
\]

Hence we exactly have to consider the set defined in the assertion above. But this means that if more \( f_j \in [\text{Sp}_2(O), k, v]_0 \) are given than we have elements in that set, the linear equation system in the Fourier-coefficients of these \( f_j \) with respect to the exponents \( T[U_T] \) associated to the \((m, n, a_0, a_1, a_2, a_3)\) from that set is under-determined. This means that there exists a non-trivial linear combination of the \( f_j \) such that the Fourier-coefficients of the resulting quaternionic modular form \( f \) vanish for all \( T[U_T] \). We have seen that this is equivalent to say that its Fourier-coefficients vanish for all \( T \in \text{Her}_2^2(O), T > 0 \) with \( \text{tr}(T) \leq b_T \). But then this is equivalent to \( f \equiv 0 \). Hence the \( f_j \) are linear dependent, which yields the assertion.

Note that this upper bound which only takes (1.67) into account can be improved a bit, but not much: If \( T_1[U_{T_1}] \) and \( T_2[U_{T_2}] \) lie in the interior of \( \mathcal{R}(O) \) then they are not congruent modulo \( \text{GL}_2(O) \) (cf. 1.30). But this also means that a priori there is no correlation between \( \alpha_f(T_1) \) and \( \alpha_f(T_2) \). Hence we have to take both into account (without making use of any further considerations, like we did in (3.36)). But the upper bound can be improved regarding the
boundary of \( \mathcal{R}(\mathcal{O}) \). Furthermore, one could check whether every matrix associated to one of the \((n, m, a_0, a_1, a_2, a_3)\) actually is congruent to an \( T \in \text{Her}_3^2(\mathcal{O}) \), \( T > 0 \) with \( \text{tr}(T) \leq b_{tr} \).

Doing some calculations with the formula above yields the following upper bounds:

\[
\begin{array}{c|ccccccccc}
  k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \text{dim}[\Gamma, k, \nu]_0 & 0 & 0 & 0 & 0 & 4 & 4 & 8 & 8 & 28
\end{array}
\]

Note that these bounds are not very satisfying, indeed. But after all, they only take into account the number of modulo \( \text{GL}_2(\mathcal{O}) \) incongruent matrices in \( \text{Her}_3^2(\mathcal{O}) \). For \( k \leq 7 \) we obtained much better (and exact) bounds in (3.36). But at least the bounds obtained by the preceding proposition give an upper bound and estimate in complexity on how many Fourier-coefficients have to be compared to verify that two quaternionic modular forms coincide. And even taking the adjustments from above into account yields that for \( k = 8 \) we obtain a bound that is greater than ten. We could also regard the considerations from (3.36)) taking Maaß lifts into account. But even then one obtains no satisfying bound, since there are too many incongruent matrices with diagonal \((2, 2)\). Hence in particular the dimension of \([\text{Sp}_2(\mathcal{O}), 8, 1] \) is not clear.

### 4.2 Investigation of \([\Gamma_0[3], k, 1]_{tr} \) and \( \mathcal{M}(k; \mathcal{O}) \)

As stated before, we want to show that the quaternionic Eisenstein-series \( E_k \) are Maaß lifts, indeed. Therefore, we have to investigate \([\Gamma_0[3], k, 1]_{tr} \). In particular, its definition does not yield explicit elliptic modular forms. But this is something we will need. The methods used in this section are similar to those in section 3.3.

First, let us briefly recall the definition of \([\Gamma_0[3], k, 1]_{tr} \) and the quaternionic Maaß lift for the trivial character (see (3.4) and (3.7)). By definition, we have \( \Gamma[1] = \text{SL}_2(\mathbb{Z}) \). We will use this abbreviation from now on. To stay close to section 3.3 we define the following operator:

\[
\text{tr}_{\Gamma_0[3]}^{[1]} : [\Gamma_0[3], k, 1] \rightarrow [\Gamma[1], k, 1], \quad g \mapsto \text{tr}_{\Gamma_0[3]}^{[1]}(g) := \sum_{M : \Gamma_0[3] \backslash \Gamma[1]} g |_M . \tag{4.4}
\]

Completely analogous to the operator \( \text{tr}_{\Gamma_0[9]}^{[1]} \) (see 3.4.4), \( \text{tr}_{\Gamma_0[3]}^{[1]} \) is well-defined (which means independent of the special choice of a transversal) and yields elliptic modular forms for the whole modular group, indeed. Note that, according to [Kl98, Le.1.24], a transversal of the cosets \( \Gamma_0[3] \backslash \Gamma[1] \) is given by \( \{ I, J_1, J_1 T_1, J_1 T_1^2 \} \). So we have

\[
\text{tr}_{\Gamma_0[3]}^{[1]}(g) = g + g |_k I_1 + g |_k J_1 T_1 + g |_k J_1 T_1^2
\]

\[
= g + g |_k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + g |_k \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + g |_k \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} .
\]

\([\Gamma_0[3], k, 1]_{tr} \) is then defined to be the kernel of this operator, i.e.

\[
[\Gamma_0[3], k, 1]_{tr} := \{ g \in [\Gamma_0[3], k, 1] : \text{tr}_{\Gamma_0[3]}^{[1]}(g) = 0 \} . \tag{4.6}
\]

Finally, according to (3.7), the quaternionic Maaß lift for the trivial character is given as follows:
Let $k \in 2\mathbb{N}$ ($k \geq 4$, or else the lift is trivial), $g_0 \in [\Gamma_0[3], k-2, 1]_{tr}$ and $\rho = e^{3\pi i}$. Define
\[
g_1 := -\frac{1}{5}(g_0|_{k-2}l_2 + \rho g_0|_{k-2}T_1 + \rho^2 g_0|_{k-2}T_1^2) \in [\Gamma[3], k-2, 1],
g_4 := -\frac{1}{5}(g_0|_{k-2}l_2 + \rho^2 g_0|_{k-2}T_1 + \rho g_0|_{k-2}T_1^2) \in [\Gamma[3], k-2, 1],
\]
and suppose the Fourier-expansions are given by
\[
g_0(\tau) = \sum_{n \in \mathbb{N}_0} \beta(3n)e^{2\pi in\tau},
g_1(\tau) = \sum_{n \in \mathbb{N}_0} \gamma(3n + 2)e^{\pi i(2n + \frac{1}{3})\tau},
g_4(\tau) = \sum_{n \in \mathbb{N}_0} \delta(3n + 1)e^{\pi i(2n + \frac{2}{3})\tau}
\]
for all $\tau \in \mathcal{H}$. Define $\alpha^* : \mathbb{N}_0 \rightarrow \mathbb{C}$ by
\[
\alpha^*(n) = \begin{cases} 
\beta(n), & \text{if } n \equiv 0 \pmod{3}, \\
\delta(n), & \text{if } n \equiv 1 \pmod{3}, \\
\gamma(n), & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]
Then the quaternionic Maaß lift $M_{g_0}$ for the trivial character is given by
\[
M_{g_0}(Z) = -\frac{B_k}{2k} \alpha^*(0) + \sum_{0 \neq T \in \text{Her}_2^*(O), T \geq 0} \left( \sum_{d \in \mathbb{N}, d|T} d^{k-1}\alpha^*(3 \det(T)/d^2) \right) e^{2\pi i(T, Z)}
\]
for all $Z \in \mathcal{H}_0(\mathbb{H})$, where $B_k$ denotes the $k$-th Bernoulli number and $\epsilon(T) := \max\{d \in \mathbb{N} ; \frac{1}{d}T \in \text{Her}_2^*(O)\}$.

Our goal now is to get a more explicit description of this Maaß lift. So far, it is not clear how to obtain explicit elliptic modular forms $g_0$ in $[\Gamma_0[3], k-2, 1]_{tr}$, and even more, how to explicitly determine $g_1$ and $g_4$, especially in terms of Fourier-expansions. So, analogous to what we did in section 3.3 concerning the Maaß lifts of odd weight, we need to find isomorphic spaces of modular forms that can be described more explicitly. The approach will be quite similar to the one found in [Kr90] (and even to the procedures in section 3.3). But it will quite differ in some details. Again, we will need Hecke-operators and newforms. And we will need further generators, since we have only seen how to obtain generators for the subspace of cusp forms, so far. But again, [Mi89] will yield appropriate elliptic modular forms.

So let us start by studying $[\Gamma_0[3], k, 1]$, where $k \in 2\mathbb{N}$. Since we will always have to deal with the trivial character this time, we use the abbreviation
\[
[\Gamma, k] := [\Gamma, k, 1]
\]
(4.7)
from now on, where $\Gamma \leq \Gamma[1]$, and the same for $[\Gamma, k]_0$, $[\Gamma, k]_0^{\text{new}}$ (see 3.40), $[\Gamma, k]_0^{\text{old}}$ (see 3.38) and so on. (Confer section 3.3 for some basic facts about spaces of elliptic modular forms.) And since we will also have to deal with non-cusp forms this time, we need another subspace of
modular forms. Recall the definition of the Peterson inner product 3.39:

\[
(f, g) := v(\Gamma_0[N] \backslash \mathcal{H}) \int_{\Gamma_0[N] \backslash \mathcal{H}} f(z) \overline{g(z)} \Im(z)^k \frac{dx \, dy}{y^2}.
\]

Note that we only defined it for cusp forms in the first place. But according to [Mi89, p.44] it suffices that either \( f \) or \( g \) is a cusp form. Hence, for \( N \in \mathbb{N} \) we can define

\[
[\Gamma_0[N], k] := \{ f \in [\Gamma_0[N], k] ; (f, g) = 0 \text{ for all } g \in [\Gamma_0[N], k]_0 \}.
\]

(4.8)

\[ [\Gamma_0[N], k]_1 := [\Gamma_0[N], k]_0^1 = \{ f \in [\Gamma_0[N], k] ; (f, g) = 0 \text{ for all } g \in [\Gamma_0[N], k]_0 \} . \]

(4.8)

Just analogous to (3.26), we can decompose \([\Gamma_0[3], k]_0^\text{old}\). By definition (see 3.38, minding that \( \Gamma_0[1] = \Gamma[1] \)), \([\Gamma_0[3], k]_0^\text{old}\) is given by

\[
[\Gamma_0[3], k]_0^\text{old} = [\Gamma[1], k]_0 + [\Gamma[1], k]_0|_k \delta_3.
\]

Hence again, we will show that \([\Gamma[1], k]_0 \cap [\Gamma[1], k]_0|_k \delta_3 = \{0\}\) holds. The proof of this is completely the same as the one of (3.26). So we omit the details and refer to this corollary. In short, we have: \( g \in [\Gamma[1], k]_0 \cap [\Gamma[1], k]_0|_k \delta_3 \) implies that \( g \) has a Fourier-expansion of the shape

\[
g(\tau) = \sum_{n \in \mathbb{N}_0} \beta(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]

for some appropriate Fourier-coefficients \( \beta(n) \) that fulfill \( \beta(n) = 0 \) for all \( n \in \mathbb{N}_0 \setminus \mathfrak{I} \mathbb{N}_0 \). Hence \( \beta(n) = 0 \) for all \( n \) prime to \( l = 3 \). The conductor of the trivial character is 1 and we are considering an elliptic modular form in \( \Gamma[1] = \Gamma_0[1] \). And since \( \gcd(l, \frac{1}{2}) = 1 \), [Mi89, thm.4.6.8] yields \( g \equiv 0 \). Thus we have shown

\[
[\Gamma_0[3], k]_0^\text{old} = [\Gamma[1], k]_0 + [\Gamma[1], k]_0|_k \delta_3.
\]

Next, let us have a look at \([\Gamma_0[3], k]_0^\text{new}\). Let \( f \in [\Gamma_0[3], k]_0^\text{new} \) be a primitive newform. Then in virtue of [Mi89, thm.4.6.15], \( f_\rho \) is also a primitive newform in \([\Gamma_0[3], k]_0^\text{new}\). Let

\[
f(\tau) = \sum_{n \in \mathbb{N}} \alpha(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]

be the Fourier-expansion of \( f \). Since we are considering the trivial character, we have \( \alpha(n) = \overline{\alpha(n)} \) for all \( n \in \mathbb{N} \), \( \gcd(n, 3) = 1 \) according to [Mi89, p.165], and thus \( \alpha(n) \in \mathbb{R} \) for all \( n \in \mathbb{N} \), \( \gcd(n, 3) = 1 \). Define \( g = f - f_\rho \). Then \( g \) obviously is again a newform and has a Fourier-expansion of the shape

\[
g(\tau) = \sum_{n \in \mathbb{N}} \beta(n)e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]
for some appropriate Fourier-coefficients \( \beta(n) \) that fulfill \( \beta(n) = 0 \) for all \( n \in \mathbb{N} \setminus 3\mathbb{N} \). Again, we apply theorem 4.6.8 in [Mi89]: \( \beta(n) = 0 \) for all \( n \) prime to \( l = 3 \). This time, we are considering an elliptic modular form in \( \Gamma_0(3), k \), and the conductor of the trivial character is 1. We have \( \gcd(l, \frac{3}{l}) = 3 \). So according to this theorem, there exists \( h \in \Gamma(1), k \) such that \( g = h|_k \delta_3 \). But \( h \not\equiv 0 \) and thus \( g \not\equiv 0 \) would contradict the condition that \( g \) is a newform. So we have shown

\[
f = f_0 .
\]

Just like in (3.29), in virtue of [Mi89, thm.4.6.15] there exists \( c \in \mathbb{C} \) such that

\[
f|_k \omega_3 = c \cdot f_0 = c \cdot f .
\]

And this time, we can determine this constant even explicitly: Combining the theorems 4.6.16 and 4.6.17 and corollary 4.6.18 in [Mi89] yields \( c = \pm 1 \). More precisely, if \( \alpha(3) > 0 \), then \( c = -1 \), and if \( \alpha(3) < 0 \), then \( c = 1 \) (note that \( \alpha(3) \) is always non-zero due to [Mi89, thm.4.6.17]). We will not give the details here, since we would need even more definitions to do so. But \( c \) is explicitly determined in [Mi89, cor.4.6.18] (which is possible for our special case – for the case considered to determine \( M_k^{3+} \) in section 3.3 this corollary yields no answer). The other two theorems are just needed to explicitly determine \( c \) in dependence of \( \alpha(3) \). So we have

\[
f|_k \omega_3 = -\text{sgn}(\alpha(3)) \cdot f
\]

for all primitive newforms \( f \) in \( \Gamma_0(3), k \)\textsuperscript{new}0. And since \( \Gamma_0(3), k \)\textsuperscript{new} possesses a basis consisting of primitive newforms (see (3.28)), we can decompose the subspace of newforms as follows:

\[
\Gamma_0(3), k \textsuperscript{new}_0 = \Gamma_0(3), k \textsuperscript{new,1}_0 \oplus \Gamma_0(3), k \textsuperscript{new,-1}_0 ,
\]

where

\[
\Gamma_0(3), k \textsuperscript{new,\epsilon}_0 := \{ f \in \Gamma_0(3), k \textsuperscript{new}_0 ; f|_k \omega_3 = \epsilon \cdot f \} , \quad \epsilon \in \{ \pm 1 \} .
\]

And finally, we need to determine \( \Gamma_0(3), k \textsuperscript{1}_1 \). Note that we have to consider Dirichlet characters mod 3. So speaking of the trivial character for \( \Gamma_0(3) \) does indeed mean considering the trivial Dirichlet character \( \chi_0^{(3)} \) mod 3, given by \( \chi_0^{(3)}(n) = 0 \) for \( n \in 3\mathbb{Z} \) and \( \chi_0^{(3)}(n) = 1 \) for \( n \in \mathbb{Z} \setminus 3\mathbb{Z} \). This character is not primitive, since its conductor is 1 and it is induced by the trivial character mod 1, i.e. \( \chi_0^{(1)} \equiv 1 \). Define

\[
f_2(\tau) := \frac{1}{12} + \sum_{n \in \mathbb{N}} \left( \sum_{d \in \mathbb{N}, d|n} \chi_0^{(1)}(n/d) \chi_0^{(3)}(d) \cdot d \right) e^{2\pi i n \tau} , \quad \tau \in \mathcal{H} .
\]

(Note that \( \frac{1}{12} = \frac{B_2}{2} \) and \( \chi_0^{(1)} \equiv 1 \).) Then according to [Mi89, thm.4.7.1], \( f_2 \in \Gamma_0(3), 2 \) holds. Again without giving details (they are not necessary since the theorems are quite explicit), combining theorems 4.7.1 and 4.7.2 in [Mi89] and as well [KK07, ch.III, p.161] (which deals with the Fourier-expansion of Eisenstein-series of degree 1) we obtain

\[
\Gamma_0(3), 2 \textsuperscript{1}_1 = \langle f_2 \rangle
\]
and

\[ [\Gamma_0[3], k] = \langle G_k, G_k \rangle \delta_3 \]

(4.14)

for all even \( k \geq 4 \). The same proof we used to verify the decomposition of \([\Gamma_0[3], k]^{\text{old}}\) also yields that \( G_k \) and \( G_k \delta_3 \) are linearly independent, and thus we have \( \dim[\Gamma_0[3], 2]_1 = 1 \) and \( \dim[\Gamma_0[3], k]_1 = 2 \) for \( k \geq 4 \).

Note that \([\Gamma[1], 2] = \{0\}\) in view of [KK07, ch.III]. So by definition we have \([\Gamma_0[3], 2]_\Gamma = [\Gamma_0[3], 2]_\Gamma\), and thus \( \dim[\Gamma_0[3], 2]_1 = 1 \) in view of (3.5). And furthermore, \([\Gamma[1], k] = CG_k \oplus [\Gamma[1], k]_0\) holds for all even \( k \geq 4 \) according to [KK07, ch.III]. We summarize all these facts in the following

\[ [\Gamma_0[3], 2] = C f_2 \]

and

\([\Gamma_0[3], 2]_0 = \{0\}\)

as well as

\[ [\Gamma_0[3], k] = [\Gamma_0[3], k]^{\text{new,1}} \oplus [\Gamma_0[3], k]^{\text{new,0}} \oplus [\Gamma_0[3], k]_0 \oplus [\Gamma_0[3], k]_0 \oplus C G_k \oplus C G_k \delta_3 \]

\[ = [\Gamma_0[3], k]^{\text{new,1}} \oplus [\Gamma_0[3], k]^{\text{new,0}} \oplus [\Gamma_0[3], k]_0 \oplus [\Gamma_0[3], k]_0 \oplus [\Gamma_0[3], k]_0 \delta_3 \]

\[ [\Gamma_0[3], k]_0 = [\Gamma_0[3], k]^{\text{new,1}} \oplus [\Gamma_0[3], k]^{\text{new,0}} \oplus [\Gamma_0[3], k]_0 \oplus [\Gamma_0[3], k]_0 \oplus [\Gamma_0[3], k]_0 \delta_3 \]

for all even \( k \geq 4 \).

We will need an analog of (3.32) about the characterization of newforms in \([\Gamma_0[3], k]_0\).

\[ f \in [\Gamma_0[3], k]^{\text{new}} \iff (\text{tr}^{[\Gamma_0[3]} \langle f \rangle = 0 \quad \text{and} \quad \text{tr}^{[\Gamma_0[3]} \langle f \rangle = 0 \rangle. \]

\[ \text{Proof:}\]

The proof of this proposition will be quite similar to the one of (3.32). Let \( f \in [\Gamma_0[3], k]^{\text{new}} \) be a primitive newform with Fourier-expansion

\[ f(\tau) = \sum_{n \in \mathbb{N}} a(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}. \]

(Note that \( a(n) \in \mathbb{R} \) holds for all \( n \in \mathbb{N} \) since above we verified \( f = f_p \)). Because \( f|\omega_3 = \pm f \) in virtue of 4.10, we only have to verify \( \text{tr}^{[\Gamma_0[3]} \langle f \rangle = 0 \). Note that we have

\[ J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_1 T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad J_1 T_1^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}. \]
We compute
\[
(f|_k \omega_3)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) + (f|_k \omega_5)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array} \right) + (f|_k \omega_3)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array} \right) \\
f|_k \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right) + f|_k \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + f|_k \left( \begin{array}{cc} 1 & -1 \\ 0 & -3 \end{array} \right) \\
f|_k \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) + f|_k \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) + f|_k \left( \begin{array}{cc} 1 & 2 \\ 0 & 3 \end{array} \right) \\
= 3^{-k/2 + 1} f|_k T(3) = 3^{-k/2 + 1} \alpha(3) \cdot f
\]
in virtue of (3.29). Because the conductor of the trivial character (or rather $\chi_0^{(3)}$) is 1 and because we are considering elliptic modular forms with respect to $\Gamma_0[3]$ (hence “$N = 3$”), [Mi89, thm.4.6.17] yields $\alpha(3)^2 = 3^{k/2 - 1}$, or in other words one could say $\alpha(3) = \text{sgn}(\alpha(3))3^{k/2 - 1}$. So in view of 4.10 we get
\[
\text{tr}_{\Gamma_0[3]}^{[1]}(f) = f - \text{sgn}(\alpha(3)) \cdot \left( (f|_k \omega_3)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) + (f|_k \omega_5)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array} \right) + (f|_k \omega_3)|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array} \right) \right) \\
= f - \text{sgn}(\alpha(3))3^{-k/2 + 1} \cdot f \equiv 0.
\]
Therefore, because $[\Gamma_0[3], k]_0^{\text{new}}$ possesses a basis consisting of primitive newforms, we have verified
\[
f \in [\Gamma_0[3], k]_0^{\text{new}} \quad \Rightarrow \quad ( \text{tr}_{\Gamma_0[3]}^{[1]}(f) \equiv 0 \quad \text{and} \quad \text{tr}_{\Gamma_0[3]}^{[1]}(f|_k \omega_3) \equiv 0 ) .
\]
On the other hand, assume there is $f \in [\Gamma_0[3], k]_0$ satisfying $\text{tr}_{\Gamma_0[3]}^{[1]}(f) \equiv 0$ and $\text{tr}_{\Gamma_0[3]}^{[1]}(f|_k \omega_3) \equiv 0$. According to (4.10), there exist $f_1, f_2 \in [\Gamma[1], k]_0$ and $f_3 \in [\Gamma_0[3], k]_0^{\text{new}}$ such that $f = f_1 + f_2|_k \omega_3 + f_3$. Because we already know that $\text{tr}_{\Gamma_0[3]}^{[1]}(f_3) = \text{tr}_{\Gamma_0[3]}^{[1]}(f|_k \omega_3) \equiv 0$ holds, we obtain
\[
\text{tr}_{\Gamma_0[3]}^{[1]}(f_1) = -\text{tr}_{\Gamma_0[3]}^{[1]}(f_2|_k \omega_3) \quad \text{and} \quad \text{tr}_{\Gamma_0[3]}^{[1]}(f_1|_k \omega_3) = -\text{tr}_{\Gamma_0[3]}^{[1]}(f_2|_k \omega_3) \cdot f_3 .
\]
Note that $f_1 \in [\Gamma[1], k]_0$ implies $\text{tr}_{\Gamma_0[3]}^{[1]}(f_1) = 4 f_1$. On the other hand, since $f_2|_k(-H_1) = f_2$ and thus $f_2|_k(a \ b \\
\ c \ d) = f_2|_k(-a \ b \\
\ c \ d)$ for all $(a \ b \\
\ c \ d) \in \text{GL}_2^+(\mathbb{R})$, we have
\[
\text{tr}_{\Gamma_0[3]}^{[1]}(f_2|_k \omega_3) = f_2|_k \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 0 & -3 \\ 1 & 0 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 0 & 3 \\ 1 & -3 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array} \right) \\
= f_2|_k \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 1 & 1 \\ 0 & 3 \end{array} \right) + f_2|_k \left( \begin{array}{cc} 1 & 2 \\ 0 & 3 \end{array} \right) .
\]
Note that $f_2$ is an elliptic modular form for $\Gamma[1] = \Gamma_0[1]$. So if we want to consider the Hecke-operator $T(3)$ for $f_2$, we need a transversal of $\Gamma_0[1] \backslash \Gamma_0[1](\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array}) \Gamma_0[1]$. Note that such a transversal does not coincide with that given in (3.23) for $\Gamma_0[3]$ and $\Gamma_0[9]$. Instead, according to [Mi89, le.4.5.6], a transversal is given by
\[
\left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 1 \\ 0 & 3 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 2 \\ 0 & 3 \end{array} \right) .
\]
Note that this coincides with the transversal of $\Gamma[1] \backslash T(3)$ in (3.16) (and also the definitions of $T(m)$ and $T(m)$ with respect to $\Gamma[1]$ coincide for all $m \in \mathbb{N}$ due to [Mi89, le.4.5.2]). And thus, for to not mix things up, we write $T(3)$ instead of $T(3)$ when we want to apply the Hecke-operator $T(3)$ (in the sense of Miyake) with respect to $\Gamma[1]$ instead of $\Gamma_0[3]$ or $\Gamma_0[9]$ – and also $T(m)$
instead of $T(m)$ for all $m \in \mathbb{N}$. So we obtain
\[
\text{tr}_{\Gamma[1]}^\Gamma(f_2 | \delta_3) = 3^{-k/2+1} f_2 | T(3),
\]
and thus
\[
f_1 = -4^{-1} 3^{-k/2+1} f_2 | k T(3).
\]
has to be fulfilled. Next, let us have a look at the second identity. We have
\[
f_2 | k \delta_3 | k \omega_3 = f_2 | k \left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right) | k \left( \begin{array}{cc} 0 & -1 \\ 3 & 0 \end{array} \right) = f_2 | k \left( \begin{array}{cc} 0 & -3 \\ 3 & 0 \end{array} \right) = f_2 | k f_1 = f_2,
\]
and thus
\[
\text{tr}_{\Gamma[1]}^\Gamma(f_2 | \delta_3 | \omega_3) = 4 f_2.
\]
On the other hand, like above, $f_1 | k (\text{tr}) = f_1$ yields
\[
f_1 | k \omega_3 = f_1 | k \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) | k \left( \begin{array}{cc} 0 & -1 \\ 3 & 0 \end{array} \right) = f_1 | k \left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right) = f_1 | \delta_3.
\]
And so the same calculation as above leads to
\[
\text{tr}_{\Gamma[1]}^\Gamma(f_1 | k \omega_3) = 3^{-k/2+1} f_1 | k T(3),
\]
and therefore
\[
f_2 = -4^{-1} 3^{-k/2+1} f_1 | k T(3).
\]
Combining this identity with the one from above and using [Mi89, le.4.5.7] (and also [Mi89, p.143]) we obtain
\[
f_1 = 4^{-2} 3^{-k/2} \cdot (f_1 | k T(3)) | k T(3) = 4^{-2} 3^{-k/2} \cdot (f_1 | k T(9) + 3 \cdot 3^{k-2} f_1),
\]
or equivalently
\[
f_1 | k T(9) = 13 \cdot 3^{k-2} \cdot f_1.
\]
By definition, we have $[\Gamma[1], k]_0 = [\Gamma[1], k]_0^{\text{new}}$. And thus, according to (3.28), $[\Gamma[1], k]_0$ possesses a basis consisting of primitive newforms $g_1, \ldots, g_n$. Clearly, $T(9)$ is a linear operator on the $n$-dimensional vector space $[\Gamma[1], k]_0$. Its eigenvalues are given by $t_j(9)$, $j = 1, \ldots, n$, where $g_j | k T(9) = t_j(9) g_j$, since primitive newforms are common eigenfunctions for all $T(m)$, $m \in \mathbb{N}$ (see (3.28)). Hence, $f_1$ (and thus $f_2 = -4^{-1} 3^{-k/2+1} f_1 | k T(3)$) can only exist non-identically vanishing if one of the $t_j(9)$ fulfills $t_j(9) = 13 \cdot 3^{k-2}$. So suppose $g$ is any primitive newform in $[\Gamma[1], k]_0$ satisfying $g_j | k T(9) = t(9) g$ with $t(9) = 13 \cdot 3^{k-2} = 13/9^{k/2}$. There is $t(3) \in \mathbb{C}$ satisfying $g | k T(3) = t(3) g$. Applying [Mi89, le.4.5.7] we obtain
\[
t(3)^2 g = g | k T(3) | k T(3) = g | k T(9) + 3 \cdot 3^{k-2} \cdot g = (13 \cdot 3^{k-2} + 3 \cdot 3^{k-2}) g = 16 \cdot 3^{k-2} \cdot g,
\]
and thus
\[
t(3) = \delta \cdot 43^{k/2},
\]
where $\delta = \text{sgn}(t(3))$. Furthermore, there exist $t(3^n) \in \mathbb{C}$ such that $g | k T(3^n) = t(3^n) g$ for all
For all \( n \in \mathbb{N} \), where \( \sigma_{-1}(3^n) = \sum_{j=0}^{n} 3^{-j} \), and prove this via induction: The assertion already holds for \( n = 1 \) and \( n = 2 \). So let \( n \in \mathbb{N} \) and suppose the assertion already holds true for all integers less or equal to \( n \). Making use of [Mi89, le.4.5.7], we compute

\[
\delta^{n+1} \frac{4}{3} \sigma_{-1}(3^n)(3^{n+1})^{k/2} = (\delta \cdot \frac{4}{3} 3^{k/2}) \cdot (\delta^{n} \sigma_{-1}(3^n)(3^n)^{k/2}) \cdot g = g|_k T(3)|_k T(3^n)
\]

which leads to

\[
(t(3^n) + \delta^{n-1} \sigma_{-1}(3^{n-1})3^{k-1}(3^{n-1})^{k/2}) \cdot g
\]

Furthermore, we get

\[
\delta^{n+1} \frac{4}{3} \sigma_{-1}(3^n) - 3^{-1} \sigma_{-1}(3^{n-1}) = 4 \sum_{j=0}^{n} 3^{-j-1} - \sum_{j=0}^{n-1} 3^{-j-1} = (3 + 1)3^{-n-1} + \sum_{j=0}^{n-1} 3 \cdot 3^{-j-1} = \sum_{j=0}^{n+1} 3^{-j} = \sigma_{-1}(3^n)
\]

and this yields the assertion. Let

\[
g(\tau) = \sum_{n \in \mathbb{N}} a(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}
\]

be the Fourier-expansion of \( g \). According to [Mi89, le.4.5.15], \( a(3^n) = t(3^n) \) holds, and thus

\[
a(3^n) = \delta^{n} \sigma_{-1}(3^n)(3^n)^{k/2}
\]

But the Ramanujan-Petersson conjecture states that the Fourier-coefficients have to fulfill \( a(n) = O(n^{k-1}/2) \) (and not only \( a(n) = O(n^{k/2}) \), which we already used before). The conjecture can, for example, be found in [Mi89, thm.4.5.17]. We are considering \( k \geq 4 \). In this case, the Ramanujan-Petersson conjecture can be reduced to the Weil conjectures, which were proved by Deligne in [Del74]. For details, confer [Mi89, thm.4.5.17]. So this is a contradiction, and \( f_1 = f_2 \equiv 0 \) follows, which completes the proof. \( \square \)

Of course, 4.15 from the preceding theorem would also hold if we would have chosen \( f_1 \) or \( f_2 \) to be a non-cusp form, since \( f_1 \) and \( f_2 \) being cusp forms was not used for the calculations. And thus, combining (4.10), (4.11) and 4.15 immediately yields

**4.12 Proposition.** \( \Gamma_0[3], 2]_v = [\Gamma_0[3], 2] v = C f_2 \) with \( f_2 \) from 4.12. And if \( k \in 2\mathbb{N}, k \geq 4 \), one has

\[
[\Gamma_0[3], k]_v = [\Gamma_0[3], k]^{new,1}_0 \oplus [\Gamma_0[3], k]^{new,-1}_0 \oplus \{ g|_k T(3) - 4 \cdot 3^{k/2-1}g|_k \delta_3 : g \in [\Gamma[1], k] \}.
\]
4.2 Investigation of $[\Gamma_0[3], k, 1]_{tr}$ and $\mathcal{M}(k; \mathcal{O})$

Let us get back to Maass lifts, now. Suppose $f \in [\text{Sp}_2(\mathcal{O}), k, 1]$. Then, according to [KL98, Le.4.2, Le.4.4] (and also confer the beginning of this section), $f$ is a quaternionic Maass lift of even weight $k \geq 4$ for the trivial character (denoted by $f \in \mathcal{M}(k; \mathcal{O})$) if and only if the Fourier-expansion of $f$ is given by

$$
 f(Z) = -\frac{B_k}{2k} \alpha^*(0) + \sum_{0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0} \left( \sum_{d \in \mathbb{N}, d|d(T)} d^{k-1} \alpha^*(3\det(T)/d^2) \right) e^{2\pi i \epsilon(T, Z)}
$$

(4.16)

for all $Z \in \mathcal{H}_n(\mathbb{H})$, where $\alpha^*: \mathbb{N}_0 \to \mathbb{C}$ is some appropriate map, called the attached function (or also Maass function) of $f$, and

$$
 \epsilon(T) := \max\{d \in \mathbb{N} : \frac{1}{d} T \in \text{Her}_2(\mathcal{O})\}.
$$

(4.17)

($\epsilon(T)$ is called the content of $0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0$.) Furthermore,

$$
 \Psi: \mathcal{M}(k; \mathcal{O}) \to [\Gamma_0[3], k-2]_{tr}, \quad \Psi(f) = f_0, \quad f_0(\tau) = \sum_{n \in \mathbb{N}_0} \alpha^*(3n)e^{2\pi in\tau}, \tau \in \mathcal{H}
$$

(4.18)

is an isomorphism. Moreover, if we define

$$
 f_1(\tau) = \sum_{n \in \mathbb{N}_0} \alpha^*(3n+2)e^{2\pi i(n+\frac{1}{2})\tau}, \tau \in \mathcal{H},
$$

(4.19)

$$
 f_4(\tau) = \sum_{n \in \mathbb{N}_0} \alpha^*(3n+1)e^{2\pi i(n+\frac{1}{2})\tau}, \tau \in \mathcal{H},
$$

then

$$
 f_1, f_4 \in [\Gamma[3], k-2]
$$

and as well

$$
 f_1 = -\frac{1}{4}(f_0|_{k-2}J_1 + \rho f_0|_{k-2}J_1T_1 + \rho^2 f_0|_{k-2}J_1T_1^2),
$$

$$
 f_4 = -\frac{1}{4}(f_0|_{k-2}J_1 + \rho^2 f_0|_{k-2}J_1T_1 + \rho f_0|_{k-2}J_1T_1^2)
$$

hold, where $\rho = e^{\frac{2\pi i}{3}}$ (see (3.4), and all of the above can be found in [KL98]). $f_0 \in [\Gamma_0[3], k-2]_{tr}$ implies $f_0|_{k-2}J_1 = -f_0 - f_0|_{k-2}J_1T_1 - f_0|_{k-2}J_1T_1^2$. Keeping $\rho + \rho^2 = -1$ in mind we compute

$$
 -\frac{1}{3}(f_0 + 4f_1 + 4f_4) = -\frac{1}{3}(f_0 - 2f_0|_{k-2}J_1 + f_0|_{k-2}J_1T_1 + f_0|_{k-2}J_1T_1^2) = f_0|_{k-2}J_1.
$$

This leads to

$$
 f_0|_{k-2}\omega_3(\tau) = f_0|_{k-2}J_1|_{k-2}\delta_3(\tau) = -\frac{1}{2}(f_0 + 4f_1 + 4f_4)|_{k-2}\delta_3(\tau)
$$

$$
 = -3^{k/2-2}(f_0(3\tau) + 4f_1(3\tau) + 4f_4(3\tau)) = -3^{k/2-2}\sum_{n \in \mathbb{N}_0} c(n)e^{2\pi in\tau},
$$

(4.20)

where

$$
 c(n) = \begin{cases} 
 \alpha^*(n), & \text{if } n \equiv 0 \pmod{3}, \\
 4\alpha^*(n), & \text{if } n \not\equiv 0 \pmod{3}.
\end{cases}
$$
Now, suppose $k \geq 6$ and $f_0 \in [\Gamma_0[3], k-2]_{\text{new}, \epsilon}$, where $\epsilon = \pm 1$. Then by definition, we have $f_0|_{k-2}\omega_3 = \epsilon f$, and we obtain

$$\epsilon \sum_{n \in \mathbb{N}_0} a^*(3n) e^{2\pi i n \tau} = -3^{k/2-2} \sum_{n \in \mathbb{N}_0} c(n) e^{2\pi i n \tau}.$$ 

Comparing Fourier-coefficients yields

$$a^*(3n) = -\epsilon \cdot 3^{k/2-2} \cdot a^*(n)$$

for all $n \in 3\mathbb{N}$ and

$$a^*(3n) = -\epsilon \cdot 4 \cdot 3^{k/2-2} \cdot a^*(n)$$

for all $n \in \mathbb{N} \setminus 3\mathbb{N}$ (as well as $a^*(0) = 0$, of course). Induction then gives the following: Suppose $j,l \in \mathbb{N}$, $3 \mid l$. We obtain

$$a^*(3l) = a^*(3 \cdot 3^{j-1}l) = -\epsilon \cdot 3^{k/2-2} \cdot a^*(3^{j-1}l) = \ldots = (-\epsilon \cdot 3^{k/2-2})^{j-1} a^*(3l) = 4(-\epsilon \cdot 3^{k/2-2})^{j} a^*(l).$$

On the other hand, if $a^*(0) = 0$ and the formula from above hold, one clearly obtains $f_0|_{k-2}\omega_3 = \epsilon f$ by simply doing all steps backwards. Furthermore, according to (4.12), $f = h_1 + h_{-1} + g|_{k-2}T(3) - 4 \cdot 3^{k/2-2} g|_{k-2}\delta_3$ for some appropriate $h \in [\Gamma_0[3], k-2]_{\text{new}, \epsilon}$ and $g \in [\Gamma[1], k-2]$. $a^*(0) = 0$ implies that the constant term in the Fourier-expansion of $g|_{k-2}T(3) - 4 \cdot 3^{k/2-2} g|_{k-2}\delta_3$ vanishes. Denote the constant term in the Fourier-expansion of $g$ by $c$. Then [Mi89, le.4.5.14] (where the Fourier-expansion of $g|_{k-2}T(3)$ is discussed) yields

$$(1 + 3^{k-3} - 4 \cdot 3^{k/2-2} \cdot 3^{k/2-1})c = (1 - 3^{k-2})c = 0,$$

which implies $c = 0$, since $k \geq 6$. Therefore, $g$ and thus $f_0$ are cusp forms, and $f_0$ fulfills $f_0|_{k-2}\omega_3 = \epsilon f_0$. So due to (4.11), $f_0 \in [\Gamma_0[3], k]_{\text{new}, \epsilon}$ follows, which can be verified as follows: Due to (4.10), $f_0$ is given by $f_0 = h_1 + h_{-1} + g_1 + g_2|_{k-2}\delta_3$ for some appropriate $h \in [\Gamma_0[3], k-2]_{\text{new}, \epsilon}$ and $g_1, g_2 \in [\Gamma[1], k-2]$. Note that $g|_{k-2}(-f_1) = g_1$, and thus $g|_{k-2}\omega_3 = g|_{k-2}\delta_3$, like we have already seen before. And for the same reason, $g_2|_{k-2}\delta_3|_{k-2}\omega_3 = g_2$ holds since $k-2$ is even. Hence we obtain

$$\epsilon f_0 = f_0|_{k-2}\omega_3 = h_1 - h_{-1} + g_1|_{k-2}\delta_3 + g_2,$$

or equivalently

$$(\epsilon h_1 - h_1) + (\epsilon h_{-1} + h_{-1}) + (\epsilon g_1 - g_2) + (\epsilon g_2 - g_1)|_{k-2}\delta_3 \equiv 0.$$ 

$[\Gamma_0[3], k-2]$ is given by the direct sum $[\Gamma_0[3], k-2]_0 = [\Gamma_0[3], k-2]_{\text{new}, 1} \oplus [\Gamma_0[3], k-2]_{\text{new}, -1} \oplus [\Gamma[1], k-2]_0 \oplus [\Gamma[1], k-2]_0|_{k-2}\delta_3$, thus the equation above leads to

$$\epsilon h_1 = h_1, \quad \epsilon h_{-1} = -h_{-1}, \quad \epsilon g_1 = g_2.$$ 

So we immediately get $h_1 \equiv 0$ (if $\epsilon = -1$) or $h_{-1} \equiv 0$ (if $\epsilon = 1$). Of course, $\epsilon f_0 \in [\Gamma_0[3], k-2]_\epsilon$ holds. Therefore, due to exactly the same methods used in the proof of (4.11), $g_1|_{k-2}T(3) = -\epsilon^4 3^{(k-2)/2} g_1$ has to hold. But, again according to the proof of (4.11), $-\epsilon^4 3^{(k-2)/2}$ cannot be
an eigenvalue of the Hecke-operator $T(3)$ (in the space of cusp forms $[\Gamma[1], k-2]_0$). Thus $g_1 = g_2 \equiv 0$ follows, and we have shown $f_0 \in [\Gamma_0[3], k-2]^{new, \epsilon}$. We state this fact again in the following

(4.13) Proposition. Let $k \in 2\mathbb{N}$, $k \geq 6$ and $f \in \mathcal{M}(k; \mathcal{O})$ with attached function $\alpha^*$ (see 4.16). Then $\Psi(f)$ (see 4.18) belongs to $[\Gamma_0[3], k-2]^{new, \epsilon}$, $\epsilon = \pm 1$, if and only if $\alpha^*$ satisfies $\alpha^*(0) = 0$ as well as

$$\alpha^*(3l) = 4(-e^{3k/2-2})^j \alpha^*(l)$$

for all $j \in \mathbb{N}$ and all $l \in \mathbb{N}$ with $3 \mid l$.

We will also have to consider another map. Given $f \in \mathcal{M}(k; \mathcal{O})$ like in 4.16, we define

$$\Omega(f)(\tau) := f_0(3\tau) + f_1(3\tau) + f_4(3\tau) = \sum_{n \in \mathbb{N}_0} \alpha^*(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}. \quad (4.21)$$

Note that in view of 4.20 we have

$$f_0|_{k-2}\delta_3(\tau) - f_0|_{k-2}\omega_3(\tau) = 3^{k-2}/2 \sum_{n \in \mathbb{N}_0} \alpha^*(3n)e^{2\pi in3\tau} + 3^{k/2-2} \sum_{n \in \mathbb{N}_0} c(n)e^{2\pi in\tau}$$

which leads to

$$\Omega(f) = 4^{-1}3^{-k/2+2}(f_0|_{k-2}\delta_3 - f_0|_{k-2}\omega_3). \quad (4.22)$$

Because $f_0 \in [\Gamma_0[3], k-2]$, we have $f_0|_{k-2}\omega_3 \in [\Gamma_0[3], k-2]$ (cf. [Mi89, le.4.3.2]) and $f_0|_{k-2}\delta_3 \in [\Gamma_0[9], k-2]$ (cf. [Mi89, le.4.6.1]), and therefore $\Omega(f) \in [\Gamma_0[9], k-2]$. Of course, $\Omega(f)$ is the “right” elliptic modular form to lift, since it contains all relevant information for the lift, namely the attached function $\alpha^*$. And of course, $\Omega$ is injective (since again, $f$ is uniquely determined by the attached function). So all we need now is the image of $\mathcal{M}(k; \mathcal{O})$ in $[\Gamma_0[9], k-2]$ under $\Omega$. We are going to verify that this image is the following subspace of $[\Gamma_0[9], k-2]$:

$$\mathcal{M}_{k-2} := \{g \in [\Gamma_0[9], k-2] ; \ g|_{k-2}\omega_9 = -g, \ (\text{tr}_{\Gamma_0[9]}^{\Gamma_0[3]}(g))|_{k-2}\omega_9 = \text{tr}_{\Gamma_0[9]}^{\Gamma_0[3]}(g) - 4g \}. \quad (4.23)$$

Note that we choose the denotation $\mathcal{M}_{k-2}$ in analogy to $\mathcal{M}_{k-2}^3$ in chapter 3, since the attached function of any Maaß lift will be the Fourier-coefficient function of an element in $\mathcal{M}_{k-2}$, which is quite similar to the situation concerning quaternionic Maaß lifts of odd weight. And note that $[\Gamma_0[9], k-2]|_{k-2}\omega_9 = [\Gamma_0[9], k-2]$ and $\text{tr}_{\Gamma_0[9]}^{\Gamma_0[3]}(g) \in [\Gamma_0[3], k-2] \subset [\Gamma_0[9], k-2]$, so that the definition makes sense regarding the types of modular forms that have to coincide.

Furthermore, one can write the conditions of $\mathcal{M}_{k-2}$ in a different way: Let $g \in \mathcal{M}_{k-2}$, hence $g|_{k-2}\omega_9 = -g$. In (3.25) we determined a transversal of $\Gamma_0[9]\setminus \Gamma_0[3]$. It is given by $I$, $(\frac{1}{3} \frac{0}{1})$, $(\frac{1}{3} \frac{0}{1})$. Minding that $g \in [\Gamma_0[9], k-2]$ and $g|_{k-2}\omega_9 = -g$ (which means that we can multiply with
appropriate matrices from the left, like we have done several times before), we compute
\[
(tr_{\Gamma_0[9]}(g))|_{\omega_9} = \left(g + g|_{k-2}(\frac{1}{3} \frac{0}{1}) + g|_{k-2}(\frac{1}{6} \frac{0}{1})\right)|_{k-2}(\frac{9}{9} \frac{1}{1})
\]
\[
= g|_{k-2}\omega_9 + g|_{k-2}(\frac{9}{6} \frac{1}{1}) + g|_{k-2}(\frac{9}{6} \frac{1}{1})
\]
\[
= g + g|_{k-2}(\frac{0}{9} \frac{1}{6}) + g|_{k-2}(\frac{0}{9} \frac{1}{6})
\]
and
\[
tr_{\Gamma_0[9]}(g) = -tr_{\Gamma_0[9]}(g)|_{\omega_9} = -g|_{k-2}\omega_9 - g|_{k-2}(\frac{9}{6} \frac{1}{1}) - g|_{k-2}(\frac{9}{6} \frac{1}{1})
\]
\[
= g - g|_{k-2}(\frac{3}{9} \frac{1}{1}) - g|_{k-2}(\frac{6}{9} \frac{1}{1})
\]
\[
= g - g|_{k-2}(\frac{3}{9} \frac{1}{1}) - g|_{k-2}(\frac{6}{9} \frac{1}{1})
\]
Therefore, defining operators \(T_1\) and \(T_2\) on \([\Gamma_0[9], k-2]\) by
\[
g|_{k-2}T_1 := g + g|_{k-2}\omega_9 = g|_{k-2}(\frac{3}{9} \frac{0}{1}) + g|_{k-2}(\frac{0}{9} \frac{1}{1})
\]
\[
g|_{k-2}T_2 := (tr_{\Gamma_0[9]}(g))|_{k-2}\omega_9 - tr_{\Gamma_0[9]}(g) = 4g
\]
we can write \(M_{k-2}\) as
\[
M_{k-2} = \{g \in [\Gamma_0[9], k-2] ; g|_{k-2}T_1 = g|_{k-2}T_2 \equiv 0\} . \tag{4.25}
\]
Note that, like we discussed before, \(T_1\) and \(T_2\) are indeed operators on \([\Gamma_0[9], k-2]\). In a more general, but not in our setting, \(T_1\) and \(T_2\) are Hecke-operators for \(\Gamma_0[9]\), and are quite similar to those used in [Kr90]. That \(M_{k-2}\) is the “right” subspace, indeed, is proven in the following

\((4.14)\) Proposition. Given \(k \in 2\mathbb{N}\) the mapping
\[
\Omega : M(k + 2; \mathcal{O}) \rightarrow M_k
\]
defined in 4.21 is an isomorphism.

Proof: Above, we have already seen that \(\Omega : M(k + 2; \mathcal{O}) \rightarrow [\Gamma_0[9], k]\) is an injective homomorphism. Hence we only need to prove that \(\Omega(f) \in M_k \subseteq [\Gamma_0[9], k]\) holds for every \(f \in M(k + 2; \mathcal{O})\), indeed, which means \(\Omega(f)|_{kT_1} = \Omega(f)|_{kT_2} \equiv 0\), and that \(\Omega\) is surjective. Define
\[
\varphi : [\Gamma_0[6], k]_{\mathcal{O}} \rightarrow [\Gamma_0[9], k], \quad f_0 \mapsto f_0|_k\delta_3 - f_0|_k\omega_9 .
\]
Then according to 4.18 and 4.22, we have
\[
\Omega = 4^{-1}3^{-k/2+1} : \varphi \circ \Psi .
\]
And because \(\Psi\) is an isomorphism and \(\Omega\) is injective, \(\varphi\) has to be injective, too. And thus proving that \(\Omega : M(k + 2; \mathcal{O}) \rightarrow M_k\) is an isomorphism is equivalent to verifying \(\varphi([\Gamma_0[6], k]_{\mathcal{O}}) = M_k\). First, let us prove that \(\varphi([\Gamma_0[9], k]_{\mathcal{O}}) \subseteq M_k\) holds, which means that we have \(\varphi(f_0)|_{kT_1} = \varphi(f_0)|_{kT_2} \equiv 0\), indeed, is proven in the following
\( \varphi(f_0)|_{T_2} \equiv 0 \) for all \( f_0 \in [\Gamma_0[3],k]_\text{tr} \), since \( \varphi(f_0) \in [\Gamma_0[9],k] \) already holds, like we have seen above. We compute

\[
\varphi(f_0)|_{T_2} = \left( f_0|_{k}\left( \frac{3}{0} \right) - f_0|_{k}\left( \frac{3}{0} \right) \right)|_{k}\left( \frac{3}{0} \right) + \left( f_0|_{k}\left( \frac{3}{0} \right) - f_0|_{k}\left( \frac{3}{0} \right) \right)|_{k}\left( \frac{3}{0} \right)
\]

because \( f_0|_{k}\left( -I \right) = f_0 \). For \( T_2 \), we obtain

\[
\varphi(f_0)|_{T_2} = \left( f_0|_{k}\left( \frac{3}{0} \right) \right)|_{k}\left( \frac{3}{0} \right) - \left( f_0|_{k}\left( \frac{3}{0} \right) \right)|_{k}\left( \frac{3}{0} \right)
\]

since \( \text{tr}_{\Gamma_0[3]}(f_0) = 0 \) due to the definition of \([\Gamma_0[3],k]_\text{tr}\). Next, to verify that actually \( \varphi([\Gamma_0[3],k)_\text{tr} = M_k \) holds, we will prove that

\[
\psi : M_k \rightarrow [\Gamma_0[3],k]_\text{tr}, \quad g \rightarrow \frac{1}{4} \left( g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) \right)
\]

is the inverse of \( \varphi \). First, let us prove that \( \psi \) is well defined, which means that \( [\Gamma_0[3],k]_\text{tr} \) is the correct co-domain. We start with showing \( \psi(g) \in [\Gamma_0[3],k] \) for all \( g \in M_k \subset [\Gamma_0[9],k] \). According to [De96], \( \Gamma_0[3] \) is generated by \(-I, T_1 \) and \( I_1^{-1}T_1I_1 = \left( \frac{1}{0} \right) \). Of course, like we have mentioned so often before, it suffices to prove the invariance under these generators. The invariance under \(-I\) is obvious, since \( k \) is even. For \( T_1 \) we obtain

\[
\psi(g)|_{T_1} = \frac{1}{4} \left( g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) \right) = \psi(g),
\]

because

\[
g|_{k}\left( \frac{1}{0} \right) = g|_{k}\left( \frac{1}{0} \right) = g|_{k}\left( \frac{1}{0} \right).
\]

For \( I_1^{-1}T_1I_1 \) we compute

\[
4\psi(g)|_{k}\left( \frac{1}{0} \right) = g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{4}{3} \right) + g|_{k}\left( \frac{7}{3} \right)
\]

\[
= g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{4}{3} \right) + g|_{k}\left( \frac{7}{3} \right)
\]

\[
= g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) + g|_{k}\left( \frac{1}{0} \right) = 4\psi(g)
\]

due to \( g \in [\Gamma_0[9],k] \). Next, we will have to prove that \( \text{tr}_{\Gamma_0[3]}(\psi(g)) = 0 \) holds. The following calculation is just an appropriate collection of terms occurring in \( \text{tr}_{\Gamma_0[3]}(\psi(g)) \). The numbers
behind the brackets simply serve as indicators for later use.

\[ 4 \text{tr}_{10|3}^{[1]}(\psi(g)) = 4\psi(g) + 4\psi(g)k(\frac{0}{1}) + 4\psi(g)k(\frac{0}{1}) + 4\psi(g)k(\frac{0}{1}) \]

\[ = gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \]

\[ = \left( gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \right) \]

\[ + \left( gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \right) \]

\[ + \left( gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \right) \]

Due to \( g|\mathcal{T}_1 \equiv 0 \) and \( g|\mathcal{T}_2 \equiv 0 \) we compute

\[ \left( \cdots \right)_{(1)} = \left( gk(\frac{3}{0}) + gk(\frac{3}{0}) + gk(\frac{3}{0}) + gk(\frac{3}{0}) \right) k(\frac{1}{3}0) \]

\[ = \left( -gk(\frac{3}{0}) \right) k(\frac{1}{3}0) = \left( gk(\frac{0}{0}) \right) k(\frac{1}{3}0) = gk(\frac{0}{0}) \]

and

\[ \left( \cdots \right)_{(1)} + \left( \cdots \right)_{(2)} = gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \]

\[ = \left( 2gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \right) k(\frac{1}{3}0) \]

\[ = \left( 2gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) + gk(\frac{0}{0}) \right) k(\frac{1}{3}0) \]

\[ = \left( gk(\frac{0}{0}) \right) k(\frac{1}{3}0) = gk(\frac{0}{0}) \]

\[ = gk(\frac{0}{0}) \]

\[ = gk(\frac{0}{0}) \]

since \( g \in [\Gamma [9], k] \) (which means we can multiply with appropriate matrices from the left). Thus we have already reduced \( 4 \text{tr}_{10|3}^{[1]}(\psi(g)) \) to

\[ 4 \text{tr}_{10|3}^{[1]}(\psi(g)) = gk(\frac{3}{3}) + gk(\frac{3}{3}) + gk(\frac{3}{3}) + gk(\frac{3}{3}) \]

But we have

\[ gk(\frac{2}{3}) + gk(\frac{1}{3}) = \left( gk(\frac{2}{3}) + gk(\frac{2}{3}) \right) k(\frac{1}{3}0) \]

\[ = \left( gk(\frac{2}{3}) + gk(\frac{2}{3}) \right) k(\frac{1}{3}0) \]

\[ = \left( g + gk(\frac{0}{0}) \right) k(\frac{1}{3}0) \]

\[ = 0 \]

and

\[ gk(\frac{2}{3}) + gk(\frac{1}{3}) = -gk(\frac{0}{0}) k(\frac{2}{3}) + gk(\frac{1}{3}) = -gk(\frac{2}{3}) + gk(\frac{1}{3}) \]

\[ = -gk(\frac{2}{3}) + gk(\frac{1}{3}) = -gk(\frac{1}{3}) + gk(\frac{1}{3}) = -gk(\frac{1}{3}) + gk(\frac{1}{3}) \]

\[ = 0 . \]
Hence $\text{tr}_{[\Gamma_0[3], 1]}(\phi(g)) \equiv 0$ and $\phi$ is well-defined. So all that remains to be done is showing $\varphi \circ \psi = \text{id}$. So let $g \in \mathcal{M}_k$. We compute

$$
\varphi(\psi(g)) = \frac{1}{4} \left( g|k\left(\frac{1}{2} \frac{0}{3}\right) + g|k\left(\frac{1}{3} \frac{0}{2}\right) + g|k\left(\frac{1}{3} \frac{2}{6}\right) \right)|k\omega_3 - \frac{1}{4} \left( g|k\left(\frac{0}{3} \frac{0}{3}\right) + g|k\left(\frac{1}{3} \frac{1}{3}\right) + g|k\left(\frac{1}{2} \frac{0}{6}\right) \right)|k\omega_3
$$

$$
= \frac{1}{4} \left( g|k\left(\frac{0}{3} \frac{0}{3}\right) + g|k\left(\frac{3}{3} \frac{0}{0}\right) + g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{3}{0} \frac{1}{0}\right) - g|k\left(\frac{3}{0} \frac{0}{1}\right) - g|k\left(\frac{0}{0} \frac{1}{0}\right) \right)
$$

$$
= \frac{1}{4} \left( 2g - g|k\left(\frac{2}{3} \frac{0}{3}\right) - g|k\left(\frac{2}{27} \frac{3}{27}\right) - g|k\left(\frac{2}{6} \frac{3}{6}\right) - g|k\left(\frac{2}{0} \frac{3}{0}\right) - g|k\left(\frac{2}{0} \frac{3}{0}\right) - g|k\left(\frac{0}{3} \frac{3}{0}\right) - g|k\left(\frac{0}{3} \frac{3}{0}\right) - g|k\left(\frac{0}{3} \frac{3}{0}\right) - g|k\left(\frac{0}{3} \frac{3}{0}\right) \right) = \frac{1}{4} \left( 2g + 2g \right) = g .
$$

Thus, $\varphi$ is surjective and this completes the proof. \qed

So we finally found the “right” space of elliptic modular forms for the quaternionic Maass lift for the trivial character, in analogy to $\mathcal{M}^3_k$. But of course, it is still quite difficult to find explicit elements in $\mathcal{M}_k$, since it is only given by some vanishing conditions with respect to some operators (again, like it was the case for $\mathcal{M}^3_k$). So we need to find another appropriate space of elliptic modular forms that can be described explicitly – like $[\Gamma_0[9], k, \chi_{30}^{\text{new}}, (2)]$ in the case of the quaternionic Maass lift of odd weight in chapter 3. In the case we are considering now, we will see that this explicit space is given by $[\Gamma_0[3], k_0^{\text{new}} \oplus [\Gamma[1], k]$ for $k \geq 4$ and $C_{f_2}$ for $k = 2$.

The following proposition is again in analogy with [Kr90]:

(4.15) Proposition. Given $k \in 2\mathbb{N}$ the mapping

$$
\Lambda : [\Gamma_0[3], k] \rightarrow \mathcal{M}_k , \ g \mapsto g - g|k\omega_9
$$

is a surjective homomorphism with the kernel

$$
\ker(\Lambda) = \{ g(3\tau) ; g(\tau) \in [\Gamma[1], k] \} = [\Gamma[1], k] |k\delta_3 .
$$

Proof: First, let us prove that $\Lambda$ is well defined, which means $\Lambda(g) \in [\Gamma_0[9], k]$ and $\Lambda(g)|kT_1 = \Lambda(g)|kT_2 \equiv 0$ for all $k \in [\Gamma_0[3], k]$:

Because $g \in [\Gamma_0[3], k]$ implies $g \in [\Gamma_0[9], k], g|k\omega_9 \in [\Gamma_0[9], k]$ follows due to [Mi89, le.4.3.2], and thus $\Lambda(g) \in [\Gamma_0[9], k]$. Moreover, since $\omega_9 = -9I$ and $k$ is even, we have

$$
\Lambda(g)|kT_1 = (g - g|k\omega_9) + (g - g|k\omega_9)|k\omega_9 = g - g|k\omega_9 + g|k\omega_9 - g \equiv 0 .
$$

For $T_2$ we compute

$$
\Lambda(g)|kT_2 = g|kT_2 - g|k\omega_9|kT_2
$$

$$
= 2g|k\left(\frac{3}{0} \frac{0}{0}\right) + g|k\left(\frac{0}{0} \frac{0}{0}\right) + g|k\left(\frac{0}{0} \frac{0}{0}\right) + g|k\left(\frac{0}{0} \frac{0}{0}\right) - 2g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right)
$$

$$
= g|k\left(\frac{1}{3} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right) + g|k\left(\frac{1}{0} \frac{0}{0}\right)
$$

$$
= 2g - 2g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) - g|k\left(\frac{0}{0} \frac{0}{0}\right) .
$$
We saw before that we are not only looking at cusp forms. Making use of the dimension formula \[ \text{(Mi89, thm.2.5.2)} \]
and noting the considerations in the proof of (3.35) we obtain
and thus \([\Gamma[1],k]|_{k}\delta_{3} \subset \ker(\Lambda)\). On the other hand, let \(g \in \ker(\Lambda)\), i.e.
\[
g|_{k\Omega_{9}} = g .
\]
We have to show that \(g|_{k}(\frac{1}{0\ 3}) \in [\Gamma[1],k]\) holds, because this will imply
\[
g = g|_{k}(\frac{1}{0\ 3})|_{k}\delta_{3} \in [\Gamma[1],k]|_{k}\delta_{3}.
\]
Note that
\[
g|_{k}(\frac{1}{0\ 3}) = g|_{k}(\frac{0\ -1}{9\ 0})|_{k}(\frac{1}{0\ 3}) = g|_{k}(\frac{0\ -3}{9\ 0}) = g|_{k}\omega_{3}.
\]
Clearly, we only have to verify the invariance for the two generators of \(\Gamma[1]\), namely \(T_{1}\) and \(J_{1}\). We compute
\[
g|_{k}\omega_{3}|_{k}T_{1} = g|_{k}(\frac{0\ -1}{3\ 3}) = g|_{k}(\frac{1\ 0}{3\ 3})|_{k}(\frac{0\ -1}{3\ 3}) = g|_{k}\omega_{3},
\]
and for \(J_{1}\) we get
\[
g|_{k}\omega_{3}|_{k}J_{1} = g|_{k}(\frac{-1\ 0}{0\ -3}) = g|_{k}\omega_{3}.
\]
So we verified \(\ker(\Lambda) = [\Gamma[1],k]|_{k}\delta_{3}\). We still have to consider the surjectivity. We do this via a dimension argument. A simple fact from linear algebra is that \([\Gamma_{0}[3],k]/[\Gamma[1],k]|_{k}\delta_{3}\) is isomorphic to \(\Lambda([\Gamma_{0}[3],k])\), and thus
\[
\dim(\Lambda([\Gamma_{0}[3],k])) = \dim([\Gamma_{0}[3],k]) - \dim([\Gamma[1],k]).
\]
We saw before that
\[
\dim([\Gamma[1],k]) = \begin{cases} k/12, & \text{if } k \equiv 2 \mod 12, \\ k/12 + 1, & \text{if } k \not\equiv 2 \mod 12 \end{cases}
\]
holds for all even \(k \in \mathbb{N}\) in view of [KK07, pp.174]. Furthermore, a dimension formula for \([\Gamma_{0}[3],k]\) is given by [Mi89, thm.2.5.2]. Recall that in (3.35) we had to determine \(\dim([\Gamma_{0}[3],k,\chi_{3}]_{0})\) (deriving from [Mi89, thm.2.5.3]). It actually turns out that the dimension formulas are the same in this special case, with the only difference that for \(k \geq 4\) we have to add 2 to this formula, since we are not only looking at cusp forms. Making use of the dimension formula [Mi89, thm.2.5.2] and noting the considerations in the proof of (3.35) we obtain
\[
\dim([\Gamma_{0}[3],k]) = \left\lfloor \frac{k}{3} \right\rfloor + 1.
\]
Moreover, one easily verifies that
\[
\dim(\Lambda([\Gamma_{0}[3],k])) = \left\lfloor \frac{k+2}{4} \right\rfloor.
\]
follows. Just like in the proof of (3.35) one simply has to go through all possible cases $k = 12n + m$, where $n \in \mathbb{N}_0$ and $m \in \{4, 6, 8, 10, 12, 14\}$ and verify the assertion with help of a straightforward calculation. According to (3.5), (3.7) and (4.14) we have $\dim (\mathcal{M}_k) = \left[\frac{k+2}{4}\right]_2$, too. This proves the surjectivity. (Also in the case of $k = 2$ since we already know that both $\mathcal{M}_2$ and $[\Gamma_0[3], 2]$ are one-dimensional.)

\(\square\)

### (4.16) Remark.

a) In the preceding proposition, we saw that $\Lambda : \Gamma_0[3], k \to \mathcal{M}_k$ is a surjective homomorphism with kernel $[\Gamma[1], k][k]\delta_3$. Again, a simple fact from linear algebra is that $[\Gamma_0[3], k][\Gamma[1], k][k]\delta_3$ is isomorphic to $\mathcal{M}_k$, and $\Lambda$ factors through $[\Gamma[1], k][k]\delta_3$. According to (4.10) we have $\Gamma_0[3], k = \Gamma_0[3], k_{0}^{\text{new}, 1} \oplus \Gamma_0[3], k_{0}^{\text{new}, -1} \oplus \Gamma[1], k \oplus \Gamma[1], k[3]\delta_3$ for all even $k \geq 4$. In this case, $\Gamma_0[3], k[\Gamma[1], k][k]\delta_3 = \Gamma_0[3], k_{0}^{\text{new}, 1} \oplus \Gamma_0[3], k_{0}^{\text{new}, -1} \oplus \Gamma[1], k$, and therefore

$$\Lambda : \Gamma_0[3], k_{0}^{\text{new}, 1} \oplus \Gamma_0[3], k_{0}^{\text{new}, -1} \oplus \Gamma[1], k \to \mathcal{M}_k$$

is an isomorphism. Elements in $\Gamma_0[3], k_{0}^{\text{new}, 1} \oplus \Gamma_0[3], k_{0}^{\text{new}, -1} \oplus \Gamma[1], k$ can be constructed explicitly. $\Gamma[1], k = \langle \mathcal{G}_m^{\text{new}} ; n, m \in \mathbb{N}, 4n + 6m = k \rangle$ according to [KK07, ch.III], while primitive newforms in $\Gamma_0[3], k_{0}^{\text{new}}$ can be constructed explicitly (in terms of Fourier-coefficients) using [SAGE]. On the other hand, for $k = 2$, we have $\Gamma[1], 2 = \{0\}$. So again in view of (4.10), $\mathcal{M}_2$ is given by

$$\mathcal{M}_2 = C(f_2 - f_2|k\omega_9)$$

with $f_2$ defined in 4.12.

b) In the case of quaternionic Maass lifts of odd weight, where we had to map elliptic modular forms in $\Gamma_0[9], k, \chi_3^{\text{new}(2)}$ to $\mathcal{M}_3^{\pm}$ via $\omega_9$, the action of $\omega_9$ could not be determined explicitly in a general sense. Thus, elements in $\mathcal{M}_3^{\pm}$ (which consists of those elliptic modular forms exactly containing the relevant information for the Maass lifts, namely the attached function) could be determined explicitly in a numerical sense, only. But in the case we are considering now, such a problem does not occur. According to a), for $k \geq 4$ we have to map elliptic modular forms $f$ in $\Gamma_0[3], k_{0}^{\text{new}, 1} \oplus \Gamma_0[3], k_{0}^{\text{new}, -1} \oplus \Gamma[1], k$ to $\mathcal{M}_k$ (again the “correct” space for the Maass lifts) via $\Lambda$, i.e. $f \mapsto f|k\omega_9$. But this time, the action of $\omega_9$ is given explicitly: For $g \in \Gamma_0[3], k_{0}^{\text{new}, 1}$, we have

$$\Lambda(g)(\tau) = g(\tau) - g_k\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)k\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\tau) = g(\tau) - g_k\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\tau) = g(\tau) - 3^{k/2}g(3\tau)$$

by definition. Completely analogous, for $g \in \Gamma_0[3], k_{0}^{\text{new}, -1}$ we obtain

$$\Lambda(g)(\tau) = g(\tau) + 3^{k/2}g(3\tau)$$

And for $g \in \Gamma[1], k$ we compute

$$\Lambda(g)(\tau) = g(\tau) - g_k\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)k\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\tau) = g(\tau) - g_k\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\tau) = g(\tau) - 3^k g(9\tau)$$

For $k = 2$ we have $\mathcal{M}_2 = C(f_2 - f_2|k\omega_9)$ in virtue of a). Again, we compute

$$\Lambda(f_2)(\tau) = f_2(\tau) - f_2|2\omega_3|2\delta_3(\tau) = f_2(\tau) - 3^{2/2}f_2|2\omega_3(3\tau)$$
so we need to determine \( f_2|_k \omega_3 \). According to (4.10), \([\Gamma[3],2] = C f_2 \). And thus, since \( f_2|_k \omega_3 \in [\Gamma[3],2] \) (again, cf. [Mi89, le.4.3.2]), there has to exist \( c \in \mathbb{C} \) such that \( f_2|_k \omega_3 = c f_2 \) holds. Note that \( \omega_3^2 = -3I \), and since \( k = 2 \) is even, we obtain

\[
f_2 = f_2|_k \omega_3|_k \omega_3 = c^2 f_2 ,
\]
hence \( c = \pm 1 \). By definition (see 4.12), the Fourier-expansion of \( f_2 \) is given by

\[
f_2(\tau) = \frac{1}{12} + \sum_{n \in \mathbb{N}} a(n) e^{2\pi i n \tau} , \quad \tau \in \mathcal{H}
\]
for some appropriate Fourier-coefficients \( a(n) \) that fulfill \( a(n) \geq 1 \) for all \( n \in \mathbb{N} \). Therefore we have

\[
f_2(iy) = \frac{1}{12} + \sum_{n \in \mathbb{N}} a(n) e^{-2\pi ny} > 0
\]
for all \( y > 0 \). On the other hand,

\[
f_2|_k \omega_3(\tau) = \frac{1}{3} \tau^{-2} f_2(-3\tau^{-1})
\]
holds by definition, and therefore

\[
f_2|_k \omega_3(iy) = -\frac{1}{3} y^{-2} f_2(i(3y)^{-1}) < 0
\]
for all \( y > 0 \). Hence \( c = -1 \) follows, which leads to

\[
\Lambda(f_2)(\tau) = f_2(\tau) + 3f_2(3\tau) .
\]
Thus, for all \( k \in 2\mathbb{N} \) every \( f \in \mathcal{M}_k \) can be determined explicitly in terms of Fourier-coefficients.

For later purposes, we need an other description of \( \mathcal{M}(k;\mathcal{O}) \) than the one we used so far (see 4.16). The following lemma is mainly due to [Ad79].

(4.17) Lemma. Let \( k \in 2\mathbb{N}, k \geq 4 \) and \( f \in [\text{Sp}_2(\mathcal{O}), k, 1] \) with Fourier-expansion

\[
f(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i T(Z)} , \quad Z \in \mathcal{H} \).
\]
Then \( f \) belongs to \( \mathcal{M}(k;\mathcal{O}) \) (i.e. \( f \) is a quaternionic Maass lift for the trivial character) if and only if there exists a function

\[
\alpha : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C} ,
\]
which satisfies:

\[
\alpha_f(T) = \alpha(\epsilon(T), 3 \det(T)/\epsilon(T)^2) \quad \text{for all } 0 \neq T \in \text{Her}_2^+(\mathcal{O}), T \geq 0
\]
(4.26)
4.2 Investigation of $[\Gamma_0[3], k, 1]_{\text{tr}}$ and $\mathcal{M}(k; \mathcal{O})$

(with $\varepsilon(T)$ from 4.17) and

\[
(1 - p^{k-1}W) \sum_{j \in \mathbb{N}_0} \alpha(p^j q, d) W^j = \sum_{j \in \mathbb{N}_0} \alpha(q, p^{2j} d) W^j \tag{4.27}
\]

holds as a formal power series in $W$ for all $d \in \mathbb{N}_0$, $q \in \mathbb{N}$ and all prime numbers $p$ with $p \nmid q$.

Since we will need it later on, it is convenient to extend the definition of $\alpha$ to $\mathbb{C} \times \mathbb{C}$ with

\[
\alpha(u, v) = 0, \quad \text{if } u \notin \mathbb{N} \lor v \notin \mathbb{N}_0. \tag{4.28}
\]

And in the course of this, we also define the characteristic function of $\mathbb{Z}$ by

\[
\chi_{\mathbb{Z}}(u) := \begin{cases} 1, & \text{if } u \in \mathbb{Z}, \\ 0, & \text{if } u \notin \mathbb{Z}. \end{cases} \tag{4.29}
\]

**Proof:** In virtue of 4.16, $f$ belongs to $\mathcal{M}(k; \mathcal{O})$ if and only if there exists a function $\alpha^*: \mathbb{N}_0 \to \mathbb{C}$ (the attached function of $f$) satisfying

\[
\alpha_f(T) = \sum_{d \in \mathbb{N}, d | \varepsilon(T)} d^{k-1} \alpha^*(3 \det(T)/d^2) = \alpha(\varepsilon(T), 3 \det(T)/\varepsilon(T)^2) \tag{4.30}
\]

for all $0 \neq T \in \text{Her}_2^\tau(\mathcal{O})$, $T \geq 0$, if we define

\[
\alpha : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C}, \quad \alpha(t, D) := \sum_{d \in \mathbb{N}, d | t} d^{k-1} \alpha^*(t^2 D/d^2). \tag{4.31}
\]

Note that for all $t, d \in \mathbb{N}$ with $d | t$ and $D \in \mathbb{N}_0$ we have $\alpha(1, t^2 D/d^2) = \alpha^*(t^2 D/d^2)$ by definition. This leads to

\[
\alpha(t, D) = \sum_{d \in \mathbb{N}, d | t} d^{k-1} \alpha^*(t^2 D/d^2) = \sum_{d \in \mathbb{N}, d | t} d^{k-1} \alpha(1, t^2 D/d^2) \tag{4.32}
\]

for all $t \in \mathbb{N}$ and $D \in \mathbb{N}_0$.

On the other hand, suppose there exists some function $\alpha : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C}$ satisfying

\[
\alpha(t, D) = \sum_{d \in \mathbb{N}, d | t} d^{k-1} \alpha(1, t^2 D/d^2) \tag{4.33}
\]

for all $t \in \mathbb{N}$ and $D \in \mathbb{N}_0$ with

\[
\alpha_f(T) = \alpha(\varepsilon(T), 3 \det(T)/\varepsilon(T)^2) \tag{4.34}
\]

for all $0 \neq T \in \text{Her}_2^\tau(\mathcal{O})$, $T \geq 0$. Then define

\[
\alpha^* : \mathbb{N}_0 \to \mathbb{C}, \quad \alpha^*(n) := \alpha(1, n). \tag{4.35}
\]
We obtain
\[ \alpha_f(T) = \alpha(\epsilon(T), 3 \det(T) / \epsilon(T)^2) = \sum_{d \in \mathbb{N}, d|\epsilon(T)} d^{k-1} \alpha(1, 3 \epsilon(T)^2 \det(T) / (\epsilon(T)^2 d^2)) \]
\[ = \sum_{d \in \mathbb{N}, d|\epsilon(T)} d^{k-1} \alpha^*(3 \det(T) / d^2), \]
and thus \( f \in \mathcal{M}(k; \mathcal{O}) \). Hence, we have verified that \( f \in \mathcal{M}(k; \mathcal{O}) \) holds if and only if there exists a function \( \alpha : \mathbb{N} \times \mathbb{N}_0 \rightarrow \mathbb{C} \) satisfying 4.30 and 4.31. So all that remains to be proven is that 4.30 is equivalent to 4.27. But actually, this is true due to [Ad79, le.2] and (11) in [Ad79, p.270]. (Note that the proofs concerning this equivalence do not depend on \( D \) fulfilling any conditions.)

In fact, (4.14) and (4.15) are the most important propositions of this section concerning the Fourier-expansions of the quaternionic Eisenstein-series \( E_k \). Ultimately, they will be the key to determine the Fourier-expansions, later on, since we will see that \( E_k = \Omega^{-1}(\frac{2k}{\prod_{j \neq k} \Lambda(G_k-2)}) \) holds for all even \( k \geq 6 \). Most of the rest of this section can be seen as a description of \( \Gamma_0[3], k, 1 \)\_tr to better understand the Maass lift.

But before we can prove this assertion, there is still a lot of pre-work to do. Mainly, we will have to deal with Hecke-operators for \( \text{Sp}_2(\mathcal{O}) \). So first, we will have to introduce them. Later, we will see that \( \mathcal{M}(k; \mathcal{O}) \) is invariant under certain Hecke-operators. (Indeed, one could show that it is invariant under the whole Hecke-algebra, but we will not prove this here, because this would need a lot of further background information – but actually, we will already give the main keys here so that working out this problem should be quite straightforward.) Even more, we will show that transforming a Maass lift \( f \) with certain Hecke-operators does indeed mean transforming \( \Omega(f) \) with certain other Hecke-operators. And therefore, if \( \Omega(f) \) is an eigenform for these operators, then so is \( f \) for the original ones. To prove this, (4.17) actually is the crucial key.

### 4.3 Number theoretical background of \( \mathcal{O} \)

Before we can actually talk about Hecke-operators, we need some number theoretical background concerning \( \mathcal{O} \). Most of the work in this section is somehow analogous to [Hu19] and [Kr87].

But before we start, let us give a full proof of (1.7)h) now, like we announced in chapter 1. Note that in principal, the proof is completely analogous to the one concerning the Hurwitz order in [Hu19]. But since it is so important for the current chapter, we should not omit a complete proof. And moreover, we will keep the proof quite basic, since we already stated this proposition at the beginning of chapter 1.

**Proposition 4.18.** For \( 0 \neq a \in \mathcal{O} \) with \( N(a) = \prod_{j=1}^n p_j \) where the \( p_j \) are primes, there exist \( a_j \in \mathcal{O} \) with \( N(a_j) = p_j \) and \( a = a_1 \cdot \ldots \cdot a_n \).

**Proof:** According to (1.7), for every prime number \( p \) there actually exists \( a \in \mathcal{O} \) such that \( N(a) = p \). So suppose that \( p \) is a prime divisor of \( N(a) \), but that there exists no \( q \in \mathcal{O} \) with \( N(q) = p \) such that \( a \in q\mathcal{O} \) (or even \( a \in p\mathcal{O} = (q\mathcal{O}) \mathcal{O} \subset q\mathcal{O} \)). Note that, since \( N(p) = p^2 \), every
left-divisor \( d \) of \( p \) fulfills \( d \in \mathcal{E}, N(d) = p \) or \( d \in p\mathcal{E} \). Hence it follows that \( a \) and \( p \) have to be relatively left-prime. Let \( I \) denote the right-sided ideal generated by \( p \) and \( a \), i.e.

\[
I = \{ pu + av ; u, v \in \mathcal{O} \} .
\]

Completely analogous to what we did in the proof of (1.9), \( I \) is a right-sided ideal, indeed, and \( I = c\mathcal{O} \), where \( 0 \neq c \in I \) is chosen such that \( N(c) \) is minimal. Thus, every \( d \in I \) fulfills \( N(d) \in N(c)N_0 \). But since \( p \in I \), \( N(c) \in \{1, p, p^2\} \) follows. If \( N(c) = p \), then \( a \in c\mathcal{O} \) is a contradiction. If \( N(c) = p^2 \), then \( p \in c\mathcal{O} \) implies \( c \in p\mathcal{E} \), but then \( a \in c\mathcal{O} = p\mathcal{O} \) is a contradiction, again. Hence \( c \in \mathcal{E} \) and \( I = \mathcal{O} \) follows. So by definition, there exist \( x, y \in \mathcal{O} \) such that

\[
1 = px + ay ,
\]

or in other words \( ay = 1 - px \). (Note that this already follows form \( a \) and \( p \) being relatively left-prime, but we wanted to keep it quite basic, here.) This gives

\[
N(ay) = N(1 - px) = 1 + p^2 N(x) - px - p\bar{x} = 1 + p^2 N(x) - 2p \text{Re}(x)
\]

But note that this is a contradiction. According to the assumption, \( N(a) \in p\mathcal{N} \) holds, and thus also \( N(ay) \in p\mathcal{N}_0 \), but \( 1 + p^2 N(x) - 2p \text{Re}(x) \in 1 + p\mathbb{Z} \). So there exist \( q \in \mathcal{O} \) with \( N(q) = p \) and \( b \in \mathcal{O} \) such that \( a = qb \). Note that \( N(b) = p^{-1} N(a) \). Therefore, we can decompose \( a \) inductively like we asserted.

\[\Box\]

Now, we start with defining so-called greatest invariant divisors and completely primitive elements in \( \mathcal{O} \). Recall that in view of (1.7) the so-called set of invariant elements of \( \mathcal{O} \) is given by

\[
\mathcal{I}(\mathcal{O}) = \{ 0 \neq a \in \mathcal{O} ; a\mathcal{O} = \mathcal{O}a \} = \{ ne, ni\sqrt{3} e ; n \in \mathbb{N}, e \in \mathcal{E} \} .
\]

In the light of that, we have the following

(4.19) **Proposition.** Each \( 0 \neq a \in \mathcal{O} \) possesses a unique representation

\[
a = (i_1 \sqrt{3})^j nb ,
\]

where \( j \in \mathbb{N}_0, n \in \mathbb{N} \) with \( 3 \mid n \), and \( b \in \mathcal{O} \) with \( 3 \mid N(b) \) and \( m^{-1}b \notin \mathcal{O} \) for all \( m \in \mathbb{N}, m \geq 2 \).

**Proof:** Suppose there is \( c \in \mathcal{O} \) with \( 3 \mid N(c) \). Then according to (1.9), there exists \( d \in \mathcal{O} \) such that \( c = i_1 \sqrt{3}d \). Therefore, by induction, there exists \( u \in \mathcal{O} \) with \( 3 \mid N(u) \) such that

\[
a = (i_1 \sqrt{3})^j u ,
\]

where \( j \) is the multiplicity of \( 3 \) in the prime factorization of \( N(a) \). Now, simply choose \( n := \max\{m \in \mathbb{N} ; m^{-1}u \in \mathcal{O} \} \) and \( b = n^{-1}x \), and we obtain

\[
a = (i_1 \sqrt{3})^j nb .
\]

Now about the uniqueness: Obviously, \( j \) is already uniquely determined by the norm of \( a \)
due to the assertion and the construction from above. So suppose we have $a = (i_1 \sqrt{3}) n_1 b_1 = (i_1 \sqrt{3}) n_2 b_2$, with $n_1, b_1$ and $n_2, b_2$ fulfilling the requirements. Hence $n_1 b_1 = n_2 b_2$. Let $c = c_0 + c_1 \frac{1+i \sqrt{3}}{2} + c_2 i_2 + c_3 \frac{1+i \sqrt{3}}{2} i_2 \in \mathcal{O}$, then for $m \in \mathbb{N}$ we obviously have the equivalence: $m^{-1} c \in \mathcal{O}$ if and only if $c_0, \ldots, c_3 \in m \mathbb{Z}$. Thus a simple algebraical fact is: If $m_1^{-1} c \in \mathcal{O}$ and $m_2^{-1} c \in \mathcal{O}$ for $m_1, m_2 \in \mathbb{Z}$, then also $m^{-1} c \in \mathcal{O}$ holds, where $m$ is the least common multiple of $m_1$ and $m_2$. The left-hand side of $n_1 b_1 = n_2 b_2$ is divisible by $n_1$, the right-hand side by $n_2$, and thus both sides are divisible by $n$, where $n$ is the least common multiple of $n_1$ and $n_2$, and thus $(n/n_1)^{-1} b_1 \in \mathcal{O}$ as well as $(n/n_2)^{-1} b_2 \in \mathcal{O}$. This implies $n_1 = n = n_2$ because of the requirement $b_1$ and $b_2$ have to fulfill. Of course, this immediately leads to $b_1 = b_2$, too. \(\square\)

The preceding proposition leads to the definition of so-called completely primitive elements in $\mathcal{O}$.

**Definition.**

a) Given $m \in \mathbb{N}$ set

$$N(m) := \{a \in \mathcal{O}; N(a) = m\}.$$ 

b) Let $0 \neq a \in \mathcal{O}$ with the unique decomposition $a = (i_1 \sqrt{3}) n b$ given by (4.19). We define

$$\text{inv}(a) := (i_1 \sqrt{3})^{n}$$

as the greatest invariant divisor of $a$.

c) An element $0 \neq a \in \mathcal{O}$ is said to be completely primitive if

$$\text{inv}(a^m) = 1$$

holds for all $m \in \mathbb{N}$.

d) Given $a, b \in \mathcal{O} \setminus \{0\}$ we say that $a$ is a total divisor of $b$ and write $a || b$ if

$$a | \text{inv}(b)$$

holds, i.e. $\text{inv}(b) \in a \mathcal{O}$ and $\text{inv}(b) \in \mathcal{O}a$.

The next proposition gives a characterization of completely primitive elements.

**Proposition.**

a) An element $0 \neq a \in \mathcal{O}$ is completely primitive if and only if $3 \nmid N(a)$ and $\gcd(N(a), 2 \text{Re}(a)) = 1$.

b) Let $p \neq 3$ be a prime number, then $N(p)$ contains a completely primitive element.

c) Given $a, b \in \mathcal{O} \setminus \{0\}$, then $a || b$ holds if and only if there exists $c \in \mathcal{I}(\mathcal{O})$ such that $a$ is a left divisor of $c$ and $c$ is a left divisor of $b$, i.e. $c \in a \mathcal{O}$, $b \in c \mathcal{O}$.

Note that by definition $N(3n)$ ($n \in \mathbb{N}$) does not contain completely primitive elements.
4.3 Number theoretical background of $\mathcal{O}$

**Proof:** a) Note that we have the following identity:

$$a^2 = 2\text{Re}(a)a - N(a),$$

because $\overline{a} = -a + 2\text{Re}(a)$ and $a\overline{a} = N(a)$. So suppose $a$ is completely primitive. Then by definition and the construction of $	ext{inv}(a)$ in view of (4.19), $3 \nmid N(a)$ holds. Furthermore, setting $n = \gcd(N(a), 2\text{Re}(a))$ we have

$$n^{-1}a^2 = (n^{-1}2\text{Re}(a))a - n^{-1}N(a) \in \mathcal{O},$$

and thus $n = 1$ by definition. On the other hand, suppose $3 \nmid N(a)$ and $\gcd(N(a), 2\text{Re}(a)) = 1$ hold. Of course, this implies $3 \nmid N(a^m)$ for all $m \in \mathbb{N}$. So, according to the proof of (4.19),

$$\text{inv}(a^m) = n_m$$

holds for all $m \in \mathbb{N}$ for some appropriate $n_m \in \mathbb{N}$. Note that the definition implies

$$n_m^{-1}a^m \in \mathcal{O}.$$ 

Let $j$ be the smallest integer such that $n_j \neq 1$. Then $j > 1$ holds: Suppose $n_1 > 1$, then $a = n_1b$ for some $b \in \mathcal{O}$. Obviously, this leads to $\text{Re}(a) = n_1\text{Re}(b) \in n_1\mathbb{Z}$ and $N(a) = n_1^2N(b) \in n_1\mathbb{N}$. But then $n_1 > 1$ contradicts $\gcd(N(a), 2\text{Re}(a)) = 1$, and thus $j > 1$. Let $p$ be some prime-divisor of $n_j$. Again, by definition we have $a^j = pb$ for some $b \in \mathcal{O}$, and thus $N(a^j) = p^2N(b) \in p\mathbb{N}$, which implies $N(a) \in p\mathbb{N}$, since $p$ is a prime number. According to the identity above, we have

$$a^j = 2\text{Re}(a)a^{j-1} - N(a)a^{j-2} \iff 2\text{Re}(a)a^{j-1} = a^j + N(a)a^{j-2}$$

where $a^0 := 1$. This leads to

$$p^{-1}(2\text{Re}(a)a^{j-1}) \in \mathcal{O},$$

because $p^{-1}a^j \in \mathcal{O}$ and as well $p^{-1}N(a) \in \mathbb{N}$. So let $a^{j-1} = a_0 + a_1\frac{1+i\sqrt{3}}{2} + a_2i_2 + a_3\frac{1+i\sqrt{3}}{2}i_2$. Thus, by definition, $a_1 \notin p\mathbb{Z}$ has to hold for some $l \in \{0, \ldots, 3\}$, since $\text{inv}(a^{j-1}) = 1$. Furthermore, $\gcd(N(a), 2\text{Re}(a)) = 1$ yields $2\text{Re}(a) \notin p\mathbb{Z}$, and thus $2\text{Re}(a)a_1 \notin p\mathbb{Z}$. This contradicts $(2\text{Re}(a)a^{j-1}) \in p\mathcal{O}$, which completes the proof.

b) According to a) we have to find $a \in \mathcal{N}(p)$ such that $\gcd(p, 2\text{Re}(a)) = 1$. So assume every $b \in \mathcal{N}(p)$ fulfills $\gcd(p, 2\text{Re}(b)) > 1$, which is equivalent to $2\text{Re}(b) \in p\mathbb{Z}$, since $p$ is a prime number. Let $a = a_0 + a_1\frac{1+i\sqrt{3}}{2} + a_2i_2 + a_3\frac{1+i\sqrt{3}}{2}i_2 \in \mathcal{N}(p)$. According to the assumption, $2\text{Re}(a) = 2a_0 + a_1 \in p\mathbb{Z}$ holds. Note that we have $ae \in \mathcal{N}(p)$ for all $e \in \mathcal{E}$, and thus $2\text{Re}(ae) \in p\mathbb{Z}$. A straightforward calculation shows that choosing $e = \frac{1-i\sqrt{3}}{2}$, $e = -i_2$ and $e = -i_2\frac{1-i\sqrt{3}}{2}$ then yields $2a_1 + a_0, 2a_2 + a_3, 2a_3 + a_2 \in p\mathbb{Z}$, too. But this gives

$$(2a_0 + a_1) - 2(2a_1 + a_0) = -3a_1 \in p\mathbb{Z}, \quad (2a_1 + a_0) - 2(2a_0 + a_1) = -3a_0 \in p\mathbb{Z},$$

and thus $a_0, a_1 \in p\mathbb{Z}$, since $p \neq 3$ is a prime. Of course, the same calculation yields $a_2, a_3 \in p\mathbb{Z}$, too. In other words, we have $a \in p\mathcal{O}$, but this contradicts $a \in \mathcal{N}(p)$, because
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$a \in p\mathcal{O}$ implies $N(a) \in p^{2}\mathbb{N}$. Hence the assertion follows.

c) First, note that in view of (1.7), $\text{inv}(r) \in \mathcal{I}(\mathcal{O})$ holds for every $r \in \mathcal{O}$, since $(i_1 \sqrt{3})^2 = -3$. So suppose $a||b$, which means $a|\text{inv}(b)$. By definition, there exists $d \in \mathcal{O}$ such that $b = \text{inv}(b)d$. So define $c = \text{inv}(b)$. Then obviously we have $c \in \mathcal{I}(\mathcal{O})$, $c \in a\mathcal{O}$ (since $a|c$) and $c \in b\mathcal{O}$.

On the other hand, suppose there is $c \in \mathcal{I}(\mathcal{O})$ with $c \in a\mathcal{O}$ and $b \in c\mathcal{O}$. Then according to (1.7) (where one has to note again that $(i_1 \sqrt{3})^2 = -3$) there exist $\varepsilon \in E$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $3 \nmid m$ such that $c = (i_1 \sqrt{3})^m \varepsilon$. By definition, there exists $d$ such that $b = cd = (i_1 \sqrt{3})^m \varepsilon d$. But then, due to the construction of $\text{inv}(-)$, $\text{inv}(b) = (i_1 \sqrt{3})^m \text{inv}(d)$ holds. Of course, this implies $\text{inv}(b) = c\varepsilon \text{inv}(d) \in a\mathcal{O}$, since $c \in a\mathcal{O}$. Note that $\text{inv}(b) \in \mathcal{I}(\mathcal{O})$ implies that $\text{inv}(b)\text{inv}(b)^{-1} \in \mathcal{O}$ holds for all $r \in \mathcal{O}$. By definition, there exists $r \in \mathcal{O}$ such that $\text{inv}(b) = ar$. There is $\tilde{r} \in \mathcal{O}$ such that $\text{inv}(b)\text{inv}(b)^{-1} = \tilde{r}$, or equivalently $r \text{inv}(b)^{-1} = \text{inv}(b)^{-1}\tilde{r}$. Since $a \neq 0$ and $c \neq 0$, we have

$$\text{inv}(b) = ar \iff a^{-1} = r\text{inv}(b)^{-1} = \text{inv}(b)^{-1}\tilde{r} \iff \text{inv}(b) = \tilde{r}a,$$

and thus $\text{inv}(b) \in Oa$, which completes the proof.

Note that in the proof of c) we have seen that the condition $c \in a\mathcal{O}$ and $b \in c\mathcal{O}$ is equivalent to $a|c$ and $c|b$, if $c \in \mathcal{I}(\mathcal{O})$.

The next theorem is an analog of [Kr85, ch.I, thm.2.3]. But note that we omit the proof, here. Although the theorem was proved for the Hurwitz order only, the proof for the order we are considering would be literally the same. Indeed, the only needed properties are the following:

The Hurwitz order is euclidean (and so is $\mathcal{O}$), and every two-sided ideal is generated by an invariant element, which is the same for $\mathcal{O}$ (see (1.13), whereas this proposition is due to $\mathcal{O}$ being euclidean, too). And because of (4.21), our definition of total divisors coincides with the one from [Kr85]. Therefore, we can simply cite this theorem as

(4.22) Theorem. Elementary divisor theorem

Suppose that $0 \neq A \in \mathcal{O}^{m\times n}$ and let $q > 0$ denote the rank of $A$. There exist $U \in \text{GL}_m(\mathcal{O})$, $V \in \text{GL}_n(\mathcal{O})$ and a digonal matrix $D = \text{diag}(d_1, \ldots, d_q) \in \mathcal{O}^{n\times q}$ such that $d_1||d_2||\ldots||d_q$ and

$$UAV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Just like in [Kr873], one could even formulate a stronger version of the preceding theorem: Given diagonal matrices $D = \text{diag}(d_1, \ldots, d_q)$, $\tilde{D} \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_q) = \in \mathcal{O}^{n\times q}$ of full rank, such that $d_1||d_2||\ldots||d_q$ and $\tilde{d}_1||\tilde{d}_2||\ldots||\tilde{d}_q$, suppose $N(d_j) = N(\tilde{d}_j)$ as well as $\text{inv}(d_j) = \text{inv}(\tilde{d}_j)$ hold for all $j \in \{1, \ldots, q\}$, then there exist $U, V \in \text{GL}_q(\mathcal{O})$ with $ADV = \tilde{D}$. But to prove this theorem, one would have to adapt [Kr873] to our order $\mathcal{O}$, completely. Again, this seems unnecessary, since we will not need this theorem in its entire form. But note that it could be adapted to our case, indeed. The proofs would be quite straightforward (given their originals), but working out the whole theory would result in a lot of work, so that it would go beyond the scope of what we want to do here.

Indeed, we will only need a very special case of this theorem, namely $q = 2$, $d_1 = \tilde{d}_1 = 1$ and...
would choose \( m \) automatically holds, because \( 3 \nmid N(d_2) = p \) implies \( \text{inv}(d_2) = m \in \mathbb{N} \), but for \( m > 1 \) we would have the contradiction \( p = N(d_2) \in m \mathbb{N} \).

We begin by showing that for a prime number \( p \neq 3 \) we have \( O/pO \simeq (\mathbb{F}_p)^{2 \times 2} \), where \( \mathbb{F}_p \) denotes the field with \( p \) elements (hence \( \mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z} \)). The basic idea to prove this is due to [Hu19], where the Hurwitz order is considered, but note that the details differ. Furthermore, this fact is crucial to prove the aforementioned theorem. And note that an analog holds if we would choose \( m \in \mathbb{N} \) with \( 3 \nmid m \). In this case we would get \( O/mO \simeq (\mathbb{Z}/m\mathbb{Z})^{2 \times 2} \). This also would be the main idea behind adapting [Kr87] to \( O \). But again, \( m \) being a prime number suffices for our considerations, whereas the general case could be proven in an analogous way.

First, we need the following

(4.23) Lemma. Given a prime number \( p \neq 3 \), there exist \( r, s \in \mathbb{Z} \) such that

\[
1 + r^2 + s^2 - rs \equiv 0 \mod p
\]

holds.

Note that one easily verifies that the assertion does not hold true for \( p = 3 \).

Proof: For simplicity, given \( m, n \in \mathbb{Z} \) with \( mn \equiv 1 \mod p \) (which implies \( \gcd(m, p) = \gcd(n, p) = 1 \)), write \( m^{-1} = n \) (as long as it is clear that we are considering equations \( \mod p \)).

The case \( p = 2 \) is trivial, since one simply chooses \( r = 1 \) and \( s = 0 \). So let \( p \) be a prime number with \( p \geq 5 \). Suppose we already found a solution for the equation

\[
x^2 + 4^{-1}3s^2 + 1 \equiv 0 \mod p.
\]

Then we obtain

\[
(r - 2^{-1}s - x) \cdot (r - 2^{-1}s + x) \equiv r^2 + 4^{-1}s^2 - x^2 - rs
\]

\[
\equiv r^2 + 4^{-1}s^2 + (4^{-1}3s^2 + 1) - rs \equiv 1 + r^2 + s^2 - rs
\]

mod \( p \). Hence choosing \( r \in \mathbb{Z} \) such that \( r \equiv 2^{-1}s + x \) or \( r \equiv 2^{-1}s - x \mod p \) yields the assertion. So let us find a solution \( (x, s) \). But finding such a solution is completely analogous to [Hu19]: The \( \frac{p-1}{2} + 1 \) elements in \( \{1 + x^2; x \in \{0, 1, \ldots, \frac{p-1}{2}\}\} \) are pairwise incongruent mod \( p \): Given \( x_1 \in \{0, \ldots, p - 1\} \), \( x_1^2 \equiv x_1 \mod p \) possesses at most 2 solutions in \( \{0, \ldots, p - 1\} \) (which is a simple fact from linear algebra). Thus, the solutions in \( \{0, \ldots, p - 1\} \) of \( 1 + x_1^2 \equiv 1 + x_2^2 \), where \( x_1 \in \{0, \ldots, \frac{p-1}{2}\} \), are exactly given by \( x_1 \) and \( -x_1 + p \notin \{0, \ldots, \frac{p-1}{2}\} \) (if \( x_1 \neq 0 \)). Hence the elements of this set are pairwise incongruent mod \( p \). The same holds for \( \{-3ns^2; s \in \{0, \ldots, \frac{p-1}{2}\}\} \) (where \( n \) is chosen such that \( 4n \equiv 1 \mod p \)), because \( -4^{-1}3s_1^2 \equiv -4^{-1}3s_2^2 \mod p \) is equivalent to \( s_1^2 \equiv s_2^2 \mod p \), since \( p \neq 2 \) and \( p \neq 3 \). The union of these two sets contains \( p + 1 \) elements (given there exist no \( x, s \) such that even \( x^2 + 3ns^2 + 1 = 0 \) holds), and thus they cannot be pairwise incongruent mod \( p \). This completes the proof. \( \square \)

Now we get to the aforementioned isomorphism \( O/pO \simeq (\mathbb{F}_p)^{2 \times 2} \):
(4.24) Proposition. Let \( p \neq 3 \) be a prime number. In view of (4.23), choose \( r, s \in \mathbb{Z} \) such that \( 1 + r^2 + s^2 - rs \equiv 0 \mod p \). Then \( \psi_p : \mathcal{O} / p\mathcal{O} \to (\mathcal{Z} / p\mathcal{Z})^{2 \times 2} \simeq (\mathbb{F}_p)^{2 \times 2} \) defined by

\[
\psi_p(a + p\mathcal{O}) := \begin{pmatrix}
a_0 + a_2r + a_3s & -a_1 + (a_2 + a_3)r - a_2s \\
a_1 + a_3r - (a_2 + a_3)s & a_0 + a_1 - a_2r - a_3s
\end{pmatrix} + p\mathbb{Z}^{2 \times 2},
\]

where \( a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \in \mathcal{O} \), is a ring isomorphism. Moreover, defining

\[
A^* := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

to be the adjoint matrix of \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2} \) (thus \( A^{-1} = \det(A)^{-1}A^* \) for \( A \in \text{GL}_2(\mathbb{R}) \)) as well as

\[
g(A) := \gcd(a, b, c, d)
\]

one has for \( 0 \neq a \in \mathcal{O} \):

- \( \psi_p(\overline{a}) \equiv \psi_p(a)^* \mod p \),
- \( 2 \text{Re}(a) \equiv \text{tr}(\psi_p(a)) \mod p \),
- \( N(a) \equiv \det(\psi_p(a)) \mod p \),
- \( \gcd(\text{inv}(a) \cdot p) = \gcd(g(\psi_p(a)), p) \).

Proof: Note that \( \psi_p \) is well-defined, because given \( a, b \in \mathcal{O} \), then obviously \( a + p\mathcal{O} = b + p\mathcal{O} \) holds if and only if the coefficients of \( a \) and \( b \) in the standard basis of \( \mathcal{O} \) are congruent modulo \( p \). For simplicity, we will omit the “\( +p\mathcal{O} \)” and simply write \( \psi_p(a) := \psi_p(a + \mathcal{O}) \) for \( a \in \mathcal{O} \), and we consider \( \psi_p(a) \in \mathbb{Z}^{2 \times 2} \) (instead of \( \psi_p(a) \in (\mathcal{Z} / p\mathcal{Z})^{2 \times 2} \)), but remember that we can compute modulo \( p \).

By definition we have \( \psi_p((ma + b)) \equiv m\psi_p(a) + \psi_p(b) \mod p \) for all \( m \in \mathbb{Z} \) and \( a, b \in \mathcal{O} \) as well as \( \psi_p(c) \equiv 0 \mod p \) for all \( c \in \mathcal{O} \), of course. Hence \( \psi_p \) is already a \( \mathbb{Z} \)-module homomorphism. (But of course we even want to show that it is a ring homomorphism.)

We will now show that \( \psi_p \) is surjective. Since \( \psi_p \) is a \( \mathbb{Z} \)-module homomorphism, it suffices to prove that \( (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) + p\mathbb{Z}^{2 \times 2}, (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) + p\mathbb{Z}^{2 \times 2}, (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) + p\mathbb{Z}^{2 \times 2} \) and \( (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) + p\mathbb{Z}^{2 \times 2} \) belong to the image of \( \psi_p \). Like said before, we will omit the “\( +p\mathbb{Z}^{2 \times 2} \)” for the sake of simplicity. Choosing \( a \in \{ 1, \frac{1+i\sqrt{3}}{2}, i_2, \frac{1+i\sqrt{3}}{2} i_2 \} \), the following matrices already belong to the image:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} r & -s \\ s & r \end{pmatrix}, \quad \begin{pmatrix} r & s \\ s & r \end{pmatrix}.
\]

Again, since \( \psi_p \) is a \( \mathbb{Z} \)-module homomorphism, we can add multiples of these matrices to each other, and thus the following matrices belong to the image, too:

\[
\begin{pmatrix} r-s & 0 \\ 0 & r \end{pmatrix}, \quad \begin{pmatrix} r & -s \\ 1 & 1 \end{pmatrix} - r\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} r & -s \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r-s & r \\ s & r \end{pmatrix} = \begin{pmatrix} r & s \\ 2r-s & 2r \end{pmatrix},
\]

\[
\begin{pmatrix} r & s \\ 2r-s & 2r \end{pmatrix} - \begin{pmatrix} r-s & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} r & s \\ 3r-3s & 3s \end{pmatrix}.
\]
4.3 Number theoretical background of \( \mathcal{O} \)

If \( s \in p\mathbb{Z} \), then \( \gcd(r, p) \) and \( \gcd(3r, p) = 1 \) have to hold, since \( p \neq 3 \) and \( 1 + r^2 + s^2 - rs \equiv 0 \) has to hold. So in this case the last matrix from above yields that \( \left( \begin{smallmatrix} s & s \end{smallmatrix} \right) \equiv \left( \begin{smallmatrix} 0 & 0 \end{smallmatrix} \right) \mod p \). Then we obtain

\[
\begin{pmatrix}
0 & 0 \\
3r - 3s & -3s
\end{pmatrix}
\]

belongs to the image, as well as

\[
(r \, s)(\begin{pmatrix} 0 & 0 \\ 3nr + 1 & 0 \end{pmatrix}) + (0 \, 0) \equiv (r - 2s + (r + s)(3nr + 1) 0) .
\]

We have

\[
r - 2s + (r + s)(3nr + 1) \equiv r - 2s + 3nr^2 + r - r + s \equiv 3n(-rs + s^2 + r^2) \equiv -3n ,
\]

and so again \( \left( \begin{smallmatrix} 0 & 0 \end{smallmatrix} \right) \) belongs to the image, since \( \gcd(-3n, p) = 1 \). The rest follows like in the case of \( s \in p\mathbb{Z} \). Thus, \( \psi_p \) is surjective. Obviously, we have \( \#\mathcal{O}/p\mathcal{O} = \#(\mathbb{Z}/p\mathbb{Z})^{2 \times 2} = p^4 \), because we already saw that \( a + p\mathcal{O} = b + p\mathcal{O} \) if and only if the coefficients of \( a \) and \( b \) with respect to the standard basis of \( \mathcal{O} \) are congruent mod \( p \). Therefore, \( \psi_p \) has to be injective, too, and turns out to be a \( \mathbb{Z} \)-module isomorphism.

And finally, let us prove that \( \psi_p \) is a ring homomorphism, indeed. Let \( f_0 = 1, f_1 = \frac{1+i\sqrt{3}}{2} \), \( f_2 = i_2 \) and \( f_3 = \frac{1+i\sqrt{3}}{2}i_2 \). Furthermore, let us assume that we already verified \( \psi_p(f_jf_k) \equiv \psi_p(f_j)\psi_p(f_k) \mod p \) for all \( j, k \in \{0, \ldots, 3\} \), and suppose \( a = \sum_{j=0}^{3} a_jf_j \) and \( b = \sum_{j=0}^{3} b_jf_j \) are given, then we obtain

\[
\psi_p(ab) \equiv \psi_p \left( \sum_{j=0}^{3} a_jb_jf_jf_k \right) \equiv \sum_{j=0}^{3} a_jb_j\psi_p(f_jf_k) \\
\equiv \sum_{j=0}^{3} a_j\psi_p(f_j) \sum_{k=0}^{3} b_k\psi_p(f_k) \equiv \psi_p \left( \sum_{j=0}^{3} a_jf_j \right) \psi_p \left( \sum_{k=0}^{3} b_kf_k \right) \equiv \psi_p(a)\psi_p(b)
\]

mod \( p \), since we already know that \( \psi_p \) is a \( \mathbb{Z} \)-module homomorphism. Hence the multiplicativity follows.

So we simply have to prove \( \psi_p(f_jf_k) \equiv \psi_p(f_j)\psi_p(f_k) \mod p \) for all \( j, k \in \{0, \ldots, 3\} \), where the assumption for \( j = 0 \) or \( k = 0 \) is trivial with regard to \( \psi_p(f_0) = 1 \). Hence there remain nine equalities to verify. But this is just a simple recalculation. For \( j = 1 \) or \( k = 1 \) one even calculates \( \psi_p(f_jf_k) = \psi_p(f_j)\psi_p(f_k) \), while for \( j, k \in \{2, 3\} \) one has to make use of the precondition \( 1 + r^2 + s^2 - rs \equiv 0 \mod p \), but nevertheless it is a straightforward calculation. So the details can be omitted here, and \( \psi_p \) turns out to be a ring isomorphism.

So let us verify the asserted properties of \( \psi_p \). Given \( a = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2i_2 + a_3 \frac{1+i\sqrt{3}}{2}i_2 \in \mathcal{O} \), one computes \( \bar{a} = (a_0 + a_1) - a_1 \frac{1+i\sqrt{3}}{2} - a_2i_2 - a_3 \frac{1+i\sqrt{3}}{2}i_2 \in \mathcal{O} \), and thus

\[
\psi_p(\bar{a}) = \left(\begin{array}{ccc}
0 & a_1 - a_2 & -a_3 \\
-a_1 + a_2 & 0 & a_3 \\
-a_1 + a_2 & 0 & 0
\end{array}\right) = \psi_p(a)'.
\]
Furthermore, we have
\[
\text{tr}(\psi_p(a)) = a_0 + a_2 + a_3s + a_0 + a_1 - a_2r - a_3s = 2a_0 + a_1 = 2\Re(a)
\]
and
\[
\det(\psi_p(a)) \\
= (a_0 + a_2 + a_3s)(a_0 + a_1 - a_2r - a_3s) - (-a_1 + (a_2 + a_3)r - a_2s)(a_1 + a_3r - (a_2 + a_3)s) \\
= a_0^2 + a_1^2 + (-r^2 + rs - s^2)a_2^2 + (-s^2 - r^2 + rs)a_3^2 + a_0a_1 + (-r + s)a_0a_3 \\
+ (r - s + s)a_1a_2 + (s + r - s - r)a_1a_3 + (-rs - rs + rs + s^2)a_2a_3 \\
\equiv a_0^2 + a_0a_1 + a_1^2 + a_2^2 + a_3a_2 + s^2 = N(a)
\]
mod \( p \). The last property is quite trivial, since we are only considering prime numbers \( p \) (and not the general case \( m \in \mathbb{N}, 3 \nmid m \)): Let \( \text{inv}(a) = (i_1\sqrt{3})/m \), where \( j \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \). If \( p \mid m \), then we have \( a \in p\mathcal{O} \) by definition. But then we have \( (\psi_p(a) \equiv 0 \mod p) \), and thus \( \gcd(\psi_p(a)), p) = p = \gcd(\text{inv}(a), p) \). On the other hand, assume \( p \nmid m \). This gives \( a \notin p\mathcal{O} \) due to the definition of \( \text{inv}(\cdot) \). But then \( \psi_p(a) \not\equiv 0 \mod p \), since \( \psi_p \) is injective. So in particular \( \gcd(\psi_p(a)), p) = 1 \) has to hold. Furthermore, we have \( N(\text{inv}(a)) = 3/m^2 \notin p\mathbb{N} \), since \( p \neq 3 \). But every divisor of \( p \) that is not a unit is an element of \( \mathcal{N}(p) \) or \( \mathcal{N}(p^2) \), of course. Thus we have \( \gcd(\text{inv}(a), p) = 1 \), too.

To prove the announced proposition from above, we need the following lemma, which is a consequence of the preceding proposition.

(4.25) Lemma. Suppose \( p \neq 3 \) is a prime number and let \( a, b \in \mathcal{N}(p) \). Then there exist \( x, y \in \mathcal{O} \) such that
\[
\gcd(N(x), p) = 1 \quad \text{and} \quad ax = yb.
\]

Proof: First, let us prove that, given \( A, B \in \mathbb{F}_p^{2 \times 2} \), where both \( A \) and \( B \) have rank 1, we can find \( X, Z \in (\mathbb{Z}/p\mathbb{Z})^{2 \times 2} \) such that
\[
X \in \text{GL}_2(\mathbb{F}_p) \quad \text{and} \quad AX = ZB
\]
holds. Of course, given \( U_1, U_2, U_3, U_4 \in \text{GL}_2(\mathbb{F}_p) \), then \( AX = ZB \) holds if and only if
\[
(U_1AU_2)(U_2^{-1}XU_4) = (U_1ZU_3^{-1})(U_3BU_4)
\]
holds, and \( X \in \text{GL}_2(\mathbb{F}_p) \) if and only if \( U_2^{-1}XU_4 \in \text{GL}_2(\mathbb{F}_p) \). It is a simple fact from linear algebra that we can choose \( U_1, U_2, U_3, U_4 \in \text{GL}_2(\mathbb{F}_p) \) such that
\[
U_1AU_2 = U_3BU_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
because both have rank 1. Now, choose \( X = U_2U_4^{-1} \in \text{GL}_2(\mathbb{F}_p) \) and \( Z = U_1^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U_3 \), and we obtain
\[
(U_1AU_2)(U_2^{-1}XU_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (U_1ZU_3^{-1})(U_3BU_4),
\]
and thus the assertion holds true. Now, define \( A = \psi_p(a) \) and \( B = \psi_p(b) \), and choose \( X, Z \) like above (noting that both \( A \) and \( B \) have rank one in view of (4.24)). According to (4.24), \( \psi_p \) is surjective, and thus there exist \( x, z \in \mathcal{O} \) such that \( \psi_p(x) = X \) and \( \psi_p(z) = Z \). And because \( \psi_p \) is also injective, \( \psi_p(ax) = AX = ZB = \psi_p(zb) \) yields \( ax + p\mathcal{O} = zb + p\mathcal{O} \), which means there exists \( c \in \mathcal{O} \) such that

\[
ax = zb + pc = (z + cb)b
\]

holds. So define \( y = z + cb \). Furthermore, in view of (4.24) \( \gcd(N(x), p) = p \) would imply \( \gcd(\det(\psi_p(x)), p) = p \), which contradicts \( X = \psi_p(x) \in \text{GL}_2(\mathbb{F}_p) \). Therefore, \( \gcd(N(x), p) = 1 \) follows, which completes the proof. \( \square \)

Finally, we get to the aforementioned proposition, which is a very special case of [Kr87, thm.4] (where the Hurwitz order is considered). We will need it in the next chapter to determine some special Hecke-operator.

(4.26) Proposition. Suppose \( p \neq 3 \) is a prime number and let \( a, b \in \mathcal{N}(p) \). Then there exist \( U, V \in \text{GL}_2(\mathcal{O}) \) such that

\[
U \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} V = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.
\]

Proof: In view of (4.25), choose \( x, y \in \mathcal{O} \) such that \( \gcd(N(x), p) = 1 \) and \( ax = yb \) hold. Because \( \gcd(N(x), p) = 1 \), we can choose \( \alpha, \beta \in \mathbb{Z} \) such that \( \alpha N(x) - \beta p = 1 \). Define

\[
V = \begin{pmatrix} ax & \beta p \\ 1 & \overline{x} \end{pmatrix}.
\]

Then

\[
\begin{pmatrix} ax & \beta p \\ 1 & \overline{x} \end{pmatrix} \begin{pmatrix} \overline{x} & -\beta p \\ -1 & ax \end{pmatrix} = \begin{pmatrix} ax\overline{x} - \beta p & -ax\beta p + \beta pax \\ \overline{x} - \overline{x} & -\beta p + \overline{x}ax \end{pmatrix} = I
\]

yields \( V \in \text{GL}_2(\mathcal{O}) \). Furthermore, define

\[
U = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} V^{-1} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}^{-1}.
\]

In view of (1.34) we have

\[
\det(\hat{U}) = N(a) \cdot 1 \cdot N(b)^{-1} = p \cdot p^{-1} = 1.
\]

Moreover, we compute

\[
U = \begin{pmatrix} \overline{x} & -\beta pb^{-1} \\ -a & aaxb^{-1} \end{pmatrix} = \begin{pmatrix} \overline{x} & -\beta b \\ -a & ay \end{pmatrix} \in \mathcal{O}^{2 \times 2},
\]

...
and thus $U \in \text{GL}_2(\mathcal{O})$ due to (1.34). Finally, we obtain

$$U \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} V = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} V^{-1} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} V = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$  

For later purposes, we need some further lemmata:

**4.27 Lemma.** Let $p$ be a prime number and $\pi \in \mathcal{N}(p)$. Then any transversal of $\mathcal{O}/\pi\mathcal{O}$ or $\mathcal{O}/\mathcal{O}\pi$ has length $p^2$.

**Proof:** The assertion already holds for $p = 3$ in view of (1.10), since we have $\mathcal{N}(3) = E_1\sqrt{3} = i_1\sqrt{3}$. So let us assume $p \neq 3$. We will now reduce this issue to $\mathcal{O}/p\mathcal{O}$: Suppose we have $a, b \in \mathcal{O}$ with $a + \pi\mathcal{O} = b + \pi\mathcal{O}$, then there is $c \in \mathcal{O}$ such that $a = b + \pi c$. But then also $\psi_p(a) = \psi_p(b) + \psi_p(\pi)\psi_p(c)$ holds (in $F_p^{2 \times 2}$). On the other hand, suppose we have $c \in \mathcal{O}$ such that $\psi_p(a) = \psi_p(b) + \psi_p(\pi)\psi_p(c)$ holds (in $F_p^{2 \times 2}$). Then in view of (4.24), there exists $d \in \mathcal{O}$ such that $a = b + \pi c + pd = b + \pi(c + \pi d)$. Therefore, we have

$$a + \pi\mathcal{O} = b + \pi\mathcal{O} \iff \psi_p(a) + \psi_p(\pi)F_p^{2 \times 2} = \psi_p(b) + \psi_p(\pi)F_p^{2 \times 2}.$$  

Hence, defining $P = \psi_p(\pi)$, we have

$$\#\mathcal{O}/\pi\mathcal{O} = \#F_p^{2 \times 2}/PF_p^{2 \times 2}.$$  

$\pi \notin p\mathcal{O}$, but $p|\mathcal{N}(\pi)$ implies that $P$ has rank one in $F_p^{2 \times 2}$ in virtue of (4.24). But then it is a simple fact from linear algebra, that $\#PF_p^{2 \times 2} = p^2$, and thus $\#F_p^{2 \times 2}/PF_p^{2 \times 2} = p^2$. The assertion for $\mathcal{O}/\mathcal{O}\pi$ follows analogously. 

**4.28 Lemma.** Let $p \neq 3$ be a prime number and $\pi \in \mathcal{N}(p)$.

a) Suppose $x \notin p\mathcal{O}$, and define $t = \frac{i_1}{\sqrt{3}} x$. Then

$$\varphi : \mathcal{O}/p\mathcal{O} \to \mathbb{Z}/p\mathbb{Z}, \quad a + p\mathcal{O} \mapsto 2 \text{Re}(\overline{t} \bar{a}) + p\mathbb{Z}$$

is a surjective group-homomorphism (with respect to “+”).

b) Suppose $x \in (-i_1\pi i_1)\mathcal{O}$, but $x \notin p\mathcal{O}$, and define $t = \frac{i_1}{\sqrt{3}} x$. Then

$$\varphi : \mathcal{O}/\pi\mathcal{O} \to \mathbb{Z}/p\mathbb{Z}, \quad a + \pi\mathcal{O} \mapsto 2 \text{Re}(\overline{t} \bar{a}) + p\mathbb{Z}$$

is a surjective group-homomorphism (with respect to “+”), too. The analog holds for $x \in \mathcal{O}\pi$ (and still $x \notin p\mathcal{O}$): In this case,

$$\varphi : \mathcal{O}/\mathcal{O}\pi \to \mathbb{Z}/p\mathbb{Z}, \quad a + \mathcal{O}\pi \mapsto 2 \text{Re}(\overline{t} \bar{a}) + p\mathbb{Z}$$

is a surjective group-homomorphism.
4.3 Number theoretical background of $O$

**Proof:** First, note that we have $t \in \frac{i}{\sqrt{3}}O = \frac{1}{2}O^2$ in both cases (see (1.7)). Therefore, we have $2\Re(ta) \in \mathbb{Z}$ for all $a \in O$ by definition. Furthermore, $\Re(\cdot)$ obviously is $\mathbb{R}$-linear, hence it is clear that $\phi$ is a homomorphism in both cases – once we actually verified the well-definedness.

a) Suppose $a, b \in O$ with $a + pO = b + pO$. Hence there exists $c \in O$ such that $b = a + pc$. We calculate

$$2\Re(ib) = 2\Re(i\pi) + p \cdot 2\Re(i\pi) ,$$

where again $2\Re(i\pi) \in \mathbb{Z}$. So we have $2\Re(i\pi) + p\mathbb{Z} = 2\Re(ib) + p\mathbb{Z}$, and thus the well-definedness. Now, since $\mathbb{Z}/p\mathbb{Z}$ is a cyclic group and $\phi$ is a homomorphism, it suffices to find one $a \in O$ such that $2\Re(i\pi) \notin p\mathbb{Z}$ holds to prove the surjectivity. So let $a = -\epsilon_1 \sqrt{3}$, where $\epsilon \in \mathcal{E}$. Due to the calculation rules for $\Re(\cdot)$ (cf. [Kr85, ch.I, prop.1.1]) we have

$$2\Re(i\pi) = 2\Re(-\frac{i}{\sqrt{3}}) = 2\Re(-i\sqrt{3} \frac{i}{\sqrt{3}}x) = 2\Re(e\pi) .$$

But since $p \neq 3$, exactly the same considerations like in the proof of (4.21)b) yield that if $2\Re(e\pi) \in p\mathbb{Z}$ would hold for all $\epsilon \in \mathcal{E}$, then $x \in pO$ would have to follow, which is a contradiction.

b) Again, suppose we have $a, b \in O$ with $a + \pi O = b + \pi O$. Hence there exists $c \in O$ such that $b = a + \pi c$. By definition, there is $y \in O$ such that $t = \frac{1}{\sqrt{3}}(-i\pi i_1)y$. Using the calculation rules again, we get

$$2\Re(it) = 2\Re(i\pi) + 2\Re(\frac{i}{\sqrt{3}}(-i\pi i_1)y\pi\pi) = 2\Re(i\pi) + 2\Re(\frac{1}{\sqrt{3}}\pi\pi i_1 y\pi) = 2\Re(it) + p \cdot 2\Re(\frac{1}{\sqrt{3}}y\pi)$$

where again $2\Re(\frac{1}{\sqrt{3}}y\pi) \in \mathbb{Z}$. So the well-definedness also follows in this case, and like in a) we only have to find one $a \in O$ such that $2\Re(i\pi) \notin p\mathbb{Z}$ holds. But since we have exactly the same situation here ($t = \frac{1}{\sqrt{3}}y, x \notin pO$), this claim holds due to a).

$x \in O\pi$ is handled the same way. Given $a, b, c \in O$ with $b = a + c\pi$ and $y \in O$ such that $t = \frac{i}{\sqrt{3}}y\pi$, we obtain

$$2\Re(it) = 2\Re(i\pi) + 2\Re(\frac{i}{\sqrt{3}}y\pi\pi\pi) = 2\Re(it) + p \cdot 2\Re(\frac{i}{\sqrt{3}}y\pi)$$

again. So the well-definedness and the surjectivity follow accordingly. □

**Lemma.** Let $p \neq 3$ be a prime number, $\pi \in \mathcal{N}(p)$, $a \in O$ and $n \in \mathbb{Z}$ with $p \nmid n$.

a) We have

$$ni_1\sqrt{3}a \in pO \iff a \in pO .$$

b) We also have

$$ni_1\sqrt{3}a \in O\pi \iff a \in O\pi$$

and

$$ni_1\sqrt{3}a \in (-i_1\pi i_1)O \iff a \in \pi O .$$
Proof: a) Let \( a = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2i + a_3 \frac{1+i\sqrt{3}}{2}i_2 \). Of course, \( ni_1 \sqrt{3}a \in p\mathcal{O} \) implies \( 3na \in p(-i\sqrt{3})\mathcal{O} \subset p\mathcal{O} \) (by multiplying with \(-i\sqrt{3}\)), or
\[
\frac{3n}{p}a_0 + \frac{3n}{p}a_1 \frac{1+i\sqrt{3}}{2} + \frac{3n}{p}a_2i + \frac{3n}{p}a_3 \frac{1+i\sqrt{3}}{2}i_2 \in \mathcal{O} ,
\]
which implies \( a_0, \ldots, a_3 \in p\mathbb{Z} \), since \( p \neq 3n \). Thus \( a \in p\mathcal{O} \) follows. On the other hand, \( a \in p\mathcal{O} \) obviously implies \( ni_1 \sqrt{3}a \in p\mathcal{O} \).

b) By simply applying a), we obtain
\[
i_1 \sqrt{3}a \in \mathcal{O} \pi \iff ni_1 \sqrt{3}a\pi \in p\mathcal{O} \iff a\pi \in p\mathcal{O} \iff a \in \mathcal{O} \pi .
\]
Furthermore, we have
\[
i_1 \sqrt{3}a \in (-i\pi i_1)\mathcal{O} \iff (-i\pi i_1)ni_1 \sqrt{3}a = ni_1 \sqrt{3}\pi a \in p\mathcal{O} \iff \pi a \in p\mathcal{O} \iff a \in \pi \mathcal{O} .
\]
due to a), again. \( \square \)

(4.30) Lemma. Let \( p \neq 3 \) be a prime number, \( n \in \mathbb{Z} \) with \( p \nmid n \) and \( \pi \in \mathcal{N}(p) \).

a) Suppose that \( \{r_1, \ldots, r_\pi 1\} \) is a transversal of \( \mathcal{O}/p\mathcal{O} \). Then
\[
\{ni_1 \sqrt{3}r_1, \ldots, ni_1 \sqrt{3}r_\pi 1\}
\]
is a transversal of \( \mathcal{O}/p\mathcal{O} \), too.

b) Suppose that \( \{s_1, \ldots, s_\pi 2\} \) is a transversal of \( \mathcal{O}/\mathcal{O}\pi \). Then
\[
\{ni_1 \sqrt{3}s_1, \ldots, ni_1 \sqrt{3}s_\pi 2\}
\]
is a transversal of \( \mathcal{O}/\mathcal{O}\pi \), too.

c) Suppose that \( \{t_1, \ldots, t_\pi 2\} \) is a transversal of \( \mathcal{O}/\pi \mathcal{O} \). Then
\[
\{ni_1 \sqrt{3}t_1, \ldots, ni_1 \sqrt{3}t_\pi 2\}
\]
is a transversal of \( \mathcal{O}/(-i\pi i_1)\mathcal{O} \).

Proof: a) Of course, since \( \#\{ni_1 \sqrt{3}r_1, \ldots, ni_1 \sqrt{3}r_\pi 1\} = p^4 \), we only have to verify that the elements in this set are pairwise incongruent modulo \( p\mathcal{O} \). So suppose there is \( j \neq k \) such that
\[
i_1 \sqrt{3}r_j + p\mathcal{O} = ni_1 \sqrt{3}r_k + p\mathcal{O} \iff ni_1 \sqrt{3}(r_j - r_k) \in p\mathcal{O} .
\]
In view of (4.29), this is equivalent to \( (r_j - r_k) \in p\mathcal{O} \), but this is a contradiction to \( \{r_1, \ldots, r_\pi 1\} \) being a transversal of \( \mathcal{O}/p\mathcal{O} \). Hence the claim follows.
b) Note that according to (4.27), the length of a transversal of \( \mathcal{O}/\mathcal{O}_\pi \) is \( p^2 \), indeed. And like in a), we only have to verify that the elements in \( \{ni_1\sqrt{3}s_1, \ldots, ni_1\sqrt{3}s_{p^2}\} \) are pairwise incongruent modulo \( \mathcal{O}_\pi \). So suppose there is \( j \neq k \) such that

\[
ni_1\sqrt{3}s_j + \mathcal{O}_\pi = ni_1\sqrt{3}s_k + \mathcal{O}_\pi \iff ni_1\sqrt{3}(s_j - s_k) \in \mathcal{O}_\pi.
\]

Again in view of (4.29), this is equivalent to \( (s_j - s_k) \in \mathcal{O}_\pi \), but this is a contradiction to \( \{s_1, \ldots, s_{p^2}\} \) being a transversal of \( \mathcal{O}/\mathcal{O}_\pi \). Hence the claim follows just like in a).

c) The proof is completely analogous to b). Again, we only have to verify that the elements in \( \{ni_1\sqrt{3}t_1, \ldots, ni_1\sqrt{3}t_{p^2}\} \) are pairwise incongruent modulo \( (-i_1\pi i_1)\mathcal{O} \). So let \( j \neq k \) with

\[
ni_1\sqrt{3}t_j + (-i_1\pi i_1)\mathcal{O} = ni_1\sqrt{3}t_k + (-i_1\pi i_1)\mathcal{O} \iff ni_1\sqrt{3}(t_j - t_k) \in (-i_1\pi i_1)\mathcal{O}.
\]

So again in view of (4.29), this is equivalent to \( (t_j - t_k) \in \pi\mathcal{O} \), which is a contradiction to \( \{t_1, \ldots, t_{p^2}\} \) being a transversal of \( \mathcal{O}/\pi\mathcal{O} \), once more.

(4.31) Lemma. Let \( p \neq 3 \) be a prime number, \( j \in \mathbb{N}_0, q \in \mathbb{N} \) with \( p \nmid q \) and \( a \in \mathcal{N}(p) \). Then

\[
\mathbb{Z} \cap p^j q^{\frac{j}{3}} \mathcal{O} a = p^{j+1} q^{\frac{j}{3}} \mathbb{Z}
\]

holds true.

Proof: Of course, we have \( p^{j+1} q^{\frac{j}{3}} \mathbb{Z} \subset p^j q^{\frac{j}{3}} \mathcal{O} a \), since \( p^j q^{\frac{j}{3}} (-i_1\sqrt{3}n\overline{a})a = p^{j+1} qn \) for all \( n \in \mathbb{Z} \).

So let \( 0 \neq t \in \mathbb{Z} \cap p^j q^{\frac{j}{3}} \mathcal{O} a \), where \( t = p^j q^{\frac{j}{3}} x a \) for some appropriate \( x \in \mathcal{O} \). We obtain

\[
t^2 = N(t) = p^{2j+2} q^{2 \frac{j}{3}} N(x)p.
\]

Therefore, \( p^{2j+2} q^{2 \frac{j}{3}} N(x)p \) has to be a square in \( \mathbb{N} \). So obviously, \( N(x) \) has to be of the shape

\[
N(x) = 3pn^2
\]

for some \( n \in \mathbb{Z} \), since \( p \neq 3 \). This leads to

\[
t^2 = p^{2j+2} q^2 n^2,
\]

and thus \( t = \pm p^{j+1} qn \in p^{j+1} q\mathbb{Z} \). \qed

(4.32) Lemma. Let \( p \neq 3 \) be a prime number, \( x \in \mathcal{O} \), but \( x \notin p\mathcal{O} \), and suppose a transversal of \( \mathcal{E}\setminus\mathcal{N}(p) \) is given by \( \{a_1, \ldots, a_{p+1}\} \). Then the following holds true:

- If \( p \nmid N(x) \), then for every \( j \in \{1, \ldots, p+1\} \) there exists exactly one \( k \in \{1, \ldots, p+1\} \) such that

\[
(-i_1a_ji_1)x \in \mathcal{O} a_k
\]

holds.
If \( p \mid N(x) \). Then there exists exactly one \( l \in \{1, \ldots, p + 1\} \) such that
\[
(-i_1a_{ij_1})x \in \mathcal{O}a_m
\]
holds for all \( m \in \{1, \ldots, p + 1\} \), while for every \( j \in \{1, \ldots, p + 1\} \setminus \{l\} \) there exists only one \( k \in \{1, \ldots, p + 1\} \) such that
\[
(-i_1a_{ij_1})x \in \mathcal{O}a_k
\]
holds. Furthermore, we have
\[
x \in (-i_1\bar{a}_{ij_1})\mathcal{O}.
\]

**Proof:** First, note that in virtue of (1.7), a transversal of \( \mathcal{E} \setminus \mathcal{N}(p) \) has length \( p + 1 \), indeed. And some easy considerations yield that the following obviously holds true (since \(-i_1\mathcal{O}_{i_1} = \mathcal{O} \) and \( \overline{ab} = \overline{b}\overline{a} \) for \( a, b \in \mathcal{O} \)): If \( \{a_1, \ldots, a_{p+1}\} \) is a transversal of \( \mathcal{E} \setminus \mathcal{N}(p) \), then \( \{-i_1a_{ij_1}, \ldots, -i_1a_{ij_{p+1}}\} \) is a transversal of \( \mathcal{E} \setminus \mathcal{N}(p) \), too, while \( \{-i_1\bar{a}_{ij_1}, \ldots, -i_1\bar{a}_{ij_{p+1}}\} \) are transversals of \( \mathcal{N}(p) / \mathcal{E} \). Therefore, in view of (4.18), if \( p \mid N(x) \) (hence \( N(x) \in p\mathcal{N}, \) since \( x \neq 0 \) due to \( x \notin p\mathcal{O} \)), then there has to exist \( l \in \{1, \ldots, p + 1\} \) such that \( x \in (-i_1\bar{a}_{ij_1})\mathcal{O} \) holds.

So let us get to the actual proof. Let \( j \in \{1, \ldots, p + 1\} \). Then we obviously have \( N((-i_1a_{ij_1})x) \in p\mathcal{N} \). So due to (4.18), there has to exist \( k \in \{1, \ldots, p + 1\} \) such that \( (-i_1a_{ij_1})x \in \mathcal{O}a_k \). Now, suppose there exists \( n \in \{1, \ldots, p + 1\} \setminus \{k\} \) such that \( (-i_1a_{ij_1})x \in \mathcal{O}a_n \) holds, too. This means we have
\[
(-i_1a_{ij_1})x \in I := \mathcal{O}a_k \cap \mathcal{O}a_n.
\]

Of course, \( I \) is a left-sided ideal, since \( \mathcal{O}a_k \) and \( \mathcal{O}a_n \) are. So according to (1.13) (and like we have seen several times now due to \( \mathcal{O} \) being euclidean), there has to exist \( 0 \neq c \in I \) (which is of minimal norm in \( I \)) such that \( I = \mathcal{O}c \). Note that, due to \( p = \overline{a}_ka_k = \overline{a}_na_n \), we have \( p \in I \).

So there is \( s \in \mathcal{O} \) such that \( p = sc \) holds, which leads to \( p^2 = N(p) = N(s)N(c) \), and thus \( N(c) \in \{1, p, p^2\} \). \( N(c) = 1 \) would be a contradiction, since obviously \( 1 \notin I \). But also \( N(c) = p \) would be a contradiction: If \( N(c) = p \), then \( a_k, a_n \in I \) would imply that there exist \( \varepsilon_k, \varepsilon_n \in \mathcal{E} \) such that \( a_k = \varepsilon_kc \) and \( a_n = \varepsilon_nc \). But this leads to \( a_k = \varepsilon_k\varepsilon_n^{-1}a_n \), which contradicts the preconditions \( \mathcal{E}a_k \cap \mathcal{E}a_n = \emptyset \). Hence we obtain \( N(c) = p^2 \), and \( p \in I \) leads to \( c = \varepsilon p, \varepsilon \in \mathcal{E} \). Thus we have
\[
(-i_1a_{ij_1})x \in I = p\mathcal{O} \iff px \in (p(-i_1\bar{a}_{ij_1})\mathcal{O} \iff x \in (-i_1\bar{a}_{ij_1})\mathcal{O}.
\]

This immediately implies \( N(x) \in p\mathcal{N} \). Hence the assertion already follows in case \( p \nmid N(x) \). So suppose we actually have \( p \mid N(x) \). Then we already fixed \( l \in \{1, \ldots, p + 1\} \) such that \( x \in (-i_1\bar{a}_{ij_1})\mathcal{O} \) holds (see above). Hence there is \( y \in \mathcal{O} \) such that \( x = (-i_1\bar{a}_{ij_1})y \), and we compute
\[
(-i_1a_{ij_1})x = (-i_1a_{ij_1})(-i_1\bar{a}_{ij_1})y = py = y\overline{a}_ma_m \in \mathcal{O}a_m
\]
for all \( m \in \{1, \ldots, p + 1\} \). Of course, the same holds for \( (-i_1a_{ij_1})x \), where \( j \) was fixed above, if there actually exist any \( k, n \in \{1, \ldots, p + 1\}, k \neq n \), such that \( (-i_1a_{ij_1})x \in \mathcal{O}a_k \) and \( (-i_1a_{ij_1})x \in \mathcal{O}a_n \) hold (hence \( x \in (-i_1\bar{a}_{ij_1})\mathcal{O} \). Suppose we had \( j \neq l \). Then the same considerations like above yield
\[
x \in (-i_1\bar{a}_{ij_1})\mathcal{O} \cap (-i_1\bar{a}_{ij_1})\mathcal{O} = p\mathcal{O}.
\]
This is a contradiction, since we assumed \( x \notin p\mathcal{O} \). Hence the claim follows. \( \square \)
4.4 The quaternionic Hecke-operators \( T_2(p) \)

In this section, we will introduce special Hecke-operators for \( \text{Sp}_2(\mathcal{O}) \). But since we are mainly interested in the Fourier-expansions of the \( E_k \), we will not present the complete theory of Hecke-algebras. Instead, we will only discuss the special Hecke-operators \( T_2(p) \) for prime numbers \( p \neq 3 \). They are defined analogous to the case of degree 1. The most important issue in the first part of this section is determining a transversal of \( \text{Sp}_2(\mathcal{O}) \backslash T_2(p) \) (like in (3.23)), such that we know how these Hecke-operators act on quaternionic modular forms. Afterwards, we will examine how the \( T_2(p) \) act on the space of quaternionic Maaß lifts for the trivial character. In the case of the Hurwitz order, a complete theory of the Hecke-algebras and Hecke-operators can be found in [Kr87] and [Kr90]. Of course, both works could be adapted to our case, but as mentioned, we are mainly interested in the special Hecke-operators \( T_2(p) \), and yet this alone will take a lot of effort.

We start by fixing some notation. To keep it simple, we will drop the index “\( \mathcal{O} \)” for now, since it should be clear that we are considering this order. Later on, we will come back to our standard notation, but in the current section it seems to be a bit redundant. And again, note that we will only consider the case of degree two, since we simply do not need the higher degrees, here (while the case of degree one was already discussed rudimentarily). The general case could be worked out in analogy to [Kr90].

\[
\begin{align*}
\text{(4.33) Definition.} \quad & \text{Let } k \in \mathbb{Z}, f \in [\text{Sp}_2(\mathcal{O}), k, 1] \text{ and } q \in \mathbb{N}. \text{ We fix the following notation:} \\
& \bullet \quad \Gamma_2 := \text{Sp}_2(\mathcal{O}). \\
& \bullet \quad \Delta_2(q) := \{ M \in \mathcal{O}^{4 \times 4} ; J[M] = qJ \}. \\
& \bullet \quad \Delta_2 := \bigcup_{q \in \mathbb{N}} \Delta_2(q). \\
& \bullet \quad \text{For } L = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Delta_2, \text{ define} \\
& \quad f|_k L(Z) := (\det(L[Z]))^{-k/2} f(L(Z)) = (\det(\tilde{C} \tilde{Z} + \tilde{D}))^{-k/2} f((AZ + B)(CZ + D)^{-1}) \\
& \quad \text{for } Z \in \mathcal{H}(\mathbb{H}) \text{ as usual.} \\
& \bullet \quad \text{For } K \in \Delta_2 \text{ define} \\
& \quad f|_k (\Gamma_2 K \Gamma_2) := \sum_{L \in \Gamma_2 \backslash \Gamma_2 K \Gamma_2} f|_k L, \\
& \quad \text{where } L : \Gamma_2 \backslash \Gamma_2 K \Gamma_2 \text{ means that } L \text{ runs through a transversal of the right cosets relative to } \Gamma_2 \text{ in } \Gamma_2 K \Gamma_2, \text{ as usual.} \\
& \bullet \quad \text{Given a formal sum } T = \sum_{j=1}^{n} c_j \Gamma_2 K_j \Gamma_2, \text{ where } c_1, \ldots, c_n \in \mathbb{C}, K_1, \ldots, K_n \in \Delta_2, \text{ define} \\
& \quad f|_k T := \sum_{j=1}^{n} c_j f|_k (\Gamma_2 K_j \Gamma_2). \\
\end{align*}
\]

The operator \( T \) (or more precisely \( |_k T \)) is called a (quaternionic) Hecke-operator. If \( c_j \in \mathbb{Z} \) for all
$j \in \{1, \ldots, n\}$ then the degree of such a Hecke-operator $T$ is defined as

$$\deg(T) := \sum_{j=1}^{n} c_j \cdot \#(\Gamma_2 \setminus (\Gamma_2 K_j \Gamma_2)) .$$

- A special Hecke-operator is given by

$$T_2(q) := \sum_{M: \Gamma_2 \Delta_2(q)/\Gamma_2} \Gamma_2 M \Gamma_2 .$$

**Remark.** a) By definition, we have $L \in \Delta_2(q)$ if and only if $M := q^{-1/2}L \in \text{Sp}_2(\mathbb{H})$ ($q \in \mathbb{N}$). Suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and let $f \in [\Gamma_2, k, 1]$ ($k \in 2\mathbb{Z}$). We obtain

$$f|_k L(Z) = f|_k(q^{1/2}M)(Z)$$

$$= (\det((q^{1/2}I_4)(\tilde{C} \tilde{Z} + (q^{1/2}I_4)\tilde{D}))^{-k/2}f((q^{1/2}AZ + q^{1/2}B)(q^{1/2}CZ + q^{1/2}D)^{-1})$$

$$= q^{-k}(\det(\tilde{C} \tilde{Z} + \tilde{D}))^{-k/2}f((AZ + B)(CZ + D)^{-1}) = q^{-k}f|M(Z)$$

for all $Z \in \mathcal{H}(\mathbb{H})$. So let $q_1, q_2 \in \mathbb{N}$, $L_j \in \Delta_2(q_j)$, $M_j = q_j^{-1/2}L_j$ ($j \in \{1, 2\}$), and thus $M_1 M_2 = (q_1 q_2)^{-1/2}(L_1 L_2)$, where of course $L_1 L_2 \in \Delta_2(q_1 q_2)$ by definition. We compute

$$(f|_k L_1)|_k L_2 = (q_1 q_2)^{-k}(f|M_1)|_k M_2 = (q_1 q_2)^{-k}f|M_1 M_2 = f|_k (L_1 L_2)$$

in view of 1.14. We will use this fact from now on without referring to this current remark. Furthermore, applying (1.3) (i.e. the fundamental relations) to $M = q^{-1/2}L$, where $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_2(q)$, immediately gives the following equivalences:

(i) $L \in \Delta_2(q)$.
(ii) $L' \in \Delta_2(q)$.
(iii) $\overline{A'}C - \overline{C'}A = \overline{B'}D - \overline{D'}B = 0$, $\overline{A'}D - \overline{C'}B = qI$.
(iv) $A\overline{B'} - B\overline{A'} = C\overline{D'} - D\overline{C'} = 0$, $A\overline{D'} - B\overline{C'} = qI$.

Note that in this case $L \in \text{GL}_4(\mathbb{H})$ holds with

$$L^{-1} = \frac{1}{q} \begin{pmatrix} \overline{D'} & -\overline{B'} \\ -\overline{C'} & \overline{A'} \end{pmatrix}$$

b) Note that our definition of the slash-operator for $\Delta_2$ in degree two differs from our definition of the slash-operator for $\text{GL}_2^+(\mathbb{R})$ in chapter 3. The actual difference lies in the normalization. In degree one we used the normalization factor $\det(M)^{k/2}$. This does not change the multiplicative property of the slash-operator, but we can no longer multiply the occurring matrices with scalars without changing the operator. Because of the normalization in degree one we had $f|_k(rM) = f|_k M$ for all $M \in \text{GL}_2^+(\mathbb{R})$ and $r > 0$ (if $k \in 2\mathbb{Z}$ then even for all $r \in \mathbb{R}^+$). But this time, without a normalization factor, we obtain

$$f|_k(qM) = q^{-2k}f|_k M$$
for all \( q \in \mathbb{N} \) and \( M \in \Delta_2 \). The reason why we chose the slash-operator to be normalized in degree one was that we needed a lot of results from [Mi89]. So in order to not have to deal with occurring pre-factors, we chose to adopt the normalization used by Miyake.

Now, in degree two, we choose to not normalize the slash-operator to stay close to the work of Krieg in [Kr87, Kr90] – although we will not need the results of these papers in a direct sense, but of course the approach we use originates from them, and moreover a comparison of the results is much easier this way. Furthermore, note that the Hecke-operators in degree one were normalized, while the ones we use now are not. The reason is the same like for the slash-operators.

c) Like in the case of degree one, the definition of \( f|_k(\Gamma_2 K \Gamma_2) \) (where \( K \in \Delta_2, f \in [\Gamma_2, k, 1] \)) does not depend on the special choice of a transversal, of course – which also means that the Hecke-operators are well-defined. Let \( \{L_j\} \) be a transversal of \( \Gamma_2 \backslash \Gamma_2 K \Gamma_2 \), and \( \{M_j L_j\} \) a second one, where \( M_j \in \Gamma_2 \). In view of a) and due to \( f \in [\Gamma_2, k, 1] \) we have

\[
\sum_j f|_k(M_j L_j) = \sum_j (f|_k M_j)|_k L_j = \sum_j f|_k L_j ,
\]

and thus the well-definedness. But note that so far, we did not show that there actually exists a finite transversal of \( \Gamma_2 \backslash \Gamma_2 K \Gamma_2 \) – and the same concerning \( \Gamma_2 \backslash \Delta_2(q) / \Gamma_2 \) with regard to \( T_2(q) \). Note that one could adapt [Kr87, le.6] to our case, which would yield the finiteness. But again, since we will only have to consider the special Hecke-operators \( T_2(p) \), where \( p \) is a prime number, we will content ourselves with proving that \( \Gamma_2 \backslash \Delta_2(p) / \Gamma_2 \) possesses a finite transversal \( K_1, \ldots, K_n \), as well as \( \Gamma_2 \backslash \Gamma_2 K_i \Gamma_2, j = 1, \ldots, n \). But to say it again, with quite some effort [Kr87, le.6] could be worked out for our case.

d) Of course, again like in the case of degree one, the Hecke-operators actually act on the space of quaternionic modular forms for the trivial character of fixed even weight \( k \):

Once more, a simple fact from linear algebra is that if \( \{L_j\} \) is a transversal of \( \Gamma_2 \backslash \Gamma_2 K \Gamma_2 \) (where \( K \in \Delta_2 \)), then obviously so is \( \{L_j M\} \), where \( M \in \Gamma_2 \). So if \( f \in [\Gamma_2, k, 1] \), then \( f|_k(\Gamma_2 K \Gamma_2) \in [\Gamma_2, k, 1] \) holds, too, because

\[
(f|_k(\Gamma_2 K \Gamma_2))|_k M = \sum_{L, j \in \Gamma_2 \backslash \Gamma_2 K \Gamma_2} f|_k(L M) = f|_k(\Gamma_2 K \Gamma_2)
\]

for all \( M \in \Gamma_2 \) in view of a). And of course, Hecke-operators are linear, and thus endomorphisms of \([\Gamma_2, k, 1]\), indeed.

e) Of course, we have

\[
\Delta_2(q) = \bigcup_{M : \Gamma_2 \backslash \Delta_2(q) / \Gamma_2} \Gamma_2 M \Gamma_2
\]

as a disjoint union, and thus also

\[
\Gamma_2 \backslash \Delta_2(q) = \bigcup_{M : \Gamma_2 \backslash \Delta_2(q) / \Gamma_2} \Gamma_2 \backslash (\Gamma_2 M \Gamma_2) .
\]
And obviously, this gives rise to
\[
f|_k T_2(q) = \sum_{M: \Gamma_2 \setminus \Delta_2(q)/T_2} f|_k (\Gamma_2 M G_2)
= \sum_{M: \Gamma_2 \setminus \Delta_2(q)/T_2} \sum_{L: \Gamma_2 \setminus (\Gamma_2 M G_2)} f|_k L
= \sum_{K: \Gamma_2 \setminus \Delta_2(q)} f|_k K.
\]

Since the following considerations are quite involving, we use subsections now in order to get it well structured.

4.4.1 The structure of the Hecke-operators $T_2(p)$

Before we can actually examine how the Hecke-operators $T_2(p)$ act on the space $\mathcal{M}(k; \mathcal{O})$ of quaternionic Maaß lifts, we need to determine the actual structure of $T_2(p)$. This means that first, we have to determine a transversal $K_1, \ldots, K_n$ of $\Gamma_2 \setminus \Delta_2(p)/T_2$, and afterwards a transversal of $\Gamma_2 \setminus \Gamma_2 K_i \Gamma_2$ for each $K_i$. Note that the approach in doing so is inspired by [Kr87]

So let $p \neq 3$ be a prime number. Fix a completely primitive element $\pi \in \mathcal{N}(p)$ (which exists according to (4.21)), and define
\[
S_2(p) := \Gamma_2 \text{diag}(1, 1, p, p) \Gamma_2, \quad S_2^*(p) := \Gamma_2 \text{diag}(1, \pi, p, \pi) \Gamma_2.
\]

Note that, in view of (4.34), $\text{diag}(1, 1, p, p) \in \Delta_2(p)$ and $\text{diag}(1, \pi, p, \pi) \in \Delta_2(p)$ hold. The first step in examining the structure of $T_2(p)$ is the following

(4.35) Proposition. Let $p \neq 3$ be a prime number and fix a completely primitive element $\pi \in \mathcal{N}(p)$. Then a transversal of $\Gamma_2 \setminus \Delta_2(p)/T_2$ is given by $\{\text{diag}(1, 1, p, p), \text{diag}(1, \pi, p, \pi)\}$. In particular, we have
\[
T_2(p) = S_2(p) + S_2^*(p)
\]

Proof: The beginning of the proof will be quite similar to (1.21). Let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_2(p)$. Multiplying with $\text{Rot}(U), U \in \text{GL}_2(\mathcal{O})$ gives
\[
L_1 = \text{Rot}(U)L = \begin{pmatrix} U A & U B \\ U C & U D \end{pmatrix} =: \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.
\]

Just like in the proof of (1.21), by applying [Kr85, ch.I, cor.2.4] to our order $\mathcal{O}$, we can choose $U \in \text{GL}_2(\mathcal{O})$ such that the first column of $U A$ possesses the form $(a, 0)'$. Denote the first column of $C_1$ by $(c_1, c_3)'$. If $a = c_1 = 0$ holds, then one easily verifies that the first column of $-J \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) L_1$ is $(c_3, 0, 0, 0)'$. Otherwise, define $u$ to be a greatest common right divisor of $a$ and $c_1$ (cf. [Kr85, ch.I, p.10] for details), which implies that there exist $a, \gamma \in \mathcal{O}$ such that $a = au$.
4.4 The quaternionic Hecke-operators $T_2(p)$

$c_1 = \gamma u$. The fundamental relations (1.3) yield

$$\begin{pmatrix} \bar{a}c_1 & * \\ * & * \end{pmatrix} = \mathbf{A}_1 c_1 = \overline{\mathbf{C}_1} A_1 = \begin{pmatrix} \bar{e}_1 a & * \\ * & * \end{pmatrix},$$

hence $\bar{a}c_1 = \bar{e}_1 a$. And since $u \neq 0$, this implies

$$\bar{a} \gamma = \bar{u}^{-1} \bar{a} c_1 u^{-1} = \bar{u}^{-1} \bar{e}_1 a u^{-1} = \bar{\gamma} a.$$

Furthermore, note that $\alpha$ and $\gamma$ have to be relatively right-prime by definition, because $u$ is defined to be a greatest common right divisor. Again like in (1.21), adopting [Kr85, ch.II, cor.2.4] for our case yields that there is $K \in \text{Sp}_1(\mathcal{O})$ having $(\alpha, \gamma)'$ as its first column. Let $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Her}_2(\mathcal{O})$ and

$$L_2 = (J^{-1} \text{Trans}(S)) \text{Trans}(S)(K^{-1} \times I) L_1.$$

A straightforward calculation shows that the first column of $L_2$ is given by $(u, c_3, 0, 0)'$. Once more, we choose $V \in \text{GL}_2(\mathcal{O})$ such that the first column of

$$L_3 = \text{Rot}(V) L_2 = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}$$

is given by $(x, 0, 0, 0)'$ like we did above (which is possible because $V^{-1}(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}) = (\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix})$). Note that $x \neq 0$ holds since $L_3 \in \Delta_2(p)$ is invertible in view of (4.34). Furthermore, like we have done several times before (for example in (1.21)), the “fundamental relations” in (4.34) for $L_3 \in \Delta_2(p)$ yield that $L_3$ has to be given by

$$L_3 = \begin{pmatrix} x & * & * \\ 0 & a_4 & * \\ 0 & 0 & 0 \\ 0 & c_4 & * \end{pmatrix},$$

and

$$\begin{pmatrix} * & * \\ * & \bar{a}_4 c_4 \end{pmatrix} = \mathbf{A}_3 c_3 = \overline{\mathbf{C}_3} A_3 = \begin{pmatrix} * & * \\ * & \bar{e}_4 a_4 \end{pmatrix},$$

which implies $\bar{a}_4 c_4 = \bar{e}_4 a_4$. Moreover, $a_4 = c_4 = 0$ would be a contradiction to $L_3$ being invertible. So let $y$ be a greatest common right divisor of $a_4$ and $c_4$. Exactly the same arguments as above yield that there is $K \in \text{Sp}_1(\mathcal{O})$ such that (with the help of a simple calculation)

$$L_4 := (I \times \bar{K}^{-1}) L_3 = \begin{pmatrix} x & * & * \\ 0 & y & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_4 & B_4 \\ 0 & D_4 \end{pmatrix}.$$

Note that in view of (4.34), $A_4 \overline{D}_4 = pI$, which implies that both $A_4$ and $D_4$ have full rank. So according to (4.22), we can choose $U_1, U_2 \in \text{GL}_2(\mathcal{O})$ such that $U_1^{-1} D_4 U_2^{-1} = \text{diag}(d_1, d_4)$ ($d_1 \neq 0$, $d_4 \neq 0$) with $d_1 || d_4$. So another calculation gives (keeping in mind that $A_5 = p(\overline{D}_5)^{-1}$ has to
Thus $N$ be completely primitive. In the remaining case, i.e. $N > 5$, which implies

$$
\begin{align*}
\text{Note that, as we have seen in the proof of (4.21), there exists } \varepsilon \in \mathcal{E} \text{ such that } \varepsilon r \text{ is completely primitive. And thus, since we can multiply with } \text{Rot(diag}(1, \varepsilon)) \text{ from the left, we can assume } r \text{ to be completely primitive. In the remaining case, i.e. } N(d_1) = 1, N(d_4) = p \text{ is handled completely}
\end{align*}
$$
analogous. We can assume that $D_5 = \text{diag}(1,r)$, where $r \in \mathcal{N}(p)$ is completely primitive, and $A_5 = \text{diag}(p,r)$, and get

$$J^{-1}\left(\text{Trans}\left(\begin{pmatrix} -b_1 & -b_3 \\ -b_2 & -b_4 \end{pmatrix}\right)L_5\right) J = \text{diag}(1,r,p,r),$$

where $b_4 = mr$. And finally, in both of the last cases, we have $N(r) = \text{N}(\pi) = p$. And thus, according to (4.26), there exist $U, V \in \text{GL}_2(O)$ such that $\overline{U} \text{ diag}(1,r) \overline{V}' = \text{diag}(1,\pi)$, and we obtain

$$\text{Rot}(U) \text{ diag}(1,r,p,r) \text{ Rot}(V) = \text{diag}(1,\pi,p,\pi) \cdot$$

Therefore, $\Gamma_2 \backslash \Delta_2(p)/\Gamma_2$ consists of at most two cosets, given by $S_2(p)$ and $S'_2(p)$. So assume these two cosets coincide. This means there exist $M_1 = (A_1 B_1 \begin{pmatrix} C_1 & D_1 \end{pmatrix}) \in \Gamma_2$ and $M_2 = (A_2 B_2 \begin{pmatrix} C_2 & D_2 \end{pmatrix}) \in \Gamma_2$ satisfying

$$M_1 \text{ diag}(1,1,p,p) = \text{diag}(1,\pi,p,\pi) M_2.$$ 

Applying [Kr85, ch.I, cor.2.4] to our order $O$, again, we can choose $U \in \text{GL}_2(O)$ such that the first row of $A_2\overline{U}′$ possesses the form $(0,a_2)$. Defining $M_3 = M_1 \text{ Rot}(U) = (A_3 B_3 \begin{pmatrix} C_3 & D_3 \end{pmatrix})$ and $M_4 = M_2 \text{ Rot}(U) = (A_4 B_4 \begin{pmatrix} C_4 & D_4 \end{pmatrix})$, we obtain

$$\text{diag}(1,\pi,p,\pi)M_4 = M_1 \text{ diag}(1,1,p,p) \text{ Rot}(U)$$

$$= M_1 \text{ Rot}(U) \text{ diag}(1,1,p,p) = M_3 \text{ diag}(1,1,p,p),$$

where $A_4 = (\begin{pmatrix} 0 & * \\ a_3 & * \end{pmatrix})$ and $C_4 = (\begin{pmatrix} c_3 & * \end{pmatrix})$. The fundamental relations (1.3) imply $\pi_3 c_3 = \pi_3 a_3$. Suppose $a_3 \neq 0$ or $c_3 \neq 0$ and let $v$ denote a greatest common right divisor of $a_3$ and $c_3$. Then there exist $a, \gamma \in O$ with $a_3 = av$ and $c_3 = \gamma v$. Since $v \neq 0$, we obtain $\pi = v^{-1}\pi_\gamma v = v^{-1}\pi_\gamma v = v\pi$. Of course, $a$ and $\gamma$ have to be relatively right-prime by definition, because $v$ is defined to be a greatest common right divisor. Again like in (1.21), adopting [Kr85, ch.II, cor.2.4] for our case yields that there is $K \in \text{Sp}_1(O)$ having $(a,\gamma)'$ as its first column. But then, in view of (1.18), there exist $\epsilon \in \mathcal{E}$ and $m, n \in \mathbb{Z}$ such that $a = me$ and $\gamma = ne$. Thus the first column of $\text{diag}(1,\pi,p,\pi)M_4$ turns out to be $(0,mp\pi ev,pc_1,n\pi ev)'$. Note that $N(\pi ev), N(\pi ev) \in p\mathbb{Z}$. Therefore, according to (4.18), there exist $\pi \in \mathcal{N}(p)$ and $u \in O$ such that $\pi ev = u\pi$. Hence the first column of $\text{diag}(1,\pi,p,\pi)M_4$ equals $(0,nu\pi,\pi c_1 p,nu\pi)'$, and $\pi$ turns out to be a common right divisor of this column (since $p = \pi\pi$). On the other hand, the first column of $M_3 \text{ diag}(1,1,p,p)$ coincides with that of $M_3$, and according to the identity above, also coincides with $(0,nu\pi,\pi c_1 p,nu\pi)'$. This is a contradiction to [Kr85, ch.II, cor.2.4], because the first column of any element in the quaternionic modular group has to be relatively right-prime, since they are invertible. (Note that we get the same contradiction if we would have had $a_3 = c_3 = 0$.) This completes the proof. \hfill $\square$

To continue, we will need a criterion that determines whether a matrix $L \in \Delta_2(p)$ belongs to the double coset $S_2(p)$ or $S'_2(p)$. Furthermore, we will have to look at the cosets $\Gamma_2 L$, where $L \in \Delta_2(p)$. In view of the proof of the preceding proposition, every such coset has a representative $(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$, since we only needed to multiply from the left to transform $L$ to this shape. So we need a criterion so determine if $\Gamma_2 (\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}) = \Gamma_2 (\begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$ holds. Note that given such a shape, $A$ is uniquely determined, since $\overline{A D}' = pI$ holds. This motivates the following
(4.36) Definition. Let \( p \neq 3 \) be a prime number and \( L \in \Delta_2(p) \). We define
\[
\mathcal{D}(\Gamma_2 \Gamma_2) := \{ D ; \text{there exists } \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right) \in \Gamma_2 \Gamma_2 \}
\]
and
\[
\mathcal{B}(D, \Gamma_2 \Gamma_2) := \{ B ; \text{there exists } \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right) \in \Gamma_2 \Gamma_2 \},
\]
where \( D \in \mathcal{D}(\Gamma_2 \Gamma_2) \). Furthermore, given \( B, \tilde{B} \in \mathcal{B}(D, \Gamma_2 \Gamma_2) \), we define
\[
B \equiv \tilde{B} \mod D
\]
if there exists \( S \in \text{Her}_2(O) \) such that
\[
B = \tilde{B} + SD.
\]

We obtain the following criterions:

(4.37) Proposition. Let \( p \neq 3 \) be a prime number and \( L \in \Delta_2(p) \).

a) One has the criterion
\[
L \in S_2(p) \iff \{ \text{det}(\tilde{D}) ; D \in \mathcal{D}(\Gamma_2 \Gamma_2) \} \subset \{ 1, p^2, p^4 \},
\]
\[
L \in S'_2(p) \iff \{ \text{det}(\tilde{D}) ; D \in \mathcal{D}(\Gamma_2 \Gamma_2) \} \subset \{ p, p^3 \}.
\]

In particular, assume \( L = \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right) \). Then we have
\[
\text{det}(\tilde{D}) \in \{ 1, p^2, p^4 \} \Rightarrow L \in S_2(p),
\]
\[
\text{det}(\tilde{D}) \in \{ p, p^3 \} \Rightarrow L \in S'_2(p).
\]

b) Given \( D, \tilde{D} \in \mathcal{D}(\Gamma_2 \Gamma_2) \) (with corresponding \( A, \tilde{A} \)), \( B \in \mathcal{B}(D, \Gamma_2 \Gamma_2) \) and \( \tilde{B} \in \mathcal{B}(\tilde{D}, \Gamma_2 \Gamma_2) \), one has
\[
\Gamma_2(\left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right)) = \Gamma_2(\left( \begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array} \right)) \Rightarrow GL_2(O)D = GL_2(O)\tilde{D}.
\]

c) Given \( D \in \mathcal{D}(\Gamma_2 \Gamma_2) \) and \( B, \tilde{B} \in \mathcal{B}(D, \Gamma_2 \Gamma_2) \), one has
\[
\Gamma_2(\left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right)) = \Gamma_2(\left( \begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array} \right)) \iff B \equiv \tilde{B} \mod D.
\]

d) A transversal of \( \Gamma_2 \backslash \Gamma_2 \Gamma_2 \) is given by the matrices
\[
K = \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right), \quad A = p(D')^{-1},
\]
where
\begin{itemize}
  \item \( D \) runs through a transversal of \( GL_2(O) \backslash \mathcal{D}(\Gamma_2 \Gamma_2) \),
  \item whenever \( D \) is given, \( B \) runs through a transversal of mod \( D \) incongruent matrices in \( \mathcal{B}(D, \Gamma_2 \Gamma_2) \).
\end{itemize}
4.4 The quaternionic Hecke-operators $T_2(p)$

Proof: a) So let $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_2 \Gamma_2$. Then just like in the proof of (4.35), there exist $U, V \in \text{GL}_2(\mathcal{O})$ such that

$$\text{Rot}(U) \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \text{Rot}(V) = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix},$$

where

$$D_1 \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \right\}$$

for an appropriate $r \in \mathcal{N}(p)$, where of course

$$\det(D) = \det(\hat{U}\hat{D}\hat{V}) = \det(D_1)$$

in view of (1.34). Furthermore, again according to the proof of (4.35) (where one simply omits the multiplication with the “permutation matrices” $J, J^{-1} \times I, \text{Rot}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ etc.), there exist $S, T \in \text{Her}_2(\mathcal{O})$ such that

$$\text{Trans}(S) \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \text{Trans}(T) = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}.$$

And therefore, if one looks at the final details of the proof of (4.35), $L \in S_2(p)$ (and thus $\left( \begin{pmatrix} A_1 & 0 \\ D_1 & 1 \end{pmatrix} \right) \in S_2(p)$) implies $D_1 \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \right\}$, hence det($\hat{D}$) $\in \{1, p^2, p^4\}$, while $L \in S_2^2(p)$ (and thus $\left( \begin{pmatrix} A_1 & 0 \\ D_1 & 0 \end{pmatrix} \right) \in S_2^2(p)$) implies $D_1 \in \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & p \end{pmatrix} \right\}$, hence det($\hat{D}$) $\in \{p, p^3\}$. On the other hand, in virtue of (4.35) $L \in S_2(p)$ or $L \in S_2^2$ has to hold, so there exist $M_1, M_2$ such that either $M_1LM_2 = \text{diag}(1, 1, p, p)$ or $M_1LM_2 = \text{diag}(1, \pi, p, \pi)$ where $\pi \in \mathcal{N}(p)$ is completely primitive. This implies $D_1 = \text{diag}(p, p) \in D(\Gamma_2 \Gamma_2)$ or $D_2 = \text{diag}(\pi, p) \in D(\Gamma_2 \Gamma_2)$, where det($\hat{D}_1$) $= p^4$ and det($\hat{D}_2$) $= p^3$. Therefore, $\{\{\{D_1, D_2 \in \mathcal{D}(\Gamma_2 \Gamma_2) \} \subset \{1, p^2, p^4\}\}$ implies $L \in S_2(p)$, while $\{\{D_1, D_2 \in \mathcal{D}(\Gamma_2 \Gamma_2) \} \subset \{p, p^3\}\}$ yields $L \in S_2^2(p)$.

Note that the second assertion is just a special case. If $L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, then $D \in \mathcal{D}(\Gamma_2 \Gamma_2)$ holds by definition. But then the determinant of $D$ already determines which one of the two cases is given (since we already know that $L \in S_2(p)$ or $L \in S_2^2(p)$ has to hold).

b) Suppose $\Gamma_2(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}) = \Gamma_2(\begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix})$, which means there is $M = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma_2$ such that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} A_1 & \tilde{A} & A_1 \tilde{B} + B_1 \tilde{D} \\ C_1 & C_1 \tilde{A} & C_1 \tilde{B} + D_1 \tilde{D} \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & \tilde{B} + B_1 D_1 \tilde{D} \\ C_1 & C_1 \tilde{B} + D_1 \tilde{D} \end{pmatrix} = \begin{pmatrix} A & \tilde{B} + B_1 D_1 \tilde{D} \\ C_1 & C_1 \tilde{B} + D_1 \tilde{D} \end{pmatrix}.$$
Then \( A, D \in GL_2(\mathbb{H}) \) implies \( C_1 = 0, A_1 = D_1 = I \), and the fundamental relations (1.3) yield \( B_1 \in \text{Her}_2(O) \). This gives \( B = B_1D \), and therefore \( B \equiv B \mod D \).

On the other hand, \( B \equiv B \mod D \) means that there exists \( S \in \text{Her}_2(O) \) such that \( B = B + SD \). One easily computes

\[
\text{Trans}(S) \begin{pmatrix} A & \tilde{B} \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & B + SD \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},
\]

and thus \( \text{Trans}(S) \in \Gamma_2 \) yields \( \Gamma_2(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}) = \Gamma_2(\begin{pmatrix} A & \tilde{B} \\ 0 & B \end{pmatrix}) \).

\[\text{d) First, let us prove that the quoted matrices pairwise belong to different cosets. So suppose } K = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array}\right) \text{ and } \bar{K} = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array}\right) \text{ are both of the stated shape with } \Gamma_2 K = \Gamma_2 \bar{K}. \text{ But then b) already yields } GL_2(O)D = GL_2(O)\tilde{D}, \text{ and since the } \{D\}'s \text{ were chosen to run through a transversal of } GL_2(O) \setminus D(\Gamma_2 L_2), \text{ D follows, as well as } A = p(\tilde{D})^{-1} = \tilde{A} \text{ due to the fundamental relations in (4.34)}. \text{ But now c) implies } B \equiv B \mod D. \text{ Again, the } \{B\}'s \text{ were chosen to run through a transversal of mod } D \text{ incongruent matrices in } B(D, \Gamma_2 L_2), \text{ hence } B = \tilde{B}. \]

So let \( K \in \Gamma_2 L_2 \). We have to prove that there exists \( M \in \Gamma_2 \) such that \( MK \) is equal to one of the matrices quoted in the assertion. Like we have seen in the proof of (4.35), there exists \( M_1 \in \Gamma_2 \) such that \( MK \) is of the shape \( MK = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array}\right) \). Note that \( K \in \Gamma_2 L_2 \) means that there exist \( N_1, N_2 \in \Gamma_2 \) such that \( K = N_1 L_2 N_2 \). This gives \( MN_1 L_2 N_2 = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array}\right) \), and thus \( D \in D(\Gamma_2 L_2) \) by definition. The \{D\}'s in the assertion were chosen to run through a transversal of \( GL_2(O) \setminus D(\Gamma_2 L_2) \). Hence there exists \( U \in GL_2(O) \) such that \( U^{-1} D = \tilde{D} \) belongs to this transversal. Defining \( M_2 = \text{Rot}(U) \) we obtain \( M_2 M_1 K = \left(\begin{array}{cc} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{array}\right) \) for appropriate \( \tilde{A}, \tilde{B} \). Due to the same reason like above, \( \tilde{D} \in D(\Gamma_2 L_2) \) and \( \tilde{B} \in B(\tilde{D}, \Gamma_2 L_2) \) have to hold. Now, since the \{B\}'s in the assertion were chosen to run through a transversal of mod \( \tilde{D} \) incongruent matrices in \( B(\tilde{D}, \Gamma_2 L_2) \), we can find \( S \in \text{Her}_2 \) such that \( \tilde{B} + SD = B^* \) belongs to this transversal. Therefore, choosing \( M_3 = \text{Trans}(S) \) and \( M = M_3 M_2 M_1 \) finally yields that \( MK = \left(\begin{array}{cc} \tilde{A} & 0 \\ 0 & \tilde{B} \end{array}\right) \) is of the quoted shape, because \( \tilde{A} = p(\tilde{D})^{-1} \) has to hold due to the fundamental relations. \( \square \)

So the actual problem in determining the exact shape of the Hecke-operators \( T_2(p) \) lies in determining \( GL_2(O) \setminus D(S_2(p)) \) and \( GL_2(O) \setminus D(S_2^*(p)) \) as well as finding a complete set of representatives of mod \( D \) incongruent matrices in \( B(D, S_2(p)) \) and \( B(D, S_2^*(p)) \), respectively, for each \( D \) in one of the first transversals.

We begin by determining \( GL_2(O) \setminus D(S_2(p)) \) and \( GL_2(O) \setminus D(S_2^*(p)) \).

\((4.38) \text{ Proposition. Let } p \neq 3 \text{ be a prime number. A transversal of } GL_2(O) \setminus D(S_2(p)) \text{ is given by }

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}, \quad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad \begin{pmatrix} \pi & qs \\ 0 & \pi \end{pmatrix},
\]

\text{ where }

\begin{itemize}
\item \( r \) runs through a transversal of \( \mathcal{O} / p\mathcal{O} \),
\end{itemize}
• both \( \pi \) and \( \bar{\pi} \) run through a transversal of \( \mathcal{E}\backslash\mathcal{N}(p) \),

• whenever \( \pi \) and \( \bar{\pi} \) are given, choose (exactly one) \( s \in \mathcal{O} \) such that \( \gcd(N(s), p) = 1 \) and \( \bar{\pi}s \in \mathcal{O}\bar{\pi} \) (which exists in virtue of (4.25)),

• \( q \) runs through a transversal of \( \mathbb{Z}/p\mathbb{Z} \).

A transversal of \( \text{GL}(2) \backslash \mathcal{D}(S_2(p)) \) is given by

\[
\begin{pmatrix} 1 & r \\ 0 & \pi \end{pmatrix}, \quad \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}, \quad \begin{pmatrix} p & 0 \\ 0 & \pi t \end{pmatrix},
\]

where

• \( \pi \) runs through a transversal of \( \mathcal{E}\backslash\mathcal{N}(p) \),

• whenever \( \pi \) is given, \( r \) runs through a transversal of \( \mathcal{O}/\mathcal{O}\pi \), and \( t \) runs through a transversal of \( \mathcal{O}/\pi\mathcal{O} \).

**Proof:** First, let us verify that every matrix from above actually belongs to \( \mathcal{D}(S_2(p)) \) or \( \mathcal{D}(S_2(p)) \), respectively, and that they pairwise belong to different cosets. Concerning the first point, let \( D \) be one of the matrices from the assertion. Due to the fundamental relations in (4.34), it suffices to verify that \( A = p(D')^{-1} \in \mathcal{O}^{2\times 2} \) holds, because given that this is true, we have \( L = (A 0 \ 0 D) \in \Delta_2(p) \), and the correct determinant condition from (4.37) then yields \( L \in S_2(p) \) or \( L \in S_2^2(p) \), respectively. We already have

\[
D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow p(D')^{-1} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad D = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \Rightarrow p(D')^{-1} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
D = \begin{pmatrix} 1 & r \\ 0 & \pi \end{pmatrix} \Rightarrow p(D')^{-1} = \begin{pmatrix} p & 0 \\ 0 & \pi \end{pmatrix}, \quad D = \begin{pmatrix} p & 0 \\ 0 & \pi \end{pmatrix} \Rightarrow p(D')^{-1} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.
\]

The remaining case is \( D = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} \). By definition, there is \( t \in \mathcal{O} \) such that

\[
\bar{\pi}s = t\bar{\pi} \quad \Leftrightarrow \quad \bar{\pi} = \bar{\pi}t \quad \Leftrightarrow \quad \bar{\pi}s = p\bar{\pi}t \quad \Leftrightarrow \quad \bar{\pi}s = \bar{\pi}t.
\]

Hence we compute

\[
\begin{pmatrix} \pi & 0 \\ -q\bar{t} & \bar{\pi} \end{pmatrix} \begin{pmatrix} \bar{\pi} & 0 \\ q\bar{s} & \bar{\pi} \end{pmatrix} = \begin{pmatrix} \pi\bar{\pi} & 0 \\ -q\bar{t}\bar{\pi} + q\bar{\pi}\bar{s} & \bar{\pi}\bar{\pi} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix},
\]

and thus \( p(D')^{-1} = \begin{pmatrix} \pi & 0 \\ -q\bar{t} & \bar{\pi} \end{pmatrix} \).

So let us get to verifying that the cosets belonging to the quoted matrices are indeed pairwise distinct. Let \( C = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) and \( D = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \) be given like in the assertion, and suppose there is \( U = \begin{pmatrix} g & h \\ j & k \end{pmatrix} \in \text{GL}(\mathcal{O}) \) such that

\[
UD = \begin{pmatrix} gd & g\bar{e} + h\bar{f} \\ jd & j\bar{e} + k\bar{f} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = C.
\]
Because $d \neq 0$ holds, $j = 0$ follows. But then (1.34) yields $g, k \in \mathcal{E}$, and thus $\mathcal{E}a = \mathcal{E}d$ as well as $\mathcal{E}c = \mathcal{E}f$. Note that all digonal entries of the matrices defined in the assertion are 1, $p$ or of norm $p$, while in the case of norm $p$ they were chosen from a transversal of $\mathcal{E} \setminus \mathcal{N}(p)$. Therefore, $a = d$, $c = f$ and $g = k = 1$ follows, and we are left with
\[
\begin{pmatrix}
a \epsilon + hf \\
0 \\
c
\end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},
\]
and therefore $b = \epsilon + hf$. Note that we only have to consider those matrices defined above whose entry of index $(1,2)$ is not zero, because otherwise $b = \epsilon = 0$ and $U = I$ immediately follows.

So let us check those four cases. First, let $C = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$. Due to $r = \tilde{r} + hp$, we have $r + p\mathcal{O} = \tilde{r} + p\mathcal{O}$, and thus $r = \tilde{r}$ by definition. Next, let $C = \begin{pmatrix} \pi \tilde{q} & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \pi \tilde{q} & 0 \\ 0 & 0 \end{pmatrix}$. Again, we have $qs = \tilde{q}s + h\pi$, which is equivalent to $(q - \tilde{q})s = h\pi$. If $h \neq 0$, then we have $\mathcal{N}(h\pi) \in p\mathcal{N}$ and thus $(q - \tilde{q})^2 \mathcal{N}(s) \in p\mathcal{N}$, which also implies $(q - \tilde{q}) \in p\mathcal{Z}$. But the “$q’s$” were chosen to be pairwise incongruent mod $p$, hence $q = \tilde{q}$ and $h = 0$. Now, we consider $C = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, and again we get $r = \tilde{r} + h\pi$, which implies $r = \tilde{r}$, since this time the “$r’s$” were chosen from a transversal of $\mathcal{O}/\mathcal{O}\pi$. And finally, let $C = \begin{pmatrix} \pi \tilde{q} & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \pi \tilde{q} & 0 \\ 0 & 1 \end{pmatrix}$, with $\pi r = \pi \tilde{r} + hp = \pi(\tilde{r} + \pi h)$, which is equivalent to $t = \tilde{t} + \pi h$.

Again, the definition yields $t = \tilde{t}$.

Now, let $D \in \mathcal{D}(S_2(p))$ or $D \in \mathcal{D}(S_2^{*}(p))$. We have to verify that there exists $U \in \text{GL}_2(\mathcal{O})$ such that $UD$ is of one of the shapes specified above. Note that by definition there exist $A, B \in \mathcal{O}^{2 \times 2}$ such that $L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in S_2(p)$ (or $S_2^{*}(p)$). But then also $\text{Rot}(U^{-1})L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ holds for all $U \in \text{GL}_2(\mathcal{O})$, and thus $UD \in \mathcal{D}(S_2(p))$ (or $D(S_2^{*}(p))$). Moreover, due to the fundamental relations for $\Delta_2$ we have $D \in \text{GL}_2(\mathbb{H})$. So the first column of $D$ is non-identically zero. Therefore, like we have done several times before, the euclidean algorithm yields that there exists $U_1 \in \text{GL}_2(\mathcal{O})$ such that $U_1D = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. $U_1D \in \mathcal{D}(S_2(p))$ (or $D(S_2^{*}(p))$) implies
\[
\det(U_1D) = N(a)N(d) \in \{1, p^2, p^4\} \quad \text{or} \quad N(a)N(d) \in \{p, p^3\}
\]
in view of (4.37). On the other hand $p(U_1D)^{-1} \in \mathcal{O}^{2 \times 2}$ has to hold, again due to the fundamental relations. Note that we have
\[
p(U_1D)^{-1} = \begin{pmatrix} p & p \frac{a}{N(a)} \\ \frac{N(a)}{p} & 0 \end{pmatrix}.
\]
Therefore, $N(a) \in \{p^3, p^4\}$ or $N(d) \in \{p^3, p^4\}$ is impossible: Suppose $N(a) = p^j$, $j \geq 3$, then we obtain $N(\frac{p}{N(a)}) = p^{2j-2}$ and $N(a) = p^{2-j} \notin N_0$, which contradicts $\frac{p}{N(a)} \in \mathcal{O}$. On the other hand, if $N(a) = 1$ or $N(d) = 1$ (hence $a \in \mathcal{E}$ or $d \in \mathcal{E}$), we can multiply with $\text{diag}(1,\epsilon)$ from the left, such that $\epsilon a = 1$ or $\epsilon d = 1$. And if $N(a) = p^2$, we obtain $N(\frac{p}{N(a)}) = p^{2-2} = 1$ like above, and thus (since $\frac{p}{N(a)} \in \mathcal{O}$ has to be fulfilled) $\frac{p}{p} \in \mathcal{E}$. Hence there exists $a \in \mathcal{E}$ such that $a = \epsilon p$. Of course, the same would hold for $d$. So in this case, we can again multiply with $\text{diag}(1,\epsilon)$ or $\text{diag}(1,\epsilon)$ from the left, such that $\epsilon a = p$ or $\epsilon d = p$. And finally, if $N(a) = p$ or $N(d) = p$, we can choose $\epsilon \in \mathcal{E}$ such that $\epsilon a$ or $\epsilon d$ belongs to the transversal of $\mathcal{E} \setminus \mathcal{N}(p)$ we fixed in the assertion.
So there exists \( U_2 \in \text{GL}_2(\mathcal{O}) \) such that \( U_2 U_1 D \) is one of the following matrices:

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & t \\
0 & p
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & p
\end{pmatrix}, \quad \begin{pmatrix}
\pi & t \\
0 & \tilde{\pi}
\end{pmatrix},
\]

if \( D \in D(S_2(p)) \), or

\[
\begin{pmatrix}
1 & t \\
0 & \pi
\end{pmatrix}, \quad \begin{pmatrix}
\pi & t \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & \pi
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & p
\end{pmatrix},
\]

if \( D \in D(S_2^*(p)) \), where \( \pi \) and \( \tilde{\pi} \) belong to the fixed transversal of \( \mathcal{E} \setminus \mathcal{N}(p) \), and \( t \) is an appropriate element of \( \mathcal{O} \) such that (see above) \( \frac{p}{\mathcal{N}(a) \mathcal{N}(d)} d \mathfrak{a} t \in \mathcal{O} \) (where \( a \) and \( d \) are the new diagonal entries). Next, given \( u \in \mathcal{O} \), we have \( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{O}) \) and

\[
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\tilde{a} & t \\
0 & \tilde{d}
\end{pmatrix} = \begin{pmatrix}
\tilde{a} & t + u \tilde{d} \\
0 & \tilde{d}
\end{pmatrix}.
\]

We can choose \( u \) such that \( t + u \tilde{d} \) belongs to a fixed transversal of \( \mathcal{O} / \mathcal{O} \tilde{d} \). So in most of the cases above we are done: There exists \( U_3 \in \text{GL}_2(\mathcal{O}) \) (where in some of the cases above we choose \( U_3 = I \) for now) such that \( U_3 U_2 U_1 D \) is one of the following matrices:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & r \\
0 & p
\end{pmatrix}, \quad \begin{pmatrix}
p & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & p
\end{pmatrix}, \quad \begin{pmatrix}
\pi & t \\
0 & \tilde{\pi}
\end{pmatrix},
\]

if \( D \in D(S_2(p)) \), where \( r \) belongs to the fixed transversal of \( \mathcal{O} / p \mathcal{O} \), or

\[
\begin{pmatrix}
1 & r \\
0 & \pi
\end{pmatrix}, \quad \begin{pmatrix}
\pi & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & \pi
\end{pmatrix}, \quad \begin{pmatrix}
p & t \\
0 & p
\end{pmatrix},
\]

if \( D \in D(S_2^*(p)) \), where this time \( r \) belongs to the fixed transversal of \( \mathcal{O} / \mathcal{O} \pi \). So in the cases where no \( t \) appears on the secondary diagonal we are done, already. So let us consider the four remaining cases. Note that \( \frac{p}{\mathcal{N}(a) \mathcal{N}(d)} d \mathfrak{a} t \in \mathcal{O} \) has to hold (see above).

The first one is \( \begin{pmatrix} p & t \\ 0 & p \end{pmatrix} \). We obtain

\[
\frac{p}{\mathcal{N}(p) \mathcal{N}(p)} p t p = p^{-1} t \in \mathcal{O}
\]

if and only if \( t \in p \mathcal{O} \). So there exists \( x \in \mathcal{O} \) such that \( t = px \), and we get

\[
\begin{pmatrix}
1 & -x \\
0 & 1
\end{pmatrix} \begin{pmatrix}
p & t \\
0 & p
\end{pmatrix} = \begin{pmatrix}
p & 0 \\
0 & p
\end{pmatrix}.
\]

Next, let us consider \( \begin{pmatrix} \pi & t \\ 0 & \tilde{\pi} \end{pmatrix} \). We have \( \frac{p}{\mathcal{N}(p)} \pi t \pi \mathcal{O} \) if and only if

\[
\pi t \pi \mathcal{O} \iff \pi t p \mathcal{O} = \pi t \mathcal{O} \pi \iff \pi t \in \mathcal{O} \pi.
\]
Note that in the assertion we chose \( s \in \mathcal{O} \) (fixed for given \( \pi, \pi \)) such that \( \gcd(N(s), p) = 1 \) and \( \pi s \in \mathcal{O} \pi \). So there exist \( x, y \in \mathcal{O} \) such that \( \pi t = x \pi \) and \( \pi s = y \pi \). Let \( P = \psi_p(\pi), \bar{P} = \psi_p(\pi) \), \( S = \psi_p(s), \bar{T} = \psi_p(t), X = \psi_p(x) \) and \( Y = \psi_p(y) \). Note that due to \((4.24)\), \( S \) is invertible in \( \mathbb{F}_p^{2 \times 2} \), while \( P \) and \( \bar{P} \) are of rank 1, because \( \gcd(N(s), p) = 1 \) and \( N(\pi) = N(\pi) = p \). So in \( \mathbb{F}_p^{2 \times 2} \) we have the identities

\[
PT = X \bar{P}, \quad PS = Y \bar{P}.
\]

Just like we did in the proof of \((4.25)\), choose \( U_1, U_2, U_3, U_4 \in \text{GL}_2(\mathbb{F}_p) \) such that \( U_1 P U_2 = U_3 \bar{P} U_4 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \), and let \( U_2^{-1} T U_4 = \left( \begin{array}{cc} t_1 & t_2 \\ t_3 & t_4 \end{array} \right), U_2^{-1} S U_4 = \left( \begin{array}{cc} s_1 & s_2 \\ s_3 & s_4 \end{array} \right), U_1 X U_3^{-1} = \left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right) \), \( U_1 Y U_3^{-1} = \left( \begin{array}{cc} y_1 & y_2 \\ y_3 & y_4 \end{array} \right) \). Then the identities from above are equivalent to

\[
(U_1 P U_2)(U_2^{-1} T U_4) = (U_1 X U_3^{-1})(U_3 \bar{P} U_4), \quad (U_1 P U_2)(U_2^{-1} S U_4) = (U_1 Y U_3^{-1})(U_3 \bar{P} U_4),
\]

and thus

\[
\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_3 & 0 \end{pmatrix}, \quad \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ y_3 & 0 \end{pmatrix}.
\]

So the only restrictions (in \( \mathbb{F}_p \)) are \( t_2 = s_2 = x_3 = x_4 = 0, t_1 = x_1 \) and \( s_1 = y_1 \), but apart from that we can freely choose \( t_1, t_2, t_3, t_4 \) (while \( s_1, s_3, s_4 \) are fixed, already). Note that \( s_4 \neq 0 \), because \( U_2^{-1} S U_4 \) has to be invertible in \( \mathbb{F}_p^{2 \times 2} \). Of course, there is \( q \in \{0, \ldots, p - 1\} \) (in \( \mathbb{F}_p \)), whereas one could also choose other representatives modulo \( p \) such that \( t_4 = qs_4 \). Now, choose \( z_1, z_3 \in \mathbb{F}_p \) such that \( t_1 + z_1 = qs_1 \) and \( t_3 + z_3 = s_3 \), and define \( Z = U_2 \left( \begin{array}{cc} z_1 & 0 \\ z_3 & 0 \end{array} \right) U_3 \). We obtain

\[
(U_2^{-1} T U_4) + (U_2^{-1} Z U_3^{-1})(U_3 \bar{P} U_4) = \begin{pmatrix} t_1 & 0 \\ t_3 & qs_4 \end{pmatrix} + \begin{pmatrix} z_1 & 0 \\ z_3 & 0 \end{pmatrix} = \begin{pmatrix} q s_1 & 0 \\ q s_3 & q s_4 \end{pmatrix} = q U_2^{-1} S U_4
\]

or equivalently

\[
T + Z \bar{P} = q S.
\]

\( \psi_p \) is bijective (see \((4.24)\)), so choose \( z \in \mathcal{O} \) such that \( \psi_p(z) = Z \). And note that \( \psi_p(q) = q I \). According to the definition of \( \psi_p \) and because \( \psi_p \) is bijective, the identity above yields that there exists \( c \in \mathcal{O} \) such that \( t + (z + ct \pi) \pi = t + z \pi + cp = qs, \) and thus

\[
\begin{pmatrix} 1 & z + c \pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & t \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} \pi & qs \\ 0 & \pi \end{pmatrix}
\]

like asserted. The next case to consider is \( \begin{pmatrix} p & t \\ 0 & \pi \end{pmatrix} \). Here, we have

\[
\frac{p}{N(p)} N(\pi) \pi t p \in \mathcal{O} \iff t \pi p \in p \mathcal{O} \iff t p \in p \mathcal{O} \pi \iff t \in \mathcal{O} \pi.
\]

Hence, there exists an appropriate \( u \in \mathcal{O} \) such that \( t = u \pi \), and this gives

\[
\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & t \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & \pi \end{pmatrix}.
\]
The last case is \( \left( \begin{array}{cc} \pi & t \\ 0 & p \end{array} \right) \), and an analogous calculation yields

\[
\frac{p}{N(\pi)N(p)} p \bar{t} \pi \in \mathcal{O} \iff t \in \pi \mathcal{O},
\]

so there exists \( u \in \mathcal{O} \) such that \( t = \pi u \). Now, choose \( v \in \mathcal{O} \) such that \( \bar{t} = u + \pi v \) belongs to the transversal of \( \mathcal{O} / \pi \mathcal{O} \) we fixed in the assertion. We get

\[
\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & \pi u \\ 0 & p \end{pmatrix} = \begin{pmatrix} \pi & \pi(u + \pi v) \\ 0 & p \end{pmatrix} = \begin{pmatrix} \pi & \pi \bar{t} \\ 0 & p \end{pmatrix},
\]

and this completes the proof. \( \square \)

After we explicitly determined \( \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2(p)) \) and \( \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2^*(p)) \), we need to do the same for \( \mathcal{B}(D, S_2(p)) \) and \( \mathcal{B}(D, S_2^*(p)) \) for each \( D \in \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2(p)) \) or \( D \in \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2^*(p)) \), respectively, where this time the cosets are given by \( \{ B + SD ; S \in \text{Her}_2(\mathcal{O}) \} \) for \( \mathcal{B}(D, S_2(p)) \) or \( B \in \mathcal{B}(D, S_2^*(p)) \), respectively.

(4.39) Proposition. Let \( D \) always be one of the matrices determined in (4.38), hence an element of the explicitly given transversals of \( \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2(p)) \) or \( \text{GL}_2(\mathcal{O}) \backslash \mathcal{D}(S_2^*(p)) \), and use the notation of that proposition.

a) Concerning \( S_2(p) \), a transversal of \( \mathcal{B}(D, S_2(p)) \) with respect to the cosets \( \{ B + SD ; S \in \text{Her}_2(\mathcal{O}) \} \), where \( B \in \mathcal{B}(D, S_2(p)) \), is given by:

\[
\begin{align*}
\bullet & \ D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\bullet & \ D = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \text{ where } q \text{ runs through a transversal of } \mathcal{O} / p \mathcal{O}, \\
\bullet & \ D = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } q \text{ runs through a transversal of } \mathcal{O} / p \mathcal{O}, \\
\bullet & \ D = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} : \begin{pmatrix} b_1 & b_2 \\ b_2 & b_4 \end{pmatrix}, \text{ where } b_1 \text{ and } b_4 \text{ run through a transversal of } \mathcal{O} / p \mathcal{O}, \\
\bullet & \ D = \begin{pmatrix} \pi & qs \\ 0 & \bar{\pi} \end{pmatrix} : \begin{pmatrix} 0 & \bar{\pi}^{-1} \bar{\pi} s \pi \\ \bar{\pi} p^{-1} \bar{\pi} s \pi & 0 \end{pmatrix}, \text{ where } \bar{\pi} \text{ runs through a transversal of } \mathcal{O} / p \mathcal{O}.
\end{align*}
\]

(Note that by definition there exists \( t \in \mathcal{O} \) such that \( \bar{\pi} s = t \bar{\pi} \), and thus \( p^{-1} \bar{\pi} s \pi = \bar{t} \in \mathcal{O} \), so the last matrix equals \( \bar{\pi} \begin{pmatrix} 0 & s \\ 1 & \bar{\pi} \end{pmatrix} \).

b) Concerning \( S_2^*(p) \), a transversal of \( \mathcal{B}(D, S_2^*(p)) \) with respect to the cosets \( \{ B + SD ; S \in \text{Her}_2(\mathcal{O}) \} \), where \( B \in \mathcal{B}(D, S_2^*(p)) \), is given by:

\[
\begin{align*}
\bullet & \ D = \begin{pmatrix} 1 & r \\ 0 & \pi \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\bullet & \ D = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]
• \( D = \begin{pmatrix} p & 0 \\ 0 & \pi \end{pmatrix} : \begin{pmatrix} q & b_2 \\ \pi b_2 & 0 \end{pmatrix} \), where \( q \) runs through a transversal of \( \mathbb{Z}/p\mathbb{Z} \), while \( b_2 \) runs through a transversal of \( \mathcal{O}/\mathcal{O}\pi \).

• \( D = \begin{pmatrix} \pi & \pi \tau \\ 0 & p \end{pmatrix} : \begin{pmatrix} 0 & \pi b_3 \\ b_3 & q - \overline{\pi}b_3 \end{pmatrix} \), where \( q \) runs through a transversal of \( \mathbb{Z}/p\mathbb{Z} \), while \( b_3 \) runs through a transversal of \( \mathcal{O}/\mathcal{O}\pi \).

**Proof:** Before we are going to look at each case, note the following general fact: Let \( L = (A B D) \in \Delta_2(p) \). Then the fundamental relations in (4.34) yield \( \overline{B}D = \overline{D}B \), hence

\[
\overline{b}_1a = ab_1, \quad \overline{b}_1b + \overline{b}_3d = ab_2, \quad \overline{b}_2b + \overline{b}_4d = \overline{ab}_2 + \overline{ab}_4.
\]  

(4.34)

And in particular, if \( a, d \in \mathbb{N} \) and \( b = 0 \), \( B \) has to be of the shape

\[
B = \begin{pmatrix} b_1 & b_2 \\ \frac{d}{b_2} & b_4 \end{pmatrix},
\]

(4.35)

where \( b_1, b_2 \in \mathbb{Z} \) and \( b_2 \in \mathcal{O} \) such that \( \frac{d}{b_2} \in \mathcal{O} \). Moreover, if the identities 4.34 hold true for some \( B = (b_1 b_2) \in \mathbb{O}^{2 \times 2} \), then the fundamental relations also yield \( B \in B(D, S_2(p)) \) (or \( B \in B(D, S_2^+(p)) \), respectively). Furthermore, note that we have \( B + SD \in B(D, S_2(p)) \) (or \( B + SD \in B(D, S_2^+(p)) \), resp.) for every \( S = (s_1 s_2) \in \text{Her}_2(\mathcal{O}) \), of course, where

\[
B + SD = \begin{pmatrix} b_1 + s_1a & b_2 + s_1b + s_2d \\ b_3 + s_2a & b_4 + s_2b + s_3d \end{pmatrix}.
\]

(4.36)

We will now stick to this notation and go through each of the appearing cases.

a) Let \( D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then 4.35 yields \( B \in \text{Her}_2(\mathcal{O}) \). But then \( B(D, S_2(p)) \) is given by the single coset \( \text{Her}_2(\mathcal{O}) \), of course, with representative 0.

Next, let \( D = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \). Then 4.34 gives \( b_1 \in \mathbb{Z} \) and \( b_1r + p\overline{b}_3 = b_2 \), and we compute \( B + S_1D = \begin{pmatrix} 0 & 0 \\ 0 & \overline{b}_3 \end{pmatrix} \) for \( S_1 = \begin{pmatrix} -b_1 \overline{b}_3 \\ 0 \end{pmatrix} \in \text{Her}_2(\mathcal{O}) \). Now, we have \( \overline{b}_4 \in \mathbb{Z} \), whereas \( B + (S_2 + S_1)D = \begin{pmatrix} 0 & 0 \\ 0 & b_4 + np \end{pmatrix} \) with \( S_2 = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix} \), which means that we can choose \( n \in \mathbb{Z} \) such that \( \overline{b}_4 + np \) belongs to a fixed transversal of \( \mathbb{Z}/p\mathbb{Z} \). On the other hand, 4.34 yields that \( \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \) belongs to \( B(D, S_2(p)) \) for every \( q \in \mathbb{Z} \), and if \( \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} = SD \), then 4.36 implies \( s_1 = s_2 = 0 \) and \( q = \overline{a} + s_3p \), and thus the assertion follows for \( D \).

Now, we consider the case \( D = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \). Here, 4.35 yields \( b_3 = p\overline{b}_2 \) and \( b_4 \in \mathbb{Z} \), and again we compute \( B + S_1D = \begin{pmatrix} b_1 & 0 \\ 0 & 0 \end{pmatrix} \) for \( S_1 = \begin{pmatrix} -b_1 \overline{b}_3 \\ 0 \end{pmatrix} \in \text{Her}_2(\mathcal{O}) \). The rest is completely analogous to the previous case.

If \( D = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \), then 4.36 leads to \( B \in \text{Her}_2(\mathcal{O}) \) (and even \( \text{Her}_2(\mathcal{O}) = B(D, S_2(p)) \)), and we have

\[
B + SD = \begin{pmatrix} b_1 + ps_1 & b_2 + ps_2 \\ \overline{b}_2 + ps_2 & b_4 + ps_3 \end{pmatrix}.
\]

Hence the assertion is obvious in this case.
4.4 The quaternionic Hecke-operators $T_2(p)$

So finally, let $D = \left( \begin{array}{cc} \pi & qs \\ 0 & \pi \end{array} \right)$. Again, \(4.34\) leads to

$$b_1 \tau = \pi b_1 \iff \pi b_1 p = pb_1 \tau \iff \pi b_1 = b_1 \tau ,$$

which means $b_1 \tau = \overline{b_1} \tau \in \mathbb{Z}$. Hence, there exists $n \in \mathbb{Z}$ such that $b_1 \tau = n$, or equivalently $b_1 = \frac{n}{p} \pi \in \mathcal{O}$. But then $N(b) = p^{-1} n^2 \in \mathbb{N}_0$ yields $p | n$ (since $p$ is a prime number), and therefore $b_1 = mn$, where $m \in \mathbb{Z}$. We compute $B + S_1 D = \left( \begin{array}{cc} 0 & \tilde{b}_2 \\ \tilde{b}_3 & b_4 \end{array} \right) \in B(D, S_2(p))$ for $S_1 = \left( \begin{array}{cc} -m & 0 \\ 0 & 0 \end{array} \right) \in \text{Her}_2(\mathcal{O})$. Applying $4.34$ to this matrix gives

$$\tilde{b}_3 \tau = \pi \tilde{b}_2 .$$

Note that this is exactly the same situation which occurred in the proof of (4.38), where $b_2$ takes the role of “\(t\)”, now (see 4.33). Therefore, we can choose $z, c \in \mathcal{O}$, again, such that $b_2 + (z + c \tau) \tau = \tilde{q}s$, where $\tilde{q}$ belongs the transversal of $\mathbb{Z}/p\mathbb{Z}$. We fixed in the assertion. Therefore, let $r = z + c \tau$ and $S_2 = \left( \begin{array}{cc} r & 0 \\ 0 & 0 \end{array} \right) \in \text{Her}_2(\mathcal{O})$. The recent considerations, 4.34 and 4.36 then yield

$$B + (S_2 + S_1) D = \left( \begin{array}{cc} 0 & \tilde{q}s \\ \tilde{q}p^{-1} \tau \pi & b \end{array} \right)$$

for some appropriate $b \in \mathcal{O}$. Again due to 4.34, we have $\tilde{q}\tilde{s}qs + \tilde{b} \tau = \tilde{q}\tilde{s}qs + \pi b \in \mathbb{Z}$, and thus

$$\tilde{b} \tau = n - q\tilde{q} N(s) =: m \in \mathbb{Z}$$

for some appropriate $n \in \mathbb{Z}$. Note that we have $m^2 = N(\tilde{b} \tau) = p N(b)$, which leads to $p | m$, and therefore

$$b = \frac{m}{p} \tilde{b} ,$$

where $x = \frac{m}{p} \in \mathbb{Z}$. So choosing $S_3 = \left( \begin{array}{cc} 0 & 0 \\ 0 & -x \end{array} \right) \in \text{Her}_2(\mathcal{O})$ finally gives

$$B + (S_3 + S_2 + S_1) D = \left( \begin{array}{cc} 0 & \tilde{q}s \\ \tilde{q}p^{-1} \tau \pi & 0 \end{array} \right) .$$

On the other hand, like we already remarked in the assertion, those matrices belong to $\mathcal{O}^{2 \times 2}$, indeed, and 4.34 is obviously fulfilled, so that they are elements of $B(D, S_2(p))$. So suppose we have

$$\left( \begin{array}{cc} 0 & \tilde{q}_1 s \\ \tilde{q}_1 p^{-1} \tau \pi & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & \tilde{q}_2 s \\ \tilde{q}_2 p^{-1} \tau \pi & 0 \end{array} \right) + SD$$

for $\tilde{q}_1, \tilde{q}_2 \in \mathbb{Z}$. 4.36 leads to $s_1 = 0$ and $\tilde{q}_1 s = \tilde{q}_2 s + s_2 \tau$, or $(\tilde{q}_1 - \tilde{q}_2)s = s_2 \tau$, which implies $p N(s_2) = (\tilde{q}_1 - \tilde{q}_2)^2 N(s)$, and thus $p | (\tilde{q}_1 - \tilde{q}_2)$, since $s$ was chosen to fulfill $\gcd(N(s), p) = 1$. So by definition we have $\tilde{q}_1 \equiv \tilde{q}_2 \mod p$, and therefore the assertion follows.

b) We begin with $D = \left( \begin{array}{cc} 1 & r \\ 0 & \pi \end{array} \right)$. 4.34 gives rise to $b_1 \in \mathbb{Z}$ and $b_1 r + \overline{b}_3 \pi = b_2$. Hence we choose $S_1 = \left( \begin{array}{cc} -b_1 & -\overline{b}_3 \\ -\overline{b}_3 & 0 \end{array} \right) \in \text{Her}_2(\mathcal{O})$ and obtain $B + S_1 D = \left( \begin{array}{cc} 0 & 0 \\ 0 & b_4 \end{array} \right) \in B(D, S_2(p))$ (see 4.36).
Applying 4.34 again gives \( \tilde{b}_4 \pi = \pi \tilde{b}_4 = n \in \mathbb{Z} \), or equivalently \( \tilde{b}_4 = \frac{n}{p} \pi \in \mathcal{O} \). We have \( N(b_4) = \frac{n^2}{p} \in \mathbb{N}_0 \), and thus \( p | n \). So let \( m = \frac{n}{p} \in \mathbb{Z} \) and \( S_1 = \left( \frac{0}{-m} \right) \in \text{Her}_2(\mathcal{O}) \). Then we obtain \( B + (S_2 + S_1)D = 0 \), hence the assertion.

Next, let \( D = \left( \frac{0}{0} \right) \). This time, 4.34 yields \( \tilde{b}_1 \pi = \pi b_1 = n \in \mathbb{Z} \), \( \tilde{b}_3 = \pi b_2 \) (or \( b_3 = \tilde{b}_2 \pi \)) and \( \tilde{b}_4 \in \mathbb{Z} \). Completely the same reason like in the previous case leads to \( b_1 = m \pi \), where \( m = \frac{n}{p} \in \mathbb{Z} \). Hence we choose \( S_1 = \left( \frac{-m}{-b_2} \right) \in \text{Her}_2(\mathcal{O}) \), and in view of 4.36 we compute \( B + S_1D = 0 \), which proves the assertion.

So let us get to \( D = \left( \frac{p}{0} \right) \). Again, we have \( \tilde{b}_4 \pi = \pi b_4 \in \mathbb{Z} \) due to 4.34, and therefore \( b_4 = m \pi \) with \( m \in \mathbb{Z} \), like we have seen in the two previous cases. We define \( S_1 = \left( \frac{-m}{b_2} \right) \in \text{Her}_2(\mathcal{O}) \) and obtain \( B + S_1D = \left( \frac{b_1}{b_3} \right) \in B(D,S_2^r(p)) \), whereas 4.34 leads to \( b_1 \in \mathbb{Z} \) and \( \tilde{b}_3 \pi = p b_2 \), or equivalently \( b_3 = \pi b_2 \). Given \( S_2 = \left( \frac{0}{r} \right) \in \text{Her}_2(\mathcal{O}) \) (where \( n \in \mathbb{Z} \), \( r \in \mathcal{O} \)) we compute

\[
B + (S_2 + S_1)D = \begin{pmatrix}
\frac{b_1 + np}{\pi (\tilde{b}_2 + \tilde{r} \pi)} & b_2 + r \pi \\
\pi (\tilde{b}_2 + \tilde{r} \pi) & 0
\end{pmatrix}.
\]

Hence we can choose \( n \) and \( r \) such that \( b_1 + np \) and \( b_2 + r \pi \) each belong to the transversal of \( \mathbb{Z}/p\mathbb{Z} \) and \( \mathcal{O}/\mathcal{O} \pi \), respectively, which we fixed in the assertion. On the other hand, due to 4.34 we have \( \left( \frac{q}{\pi b_2} \frac{b_2}{0} \right) \in B(D,S_2^r(p)) \) for all \( q \in \mathbb{Z} \) and all \( b_2 \in \mathcal{O} \). So suppose we have

\[
\begin{pmatrix}
\frac{q}{\pi b_2} & b_2 \\
\pi b_2 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{q}{\pi b_2} & \tilde{b}_2 \\
\pi b_2 & 0
\end{pmatrix} + SD,
\]

then 4.36 yields \( s_3 = 0 \), \( q = \tilde{q} + ps_1 \) and \( b_2 = \tilde{b}_2 + s_2 \pi \), which obviously proves the assertion.

And finally, let \( D = \left( \frac{0}{\pi b_4} \right) \). Once more, we have \( \tilde{b}_1 \pi = \pi b_1 \in \mathbb{Z} \) due to 4.34, and therefore \( b_1 = m \pi \) with \( m \in \mathbb{Z} \), which leads to \( B + S_1D = \left( \frac{0}{b_3 b_4} \right) \in B(D,S_2^r(p)) \) for \( S_1 = \left( \frac{-m}{0} \right) \in \text{Her}_2(\mathcal{O}) \). And according to 4.34, again, we obtain \( \tilde{b}_3 p = \pi \tilde{b}_2 \) and \( \tilde{b}_2 \pi t + \tilde{b}_4 p = \pi \tilde{b}_2 + p \tilde{b}_4 \in \mathbb{Z} \), or equivalently \( \tilde{b}_2 = \pi b_3 \) and \( p \tilde{b}_4 = n - Ip \tilde{b}_3 \) - and thus \( \tilde{b}_4 = \frac{n}{p} - \tilde{b}_3 \). Note that \( \frac{n}{p} = \tilde{b}_4 + \tilde{b}_3 \in \mathcal{O} \) implies \( x = \frac{n}{p} \in \mathbb{Z} \). Therefore, given \( S_2 = \left( \frac{r}{m} \right) \in \text{Her}_2(\mathcal{O}) \) (where \( m \in \mathbb{Z} \), \( r \in \mathcal{O} \)) we compute

\[
B + (S_2 + S_1)D = \begin{pmatrix}
0 & \pi (\tilde{b}_3 + \tilde{r} \pi) \\
\tilde{b}_3 + r \pi & (x + 2 \text{Re}(r \pi t)) + mp - \tilde{r} (\tilde{b}_3 + \tilde{r} \pi)
\end{pmatrix}.
\]

So we can choose \( m \) and \( r \) appropriately such that

\[
B + (S_2 + S_1)D = \begin{pmatrix}
0 & \pi n \\
u & -\pi n
\end{pmatrix},
\]

where \( v \) und \( u \) each belong to the transversal of \( \mathbb{Z}/p\mathbb{Z} \) and \( \mathcal{O}/\mathcal{O} \pi \), respectively, which we fixed in the assertion. On the other hand, by construction and by what we computed
above, all these matrices belong to \( B(D, S_2^+(p)) \) in virtue of 4.34. So suppose we have

\[
\begin{pmatrix}
0 & \pi u_1 \\
\pi u_1 & v_1 - \overline{u}_1
\end{pmatrix} = \begin{pmatrix}
0 & \pi u_2 \\
\pi u_2 & v_2 - \overline{u}_2
\end{pmatrix} + SD,
\]

then we get \( s_1 = 0, u_1 = u_2 + \overline{s}_2\pi \) and \( v_1 - \overline{u}_1 = v_2 - \overline{u}_2 + \overline{s}_2\pi t + p s_3 \). Therefore, since we have chosen the “\( u \)'s” from a transversal of \( \mathcal{O}/\mathcal{O}\pi, u_1 = u_2 \) and \( s_2 = 0 \) follow. But then we also have \( v_1 = v_2 + p s_3 \), and this completes the proof. \( \square \)

We collect the results from (4.37), (4.38) and (4.39) in the following

(4.40) Theorem. Let \( p \neq 3 \) be a prime number. A transversal of \( \Gamma_2 \backslash S_2^+(p) \) is given by

\[
\begin{pmatrix}
p & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 1 & r \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & q & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & b_1 & b_2 & 0 \\
0 & \overline{b}_2 & b_4 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix},
\]

where

- \( r \) runs through a transversal of \( \mathcal{O}/p\mathcal{O} \),
- both \( q \) and \( \overline{q} \) run through a transversal of \( \mathbb{Z}/p\mathbb{Z} \),
- \( b_1 \) and \( b_4 \) run through a transversal of \( \mathcal{Z}/p\mathcal{Z} \), while \( b_2 \) runs through a transversal of \( \mathcal{O}/p\mathcal{O} \),
- both \( \pi \) and \( \overline{\pi} \) run through a transversal of \( \mathcal{E}\backslash\mathcal{N}(p) \),
- whenever \( \pi \) and \( \overline{\pi} \) are given, choose (exactly one) \( s \in \mathcal{O} \) such that \( \gcd(N(s), p) = 1 \) and \( \overline{\pi}s \in \mathcal{O}\overline{\pi} \) (which exists in virtue of (4.25)),

(And note that \( p^{-1}\overline{\pi}s\pi = \overline{t} \in \mathcal{O} \) holds just like in (4.39), where \( \overline{\pi}s = t\overline{\pi} \).) Furthermore, a transversal of \( \Gamma_2 \backslash S_2^+(p) \) is given by

\[
\begin{pmatrix}
p & 0 & 0 & 0 \\
-\pi \overline{t} & \pi & 0 & 0 \\
0 & 0 & 1 & r \\
0 & 0 & 0 & \pi
\end{pmatrix}, \quad \begin{pmatrix}
\pi & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & \pi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & q & b_2 \\
0 & \pi & \pi \overline{b}_2 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & \pi t
\end{pmatrix}, \quad \begin{pmatrix}
\pi & 0 & 0 & \pi \overline{b}_3 \\
-\overline{t} & 1 & b_3 & q - \overline{t} \overline{b}_3 \\
0 & 0 & \pi & \pi t \\
0 & 0 & 0 & p
\end{pmatrix},
\]

where

- \( \pi \) runs through a transversal of \( \mathcal{E}\backslash\mathcal{N}(p) \),
- whenever \( \pi \) is given, \( r \) runs through a transversal of \( \mathcal{O}/\mathcal{O}\pi \), and \( t \) runs through a transversal of \( \mathcal{O}/\pi\mathcal{O} \),
• $q$ runs through a transversal of $\mathbb{Z}/p\mathbb{Z}$,
• both $b_2$ and $b_5$ run through a transversal of $O/O\pi$.

So in particular, we have

$$\deg(T_2(p)) = \prod_{j=0}^{3} (p^j + 1).$$

**Proof:** Of course, only the assertion about the degree of $T_2(p)$ remains to be verified. Noting $\#\mathbb{Z}/p\mathbb{Z} = p$, $\#O/pO = p^4$ (which is obvious, since a transversal is given by letting the coefficients with respect to the standard basis of $O$ run through a transversal of $\mathbb{Z}/p\mathbb{Z}$), $\#E\setminus N(p) = p + 1$ (see (1.7)) and $\#O/\pi O = \#O/O\pi = p^2$ (according to (4.27)), we obviously have

$$\deg(T_2(p)) = 1 + p^5 + p^6 + p^2(p + 1)^2 + p^2(p + 1) + p^3(p + 1) + p^5(p + 1)$$

$$= \prod_{j=0}^{3} (p^j + 1),$$

where the second identity can be verified by a straightforward calculation (for example with the help of a computer algebra system like [SAGE]).

**Remark.** Note that we are only interested into a very particular and special case, namely the degree two and only the special Hecke-operators $T_2(p)$. Although we will not go any further in that direction (simply because we do not need it for this thesis and because it would take several more pages to actually work out the details), one could determine the structure of the whole Hecke-algebra like it was done in [Kr87].

We did not really work out an elementary divisor theory for our order $O$ like it was done for the Hurwitz order (cf. [Kr87]). Such a theory would be needed to analyze the Hecke-algebra of arbitrary degree. M. Raum developed a general elementary divisor theory for quaternion algebras in [Rau10], but more in an abstract and non-explicit sense. Nevertheless, combining this theory with the number theoretical background in section 4.3 one can develop an explicit elementary divisor theory for matrices over $O$. Doing so would result in the same theorem like [Kr87, thm.4] (which is for the Hurwitz order), where the greatest invariant divisor concerning the Hurwitz order (which is also denoted by $\text{inv}(a)$) has to be replaced by the greatest invariant divisor $\text{inv}(a)$ concerning $O$. The theorem would state the following: Let $A_1, A_2 \in O^{m \times n}$, where $A$ and $B$ have the same rank $q$. According to (4.22) there exist $U_1, U_2 \in \text{GL}_m(O)$, $V_1, V_2 \in \text{GL}_n(O)$ and $C = \text{diag}(c_1, \ldots, c_q), D = \text{diag}(d_1, \ldots, d_q) \in O^{q \times q}$ such that $c_1||c_2|\ldots||c_q$, $d_1||d_2|\ldots||d_q$ and $U_1A_1V_1 = (\begin{smallmatrix} C & 0 \\ \hline 0 & D \end{smallmatrix})$, $U_2A_2V_2 = (\begin{smallmatrix} 0 & C' \\ \hline D' & 0 \end{smallmatrix})$. If $\text{inv}(c_j) = \text{inv}(d_j)$ and $N(c_j) = N(d_j)$, then there exist $U \in \text{GL}_m(O)$ and $V \in \text{GL}_n(O)$ such that $UA_1V = A_2$.

Afterwards, one can develop the theory of Hecke-algebras corresponding to [Kr87]. Again, Hecke-algebras for quaternion algebras were analyzed in [Rau11] in general, but once more not in an explicit way. But putting together the work done in [Kr87] and [Rau11], and as well the calculations done above, one should obtain the following result which corresponds to the analog concerning the Hurwitz order (cf. [Kr87, thm.7, thm.8]):

For $n \geq 2$ define $\Delta_n(q)$ and $\Delta_n$ analogous to $\Delta_2(q)$ and $\Delta_2$ in (4.33), but for higher degrees.
4.4 The quaternionic Hecke-operators $T_2(p)$

Concerning the definition of the Hecke-algebra $\mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_n)$ confer [Kr87] (it is generated by all double cosets $\text{Sp}_n(\mathcal{O})K\text{Sp}_n(\mathcal{O})$, where $K \in \Delta_n$, equipped with a special composition defining the product of two such cosets). The Hecke-algebra $\mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_n)$ turns out to be commutative and decomposes into its primary components, i.e. $\mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_n) = \bigotimes_p \mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_{n,p})$ where $p$ runs through all prime numbers and $\Delta_{n,p} := \bigcup_{l \in \mathbb{N}_0} \Delta_n(p^l)$. Furthermore, define the following special double cosets:

$$T_n(i_1 \sqrt{3}, l) := \text{Sp}_n(\mathcal{O}) \text{diag}(I_{n-1}, i_1 \sqrt{3}I_l, 3I_{n-1}, i_1 \sqrt{3}I_l) \text{Sp}_n(\mathcal{O}) \subset \Delta_n(3)$$

for $l = 0, \ldots, n$, and

$$S_n(p) := \text{Sp}_n(\mathcal{O}) \text{diag}(I_n, pI_n) \text{Sp}_n(\mathcal{O}) \subset \Delta_n(p),$$

$$S_n^*(p) := \text{Sp}_n(\mathcal{O}) \text{diag}(I_{n-1}, \pi, pI_{n-1}, \pi) \text{Sp}_n(\mathcal{O}) \subset \Delta_n(p),$$

$$R_n(p, 2l) := \text{Sp}_n(\mathcal{O}) \text{diag}(I_{n-1}, pI_l, p^2I_{n-1}, pI_l) \text{Sp}_n(\mathcal{O}) \subset \Delta_n(p^2),$$

$$R_n(p, 2l - 1) := \text{Sp}_n(\mathcal{O}) \text{diag}(I_{n-1}, pI_l, p^2I_{n-1}, p\pi, pI_{l-1}) \text{Sp}_n(\mathcal{O}) \subset \Delta_n(p^2)$$

for $p \neq 3$ a prime number, $\pi \in \mathcal{N}(p)$ a fixed completely primitive element, and $l = 1, \ldots, n$. Then $\mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_{n,3})$ is the polynomial ring over $\mathbb{Z}$ in $n+1$ elements

$$T_n(i_1 \sqrt{3}, 0), \ldots, T_n(i_1 \sqrt{3}, n),$$

which are algebraically independent, and for $p \neq 3$ a prime number $\mathcal{H}(\text{Sp}_n(\mathcal{O}), \Delta_{n,p})$ is the polynomial ring in $2n+1$ elements

$$S_n(p), S_n^*(p), R_n(p, 2), R_n(p, 3), \ldots, R_n(p, 2n),$$

which are algebraically independent, too.

Of course, this is just a conjecture. But working out the details should result in such a theorem, regarding [Kr87], [Rau11] and the similarities between the Hurwitz order and $\mathcal{O}$. Furthermore, like for $S_2(p)$ and $S_2^*(p)$, the explicit shape of the cosets above can be worked out using the methods presented in this thesis (or also confer [Kr87]).

So we finally determined the exact shape of the Hecke-operators $T_2(p)$ for prime numbers $p \neq 3$. Therefore, we are now able to properly analyze the action of these Hecke-operators on quaternionic modular forms of even weight and trivial character. In particular, we are interested in how the $T_2(p)$ act on the spaces of quaternionic Maaß lifts of even weight.

So let $f \in \mathcal{M}(k; \mathcal{O})$ with attached function $\alpha^* : \mathbb{N}_0 \rightarrow \mathbb{C}$ and Fourier-expansion

$$f(Z) = -\frac{B_k}{2k} \alpha^*(0) + \sum_{0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0} \left( \sum_{d \in \mathbb{N}, d|\epsilon(T)} d^{k-1} \alpha^*(3 \det(T)/d^2) \right) e^{2 \pi i \epsilon(T)Z},$$

for all $Z \in \mathcal{H}_n(\mathbb{H})$, with $B_k$ being the $k$-th Bernoulli number and

$$\epsilon(T) = \max\{d \in \mathbb{N} ; \frac{1}{3} T \in \text{Her}_2(\mathcal{O})\}.$$
Or in view of (4.17) equivalently

\[ f(Z) = \sum_{T \in \text{Her}_2(O), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}), \]

with

\[ \alpha_f(T) = \alpha(\epsilon(T), 3 \det(T)/\epsilon(T)^2) \quad \text{for all } T \neq \text{Her}_2(O), T \geq 0, \]

where

\[ \alpha : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C}, \]

satisfying

\[ (1 - p^{k-1}W) \sum_{j \in \mathbb{N}_0} \alpha(p^j q, d)W^j = \sum_{j \in \mathbb{N}_0} \alpha(q, p^{2j}d)W^j \]

as a formal power series in \( W \) for all \( d \in \mathbb{N}_0, q \in \mathbb{N} \) and all prime numbers \( p \) with \( p \nmid q \).

Note that we also extended \( \alpha \) in 4.28 via \( \alpha(u, v) = 0 \) if \( u \not\in \mathbb{N} \) or \( v \not\in \mathbb{N}_0 \). Furthermore, let \( L = (\begin{smallmatrix} A & B \\ D & \frac{1}{p} \end{smallmatrix}) \in \Delta_2(p) \) with \( D = (d_1 \quad d_2) \). Due to the fundamental relations 4.34 we have \((D')^{-1} = \frac{1}{p} A\) (or \( A^{-1} = \frac{1}{p} D' \)) and \( A\overline{B} = B\overline{A} \in \text{Her}_2(O) \). By definition (and in view of (1.34)) we get

\[ f|_k L(Z) = (\det(D))^{-k/2}f((AZ + B)D^{-1}) = (N(d_1)N(d_4))^{-k/2}f(AZD^{-1} + BD^{-1}) \]

\[ = (N(d_1)N(d_4))^{-k/2} \sum_{T \in \text{Her}_2(O), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T, AZD^{-1} + BD^{-1})}, \quad Z \in \mathcal{H}(\mathbb{H}), \]

where

\[ \tau(T, AZD^{-1} + BD^{-1}) = \tau(T, AZD^{-1}) + \tau(T, BD^{-1}) = \tau(\overline{A}'T(D')^{-1}, Z) + \tau(T, BD^{-1}) \]

\[ = \frac{1}{p} \tau(\overline{A}'TA, Z) + \frac{1}{p} \tau(T, B\overline{A}') = \frac{1}{p} \left( \tau(T[A], Z) + \tau(T, B\overline{A}') \right) \]

in virtue of (1.46), and thus

\[ f|_k L(Z) = (N(d_1)N(d_4))^{-k/2} \sum_{T \in \text{Her}_2(O), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T, B\overline{A}')/p + 2\pi i \tau(T[A], Z)/p} \]

\[ = (N(d_1)N(d_4))^{-k/2} \sum_{\overline{T} \in \text{Her}_2(O)|A}, \overline{T} \geq 0} \alpha_f(\frac{1}{p}, \overline{T}|D')e^{2\pi i \tau(\overline{T}, D')/p + 2\pi i \tau(Z)} . \]

Note that if \( \overline{T} \in \frac{1}{p} \text{Her}_2(O)|A \), \( \overline{T} \geq 0 \) – hence there is \( T \in \text{Her}_2(O), T \geq 0 \) with \( \overline{T} = \frac{1}{p} T[A] \) – then we obviously have \( \overline{T} \in \text{Her}_2(\mathbb{H}) \) and \( \frac{1}{p} \overline{T}|D' = \frac{1}{p} T[A\overline{D}'] = T \in \text{Her}_2(O) \).

Our goal now is to determine the Fourier-expansion of \( f|_k S_2(p) \) and \( f|_k S_2^*(p) \) (and thus of \( f|_k T_2(p) \)) in terms of the original one. We have

\[ \alpha_f(\frac{1}{p}, \overline{T}|D') = \alpha(\epsilon(\frac{1}{p}, \overline{T}|D'), 3 \det(\frac{1}{p}, \overline{T}|D')/\epsilon(\frac{1}{p}, \overline{T}|D'))^2, \]

so in particular we will always have to determine \( \epsilon(\frac{1}{p}, \overline{T}|D') \) as well as \( 3 \det(\frac{1}{p}, \overline{T}|D') \) for all \( \overline{T} \).
occuring in the Fourier-expansions. Note that, due to $f|kS_2(p), f|kS_2^*(p) \in [\Gamma_2, k, 1]$ (see (4.34)), every $\tilde{T}$ occurring in these Fourier-expansions (with non-vanishing Fourier-coefficient) has to be an element of $\text{Her}_T^2(O)$, which is in general not the case for just a single summand $f|kL$. So these Fourier-coefficients will (in general) not vanish until we sum over the whole transversals determined in (4.40) – but they actually have to when we do so.

To keep it well-arranged, we will not formulate one theorem concerning this issue and then prove it, since the proof would fill many pages. Instead, we will look at each “type” of matrices occuring in the transversals of $S_2(p)$ and $S_2^*(p)$ and determine the Fourier-coefficients with respect to the sums over these types of matrices, and then formulate it in a lemma. Collecting all these conclusions will then result in the theorems we need. Therefore, we use indices for these partial sums. So numerate the types of matrices occuring in (4.40) in the order given in this theorem, which means

$$f|kS_2(p)_1 := f|k \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(4.40)

$$f|kS_2(p)_2 := \sum_{r:O/pO \atop q:Z/Z} f|k \begin{pmatrix} p & 0 & 0 \\ -\tau & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & p \end{pmatrix}$$

e tc., and the same for $S_2^*(p)_j$, so that $f|kS_2(p) = \sum_{j=1}^5 f|kS_2(p)_j$ and $f|kS_2^*(p) = \sum_{j=1}^4 f|kS_2^*(p)_j$.

And before we start, note that we have

$$T = \begin{pmatrix} m \\ t \\ n \end{pmatrix} \in \text{Her}_T^2(O), \ T \geq 0 \iff m, n \in \mathbb{N}_0, t \in \frac{\sqrt{3}}{2}O = \frac{1}{2}O^2 \supset O, \ mn - N(t) \geq 0,$$

(4.41)

and furthermore

$$2\text{Re}(ta) \in \mathbb{Z} \ \text{holds for all } t \in \frac{\sqrt{3}}{2}O \ \text{and all } a \in O,$$

(4.42)

so in particular $2\text{Re}(t) \in \mathbb{Z}$.

### 4.4.2 The action of the Hecke-operators $S_2(p)$ on Maaß lifts

Again, we will consider $S_2(p)$ first, and begin with the most trivial case $S_2(p)_1$. Here, we have $A = pI, D = I$ and $B = 0$. Let $T = \begin{pmatrix} m \\ t \\ n \end{pmatrix} \in \text{Her}_T^2(O)$. We obtain $\tilde{T} = \frac{1}{p}T[A] = \begin{pmatrix} pm \\ pt \\ pn \end{pmatrix} = pT \in p\text{Her}_T^2(O)$. So every $\tilde{T}$ occurring in the Fourier-expansion of $f|kS_2(p)_1$ fulfills $\tilde{T} \in p\text{Her}_T^2(O)$, and thus also $p|\epsilon(\tilde{T})$ and $p^2 | \det(\tilde{T})$. Of course, we have $\epsilon(\frac{1}{p}T[\tilde{T}']) = \epsilon(\frac{1}{p}\tilde{T}) = \epsilon(T) = p^{-1}\epsilon(T)$
and $3 \det\left(\frac{1}{p} \mathcal{T}[\mathcal{D}']\right) = 3 \det\left(\frac{1}{p} \mathcal{T}\right) = 3 \det(T) = 3p^{-2} \det(\mathcal{T})$. This gives

$$a_f\left(\frac{1}{p} \mathcal{T}[\mathcal{D}']\right) = a\left(\varepsilon\left(\frac{1}{p} \mathcal{T}[\mathcal{D}']\right), 3 \det\left(\frac{1}{p} \mathcal{T}[\mathcal{D}']\right) / \varepsilon\left(\frac{1}{p} \mathcal{T}[\mathcal{D}']\right)^2\right) = a\left(p^{-1} \varepsilon(\mathcal{T}), 3 \det(\mathcal{T}) / (p^{-1} \varepsilon(\mathcal{T}))^2\right)$$

in virtue of 4.39. Note that if we have $\mathcal{T} \in \text{Her}_2^\tau(O)$ such that $p \nmid \varepsilon(\mathcal{T})$. Then we obviously have $\mathcal{T} \notin p \text{Her}_2^\tau(O)$. Hence $\mathcal{T}$ does not occur in the Fourier-expansion of $f|_k S_2(p)_1$. But on the other hand we have $a(p^{-1} \varepsilon(\mathcal{T}), 3 \det(\mathcal{T}) / \varepsilon(\mathcal{T})^2) = 0$ by definition. Thus, by applying 4.38 we proved

**(4.41) Lemma.** Suppose $p \neq 3$ is a prime number and $f \in \mathcal{M}(k; O)$ with associated function $a$ according to (4.17). Then we get

$$f|_k S_2(p)_1(Z) = \sum_{\mathcal{T} \in \text{Her}_2^\tau(O), \mathcal{T} \geq 0} \beta(\mathcal{T}) e^{2\pi i \tau(\mathcal{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),$$

where

$$\beta(\mathcal{T}) = a(p^{-1} \varepsilon(\mathcal{T}), 3 \det(\mathcal{T}) / \varepsilon(\mathcal{T})^2) \quad \text{for all } 0 \neq \mathcal{T} \in \text{Her}_2^\tau(O), \mathcal{T} \geq 0.$$

To point out how the $\mathcal{T}$ not belonging to $\text{Her}_2^\tau(O)$ vanish in the Fourier-expansions once we actually sum up certain $f|_k L$, we will now consider $S_2(p)_4$ before we come to $S_2(p)_2$ and $S_2(p)_3$, because it is still a quite easy case, but the main aspects can be made clear. So we have $A = I$, $D = pI$ and $B = (b_1 b_4)$, where $b_1$ and $b_4$ run through a transversal of $Z/pZ$, while $b_2$ runs through a transversal of $O/pO$. Let $T = (\frac{m}{\mathcal{T}})$ in $\text{Her}_2^\tau(O)$. This time, we obtain

$$\mathcal{T} = \frac{1}{p} T[A] = \left(\frac{p^{-1} m}{p^{-1} \mathcal{T}}\right) = p^{-1} T \in p^{-1} \text{Her}_2^\tau(O),$$

so 4.38 yields

$$f|_k S_2(p)_4(Z) = p^{-2k} \sum_{\mathcal{T} \in \text{Her}_2^\tau(O), \mathcal{T} \geq 0} \left(\sum_{B = (b_1 b_2) / \mathcal{O}/pO} e^{2\pi i \tau(T, B)} / p\right) \varepsilon_f(T) e^{2\pi i \tau(T, Z) / p}.$$

According to (1.46), we have

$$\tau(T, B) = mb_1 + nb_4 + 2 \text{Re}(\overline{b_2}),$$

and thus

$$\sum_{B = (b_1 b_2) / \mathcal{O}/pO} e^{2\pi i \tau(T, B) / p} = \sum_{b_1 = 0}^{p-1} \left(\sum_{b_4 = 0}^{p-1} e^{2\pi i b_4 / p}\right) b_1 \sum_{b_4 = 0}^{p-1} \left(\sum_{b_2 / \mathcal{O}/pO} e^{2\pi i b_2 / p}\right) b_4 = e^{2\pi i 2 \text{Re}(\overline{b_2}) / p}.$$

If $p \nmid m$, then the formula for (finite) geometric series gives

$$\sum_{b_1 = 0}^{p-1} \left(\frac{e^{2\pi i b_1 / p}}{1 - e^{2\pi i b_1 / p}}\right)^{b_1} = \left(1 - \frac{e^{2\pi i b_1 / p}}{1 - e^{2\pi i b_1 / p}}\right)^{b_1} = 0.$$
Hence we have
\[
\sum_{b_1=0}^{p-1} \left( e^{2\pi i b_1/p} \right)^{b_1} = \begin{cases} 
0, & \text{if } p \nmid m, \\
 p, & \text{if } p|m. 
\end{cases}
\]  
(4.43)

Of course, the same holds for the sum over \( b_4 \). But we also get an analogous result for the sum over \( b_2 \): Let \( b_2 = a_0 + a_1 \frac{1+i\sqrt{3}}{2} + a_2 i_2 + a_3 \frac{1+i\sqrt{3}}{2} i_2 \), then the linearity property of \( \text{Re}(\cdot) \) gives
\[
\text{Re}(Tb_2) = a_0 \text{Re}(T) + a_1 \text{Re}(T \frac{1+i\sqrt{3}}{2}) + a_2 \text{Re}(iT_2) + a_3 \text{Re}(T \frac{1+i\sqrt{3}}{2} i_2). 
\]

And note again that \( 2 \text{Re}(Ta) \in \mathbb{Z} \) for all \( a \in \mathcal{O} \). So we have
\[
\sum_{b_2:O/p\mathcal{O}} \left( e^{2\pi i 2 \text{Re}(Tb_2)/p} \right)^{a_0} \sum_{a_1=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(T \frac{1+i\sqrt{3}}{2})/p} \right)^{a_1} \sum_{a_2=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(iT_2)/p} \right)^{a_2} \sum_{a_3=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(T \frac{1+i\sqrt{3}}{2} i_2)/p} \right)^{a_3}
\]
But then 4.43 yields that this sum is \( p^4 \) if \( p|2 \text{Re}(\tau \epsilon) \) holds for all \( \epsilon \in \{ 1, \frac{1+i\sqrt{3}}{2}, i_2, \frac{1+i\sqrt{3}}{2} i_2 \} \), and 0 otherwise. Of course, the linearity of \( \text{Re}(\cdot) \) leads to \( p|2 \text{Re}(\tau \epsilon) \) for all \( a \in \mathcal{O} \) if \( p|2 \text{Re}(\tau \epsilon) \) holds, indeed. By definition (and since \( \frac{1+i\sqrt{3}}{2} \mathcal{O} = \mathcal{O} \frac{1+i\sqrt{3}}{2} \)), there exists \( x \in \mathcal{O} \) such that \( T = x \frac{1+i\sqrt{3}}{2} \). So choosing \( a = -\epsilon_1 i_1 \sqrt{3} \), where \( \epsilon \in \mathcal{E} \), we have \( p|2 \text{Re}(\tau \epsilon \alpha) \), in particular. But then exactly the same considerations like in the proof of (4.21)b yield \( \tau \in p \frac{1+i\sqrt{3}}{2} \mathcal{O} \) (which also implies \( t \in p \frac{1+i\sqrt{3}}{2} \mathcal{O}, \) of course). Therefore, we finally obtain
\[
\sum_{b_2:O/p\mathcal{O}} e^{2\pi i 2 \text{Re}(Tb_2)/p} = \begin{cases} 
0, & \text{if } t \notin p \frac{1+i\sqrt{3}}{2} \mathcal{O}, \\
p^4, & \text{if } t \in p \frac{1+i\sqrt{3}}{2} \mathcal{O}, 
\end{cases}
\]  
(4.44)
as thus
\[
\sum_{b_1,b_4:Z/p\mathcal{O},b_2:O/p\mathcal{O}} \left( e^{2\pi i (T,B)/p} \right)^{a_0} \sum_{a_1=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(T \frac{1+i\sqrt{3}}{2})/p} \right)^{a_1} \sum_{a_2=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(iT_2)/p} \right)^{a_2} \sum_{a_3=0}^{p-1} \left( e^{2\pi i 2 \text{Re}(T \frac{1+i\sqrt{3}}{2} i_2)/p} \right)^{a_3}
\]
So we can continue the calculation from above and obtain
\[
f|k S_2(p) e(Z) = p^{6-2k} \sum_{T \in p \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i (T,B)} = p^{6-2k} \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(pT) e^{2\pi i (T,B)}.
\]

Of course, we have \( \epsilon(pT) = p \epsilon(T) \) and \( 3 \text{det}(pT) = 3p^2 \text{det}(T) \), which leads to
\[
\alpha_f(pT) = \alpha(p \epsilon(T), 3 \text{det}(T)/\epsilon(T)^2)
\]
for all \( 0 \neq T \in \text{Her}_2(\mathcal{O}), T \geq 0 \) (see 4.37). So we proved
(4.42) Lemma. Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we get
\[
f_{|k} S_2(p) |4(Z) = \sum_{\tilde{T} \in \text{Her}_2(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2 \pi i \tau(\tilde{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]
where
\[
\beta(\tilde{T}) = p^{6-2k} \alpha(p \varepsilon(\tilde{T}), 3 \det(\tilde{T})/e(\tilde{T})^2) \quad \text{for all } 0 \neq \tilde{T} \in \text{Her}_2(\mathcal{O}), \tilde{T} \geq 0.
\]

Before we get to the quite complicated cases \( S_2(p)_2 \) and \( S_2(p)_3 \), we will consider the easier case \( S_2(p)_3 \), first – mainly because this time we have an easy example of what we have to pay attention to when determining the needed content of \( \frac{1}{p} \tilde{T}[\mathcal{D}] \) (see 4.39). So we have
\[
A = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad D = \left( \begin{array}{cc} 0 & q \\ 0 & 0 \end{array} \right), \quad B = \left( \begin{array}{cc} q & 0 \\ 0 & 0 \end{array} \right),
\]
where \( q \) will run through a transversal of \( \mathbb{Z}/p \mathbb{Z} \).

This time, given \( T = \left( \begin{array}{cc} m & i \\ i^n & n \end{array} \right) \in \text{Her}_2(\mathcal{O}) \), we have \( \tilde{T} = \frac{1}{p} T[A] = \left( \begin{array}{cc} n^{-1}m & i \\ i & pn \end{array} \right), \quad B \tilde{T} = \left( \begin{array}{cc} q & 0 \\ 0 & 0 \end{array} \right) \) and \( \tau(T, B \tilde{T})/p = p^{-1}q \), so 4.38 yields
\[
f_{|k} S_2(p)_3(Z) = p^{-k} \sum_{\tilde{T} = \left( \begin{array}{cc} \tilde{m} & \tilde{i} \\ \tilde{i} & \tilde{n} \end{array} \right) \in \text{Her}_2(\mathcal{O}), \tilde{T} \geq 0} \left( \sum_{q \mathbb{Z}/p \mathbb{Z}} \left( e^{2 \pi i q/m} \right)^n \right) \alpha_f(T) e^{2 \pi i \tau(\tilde{T}[\mathcal{D}]).}
\]

Again, in view of 4.43 the Fourier-coefficient of \( \tilde{T} \) vanishes if \( p \nmid m \), and equals \( p \alpha_f(T) \) otherwise. So the Fourier-expansion ranges over \( \tilde{T} \in \text{Her}_2(\mathcal{O}) \) as desired. On the other hand, we have seen that every \( \tilde{T} \) occurring in the Fourier-expansion is of the shape
\[
\tilde{T} = \begin{pmatrix} \tilde{m} & \tilde{i} \\ \tilde{i} & \tilde{n} \end{pmatrix},
\]
where \( \tilde{m} \in \mathbb{N}_0, \tilde{n} \in p \mathbb{N}_0 \) and \( \tilde{i} \in \frac{i}{\sqrt{3}} \mathcal{O} \). According to 4.39, we have to determine \( \varepsilon(\frac{1}{p} \tilde{T}[\mathcal{D}]) \) and as well 3 \det(\frac{1}{p} \tilde{T}[\mathcal{D}]), where
\[
\frac{1}{p} \tilde{T}[\mathcal{D}] = \begin{pmatrix} pm & \tilde{t} \\ \tilde{t} & p^{-1} \tilde{n} \end{pmatrix}.
\]
3 \det(\frac{1}{p} \tilde{T}[\mathcal{D}]) = 3 \det(\tilde{T}) is obvious, so let us determine the content of \( \frac{1}{p} \tilde{T}[\mathcal{D}] \) (in terms of the content of \( \tilde{T} \)).

So let \( p \mid q = \varepsilon(\tilde{T}) \), where \( p \nmid q \). This means we have \( p \mid q | \tilde{m}, \tilde{n} \) and \( \tilde{i} \in p \mathcal{O} \), but the claim would be wrong for every non-trivial multiple of \( p \mid q \). And note that \( \tilde{i} \in p \mathcal{O} \) is a condition concerning the coefficients: \( \tilde{i} = \frac{x}{\sqrt{3}} \) for some \( x \in \mathcal{O} \), and \( \tilde{i} \in p \mathcal{O} \) is equivalent to \( x \in p \mathcal{O} \), and this in turn is equivalent to say that the coefficients of \( x \) in the standard basis of \( \mathcal{O} \) are elements of \( p \mathcal{O} \).

So suppose \( x = x_0 + x_1 \frac{1+i\sqrt{3}}{2} + x_2 i_2 + x_3 \frac{1+i\sqrt{3}}{2} i_2 \), then we have
\[
\varepsilon(\tilde{T}) = \gcd(\tilde{m}, \tilde{n}, x_0, x_1, x_2, x_3).
\]
But obviously, this leads to
\[ \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = \gcd(p \bar{m}, p^{-1} \bar{n}, x_0, x_1, x_2, x_3) \in \{ p^{i-1} q, p^i q, p^{i+1} q \} . \]
And of course we have \( \gcd(p \bar{m}, x_0, x_1, x_2, x_3) \geq p^i q \) by assumption, thus
\[ \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^{i-1} q \iff p^i q \nmid p^{-1} \bar{n} \iff p^{i+1} q \nmid \bar{n} . \]
(Note that this implies \( j \geq 1 \), since \( p \mid \bar{n} \) holds by assumption.) So let us assume \( p^{i+1} q \nmid \bar{n} \) (so \( p^i q \mid p^{-1} \bar{n} \)). Note that we have \( p^{i+1} q \nmid p \bar{m} \), and still \( x \in p^i q \mathcal{O} \). Therefore,
\[ \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^i q \iff p^{i+2} q \nmid \bar{n} \lor x \notin p^{i+1} q \mathcal{O} \]
follows, and so finally
\[ \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^{i+1} q \iff p^{i+2} q \nmid \bar{n} \land x \in p^{i+1} q \mathcal{O} . \]
Furthermore, we have
\[ 3 \det(\tilde{T}) = 3 \bar{m} \bar{n} - N(x) \in p^{2j+2} q^2 \mathbb{N}_0 , \]
since \( \bar{m}, \bar{n} \in p^i q \mathbb{Z} \) and \( x \in p^i q \mathcal{O} \) (and thus \( N(x) \in p^{2j+2} q^2 \mathbb{N}_0 \)) by assumption. So let us go through the cases, again.

If we have \( p^{i+2} q \nmid \bar{n} \) and \( x \in p^{i+1} q \mathcal{O} \) (hence \( \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^{i+1} q \)), then \( p^{i+1} q \nmid \bar{m} \) to hold, or otherwise \( \epsilon(\tilde{T}) = p^{i+1} q \) would have to follow. And moreover, we have \( 3 \det(\tilde{T}) = 3 \bar{m} \bar{n} - N(x) \in p^{2j+2} q^2 \mathbb{N}_0 \).

Next, let us assume \( p^{i+1} q \nmid \bar{n} \) and \( p^{i+2} q \nmid \bar{n} \) or \( x \notin p^{i+1} q \mathcal{O} \) (hence \( \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^i q \)). If \( p^{i+2} q \nmid \bar{n} \) holds, then \( p^{i+1} q \nmid \bar{m} \) or \( x \notin p^{i+1} q \mathcal{O} \) has to hold, since we have \( \epsilon(\tilde{T}) = p^i q \) by assumption. Note that if we actually have \( p^{i+2} q \nmid \bar{n} \), \( p^{i+1} q \nmid \bar{m} \) (but \( p^{i+1} \nmid \bar{n} \) and \( p^i q \nmid \bar{m} \) by assumption) and \( x \in p^{i+1} q \mathcal{O} \), then we obviously obtain \( 3 \det(\tilde{T}) = 3 \bar{m} \bar{n} - N(x) \in p^{2j+2} q^2 \mathbb{N}_0 \), but \( 3 \det(\tilde{T}) = 3 \bar{m} \bar{n} - N(x) \notin p^{2j+2} q^2 \mathbb{N}_0 \). On the other hand, \( x \notin p^{i+1} q \mathcal{O} \) yields no further insights in any case.

And finally, assume \( p^{i+1} q \nmid \bar{n} \) (but still \( p^i q \nmid \bar{n} \) by assumption). We already remarked that this implies \( j \geq 1 \), since \( p \nmid \bar{n} \) to hold. But note that we can actually allow \( j = 0 \), hence \( p \nmid \bar{n} \) in this case: The considerations above yield that if the case \( p^{i+1} q \nmid \bar{n} \) is given, then \( \epsilon \left( \frac{1}{p} \tilde{T} [D'] \right) = p^{-1} \epsilon(\tilde{T}) \) holds, whereas we always have \( 3 \det(\tilde{T}) = 3 \det(\tilde{T}) \). So according to \( 4.39 \), the Fourier-coefficient attached to \( \tilde{T} \) equals \( a(p^{-1} \epsilon(\tilde{T}), 3p^2 \det(\tilde{T})/\epsilon(\tilde{T})^2) \). But this also holds for \( j = 0 \), since then we have \( p^{-1} \epsilon(\tilde{T}) \notin \mathbb{N} \) and thus \( a(p^{-1} \epsilon(\tilde{T}), 3p^2 \det(\tilde{T})/\epsilon(\tilde{T})^2) = 0 \), which actually has to hold since no \( \tilde{T} \) with \( p \nmid \bar{n} \) appears in the Fourier-expansion. And as a final note, this case yields no further insights in the shape of \( \tilde{n}, x \) or \( \det(\tilde{T}) \).

Collecting all results from above and applying \( 4.38 \) and \( 4.39 \) gives us

**Lemma.** Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to \( 4.17 \). Then we get
\[
{f \mid}_k S_2(p)_3(Z) = \sum_{\tilde{T} \in \text{Her}_2(\mathcal{O}), T \geq 0} \beta(\tilde{T}) e^{2\pi i \gamma(\tilde{T}, Z)} , \quad Z \in \mathcal{H}(\mathbb{H}) ,
\]
where we have the following cases for \( 0 \neq \tilde{T} = (\frac{m}{i} \frac{t}{n}) \in \text{Her}_2^\epsilon(\mathcal{O}), \tilde{T} \geq 0 \), with \( \tilde{r} = \frac{i}{\sqrt{3}}x \) for some appropriate \( x \in \mathcal{O} \):

- If \( \text{pe}(\tilde{T}) \nmid \tilde{n} \), then
  \[
  \beta(\tilde{T}) = p^{1-k} \alpha(p^{-1} \text{e}(\tilde{T}), 3p^2 \det(\tilde{T}) / \text{e}(\tilde{T})^2).
  \]

- If \( \text{pe}(\tilde{T}) | \tilde{n} \), but \( p^2 \text{e}(\tilde{T}) \nmid \tilde{n} \) or \( x \notin \text{pe}(\tilde{T}) \mathcal{O} \), then
  \[
  \beta(\tilde{T}) = p^{1-k} \alpha(\text{e}(\tilde{T}), 3 \det(\tilde{T}) / \text{e}(\tilde{T})^2).
  \]

Furthermore, \( \text{pe}(\tilde{T}) \nmid \tilde{m} \) or \( x \notin \text{pe}(\tilde{T}) \mathcal{O} \) holds in all cases, and if \( p^2 \text{e}(\tilde{T}) \nmid \tilde{n} \), \( \text{pe}(\tilde{T}) \nmid \tilde{m} \) and \( x \in \text{pe}(\tilde{T}) \mathcal{O} \), then
\[
3 \det(\tilde{T}) \in p^2 \text{e}(\tilde{T})^2 \mathbb{N}_0
\]
holds.

- If \( p^2 \text{e}(\tilde{T}) \nmid \tilde{n} \) and \( x \in \text{pe}(\tilde{T}) \mathcal{O} \), then
  \[
  \beta(\tilde{T}) = p^{1-k} \alpha(\text{e}(\tilde{T}), 3p^{-2} \det(\tilde{T}) / \text{e}(\tilde{T})^2),
  \]
  where \( 3p^{-2} \det(\tilde{T}) / \text{e}(\tilde{T})^2 \in \mathbb{N}_0 \) holds, indeed.

Next, we have to consider the more complicated cases and start with \( S_2(p)_2 \). So we have \( A_r = (\frac{p}{r T} \frac{0}{1}) \), \( D_r = (\frac{1}{0} \frac{r}{p}) \) (where \( r \) will run through a transversal of \( O/pO \)) and \( B = (\frac{0}{0} \frac{0}{q}) \) (where \( q \) will run through a transversal of \( Z/pZ \)). So given \( T = (\frac{m}{n} \frac{t}{i} \frac{l}{n}) \in \text{Her}_2^\epsilon(\mathcal{O}) \), we have
\[
\tilde{T} = \frac{1}{p} T[A_r] = \begin{pmatrix} pm - rT - r\tau + N(r)p^{-1}n & t - rp^{-1}n \\ i - r\tau p^{-1}n & p^{-1}n \end{pmatrix},
\]
\[
B\tilde{A}_r = (\frac{0}{0} \frac{0}{q}) \quad \text{and} \quad \tau(T, B\tilde{A}_r)/p = p^{-1}qn, \]
so 4.38 yields again
\[
f|_k S_2(p)_2(Z) = p^{-k} \sum_{r:O/pO} \sum_{T=(\frac{m}{n} \frac{t}{i} \frac{l}{n}) \in \text{Her}_2^\epsilon(\mathcal{O}), T \geq 0} \left( \sum_{q:Z/pZ} \left( e^{2\pi in/p} \right)^q \right) \alpha_f(T) e^{2\pi i \frac{k}{p} T[A_r], Z}.
\]

And so once more in view of 4.43, the Fourier-coefficient of \( \tilde{T} \) vanishes if \( p \nmid n \), and equals \( p \alpha_f(T) \) otherwise. So the Fourier-expansion ranges over \( \tilde{T} \in \text{Her}_2^\epsilon(\mathcal{O}) \) as desired. Now, let \( 0 \neq \tilde{T} = (\frac{1}{p} T[A_r] \), where \( p \nmid n \), and \( \tilde{T} \in \text{Her}_2^\epsilon(\mathcal{O}) \), \( \tilde{T} \geq 0 \) be one of those matrices occuring in the Fourier-expansion, given by
\[
\tilde{T} = \begin{pmatrix} \tilde{m} & \tilde{i} \\ \tilde{t} \ & \tilde{n} \end{pmatrix}.
\]

Like in the other cases, we need to analyze \( \frac{1}{p} \tilde{T}[D_r] \). A simple calculation yields
\[
T_r := \frac{1}{p} \tilde{T}[D_r] = \begin{pmatrix} p^{-1}(\tilde{m} + \tilde{t} + \tilde{r} + N(r)\tilde{n}) & \tilde{t} + r\tilde{n} \\ \tilde{r} + \tilde{m} & p\tilde{n} \end{pmatrix}.
\]
Note that we have \( \frac{1}{p} A_r = p^{-2} \tilde{T}[D^r(A)] = p^{-2} \tilde{T}[pI] = \tilde{T} \), and the second diagonal entry is divisible by \( p \), indeed. So due to what we have seen above, \( \tilde{T} \) occurs in the Fourier-expansion of \( f|_k S_2(p) \) if and only if \( T_r \in \text{Herm}_2(\mathcal{O}) \) for at least one \( r \in \mathcal{O} \), and thus the only condition that has to be fulfilled is

\[
\tilde{m} + r\tilde{t} + i\tilde{r} + N(r)\tilde{n} \in p\mathbb{Z}.
\]

Note that we have to sum over \( r \), hence there might exist several \( T_r \) such that \( T_r[A_r] = \tilde{T} \). We already have

\[
det(T_r) = \tilde{m}\tilde{n} + r\tilde{t}\tilde{n} + i\tilde{r}\tilde{n} + N(r)\tilde{n}^2 - N(\tilde{t}) - N(r)\tilde{n}^2 - i\tilde{r}\tilde{n} - r\tilde{t}\tilde{n} = \tilde{m}\tilde{n} - N(\tilde{t}) = \det(\tilde{T}).
\]

So let us determine \( \varepsilon(T_r) \) in terms of \( \varepsilon(\tilde{T}) \), now. Suppose \( \varepsilon(\tilde{T}) = p^i q \), where \( p \nmid q \). Choose \( x = x_0 + x_1 \frac{1+i\sqrt{3}}{2} + x_2 i_2 + x_3 \frac{1+i\sqrt{3}}{2} i_2 \in \mathcal{O} \) such that \( \tilde{t} = \frac{i}{\sqrt{3}} x \). Again, we have

\[
\varepsilon(\tilde{T}) = \gcd(\tilde{m}, \tilde{n}, x_0, x_1, x_2, x_3)
\]

like in 4.45. Note that we have \( x \in p^i q \mathcal{O} \), so there exists \( y \in \mathcal{O} \) such that \( x = p^i q y \). This gives

\[
r\tilde{t} + i\tilde{r} = 2 \text{Re}(i\tilde{r}) = p^i q \cdot 2 \text{Re}(\frac{i}{\sqrt{3}} y\tilde{r}) \in p^i q \mathbb{Z},
\]

since we have \( 2 \text{Re}(ta) \in \mathbb{Z} \) for all \( t \in \frac{i}{\sqrt{3}} \mathcal{O} \) and all \( a \in \mathcal{O} \) (see 4.42). So we obviously get

\[
\varepsilon(T_r) = p^k p^i q^{-1}
\]

for some \( k \in \mathbb{N}_0 \). Suppose \( k \geq 3 \). Then this implies \( p^{i+2} q | \tilde{m} \tilde{n} \), or \( p^{i+1} q | \tilde{n} \), and thus \( \tilde{t} + r\tilde{n} \in p^{i+2} q \frac{i}{\sqrt{3}} \mathcal{O} \subset p^{i+1} q \frac{i}{\sqrt{3}} \mathcal{O} \) leads to \( \tilde{t} \in p^{i+1} q \frac{i}{\sqrt{3}} \mathcal{O} \), since we have \( r\tilde{n} \in p^{i+1} q \frac{i}{\sqrt{3}} \mathcal{O} \) by assumption.

This is equivalent to \( x \in p^{i+1} q \mathcal{O} \). So the same calculation like above yields \( r\tilde{t} + i\tilde{r} \in p^{i+1} q \mathbb{Z} \).

But then \( p^{-1} (\tilde{m} + r\tilde{t} + i\tilde{r} + N(r)\tilde{n}) \in p^{i+2} q \mathbb{Z} \) (and thus \( \tilde{m} + r\tilde{t} + i\tilde{r} + N(r)\tilde{n} \in p^{i+3} q \mathbb{Z} \subset p^{i+1} q \mathbb{Z} \)) implies \( \tilde{m} \in p^{i+1} q \mathbb{Z} \), and we obtain \( \varepsilon(\tilde{T}) \geq p^{i+1} q \) as a contradiction. So we get

\[
\varepsilon(T_r) \in \{ p^{i-1} q, p^i q, p^{i+1} q \}.
\]

But before we can actually determine \( \varepsilon(T_r) \), we have to analyze the “existence” of \( T_r \), first, which means we have to determine the conditions that \( \tilde{m}, \tilde{n} \) and \( \tilde{t} \) (or \( x \)) have to fulfill such that \( \tilde{m} + 2 \text{Re}(i\tilde{r}) + N(r)\tilde{n} \in p\mathbb{Z} \) holds true (see above). There are two cases:

- Assume \( \tilde{n} \in p\mathbb{Z} \). Then the condition is equivalent to

\[
\tilde{m} + 2 \text{Re}(i\tilde{r}) \in p\mathbb{Z}.
\]

Again, note that \( x \in l\mathcal{O} \) for some \( l \in \mathbb{N} \) implies \( 2 \text{Re}(i\tilde{r}) \in l\mathbb{Z} \). So if we have \( x \in p\mathcal{O} \), then this implies \( \tilde{m} \in p\mathbb{Z} \) (and all \( r \) are possible, which are \( p^4 \) in total). So let us assume \( x \notin p\mathcal{O} \). But then according to (4.28),

\[
\varphi : \mathcal{O} / p\mathcal{O} \to \mathbb{Z} / p\mathbb{Z}, \quad a + p\mathcal{O} \mapsto 2 \text{Re}(i\tilde{r}) + p\mathbb{Z}
\]
is a surjective group-homomorphism (with respect to “+”). And therefore, because 
\#O/pO = p^3 and \#Z/pZ = p, every element in Z/pZ is the image under r \mapsto 2 \text{Re}(iT) of exactly p^3 of the r running through a transversal of O/pO. And thus, given m (where it does not matter if \( m \in pZ \) or not), exactly \( p^3 \) of the r running through a transversal of O/pO (which are \( p^4 \) in total) fulfill \( m + 2 \text{Re}(iT) \in pZ \). Therefore, we have:

- \( x \in pO \): All \( p^4 \) of the \( T_r \) “exist” and fulfill \( T_r[A_r] = \tilde{T} \) if and only if \( \tilde{m} \in pN_0 \).
- \( x \notin pO \): For every given \( \tilde{m} \), only \( p^3 \) of the \( T_r \) “exist” and fulfill \( T_r[A_r] = \tilde{T} \). If \( r \in O/pO \) yields a proper \( T_r \) or not is completely determined by \( \tilde{m} \mod p \).

- Now, assume \( \tilde{n} \notin pZ \). Note that this implies that there is \( n \in Z \) such that \( n\tilde{n} \equiv 1 \mod p \).

Here, we have to differ between a lot of cases. First, assume \( x \in pO \), then just like above the condition is reduced to \( \tilde{m} + N(r)\tilde{n} \in pZ \) or equivalently

\[
N(r) \equiv -n\tilde{m} \mod p .
\]

But this issue is solved by (4.24): \( \psi_p : O/pO \to (Z/pZ)^{2\times 2} \simeq \mathbb{F}_p^{2\times 2} \) is an isomorphism and fulfills \( N(r) \equiv \det(\psi_p(r)) \mod p \) for all the \( r \) running through the fixed transversal of \( O/pO \). Hence, given \( \tilde{n} \) (and thus \( n \)) and \( \tilde{m} \) the issue is reduced to determine the number of \( r \) fulfilling

\[
\det(\psi_p(r)) \equiv -n\tilde{m} \mod p .
\]

If \( \tilde{m} \in pZ \), then the identity above means that \( \psi_p(r) \) has to be singular in \( \mathbb{F}_p^{2\times 2} \). A simple consideration yields that there exist exactly \( p^2 + (p^2 - 1) \cdot p = p^3 + p^2 - p \) singular matrices in \( \mathbb{F}_p^{2\times 2} \). On the other hand, if \( \tilde{m} \notin pZ \), then \( -n\tilde{m} \) is invertible in \( Z/pZ \), hence we have \( \psi_p(r) \in \text{GL}_2(\mathbb{F}_p) \). Another easy consideration yields \# \text{GL}_2(\mathbb{F}_p) = \frac{(p^2 - 1)(p^2 - p)}{2} \). And since \( \det : \text{GL}_2(\mathbb{F}_p) \to \mathbb{F}_p^* \) is a surjective homomorphism, exactly \( (p^2 - 1)(p^2 - p)/(p - 1) = p^3 - p \) of them fulfill \( \det(\psi_p(r)) \equiv -n\tilde{m} \mod p \). And again, given \( \tilde{n} \), if \( r \in O/pO \) yields a proper \( T_r \) or not is completely determined by \( \tilde{m} \mod p \).

Next, suppose \( x \notin pO \), but \( p | N(x) \). We have \( 2 \text{Re}(iT) = \frac{1}{2} \cdot 2 \text{Re}(i_1\sqrt{3}x) \). Note that \( p \neq 3 \) implies that 3 is invertible in \( Z/pZ \). For to not get confused by too many variables, let \( 3^{-1} \) stand for an \( l \in Z \) such that \( 3l \equiv 1 \mod p \), for now. So \( \tilde{m} + 3^{-1} \cdot 2 \text{Re}(i_1\sqrt{3}x) + N(r)\tilde{n} \equiv 0 \mod p \).

Define \( R = \psi_p(T) \) and \( X = \psi_p(i_1\sqrt{3}x) \). Then according to (4.24) this identity is equivalent to

\[
3^{-1} \text{tr}(XR) + \tilde{n} \det(R) \equiv -\tilde{m} \mod p .
\]

Note that we have \( N(i_1\sqrt{3}x) \in pN_0 \) by assumption, and thus \( \det(X) = 0 \in \mathbb{F}_p \). On the other hand, \( x \notin pO \) implies \( i_1\sqrt{3}x \notin pO \), or otherwise \( 3x \in p(-i_1\sqrt{3})O \subset pO \) follows. But \( x \notin pO \) means that not all coefficients of \( x \) in the standard basis of \( O \) are divisible by \( p \), and the same holds for \( 3x \) then, since \( p \neq 3 \). So we have \( \det(X) = 0 \in \mathbb{F}_p \), but \( X \neq 0 \in \mathbb{F}_p^{2\times 2} \), hence \( X \) has rank 1. But then simple linear algebraic calculus yields that there exists \( U \in \text{GL}_2(\mathbb{F}_p) \) with \( \det(U) = 1 \in \mathbb{F}_p^* \) such that \( UXU^{-1} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \in \mathbb{F}_p^{2\times 2} \), where \( (a,c) \neq (0,0) \in \mathbb{F}_p^2 \). Note that \( R \mapsto U^{-1}R =: \tilde{R} \) is a bijection, and we have

\[
3^{-1} \text{tr}(XR) + \tilde{n} \det(R) \equiv 3^{-1} \text{tr}(XUU^{-1}R) + \tilde{n} \det(U^{-1}R) \mod p ,
\]
and thus we have to solve
\[ 3^{-1} \text{tr} \left( \left( \begin{array}{cc} a & 0 \\ c & 0 \end{array} \right) \tilde{R} \right) + \tilde{n} \det(\tilde{R}) \equiv -\tilde{m} \mod p. \]

Let \( \tilde{R} = \left( \begin{array}{cc} r_1 & r_2 \\ r_3 & r_4 \end{array} \right) \). One computes \( \text{tr}(XU\tilde{R}) = ar_1 + cr_2 \). If \( a \neq 0 \in F_p \) or \( c \neq 0 \in F_p \), then \( r_1 \mapsto ar_1 + cr_2 \) (or \( r_2 \mapsto ar_1 + cr_2 \)) is a bijection in \( F_p \), hence running through all \( p^2 \) possible combinations of \( (r_1, r_2) \) means that \( \text{tr}(XU\tilde{R}) \) takes every value in \( F_p \) \( p \)-times. Now, we have two cases: If \( r_1 = r_2 = 0 \in F_p \), then \( \det(\tilde{R}) = 0 \in F_p \) holds for all \( p^2 \) possible combinations of \( (r_3, r_4) \). But if \( (r_1, r_2) \neq (0, 0) \in F_p \), then a simple consideration yields that exactly \( p \) combinations of \( (r_3, r_4) \) yield a singular matrix \( \tilde{R} \), while \( p^2 - p \) yield \( \tilde{R} \in GL_2(F_p) \). And given one such \( \tilde{R} = \left( \begin{array}{cc} r_1 & r_2 \\ r_3 & r_4 \end{array} \right) \in GL_2(F_p) \), we also have \( \tilde{R}_q := \left( \begin{array}{cc} q & q \\ r_3 & r_4 \end{array} \right) \in GL_2(F_p) \) for all \( q \in \{1, \ldots, p-1\} \), with \( \det(\tilde{R}_q) = q \det(\tilde{R}) \). And since \( p \nmid \tilde{n} \), \( \tilde{n} \det(\tilde{R}) \) takes every value in \( F_p^* \) exactly \( (p^2 - p)/(p - 1) = p \) times. Hence we obtain: When \( \tilde{R} = \left( \begin{array}{cc} r_1 & r_2 \\ r_3 & r_4 \end{array} \right) \) runs through all \( p^4 \) possibilities, then \( 3^{-1} \text{tr}(\left( \begin{array}{cc} a & 0 \\ c & 0 \end{array} \right) \tilde{R}) + \tilde{n} \det(\tilde{R}) \) takes the value \( 0 \in F_p \) exactly \( p^2 \)-times for \( r_1 = r_2 = 0 \in F_p \), \( p \)-times for \( \text{tr}(XU\tilde{R}) = 0 \in F_p \) and \( (r_1, r_2) \neq (0, 0) \in F_p^2 \) (since then \( \det(\tilde{R}) = 0 \in F_p \) has to), and \( (p - 1)p \cdot p \) times regarding every other value of \( \text{tr}(XU\tilde{R}) \) (since there are \( p - 1 \) other values, taken by \( p \) combinations of \( (r_1, r_2) \), combined with \( p \) combinations of \( (r_3, r_4) \) for \( \tilde{n} \det(\tilde{R}) \equiv 3^{-1} \text{tr}(XU\tilde{R}) \mod p ) \), and thus exactly \( p^3 + p^2 - p \) possible combinations of the \( r_i \) yield the value \( 0 \in F_p \). The same holds for all other values \( q \in \{1, \ldots, p-1\} \), where simply the combinations for \( r_1 = r_2 = 0 \in F_p \) are missing, hence \( p^3 - p \) possible combinations such that \( 3^{-1} \text{tr}(\left( \begin{array}{cc} a & 0 \\ c & 0 \end{array} \right) \tilde{R}) + \tilde{n} \det(\tilde{R}) \equiv q \mod p \) for every \( q \in \{1, \ldots, p-1\} \). So we conclude:

\[
p | m : \text{p}^3 + \text{p}^2 - \text{p} \text{ of } \text{p}^4 \text{ possible } r \text{ yield an appropriate } \text{T}_r ,
\]

\[
p \nmid m : \text{p}^3 - \text{p} \text{ of } \text{p}^4 \text{ possible } r \text{ yield an appropriate } \text{T}_r ,
\]

and again, given \( \tilde{n} \), if \( r \in \mathcal{O}/p\mathcal{O} \) yields a proper \( \text{T}_r \) or not is completely determined by \( \tilde{m} \mod p \). Furthermore, note that the number of possible \( \text{T}_r \) coincides with the number of possible \( \text{T}_r \) for the previous case \( (x \in p\mathcal{O}) \), whereas this number only depends on \( \tilde{m} \). So these two cases are identical in terms of the number of possible \( \text{T}_r \).

So finally, assume \( p \nmid N(x) \). Doing exactly the same considerations like in the previous case, yields that we have to solve the equation

\[
\tilde{m} + \text{tr}(3^{-1}XR) + \tilde{n} \det(R) \equiv 0 \mod p .
\]

But this time, we have \( 3^{-1}X \in GL_2(F_p) \) in virtue of \((4.24) \), since \( p \nmid N(i_1\sqrt{3}x) \), so this time we define \( U = (3^{-1}X) \in F_p^{2 \times 2} \) and \( \tilde{R} = \left( \begin{array}{cc} r_1 & r_2 \\ r_3 & r_4 \end{array} \right) := UR \), where \( R \mapsto UR \) is again a bijection. Note that \( \det(U^{-1}) \in F_p^* \) (and so is \( \tilde{n} \)), so let \( n := \tilde{n} \det(U^{-1}) \in F_p^* \). Therefore, we have to solve

\[
r_1 + r_4 + \text{n}(r_1r_4 - r_2r_3) + \tilde{m} \equiv 0 \mod p
\]

\[
\Leftrightarrow nr_2r_3 \equiv r_1 + r_4 + nr_1r_4 + \tilde{m} \mod p .
\]

Suppose the right-hand side is 0, then \( r_2 = 0 \) or \( r_3 = 0 \) has to follow, hence there are \( 2p - 1 \) possibilities once the right-hand side is fixed and equals 0 (in \( F_p \)). If the right-hand
side equals $q \in \mathbb{F}_p^*$, then there exist exactly $p - 1$ possibilities for the left-hand side (if $r_2 \neq 0$ is fixed, then $r_3 = r_2^{-1} q$ is the only possible choice). Hence we have to find out how many combinations of $(r_1, r_4)$ there are for the right-hand side to be 0 or $q \in \mathbb{F}_p^*$. So let us consider

$$r_1 + r_4 + nr_4 + \tilde{m} \equiv 0 \mod p,$$

first. Suppose $p \nmid \tilde{m}$, then we have to solve

$$r_1(1 + nr_4) \equiv -r_4 \mod p.$$ 

Note that given $r_4$, there exists exactly one solution for $r_1$, as long as $1 + nr_4 \not\equiv 0 \mod p$. But the equation is unsolvable if $nr_4 \equiv -1 \mod p$, since this implies $r_4 \not\equiv 0 \mod p$, but $r_1(1 + nr_4) \equiv 0 \mod p$. So there exist exactly $p - 1$ possible combinations for $(r_1, r_4)$. On the other hand, suppose $p \nmid \tilde{m}$. So we have to solve

$$r_1(1 + nr_4) \equiv -r_4 - \tilde{m} \mod p.$$ 

So again, given $r_4$ such that $1 + nr_4 \not\equiv 0 \mod p$, there exists exactly one solution for $r_1$. So suppose $nr_4 \equiv -1 \mod p$. Again by abuse of notation, let $n^{-1}$ denote the inverse of $n$ in $\mathbb{F}_p$, so we have $r_4 \equiv -n^{-1} \mod p$. But then, all $r_1$ are solutions ($p$ in total) if

$$-r_4 - \tilde{m} \equiv 0 \iff \tilde{m} \equiv n^{-1} \mod p,$$

and there is no solution in any other case. Note that, again by abuse of notation (with "$(\cdot)^{-1}$" always standing for the inverse in $\mathbb{F}_p$), we have

$$n^{-1} \equiv \tilde{n}^{-1} \det(U) \equiv \tilde{n}^{-1}(3^{-1})^2 \det(X) \equiv \tilde{n}^{-1}3^{-1} N(x) \mod p,$$

and thus

$$\tilde{m} \equiv n^{-1} \iff 3\tilde{m} \tilde{n} - N(x) = 3 \det(\bar{T}) \equiv 0 \mod p.$$ 

So if $p \nmid \tilde{m}$, there are exactly $p - 1$ possible combinations of $(r_1, r_4)$ such that the right-hand side (of the equation above) equals $0 \in \mathbb{F}_p$ if $3 \det(\bar{T}) \not\equiv p\mathbb{N}_0$, and there exist $2p - 1$ possible combinations if $3 \det(\bar{T}) \equiv p\mathbb{N}_0$. Next, let us consider

$$r_1 + r_4 + nr_4 + \tilde{m} \equiv q \mod p,$$

where $q \in \mathbb{F}_p^*$. First, let us assume $p \nmid \tilde{m}$, again. So we have to solve

$$r_1(1 + nr_4) \equiv -r_4 + q \mod p.$$ 

Again, there is exactly one solution for $r_1$ if for a given $r_4$ we have $1 + nr_4 \not\equiv 0 \mod p$. If we actually have $1 + nr_4 \equiv 0 \mod p$, then the equation is solvable if and only if the right-hand side is $0 \in \mathbb{F}_p$, too (and then every $r_1$ is a solution), so if and only if $r_4 \equiv q \mod p$. This is fulfilled for exactly one of the $q$ we are considering (which are $p - 1$ in total). So we have either $p - 1$ possible combinations of $(r_1, r_4)$, or $2p - 1$ (which happens exactly
once). And finally, let us assume \( p \not| \tilde{m} \), so we have the equation
\[
\begin{align*}
    r_1(1 + nr_4) &\equiv -r_4 - \tilde{m} + q \mod p.
\end{align*}
\]

Once more, there are at least \( p - 1 \) possible combinations for \((r_1, r_4)\), and we have to consider the case \( r_4 \equiv -n^{-1} \mod p \) separately, again. The equation is solvable (and then every \( r_1 \) is possible) if and only if
\[
    n^{-1} - \tilde{m} + q \equiv 0 \mod p
\]
is fulfilled. But since we have \( q \in \mathbb{F}_{p^r}^* \), this never holds true if \( n^{-1} - \tilde{m} \equiv 0 \mod p \) \( (\iff 3 \det(\tilde{T}) \in p\mathbb{N}_0) \). Otherwise, the requirement is met for exactly one of the \( q \) we are looking at (which are \( p - 1 \) in total). Hence we get for \( p \not| \tilde{m} \): If \( 3 \det(\tilde{T}) \in p\mathbb{N}_0 \), then for every \( q \in \mathbb{F}_p \) there are exactly \( p - 1 \) possible combinations for \((r_1, r_4)\) that solve the equation above. Otherwise, we have either \( p - 1 \) possible combinations of \((r_1, r_4)\), or \( 2p - 1 \) (which again happens exactly once). Combining this with the number of possible solutions for \( r_2 \) and \( r_3 \) if we have either \( r_1 + r_4 + nr_4 + \tilde{m} \equiv 0 \) or \( r_1 + r_4 + nr_4 + \tilde{m} \equiv q \mod p \), we conclude: If \( p | \tilde{m} \), then exactly
\[
\begin{align*}
    1 \cdot (2p - 1) \cdot (p - 1) &+ 1 \cdot ((p - 1) \cdot (2p - 1)) + (p - 2) \cdot ((p - 1) \cdot (p - 1)) = p^3 - p
\end{align*}
\]
of \( p^4 \) possible \( r \) yield an appropriate \( T_r \). (A short explanation: The first factor inside the brackets always denotes the number of possible \((r_2, r_3)\), the second the number of possible \((r_1, r_4)\). The first summand is for the case of the right-hand side in the equation above being 0, the second for the one \( q \in \mathbb{F}_p \) such that there exist \( 2p - 1 \) possible solutions for \((r_1, r_4)\), and the last for the other \( p - 2 \) choices of \( q \)). Doing the same considerations for the other cases, we have: If \( p \not| \tilde{m} \), but \( 3 \det(\tilde{T}) \in p\mathbb{N}_0 \), then exactly
\[
\begin{align*}
    1 \cdot (2p - 1) \cdot (2p - 1) &+ (p - 1) \cdot ((p - 1) \cdot (p - 1)) = p^3 + p^2 - p
\end{align*}
\]
of \( p^4 \) possible \( r \) yield an appropriate \( T_r \). And if \( p \not| \tilde{m} \) and \( 3 \det(\tilde{T}) \not\in p\mathbb{N}_0 \), then exactly
\[
\begin{align*}
    1 \cdot (2p - 1) \cdot (p - 1) &+ 1 \cdot ((p - 1) \cdot (2p - 1)) + (p - 2) \cdot ((p - 1) \cdot (p - 1)) = p^3 - p
\end{align*}
\]
of \( p^4 \) possible \( r \) yield an appropriate \( T_r \).

For to not lose track of these results, we summarize the considerations above in the following lemma, although we have not yet determined \( \epsilon(T_r) \), only the number of “possible” \( T_r \), depending on the shape of \( \tilde{T} \).

**Lemma.** Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we have
\[
    f|_k S_2(p) \mathcal{H}(Z) = \sum_{\tilde{T} \in \text{Her}_2^*(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2\pi i \tau(\tilde{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}).
\]

Given \( r \in \mathcal{O} \) and \( 0 \neq \tilde{T} = (\frac{m}{n}, \frac{l}{n}) \in \text{Her}_2^*(\mathcal{O}), \tilde{T} \geq 0 \), with \( \tilde{T} = \frac{i}{\sqrt{3}}x \) for some appropriate \( x \in \mathcal{O} \),
define $T_r = \frac{1}{p} T_r[D_r]$, where $D_r = \left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right)$. Then we have

$$
\beta (\bar{T}) = p^{1-k} \sum_{r: O \mid pO, T_r \in \text{Her}_2(O)} \alpha(\varepsilon(T_r), 3 \det(\bar{T})/\varepsilon(T_r)^2).
$$

Let $n(\bar{T}) = \# \{ r : O \mid pO ; T_r \in \text{Her}_2(O) \}$. Then we have the following cases:

- $p \mid \bar{n}$:
  - $x \in pO$:
    * $p \mid \bar{m}$: $n(\bar{T}) = p^4$. We have $3 \det(\bar{T}) \in p^2 N_0$ in this case.
    * $p \not\mid \bar{m}$: $n(\bar{T}) = 0$.
  - $x \not\in pO$: $n(\bar{T}) = p^3$ (independent of $\bar{m}$).

- $p \nmid \bar{n}$:
  - $p \mid N(x)$:
    * $p \mid \bar{m}$: $n(\bar{T}) = p^3 + p^2 - p$. We have $3 \det(\bar{T}) \in pN_0$ in this case.
    * $p \nmid \bar{m}$: $n(\bar{T}) = p^3 - p$. We have $3 \det(\bar{T}) \not\in pN_0$ in this case.
  - $p \nmid N(x)$:
    * $p \mid \bar{m}$: $n(\bar{T}) = p^3 - p$. We have $3 \det(\bar{T}) \not\in pN_0$ in this case.
    * $p \nmid \bar{m}$:
      - $p \mid 3 \det(\bar{T})$: $n(\bar{T}) = p^3 + p^2 - p$.
      - $p \nmid 3 \det(\bar{T})$: $n(\bar{T}) = p^3 - p$.

Or in other words:

- $p \mid \bar{n}$:
  - $x \in pO$:
    * $p \mid \bar{m}$: $n(\bar{T}) = p^4$. We have $3 \det(\bar{T}) \in p^2 N_0$ in this case.
    * $p \nmid \bar{m}$: $n(\bar{T}) = 0$.
  - $x \not\in pO$: $n(\bar{T}) = p^3$ (independent of $\bar{m}$).

- $p \nmid \bar{n}$:
  - $p \mid 3 \det(\bar{T})$: $n(\bar{T}) = p^3 + p^2 - p$.
  - $p \nmid 3 \det(\bar{T})$: $n(\bar{T}) = p^3 - p$.

Now, let us get back to determining $\varepsilon(T_r)$ in terms of $\varepsilon(\bar{T})$, where $\bar{T} = \left( \begin{array}{cc} \bar{m} & \bar{v} \\ \bar{v} & \bar{n} \end{array} \right)$, with $\bar{v} = \frac{i}{\sqrt{3}} x$, $x \in O$. Again, suppose $\varepsilon(\bar{T}) = p^i q$, where $p \nmid q$. Recall that we already found out that

$$
\varepsilon(T_r) \in \{ p^{i-1} q, p^i q, p^{i+1} q \}
$$

holds, and $T_r$ is given by

$$
T_r = \left( \begin{array}{cc} p^{-1}(\bar{m} + 2 \text{Re}(\bar{v}) + N(r)\bar{n}) & \bar{v} + r\bar{n} \\ \bar{v} + r\bar{n} & p\bar{n} \end{array} \right).
$$
We will now go through each of the cases determined in the preceding lemma. So first, let us with $x$.

According to (4.30), (4.44) yields $\beta$.

So we have to distinguish between these two cases. Note that we will keep the notation of the preceding lemma, so $\beta$ denotes the Fourier-coefficients of $f|_{S_2(p)}$.

We start by assuming $j = 0$, hence $\epsilon(\tilde{T}) = q$ and $p \nmid \epsilon(\tilde{T})$, which also implies $\epsilon(T_r) \in \{q, pq\}$.

We will now go through each of the cases determined in the preceding lemma. So first, let us assume $p \nmid \tilde{n}$. Here, the first case is $x \in pO$. But then $p \nmid \tilde{m}$ holds due to the assumption, and (4.44) yields $\beta(\tilde{T}) = 0$.

So suppose $x \notin pO$. Since we assumed $n \in pN_0 \subset p\sqrt[3]{3}O$, we obtain

$$\tilde{t} + r\tilde{n} \in p\sqrt[3]{3}O \iff \tilde{t} \in p\sqrt[3]{3}O \iff x \in pO,$$

and thus $\tilde{t} + r\tilde{n} \notin p\sqrt[3]{3}O$, which gives rise to $\epsilon(T_r) = q = \epsilon(\tilde{T})$ by definition. Therefore, (4.44) yields

$$\beta(\tilde{T}) = p^{1-k}a(\epsilon(\tilde{T}), 3 \det(T)/\epsilon(\tilde{T})^2).$$

So let us assume $p \mid \tilde{n}$ (and still $\epsilon(\tilde{T}) = q$), now. We have

$$\tilde{t} + r\tilde{n} \in p\sqrt[3]{3}O \iff x - i\sqrt{3}r\tilde{n} \in pO.$$

According to (3.30), $i\sqrt{3}r\tilde{n}$ runs through a transversal of $O/pO$ (since $r$ does). Therefore, there exists exactly one $r$ in the transversal we fixed, such that $x - i\sqrt{3}r\tilde{n} \in pO$ holds. For all the other $r$ (yielding an appropriate $T_r$) we already have $\epsilon(T_r) = q = \epsilon(\tilde{T})$. So let us consider this one $r$ with $x - i\sqrt{3}r\tilde{n} \in pO$, and thus $\tilde{t} + r\tilde{n} \in p\sqrt[3]{3}O$, which leads to $N(\tilde{t} + r\tilde{n}) \in \mathbb{Z}$. We have

$$N(\tilde{t} + r\tilde{n}) = N(\tilde{t}) + \tilde{n}^2N(r) + 2\tilde{n} \text{Re}(i\tilde{r}),$$

and this leads to

$$N(\tilde{t}) + \tilde{n}^2N(r) + 2\tilde{n} \text{Re}(i\tilde{r}) \in \mathbb{Z}.$$

So let $y \in \mathbb{N}$ such that

$$N(\tilde{t}) + \tilde{n}^2N(r) + 2\tilde{n} \text{Re}(i\tilde{r}) = \frac{y}{3}.$$

holds. Note that this leads to

$$\frac{p^2}{3}y - N(\tilde{t}) = (\tilde{n}N(r) + 2\text{Re}(i\tilde{r}))\tilde{n} \in \tilde{n}\mathbb{Z} \implies p^2y - 3N(\tilde{t}) \in 3\tilde{n}\mathbb{Z},$$
and so we finally obtain (taking note of $p \nmid 3\bar{n}$ and $3\bar{m}\bar{n} + p^2y - 3N(\bar{t}) \in 3\bar{n}Z$)
\begin{align*}
\bar{m} + 2\text{Re}(\bar{r}) + N(r)\bar{n} &= \bar{m} + \frac{1}{36}(p^2y - 3N(\bar{t})) \in p^iZ \\
\Leftrightarrow 3\bar{m}\bar{n} + p^2y - 3N(\bar{t}) &\in 3\bar{n}p^iZ \\
\Leftrightarrow 3\bar{m}\bar{n} + p^2y - 3N(\bar{t}) &\in p^iZ \\
\Leftrightarrow 3\det(\bar{T}) &= 3\bar{m}\bar{n} - 3N(\bar{t}) \in p^iZ
\end{align*}

for $j \in \{1, 2\}$. Therefore, this special $r$ yields an appropriate $T_r$ if and only if $3\det(\bar{T}) \in pZ$, and then $\epsilon(T_r) = pq$ holds if and only if $3\det(\bar{T}) \in p^2Z$. Hence, going through all three cases, (4.44) yields
\begin{equation*}
\beta(\bar{T}) = \begin{cases} 
p^{1-k}(p^3 - p_\alpha(\epsilon(\bar{T}), d), & \text{if } 3\det(\bar{T}) \in Z \setminus pZ, \\
p^{1-k}(p^3 + p^2 - p_\alpha(\epsilon(\bar{T}), d), & \text{if } 3\det(\bar{T}) \in pZ \setminus p^2Z, \\
p^{1-k}\left((p^3 + p^2 - p - 1)\alpha(\epsilon(\bar{T}), d) + \alpha(\epsilon(\bar{T}), 3p^2d)\right), & \text{if } 3\det(\bar{T}) \in p^2Z, 
\end{cases}
\end{equation*}

where $d = 3\det(\bar{T})/\epsilon(\bar{T})^2$. Or in other words, noting that $\alpha(\cdot, p^{-2}d) = 0$ if $d \notin p^2N_0$, we can also write
\begin{equation*}
\beta(\bar{T}) = p^{1-k}\left((p^3 - p + \chi_3(p^{-1}d)p^2 - \chi_3(p^{-2}d))\alpha(\epsilon(\bar{T}), d) + \alpha(\epsilon(\bar{T}), p^{-2}d)\right).
\end{equation*}

Now, we have to consider the other case, which means $\epsilon(\bar{T}) = p^iq$, where $j \geq 1$. We will trace this case back to the one we considered first. By definition, we have $S := \frac{1}{p^i}\bar{T} \in \text{Her}_2(\mathcal{O})$, with $\epsilon(S) = 1$, and $3\det(S) = 3\det(S)/\epsilon(S)^2 = 3\det(\bar{T})/\epsilon(\bar{T})^2 =: d$, and denote $S = \left(\begin{array}{cc} n & m \\ t & u \end{array}\right)$, with $t = \frac{n}{\sqrt{3}}y$, $y \in \mathcal{O}$. Like we remarked above, all of the $p^4$ possible $r$ yield an appropriate $T_r$, and
\begin{equation*}
\epsilon(T_r) \in \{p^{i-1}q, p^iq, p^{i+1}q\}.
\end{equation*}

Suppose that we have $\epsilon(T_r) \neq p^{i-1}q$, then $S_r := \frac{1}{p^i}T_r \in \text{Her}_2(\mathcal{O})$ holds by definition, and we obviously have
\begin{equation*}
\epsilon(T_r) = p^i q \epsilon(S_r)
\end{equation*}
and
\begin{equation*}
S_r = \frac{1}{p}S[D_r].
\end{equation*}

But note that this is exactly the situation we had to consider in the previous case, so we already know which (and how many) of the $r$ yield an appropriate $S_r$ and under which conditions we have either $\epsilon(S_r) = 1$ or $\epsilon(S_r) = p$. Keeping that in mind, we go through all three cases we had to consider above.

So suppose we have $p^{i+1}q|\bar{n}$ ($\iff p|n$) and $x \in p^{i+1}q\mathcal{O}$ ($\iff y \in p\mathcal{O}$). But we have seen above that the existence of $S_r$ would imply $p|m$ ($\iff p^{i+1}q|\bar{m}$), which would contradict $\epsilon(\bar{T}) = p^i q$. Hence all of the $T_r$ (which are $p^4$ in total) have to fulfill $\epsilon(T_r) = p^{i-1}q = p^{-1}\epsilon(\bar{T})$, and thus (4.44) leads to
\begin{equation*}
\beta(\bar{T}) = p^{5-k}\alpha(p^{-1}\epsilon(\bar{T}), p^2d).
\end{equation*}
And finally, if we have $p^{i+1}q|\bar{n}$ ($\Leftrightarrow p|n$) and $x \notin p^{i+1}q\mathcal{O}$ ($\Leftrightarrow y \notin p\mathcal{O}$), then only $p^3$ of the $S_t$ fulfill $S_t \in \text{Her}_2^t(\mathcal{O})$, and the result from above yields

$$\beta(\bar{T}) = p^{1-k}\left((p^4 - p^3)x(p^{-1}\epsilon(\bar{T}), p^2d) + p^3x(\epsilon(\bar{T}), d)\right).$$

And finally, if we have $p^{i+1}q \nmid \bar{n}$ ($\Leftrightarrow p \nmid n$), then the considerations from above give rise to

$$\beta(\bar{T}) = p^{1-k}\left((p^4 - p^3 + p - \chi Z(p^{-1}d)p^2)x(p^{-1}\epsilon(\bar{T}), p^2d)
\quad + (p^3 - p + \chi Z(p^{-1}d)p^2 - \chi Z(p^{-2}d))x(\epsilon(\bar{T}), d) + x(p(\epsilon(\bar{T}), p^2d))\right).$$

Note that all three formulas also hold true if we allow $j = 0$, since we have $x(p^{-1}q, \ast) = 0$ by definition. We formulate this result in the following lemma, again.

**4.45 Lemma.** Suppose $p \neq 3$ is a prime number and $f \in \mathcal{M}(k;\mathcal{O})$ with associated function $x$ according to (4.17). Then we get

$$f|_k S_2(p)_2(Z) = \sum_{\bar{T} \in \text{Her}_2^t(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T})e^{2\pi i \tau(\bar{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),$$

where we have the following cases for $0 \neq \bar{T} = (\bar{n} \bar{\tau} \bar{x}) \in \text{Her}_2^t(\mathcal{O}), \bar{T} \geq 0$, with $\bar{\tau} = \frac{\bar{\eta}}{\sqrt{3}}$ for some appropriate $x \in \mathcal{O}$, where $d := 3\det(\bar{T})/\epsilon(\bar{T})^2$:

- If $pe(\bar{T})|\bar{n}$ and $x \in pe(\bar{T})\mathcal{O}$, then

$$\beta(\bar{T}) = p^{5-k}x(p^{-1}\epsilon(\bar{T}), p^2d).$$

Furthermore, we have

$$d \in p\mathbb{N}_0$$

in this case.

- If $pe(\bar{T})|\bar{n}$ and $x \notin pe(\bar{T})\mathcal{O}$, then

$$\beta(\bar{T}) = p^{1-k}\left((p^4 - p^3)x(p^{-1}\epsilon(\bar{T}), p^2d) + p^3x(\epsilon(\bar{T}), d)\right).$$

- If $pe(\bar{T}) \nmid \bar{n}$, then

$$\beta(\bar{T}) = p^{1-k}\left((p^4 - p^3 + p - \chi Z(p^{-1}d)p^2)x(p^{-1}\epsilon(\bar{T}), p^2d)
\quad + (p^3 - p + \chi Z(p^{-1}d)p^2 - \chi Z(p^{-2}d))x(\epsilon(\bar{T}), d) + x(p(\epsilon(\bar{T}), p^2d))\right).$$

So finally, we have to consider $S_2(p)_5$. Thus, we have $A_{\pi, \bar{n}, \varphi} = \left(\begin{smallmatrix} \pi & 0 \\ -q & \pi \end{smallmatrix}\right)$, $D_{\pi, \bar{n}, \varphi} = \left(\begin{smallmatrix} \pi & q \\ 0 & \pi \end{smallmatrix}\right)$ (where $\pi$ and $\bar{n}$ will run through a transversal of $\mathcal{E}\setminus\mathcal{N}(p)$, $s$ and $c$ are chosen such that $p \nmid \mathbb{N}(s)$, $p \nmid \mathbb{N}(c)$, $\pi s = c\bar{n}$, and $q$ will run through a transversal of $\mathbb{Z}/p\mathbb{Z}$) and $B = \bar{q}(\varphi \pi )$ (where $\bar{q}$ will
run through a transversal of \( \mathbb{Z}/p\mathbb{Z} \). Note that we have
\[
\pi \cdot s = c \cdot \tilde{\pi} \iff \mathfrak{s} \cdot \pi = \pi \cdot \mathfrak{s} \iff s \cdot \tilde{\pi} = \pi \cdot c \iff \tilde{\pi} \cdot \mathfrak{s} = \mathfrak{s} \cdot \pi.
\] (4.46)

Given \( T = \begin{pmatrix} m & t \\ -t & n \end{pmatrix} \in \text{Her}_2(\mathcal{O}) \), we compute
\[
\tilde{T} = \frac{1}{p} T[A_{\pi, \tilde{\pi}, \mathfrak{s}}] = \begin{pmatrix} m - 2p^{-1} q \Re(\tilde{\tau} \mathfrak{c} \pi c) + p^{-1} q \Re(\mathfrak{c} \pi n) & n \mathfrak{c} \pi \tilde{\tau} - p^{-1} q \Re(\mathfrak{c} \pi n) \\ p^{-1} \mathfrak{c} \pi \tilde{\tau} - p^{-1} \mathfrak{c} \pi n & n \end{pmatrix}
\]
and
\[
B\tilde{\mathcal{A}}_{\pi, \tilde{\pi}, \mathfrak{s}} = \begin{pmatrix} 0 & \tilde{q} s \pi \mathfrak{c} \\ -q \tilde{q} \pi \mathfrak{s} & -q \tilde{q} \pi \mathfrak{n} \end{pmatrix} = \begin{pmatrix} 0 & \tilde{q} \pi \mathfrak{c} \\ -q \mathfrak{c} \pi \mathfrak{s} & -q \mathfrak{c} \pi \mathfrak{n} \end{pmatrix}.
\]
Thus we have \( \tau(T, B\tilde{\mathcal{A}}_{\pi, \tilde{\pi}, \mathfrak{s}})/p = p^{-1}(-q \tilde{q} \mathfrak{n} \mathfrak{c} \pi n + 2q \Re(\tilde{\tau} \mathfrak{c} \pi c)) \), so 4.38 yields
\[
f|_k S_2(p)_5(\mathbb{Z}) = p^{-k} \sum_{\pi, \tilde{\pi} \in \mathcal{N}(p), \mathfrak{s} \in \mathcal{N}(\mathcal{O}), T \geq 0} \left( \sum_{\tilde{\mathcal{A}}_{\pi, \tilde{\pi}, \mathfrak{s}} / p\mathbb{Z}} (e^{2\pi ip^{-1}(-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c)))} \right) \tilde{T}.
\]
According to 4.43, we have
\[
\sum_{q=0}^{p-1} \left( e^{2\pi ip^{-1}(-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c)))} \right) \tilde{T} = \begin{cases} 0, & \text{if } p \nmid (-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c)) \\ p, & \text{if } p | (-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c)) \end{cases},
\]
where
\[
-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c) = -q \mathfrak{c} \pi n + \mathfrak{c} \pi \tilde{\tau} = q \mathfrak{c} \pi n & \text{in } p\mathbb{Z}.
\]
Thus, we have the equivalence \( x N(s) \in p^{1/3} \mathcal{O} \iff y N(s) \in p\mathcal{O} \iff x = y \mathfrak{c} \mathfrak{s} n \mathfrak{c} \pi n \), like we have seen several times before. And therefore, since \( p \mathfrak{n} (-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c) \) also implies \( p \mathfrak{n} (-2q \Re(\tilde{\tau} \mathfrak{c} \pi c) + q^2 \mathfrak{n} \mathfrak{c} \pi n) \), we obtain
\[
p \mathfrak{n} (-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c) \) \Rightarrow \mathfrak{T} \in \text{Her}_2(\mathcal{O}),
\]
which means that the Fourier-expansion of \( f|_k S_2(p)_5 \) actually ranges over \( \mathfrak{T} \in \text{Her}_2(\mathcal{O}) \), again. Now, let \( 0 \neq \mathfrak{T} = \frac{1}{p} T[A_{\pi, \tilde{\pi}, \mathfrak{s}}] \), where \( p \mathfrak{n} (-q \mathfrak{c} \pi n + 2\Re(\tilde{\tau} \mathfrak{c} \pi c) \), and \( \mathfrak{T} \in \text{Her}_2(\mathcal{O}), \mathfrak{T} \geq 0 \) be
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one of those matrices occuring in the Fourier-expansion, given by

$$\tilde{T} = \begin{pmatrix} \bar{m} & \tilde{t} \\ \tilde{t} & \bar{n} \end{pmatrix}, \quad \tilde{t} = \frac{i}{\sqrt{3}}x \quad \text{where } x \in O.$$ 

Just like in all the other cases, we need to analyze $\frac{1}{p} \tilde{T}[\tilde{D}_{\pi, \tilde{\pi}, q}]$. A simple calculation yields

$$T_{\pi, \tilde{\pi}, q} := \frac{1}{p} \tilde{T}[\tilde{D}_{\pi, \tilde{\pi}, q}] = \begin{pmatrix} \bar{m} + p^{-1}q(2 \text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n}) & p^{-1}(\pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n}) \\ p^{-1}(\pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n}) & \bar{n} \end{pmatrix}.$$ 

Note that we have $\frac{1}{p} T_{\pi, \tilde{\pi}, q}[A_{\pi, \tilde{\pi}, q}] = p^{-2} \tilde{T}[\tilde{D}_{\pi, \tilde{\pi}, q}A_{\pi, \tilde{\pi}, q}] = p^{-2}\tilde{T}[pI] = \tilde{T}$, again, and so due to that and by what we have seen above, $\tilde{T}$ occurs in the Fourier-expansion of $f|_{S_2(p)}$ if and only if $T_{\pi, \tilde{\pi}, q} \in \text{Her}_2^\pi(O)$ for at least one combination $(\pi, \tilde{\pi}, q)$. So the conditions that have to be fulfilled are

$$2 \text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n} \in pZ \quad (if \ q \not\in pZ)$$

and

$$\pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p\frac{i}{\sqrt{3}}O.$$ 

But we will show now that these conditions are equivalent. First, assume $2 \text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n} \in pZ$. Hence there exists $y \in Z$ such that

$$\tilde{t}c\tilde{\pi} + \pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n} = py$$

$$\Rightarrow q\tilde{s}n\tilde{v}\tilde{\pi} + \pi\tilde{v}\tilde{\pi} + \pi\tilde{v}\tilde{\pi} = q\tilde{v}n\pi c + \pi\tilde{v}\tilde{\pi} = p\tilde{y}\pi - \tilde{tc}p$$

$$\Rightarrow N(s)(\tilde{v}n\pi c + \pi\tilde{v}\tilde{\pi}) = p(s\tilde{y}\pi - \tilde{tc}c) \in p\frac{i}{\sqrt{3}}O$$

$$\Rightarrow \pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p\frac{i}{\sqrt{3}}O,$$

since $p \nmid N(s)$ (so the last step follows like we have seen several times before). On the other hand, suppose $\pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p\frac{i}{\sqrt{3}}O$. Then also $\tilde{s}(\pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n}) + \tilde{tc}p \in p\frac{i}{\sqrt{3}}O$ holds. So there exists $y \in O$ such that we have (using the calculation rules for Re($\cdot$) and 4.46, hence $\tilde{s}\pi q\tilde{c}\tilde{\pi} = \tilde{ss}\pi q\tilde{n} = q\pi\tilde{c}\tilde{\pi}\pi$ and $\pi\tilde{v}\tilde{\pi} = \pi\tilde{v}\tilde{\pi}$)

$$q\pi\tilde{c}\tilde{\pi} + \pi\tilde{v}\tilde{\pi} + \tilde{tc}p = p\frac{i}{\sqrt{3}}y$$

$$\Rightarrow q\pi\tilde{c}\tilde{\pi}p + \pi\tilde{v}\tilde{\pi}p + \tilde{tc}p\tilde{\pi} = p\frac{i}{\sqrt{3}}y\tilde{\pi}$$

$$\Rightarrow q\pi\tilde{c}\tilde{\pi} + \pi\tilde{v}\tilde{\pi} + \tilde{tc}\tilde{\pi} = 2\text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n} = \frac{i}{\sqrt{3}}y\tilde{\pi}$$

$$\Rightarrow 2\text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n} \in Z \cap \frac{i}{\sqrt{3}}O\tilde{n} = pZ,$$

since $Z \cap \frac{i}{\sqrt{3}}O\tilde{n} = pZ$ holds in virtue of (4.31). Hence we obtain

$$T_{\pi, \tilde{\pi}, q} \in \text{Her}_2^\pi(O) \iff 2\text{Re}(\tilde{t}c\tilde{\pi}) + qN(c)\tilde{n} \in pZ \iff \pi\tilde{v}\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p\frac{i}{\sqrt{3}}O. \quad (4.47)$$
If we actually have \( q \in p\mathbb{Z} \) or \( \tilde{n} \in p\mathbb{N}_0 \), then this condition is reduced to
\[
(-i_1\pi i_1)x \in \mathcal{O}_{\tilde{\pi}}.
\]

Note that we have to sum over \( \pi, \tilde{\pi}, \) and \( q \), hence there might exist several \( T_{\pi,\tilde{\pi},q} \) such that \( T_{\pi,\tilde{\pi},q} \in \text{Her}_2^\pi(\mathcal{O}) \) and \( T_{\pi,\tilde{\pi},q}[A_{\pi,\tilde{\pi},q}] = \tilde{T} \). And since we have to sum twice over a transversal of \( \mathcal{E}\setminus\mathcal{N}(p) \) and once over a transversal of \( \mathbb{Z}/p\mathbb{Z} \), there exist \( p(p+1)^2 \) of these \( T_{\pi,\tilde{\pi},q} \), at most (see (1.7)). Therefore, just like we had to do for \( f|_1 S_2(p) \), we have to find out how many of the \( T_{\pi,\tilde{\pi},q} \) actually fulfill \( T_{\pi,\tilde{\pi},q} \in \text{Her}_2^\pi(\mathcal{O}) \) (hence \( \pi \tilde{t} \tilde{\pi} + \pi q \tilde{c} \tilde{n} \in p\frac{i}{\sqrt{3}}\mathcal{O} \)), depending on the shape of \( T \). So let
\[
\begin{align*}
n(\tilde{T}) & := \#\{ \pi, \tilde{\pi} : \mathcal{E}\setminus\mathcal{N}(p), \ q : \mathbb{Z}/p\mathbb{Z}; \ T_{\pi,\tilde{\pi},q} \in \text{Her}_2^\pi(\mathcal{O}) \} \\
& = \#\{ \pi, \tilde{\pi} : \mathcal{E}\setminus\mathcal{N}(p), \ q : \mathbb{Z}/p\mathbb{Z}; \ 2 \text{Re}(\tilde{t} c \tilde{n}) + q N(c) \tilde{n} \in p\mathbb{Z} \} \\
& = \#\{ \pi, \tilde{\pi} : \mathcal{E}\setminus\mathcal{N}(p), \ q : \mathbb{Z}/p\mathbb{Z}; \ \pi \tilde{t} \tilde{\pi} + \pi q \tilde{c} \tilde{n} \in p\frac{i}{\sqrt{3}}\mathcal{O} \}.
\end{align*}
\]

Note that \( n(\tilde{T}) \) is completely independent of \( \tilde{m} \).

We will now go through all possible cases that might occur, and start by assuming \( x \in p\mathcal{O} \) (hence \( \tilde{t} \in p\frac{i}{\sqrt{3}}\mathcal{O} \)). We have seen before that this implies \( 2 \text{Re}(\tilde{t}a) \in p\mathbb{Z} \) for all \( a \in \mathcal{O} \), thus
\[
q N(c) \tilde{n} \in p\mathbb{Z} \iff q \tilde{n} \in p\mathbb{Z}
\]
has to be fulfilled, since \( p \nmid N(c) \). If \( p \nmid \tilde{n} \), then \( q \in p\mathbb{Z} \) has to hold, which is fulfilled for exactly one of the \( q \) running through a transversal of \( \mathbb{Z}/p\mathbb{Z} \). So we obviously get
\begin{itemize}
\item \( x \in p\mathcal{O}, \ p|\tilde{n}: n(\tilde{T}) = p(p+1)^2 \).
\item \( x \in p\mathcal{O}, \ p \nmid \tilde{n}: n(\tilde{T}) = (p+1)^2 \).
\end{itemize}

Next, assume \( x \notin p\mathcal{O} \). If \( p|\tilde{n} \), then (see above)
\[
(-i_1\pi i_1)x \in \mathcal{O}_{\tilde{\pi}}
\]
has to hold. But then (4.32) yields that there exist \( p + 1 \) combinations of \( (\pi, \tilde{\pi}) \) (exactly one \( \tilde{\pi} \) for each given \( \pi \)) such that the condition above holds, if \( p \nmid N(x) \). On the other hand, if \( p|N(x) \), then there exist \( (p+1) + p = 2p + 1 \) such combinations (for exactly one \( \pi \), all of the \( \tilde{\pi} \) are possible, which are \( p + 1 \) in total, and for every other \( \pi \) only one \( \tilde{\pi} \) exists such that the condition above is met). Noting that there are no conditions on \( q \), we summarize:
\begin{itemize}
\item \( x \notin p\mathcal{O}, \ p|\tilde{n}, \ p|N(x): n(\tilde{T}) = 2p(p+1) \).
\item \( x \notin p\mathcal{O}, \ p|\tilde{n}, \ p \nmid N(x): n(\tilde{T}) = p(p+1) \).
\end{itemize}

So assume \( p \nmid \tilde{n} \), now. Given \( \pi \) and \( \tilde{\pi} \), then \( 2 \text{Re}(\tilde{t} c \tilde{\pi}) \) is some fixed value in \( \mathbb{Z} \). Hence, noting that \( p \nmid N(c) \tilde{n} \) (thus \( a \mapsto a N(c) \tilde{n} \) is a bijection in \( \mathbb{Z}/p\mathbb{Z} \)), there exists exactly one \( q \) in the transversal we fixed, such that
\[
2 \text{Re}(\tilde{t} c \tilde{\pi}) + q N(c) \tilde{n} \in p\mathbb{Z}.
\]
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So we have $(p+1)^2$ possible combinations of $(\pi, \bar{\pi}, q)$, not matter if $p|N(x)$ or not. But note that whether $p|N(x)$ holds or not determines for how many combinations of $(\pi, \bar{\pi})$ we have to choose the one $q$ to satisfy $q \in p\mathbb{Z}$, since for $q \in p\mathbb{Z}$ the condition is again reduced to

$$(-i_1\pi i_1)x \in \mathcal{O}\bar{\pi}.$$ 

And note that above, we verified

$$2\Re(\bar{t}c\bar{\pi}) + q \, N(c)\bar{n} \in p\mathbb{Z} \iff \pi\bar{t}\bar{\pi} + \pi q c\bar{n} \in p\frac{i}{\sqrt{3}}\mathcal{O},$$

which holds for every $q \in \mathbb{Z}$, so in particular for $q = 0$. Therefore, we obtain

$$2\Re(\bar{t}c\bar{\pi}) \in p\mathbb{Z} \iff \pi\bar{t}\bar{\pi} \in p\frac{i}{\sqrt{3}}\mathcal{O} \iff (-i_1\pi i_1)x \in \mathcal{O}\bar{\pi}.$$ 

Combining this with the results from above immediately yields:

- $x \notin p\mathcal{O}$, $p \nmid \bar{n}$, $p \nmid N(x)$: $n(\bar{T}) = (p+1)^2$, where for $2p+1$ combinations of $(\pi, \bar{\pi})$ we have to choose $q \in p\mathbb{Z}$.

- $x \notin p\mathcal{O}$, $p \nmid \bar{n}$, $p \nmid N(x)$: $n(\bar{T}) = (p+1)^2$, where for $p+1$ combinations of $(\pi, \bar{\pi})$ we have to choose $q \in p\mathbb{Z}$.

Again, we formulate this intermediate result in the following

\textbf{(4.46) Lemma.} Suppose $p \neq 3$ is a prime number and $f \in \mathcal{M}(k;\mathcal{O})$ with associated function $\alpha$ according to (4.17). Then we have

$$f|_k S_2(p)_{\mathcal{S}}(\mathcal{Z}) = \sum_{\bar{T} \in \text{Her}_2^{(\mathcal{O})}, \bar{T} \geq 0} \beta(\bar{T}) e^{2\pi i \bar{T} \mathcal{Z}}, \quad \mathcal{Z} \in \mathcal{H}(\mathbb{H}).$$

Given $\pi, \bar{\pi} \in \mathcal{N}(p)$, $q \in \mathbb{Z}$ and $0 \neq \bar{T} = \left( \begin{array}{c} \bar{\pi} \\ \bar{\pi} \end{array} \right) \in \text{Her}_2^{(\mathcal{O})}$, $\bar{T} \geq 0$, with $\bar{t} = \frac{i}{\sqrt{3}}t$ for some appropriate $x \in \mathcal{O}$, define $T_{\pi, \bar{\pi}, q} = \frac{i}{\sqrt{3}} \bar{T} \{ D_{\pi, \bar{\pi}, q} \}$, where $D_{\pi, \bar{\pi}, q} = \left( \begin{array}{c} \pi \\ \bar{\pi} \\ q \\ 0 \end{array} \right)$ (with $s \in \mathcal{O}$ chosen such that $p \nmid N(s)$ and $\bar{\pi}s \in \mathcal{O}\bar{\pi}$). Then we have

$$\beta(\bar{T}) = p^{1-k} \sum_{\pi, \bar{\pi} \in \mathcal{N}(p), q \in \mathbb{Z}/p\mathbb{Z}, T_{\pi, \bar{\pi}, q} \in \text{Her}_2^{(\mathcal{O})}} \alpha(e(T_{\pi, \bar{\pi}, q}), 3 \det(\bar{T})/e(T_{\pi, \bar{\pi}, q})^2).$$

Let $n(\bar{T}) = \#\{\pi, \bar{\pi} : E \setminus \mathcal{N}(p), q : \mathbb{Z}/p\mathbb{Z} ; T_{\pi, \bar{\pi}, q} \in \text{Her}_2^{(\mathcal{O})}\}$ (see 4.48). Then we have the following cases:

- $p|\bar{n}$:
  - $x \in p\mathcal{O}$: $n(\bar{T}) = p(p+1)^2$.
    (All $\pi$, $\bar{\pi}$ and $q$ are possible.)
  - $x \notin p\mathcal{O}$, $p \nmid N(x)$: $n(\bar{T}) = p(2p+1)$.
    (For one $\pi$ all $\bar{\pi}$ are possible, for all other $\pi$ exactly one $\bar{\pi}$ is possible, only, while all $q$ are possible in all cases.)
- \( x \notin pO, \ p \nmid N(x): n(\tilde{T}) = p(p + 1). \)
  (For all \( \pi \) there exists exactly one possible \( \tilde{\pi} \), while all \( q \) are possible.)

- \( p \nmid \tilde{n}: \)
  - \( x \in pO: n(\tilde{T}) = (p + 1)^2. \)
    (All \( \pi \) and \( \tilde{\pi} \) are possible, but \( q \) always has so satisfy \( q \in p\mathbb{Z} \).)
  - \( x \notin pO, \ p|N(x): n(\tilde{T}) = (p + 1)^2. \)
    (All \( \pi \) and \( \tilde{\pi} \) are possible. For one \( \pi \) we have that for every choice of \( \tilde{\pi}, q \) has to satisfy \( q \in p\mathbb{Z} \). For all other \( \pi \) there exists exactly one \( \tilde{\pi} \) such that \( q \) has to satisfy \( q \in \mathbb{Z} \), while for all \( \tilde{\pi} \) there exists exactly one possible \( q \), too, but this one has to satisfy \( q \notin p\mathbb{Z} \).
  - \( x \notin pO, \ p|N(x): n(\tilde{T}) = (p + 1)^2. \)
    (All \( \pi \) and \( \tilde{\pi} \) are possible. For every \( \pi \) there exists exactly one \( \tilde{\pi} \) such that \( q \) has to satisfy \( q \in p\mathbb{Z} \). For every other combination of \( \pi \) and \( \tilde{\pi} \) there exists exactly one possible \( q \), too, but this one has to satisfy \( q \notin p\mathbb{Z} \).

Again, we will stick to the notation of the preceding lemma, which means that \( \beta \) shall denote the Fourier-coefficient function of \( f|_kS_2(p) \). So next, we have to determine \( \varepsilon(T_{\pi,\tilde{\pi}}) \) and \( 3\det(T_{\pi,\tilde{\pi}}) \) in terms of \( \varepsilon(\tilde{T}) \) and \( 3\det(\tilde{T}) \), again. Recall that we have

\[
T_{\pi,\tilde{\pi}} = \frac{1}{p} \tilde{T}[D_{\pi,\tilde{\pi}}] = \left( \begin{array}{c} \tilde{m} + p\varepsilon(2\text{Re}(\tilde{t}c\tilde{n}) + qN(c)\tilde{n}) \varepsilon^{-1}(\pi t\tilde{n} + \pi q\tilde{n}) \\ p^{-1}(\pi t\tilde{n} + \pi q\tilde{n}) \end{array} \right). 
\]

In view of (1.34), we have

\[
\det(T_{\pi,\tilde{\pi}})^2 = \det(\tilde{T})^2 = p^{-4} \cdot (N(\pi)N(\tilde{n})) \cdot \det(\tilde{T}) \cdot (N(\pi)N(\tilde{n})) = \det(\tilde{T}) = \det(T)^2,
\]

and thus \( \det(T_{\pi,\tilde{\pi}}) = \det(\tilde{T}) \), since both \( T_{\pi,\tilde{\pi}} \) and \( \tilde{T} \) are positive semi-definite.

So let us consider \( \varepsilon(T_{\pi,\tilde{\pi}}) \). Just like before, suppose \( \varepsilon(\tilde{T}) = p'|r \), where \( r \in \mathbb{N} \) with \( p \nmid r \). Hence \( \tilde{n}, \tilde{m} \in p'/r\mathbb{N}_0 \) and \( t \in p'/r\mathbb{O}/\sqrt{3} \) (or equivalently \( x \in p'/r\mathbb{O} \)), but not all of these three properties can hold true for every non-trivial multiple of \( p'r \). Note again that \( 2\text{Re}(\tilde{t}a) \in l\mathbb{Z} \) holds for all \( a \in \mathcal{O} \) in case \( x \in l\mathcal{O} \).

First, suppose there is \( u \in \mathbb{N}, \ p \nmid u \) and \( ru|\varepsilon(T_{\pi,\tilde{\pi}}) \). This implies \( ru|\tilde{n} \). Furthermore, this also leads to \( q\tilde{n}\tilde{n} \in ru\frac{1}{\sqrt{3}}\mathcal{O} \), and we get

\[
\pi t\tilde{n} + \pi q\tilde{n} \in pru\frac{1}{\sqrt{3}}\mathcal{O}
\]

\[
\Rightarrow p\tilde{t} + pq\tilde{n} \tilde{n} \in pru\frac{1}{\sqrt{3}}\mathcal{O} \tilde{n} \subset pru\frac{1}{\sqrt{3}}\mathcal{O}
\]

\[
\Rightarrow p\tilde{t} \in ru\frac{1}{\sqrt{3}}\mathcal{O} - q\tilde{n}\tilde{n} = ru\frac{1}{\sqrt{3}}\mathcal{O}
\]

\[
\Rightarrow \tilde{t} \in ru\frac{1}{\sqrt{3}}\mathcal{O},
\]
where the last step follows due to $p \nmid ru$, like we seen several times before. So $2 \text{Re}(\tilde{t}c\tilde{n}) + q N(c)\tilde{n} \in ru\mathbb{Z}$ follows, and noting $p \nmid ru$, this gives rise to

$$p\tilde{m} + q(2 \text{Re}(\tilde{t}c\tilde{n}) + q N(c)\tilde{n}) \in pru\mathbb{Z} \subset ru\mathbb{Z}$$

$$\Rightarrow p\tilde{m} \in ru\mathbb{Z}$$

$$\Rightarrow \tilde{m} \in ru\mathbb{Z}.$$

Therefore, $ru|\varepsilon(\tilde{T}) = p^j r$ has to follow, and this implies $u = 1$. On the other hand, we have $\tilde{n}, \tilde{m} \in p^i r\mathbb{N}_0$ and $t \in p^i r\frac{1}{\sqrt{3}}\mathcal{O}$, and this immediately implies $\tilde{n} \in p^{i-1}r\mathbb{N}_0, \ p^{-1}(\pi\tilde{n} + \pi q\tilde{n}) \in p^{j-1}r\frac{1}{\sqrt{3}}\mathcal{O}$ and (since $p^{-1}q(2 \text{Re}(\tilde{t}c\tilde{n}) + q N(c)\tilde{n}) \in p^{j-1}r\mathbb{Z}$ also $\tilde{m} + p^{-1}q(2 \text{Re}(\tilde{t}c\tilde{n}) + q N(c)\tilde{n}) \in p^{j-1}r\mathbb{Z}$. So we obtain

$$\varepsilon(T_{\pi,\pi,\tilde{a}}) = p^k p^{j-1}r,$$

for some $k \in \mathbb{N}_0$ (where of course $k \geq 1$ if $j = 0$). Now, assume we had $k \geq 3$, which means $p^{j+2}r|\varepsilon(T_{\pi,\pi,\tilde{a}})$. First, this implies $p^{j+2}r|\tilde{m}$. This would lead to

$$\pi\tilde{n} + \pi q\tilde{n} \in p^{j+3}r\frac{1}{\sqrt{3}}\mathcal{O}$$

$$\Rightarrow p^2\tilde{t} + q\pi q\tilde{n} \in p^{j+3}r\pi\frac{1}{\sqrt{3}}\tilde{n} \subset p^{j+3}r\frac{1}{\sqrt{3}}\mathcal{O}$$

$$\Rightarrow p\tilde{t} \in p^{j+2}r\frac{1}{\sqrt{3}}\mathcal{O} - q\pi q\tilde{n} = p^{j+2}r\frac{1}{\sqrt{3}}\mathcal{O}$$

$$\Rightarrow \tilde{t} \in p^{j+1}r\frac{1}{\sqrt{3}}\mathcal{O}.$$
Of course, if actually \( j = 0 \) holds, then we have
\[
\epsilon(T_{\pi, \bar{n}, q}) \in \{ r, pr \}
\]
for those \( T_{\pi, \bar{n}, q} \) that fulfill \( T_{\pi, \bar{n}, q} \in \text{Her}_2^1(\mathcal{O}) \). On the other hand, if \( j \geq 1 \), then the shape of the \( T_{\pi, \bar{n}, q} \) yields that all \( p(p+1)^2 \) combinations of \( (\pi, \bar{n}, q) \) yield an appropriate \( T_{\pi, \bar{n}, q} \) (which we already saw in (4.46), where \( p|\bar{n} \) even does not have to hold). So just like we did for \( S_2(p)_2 \), we have to distinguish between \( j = 0 \) and \( j \geq 1 \), where the second case can be led back to the case \( j = 0 \). Therefore, we start by assuming \( j = 0 \). In each of the following cases, we have to determine under which conditions \( p|\epsilon(T_{\pi, \bar{n}, q}) \) holds (hence \( \epsilon(T_{\pi, \bar{n}, q}) = pr \)).

Like before, let \( d = 3 \det(\bar{T})/\epsilon(\bar{T})^2 = 3 \det(T_{\pi, \bar{n}, q})/\epsilon(\bar{T})^2 \). Of course, we only have to consider \( p|\bar{n} \) here, since \( p \nmid \bar{n} \) immediately leads to \( \epsilon(T_{\pi, \bar{n}, q}) = r = \epsilon(\bar{T}) \). So first, let us tick off the case \( p \nmid \bar{n} \), hence \( \epsilon(T_{\pi, \bar{n}, q}) = \epsilon(\bar{T}) \). In this case, according to (4.46), exactly \( (p+1)^2 \) of the \( p(p+1)^2 \) possible combinations of \( (\pi, \bar{n}, q) \) yield an appropriate \( T_{\pi, \bar{n}, q} \), independent of the shape of \( x \). (This only determines which combinations are possible, but not how many). Therefore, we obtain
\[
\beta(\bar{T}) = p^{1-k}(p+1)^2\alpha(\epsilon(\bar{T}), d)
\]
in virtue of (4.46). So let us consider the cases where \( p|\bar{n} \).

We begin by assuming \( x \in p\mathcal{O} \). So due to (4.46), all of the \( p(p+1)^2 \) possible combinations of \( (\pi, \bar{n}, q) \) yield an appropriate \( T_{\pi, \bar{n}, q} \). And note that 3 \( \det(\bar{T}) \in p\mathbb{N}_0 \) holds in this case (hence \( d \in p\mathbb{N}_0 \)). In view of 4.47 we have the equivalence
\[
\pi(p^{-1}\bar{T}) + \pi\bar{q}c(p^{-1}\bar{n}) \in \frac{1}{3\sqrt{3}}\mathcal{O} \iff 2 \Re((p^{-1}\bar{T}c\bar{n}) + \bar{q}N(c)(p^{-1}\bar{n}) \in p\mathbb{Z}
\]
by applying the equivalence 4.47 to \( n = p^{-1}\bar{n} \in \mathbb{N}_0 \) and \( t = p^{-1}\bar{T} \in \frac{1}{3\sqrt{3}}\mathcal{O} \). Therefore, \( p^{-1}T_{\pi, \bar{n}, q} \in \text{Her}_2^1(\mathcal{O}) \) (and thus \( p|\epsilon(T_{\pi, \bar{n}, q}) \)) would imply
\[
\bar{m} + p^{-1}q(2 \Re(\bar{T}c\bar{n}) + \bar{q}N(c)\bar{n}) \in p\mathbb{Z}
\Rightarrow \bar{m} \in p\mathbb{Z}
\]
hence \( p|\epsilon(\bar{T}) \), which contradicts the assumption. Therefore, \( \epsilon(T_{\pi, \bar{n}, q}) = r = \epsilon(\bar{T}) \) has to hold in all cases, and (4.46) yields
\[
\beta(\bar{T}) = p^{1-k}(p+1)^2\alpha(\epsilon(\bar{T}), d).
\]

Before we continue, we need to have a look at some (however fundamental) consideration: Suppose we actually have \( p|\epsilon(T_{\pi, \bar{n}, q}) \) for some combination \( (\pi, \bar{n}, q) \). Hence there is \( T \in \text{Her}_2^1(\mathcal{O}) \) such that \( T_{\pi, \bar{n}, q} = pT \). But then we obtain (see above) \( 3 \det(\bar{T}) = 3 \det(T_{\pi, \bar{n}, q}) = 3p^2 \det(T) \in p^2\mathbb{N}_0 \). Thus we get
\[
\epsilon(T_{\pi, \bar{n}, q}) = pr \Rightarrow 3 \det(\bar{T}) \in p^2\mathbb{N}_0.
\]
So in the following considerations we only have to look at those \( \bar{T} \) (hence \( \bar{m}, \bar{n} \) and \( \bar{t} \)) such that \( 3 \det(\bar{T}) = 3\bar{m}\bar{n} - 3N(\bar{t}) = 3\bar{m}\bar{n} - N(x) \in p^2\mathbb{N}_0 \) holds, since in all other cases \( \epsilon(T_{\pi, \bar{n}, q}) = \epsilon(\bar{T}) \) immediately follows. Again, we tick off this case first. If \( 3 \det(\bar{T}) \notin p^2\mathbb{N}_0 \), then (4.46) yields (where still \( p|\bar{n} \))
4.4 The quaternionic Hecke-operators $T_2(p)$

- $x \not\in p\mathcal{O}$, but $p \mid N(x)$: $\beta(\tilde{T}) = p^{1-k}p(2p + 1)\alpha(\epsilon(\tilde{T}), d)$
- $p \nmid N(x)$: $\beta(\tilde{T}) = p^{1-k}p(p + 1)\alpha(\epsilon(\tilde{T}), d)$

So from now on, assume $3 \det(\tilde{T}) \in p^2\mathbb{N}_0$.

We start with the easier case and suppose $p \nmid N(x)$. We assumed $p|\tilde{n}$, and so we have $3\tilde{m}\tilde{n} \in p\mathbb{N}_0$ (no matter how $\tilde{m}$ is chosen). But then $p \nmid N(x)$ yields $3 \det(\tilde{T}) = 3\tilde{m}\tilde{n} - N(x) \not\in p\mathbb{Z}$, so in particular $3 \det(\tilde{T}) \not\in p^2\mathbb{N}_0$. Hence once more, (4.46) yields

$$\beta(\tilde{T}) = p^{1-k}p(p + 1)\alpha(\epsilon(\tilde{T}), d)$$

in this case.

So let us consider the most involved situation now, i.e. $x \not\in p\mathcal{O}$, but $p \mid N(x)$. Suppose we found some combination $(\pi, \tilde{n}, q)$ such that

$$\pi\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p^2\frac{1}{\sqrt{3}}\mathcal{O}$$

holds (which is necessary if we want $p|\epsilon(T_{\pi,\tilde{n},q})$ to be fulfilled). Note that this implies

$$3N(p^{-1}(\pi\tilde{\pi} + \pi q\tilde{c}\tilde{n})) \in p^2\mathbb{N}_0 .$$

We have to distinguish between two cases: First, suppose $p^2 \nmid \tilde{n}$ (but still $p|\tilde{n}$). Then

$$3 \det(\tilde{T}) = 3 \det(T_{\pi,\tilde{n},q}) = \tilde{n}\left(\tilde{m} + p^{-1}q(2 \text{Re}(\tilde{c}\tilde{n}) + qN(c)\tilde{n})\right) - 3N\left(p^{-1}(\pi\tilde{\pi} + \pi q\tilde{c}\tilde{n})\right) \in p^2\mathbb{N}_0$$

implies

$$\tilde{m} + p^{-1}q(2 \text{Re}(\tilde{c}\tilde{n}) + qN(c)\tilde{n}) \in p\mathbb{N}_0 ,$$

so that we automatically obtain $p|\epsilon(T_{\pi,\tilde{n},q})$. On the other hand, if $p^2|\tilde{n}$ (hence $\pi q\tilde{c}\tilde{n} \in p^2\frac{1}{\sqrt{3}}\mathcal{O}$), then $\pi\tilde{\pi} + \pi q\tilde{c}\tilde{n} \in p^2\frac{1}{\sqrt{3}}\mathcal{O}$ is equivalent to

$$\pi\tilde{\pi} \in p^2\frac{1}{\sqrt{3}}\mathcal{O} .$$

Note that this condition is independent of $q$. So if the condition above holds, then there is $t \in \frac{1}{\sqrt{3}}\mathcal{O}$ such that $\pi\tilde{\pi} = p^2t$. Now, assume that $2 \text{Re}(\tilde{c}\tilde{n}) \in p^2\mathbb{Z}$ would hold. But then there exists $y \in \mathbb{Z}$ such that

$$\tilde{c}\tilde{n} + \pi\tilde{c}\tilde{t} = p^2y$$

$$\Rightarrow \tilde{c}p = p^2y\tilde{\pi} - \pi\tilde{c}\tilde{t}\tilde{\pi} = p^2y\tilde{\pi} - \pi\tilde{c}\tilde{t}\tilde{\pi} = p^2y\tilde{\pi} - \tilde{c}p^2t$$

$$\Rightarrow \tilde{t}N(c) = py\tilde{\pi} - \tilde{c}p\tilde{t} \in p\frac{1}{\sqrt{3}}\mathcal{O}$$

$$\Rightarrow \tilde{t} \in p\frac{1}{\sqrt{3}}\mathcal{O} ,$$

since $p \nmid N(c)$. But this is a contradiction, and thus

$$p^{-1}2 \text{Re}(\tilde{c}\tilde{n}) \not\in p\mathbb{Z} .$$
(But it is still an integer due to the assumption.) So we obtain
\[ \tilde{m} + p^{-1}q(2\Re(\tilde{t}c\tilde{n})) + qN(c)\tilde{n}) \in p\mathbb{N}_0 \]
\[ \Leftrightarrow \tilde{m} + q(p^{-1}2\Re(\tilde{t}c\tilde{n})) \in p\mathbb{N}_0 , \]
and this is fulfilled for exactly one of the \( q \) running through a transversal of \( \mathbb{Z}/p\mathbb{Z} \) (depending on \( \tilde{m} \)). So in both cases, it suffices to find a combination \((\pi, \tilde{n}, q)\) such that
\[ \pi\tilde{t}\tilde{n} + \pi qc\tilde{n} \in p^2\frac{1}{\sqrt{3}}\mathcal{O} \]
holds. If \( p^2 \nmid \tilde{n} \), then \( p|\varepsilon(T_{\pi,\tilde{n},q}) \) automatically follows, and if \( p^2|\tilde{n} \), then this combination is independent of \( q \), but \( p|\varepsilon(T_{\pi,\tilde{n},q}) \) holds for only exactly one choice of \( q \). Keeping that in mind (and \( \tilde{n} \in \mathbb{N}_0 \)), we will now search for those combination \((\pi, \tilde{n}, q)\) fulfilling
\[ \pi\tilde{t}\tilde{n} + \pi qc\tilde{n} \in p^2\frac{1}{\sqrt{3}}\mathcal{O} \]
\[ \Leftrightarrow p\tilde{t}\tilde{n} + pqc\tilde{n}\tilde{t} \in \pi p^2\frac{1}{\sqrt{3}}\mathcal{O}\tilde{n} \]
\[ \Leftrightarrow \tilde{t} + qc\frac{\tilde{n}}{p}\tilde{t} \in \frac{\pi}{\sqrt{3}}\mathcal{O}\tilde{n} \]
\[ \Leftrightarrow x - i_1\sqrt{3}qc\frac{\tilde{n}}{p}\tilde{t} \in (-i_1\tilde{\Pi}i_1)\mathcal{O}\tilde{n} . \]

In particular, this implies
\[ x \in \mathcal{O}\tilde{n} . \]
We assumed \( p|\mathbb{N}(x) \), so according to (4.18) (and since \( \tilde{n} \) runs through a transversal of \( \mathcal{E}\setminus \mathcal{N}(p) \)) we can choose \( \tilde{n} \) such that there exists \( y \in \mathcal{O} \) with \( x = y\tilde{n} \), and thus \( x \in \mathcal{O}\tilde{n} \). But note that this is the only possible choice for \( \tilde{n} \). If there was a combination \((\pi^+, \tilde{n}^+, q^+)\) such that also \( \pi^+\tilde{t}\tilde{n}^+ + \pi^+ q^+ c\tilde{n}^+ \in p^2\frac{1}{\sqrt{3}}\mathcal{O} \) holds, then \( x \in \mathcal{O}\tilde{n}^+ \) follows accordingly. But if \( \mathcal{E}\tilde{n} \cap \mathcal{E}\tilde{n}^+ = \emptyset \), then the same arguments used in (4.32) yield \( \mathcal{O}\tilde{n} \cap \mathcal{O}\tilde{n}^+ = p\mathcal{O} \), which leads to the contradiction \( x \in p\mathcal{O} \). So there is at most one possible choice for \( \tilde{n} \). So let \( \tilde{n} \) be the one special choice such that \( x \in \mathcal{O}\tilde{n} \), then the condition is reduced to
\[ y - i_1\sqrt{3}qc\frac{\tilde{n}}{p} \in (-i_1\Pi i_1)\mathcal{O} \].

On the other hand, given this fixed \( \tilde{n} \) and some arbitrary \( \pi \), we have \( c\tilde{n} = \pi s \), and thus
\[ -i_1\sqrt{3}qc\frac{\tilde{n}}{p} = -(-i_1\Pi i_1)(i_1\sqrt{3}qs\frac{\tilde{n}}{p}) \].
So the condition from the beginning can also be written as
\[ x - (-i_1\Pi i_1)(i_1\sqrt{3}qs\frac{\tilde{n}}{p}) \in (-i_1\Pi i_1)\mathcal{O}\tilde{n} . \]

So again, this condition implies
\[ x \in (-i_1\Pi i_1)\mathcal{O} , \]
in particular. Once more, we assumed \( p|\mathbb{N}(x) \), so according to (4.18) (and since \(-i_1\Pi i_1 \) obviously runs through a transversal of \( \mathcal{N}(p)/\mathcal{E} \)) we can choose \( \pi \) such that there exists \( z \in \mathcal{O} \) with \( x = (-i_1\Pi i_1)z \). Therefore, exactly the same reason like above yields that there is at most one
possible choice for \( \pi \). Note that the condition can therefore be reduced to

\[
z - i_1 \sqrt{3}qs \frac{n}{p} \in \mathcal{O}\bar{\pi}.
\]

So let us fix this one \( \pi \) satisfying \( x \in (-i_1\bar{\pi}i_1)\mathcal{O} \) and get back to the condition

\[
y - i_1 \sqrt{3}qc \frac{n}{p} \in (-i_1\pi i_1)\mathcal{O}.
\]

Again, if \( p^2 | \bar{n} \), then this condition is independent of \( q \), but we have seen that (if the \( \pi \) and \( \bar{n} \) we fixed actually fulfill the condition above) there is only one choice for \( q \) such that also \( p|\varepsilon(T,\pi,\pi,q) \) holds. On the other hand, suppose \( p^2 \not| \bar{n} \), and assume there are \( q_1, q_2 \in \mathbb{Z} \) with \( q_1 - q_2 \notin p\mathbb{Z} \), such that we have \( y - i_1 \sqrt{3}q_1c \frac{n}{p} \in (-i_1\pi i_1)\mathcal{O} \) and \( y - i_1 \sqrt{3}q_2c \frac{n}{p} \in (-i_1\pi i_1)\mathcal{O} \). This yields

\[
i_1 \sqrt{3}(q_1 - q_2)c \frac{n}{p} \in (-i_1\pi i_1)\mathcal{O},
\]

and thus

\[
N\left(i_1 \sqrt{3}(q_1 - q_2)c \frac{n}{p}\right) = 3(q_1 - q_2)^2 N(c) \frac{n^2}{p^2} \in p\mathbb{N}_0,
\]

which is a contradiction, since \( p \) divides non of the factors due to the assumption. So finally, we verified that exists at most one possible combination \( (\pi, \bar{n}, q) \) such that \( p|\varepsilon(T,\pi,\pi,q) \) holds. So let us prove that there exists \( q \) with

\[
z - i_1 \sqrt{3}qs \frac{n}{p} \in \mathcal{O}\bar{\pi},
\]

where we had \( x = y\bar{\pi} = (-i_1\bar{\pi}i_1)z \). So in particular

\[
(-i_1\pi i_1)z \in \mathcal{O}\bar{\pi}.
\]

Now, note that we verified the following in the proof of (4.38): Let \( a, b \in \mathcal{N}(p) \) and suppose there is \( u \in \mathcal{O} \) with \( p \not| \mathcal{N}(u) \) and \( au \in \mathcal{O}b \) (which exists according to (4.25)). Then for every \( v \in \mathcal{O} \) satisfying \( av \in \mathcal{O}b \), there is \( w \in \mathcal{O} \) and \( q \in \mathbb{Z} \), where \( q \) can be chosen from a fixed transversal of \( \mathbb{Z} / p\mathbb{Z} \), such that \( v = qu + wb \).

But this is exactly the situation we have given, here: \( \pi s \in \mathcal{O}\bar{\pi} \) holds by construction. But then we also have

\[
(-i_1\pi i_1)(i_1 \sqrt{3}s \frac{n}{p}) = i_1 \sqrt{3}s \frac{n}{p} \pi s \in \mathcal{O}\bar{\pi}.
\]

First, suppose \( p^2 \not| \bar{n} \). Then \( N(-i_1 \sqrt{3}s \frac{n}{p}) = 3 N(s) \frac{n^2}{p^2} \notin p\mathbb{N}_0 \). Therefore, the considerations above yield that there exists \( q \) in the transversal of \( \mathbb{Z} / p\mathbb{Z} \) we fixed and \( w \in \mathcal{O} \) such that

\[
z = i_1 \sqrt{3}qs \frac{n}{p} + w\bar{\pi},
\]

which implies

\[
z - i_1 \sqrt{3}qs \frac{n}{p} \in \mathcal{O}\bar{\pi}.
\]

On the other hand, suppose \( p^2 | \bar{n} \). Then the condition \( z - i_1 \sqrt{3}qs \frac{n}{p} \in \mathcal{O}\bar{\pi} \) is reduced to verifying

\[
z \in \mathcal{O}\bar{\pi},
\]
since \( i_1 \sqrt{3} q \frac{n}{p} = i_1 \sqrt{3} q \frac{n}{p} \pi \tilde{\pi} \in \mathcal{O} \tilde{\pi} \). But this is clear due to the assumption: We had \( 3 \det(T) = 3 \tilde{\pi} \pi - N(x) \in p^2 \mathcal{N}_0 \), but then \( p^2 | \tilde{\pi} \pi \) implies \( p^2 | N(x) = N((-i_1 \pi i_1)z) = p N(z) \), and thus \( p | N(z) \). So again, there is \( \pi^* \) from the transversal we fixed such that \( z \in \mathcal{O} \pi^* \). But then \( x = y \tilde{\pi} = (-i_1 \pi i_1)z \) implies

\[
x \in \mathcal{O} \tilde{\pi} \cap \mathcal{O} \pi^*
\]

and thus \( \pi^* = \tilde{\pi} \), or otherwise \( \mathcal{O} \tilde{\pi} \cap \mathcal{O} \pi^* = \mathcal{O} \) yields a contradiction, like we have seen before.

So finally, (4.46) gives

\[
\beta(\tilde{T}) = p^{1-k}(p(2p + 1) - 1)\alpha(\epsilon(\tilde{T}), d) + p^{1-k} \alpha(\epsilon(\tilde{T}), p^{-2}d)
\]

if \( p | \tilde{n}, x \notin \mathcal{O} \), but \( p | N(x) \), and \( 3 \det(\tilde{T}) \in p^2 \mathcal{N}_0 \).

We will need the results we found for \( j = 0 \) for the case \( j \geq 1 \), so we summarize them:

- \( p | \tilde{n} \):
  - \( x \in \mathcal{O} \): \( \beta(\tilde{T}) = p^{1-k}p(p + 1)^2 \alpha(\epsilon(\tilde{T}), d) \).
  - \( x \notin \mathcal{O} \), \( p | N(x) \): \( \beta(\tilde{T}) = p^{1-k}\left((p(2p + 1) - \chi_Z(p^{-2}d)) \alpha(\epsilon(\tilde{T}), d) + \alpha(\epsilon(\tilde{T}), p^{-2}d)\right) \).
  - \( p \nmid N(x) \): \( \beta(\tilde{T}) = p^{1-k}p(p + 1)^2 \alpha(\epsilon(\tilde{T}), d) \).

- \( p \nmid \tilde{n} \): \( \beta(\tilde{T}) = p^{1-k}(p + 1)^2 \alpha(\epsilon(\tilde{T}), d) \).

Here, one should keep in mind that we have \( p^{-2}d \notin \mathcal{N}_0 \) if and only if \( 3 \det(\tilde{T}) \in p^2 \mathcal{N}_0 \), and \( \alpha(\cdot, p^{-2}d) = 0 \) for \( p^{-2}d \notin \mathcal{N}_0 \).

The case \( j \geq 1 \) is handled in exactly the same way we did for \( S_2(p) \). So let \( \epsilon(\tilde{T}) = p^j r \), where now \( j \geq 1 \). Again, we will trace this case back to the one we considered first. By definition, we have \( S := \frac{1}{p^j} \tilde{T} \in \text{Her}_2^x(\mathcal{O}) \), with \( \epsilon(S) = 1 \), and \( 3 \det(S) = 3 \det(S)/\epsilon(S)^2 = 3 \det(\tilde{T})/\epsilon(\tilde{T})^2 = d \), and denote \( S = (\frac{y}{y}, y) \), with \( t = \frac{i}{\sqrt{3}} y, y \in \mathcal{O} \). Like we remarked above, all of the \( p(p + 1)^2 \) possible combinations \( (\pi, \tilde{\pi}, q) \) yield an appropriate \( T_{\pi, \tilde{\pi}, q} \), and

\[
\epsilon(T_{\pi, \tilde{\pi}, q}) \in \{ p^j r, p^{j+1} r \}.
\]

Again, suppose that we have \( \epsilon(T_{\pi, \tilde{\pi}, q}) \neq p^j r \), then \( S_{\pi, \tilde{\pi}, q} := \frac{1}{p^j} T_{\pi, \tilde{\pi}, q} \in \text{Her}_2^x(\mathcal{O}) \) holds by definition, and we obviously have

\[
\epsilon(T_{\pi, \tilde{\pi}, q}) = p^j re(S_{\pi, \tilde{\pi}, q})
\]

and

\[
S_{\pi, \tilde{\pi}, q} = \frac{1}{p} S[\tilde{D}_{\pi, \tilde{\pi}, q}].
\]

And once more, note that this is exactly the situation we had to consider above (where \( j = 0 \), and now even \( r = 1 \)), so we already know which (and how many) of the combinations \( (\pi, \tilde{\pi}, q) \) yield an appropriate \( S_{\pi, \tilde{\pi}, q} \) and under which conditions we have either \( \epsilon(S_{\pi, \tilde{\pi}, q}) = 1 \) or \( \epsilon(S_{\pi, \tilde{\pi}, q}) = p \). Keeping that in mind, we go through all cases we had to consider above.

So suppose we have \( p^{j+1} | \tilde{n} \) (\( p | \tilde{n} \)) and \( x \in p^{j+1} \mathcal{O} \) (\( \epsilon(x) \in \mathcal{O} \)). We have seen above that all \( S_{\pi, \tilde{\pi}, q} \) exist in this case, and fulfill \( \epsilon(S_{\pi, \tilde{\pi}, q}) = 1 \). Hence all of the \( T_{\pi, \tilde{\pi}, q} \) (which are \( p(p + 1)^2 \)
in total) have to fulfill \( \epsilon(T_r) = p r = \epsilon(\tilde{T}) \), and thus (4.46) leads to
\[
\beta(\tilde{T}) = p^{1-k} p(p + 1)^2 \alpha(\epsilon(\tilde{T}), d) .
\]
Furthermore, \( d \in p \mathbb{N}_0 \) holds in this case. Next, if we have \( p^{l+1} r | \tilde{n} \) (\( \Leftrightarrow p | n \) and \( x \not\in p^{l+1} \mathcal{O} \) (\( \Leftrightarrow y \not\in p \mathcal{O} \)), but \( p^{l+1} r^2 | N(x) \) (\( \Leftrightarrow p | N(y) \)), then only \( p(2p + 1) \) of the \( S_{\pi, \pi, q} \) fulfill \( S_{\pi, \pi, q} \in \text{Her}_2^r(\mathcal{O}) \). Moreover, we have \( 3 \det(S) \in p^2 \mathbb{N}_0 \) if and only if \( d \in p^2 \mathbb{N}_0 \) (since \( 3 \det(S) = d \)) and the result from above yields
\[
\beta(\tilde{T}) = p^{1-k} \left( p^2(p + 1) \alpha(\epsilon(\tilde{T}), p^2d) + (p^2 + 1) - \chi z(p^{-2}d) \alpha(\epsilon(\tilde{T}), d) + \alpha(\epsilon(\tilde{T}), p^{-2}d) \right),
\]
since \( p(p + 1)^2 - p(2p + 1) = p^3 \). Again, \( d \in p \mathbb{N}_0 \) holds in this case. Now, suppose \( p^{l+1} r | \tilde{n} \) (\( \Leftrightarrow p | n \)) and \( p^{l+1} r^2 \not\in \mathbb{N}(x) \) (\( \Leftrightarrow p \not\in \mathbb{N}(y) \)), then only \( p(2p + 1) \) of the \( S_{\pi, \pi, q} \) fulfill \( S_{\pi, \pi, q} \in \text{Her}_2^r(\mathcal{O}) \), and we obtain
\[
\beta(\tilde{T}) = p^{1-k} \left( p^2(p + 1) \alpha(\epsilon(\tilde{T}), p^2d) + p + 1) \alpha(\epsilon(\tilde{T}), d) \right),
\]
since \( p(p + 1)^2 - p(p + 1) = p^2(p + 1) \). This time, \( d \not\in p \mathbb{N}_0 \) obviously holds. And finally, if we have \( p^{l+1} r \not\in \tilde{n} \) (\( \Leftrightarrow p \not\in n \)), then the considerations from above give rise to
\[
\beta(\tilde{T}) = p^{1-k} \left( (p^2 + p^2 - p - 1) \alpha(\epsilon(\tilde{T}), p^2d) + (p + 1)^2 \alpha(\epsilon(\tilde{T}), d) \right),
\]
since \( p(p + 1)^2 - (p + 1)^2 = p^3 + p^2 - p - 1 \).

Note again that all these formulas also hold true if we allow \( j = 0 \), since we have \( \alpha(\epsilon^{-1} r, \ast) = 0 \) by definition. Once more, we formulate this result in the following lemma. The formula in this lemma can be verified by simply going through each of the cases above and compare the resulting expressions.

**Lemma.** Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we get
\[
f|_k S_2(p)s(Z) = \sum_{\tilde{T} \in \text{Her}_2^r(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2\pi i \tau(\tilde{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]
where we have the following cases for \( 0 \neq \tilde{T} = \left( \begin{array}{c} \tilde{n} \\ \tilde{r} \end{array} \right) \in \text{Her}_2^r(\mathcal{O}), \quad \tilde{T} \geq 0, \) with \( \tilde{r} = \frac{p}{\sqrt{3}} X \) for some appropriate \( x \in \mathcal{O} \), where \( d = 3 \det(\tilde{T})/\epsilon(\tilde{T})^2 \):

- If \( pe(\tilde{T})|\tilde{n} \) and \( x \in pe(\tilde{T})\mathcal{O} \), then
  \[
  \beta(\tilde{T}) = p^{1-k} p(p + 1)^2 \alpha(\epsilon(\tilde{T}), d) .
  \]
  Furthermore, we have
  \[
  d \in p \mathbb{N}_0
  \]
in this case.
Using the same notation like in all the considerations above, we obtain:

\[ \beta(\bar{T}) = p^{1-k} \left( p^3 \alpha(p^{-1} \epsilon(\bar{T}), p^2 d) + (p(2p + 1) - \chi_Z(p^{-2}d)) \alpha(\epsilon(\bar{T}), d) + \alpha(p\epsilon(\bar{T}), p^{-2}d) \right). \]

Furthermore, we have

\[ d \in p\mathbb{N}_0 \]

in this case, again.

• If \( pe(\bar{T}) \nmid \bar{n} \) and \( pe(\bar{T})^2 \not\mid N(x) \), then

\[ \beta(\bar{T}) = p^{1-k} \left( p^2(p + 1) \alpha(p^{-1} \epsilon(\bar{T}), p^2 d) + p(p + 1) \alpha(\epsilon(\bar{T}), d) \right). \]

This time, \( d \not\in p\mathbb{N}_0 \) holds.

• If \( pe(\bar{T}) \nmid \bar{n} \), then

\[ \beta(\bar{T}) = p^{1-k} \left( (p^3 + p^2 - p - 1) \alpha(p^{-1} \epsilon(\bar{T}), p^2 d) + (p + 1)^2 \alpha(\epsilon(\bar{T}), d) \right). \]

Finally, we collect all results from (4.41), (4.42), (4.43), (4.45) and (4.47). But since there is no dependency on \( \bar{T} \) when considering \( S_2(p)_1 \) and \( S_2(p)_4 \), we omit them, for now. So let

\[ (f|_kS_2(p)_2 + f|_kS_2(p)_3 + f|_kS_2(p)_5)(Z) = \sum_{\bar{T} \in \text{Her}_\mathbb{Z}(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T})e^{2\pi i \tau(\bar{T},Z)}, \quad Z \in \mathcal{H}(\mathbb{I}). \]

Using the same notation like in all the considerations above, we obtain:

• \( pe(\bar{T}) \nmid \bar{n} \):

\[ \beta(\bar{T}) = p^{1-k} \left( (p^4 + (1 - \chi_Z(p^{-1}d))p^2) \cdot \alpha(p^{-1} \epsilon(\bar{T}), p^2 d) \\
+ (p^3 + (1 + \chi_Z(p^{-1}d))p^2 + p + 1 - \chi_Z(p^{-2}d)) \cdot \alpha(\epsilon(\bar{T}), d) \\
+ \alpha(p\epsilon(\bar{T}), p^{-2}d) \right). \]

• \( pe(\bar{T}) | \bar{n} \), but \( p^2 \epsilon(\bar{T}) \nmid \bar{n} \):
  - \( pe(\bar{T})^2 \not\mid N(x) \) (hence also \( p \nmid d \)):

\[ \beta(\bar{T}) = p^{1-k} \left( (p^4 + p^2) \cdot \alpha(p^{-1} \epsilon(\bar{T}), p^2 d) \\
+ (p^3 + p^2 + p + 1) \cdot \alpha(\epsilon(\bar{T}), d) \right). \]
- \( p\epsilon(\tilde{T})^2 \mid N(x) \), but \( x \notin p\epsilon(\tilde{T})\mathcal{O} \) (which implies \( p\mid d \)):

\[
\beta(\tilde{T}) = p^{1-k} \left( p^4 \cdot \alpha(p^{-1}\epsilon(\tilde{T}), p^2d) \\
+ (p^3 + 2p^2 + p + 1 - \chi_{\mathbb{Z}}(p^{-2}d)) \cdot \alpha(\epsilon(\tilde{T}), d) \\
+ \alpha(p\epsilon(\tilde{T}), p^{-2}d) \right)
\]

- \( x \in p\epsilon(\tilde{T})\mathcal{O} \) (which implies \( p\mid d \)):

\[
\beta(\tilde{T}) = p^{1-k} \left( p^4 \cdot \alpha(p^{-1}\epsilon(\tilde{T}), p^2d) \\
+ (p^3 + 2p^2 + p + 1) \cdot \alpha(\epsilon(\tilde{T}), d) \right).
\]

- \( p^2\epsilon(\tilde{T}) \mid \tilde{n} \):

- \( p\epsilon(\tilde{T})^2 \nmid N(x) \) (hence also \( p \nmid d \)):

\[
\beta(\tilde{T}) = p^{1-k} \left( (p^4 + p^2) \cdot \alpha(p^{-1}\epsilon(\tilde{T}), p^2d) \\
+ (p^3 + 2p^2 + p + 1) \cdot \alpha(\epsilon(\tilde{T}), d) \right).
\]

- \( p\epsilon(\tilde{T})^2 \mid N(x) \), but \( x \notin p\epsilon(\tilde{T})\mathcal{O} \) (which implies \( p\mid d \)):

\[
\beta(\tilde{T}) = p^{1-k} \left( p^4 \cdot \alpha(p^{-1}\epsilon(\tilde{T}), p^2d) \\
+ (p^3 + 2p^2 + p + 1 - \chi_{\mathbb{Z}}(p^{-2}d)) \cdot \alpha(\epsilon(\tilde{T}), d) \\
+ \alpha(p\epsilon(\tilde{T}), p^{-2}d) \right)
\]

- \( x \in p\epsilon(\tilde{T})\mathcal{O} \) (which implies \( p^2\mid d \)):

\[
\beta(\tilde{T}) = p^{1-k} \left( p^4 \cdot \alpha(p^{-1}\epsilon(\tilde{T}), p^2d) \\
+ (p^3 + 2p^2 + p) \cdot \alpha(\epsilon(\tilde{T}), d) \\
+ \alpha(p\epsilon(\tilde{T}), p^{-2}d) \right).
\]

Note that the cases \( \tilde{n} \in p^2\epsilon(\tilde{T})\mathbb{N}_0 \) and \( \tilde{n} \in p\epsilon(\tilde{T})\mathbb{N}_0 \setminus p^2\epsilon(\tilde{T})\mathbb{N}_0 \) are almost identical, except for when \( x \in p\epsilon(\tilde{T})\mathcal{O} \) holds. But note that if \( \tilde{n} \in p\epsilon(\tilde{T})\mathbb{N}_0 \setminus p^2\epsilon(\tilde{T})\mathbb{N}_0 \) and \( x \in p\epsilon(\tilde{T})\mathcal{O} \), then \( p\epsilon(\tilde{T}) \nmid \tilde{m} \) has to hold due to the definition of \( \epsilon(\cdot) \). But this implies \( d \nmid p^2\mathbb{N}_0 \). Note again that we have \( \alpha(\cdot, p^{-2}d) = 0 \) if \( p^{-2}d \notin \mathbb{N}_0 \). Therefore, we can combine the two cases and obtain
\[ p \varepsilon(\bar{T}) | n: \]
- \( p \varepsilon(\bar{T})^2 \nmid N(x) \) (hence also \( p \nmid d \)):
  \[
  \beta(\bar{T}) = p^{1-k} \left( (p^4 + p^2) \cdot a(p^{-1} \varepsilon(\bar{T}), p^2 d) + (p^3 + p^2 + p + 1) \cdot a(\varepsilon(\bar{T}), d) \right).
  \]
- \( p \varepsilon(\bar{T})^2 \mid N(x) \), but \( x \notin p \varepsilon(\bar{T}) \mathcal{O} \) (which implies \( p \mid d \)):
  \[
  \beta(\bar{T}) = p^{1-k} \left( p^4 \cdot a(p^{-1} \varepsilon(\bar{T}), p^2 d) + (p^3 + 2p^2 + p + 1 - \chi_Z(p^{-2}d)) \cdot a(\varepsilon(\bar{T}), d) + a(p \varepsilon(\bar{T}), p^{-2}d) \right).
  \]
- \( x \in p \varepsilon(\bar{T}) \mathcal{O} \) (which implies \( p \mid d \)):
  \[
  \beta(\bar{T}) = p^{1-k} \left( p^4 \cdot a(p^{-1} \varepsilon(\bar{T}), p^2 d) + (p^3 + 2p^2 + p + 1 - \chi_Z(p^{-2}d)) \cdot a(\varepsilon(\bar{T}), d) + a(p \varepsilon(\bar{T}), p^{-2}d) \right).
  \]

Moreover, going through all cases concerning \( p \varepsilon(\bar{T}) | n \) (where we also specified if \( d \in p \mathcal{N}_0 \) or not) shows that all cases coincide with the one where \( p \varepsilon(\bar{T}) \nmid n \). Therefore, we finally have the following result:

**4.48 Theorem.** Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( a \) according to (4.17). Then we get
\[
f|_k S_2(p)(Z) = \sum_{\bar{T} \in \text{Her}_2^+(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T}) e^{2\pi i \tau(\bar{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]
where for \( 0 \neq \bar{T} \in \text{Her}_2^+(\mathcal{O}), \bar{T} \geq 0 \), we have
\[
\beta(\bar{T}) = \gamma_p(\varepsilon(\bar{T}), 3 \det(\bar{T}) / \varepsilon(\bar{T})^2)
\]
for a map \( \gamma_p : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C} \). Given \( j, d \in \mathbb{N}_0 \) and \( q \in \mathbb{N} \) with \( p \nmid q \), we have
\[
\gamma_p(p^j q, d) = p^{6-2k} a(p^j+1 q, d) + (p^2 + 1 - \chi_Z(p^{-1}d)) p^{3-k} a(p^j+1 q, p^2 d) + p^{1-k} a(p^j+1 q, p^{-2}d) + (p^3 + (1 + \chi_Z(p^{-1}d)) p^2 + p + 1 - \chi_Z(p^{-2}d)) p^{1-k} a(p^j q, d) + a(p^j q, d).
\]

Note that this is exactly the same result that came up in [Kr90, thm.1] for the Hurwitz order, but all the calculations were omitted. Furthermore, we will see that we will even get the same result for \( S_2^+(p) \). But we will not consider the other generators of the Hecke algebra, here. But note that \( S_2(p) \) and \( S_2^+(p) \) are the most involved ones, and the results for the other generators would follow accordingly. So in principle, one could adopt [Kr90, thm.1] to \( \mathcal{O} \), stating that the
4.4 The quaternionic Hecke-operators $T_2(p)$

The space of Maaß lifts is invariant under all Hecke-operators. But since this would need a lot more (yet straightforward) pre-work, we are not going to prove this. Yet still, the most involved results can already be found here.

### 4.4.3 The action of the Hecke-operators $S_2^2(p)$ on Maaß lifts

Of course, we have to consider $S_2^2(p)$, now. The process of determining the action of $S_2^2(p)$ on Maaß lifts will be exactly the same like the one concerning $S_2(p)$. Yet there are some critical points that work a bit different, so we should not omit the details, here. But we will go through each case a bit faster without prompting out every small detail, since the procedures are quite the same like the ones concerning $S_2(p)$.

Just like we did with $S_2(p)$, we will look at $S_2^2(p)_1$, $S_2^2(p)_2$, $S_2^2(p)_3$ and $S_2^2(p)_4$ separately (see 4.40). Here, $S_2^2(p)_2$ and $S_2^2(p)_3$ are the easier cases, while $S_2^2(p)_1$ and $S_2^2(p)_4$ are more involved.

So let us start by considering $S_2^2(p)_2$, hence $A_\pi = (\begin{smallmatrix} 0 & 0 \\ \pi & 0 \end{smallmatrix})$, $D_\pi = (\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix})$ (where $\pi$ will run through a transversal of $E \setminus \mathcal{N}(p)$) and $B = 0$. Given $T = (\begin{smallmatrix} m & t \\ \pi & n \end{smallmatrix}) \in \text{Her}_2(\mathcal{O})$, we compute

$$
\widetilde{T} = \frac{1}{p} T[A_\pi] = \left( \begin{array}{cc} m & \pi t \\ \pi & pn \end{array} \right).
$$

Hence $\widetilde{T}$ occurring in the Fourier-expansion of $f|_k S_2^2(p)_2$ will belong to $\text{Her}_2(\mathcal{O})$, indeed, and 4.38 yields again

$$
f|_k S_2^2(p)_2(Z) = p^{-k/2} \sum_{\pi: E \setminus \mathcal{N}(p)} \sum_{\pi: A_\pi, T = (\begin{smallmatrix} m & t \\ \pi & n \end{smallmatrix}) \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i t (\frac{1}{p} T[A_\pi], Z)}.
$$

Now, let $0 \neq \widetilde{T} = (\frac{1}{p} T[A_\pi])$, where $\widetilde{T} \in \text{Her}_2(\mathcal{O})$, $\widetilde{T} \geq 0$ be one of those matrices occurring in the Fourier-expansion, given by

$$
\widetilde{T} = \left( \begin{array}{cc} \pi & t \\ t & \pi \end{array} \right),
$$

where again we choose $x \in \mathcal{O}$ such that $\tilde{t} = \frac{\pi}{\sqrt{3}} x$. We already know that we have $\tilde{n} \in p\mathbb{N}_0$ and $\tilde{t} = \frac{i}{\sqrt{3}} x \in \pi \frac{i}{\sqrt{3}} \mathcal{O}$ (or equivalently $x \in (-i, \pi i)\mathcal{O}$), then. Like we did when we considered $S_2(p)$, we need to analyze $\frac{1}{p} T[D_\pi']$. A simple calculation yields

$$
T_\pi := \frac{1}{p} T[D_\pi'] = \left( \begin{array}{cc} \pi & \pi t \\ p^{-1} \pi & p^{-1} n \end{array} \right).
$$

Again, we have $\frac{1}{p} T_\pi[A_\pi] = p^{-2} T[D_\pi'] A_\pi = p^{-2} T[pI] = \widetilde{T}$, and so once more the question arises under which conditions $T_\pi \in \text{Her}_2(\mathcal{O})$ holds. Of course, $p|\tilde{n}$ has to be fulfilled. Furthermore, we have

$$
p^{-1} \pi t \in \frac{i}{\sqrt{3}} \mathcal{O} \iff p^{-1} \pi \tilde{t} = \tilde{t} \in \pi \frac{i}{\sqrt{3}} \mathcal{O} \iff x \in (-i, \pi i)\mathcal{O}.
$$

This is exactly the conditions from above. Therefore, $\widetilde{T}$ occurs in the Fourier-expansion of
f|₄S₂°(p)₂ if and only if T₁ ∈ Her₂(𝒪) for at least one π ∈ N(p), and thus

\( p|\tilde{n} \quad \text{and} \quad x ∈ (-i₁\tilde{π}i₁)𝒪 \)

has to be fulfilled (for at least one π ∈ N(p)). Note that the condition \( x ∈ (-i₁\tilde{π}i₁)𝒪 \) for at least one π ∈ N(p) is equivalent to \( p|N(x) \) in view of (4.18). Moreover, we have to sum over π from a fixed transversal of \( E\setminus N(p) \) (which implies that \( -i₁\tilde{π}i₁ \) runs through a transversal of \( N(p)/E \) – see (4.30)), hence there might exist several \( T₁ \) (of which we have \( p + 1 \) in total in view of (1.7)) such that \( T₁ ∈ Her₂(𝒪) \) and \( T₁[\mathcal{A}_π] = \tilde{T} \) – and at least one if and only if \( p|\tilde{n} \) and \( p|N(x) \). We already have

\[
3 \det(T₁) = 3p^{-1}\tilde{m}n - N(p^{-1}πx) = p^{-1}(3\tilde{m}n - N(x)) = p^{-1}3\det(\tilde{T}) .
\]

Note that \( p|\tilde{n} \) and \( p|N(x) \) implies \( 3\det(T₁) ∈ \mathbb{N}_0 \). Before we determine \( ε(T₁) \), we have to analyze under which conditions \( T₁ ∈ Her₂(𝒪) \) holds (where π runs through the transversal of \( E\setminus N(p) \) we fixed). We already have the necessary conditions \( p|\tilde{n} \) and \( p|N(x) \). So we can choose a π such that \( x ∈ (-i₁\tilde{π}i₁)𝒪 \) holds (since \( -i₁\tilde{π}i₁ \) runs through a transversal of \( N(p)/E \)). Now, suppose that there exists another \( \tilde{π} \) in this transversal such that \( Eπ \cap E\tilde{π} = \emptyset \) (hence equivalently \( (-i₁\tilde{π}i₁)E \cap (-i₁\tilde{π}i₁)E = \emptyset \)) and \( x ∈ (-i₁\tilde{π}i₁)𝒪 \). Then we get \( x ∈ (-i₁\tilde{π}i₁)𝒪 \cap (-i₁\tilde{π}i₁)𝒪 = p𝒪 \), like we have seen several times before (or confer the proof of (4.30)). On the other hand, we have \( p𝒪 = (-i₁\tilde{π}i₁)(-i₁\tilde{π}i₁)𝒪 \subset (-i₁\tilde{π}i₁)𝒪 \) for all π ∈ N(p). Therefore we conclude:

- p \( \not| \) \( \tilde{n} \) or p \( \not| \) \( N(x) \), then none of the \( p + 1 \) possible π yields an appropriate \( T₁ ∈ Her₂(𝒪) \).
- p \( | \) \( \tilde{n} \) and \( p|N(x) \), but \( x \not∈ p𝒪 \), then exactly one of the \( p + 1 \) possible π yields an appropriate \( T₁ ∈ Her₂(𝒪) \).
- p \( | \) \( \tilde{n} \) and \( x ∈ p𝒪 \), then all \( p + 1 \) possible π yield an appropriate \( T₁ ∈ Her₂(𝒪) \).

So let us determine \( ε(T₁) \) in terms of \( ε(\tilde{T}) \), now. Suppose \( ε(\tilde{T}) = pq \), where p \( \not| \) q. So we have \( \tilde{m}, \tilde{n} ∈ pq\mathbb{N}_0 \) and \( \tilde{t} ∈ pq\sqrt{-3}O \). This gives \( p^{-1}\tilde{n} ∈ p^{-1}q\mathbb{N}_0 \) and \( p^{-1}\pi\tilde{t} ∈ p^{-1}qπ\sqrt{-3}O \subset p^{-1}q\sqrt{-3}O \), hence \( (p^{-1}q)^{-1}T₁ ∈ Her₂(𝒪) \). Now, suppose there is \( r ∈ \mathbb{N} \) with \( (p^{-1}q)^{-1}T₁ ∈ Her₂(𝒪) \). This yields \( \tilde{m} ∈ p^{-1}qr\mathbb{N}_0 , \tilde{n} ∈ p^{-1}qr\mathbb{N}_0 \) and

\[
p^{-1}\pi\tilde{t} ∈ p^{-1}qr\frac{1}{\sqrt{-3}}O \quad ⇒ \quad \tilde{t} ∈ p^{-1}qr\pi\frac{1}{\sqrt{-3}}O \subset p^{-1}qr\frac{1}{\sqrt{-3}}O ,
\]

and thus \( (p^{-1}q)^{-1}T₁ ∈ Her₂(𝒪) \), which implies \( r ∈ \{1,p\} \) due to the definition of \( ε(\cdot) \). So we obtain

\[
ε(T₁) ∈ \{p^{-1}q,pq\}
\]

for all appropriate \( T₁ ∈ Her₂(𝒪) \) (where of course \( ε(T₁) = q \) if \( j = 0 \)). So we have to find out in which cases \( ε(T₁) = pq = ε(\tilde{T}) \) actually holds. Of course, \( pq|p^{-1}\tilde{n} \) is equivalent to \( pq|p^{j+1}\tilde{n} \). Furthermore, we have

\[
p^{-1}\pi\tilde{t} ∈ pq\frac{1}{\sqrt{-3}}O \quad ⇔ \quad (pq)^{-1}x ∈ (-i₁\tilde{π}i₁)𝒪 \).
\]

Note that \( y = (pq)^{-1}x ∈ O \) holds by assumption. Therefore, the same considerations like above yield that \( p|N(y) \) is a necessary condition, and if \( y ∈ p𝒪 \), then \( y ∈ (-i₁\tilde{π}i₁)𝒪 \) holds for all π,
while for \( p \mid N(y) \), but \( y \not\in p\mathcal{O} \), \( y \in (-i_1 \pi i_1)\mathcal{O} \) holds for exactly one of the \( \pi \) from the fixed transversal. So again, we actually reduced the case \( j \geq 1 \) to the considerations for \( j = 0 \) like we did in some cases concerning \( S_2(p) \) — without explicitly pointing it out here, since this first case is still quite easy and straightforward. Therefore, noting that \( \alpha(z, \cdot) = 0 \) if \( z \not\in \mathbb{N} \), 4.38 and 4.39 give:

(4.49) Lemma. Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we get

\[
f|_k S_2^k(p)z(Z) = \sum_{\tilde{T} \in \text{Her}_2^z(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T})e^{2\pi i r(\tilde{T},Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]

where we have the following cases for \( 0 \neq \tilde{T} = \left( \frac{\alpha \tilde{t}}{\tilde{n}} \right) \in \text{Her}_2^z(\mathcal{O}), \tilde{T} \geq 0 \), with \( \tilde{t} = \frac{\sqrt{3}}{2} x \) for some appropriate \( x \in \mathcal{O} \), where \( d := 3 \det(\tilde{T})/\epsilon(\tilde{T})^2 \):  

- If \( p \epsilon(\tilde{T}) \nmid \tilde{n} \) or \( p \epsilon(\tilde{T})^2 \nmid N(x) \), then
  \[
  \beta(\tilde{T}) = p^{-k/2}(p + 1)\alpha(p^{-1}\epsilon(\tilde{T}), pd). 
  \]

- If \( p \epsilon(\tilde{T})\tilde{n} \) and \( p \epsilon(\tilde{T})^2 \mid N(x) \), but \( x \not\in p\epsilon(\tilde{T})\mathcal{O} \), then
  \[
  \beta(\tilde{T}) = p^{-k/2}\left(p\alpha(p^{-1}\epsilon(\tilde{T}), pd) + \alpha(\epsilon(\tilde{T}), p^{-1}d)\right). 
  \]

Furthermore, we have

\[
d \in p\mathbb{N}_0
\]

in this case.

- If \( p \epsilon(\tilde{T})\tilde{n} \) and \( x \in p\epsilon(\tilde{T})\mathcal{O} \), then
  \[
  \beta(\tilde{T}) = p^{-k/2}(p + 1)\alpha(\epsilon(\tilde{T}), p^{-1}d). 
  \]

Furthermore, we have

\[
d \in p\mathbb{N}_0
\]

in this case, again.

Next, we have a look at \( S_2(p)_3 \), since it is also quite straightforward, while another important consideration comes into play. So we have \( A_\pi = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \), \( D_\pi = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \) (where \( \pi \) will run through a transversal of \( \mathcal{E}/\mathcal{N}(p) \)) and \( B = \left( \frac{q}{\pi b_2} 0 \right) \) (where \( q \) will run through a transversal of \( \mathbb{Z}/p\mathbb{Z} \), while \( b_2 \) will run through a transversal of \( \mathcal{O}/\mathcal{O}\pi \)). So given \( T = \left( \begin{smallmatrix} a & t \\ \pi & n \end{smallmatrix} \right) \in \text{Her}_2^z(\mathcal{O}) \), we have

\[
\tilde{T} = \frac{1}{p} T A_\pi = \left( \begin{array}{cc} p^{-1}m & p^{-1}t \pi \\ p^{-1}\pi t & n \end{array} \right)
\]

and

\[
B\tilde{T} A_\pi' = \left( \begin{array}{cc} q & b_2 \pi \pi \\ \pi b_2 & 0 \end{array} \right).
\]
Thus \( \tau(T, B\overline{\mathcal{A}}) / p = p^{-1}q + p^{-1}2\Re(\tilde{b}_2\pi) \), so (4.38) yields again

\[
\sum_{\pi \in \mathcal{N}(p)} \sum_{T = \left( \frac{m}{n} \right)} \left( \sum_{q \in \mathbb{Z} / p\mathbb{Z}} \left( \frac{2\pi \lambda m / p}{q} \right)^{\frac{1}{2}} \sum_{b_2 \in \mathcal{O} / \mathcal{O}_\pi} \right) e^{2\pi i 2\Re(\overline{\tilde{b}_2}\pi)/p} \cdot \alpha_f(T) e^{2\pi i \tau(\frac{1}{p}T[A_\pi], Z)}. 
\]

Again, in view of (4.43), the Fourier-coefficient of \( \overline{\mathcal{A}} \) vanishes if \( p \nmid m \). Otherwise, a factor \( p \) is gained. Now, suppose \( \{a_1, \ldots, a_p\} \) is the transversal of \( \mathcal{O} / \mathcal{O}_\pi \) we fixed, where one should note that it has length \( p^2 \) according to (4.27), indeed. Furthermore, let \( \{c_1, \ldots, c_{p^2}\} \) be a transversal of \( \mathcal{O} / \mathcal{O}_\pi \). We assert now that \( \{a_j + c_k\pi; j, k \in \{1, \ldots, p^2\}\} \) is a transversal of \( \mathcal{O} / p\mathcal{O} \). We will show that the elements of this set are pairwise incongruent modulo \( p\mathcal{O} \) (hence they are also pairwise distinct), which implies that this set is a transversal, since it consists of \( p^4 \) elements. So suppose we have \( j_1, j_2, k_1, k_2 \in \{1, \ldots, p^2\} \) and \( d \in \mathcal{O} \) such that

\[
a_{j_1} + c_{k_1}\pi = a_{j_2} + c_{k_2}\pi + pd = a_{j_2} + (c_{k_2} + d\pi)\pi. 
\]

Then \( j_1 = j_2 \) immediately follows. But then we obtain

\[
c_{k_1}\pi = c_{k_2}\pi + pd \iff c_{k_1} = c_{k_2} + d\pi,
\]

which implies \( k_1 = k_2 \), hence the assertion. Note that we have

\[
\sum_{k = 1}^{p^2} e^{2\pi i 2\Re(\tilde{c}_k\pi\pi)/p} = \sum_{k = 1}^{p^2} e^{2\pi i 2\Re(\overline{\tilde{c}}_k)/p} = p^2,
\]

because \( 2\Re(\tilde{a}) \in \mathbb{Z} \) holds for all \( a \in \mathcal{O} \). Moreover, noting the calculation rules for \( \Re(\cdot) \) (see [Kr85, ch.I, prop.1.1]) we have

\[
\Re(\tilde{b}_2\pi) + \Re(\tilde{c}_k\pi\pi) = \Re(\tilde{f}(b_2 + c_k\pi\pi)) = \Re(\overline{\tilde{f}}(b_2 + c_k\pi)) .
\]

Therefore, we obtain

\[
\sum_{b_2 \in \mathcal{O} / \mathcal{O}_\pi} e^{2\pi i 2\Re(\overline{\tilde{b}}_2\pi)/p} = \frac{1}{p^2} \sum_{b \in \mathcal{O} / p\mathcal{O}} \sum_{c \in \mathcal{O} / \mathcal{O}_\pi} e^{2\pi i 2\Re(\overline{\tilde{f}}(b + c\pi))/p} = \frac{1}{p^2} \sum_{b \in \mathcal{O} / p\mathcal{O}} e^{2\pi i 2\Re(\overline{\tilde{f}}(b))/p}. 
\]

Thus, by making use of (4.44), this gives

\[
\sum_{b_2 \in \mathcal{O} / \mathcal{O}_\pi} e^{2\pi i 2\Re(\overline{\tilde{b}}_2\pi)/p} = \begin{cases} 
0, & \text{if } t\pi \notin \frac{1}{\sqrt{3}}\mathcal{O}, \\
p^2, & \text{if } t\pi \in \frac{1}{\sqrt{3}}\mathcal{O}.
\end{cases} \tag{4.49}
\]

Therefore, the Fourier-coefficient of \( \frac{1}{p}T[A_\pi] \) vanishes if \( p \nmid m \) or \( t\pi \notin \frac{1}{\sqrt{3}}\mathcal{O} \), and equals \( p^3\alpha_f(T) \) otherwise. (Note that we have \( t\pi \in \frac{1}{\sqrt{3}}\mathcal{O} \) if and only if \( t \in \frac{1}{\sqrt{3}}\mathcal{O}_\pi \). Hence again, only those \( \overline{T} \)
satisfying $\tilde{T} \in \text{Her}_2^1(\mathcal{O})$ occur in the Fourier-expansion of $f|_kS_2^+(p)_3$, and we have
\[ f|_kS_2^+(p)_3(Z) = p^{3-3k/2} \sum_{\pi : x \in \mathbb{N}(p) \cap \text{Her}_2^1(\mathcal{O}), T} \sum_{\pi \in \text{Her}_2^1(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(T[A_\pi], Z)}. \]

Now, let $0 \neq \tilde{T} = \frac{1}{p} T[A_\pi]$, where $p|m, t \in \mathbb{N}^\mathcal{O}_3 \mathcal{O}_\pi$, and $\tilde{T} \in \text{Her}_2^1(\mathcal{O})$, $\tilde{T} \geq 0$ be one of those matrices occurring in the Fourier-expansion, given by
\[ \tilde{T} = \begin{pmatrix} \tilde{m} & \tilde{t} \\ \tilde{t} & \tilde{n} \end{pmatrix} \]
with $\tilde{t} = \frac{i}{\sqrt{3}} x$ for some $x \in \mathcal{O}$. Just like we always did, we need to analyze $\frac{1}{p} \tilde{T}[\mathcal{D}_\pi']$. A simple calculation yields
\[ T_\pi := \frac{1}{p} \tilde{T}[\mathcal{D}_\pi'] = \begin{pmatrix} p\tilde{m} & i\tilde{n} \\ \pi \tilde{t} & \tilde{n} \end{pmatrix}. \]
Just like in all other cases, we have $\frac{1}{p} T_r[A_\pi] = p^{-2} \tilde{T}[\mathcal{D}_\pi A_\pi] = p^{-2} \tilde{T}[pI] = \tilde{T}$, while the first diagonal entry is divisible by $p$, and the secondary diagonal entry is an element of $\mathbb{N}^\mathcal{O}_3 \mathcal{O}_\pi$, indeed. So this time, all of the $p+1$ possible $\pi$ yield an appropriate $T_\pi \in \text{Her}_2^1(\mathcal{O})$. We compute
\[ 3 \det(T_\pi) = 3\tilde{m}\tilde{n} - p N(x) = 3p \det(\tilde{T}), \]
and again we need to determine $\varepsilon(T_\pi)$ in terms of $\varepsilon(\tilde{T})$. So let $\varepsilon(\tilde{T}) = p/q$, where $p \nmid q$. Obviously, $(p/q)^{-1} T_\pi \in \text{Her}_2^1(\mathcal{O})$ already holds for all $\pi$. Now, suppose there is $r \in \mathbb{N}$ with $(p/q)^{-1} T_\pi \in \text{Her}_2^1(\mathcal{O})$. This yields $\tilde{m} \in p^{-1} qr \mathcal{O}_0, \tilde{n} \in p^{-1} qr \mathcal{O}_0$ and
\[ \tilde{t} \in p^{-1} qr \frac{i}{\sqrt{3}} \mathcal{O} \subset p^{-1} qr \frac{i}{\sqrt{3}} \mathcal{O}, \]
and thus $(p^{-1} qr)^{-1} T \in \text{Her}_2^1(\mathcal{O})$, which implies $r \in \{1, p\}$ due to the definition of $\varepsilon(\cdot)$. So we obtain
\[ \varepsilon(T_\pi) \in \{p/q, p^{i+1}q\} \]
for all $T_\pi$. So again, we have to find out in which cases $\varepsilon(T_\pi) = p^{i+1}q = pe(\tilde{T})$ actually holds. Of course, $p^{i+1}q|\tilde{n}$ must be fulfilled (and $p^{i+1}q|\tilde{m}$ holds in all cases). Furthermore, we get
\[ \tilde{n} \in p^{i+1}q \frac{i}{\sqrt{3}} \mathcal{O} \iff (p/q)^{-1} x \in \mathcal{O} \pi, \]
where $y = (p/q)^{-1} x \in \mathcal{O}$ holds by assumption. Therefore, the same considerations used for $S_2^+(p)_2$ yield that $p|N(y)$ is a necessary condition, and if $y \in \mathcal{O}\pi$, then $y \in \mathcal{O}\pi$ holds for all $\pi$, while for $p|N(y)$, but $y \notin \mathcal{O}$, $y \in \mathcal{O}\pi$ holds for exactly one of the $\pi$ from the fixed transversal. Therefore, 4.38 and 4.39 yield:
(4.50) Lemma. Suppose $p \neq 3$ is a prime number and $f \in \mathcal{M}(k; \mathcal{O})$ with associated function $\alpha$ according to (4.17). Then we get

$$f|_k S_2^2(p)_3(Z) = \sum_{\tilde{T} \in \text{Her}_2^2(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2\pi i \tilde{T} Z}, \quad Z \in \mathcal{H}(\mathbb{H}),$$

where we have the following cases for $0 \neq \tilde{T} = \left( \begin{smallmatrix} \tilde{m} & \tilde{l} \\ \tilde{r} & \tilde{n} \end{smallmatrix} \right) \in \text{Her}_2^2(\mathcal{O}), \tilde{T} \geq 0$, with $\tilde{t} = \frac{\tilde{m}}{\sqrt{3}} x$ for some appropriate $x \in \mathcal{O}$, where $d := 3 \det(\tilde{T})/\varepsilon(\tilde{T})^2$:

- If $p \varepsilon(\tilde{T}) \nmid \tilde{n}$ or $p \varepsilon(\tilde{T})^2 \nmid N(x)$, then
  $$\beta(\tilde{T}) = p^{3 - 3k/2}(p + 1)\alpha(\varepsilon(\tilde{T}), pd).$$

- If $p \varepsilon(\tilde{T})|\tilde{n}$ and $p \varepsilon(\tilde{T})^2|N(x)$, but $x \notin p \varepsilon(\tilde{T}) \mathcal{O}$, then
  $$\beta(\tilde{T}) = p^{3 - 3k/2}(p\alpha(\varepsilon(\tilde{T}), pd) + \alpha(p \varepsilon(\tilde{T}), p^{-1}d)).$$

Furthermore, we have

$$d \in p\mathbb{N}_0$$

in this case.

- If $p \varepsilon(\tilde{T})|\tilde{n}$ and $x \in p \varepsilon(\tilde{T}) \mathcal{O}$, then
  $$\beta(\tilde{T}) = p^{3 - 3k/2}(p + 1)\alpha(p \varepsilon(\tilde{T}), p^{-1}d).$$

Furthermore, we have

$$d \in p\mathbb{N}_0$$

in this case, again.

Next, we have to get to the more complicated cases and start by examining the action of $S_2^2(p)_1$. So let $A_{\pi,r} = \left( \begin{smallmatrix} p & 0 \\ -\pi r & \pi \end{smallmatrix} \right)$, $D_{\pi,r} = \left( \begin{smallmatrix} 1 & r \\ 0 & \pi \end{smallmatrix} \right)$ (where $\pi$ will run through a transversal of $\mathcal{E}\setminus \mathcal{N}(p)$, while $r$ will run through a transversal of $\mathcal{O}/\mathcal{O}\pi$ for each given $\pi$) and $B = 0$. Given $T = \left( \begin{smallmatrix} m & l \\ t & n \end{smallmatrix} \right) \in \text{Her}_2^2(\mathcal{O})$, we compute

$$\tilde{T} = \frac{1}{p} T[A_{\pi}] = \left( \begin{array}{cc} pm - 2 \text{Re}(t\pi \bar{r} + n N(r)) & t\pi - rn \\ \pi \bar{t} - n \bar{r} & n \end{array} \right).$$

Hence all $\tilde{T}$ occurring in the Fourier-expansion of $f|_k S_2^2(p)_1$ will belong to $\text{Her}_2^2(\mathcal{O})$, indeed, and once more 4.38 yields

$$f|_k S_2^2(p)_1(Z) = p^{-k/2} \sum_{\pi \in \mathcal{E}\setminus \mathcal{N}(p), r : \mathcal{O}/\mathcal{O}\pi} \sum_{T \in \left( \begin{smallmatrix} m & l \\ t & n \end{smallmatrix} \right) \in \text{Her}_2^2(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tilde{T} Z(A_{\pi}, Z)}.$$

So let $0 \neq \tilde{T} = \frac{1}{p} T[A_{\pi,r}]$, where $\tilde{T} \in \text{Her}_2^2(\mathcal{O}), \tilde{T} \geq 0$ be one of those matrices occurring in the
Fourier-expansion, given by

\[
\tilde{T} = \begin{pmatrix}
\tilde{m} & \tilde{t} \\
\tilde{t} & \tilde{n}
\end{pmatrix},
\]

where \(\tilde{t} = \frac{i}{\sqrt{3}} x\) for some \(x \in \mathcal{O}\). Of course, we need to analyze \(\frac{1}{p} \tilde{T}[D_{\pi,r}']\) this time, too. Another simple calculation yields

\[
T_{\pi,r} := \frac{1}{p} \tilde{T}[D_{\pi,r}'] = \begin{pmatrix}
p^{-1}(\tilde{m} + 2 \text{Re}(\tilde{t}) + \tilde{n} N(r)) & p^{-1}(\tilde{t} \tilde{t} + \tilde{n} r \tilde{t}) \\
p^{-1}(\tilde{t} \tilde{t} + \tilde{n} r \tilde{t}) & \tilde{n}
\end{pmatrix}.
\]

So since we have \(\frac{1}{p} T_{\pi,r}[A_{\pi,r}] = p^{-2} \tilde{T}[D_{\pi,r}'A_{\pi,r}] = p^{-2} \tilde{T}[pI] = \tilde{T}\), once more the question arises under which conditions \(T_{\pi,r} \in \text{Her}_2^r(\mathcal{O})\) holds, since these are exactly those matrices from which a \(\tilde{T}\) can originate in the Fourier-expansion. In view of (1.7) and (4.27) there are exactly \(p^2(p+1)\) combinations \((\pi, r)\). But first, note that we have

\[
\det(T_{\pi,r})^2 = \det(\tilde{T}_{\pi,r}) = p^{-4}(N(\pi))^2 \det(\tilde{T}) = p^{-2} \det(\tilde{T}) = (p^{-1} \det(\tilde{T}))^2
\]

in virtue of (1.34), and thus

\[
\det(T_{\pi,r}) = p^{-1} \det(\tilde{T})
\]

due to \(T_{\pi,r}\) and \(\tilde{T}\) being positive semi-definite. Of course, \(T_{\pi,r} \in \text{Her}_2^r(\mathcal{O})\) implies \(3 \det(T_{\pi,r}) \in \mathbb{N}_0\), and thus a necessary condition is

\[
3 \det(\tilde{T}) \in p\mathbb{N}_0,
\]
or otherwise no combination \((\pi, r)\) yields an appropriate \(T_{\pi,r} \in \text{Her}_2^r(\mathcal{O})\).

Therefore, let us assume \(3 \det(\tilde{T}) \in p\mathbb{N}_0\) henceforth. Now, suppose that we already found a combination such that \(\tilde{t} := p^{-1}(\tilde{t} \tilde{t} + \tilde{n} r \tilde{t}) \in \frac{i}{\sqrt{3}} \mathcal{O}\) holds (and still \(3 \det(\tilde{T}) \in p\mathbb{N}_0\)). Then this implies \(3 N(t) \in \mathbb{N}_0\), and thus

\[
3 \det(T_{\pi,r}) = 3 \left(p^{-1}(\tilde{m} + 2 \text{Re}(\tilde{t}) + \tilde{n} N(r))\right) \tilde{n} - 3 N(t) \in \mathbb{Z}
\]

implies

\[
3 \tilde{n} \left(\tilde{m} + 2 \text{Re}(\tilde{t}) + \tilde{n} N(r)\right) \in p\mathbb{Z}.
\]

But assuming that \(p \nmid \tilde{n}\) holds then leads to

\[
\tilde{m} + 2 \text{Re}(\tilde{t}) + \tilde{n} N(r) \in p\mathbb{Z},
\]

and thus automatically \(T_{\pi,r} \in \text{Her}_2^r(\mathcal{O})\). Therefore, if \(p \nmid \tilde{n}\) and \(3 \det(\tilde{T}) \in p\mathbb{N}_0\) hold, then we have

\[
T_{\pi,r} \in \text{Her}_2^r(\mathcal{O}) \iff \tilde{t} \tilde{t} + \tilde{n} r \tilde{t} \in p\frac{i}{\sqrt{3}} \mathcal{O}.
\]

So we start with this case and search for all combinations fulfilling \(\tilde{t} \tilde{t} + \tilde{n} r \tilde{t} \in p\frac{i}{\sqrt{3}} \mathcal{O}\), which is equivalent to

\[
x - i_1 \sqrt{3} \tilde{n} r \in \mathcal{O}\pi.
\]
According to (4.30), when \( r \) runs through a transversal of \( \mathcal{O}/\mathcal{O}_\pi \), then so does \( i_1 \sqrt{3} \tilde{n} r \) (since \( p \nmid \tilde{n} \)). Therefore, there exists exactly one \( r \) in the transversal we fixed fulfilling this condition. So we have \( p + 1 \) possible combinations \( (\pi, r) \) (one \( r \) for each of the \( p + 1 \) possible \( \pi \)) in case \( p \nmid \tilde{n} \).

Next, assume \( p|\tilde{n} \). Then the condition \( x - i_1 \sqrt{3} \tilde{n} r \in \mathcal{O}_\pi \) is reduced to
\[
x \in \mathcal{O}_\pi.
\]
Of course, this can be fulfilled if and only if \( p|N(x) \). But note that this is automatically fulfilled, since we are still assuming \( 3 \det(\tilde{T}) = 3\tilde{n}\tilde{n} - N(x) \in p\mathcal{N}_0 \). First, suppose \( x \notin p\mathcal{O} \) (but still \( p|N(x) \)). Then, due to (4.18) there actually exists a \( \pi \) comming from the transversal of \( \mathcal{E} \setminus \mathcal{N}(p) \) we fixed fulfilling \( x \in \mathcal{O}_\pi \). But like we have seen several times before, there cannot exists a second \( \tilde{n} \) fulfilling \( x \in \mathcal{O}_\tilde{n} \), since otherwise \( x \in \mathcal{O}_\pi \cap \mathcal{O}_\tilde{n} = p\mathcal{O} \) would have to follow (see the proof of (4.32)). So let us fix this one \( \pi \) fulfilling \( x \in \mathcal{O}_\pi \). Next, \( p^{-1}(\tilde{m} + 2 \text{Re}(\tilde{r}r) + \tilde{n}N(r)) \in \mathcal{N}_0 \) has to hold. So since \( p|\tilde{n} \), we have to find all \( r \) in the transversal of \( \mathcal{O}/\mathcal{O}_\pi \) satisfying
\[
\tilde{m} + 2 \text{Re}(\tilde{r}r) \in p\mathcal{Z}.
\]
According to (4.28),
\[
\varphi: \mathcal{O}/\mathcal{O}_\pi \to \mathcal{Z}/p\mathcal{Z} \ , \ a + \mathcal{O}_\pi \mapsto 2 \text{Re}(\tilde{r}a) + p\mathcal{Z}
\]
is a surjective homomorphism. Again, we have \#\( \mathcal{O}/\mathcal{O}_\pi = p^2 \) in view of (4.27), while \#\( \mathcal{Z}/p\mathcal{Z} = p \).
Therefore, obviously exactly \( p = p^2/p \) of the \( r \) from the transversal of \( \mathcal{O}/\mathcal{O}_\pi \) fulfill
\[
2 \text{Re}(\tilde{r}r) \in -\tilde{m} + p\mathcal{Z}.
\]
Therefore, we have \( p \) possible combinations \( (\pi, r) \) (since \( \pi \) is fixed) in case \( p|\tilde{n} \), \( p|N(x) \), but \( x \notin p\mathcal{O} \).

And finally, assume \( p|\tilde{n} \) and \( x \in p\mathcal{O} \) (so in particular \( 3 \det(\tilde{T}) \in p\mathcal{N}_0 \) is fulfilled.) Since \( p = \pi \pi \),

\[
x \in \mathcal{O}_\pi
\]
holds for every choice of \( \pi \). Furthermore, we already used several times before that \( x \in p\mathcal{O} \) implies \( 2 \text{Re}(t\bar{a}) \in p\mathcal{Z} \) for all \( a \in \mathcal{O} \). Therefore, the condition on the first diagonal entry of \( T_{\pi,r} \) is reduced to
\[
\tilde{m} \in p\mathcal{Z}.
\]
But since \( p|\tilde{n} \) and \( x \in p\mathcal{O} \) already hold by assumption, this is equivalent to \( p|\epsilon(\tilde{T}) \). Therefore, if we assume \( p|\tilde{n} \) and \( x \in p\mathcal{O} \), then no combination \( (\pi, r) \) yields an appropriate \( T_{\pi,r} \) if \( p \nmid \epsilon(\tilde{T}) \).
Otherwise, every combination \( (\pi, r) \) (which are \( p^2(p + 1) \) in total) yields an appropriate \( T_{\pi,r} \). Again, we summarize these results in the following

(4.51) Lemma. Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k;\mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we have
\[
f|_k S_2(p)_1(Z) = \sum_{\tilde{T} \in \text{Her}_2(\mathcal{O}), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2\pi i \tau(\tilde{T},Z)} , \quad Z \in \mathcal{H}(\mathbb{H}) .
\]
4.4 The quaternionic Hecke-operators $T_2(p)$

Given $\pi \in \mathcal{N}(p)$ and $r \in \mathcal{O}$ and $0 \neq \tilde{T} = (\frac{m}{n} \ i \ x) \in \text{Her}^2_{\mathbb{H}}(O)$, $\tilde{T} \geq 0$, with $\tilde{t} = \frac{i}{\sqrt{3}}x$ for some appropriate $x \in \mathcal{O}$, define $T_{\pi,r} = \frac{1}{p} \tilde{T}[D_{\pi,r}]$, where $D_{\pi,r} = (\frac{1}{0} \ i \ n)$. Then we have

$$\beta(\tilde{T}) = p^{-k/2} \sum_{\begin{subarray}{l} \pi \in \mathcal{E} \setminus \mathcal{N}(p), \ r \mathcal{O}/\mathcal{O} \pi \\ T_{\pi,r} \in \text{Her}^2_{\mathbb{H}}(O) \end{subarray}} \alpha(\varepsilon(T_{\pi,r}), p^{-1}3\det(\tilde{T})/\varepsilon(T_{\pi,r})^2).$$

Let $n(\tilde{T}) = \#\{\pi : \mathcal{E} \setminus \mathcal{N}(p), \ r : \mathcal{O}/\mathcal{O} \pi ; T_{\pi,r} \in \text{Her}^2_{\mathbb{H}}(O)\}$. Then we have the following cases:

- $p \nmid \tilde{n}$:
  - $3\det(\tilde{T}) \notin p\mathbb{N}_0$: $n(\tilde{T}) = 0$.
  - $3\det(\tilde{T}) \in p\mathbb{N}_0$: $n(\tilde{T}) = p + 1$.

- $p | \tilde{n}$:
  - $p | \mathcal{N}(x)$: $n(\tilde{T}) = 0$.
    (Note that $p | \mathcal{N}(x)$ is equivalent to $3\det(\tilde{T}) \notin p\mathbb{N}_0$ in this case.)
  - $p | \mathcal{N}(x)$, but $x \notin \mathcal{O}$: $n(\tilde{T}) = p$.
    (And note again that $p | \mathcal{N}(x)$ is equivalent to $3\det(\tilde{T}) \in p\mathbb{N}_0$ in this case.)
  - $x \in \mathcal{O}$:
    * $p \nmid \varepsilon(\tilde{T})$: $n(\tilde{T}) = 0$.
    * $p | \varepsilon(\tilde{T})$: $n(\tilde{T}) = p^2(p + 1)$.
    (And we have $3\det(\tilde{T}) \in p^2\mathbb{N}_0$ in this case.)

Let us determine $\varepsilon(T_{\pi,r})$ in terms of $\varepsilon(\tilde{T})$, now. So suppose $\varepsilon(\tilde{T}) = p^l q$ with $p \nmid q$. Of course, this implies $p^{l-1}q | \tilde{n}$, $p^{-1}(i\pi + \tilde{n}r\pi) \in p^{-1}q \frac{i}{\sqrt{3}}\mathcal{O}$ and $p^{l-1}q | p^{-1}(\tilde{m} + 2\Re(\tilde{r}) + \tilde{n}\mathcal{N}(r))$, as well. So we already obtain $p^{l-1}q | \varepsilon(T_{\pi,r})$. On the other hand, suppose there exists $r \in \mathbb{N}$ such that $(p^{l-1}q)^{-1}T_{\pi,r} \in \text{Her}^2_{\mathbb{H}}(O)$. Of course, this implies $p^{l-1}q | \tilde{m}$, and also $\tilde{\pi} + \tilde{n}r\pi \in p^{l-1}q \frac{i}{\sqrt{3}}\mathcal{O}$, hence (by simply multiplying with $\pi$ and noting $\tilde{n}r \in p^{l-1}q \frac{i}{\sqrt{3}}\mathcal{O}$)

$$\tilde{t} + \tilde{n}r \in p^{l-1}q \frac{i}{\sqrt{3}}\mathcal{O} \pi \subset p^{l-1}q \frac{i}{\sqrt{3}}\mathcal{O} \quad \Rightarrow \quad \tilde{t} \in p^{l-1}q \frac{i}{\sqrt{3}}\mathcal{O}. $$

And finally $p^{l}q | (\tilde{m} + 2\Re(\tilde{r}) + \tilde{n}\mathcal{N}(r))$ holds. But this also implies

$$p^{l-1}q | (\tilde{m} + 2\Re(\tilde{r}) + \tilde{n}\mathcal{N}(r)), $$

and thus $p^{l-1}q | \tilde{m}$, since $p^{l-1}q | \tilde{n}\mathcal{N}(r)$ and $p^{l-1}q | 2\Re(\tilde{r})$ already hold. Therefore, $r \in \{1, p\}$ follows due $\varepsilon(\tilde{T}) = p^l q$, hence

$$\varepsilon(T_{\pi,r}) \in \{p^{l-1}q, p^l q\}$$

for all appropriate $T_{\pi,r} \in \text{Her}^2_{\mathbb{H}}(O)$ (where of course $T_{\pi,r} = q$ if $j = 0$). But then (4.51) almost automatically yields the Fourier-expansion of $f_{\sqrt{2}}S_2^2(p)_{1}$: By definition, we have $S := \frac{1}{p^l q} \tilde{T} \in \text{Her}^2_{\mathbb{H}}(O)$, with $\varepsilon(S) = 1$, and $3\det(S) = 3\det(\tilde{T})/\varepsilon(S)^2 = 3\det(\tilde{T})/\varepsilon(\tilde{T})^2 =: d$. Denote $S = (\frac{m}{n} \ t \ y)$, with $t = \frac{i}{\sqrt{3}}y \ y \in \mathcal{O}$. Like we have seen in (4.51), all of the $p^2(p + 1)$ possible
combinations \((\pi,r)\) yield an appropriate \(T_{\pi,r}\) if and only if \(j \geq 1\), and
\[
\varepsilon(T_{\pi,r}) \in \{ p^{j-1}q, p^j q \}.
\]
So suppose that we actually have \(\varepsilon(T_{\pi,r}) = p^j q\), then \(S_{\pi,r} := \frac{1}{p^j q} T_{\pi,r} \in \text{Her}_2^2(\mathcal{O})\) holds by definition. Of course, this holds the other way round, too, so that
\[
\varepsilon(T_{\pi,r}) = p^j q \quad \Leftrightarrow \quad S_{\pi,r} \in \text{Her}_2^2(\mathcal{O}).
\]
We obviously have
\[
S_{\pi,r} = \frac{1}{p} S[D_{\pi,r}].
\]
But then (4.51) gives the answer on how many combinations \((\pi,r)\) yield such an \(S_{\pi,r} \in \text{Her}_2^2(\mathcal{O})\) (in terms of \(n, y, \varepsilon(S)\) and \(d = 3 \det(S)\), which can be traced back to \(\bar{n}, x, \varepsilon(\bar{T})\) and \(d = 3 \det(\bar{T})/\varepsilon(\bar{T})^2\)). Furthermore, note again that \(\alpha(a,b) = 0\) if \(a \notin \mathbb{N}\) or \(b \notin \mathbb{N}_0\). Therefore, simply going through each case in (4.51) immediately yields:

(4.52) Lemma. Suppose \(p \neq 3\) is a prime number and \(f \in \mathcal{M}(k; \mathcal{O})\) with associated function \(\alpha\) according to (4.17). Then we get
\[
f|k S_2^2(p)_1(Z) = \sum_{\bar{T} \in \text{Her}_2^2(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T}) e^{2\pi i \varepsilon(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]
where we have the following cases for \(0 \neq \bar{T} = (\bar{m} \bar{u} \bar{n}) \in \text{Her}_2^2(\mathcal{O}), \bar{T} \geq 0\), with \(\bar{t} = \frac{4}{\sqrt{3}} x\) for some appropriate \(x \in \mathcal{O}\), where \(d := 3 \det(\bar{T})/\varepsilon(\bar{T})^2\):

- If \(pe(\bar{T}) \nmid \bar{n}\), then
  \[
  \beta(\bar{T}) = p^{-k/2} \left( (p^3 + p^2 - \chi_Z(p^{-1}d) \cdot (p + 1)) \alpha(p^{-1} \varepsilon(\bar{T}), pd) + (p + 1) \alpha(\varepsilon(\bar{T}), p^{-1}d) \right).
  \]
- If \(pe(\bar{T}) \mid \bar{n}\) and \(x \notin pe(\bar{T}) \mathcal{O}\), then
  \[
  \beta(\bar{T}) = p^{-k/2} \left( (p^3 + p^2 - \chi_Z(p^{-1}d) \cdot p) \alpha(p^{-1} \varepsilon(\bar{T}), pd) + p \alpha(\varepsilon(\bar{T}), p^{-1}d) \right).
  \]
- If \(pe(\bar{T}) \mid \bar{n}\) and \(x \in pe(\bar{T}) \mathcal{O}\), then
  \[
  \beta(\bar{T}) = p^{-k/2} (p^3 + p^2) \alpha(p^{-1} \varepsilon(\bar{T}), pd).
  \]

And finally, we have to consider the last of all cases, hence \(S_2^2(p)_4\). So we have \(A_{\pi,u} = (\pi_{\pi_{-1}})\), \(D_{\pi,u} = (\pi_{\pi_{1}})\) (where \(\pi\) will run through a transversal of \(\mathcal{E} \setminus \mathcal{N}(p)\), while \(u\) will run through a transversal of \(\mathcal{O}/\pi \mathcal{O}\) and \(B = (0_{b_3} \pi_{b_3} \pi_{b_3})\) (where \(q\) will run through a transversal of \(\mathcal{Z}/p \mathcal{Z}\), while \(b_3\) will run through a transversal of \(\mathcal{O}/\mathcal{O} \pi\)). So given \(T = (\pi_{b_3}) \in \text{Her}_2^2(\mathcal{O})\), we compute
\[
\bar{T} = \frac{1}{p} T A_{\pi,u} = \left( \begin{array}{ccc} m + p^{-1}(-2 \text{Re}(\bar{T} \pi u) + n \mathcal{N}(u)) & p^{-1} \mathcal{N}(\pi u - nu) \\ p^{-1}(\bar{T} \pi - nu) & p^{-1}n \end{array} \right).
\]
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and

$$B\overline{A}_{\pi,u} = \left( \begin{array}{cc} 0 & \pi b_3 \\ b_3 \pi & q - 2 \text{Re}(b_3 u) \end{array} \right),$$

thus \( \tau(T, B\overline{A}_{\pi,u})/p = p^{-1}(q - 2 \text{Re}(b_3 u))n + p^{-1}2 \text{Re}(\pi b_3) \), so 4.38 yields again

$$f|kS_2^\pi(p)_4(Z) = p^{-3k/2} \sum_{\pi: \mathcal{E} \backslash \mathcal{N}(p), u: \pi \mathcal{O} / \pi \mathcal{O} T \in \text{Her}_2^\pi(O), T \geq 0} \sum_{T = \left( \begin{array}{cc} m & t \\ m & n \end{array} \right)} \left( \sum_{q \in Z/pZ} (e^{2\pi i n / p})^q \right) \cdot \sum_{b_3 \mathcal{O} / \pi \mathcal{O}} e^{2\pi i (2 \text{Re}(\pi b_3) - 2n \text{Re}(b_3 u))/p} \cdot \alpha_f(T)e^{2\pi i \nu (\pi \mathcal{N}(1T[A_{\pi,u}]/Z)}.$$

So again in view of 4.43, the Fourier-coefficient of \( \frac{1}{p} T[A_{\pi,u}] \) vanishes if \( p \nmid n \). Otherwise, a factor \( p \) is gained. But if \( p|n \) holds, then we have

$$e^{2\pi i (2 \text{Re}(\pi b_3) - 2n \text{Re}(b_3 u))/p} = e^{2\pi i 2 \text{Re}(\pi b_3)/p}e^{-2\pi i \nu_2 2 \text{Re}(b_3 u)} = e^{2\pi i 2 \text{Re}(\pi b_3)/p} = e^{2\pi i 2 \text{Re}(b_3 \pi)/p},$$

and 4.49 yields

$$\sum_{b_3 \mathcal{O} / \pi \mathcal{O}} e^{2\pi i 2 \text{Re}(b_3 \pi)/p} = \begin{cases} 0, & \text{if } \pi \mathcal{O} \nmid \nu_3 \mathcal{O} \\ p^2, & \text{if } \pi \mathcal{O} \mid \nu_3 \mathcal{O}. \end{cases}$$

Therefore, the Fourier-coefficient of \( \frac{1}{p} T[A_{\pi,u}] \) vanishes if \( p \nmid n \) or \( \pi \mathcal{O} \nmid \nu_3 \mathcal{O} \) (which is equivalent to \( \pi t \nmid \nu_3 \mathcal{O} \)), and equals \( p^2\alpha_f(T) \) otherwise. (Note again that we have \( \pi t \in \nu_3 \mathcal{O} \) if and only if \( t \in \nu_3 \mathcal{O} \).) Hence, also in the case of \( S_2^\pi(p) \) only those \( T \) satisfying \( \tilde{T} \in \text{Her}_2^\pi(O) \) occur in the Fourier-expansion of \( f|kS_2^\pi(p)_4 \), and we get

$$f|kS_2^\pi(p)_4(Z) = p^{3-3k/2} \sum_{\pi: \mathcal{E} \backslash \mathcal{N}(p), u: \pi \mathcal{O} / \pi \mathcal{O}} \sum_{T = \left( \begin{array}{cc} m & t \\ m & n \end{array} \right) \in \text{Her}_2^\pi(O), T \geq 0} \frac{p|n, t \in \pi \mathcal{O} \nu_3 \mathcal{O} \mathcal{O}}{\alpha_f(T)e^{2\pi i \nu (\pi \mathcal{N}(1T[A_{\pi,u}]/Z)}.$$

So once more, let \( 0 \neq \tilde{T} = \left( \begin{array}{cc} \tilde{m} & \tilde{t} \\ \tilde{m} & \tilde{n} \end{array} \right) \), where \( p|n, t \in \pi \mathcal{O} \nu_3 \mathcal{O} \), and \( \tilde{T} \in \text{Her}_2^\pi(O), \tilde{T} \geq 0 \) be one of those matrices occurring in the Fourier-expansion, given by

$$\tilde{T} = \left( \begin{array}{cc} \tilde{m} & \tilde{t} \\ \tilde{m} & \tilde{n} \end{array} \right)$$

with \( \tilde{t} = \frac{\nu_3}{\pi} x \) for some \( x \in \mathcal{O} \). We need to analyze \( \frac{1}{p} \tilde{T}[\mathcal{D}'_{\pi,u}] \), just like in all previous cases. Another calculation yields

$$T_{\pi,u} := \frac{1}{p} \tilde{T}[\mathcal{D}'_{\pi,u}] = \left( \begin{array}{cc} \tilde{m} + 2 \text{Re}(\tilde{m} \tilde{u}) + \tilde{t} N(u) & \pi \tilde{t} + \pi u \tilde{n} \\ \tilde{m} \pi + \tilde{t} \pi \tilde{n} & \pi \tilde{n} \end{array} \right).$$

Once more, we have \( \frac{1}{p} T[A_{\pi,u}] = p^{-2} \tilde{T}[\mathcal{D}'_{\pi,u} A_{\pi,u}] = p^{-2} \tilde{T}[pI] = \tilde{T} \), while the second diagonal entry is divisible by \( p \), and the secondary diagonal entry is an element of \( \pi \mathcal{O} \nu_3 \), indeed. So this time, like for \( S_2^\pi(p)_3 \), all of the \( p^2(p+1) \) possible combinations \( (\pi, u) \) (see (1.7) and (4.27)) yield
an appropriate $T_{\pi,u} \in \text{Her}_2^+(\mathcal{O})$. We compute

$$\det(T_{\pi,u})^2 = \det(\tilde{T}_{\pi,u}) = p^{-4} (N(\pi) N(p))^2 \det(\tilde{T}) = p^2 \det(\tilde{T}) = (p \det(\tilde{T}))^2$$

in virtue of (1.34), and thus

$$\det(T_{\pi,u}) = p \det(\tilde{T})$$

due to $T_{\pi,u}$ and $\tilde{T}$ being positive semi-definite. Next, we need to determine $\epsilon(T_{\pi,u})$ in terms of $\epsilon(\tilde{T})$, again. So let $\epsilon(\tilde{T}) = p^j q$, where $p \nmid q$. Obviously, the shape of $T_{\pi,u}$ then yields $(p^j q)^{-1} T_{\pi,u} \in \text{Her}_2^+(\mathcal{O})$. So once more, suppose there exists $r \in \mathbb{N}$ satisfying $(p^j q)^{-1} T_{\pi,u} \in \text{Her}_2^+(\mathcal{O})$. Then this implies $p^j q | p\tilde{n}$, or equivalently $p^{-j} q | p \tilde{n}$ (if $j \geq 1$). Furthermore, we have $\tilde{n} + \pi u \tilde{n} \in p^j q \frac{1}{\sqrt{3}} \mathcal{O}$, and thus (by multiplying with $\pi$ and noting $u \tilde{n} \in p^{-j} q \frac{1}{\sqrt{3}} \mathcal{O}$)

$$\tilde{t} + u \tilde{n} \in p^{-j} q \pi \frac{1}{\sqrt{3}} \mathcal{O} \subset p^{-j} q \frac{1}{\sqrt{3}} \mathcal{O} \implies \tilde{t} \in p^{-j} q \frac{1}{\sqrt{3}} \mathcal{O}.$$  

And finally, this leads to

$$\tilde{m} + 2 \text{Re}(\tilde{t} \tilde{n}) + \tilde{n} N(u) \in p^j q r \mathcal{Z} \subset p^j q r \mathcal{Z} \implies \tilde{m} \in p^j q r \mathcal{Z},$$

since $2 \text{Re}(\tilde{t} a) \in p^j q r \mathcal{Z}$ holds for all $a \in \mathcal{O}$. So once again, $\epsilon(\tilde{T}) = p^j q$ implies $r \in \{1, p\}$, and thus

$$\epsilon(T_{\pi,u}) \in \{ p^j q, p^{j+1} q \}.$$  

This time, since we do not have to determine which combinations $(\pi, u)$ yield an appropriate $T_{\pi,u}$, we directly start by defining $S := \frac{1}{p^j q} \tilde{T}$ and denote $S = (\frac{m}{\tilde{t}} \frac{l}{\tilde{n}})$, with $t = \frac{\tilde{t}}{\sqrt{3}} y, y \in \mathcal{O}$. Of course, we have $S \in \text{Her}_2^+(\mathcal{O})$, with $\epsilon(S) = 1$, and $3 \det(S) = 3 \det(S) / \epsilon(S)^2 = 3 \det(\tilde{T}) / \epsilon(\tilde{T})^2 =: d$ by definition. And since $\epsilon(T_{\pi,u}) \in \{ p^j q, p^{j+1} q \}$ holds, we can define $S_{\pi,u} := \frac{1}{p^j q} T_{\pi,u} \in \text{Her}_2^+(\mathcal{O})$, and we obviously have

$$\epsilon(T_{\pi,u}) = p^j q \epsilon(S_{\pi,u}),$$

thus

$$\epsilon(S_{\pi,u}) \in \{1, p\},$$

and of course

$$S_{\pi,u} = \frac{1}{p^j q} \tilde{T}_{\pi,u} = \begin{pmatrix} m + 2 \text{Re}(\tilde{t} \tilde{n}) + n N(u) & \pi t + \pi u n \\ \tilde{t} \tilde{n} + \pi \pi n & pn \end{pmatrix},$$

with

$$3 \det(S_{\pi,u}) = 3 p \det(S) = pd.$$  

Once more, we have to determine in which cases $\epsilon(S_{\pi,u}) = p$ holds, indeed.

So we have to go through all possible and relevant cases, and start by assuming $p | n$. In this case, $\pi t + \pi u n \in p^{1/2} \mathcal{O}$ is equivalent to

$$\pi t \in p^{1/2} \mathcal{O} \iff y \in (-i_1 \pi i_1) \mathcal{O}.$$
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Of course, this implies $p \mid N(y)$. Therefore, we have: If $p \nmid N(y) \iff p \varepsilon(\tilde{T})^2 \mid N(x)$, then

$$
\varepsilon(S_{\pi,u}) = 1
$$

holds in all $p^2(p + 1)$ possible cases. And note that (assuming $p \mid n$) we have $p \mid N(y)$ if and only if $d = 3 \det(S) \in pN_0$. So assume $p \mid N(y)$, but $y \notin pO$. Note that we have seen several times before that $\pi$ running through a transversal of $E/N(p)$ implies that $(-i_1 \pi i_1)$ runs through a transversal of $N(p)/E$. Therefore, there has to exist $\pi$ in the transversal we fixed such that $y \in (-i_1 \pi i_1)O$ holds, indeed. But again, there exists no second $\pi$ such that $y \in (-i_1 \pi i_1)O$ holds, too, since otherwise $y \in (-i_1 \pi i_1)O \cap (-i_1 \pi i_1)O = pO$ would have to follow (confer the proof of (4.32)). So let us fix this one $\pi$ satisfying

$$
y \in (-i_1 \pi i_1)O.
$$

Now, $p \mid n$ yields that $m + 2 \Re(t\pi) + nN(u) \in pZ$ is equivalent to

$$
m + 2 \Re(t\pi) \in pZ.
$$

But according to (4.28)

$$
\varphi : O/\pi O \to Z/pZ, \quad a + \pi O \mapsto 2 \Re(t\pi) + pZ
$$

is a surjective group-homomorphism. Therefore, analogous to the considerations concerning $S_2(p)_1$, exactly $p$ of the $p^2$ possible $u$ running through the transversal of $O/\pi O$ we fixed actually satisfy

$$
2 \Re(t\pi) \in -m + pZ.
$$

Hence we obtain: If $p \mid N(y)$, but $y \notin pO$, then

$p$ combinations $(\pi, u)$ fulfill $\varepsilon(S_{\pi,u}) = p$, and

$$p^2(p + 1) - p = p^3 + p^2 - p$$ combinations $(\pi, u)$ fulfill $\varepsilon(S_{\pi,u}) = 1$.

Next, assume $y \in pO$. But then $m + 2 \Re(t\pi) + nN(u) \in pZ$ is equivalent to

$$
m \in pZ,
$$

(since $2 \Re(ta) \in pZ$ holds under this assumption) which is a contradiction to $\varepsilon(S) = 1$. Therefore, $p \nmid m$ has to hold, and thus we have $\varepsilon(S_{\pi,u}) = p$ for none of the $p^2(p + 1)$ possible combinations $(\pi, u)$. Therefore, assuming $p \mid n$ and $y \in pO$, we obtain

$$
\varepsilon(S_{\pi,u}) = 1
$$

for all $S_{\pi,u}$. So suppose $p \nmid n$, now. Again, we have to find all possible combinations satisfying

$$
y - i_1 \sqrt{3}nu \in (-i_1 \pi i_1)O.
$$

According to (4.30) (since all prerequisites are met), if $u$ runs through a transversal of $O/\pi O$, then $i_1 \sqrt{3}nu$ runs through a transversal of $O/(-i_1 \pi i_1)O$. Therefore, given $\pi$, there exists exactly
one $u$ in the transversal of $O/\pi O$ we fixed such that

$$i_1 \sqrt{3} nu \in y + (-i_1 \pi_i) O$$

holds. So let $(\pi, u)$ be one of these $(p + 1)$ appropriate combinations. We still have to find out under which circumstances

$$m + 2 \text{Re}(t\pi) + n N(u) \in p \mathbb{Z}$$

holds. Note that we assumed $p \nmid n$, while we are looking at those combinations fulfilling $\pi t + \pi un \in p \frac{1}{\sqrt{3}} O$ (which means $N(\pi t + \pi un) \in \frac{p^2}{3} N_0$), now. Therefore, we obtain the following equivalence:

$$3 \det(S_{\pi, u}) = 3pn(m + 2 \text{Re}(t\pi) + n N(u)) - 3 N(\pi t + \pi un) \in p^2 \mathbb{Z}$$

$$\Leftrightarrow 3n(m + 2 \text{Re}(t\pi) + n N(u)) \in p \mathbb{Z}$$

$$\Leftrightarrow m + 2 \text{Re}(t\pi) + n N(u) \in p \mathbb{Z},$$

while

$$3 \det(S_{\pi, u}) = p \det(S) \in p^2 \mathbb{Z} \quad \Leftrightarrow \quad \det(S) \in p \mathbb{Z}. $$

Hence $\varepsilon(S_{\pi, u}) = p$ holds for all $(p + 1)$ appropriate combinations $(\pi, u)$ if $d = \det(S) \in p \mathbb{Z}$, and for none otherwise. Thus, assuming $p \nmid n$ and $p \nmid d$, then all $p^2(p + 1)$ possible $S_{\pi, u}$ satisfy

$$\varepsilon(S_{\pi, u}) = 1.$$ 

On the other hand, assuming $p \nmid n$ and $p \mid d$, then

$$p + 1 \text{ combinations } (\pi, u) \text{ fulfill } \varepsilon(S_{\pi, u}) = p, \text{ and }$$

$$p^2(p + 1) - (p + 1) = p^3 + p^2 - p - 1 \text{ combinations } (\pi, u) \text{ fulfill } \varepsilon(S_{\pi, u}) = 1.$$

All these results (and making use of 4.38 and 4.39) finally give rise to

\textbf{(4.53) Lemma.} Suppose $p \neq 3$ is a prime number and $f \in \mathcal{M}(k; O)$ with associated function $\alpha$ according to (4.17). Then we get

$$f|_k S_2^+(p)4(Z) = \sum_{\tilde{T} \in \text{Her}_1^2(O), \tilde{T} \geq 0} \beta(\tilde{T}) e^{2\pi i (\tilde{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),$$

where we have the following cases for $0 \neq \tilde{T} = (\tilde{m} \; \tilde{T}) \in \text{Her}_1^2(O)$, $\tilde{T} \geq 0$, with $\tilde{r} = \frac{4}{\sqrt{3}} x$ for some appropriate $x \in O$, where $d := 3 \det(\tilde{T}) / \varepsilon(\tilde{T})^2$:

- If $p \varepsilon(\tilde{T}) \nmid \tilde{n}$, then

$$\beta(\tilde{T}) = p^{3-3k/2} \left( (p^3 + p^2 - \chi_Z(p^{-1}d) \cdot (p + 1))\alpha(\varepsilon(\tilde{T}), pd) + (p + 1)\alpha(p \varepsilon(\tilde{T}), p^{-1}d) \right).$$
4.4 The quaternionic Hecke-operators $T_2(p)$

- If $pe(\bar{T})|\bar{n}$ and $x \notin pe(\bar{T})\mathcal{O}$, then
  \[
  \beta(\bar{T}) = p^{3-3k/2} \left( (p^3 + p^2 - \chi_\mathcal{O}(p^{-1}d) \cdot p)\alpha(\varepsilon(\bar{T}), pd) + p\alpha(\varepsilon(\bar{T}), p^{-1}d) \right).
  \]

- If $pe(\bar{T})|\bar{n}$ and $x \in pe(\bar{T})\mathcal{O}$, then
  \[
  \beta(\bar{T}) = p^{3-3k/2}(p^3 + p^2)\alpha(\varepsilon(\bar{T}), pd).
  \]

So finally, like we did for $S_2(p)$, we collect all results from (4.49), (4.50), (4.52) and (4.53). Let
\[
(f|_hS_2^+(p)_1 + f|_hS_2^+(p)_2 + f|_hS_2^+(p)_3 + f|_hS_2^+(p)_4)(Z) = \sum_{\bar{T} \in \text{Her}_2(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T})e^{2\pi i(T\bar{T})}
\]
for $Z \in \mathcal{H}(\mathcal{H})$. Using the same notation like in all the considerations above, we obtain:

- $pe(\bar{T}) \nmid \bar{n}$:
  \[
  \beta(\bar{T}) = (p+1)(p^2 + 1 - \chi_\mathcal{O}(p^{-1}d))p^{-k/2} \left( \alpha(p^{-1}\varepsilon(\bar{T}), pd) + p^{3-k}\alpha(\varepsilon(\bar{T}), pd) \right)
  + (p+1)p^{-k/2} \left( \alpha(\varepsilon(\bar{T}), p^{-1}d) + p^{3-k}\alpha(p\varepsilon(\bar{T}), p^{-1}d) \right)
  \]

- $pe(\bar{T})|\bar{n}$, $x \notin pe(\bar{T})\mathcal{O}$:
  \[
  \beta(\bar{T}) = (p+1)(p^2 + 1 - \chi_\mathcal{O}(p^{-1}d))p^{-k/2} \left( \alpha(p^{-1}\varepsilon(\bar{T}), pd) + p^{3-k}\alpha(\varepsilon(\bar{T}), pd) \right)
  + (p+1)p^{-k/2} \left( \alpha(\varepsilon(\bar{T}), p^{-1}d) + p^{3-k}\alpha(p\varepsilon(\bar{T}), p^{-1}d) \right)
  \]

- $pe(\bar{T})|\bar{n}$, $x \in pe(\bar{T})\mathcal{O}$:
  \[
  \beta(\bar{T}) = (p+1)p^2p^{-k/2} \left( \alpha(p^{-1}\varepsilon(\bar{T}), pd) + p^{3-k}\alpha(\varepsilon(\bar{T}), pd) \right)
  + (p+1)p^{-k/2} \left( \alpha(\varepsilon(\bar{T}), p^{-1}d) + p^{3-k}\alpha(p\varepsilon(\bar{T}), p^{-1}d) \right)
  \]

Of course, we already collected the terms from the lemmata above and rearranged the resulting sums in an appropriate way. But nevertheless, one simply obtains the results above by adding up the expressions found in the lemmata. Moreover, the last case $(pe(\bar{T})|\bar{n}$ and $x \in pe(\bar{T})\mathcal{O})$ also coincides with the first ones, since $d \in p\mathbb{N}_0$ is always fulfilled, then. Therefore, we finally have the following result:
(4.54) Theorem. Suppose \( p \neq 3 \) is a prime number and \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). Then we get

\[
f|_k S_2^\pm(p)(Z) = \sum_{\bar{T} \in \text{Her}_2^\pm(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T}) e^{2\pi i \tau(\bar{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{I}),
\]

where for \( 0 \neq \bar{T} \in \text{Her}_2^\pm(\mathcal{O}), \bar{T} \geq 0, \) we have

\[
\beta(\bar{T}) = \delta_p(\varepsilon(\bar{T}), 3 \det(\bar{T})/\varepsilon(\bar{T})^2)
\]

for a map \( \delta_p : \mathbb{N} \times \mathbb{N}_0 \to \mathbb{C} \). Given \( j, d \in \mathbb{N}_0 \) and \( q \in \mathbb{N} \) with \( p \nmid q \), we have

\[
\delta_p(p^j q, d) = (p + 1)(p^2 + 1 - \chi_Z(p^{-1}d))p^{-k/2}\left(\alpha(p^{j-1}q, pd) + p^{3-k}\alpha(p^j q, pd)\right)
\]

\[
+ (p + 1)p^{-k/2}\left(\alpha(p^j q, p^{-1}d) + p^{3-k}\alpha(p^{j+1}q, p^{-1}d)\right)
\]

So just like for \( S_2(p) \), note that this is exactly the same result that came up in [Kr90, thm.1] for the Hurwitz order, where one has to say again that all the calculations were omitted in that work.

4.4.4 The action of the Hecke-operators \( T_2(p) \) on the input functions of Maaß lifts

So we finally determined the Fourier-expansions of \( f|_k S_2(p) \) and \( f|_k S_2^\pm(p) \) in terms of the original one (see (4.48) concerning \( S_2(p) \)). But we are not going to prove that \( \gamma_p \) and \( \delta_p \) satisfy the condition about the formal power series from (4.17), since we will see further below that this is not necessary for our purposes (since our main goal is to determine the Fourier-expansion of the Eisenstein-series). But note that, since we obtained exactly the same results like in [Kr90, thm.1] (where statements about this formal power series can be found), it is clear that the space of Maaß lifts \( \mathcal{M}(k; \mathcal{O}) \) is invariant under \( S_2(p) \) and \( S_2^\pm(p) \) (and thus under \( T_2(p) \)), because \( \gamma_p \) and \( \delta_p \) actually do meet the condition concerning the formal power series. So without any further proof, we state the following corollary:

(4.55) Corollary. \( \mathcal{M}(k; \mathcal{O}) \) is invariant under the Hecke-operators \( S_2(p), S_2^\pm(p) \) and \( T_2(p) \) for every prime number \( p \neq 3 \). This means, if \( T \) is one of those operators and given \( f \in \mathcal{M}(k; \mathcal{O}) \), then also \( f|_k T \in \mathcal{M}(k; \mathcal{O}) \) holds.

So instead of looking at the formal power series in (4.17), we are going analyze what happens to the underlying elliptic modular form, which is lifted to \( f \in \mathcal{M}(k; \mathcal{O}) \). We are going to prove the following: According to (4.14), given \( f \in \mathcal{M}(k; \mathcal{O}) \), there exists \( g \in \mathcal{M}_{k-2} \) with \( f = \Omega^{-1}(g) \). Suppose \( p \neq 3 \) is a prime number, then there exists a Hecke-operator \( T \) satisfying

\[
f|_k T_2(p) = \Omega^{-1}(g|_{k-2} T).
\]
So let \( f \in \mathcal{M}(k; \mathcal{O}) \) with associated function \( \alpha \) according to (4.17). In virtue of (4.17) we have

\[
\alpha(t, d) = \sum_{r \in \mathbb{N}, r \mid \tau} r^{-1} \alpha^*(t^2 d / r^2),
\]

where \( \alpha^* \) is the attached function of \( f \), and (for \( \tau \in \mathcal{H} \))

\[
g(\tau) := \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2\pi i n \tau} \in \mathcal{M}_k(\mathbb{Q})
\]

in view of (4.14).

We examine the action of \( S_2(p) \) and \( S_2^*(p) \) seperately, and start with \( S_2^*(p) \), this time. So let

\[
f|S_2^*(p)(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), \bar{T} \geq 0} \beta(\bar{T}) e^{2\pi i \tau(\bar{T}, Z)}, \quad Z \in \mathcal{H}(\mathbb{H}).
\]

Then according to (4.54) and the consideration above, we have for \( 0 \neq \bar{T} \in \text{Her}_2(\mathcal{O}), \bar{T} \geq 0 \), where \( \epsilon(\bar{T}) = p^j q \) (with \( p \nmid q \)), \( d := 3 \det(\bar{T}) / \epsilon(\bar{T})^2 \) and \( D := 3 \det(\bar{T}) = \epsilon(\bar{T})^2 d \):

\[
\beta(\bar{T}) = (p + 1)(p^2 + 1 - \chi_Z(p^{-1} d)) p^{-k/2} \left( \sum_{r \mid p^{j-1} q} r^{-1} \alpha^*(p^{-1} D / r^2) + p^{3-k} \sum_{r \mid p^j q} r^{-1} \alpha^*(p D / r^2) \right)
\]

\[
+ \chi_Z(p^{-1} d) \cdot (p + 1)p^{-k/2} \left( \sum_{r \mid p^j q} r^{-1} \alpha^*(p^{-1} D / r^2) + p^{3-k} \sum_{r \mid p^{j+1} q} r^{-1} \alpha^*(p D / r^2) \right),
\]

where one should note that the first sum is empty if \( j = 0 \). Now, note that the first and the third sum are almost the same, with the only difference that in the third sum we also have to sum over \( r = p^{j+1} \bar{q} \), with \( \bar{q} \) running through the divisors of \( q \). The same holds for the second and the fourth sum, where in the fourth sum we additionally have to sum over \( r = p^{j+1} \bar{q} \). Therefore, adding up the first and the third sum yields

\[
(p + 1)(p^2 + 1)p^{-k/2} \sum_{r \mid p^{j-1} q} r^{-1} \alpha^*(p^{-1} D / r^2)
\]

\[
+ \chi_Z(p^{-1} d) \cdot (p + 1)p^{-k/2} \sum_{r \mid p^j q} (p r)^{-1} \alpha^*(p^{-1} D / (p r)^2),
\]

while adding up the second and the fourth sum gives

\[
(p + 1)(p^2 + 1)p^{3-3k/2} \sum_{r \mid p^j q} r^{-1} \alpha^*(p D / r^2)
\]

\[
+ \chi_Z(p^{-1} d) \cdot (p + 1)p^{3-3k/2} \sum_{r \mid p^{j+1} q} (p^{j+1} r)^{-1} \alpha^*(p D / (p^{j+1} r)^2),
\]

Furthermore, we calculate

\[
\sum_{r \mid q} (p r)^{-1} \alpha^*(D / (p^{2j+1} r^2)) + p^2 \sum_{r \mid q} (p r)^{-1} \alpha^*(D / (p^{2j+1} r^2))
\]

\[
= (p^2 + 1) \sum_{r \mid q} (p r)^{-1} \alpha^*(D / (p^{2j+1} r^2)),
\]
and thus (by multiplying with \((p + 1)p^{-k/2}\))

\[
(p + 1)p^{-k/2} \sum_{r|q} (p^r)^{k-1} \alpha^*(p^{-1}D/(p^r)^2) + p^3 - 3k/2 \sum_{r|q} (p^r)^{k-1} \alpha^*(pD/(p^r)^2) = (p + 1)(p^2 + 1)p^{-k/2} \sum_{r|q} (p^r)^{k-1} \alpha^*(D/(p^2 + 1r^2)) .
\]

Moreover, note that by definition \(D/(p^2 + 1r^2) = (q/r)^2d/p \in \mathbb{N}_0\) if and only if \(p|d\). Therefore, defining

\[
\alpha^*(x) := 0 \quad \text{for all } x \notin \mathbb{N}_0
\]

we obtain

\[
\beta(\tilde{T}) = (p + 1)(p^2 + 1)p^{-k/2} \left( \sum_{r|p|q} r^{k-1} \alpha^*(p^{-1}D/r^2) + p^3 - k \sum_{r|p|q} r^{k-1} \alpha^*(pD/r^2) \right) = \sum_{r \in \tilde{T}} r^{k-1} \delta^*(3 \text{det}(\tilde{T})/r^2),
\]

where

\[
\delta^*(l) = (p + 1)(p^2 + 1)p^{-k/2} \left( \alpha^*(p^{-1}l) + p^3 - k \alpha^*(pl) \right)
\]

for \(l \in \mathbb{N}_0\). Note that according to [Kl98, Le.4.2], the existence of such a function \(\delta^*: \mathbb{N}_0 \to \mathbb{C}\) satisfying

\[
\beta(\tilde{T}) = \sum_{r \in \tilde{T}} r^{k-1} \delta^*(3 \text{det}(\tilde{T})/r^2)
\]

for all \(0 \neq \tilde{T} \in \text{Her}_2^+(O), \tilde{T} \geq 0\) is equivalent to \(f|_k S^*_2(p) \in \mathcal{M}(k; O)\), so this actually proves (one part of) (4.55) without the need to look at the formal power series.

Next, we have to make the same considerations concerning \(S_2(p)\). But the calculations are slightly more difficult, here. So let

\[
f|k S_2(p)(Z) = \sum_{\tilde{T} \in \text{Her}_2^+(O), \tilde{T} \geq 0} \beta(\tilde{T})e^{2\pi i r(\tilde{T}, Z)}, \ Z \in \mathcal{H}(H),
\]

now. According to (4.48) we have for \(0 \neq \tilde{T} \in \text{Her}_2^+(O), \tilde{T} \geq 0\), where again \(\epsilon(\tilde{T}) = p^r q \) (with \(p \nmid q\), \(d := 3 \text{det}(\tilde{T})/\epsilon(\tilde{T})^2\) and \(D := 3 \text{det}(\tilde{T}) = \epsilon(\tilde{T})^2d\):

\[
\beta(\tilde{T}) = p^{6 - 2k} \alpha(p^{i+1}q, d) + (p^2 + 1 - \chi_Z(p^{-1}d))p^{3 - k} \alpha(p^{i-1}q, p^2d) + p^3 - k \alpha(p^{i+1}q, p^{-2}d) + (p^3 + (1 + \chi_Z(p^{-1}d))p^2 + p + 1 - \chi_Z(p^{-2}d))p^{1 - k} \alpha(p^l, d) + \alpha(p^{i+1}q, d)
\]

\[
= p^{6 - 2k} \sum_{r|p^{i+1}q} r^{k-1} \alpha^*(p^2D/r^2) + (p^2 + 1 - \chi_Z(p^{-1}d))p^{3 - k} \sum_{r|p^{i-1}q} r^{k-1} \alpha^*(D/r^2)
\]

\[
+ \chi_Z(p^{-2}d) p^{1 - k} \sum_{r|p^{i+1}q} r^{k-1} \alpha^*(D/r^2) + \sum_{r|p^{i-1}q} r^{k-1} \alpha^*(p^{-2}D/r^2)
\]

\[
+ (p^3 + (1 + \chi_Z(p^{-1}d))p^2 + p + 1 - \chi_Z(p^{-2}d))p^{1 - k} \sum_{r|p^{i}q} r^{k-1} \alpha^*(D/r^2)
\]

now.
4.4 The quaternionic Hecke-operators $T_2(p)$

To not trail away from a clear overview, is seems necessary to actually look at the cases $d \not\in p\mathbb{N}_0$, $d \in p\mathbb{N}_0 \setminus p^2\mathbb{N}_0$ and $d \in p^2\mathbb{N}_0$ separately. So let us assume $p \nmid d$, first. Note that this is equivalent to $p^{2j+1} \nmid D$. We defined $\alpha^*(z)$ to be zero if $z \not\in \mathbb{N}_0$. Hence $\alpha^*(p^{-2}D/(p\tilde{q})) = 0$ for all $\tilde{q}|q$. So we obtain

$$
\beta(\tilde{T}) = p^{6-2k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^2D/r^2) + (p^2 + 1)p^{3-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
+ \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^{-2}D/r^2) + (p^3 + p^2 + p + 1)p^{1-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
= p^{6-2k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^2D/r^2) + (p^4 + p^3 + p^2 + p + 1)p^{1-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
+ \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^{-2}D/r^2) + p^{3-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

because

$$
p^{6-2k} \sum_{r|q} (p^{j+1})^{k-1}\alpha^*(p^2D/(p^{j+1}r)^2) - p^{5-k} \sum_{r|q} (p^j)^{k-1}\alpha^*(p^2D/(p^jr)^2)
$$

$$
= p^{5-k} \sum_{r|q} (p^j)^{k-1}\alpha^*(D/(p^jr)^2) - p^{5-k} \sum_{r|q} (p^j)^{k-1}\alpha^*(D/(p^jr)^2) = 0 .
$$

Therefore, we get

$$
\beta(\tilde{T}) = \sum_{r|e(\tilde{T})} r^{k-1}\gamma^*(3 \det(\tilde{T})/r^2)
$$

in this case, where

$$
\gamma^*(l) = p^{6-2k}\alpha^*(p^2l) + \frac{p^{3-1}}{p-1}p^{1-k}\alpha^*(l) + \alpha^*(p^{-2}l) + \chi_{Z}(p^{-1}l)p^{3-k}\alpha^*(l)
$$

for $l \in \mathbb{N}_0$. (Since, if $r = p^l\tilde{q}$ for $\tilde{q}|q$, then $p \nmid D/r^2$ holds by assumption – and of course the same holds the other way round.)

Next, assume $p|d$, but $p^2 \nmid d$, which is equivalent to $p^{2j+1}|D$, but $p^{2j+2} \nmid D$. Again, we have $\alpha^*(p^{-2}D/(p\tilde{q})) = 0$ for all $\tilde{q}|q$. So for the same reason like above we compute

$$
\beta(\tilde{T}) = p^{6-2k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^2D/r^2) + p^2p^{3-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
+ \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^{-2}D/r^2) + (p^3 + 2p^2 + p + 1)p^{1-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
= p^{6-2k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^2D/r^2) + (p^4 + p^3 + p^2 + p + 1)p^{1-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

$$
+ \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(p^{-2}D/r^2) + p^{3-k} \sum_{r|p^{l+1}q} r^{k-1}\alpha^*(D/r^2)
$$

where only the last sum differs from the expression above. But note that due to the assumption, we have $p|D/r^2$ for every $r|p^lq$. Therefore, the “$\chi_{Z}(p^{-1}l)$” in $\gamma^*(l)$ is irrelevant in this case,
hence we obtain

$$ \beta(\bar{T}) = \sum_{r \mid \det(\bar{T})} r^{k-1} \gamma^*(3 \det(\bar{T})/r^2) , $$

again, with $\gamma^*(l)$ defined like above. So finally, assume $p^2 \mid d$, or equivalently $p^{2i+2} \mid D$. Because of the same reason like above and

$$ p^{1-k} \sum_{r \mid q} (p^{i+1}r)^{k-1} \alpha^*(D/(p^{i+1}r)^2) - \sum_{r \mid q} (p^i r)^{k-1} \alpha^*(p^{-2} D/(p^i r)^2) $n

$$ = \sum_{r \mid q} (p^i r)^{k-1} \alpha^*(D/(p^{i+1}r)^2) - \sum_{r \mid q} (p^i r)^{k-1} \alpha^*(D/(p^{i+1}r)^2) = 0 $$

as well, we calculate

$$ \beta(\bar{T}) = p^{6-2k} \sum_{r \mid p^{i+1}q} r^{k-1} \alpha^*(p^2 D/r^2) + p^2 p^{3-k} \sum_{r \mid p^{i+1}q} r^{k-1} \alpha^*(D/r^2) $n

$$ + p^{1-k} \sum_{r \mid p^{i+1}q} r^{k-1} \alpha^*(D/r^2) + \sum_{r \mid p^{i+1}q} r^{k-1} \alpha^*(p^{-2} D/r^2) $n

$$ + (p^3 + 2p^2 + p)p^{1-k} \sum_{r \mid p^i q} r^{k-1} \alpha^*(D/r^2) $n

$$ = p^{6-2k} \sum_{r \mid p^i q} r^{k-1} \alpha^*(p^2 D/r^2) + (p^4 + p^3 + p^2 + p + 1)p^{1-k} \sum_{r \mid p^i q} r^{k-1} \alpha^*(D/r^2) $n

$$ + \sum_{r \mid p^i q} r^{k-1} \alpha^*(p^{-2} D/r^2) + p^3 - k \sum_{r \mid p^i q} r^{k-1} \alpha^*(D/r^2) . $$

Thus, once more we obtain

$$ \beta(\bar{T}) = \sum_{r \mid \det(\bar{T})} r^{k-1} \gamma^*(3 \det(\bar{T})/r^2) , $$

with the same $\gamma^*(l)$, again. We collect these results in the following

\textbf{(4.56) Theorem.} Suppose $p \neq 3$ is a prime number and let $f \in M(k; O)$ with attached function $\alpha^*$. Then $f \mid S_2(p)$, $f \mid S_2^2(p)$ and $f \mid T_2(p)$ belong to $M(k; O)$, too. If we denote the attached functions of $f \mid S_2(p)$, $f \mid S_2^2(p)$ and $f \mid T_2(p)$ by $\gamma^*$, $\delta^*$ and $\beta^*$, respectively, then we have

- $\gamma^*(l) = p^{6-2k} \alpha^*(p^2 l) + \frac{p^3 - 1}{p - 1} p^{1-k} \alpha^*(l) + \alpha^*(p^{-2} l) + \chi_2(p^{-1}) p^{3-k} \alpha^*(l) ,$

- $\delta^*(l) = (p + 1)(p^2 + 1)p^{-k/2} \left( \alpha^*(p^{-1} l) + p^{3-k} \alpha^*(pl) \right) ,$

- $\beta^*(l) = \gamma^*(l) + \delta^*(l)$

for all $l \in \mathbb{N}_0$.

Next, note again that, given $f \in M(k; O)$ with attached function $\alpha^*$, then there exists an elliptic modular form $g \in M_{k-2}$ with

$$ g(\tau) = \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2\pi i n \tau} , \quad \tau \in \mathcal{H} $$
and \( f = \Omega^{-1}(g) \). And due to the last theorem, we know that
\[
h(\tau) := \sum_{n \in \mathbb{N}_0} \beta^*(n) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}
\]
has to be an elliptic modular form in \( \mathcal{M}_{k-2} \), too, and we have
\[
f|_{k} T_2(p) = \Omega^{-1}(h).
\]
So now, the last step is to describe \( h \) in terms of \( g \). To be more precise, we want to determine a Hecke-operator \( T \) such that \( h = g|_{k-2} T \) holds, and thus \( f|_{k} T_2(p) = \Omega^{-1}(g|_{k-2} T) \).

Note that, since we are considering elliptic modular forms in \( \Gamma_0[9], k \), we will again stick to the notation of Hecke-operators used in [Mi89] – so in particular concerning normalization factors for “\( \Gamma M \)” and the Hecke-operators. We will need the Hecke-operators \( T(1) \) (which is simply the identity), \( T(p) \) and \( T(p^2) \). Note that we still have \( p \neq 3 \). So according to [Mi89, le.4.56] (where \( N = 9 \)), a transversal of \( \Gamma_0[9] \backslash \mathcal{T}(p) \) is given by
\[
\left( \begin{array}{cc} d & b \\ 0 & \frac{p}{\pi} \end{array} \right), \quad d \in \{1, p\}, \quad b = 0, \ldots, \frac{p^2}{\pi} - 1,
\]
while a transversal of \( \Gamma_0[9] \backslash \mathcal{T}(p^2) \) is given by
\[
\left( \begin{array}{cc} d & b \\ 0 & \frac{p^2}{\pi} \end{array} \right), \quad d \in \{1, p, p^2\}, \quad b = 0, \ldots, \frac{p^2}{\pi} - 1.
\]
Note that this coincides with the result for \( T(p) \) and \( T(p^2) \) in (3.16). So we are actually talking about the “ordinary” Hecke-operators for \( \Gamma[1] \).

By definition we have (noting \( \alpha^*(z) := 0 \) for \( z \notin \mathbb{N}_0 \) and applying 4.43)
\[
g|_{k-2} T_2(p)(\tau) = p^{(k-2)/2-1} \left( p^{(k-2)/2} g(p\tau) + p^{(k-2)/2} p^{-(k-2)} \sum_{b=0}^{p-1} g(p^{-1}(\tau + b)) \right)
\]
\[
= p^{k-3} \left( \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2\pi i (np)p^2\tau} + p^{-k+2} \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2\pi i n/p} \sum_{b=0}^{p-1} \left( e^{2\pi i n/p} \right)^b \right)
\]
\[
= p^{k-3} \left( \sum_{n \in \mathbb{N}_0} \alpha^*(p^{-1}n) e^{2\pi i n/p} + p^{-k+3} \sum_{n \in \mathbb{N}_0} \alpha^*(pn) e^{2\pi i n/p} \right)
\]
\[
= \sum_{n \in \mathbb{N}_0} \left( p^{k-3} \alpha^*(p^{-1}n) + \alpha^*(pn) \right) e^{2\pi i n/p}.
\]
Therefore, in view of (4.56), we already verified
\[
f|_{k} S_2(p) = \Omega^{-1}((p + 1)(p^2 + 1)p^{-k/2} p^{3-k} g|_{k-2} T_2(p)).
\]
Analogously, we calculate
\[
g|_{k-2} T_2(p^2)(\tau) = p^{2k-6} \left( g(p^2\tau) + p^{-k+2} \sum_{b=0}^{p-1} g(\tau + p^{-1}b) + p^{-2k+4} \sum_{b=0}^{p^2-1} g(p^{-2}(\tau + b)) \right)
\]
\[
\begin{align*}
&= p^{2k-6} \left( \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2 \pi i (np^2) \tau} + p^{-k+2} \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2 \pi i n \tau} \sum_{b=0}^{p-1} \left( e^{2 \pi i n / p} \right)^b \right) \\
&\quad + p^{-2k+4} \sum_{n \in \mathbb{N}_0} \alpha^*(n) e^{2 \pi i n \tau} / p^2 \sum_{b=0}^{p-1} \left( e^{2 \pi i n / p^2} \right)^{b^2} \\
&= p^{2k-6} \left( \sum_{n \in \mathbb{N}_0} \alpha^*(p^{-2}n) e^{2 \pi i n \tau} + p^{-k+2} \sum_{n \in \mathbb{N}_0} \chi_{\mathbb{Z}}(p^{-1}n) \alpha^*(n) e^{2 \pi i n \tau} \\
&\quad + p^{-2k+6} \sum_{n \in \mathbb{N}_0} \alpha^*(p^2n) e^{2 \pi i n \tau} \right) \\
&= \sum_{n \in \mathbb{N}_0} \left( p^{2k-6} \alpha^*(p^{-2}n) + p^{k-3} \chi_{\mathbb{Z}}(p^{-1}n) \alpha^*(n) + \alpha^*(p^2n) \right) e^{2 \pi i n \tau},
\end{align*}
\]

and thus in virtue of (4.56), we obtain

\[
f|_{k} S_2(p) = \Omega^{-1}(\frac{p^6-1}{p-1} p^{1-k} g|_{k-2} T(1) + p^{6-2k} g|_{k-2} T(p^2)).
\]

So we summarize:

\[\textbf{(4.57) Theorem.}\] Suppose \( p \neq 3 \) is a prime number and \( k \in 2\mathbb{N}, k \geq 4 \). Let \( f \in \mathcal{M}(k; \mathcal{O}) \) and \( g = \Omega(f) \in \mathcal{M}_{k-2} \). Then we have

- \( f|_{k} S_2(p) = \Omega^{-1}(\frac{p^6-1}{p-1} p^{1-k} g|_{k-2} T(1) + p^{6-2k} g|_{k-2} T(p^2)) \),
- \( f|_{k} S_2'(p) = \Omega^{-1}((p+1)(p^2+1)p^{-k/2} p^{3-k} g|_{k-2} T(p)) \),
- \( f|_{k} T_2(p) = \Omega^{-1}(g|_{k-2} T_{\mathcal{M}}(p)) \), where

\[
T_{\mathcal{M}}(p) = \frac{p^5-1}{p-1} p^{1-k} T(1) + (p+1)(p^2+1)p^{-k/2} p^{3-k} T(p) + p^{6-2k} T(p^2).
\]

In a final step, we want to apply the preceding theorem to (4.15), which says that

\[
\Lambda : \Gamma_0[3], k \to \mathcal{M}_k, \ g \mapsto g - g|_{k} \omega_9
\]
is a surjective homomorphism. Therefore, given \( f \in \mathcal{M}(k; \mathcal{O}) \), there exists \( g \in [\Gamma_0[3], k-2] \) such that

\[
f = \Omega^{-1}(g - g|_{k-2} \omega_9),
\]

and thus

\[
f|_{k} T_2(p) = \Omega^{-1}(g|_{k-2} T_{\mathcal{M}}(p) - g|_{k-2} \omega_9|_{k-2} T_{\mathcal{M}}(p)).
\]

But if we even assume \( g \in [\Gamma[1], k-2] \), then \( g|_{k-2} \omega_9 = g|_{k-2}(\frac{9}{0} \frac{0}{1}) \), since \( g|_{k-2}(-J_1) = g \). And note that given \( M = (\frac{a}{b} \frac{b}{d}) \), then

\[
\left( \frac{9}{0} \frac{0}{1} \right) (\frac{a}{b} \frac{b}{d}) = \left( \frac{9a}{0} \frac{0}{d} \right) = \left( \frac{a}{0} \frac{9b}{d} \right) (\frac{9}{0} \frac{0}{1}),
\]

and if \( b \) runs through a transversal of \( \mathbb{Z}/d\mathbb{Z} \), then so does \( 9b \), given that \( 3 \nmid d \) holds. Therefore, the operators \( T(p^j), j \in \{0,1,2\} \) obviously commute with the operator “\( |_{k-2}(\frac{9}{0} \frac{0}{1}) \)”. So we have
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the following

(4.58) Corollary. Let \( p \neq 3 \) be a prime number, \( k \in 2\mathbb{N}, k \geq 6, g \in [\Gamma[1], k-2] \) and

\[
f := \Omega^{-1}(g - g|_{k-2}\omega_9) = (g - g|_{k-2} \left( \begin{smallmatrix} 0 & 1 \\ \omega_9 & 0 \end{smallmatrix} \right)) .
\]

Then

\[
f|_k T_2(p) = \Omega^{-1}(g|_{k-2} T_M(p) - g|_{k-2} T_M(p)|_{k-2}(\begin{smallmatrix} 0 & 1 \\ \omega_9 & 0 \end{smallmatrix}))
= \Omega^{-1}(g|_{k-2} T_M(p)(\tau) - 3^{k-2}g|_{k-2} T_M(p)(9\tau))
\]

holds. Furthermore, in view of (4.16) we also get for \( k = 4 \):

\[
f := \Omega^{-1}(f_2(\tau) + 3f_2(3\tau))
\]

is the only Maass lift of weight 4, and we have

\[
f|_4 T_2(p) = \Omega^{-1}(f_2|_2 T_M(p)(\tau) + 3f_2|_2 T_M(p)(3\tau))
\]

There still remain some facts about the Hecke-operators and Maass lifts to be considered, but these are more of the shape to fit into the context of determining the Fourier-expansion of the Eisenstein-series, so we will delay them until the next section. But the exact shape of the action of the Hecke-operators \( S_2(p), S_2^*(p) \) and \( T_2(p) \) has been determined completely.

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Finally, we are going to explicitly determine the Fourier-expansions of the quaternionic Eisenstein-series, analogous to [Kr90]. Like announced, we will verify that the Eisenstein-series \( E_k \) are quaternionic Maass Lifts for the trivial character, and moreover, we will determine their Fourier-expansions by explicitly specifying the input of the Maass lift, i.e. the appropriate elliptic modular forms. Note that \( E_4 \) and \( E_6 \) are already uniquely given as Maass lifts by definition (see 4.2).

To be more precise, we will verify that the quaternionic Eisenstein-series are eigenforms for the Hecke-operators \( T_2(p) \) (where \( p \neq 3 \) is a prime number). Furthermore, we will construct Maass lifts that are eigenforms for the \( T_2(p) \) with respect to the same eigenvalues. And finally, we will prove that if a Maass lift of weight \( k \geq 8 \) is an eigenform for \( T_2(p) \) (where even one \( p \) suffices) with respect the same eigenvalue like \( E_k \) and possesses the same constant term in the Fourier-expansion (which equals 1 in view of (4.8)), then it actually has to be identical to \( E_k \).

So let us start by defining those Maass lifts we just spoke of. Note that we will normalize them later on, but skip the normalization for now (so that we do not have to care about this factor).
(4.59) Definition. Let $k \in 2\mathbb{N}, k \geq 6$. Then we define

$$\tilde{E}_k := \Omega^{-1}(G_{k-2}(\tau) - 3^{k-2}G_{k-2}(9\tau)) .$$

Note that the $\tilde{E}_k$ are well-defined in view of (4.15) (and by what we saw in (4.58)). And like announced, they turn out to be eigenforms for the Hecke-operators $T_2(p)$:

(4.60) Proposition. Suppose $p \neq 3$ is a prime number and let $k \in 2\mathbb{N}, k \geq 6$. Then

$$\tilde{E}_k|_{k} T_2(p) = \lambda_{k,p} \tilde{E}_k$$

holds, where

$$\lambda_{k,p} = \frac{p^{5-1}}{p-1} p^{1-k} + (p + 1)(p^2 + 1)p^{3-3k/2}\sigma_{k-3}(p) + p^{6-2k}\sigma_{k-3}(p^2)$$

$$= \prod_{j=0}^{3} (p^{j-k/2} + 1) ,$$

where $\sigma_{a}(b) = \sum_{d \mid b} d^a$ as usual.

Proof: In virtue of (4.58) we get

$$\tilde{E}_k|_{k} T_2(p) = \Omega^{-1}(G_{k-2}|_{k} T_2(M)(p)(\tau) - 3^{k-2}G_{k-2}|_{k} T_2(M)(p)(9\tau)) ,$$

where

$$T_2(M)(p) = \frac{p^{5-1}}{p-1} p^{1-k}T(1) + (p + 1)(p^2 + 1)p^{-k/2}p^{3-k}T(p) + p^{6-2k}T(p^2) .$$

And according to [KK07, Kap.IV, Prop.2.4] we have

$$G_{k-2}|_{k-2} T(p^j) = \sigma_{k-3}(p^j)G_{k-2}$$

for all $j \in \mathbb{N}_0$. Therefore, we obtain

$$G_{k-2}|_{k-2} T(M)(p)(\tau) = \lambda_{k,p} G_{k-2} ,$$

and thus

$$\tilde{E}_k|_{k} T_2(p) = \lambda_{k,p} \tilde{E}_k$$

due to the linearity of the lift $\Omega^{-1}$. The representation of $\lambda_{k,p}$ as the product from the assertion above can be verified by a straightforward calculation (for example with the help of a computer algebra system like [SAGE]).

We already mentioned above that the quaternionic Eisenstein-series are eigenforms of the Hecke-operators $T_2(p)$, and it will turn out in the next theorem that they are eigenforms for the eigenvalue $\lambda_{k,p}$ indeed. Note that $E_4$ and $E_6$ have to be eigenforms, too, since the spaces $[\text{Sp}_2(O), 4]$ and $[\text{Sp}_2(O), 6]$ are one dimensional (see (3.36) in combination with (3.7)).

We already defined the abbreviation $\Gamma_2 = \text{Sp}_2(O)$. Furthermore, we will also use the following
one:
\[ \Gamma_{2,0} := \text{Sp}_2(\mathcal{O})_0, \]  
(4.50)

where \( \text{Sp}_2(\mathcal{O})_0 \) was defined in (4.1). So let us get to the announced theorem:

(4.61) Theorem. Let \( k \in \mathbb{N}, k \geq 8 \). Then
\[ E_k|_k T_2(p) = \lambda_{k,p} E_k \]
holds, where \( \lambda_{k,p} \) is defined in (4.60).

Proof: By definition (see (4.3)), the quaternionic Eisenstein-series are given as
\[ E_k(Z) = \sum_{M : \Gamma_{2,0} \setminus \Gamma_2} \det(\tilde{M}\{\tilde{Z}\})^{-k/2} = \sum_{M : \Gamma_{2,0} \setminus \Gamma_2} 1|_k M(Z), \quad Z \in \mathcal{H}(\mathbb{H}). \]

A transversal of \( \Gamma_2 \setminus \Delta_2(p) \) is given by certain \( L_1, \ldots, L_{\text{deg}(T_2(p))} \in \Delta_2(p) \) (where the \( L_j \) were specified in (4.40), confer (4.34), too). Denote \( S = \{1, \ldots, \text{deg}(T_2(p))\} \). Furthermore, let some (infinite) transversal of \( \Gamma_{2,0} \setminus \Gamma_2 \) be denoted by \( \{M_l ; l \in I_0\} \) for some index set \( I_0 \). This means we have
\[ \Delta_2(p) = \bigcup_{j \in S} \Gamma_2 L_j \quad \text{and} \quad \Gamma_2 = \bigcup_{l \in I_0} \Gamma_{2,0} M_l \]
as disjoint unions, and thus also
\[ \Delta_2(p) = \bigcup_{l \in I_0} \bigcup_{j \in S} \Gamma_{2,0} M_l L_j. \]
Moreover,
\[ E_k|_k T_2(p) = \sum_{l \in I_0} \sum_{j \in S} 1|_k(M_l L_j) \]
holds in view of (4.34). Therefore, we have to find a way to somehow commute \( M_l \) and \( L_j \). Of course \( L_j M_l \in \Delta_2(p) \) holds. Therefore, due to the disjoint union above, if \( (j, l) \in S \times I_0 \), there exists a unique pair \( (j_0, l_0) \in S \times I_0 \) and some \( N \in \Gamma_{2,0} \) such that
\[ L_j M_l = N M_{l_0} L_{j_0} \]
holds. Therefore,
\[ \varphi : S \times I_0 \to S \times I_0, \quad (j, l) \mapsto (j_0, l_0) \]
is a well-defined map, and we are going to prove that \( \varphi \) is bijective.

Let us verify the injectivity, first. So suppose we have \( \varphi(j, l) = \varphi(\tilde{j}, \tilde{l}) \). This means there exist \( N, \tilde{N} \in \Gamma_{2,0} \) such that
\[ L_j M_l = N M_{l_0} L_{j_0} \quad \text{and} \quad L_{\tilde{j}} M_{\tilde{l}} = \tilde{N} M_{l_0} L_{j_0} \]
hold. This implies $M_{lk}L_{j0} = N^{-1}L_jM_{lk}$, and thus
\[ L_j = \tilde{N}N^{-1}L_jM_{lk}^{-1} \]
But since we chose $\{L_j; j \in S\}$ to be the transversal specified in (4.40), the lower left $2 \times 2$ block of each of the $L_j$ equals 0. The same holds for $N$ and $\tilde{N}$ by definition. Therefore, we have
\[ L_j = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} , \quad \tilde{N}N^{-1}L_j = \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix} , \quad M_{lk}^{-1} = \begin{pmatrix} A_3 & B_3 \\ 0 & D_3 \end{pmatrix} \]
for some appropriate $2 \times 2$ blocks, where, in particular, $D_2 \in \text{GL}_2(\mathbb{H})$ holds. Due to the assumption, we have
\[ \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix} \cdot \begin{pmatrix} A_3 & B_3 \\ 0 & D_3 \end{pmatrix} = \begin{pmatrix} A_4 & B_4 \\ 0 & D_4 \end{pmatrix} \]
and thus $C_3 = 0$, since $D_2$ is invertible. But then $M_{lk}M_{lk}^{-1} \in \Gamma_{2,0}$ holds by definition, hence $\tilde{I} = I$. So we get $L_j = \tilde{N}N^{-1}L_j$, where $\tilde{N}N^{-1} \in \Gamma_2$, which implies $\tilde{j} = j$. Hence the injectivity.
Next, let us get to the surjectivity. So suppose $(j_0,l_0) \in S \times I_0$. We have to find $(j,l) \in S \times I_0$ and $N \in \Gamma_{2,0}$, such that $L_jM_l = NM_{j0}L_{j0}$ holds. Of course, we have $M_{j0}L_{j0} \in \Delta_2(p)$. So due to the same considerations like in the proof of (4.35) (where we operate on the columns now instead of operating on the rows – but the argumentation is completely the same) there exists $M \in \Gamma_2$ such that we have
\[ M_{lk}L_{j0}M = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \]
for some appropriate $A_1, B_1, D_1 \in \mathcal{O}^{2 \times 2}$. By construction there exist $N_1 \in \Gamma_{2,0}$ and $l \in I_0$ such that $M^{-1} = N_1M_l$, or equivalently $M = M_lN_1^{-1}$. But since the lower left $2 \times 2$ block of $N_1$ equals 0, we get
\[ \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} A_3 & B_3 \\ 0 & D_3 \end{pmatrix} = M_{lk}L_{j0}M_{lk}^{-1} \]
for some appropriate $A_2, B_2, D_2 \in \mathcal{O}^{2 \times 2}$. Again, we have $M_{lk}L_{j0}M_{lk}^{-1} \in \Delta_2(p)$. So there exist $j \in S$ and $N_2 \in \Gamma_2$ such that $M_{lk}L_{j0}M_{lk}^{-1} = N_2L_j$. Denote $N_2 = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}$ and $L_j = \begin{pmatrix} A_4 & B_4 \\ 0 & D_4 \end{pmatrix}$. Again, we have
\[ \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \cdot \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} = \begin{pmatrix} A_4 & B_4 \\ 0 & D_4 \end{pmatrix} \]
where $A_4 \in \text{GL}_2(\mathbb{H})$, hence $C_3 = 0$, which means $N_2 \in \Gamma_{2,0}$. So we finally obtain
\[ N_2^{-1}M_{lk}L_{j0} = L_jM_l , \]
and thus the surjectivity. So noting
\[ 1|_k(L_jM_l) = 1|_k(NM_{lk}L_{j0}) = 1|_k(M_{lk}L_{j0}) \]
in virtue of (4.34) (since $1|kN = 1$ holds by definition), we conclude
\[ E_k|_k\mathcal{T}_2(p) = \sum_{l \in I_0} \left( \sum_{j \in S} 1|_k(L_j) \right) |_kM_l \]
by simply rearranging the infinite sum from above (and making use of (4.34) and the linearity of the slash-operator). Let $L_j = \begin{pmatrix} \ast & \ast \\ 0 & D_j \end{pmatrix}$, $j \in S$. Making use of the explicit structure of the $D_j$.
specified in (4.40), we compute

\[
\sum_{j \in S} 1_k(L_j) = \sum_{j \in S} \det(\tilde{D}_j)^{-k/2} \\
= 1 + p^{-k}p^5 + p^{-k}p + p^{-2k}p^6 + p^{-k}p^2(p + 1)^2 \\
+ p^{-k/2}p^2(p + 1) + p^{-k/2}(p + 1) + p^{-3k/2}p^3(p + 1) + p^{-3k/2}p^5(p + 1) \\
= \prod_{j=0}^3 (p^{j-k/2} + 1) = \lambda_{k,p},
\]

where the last identity can be verified by a straightforward calculation, again. Thus,

\[ E_k|_{T_2(p)} = \lambda_{k,p} \]

follows. \(\square\)

Before we continue, let us give a little remark on possible eigenvalues for quaternionic modular forms whose constant terms in the Fourier-expansion is non-zero.

(4.62) Lemma. Let \( f \in \Gamma_2, k \) (where \( k \in 2\mathbb{N} \)) with Fourier-expansion

\[ f(Z) = \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,Z)} , \quad Z \in \mathcal{H}(\mathbb{H}), \]

such that \( \alpha_f(0) \neq 0 \), and suppose \( f \) is an eigenform of \( T_2(p) \) for some prime number \( p \neq 3 \) with

\[ f|_k T_2(p) = \lambda f . \]

Then

\[ \lambda = \lambda_{k,p} \]

has to hold.

Proof: Like in (4.61), let \( L_1, \ldots, L_{\text{deg}(T_2(p))} \in \Delta_2(p) \) be the transversal of \( \Gamma_2 \setminus \Delta_2(p) \) specified in (4.40), and denote \( S = \{ 1, \ldots, \text{deg}(T_2(p)) \} \). For \( j \in S \), let \( L_j = \left( \begin{array}{cc} A_j & B_j \\ 0 & D_j \end{array} \right) \). Like in the previous section, we have

\[
f|_k L_j(Z) = \det(\tilde{D}_j)^{-k/2} \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,(A_jZ+B_j)D_j^{-1})} \\
= \det(\tilde{D}_j)^{-k/2} \sum_{T \in \text{Her}_2(\mathcal{O}), T \geq 0} \alpha_f(T)e^{2\pi i \tau(T,B_j/A_j)/p} e^{2\pi i \tau(T[A_j],Z)/p}
\]

for \( Z \in \mathcal{H}(\mathbb{H}) \), and this is again some Fourier-expansion. And since \( A_j \in \text{GL}_2(\mathbb{H}) \), we have \( T[A_j] = 0 \) if and only if \( T = 0 \). Therefore, the constant term in the Fourier-expansion of \( f|_k L_j \)
equals \( \text{det}(\tilde{D}_j)^{-k/2} \alpha_f(0) \). Thus,
\[
\sum_{j \in S} \text{det}(\tilde{D}_j)^{-k/2} \alpha_f(0) = \lambda_{k,p} \alpha_f(0)
\]
(see the proof of (4.61)) has to be the constant term in the Fourier-expansion of \( f_k T_2(p) \), while the constant term in the Fourier-expansion of \( \lambda f \) equals \( \lambda \alpha_f(0) \). Thus, \( \alpha_f(0) \neq 0 \) implies

\[
\lambda = \lambda_{k,p} .
\]

Next, we need to analyze \( (f_k T(p))|\Phi \). To be more precise, we want to show that there is some Hecke-operator \( T \) (for \( \text{SL}_2(\mathbb{Z}) \)) such that \( (f_k T(p))|\Phi = (f|\Phi)|k T \).

(4.63) Proposition. Let \( k \in 2\mathbb{N} \) and \( f \in \Gamma_2, k \). Then
\[
(f_k T(p))|\Phi = (f|\Phi)|k T
\]
holds, where the Hecke-operator \( T \) is defined by
\[
T = \left( p^{1-k}((p^{3-k/2} + 1)(p^{2-k/2} + 1) \right) T(p)
+ \left( p^{-3k/2+5} (p+1) + p^{-k/2}(p+1) + p^{-k+2}(p+1)^2 \right) T(1)
\]

Proof: Let the Fourier-expansion of \( f \) be given by
\[
f(Z) = \sum_{T \in \text{Her}_1(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(Z, T)} , \quad Z \in \mathcal{H} \left( \mathbb{H} \right),
\]
then according to the proof of (1.59), the Fourier-expansion of (the elliptic modular form) \( f|\Phi \) is given by
\[
f|\Phi(\tau) = \sum_{m \in \mathbb{N}_0} \alpha_f(\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}) e^{2\pi i m \tau} , \quad \tau \in \mathcal{H}.
\]
Again, let \( L_1, \ldots, L_{\deg(T_2(p))} \in \Delta_2(p) \) be the transversal of \( \Gamma_2 \backslash \Delta_2(p) \) specified in (4.40), and denote \( S = \{ 1, \ldots, \deg(T_2(p)) \} \). For \( L_j = \begin{bmatrix} A_j & B_j \\ 0 & D_j \end{bmatrix} \), where \( j \in S \), the same calculations like in the proof of (1.59) and the same considerations like in (4.62) yield
\[
(f|\Phi(L_j))|\Phi(\tau) = \text{det}(\tilde{D}_j)^{-k/2} \sum_{T \in \text{Her}_1(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(Z, T, B_j, \tilde{A}_j)}/e^{2\pi i \tau(1/2 T(A_j), Z)},
\]

hence
\[
(f_k T_2(p))|\Phi(\tau) = \sum_{j \in S} \text{det}(\tilde{D}_j)^{-k/2} \sum_{T \in \text{Her}_1(\mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(Z, T, B_j, \tilde{A}_j)}/e^{2\pi i \tau(1/2 T(A_j), Z)} .
\]

Now, we have to go through each case we considered in the previous section, hence we consider
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\[ S_2(p)_1, S_2(p)_2, \ldots \text{ seperately. One easily checks (by going through each case) that} \]

\[ T[A_j] = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \iff T = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \text{ for some } m \in \mathbb{N}_0, \]

so we can also write

\[ (f|kT_2(p))\Phi(\tau) = \sum_{m \in \mathbb{N}_0} \alpha_f(\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}) \sum_{j \in \mathcal{S}} \det(\tilde{D}_j)^{-k/2} e^{2\pi i \tau(\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}) \cdot \tilde{\mathfrak{N}}_j / p} e^{2\pi i \tau(\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}) [A_j]_{\tilde{Z}}}. \]

Once we are seperately looking at \( S_2(p)_1 \) etc., \( \frac{1}{p} (\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}) [A_j] \) is independent of the special shape of \( A_j \). Moreover, we have seen in the previous section that, when seperating the \( S_2(p)_1 \) etc.,

the remaining sum over \( e^{2\pi i \tau(\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}) \cdot \tilde{\mathfrak{N}}_j / p} \) yields a special prefactor and certain conditions on the occuring \( \tilde{m} \), where \( \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{p} (\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}) [A_j] \). For example, every \( A_j \) coming from \( S_2(p)_1 \) yields \( \tilde{m} = pm \). Let the Fourier-expansion of \( (f|kT_2(p))\Phi \) be given by

\[ (f|kT_2(p))\Phi(\tau) = \sum_{m \in \mathbb{N}_0} \gamma(m) e^{2\pi i m \tau}, \quad \tau \in \mathcal{H}. \]

Note that we will not go through each of the cases we have to consider seperately, since one simply has to read off the results from the previous section. Denoting

\[ \beta(m) = \alpha_f(\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}), \quad m \in \mathbb{N}_0, \]

defining

\[ \beta(z) = 0, \quad \text{if } z \notin \mathbb{N}_0, \]

and going through each of the cases we had to consider in the previous section, we obtain for \( n \in \mathbb{N}_0 \) by collecting all \( \beta(m) \) (including the appropriate pre-factors determined in the previous section) with \( \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{p} (\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}) [A_j]:\)

\[ \gamma(n) = \beta(p^{-1}n) + p^{-k}p^5 \beta(p^{-1}n) + p^{-k}p\beta(pn) + p^{-2k}p^6 \beta(pn) + p^{-k}p^2(p + 1)^2 \beta(n) \]
\[ + p^{-k/2}p^2(p + 1) \beta(p^{-1}n) + p^{-k/2}(p + 1)^2 \beta(n) + p^{-3k/2}p^3(p + 1) \beta(pn) \]
\[ + p^{-3k/2}p^2(p + 1) \beta(n) \]
\[ = \left(p^{1-k}(p^{3-k/2} + 1)(p^{2-k/2} + 1)\right) \cdot \left(p^{k-1} \beta(p^{-1}n) + \beta(pn)\right) \]
\[ + \left(p^{-3k/2}p^2(p + 1) + p^{-k/2}(p + 1) + p^{-3k/2}(p + 1)^2\right) \beta(n). \]

But then the calculations done to prove (4.57) immediately yield

\[ (f|kT(p))\Phi = (f|\Phi)|kT. \]

A direct consequence of the preceding proposition is the following corollary:
(4.64) Corollary. Let \( p \neq 3 \) be a prime number, \( k \in 2\mathbb{N} \) and \( f \in [\Gamma_2, k] \). Suppose that \( f \) is an eigenform of \( T_2(p) \) and that the constant term in the Fourier-expansion is not zero. Then

\[
(f|\Phi)|_kT(p) = \sigma_{k-1}(p)(f|\Phi) = (1 + p^{k-1})(f|\Phi)
\]

holds.

Proof: In view of (4.62),

\[
f|_kT_2(p) = \lambda_{k,p}f
\]

has to hold. This implies

\[
\lambda_{k,p}(f|\Phi) = (f|_kT(p))|\Phi = (f|\Phi)|_kT
\]

where \( T \) is the special Hecke-operator from (4.63). Rearranging the terms in this identity yields

\[
(f|\Phi)|_kT(p) = \frac{\lambda_{k,p} - (p^{-3k/2+5}(p+1) + p^{-k/2}(p+1) + p^{-k+2}(p+1)^2)}{p^{1-k}(p^{3-k/2}+1)(p^{2-k/2}+1)}(f|\Phi)
\]

\[
= (1 + p^{k-1})(f|\Phi),
\]

where the second identity is again just a straightforward calculation (and could be verified by using [SAGE], for example). \( \square \)

Furthermore, a consequence of this corollary is

(4.65) Corollary. Let \( p \neq 3 \) be a prime number, \( k \in 2\mathbb{N}, k \geq 8 \) and \( f \in [\Gamma_2, k] \). Suppose that \( f \) is an eigenform of \( T_2(p) \) and that the constant term in the Fourier-expansion of \( f \) is given by

\[
f(Z) = \sum_{T \in \text{Her}_2(\mathbb{D}), T \geq 0} \alpha_f(T)e^{2\pi i(T,Z)}, \quad Z \in \mathcal{H}(\mathbb{H}),
\]

with \( \alpha_f(0) \neq 0 \). Then

\[
E_k - \alpha_f(0)^{-1}f \in [\Gamma_2, k]_0
\]

holds true.

Proof: According to (4.8), the constant term in the Fourier-expansion of \( E_k \) equals 1, hence the constant term in the Fourier-expansion of \( g := E_k - \alpha_f(0)^{-1}f \) equals 0. Thus, \( g|\Phi \) is an elliptic cusp form in view of the proof of (1.59). Furthermore, \( E_k|\Phi = G_k \) holds in virtue of (4.8). So applying [KK07, Kap.IV, Prop.2.4] and (4.64) yields that \( g|\Phi \) is an eigenform of \( T(p) \) with

\[
(g|\Phi)|_kT(p) = (1 + p^{k-1})(g|\Phi).
\]

But then, according to [KK07, Kap.IV, Prop.1.4], \( (g|\Phi) \neq 0 \) would be a contradiction, since otherwise \( g|\Phi \) being an elliptic cusp form and an eigenform of \( T(p) \) with eigenvalue \( 1 + p^{k-1} \) would imply

\[
1 + p^{k-1} \leq p^{k/2}(1 + p^{-1}),
\]
which is obviously not true, since this would also imply
\[ p^{k/2-1} \leq 2 , \]
which is a contradiction for \( p \geq 2 \) and \( k \geq 8 \). Thus, \( (g|\Phi) = 0 \) follows, which means that \( E_k - \alpha f(0)^{-1} f \) is a cusp form, like asserted. \( \square \)

Of course (in view of (4.62)), \( g = E_k - \alpha f(0)^{-1} f \) is again an eigenform of \( T_2(p) \) (if we allow identically vanishing functions to be called eigenforms, too) with \( g|\kappa T_2(p) = \lambda_k p g \). But we will show now that such a cusp form has to vanish identically, and thus \( f = \alpha f(0) E_k \).

(4.66) Lemma. Let \( p \neq 3 \) be a prime number, \( k \in 2\mathbb{N} \) and \( 0 \neq f \in [\Gamma_2, k]_0 \). Suppose that \( f \) is an eigenform of \( T_2(p) \) with \( f|\kappa T_2(p) = \lambda f \). Then
\[ |\lambda| \leq p^{-k} \prod_{j=0}^{3} (p^j + 1) \]
has to hold.

Proof: According to (1.64), the function
\[ \tilde{f}(Z) := (\det(Y))^{k/2} |f(Z)|, \quad Z = X + iY \in \mathcal{H}(\mathbb{H}) \]
is invariant under modular transformations with respect to \( \Gamma_2 \) (which means \( \tilde{f}(M(Z)) = \tilde{f}(Z) \) for all \( M \in \Gamma_2 \) and all \( Z \in \mathcal{H}(\mathbb{H}) \)). Moreover, since \( f \) is assumed to be a cusp form, \( \tilde{f} \) is bounded and there exists \( Z_0 = X_0 + iY_0 \in \mathcal{F}_2(\mathcal{O}) \) satisfying
\[ \tilde{f}(Z_0) = \max \{ \tilde{f}(Z) ; Z \in \mathcal{H}(\mathbb{H}) \} > 0. \]
Once more, let \( L_1, \ldots, L_{\deg(T_2(p))} \in \Delta_2(p) \) be the transversal of \( \Gamma_2 \setminus \Delta_2(p) \) specified in (4.40), denote \( S = \{ 1, \ldots, \deg(T_2(p)) \} \), and let \( L_j = (\begin{smallmatrix} A_j & B_j \\ 0 & D_j \end{smallmatrix}) \) for \( j \in S \). Due to (4.34), we have \( A_j = p(D_j)^{-1} \) for all \( j \in S \), and thus \( \det(A_j) = p^4(\det(D_j))^{-1} \). This leads to
\[ \left| \det(A_j)^{-k/4} \det(D_j)^{k/4} \det(D_j)^{-k/2} \right| = p^{-k} \cdot \]
Note that we have \( (\det(Y_0))^{k/2} = (\det(\tilde{Y}_0))^{k/4} \) in view of (1.34) due to \( Y_0 \) being positive definite. Furthermore, \( \Im((A_j Z_0 + B_j)D_j^{-1}) = A_j Y_0 D_j^{-1} \) holds, whereas
\[ \left| \det(A_j Y_0 D_j^{-1})^{k/2} \right| = \det(A_j Y_0 D_j^{-1})^{k/2} = \det(\tilde{A}_j)^{k/4} \det(\tilde{Y}_0)^{k/4} \det(D_j)^{-k/4}. \]
And finally, recall that \( f|\kappa T_2(p) = \lambda f \) holds due to the assumption. Keeping all that in mind, we
calculate
\[
|\lambda \tilde{f}(Z_0)| = |f_k T_2(p)(Z_0)| = \left| \det(\tilde{Y}_0)^{k/4} \sum_{j \in S} \det(\tilde{D}_j)^{-k/2} f((A_j Z_0 + B_j) D_j^{-1}) \right|
\leq p^{-k} \sum_{j \in S} \det(A_j Y_0 D_j^{-1})^{k/2} |f((A_j Z_0 + B_j) D_j^{-1})|
= p^{-k} \sum_{j \in S} \tilde{f}((A_j Z_0 + B_j) D_j^{-1}) \leq p^{-k} \sum_{j \in S} \tilde{f}(Z_0) = p^{-k} \deg(T_2(p)) \tilde{f}(Z_0).
\]

We have \( \tilde{f}(Z_0) > 0 \) due to the assumption, while \( \deg(T_2(p)) = \prod_{j=0}^{3} (p^j + 1) \) in virtue of (4.40). Thus
\[
|\lambda| \leq p^{-k} \prod_{j=0}^{3} (p^j + 1)
\]
follows. \( \square \)

We combine all the results from above and obtain the following crucial theorem:

(4.67) Theorem. Let \( k \in 2 \mathbb{N}, k \geq 8 \) and \( f \in \Gamma_2, k \). Suppose that \( f \) is an eigenform of \( T_2(p) \) for some prime number \( p \neq 3 \) and that the Fourier-expansion of \( f \) is given by
\[
f(Z) = \sum_{T \in \text{Her}_2(O), T \geq 0} \alpha_f(T) e^{2\pi i T \cdot Z}, \quad Z \in \mathcal{H} \),
\]
with \( \alpha_f(0) \neq 0 \). Then
\[
f = \alpha_f(0) E_k
\]
follows.

Proof: Let \( g = E_k - \alpha_f(0)^{-1} f \). According to (4.61), (4.62) and (4.65), \( g \) is a cusp form and fulfills
\[
g | k T_2(p) = \lambda_{k,p} g.
\]
Suppose \( g \neq 0 \). But then, in virtue of (4.66)
\[
\lambda_{k,p} = \prod_{j=0}^{3} (p^{-k/2} + 1) \leq p^{-k} \prod_{j=0}^{3} (p^j + 1) = \prod_{j=0}^{3} (p^{j-k} + p^{-k})
\]
would have to hold. But obviously, this not true for \( p \geq 2 \) and \( k \geq 8 \), since we obviously have \( p^{j-k/2} > p^{j-k} \) and \( 1 > p^{-k} \). Therefore, \( g \equiv 0 \) has to follow, which completes the proof. \( \square \)

So let us get back to the Maaß lifts \( \tilde{E}_k \) we defined at the beginning of this section (see (4.59)). According to (4.60), they are eigenforms for all \( T_2(p) \) (\( p \neq 3 \) a prime number) with respect to the eigenvalue \( \lambda_{k,p} \). Therefore, according to the preceding theorem, they have to be multiples of the quaternionic Eisenstein-series – given the constant term in their Fourier-expansions does not vanish. So let us determine this constant term.

According to [Kl98, Le.4.4], the constant term of a Maaß lift \( f \) of weight \( k \) is given by \(-\frac{B_k}{2\pi} \alpha^*(0),\)
4.5 The Fourier-expansion of the quaternionic Eisenstein-series

where \( \alpha^* \) is the attached function of \( f \), which means that \( \alpha^*(0) \) is the constant term in the Fourier-expansion of the elliptic modular form \( g \), where \( f = \Omega^{-1}(g) \).

The constant term in the Fourier-expansion of the elliptic Eisenstein-series \( G_k \) equals 1 (see (4.8)). Therefore, the constant term in the Fourier-expansion of

\[
G_{k-2}(\tau) - 3^{k-2}G_{k-2}(9\tau)
\]

equals \( 1 - 3^{k-2} \). Thus, the constant term of \( E_k \) equals

\[
-\frac{B_k}{2k}(1 - 3^{k-2})
\]

So we finally obtain

\[
E_k = \frac{2k}{B_k(3^{k-2}-1)} \tilde{E}_k
\]

for all \( k \in 2N, k \geq 6 \) in view of (4.67) (where for \( k = 6 \) this is already clear by definition.) Note that according to [KK07, Kap.III, p.161] the Fourier-expansion of \( G_k - 2 \) is given by

\[
G_k(\tau) = 1 - \frac{2(k-2)}{B_k} \sum_{n \in \mathbb{N}} \sigma_{k-3}(n)e^{2\pi in\tau}, \quad \tau \in \mathcal{H}.
\]

Therefore, defining

\[
(4.51)
\]

for \( a, b \in \mathbb{C} \), we obviously obtain

\[
G_{k-2}(\tau) - 3^{k-2}G_{k-2}(9\tau) = 1 - 3^{k-2} + \sum_{n \in \mathbb{N}} \frac{-2(k-2)}{B_k} \left( \sigma_{k-3}(n) - 3^{k-2}\sigma_{k-3}(n/9) \right)e^{2\pi in\tau}
\]

for all \( \tau \in \mathcal{H} \).

And finally, we want to determine the Fourier-expansion of \( E_4 \), too. By definition and in view of (4.16) we have

\[
E_4 = \Omega^{-1}(c(f_2(\tau) + 3f_2(3\tau)))
\]

where \( c \in \mathbb{C} \) is some normalizing factor such that the constant term in the Fourier-expansion of \( E_4 \) equals 1. \( f_2 \) was defined as (see 4.12)

\[
f_2(\tau) = \frac{B_2}{2} + \sum_{n \in \mathbb{N}} \left( \sum_{d|n, d|n/3} d \right)e^{2\pi in\tau}, \quad \tau \in \mathcal{H},
\]

and thus the constant term in the Fourier-expansion of \( f_2(\tau) + 3f_2(3\tau) \) equals \( c\frac{B_2}{2}(1 + 3) \). So we obtain

\[
-\frac{B_4}{2 \cdot 4} \cdot (2cB_2) = 1 \quad \Leftrightarrow \quad c = -\frac{4}{B_6 B_4} = -\frac{4(k-2)}{(3^{k-2}-1)B_k B_{k-2}},
\]

where \( k = 4 \). And note that the constant term in the Fourier-expansion of \( c(f_2(\tau) + 3f_2(3\tau)) \)
We summarize these results in the following final theorem, which is the pinnacle of all the preliminary work in this chapter:

\( \sum_{d \in \mathbb{N}, d|n, 3|d} d = \sum_{d \in \mathbb{N}, d|n} d - \chi_Z(n/3) \sum_{d \in \mathbb{N}, d|n/3} 3d = \sigma_1(n) - 3\sigma_1(n/3) , \)

hence

\[ f_2(\tau) = \frac{B_2}{2} + \sum_{n \in \mathbb{N}} \left( \sigma_1(n) - 3\sigma_1(n/3) \right) e^{2\pi i n \tau} , \quad \tau \in \mathcal{H} . \]

Furthermore, we have for \( \tau \in \mathcal{H} : \)

\[ f_2(3\tau) = \frac{B_2}{2} + \sum_{n \in \mathbb{N}} \left( \sigma_1(n) - 3\sigma_1(n/3) \right) e^{2\pi i (3n) \tau} = \frac{B_2}{2} + \sum_{n \in \mathbb{N}} \left( \sigma_1(n/3) - 3\sigma_1(n/9) \right) e^{2\pi i n \tau} , \]

and thus

\[ f_2(\tau) + 3f_2(3\tau) = 2B_2 + \sum_{n \in \mathbb{N}} \left( \sigma_1(n) - 9\sigma_1(n/9) \right) e^{2\pi i n \tau} = 2B_2 + \sum_{n \in \mathbb{N}} \left( \sigma_{k-3}(n) - 3^{k-2}\sigma_{k-3}(n/9) \right) e^{2\pi i n \tau} . \]

We summarize these results in the following final theorem, which is the pinnacle of all the preliminary work in this chapter:

**Theorem.** We have

\[ E_4 = -\frac{4}{B_2B_4} \Omega^{-1}(f_2(\tau) + 3f_2(3\tau)) \]

and

\[ E_k = \frac{2k}{B_k(3^{k-2}-1)} \Omega^{-1}(G_{k-2}(\tau) - 3^{k-2}G_{k-2}(9\tau)) \]

for all \( k \in 2\mathbb{N} , k \geq 6 \). The Fourier-expansion of \( E_k \) for \( k \in 2\mathbb{N} , k \geq 4 \) is given by

\[ E_k(Z) = 1 + \sum_{0 \neq T \in \text{Her}_2^1(\mathcal{O}), T \geq 0} \left( \sum_{d \in \mathbb{N}, d|\det(T)} d^{k-1}a_k^+(\det(T)/d^2) \right) e^{2\pi i T(Z)} , \quad Z \in \mathcal{H}(\mathbb{H}) , \]

where

\[ a_k^+(n) = \begin{cases} \frac{2k}{B_k} , & \text{if } n = 0 , \\ \frac{4k(k-2)}{(3^{k-2}-1)B_kB_{k-2}} \left( \sigma_{k-3}(n) - 3^{k-2}\sigma_{k-3}(n/9) \right) , & \text{if } n \in \mathbb{N} , \end{cases} \]

with \( B_n \) being the Bernoulli numbers (cf. [Mi89, p.89]) and \( \sigma_n(\cdot) \) defined in 4.51.

Note that the preceding theorem implies that the Fourier-coefficients \( a(T) \) of the quaternionic Eisenstein-series \( E_k \) (where \( k \in 2\mathbb{N} , k \geq 8 \)) fulfill

\[ a(T) = O(\det(T)^{k-3}) \quad \text{for all } T \in \text{Her}_2^1(\mathcal{O}), T > 0 , \]

just like in [Kr90, thm.3] (and due to the same reason). Furthermore, it is clear that the
Fourier-coefficients \( a(T) \) (where \( k \geq 4 \) is allowed, now) are rational numbers with bounded denominators. This denominator is given by the common denominator of

\[
\frac{2k}{B_k} \quad \text{and} \quad \frac{4(k-2)}{(3^{k-2}-1)B_kB_{k-2}}.
\]

So in particular, one easily checks that the Fourier-coefficients of \( E_4 \) and \( E_6 \) are integral.

In (4.4) we saw that \( E_k \in [\text{Sp}_2(\mathcal{O}), k, 1] \) for \( k \in 2\mathbb{N}, k \geq 8 \) (and the same holds for \( E_4 \) and \( E_6 \) since they are defined as Maass lifts). Moreover, we already saw several times before (for example confer (2.22) or (1.55)) that if the Fourier-coefficients of \( f \in [\text{Sp}_2(\mathcal{O}), k, 1] \) are given by \( \alpha_f(T) \) (where \( T \in \text{Her}_2(\mathcal{O}), T \geq 0 \)), then the Fourier-coefficients of \( f(Z') \) and of \( f(Z[iI]) \) are given by \( \alpha_f(T') \) and \( \alpha_f(T[iI]) \), respectively. But for \( f \in \mathcal{M}(k; \mathcal{O}) \) the Fourier-coefficient \( \alpha_f(T) \) only depends on \( \epsilon(T) \) and \( \det(T) \), which stay invariant under \( T \mapsto T' \) and \( T \mapsto T[iI] \). So noting that all quaternionic Eisenstein-series are Maass lifts we obviously get

\[ (4.69) \textbf{Corollary.} \] Let \( k \in 2\mathbb{N}, k \geq 4 \). We have

\[ \mathcal{M}(k; \mathcal{O}) \subset [\Gamma(\mathcal{O}), k, 1] , \]

and in particular

\[ E_k \in [\Gamma(\mathcal{O}), k, 1] . \]
5 Orthogonal Modular Forms

We will now introduce so-called orthogonal modular forms. We will need a completely different setting, here. We will give an introduction to (some special) orthogonal groups and their attached half-spaces, orthogonal modular groups and finally orthogonal modular forms. But although the setting is completely different to what we had before, there are links between this orthogonal world and the symplectic one we considered so far. Indeed, certain orthogonal modular groups (and all those things attached to them) can be identified with certain symplectic modular groups, be it Siegel, Hermitian, or quaternionic modular groups (or other similar objects).

So our first goal will be to briefly introduce the orthogonal world. Afterwards, we will give the link to the quaternionic setting we are interested in. We will work out a dictionary which will give the answer on how important objects in both worlds are transferred to each other, for example concerning the abelian characters attached to both the quaternionic and the orthogonal modular group. Note that a good introduction to the orthogonal setting can be found in [Kl06]. Furthermore, the orthogonal modular groups and orthogonal modular forms related to the quaternionic modular forms with respect to the Hurwitz order (and restrictions of these to submanifolds) were investigated in full detail in [FH00], [FS07] and [FSM].

The main point why we consider orthogonal modular forms and their relations to our quaternionic modular forms is that a very important tool concerning the determination of the graded rings of modular forms arises – so-called Borcherds products (cf. [Bo98], [Bo99], and also [Bu01] for a good and quite comprehensible and more explicit summary). These are orthogonal modular forms (and hence, with the help of our identification, also symplectic modular forms) with prescribed zero sets (or divisors). With help of these Borcherds products the problem of determining generators for the graded rings of modular forms can be (under certain appropriate circumstances) reduced to lower dimensional settings, which already may have been investigated in detail. We will get to that issue later. But for the moment, one just has to keep in mind that we are introducing the orthogonal world in order to be able to start such a reduction process – although we will ultimately end up in a setting of certain modular forms that could not be determined completely in this thesis. But at least we will find out which problems have to be solved first before this reduction process could finally yield the desired structures of our graded rings of quaternionic modular forms for $\text{Sp}_2(\mathcal{O})$.

5.1 Introduction to the orthogonal setting

This first section serves as an introduction to the orthogonal world, which means we will define the orthogonal groups, their attached half-spaces, and orthogonal modular forms. Furthermore, we will determine all abelian characters for the orthogonal modular groups. Note that we will try to keep this introduction quite brief. Details can be found in [Kl06]. Moreover, we will not give any proofs here, since this first part is just a summary of the first chapter from [Kl06]. In
some important cases, we will cite explicitly where the result can be found in that thesis, but in most cases we will not. Simply confer the said first chapter of that work for details.

We begin by giving the most basic definitions. Note that we stick to the notation found in [Kl06]. First, let us consider lattices (although we already used the term, but without giving further insights since it was not needed):

(5.1) Definition. a) A lattice is a free \( \mathbb{Z} \)-module of finite rank equipped with a \( \mathbb{Z} \)-valued bilinear form \( (\cdot, \cdot) \). Given a lattice \( \Lambda \), then the associated quadratic form \( q \) is defined by

\[
q(\lambda) = \frac{1}{2}(\lambda, \lambda), \quad \lambda \in \Lambda.
\]

\( \Lambda \) is said to be even, if \( q(\lambda) \in \mathbb{Z} \) holds for all \( \lambda \in \Lambda \). Furthermore, we will always assume \( \Lambda \) to be non-degenerate, and we set

\[
V := \Lambda \otimes \mathbb{R}.
\]

\( (\cdot, \cdot) \) induces a bilinear form on \( V \times V \), again denoted by \( (\cdot, \cdot) \), and the same holds for \( q \). We keep this notation for the next definitions:

b) The dual lattice \( \Lambda^\sharp \) is defined by

\[
\Lambda^\sharp := \{ \mu \in V ; (\mu, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \}.
\]

Note that \( \Lambda \subset \Lambda^\sharp \) holds.

c) The finite abelian group

\[
\text{Dis}(\Lambda) := \Lambda^\sharp / \Lambda
\]

is called the discriminant group of \( \Lambda \).

d) The level of \( \Lambda \) is defined by

\[
\min \{ n \in \mathbb{N} ; nq(\mu) \in \mathbb{Z} \text{ for all } \mu \in \Lambda^\sharp \}.
\]

e) \( \mu \in \Lambda^\sharp \) is called primitive if \( \max \{ n \in \mathbb{N} ; \frac{1}{n} \mu \in \Lambda^\sharp \} = 1. \)

f) If \( \Lambda \) is even, then \( q \) induces a well-defined map \( \overline{q} \) on \( \text{Dis}(\Lambda) \) given by

\[
\overline{q} : \text{Dis}(\Lambda) \rightarrow \mathbb{Q} / \mathbb{Z}, \mu + \Lambda \mapsto q(\mu) + \mathbb{Z}.
\]

Next, we get to the orthogonal groups:

(5.2) Definition. a) Let \( l \in \mathbb{N} \), \( \Lambda = \mathbb{Z}^l \), \( V = \Lambda \otimes \mathbb{R} \cong \mathbb{R}^l \), and \( S \in \text{Sym}(\mathbb{Z}) \cap \text{GL}_l(\mathbb{R}) \). The symmetric bilinear form \( (\cdot, \cdot)_S \) on \( V \) associated to \( S \) is given by

\[
(x, y)_S := x^tSy, \quad x, y \in V,
\]

with corresponding quadratic form

\[
q_S(x) = \frac{1}{2}S[x], \quad x \in V.
\]
If it is clear what $S$ is, then we also omit the index $S$. Furthermore, $\Lambda$ together with $(\cdot,\cdot)_S$ is a lattice of rank $l$. The dual lattice is given by

$$\Lambda^\vee = S^{-1} \Lambda,$$

and

$$\# \text{Dis}(\Lambda) = \det(S).$$

$S$ is called even if the associated lattice $\Lambda$ is even. Again, we stick to this notation for the next definitions:

b) The real orthogonal group $O(S; \mathbb{R})$ with respect to $S$ is defined by

$$O(S; \mathbb{R}) := \{ M \in \mathbb{R}^{l \times l} ; S[M] = S \} = \{ M \in \mathbb{R}^{l \times l} ; q_S(Mx) = q_S(x) \text{ for all } x \in \mathbb{R}^l \}.$$

Up to isomorphism, $O(S; \mathbb{R})$ only depends on the signature $(b^+, b^-)$ of $S$, where $b^+$ and $b^-$ are the numbers of positive and negative eigenvalues of $S$ (counted with multiplicities). And note that $\det(M) = \pm 1$ holds for all $M \in O(S; \mathbb{R})$.

c) The stabilizer of $\Lambda$ in $O(S; \mathbb{R})$ is denoted by

$$O(\Lambda) = \{ M \in O(S; \mathbb{R}) ; M \Lambda = \Lambda \} = O(S; \mathbb{R}) \cap \text{GL}_l(\mathbb{Z}).$$

d) Since $M \Lambda^\vee = \Lambda^\vee$ holds for all $M \in O(\Lambda)$, $O(\Lambda)$ acts on $\text{Dis}(\Lambda)$. The kernel of this action is called the discriminant kernel and denoted by $O_d(\Lambda)$.

e) The lattice $\Lambda$ is called euclidean if for all $x \in V$ there exists $\lambda \in \Lambda$ such that

$$q_S(x + \lambda) < 1.$$

In particular, we are interested in symmetric matrices over $\mathbb{Z}$ of full rank possessing a certain shape. And for these special matrices we also define the attached half-spaces and the orthogonal modular groups:

(5.3) Definition. a) Let $l \in \mathbb{N}$ and $S \in \text{Pos}_l(\mathbb{Z})$ be even. We set

$$S_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that $S$ is of signature $(l,0)$, while $S_0$ and $S_1$ are of signature $(1,l+1)$ and $(2,l+2)$,
respectively. We use the following abbreviations:

\[(\cdot, \cdot) = (\cdot, \cdot)_S, \quad q = q_S,\]
\[(\cdot, \cdot)_0 = (\cdot, \cdot)_{S_0}, \quad q_0 = q_{S_0},\]
\[(\cdot, \cdot)_1 = (\cdot, \cdot)_{S_1}, \quad q_1 = q_{S_1}.\]

b) Set

\[e := (1, 0, \ldots, 0, 1) \in \mathbb{R}^{l+2},\]

and define

\[\mathcal{H}_S := \{u + iv \in \mathbb{C}^{l+2}; \; v \in \mathcal{P}_S\},\]

where

\[\mathcal{P}_S := \{v \in \mathbb{R}^{l+2}; \; q_0(v) > 0, \; (v, e)_0 > 0\}
\]
\[= \{(v_0, \tilde{v}, v_{l+1}) \in \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}; \; v_0v_{l+1} > q_S(\tilde{v}), \; v_0 > 0\}.
\]

\[\mathcal{H}_S\] is called the (orthogonal) half-space attached to \(S\). Note that we have

\[\mathcal{H}_S \subset \mathcal{H} \times \mathbb{C}^l \times \mathcal{H}.
\]

So usually, we will write elements if \(\mathcal{H}_S\) in the form \(w = (\tau_1, z, \tau_2)\), where \(\tau_1, \tau_2 \in \mathcal{H}, \; z \in \mathbb{C}^l\). Once more, stick to this notation for the next definitions.

c) In the orthogonal setting, we will always write \(M \in \mathbb{R}^{(l+4) \times (l+4)}\) in the form

\[M = \begin{pmatrix}
\alpha & a' & b \\
b & A & c \\
\gamma & d' & \delta
\end{pmatrix}, \quad \text{where } A \in \mathbb{R}^{(l+2) \times (l+2)}.
\]

Note that we have \(M \in O(S_1; \mathbb{R})\) if and only if

\[
\begin{pmatrix}
2a\gamma + S_0[b] & ad' + b'S_0A + \gamma a' & a\delta + b'S_0c + \beta\gamma \\
ad + A'S_0b + \gamma a & ad' + S_0[A] + da' & \beta d' + A'S_0c + \delta a \\
a\delta + b'S_0c + \beta\gamma & \beta d' + c'S_0A + \delta a' & 2\beta \delta + S_0[c]
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
0 & S_0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

d) \(O(S_1, \mathbb{R})\) acts transitively on \(H^S := H_S \cup (-H_S)\) as a group of biholomorphic automorphisms via

\[w \mapsto M\{w\} := (-q_0(w)b + Aw + c) \cdot (M\{w\})^{-1},\]

where

\[M\{w\} := -\gamma q_0(w) + d'w + \delta\]

for \(M \in O(S_1, \mathbb{R})\) and \(w \in H^S\). Here, \(q_0\) was extended to \(\mathbb{C}^{l+2}\) in a natural way via \(q_0(w) := \frac{1}{2} w'S_0\tilde{w}w\). Note that \(M\) acts trivially on \(H^S\) if and only if \(M = \pm I\). Furthermore, one can verify

\[M_1\{M_2\{w\}\} = (M_1M_2)\{w\} \quad \text{for all } M_1, M_2 \in O(S_1, \mathbb{R}), \; w \in H^S.\]
5.1 Introduction to the orthogonal setting

\[ O^+(S_1; \mathbb{R}) := \{ M \in O(S_1; \mathbb{R}) ; M(\mathcal{H}_S) = \mathcal{H}_S \} \]

as the subgroup of \( O(S_1; \mathbb{R}) \) stabilizing \( \mathcal{H}_S \). Note that for \( M = (A_{\pm} \pm A_{\mp}) \), where \( C, D \in \mathbb{R}^{2 \times 2} \), we have

\[ M \in O^+(S_1; \mathbb{R}) \iff \det \left( \frac{0}{1} \frac{1}{0} \frac{1}{0} \frac{1}{0} \right) + D > 0 \]

\[ \Gamma_S := O(\Lambda_1) \cap O^+(S_1; \mathbb{R}) \].

\( \Gamma_S \) acts on \( \text{Dis}(\Lambda_1) \simeq \text{Dis}(\Lambda_0) \simeq \text{Dis}(\Lambda) \). Even more, \( \Gamma_S \) acts on the sets of \( \text{Dis}(\Lambda_1) \) with the same value of \( \Gamma_1 \), and for

\[ M = \left( \begin{array}{ccc} * & * & * \\ * & A & * \\ * & * & * \end{array} \right) \in \Gamma_S, \quad \text{where } A \in \mathbb{Z}^{l \times l}, \]

both, the action of \( M \) on \( \text{Dis}(\Lambda_1) \) and the action of \( M \) on the sets of elements of \( \text{Dis}(\Lambda_1) \) with the same value of \( \Gamma_1 \), only depend on \( A \).

For further considerations, we will need the following matrices, which all belong to \( \Gamma_S \). (And note that we have \( \pm l \in \Gamma_S \), of course.) We will also give the action of these matrices on \( w = (\tau_1, z, \tau_2) \in \mathcal{H}_S \).

- \( J_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \tilde{I} & 0 \\ -1 & 0 & 0 \end{pmatrix} \), where \( \tilde{I} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{pmatrix} \),
  \[ J_0(w) = -q_0(w)^{-1}(\tau_2, -z, \tau_1) \],

- \( T_g = \begin{pmatrix} 1 & -g' S_0 & -q_0(g) \\ 0 & I & g \\ 0 & 0 & 1 \end{pmatrix} \), where \( g \in \Lambda_0 \),
  \[ T_g(w) = w + g \],

- \( U_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{U}_\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \), where \( \tilde{U}_\lambda = \begin{pmatrix} 1 & \lambda' S & q(\lambda) \\ 0 & I & \lambda \\ 0 & 0 & 1 \end{pmatrix} \), \( \lambda \in \Lambda \),
  \[ U_\lambda(w) = (\tau_1 + \lambda' S z + q(\lambda) \tau_2, z + \lambda \tau_2, \tau_2) \],

- \( R_g = \begin{pmatrix} \varepsilon_g & 0 & 0 \\ 0 & \tilde{R}_g & 0 \\ 0 & 0 & \varepsilon_g \end{pmatrix} \), where \( \tilde{R}_g = (I_{l+2} - \varepsilon_g g' S_0)\tilde{I} \), if \( g \in \Lambda_0 \) such that \( \varepsilon_g = q_0(g) = \pm 1 \),
  \[ R_g(w) = q_0(g)\tilde{R}_g w \],
Here the following proposition is in analogy with [Kl06, Prop.1.17]:

Let $S$ be one of the matrices specified in 5.1, and let $l \in \mathbb{N}$ be the size (and rank) of $S$. Then the quadratic space $(\mathbb{R}^l, q_S)$ with lattice $\Lambda = \mathbb{Z}^l$ is isomorphic to the quadratic space $(\mathbb{H}_S, N_S)$ with lattice $\mathcal{O}_S$, where $\mathbb{H}_S$ is some subspace of $\mathbb{H}$, $N_S$ is the restriction of the norm $N$ to $\mathbb{H}_S$, and $\mathcal{O}_S = \mathcal{O} \cap \mathbb{H}_S$ is the sublattice of $\mathcal{O}$ in $\mathbb{H}_S$. The following list contains the subspaces $\mathbb{H}_S$, the corresponding lattices $\mathcal{O}_S$, a possible isomorphism $\iota_S : \mathbb{R}^l \to \mathbb{H}_S$ and the quadratic forms $q_S = N_S \circ \iota_S$. Here $\mathbb{Z}_n$ is defined to be the additive group $\mathbb{Z}/n\mathbb{Z}$, and $x_j$ will always refer to the identification in $a$.

\begin{itemize}
\item $\mathbb{H}_{A_2^{(2)}} = \mathbb{H}$, $\mathcal{O}_{A_2^{(2)}} = \mathcal{O}$,
\end{itemize}

\begin{align*}
\iota_{A_2^{(2)}}(x_1, x_2, x_3, x_4) & \mapsto x_1 + x_2 \frac{1+i\sqrt{3}}{2} + x_3 i_2 + x_4 \frac{1+i\sqrt{3}}{2} i_2, \\
q_{A_2^{(2)}}(x) & = x_1^2 + x_1 x_2 + x_2^2 + x_3 x_4 + x_4^2,
\end{align*}

\begin{itemize}
\item $\mathbb{H}_{A_{2,1}} = \{x \in \mathbb{H} ; \ x_4 = 0\}$, $\mathcal{O}_{A_{2,1}} = \mathbb{Z} + \mathbb{Z} \frac{1+i\sqrt{3}}{2} + \mathbb{Z} i_2$,
\end{itemize}

\begin{align*}
\iota_{A_{2,1}}(x_1, x_2, x_3) & \mapsto x_1 + x_2 \frac{1+i\sqrt{3}}{2} + x_3 i_2, \\
q_{A_{2,1}}(x) & = x_1^2 + x_1 x_2 + x_2^2 + x_3^2,
\end{align*}

\begin{itemize}
\item $\mathbb{H}_{T_3} = \{x \in \mathbb{H} ; \ x_2 = x_4\}$, $\mathcal{O}_{T_3} = \mathbb{Z} + \mathbb{Z} \frac{1+i\sqrt{3}+i+2\sqrt{3}}{2} + \mathbb{Z} i_2$,
\end{itemize}
Proof: All claims can be verified by some explicit calculations, and it seems unnecessary to actually do them here. Even the assertions about the discriminant groups are simple calculations by taking $\Lambda^2 = S^{-1}Z^l$ into account, where one simply has to find appropriate generators. \qed

(5.5) Remark. a) Note that the lattice associated to $D_2^1$ is also a sublattice of the lattice associated to $A_{2,1}$ (and not only $T_3$), since we could have chosen the identification $H_{A_{2,1}} = \{ x \in H ; x_3 = 0 \}$.

b) There are two further important lattices, which are induced by the matrices

$$A_1^{(2)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. $$

They correspond to $H_{A_1^{(2)}} = R + R i_2 \simeq \mathbb{C}$, with $O_{A_1^{(2)}} = Z + Z i_2$, and $H_{A_2} = R + R \frac{1 + i \sqrt{3}}{2} \simeq \mathbb{C}$, with $O_{A_2} = Z + Z \frac{1 + i \sqrt{3}}{2}$, or in other words the ring of integers in $\mathbb{Q}(i)$ and $\mathbb{Q}(i \sqrt{3})$, respectively. They were analyzed in [Kl06], while the modular forms corresponding to them were investigated in [De01]. Analogous to (5.4), we have the isomorphisms

- $I_{A_1^{(2)}} : (x_1, x_2) \mapsto x_1 + x_2 i_2,$
- $I_{A_2} : (x_1, x_2) \mapsto x_1 + x_2 \frac{1 + i \sqrt{3}}{2}.$

c) $O_{D_2^1}$ is isomorphic to the ring of integers in $\mathbb{Q}(i \sqrt{7})$: We have $H_{D_2^1} = R + R (i \sqrt{3} + i_2 + i_1 i_2 \sqrt{3})$, so let $R = Q + Q (i \sqrt{3} + i_2 + i_1 i_2 \sqrt{3})$. One easily verifies $(i_1 \sqrt{3} + i_2 + i_1 i_2 \sqrt{3})^2 = -7$. Note that the considerations at the beginning of section 1.5 yield that $H_{D_2^1}$ is isomorphic to $C$, and thus $R$ is isomorphic to $\mathbb{Q}(i \sqrt{7})$, when identifying $(i_1 \sqrt{3} + i_2 + i_1 i_2 \sqrt{3})$ with $i \sqrt{7}$. According to [De01], the ring of integers in $\mathbb{Q}(i \sqrt{7})$ is given by $Z + Z \frac{1 + i \sqrt{7}}{2}$, and this yields the assertion.

Now, we will have to find generators for the orthogonal groups above. The most important fact concerning this issue is that the attached lattices are euclidean:

(5.6) Proposition. The lattices attached to the matrices from 5.1 are euclidean.

Proof: In all cases, we can make use of the identifications from (5.4): The lattice attached to $A_2^{(2)}$ is euclidean in view of (1.24). Note that due to that proposition, given $x \in R^4$, there exists
\( \lambda \in \mathbb{Z}^4 \) such that \( x + \lambda = (y_1, y_2, y_3, y_4)' \), with \( y_1^2 + y_1 y_2 + y_2^2 \leq \frac{1}{3} \) and \( y_3^2 + y_3 y_4 + y_4^2 \leq \frac{1}{3} \). Thus, the lattice attached to \( A_{2,1} \) obviously is euclidian, too: Given \( x \in \mathbb{R}^3 \), choose \( \lambda \in \mathbb{Z}^3 \) such that \( x + \lambda = (y_1, y_2, y_3)' \) with \( y_1^2 + y_1 y_2 + y_2^2 \leq \frac{1}{3} \) and \( |y_3| \leq \frac{1}{3} \), which implies \( q_{A_{2,1}}(y) \leq \frac{1}{3} + \frac{1}{9} = \frac{7}{12} \).

For \( T_3 \), let \( x = x_1 + x_2 \frac{1+i\sqrt{3}+i2+i\sqrt{3}}{2} + x_3 \in \mathbb{H}_{T_3} \), and choose \( \lambda \in \mathcal{O}_{T_3} \) such that \( y = x + \lambda = y_1 + y_2 \frac{1+i\sqrt{3}+i2+i\sqrt{3}}{2} + y_3 \) fulfills: \( |y_2| \leq \frac{1}{3} \) and (after \( y_2 \) is given that way) \( |y_1 + \frac{y_2}{2}| \leq \frac{1}{2} \) as well as \( |y_3 + \frac{y_2}{2}| \leq \frac{1}{2} \). Then we obtain \( N(y) \leq 2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{16} = \frac{7}{8} \). And finally, setting \( y_3 = 0 \) in this inequality yields the assertion for \( D_2^2 \).

We need some further special matrices for each of the four orthogonal modular groups. First, note that for each of the matrices \( S \) specified in 5.1, \( \mathbb{H}_S \) and \( \mathcal{O}_S \) are self-contained under conjugation. Furthermore, there exists a matrix \( A_{2,0}^{(2)} \) such that \( i_S(x) = i_S(A_{2,0}^{(2)}x) \) holds for all \( x \in V \). A straightforward calculation shows that \( A_{2,0}^{(2)} \) is given as follows:

\[
A_{2,0}^{(2)} : \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad A_{2,1} : \begin{pmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad T_3 : \begin{pmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad D_2^2 : \begin{pmatrix}
1 & 1 \\
0 & -1
\end{pmatrix}
\]

and

\[
\mathcal{O}_S(A) = R_{\mathcal{O}_S}(A) = R_{\mathcal{O}_S}(A) \quad \text{for all} \quad A \in \mathcal{O}_S.
\]

Next, analogous to [Kl06, cor.1.23] we get:

**Proposition (5.7)** If \( S \) is one of the matrices listed in 5.1, then \( \Gamma_S \) is generated by

\[
J_\mathcal{O}, T_g (g \in \Lambda_0), R_A (A \in \mathcal{O}(\Lambda))
\]

**Proof:** According to (5.6), the lattice \( \Lambda \) attached to \( S \) is euclidean. Therefore, applying propositions 1.19 and 1.22 from [Kl06] yields that \( \Gamma_S \) is generated by

\[
J_\mathcal{O}, T_g (g \in \Lambda_0), P, U_\lambda (\lambda \in \Lambda), R_A (A \in \mathcal{O}(\Lambda))
\]

According to [Kr96, pp.249] (or also [KrO, prop.2]), \( U_\lambda \) and \( R_\lambda \) (see above) can be written as a product of \( J_\mathcal{O} \) and \( T_h \) for certain \( h \in \Lambda_0 \). Furthermore, some straightforward calculations show that \( P = R_{(1,0,...,0,-1)} R_{(0,1,0,...,0)} M_{\mathcal{O}_S} \) holds for all \( S \) we are considering, which completes the proof.

Note that the preceding proposition was just some first, general statement about the generators of the orthogonal modular groups we have to consider. But for further examinations we will need it in much greater detail. So let us start with some arbitrary \( M \in \Gamma_S \), where \( S \) is one of the
matrices listed in 5.1. Going through the proof of [KrO, thm.1] in detail yields that there exist

\[ M_1 \in \langle T_g, I_O ; g \in \Lambda_0 \rangle, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_S, \text{ with } A \in O^+(S_0), \]

where \( O^+(S_0) := \{ A \in O(\Lambda_0) ; A \cdot \mathcal{H}_S = \mathcal{H}_S \} \), such that

\[ M = A_1 M_1. \]

Next, going through the proof of [Kl06, prop.1.22] yields that there exist

\[ N_1, N_2 \in \langle U_{\lambda}, P ; \lambda \in \Lambda \rangle, \quad A \in O(\Lambda) \]

such that

\[ A_1 = N_1 R_A N_2, \]

and thus

\[ M = N_1 R_A N_2 M_1. \]

Just like in the proof of (5.7), \( U_{\lambda} \) and \( R_g \) (where \( \lambda \in \Lambda \) and \( g \in \Lambda_0 \) like above) can be written as a product of \( I_O \) and \( T_h \) for certain \( h \in \Lambda_0 \) (cf. [Kr96, pp.249] or [KrO, prop.2]). And again, some straightforward calculations show that \( P = R_{(0,1,0,...,0)} \cdot (-M^S_m) \). This means we have

\[ N_1, N_2 \in \langle T_g, I_O, -M^S_m ; g \in \Lambda_0 \rangle. \]

Therefore, we verified

\[ (5.8) \text{ Proposition.} \text{ Let } S \text{ be one of the matrices listed in 5.1, and suppose } M \in \Gamma_S. \text{ Then } M \text{ can be written as } \]

\[ M = M_1 R_A M_2, \]

where

\[ M_1, M_2 \in \langle T_g, I_O, -M^S_m ; g \in \Lambda_0 \rangle \text{ and } A \in O(\Lambda), \]

with \( M^S_m \) defined in 5.2.

In particular, this proposition immediately gives rise to the following corollary about the action of \( \Gamma_S \) on \( \text{Dis}(\Lambda_1) \) and about the discriminant kernel:

\[ (5.9) \text{ Corollary.} \text{ Let } S \text{ be one of the matrices listed in 5.1, and suppose } M \in \Gamma_S \text{ with decomposition } M = M_1 R_A M_2 \text{ according to (5.8). Then for } \mu_1 = (0,0,\mu,0,0) + \Lambda_1 \in \text{Dis}(\Lambda_1), \text{ where } \mu \in \Lambda^2, \text{ we have } \]

\[ M \mu_1 = (0,0,A\mu,0,0) + \Lambda_1. \]

In particular, we get

\[ M \in O_d(\Lambda_1) \cap \Gamma_S \iff A \in O_d(\Lambda). \]
Proof: Let \( l \) be the size (and rank) of \( S \). In view of (5.3), given

\[
N_1 = \begin{pmatrix} * & * & * \\ * & A_1 & * \\ * & * & * \end{pmatrix} \in \Gamma_S , \quad N_2 = \begin{pmatrix} * & * & * \\ * & A_2 & * \\ * & * & * \end{pmatrix} \in \Gamma_S , \quad \text{where } A_1, A_2 \in \mathbb{Z}^{l \times l},
\]

we have \( N_1 \mu_1 = (0, 0, A_1 \mu, 0, 0) + \Lambda_1 \), and thus also \((N_1 N_2) \mu_1 = (0, 0, (A_1 A_2) \mu, 0, 0) + \Lambda_1 \). So let us have a closer look at the matrices involved in the decomposition of \( M \). By definition, both that middle block of \( J_O \) and of \( T_g \) (where \( g \in \Lambda_0 \)) we have to consider equal \( I \), and thus \( J_O \) and \( T_g \) act trivially on \( \text{Dis}(\Lambda_1) \). So let us consider \(-M_{tr}^S\) for each of the four cases.

First, let \( S = A_2^{(2)} \). Then, given \( \mu = (x_1, x_2, x_3, x_4) \), we have \(-A_{tr}^S\mu = (-x_1 - x_2, x_2, x_3, x_4)\). So for the generators of \( \text{Dis}(\Lambda_1) \) specified in (5.4) we have modulo \( \Lambda_1 \): \(-A_{tr}^S(\frac{1}{2}, \frac{1}{7}, 0, 0) = (-\frac{7}{2}, 0, 0, 0) \equiv (\frac{1}{2}, \frac{1}{7}, 0, 0) \) and \(-A_{tr}^S(0, 0, \frac{1}{7}, 0) = (0, 0, \frac{1}{7}, \frac{1}{7}) \). Hence, \(-M_{tr}^S\) acts trivially on \( \text{Dis}(\Lambda_1) \), too. Exactly the same calculation yields the according claim for \( S = A_2^{(1)} \), so we skip that and suppose \( S = T_3 \), next. Again, for the generators specified in (5.4), we compute \(-A_{tr}^S(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}) = (-\frac{2}{3}, 0, 0, 0) \equiv (\frac{1}{2}, \frac{1}{3}, \frac{1}{3}) \) and \(-A_{tr}^S(\frac{1}{3}, \frac{1}{2}, \frac{1}{3}) = (-\frac{1}{3}, \frac{1}{2}, 0) \equiv (\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}) \). And finally, for \( S = D_2 \) we get \(-A_{tr}^S(\frac{1}{2}, -\frac{1}{2}) = (\frac{1}{2}, -\frac{1}{2}) \).

Therefore, in all cases

\[
(T_g, J_O, -M_{tr}^S; g \in \Lambda_0) \subset O_d(\Lambda_1) \cap \Gamma_S
\]

holds. Thus, due to the considerations from above, the assertion follows.

\[\square\]

Regarding (5.8), we have to determine generators for \( O(\Lambda) \). Note that the matrices \( S \) defined in 5.1 are positive definite, so we can always find a diagonal matrix \( D = (d_1, \ldots, d_l) > 0 \) satisfying \( S \geq D \) (where \( l \) is the size (and rank) of \( S \)). Therefore, just like in (2.29) we obtain

\[
S[M] - D[M] \geq 0 \Rightarrow \text{tr}(S[M] - D[M]) \geq 0 \\
\Rightarrow \text{tr}(S[M]) \geq \text{tr}(D[M]) = \sum_{j=1}^{l} d_j \left( \sum_{k=1}^{l} m_{jk}^2 \right)
\]

for \( M = (m_{jk}) \in \mathbb{Z}^{l \times l} \). Thus, \( M \in O(\Lambda) \) (which means \( S[M] = S \)) implies

\[
\sum_{j=1}^{l} d_j \left( \sum_{k=1}^{l} m_{jk}^2 \right) \leq \text{tr}(S) .
\]

Of course, there are only finitely many \( M = (m_{jk}) \in \mathbb{Z}^{l \times l} \) fulfilling this condition, hence \( O(\Lambda) \) turns out to be a finite group. In particular, finding some maximal \( d_j \) for each of the \( S \) means that we can determine \( O(\Lambda) \), since we only have to check finitely many matrices. This was done indeed, using [MAGMA] and [SAGE]. (Note that there even already exist implemented procedures to determine the finite orthogonal group for positive definite matrices. So this only has to be done for verification issues.) And once \( O(\Lambda) \) is given as a finite set of matrices, it is also possible to determine generators for \( O(\Lambda) \) as a group utilizing the same computer algebra systems. The same holds for \( O_d(\Lambda) \), since we only have to check on every matrix in the finite set
$O(\Lambda)$. Moreover, we can also determine the commutator subgroup $O(\Lambda)'$ and the commutator factor group $O(\Lambda)^{ab}$.

Therefore, we will not give a manual proof of the next proposition. We only write down the result one obtains using the method described above. $C_n$ stands for the cyclic group of order $n$, here. Note that we will not write down the generators of $O(\Lambda)'$, since this seems unnecessary for further considerations. But we will give the structure of $O(\Lambda)^{ab}$.

(5.10) Proposition.  a) Let $S = A_2^{(2)}$ and

$$A_{tr}^S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_{i_1}^S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_{i_1 \sqrt{3}}^S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$A_{i_2}^S = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_{-i_2,i_2}^S = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$O(\Lambda) = \langle A_{tr}^S, A_{i_1}^S, A_{i_1 \sqrt{3}}^S, A_{i_2}^S \rangle = \langle O_d(\Lambda), A_{i_1}^S, A_{i_2}^S \rangle, \quad O_d(\Lambda) = \langle -A_{tr}^S, -A_{i_1 \sqrt{3}}^S, A_{i_2}^S \rangle, \quad O(\Lambda)^{ab} \simeq C_2 \times C_2 \times C_2.$$

b) Let $S = A_{2,1}$ and

$$A_{i_1}^S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_{i_1}^S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_{i_1 \sqrt{3},i_2}^S = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$O(\Lambda) = \langle A_{tr}^S, A_{i_1}^S, -I, A_{-i_2,i_2}^S \rangle = \langle O_d(\Lambda), -I \rangle, \quad O_d(\Lambda) = \langle -A_{tr}^S, A_{i_1}^S, A_{-i_1 \sqrt{3},i_1 \sqrt{3}}^S \rangle, \quad O(\Lambda)^{ab} \simeq C_2 \times C_2 \times C_2.$$

c) Let $S = T_3$ and

$$A_{tr}^S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_{i_1,i_2}^S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{-i_2,i_2}^S = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$
Then we have
\[
\begin{align*}
O(\Lambda) &= \langle A^S_{tr}, -I, A_{n,j i i} \rangle = \langle O_d(\Lambda), -I, A_{n,j i i} \rangle, \\
O_d(\Lambda) &= \langle -A^S_{tr}, A_{-i j i} \rangle, \\
O(\Lambda)^{ab} &\simeq C_2 \times C_2 \times C_2.
\end{align*}
\]

d) Let \( S = D_2^* \) and
\[
A^S_{tr} = \begin{pmatrix} 1 & 1 \\
0 & -1 \end{pmatrix}.
\]

Then we have
\[
\begin{align*}
O(\Lambda) &= \langle A^S_{tr}, -I \rangle, \\
O_d(\Lambda) &= \langle -A^S_{tr} \rangle, \\
O(\Lambda)^{ab} &\simeq C_2 \times C_2.
\end{align*}
\]

Next, we need to consider the abelian characters of \( \Gamma_S \). There are exactly two kinds of abelian characters we will need. The first one is the determinant, since due to \( S_1[M] = S_1 \) we have \( \det(M) = \pm 1 \) for all \( M \in \Gamma_S \). Note that \( \det(M) = 1 \) holds for all \( M \in \langle T_2, J_0, ; g \in \Lambda_0 \rangle \). Therefore, \( \det(M) \) only depends on the \( R_A \) (with \( \det(R_A) = \det(A) \)) from the generators of \( \Gamma_S \) (see 5.7). We denote this abelian character by
\[
\chi^{S}_{\det} : \Gamma_S \to \{ \pm 1 \}, \quad M \mapsto \det(M).
\]

The second kind of abelian character is given as follows: In virtue of (5.3), \( \Gamma_S \) acts on the sets of elements of \( \text{Dis}(\Lambda_1) \) with the same value of \( \eta_1 \) (in \( \mathbb{Q}/\mathbb{Z} \)), and for
\[
M = \begin{pmatrix} * & * & * \\
* & A & * \\
* & * & * \end{pmatrix} \in \Gamma_S, \quad \text{where} \ A \in \mathbb{Z}^{3 \times 3},
\]
the action only depends on \( A \). In (5.9) we have even seen that for \( M \in \Gamma_S \) with decomposition \( M = M_1 R_A M_2 \) according to (5.8) and for \( \mu_1 = (0, 0, \mu, 0, 0) + \Lambda_1 \in \text{Dis}(\Lambda_1) \), where \( \mu \in \Lambda_S^* \), we have
\[
M \mu_1 = (0, 0, A \mu, 0, 0) + \Lambda_1.
\]

Of course, the signs of the permutations of non-trivial sets of elements of \( \text{Dis}(\Lambda_1) \) with the same value of \( \eta_1 \) are abelian characters, and again these only depend on \( A \), with \( A \) from the decomposition \( M = M_1 R_A M_2 \). Here, \( M_1 \) and \( M_2 \) both belong to the kernel of these abelian characters. Note that for \( A_2^{(2)} \) and \( T_3 \) we have two different such abelian characters, while for \( A_{2,1} \) and \( D_2^* \) only one arises.

So concerning both these kinds of abelian characters we only have to consider their values for \( R_A, A \in O(\Lambda) \) – and even only for the generators of \( O(\Lambda) \) specified in (5.10).

For \( A_2^{(2)} \), the set of elements in \( \text{Dis}(\Lambda) \) of norm \( \frac{1}{3} \) (mod \( \mathbb{Z} \)) is given by \( \{ \pm (\frac{1}{3}, \frac{1}{3}, 0, 0), \pm (0, 0, \frac{1}{3}, \frac{1}{2}) \} \), while the set for norm \( \frac{2}{3} \) is given by \( \{ \pm (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}), \pm (\frac{1}{3}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}) \} \). The abelian character at-
tached to the first set shall be denoted by $\chi^S_{\pi^1}$, the other one $\chi^S_{\pi_2}$. For $R_A$, where $A$ is chosen from the generators of $O(\Lambda)$ (see (5.10)), $\chi^S_{\pi^1}, \chi^S_{\pi_2}$ and $\chi^S_{\det}$ are given by

\[
\begin{array}{cccccc}
A_2^{(2)} & A^S_{\tr} & A^S_{i_1} & A^S_{i_1,3} & A^S_{i_2} \\
\chi^S_{\det} & -1 & 1 & 1 & 1 \\
\chi^S_{\pi^1} & 1 & -1 & 1 & -1 \\
\chi^S_{\pi_2} & 1 & 1 & 1 & -1 \\
\end{array}
\]  

(5.4)

For $A_{2,1}$, there are several sets of a fixed norm (in $\mathbb{Q}/\mathbb{Z}$). $\frac{1}{4}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, 0\right)\}$. $\frac{7}{12}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right)\}$. The sets for norm 0 and $\frac{1}{4}$ are of order 1, and thus irrelevant. And it is obvious that $\Gamma_S$ acts identically on the first two sets. The abelian character attached to them shall be denoted by $\chi^S_{\pi}$. Again, for $R_A$, where $A$ is chosen from the generators of $O(\Lambda)$, the abelian characters $\chi^S_{\pi^1}$ and $\chi^S_{\det}$ are given by

\[
\begin{array}{cccccc}
A_{2,1} & A^S_{\tr} & A^S_{i_1} & -I & A_{-i_1,3, i_1,3} \\
\chi^S_{\det} & 1 & -1 & -1 & 1 \\
\chi^S_{\pi^1} & -1 & 1 & -1 & 1 \\
\chi^S_{\pi_2} & -1 & 1 & -1 & 1 \\
\end{array}
\]  

(5.5)

Next, let us consider $T_3$. Again, there are several sets of a fixed norm (in $\mathbb{Q}/\mathbb{Z}$). We only list those which consist of more than one element. $\frac{7}{3}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$. $\frac{1}{3}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$. $\frac{5}{3}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$. $\frac{7}{12}$: $\{\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\}$. Some calculations yield that the associated abelian characters coincide for the first and the second set, but differ from the one associated to the third one, while the abelian character associated to the fourth set is trivial. Therefore, let us denote the abelian character attached to the first two sets by $\chi^S_{\pi^1}$, while the abelian character attached to the third set shall be denoted by $\chi^S_{\pi_2}$. Then for $R_A$, where $A$ is chosen from the generators of $O(\Lambda)$, the abelian characters $\chi^S_{\pi^1}, \chi^S_{\pi_2}$ and $\chi^S_{\det}$ are given by

\[
\begin{array}{cccc}
T_3 & A^S_{\tr} & -I & A_{i_1, i_1,2} \\
\chi^S_{\det} & 1 & -1 & -1 \\
\chi^S_{\pi^1} & -1 & -1 & 1 \\
\chi^S_{\pi_2} & -1 & -1 & -1 \\
\end{array}
\]  

(5.6)

And finally, let us have a look at $D_3^*$. We have the following sets of fixed norm: $\frac{1}{2}$: $\{\pm\left(\frac{1}{2}, -\frac{3}{2}\right)\}$. $\frac{2}{3}$: $\{\pm\left(\frac{1}{2}, -\frac{1}{3}\right)\}$. $\frac{4}{3}$: $\{\pm\left(\frac{1}{2}, -\frac{4}{3}\right)\}$. $\Gamma_S$ acts on these in completely the same way, hence only one abelian character arises that way, and it shall be denoted by $\chi^S_{\pi}$ again. And once more, for $R_A$ with $A$ from the generators of $O(\Lambda)$ the abelian characters $\chi^S_{\pi^1}$ and $\chi^S_{\det}$ are given by

\[
\begin{array}{cccc}
D_3^* & A^S_{\tr} & -I & A_{i_1, i_1,2} \\
\chi^S_{\det} & -1 & 1 & -1 \\
\chi^S_{\pi^1} & -1 & -1 & -1 \\
\end{array}
\]  

(5.7)

In the next proposition we will see that we actually found all abelian characters. Note that the proof is mostly based on the ideas already found in [Kl06, prop.1.25], but some parts are new for the lattices we have to consider.
(5.11) Proposition. If $S \in \{A_2^{(2)}, T_3\}$ then $\Gamma_{S}^{ab} \simeq C_2 \times C_2 \times C_2$, while for $S \in \{A_{2,1}, D_2^1\}$ we have $\Gamma_{S}^{ab} \simeq C_2 \times C_2$. In all of the four cases, $\Gamma_{S}^{ab}$ is generated by the abelian characters described in 5.4, 5.5, 5.6 and 5.7.

Proof: First, we will determine an upper bound on $[\Gamma_S : \Gamma_S']$ for all of the four cases we have to consider. The first steps are identical for all $S$ we have to consider. So let $S$ be one of the matrices described in 5.1. $l$ shall denote the size (and rank) of $S$.

A fact from algebra is that commutator factor groups are abelian, hence also $\Gamma_S / \Gamma_S'$. So let $M \in \Gamma_S$. According to (5.7), $M$ can be written as a (finite) product of $J_O$, certain $T_g$ (with $g \in \Lambda_0$), several $R_A$ (with $A \in O(\Lambda)$) and the inverses of these matrices. Note that one easily verifies $T_g T_h = T_{g+h}$ for all $g,h \in \Lambda_0$ and $R_A R_B = R_{AB}$ for all $A,B \in O(\Lambda)$. Furthermore, we have $J_O^3 = I$. Because of these considerations, there exist $\varepsilon \in \{0,1\}$, $g \in \Lambda_0$ and $A \in O(\Lambda)$ such that

$$M \Gamma'_S = R_A J_O^\varepsilon T_g \Gamma'_S,$$

or in other words $M \equiv R_A J_O^\varepsilon T_g$ modulo $\Gamma'_S$.

So let us construct some elements belonging to the commutator subgroup $\Gamma'_S$. Just like in [Kl06, prop.1.25], a straightforward calculation yields

$$[U_{\lambda}, T_g] = T_{\lambda S \tilde{g} + \eta(\lambda) \xi_{1+1}, \lambda \xi_{1+1}, 0}$$

for $\lambda \in \Lambda$ and $g = (\xi_0, \tilde{g}, \xi_{1+1}) \in \Lambda_0$ (with $\tilde{g} \in \Lambda$). Thus for the standard basis $(e_1, \ldots, e_l)$ of $\Lambda$ we obtain

$$[U_{\lambda}, T_{(0, \ldots, 0, 1)}] = T_{(\eta(\xi_j), 1, 0, 0)} = \begin{cases} T_{(2, 0, 1, 0, 0)}, & \text{if } S = T_3 \text{ and } j = 2, \\ T_{(2, 0, 1, 0)}, & \text{if } S = D_2^1 \text{ and } j = 2, \\ T_{(1, e_j, 0)}, & \text{otherwise}, \end{cases}$$

and

$$[U_{\xi_1}, T_{(0, \ldots, 0, 2)}] = T_{(\xi_j S_{e_j} \xi_{e_j}, 0, \ldots, 0)} = T_{(S_{e_j} \xi_{e_j}, 0, \ldots, 0)} = T_{(1, \ldots, 0)}.$$ 

Another straightforward calculation gives

$$[R_{(0, e_1, 0), T_{(0, \ldots, 0, 1)}}] = T_{(1, 0, \ldots, 0, -1)}$$

in all cases. Due to these matrices belonging to $\Gamma'_S$ (as well as powers of them and their inverses), every $T_h$, where $h \in \Lambda_0$, is congruent to $I = T_0$ modulo $\Gamma'_S$. Or in other words, we have $T_h \in \Gamma'_S$ for all $h \in \Lambda_0$. A further computation shows that $(J_O T_{(1, 0, \ldots, 0, 1)})^3 = 1$ holds, and thus (again due to the fact that the commutator factor group is abelian) $J_O^3 = T_{(-3, 0, \ldots, 0, -3)} \in \Gamma'_S$ holds modulo $\Gamma'_S$. Hence $J_O = J_O^3 \in \Gamma'_S$. This means that for the $M \in \Gamma_S$ we fixed at the beginning we obtain

$$M \Gamma'_S = R_A \Gamma'_S.$$ 

And finally, since $R_B R_C = R_{BC}$ for all $B, C \in O(\Lambda)$, we have

$$R_B \in \Gamma'_S \text{ for all } B \in O(\Lambda)'.$$
Therefore, we immediately obtain
\[ [\Gamma_S : \Gamma'_S] \leq 8 \quad \text{for} \ S \in \{ A_2^{(2)}, A_{2,1}, T_3 \} \quad \text{and} \quad [\Gamma_{D_2^S} : \Gamma'_{D_2^S}] \leq 4 \]
in virtue of (5.10). So the assertion follows for all \( S \) except \( A_{2,1} \), since above we saw that there exist at least eight abelian characters for \( S \in \{ A_2^{(2)}, T_3 \} \), and four for \( S = D_2^S \). Hence, only the case of \( S = A_{2,1} \) remains to be considered: We just verified \([\Gamma_S : \Gamma'_S] \leq 8\) so far, but only found four abelian characters in 5.5. Of course, \([\Gamma_S : \Gamma'_S] \) has to be a multiple of 4 regarding 5.5, thus \([\Gamma_S : \Gamma'_S] \in \{4,8\}\). We compute
\[ R_{(0,\pm 1,0)} R_{A_6^S} = R_{(0,\pm 3,0)} R_{-A_1^S}. \]

Once more, note that according to [Kr96, pp.249] every \( R_8 \) (with \( g \in \Lambda_0 \), such that \( q_0(g) = \pm 1 \)) can be written as a product of \( J_0 \) and \( T_h \) for certain \( h \in \Lambda_0 \), hence \( R_8 \in \Gamma'_S \) follows. Noting \((-A_1^S)^{-1} = -A_1^S\), this leads to
\[ R_{A_6^S} R_{-A_1^S} \in \Gamma'_S, \]
where
\[ A := -A_1^S A_1^S = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

Like we mentioned right above (5.10), one can determine \( O(\Lambda)' \) explicitly, for example using [MAGMA]. Doing so, we obtain
\[ O(\Lambda)' = \begin{cases} I, \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}, \]
and thus \( A \notin O(\Lambda)' \). So due to the considerations above, this immediately implies \([\Gamma_S : \Gamma'_S] \leq [O(\Lambda) : O(\Lambda)'] = 8\), and thus \([\Gamma_S : \Gamma'_S] = 4\). Hence the assertion follows for \( S = A_{2,1} \), too. \( \square \)

Further below, we will see that \( \Gamma_S \) for \( S = A_2^{(2)} \) can be identified with the extended quaternionic modular group, or to be more precise \( \Gamma_S / \{ \pm I \} \cong \Gamma(O) \) (where one should note that also in \( \Gamma(O) \) we have \(-I = I\), since \((-I)(Z) = I(Z)\)). But like we have seen, there exist no quaternionic modular forms of odd weight for the whole extended quaternionic modular group. For example, Maass lifts of odd weight are quaternionic modular forms for \( \text{Sp}_2(O) \) and they also possess a transformation behavior for \( Z \mapsto Z' \), but not for \( Z \mapsto Z[i_1 I] \). And we will get similar results for certain Borcherds products, where certain automorphisms have to be excluded. But still, we want to identify quaternionic modular froms for these subgroups with orthogonal modular forms with respect to certain subgroups.

This is why we need to consider some further groups, now. To be more precise two subgroups of \( \Gamma_S \) of index two, where \( S = A_2^{(2)} \). But first, let us fix some notation: Suppose \( \Delta \) is some subgroup of \( \Gamma_S \), and let \( \chi \) be an abelian character for \( \Delta \). Of course, \( \bar{\Delta} := \{ A \in O(\Lambda) ; R_A \in \Delta \} \)
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is a subgroup of $O(\Lambda)$, and

$$\bar{\chi} : \tilde{\Delta} \to \mathbb{C}, \quad A \mapsto \chi(R_A)$$  \hspace{1cm} (5.8)

is an abelian character of $\tilde{\Delta}$. Next, define

$$\chi_{\text{Maaß}} := \chi_{\tilde{\Delta}}^{\mathcal{S}} \chi_{\tilde{\Delta}}^{\mathcal{S}}, \quad \chi_{\text{Bor}} := \chi_{\tilde{\Delta}}^{\mathcal{S}}.$$  \hspace{1cm} (5.9)

With the help of these characters we define subgroups

$$\Gamma_{\text{Maaß}} := \ker(\chi_{\text{Maaß}}), \quad \Gamma_{\text{Bor}} := \ker(\chi_{\text{Bor}})$$  \hspace{1cm} (5.10)

of $\Gamma_{\tilde{\Delta}}$ and subgroups

$$O(\Lambda)_{\text{Maaß}} := \{ A \in O(\Lambda) ; R_A \in \Gamma_{\text{Maaß}} \}, \quad O(\Lambda)_{\text{Bor}} := \{ A \in O(\Lambda) ; R_A \in \Gamma_{\text{Bor}} \}$$  \hspace{1cm} (5.11)

of $O(\Lambda)$. Note that it is a fact from linear algebra that $[\Gamma_{\tilde{\Delta}} : \Gamma_{\text{Maaß}}] = [\Gamma_{\tilde{\Delta}} : \Gamma_{\text{Bor}}] = 2$ holds, because $\chi_{\text{Maaß}}(\Gamma_{\tilde{\Delta}}) = \chi_{\text{Bor}}(\Gamma_{\tilde{\Delta}}) = \{ \pm 1 \}$. Furthermore, one easily verifies $-M_{tr}^{\mathcal{S}} \in \Gamma_{\text{Maaß}}$ as well as $-M_{tr}^{\mathcal{S}} \in \Gamma_{\text{Bor}}$, hence

$$\langle T_g, J_{O}, -M_{tr}^{\mathcal{S}} ; g \in \Lambda_0 \rangle \subset \Delta \quad \text{for } \Delta \in \{ \Gamma_{\text{Maaß}}, \Gamma_{\text{Bor}} \}.$$  

Therefore, (5.8) immediately leads to

**Proposition.** Let $M \in \Gamma_{\tilde{\Delta}}$ with decomposition $M = M_1 R_A M_2$ according to (5.8). Using the notations from above, we have

$$M \in \Gamma_{\text{Maaß}} \iff A \in \ker(\bar{\chi}_{\text{Maaß}}) = O(\Lambda)_{\text{Maaß}}$$

and

$$M \in \Gamma_{\text{Bor}} \iff A \in \ker(\bar{\chi}_{\text{Bor}}) = O(\Lambda)_{\text{Bor}}.$$  

In particular, every $N \in \Gamma_{\text{Maaß}}$ (or $N \in \Gamma_{\text{Bor}}$) can be written as

$$N = N_1 R_B N_2,$$

where

$$N_1, N_2 \in \langle T_g, J_{O}, -M_{tr}^{\mathcal{S}} ; g \in \Lambda_0 \rangle, \quad B \in O(\Lambda)_{\text{Maaß}} \quad \text{(or } B \in O(\Lambda)_{\text{Bor}} \text{)}.$$  

Moreover, noting $-I = \text{diag}(-1,-1,\ldots,1,-1,-1) \cdot R_{-I} \in \Delta$ (where $\Delta \in \{ \Gamma_{\text{Maaß}}, \Gamma_{\text{Bor}} \}$), with $\text{diag}(-1,-1,\ldots,1,-1,-1)$ being a product of $J_{O}$ and certain $T_g$ ($g \in \Lambda_0$) in view of [Kr96, pp.249] (or [KrO, prop.2]), and $R_{-I} \in \Delta$ as well as $M_{tr}^{\mathcal{S}} \in \Delta$,

$$\Gamma_{\text{Maaß}} = \langle J_{O}, T_g, R_A ; g \in \Lambda_0, \ A \in O(\Lambda)_{\text{Maaß}} \rangle,$$

$$\Gamma_{\text{Bor}} = \langle J_{O}, T_g, R_A ; g \in \Lambda_0, \ A \in O(\Lambda)_{\text{Bor}} \rangle$$  \hspace{1cm} (5.12)

immediately follows due to the preceding proposition.

Again using [MAGMA], we obtain the following properties of $O(\Lambda)_{\text{Maaß}}$ and $O(\Lambda)_{\text{Bor}}$:
(5.13) Proposition. Let \( S = A_2^{(2)} \). Using the notation from above and from (5.10) we have

\[
\begin{align*}
O(\Lambda)_{\text{Maaß}} &= \langle A_{\text{tr}}^S, A_{i_1 \sqrt{3}}^S, A_{i_2}^S \rangle, \\
O(\Lambda)_{\text{Bor}} &= \langle A_{\text{tr}}^S, A_{i_1 \sqrt{3}}^S, A_{-i_2,i_1}^S, A_{i_1}^S \cdot A_{i_2}^S \rangle, \\
O(\Lambda)_{\text{Maaß}}^{\text{ab}} &\simeq C_4 \times C_2, \\
O(\Lambda)_{\text{Bor}}^{\text{ab}} &\simeq C_2 \times C_2 \times C_2.
\end{align*}
\]

Like for the other groups, we have to determine all abelian characters. Of course, \( \chi^S_{\text{det}} \) is an abelian character for both \( \Gamma_{\text{Maaß}} \) and \( \Gamma_{\text{Bor}} \), again. And just like for the other groups, abelian characters will arise from the action of \( \Gamma_{\text{Maaß}} \) and \( \Gamma_{\text{Bor}} \) on \( \text{Dis}(\Lambda_1) \), since both groups are subgroups of \( \Gamma_S \).

Let us consider \( \Gamma_{\text{Maaß}} \) first. We have a closer look at the operation on the set of \( \text{Dis}(\Lambda) \) with elements of norm \( \frac{1}{2} \): \( (\frac{1}{2}, \frac{1}{3}, 0, 0), (-\frac{1}{3}, -\frac{1}{3}, 0, 0), (0, 0, \frac{1}{3}, \frac{1}{3}), (0, 0, -\frac{1}{3}, -\frac{1}{3}) \). Again, the action of \( M \in \Gamma_{\text{Maaß}} \) only depends on \( A \) in the decomposition \( M = M_1 R_A M_2 \) from (5.12). And of course, the action of \( \Gamma_{\text{Maaß}} \) is completely determined by the action of \( A \) on the tuple above, where \( A \) runs through the set of generators specified in (5.13). If the elements in the tuple are numerated by 1 to 4, then the action of the generators is given by the following permutations:

\[
\begin{align*}
A_{\text{tr}}^S : (1, 2)(3, 4), & \quad A_{i_1 \sqrt{3}}^S : (1, 2)(3, 4), & \quad A_{i_2}^S : (1, 3, 2, 4)
\end{align*}
\]

This gives rise to a group homomorphism

\[
\Gamma_{\text{Maaß}} \to \langle (1, 3, 2, 4) \rangle \simeq C_4,
\]

given by the action of \( \Gamma_{\text{Maaß}} \) on the tuple from above. Obviously, \( C_4 \) possesses four abelian characters, and the group of abelian characters is isomorphic to \( C_4 \), generated by the abelian character induced by \( (1, 3, 2, 4) \mapsto i \). Of course, we obtain four abelian characters for \( \Gamma_{\text{Maaß}} \) that way. Let the character corresponding to \( (1, 3, 2, 4) \mapsto i \) be denoted by \( \chi^S_i \). Then for \( R_A \), where \( A \) is chosen from the generators of \( O(\Lambda)_{\text{Maaß}}^{\text{ab}} \), the abelian characters \( \chi^S_i \) and \( \chi^S_{\text{det}} \) are given by

\[
\begin{array}{c|ccc}
\chi^S_{\text{det}} & A_{\text{tr}}^S & A_{i_1 \sqrt{3}}^S & A_{i_2}^S \\
\hline
\chi^S_i & -1 & 1 & 1 \\
\end{array}
\]

(5.13)

Note that \( \chi^S_i \) is of order four.

Next, let us have a look at \( \Gamma_{\text{Bor}} \). This time, we consider the operation on the set of \( \text{Dis}(\Lambda) \) with elements of norm \( \frac{3}{2} \): \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Here, the action of the generators is given by the following permutations:

\[
\begin{align*}
A_{\text{tr}}^S : (1, 2)(3, 4), & \quad A_{i_1 \sqrt{3}}^S : (1, 2)(3, 4), & \quad A_{-i_2, i_1}^S : \text{id}, & \quad A_{i_1}^S, A_{i_2}^S : (1, 2)
\end{align*}
\]

So in this case, this gives rise to a group homomorphism

\[
\Gamma_{\text{Bor}} \to \langle (1, 2)(3, 4), (1, 2) \rangle \simeq C_2 \times C_2,
\]
and two characters of order two arise. The one corresponding to \((1,2)(3,4) \mapsto -1, (1,2) \mapsto 1\) shall be denoted by \(\chi_{i_1 \sqrt{3}}^S\) and the one corresponding to \((1,2)(3,4) \mapsto 1, (1,2) \mapsto -1\) shall be denoted by \(\chi_{i_2}^S\). For \(R_A\), where \(A\) is chosen from the generators of \(O(\Lambda)_{\text{Bor}}\), the abelian characters \(\chi_{i_1 \sqrt{3}}, \chi_{i_2}\) and \(\chi_{\det}^S\) are given by

\[
\begin{array}{cccc}
\Gamma_{\text{Bor}} & A^S_{\text{tr}} & A^S_{i_1 \sqrt{3}} & A^S_{-i_2, i_2} & A^S_{i_1} \cdot A^S_{i_2} \\
\chi_{\det}^S & -1 & 1 & 1 & 1 \\
\chi_{i_1 \sqrt{3}}^S & -1 & -1 & 1 & 1 \\
\chi_{i_2}^S & 1 & 1 & 1 & -1 \\
\end{array}
\]

(5.14)

Just like for the other groups, we found all abelian characters:

**Proposition.** We have \(\Gamma_{\text{Maas}}^{ab} \simeq C_4 \times C_2\), while \(\Gamma_{\text{Bor}}^{ab} \simeq C_2 \times C_2 \times C_2\). In both cases, the group of abelian characters is generated by the abelian characters described in 5.13 and 5.14, respectively.

**Proof:** Let \(S = A_2^{(2)}\). We already mentioned that \(\langle T_g, J_0, -M^S_{\text{tr}} : g \in \Lambda_0 \rangle \subset \Delta\) holds for \(\Delta \in \{\Gamma_{\text{Maas}}, \Gamma_{\text{Bor}}\}\). Hence according to [Kr96, pp.249] (or [KrO, prop.2]) we also have \(U_\lambda \in \Delta\) and \(R_\lambda \in \Delta\) for all \(\lambda \in \Lambda_0\) and \(g \in \Lambda_0\) with \(q_0(g) = \pm 1\). Now, note that we only needed these matrices (and of course \(T_g\) and \(J_0\)) in the proof of (5.11) to verify \(J_0 \in \Gamma_\lambda^\prime\) and \(T_g \in \Gamma_\lambda^\prime\) for all \(g \in \Lambda_0\). Hence we also obtain \(J_0 \in \Delta^\prime\) and \(T_g \in \Delta^\prime\) for all \(g \in \Lambda_0\), this time. Therefore, due to the same argument like in (5.11) we obtain

\[
[\Gamma_{\text{Maas}} : \Gamma_{\text{Maas}}^\prime] \leq [O(\Lambda)_{\text{Maas}} : O(\Lambda)_{\text{Maas}}^\prime] = 8 \quad \text{and} \\
[\Gamma_{\text{Bor}} : \Gamma_{\text{Bor}}^\prime] \leq [O(\Lambda)_{\text{Bor}} : O(\Lambda)_{\text{Bor}}^\prime] = 8
\]

in virtue of (5.13), hence the assertion follows. \(\square\)

### 5.2 Introduction to orthogonal modular forms

Now that the background concerning the orthogonal modular groups has been worked out, we can finally define orthogonal modular forms. Like for the definition of quaternionic modular forms, we will need automorphy factors and multiplier systems. But since no roots will occur this time, the multiplier systems will be given by the abelian characters of the orthogonal modular forms.

So let \(S\) be an even positive definite matrix of size (and rank) \(l\). (Note that the matrices defined in 5.1 fulfill this condition). Then (according to [Kl06])

\[
j_0 : O^+(S; \mathbb{R}) \times \mathcal{H}_S \to \mathbb{C}^*, \quad (M, w) \mapsto M\{w\}
\]

(5.15)

is an automorphy factor, i.e. \(j_0(M, \cdot)\) is holomorphic for all \(M \in O^+(S; \mathbb{R})\), and \(j_0\) satisfies the so-called cocycle relation

\[
j_0(M_1M_2, w) = j_0(M_1, M_2\{w\}) \cdot j_0(M_2, w)
\]

(5.16)
for all \( M_1, M_2 \in O^+(S_1; \mathbb{R}) \) and all \( w \in \mathcal{H}_S \). Next, completely analogous to the symplectic setting, given \( f : \mathcal{H}_S \to \mathbb{C}, M \in O^+(S_1; \mathbb{R}) \) and \( k \in \mathbb{Z} \), we define the so-called slash-operator by
\[
f|_k M(w) := j_\Omega(M, w)^{-k} f(M\langle w \rangle) = (M\{w\})^{-k} f(M\langle w \rangle) \quad \text{for all } w \in \mathcal{H}_S.
\]
Of course, \( f|_k M : \mathcal{H}_S \to \mathbb{C} \) is holomorphic, again, and due to the cocycle relation and (5.3) we have
\[
(f|_k M_1)|_k M_2 = f|_k (M_1 M_2)
\]
for all \( M_1, M_2 \in O^+(S_1; \mathbb{R}) \). Keeping this notation, we have the following definition of orthogonal modular forms:

(5.15) Definition. Let \( k \in \mathbb{Z}, \Gamma \) a subgroup of \( \Gamma_S \) of finite index and \( \chi \in \Gamma^\text{ab} \) an abelian character of \( \Gamma \) of finite order. A holomorphic function \( f : \mathcal{H}_S \to \mathbb{C} \) is called an orthogonal modular form of weight \( k \) with respect to \( \Gamma \) and \( \chi \) if it satisfies
\[
f|_k M = \chi(M) \cdot f \quad \text{for all } M \in \Gamma.
\]
We denote the vector space of orthogonal modular forms of weight \( k \) with respect to \( \Gamma \) and \( \chi \) by \([\Gamma, k, \chi]\).

Note that there should occur no confusion concerning the notation of the spaces of orthogonal modular forms and quaternionic modular forms (or “symplectic” modular forms in general), since the group will indicate which ones we are speaking of.

Obviously,
\[
f \cdot g \in [\Gamma, k_1 + k_2, \chi_1 \chi_2]
\]
holds for \( f \in [\Gamma, k_1, \chi_1] \) and \( g \in [\Gamma, k_2, \chi_2] \), again.

Next, in order to give some basic facts about orthogonal modular forms, we need the following definition:

(5.16) Definition. For \( a \in \mathbb{R}^{l+2} \) we write \( a > 0 \), if \( a \) belongs to \( \mathcal{P}_S \) (see (5.3)), and we write \( a \geq 0 \), if \( a \) belongs to the closure
\[
\overline{\mathcal{P}}_S = \{ v = (v_0, \ldots, v_{l+1}) \in \mathbb{R}^{l+2}; q_0(v) \geq 0, v_0 \geq 0 \}
\]
of \( \mathcal{P}_S \). Moreover, given \( a, b \in \mathbb{R}^{l+2} \) we define as usual
\[
a > b \iff a - b > 0,
\]
\[
a \geq b \iff a - b \geq 0.
\]
Note that if \( u, v \in \mathbb{R}^{l+2} \) with \( u \geq 0, v > 0 \) and \( u \neq 0 \), then we have (cf. [Kl06, Prop.2.7])
\[
u^\prime S_0 v > 0.
\]

Now, we cite the following facts from [Kl06, pp.29] about orthogonal modular forms:
(5.17) **Proposition.** Let $S$ be an even positive definite matrix of size (and rank) $l$, $k \in \mathbb{Z}$, $\Gamma$ a subgroup of $\Gamma_S$ of finite index and $\chi \in \Gamma_{ab}$ an abelian character of $\Gamma$ of finite order.

a) If $-I \in \Gamma$ and $\chi(-I) \neq (-1)^k$ then $[\Gamma, k, \chi] = \{0\}$.

b) Suppose $\chi$ is of order $h \in \mathbb{N}$. Then

$$[\Gamma_S, 0, \chi] = \begin{cases} \mathbb{C}, & \text{if } \chi \equiv 1, \\ \{0\}, & \text{if } \chi \not\equiv 1, \end{cases}$$

and

$$[\Gamma_S, k, \chi] = \{0\}, \quad \text{if } k < \frac{l}{2h}, \ k \neq 0.$$

c) Each $f \in [\Gamma, k, \chi]$ possesses an absolutely and locally uniformly convergent Fourier-expansion of the form

$$f(w) = \sum_{\mu \in \Lambda_0^z} \alpha_f(\mu) e^{2\pi i \mu' S_0 w/r}, \quad w \in \mathcal{H}_S$$

for some $r \in \mathbb{N}$, which only depends on $\Gamma$ and the order of $\chi$. In particular, if $T_S \in \Gamma$ holds for all $g \in \Lambda_0$ and if the order of $\chi$ is $h \in \mathbb{N}$, then

$$f(w) = \sum_{\mu \in \Lambda_0, \mu \geq 0} \alpha_f(\mu) e^{2\pi i \mu' S_0 w/h}, \quad w \in \mathcal{H}_S,$$

where the condition $\mu \geq 0$ is due to the Koecher’s principle in the orthogonal setting (cf. [Kl06, thm.2.8]). Moreover, if the conditions from above hold and given $\beta > 0$, then $f$ is bounded in the domain $\{w = u + iv \in \mathcal{H}_S : v \geq (\beta, 0, \ldots, 0, \beta)'\}$, and its Fourier-expansion converges uniformly in this domain. Furthermore (even in the general setting), if $\tilde{M} \in \text{O}^+(\Lambda_0)$ such that $\tilde{M} = \text{diag}(1, \tilde{M}, 1) \in \Gamma$, then we have

$$\alpha_f(\tilde{M}\mu) = \chi(M) \cdot \alpha_f(\mu) \quad \text{for all } \mu \in \Lambda_0^z.$$

Applying part a) of the preceding proposition to the orthogonal groups we are considering, and going through the abelian characters we determined, we obtain the following

(5.18) **Corollary.** Let $k, l, m, n \in \mathbb{Z}$. We omit the index “$S$” for the abelian characters in the following assertions:

a) $[\Gamma_{A_2^2}, k, \chi] = \{0\}$ for all $\chi \in \Gamma_{ab}^2$ if $k$ is odd.

b) $[\Gamma_{Maaß}, k, \chi_{det}^l \chi_m^l] = \{0\}$ if $k + m \equiv 1 \mod 2$.

c) $[\Gamma_{Bor}, k, \chi_{det}^l \chi_{i_1 i_2}^m \chi_{n_1}^n \chi_{n_2}^n] = \{0\}$ if $k + m \equiv 1 \mod 2$.

d) $[\Gamma_{A_2^2}, k, \chi_{det}^l \chi_{n_1}^m \chi_{l_1}^n] = \{0\}$ if $k + l + m \equiv 1 \mod 2$.

e) $[\Gamma_{T_2}, k, \chi_{det}^l \chi_{n_1}^m \chi_{n_2}^m \chi_{n_2}^n] = \{0\}$ if $k + l + m + n \equiv 1 \mod 2$. 
Next, note that just like in the symplectic setting, the existence and shape of the Fourier-expansion is due to $f|_k T_g(w) = f(w + g)$. This means the factor $r^{-1}$ in \( e^{2\pi i \mu' S_{w/r}w} \) comes from not all $T_g$ being elements of $\Gamma$ or because $\chi(T_g) = 1$ does not hold for all $T_g$. But in the cases we have to consider, $T_g \in \Gamma$ and $\chi(T_g) = 1$ holds for all $g \in \Lambda_0$ and all abelian characters $\chi$. Hence we obtain:

\[(5.19) \text{ Corollary.} \text{ Let } \Gamma \in \{ \Gamma_S, \Gamma_{\text{Maaß}}, \Gamma_{\text{Bor}} \}; \ S \in \{ A_2^{(2)}, A_{2,1}, T_3, D_2^* \}, \ \chi \in \Gamma^{\text{ab}} \text{ and } k \in \mathbb{N}_0. \text{ Then each } f \in [\Gamma, k, \chi] \text{ possesses an absolutely and locally uniformly convergent Fourier-expansion of the form}
\]
\[f(w) = \sum_{\mu \in \Lambda_0^* \mu \geq 0} a_f(\mu) e^{2\pi i \mu' S_{w/r}w}, \ w \in H_S.
\]

Just like in the symplectic setting, we can define orthogonal cusp forms. But since we are only interested in those groups we have been considering so far, we will not give a definition in a general sense.

\[(5.20) \text{ Definition.} \text{ Let } \Gamma \in \{ \Gamma_S, \Gamma_{\text{Maaß}}, \Gamma_{\text{Bor}} \}; \ S \in \{ A_2^{(2)}, A_{2,1}, T_3, D_2^* \}, \ \chi \in \Gamma^{\text{ab}} \text{ and } k \in \mathbb{N}_0. \text{ Then an orthogonal modular from } f \in [\Gamma, k, \chi] \text{ with Fourier-expansion}
\]
\[f(w) = \sum_{\mu \in \Lambda_0^* \mu \geq 0} a_f(\mu) e^{2\pi i \mu' S_{w/r}w}, \ w \in H_S.
\]
is called an orthogonal cusp from if $a_f(\mu) \neq 0$ implies $\mu > 0$. We denote the space of cusp forms in $[\Gamma, k, \chi]$ by $[\Gamma, k, \chi]_0$.

\[\text{Remark.} \text{ Note that the definition of cusp forms actually makes sense. In general, we would have to demand that not only } f, \text{ but also } f|_k M \text{ for all } M \in \Gamma_S \text{ possesses such a special Fourier-expansion. Of course, this can be omitted for } \Gamma = \Gamma_S, \text{ since } f|_k M \text{ is just a multiple of } f, \text{ then. But the question arises if this single condition also suffices for } \Gamma \in \{ \Gamma_{\text{Maaß}}, \Gamma_{\text{Bor}} \}. \text{ But note that in this case we have } [\Gamma_S : \Gamma] = 2, \text{ with}
\]
\[\Gamma_S/\Gamma_{\text{Maaß}} = \Gamma_S/\Gamma_{\text{Bor}} = \{ 1, R_{A_1^*} \}
\]
by construction. Therefore, we only have to check whether the condition concerning the Fourier-expansion holds for $f|_k R_{A_1^*}$ if it already holds for $f$. In general, we compute for $A \in O(\Lambda)$ and $w = (\tau_1, z, \tau_2)$, where $\mu \in \Lambda_0^*$ shall be decomposed as $\mu = (\mu_0, \tilde{\mu}', \mu_{t+1})'$, with $\tilde{\mu} \in \Lambda^*$:

\[f|_k R_A(w) = f(R_A(w)) = f((\tau_1, A z, \tau_2)) = \sum_{\mu \in \Lambda_0^*} a_f(\mu) e^{2\pi i(\mu_0 A^{-1} \tilde{\mu}' + \mu_{t+1}) S_{0}(\tau_1, z, \tau_2)}
\]
\[= \sum_{\mu \in \Lambda_0^3} \alpha_f((\mu_0, (A\mu)\lambda, \mu_1+1)') e^{2\pi i p' S_0 \mu} \cdot \]

Of course, \(\mu_0 > 0\) and \(\mu_0\mu_{l+1} > q(\tilde{\mu})\) holds if and only if \(\mu_0 > 0\) and \(\mu_0\mu_{l+1} > q(A\mu)\), since \(A \in O(\Lambda)\). Therefore, it suffices to check whether \(f\) fulfills the condition concerning the Fourier-expansion, only.

Alternatively, as long as \(\Lambda\) is euclidean, we can define cusp forms via Siegel’s \(\Phi\)-operator: Let the setting be given like in (5.20). Then just like in [Kl06, prop.2.10], due to (5.19) and because \(\chi(Tg) = \chi(JO) = 1\) for all \(g \in \Lambda_0\), the map

\[
\Phi_O: \Gamma, k, \chi \rightarrow SL_2(\mathbb{Z}), f \mapsto f|_{\Phi_O} \\
(f|_{\Phi_O})(\tau) := \lim_{y \rightarrow \infty} f(iy, 0, \tau) \quad \text{for } \tau \in \mathcal{H}
\]

is a homomorphism. And combining [Kl06, prop.1.22], the proof of [Kl06, prop.2.10] and the fact that \(P\) and \(U_\lambda\) (with \(\lambda \in \Lambda\)) are also elements of \(\Gamma_{\text{Maaß}}\) and \(\Gamma_{\text{Bor}}\), we obtain:

**5.21 Proposition.** Suppose the setting in (5.20) is given. Then we have

\[ [\Gamma, k, \chi]_0 = \ker(\Phi_O) \cdot \]

Furthermore, exactly the same arguments like in the proof of [Kl06, cor.2.11] yield:

**5.22 Corollary.** Again, suppose the setting in (5.20) is given. If \(k\) is odd or \(k = 2\) or \(\chi \neq 1\), then

\[ [\Gamma, k, \chi] = [\Gamma, k, \chi]_0 \cdot \]

A final topic in this introductory section on the orthogonal setting shall be the restriction of orthogonal modular forms to submanifolds. So let \(S\) and \(T\) be some even positive definite matrices, and suppose the rank of \(S\) is \(l \geq 2\), while the rank of \(T\) is \(l' < l\). The lattice attached to \(S\) shall be denoted by \(\Lambda (= \mathbb{Z}^l)\), and the one attached to \(T\) by \(\Lambda_T (= \mathbb{Z}^{l'})\). Furthermore, suppose there exists an isometric embedding

\[i_T^S: \Lambda_T \rightarrow \Lambda, \quad \lambda_T \mapsto M_T^S \lambda_T \]

for some matrix \(M_T^S \in \mathbb{Z}^{l \times l'}\), which means that \(M_T^S\) has full rank and we have

\[(a, b)_T = (i_T^S(a), i_T^S(b))_S \]

for all \(a, b \in \Lambda_T\). Of course, this induces an embedding of \(\mathcal{H}_T\) into \(\mathcal{H}_S\), and by abuse of notation, this embedding shall be denoted by \(i_T^S\) again (and the same concerning \(\Lambda_{T,0}\) and \(\Lambda_{T,1}\)). Note that those elements of \(\Gamma_S\) stabilizing the embedded lattice \(i_T^S(\Lambda_{T,1})\) can be viewed as elements of \(\Gamma_T\), and this yields a homomorphism

\[ \varphi_T : \text{Stab}_{\Gamma_S}(i_T^S(\Lambda_{T,1})) \rightarrow \Gamma_T. \]
Of course, a priori it is not clear (but necessary for further analysis) whether \( \varphi_T \) is surjective. But if both \( \Gamma_S \) and \( \Gamma_T \) are generated by the matrices \( J_0, T_g \) (\( g \in \Lambda_0 \), or \( g \in \Lambda_{T,0} \), resp.) and \( R_A \) (with \( A \in O(\Lambda) \) or \( A \in O(\Lambda_T) \), resp.), then according to [FH00, sec.4] it suffices to check whether

\[
\varphi_T : \text{Stab}_{O(\Lambda)}(I_T^S(\Lambda_T)) \to O(\Lambda_T)
\]

is surjective. Note that this condition can easily be checked for the cases we are interested in using some computer algebra system ([MAGMA] was used, here). Furthermore, every abelian character \( \chi \) of \( \Gamma_S \) yields an abelian for \( \varphi_T(\text{Stab}_{T_S}(I_T^S(\Lambda_{T,1}))) \) if \( \ker(\varphi_T) \subset \ker(\chi) \) holds. But before considering the special cases we are interested in, let us cite the following general theorem from [Kl06, thm.2.25]:

\(5.23\) Theorem. Let the setting described above be given (also concerning the condition on the generators of \( \Gamma_S \) and \( \Gamma_T \), and suppose that \( \varphi_T \) is surjective. Let \( k \in \mathbb{Z} \). If \( \chi \in \Gamma_S^{ab} \) is the continuation of an abelian character of \( \Gamma_T \) and \( f \in [\Gamma_S, k, \chi] \), then

\[
f|_{\mathcal{H}_T} \in [\Gamma_T, k, \chi|_{\mathcal{H}_T}].
\]

If \( f \) possesses the Fourier-expansion

\[
f(w) = \sum_{m,n \in \mathbb{N}_0} \sum_{\mu \in \mathcal{N}, q_S(\mu) \leq \eta n} \alpha_f((m, \mu, n))e^{2\pi i (mnt_1 + mnt_2 - \mu Sz)}
\]

for \( w = (t_1, z, t_2) \in \mathcal{H}_S \), then the Fourier-expansion of \( f|_{\mathcal{H}_T} \) is given by

\[
f|_{\mathcal{H}_T}(w_T) = \sum_{m,n \in \mathbb{N}_0} \sum_{\mu_T \in \mathcal{N}_T, q_T(\mu_T) \leq \eta n} \beta_f((m, \mu_T, n))e^{2\pi i (mnt_1 + mnt_2 - \mu_T Sz)}
\]

for \( w_T = (t_1, z_T, t_2) \in \mathcal{H}_T \), where

\[
\beta_f((m, \mu_T, n)) = \sum_{\mu \in \mathcal{N}, q_S(\mu) \leq \eta n} \alpha_f((m, \mu, n))
\]

for \( m, n \in \mathbb{N}_0 \) and \( \mu_T \in \mathcal{N}_T^1 \) with \( q_T(\mu_T) \leq \eta n \).

Regarding the proof of this theorem (cf. [Kl06, thm.2.25]), the same statements hold if we have \( S = A_2^{(2)} \), but look at orthogonal modular forms with respect to \( \Gamma_{Maaß} \) of \( \Gamma_{Bor} \), since all necessary conditions are met. Of course, \( f|_{\mathcal{H}_T} \) might then be an orthogonal modular form with respect to a certain subgroup of \( \Gamma_T \), only.

Now, we will go through the cases we are interested in. First, we will determine the matrices \( M_T^S \). Then, we will state whether \( \varphi_T \) is surjective, and if not, then we will specify the image of \( \varphi_T \). Furthermore, we will analyze which characters arise when restricting to \( \mathcal{H}_T \). And finally, we will see that under certain conditions (concerning the weight of an orthogonal modular form and their associated abelian characters) an orthogonal modular form (living on \( \mathcal{H}_S \)) has to vanish identically when restricting to some submanifold \( \mathcal{H}_T \).
So let us start with \( S = A_2^{(2)} \). Here, we will have to consider some further lattice, which is attached to
\[
S_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.
\]

The embedding into \( \mathbb{H} \) is given by
\[
\begin{align*}
\mathbb{H}_{S_3} &= \{ x \in \mathbb{H} : x_4 = -2x_3 \} = \mathbb{R} + \mathbb{R} \frac{1+i\sqrt{3}}{2} + \mathbb{R} i(1+2\sqrt{3}), \\
\mathcal{O}_{S_3} &= \mathbb{Z} + \mathbb{Z} \frac{1+i\sqrt{3}}{2} + \mathbb{Z} i(1+2\sqrt{3}), \\
\iota_{S_3} : (x_1,x_2,x_3) &\mapsto x_1 + x_2 \frac{1+i\sqrt{3}}{2} + x_3 i(1+2\sqrt{3}), \\
q_{S_3}(x) &= x_1^2 + x_1x_2 + x_2^2 + 3x_3^2.
\end{align*}
\]

The lattice attached to \( S_3 \) is not euclidean. For \( x_1 = (0,0,\frac{1}{2}) \) we have \( q_{S_3}(x_1+y) \geq \frac{3}{4} \) for all \( y \in \Lambda \). Furthermore, since the block \( (\frac{1}{2}, \frac{1}{2}) \) corresponds to \( \mathbb{Z} \frac{1+i\sqrt{3}}{2} \) (the ring of integers for \( \mathbb{Q}(i\sqrt{3}) \)), one can show that there exists \( x_2 = (\ast, \ast, 0) \) such that \( q_{S_3}(x_2+y) \geq \frac{1}{4} \) for all \( y \in \Lambda \). Hence we obtain \( q_{S_3}(x_1 + x_2 + y) > 1 \) for all \( y \in \Lambda \). This is why we did not investigate \( S_3 \) so far, since we even cannot determine a “nice” set of generators for \( \Gamma_{S_3} \). Nevertheless, this lattice will occur later on, and under a certain condition orthogonal modular forms living on \( \mathcal{H}_S \) vanish identically when restricting to \( \mathcal{H}_{S_3} \).

Some easy considerations using (5.4) yield that we have the following embeddings:
\[
\begin{align*}
\iota_{A_2^{(2)}}^{(1)} : A_2^{(2)} &\rightarrow \Lambda, \quad (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,0) \\
\iota_{A_2^{(2)}}^{(2)} : A_2^{(2)} &\rightarrow \Lambda, \quad (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,x_2) \\
\iota_{T_3,(1)} : T_3 &\rightarrow \Lambda, \quad (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,x_2) \\
\iota_{T_3,(2)} : T_3 &\rightarrow \Lambda, \quad (x_1,x_2,x_3) \mapsto (x_1,x_2,-x_3,-x_2) \\
\iota_{T_3}^{(2)} : T_3 &\rightarrow \Lambda, \quad (x_1,x_2,x_3) \mapsto (x_1,x_2,x_3,-2x_3)
\end{align*}
\]

The reason why we have to consider two embeddings for \( T_3 \) will become clear later on. In the following proposition, we omit the index “\( S \)” for the abelian characters, again. We obtain (noting we only have to consider \( k \in 2\mathbb{N}_0 \) in view of (5.18), since otherwise \( [\Gamma_A^{(2)}(k,\chi) = \{0\}) ] \):

\[ (5.24) \text{Proposition.} \text{ Let } k \in 2\mathbb{N}_0 \text{ and } S = A_2^{(2)}. \]

a) \( \varphi_{A_2^{(2)}} \), \( \varphi_{T_3,(1)} \) and \( \varphi_{T_3,(2)} \) are surjective.

b) If \( f \in [\Gamma_S,k,\chi \cdot \chi_{\alpha_2}^m] \) with \( \chi \in \{ \chi_{\det}\chi_{\alpha_2}^m \} \) and \( m \in \mathbb{Z} \), then \( f \) vanishes identically on \( \mathcal{H}_{A_2^{(2)}} \).

c) If \( f \in [\Gamma_S,k,\chi_{\det}\chi_{\alpha_2}^m] \) with \( m,n \in \mathbb{Z} \), then \( f \) vanishes identically on \( \mathcal{H}_{S_3} \).

d) If \( f \in [\Gamma_S,k,(\chi_{\det}\chi_{\alpha_2}^m)^n] \) with \( m,n \in \mathbb{Z} \), then \( f|_{\mathcal{H}_{S_3}} \in [\Gamma_{A_2^{(2)},k,\tilde{\chi}}] \) holds for some abelian character \( \tilde{\chi} \) of \( \Gamma_{A_2^{(2)}} \).

If \( f \in [\Gamma_S,k,\chi] \) with \( \chi \) some abelian character of \( \Gamma_S \), then \( f|_{\mathcal{H}_{T_3}} \in [\Gamma_{T_3,k,\tilde{\chi}}] \) holds for some abelian character \( \tilde{\chi} \) of \( \Gamma_{T_3} \).
The corresponding characters can be read off the following table (where it does not matter which embedding we choose regarding $T_3$):

<table>
<thead>
<tr>
<th>$A_2^{(2)}$</th>
<th>$\chi_{\det}$</th>
<th>$\chi_{\pi_1}$</th>
<th>$\chi_{\pi_2}$</th>
<th>$\chi_{\det \chi_{\pi_1}}$</th>
<th>$\chi_{\det \chi_{\pi_2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{2,1}$</td>
<td>$-\chi_{\det}$</td>
<td>$-\chi_{\pi_1}$</td>
<td>1</td>
<td>$\chi_{\det \chi_{\pi_1}}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td>1</td>
<td>$\chi_{\pi_1} \chi_{\pi_2}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td></td>
</tr>
</tbody>
</table>

**Proof:**

a) Like mentioned above, it suffices to verify that $\varphi_T : \text{Stab}_{O(A)}(i_1^S(\Lambda_T)) \to O(\Lambda_T)$ is surjective. But this can be done using some computer algebra system. (Here, [SAGE] was used.)

b) Let

$$A = -A_{tr}^S A_{-i_2, i_2}^S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and $M = R_A$. One easily verifies $M(w) = w$ for all $w \in i_{A_{2,1}}^S(\mathcal{H}_{A_{2,1}})$. Let $\bar{\chi} = \chi \cdot \chi_{\pi_2}^m$ with $\chi \in \{\chi_{\det}, \chi_{\pi_1}\}$. Then $\bar{\chi}(M) = -1$, and thus $f \in [\mathcal{I}, k, \bar{\chi}]$ yields $f(w) = f\chi M(w) = \bar{\chi}(M) \cdot f(w) = -f(w)$ for all $w \in i_{A_{2,1}}^S(\mathcal{H}_{A_{2,1}})$, hence the assertion follows.

c) Let

$$A = -A_{tr}^S A_{-i_2, i_2}^S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $M = R_A$. Again, one easily verifies $M(w) = w$ for all $w \in i_{T_3}^S(\mathcal{H}_{T_3})$. Let $\bar{\chi} = \chi_{\det \chi_{\pi_2}}^m \chi_{\pi_1}^n$ with $m, n \in \mathbb{Z}$. Then $\bar{\chi}(M) = -1$ holds, too, and thus we obtain $f(w) = f\chi M(w) = \bar{\chi}(M) \cdot f(w) = -f(w)$ for all $w \in i_{T_3}^S(\mathcal{H}_{T_3})$, again.

d) Using [SAGE], one verifies that $\varphi_{T_3(1)}$ and $\varphi_{T_3(2)}$ are even injective, hence every abelian character $\chi$ of $\Gamma_S$ yields an abelian character for $O(\Lambda_{T_3}) = \varphi_{T_3(1/2)}(\text{Stab}_{O(A)}(i_1^S(\Lambda_{T_3}))) \simeq \text{Stab}_{O(A)}(i_1^S(\Lambda_{T_3}))$, which can be continued to an abelian character of $\Gamma_{T_3}$ regarding the results from the previous section. Furthermore, one computes that this abelian character does not depend on the embedding we choose (which is mainly due to $\chi(-I) = -1$ for all $\chi$). On the other hand, $\varphi_{A_{2,1}}$ is not injective. Again using [SAGE] one verifies that its kernel is given by $\{I, M\}$, with $M = -A_{tr} A_{-i_2, i_2}^S$ from b). Therefore, an abelian character $\chi$ of $\Gamma_S$ yields an abelian character for $O(\Lambda_{A_{2,1}}) = \varphi_{A_{2,1}}(\text{Stab}_{O(A)}(i_1^S(\Lambda_{A_{2,1}}))) \simeq \text{Stab}_{O(A)}(i_1^S(\Lambda_{A_{2,1}})) \cap \{I, M\}$ (and thus for $\Gamma_{A_{2,1}}$), if $\chi(M) = 1$ holds. And this is fulfilled for the abelian characters specified in the assertion (while orthogonal modular forms for the remaining ones vanish identically when restricting to $\mathcal{H}_{A_{2,1}}$ according to b), anyways). Therefore, the first part of the assertion is due to a) and (5.23), while the second part about the exact shape of the induced abelian characters can be verified by simply finding
Analogous to the previous section, let us analyze \( \Gamma \) \((5.25)\) Proposition. Let \( k \)

Now, we have the following characters of \( \Gamma \) and thus 

\[ C \]

and the commutator factor group is isomorphic to \( C_2 \times C_2 \times C_2 \). But actually, it turns out that 

\[ \Gamma_{\text{Maaß},3} \simeq C_2 \times C_2 \]: We have

\[
M := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \langle A_{T_3}^{0}, -I, A_{T_3}^{T_3}, I, T_{T_3}^{a, b} \rangle ,
\]

and analogous to a similar problem in the previous section one computes

\[
R_{(0, e_1, 0)} R_{A_{12}}^5 = R_{(0, e_3, 0)} R_M ,
\]

and thus \( \# \Gamma_{\text{Maaß},3} \leq 8 \) has to follow since \( R_8 \in \Gamma_{\text{Maaß},3} \) just like in the previous section, because 

\[ A_{12}^5 M^{-1} \notin \langle A_{T_3}^{0}, -I, A_{T_3}^{T_3}, I, T_{T_3}^{a, b} \rangle \] according to the results from [SAGE]. Of course, the abelian characters of \( \Gamma_{T_3} \) are also abelian characters for \( \Gamma_{\text{Maaß},3} \) (while \( \chi_{\text{Maaß}}^{T_1} \) and \( \chi_{\text{Maaß}}^{T_3} \) are identical on \( \Gamma_{\text{Maaß},3} \)). Therefore, we have the following set of generators for \( \Gamma_{\text{Maaß},3}^{ab} \):

\[
\begin{array}{ccc}
\Gamma_{\text{Maaß},3} & A_{12}^5 \chi_{\text{det}} & -I \chi_{\text{det}} \\
\chi_{T_1}^5 & -1 & 1 \\
\chi_{T_1}^5 & -1 & 1 \\
\end{array}
\]

(5.23)

Now, we have the following

(5.25) Proposition. Let \( k \in \mathbb{N}_0 \) and \( S = A_{12}^{(2)} \).

a) \( \varphi_{A_{12}} \mid \text{Stab}_{\text{Maaß}}(\xi_{A_{12}, 1}(\Lambda_{A_{12}, 1})) \) \( \varphi_{A_{12}} \mid \text{Stab}_{\text{Bor}}(\xi_{A_{12}, 1}(\Lambda_{A_{12}, 1})) \) and \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Maaß}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) are surjective, while we have \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Bor}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) \( \Gamma_{\text{Maaß},3} \).

b) If \( f \in \Gamma_{\text{Maaß},3, k, \chi} \) and \( g \in \Gamma_{\text{Bor}, k, \chi} \) with \( \chi \) some abelian character of \( \Gamma_{\text{Maaß}} \) or \( \Gamma_{\text{Bor}} \), respectively, then \( f |_{\text{H}_{A_{12}, 1}} \in \Gamma_{A_{12}, 1, k, \chi} \), \( f |_{\text{H}_{T_3}} \in \Gamma_{T_3, 3, k, \chi} \), \( g |_{\text{H}_{A_{12}, 1}} \in \Gamma_{A_{12}, 1, k, \chi} \) and \( g |_{\text{H}_{T_3}} \in \Gamma_{T_3, 3, k, \chi} \) hold for certain abelian characters \( \chi \) of \( \Gamma_{A_{12}, 1, k, \chi} \), \( \Gamma_{T_3, 3, k, \chi} \)

(c) \( \varphi_{A_{12}} \mid \text{Stab}_{\text{Maaß}}(\xi_{A_{12}, 1}(\Lambda_{A_{12}, 1})) \) and \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Bor}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) are surjective, while we have \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Maaß}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) \( \Gamma_{\text{Maaß},3} \).

b) If \( f \in \Gamma_{\text{Maaß},3, k, \chi} \) and \( g \in \Gamma_{\text{Bor}, k, \chi} \) with \( \chi \) some abelian character of \( \Gamma_{\text{Maaß}} \) or \( \Gamma_{\text{Bor}} \), respectively, then \( f |_{\text{H}_{A_{12}, 1}} \in \Gamma_{A_{12}, 1, k, \chi} \), \( f |_{\text{H}_{T_3}} \in \Gamma_{T_3, 3, k, \chi} \), \( g |_{\text{H}_{A_{12}, 1}} \in \Gamma_{A_{12}, 1, k, \chi} \) and \( g |_{\text{H}_{T_3}} \in \Gamma_{T_3, 3, k, \chi} \) hold for certain abelian characters \( \chi \) of \( \Gamma_{A_{12}, 1, k, \chi} \), \( \Gamma_{T_3, 3, k, \chi} \)

(c) \( \varphi_{A_{12}} \mid \text{Stab}_{\text{Maaß}}(\xi_{A_{12}, 1}(\Lambda_{A_{12}, 1})) \) and \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Bor}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) are surjective, while we have \( \varphi_{T_3}(1/2) \mid \text{Stab}_{\text{Maaß}}(\xi_{T_3, 1}(1/2)(\Lambda_{T_3, 1})) \) \( \Gamma_{\text{Maaß},3} \).
5.2 Introduction to orthogonal modular forms

restriction regarding $T_3$ does depend on the choice of the embedding.
The corresponding characters can be read off the following tables:

<table>
<thead>
<tr>
<th>$\Gamma_{\text{Maaß}}$</th>
<th>$\chi_{\text{det}}$</th>
<th>$\chi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{A_{2,1}}$</td>
<td>$\chi_{\text{det}}\chi_{\pi}$</td>
<td>$\chi_{\pi}$</td>
</tr>
<tr>
<td>$\Gamma_{\text{Maaß,3}}$</td>
<td>$\chi_{\text{det}}\chi_{\pi_1}$</td>
<td>$\chi_{\pi_1}$</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>$\Gamma_{\text{Bor}}$</th>
<th>$\chi_{\text{det}}$</th>
<th>$\chi_{i,\sqrt{3}}$</th>
<th>$\chi_{i_1i_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{A_{2,1}}$</td>
<td>$\chi_{\text{det}}\chi_{\pi}$</td>
<td>$\chi_{\pi}$</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_{T_3, (1)}$</td>
<td>$\chi_{\text{det}}\chi_{\pi_2}$</td>
<td>$\chi_{\pi_2}$</td>
<td>$\chi_{\pi_1}\chi_{\pi_2}$</td>
</tr>
<tr>
<td>$\Gamma_{T_3, (2)}$</td>
<td>$\chi_{\text{det}}\chi_{\pi_2}$</td>
<td>$\chi_{\pi_1}$</td>
<td>$\chi_{\pi_1}\chi_{\pi_2}$</td>
</tr>
</tbody>
</table>

**Proof:**

a) Again, it suffices to verify that $\varphi_T$ restricted to $\text{Stab}_{O(\Lambda)_{\text{Maaß}}} (t_T^3(\Lambda_T))$ (or restricted to $\text{Stab}_{O(\Lambda)_{\text{Maaß}}} (t_T^3(\Lambda_T))$) is surjective, which can be done using [SAGE], for example.

b) Using [SAGE], one verifies that $\varphi_T$ (or better its restriction) is injective in all cases, hence every abelian character $\chi$ of $\Gamma_{\text{Maaß}}$ or $\Gamma_{\text{Bor}}$ yields an abelian character for the image of $\varphi_T$ restricted to $O(\Lambda)_{\text{Maaß}}$ or $O(\Lambda)_{\text{Maaß}}$, which can again be continued to an abelian character of $\Gamma_{A_{2,1}}, \Gamma_{T_3}$ or $\Gamma_{\text{Maaß,3}}$, respectively.

Therefore, the first part of the assertion is due to a) and (5.23), while the second part about the exact shape of the induced abelian characters can be verified like in (5.24). Here one has to note that this time the chosen embedding regarding $T_3$ influences the shape of the abelian character, which is mainly due to $\chi_{i,\sqrt{3}}(I) = -1$.

For the next two propositions (which deal with $A_{2,1}$ and $T_3$), recall that we defined two further lattices in (5.5), namely those attached to $A_{1}[(2)] = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which have been investigated in [Kl06]. Hence, we will not give details on these lattices (including their abelian characters), but we will use the notation found in that work.

We have the following further embeddings:

\[
\begin{align*}
\iota_{A_{1}[(2)]}^{A_{2,1}} &: \Lambda_{A_{1}[(2)]} \rightarrow \Lambda_{A_{2,1}}, \\
& (x_1, x_2) \mapsto (x_1, 0, x_2) \\
\iota_{A_2}^{A_{2,1}} &: \Lambda_{A_2} \rightarrow \Lambda_{A_{2,1}}, \\
& (x_1, x_2) \mapsto (x_1, x_2, 0) \\
\iota_{D_2}^{A_{2,1}} &: \Lambda_{D_2} \rightarrow \Lambda_{A_{2,1}}, \\
& (x_1, x_2) \mapsto (x_1, x_2, x_2) \\
\iota_{T_3}^{A_{1}[(2)]} &: \Lambda_{A_{1}[(2)]} \rightarrow \Lambda_{T_3}, \\
& (x_1, x_2) \mapsto (x_1, 0, x_2) \\
\iota_{T_3}^{D_2} &: \Lambda_{D_2} \rightarrow \Lambda_{T_3}, \\
& (x_1, x_2) \mapsto (x_1, x_2, 0)
\end{align*}
\]

Once more, one of the restrictions is not surjective, and we need the according subgroup of $\Gamma_{A_{1}[(2)]}$. So let

\begin{equation}
\Gamma_{A_{1}[(2)]} \chi_{\pi} := \ker(\chi_{\pi})
\end{equation}
with the abelian character \( \pi \) from [Kl06, p.22]. Again, \( \Gamma_{A_1^{[2]}}^{[\pi]} \) is a subgroup of \( \Gamma_{A_1^{[2]}} \) of index two, since \( \chi_\pi \) is of order two. Noting \(-M_{tr}^{A_1^{[2]}} = -R_0 (\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} ) \in \Gamma_{A_1^{[2]}}^{[\pi]} \) (see [Kl06, p.20]), one obtains that \( \Gamma_{A_1^{[2]}}^{[\pi]} \) is generated by \( \Gamma_0 \), \( \Gamma_S \) \((g \in A_1^{[2]})\) and \( R_A \) with \( A \in \ker(\chi_\pi|_{O(A_1^{[2]}))} \) (cf. [Kl06, cor.1.23]). Using [SAGE] one verifies

\[
\{ A \in O(A_1^{[2]}); \chi_\pi(R_A) = 1 \} = \langle A_{tr}^{A_1^{[2]}}, -I \rangle,
\]

where \( A_{tr}^{A_1^{[2]}} = (\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} ) \), and the commutator factor group is isomorphic to \( C_2 \times C_2 \). But we have

\[
R_{(0,0,0)} R_{A_{tr}^{A_1^{[2]}}} R_{(0,0,0)} R_{-A_{tr}^{A_1^{[2]}}}.
\]

where, regarding the proof of [Kr96, p.249, prop.4] and [Kl06, prop.1.25], \( R_{(0,0,0)} \) and \( R_{(0,0,0)} \) belong to the same coset modulo \( \Gamma_{A_1^{[2]}} \). Thus also \( R_A \) and \( R_B \) belong to the same coset for \( A = A_{tr}^{A_1^{[2]}} \) and \( B = -A_{tr}^{A_1^{[2]}} \), while \( A \) and \( B \) are from different cosets modulo \( \langle A_{tr}^{A_1^{[2]}}, -I \rangle \) (according to the results from [SAGE]). Combining this with [Kl06, prop.1.25] we obtain \( \Gamma_{A_1^{[2]}} \simeq C_2 \times C_2 \), where one should note that this time not all \( T_g \) belong to \( \Gamma_{A_1^{[2]}} \) (cf. [Kl06, prop.1.25]) – which means that one character arises from the \( T_g \), while the second arises from the \( R_A \) (but not two arise from \( \langle A_{tr}^{A_1^{[2]}}, -I \rangle \), like its commutator factor group being isomorphic to \( C_2 \times C_2 \) might suggest; see above). We have the following set of generators for \( \Gamma_{A_1^{[2]}}^{ab} \):

\[
\begin{array}{c|c|c|c}
\Gamma_{A_1^{[2]}} & M_{tr}^{A_1^{[2]}} & R_{-I} & T_{(0,1,0,0)} \\
\hline
\chi_{\det} & -1 & 1 & 1 \\
v_2 & 1 & 1 & -1 \\
\end{array}
\]

(Confer [Kl06, pp.22] concerning the abelian character \( v_2 \).) In the following proposition, we omit the index “\( S \)” for the abelian characters, again.

\(5.26 \) Proposition. Let \( k \in \mathbb{N}_0 \) and \( S = A_{2,1} \).

a) \( \varphi_{A_2} \) and \( \varphi_{D_2} \) are surjective, while we have \( \varphi_{A_1^{[2]}}(\text{Stab}_S(\mathcal{I}\mathcal{S}(\Lambda_{A_1^{[2]}}^{[1]}))) = \Gamma_{A_1^{[2]}}^{[\pi]} \).

b) If \( f \in [\Gamma_S,k,\chi] \) with \( \chi \in \chi_{\det,\chi_\pi} \), then \( f \) vanishes identically on \( \mathcal{H}_{A_1^{[2]}} \).

c) If \( f \in [\Gamma_S,k,v_2,\chi_{\det,\chi_\pi}] \) with \( m \in \mathbb{Z} \), then \( f \) vanishes identically on \( \mathcal{H}_{A_2} \).

d) If \( f \in [\Gamma_S,k,(\chi_{\det,\chi_\pi})^m] \) with \( m \in \mathbb{Z} \), then \( f|_{\mathcal{H}_{A_1^{[2]}}} \in [\Gamma_{A_1^{[2]}}^{[\pi]},k,\tilde{\chi}] \) holds for some abelian character \( \tilde{\chi} \) of \( \Gamma_{A_1^{[2]}}^{[\pi]} \).

If \( f \in [\Gamma_S,k,\chi_{\det,\chi_\pi}] \) with \( m \in \mathbb{Z} \), then \( f|_{\mathcal{H}_{A_2}} \in [\Gamma_{A_2},k,\chi] \) holds for some abelian character \( \chi \) of \( \Gamma_{A_2} \).

If \( f \in [\Gamma_S,k,\chi] \), where \( \chi \) is some abelian character of \( \Gamma_S \), then \( f|_{\mathcal{H}_{D_2}} \in [\Gamma_{D_2},k,\tilde{\chi}] \) holds for some abelian character \( \tilde{\chi} \) of \( \Gamma_{D_2} \).
The corresponding characters can be read off the following table:

<table>
<thead>
<tr>
<th>$\Gamma_{A_2,1}$</th>
<th>$\chi_{\text{det}}$</th>
<th>$\chi_{\pi}$</th>
<th>$\chi_{\text{det}}\chi_{\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{A_2}^{(2)}\pi$</td>
<td>-</td>
<td>-</td>
<td>$\chi_{\text{det}}$</td>
</tr>
<tr>
<td>$\Gamma_{A_2}$</td>
<td>-</td>
<td>$\chi_{\pi}$</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_{D_2}^\pi$</td>
<td>$\chi_{\text{det}}\chi_{\pi}$</td>
<td>$\chi_{\pi}$</td>
<td>$\chi_{\text{det}}$</td>
</tr>
</tbody>
</table>

**Proof:**

a) Just like in (5.24) and in (5.25), the assertion can be verified using [SAGE], for example.

b) Let

$$A = A_{i_2}^S A_{i_1}^S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $M_1 = R_A$. One easily verifies $M_1(w) = w$ for all $w \in i_{A_2}^{A_2,1}(\mathcal{H}_{A_2})$. If $\chi$ is given like in the assertion, then $\chi(M_1) = -1$, and thus $f(w) = f|_M M_1(w) = \chi(M_1) \cdot f(w) = -f(w)$ for all $w \in i_{A_2}^{A_2,1}(\mathcal{H}_{A_2})$, hence the assertion follows.

c) This time, let $M_2(w) = w$ for all $w \in i_{A_2}^{A_2,1}(\mathcal{H}_{A_2})$. Let $\tilde{\chi} = \chi_{\text{det}}\chi_{\pi}^m$ with $m \in \mathbb{Z}$. Again, $\tilde{\chi}(M_2) = -1$ holds, and thus we obtain $f(w) = f|_M M_2(w) = \tilde{\chi}(M_2) \cdot f(w) = -f(w)$ for all $w \in i_{A_2}^{A_2,1}(\mathcal{H}_{A_2})$.

d) Using [SAGE], one verifies that $\varphi_{D_2}^S$ is injective, hence every abelian character $\chi$ of $\Gamma_S$ yields an abelian character for $O(\Lambda_{D_2}) = \varphi_{D_2}^S(\text{Stab}_{O(\Lambda)}(i_{D_2}^S(\Lambda_{D_2}))) \simeq \text{Stab}_{O(\Lambda)}(i_{D_2}^S(\Lambda_{D_2})))$, which can be continued to an abelian character of $\Gamma_{D_2}^\pi$ regarding the results from the previous section. On the other hand, $\varphi_{A_2}^{A_2,1}$ and $\varphi_{A_2}$ are not injective. Using [SAGE] one can show that their kernels are given by $\{I, M_1\}$ from b) and $\{I, M_2\}$ from c), respectively. Therefore, an abelian character $\chi$ of $\Gamma_S$ yields an abelian character for $O(\Lambda_{A_2}^{S}) = \varphi_{A_2}^{A_2,1}(\text{Stab}_{O(\Lambda)}(i_{A_2}^S(\Lambda_{A_2}))) \simeq \text{Stab}_{O(\Lambda)}(i_{A_2}^S(\Lambda_{A_2}))) / \{I, M_1\}$ (and thus for $\Gamma_{A_2}^{(2)}$, if $\chi(M_1) = 1$ holds. Of course, we have the same for $A_2$ instead of $A_1^{(2)}$. And this is fulfilled for the abelian characters specified in the assertions (while orthogonal modular forms for the remaining ones vanish identically when restricting to $\mathcal{H}_{A_2}$ or $\mathcal{H}_{A_2}$ according to b) and c), anyways).

Therefore, the first part of the assertion is due to a) and (5.23) again, and the second part follows accordingly, like we have seen twice before. □

And finally, we consider $T_3$:
(5.27) Proposition. Let $k \in \mathbb{N}_0$ and $S = T_3$.

a) Both $\varphi_{A^1_2}$ and $\varphi_{D^*_2}$ are surjective.

b) If $f \in [\Gamma_S, k, \chi_{\det \chi_{\pi_1}}]$ with $l, m, n \in \mathbb{Z}$ such that $l + m + n \equiv 1 \mod 2$, then $f$ vanishes identically on $\mathcal{H}_{A^1_2}$.

c) If $f \in [\Gamma_S, k, \chi_{\det \chi_{\pi_1}}]$ with $l, m, n \in \mathbb{Z}$ such that $l + m + n \equiv 0 \mod 2$, then $f|_{\mathcal{H}_{A^1_2}} \in [\Gamma_{A^1_2}, k, \tilde{\chi}]$ holds for some abelian character $\tilde{\chi}$ of $\Gamma_{A^1_2}$.

The corresponding characters can be read off the following table:

<table>
<thead>
<tr>
<th>$T_3$</th>
<th>$\chi_{\det \chi_{\pi_1}}$</th>
<th>$\chi_{\pi_1}$</th>
<th>$\chi_{\det \chi_{\pi_2}}$</th>
<th>$\chi_{\det \chi_{\pi_1}}$</th>
<th>$\chi_{\det \chi_{\pi_2}}$</th>
<th>$\chi_{\pi_1, \chi_{\pi_2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^1_2$</td>
<td>$- \chi_{\det \chi_{\pi_1}}$</td>
<td>$\chi_{\pi_1}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td>$\chi_{\det \chi_{\pi_1}}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td>$\chi_{\pi_1, \chi_{\pi_2}}$</td>
</tr>
<tr>
<td>$D^*_2$</td>
<td>$\chi_{\det \chi_{\pi_1}}$</td>
<td>$\chi_{\pi_1}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td>$\chi_{\det \chi_{\pi_1}}$</td>
<td>$\chi_{\det \chi_{\pi_2}}$</td>
<td>$\chi_{\pi_1, \chi_{\pi_2}}$</td>
</tr>
</tbody>
</table>

Proof: The proof of this proposition is completely analogous to the proofs of the three corresponding propositions above. This time, define

$$A = -A_{-i, i}^S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and $M = R_A$. Then for the characters $\chi$ specified in b) we have $\chi(M) = -1$, while $M(w) = w$ for all $w \in \Gamma_{A^1_2} \setminus (\mathcal{H}_{A^1_2})$. In c) one has to note again that $\varphi_{D^*_2}$ turns out to be injective, while the kernel of $\varphi_{A^1_2}$ is given by $\{I, M\}$. The rest of the proof is completely analogous to what we have seen before.

We will also have to restrict orthogonal modular forms living on $\mathcal{H}_{D^*_2}$ to submanifolds, but we will get to that in detail later, since doing so will need some further mechanisms.

5.3 A dictionary between the symplectic and the orthogonal world

Now that we introduced the orthogonal setting and orthogonal modular forms (along with some important results which we will need later on), we will now compare the symplectic and the orthogonal world. Like announced before and indicated in (5.4), it turns out that orthogonal modular forms can be identified with quaternionic modular forms, or in the lower degree cases, with restrictions of these and with certain Hermitian modular forms. This means that in some sense it does not make a difference if we are talking about the symplectic or the orthogonal world, and can benefit from the mechanisms of both. For example, theta series, Maaß lifts and Eisenstein-series primarily arise from the symplectic setting, while the orthogonal setting provides the important machinery of (yet to introduce) Borcherds products.
So first, let us demonstrate how to identify the symplectic and the orthogonal setting. Recall that there exist isomorphisms $i$ mapping the quadratic spaces $(\mathbb{R}^l, q_S)$ to the quadratic spaces $(\mathbb{H}_S, N_S)$ (where $\mathbb{H}_S$ is some subspace of $\mathbb{H}$) and the attached lattices $\Lambda = \mathbb{Z}^l$ to the sublattices $O_S = O \cap \mathbb{H}_S$ of $O$ (see (5.4) and (5.5)). We use these isomorphisms to identify the half-spaces we have to consider in the symplectic setting on the one hand, and in the orthogonal setting on the other hand.

Note that we already discussed restrictions of quaternionic modular forms to submanifolds in (1.72). Therefore, we will discuss the identification of orthogonal modular forms for $A_2^{(2)}$ with quaternionic modular forms, first. Once this identification is clear, we can “leave” the symplectic setting and consider orthogonal modular forms, only, since it will be clear from the construction (and the explicitly given isomorphisms $i$) how to identify restrictions of quaternionic modular forms with orthogonal modular forms attached to lattices of lower degree. Furthermore, for orthogonal modular forms attached to lattices of degree two (hence $A_1^{(2)}$, $A_2$ and $D_4^*$) it was already worked out in [Kl06, pp.53] how to identify these with Hermitian modular forms.

So before we actually get to the mentioned identification for $A_2^{(2)}$, let us summarize the setting we will have to analyze: We start with quaternionic modular forms, which can be identified with orthogonal modular forms for $A_2^{(2)}$. Next, we restrict these to certain submanifolds (in standard notation, if $z = z_0 + z_1 \frac{1+i\sqrt{3}}{2} + z_2 i_2 + z_3 \frac{1+i\sqrt{3}}{2} i_2 \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ is the second diagonal entry of $Z \in \mathcal{H}(\mathbb{H})$, we restrict to the submanifolds given by $z_3 = 0$ or $z_1 = z_3$). We have not worked out any theory concerning quaternionic modular forms on these submanifolds, but the restrictions can be identified with orthogonal modular forms for $A_{2,1}$ and $T_3$, respectively. And in contrast to the symplectic setting, we worked out the whole background concerning orthogonal modular forms for these lattices. Hence we ignore the symplectic world in this “transitional stage” and only work in the orthogonal setting. Next, we can restrict orthogonal modular forms for $A_{2,1}$ and $T_3$ to those submanifolds that yield orthogonal modular forms for $A_1^{(2)}$, $A_2$ and $D_4^*$. But now, the symplectic setting is worked out completely, again, since we already mentioned that talking about orthogonal modular forms for $A_1^{(2)}$, $A_2$ and $D_4^*$ means investigating Hermitian modular forms, and the whole setting concerning that can be found in [De01]. Of course, we could have restricted quaternionic modular forms to the Hermitian half-spaces directly (like discussed in section 1.5), but then we would lose the information coming from the “transitional stages” $A_{2,1}$ and $T_3$.

However, determining the spaces or graded rings of quaternionic modular forms (with respect to our order $O$) is identical to determining the spaces or graded rings of orthogonal modular forms with respect to $A_2^{(2)}$ – with the difference that a reduction process concerning submanifolds is more “natural” in the orthogonal world than it is in the symplectic world, in a way.

So finally, let us show how to compare the two settings. Note that for the Hurwitz order instead of $O$ this can already be found in [Kl06, pp.57], while in this case the issue concerning quaternionic modular forms of odd weight does not occur.
(5.28) Proposition. The orthogonal half-space $\mathcal{H}_{A_2^{(2)}}$ is biholomorphically mapped to $\mathcal{H}(\mathbb{H})$ by

$$\varphi_H : \mathcal{H}_{A_2^{(2)}} \to \mathcal{H}(\mathbb{H}), \quad (x_1, u, x_2) + iy_1(v, y_2) \mapsto \left( \begin{array}{c} x_1 + iy_1 \\ iA_2^{(2)}(u) + iv_2 \end{array} \right).$$

Proof: Let $S = A_2^{(2)}$, $w = (x_1, u, x_2) + iy_1(v, y_2) \in S$ and $Z = X + iY = \varphi_H(w)$. Obviously, $Z \in \mathbb{H}^{2x2} \otimes_{\mathbb{R}} \mathbb{C}$ and $Z = \overline{Z}$ hold by definition, and $\varphi_H : S \to \operatorname{Her}_2(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ is an injective homomorphism (since $\iota_S$ is). Hence only $\varphi_H(S) = \mathcal{H}(\mathbb{H})$ remains to be shown. We defined $P_S = \{(v_0, \tilde{v}, v_{i+1}) \in \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R} ; v_0v_{i+1} > q_S(\tilde{v}), v_0 > 0\}$, and $(y_1, v, y_2) \in P_S$ is mapped to

$$Y = \left( \begin{array}{c} y_1 \\ i\iota_S(v) \\ y_2 \end{array} \right),$$

with $y_1 > 0$ and $\det(Y) = y_1y_2 - N(i\iota_S(v)) = y_1y_2 - q_S(v) > 0$,

hence $Y > 0$. This means that $\varphi_H(S) \subseteq \mathcal{H}(\mathbb{H})$ holds, indeed. On the other hand, since $\iota_S$ is bijective and because $Y > 0$ implies $(y_1, v, y_2) \in P_S$, the bijectivity of $\varphi_H$ is obvious.

Since $\mathcal{H}_{A_2^{(2)}}$ and $\mathcal{H}(\mathbb{H})$ are isomorphic, the groups consisting of all automorphisms of $\mathcal{H}_{A_2^{(2)}}$ and of $\mathcal{H}(\mathbb{H})$ have to be isomorphic, too. In particular, we are going to show now that $\Gamma(\mathcal{O})$ is isomorphic to $\Gamma_S/\{\pm I\}$. Note that according to (5.3) $M \in \Gamma_S$ acts trivially on $S$ if and only if $M = \pm I$. Therefore, we have the identification

$$\Gamma_S/\{\pm I\} \simeq \{\mathcal{H} \to S, w \mapsto M(w) ; M \in \Gamma_S\}.$$

By abuse of notation, writing $\Gamma_S/\{\pm I\}$ should refer to the factor group itself, or the group of automorphisms from above, which ever is more suitable. Keeping that in mind, we obtain:

(5.29) Proposition. Let $S = A_2^{(2)}$. We have

$$\Gamma_S/\{\pm I\} \simeq \Gamma(\mathcal{O}).$$

For the generators of $\Gamma_S/\{\pm I\}$, the isomorphism is given by:

- $(w \mapsto I_0\langle w \rangle) \mapsto (Z \mapsto J(Z))$
- $(w \mapsto T_g\langle w \rangle)$, where $g = (g_0, \tilde{g}, g_5) \in \mathbb{Z}^6$ $\mapsto (Z \mapsto \operatorname{Trans}(T)\langle Z \rangle)$, where $T = (g_0, i\iota_S(\tilde{g}))$
- $(w \mapsto M_2^{ht}\langle w \rangle) \mapsto (Z \mapsto Z')$
- $(w \mapsto R_A^{ij}\langle w \rangle) \mapsto (Z \mapsto Z[i_1I])$
- $(w \mapsto R_A^{ij}\langle w \rangle) \mapsto (Z \mapsto \operatorname{Rot}(U)\langle Z \rangle)$, where $U = (\frac{1}{2}(1 + i\sqrt{3}) 0 0)$
- $(w \mapsto R_A^{ij}\langle w \rangle) \mapsto (Z \mapsto \operatorname{Rot}(U)\langle Z \rangle)$, where $U = (i_2 0 0)$
5.3 A dictionary between the symplectic and the orthogonal world

**Proof:** Like mentioned above, the isomorphism \( \varphi_H \) induces an isomorphism \( \varphi \) mapping the group of automorphisms of \( H \) to the group of automorphisms of \( \mathcal{H}(H) \). Note that it is given by

\[
\psi \mapsto \varphi_H \circ \psi \circ \varphi_H^{-1}
\]

for all automorphisms \( \psi \) of \( H \). Of course, \( \varphi|_{\Gamma_S/(\pm I)} \) is still an injective homomorphism. Hence we only have to determine its image. According to (5.7) and (5.10), \( \Gamma_S/(\pm I) \) is generated by the automorphisms specified in the assertion, indeed. So let us have a closer look at them. Note that \( \iota_S : \mathbb{R}^4 \to H \) can be extended to \( \mathbb{R}^4 \otimes_{\mathbb{R}} C \to H \otimes_{\mathbb{R}} C \) via \( x + iy \mapsto i(x) + i(x) \). We denote it again by \( \iota_S \).

Suppose \( w = (\tau_1, z, \tau_2) \in H \). By definition, we have

\[
J_0(w) = -q_0(w)^{-1}(\tau_2, -z, \tau_1) = -(\tau_1 - q_5(z)^{-1}(\tau_2, -z, \tau_1).
\]

If this automorphism is simply denoted by \( J_0 \), again (which we will do accordingly for the other automorphisms we have to consider), then we compute for \( Z = (z_{1,2}) \in \mathcal{H}(H) \):

\[
\varphi_H \circ J_0 \circ \varphi_H^{-1}(Z) = -(z_{1,1}z_{2,2} - z_{1,2}z_{2,1})^{-1} \left( \begin{array}{cc} z_{2,2} & -z_{1,2} \\ -z_{1,2} & z_{1,1} \end{array} \right) = -Z^{-1} = J(Z).
\]

Next, for \( g = (g_0, \bar{g}, g_5) \in \mathbb{Z}^5 \), we have \( T_g(w) = w + g \), hence we obviously obtain

\[
\varphi_H \circ T_g \circ \varphi_H^{-1}(Z) = \left( \begin{array}{c} z_{1,1} + g_0 \zeta_{1,2} + g_5 \\ z_{1,2} + g_5 \zeta_{1,2} \end{array} \right) = \text{Trans}(T)(Z) \quad \text{for} \quad T = \left( \begin{array}{cc} g_0 & \iota_S(\bar{g}) \\ \iota_S(g) & g_5 \end{array} \right),
\]

where \( T \in \text{Her}_2(\mathcal{O}) \). So let us consider \( M^S_{it} \), with \( M^S_{it}(w) = (\tau_1, A_{it}^S z, \tau_2) \). By construction (see 5.2), we have

\[
\iota_S(z) = \iota_S(A_{it}^Sz).
\]

This leads to

\[
\varphi_H \circ M^S_{it} \circ \varphi_H^{-1}(Z) = \left( \begin{array}{cc} z_{1,1} & z_{1,2} \\ z_{1,2} & z_{2,2} \end{array} \right) = Z'.
\]

For \( A = A_{it}^S \), we have \( R_A(w) = (\tau_1, z_0, z_1, -z_2, -z_3, \tau_2) \) for \( z = (z_0, z_1, z_2, z_3) \), whereas

\[
-i_1(z_0 + z_1 + i_2 \sqrt{3} + z_2 i_2 + z_3 + i_2 \sqrt{3} i_2) i_1 = z_0 + z_1 + i_2 \sqrt{3} + z_2 i_2 + z_3 - z_2 + z_3 - z_2 \sqrt{3} i_2.
\]

Hence we obtain

\[
\varphi_H \circ R_A \circ \varphi_H^{-1}(Z) = \left( \begin{array}{cc} z_{1,1} & -i_1 z_{1,2} i_1 \\ -i_1 z_{1,2} i_1 & z_{2,2} \end{array} \right) = Z[i_1 i_1].
\]

Next, let \( A = A_{i1}^S \), with \( R_A(w) = (\tau_1, z_0 + z_1, -z_0, z_2 + z_3, -z_2, \tau_2) \). One computes

\[
\frac{1 - i_2 \sqrt{3}}{2} (z_0 + z_1 + i_2 \sqrt{3} + z_2 i_2 + z_3 + i_2 \sqrt{3} i_2) i_1 = z_0 + z_1 - z_0 - z_0 \frac{1 + i_2 \sqrt{3}}{2} + (z_2 + z_3)i_2 - z_2 - z_2 \frac{1 + i_2 \sqrt{3}}{2} i_2,
\]
and thus
\[ \varphi_{\mathbb{H}} \circ R_A \circ \varphi_{\mathbb{H}}^{-1}(Z) = \begin{pmatrix} z_{1,1} & \frac{1-i\sqrt{3}}{2}z_{1,2} \\ \frac{z_{1,2}}{1+i\sqrt{3}} & z_{2,2} \end{pmatrix} = \text{Rot}(U)\langle Z \rangle \text{ for } U = \begin{pmatrix} 1+i\sqrt{3} & 0 \\ 0 & 1 \end{pmatrix}, \]
where \( U \in \text{GL}_2(\mathcal{O}) \). And finally, suppose \( A = A_{i_2}^{S} \). We have \( R_A\langle w \rangle = (\tau_1, z_2 + z_3, -z_3, -z_0 - z_1, z_1, \tau_2) \) and
\[
-i_2(z_0 + z_1 \frac{1+i\sqrt{3}}{2} + z_2 i_2 + z_3 \frac{1+i\sqrt{3}}{2}i_2) = z_2 + z_3 - z_3 \frac{1+i\sqrt{3}}{2} + (-z_0 - z_1)i_2 + z_1 \frac{1+i\sqrt{3}}{2}i_2,
\]
which implies
\[ \varphi_{\mathbb{H}} \circ R_A \circ \varphi_{\mathbb{H}}^{-1}(Z) = \begin{pmatrix} z_{1,1} & -i_2z_{1,2} \\ z_{1,2}i_2 & z_{2,2} \end{pmatrix} = \text{Rot}(U)\langle Z \rangle \text{ for } U = \begin{pmatrix} i_2 & 0 \\ 0 & 1 \end{pmatrix}, \]
where \( U \in \text{GL}_2(\mathcal{O}) \).

According to (1.12) and (1.17), \( \Gamma(\mathcal{O}) \) is generated by these images in the group of automorphisms of \( \mathcal{H}(\mathbb{I}_1) \). Hence the assertion follows. \( \Box \)

Next, note that according to 5.4 and (5.11), every abelian character \( \chi \) of \( \Gamma_S \) fulfills \( \chi(-1) = 1 \), hence \( \chi(M) = \chi(-M) \) for all \( M \in \Gamma_S \). Therefore, the abelian characters of \( \Gamma_S \) and \( \Gamma_S/\{ \pm 1 \} \) coincide (or if one wants to be precise, the groups of abelian characters are isomorphic). And since \( \Gamma_S/\{ \pm 1 \} \) and \( \Gamma(\mathcal{O}) \) are isomorphic, their groups of abelian characters have to be isomorphic, too. In the next proposition we are going to state how this correspondence looks like. Note that the result can be verified by simply using the explicit isomorphism given in (5.29) and analyzing the values of the abelian characters on the generators of \( \Gamma(\mathcal{O}) \) and \( \Gamma_S \), respectively (see (1.19), (1.20) and 5.4).

**(5.30) Proposition.** The abelian characters of \( \Gamma(\mathcal{O}) \) and \( \Gamma_{A_i^S}^{(2)} \) correspond to each other in the following way:

<table>
<thead>
<tr>
<th>( \Gamma(\mathcal{O}) )</th>
<th>( v_{\text{det}} )</th>
<th>( v_{\tau_1} )</th>
<th>( v_{\tau} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_{A_i^S}^{(2)} )</td>
<td>( \chi_{\pi_2} )</td>
<td>( \chi_{\pi_1} )</td>
<td>( \chi_{\pi_1} )</td>
</tr>
</tbody>
</table>

We also have to identify the subgroups \( \Gamma_{\text{Maass}} \) and \( \Gamma_{\text{Bor}} \). But using the definition of both and as well the preceding proposition, it is obvious how to find their counterparts in the symplectic setting. Define

\[ \Gamma(\mathcal{O})_{\text{Maass}} := \{ \varphi \in \Gamma(\mathcal{O}) \mid v_{\tau_1}(\varphi) = 1 \}, \quad \Gamma(\mathcal{O})_{\text{Bor}} := \{ \varphi \in \Gamma(\mathcal{O}) \mid v_{\text{det}v_{\tau_1}}(\varphi) = 1 \}. \]

Of course, both are subgroups of index two. Using 5.12, (5.13), (5.30) and (5.29) we can easily determine generators for these subgroups. But first, note that applying the same methods like in the proof of (5.29) and doing some easy calculations yields the following further correspondence:

- \( ( w \mapsto R_{A_i^S \tau_2} \langle w \rangle ) \mapsto ( Z \mapsto \text{Rot}(U)\langle Z \rangle \), where \( U = \begin{pmatrix} -i_2 & 0 \\ 0 & i_2 \end{pmatrix} \). \)
To keep it simple, in the following proposition we just write “M”, although we actually mean the automorphism $H(\mathbb{H}) \rightarrow H(\mathbb{H})$, $Z \mapsto M(Z)$.

(5.31) Proposition. Let $S = A_2^{(2)}$. We have

$$\Gamma_{\text{Maaß}} / \{ \pm I \} \simeq \Gamma(O)_{\text{Maaß}} \quad \text{and} \quad \Gamma_{\text{Bor}} / \{ \pm I \} \simeq \Gamma(O)_{\text{Bor}},$$

where in both cases the isomorphy is given by the isomorphism from (5.29).

$\Gamma(O)_{\text{Maaß}}$ is generated by

$$J, \text{Trans}(T), \text{Rot}((\begin{smallmatrix} 1 & \epsilon \\ 0 & 1 \end{smallmatrix})/2, i_2),$$

with $T \in \text{Her}_2(O)$ and $\epsilon \in \{1 + i\sqrt{3}/2, i\}$, or in other words

$$\Gamma(O)_{\text{Maaß}} = \langle Z \mapsto M(Z), \tau ; M \in \text{Sp}_2(O) \rangle.$$

$\Gamma(O)_{\text{Bor}}$ is generated by

$$J, \text{Trans}(T), \text{Rot}((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})), \text{Rot}((\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix})), \text{Rot}((\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix})) \text{ and } \tau,$$

with $T \in \text{Her}_2(O)$ and $\delta = \frac{1 + i\sqrt{3}}{2}$.

Finally, we show that by using the isomorphisms from above, the spaces of quaternionic modular forms coincide with the corresponding spaces of orthogonal modular forms. Note that according to (5.18) there exist no non-trivial orthogonal modular forms of odd weight with respect to the whole orthogonal modular group. (And the same holds for quaternionic modular forms with respect to the whole extended quaternionic modular group – see (1.58).)

(5.32) Theorem. Let $S = A_2^{(2)}$, $k \in 2\mathbb{N}_0$ and $\nu$ an abelian character of $\Gamma(O)$. If $f \in [\Gamma(O), k, \nu]$ with Fourier-expansion

$$f(Z) = \sum_{T = (n \ m) \in \text{Her}_2(O), T \geq 0} \alpha_f(T) e^{2\pi i (nt_1 + mt_2 + 2\text{Re}(\tau_z))}, \quad Z = (\begin{smallmatrix} \tau_1 & z \\ 0 & \tau_2 \end{smallmatrix}) \in H(\mathbb{H}),$$

then $g := f \circ \varphi_H \in [\Gamma_S, k, \chi]$, where the abelian character $\chi$ of $\Gamma_S$ depends of $\nu$ and can be read off the table in (5.30). Moreover, the Fourier-expansion of $g$ is given by

$$g(w) = \sum_{m,n \in \mathbb{N}_0} \sum_{\mu \in \Lambda^{(1)}_{\delta_S(\mu) \leq mn}} \alpha_S(m, \mu, n) e^{2\pi i (nt_1 + mt_2 + \mu(\delta_S \bar{w}))}, \quad w = (\tau_1, \bar{w}, \tau_2) \in H_S,$$

where

$$\alpha_S(m, \mu, n) = \alpha_f \left( \begin{array}{c} n \\ \mu \\ m \end{array} \right).$$

Furthermore, the map

$$[\Gamma(O), k, \nu] \rightarrow [\Gamma_S, k, \chi], \quad f \mapsto f \circ \varphi_H$$

is an isomorphism.
\[q_0(w) = \tau_1 \tau_2 - q_5(\tilde{w}) .\]

By construction, we have \(N(\iota_S(x)) = \iota_S(x) \overline{\iota_S(x)} = q_5(x)\) for all \(x \in \mathbb{R}^4\), and like we mentioned in (5.29) \(\iota_S\) can be extended to \(\mathbb{R}^4 \otimes \mathbb{R} \mathbb{C}\). Thus, due to the way we extended \(q_5\) to \(\mathbb{R}^4 \otimes \mathbb{R} \mathbb{C}\) in (5.3), we obtain
\[q_5(\tilde{w}) = \iota_S(\tilde{w}) \overline{\iota_S(\tilde{w})} ,\]
which leads to
\[q_0(w) = \tau_1 \tau_2 - \iota_S(\tilde{w}) \overline{\iota_S(\tilde{w})} = \det(\varphi_\mathbb{H}(w)) .\]

Here, for \(z = x + iy \in \mathbb{H} \otimes \mathbb{R} \mathbb{C}\), we define \(\overline{z} = \overline{x} + i\overline{y}\) as usual (and like we did in all preceding chapters). Of course, if we have \(g = f \circ \varphi_\mathbb{H}\), then \(g\) is holomorphic if and only if \(f\) is holomorphic. And if we additionally apply the results from (5.29) to the considerations above, then we obtain \(f \in [\Gamma(\mathcal{O}), k, \nu]\) if and only if \(g \in [\Gamma_S, k, \chi]\). Hence (using the fact that \(\varphi_\mathbb{H}\) is an isomorphism of the half-spaces) the map \([\Gamma(\mathcal{O}), k, \nu] \rightarrow [\Gamma_S, k, \chi]\), \(f \mapsto f \circ \varphi_\mathbb{H}\) turns out to be an isomorphism, indeed.

Regarding the Fourier-expansions, one should consider the following: Due to construction, a straightforward calculation yields
\[a' Sb = 2 \text{Re}(\iota_S(a) \overline{\iota_S(b)}) = 2 \text{Re}(\overline{\iota_S(a) \iota_S(b)}) .\]

Therefore, we also have
\[p' S\tilde{w} = 2 \text{Re}(\iota_S(p) \overline{\iota_S(\tilde{w})}) .\]
by definition (see 1.7), where \( \mu \in \Lambda^s \) and \( \tilde{w} \in \mathbb{R}^4 \otimes \mathbb{C} \) (which means \( t_2(\tilde{w}) \in \mathbb{H} \otimes \mathbb{C} \)). Therefore, given \( T = (n \ t \ m) \in \text{Her}_2(\mathcal{O}) \) and \( w = (\tau_1, \tilde{w}, \tau_2) \in \mathcal{H}_2 \) we obtain
\[
\tau(T, \varphi_\mathcal{H}(w)) = nt_1 + m\tau_2 + \mu' S\tilde{w} = (m, -\mu', n) S_0 \tilde{w},
\]
where \( \mu = t_2^{-1}(t) \). Furthermore, note that the consideration above also leads to
\[
i_\mathcal{H}(\Lambda^s) = \frac{1}{2} \mathcal{O}^s.
\]
Putting together these results yields the assertion about the Fourier-expansions. \( \square \)

Obviously, equivalent assertions hold if we replace \( \Gamma(\mathcal{O}) \) by \( \Gamma(\mathcal{O})_{\text{Maaß}} \) or \( \Gamma(\mathcal{O})_{\text{Bor}} \) on the one side, and \( i_\mathcal{H} \) by \( i_{\text{Maaß}} \) or \( i_{\text{Bor}} \) on the other side – again given that \( k \) is even. Note that in virtue of (5.18) the character \( \chi \) has to fulfill \( \chi = \chi_{\text{det}}^l \chi_{\text{A}}^{2m} \) with \( l, m \in \mathbb{Z} \) if we consider \( \Gamma_{\text{Maaß}} \), and \( \chi = \chi_{\text{det}}^l \chi_{\text{O}}^{2m} \) with \( l, m \in \mathbb{Z} \) if we consider \( \Gamma_{\text{Bor}} \), since otherwise the spaces of orthogonal modular forms are trivial. Moreover, for even weight \( k \), we have the following correspondences concerning the abelian characters, where on the symplectic side the restriction to \( \Gamma(\mathcal{O})_{\text{Maaß}} \) or \( \Gamma(\mathcal{O})_{\text{Bor}} \) is meant:

- \( i_{\text{Maaß}} : \chi_{\text{det}}^l \chi_{\text{A}}^{2m} \leftrightarrow v_{\chi_{\text{det}}}^l v_{\text{det}}^m \)
- \( i_{\text{Bor}} : \chi_{\text{det}}^l \chi_{\text{O}}^{2m} \leftrightarrow v_{\chi_{\text{det}}}^l v_{\text{det}}^m \)

But if \( k \) is odd, then the identification becomes a bit more involved: On the orthogonal side we have an abelian character. Considering \( \Gamma_{\text{Maaß}} \), we have \( \chi = \chi_{\text{det}}^l \chi_{\text{A}}^{2m+1} \) with \( l, m \in \mathbb{Z} \), while for \( \Gamma_{\text{Bor}} \) we have \( \chi = \chi_{\text{det}}^l \chi_{\text{O}}^{2m+1} \) with \( l, m \in \mathbb{Z} \). But on the symplectic side, we only have multiplier systems, and like we saw in chapter 1, these are not abelian characters if the weight \( k \) is odd.

Nevertheless, making use of 1.17 and (5.31) we have the following: Let always \( f : \mathcal{H}(\mathbb{H}) \rightarrow \mathbb{C} \) be holomorphic. Suppose \( \psi \) is a multiplier system for \( \Gamma(\mathcal{O})_{\text{Maaß}} \) or for \( \Gamma(\mathcal{O})_{\text{Bor}} \) of odd weight \( k \), then \( f \in [\Gamma(\mathcal{O})_{\text{Maaß}}, \nu] \) holds if and only if \( f \) satisfies the following transformation laws:

- \( f(-Z^{-1}) = v(J)^{-1}(\det(\tilde{Z}))^{k/2} f(Z) \),
- \( f(Z + T) = v(\text{Trans}(T)) \cdot f(Z) \) for all \( T \in \text{Her}_2(\mathcal{O}) \),
- \( f(Z') = v(\tau) \cdot f(Z) \),
- \( f(Z[U]) = v(\text{Rot}(U)) \cdot f(Z) \) for \( U = \left( \begin{array}{cc} 1 & \epsilon \\ 0 & 1 \end{array} \right) \), where \( \epsilon \in \{ 1, i, -1, -i \} \) and \( \nu \in \left\{ \frac{1+i\sqrt{3}}{2}i, 2 \right\} \).

And \( f \in [\Gamma(\mathcal{O})_{\text{Bor}}, \nu] \) holds if and only if \( f \) satisfies the following transformation laws:

- \( f(-Z^{-1}) = v(J)^{-1}(\det(\tilde{Z}))^{k/2} f(Z) \),
- \( f(Z + T) = v(\text{Trans}(T)) \cdot f(Z) \) for all \( T \in \text{Her}_2(\mathcal{O}) \),
- \( f(Z') = v(\tau) \cdot f(Z) \),
- \( f(Z[U]) = v(\text{Rot}(U)) \cdot f(Z) \) for \( U \in \left\{ \left( \begin{array}{cc} 1 & \epsilon \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} i & 0 \\ 0 & i \end{array} \right), \left( \begin{array}{cc} -i & 0 \\ 0 & -i \end{array} \right) \right\} \).
On the other hand, let always \( g : \mathcal{H}_S \to \mathbb{C} \) be holomorphic, and suppose \( \chi \) is an abelian character for \( \Gamma_{\text{Maaß}} \) or for \( \Gamma_{\text{Bor}} \). Making use of the explicit values of \( \chi \) (which always equal 1 for \( J_0 \) and \( T_g \)), we obtain: \( g \in [\Gamma_{\text{Maaß}}, k, \chi] \) holds if and only if \( g \) satisfies the following transformation laws:

\[
\begin{align*}
g(-q_0(w)^{-1}(\tau_2, -z, \tau_1)) &= (q_0(w))^k g(w), \\
g(w + g) &= g(w) \quad \text{for all } g \in \mathbb{Z}^6, \\
g((\tau_1, A^S_t \tilde{w}, \tau_2)) &= \chi(M^S_t) \cdot g(w), \\
g((\tau_1, A \tilde{w}, \tau_2)) &= \chi(R_A) \cdot g(w) \quad \text{for } A \in \{A^S_{i_1 \sqrt{3}}, A^S_{i_2}\}.
\end{align*}
\]

And \( g \in [\Gamma_{\text{Bor}}, k, \chi] \) holds if and only if \( g \) satisfies the following transformation laws:

\[
\begin{align*}
g(-q_0(w)^{-1}(\tau_2, -z, \tau_1)) &= (q_0(w))^k g(w), \\
g(w + g) &= g(w) \quad \text{for all } g \in \mathbb{Z}^6, \\
g((\tau_1, A^S_t \tilde{w}, \tau_2)) &= \chi(M^S_t) \cdot g(w), \\
g((\tau_1, A \tilde{w}, \tau_2)) &= \chi(R_A) \cdot g(w) \quad \text{for } A \in \{A^S_{i_1 \sqrt{3}}, A^S_{i_2}, A^S_{i_1}, A^S_{i_2}\}.
\end{align*}
\]

So in the light of the proof of (5.32), the only actual problem that occurs is the transformation behavior concerning \( J \): We have

\[ q_0(w) = \det(\varphi_H(w)) \]

for \( w \in \mathcal{H}_S \), but on the symplectic side we do not have to consider \( (\det(Z))^k \), but \( v(f)^{-1} \cdot \det(Z)^{k/2} \). But on the other hand,

\[ ((\det(Z))^k)^2 = (\det(Z))^{2k} = (\det(\bar{Z}))^{2k/2} = (\det(\bar{Z})^{k/2})^2 \]

holds in virtue of (1.34). Therefore, since both sides are holomorphic functions without zeros and since \( \mathcal{H}(H) \) is a convex cone, \( (\det(Z))^k \) and \( (\det(\bar{Z}))^{k/2} \) (seen as functions in \( Z \)) can only differ by a factor \( \pm 1 \). Plugging in the special choice \( Z = iI \) then yields

\[ (\det(Z))^k = - (\det(\bar{Z})^{k/2}) \quad \text{for all } Z \in \mathcal{H}(H) \text{ and all odd } k. \quad (5.28) \]

But then, the same considerations like in (5.32) lead to \( f \in [\Gamma(O)_{\text{Maaß}}, k, v] \) if and only if \( f \circ \varphi_H \in [\Gamma_{\text{Maaß}}, k, \chi] \), where the multiplier system \( v \) and the abelian character \( \chi \) have to be identified accordingly: It always suffices to give their values for the generators only, and this can be read off the transformation laws from above. And of course, the same holds for \( \Gamma_{\text{Bor}} \) and \( \Gamma_{\text{Bor}} \). Furthermore, note that due to these considerations every multiplier system \( v \) for \( \Gamma(O)_{\text{Maaß}} \) or \( \Gamma(O)_{\text{Bor}} \) of odd weight has to fulfill \( v(f) = -1 \) and \( v(\text{Trans}(T)) = 1 \) for all \( T \in \text{Her}_2(O) \) (which in the case of \( \Gamma(O)_{\text{Maaß}} \) we already saw in (1.56)).

We summarize these results in the following

**Theorem.** Let \( S = A_2^{(2)} \), \( k \in \mathbb{N}_0 \) and \( v \) a multiplier system for \( \Gamma(O)_{\text{Maaß}} \) or \( \Gamma(O)_{\text{Bor}} \) of weight \( k \) (which means that \( v \) is an abelian character if \( k \) is even). Let \( \chi \) be the abelian character of \( \Gamma_{\text{Maaß}} \) or \( \Gamma_{\text{Bor}} \) given by the correspondence below. Then the maps

\[ [\Gamma(O)_{\text{Maaß}}, k, v] \to [\Gamma_{\text{Maaß}}, k, \chi], \quad f \mapsto f \circ \varphi_H \]
and

\[ [\Gamma(\mathcal{O})_{\text{Bor}}, k, \nu] \rightarrow [\Gamma_{\text{Bor}}, k, \chi], \quad f \mapsto f \circ \varphi_H \]

are isomorphisms.

The correspondence of \( \nu \) and \( \chi \) is given as follows: If \( k \) is even, then we have:

- \( \Gamma_{\text{Maaß}} : \chi_{\det}^{l}x_{i}^{2m} \leftrightarrow \nu_{r}^{l}v_{\det}^{m}, \)
- \( \Gamma_{\text{Bor}} : \chi_{\det}^{l}x_{i_{1}i_{2}}^{m} \leftrightarrow v_{r}^{l}v_{\det}^{m} = v_{i_{1}i_{2}}^{m}, \)

where \( l, m \in \mathbb{Z} \). If \( k \) is odd, then for \( \Gamma_{\text{Maaß}} \) the correspondence is given by:

- \( \Gamma_{\text{Maaß}} : \chi_{\det}^{l}x_{i}^{2m+1} \leftrightarrow v_{r}^{l+1}v_{i}^{2m+1}, \)

where \( l, m \in \mathbb{Z} \). For \( \Gamma_{\text{Bor}} \), if \( \chi = \chi_{\det}^{l}x_{i_{1}i_{2}}^{m} \), where \( l, m \in \mathbb{Z} \), the multiplier system \( \nu \) for \( \Gamma(\mathcal{O})_{\text{Bor}} \) is given by the following values on the generators of \( \Gamma(\mathcal{O})_{\text{Bor}} \):

\[
\begin{align*}
\nu(f) &= -1, \quad v(\text{Trans}(T)) = 1, \quad v(\text{Rot}(\frac{1+i\sqrt{3}}{1+i\sqrt{3}})) = -1, \\
v(\text{Rot}(\frac{i}{0} i_{2})) &= 1, \quad v(\text{Rot}(\frac{i}{0} i_{2})) = (-1)^{m}, \quad v(\tau) = (-1)^{1+l},
\end{align*}
\]

where \( T \in \text{Her}_{2}(\mathcal{O}) \).

Note that due to chapter 3 we already know that \( \chi_{\det}^{l}x_{i}^{2m+1} \) is a multiplier system for \( \Gamma(\mathcal{O})_{\text{Maaß}} \) of odd weight. But again, the multiplier systems for \( \Gamma(\mathcal{O})_{\text{Bor}} \) specified in the preceding theorem are just hypothetical possible multiplier systems! Due to the considerations above, we know that if there exist non-identically vanishing quaternionic modular forms of odd weight with respect to \( \Gamma(\mathcal{O})_{\text{Bor}} \), then their multiplier systems have to look like noted in the theorem. This is because we only “defined” it for generators. Nevertheless, we definitely have this correspondence, and the only question that is still unanswered is if we are actually talking about trivial spaces or not. But as soon as we find non-identically vanishing orthogonal modular forms of odd weight with respect to \( \Gamma_{\text{Bor}} \) and one of the characters specified above, then in view of (1.52) we also know that the corresponding, yet only hypothetical multiplier system turns out to be a multiplier system, indeed.

For the sake of convenience, we also discuss the correspondence with Hermitian modular forms in short. Note that it can be found in full detail in [Kl06, pp.53], while the complete theory about Hermitian modular forms can be found in [De01]. We only state the results, here.

We already introduced Hermitian modular forms in section 1.5. So first, we need to fix an imaginary quadratic number field \( \mathbb{F} = \mathbb{Q}(\sqrt{-\Delta_F}) \) of class number one, where \( \Delta_F \equiv 0 \) or \( \Delta_F \equiv 3 \mod 4 \) shall hold. (Note that we only consider imaginary quadratic number fields of class number one here, since their structure of the cusps is simple – cf. [De01].) Then a classic result from algebraic number theory is that the ring of integers in \( \mathbb{F} \) is given by \( \mathfrak{o}_F = \mathbb{Z} + \mathbb{Z}\omega_F \), where

\[
\omega_F = \begin{cases} 
\frac{i\sqrt{\Delta_F}}{2}, & \text{if } \Delta_F \equiv 0 \mod 4, \\
\frac{(1+i\sqrt{\Delta_F})}{2}, & \text{if } \Delta_F \equiv 3 \mod 4.
\end{cases}
\]
Define the extended Hermitian modular group with respect to \( \mathcal{F} \) as

\[
\Gamma(\sigma_\mathcal{F}) := \langle Z \mapsto M(Z), \tau; M \in \text{Sp}_2(\sigma_\mathcal{F}) \rangle,
\]

which means it is a subgroup of the automorphisms of \( \mathcal{H}_2(\mathcal{C}) \) (or \( \tilde{\mathcal{H}}_2(\mathcal{C}) \)). Then Hermitian modular forms are defined like we described in section 1.5, where in addition we have some transformation behavior regarding \( Z \mapsto Z' \). The correspondence of Hermitian modular forms and orthogonal modular forms is given as follows: Let

\[
S_\mathcal{F} = \begin{pmatrix}
2 & 2 \text{Re}(\omega_\mathcal{F}) \\
2 \text{Re}(\omega_\mathcal{F}) & 2|\omega_\mathcal{F}|^2
\end{pmatrix}.
\]

Then

\[
\varphi_\mathcal{F} : \mathcal{H}_{S_\mathcal{F}} \to \tilde{\mathcal{H}}_2(\mathcal{C}), \quad (x_1, u_1, u_2, x_2) + i(y_1, v_1, v_2, y_2) \mapsto \left(\begin{array}{c}
x_1 + i y_1 \\
(u_1 + \omega_\mathcal{F} u_2) + i(v_1 + \omega_\mathcal{F} v_2)
\end{array}\right)
\]

biholomorphically maps the orthogonal half-space \( \mathcal{H}_{S_\mathcal{F}} \) to the Hermitian half-space \( \tilde{\mathcal{H}}_2(\mathcal{C}) \). Hence, according to the remarks in chapter 1 there also exists a biholomorphical mapping from \( \mathcal{H}_{S_\mathcal{F}} \) to \( \mathcal{H}_2(\mathcal{C}) \) (which is given by appropriately identifying \( \omega_\mathcal{F} \) in \( \mathcal{H} \)). Therefore, the group of automorphisms of \( \mathcal{H}_{S_\mathcal{F}} \) is isomorphic to the group of isomorphisms of \( \mathcal{H}_2(\mathcal{C}) \), again. And just like in (5.29), we can start identifying elements in \( \Gamma_{S_\mathcal{F}} \) with elements in \( \Gamma(\sigma_\mathcal{F}) \). Doing so, we obtain

\[
\Gamma(\sigma_{Q(\sqrt{-1})}) \simeq \Gamma_{A_1[2]} / \{ \pm I \}, \quad \Gamma(\sigma_{Q(\sqrt{-3})}) \simeq \Gamma_{A_2} / \{ \pm I \}, \quad \Gamma(\sigma_{Q(\sqrt{-7})}) \simeq \Gamma_{D_2} / \{ \pm I \}.
\]

We omit the details, here, but note that the result for \( Q(\sqrt{-1}) \) and \( Q(\sqrt{-3}) \) can already be found in [Kl06, p.55]. Furthermore, sticking to the notation of [De98], a result from that work is that the groups of abelian characters are given as follows:

\[
\Gamma(\sigma_{Q(\sqrt{-1})})^{ab} = \langle \text{det}, v_p, v_{\text{skew}} \rangle, \quad \Gamma(\sigma_{Q(\sqrt{-3})})^{ab} = \langle \text{det}, v_{\text{skew}} \rangle, \quad \Gamma(\sigma_{Q(\sqrt{-7})})^{ab} = \langle v_{\text{skew}} \rangle
\]

Analogous to [Kl06, thm.2.35] (where only the case \( Q(\sqrt{-7}) \) is new here, which can be verified accordingly), we have:

(5.34) Theorem. Let \( k \in \mathbb{N}_0 \) and \( l, m, n \in \{0, 1\} \).

a) If \( k \) is even, then \( f \in \Gamma(\sigma_{Q(\sqrt{-1})}) \), \( k, \text{det}^l v_p^m v_{\text{skew}}^{n}\) \iff \( f \circ \varphi_{Q(\sqrt{-1})} \in \Gamma_{A_1[2]} k, \chi_{\pi}^{l+k/2} v_p^m \chi_{\det}^n \), with \( v_2 \) defined in [Kl06, pp.22].

b) \( f \in \Gamma(\sigma_{Q(\sqrt{-3})}) \), \( k, \text{det}^l v_{\text{skew}}^m \) \iff \( f \circ \varphi_{Q(\sqrt{-3})} \in \Gamma_{A_2} k, \chi_{\pi}^{k} \chi_{\det}^{n+k} \).

c) \( f \in \Gamma(\sigma_{Q(\sqrt{-7})}) \), \( k, v_{\text{skew}}^m \) \iff \( f \circ \varphi_{Q(\sqrt{-7})} \in \Gamma_{A_2} k, \chi_{\pi}^{k} \chi_{\det}^{n+k} \).
And finally, note that $S = 2 \in \mathbb{Z}^{1 \times 1}$ would yield Siegel modular forms, which is a classical and well known result. Later on, we will also have to consider $S = 2n$, where $n \in \mathbb{N}$, $n > 1$. In this case, the corresponding orthogonal modular forms can be identified with paramodular forms. But we will get to that at the end of this thesis.

For now, just bear in mind that we can skip between the orthogonal and the symplectic world, whichever suits best for the current questions. In the previous chapters, the symplectic setting yielded a lot of results and important modular forms. But for further analysis, in particular concerning the exact shape of the spaces of modular forms and the graded rings of modular forms, the orthogonal setting seems to be the better choice – mainly due to the Borcherds products that will play an important role.

5.4 Introduction to vector-valued modular forms

In this last section of chapter 5 we will introduce so-called vector-valued modular forms. They will be needed to construct Borcherds products, which will play an important role in the subsequent chapter. Again, we will only present the basics, so that it is clear what objects we are talking about once we finally introduce Borcherds products. Further details can for example be found in [Kl06, ch.3].

Since we will have to deal with half-integral weights, we need the so-called metaplectic group, which is a double cover of the special linear group. Recall that for integral weight (or even weight for quaternionic modular forms) we needed to consider abelian characters. But for non-integral weight (or odd weight for quaternionic modular forms), we had to introduce the more general multiplier systems. Now, in the case of elliptic modular forms (and also for vector-valued modular forms) these multiplier systems are substituted by the metaplectic group. The reason why we had to consider the more general multiplier systems was that if the exponent $k$ in $\det(M\{Z\})^k$ (or in $\det(\tilde{M}\{\tilde{Z}\})^k$) is not integral, then problems occur when multiplying two such expressions. In the one-dimensional and the vector-valued case, the issue concerning half-integral weight is solved differently:

\[(M, \varphi)\] where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, and \(\varphi : \mathbb{H} \to \mathbb{C}\) is a holomorphic function satisfying $\varphi^2(\tau) = c\tau + d$ for all $\tau \in \mathbb{H}$, i.e. $\varphi$ is a holomorphic root of $\tau \mapsto c\tau + d$. $\text{Mp}_2(\mathbb{R})$ turns out to be a group, where the product of two elements $(M_1, \varphi_1), (M_2, \varphi_2) \in \text{Mp}_2(\mathbb{R})$ is given by

\[(M_1, \varphi_1) \cdot (M_2, \varphi_2) = (M_1M_2, \varphi_1(M_2(\cdot))(\cdot)\varphi_2).\]
Furthermore, \( \mathrm{Mp}_2(\mathbb{R}) \) acts on \( \mathcal{H} \), where for \( (M, \varphi) \in \mathrm{Mp}_2(\mathbb{R}) \) and \( \tau \in \mathcal{H} \) the action is given by

\[
(M, \varphi)(\tau) = M(\tau) = \frac{a \tau + b}{c \tau + d}.
\]

\( \mathrm{SL}_2(\mathbb{R}) \) can be embedded into \( \mathrm{Mp}_2(\mathbb{R}) \), indeed: Suppose \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{R}) \). If \( c = 0 \), then \( d \neq 0 \) has to follow, and we can choose \( \sqrt{d} = e^{\frac{1}{2}\log(d)} \), where \( \log(d) \) is the principal value of the logarithm, i.e. \( \sqrt{d} > 0 \) is the ordinary root for \( d > 0 \), and \( \sqrt{d} = i\sqrt{-d} \) for \( d < 0 \). For \( c > 0 \) we have \( ct + d \in \mathcal{H} \) for all \( \tau \in \mathcal{H} \), while for \( c < 0 \) we obtain \( ct + d \in \{ z \in \mathbb{C} \mid \text{Im}(z) < 0 \} \). So in both cases, we have \( c \tau + d \in \mathbb{C} \setminus \mathbb{R} \), hence again we can choose the principal value \( \log \) (where \( \sqrt{i} = e^{\frac{1}{4}\pi i} \) and \( \sqrt{-i} = e^{-\frac{1}{4}\pi i} \)). Therefore, just like in [Br02] we can define the embedding of \( \mathrm{SL}_2(\mathbb{R}) \) into \( \mathrm{Mp}_2(\mathbb{R}) \) as

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \left( \begin{array}{cc} \sqrt{|d|} & \phi \\ \phi & \frac{1}{\sqrt{|d|}} \end{array} \right), \quad \phi = \frac{1}{2} \log(c \tau + d).
\]

It is well known that \( \mathrm{Mp}_2(\mathbb{Z}) \) is generated by

\[
T_{\mathrm{Mp}} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \quad J_{\mathrm{Mp}} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad \sqrt{\tau} = e^{\frac{1}{2}\log(\tau)}
\]

and that the center of \( \mathrm{Mp}_2(\mathbb{Z}) \) is generated by

\[
C_{\mathrm{Mp}} := J_{\mathrm{Mp}}^2 = (J_{\mathrm{Mp}}T_{\mathrm{Mp}})^3 = \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right), i.
\]

For \( N \in \mathbb{N} \) we denote the principal congruence subgroup of \( \mathrm{Mp}_2(\mathbb{Z}) \) of level \( N \) by

\[
\mathrm{Mp}_2(\mathbb{Z})[N] := \{(M, \varphi) \in \mathrm{Mp}_2(\mathbb{Z}) \mid M \equiv I \text{ mod } N\}.
\]

And finally, we set

\[
\tilde{\Gamma}_\infty := \left\{ \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) \mid n \in \mathbb{Z} \right\} = \langle T_{\mathrm{Mp}} \rangle \leq \mathrm{Mp}_2(\mathbb{Z}).
\]
5.4 Introduction to vector-valued modular forms

function \( g : \mathbb{C} \to V \) is called vector-valued.

For a vector-valued function \( f : \mathcal{H} \to V \) and \((M, \varphi) \in \text{Mp}_2(\mathbb{Z})\) we define the slash-operator by

\[
(f|_k(M, \varphi))(\tau) := \varphi(\tau)^{-2k} f(M(\tau)), \quad \tau \in \mathcal{H}.
\]

(5.34)

Note that this defines an action of \( \text{Mp}_2(\mathbb{Z}) \) on the space of vector-valued functions \( f : \mathcal{H} \to V \). This can be verified making use of the cocycle relations, just like in 1.15. And note that for \( k \in \mathbb{Z} \), we simply have \( f|_k(M, \varphi) = f|_kM \). The slash-operator gives rise to the definition of vector-valued modular forms:

(5.36) **Definition.** Suppose that \( \rho \) is a finite representation of \( \text{Mp}_2(\mathbb{Z}) \) on \( V \) (i.e. \( \rho : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(V) \simeq \text{GL}_n(\mathbb{C}) \), where \( n = \dim(V) \), is a group homomorphism, and \( \rho(\text{Mp}_2(\mathbb{Z})) \) is finite). A (holomorphic) vector-valued modular form of weight \( k \) with respect to \( \rho \) and \( \text{Mp}_2(\mathbb{Z}) \) is a function \( f : \mathcal{H} \to V \) satisfying

(VV.1) \( f|_k g = \rho(g) f \) for all \( g \in \text{Mp}_2(\mathbb{Z}) \),

(VV.2) \( f \) is holomorphic on \( \mathcal{H} \),

(VV.3) \( f \) is bounded on \( \{ \tau \in \mathcal{H} \ ; \ \text{Im}(\tau) > y_0 \} \) for all \( y_0 > 0 \).

If in addition \( f \) satisfies \( \lim_{\text{Im}(\tau) \to \infty} f(\tau) = 0 \), then \( f \) is called a (vector-valued) cusp form. We denote that space of all (holomorphic) vector-valued modular forms of weight \( k \) with respect to \( \rho \) and \( \text{Mp}_2(\mathbb{Z}) \) by \( [\text{Mp}_2(\mathbb{Z}), k, \rho] \), and the subspace of cusp forms by \( [\text{Mp}_2(\mathbb{Z}), k, \rho]_0 \).

We cite the following remark from [Kl06, rem.3.2]:

(5.37) **Remark.**

a) Since \( \text{Mp}_2(\mathbb{Z}) \) is generated by \( T_{\text{Mp}} \) and \( J_{\text{Mp}} \) and because the slash-operator is an action in this new setting, too, condition (VV.1) is equivalent to

(VV.1') \( f(\tau + 1) = \rho(T_{\text{Mp}}) f(\tau) \) and \( f(-\tau^{-1}) = \sqrt{\tau}^{2k} \rho(J_{\text{Mp}}) f(\tau) \)

for all \( \tau \in \mathcal{H} \), where \( \sqrt{\tau}^{2k} = e^{k \Log(\tau)} \).

b) We denote the components of \( f \) by \( f_v \), so that \( f = \sum_{v \in B} f_v v \). Obviously, \( f \) is holomorphic if and only if all its components \( f_v \) are holomorphic. Furthermore, since \( \rho \) is a finite representation, there exists \( N \in \mathbb{N} \) such that \( T_{\text{Mp}}^N \in \ker(\rho) \), and thus \( f(\tau + N) = f(\tau) \) for all \( \tau \in \mathcal{H} \). Therefore, each \( f_v \) possesses a Fourier-expansion of the shape

\[
f_v(\tau) = \sum_{n \in \mathbb{Z}} c_v(n/N) e^{2\pi i n \tau/N}, \quad \tau \in \mathcal{H}.
\]

But then of course condition (VV.3) is equivalent to

(VV.3') \( f \) possesses a Fourier-expansion of the form

\[
f(\tau) = \sum_{v \in B} \sum_{n \in \mathbb{N}_0} c_v(n/N) e^{2\pi i n \tau/N} v, \quad \tau \in \mathcal{H}.
\]
Moreover, \( f \) is a cusp form if and only if \( c_v(0) = 0 \) for all \( v \in B \).

From [Kl06, prop.3.4, prop.3.5] we cite:

**(5.38) Proposition.** Suppose that \( \tilde{V} \) is another finite dimensional \( \mathbb{C} \)-vector space and \( \tilde{k} \in \frac{1}{2} \mathbb{Z} \). Let \( \rho \) and \( \tilde{\rho} \) be finite representations of \( \text{Mp}_2(\mathbb{Z}) \) on \( V \) and \( \tilde{V} \), respectively.

a) If \( f \in [\text{Mp}_2(\mathbb{Z}), k, \rho] \) and \( \tilde{f} \in [\text{Mp}_2(\mathbb{Z}), \tilde{k}, \tilde{\rho}] \), then \( f \otimes \tilde{f} \in [\text{Mp}_2(\mathbb{Z}), k + \tilde{k}, \rho \otimes \tilde{\rho}] \) holds for \( f \otimes \tilde{f} : \mathcal{H} \to V \otimes \tilde{V} \), \( \tau \mapsto f(\tau) \otimes \tilde{f}(\tau) \).

In particular, if \( f \) is scalar-valued (i.e. \( V \cong \mathbb{C} \)), then \( f \cdot \tilde{f} \in [\text{Mp}_2(\mathbb{Z}), k + \tilde{k}, \rho \cdot \tilde{\rho}] \).

Now, additionally suppose that \( \ker(\rho) \subset \text{Mp}_2(\mathbb{Z})[N] \) holds for some \( N \in \mathbb{N} \).

b) \( [\text{Mp}_2(\mathbb{Z}), k, \rho] = \{0\} \) if \( k < 0 \).

c) \( [\text{Mp}_2(\mathbb{Z}), 0, \rho] \cong \mathbb{C}^j \), where \( j \) is the multiplicity of the trivial one-dimensional representation in \( \rho \).

d) \( \dim[\text{Mp}_2(\mathbb{Z}), k, \rho] < \infty \) for all \( k \in \mathbb{Z} \).

For the construction of Borcherds products later on we need so-called weakly holomorphic (vector-valued) modular forms, where poles at infinity are permitted.

**(5.39) Definition.** Suppose that \( \rho \) is a finite representation of \( \text{Mp}_2(\mathbb{Z}) \) on \( V \). A weakly holomorphic vector-valued modular forms of weight \( k \) with respect to \( \rho \) and \( \text{Mp}_2(\mathbb{Z}) \) is a function \( f : \mathcal{H} \to V \) satisfying

(W.1) \( f|_g = \rho(g)f \) for all \( g \in \text{Mp}_2(\mathbb{Z}) \),

(W.2) \( f \) is holomorphic on \( \mathcal{H} \),

(W.3) \( f \) has at most a pole in \( \infty \), i.e. there exists \( n_0 \in \mathbb{Z} \) such that \( f \) possesses a Fourier-expansion of the shape

\[
f(\tau) = \sum_{v \in B} \sum_{n \geq n_0} c_v(n/N)e^{2\pi in\tau/N} v, \quad \tau \in \mathcal{H}.
\]

We denote the space of all weakly holomorphic vector-valued modular forms of weight \( k \) with respect to \( \rho \) and \( \text{Mp}_2(\mathbb{Z}) \) by \( [\text{Mp}_2(\mathbb{Z}), k, \rho] \). The principal part of \( f \) is defined to be

\[
\sum_{v \in B} \sum_{n \in \mathbb{Z}} c_v(n/N)e^{2\pi in\tau/N} v, \quad \tau \in \mathcal{H}.
\]

To define and construct Borcherds products, we will need a special representation, called the **Weil representation**. Confer [Kl06, pp.68] about the details.
**Definition.** Suppose that \( S \in \text{Sym}_l(\mathbb{R}) \) (where \( l \in \mathbb{N} \)) is an even matrix of signature \((b^+, b^-)\) (see 5.2). Let \( \Lambda = \mathbb{Z}^l \) be the associated lattice with bilinear form \((\cdot, \cdot)_S\) and the corresponding quadratic form \( q = q_S \). The standard basis of the group ring \( \mathbb{C}[\Lambda^\natural/\Lambda] \) shall be given by \( (e_\mu)_{\mu \in \Lambda^\natural/\Lambda} \).

Then there is a unitary representation \( \rho_S \) of \( \text{Mp}_2(\mathbb{Z}) \) on \( \mathbb{C}[\Lambda^\natural/\Lambda] \) which is defined by

\[
\rho_S(T_{\text{Mp}})e_\mu = e^{2\pi i q(\mu)} e_\mu, \\
\rho_S(J_{\text{Mp}})e_\mu = \frac{\sqrt{i^{b^- - b^+}}}{\sqrt{|\det(S)|}} \sum_{v \in \Lambda^\natural/\Lambda} e^{-2\pi i (\mu, v)} e_v.
\]

Note that this implies \( \rho_S(C_{\text{Mp}})e_\mu = i^{b^- - b^+} e_{-\mu} \).

Furthermore, we will always see \( \rho_S((M, \varphi)) \) as a matrix in \( \mathbb{C}^{n \times n} \), where \( n = \#\Lambda^\natural/\Lambda \).

According to [Kl06, p.68], \( \rho_S \) is a finite representation of \( \text{Mp}_2(\mathbb{Z}) \): If \( N \in \mathbb{N} \) denotes the level of \( \Lambda \) (see (5.1)), then \( \text{Mp}_2(\mathbb{Z})[N] \subset \ker(\rho_S) \) holds. And the dual representation \( \rho^\natural_S \) fulfills

\[
\rho^\natural_S = \overline{\rho} = \rho_S. \tag{5.35}
\]

We will have to consider vector-valued modular forms with respect to the Weil representation and its dual. According to [Kl06, p.69], the following holds: If \( f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S] \), then \( f \) possesses a Fourier-expansion of the shape

\[
f(\tau) = \sum_{\mu \in \Lambda^\natural/\Lambda} \sum_{n \in q(\mu) + \mathbb{Z}} c_\mu(n) e^{2\pi i n \tau} e_\mu, \quad \tau \in \mathcal{H}. \tag{5.36}
\]

And if \( f \in [\text{Mp}_2(\mathbb{Z}), k, \overline{\rho}_S] \), then \( f \) possesses a Fourier-expansion of the shape

\[
f(\tau) = \sum_{\mu \in \Lambda^\natural/\Lambda} \sum_{n \in -q(\mu) + \mathbb{Z}} c_\mu(n) e^{2\pi i n \tau} e_\mu, \quad \tau \in \mathcal{H}. \tag{5.37}
\]

Sticking to this notation and citing [Kl06, prop.3.7, prop.3.8, cor.3.9], we have:

**Proposition.**

a) If \( 2k \not\equiv b^+ - b^- \mod 2 \) then

\[
[\text{Mp}_2(\mathbb{Z}), k, \rho_S] = \{0\}.
\]

b) Let \( 2k \equiv b^+ - b^- \mod 2 \) and \( f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S] \) with Fourier-expansion according to 5.36. Then

\[
c_{-\mu}(n) = (-1)^{(2k + b^- - b^+)/2} c_\mu(n)
\]

for all \( \mu \in \Lambda^\natural/\Lambda \) and \( n \in q(\mu) + \mathbb{Z} \).
c) If $2k + b^- - b^+ \equiv 2 \mod 4$ and $\mu = -\mu$ for all $\mu \in \Lambda^j / \Lambda$ then 

$$[\text{Mp}_2(\mathbb{Z}), k, \rho_S] = \{0\}.$$ 

Later on, we will need to calculate the dimension of certain spaces of vector-valued modular forms. In virtue of [Kl06, thm.3.10] we have the following dimension formula, which is due to [ES95] and also [Sk84]:

(5.42) Theorem. Let $\rho$ be a finite representation of $\text{Mp}_2(\mathbb{Z})$ on $V$ such that $\rho(C_{\text{Mp}}) = e^{-\pi i k} \text{id}_V$. Then the dimension of $[\text{Mp}_2(\mathbb{Z}), k, \rho]$ is given by the following formula

$$\dim[\text{Mp}_2(\mathbb{Z}), k, \rho] - \dim[\text{Mp}_2(\mathbb{Z}), 2 - k, \rho] = \frac{k + 5}{12} n + \frac{1}{4} \text{Re}(e^{\pi i k / 2 \text{tr}(\rho(J_{\text{Mp}})))} + \frac{2}{3\sqrt{3}} \text{Re}(e^{\pi i (2k + 1) / 6 \text{tr}(\rho(J_{\text{Mp}} T_{\text{Mp}})))} - \sum_{j=1}^n \lambda_j,$$

where $n = \dim(V)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$, $0 \leq \lambda_j < 1$, such that the eigenvalues of $\rho(T_{\text{Mp}})$ are $e^{2\pi i \lambda_j}$.

(5.43) Remark. a) If $k \geq 2$ then the formula from above directly yields $\dim[\text{Mp}_2(\mathbb{Z}), k, \rho]$, since the dimension of the spaces of cusp forms of non-positive weight is zero (see (5.38)). In the cases $k = \frac{1}{2}$ and $k = \frac{3}{2}$ the dimension of $[\text{Mp}_2(\mathbb{Z}), k, \rho]$ can be calculated explicitly, too (cf. [Sk84]). But for the case $k = 1$ there seems to be no formula.

b) Unfortunately, the dimension formula is not directly applicable to the Weil representation $\rho_S$ in general. The precondition on $\rho_S(C_{\text{Mp}})$ is usually not satisfied since $\rho_S(C_{\text{Mp}}) e_\mu = i^{b^- - b^+} e_{-\mu}$ (see (5.40)). But fortunately, there is a way to evade this problem: Let $f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S]$. According to (5.41) we have $f_\mu = f_{-\mu}$ if $2k + b^- - b^+ \equiv 0 \mod 4$, and $f_\mu = -f_{-\mu}$ if $2k + b^- - b^+ \equiv 2 \mod 4$. Furthermore, note that due to the special shape of the Weil representation (see (5.40)) one easily verifies that $\rho_S$ stabilizes the subspaces spanned by $\{e_\mu + e_{-\mu} : \mu \in \Lambda^j / \Lambda\}$ on the one hand, and $\{e_\mu - e_{-\mu} : \mu \in \Lambda^j / \Lambda\}$ on the other hand. Therefore, if $\rho_S^+$ and $\rho_S^-$ denote the induced representations on these two subspaces, then $f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S^+]$ follows if $2k + b^- - b^+ \equiv 0 \mod 4$, and $f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S^-]$ if $2k + b^- - b^+ \equiv 2 \mod 4$. Moreover, due to $\rho_S(C_{\text{Mp}}) e_\mu = i^{b^- - b^+} e_{-\mu}$, the precondition of the dimension formula concerning $\rho(C_{\text{Mp}})$ is fulfilled for $\rho_S^+$ and $\rho_S^-$. Hence we simply have to calculate the dimension of $[\text{Mp}_2(\mathbb{Z}), k, \rho_S^+]$ or $[\text{Mp}_2(\mathbb{Z}), k, \rho_S^-]$ (whichever case is at hand) using the dimension formula, and this dimension coincides with the dimension of $[\text{Mp}_2(\mathbb{Z}), k, \rho_S]$.

And finally, we need two important examples of vector-valued modular forms, namely the (vector-valued) Eisenstein-series und (vector-valued) theta-series. It will turn out that in the cases of small weights we will have to consider, these suffice to span the spaces of vector-valued modular forms. Again, we only present the basic results, here. Details can be found in [Kl06, pp.73].

For the rest of this section, let $S \in \text{Sym}_1(\mathbb{R})$ be an even matrix of signature $(b^+, b^-)$, and let $\Lambda = \mathbb{Z}^j$ be the lattice attached to $S$. First, we define the Eisenstein-series.
(5.44) **Definition.** Let \( k \in \frac{1}{2}\mathbb{Z}, k > 2 \), such that \( 2k - b^+ + b^- \equiv 0 \) mod 4, and \( v \in \mathbb{C}[\Lambda^2/\Lambda] \) such that \( \rho_S(T\wp) v = v \). Then we define the Eisenstein-series \( E_k(\cdot; v, S) : \mathcal{H} \to \mathbb{C}[\Lambda^2/\Lambda] \) by

\[
E_k(\tau; v, S) = \frac{1}{2} \sum_{g \in \Gamma_\infty \setminus \mathbb{M}_2(\mathbb{Z})} \rho_S(g)^{-1}(v|kg)(\tau),
\]

where \( g \) runs through a transversal of \( \tilde{\Gamma}_\infty \setminus \mathbb{M}_2(\mathbb{Z}) \).

Note that due to \( \rho_S(T) v = v \) the definition is independent of the choice of a transversal. And like in the scalar case, the series converges normally on \( \mathcal{H} \) for \( k > 2 \), indeed.

Furthermore, let \( v = \sum_{\mu \in \Lambda^2/\Lambda} a(\mu) e_\mu \in \mathbb{C}[\Lambda^2/\Lambda] \). Then the condition \( \rho_S(T) v = v \) is equivalent to \( a(\mu) = 0 \) for all \( \mu \in \Lambda^2/\Lambda \) with \( q(\mu) \notin \mathbb{Z} \). Moreover, we obviously have \( E_k(\cdot; v, S) = \sum_{\mu \in \Lambda^2/\Lambda} a(\mu) E_k(\cdot; e_\mu, S) \) (if we ignore that \( E_k(\cdot; e_\mu, S) \) is not defined if \( q(\mu) \notin \mathbb{Z} \), since in this case we have \( a(\mu) = 0 \). Thus it suffices to consider the Eisenstein-series \( E_k(\cdot; e_\beta, S) \) for \( \beta \in \Lambda^2/\Lambda \) with \( q(\beta) \in \mathbb{Z} \). For these, we have to following proposition (cf. [Kl06, prop.3.17]):

(5.45) **Proposition.** Let \( k \in \frac{1}{2}\mathbb{Z}, k > 2 \), such that \( 2k - b^+ + b^- \equiv 0 \) mod 4, and \( \beta \in \Lambda^2/\Lambda \) with \( q(\beta) \in \mathbb{Z} \). Then

\[
E_k(\cdot; e_\beta, S) \in [\mathbb{M}_2(\mathbb{Z}), k, \rho_S].
\]

Note that in the cases we will have to consider, \( q(\beta) \in \mathbb{Z} \) will only hold for \( \beta = 0 \in \Lambda^4/\Lambda \). Hence we use the following abbreviation:

\[
E_k(\cdot; S) := E_k(\cdot; e_0, S). \tag{5.38}
\]

Moreover, suppose that \( S \) is positive definite (and still of size and rank \( l \)). Then \( S_1 \) (see (5.3)) is of signature \((2, l + 2)\). Like we have seen in (5.3), \( \Lambda^2/\Lambda \cong \Lambda_{l+2}^2/\Lambda_1 \) holds, and we have \((\mu, v)_S + \mathbb{Z} = (\mu, v)_{-S_1} + \mathbb{Z} \) for all \( \mu, v \in \Lambda^2/\Lambda \) (if for the right hand side \( \mu \) and \( v \) are identified in \( \Lambda_{l+2}^2/\Lambda_1 \) accordingly). Therefore, we obtain

\[
E_k(\cdot; -S_1) \in [\mathbb{M}_2(\mathbb{Z}), k, \rho_{-S_1}] = [\mathbb{M}_2(\mathbb{Z}), k, \rho_S]. \tag{5.39}
\]

Next, note that Bruinier and Kuss investigated these vector-valued Eisenstein-series in [BK01]. But note that instead of \( S \) they used \( -S \) in (5.44) for the definition of vector-valued Eisenstein-series. Hence our \( E_k(\cdot; -S_1) \) is identical to the vector-valued Eisenstein-series Bruinier and Kuss investigated with respect to \( S_1 \) (and not \( -S_1 \)). Note again that the size (and rank) of \( S_1 \) is \( l + 4 \) and its signature is \((2, l + 2)\). Therefore, if actually \( k = l/2 + 2 \) holds, then the preconditions of [BK01, thm.4.8] are met, and that theorem then yields an explicit formula for the Fourier-coefficients of \( E_{l/2+2}(\cdot; -S_1) \) (where, to say it again, \( -S_1 \) has to be replaced by \( S_1 \) in that theorem in order to utilize this formula, which means that the bilinear form \((\cdot, \cdot)_{S_1} \) has to be used). Furthermore, for the current thesis a program was implemented using [SAGE] which calculates the Fourier-coefficients of \( E_{l/2+2}(\cdot; -S_1) \) for small exponents \( n \).

Next, we introduce vector-valued theta-series.
**5.46 Definition.** Suppose that additionally \( S \) is positive definite (hence \( b^+ = l, b^- = 0 \)).

a) Let \( r \in \mathbb{N}_0 \), and additionally \( r \leq 1 \) if \( l = 1 \). A homogeneous spherical polynomial of degree \( r \) with respect to \( S \) is a function \( p_r : \mathbb{R}^l \to \mathbb{C} \) of the shape

\[
p_r(x) = \sum_{v \in C^l} a_v (v'Sx)^r
\]

with \( a_v \neq 0 \) for only finitely many \( v \in C \), which also have to satisfy \( S[v] = 0 \) if \( r > 1 \).

b) Let \( p_r \) be a homogeneous spherical polynomial of degree \( r \) with respect to \( S \). Then we define the vector-valued theta-series \( \Theta(\cdot; S, p_r) : \mathcal{H} \to \mathbb{C}[\Lambda^2/\Lambda] \) by

\[
\Theta(\tau; S, p_r) = \sum_{\mu \in \Lambda^2/\Lambda} \theta_{\mu}(\tau; S, p_r) e_{\mu},
\]

where

\[
\theta_{\mu}(\tau; S, p_r) = \sum_{\lambda \in \mu + \Lambda} p_r(\lambda) e^{\pi i S[\lambda] \tau}, \quad \tau \in \mathcal{H}.
\]

The vector-valued theta-series turn out to be vector-valued modular forms (cf. [Kl06, thm.3.19]).

**5.47 Proposition.** Suppose that the setting of (5.46) is given. Then

\[
\Theta(\cdot; S, p_r) \in [\text{Mp}_2(\mathbb{Z}), l/2 + r, \rho_S]
\]

holds. If \( r > 0 \) then \( \Theta(\cdot; S, p_r) \) is a cusp form. The Fourier-expansion of the components \( \theta_{\mu} \) of \( \Theta \), where \( \mu \in \Lambda^2/\Lambda \), is given by

\[
\theta_{\mu}(\tau; S, p_r) = \sum_{n \in q(\mu) + \mathbb{Z}} c_\mu(n; p_r) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H},
\]

with

\[
c_\mu(n; p_r) = \sum_{\lambda \in \mu + \Lambda} p_r(\lambda) .
\]

Note that

\[
c \cdot \Theta(\cdot; S, p_r) + \Theta(\cdot; S, \tilde{p}_r) = \Theta(\cdot; S, c \cdot p_r + \tilde{p}_r)
\]

holds for \( c \in \mathbb{C} \) and a second homogeneous spherical polynomial \( \tilde{p}_r \) of degree \( r \) with respect to \( S \). Hence we only have to consider a basis of all homogeneous spherical polynomials of degree \( r \) with respect to \( S \). Also Dern investigated vector-valued theta-series and homogeneous spherical polynomials in [De01, pp.62]. According to that, homogeneous spherical polynomials can be characterized differently: Let \( v_1, \ldots, v_l \) be an orthogonal, normalized basis of \( \mathbb{C}[\Lambda^2/\Lambda] \) (with respect to \( \langle \cdot, \cdot \rangle_S \)). Then \( p_r \) is a homogeneous spherical polynomial of degree \( r \) with respect to \( S \) if
and only if

\[ \tilde{p}_r : \mathbb{R}^l \to \mathbb{C}, \quad (x_1, \ldots, x_l) \mapsto p_r(\sum_{j=1}^l x_j v_j) \]

is homogeneous of degree \( r \) (in the classical sense) and harmonic with respect to the Laplace operator \( \sum_j \partial^2_{x_j} \). But the space of homogeneous, harmonic polynomials of degree \( r \) can be computed explicitly, of course. It is finite dimensional and a basis can be computed using some computer algebra system. For this thesis, a small program using [SAGE] was written which yields such a basis. Furthermore, a program was implemented in [MAGMA] calculating the Fourier-coefficients of \( \Theta(\cdot; S, p_r) \).
6 A Reduction Process

In this final chapter we will present some sort of reduction process. The ultimate goal would be to explicitly determine the spaces of quaternionic modular forms with respect to $\text{Sp}_2(\mathcal{O})$ or $\Gamma(\mathcal{O})$ and the graded rings attached to them. To do so, we will introduce Borcherds products. These are orthogonal modular forms (which can be identified with certain symplectic modular forms in view of the preceding chapter, if the right orthogonal groups are chosen) with an important and fascinating property: Their divisors (i.e. zero sets) are given explicitly, including multiplicities, and these can be controlled to some degree. They are lifts of certain weakly holomorphic vector-valued modular forms, and depending on how “good-natured” the associated “control spaces” are, the divisors can be dictated, where some construction rules have to be met. The reduction process is the following, then: Let $f$ be an orthogonal modular form of weight $k_1$ and $\psi$ such a Borcherds product of weight $k_2$ (and both with respect to the same group), and suppose that the zero set of $\psi$ is included in the zero set of $f$ (including multiplicities), then $f/\psi$ is again holomorphic, and thus an orthogonal modular form of weight $k_1-k_2$. Next, suppose that we found some orthogonal modular forms $g_1, \ldots, g_n$ such that for every $f$ we can find a polynomial (homogeneous with respect to the weights) such that $f-p(g_1, \ldots, g_n)$ vanishes identically when restricting to some submanifold. Now, if the divisor of $\psi$ would be exactly given by that submanifold (of order one), then due to the consideration above we have $f = p(g_1, \ldots, g_n) + \psi g$ for some orthogonal modular form $g$ of lesser weight. But then an induction yields a basis for the spaces of orthogonal modular forms and also for the graded rings. Of course, this is the most fortunate case, but the approach becomes clear, in principle.

So next, the question arises how to find such modular forms $g_1, \ldots, g_n$. But this could be done if one already knew the structure of the graded rings of the orthogonal modular forms living on that submanifold: If one can find $g_1, \ldots, g_n$ such that their restrictions are a basis for those graded rings, then we have found the orthogonal modular forms we were searching for. Therefore, we obtain some sort of reduction process: To determine a basis for the orthogonal modular forms (and hence for the symplectic modular forms) with respect to a given lattice, we first need to find a basis for the orthogonal modular forms with respect to some sublattices. For example, regarding $A_2^{(2)}$, it would be useful (and eventually crucial) to know the structure of the spaces of orthogonal modular forms attached to $A_{2,1}$ and $T_3$.

Ultimately, we will “go down” until we end up with paramodular forms, hence with lattices attached to a scalar $S$. Unfortunately, we will not be able to determine the structure of those spaces. But nevertheless, we will point out which problems have to be solved first before being able to determine the spaces attached to $A_2^{(2)}$, hence the spaces of quaternionic modular forms with respect to $\mathcal{O}$. And we will point out how a reduction process could work once the issue concerning the spaces of paramodular forms has been solved.

And finally, note that we are quite confident that this reduction process could be successful (once the issue with the paramodular forms has been solved). The same method described above
has been used in several other cases so far: [De01], [DK03] and [DK04] concerning Hermitian modular forms, [Kl06] concerning some further spaces of orthogonal modular forms, and [Kr05] regarding quaternionic modular forms with respect to the Hurwitz order. Moreover, only Eisenstein-series, Maass lifts and theta-series were needed in all cases, and we already analyzed these in the thesis. This is why we are quite confident in the reduction process being able to yield the sought answers – if only some further questions concerning lower dimensional settings were answered, first. A final result in this chapter will be the aforementioned construction of a set of seven algebraically independent quaternionic modular forms.

6.1 Borcherds products

In this first section we will introduce Borcherds products. Like mentioned before, they are lifts of certain weakly holomorphic vector-valued modular forms. They are defined as infinite products (or initially via a certain integral), and their zero and pole sets are completely determined by the principal part of the underlying vector-valued form. But before we come to the definition of Borcherds products, a there is still some necessary preliminary work. In particular, we have to define quadratic divisors.

For the rest of this section, let us fix some notation, again: $S$ shall be an even positive definite matrix of degree (and rank) $l$. $\Lambda = \mathbb{Z}^l$ shall be the associated lattice with bilinear form $(\cdot,\cdot)_S = (\cdot,\cdot)$ and quadratic form $q_S = q$. $S_0, S_1, \Lambda_0, \Lambda_1$ etc. are defined like in section 5.1.

(6.1) Definition. Suppose $0 \neq \lambda = (l_1, \lambda_0, l_1+2) \in \Lambda_1^\perp$. We define the rational quadratic divisor $\lambda^\perp$ for $\lambda$ by

$$\lambda^\perp = \{ w \in \mathcal{H}_S : l_1 + (\lambda_0, w)_0 - l_1+2q_0(w) = 0 \} .$$

Let $\lambda_p \in \mathbb{Q} \cap \Lambda_1^\perp$ be primitive (see (5.1)). Then the discriminant $\delta(\lambda^\perp)$ is defined by

$$\delta(\lambda^\perp) = -Nq_1(\lambda_p) ,$$

where $N$ is the level of $\Lambda_1$ (see (5.1)).

Note that the discriminant is well-defined since the primitive vector $\lambda_p$ corresponding to $\lambda$ is uniquely determined up to the sign. Furthermore, we have $w \in \lambda^\perp$ if and only if $(\lambda, (-q_0(w), w, 1))_1 = 0$. Therefore, we have

$$\lambda^\perp = \lambda_p^\perp ,$$

and furthermore, $\Gamma_S$ acts on the set of all rational quadratic divisors via

$$M\lambda^\perp := (M\lambda)^\perp = \{ M(w) ; w \in \lambda^\perp \} ,$$

where the last identity is due to some considerations found in [Kl06, pp.83], where $\mathcal{H}_S$ is identified with a certain projective space. Of course, the discriminant is invariant under this action. And due to the same arguments found in [Kl06, prop.4.11], we have the following
(6.2) Proposition. Let \( S \in \{ A_2^{(2)}, A_{2,1}, T_3, D_3^+, A_1^{(2)}, A_2 \} \). Then \( \Gamma_S \) acts transitively on the set of rational quadratic divisors of fixed discriminant, i.e. if \( \lambda_1, \lambda_2 \in \Lambda_1^\perp \) such that \( \delta(\lambda_1^+) = \delta(\lambda_2^+) \) then there exists \( M \in \Gamma_S \) such that \( \lambda_1^+ = M\lambda_2^+ \).

Note that this proposition is due to lemma 4.5 in [FH00] (combined with appendix C in [Kl06]), which says the following: Suppose that \( \Gamma \) is a subgroup of \( \Gamma_S \) satisfying:

- \( \langle J_O, T_g \rangle : g \in A_0 \rangle \subset \Gamma, \)

- \( \Gamma \) is generated by \( J_O, T_g \) (with \( g \in A_0 \)) and \( R_A \) for certain \( A \in \Omega(\Lambda). \)

Then \( \Gamma \) acts transitively on the sets of elements in \( \Lambda_1^\perp \) of the same norm if and only if \( \Gamma \) acts transitively on the sets of elements in \( \text{Dis}^+(\Lambda_1) \) with the same value of \( \overline{q}_1 \) (in \( \mathbb{Q}/\mathbb{Z} \)), which is given if and only if the subgroup of \( \Omega(\Lambda) \) generated by the \( A \) from above acts transitively on the sets of elements in \( \text{Dis}^+(\Lambda) \) with the same value of \( \overline{q} \) (in \( \mathbb{Q}/\mathbb{Z} \)).

One easily computes that these preconditions are fulfilled for \( \Gamma_S \), where \( S \) is one of the matrices occurring in (6.2). But like we will see further below, we will also have to consider \( \Gamma_{\text{Bor}} \). And here, the transitivity is not given! Making use of the generators of \( \Omega(\Lambda)_{\text{Bor}} \) one easily verifies that \( \Omega(\Lambda)_{\text{Bor}} \) acts transitively on \( \{ (0,0,0,0) + \Lambda \} \) and on \( \{ \pm (\frac{1}{3}, \frac{1}{3}, 0, 0) + \Lambda, \pm (0,0, \frac{1}{3}, \frac{1}{3}) + \Lambda \} \), but not on \( \{ \pm (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) + \Lambda, \pm (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) + \Lambda \} \), since there is no \( A \in \Omega(\Lambda)_{\text{Bor}} \) with \( A(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)' + \Lambda = (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})' + \Lambda. \) Thus we have, noting that the level of \( \Lambda_1 \) for \( S = A_2^{(2)} \) equals 3:

(6.3) Proposition. Let \( S = A_2^{(2)} \) and \( \lambda_1, \lambda_2 \in \Lambda_1^\perp \) be primitive with \( q_{5,1}(\lambda_1) = q_{5,1}(\lambda_2) \). If \( 3q_{5,1}(\lambda_1) \equiv 0 \) or \( 3q_{5,1}(\lambda_1) \equiv 2 \mod 3 \), then there exists \( M \in \Gamma_{\text{Bor}} \) such that \( \lambda_1^+ = M\lambda_2^+ \).

But if \( 3q_{5,1}(\lambda_1) \equiv 1 \mod 3 \), then there exists \( M \in \Gamma_{\text{Bor}} \) such that \( \lambda_1^+ = M\lambda_2^+ \) if and only if \( \lambda_1, \lambda_2 \in \pm (0,0, \frac{1}{3}, \frac{1}{3}, 0, 0) + \mathbb{Z}^6 \) or \( \lambda_1, \lambda_2 \in \pm (0,0, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}) + \mathbb{Z}^6 \).

Furthermore, we need the so-called Weyl vector associated to a weakly holomorphic vector-valued modular form. We will not give any details on the construction, here. These can be found in [Kl06, pp.77].

(6.4) Definition. a) Let \( t = (t_1, \ldots, t_l) \in \Lambda = \mathbb{Z}^l \). We write \( t > 0 \) if there exists \( j \in \mathbb{N}, 1 \leq j \leq l, \) such that \( t_1 = \ldots = t_{j-1} = 0 \) and \( t_j > 0 \). For \( \lambda = S^{-1}t \in \Lambda^l \) we write \( \lambda > 0 \) if \( t > 0 \), and for \( \lambda_0 \in (m, \lambda, n) \in \Lambda_0^\perp \) we write \( \lambda_0 > 0 \) if \( n > 0 \) or \( n = 0 \) and \( m > 0 \) or \( m = n = 0 \) and \( \lambda > 0 \).

Additionally, we write \( t < 0, \lambda < 0 \) and \( \lambda_0 < 0 \) if \( -t > 0, -\lambda > 0 \) and \( -\lambda_0 > 0 \), respectively.

b) Let \( k \in \frac{1}{2} \mathbb{Z} \) and \( f \in [M\mathcal{P}_2(\mathbb{Z}), k, \rho_2^\infty]_\infty \) with Fourier-expansion

\[
    f(\tau) = \sum_{\mu \in \Lambda^\perp/\Lambda} \sum_{\pi \in -q(\mu) + \mathbb{Z}} c_{\mu}(n)e^{2\pi i \tau}e_{\mu}, \quad \tau \in \mathcal{H}.
\]
Then we define the Weyl vector \( \varrho_f \) associated to \( f \) by
\[
\varrho_f = (\varrho_0, \varrho, \varrho_z^\dagger),
\]
where
\[
\varrho_0 = \frac{1}{24} \sum_{\lambda \in \Lambda^2} c_\lambda(-q(\lambda)),
\]
\[
\varrho = -\frac{1}{2} \sum_{\lambda \in \Lambda^1, \lambda > 0} c_\lambda(-q(\lambda))\lambda,
\]
\[
\varrho_z^\dagger = \varrho_0 - \sum_{n \in \mathbb{N}} \sigma_1(n) \sum_{\lambda \in \Lambda^2} c_\lambda(-n - q(\lambda)).
\]

and \( \sigma_1(n) = \sum \{d \mid n \} \) is the sum of divisors of \( n \).

Note that the sums occurring in the definition of the Weyl vector associated to \( f \) are finite since \( q = q_S \) is positive definite and the principal part of \( f \) is a finite sum.

Finally, we can get to the definiton of Borcherds products. The following theorem is taken from [Kl06, thm.4.12]. It is a special case of the work done in [Bo98] adjusted to our setting. Also confer [Br02].

**6.5 Theorem.** Suppose that \( S \) is an even positive definite matrix of degree \( l \). Given a weakly holomorphic vector-valued modular form \( f \in [\text{Mp}_2(\mathbb{Z}), -l/2, \rho_S^\dagger]_{0, \infty} \), where \( \rho_S^\dagger \) is the dual Weil representation attached to \( S \), with Fourier-expansion
\[
f(\tau) = \sum_{\mu \in \Lambda^2 / \Lambda} \sum_{n \in -q(\mu) + \mathbb{Z}} c_\mu(n) e^{2\pi int} e_\mu, \quad \tau \in \mathcal{H},
\]
such that \( c_0(0) \in 2\mathbb{Z} \) and \( c_\mu(n) \in \mathbb{Z} \) whenever \( n < 0 \), there exists a Borcherds product \( \psi_k : \mathcal{H}_S \to \mathbb{C} \) with the following properties:

a) \( \psi_k \) is a meromorphic orthogonal modular form of weight \( k = c_0(0)/2 \) with respect to \( O_d(\Lambda_1) \cap \Gamma_S \) and some Abelian character \( \chi \) of finite order.

b) The only zeros and poles of \( \psi_k \) lie on rational quadratic divisors \( \lambda^\perp \), where \( \lambda \in \Lambda_1^2 \) with \( q_1(\lambda) < 0 \). If \( \lambda \in \Lambda_1^2 \) is primitive with \( q_1(\lambda) < 0 \) then the order of \( \psi_k \) along \( \lambda^\perp \) is given by
\[
\sum_{r \in \mathbb{N}} c_{r\lambda}(r^2 q_1(\lambda)).
\]

c) Let \( \varrho_f \) be the Weyl vector associated to \( f \). Define \( n_0 = \min\{n \in \mathbb{Q} ; c_\mu(n) \neq 0 \) for some \( \mu \in \Lambda^2\} \), and let \( S \) be the set of poles of \( \psi_k \). Then on \( \{w = u + iv \in \mathcal{H}_S ; q_0(v) > |n_0|\} \setminus S \) the function \( \psi_k \) is given by the normally convergent product expansion
\[
\psi_k(w) = e^{2\pi i (\varrho_f(w))} \prod_{\lambda \in \Lambda_0^2 \setminus \Lambda_0} (1 - e^{2\pi i (\lambda_0,w)_{0}})^{c_{\lambda_0}(q_0(\lambda_0))}.
\]

Here, \( \lambda_0 \) is identified in \( \text{Dis}(\Lambda) = \Lambda^2 / \Lambda \simeq \Lambda_0^2 / \Lambda_0 \).
6.1 Borcherds products

(6.6) Remark. At this point, one should say a few words about divisors and sets of zeros (and poles) of holomorphic (or meromorphic) functions in several complex variables. Note that we will only deal with holomorphic Borcherds products, hence we stick to the discussion of sets of zeros of a function. Furthermore, since sets of zeros always have to be analyzed locally, and since a coordinate change does not matter, we will stick to a very special case, here. Further insights into this topic can for example be found in [Ra86] and [Mu95], as well as in [Bu98] and [Bu01] concerning our special setting.

Like we have just seen in the preceding theorem, the set of zeros of a Borcherds product is given by a (in our case countable) union of complex submanifolds of codimension 1 – where one should note that this is a phenomena occurring for all holomorphic functions in several complex variables (see [Ra86, ch.I, thm.2.9]).

a) First, let us discuss the holomorphic continuation of a quotient of two orthogonal modular forms (or of two holomorphic functions in general): Let \( \lambda_1 = (l_{-1}, l_0, \lambda_1, l_{l_1+1}, l_{l_1+2}) \in \Lambda^+_1 \) be primitive with discriminant \( N(q(\lambda)) = l_{-1}l_{l_1+2} - l_0l_{l_1+1} \). Of course, \( \lambda_1 = (0, 1, \lambda, l_{-1}l_{l_1+2} + l_0l_{l_1+1}, 0) \) has the same discriminant and is also primitive. But then in the cases that we consider, there exists \( M \in \Gamma_S \) mapping \( \lambda_1^+ \) to \( \lambda_1^+ \). Since \( w \mapsto M(w) \) is biholomorphic, this is just a coordinate change and does not matter regarding holomorphic continuation. Hence we can stick to the divisors \( \lambda_1^+ \), where \( \lambda_1 = (0, \lambda_0, 0) \in \Lambda^+_1 \). But then \( \lambda_1^+ \) is given by a single homogeneous linear equation in \( w = (z_1, \ldots, z_n) \) (with \( n = l + 2 \)). Again, without loss of generality we can assume that this equation is given by \( z_n = 0 \) by applying an appropriate linear coordinate change (although this may result in \( w \notin \mathcal{H}_S \), but this does not matter if we just want to talk about holomorphic continuation, since we can re-substitute any time). Regarding this setting, we can finally say what it means that “the zero along \( z_n = 0 \) is of order \( m \)” (and hence what it means that the order of the Borcherds product along \( \lambda_1^+ \) is \( m \)).

According to [Ra86, ch.I, thm.1.18] every holomorphic function \( f \) is locally given by its Taylor series with respect to the point \( (a_1, \ldots, a_n) \) (hence by a multivariate power series), which converges to that function in all regions \( \{(z_1, \ldots, z_n) \in \mathbb{C}^n; |z_j - a_j| < r_j \} \) that are contained in the domain of holomorphy of \( f \). Now, “the zero along \( z_n = 0 \) is of order \( m \)” means that the Taylor expansions with respect to the points \( (a_1, \ldots, a_{n-1}, 0) \) are given by

\[
f(z_1, \ldots, z_n) = \sum_{j \geq m} a_j(z_1, \ldots, z_{n-1})^j z_n^n,
\]

where \( a_m(z_1, \ldots, z_{n-1}) \neq 0 \) is holomorphic (also cf. [Bu01, pp.33]). Or to be more precise, \( a_m(z_1, \ldots, z_{n-1}) \neq 0 \) has to hold for all regular points, which in this case means that \( (z_1, \ldots, z_{n-1}, 0) \) is not contained in a further submanifold on which \( f \) vanishes. So in other words, \( f(z_1, \ldots, z_{n-1}, \cdot) \) has to be a holomorphic function in one variable, which is of order \( m \) in \( z_n = 0 \) in “most cases” (which means in the regular points), and of higher order (or even identically 0) in the others. Now, suppose that \( g \) is another holomorphic function and that \( g \) has a zero along every submanifold (of codimension 1) that \( f \) vanishes on, and that these zeros are of order greater or equal to the orders of \( f \) (in the above sense). Define \( h = g/f \). Of course, like in the one-dimensional case, \( h \) is holomorphic in all points (in the domain of holomorphy of \( f \) and \( g \)) where \( f \) does not vanish. So let us consider the zeros

...
of $f$ along a submanifold: After a suitable coordinate change we have to consider $z_n = 0$, again. In an appropriate region we have

$$h(z) = \sum_{j \geq m} b_j(z_1, \ldots, z_{n-1}) z_n^{j-m} / \sum_{j \geq m} a_j(z_1, \ldots, z_{n-1}) z_n^{j-m}.$$  

The function in the denominator is free of zeros in a small region of $(z_1, \ldots, z_{n-1}, 0)$ – given it is not an intersection point of two different submanifolds, hence if it is a regular point. In this case, $h$ can be continued holomorphically to $(z_1, \ldots, z_{n-1}, 0)$ (see also [Ra86, ch.I, thm.3.4]). So next, we have to consider the points that are not regular, hence points in the intersection of two submanifolds of codimension 1 on which $f$ vanishes. Obviously, these points belong to some submanifolds of codimension 2. But then in view of [Ra86, ch.VI, thm.4.7], $h$ also has to be holomorphically continuable to these submanifolds of codimension 2, since otherwise these were poles by definition, but the set of poles is always given by a union of submanifolds of codimension 1 (which also derives from [Ra86, ch.I, thm.2.9], saying that the set of zeros is (locally) given by a union of submanifolds of codimension 1).

Hence we have the following setting: Suppose that $\psi_k \in [\Gamma,k,\chi_1]$ is a Borcherds product, where $\Gamma \leq \Gamma_S$ is some subgroup of finite index (with $S$ one of the matrices we are considering) and $\chi_1$ an abelian character of $\Gamma$, and that the set of zeros of $\psi_k$ is given by the union of certain $\lambda_j^+$, with orders $c_j$. Furthermore, suppose we have $f \in [\Gamma,\tilde{k},\chi_2]$, such that $f$ vanishes along the $\lambda_j^+$ of order greater or equal to $c_j$, then we obtain

$$f/\psi_k \in [\Gamma,\tilde{k} - k,\chi_2\chi_1^{-1}].$$

We will use this fact several times from now on, without referring to this current remark.

b) Another important fact arises when restricting to submanifolds. Since we will only have to deal with rational quadratic divisors $\lambda_j^+$ which are given by a single homogeneous linear equation (or at least all divisors are congruent to one of these), we only consider this special case.

So suppose $\psi$ is a holomorphic Borcherds product, and that $\psi$ vanishes along two rational quadratic divisors $\lambda_1^+$ and $\lambda_2^+$ (where both are given by a single homogeneous linear equation) of orders $m_1$ and $m_2$. After a suitable linear coordinate change we can assume that $\lambda_1^+$ is given by the equation $z_n = 0$, while $\lambda_2^+$ is given by the equation $z_{n-1} = 0$ (where after the coordinate change $\psi$ is a function in $w = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with $n = l + 2$). Then by definition we get the following: In every suitable small neighborhood of $(z_1, \ldots, z_{n-1}, 0)$ in the domain of holomorphy we have

$$\psi(z_1, \ldots, z_n) = \sum_{j \geq m_1} a_j(z_1, \ldots, z_{n-1}) z_n^j,$$

and the same for $(z_1, \ldots, z_{n-2}, 0, z_n)$ with

$$\psi(z_1, \ldots, z_n) = \sum_{k \geq m_2} b_k(z_1, \ldots, z_{n-2}, z_n) z_{n-1}^k.$$
Now, suppose that \((z_1, \ldots, z_{n-2}, 0, 0)\) belongs to the domain of holomorphy, too, hence \(\psi\) also possesses a Taylor expansion in a neighborhood \(U\) of that point. And the coefficient functions \(a_{j,k}(z_1, \ldots, z_{n-2})\) and \(b_{j,k}(z_1, \ldots, z_{n-2})\) are holomorphic in 0 (as one-dimensional functions), thus they possess appropriate Taylor expansions. Furthermore, choose \((z_1, \ldots, z_{n-1}, 0)\) and \((z_1, \ldots, z_{n-2}, 0, z_n)\) in \(U\) sufficiently close to \((z_1, \ldots, z_{n-2}, 0, 0)\), such that the domains of convergence of the two Taylor expansions from above have non-empty intersection \(\tilde{U}\) containing \((z_1, \ldots, z_{n-2}, 0, 0)\). So in the neighborhood \(\tilde{U}\) of \((z_1, \ldots, z_{n-2}, 0, 0)\) we have

\[
\psi(z_1, \ldots, z_n) = \sum_{j \geq m_1} \sum_{k \geq m_2} a_{j,k}(z_1, \ldots, z_{n-2}) z_{n-1}^j z_n^k.
\]

But then an identity theorem for holomorphic functions in several complex variables concerning nonempty open sets (cf. [Ra86, ch.I, thm.1.19]) implies that the coefficient functions \(a_{j,k}\) and \(b_{j,k}\) have to coincide, and we obtain

\[
\psi(z_1, \ldots, z_n) = \sum_{j \geq m_1} \sum_{k \geq m_2} a_{j,k}(z_1, \ldots, z_{n-2}) z_{n-1}^j z_n^k.
\]

Choosing \(z_{n-1} = cz_n\) for some \(0 \neq c \in \mathbb{C}\) yields that the function \(f(z_1, \ldots, z_{n-2}, \tilde{z}) = \psi(z_1, \ldots, z_{n-2}, c\tilde{z}, \tilde{z})\) is given by

\[
f(z_1, \ldots, z_{n-2}, \tilde{z}) = \sum_{j \geq m_1} \sum_{k \geq m_2} a_{j,k}(z_1, \ldots, z_{n-2}) c^{k+1} \tilde{z}^j
\]

in some neighborhood of \((z_1, \ldots, z_{n-2}, 0)\). But this means that \(f\) (given it is non-identically zero) vanishes along the submanifold \(\tilde{z} = 0\) of order greater or equal to \(m_1 + m_2\). (And note that \(f\) is again holomorphic, since it is the restriction of a holomorphic function to a submanifold (cf. [Ra86, pp.27]).)

Therefore, we have the following important setting: Let \(\psi\) be a Borcherds product which vanishes along \(\lambda_{n-1}^1\) and \(\lambda_{n-1}^2\) (given like above) of orders \(m_1\) and \(m_2\). And without loss of generality, suppose \(\lambda_{n-1}^1\) and \(\lambda_{n-1}^2\) are given by \(z_n = 0\) and \(z_{n-1} = 0\) (where \(z_n\) and \(z_{n-1}\) can be substituted by some homogeneous linear equations). Let \(M\) be the submanifold given by \(\{(z_1, \ldots, z_n) ; z_{n-1} = cz_n\}\) (with \(0 \neq c \in \mathbb{C}\), then

\[
f(z_1, \ldots, z_{n-2}, z_n) = \psi(z_1, \ldots, z_{n-2}, cz_n, z_n) = \psi(z_1, \ldots, z_n)|_M
\]

vanishes along \(z_n = 0\) of order greater or equal to \(m_1 + m_2\). Again, we will use this fact (and analogous ones) several times from now on, without referring to this current remark. It will be helpful in the following situation: Suppose again that the Borcherds product \(\psi\) vanishes along two submanifolds \(\lambda_{n-1}^1\) and \(\lambda_{n-1}^2\) of codimension 1 with orders \(m_1\) and \(m_2\), while its order along a third one, \(\lambda_{n-1}^3\), equals 0 (which means that \(\psi\) has no zeros on \(\lambda_{n-1}^3\), except for those points that also belong to other submanifolds \(\psi\) vanishes on). Furthermore, suppose that \(\lambda_{n-1}^3 \cap \lambda_{n-1}^1 = \lambda_{n-1}^3 \cap \lambda_{n-1}^2 \neq \emptyset\). Then \(\lambda_{n-1}^3 \cap \lambda_{n-1}^1\) has codimension 1 in \(\lambda_{n-1}^3\), and \(\psi|_{\lambda_{n-1}^3}\) vanishes along this submanifold of order greater or equal to \(m_1 + m_2\).
c) Finally, note that if an orthogonal modular form $f$ vanishes on $\lambda^\perp$ of order $m$, then given $M \in \Gamma_S$, $f|_k M$ vanishes along $M\lambda^\perp$ of order $m$, too. This is because by definition the order $m$ is invariant under coordinate changes (which means under biholomorphic mappings). Hence we only need to regard rational quadratic divisors which are incongruent modulo $\Gamma \leq \Gamma_S$ if $f$ is an orthogonal modular form with respect to $\Gamma$ – in particular concerning the considerations from a) and b).

Note that a priori, Borcherds products are meromorphic orthogonal modular forms with respect to the subgroup $\Gamma = \O_{d}(\Lambda_1) \cap \Gamma_S$, only. In (5.10) we determined which further $R_A$ (with $A \in \O(\Lambda)$) we have to consider to obtain a transformation behavior with respect to the full orthogonal modular group $\Gamma_S$. Moreover, we will see that all Borcherds products are indeed orthogonal modular forms with respect to $\Gamma_S$ or, if $S = A_2^{(2)}$, in some cases for $\Gamma_{\text{Bor}}$, only. In either case, the abelian characters are completely determined by the generators of $\O(\Lambda)$ or $\O(\Lambda)_{\text{Bor}}$ specified in (5.10) and in (5.13). Hence, whichever we want to analyze (be it the abelian character attached to $\psi_k$ or the question if $\psi_k$ transforms correctly for the whole orthogonal modular group), we need to determine some formula that gives us the transformation behavior of $\psi_k$ under $R_A$ for $A \in \O(\Lambda)$.

The next proposition deals with this transformation behavior. Note that a similar proposition can be found in [Kl06, prop.4.13]. But there is one conclusion in the proof of that proposition which does not hold in general, hence that proposition only holds true under special conditions. Therefore, we had to revise it. Nevertheless, we follow the main approach of that proof. But first, we need the following preparatory lemma:

(6.7) Lemma. Suppose that the setting of (6.5) is given. Let $A \in \O(\Lambda)$, and suppose that $c_\mu(n) = c_{A\mu}(n)$ holds for all $\mu \in \Lambda^\perp/\Lambda$ and all $n < 0$. Then we have

$$f_\mu = f_{A\mu} \quad \text{for all } \mu \in \Lambda^\perp/\Lambda,$$

where the $f_\mu$ denote the components of $f$, i.e. $f = \sum_{\mu \in \Lambda^\perp/\Lambda} f_\mu \epsilon_\mu$. Or in other words $c_\mu(n) = c_{A\mu}(n)$ holds for all $n$.

Proof: First, note that $A\Lambda^\perp = \Lambda^\perp$, and $A$ acts on $\Lambda^\perp/\Lambda$ (see (5.2)). So let $\mu \in \Lambda^\perp/\Lambda$, and define $g = f_\mu - f_{A\mu}$. We have to verify $g \equiv 0$.

For all $v \in \Lambda^\perp/\Lambda$ define $g_v = f_v - f_{Av}$. Noting $q(v) = q(Av)$ (since $A \in \O(\Lambda)$), the Fourier-expansion of $g_v$ is given by

$$g_v(\tau) = \sum_{n \in q(\mu) + \mathbb{Z}} (c_v(n) - c_{Av}(n)) e^{2\pi i n \tau}, \quad \tau \in \mathcal{H}.$$ 

Hence, due to the assumption, the principal part of this Fourier-expansion vanishes. Furthermore, by definition (see (5.40)) the transformation behavior of $g_v$ with respect to $T_{Mp}$ is given by

$$g_v|_{-1/2} T_{Mp} = e^{-2\pi i q(v)} f_v - e^{-2\pi i q(Av)} f_{Av} = e^{-2\pi i q(v)} g_v.$$
while for $J_{\text{Mp}}$ we obtain (noting $(Av, \kappa) = (v, A^{-1} \kappa)$ due to $A \in O(\Lambda)\)  
\begin{align*}
g_{\mu} \mid_{-l/2} J_{\text{Mp}} &= \frac{\sqrt{i}^l}{\sqrt{\det(S)}} \left( \sum_{\kappa \in \Lambda^l/\Lambda} e^{2\pi i (v, \kappa)} f_k - \sum_{\kappa \in \Lambda^l/\Lambda} e^{2\pi i (v, A^{-1} \kappa)} f_k \right) \\
&= \frac{\sqrt{i}^l}{\sqrt{\det(S)}} \left( \sum_{\kappa \in \Lambda^l/\Lambda} e^{2\pi i (v, \kappa)} f_k - \sum_{\kappa \in \Lambda^l/\Lambda} e^{2\pi i (v, \kappa)} f_{A\kappa} \right) \\
&= \frac{\sqrt{i}^l}{\sqrt{\det(S)}} \left( \sum_{\kappa \in \Lambda^l/\Lambda} e^{2\pi i (v, \kappa)} g_k \right) .
\end{align*}

Therefore, by induction and because $T_{\text{Mp}}$ and $J_{\text{Mp}}$ generate $\text{Mp}_2(\mathbb{Z})$ (see 5.30), we have 
\[ g_{\mu} \mid_{-l/2} (M, \varphi) = \sum_{v \in \Lambda^l/\Lambda} a_v g_v \]
for all $(M, \varphi) \in \text{Mp}_2(\mathbb{Z})$, where the $a_v$ have to be chosen appropriately. Thus, regarding the considerations above, the principal part of $g_{\mu} \mid_{-l/2} (M, \varphi)$ vanishes for all $(M, \varphi) \in \text{Mp}_2(\mathbb{Z})$, which means that $g_{\mu} \mid_{-l/2} (M, \varphi)$ is holomorphic in $i \infty$.

We already mentioned that according to [Klo6, p.68] $\text{Mp}_2(\mathbb{Z})[\mathbb{N}] \subset \ker(\rho_S)$ holds if $N \in \mathbb{N}$ denotes the level of $\Lambda$. Hence we get 
\[ g_{\mu} \mid_{-l/2} (M, \varphi) = g_{\mu} \]
for all $(M, \varphi) \in \text{Mp}_2(\mathbb{Z})[\mathbb{N}]$, and by definition we have
\[ g_{\mu}^2 (\tau) = g_{\mu} \mid_{-l} (M, \varphi)(\tau) = (g_{\mu}^2 (\tau))^{-1} g_{\mu}^2 (M(\tau)) = (c\tau + d)^{-1} g_{\mu}^2 (M(\tau)) = g_{\mu}^2 \mid_{-l} M(\tau) \]
for all $(M, \varphi) \in \text{Mp}_2(\mathbb{Z})[\mathbb{N}]$, where $M = (a \ b; c \ d) \in \Gamma[\mathbb{N}]$. Furthermore, the principal part in the Fourier-expansion of $g_{\mu}^2 \mid_{-l} M = g_{\mu}^2 \mid_{-l} (M, \varphi) = (g_{\mu} \mid_{-l/2} (M, \varphi))^2$ vanishes for all $(M, \varphi) \in \text{Mp}_2(\mathbb{Z})$ (and thus for all $M \in \text{SL}_2(\mathbb{Z})$) due to the result from above. But this means that $g_{\mu}^2 \in [\Gamma[\mathbb{N}], -l]$ by definition. According to [KK07, ch.III, Satz 7.4] there exist no non-trivial elliptic modular forms of negative weight with respect to $\Gamma[\mathbb{N}]$ and trivial character. Thus $g_{\mu}^2 \equiv 0$ follows, hence also $g \equiv 0$. 

(6.8) Proposition. Suppose that the setting of (6.5) is given. Let $A \in O(\Lambda)$, and let $\psi$ be a Borcherds product with product expansion like in (6.5). Denote the set of zeros of $\psi$ by $\mathcal{N}$ and set of poles by $\mathcal{S}$. If $c_{\mu}(n) = c_{A^{-1}}(\mu(n))$ holds for all $\mu \in \Lambda^2 / \Lambda$ and all $n < 0$, then 
\[ \frac{\psi(R_A(\omega))}{\psi(\omega)} = \prod_{t \in \mathbb{Z}, t > 0 \atop \lambda = S^{-1}t} \left( e^{2\pi i (A' t - \tau) / 2} \frac{1 - e^{-2\pi i (\lambda') / 2}}{1 - e^{-2\pi i \lambda / 2}} \right) c_{\lambda}(-\eta(\lambda)) \]
holds for all $w = (\tau_1, z, \tau_2)$ in the domain of convergence such that $w \notin \mathcal{N}$ and $R_A(\omega) \notin \mathcal{S}$. 
Proof: Let \( w = u + iv = (\tau_1, z, \tau_2) \) be an element in the domain of convergence, hence \( q_0(v) > |n_0| \) and \( w \notin S \) (see (6.5)). We have \( R_A(w) = (\tau_1, Az, \tau_2) \). But then \( A \in O(\Lambda) \) implies \( q_0(\text{Im}(R_A(w))) = q_0(v) > |n_0| \), too. So due to the preconditions \( R_A(w) \) lies in the domain of convergence, too. Thus we can insert \( R_A(w) \) in the product expansion of \( \psi \) and get

\[
\frac{\psi(R_A(w))}{\psi(w)} = e^{2\pi i (q_f, R_A(w) - w)_0} \prod_{\lambda_0 \in \Lambda_0^i, \lambda_0 > 0} \left( 1 - e^{2\pi i (\lambda_0, R_A(w))_0} \right) c_{\lambda_0}(q_0(\lambda_0)).
\]

Let us have a closer look at \( (\lambda_0, R_A(w))_0 \). So suppose \( \lambda_0 = (m, S^{-1}t, n) \in \Lambda_0^i \), where \( t \in \Lambda = \mathbb{Z}' \). We obtain

\[
(S^{-1}t, Az) = (S^{-1}t)'SAz = t'Az = (A't)'z = (S^{-1}A't)'z = (S^{-1}(A't), z)
\]

and

\[
S^{-1}A' = A^{-1}S^{-1},
\]

since \( A \in O(\Lambda) \) and \( S \in \text{Sym}_1(\mathbb{R}) \). This leads to

\[
(\lambda_0, R_A(w))_0 = ((m, S^{-1}t, n), (\tau_1, Az, \tau_2))_0 = mt_2 + n\tau_1 - (S^{-1}t, Az)
\]

\[
= mt_2 + n\tau_1 - (S^{-1}(A't), z) = ((m, S^{-1}(A't), n), w)_0 = (\tilde{R}_{A^{-1}}\lambda_0, w)_0,
\]

where \( \tilde{R}_{A^{-1}} \) is given via \( R_{A^{-1}} = \text{diag}(1, \tilde{R}_{A^{-1}}, 1) \). Due to the assumption and (6.7) we have

\[
c_{\tilde{R}_{A^{-1}}\lambda_0}(n) = c_{\lambda_0}(n)
\]

for all \( n \) and not only for \( n < 0 \), thus in particular

\[
c_{\tilde{R}_{A^{-1}}\lambda_0}(q_0(\tilde{R}_{A^{-1}}\lambda_0)) = c_{\lambda_0}(q_0(\lambda_0))
\]

since \( q_0(\tilde{R}_{A^{-1}}\lambda_0) = q_0(\lambda_0) \) because of \( A^{-1} \in O(\Lambda) \). Therefore, noting that \( \Lambda_0^i \to \Lambda_0^i, \lambda_0 \mapsto \tilde{R}_{A^{-1}}\lambda_0 \) is a bijection, all terms for which \( \tilde{R}_{A^{-1}}\lambda_0 > 0 \) cancel out. This is fulfilled for \( n > 0 \) or \( n = 0, m > 0 \), in particular. Therefore, all terms in the product for \( \lambda = (m, S^{-1}t, n) \in \Lambda_0^i \), \( \lambda_0 > 0 \) with \( n \neq 0 \) or \( m \neq 0 \) cancel out. Thus we obtain

\[
\frac{\psi(R_A(w))}{\psi(w)} = e^{2\pi i (q_f, R_A(w) - w)_0} \prod_{t \in \mathbb{Z}', t > 0, \lambda_0 = (0, S^{-1}t, 0)} \left( 1 - e^{2\pi i (\lambda_0, R_A(w))_0} \right) c_{\lambda_0}(q_0(\lambda_0))
\]

\[
= e^{2\pi i (q_f, R_A(w) - w)_0} \prod_{t \in \mathbb{Z}', t > 0, \lambda = S^{-1}t} \left( 1 - e^{-2\pi i (A't)z} \right) e^{\lambda(-q(\lambda))}.
\]

So next, let us consider \( (q_f, R_A(w) - w)_0 = (q_f, (0, Az - z, 0))_0 = -(q, Az - z) \) (see (6.4)).
Inserting the explicit formula from (6.4) for $\varrho$ yields

$$
(q_f, R_A\langle w \rangle - w)_0 = \frac{1}{2} \sum_{t \in \mathbb{Z}, t > 0} \sum_{\lambda = S^{-1}t} c_\lambda(-q(\lambda)) \lambda' S(Az - z)
$$

and this completes the proof. □

**Remark.** Note that the formula in the preceding proposition yields an explicit transformation behavior. By the definition of weakly holomorphic vector-valued modular forms, the principal part in the Fourier-expansion of $f$ has to be finite, which means that there exists $n_0 \in \mathbb{Q}$ such that $c_\mu(n) = 0$ for all $\mu \in \Lambda^2 / \Lambda$ and all $n < n_0$. Hence there also exist only finitely many $\lambda \in \Lambda^2$, $\lambda > 0$ such that $c_\lambda(-q(\lambda)) \neq 0$. Therefore, the product on the right hand side of the formula is finite. On the other hand, only finitely many equations have to be checked to verify whether $A \in O(\Lambda)$ meets the preconditions of the proposition.

Concerning this issue, a program was written for this thesis using [MAPLE]. The precondition on $A$ can easily be checked manually. The program then calculates the product from (6.8) and checks whether the nominator and denominator only differ by some constant factor $c_A$. And if this holds true, then we get

$$
\psi(R_A\langle w \rangle) = c_A \psi(w)
$$

for all $w$ considered in the proposition. But then of course this holds for all $w \in \mathcal{H}_S \setminus S$ since both sides are holomorphic on this domain. Furthermore, we are only going to consider holomorphic Borcherds products, hence the transformation behavior will hold for all $w \in \mathcal{H}_S$. Moreover, this procedure immediately yields the abelian character attached to the Borcherds product, since it is determined completely by the behavior concerning the $R_A$, $A \in O(\Lambda)$.

But note that we will not give any details on the explicit calculations, since it is only a matter of computing the finite product from (6.8) – in principle, this could also be done manually, although there are quite some factors involved in this product. But of course, explicitly written down calculations would not yield any further insights. Therefore, we will only present the results obtained by the procedure described above, since their correctness could be verified easily.

Before we actually get to constructing Borcherds products for the cases we are interested in, there remains one important issue to be noted: So far, we know how to construct Borcherds products and how to obtain their sets of poles and zeros (including multiplicities) as well as their transformation behavior (given some preconditions hold) – as long as $f \in \text{Mp}_2(\mathbb{Z}), -1/2, \rho_S^\infty$ is given. So obviously the question arises how to obtain such an $f$. Even more so in the light of $\text{Mp}_2(\mathbb{Z}), -1/2, \rho_S^\infty$ being infinite dimensional: If $f \in \text{Mp}_2(\mathbb{Z}), -1/2, \rho_S^\infty$ then $j \cdot f \in \text{Mp}_2(\mathbb{Z}), -1/2, \rho_S^\infty$ holds in view of (5.38), too, where $j$ is the so-called Absolute Invariant (cf. [KK07, pp.183]) fulfilling $j \in \text{Mp}_2(\mathbb{Z}), 0, 1$ $\text{Mp}_2(\mathbb{Z}), -1/2, \rho_S^\infty$. The principal part of the Fourier-expansion grows larger and larger when multiplying with powers of $j$. 

6.1 Borcherds products
Fortunately, there exists a solution to this issue – at least partly. Note that the most important information concerning Borcherds products is contained in the principal part and the constant term of the Fourier-expansion of \( f \), be it the sets of poles and zeros or the transformation behavior. The rest of the Fourier-expansion is only needed to calculate the Fourier-expansion of the Borcherds product making use of its product expansion. And as we will see in the next proposition, we can construct explicit principal parts (plus constant term) belonging to some \( f \in \Mp_2(\mathbb{Z}), -l/2, \rho_S^\sharp \) – although the full Fourier-expansion is not known. But as long as one does not want to compute any Fourier-coefficients of the Borcherds product, this information suffices.

On the other hand, note that in principal it is even possible to determine the full Fourier-expansion of \( f \). Let \( \Delta \) denote the Modular Discriminant (cf. [KK07, pp.162]), which means that \( \Delta \in \Mp_2(\mathbb{Z}), 12 \) is the unique normalized elliptic cusp form of weight 12. And since the principal part of \( f \) is finite, there exists some \( n \in \mathbb{N} \) such that the principal part of \( \Delta^n f \) vanishes, hence \( \Delta^n f \in \Mp_2(\mathbb{Z}), -l/2 + 12n, \rho_S^\sharp \), and this space is finite dimensional. Therefore, one has to know a basis of this space. On the other hand, given \( g \in \Mp_2(\mathbb{Z}), -l/2 + 12n, \rho_S^\sharp \) then \( \Delta^{-n} g \in \Mp_2(\mathbb{Z}), -l/2, \rho_S^\sharp \) because \( \Delta \) has no zeros. Furthermore, the smallest possible exponent in the Fourier-expansion of \( \Delta^{-n} g \) is \(-n\). Therefore, we have some kind of correspondence between these spaces. But note that in general these spaces of vector-valued modular forms are not determined explicitly (only their dimensions using (5.42)). Therefore, in most cases one has to be content to know the principal parts, only – although in principal it is possible to determine the spaces from above, but this is very involved. Furthermore, note that according to [Br02, prop.1.12] \( f \) is uniquely determined by its principal part.

But for now, let us cite the following theorem from [Bu01, Thm.2.1.6]. It is taken from [Bo99, thm.3.1] and formulated in a way that fits to our setting.

**6.9 Theorem.** Suppose that \( S \) is an even positive definite matrix of degree \( l \). There exists a (unique) weakly holomorphic vector-valued modular form \( f \in \Mp_2(\mathbb{Z}), -l/2, \rho_S^\sharp \) with principal part and constant term

\[
\sum_{\mu \in \Lambda^\sharp / \Lambda} \sum_{\substack{n \in -q(\mu) + \mathbb{Z} \\
 n \leq 0}} c_\mu(n)e^{2\pi in\tau} e_\mu, \quad \tau \in \mathcal{H},
\]

if and only if

\[
c_\mu(n) = c_{-\mu}(n) \quad \text{for all } \mu \in \Lambda^\sharp / \Lambda \text{ and all } n \leq 0
\]

and

\[
\sum_{\mu \in \Lambda^\sharp / \Lambda} \sum_{\substack{n \in -q(\mu) + \mathbb{Z} \\
 n \leq 0}} c_\mu(n)a_\mu(-n) = 0
\]

holds for all holomorphic vector-valued modular forms \( g \in \Mp_2(\mathbb{Z}), 2 + l/2, \rho_S \) (the so-called obstruction space) with Fourier-expansion

\[
g(\tau) = \sum_{\mu \in \Lambda^\sharp / \Lambda} \sum_{\substack{n \in q(\mu) + \mathbb{Z} \\
 n \geq 0}} \alpha_\mu(n)e^{2\pi in\tau} e_\mu, \quad \tau \in \mathcal{H}.
\]
(6.10) Remark. So the proceeding in the next section is clear: First, given \( S \), we need to completely determine the obstruction space \([Mp_2(\mathbb{Z}), 2 + l/2, \rho_S]\), including the Fourier-expansions of the elements in a basis. To do so, we begin by calculating the dimension of this space utilizing the dimension formula from (5.42), taking (5.43) into account. Note that for this thesis a programm using [MAPLE] was written which implements this dimension formula. Therefore, we will always present the result, only. Next, a basis has to be constructed, and since we already know the dimension of the space, we only need to construct sufficiently many linear independent forms. Note that for the cases we will have to consider, it will turn out that vector-valued Eisenstein-series and theta-series (cf. (5.44) and (5.46)) suffice. Again, like mentioned before, programs have been implemented in [SAGE] for this thesis that calculate the corresponding Fourier-expansions – or at least sufficiently many Fourier-coefficients. So once more, we will only present the results of these computations.

Once the obstruction space is determined, in virtue of (6.9) we only have to solve a linear equation system to obtain a basis of all possible principal parts which will be needed for the Borcherds products. Or to be more precise, a basis of all possible principal parts satisfying \( c_\mu(n) = 0 \) for all \( \mu \in \Lambda^\sharp/\Lambda \) and all \( n < n_0 \) for some \( n_0 \), depending on how many Fourier-coefficients we calculated concerning the basis of the obstruction space. And since a programm using [SAGE] was implemented to yield such a basis for the principal part, we will only present the results.

Next, once we determined the possible principle parts, we will use the method described in (6.8) (and the remark afterwards) to analyze the transformation behavior of the Borcherds products we constructed from the principal parts, including the associated abelian character. Again, we will only present the results obtained from [MAGMA]. And finally, we use (6.5) to determine the sets of zeros (including multiplicities) – and also the sets of poles, whereas we will only construct holomorphic Borcherds products.

And as a final remark, note that due to the product expansion of Borcherds products we have the following: Let \( f_1, f_2 \in [Mp_2(\mathbb{Z}), -l/2, \rho_S^\sharp]_\infty \) with associated Borcherds products \( \psi_{f_1} \) and \( \psi_{f_2} \). Then \( \psi_{f_1 + f_2} = \psi_{f_1} \psi_{f_2} \) holds.
6.2 A possible reduction process

Like described above, we will now construct Borcherds products for the cases we have to consider and analyze them. Once this is done, we will present a possible reduction process which arises from these results – similar to what we explained at the beginning of this chapter. To keep it well-arranged, we will use subsections this time, one for each of the \( S \) we have to consider. Note that for \( S \in \{ \Lambda_1^{(2)}, \Lambda_2 \} \) the whole issue has already been analyzed in [De01]. Hence we will consider exactly four cases (and another one later on): \( S \in \{ \Lambda_2^{(2)}, \Lambda_{2,1}, T_3, D_2^* \} \). And for the sake of clearness, we will numerate the occurring Borcherds products – but to keep it well-arranged not by the index \( S \), but by \( 1, \ldots, 4 \), where \( 1 \equiv \Lambda_2^{(2)}, 2 \equiv \Lambda_{2,1}, 3 \equiv T_3, \text{ and } 4 \equiv D_2^* \).

6.2.1 Borcherds products for \( \Lambda_2^{(2)} \)

So let us consider \( \Lambda_2^{(2)} \), first. So for the rest of this subsection, let \( S = \Lambda_2^{(2)} \). For the obstruction space \( \text{Mp}_2(\mathbb{Z}), 2 + 4/2, \rho_5 \) we obtain:

**Proposition.** We have \( \dim \text{Mp}_2(\mathbb{Z}), 4, \rho_5 \) = 2. A basis is given by \( E_4(\cdot; -S_1) \) and \( \Theta(\cdot; S, p_2) \), where by defining \( v_1 = (1,0,i,0)' \) and \( v_2 = (1,0,-i,0)' \), \( p_2 \) is given by

\[
p_2(x) = \frac{1}{2}(v_1'(Sx))^2 + \frac{1}{2}(v_2'(Sx))^2 = (2x_1 + x_2)^2 - (2x_3 + x_4)^2.
\]

If we use the common abbreviation \( q = e^{2\pi i \tau} \), then we get

\[
E_4,(0,0,0,0)(\tau; -S_1) = 2 - 168q^{1} + O(q^2),
\]
\[
E_4,(\pm(\frac{1}{2},0,0,0))(\tau; -S_1) = -6q^{1/3} + O(q^{4/3}),
\]
\[
E_4,(\pm(0,\pm1,\pm1,\pm1))(\tau; -S_1) = -6q^{1/3} + O(q^{4/3}),
\]
\[
E_4,(\pm(\frac{1}{2},\pm1,\pm1,\pm1))(\tau; -S_1) = -54q^{2/3} + O(q^{5/3}),
\]
\[
E_4,(\pm(\frac{1}{2},\pm1,\pm1,\pm1))(\tau; -S_1) = -54q^{2/3} + O(q^{5/3}),
\]

and

\[
\Theta,(0,0,0,0)(\tau; S, p_2) = 0 + 0q^{1} + O(q^2),
\]
\[
\Theta,(\pm(\frac{1}{2},\pm1,0,0))(\tau; S, p_2) = 2q^{1/3} + O(q^{4/3}),
\]
\[
\Theta,(\pm(0,\pm1,\pm1,0))(\tau; S, p_2) = -2q^{1/3} + O(q^{4/3}),
\]
\[
\Theta,(\pm(\frac{1}{2},\pm1,\pm1,\pm1))(\tau; S, p_2) = 0q^{2/3} + O(q^{5/3}),
\]
\[
\Theta,(\pm(\frac{1}{2},\pm1,\pm1,\pm1))(\tau; S, p_2) = 0q^{2/3} + O(q^{5/3}).
\]

Therefore, a basis for possible principal parts and constant terms of weakly holomorphic vector-valued
modular forms satisfying \( c_\mu(n) = 0 \) for \( n < -1 \) is given by

\[
q^{-1/3}(e_{1/3_1} + e_{1/3_2}) + 12e_0,
q^{-2/3}e_{2/3_1}
q^{-2/3}e_{2/3_2}
q^{-1}e_0
\]

where we use the following abbreviations:

\[
e_0 = e_{(0,0,0,0)},
e_{1/3_1} = e_{(1/3,1/3,0,0)} + e_{-(1/3,1/3,0,0)},
e_{1/3_2} = e_{(0,0,1/3,1/3)} + e_{-(0,0,1/3,1/3)},
e_{2/3_1} = e_{(1/3,1/3,1/3)} + e_{-(1/3,1/3,1/3)},
e_{2/3_2} = e_{(1/3,-1/3,-1/3)} + e_{-(1/3,-1/3,-1/3)}.
\]

Note that the Borcherds products with respect to the second and third principal part have odd weight \( 54/2 = 27 \). Hence in view of (5.18) it is already clear that these Borcherds product cannot possess a proper transformation behavior for the full orthogonal modular group \( \Gamma_S \) since there exist no non-trivial orthogonal modular forms of odd weight with respect to the full modular group. And indeed, the precondition on \( A \) in (6.8) is not fulfilled for \( A = A_{i_2}^S \) (and also for \( A = A_{i_2}^S \)), because \( A_{i_2}^S (1/3,1/3,1/3,1/3) = (1/3,1/3,1/3,1/3) \), but the coefficients do not coincide. But one easily verifies that the condition is fulfilled for all \( A \in O(\Lambda)_{\text{Bor}} \). And furthermore, the condition is also fulfilled for all \( A \in O(\Lambda) \) when adding up the second and third principal part, as well as for the remaining principal parts.

Next, we want to apply (6.5) to these principal parts to obtain Borcherds products \( \phi_{1,k} \) which have zeros along rational quadratic divisors with discriminant \( \leq 3 \). According to (6.2) the orthogonal modular group \( \Gamma_S \) acts transitively on the set of rational quadratic divisors of fixed discriminant, while in virtue of (6.3) \( \Gamma_{\text{Bor}} \) only acts transitively on the set of rational quadratic divisors of fixed discriminant, if this discriminant is congruent to 0 or 1 modulo 3. Otherwise, it only acts transitively on the intersection with \( \pm(0,0,1/3,1/3,1/3,1/3,0,0) + \mathbb{Z}^6 \) and with \( \pm(0,0,1/3,1/3,-1/3,-1/3,0,0) + \mathbb{Z}^6 \). Therefore, it suffices to consider the following representatives \( \lambda^\perp_\delta \) of discriminant \( \delta \):

\[
\lambda^\perp_1 = \{ w \in \mathcal{H}_S ; z_4 = 0 \} \simeq \mathcal{H}_{A_2},
\lambda^\perp_{2(1)} = \{ w \in \mathcal{H}_S ; z_4 = z_2 \} \simeq \mathcal{H}_{T_3},
\lambda^\perp_{2(2)} = \{ w \in \mathcal{H}_S ; z_4 = -z_2 \} \simeq \mathcal{H}_{T_3},
\lambda^\perp_3 = \{ w \in \mathcal{H}_S ; z_4 = -2z_2 \} \simeq \mathcal{H}_{S_3},
\]

where \( w = (\tau_1, z_1, z_2, z_3, z_4, \tau_2) \). Concerning \( S_3 \) see 5.20 (where the associated lattice is not euclidean). The claims about the isomorphies can be read off 5.21. And note that \( \lambda^\perp_{2(1)} \) and \( \lambda^\perp_{2(2)} \)
are congruent modulo $\Gamma_S$ (via $R_{A^2_1}$), but not modulo $\Gamma_{\text{Bor}}$.

Proceeding like described in (6.10) yields the following theorem. Note that $\mathcal{H} \times \{0\}^4 \times \mathcal{H} \subset \lambda^{1/2}_S$ holds for all rational quadratic divisors we consider. Therefore, the Borcherds products are also cusp forms.

(6.12) Theorem. Let $S = A^2_2$. Then there exist holomorphic Borcherds products

$$\psi_{1,6} \in [\Gamma_S, 6, \chi^S_{\pi_1}]_0, \quad \psi_{1,42} \in [\Gamma_S, 42, \chi^S_{\det \chi_{\pi_1}}]_0, \quad \psi_{1,54} \in [\Gamma_S, 54, \chi^S_{\tau_2}]_0$$

and

$$\psi_{1,27,(1)} \in [\Gamma_{\text{Bor}}, 27, \chi^S_{\lambda^{1/2}_T}]_0, \quad \psi_{1,27,(2)} \in [\Gamma_{\text{Bor}}, 27, \chi^S_{\lambda^{1/2}_T}]_0.$$

The zeros of these Borcherds products are all of first order and are given by

$$\bigcup_{M \in \Gamma_S} M \langle \mathcal{H}_{A_2,1} \rangle, \quad \bigcup_{M \in \Gamma_S} M \langle \mathcal{H}_{S_1} \rangle, \quad \bigcup_{M \in \Gamma_S} M \langle \mathcal{H}_{T_3} \rangle$$

and

$$\bigcup_{M \in \Gamma_{\text{Bor}}} M \langle \lambda^{1/2}_{S_1,(1)} \rangle, \quad \bigcup_{M \in \Gamma_{\text{Bor}}} M \langle \lambda^{1/2}_{S_1,(2)} \rangle.$$

Furthermore, $\psi_{1,54} = \psi_{1,27,(1)} \psi_{1,27,(2)}$ holds.

Note that $\psi_{1,27,(1)}$ and $\psi_{1,27,(2)}$ are orthogonal modular forms with respect to $\Gamma_{\text{Bor}}$, only. But nevertheless, we have

$$\psi_{1,27,(1)} |_{\lambda^{1/2}_{S_1,(1)}}, \quad \psi_{1,27,(2)} |_{\lambda^{1/2}_{S_1,(2)}} \in [\Gamma_{T_3}, 27, \chi^T_{\pi_1}]_0,$$

$$\psi_{1,27,(1)} |_{\mathcal{H}_{A_2,1}}, \quad \psi_{1,27,(2)} |_{\mathcal{H}_{A_2,1}} \in [\Gamma_{A_2,1}, 27, \chi^A_{T_3}]_0,$$

which derives from (5.25).

6.2.2 Borcherds products for $A_{2,1}$

For the rest of this subsection, let $S = A_{2,1}$. This time, we obtain for the obstruction space $[\text{Mp}_2(\mathbb{Z}), 2 + 3/2, \rho_5]$

(6.13) Proposition. We have $\dim [\text{Mp}_2(\mathbb{Z}), 7/2, \rho_5] = 2$. A basis is given by $E_{7/2}(\cdot; -S_1)$ and $
\Theta(\cdot; S, p_2)$, where by defining $v_1 = (1, 0, i)'$ and $v_2 = (1, 0, -i)'$, $p_2$ is given by

$$p_2(x) = \frac{1}{2} (v'_1 S x)^2 + \frac{1}{2} (v'_2 S x)^2 = (2x_1 + x_2)^2 - 4x_3^2.$$

If we use the common abbreviation $q = e^{2 \pi i r}$, then we get

$$E_{7/2,(0,0,0)}(\tau; -S_1) = 2 - \frac{207}{15} q^{1/4} + O(q^2),$$

$$E_{7/2,(0,0,1/2)}(\tau; -S_1) = -\frac{56}{15} q^{1/4} + O(q^{5/4}),$$

$$E_{7/2,(1,1/2,0)}(\tau; -S_1) = -\frac{126}{15} q^{1/3} + O(q^{4/3}).$$
and

\[
E_{7/2, \pm(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})}(\tau; -S_1) = -\frac{576}{13}q^{7/12} + O(q^{19/12}),
\]

Therefore, a basis for possible principal parts and constant terms of weakly holomorphic vector-valued modular forms satisfying \(c_\mu(n) = 0\) for \(n < -1\) is given by

\[
q^{-1/3}e_{1/3} + 2q^{-1/4}e_{1/4} + 14e_0,
\]

\[
q^{-7/12}e_{7/12} + q^{-1/3}e_{1/3} + 54e_0,
\]

\[
q^{-1}e_0 + q^{-1/4}e_{1/4} + 84e_0,
\]

where we use the following abbreviations:

\[
e_0 = e_{(0,0,0)},
\]

\[
e_{1/4} = e_{(0,0,\frac{1}{2})},
\]

\[
e_{1/3} = e_{(\frac{1}{3},\frac{1}{3},0)} + e_{-(\frac{1}{3},\frac{1}{3},0)},
\]

\[
e_{7/12} = e_{(\frac{1}{3},\frac{1}{3},\frac{1}{2})} + e_{-(\frac{1}{3},\frac{1}{3},\frac{1}{2})}.
\]

This time, the precondition on \(A\) in (6.8) is fulfilled for all \(A \in O(\Lambda)\) and all principal parts from above. Therefore, we can apply (6.8) in all cases, and it will turn out that all of the three principal parts yield holomorphic Borcherds products which transform accordingly with respect to the full orthogonal modular group \(\Gamma_S\).

Again, we want to apply (6.5) to these principal parts to obtain Borcherds products \(\psi_{2,k}\) which have zeros along rational quadratic divisors with discriminant \(\leq 12\). According to (6.2) the orthogonal modular group \(\Gamma_S\) acts transitively on the set of rational quadratic divisors of fixed discriminant. So once more, it suffices to consider the following representatives \(\lambda_\delta\) of discriminant \(\delta\):

\[
\lambda_\frac{2}{3} = \{w \in \mathcal{H}_S \mid z_3 = 0\} \simeq \mathcal{H}_{A_2},
\]

\[
\lambda_\frac{4}{5} = \{w \in \mathcal{H}_S \mid z_2 = 0\} \simeq \mathcal{H}_{A_1^{(2)}},
\]

\[
\lambda_\frac{7}{12} = \{w \in \mathcal{H}_S \mid z_2 = z_3\} \simeq \mathcal{H}_{D_2},
\]

\[
\lambda_{12} = \{w \in \mathcal{H}_S \mid z_1 = z_2\} \simeq \mathcal{H}_{(\frac{2}{0},\frac{0}{6})},
\]

where \(w = (\tau_1, z_1, z_2, z_3, \tau_2)\). Note that the lattice attached to \((\frac{2}{0},\frac{0}{6})\) is not euclidean (choose \(x_1 = x_2 = \frac{1}{2}\)). Obviously, it corresponds to \(\mathbb{Z}[\sqrt{-3}]\). So this would yield Hermitian modular
forms regarding \( \mathbb{Q}(\sqrt{-3}) \), but only with respect to \( \text{Sp}_2(\mathbb{Z}|\sqrt{-3}) \) and not the full modular group, since \( \mathbb{Z}|\sqrt{-3} \) is only a subring of the ring of integers in \( \mathbb{Q}(\sqrt{-3}) \). Hence it does not seem fruitful to consider further reductions with respect to that lattice. The claims about the other isomorphies can be read off 5.24.

Again, proceeding like described in (6.10) yields the following theorem, where once more \( \mathcal{H} \times \{0\}^3 \times \mathcal{H} \subset \lambda_3^+ \) holds for all rational quadratic divisors we consider. Therefore, the Borcherds products are also cusp forms.

(6.14) Theorem. Let \( S = A_{2,1} \). Then there exist holomorphic Borcherds products

\[
\begin{align*}
\psi_{2,7} &\in [\Gamma_S, 7, \chi^S_\pi]_0, \\
\psi_{2,27} &\in [\Gamma_S, 27, \chi^S_\pi]_0, \\
\psi_{2,42} &\in [\Gamma_S, 42, \chi^S_{\det\chi^S_\pi}]_0, \\
\psi_{2,62} &\in [\Gamma_S, 62, \chi^S_{\det\chi^S_\pi}]_0.
\end{align*}
\]

The zeros of these Borcherds products are given as follows, where the number in front of the unions denotes the order along these sets of zeros:

\[
\begin{align*}
2 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}_{A_2}) &\cup 1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}^{A_2}_2), \\
1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}^{A_2}_1) &\cup 1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}^{D_2}_2), \\
3 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}_{A_2}) &\cup 1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}^{A_2}_2), \\
1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}_{A_2}) &\cup 1 \cdot \bigcup_{M \in \Gamma_S} \mathcal{M}(\mathcal{H}^{D_2}_2).
\end{align*}
\]

Furthermore

\[\psi_{2,62} = \frac{\psi_{2,27}\psi_{2,42}}{\psi_{2,7}}\]

holds.

6.2.3 Borcherds products for \( T_3 \)

Next, we consider \( S = T_3 \). For the obstruction space \( \text{dim}[\text{Mp}_2(\mathbb{Z}), 2 + 3/2, \rho_S] = 3 \). A basis is given by \( E_{7/2}(\cdot; -S_1) \), \( \Theta(\cdot; S, p_{2,(1)}) \) and \( \Theta(\cdot; S, p_{2,(2)}) \), where by defining \( v_{1,(1)} = (1,0,i)' \), \( v_{2,(1)} = (1,0,-i)' \), \( v_{1,(2)} = (1,i,-1)' \) and \( v_{2,(2)} = (-1,i,1)' \), \( p_{2,(1)} \) is given by

\[p_{2,(1)}(x) = \frac{1}{2}(v'_{1,(1)}Sx)^2 + \frac{1}{2}(v'_{2,(1)}Sx)^2 = 4x_1^2 + 4x_1x_2 - 4x_2x_3 - 4x_3^2,\]

while \( p_{2,(2)} \) is given by

\[p_{2,(2)}(x) = \frac{1}{2}(v'_{1,(2)}Sx)^2 + \frac{1}{2}(v'_{2,(2)}Sx)^2 = 3x_1^2 - 10x_1x_3 + 3x_3^2 - 16x_2^2 - 8x_1x_2 - 8x_2x_3,\]
If we use the common abbreviation \( q = e^{2\pi i \tau} \), then we get

\[
E_{7/2, (0, 0, 0)}(\tau; -S_1) = 2 - \frac{1288}{15} q^{1} + O(q^2),
\]
\[
E_{7/2, (\frac{1}{2}, 0, \frac{1}{2})}(\tau; -S_1) = -\frac{280}{15} q^{1/2} + O(q^{3/2}),
\]
\[
E_{7/2, \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}(\tau; -S_1) = -\frac{574}{15} q^{2/3} + O(q^{5/3}),
\]
\[
E_{7/2, \pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}(\tau; -S_1) = -\frac{448}{15} q^{5/8} + O(q^{13/8}),
\]
\[
E_{7/2, \pm(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})}(\tau; -S_1) = -\frac{14}{15} q^{1/6} + O(q^{7/6}),
\]
\[
E_{7/2, \pm(\frac{\tau}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; -S_1) = -\frac{64}{15} q^{7/24} + O(q^{31/24}),
\]
\[
E_{7/2, \pm(-\frac{1}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; -S_1) = -\frac{64}{15} q^{7/24} + O(q^{31/24}),
\]

and

\[
\Theta_{(0, 0, 0)}(\tau; S, p_{2,(1)}) = 0 + 0 q^{1} + O(q^2),
\]
\[
\Theta_{(\frac{1}{2}, 0, \frac{1}{2})}(\tau; S, p_{2,(1)}) = 0 q^{1/2} + O(q^{3/2}),
\]
\[
\Theta_{\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}(\tau; S, p_{2,(1)}) = 0 q^{2/3} + O(q^{5/3}),
\]
\[
\Theta_{\pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}(\tau; S, p_{2,(1)}) = 0 q^{5/8} + O(q^{13/8}),
\]
\[
\Theta_{\pm(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})}(\tau; S, p_{2,(1)}) = 0 q^{1/6} + O(q^{7/6}),
\]
\[
\Theta_{\pm(\frac{\tau}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; S, p_{2,(1)}) = 2 q^{7/24} + O(q^{31/24}),
\]
\[
\Theta_{\pm(-\frac{1}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; S, p_{2,(1)}) = -2 q^{7/24} + O(q^{31/24}),
\]

while

\[
\Theta_{(0, 0, 0)}(\tau; S, p_{2,(2)}) = 0 + 12 q^{1} + O(q^2),
\]
\[
\Theta_{(\frac{1}{2}, 0, \frac{1}{2})}(\tau; S, p_{2,(2)}) = 6 q^{1/2} + O(q^{3/2}),
\]
\[
\Theta_{\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}(\tau; S, p_{2,(2)}) = -2 q^{2/3} + O(q^{5/3}),
\]
\[
\Theta_{\pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}(\tau; S, p_{2,(2)}) = -6 q^{5/8} + O(q^{13/8}),
\]
\[
\Theta_{\pm(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})}(\tau; S, p_{2,(2)}) = -q^{1/6} + O(q^{7/6}),
\]
\[
\Theta_{\pm(\frac{\tau}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; S, p_{2,(2)}) = q^{7/24} + O(q^{31/24}),
\]
\[
\Theta_{\pm(-\frac{1}{15}, \frac{1}{6}, \frac{1}{15})}(\tau; S, p_{2,(2)}) = q^{7/24} + O(q^{31/24}),
\]

Therefore, a basis for possible principal parts and constant terms of weakly holomorphic vector-valued
modular forms satisfying \( c_\mu(n) = 0 \) for \( n < -1 \) is given by

\[
q^{-7/24} e_{7/24} + 2q^{-1/6} e_{1/6} + 12e_0,
\]

\[
q^{-1/2} e_{1/2} + 3q^{-1/6} e_{1/6} + 14e_0,
\]

\[
q^{-5/8} e_{5/8} + 2q^{-1/2} e_{1/2} + 54e_0,
\]

\[
q^{-2/3} e_{2/3} + 56e_0,
\]

\[
q^{-5/8} e_{5/8} + 84e_0,
\]

where we use the following abbreviations:

\[
e_0 = e_{(0,0,0)}, \\
e_{1/2} = e_{(\frac{1}{4},0,\frac{1}{4})}, \\
e_{1/6} = e_{(\frac{1}{6},\frac{1}{6},-\frac{1}{6})}, \\
e_{7/24} = e_{(\frac{5}{24},\frac{1}{24},-\frac{1}{24})} + e_{-(\frac{1}{24},\frac{5}{24},\frac{1}{24})}, \\
e_{5/8} = e_{(\frac{1}{4},\frac{1}{4},-\frac{1}{4})} + e_{-(\frac{1}{4},\frac{1}{4},-\frac{1}{4})}, \\
e_{2/3} = e_{(\frac{1}{3},\frac{1}{3},-\frac{1}{3})}.
\]

Again, one easily verifies that the precondition on \( A \) in (6.8) is fulfilled for all \( A \in O(\Lambda) \) and all principal parts from above. Therefore, we can apply (6.8) in all cases, and it will turn out that all principal parts yield holomorphic Borcherds products which transform accordingly with respect to the full orthogonal modular group \( \Gamma_S \).

Once more, we want to apply (6.5) to these principal parts to obtain Borcherds products \( \psi_{3k} \) which have zeros along rational quadratic divisors with discriminant \( \leq 24 \). Like in the other cases, it suffices to consider the following representatives \( \lambda_{\delta} \) of discriminant \( \delta \):

\[
\lambda_{\frac{1}{4}} = \{ w \in \mathcal{H}_S ; z_2 = 0 \} \simeq \mathcal{H}_{A_1^{(2)}}, \\
\lambda_{\frac{7}{24}} = \{ w \in \mathcal{H}_S ; z_3 = 0 \} \simeq \mathcal{H}_{D_2}, \\
\lambda_{12} = \{ w \in \mathcal{H}_S ; z_1 = z_3 \} \simeq \mathcal{H}_{(\frac{4}{2},\frac{4}{2})}, \\
\lambda_{15} = \{ w \in \mathcal{H}_S ; z_2 = z_3 \} \simeq \mathcal{H}_{(\frac{2}{1},\frac{2}{1})}, \\
\lambda_{16} = \{ w \in \mathcal{H}_S ; z_1 = -z_3 \} \simeq \mathcal{H}_{(\frac{4}{0},\frac{4}{0})}, \\
\lambda_{24} = \{ w \in \mathcal{H}_S ; z_2 = -2z_3 \} \simeq \mathcal{H}_{(\frac{0}{0},\frac{12}{12})},
\]

where \( w = (\tau_1,z_1,z_2,z_3,\tau_2) \). Note that none of the lattices attached to those matrices from above which we did not consider yet are euclidean, except for the first one (concerning \( \lambda_{\frac{1}{4}} \)). But this one does not directly correspond to a Hermitian setting (it is \( 2 \cdot A_2 \)). But note that \((\frac{1}{2})\) corresponds to \( \mathbb{Z}[\frac{1}{2}(1+i\sqrt{15})] \), hence we would obtain Hermitian modular forms for \( Q(\sqrt{-15}) \).
(while the class number is not equal to one here). But although this is the case, it does not seem
fruitful to take a closer look at reductions to this submanifold, since the lattice attached to this
matrix is not euclidean. So the orthogonal side is not easy to handle. But maybe something
could be achieved on the Hermitian side. The claims about the first two isomorphies can be
read off 5.24.

Again, proceeding like described in (6.10) yields the following theorem, where once more
the Borcherds products are also cusp forms. And since in the future they might be needed,
we list more than the the five Borcherds products associated to the basis from the preceding
proposition. They are products of these five. Note that the Borcherds product associated to the
principal part with constant term 56 is denoted by $\psi_{3,28,(1)}$. Furthermore, we will present the
divisors a bit different this time because there are more involved.

(6.16) Theorem. Let $S = T_3$. Then there exist holomorphic Borcherds products

\[ \psi_{3,6} \in [\Gamma_S, 6, 1]_0, \quad \psi_{3,7} \in [\Gamma_S, 7, \chi_{\pi_1}^S]_0, \quad \psi_{3,27} \in [\Gamma_S, 27, \chi_{\pi_1}^S]_0, \]
\[ \psi_{3,28,(1)} \in [\Gamma_S, 28, \chi_{\pi_1}^S \chi_{\pi_2}^S]_0, \quad \psi_{3,42} \in [\Gamma_S, 42, \chi_{\det}^S \chi_{\pi_2}^S]_0, \]

and

\[ \psi_{3,28,(2)} \in [\Gamma_S, 28, \chi_{\pi_1}^S \chi_{\pi_2}^S]_0, \quad \psi_{3,28,(3)} \in [\Gamma_S, 28, \chi_{\det}^S \chi_{\pi_1}^S]_0, \quad \psi_{3,48} \in [\Gamma_S, 48, 1]_0, \]
\[ \psi_{3,49} \in [\Gamma_S, 49, \chi_{\pi_2}^S]_0, \quad \psi_{3,70} \in [\Gamma_S, 70, \chi_{\det}^S \chi_{\pi_1}^S]_0. \]

Use the following abbreviation for the occurring sets of zeros:

\[ D_4 = \bigcup_{M \in \Gamma_S} M\langle H_{A_1}^{\perp} \rangle, \quad D_7 = \bigcup_{M \in \Gamma_S} M\langle H_{D_4}^{\perp} \rangle, \quad D_{12} = \bigcup_{M \in \Gamma_S} M\langle \lambda_{12}^{\perp} \rangle, \]
\[ D_{15} = \bigcup_{M \in \Gamma_S} M\langle \lambda_{15}^{\perp} \rangle, \quad D_{16} = \bigcup_{M \in \Gamma_S} M\langle \lambda_{16}^{\perp} \rangle, \quad D_{24} = \bigcup_{M \in \Gamma_S} M\langle \lambda_{24}^{\perp} \rangle. \]

The zeros of these Borcherds can be read off the following tables, where the number denotes the order along
the associated set. There are no further zeros apart from those appearing in the tables.

<table>
<thead>
<tr>
<th></th>
<th>$D_4$</th>
<th>$D_7$</th>
<th>$D_{12}$</th>
<th>$D_{15}$</th>
<th>$D_{16}$</th>
<th>$D_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_{3,6}$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,7}$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,27}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,28,(1)}$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,42}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th></th>
<th>$D_4$</th>
<th>$D_7$</th>
<th>$D_{12}$</th>
<th>$D_{15}$</th>
<th>$D_{16}$</th>
<th>$D_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_{3,28,(2)}$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,28,(3)}$</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_{3,48}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,49}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_{3,70}$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Furthermore, we have the following identities:

\[
\begin{align*}
\psi_{3,28,(2)} &= \frac{\psi_{3,7}\psi_{3,27}}{\psi_{3,6}}, & \psi_{3,28,(3)} &= \frac{\psi_{3,7}^2\psi_{3,42}}{\psi_{3,28,(1)}}, & \psi_{3,48} &= \frac{\psi_{3,27}^2}{\psi_{3,6}}, \\
\psi_{3,49} &= \frac{\psi_{3,7}\psi_{3,27}^2}{\psi_{3,6}^2}, & \psi_{3,70} &= \frac{\psi_{3,7}\psi_{3,27}^2\psi_{3,42}}{\psi_{3,6}^2\psi_{3,28,(1)}}.
\end{align*}
\]

### 6.2.4 Borcherds products for \( D_2^+ \)

Finally, we analyze the Borcherds products for \( S = D_2^+ \). Concerning the obstruction space \( \text{dim}[\text{Mp}_2(Z), 2 + 2/2, \rho_3] \) we obtain:

\textbf{(6.17) Proposition.} We have \( \text{dim}[\text{Mp}_2(Z), 3, \rho_3] = 2 \). A basis is given by \( E_3(; -S_1) \) and \( \Theta(\cdot; S, p_2) \), where by defining \( v_1 = (\frac{1}{2}(-1 + i\sqrt{3}))' \) and \( v_2 = (\frac{1}{2}(-1 - i\sqrt{3}))' \) (with \( q(v_1) = q(v_2) = 0 \), which can be verified by a straightforward calculation), \( p_2 \) is given by

\[
p_2(x) = -(v_1' S x)^2 - (v_2' S x)^2 = 14x_1^2 + 14x_1x_2 - 21x_2^2.
\]

If we use the common abbreviation \( q = e^{2\pi i \tau} \), then we get

\[
\begin{align*}
E_{3,(0,0)}(\tau; -S_1) &= 2 - \frac{125}{2} q^{1} + O(q^2), \\
E_{3, \pm\frac{1}{2}, -\frac{1}{2}}(\tau; -S_1) &= -\frac{7}{4} q^{1/7} + O(q^{8/7}), \\
E_{3, \pm\frac{1}{2}, -\frac{1}{2}}(\tau; -S_1) &= -\frac{35}{4} q^{1/7} + O(q^{9/7}), \\
E_{3, \pm\frac{1}{2}, -\frac{1}{2}}(\tau; -S_1) &= -\frac{142}{4} q^{1/7} + O(q^{11/7}),
\end{align*}
\]

and

\[
\begin{align*}
\Theta_{(0,0)}(\tau; S, p_2) &= 0 + 28q^{1} + O(q^2), \\
\Theta_{\pm\frac{1}{2} - \frac{1}{2}}(\tau; S, p_2) &= -2q^{1/7} + O(q^{8/7}), \\
\Theta_{\pm\frac{1}{2} - \frac{1}{2}}(\tau; S, p_2) &= 6q^{2/7} + O(q^{9/7}), \\
\Theta_{\pm\frac{1}{2} - \frac{1}{2}}(\tau; S, p_2) &= -10q^{4/7} + O(q^{11/7}).
\end{align*}
\]

Therefore, a basis for possible principal parts and constant terms of weakly holomorphic vector-valued modular forms satisfying \( c_\mu(n) = 0 \) for \( n < -1 \) is given by

\[
\begin{align*}
q^{-2/7} e_{2/7} + 3q^{-1/7} e_{1/7} + 14e_0, \\
q^{-4/7} e_{4/7} + 2q^{-2/7} e_{2/7} + q^{-1/7} e_{1/7} + 56e_0, \\
q^{-1} e_0 + q^{-4/7} e_{4/7} + 2q^{-1/7} e_{1/7} + 84e_0,
\end{align*}
\]

where we use the following abbreviations:

\[
e_0 = e_{(0,0)},
\]
6.2 A possible reduction process

\[ e_{1/7} = e_{\left(\frac{1}{7}, -\frac{1}{7}\right)} + e_{\left(-\frac{1}{7}, \frac{1}{7}\right)}, \]
\[ e_{2/7} = e_{\left(\frac{1}{7}, -\frac{1}{7}\right)} + e_{\left(-\frac{1}{7}, \frac{1}{7}\right)}, \]
\[ e_{4/7} = e_{\left(\frac{2}{7}, -\frac{2}{7}\right)} + e_{\left(-\frac{2}{7}, \frac{2}{7}\right)}. \]

Once more, one easily verifies that the precondition on \( A \) in (6.8) is fulfilled for all \( A \in O(\Lambda) \) and all principal parts from above. Therefore, we can apply (6.8) in all cases, again, and it will turn out that all principal parts yield holomorphic Borcherds products for the full orthogonal modular group \( \Gamma_S \).

Again, we will apply (6.5) to these principal parts to obtain Borcherds products \( \psi_{4,k} \) which have zeros along rational quadratic divisors with discriminant \( \leq 7 \). And like in all the previous cases, it suffices to consider the following representatives \( \lambda_\delta^\perp \) of discriminant \( \delta \):

\[ \lambda_1^\perp = \{ w \in H_S \ ; z_2 = 0 \} \simeq H(2), \]
\[ \lambda_2^\perp = \{ w \in H_S \ ; z_1 = 0 \} \simeq H(4), \]
\[ \lambda_4^\perp = \{ w \in H_S \ ; z_1 = z_2 \} \simeq H(8), \]
\[ \lambda_7^\perp = \{ w \in H_S \ ; z_2 = -2z_1 \} \simeq H(14), \]

where \( w = (\tau_1, z_1, z_2, \tau_2) \). Here, the "(n)" stands for the matrix \( n \in \mathbb{Z}^{1 \times 1} \). The claims about the isomorphies are obvious since one only has to determine the norm of the generating element.

Note that only the lattices attached to (2) and (4) are euclidean. Furthermore, like we will see in the next section, lattices attached to scalar matrices and the orthogonal modular forms associated to them correspond to paramodular forms, whereas for the special case (2) we obtain ordinary Siegel modular forms. Note that the spaces of Siegel modular forms are completely determined (cf. [Ig62] and [Ig64], or also [Fr65]), and as well the spaces of paramodular forms associated to (4) (cf. [Fr67]). But unfortunately, the sets of zeros of the Borcherds products which we obtain here do not allow a proper reduction to these spaces. And like we will see, a reduction with respect to (14) seems to be most promising.

Again, proceeding like described in (6.10) yields the following theorem, where once more the Borcherds products are also cusp forms. And again, we list more than the three Borcherds products associated to the basis from the preceding proposition, mostly because one of these will be needed for the yet to introduce reduction process. They are products of these three. Note that the Borcherds product associated to the principal part with constant term 56 is denoted by \( \psi_{4,28,(1)} \).
Theorem. Let \( S = \mathbb{A}^{(4)}_2 \). Then there exist holomorphic Borcherds products
\[
\psi_{4,7} \in [\Gamma_S, 7, \chi^S_{\Gamma}]_0, \quad \psi_{4,28,(1)} \in [\Gamma_S, 28, 1]_0, \quad \psi_{4,42} \in [\Gamma_S, 42, \chi^S_{\det}]_0,
\]
and
\[
\psi_{4,28,(2)} \in [\Gamma_S, 28, \chi^S_{\det}]_0, \quad \psi_{4,49} \in [\Gamma_S, 49, \chi^S_{\Gamma}]_0, \quad \psi_{4,63} \in [\Gamma_S, 63, \chi^S_{\det \chi^S_{\Gamma}}]_0, \quad \psi_{4,70} \in [\Gamma_S, 70, 1]_0.
\]
The zeros of these Borcherds products are given as follows, where the number in front of the unions denote the orders along these sets of zeros:

\[
\begin{align*}
3 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_2), \\
2 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 2 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_2) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_4), \\
3 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_4) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_7), \\
7 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_7), \\
1 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 3 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_2) \cup 2 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_4), \\
2 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_1) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_2) \cup 2 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_4) \cup 1 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_7), \\
4 & \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_2) \cup 3 \cdot \bigcup_{M \in \Gamma_S} M(\lambda^\perp_4).
\end{align*}
\]
Furthermore, we have the following identities:
\[
\psi_{4,28,(2)} = \frac{\psi_{4,7}^2 \psi_{4,42}}{\psi_{4,28,(1)}}, \quad \psi_{4,49} = \frac{\psi_{4,28,(1)}}{\psi_{4,7}}, \quad \psi_{4,63} = \frac{\psi_{4,28,(1)} \psi_{4,42}}{\psi_{4,7}}, \quad \psi_{4,70} = \frac{\psi_{4,28,(1)}^3}{\psi_{4,7}^2}.
\]

6.2.5 The reduction process

Now, we are going to present the reduction process which is developed according to what we described at the beginning of the current chapter. We start by considering \( S = \mathbb{A}^{(4)}_2 \). Like we have seen in (6.12), we have Borcherds products with “nice” divisors, which means that the sets of zeros are given by exactly one \( \lambda^\perp \) (and those that are congruent to it modulo \( \Gamma_S \)), and in particular of order one.

According to (5.24), we have the following: Let \( k \in 2\mathbb{N}_0 \). If \( f \in [\Gamma_S, k, \chi \cdot \chi^m_{\det}] \) with \( \chi \in \)
\{x_{\text{det}, \pi_1}\} and \(m \in \mathbb{Z}\), then \(f\) vanishes identically on \(\mathcal{H}_{A_{2,1}}\). And if \(f \in [\Gamma_S, k, \chi_{\text{det} \pi_1}^m \chi_{\pi_2}^n]\) with \(m, n \in \mathbb{Z}\), then \(f\) vanishes identically on \(\mathcal{H}_{T_3}\). But there are no vanishing conditions concerning \(T_3\). Using (6.6) and the explicit divisors of the Borcherds products from (6.12), we obtain:

\[
f \in [\Gamma_S, k, \chi_{\text{det} \pi_1}^m \chi_{\pi_2}^n], \quad m \in \mathbb{Z} \quad \Rightarrow \quad \frac{f}{\psi_{1,6}} \in [\Gamma_S, k - 6, \chi_{\text{det} \pi_1}^m \chi_{\pi_2}^n],
\]

and

\[
f \in [\Gamma_S, k, \chi_{\pi_1}^m \chi_{\pi_2}^n], \quad m \in \mathbb{Z} \quad \Rightarrow \quad \frac{f}{\psi_{1,6}} \in [\Gamma_S, k - 6, \chi_{\pi_2}^n],
\]

while

\[
f \in [\Gamma_S, k, \chi_{\text{det} \pi_1}^m \chi_{\pi_2}^n], \quad m \in \mathbb{Z} \quad \Rightarrow \quad \frac{f}{\psi_{1,42}} \in [\Gamma_S, k - 42, \chi_{\pi_1}^{m-1} \chi_{\pi_2}^n].
\]

Combining these results leads to

(6.19) Proposition. Let \(S = A_2^{(2)}, k \in 2\mathbb{N}_0, l, m, n \in \{0, 1\}\) and \(f \in [\Gamma_S, k, \chi_{\text{det} \pi_1}^m \chi_{\pi_2}^n]\). Then

\[
f \in \psi_{1,6}^{1-l} \cdot \psi_{1,42}^{l} [\Gamma_S, k - 6|l - m| - 42l, \chi_{\pi_2}^n].
\]

Due to this result, we only need to consider the spaces \([\Gamma_S, k, 1]\) and \([\Gamma_S, k, \chi_{\pi_2}]\), where \(k \in 2\mathbb{N}_0\). Note that we have

\[
[\Gamma_S, k, \chi_{\pi_1}^m]_{|\mathcal{H}_{A_{2,1}}} \subset [\Gamma_{A_{2,1}}, 1], \quad (6.4)
\]

\[
[\Gamma_S, k, \chi_{\pi_2}^n]_{|\mathcal{H}_{T_3}} \subset [\Gamma_{T_3}, (\chi_{\pi_1}^m \chi_{\pi_2}^n)]
\]

in view of (5.24). So the reduction process is quite simple in this case – given that we already determined the structure on the spaces of orthogonal modular forms for \(A_{2,1}\) and \(T_3\), and that we can find orthogonal modular forms for \(A_2^{(2)}\) such that the restrictions of these generate the graded rings of orthogonal modular forms for \(A_{2,1}\) and \(T_3\).

Let \(f \in [\Gamma_S, k, \chi_{\pi_2}^m]\), and suppose we found orthogonal modular forms \(g_j \in [\Gamma_S, k_j, \chi_{\pi_2}^m]\), \(j = 1, \ldots, n\) for some \(n \in \mathbb{N}\) such that there exists a polynomial \(p\) (homogeneous with respect to the weights \(k_j\)) with \(f - p(g_1, \ldots, g_n) |_{\mathcal{H}_{A_{2,1}}} \equiv 0\), then according to what we saw above we can devide twice by \(\psi_{1,6}\) and obtain

\[
f = \psi_{1,6}^2 \cdot p(g_1, \ldots, g_n) \cdot g
\]

for some \(g \in [\Gamma_S, k - 12, \chi_{\pi_2}^m]\). Then, an induction yields the structure of these spaces, and as well of the graded ring of orthogonal modular forms.

On the other hand, if we found orthogonal modular forms \(g_j \in [\Gamma_S, k_j, \chi_{\pi_2}^m]\), \(j = 1, \ldots, n\) for some \(n \in \mathbb{N}\) such that there exists a polynomial \(p\) (again homogeneous with respect to the weights \(k_j\)) with \(f - p(g_1, \ldots, g_n) |_{\mathcal{H}_{T_3}} \equiv 0\), then we can devide by \(\psi_{1,54}\) and obtain

\[
f = \psi_{1,54} \cdot p(g_1, \ldots, g_n) \cdot g
\]

for some \(g \in [\Gamma_S, k - 54, \chi_{\pi_2}^{m-1}]\). Of course, the first approach might be more fruitful due to the much smaller weight of \(\psi_{1,6}\) and since there is no change in the abelian character. Nevertheless,
we pursue both approaches. In Particular because with help of the second one (i.e. regarding $T_3$) we might get rid of the abelian character $\chi_{T_3}$.

So since for a reduction process we would need appropriate orthogonal modular forms $g_j$ whose restrictions generate the graded rings of orthogonal modular forms for $A_{2,1}$ or $T_3$, we will have a closer look at the restrictions of those Borcherds products that do not vanish identically on $H_{A_{2,1}}$ or $H_{T_3}$. We start with $A_{2,1}$: $\psi_{1,42}$ and $\psi_{1,54}$, as well as $\psi_{1,27,(1)}$ and $\psi_{1,27,(2)}$ do not vanish identically on $H_{A_{2,1}}$.

Let $w = (\tau_1, z_1, z_2, z_3, z_4, \tau_2) \in H_5$. According to (6.11), $\psi_{1,42}$ vanishes on all rational quadratic divisors $\lambda_1^\perp$ of discriminant 3, where $\lambda \in \Lambda_1^\perp$ is primitive with $\lambda \equiv 0 \mod \Lambda_1$ and $q_1(\lambda) = -1$, so in particular for $\lambda = (0, 0, 1, -1, 0, 0, 0, 0)$, with $\lambda_1^\perp = \{ w \in H_5 ; z_1 = z_2 \}$. Therefore, deriving from the considerations in (6.6), $\psi_{1,42}|_{H_{A_{2,1}}}$ vanishes along $\delta_2^1$ (which is a submanifold in $H_{A_{2,1}}$ given by $z_1 = z_2$) of order greater or equal to 1. Doing the same for $\lambda = (0, 0, 1, 0, 0, 0)$, where $\lambda \in \{(0, 0, 1, 0, 0, 0, 0, 0)\}$, we also get that $\psi_{1,42}$ vanishes of first order along $2z_3 + z_4 = 0$, $z_3 + 2z_4 = 0$ and $z_3 = z_4$. But then, again deriving from the considerations in (6.6) and because the restriction to $H_{A_{2,1}}$ given by $z_4 = 0$, $\psi_{1,42}|_{H_{A_{2,1}}}$ vanishes along $\delta_3^1$ (where the submanifold in $H_{A_{2,1}}$ is meant, given by $z_3 = 0$) of order greater or equal to 3. Therefore, because $\psi_{1,42}|_{H_{A_{2,1}}} \in \left[ \Gamma_{A_{2,1}}, 42, \chi_{A_{2,1}} \right]_0$ holds in view of (6.12) and (5.24), and since $\psi_{2,42}$ vanishes exactly along $\lambda_4^\perp$ of order 3, along $\lambda_2^\perp$ of order 1 and along the rational quadratic divisors congruent to them modulo $\Gamma_{A_{2,1}}$ (of according order), we obtain

$$\psi_{1,42}|_{H_{A_{2,1}}} \subseteq \psi_{2,42}[\Gamma_{A_{2,1}}, 0, 1] = C\psi_{2,42}$$

in view of (6.6).

Concerning $\psi_{1,27,(1)}$ and $\psi_{1,27,(2)}$, the situation is similar: Both vanish of first order along $\lambda_{2,(1)}^\perp$ or $\lambda_{2,(2)}^\perp$, which means along $z_4 = z_2$ or $z_4 = -z_2$. The restriction to $H_{A_{2,1}}$ is given by $z_4 = 0$, hence both restrictions vanish on $\lambda_4^\perp$ (where the submanifold in $H_{A_{2,1}}$ is meant, given by $z_2 = 0$). Furthermore, according to (6.11) ($\psi_{1,27,(1)}$ is the Borcherds product associated to the principal part $q^{-2/3}e_{2/3} + 54e_0$) and (6.5), $\psi_{1,27,(1)}$ vanishes along $(0, 0, 0, \frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, 0)$ of first order, where this submanifold is given by $z_2 = z_3$. Therefore, we immediately obtain that $\psi_{1,27,(1)}|_{H_{A_{2,1}}}$ vanishes along $\lambda_4^\perp$ (which is actually given by $z_2 = z_3$ in $H_{A_{2,1}}$). On the other hand, $\psi_{1,27,(2)}$ vanishes along $(0, 0, 0, 0, \frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, 0)$ of first order, hence along $z_2 = -z_3$. So $\psi_{1,27,(2)}|_{H_{A_{2,1}}}$ does so in $H_{A_{2,1}}$, too. Note that the rational quadratic divisor in $H_{A_{2,1}}$ given by $z_2 = -z_3$ is congruent to $\lambda_4^\perp$ via $A_{1,2}^{A_{2,1}}$. Therefore, because $\psi_{1,27,(1)}|_{H_{A_{2,1}}}, \psi_{1,27,(2)}|_{H_{A_{2,1}}} \in \left[ \Gamma_{A_{2,1}}, 27, \chi_{A_{2,1}} \right]_0$ holds in view of 6.3, and since $\psi_{2,27}$ vanishes exactly along $\lambda_4^\perp, \lambda_2^\perp$ and the rational quadratic divisors congruent to them modulo $\Gamma_{A_{2,1}}$, and in all cases of first order, we obtain

$$\psi_{1,27,(1)}|_{H_{A_{2,1}}}, \psi_{1,27,(2)}|_{H_{A_{2,1}}} \subseteq \psi_{2,27}[\Gamma_{A_{2,1}}, 0, 1] = C\psi_{2,27}.$$ 

And since $\psi_{1,54} = \psi_{1,27,(1)}\psi_{1,27,(2)}$ (see (6.12)), we also get

$$\psi_{1,54}|_{H_{A_{2,1}}} \subseteq C^2\psi_{2,27}^2.$$ 

Finally, note that due to their sets of zeros, none of the restrictions are identically zero. We
6.2 A possible reduction process

summarize these results in

(6.20) Proposition.  
- \( \psi_{1,6}|_{H_{A_{2,1}}} \equiv 0 \),
- \( \psi_{1,27,1}|_{H_{A_{2,1}}}, \psi_{1,27,2}|_{H_{A_{2,1}}} \in \mathbb{C}^* \psi_{2,27} \),
- \( \psi_{1,42}|_{H_{A_{2,1}}} \in \mathbb{C}^* \psi_{2,42} \),
- \( \psi_{1,54}|_{H_{A_{2,1}}} \in \mathbb{C}^* \psi_{2,27}^2 \).

Note that since \( \psi_{2,27} \) might play an important role in determining the structure of the spaces of orthogonal modular forms for \( A_{2,1} \), one might have to consider \( \Gamma_{\text{Bor}} \), first, before being able to determine the spaces of orthogonal modular forms with respect to \( \Gamma_{A_{2,1}} \). Fortunately, \( \Gamma_{\text{Bor}} \) acts transitively on the set of rational quadratic divisors of fixed discriminant, if this discriminant is congruent 0 or 1 modulo 3 (which is given for \( \lambda_{1}^\perp \) and \( \lambda_{3}^\perp \)). Therefore, there occurs no problem if we want to divide orthogonal modular forms with respect to \( \Gamma_{\text{Bor}} \) by \( \psi_{1,6} \) or by \( \psi_{1,42} \). Again, it suffices to check whether these forms vanish on \( H_{A_{2,1}} \) or on \( H_{S_3} \). In this context, note that we have

\[
\psi_{1,6} \in [\Gamma_{\text{Bor}}, 6, 1], \\
\psi_{1,42} \in [\Gamma_{\text{Bor}}, 42, \chi_{\text{det}}].
\]

Unfortunately, there are no conditions concerning orthogonal modular forms with respect to \( \Gamma_{\text{Bor}} \) such that a form has to vanish on \( H_{A_{2,1}} \). But just like in (5.24) and in (6.19) we get

\[
f \in [\Gamma_S, k, \chi_{\text{det}}^{m_i, -3i_i, n_i}] \Rightarrow f \in [\Gamma_{\text{S}}, k - 42, \chi_{i_i, i_i}^{m_i, -3i_i}]
\]

hence we only need to consider the abelian characters in \( \langle \chi_{i_i, i_i}^{1, -3} \rangle \).

To come back to the restriction to \( A_{2,1} \), we still need an orthogonal modular form living on \( H_{S_2} \) (with \( S = A_2^{(2)} \)) of weight 7, whose restriction equals \( \psi_{2,7} \), since then we would obtain the complete basis of Borcherds products for \( A_{2,1} \) specified in (6.14). And here, the quaternionic Maaß lifts of odd weight come into play: According to (3.35) \( M(7, v_i; O) \) and \( M(7, v_{-i}; O) \) are both one-dimensional. Let \( g \in M_5^{2+} \) be a generator of the one-dimensional space \( M_5^{2+} \), and define \( f_{7,i} = M_5^{(i)} \) and \( f_{7,-i} = M_5^{(-i)} \). In virtue of (3.22) \( f_{7,i}|_{k(i_1 I)} = -f_{7,-i} \) holds. Now, define

\[
f_7 = f_{7,i} - if_{7,-i} \quad (6.5)
\]

Due to the transformation behavior of the orthogonal Maaß lifts of odd weight (see (1.57)) we obtain for the generators of \( \Gamma(O)_{\text{Bor}} \) (cf. (5.31), (3.20) and (3.21)):

- \( f_7|T = -f_7 \),
- \( f_7| \text{Trans}(T) = f_7 \) for all \( T \in \text{Her}_2(O) \),
- \( f_7| \text{Rot}(( 1+i \sqrt{3} 0 \choose 0 1 )) = -f_7 \),
- \( f_7| \text{Rot}(( -i 0 \choose 0 i )) = f_7 \).
• $f_7|\gamma(i_1I \cdot \text{Rot}(\frac{0}{0} \ 1 \ 1)) = (-f_{7,-i} + if_{7,i})|\gamma \text{Rot}(\frac{0}{0} \ 1 \ 1)) = if_{7,-i} - f_{7,i} = -f_7$, \\
• $f_7(Z') = -f_7(Z)$ for all $Z \in \mathcal{H}(\mathbb{H})$.

Therefore, in virtue of (5.31) and (5.33) we get

$$\psi_{1,7} := f_7 \circ \varphi_{\mathbb{H}} \in \left[\Gamma_{\text{Bor}}, 7, \chi_{i1} \sqrt{3} \chi_{ii2}\right]. \quad (6.6)$$

In view of (3.26) Maaß lifts of odd weight vanish identically when restricting them to $\mathcal{H}(\mathbb{R} + \mathbb{R}i_2)$. Like we have seen, on the orthogonal side this means that $\psi_{1,7}$ vanishes when restricting to $\mathcal{H}_{\Gamma_0(2)}$. Moreover, restricting to $\mathcal{H}_{\Gamma_2}$ would yield an orthogonal modular form of weight 7 with respect to $\Gamma_{\Delta_2}$, or in other words, we obtain a Hermitian modular form of weight 7 for $\mathbb{Q}(\sqrt{-3})$ (cf. (5.34)). But according to [De01, p.116] (or [DK03, thm.6]) there exists no non-trivial Hermitian modular form of weight 7 (the smallest odd weight is 9). Hence $\psi_{1,7}$ also vanishes on $\mathcal{H}_{\Gamma_2}$, and we summarize:

$$\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}} \in \left[\Gamma_{\text{Bor}}, 7, \chi_{A_{21}}\right],$$
and $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ vanishes along $\lambda_3^1$ and $\lambda_4^1$ (in $\mathcal{H}_{\Gamma_2}$), while computing some Fourier-coefficients shows that $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}} \neq 0$. So $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ has a zero whenever $\psi_{2,7}$ does – thus we only have to verify that the order of $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ along $\lambda_3^1$ is always 2 at least. So suppose this is not the case. $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ vanishes along $\lambda_3^1$ of order greater or equal to 2 – and still of order greater or equal to 1 along $\lambda_4^1$. Hence $f := \psi_{2,7}|_{\mathcal{H}_{\Gamma_2}} \in \left[\Gamma_{\text{Bor}}, 7, \chi_{A_{21}}\right]$ (see (6.14) about the orders of $\psi_{2,7}$), and by assumption $f$ does not vanish identically along $\lambda_3^1$ since there are sections where $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ only vanishes of order 2 (and hence $0 \neq f|_{\mathcal{H}_{\Gamma_2}} \in \left[\Gamma_{\text{Bor}} , 7 , \chi_{A_{21}}\right]$). But we already mentioned above that this is a contradiction. Thus the order of $\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}}$ along $\lambda_3^1$ always has to be greater or equal to 2, and we get

$$\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}} \in \mathbb{C}^\ast \psi_{2,7}. \quad (6.7)$$

And finally, note the following: According to (5.25) and (5.33) we have $f_{7,i} \circ \varphi_{\mathbb{H}} \in \left[\Gamma_{\text{Maaß}}, 7, \chi_i\right]$, $f_{7,-i} \circ \varphi_{\mathbb{H}} \in \left[\Gamma_{\text{Maaß}}, 7, \chi_i\right]$ and $(f_{7,i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}} \in \left[\Gamma_{\text{Bor}}, 7, \chi_{A_{21}}\right]$, $(f_{7,-i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}} \in \left[\Gamma_{\text{Bor}}, 7, \chi_{A_{21}}\right]$. Furthermore, $A = A_{21}$ corresponds to $B = A_{21}$, and thus

$$(f_{7,i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}} = (f_{7,i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}} |\gamma R_B = (f_{7,i} \circ \varphi_{\mathbb{H}} |\gamma R_A)|_{\mathcal{H}_{\Gamma_2}} = -(f_{7,-i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}},$$

which leads to

$$\psi_{1,7}|_{\mathcal{H}_{\Gamma_2}} = (1 + i)(f_{7,i} \circ \varphi_{\mathbb{H}})|_{\mathcal{H}_{\Gamma_2}}.$$ 

Hence we could have started with $f_{7,i}$ directly instead of defining $f_7$. But $f_{7,i} \circ \varphi_{\mathbb{H}}$ is an orthogonal modular form for $\Gamma_{\text{Maaß}}$, but like we have seen it might be necessary to determine the spaces of orthogonal modular forms with respect to $\Gamma_{\text{Bor}}$. So this is why we introduced $\psi_{1,7}$.

Next, we have a closer look at the restrictions to $\mathcal{H}_{T_3}$. Here, $\psi_{1,27,(1)}$ and $\psi_{1,54}$ vanish identically when restricting to $\mathcal{H}_{T_3}$ (where we choose the first restriction to $\mathcal{H}_{T_3}$, hence $z_2 = z_4$), while $\psi_{1,6}$, $\psi_{1,27,(2)}$ and $\psi_{1,42}$ do not.
We start by analyzing the restriction of $\psi_{1,6}$. Let again $w = (\tau_1, z_1, z_2, z_3, z_4, T_2) \in \mathcal{H}_5$. According to (6.11), $\psi_{1,6}$ vanishes on all rational quadratic divisors $\lambda_1$ of discriminant 1, where $\lambda \in \Lambda_1^*$ is primitive with $q_1(\lambda) = -\frac{1}{4}$, so in particular for $\lambda = (0,0,\lambda,0,0)$, where $\lambda \in \{(\frac{1}{3}, -\frac{2}{3}, 0,0), (0,0, -\frac{1}{2}, -\frac{1}{2}), (0,0, -\frac{2}{3}, 0,0)\}$. Hence $\psi_{1,6}$ vanishes of first order along $z_2 = 0, z_4 = 0$ and $z_3 = 0$. Again deriving from the considerations in (6.6) and because the restriction to $\mathcal{H}_{T_3}$ is given by $z_2 = z_4$, $\psi_{1,6}|_{\mathcal{H}_{T_3}}$ vanishes along $\lambda_4^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_2 = 0$ of order greater or equal to 2, and along $\lambda_2^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_3 = 0$) of order greater or equal to 1. And therefore, because $\psi_{1,6}|_{\mathcal{H}_{T_3}} \in [\Gamma_{T_3}, 6, 1]$ holds in view of (6.12) and (5.24), and since $\psi_{3,6}$ vanishes exactly along $\lambda_4^\perp$ of order 2, along $\lambda_2^\perp$ of order 1 and along the rational quadratic divisors congruent to them modulo $\Gamma_{T_3}$ (of according order), we obtain in view of (6.6):

$$\psi_{1,6}|_{\mathcal{H}_{T_3}} \in \psi_{3,6}[\Gamma_{T_3}, 0, 1] = C\psi_{3,6}$$

The situation for $\psi_{1,27,(2)}$ is similar, again: In view of (6.11), $\psi_{1,27,(2)}$ vanishes on all rational quadratic divisors $\lambda_1$ of discriminant 2, where $\lambda \in \Lambda_1^*$ is primitive and has to be congruent $\pm(0,0,\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, 0,0)$ modulo $\Lambda_1$, with $q_1(\lambda) = -\frac{2}{3}$, so in particular for $\lambda = (0,0,\lambda,0,0)$, where $\lambda \in \{(\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}), (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3})\}$. Hence $\psi_{1,27,(2)}$ vanishes of first order along $z_2 = -z_4, z_3 = z_2 - z_4$ and $z_1 = -z_3$. Once more, since the restriction to $\mathcal{H}_{T_3}$ is given by $z_2 = z_4, \psi_{1,27,(2)}|_{\mathcal{H}_{T_3}}$ vanishes along $\lambda_4^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_2 = 0$), along $\lambda_2^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_3 = 0$) and along $\lambda_{16}^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_1 = -z_3$). And therefore, because $\psi_{1,27,(2)}|_{\mathcal{H}_{T_3}} \in [\Gamma_{T_3}, 27, \chi_{T_3}^1]$ holds in view of (6.3), and since $\psi_{1,27,(2)}|_{\mathcal{H}_{T_3}}$ vanishes whenever $\psi_{3,27}$ does, while the zeros of $\psi_{3,27}$ are always of first order, we obtain in view of (6.6):

$$\psi_{1,27,(2)}|_{\mathcal{H}_{T_3}} \in \psi_{3,27}[\Gamma_{T_3}, 0, 1] = C\psi_{3,27}$$

The same holds for $\psi_{1,27,(1)}$ if we choose the second restriction to $\mathcal{H}_{T_3}$ given by $z_2 = -z_4$.

And finally, let us consider $\psi_{1,42}$. Once more in view of (6.11), $\psi_{1,42}$ vanishes on all rational quadratic divisors $\lambda_1$ of discriminant 3, where $\lambda \in \Lambda_1^*$ is primitive with $\lambda \equiv 0$ modulo $\Lambda_1$, with $q_1(\lambda) = -1$, so in particular for $\lambda = (0,0,\lambda,0,0)$, where $\lambda \in \{(0,0,1,-1),(0,0,1,0)\}$. Hence $\psi_{1,42}$ vanishes of first order along $z_4 = z_3$ and along $z_4 = -2z_3$. So again, since the restriction to $\mathcal{H}_{T_3}$ is given by $z_2 = z_4, \psi_{1,42}|_{\mathcal{H}_{T_3}}$ vanishes along $\lambda_1^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_2 = z_3$) and along $\lambda_4^\perp$ (in $\mathcal{H}_{T_3}$, given by $z_2 = -2z_3$). So because $\psi_{1,42}|_{\mathcal{H}_{T_3}} \in [\Gamma_{T_3}, 42, \chi_{T_3}^1] \in [\Gamma_{T_3}, 42, \chi_{T_3}^1]$ holds in view of (6.12) and (5.24), and since $\psi_{1,42}|_{\mathcal{H}_{T_3}}$ vanishes whenever $\psi_{3,42}$ does, while again the zeros of $\psi_{3,42}$ are always of first order, we obtain

$$\psi_{1,42}|_{\mathcal{H}_{T_3}} \in \psi_{3,42}[\Gamma_{T_3}, 0, 1] = C\psi_{3,42}$$

in view of (6.6). We summarize these results in

(6.21) Proposition. If we consider the restriction to $\mathcal{H}_{T_3}$ given by $z_2 = z_4$, then the following holds:

- $\psi_{1,27,(1)}|_{\mathcal{H}_{T_3}} = \psi_{1,54}|_{\mathcal{H}_{T_3}} \equiv 0$
- $\psi_{1,6}|_{\mathcal{H}_{T_3}} \in C^\ast \psi_{3,6}$
- $\psi_{1,27,(2)}|_{\mathcal{H}_{T_3}} \in C^\ast \psi_{3,27}$
- $\psi_{1,42}|_{\mathcal{H}_{T_3}} \in C^\ast \psi_{3,42}$
To obtain the basis of Borcherds products for $\Gamma_{T_3}$ (that only have zeros on rational quadratic divisors of discriminant less or equal to 24) from (6.16), we still need to find counterparts for $\psi_{3,7}$ and $\psi_{3,28,(1)}$. Again, the solution for $\psi_{3,7}$ might be the quaternionic Maaß lifts of odd weight. Using the same notation like above, we define

$$f_7^* = f_{7,i} + if_{7,-i} .$$  \hspace{1cm} (6.8)

Of course, $f_7^*$ meets the same transformation laws for the generators of $\Gamma(\mathcal{O}_{\text{Bor}})$ like $f_7$, with the exception of

- $f_7^* \mid_{\mathcal{T}_3} = f_{7,i} - if_{7,-i} \mid_{\mathcal{T}_3} = i f_{7,i} = f_7^*$.

Therefore, in virtue of (5.31) and (5.33) we get

$$\psi_{1,7}^* := f_7^* \circ \varphi_H \in [\Gamma_{\text{Bor}}, 7, \chi_{i_1 \sqrt{3}}],$$ \hspace{1cm} (6.9)

and thus (see (5.25))

$$\psi_{1,7}^* |_{\mathcal{T}_3} \in [\Gamma_{T_3}, 7, \chi_{\pi \mathcal{T}_3}].$$

just like $\psi_{3,7}$. Again, computing a few Fourier-coefficients shows that $\psi_{1,7}^* |_{\mathcal{T}_3}$ does not vanish identically. The same considerations we did for $\psi_{1,7}^* |_{\mathcal{T}_3}$ yield that $\psi_{1,7}^* |_{\mathcal{T}_3}$ vanishes along $\lambda_{12}^\perp$ (on $\mathcal{H}_{T_3}$), but the order is not clear. And it also is not clear if $\psi_{1,7}^* |_{\mathcal{T}_3}$ actually vanishes on $\lambda_{12}^\perp$. Nevertheless, it is again reasonable that $\psi_{1,7}^* |_{\mathcal{T}_3}$ and $\psi_{3,7}$ coincide (up to some constant pre-factor) due to the small weight.

Unfortunately, there does not seem to be a natural counterpart for $\psi_{3,28,(1)}$, or at least none coming from a Borcherds product for $A_2^{(2)}$. But on the other hand, we will see further below that a reduction process involving $T_3$ might not be very fruitful, hence we might have to ignore that branch anyways.

Now, we will analyze how a reduction process starting at $\mathcal{H}_{A_{2,1}}$ might work (in order to be able to determine the spaces of orthogonal modular forms with respect to $\Gamma_{A_{2,1}}$). So let $S = A_{2,1}$. First, we need some information concerning some conditions which guarantee that an orthogonal modular form vanishes along $\lambda_{12}^\perp$ (in $\mathcal{H}_{A_{2,1}}$, given by $z_1 = z_2$). Note that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}(\Lambda),$$

and $R_A$ acts trivially on the submanifold $\lambda_{12}^\perp$. Noting $\chi_{\text{det}}(R_A) = -1$ and $\chi_\pi(R_A) = 1$ we obtain the following lemma in analogy to (5.26):

**Lemma.** Let $S = A_{2,1}$, $k \in \mathbb{N}_0$ and $f \in [\Gamma_S, k, \chi_{\text{det}}^{m}]$ for some $m \in \mathbb{Z}$. Then $f$ vanishes along $\lambda_{12}^\perp$.

Combining this lemma and (5.26) we have quite a few conditions under which the restriction of an orthogonal modular form vanishes. Furthermore, due to (5.18) we only have to consider...
the spaces \([\Gamma_5, 2k, 1], [\Gamma_5, 2k, \chi_{\text{det}}\chi_\pi]\), \([\Gamma_5, 2k + 1, \chi_{\text{det}}]\) and \([\Gamma_5, 2k + 1, \chi_\pi]\) for \(k \in \mathbb{N}_0\) since all other spaces of orthogonal modular forms for \(\Gamma_5\) are trivial. We summarize the results from (5.26) and (6.22):

- If \(f \in [\Gamma_5, 2k, \chi_{\text{det}}\chi_\pi]\), then \(f\) vanishes along \(\lambda_3^\perp\) and \(\lambda_{12}^\perp\).

- If \(f \in [\Gamma_5, 2k + 1, \chi_{\text{det}}]\), then \(f\) vanishes along \(\lambda_3^\perp, \lambda_4^\perp\) and \(\lambda_{12}^\perp\).

- If \(f \in [\Gamma_5, 2k + 1, \chi_\pi]\), then \(f\) vanishes along \(\lambda_4^\perp\).

Unfortunately, due to the combinations of the sets of zeros and their orders that occur for Borcherds products for \(A_{2,1}\), it is not possible to directly divide one of the \(f\) from above by one of the Borcherds products to obtain a first reduction step. But the following reduction process might work in analogy to the considerations done in [De01, pp.128] (or in [DK04]) concerning Hermitian modular forms for \(Q(\sqrt{-2})\):

Let \(f \in [\Gamma_5, k, \chi_{\text{det}}\chi_\pi^m]\), where \(m \in \mathbb{Z}\). Then \(f\) vanishes along \(\lambda_3^\perp\) and \(\lambda_{12}^\perp\). But this implies that \(\psi_{2,7}f\) vanishes along \(\lambda_3^\perp\) of order greater or equal to 3. Therefore, we can divide by \(\psi_{2,42}\) and obtain

\[
\frac{\psi_{2,7}f}{\psi_{2,42}} \in [\Gamma_5, k - 35, \chi_\pi^m] .
\]

And since we are already multiplying with \(\psi_{2,7}\) anyways, we can even consider the following: If \(f \in [\Gamma_5, 2k + 1, \chi_\pi]\), then

\[
\psi_{2,7}f \in [\Gamma_5, k + 7, 1] .
\]

This means we have to determine the spaces of orthogonal modular forms with respect to \(\Gamma_{A_{2,1}}\), trivial character and even weight. So let us have a look at \(f \in [\Gamma_5, k, 1]\). Like above, suppose we already found some orthogonal modular forms \(g_1, \ldots, g_n\) with respect to \(\Gamma_5\) and abelian character 1 of even weights, such that the restrictions of these generate the graded ring of orthogonal modular forms with respect to \(\Gamma_{D_2^+}\) (and the trivial character). Then we can find a polynomial \(p\) (homogeneous with respect to the weights) such that \(f - p(g_1, \ldots, g_n)\) vanishes along \(\lambda_3^\perp\) (hence on \(\mathcal{H}_{D_2^+}\)). Then for the same reasons like above we obtain

\[
\frac{\psi_{2,7}(f - p(g_1, \ldots, g_n))}{\psi_{2,27}} \in [\Gamma_5, k - 20, 1] .
\]

Hence an induction would yield that for every \(f \in [\Gamma_5, k, \chi]\) (where \(\chi\) is any abelian character) there exists \(j \in \mathbb{N}_0\) such that

\[
\psi_{2,7}^j f \in \mathbb{C}[\psi_{2,7}, \psi_{2,27}, \psi_{2,42}, g_1, \ldots, g_n] .
\]

The rest might work like in [De01, pp.133] concerning Hermitian modular forms for \(Q(\sqrt{-2})\), where at first one only had \(\psi_{2,7}^j f \in \mathbb{C}[\ldots]\), too (see [De01, Prop.6.19]). So this is how a reduction process could work concerning \(A_{2,1}\) – which means that the spaces of orthogonal modular forms for \(\Gamma_{D_2^+}\) have to be determined, first.

There exist analogous arguments concerning \(S = T_3\). Again, we need some further information concerning conditions which guarantee that an orthogonal modular form vanishes along \(\lambda_{12}^\perp\) (in
we cannot divide by \( \psi \).

Again, due to the combinations of the sets of zeros and their orders that occur for Borcherds processes where products for \( \psi \) and \( H \) orthogonal modular forms with respect to \( H \) of the standard proceeding with Borcherds products only.

Therefore, the reduction concerning \( \lambda \) supply this zero. There might exist a reduction process with respect to \( \psi \) divide by \( \psi \).

\[ \mathcal{H}_{T_3}, \text{given by } z_1 = z_3 \text{ or along } \lambda_2^\perp \text{ (given by } z_2 = -2z_3). \]

We have

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in O(\Lambda), \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \in O(\Lambda). \]

One easily verifies that \( R_A \) acts trivially on the submanifold \( \lambda_2^\perp \), while \( R_B \) acts trivially on the submanifold \( \lambda_2^\perp \). Noting \( \chi_{\text{det}}(R_A) = -1, \chi_{\tau_1}(R_A) = 1 \) and \( \chi_{\tau_1}(R_A) = -1 \), while \( \chi_{\text{det}}(R_B) = -1, \chi_{\tau_1}(R_B) = 1 \) and \( \chi_{\tau_2}(R_B) = 1 \), we obtain the following lemma in analogy to (5.27):

\[ (6.23) \text{Lemma. Let } S = T_3, \ k \in \mathbb{N}_0, \ f \in [\Gamma_S, k, \chi^m_{\text{det}}\chi^n_{\tau_1}\chi^{m-1}_{\tau_2}] \text{ and } g \in [\Gamma_S, k, \chi^m_{\text{det}}\chi^n_{\tau_1}\chi^{m}_{\tau_2}] \text{ for some } m,n \in \mathbb{Z}. \text{ Then } f \text{ vanishes along } \lambda_2^\perp, \text{ while } g \text{ vanishes along } \lambda_2^\perp. \]

Recall that \([\Gamma_{T_3}, k, \chi^m_{\text{det}}\chi^n_{\tau_1}, \chi^{m-1}_{\tau_2}] = \{0\} \) if \( k + l + m + n \equiv 1 \mod 2 \) (see (5.18)). To have an overview, we summarize the results from (5.27) and (6.23) here, too:

- If \( f \in [\Gamma_S, 2k + 1, \chi_{\text{det}}] \), then \( f \) vanishes along \( \lambda_4^\perp, \lambda_2^\perp \) and \( \lambda_2^\perp \).
- If \( f \in [\Gamma_S, 2k + 1, \chi_{\tau_1}] \), then \( f \) vanishes along \( \lambda_4^\perp \).
- If \( f \in [\Gamma_S, 2k + 1, \chi_{\tau_2}] \), then \( f \) vanishes along \( \lambda_4^\perp \) and \( \lambda_2^\perp \).
- If \( f \in [\Gamma_S, 2k, \chi_{\text{det}}\chi_{\tau_1}] \), then \( f \) vanishes along \( \lambda_2^\perp \) and \( \lambda_2^\perp \).
- If \( f \in [\Gamma_S, 2k, \chi_{\text{det}}\chi_{\tau_2}] \), then \( f \) vanishes along \( \lambda_2^\perp \).
- If \( f \in [\Gamma_S, 2k + 1, \chi_{\text{det}}\chi_{\tau_1}, \chi_{\tau_2}] \), then \( f \) vanishes along \( \lambda_4^\perp \) and \( \lambda_2^\perp \).

Again, due to the combinations of the sets of zeros and their orders that occur for Borcherds products for \( T_3 \), it is not possible to directly devide one of the \( f \) from above by one of the Borcherds products to obtain a first reduction step. Furthermore, there is no obvious reduction process where \( D_2^{\perp} \) (or even \( A_2 \)) is involved: Regarding the basis of Borcherds products only \( \psi_{3,6} \) and \( \psi_{3,27} \) vanish along \( \delta_{16}^{\perp} \). But \( \psi_{3,27} \) is the only Borcherds product that also vanishes on \( \delta_{16}^{\perp} \). So we cannot devide by \( \psi_{3,27} \) since there is no way to “generate” the zero at \( \delta_{16}^{\perp} \). And we cannot divide by \( \psi_{3,6} \) since its also vanishes along \( \delta_{4}^{\perp} \), and there is no form of lower weight which could supply this zero. There might exist a reduction process with respect to \( \lambda_6^{\perp} \) or \( \lambda_2^{\perp} \) regarding \( \psi_{3,27} \) or \( \psi_{3,28,(3)} \). But it is quite doubtful that one is able to determine the spaces of orthogonal modular forms with respect to \( (4 \ 0 \ 0) \) or \( (2 \ 0 \ 12) \), since the lattices attached to them are not even euclidean. Therefore, the reduction concerning \( T_3 \) seems to be a dead end – at least in the sense of the standard proceeding with Borcherds products only.

So we will only have a closer look at the restriction from \( \mathcal{H}_{A_2} \) to \( \mathcal{H}_{D_2} \), now, since the restriction from \( \mathcal{H}_{T_3} \) to \( \mathcal{H}_{D_2} \) does not seem to yield any answers concerning the structure of the spaces of orthogonal modular forms with respect to \( \mathcal{H}_{T_3} \).

We start by analyzing the restriction of \( \psi_{2,7} \). Let \( w = (\tau_1, z_1, z_2, z_3, \tau_2) \in \mathcal{H}_S \). In virtue of (6.14) \( \psi_{2,7} \) vanishes of order 2 along \( \lambda_3^\perp \) (given by \( z_3 = 0 \)). Furthermore, according to (6.13) \( \psi_{2,7} \) vanishes
of first order on all rational quadratic divisors $\lambda^\perp$ of discriminant 4, where $\lambda \in \Lambda^\perp_1$ is primitive with $q_1(\lambda) = -\frac{1}{4}$, so in particular for $\lambda = (0, 0, \tilde{\lambda}, 0, 0)$, where $\lambda \in \{(\frac{1}{4}, -\frac{3}{2}, 0), (-\frac{3}{2}, \frac{1}{4}, 0)\}$. Hence $\psi_{2,7}$ vanishes of first order along $z_2 = 0$ and along $z_1 = 0$. So again deriving from the considerations in (6.6) and because the restriction to $\mathcal{H}_{D_2^*}$ is given by $z_2 = z_3$, $\psi_{2,7}|_{\mathcal{H}_{D_2^*}}$ vanishes along $\lambda^\perp_1$ (in $\mathcal{H}_{D_2^*}$, given by $z_2 = 0$) of order greater or equal to 3, and along $\lambda^\perp_7$ (in $\mathcal{H}_{D_2}$, given by $z_1 = 0$) of order greater or equal to 1. And therefore, because $\psi_{2,7}|_{\mathcal{H}_{D_2^*}} \in \left[\Gamma_{D_2}, 7, \chi_{D_2^*}\right]$ holds in view of (6.14) and (5.26), and since $\psi_{4,7}$ vanishes exactly along $\lambda_1^\perp$ of order 3, along $\lambda^\perp_7$ of order 1 and along the rational quadratic divisors congruent to them modulo $\Gamma_{D_2^*}$ (of according order), we obtain

$$\psi_{2,7}|_{\mathcal{H}_{D_2^*}} \in \psi_{4,7}|_{\Gamma_{D_2}, 0, 1|} = C\psi_{4,7}$$

in view of (6.6).

The situation concerning $\psi_{2,42}$ is also satisfying. In virtue of (6.14) $\psi_{2,42}$ vanishes of order 3 along $\lambda_3^\perp$ (given by $z_3 = 0$). And in view of (6.13) $\psi_{2,42}$ also vanishes of first order on all rational quadratic divisors $\lambda^\perp$ of discriminant 12, where $\lambda \in \Lambda^\perp_1$ is primitive with $q_1(\lambda) = -1$, so in particular for $\lambda = (0, 0, \tilde{\lambda}, 0, 0)$, where $\lambda \in \{(1, -1, 0), (1, 0, 0)\}$. Hence $\psi_{2,42}$ vanishes of first order along $z_1 = z_2$ and along $z_2 = -2z_1$. So again deriving from the considerations in (6.6) and because the restriction to $\mathcal{H}_{D_2^*}$ is given by $z_2 = z_3$, $\psi_{2,42}|_{\mathcal{H}_{D_2^*}}$ vanishes along $\lambda^\perp_1$ (in $\mathcal{H}_{D_2^*}$, given by $z_2 = 0$) of order greater or equal to 3, along $\lambda^\perp_4$ (in $\mathcal{H}_{D_2^*}$, given by $z_1 = z_2$), and along $\lambda^\perp_7$ (in $\mathcal{H}_{D_2^*}$, given by $z_2 = -2z_1$), both of order greater or equal to 1. So once more, because $\psi_{2,42}|_{\mathcal{H}_{D_2^*}} \in \left[\Gamma_{D_2}, 42, \chi_{D_2^*}^{\det}\right]$ holds in view of (6.14) and (5.26), and since $\psi_{4,42}$ vanishes exactly along $\lambda_1^\perp$ of order 3, along $\lambda_4^\perp$ and $\lambda_7^\perp$ of order 1, and along the rational quadratic divisors congruent to them modulo $\Gamma_{D_2}$ (of according order), we obtain

$$\psi_{2,42}|_{\mathcal{H}_{D_2^*}} \in \psi_{4,42}|_{\Gamma_{D_2}, 0, 1|} = C\psi_{4,42}.$$ 

Once more, we shall summarize these results.

(6.24) Proposition. \hspace{1cm} \bullet \, \psi_{2,27}|_{\mathcal{H}_{D_2^*}} = \psi_{2,62}|_{\mathcal{H}_{D_2^*}} \equiv 0,$$

\bullet \, \psi_{2,7}|_{\mathcal{H}_{D_2^*}} \in C^*\psi_{4,7},$

\bullet \, \psi_{2,42}|_{\mathcal{H}_{D_2^*}} \in C^*\psi_{4,42}.$$

Unfortunately, like in the case regarding the restriction from $\mathcal{H}_{A_2^{(2)}}$ to $\mathcal{H}_{T_3}$, there does not seem to be a natural counterpart for $\psi_{4,28,(1)}$ or $\psi_{4,28,(2)}$, or at least none coming from a Borcherds product for $A_{2,1}$. Even more so, since we will see further below that $\psi_{4,28,(2)}$ might be crucial concerning the further reduction process on $\mathcal{H}_{D_2}$, or in other words $\psi_{4,28,(2)}$ might be needed to generate the graded ring of orthogonal modular forms with respect to $D_2$. So regarding future work on this topic, one might have to construct this counterpart in another way. Anyways, as long as the spaces of orthogonal modular forms with respect to $\Gamma_{D_2}$ have not been determined, one should defer this issue. So we will not consider this question further in the current thesis.

Note that using the same methods like above one can verify that when restricting the Borcherds products for $T_3$ to $\mathcal{H}_{D_2}$, then one obtains all necessary Borcherds products for $D_2^*$
Then \( f \) vanishes along \( \lambda \Gamma_k \) (with \( \lambda \Gamma_k \)) (with \( \lambda \Gamma_k \)). This means we have to determine the spaces of orthogonal modular forms with respect to \( \Gamma \). The parity of the weight depends on \( \chi \) (given by \( \chi \)). This means the parity of the weight \( k \) depends on whether \( \chi \pi \) occurs or not (at least as long as we do not want to talk about trivial spaces). Next, note

\[
A := A^S_{\chi \pi} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in O(\Lambda),
\]

and also \(-A \in O(\Lambda)\). One easily verifies that \( R_A \) acts trivially on the submanifold \( \lambda^+ \) (given by \( z_2 = 0 \)), while \( R_{-A} \) acts trivially on the submanifold \( \lambda^- \) (given by \( z_2 = -2z_1 \)).

Noting \( \chi_{\det}(R_A) = -1, \chi_\pi(R_A) = -1 \) and \( \chi_{\det}(R_{-A}) = -1, \chi_\pi(R_{-A}) = 1 \) we obtain the following lemma (for example in analogy to (5.24)):

**Lemma.** Let \( S = D_2^\# \), \( k \in \mathbb{N}_0 \), \( f \in [\Gamma_S, k, \chi_{\det}, \chi_{\pi}^{m-1}] \) and \( g \in [\Gamma_S, k, \chi_{\det}, \chi_{\pi}^m] \) for some \( m \in \mathbb{Z} \). Then \( f \) vanishes along \( \lambda^+_\pi \), while \( g \) vanishes along \( \lambda^+ \).

There are no \( M \in \Gamma_S \setminus \Gamma'_S \) acting trivially on \( \lambda^+_\pi \) or \( \lambda^+_\pi \), so we obtain no conditions concerning the abelian character such that an orthogonal modular form has to vanish along these. Combining the preceding lemma with (6.18) and the considerations in (6.6), we get (recalling that the parity of the weight depends on \( \chi \pi \))

- \( f \in [\Gamma_S, 2k, \chi_{\det}] \Rightarrow \frac{\psi_{3,7} f}{\psi_{4,28,1}} \in [\Gamma_S, 2k - 14, 1] \),

- \( f \in [\Gamma_S, 2k + 1, \chi_{\det}, \chi_{\pi}] \Rightarrow \frac{\psi_{3,7} f}{\psi_{4,28,1}} \in [\Gamma_S, 2k - 6, 1] \).

Just like we did concerning \( A_{2,1} \), since we are already multiplying with \( \psi_{3,7} \) anyways, we can even consider

- \( f \in [\Gamma_S, 2k + 1, \chi_{\pi}] \Rightarrow \psi_{4,7} f \in [\Gamma_S, 2k + 8, 1] \).

This means we have to determine the spaces of orthogonal modular forms with respect to \( \Gamma_{D_2^\#} \), trivial character and even weight. But then the next steps are analogous to what we saw for \( A_{2,1} \):

Let \( f \in [\Gamma_S, 2k, 1] \). Again, suppose we already found some orthogonal modular forms \( g_1, \ldots, g_n \) with respect to \( \Gamma_S \) and abelian character 1 of even weights, such that the restrictions of these generate the graded ring of orthogonal modular forms with respect to \( \Gamma_\chi \) (and the trivial character) – we will come back to that further below. Then again we can find a polynomial \( p \).
(homogeneous with respect to the weights) such that \( f - p(g_1, \ldots, g_n) \) vanishes along \( \lambda^\perp \) (hence on \( H(14) \)). Then for the same reasons like above we obtain

\[
\frac{\psi_{4,7}^3(f - p(g_1, \ldots, g_n))}{\psi_{4,28,(2)}} \in [\Gamma_S, 2k - 7, \chi_{\det \chi}] ,
\]

hence also

\[
\frac{\psi_{4,7}^6(f - p(g_1, \ldots, g_n))}{\psi_{4,28,(2)}^2} \in [\Gamma_S, 2k - 14, 1]
\]
due to the considerations above. So again an induction would yield that for every \( f \in [\Gamma_S, k, \chi] \) (where \( \chi \) is any abelian character) there exists \( j \in \mathbb{N}_0 \) such that

\[
\psi_{4,7}^j f \in \mathbb{C}[\psi_{4,7}, \psi_{4,28,(2)}, g_1, \ldots, g_n] .
\]

Just like for \( A_{2,1} \), the rest might work like in [De01, pp.133] concerning Hermitian modular forms for \( \mathbb{Q}(\sqrt{-2}) \). So a reduction process concerning \( D_2^2 \) could work this way – which means that the spaces of orthogonal modular forms for \( \Gamma(14) \) have to be determined, first.

So this finally is the setting we have to analyze first, before there is any chance to go all the way back up to \( A_{2}^{(2)} \) – at least if we want to follow the approach of utilizing Borcherds products in such a reduction process. So this is what the subsequent section will be about: Orthogonal modular forms with respect to \( \Gamma(14) \) (where “(14)” is the scalar matrix \( 14 \in \mathbb{Z}^{1 \times 1} \)). Note again that analyzing orthogonal modular forms with respect to \( \Gamma(2) \) would result in analyzing Siegel modular forms, which are well-known and completely classified (at least concerning the full modular group). But considering orthogonal modular forms with respect to \( \Gamma(14) \) means to deal with paramodular forms of level 7 (and degree two) – we will see in the next section what kind of objects these are. Therefore, to be able to profit from both worlds (the symplectic and the orthogonal) we will briefly introduce paramodular forms. But note that ultimately we will see that it has not been possible to determine the structure of the spaces of paramodular forms of level 7, which would be necessary to finally start the reduction process described in the current section. We will get stuck at the same point where also Marschner could not find any further results in [Mar04], who analyzed paramodular forms of level 5 (and degree two). Nevertheless, we should present the results here, so that it becomes clear where the difficulties occur. In some future work, methods might be found to circumvent these problems. Note that the spaces of paramodular forms of level 2 and 3 (and degree two) could be determined completely (cf. [Fr67] and [De02]). And finally, note that we do not consider any reduction process concerning \( H(8) \) because \( 8/2 = 4 \) is not squarefree, which means that the theory concerning paramodular forms becomes much more involved. So we leave out that branch, just like we did not consider any branches regarding non-euclidean lattices any further, since even the basic theory is not evolved enough in these cases.
6.3 Paramodular forms of level 7 and degree 2

We begin by defining paramodular forms of degree two (we will omit the “degree two” from now on, since we will not have to deal with any other degrees). Afterwards, we will show how to identify paramodular forms of level 7 with orthogonal modular forms with respect to $\Gamma_{(14)}$. The introduction of paramodular forms will be quite brief since we only want to point out how the setting looks like. Furthermore, we will restrict ourselves to paramodular forms of level $p$, where $p$ is a prime number, as we are only interested in level 7 afterwards. For example, confer [Mar04] for a good and complete overview regarding the general setting.

First, drop the notation $\Gamma_2$ for $\text{Sp}_2(\mathbb{O})$ which we used in chapter 4, because we want to stay close to the notation in [Mar04] where $\Gamma_p$ is used to denote the paramodular groups of level $p$ (and degree 2). So let us immediately get to the definition of the paramodular groups and the extended paramodular groups.

(6.26) Definition. Let $p$ be a prime number and set $P = \text{diag}(1, p)$.

a) The paramodular group of level $p$ (and degree two) is defined to be

$$\Gamma_p := \left\{ M \in \text{Sp}_2(\mathbb{Q}) ; \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \in \mathbb{Z}^{4 \times 4} \right\}. \quad (6.10)$$

b) The extended paramodular group of level $p$ is defined to be

$$\Gamma_p^{\text{max}} := \langle \Gamma_p, V_p \rangle,$$

where

$$V_p := \text{Rot}(U_p), \quad \text{with} \quad U_p = \begin{pmatrix} 0 & \sqrt{p}^{-1} \\ \sqrt{p} & 0 \end{pmatrix}. \quad (6.27)$$

Note that $\Gamma_p^{\text{max}}$ is an extension of order 2. And by definition we have $\Gamma_p^{\text{max}} \subset \text{Sp}_2(\mathbb{R})$, so it should already be clear how paramodular forms will be defined. Furthermore, in view of [Mar04, pp.3] $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Q}^{4 \times 4}$ belongs to $\Gamma_p$ if and only if $M \in \text{Sp}_2(\mathbb{Q})$ and the block entries of $M$ are of the shape

$$A = \begin{pmatrix} a_1 & p a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & p^{-1} b_4 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & p c_2 \\ p c_3 & p c_4 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & d_2 \\ p d_3 & d_4 \end{pmatrix}, \quad (6.27)$$

where $a_1, a_2, \ldots$ are integers. And note that in view of [Mar04, ex.2.1.4] a generating system of $\Gamma_p$ is given as follows:

(6.27) Proposition. Let $p$ be a prime number and set $P = \text{diag}(1, p)$. Then $\Gamma_p$ is generated by the following matrices:

$$J_p := \begin{pmatrix} 0 & -p^{-1} \\ P & 0 \end{pmatrix}, \quad \text{Trans}(T), \text{where} \ T \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & p^{-1} \end{pmatrix} \right\}. \quad (6.27)$$
Of course, we will need the group of abelian characters for \( \Gamma^\text{max}_p \) to be able to define paramodular forms. We will only cite the result for \( p \notin \{2, 3\} \) here, since we will only be interested in \( p = 7 \). According to [Mar04, ex.2.3.6] we have:

(6.28) Proposition. Let \( p \notin \{2, 3\} \) be a prime number. Then \( (\Gamma_p)^\text{ab} \simeq C_2 \) and \( (\Gamma^\text{max}_p)^\text{ab} \simeq C_2 \times C_2 \) hold. The group of abelian characters for \( (\Gamma_p) \) is generated by \( \kappa_{1,2} \), while the group of abelian characters for \( (\Gamma^\text{max}_p) \) is generated by \( \chi_p \) and \( \kappa_{1,2} \), where

\[
\chi_p(M) = 1 \text{ for all } M \in \Gamma_p, \quad \chi_p(V_p) = -1,
\]

and

\[
\kappa_{1,2}(J(p)) = 1, \quad \kappa(\text{Trans}((t_{1,k}))) = (-1)^{t_{1,1} + t_{1,2} + pt_{2,2}}, \quad \kappa_{1,2}(V_p) = 1.
\]

Now, we can immediately get to the definition of paramodular forms of level \( p \). Recall that the slash-operator for functions \( f : \mathcal{H}(\mathbb{R}) \to \mathbb{C} \) is given by

\[
f|_k M = (\det(CZ + D))^{-k} f(MZ)
\]

for all \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}_2(\mathbb{R}) \) and all \( Z \in \mathcal{H}(\mathbb{R}) \).

(6.29) Definition. Let \( p \) be a prime number, \( \Gamma \leq \Gamma^\text{max}_p \) of finite index and \( \nu \) an abelian character of \( \Gamma \). A holomorphic function \( f : \mathcal{H}(\mathbb{R}) \to \mathbb{C} \) is called a paramodular form of level \( p \) and of weight \( k \in \mathbb{Z} \) with respect to \( \Gamma \) and the abelian character \( \nu \) if

\[
f|_k M = \nu(M) \cdot f \quad \text{for all } M \in \Gamma.
\]

The space of paramodular forms of weight \( k \) (and level \( p \)) with respect to \( \Gamma \) and \( \nu \) is denoted by \( [\Gamma, k, \nu] \), as usual.

Like in all other cases of (holomorphic) modular forms we obtain \( [\Gamma, k_1, \nu_1] \cdot [\Gamma, k_2, \nu_2] \subset [\Gamma, k_1 + k_2, \nu_1 \nu_2] \) and

\[
[\Gamma, k, \nu] = \begin{cases} 
\{0\}, & \text{if } k < 0 \text{ or } (k = 0, \nu \neq 1), \\
C, & \text{if } k = 0, \nu = 1
\end{cases}
\]

here, too (see [Mar04, p.9]). Of course, paramodular forms possess an absolutely and locally uniformly convergent Fourier-expansion since \( \text{Trans}(T) \) is contained in \( \Gamma_p \) for special \( T \in \text{Sym}_2(\mathbb{R}) \). We are only interested in \( \Gamma \in \{ \Gamma_p, \Gamma^\text{max}_p \} \). So we cite [Mar04, p.9] for this special case, only (and again we restrict ourselves to \( p \geq 5 \) – whereas the only difference for \( p \in \{2, 3\} \) would be that further abelian characters arise).
(6.30) Proposition. Let $p \notin \{2, 3\}$ be a prime number, $k \in \mathbb{N}_0$, $\Gamma \in \{\Gamma_p, \Gamma^\text{max}_p\}$ and $f \in [\Gamma, k, \kappa_1^m \Lambda_p^n]$, where $m, n \in \{0, 1\}$ (and $n = 0$ in case $\Gamma = \Gamma_p$). Then $f$ possesses an absolutely and locally uniformly convergent Fourier-expansion of the shape
\[
f(Z) = \sum_{T \in \mathbb{H}} \alpha_f(T) e^{2\pi i \text{tr}(TZ)},
\]
where
\[
\Delta_p := \left\{ \left( \frac{m}{2}, \frac{t}{2} \right) ; m, n, t \in \mathbb{Z} \right\}.
\]

Next, we also need the definition of (paramodular) cusp forms. The situation is a bit more complicated, here. In all cases, cusp forms were defined to vanish in all cusps, which means that the limit of $f|_k M$ for some special sequence (which goes to infinity in one or another way) of the argument of $f|_k M$ equals zero “for all $M$”. So in all the cases we were investigating we only had to check this condition for $f$, in case $f$ was a modular form with respect to the full modular group. (This was always due to the fact that we were considering euclidean lattices.) But in the case of paramodular forms further cusps arise. In the special cases we are considering, the situation is not too complicated. And in particular, it suffices to only take the Fourier-expansion of $f$ into account.

Again, the $\Phi$-operator is defined to be
\[
\Phi : [\Gamma_p, k, \nu] \to \mathbb{C}, \quad f \mapsto f|_{\Phi},
\]
where
\[
f|_{\Phi} : \mathcal{H} \to \mathbb{C}, \quad \tau \mapsto \lim_{y \to \infty} f \left( \frac{\tau}{0 \ y} \right),
\]
and $\tilde{\nu}$ is given by $\tilde{\nu}(M \times I)$ for all $M \in \text{SL}_2(\mathbb{Z})$ (see [Mar04, p.10]). We skip the discussion about cusps and simply cite the results from [Mar04, prop.2.5.14] and [Mar04, ex.2.5.15], reformulated as a definition.

(6.31) Definition. Let $p$ be a prime number, $k \in \mathbb{N}_0$, $\nu$ an abelian character of $\Gamma_p$ and $f \in [\Gamma_p, k, \nu]$. Then $f$ is said to be a (paramodular) cusp form if
\[
f|_{\Phi} = (f|_{k \nu_p})|_{\Phi} \equiv 0.
\]
This is equivalent to demand $f$ to possess a Fourier-expansion of the shape
\[
\sum_{T > 0} \alpha_f(T) e^{2\pi i \text{tr}(TZ)},
\]
which means that all Fourier-coefficients $\alpha_f(T)$ equal zero if $T$ is not positive definite. We denote the space of all cusp forms in $[\Gamma_p, k, \nu]$ by $[\Gamma_p, k, \nu]|_0$.

Now, we immediately get to the aforementioned identification of paramodular forms with orthogonal modular forms for scalar matrices. We will not go into detail here and simply summarize the results of chapter 3 in [Mar04], without explicitly referring to the according
propositions and theorems in that thesis. But all the results we are going to present now can be found in the said chapter.

First, we get to the identification of the corresponding half-spaces. Let \( p \) be a prime number. Again meaning “\((n)\)” to be the scalar matrix \( n \in \mathbb{Z}^{1\times 1} \), the identification of the Siegel half-space \( \mathcal{H}(\mathbb{R}) \) of degree two (where paramodular forms live on) with the orthogonal half-space \( \mathcal{H}_{(2p)} \) is given as follows:

\[
\Phi_p : \mathcal{H}_{(2p)} \rightarrow \mathcal{H}(\mathbb{R}), \quad w = (\tau_1, z, \tau_2) \mapsto \left( \frac{\tau_1}{z} \frac{z}{p^{-1} \tau_2} \right)
\]

is a biholomorphic map with the property

\[
q_0(w) = \frac{1}{2} S_0[w] = \tau_1 \tau_2 - pz^2 = p \cdot \det(\Phi_p(w)),
\]

where \( S_0 \) is defined according to (5.3) with \( S = 2p \in \mathbb{Z}^{1\times 1} \).

Again, this map induces a bijection between the group of automorphisms of \( \mathcal{H}(\mathbb{R}) \) and the group of automorphisms of \( \mathcal{H}_{(2p)} \). So in particular, like we will see in the next theorem, the modular groups can be identified.

\[ (6.32) \textbf{Theorem.} \text{ Let } p \text{ be a prime number. Deriving from } \Phi_p \text{ we obtain}
\]

\[
\Gamma_p^{\text{max}} / \{ \pm I \} \simeq \Gamma_{(2p)} / \{ \pm I \}.
\]

The corresponding isomorphism shall be denoted by \( \Psi \). For the generators of \( \Gamma_p^{\text{max}} \) (see (6.27)) the correspondence is given as follows (omitting the “\(/ \{ \pm I \} \)”):

- \( J_{(p)} \leftrightarrow J_0, \)
- \( \text{Trans}(T) \leftrightarrow T_g, \text{ where } T = \left( \begin{array}{cc} a & b \\ b & p^{-1} c \end{array} \right) \text{ (with } a, b, c \in \mathbb{Z}) \text{ and } g = (a, b, c), \)
- \( V_p \leftrightarrow P. \)

In particular, this theorem yields that \( \Gamma_{(2p)} \) is generated by \( J_0, T_g \ (g \in \mathbb{Z}^3), P \) and \(-I\). (Concerning the definition of \( P \) see page 310.) This is a result not easily obtained when regarding the orthogonal side, only, since the lattice \( \Lambda = \mathbb{Z} \) attached to \( 2p \in \mathbb{Z}^{1\times 1} \) is not euclidean for \( p > 2 \). And moreover, the preceding theorem also allows us determine the group of abelian characters of \( \Gamma_{(2p)} \), given \( p \) is a prime number with \( p \geq 5 \) (which would work for \( p \in \{2,3\} \), too, but we did not introduce the abelian characters on the paramodular side for these cases since we do not need them).

In view of (6.28) and noting \( J_{(p)}^2 = -I \), every abelian character \( \nu \) of \( \Gamma_p^{\text{max}} \) fulfills \( \nu(-I) = 1 \), and this obviously implies \( (\Gamma_p^{\text{max}})^{\text{ab}} \simeq (\Gamma_p^{\text{max}} / \{ \pm I \})^\text{ab} \), thus \( (\Gamma_p^{\text{max}})^{\text{ab}} \simeq (\Gamma_{(2p)} / \{ \pm I \})^\text{ab} \). Next, note that \( \chi_{\text{det}} \) is an abelian character of \( \Gamma_{(2p)} \) with \( \chi_{\text{det}}(-I) = -1 \). Of course, the canonical epimorphism \( \Gamma_{(2p)} \twoheadrightarrow \Gamma_{(2p)} / \{ \pm I \} \) yields that every abelian character of \( \Gamma_{(2p)} / \{ \pm I \} \) induces a unique abelian character \( \chi \) of \( \Gamma_{(2p)} \) which satisfies \( \chi(-I) = 1 \). On the other hand, if \( \chi \) is any abelian character of \( \Gamma_{(2p)} \) with \( \chi(-I) = 1 \), then \( \chi \) induces a unique abelian character of \( \Gamma_{(2p)} / \{ \pm I \} \). And if \( \chi(-I) = -1 \), then so does \( \chi_{\text{det}} \). These considerations immediately lead to
\(\Gamma_{(2p)}^{ab} \simeq (\Gamma_p^{\text{max}})^{ab} \times \langle \chi_{\det} \rangle\), so we obtain

(6.33) Proposition. Let \(p \geq 5\) be a prime number. Then \(\Gamma_{(2p)}^{ab} = \langle \chi_{1,2}, \chi_{P, \det} \rangle \simeq C_2 \times C_2 \times C_2\), where on the generators of \(\Gamma_{(2p)}\) these abelian characters are given by:

- \(\chi_{1,2}(J_0) = 1, \chi_{1,2}(T_{a,b,c}) = (-1)^a + b + c\) (where \(a, b, c \in \mathbb{Z}\)), \(\chi_{1,2}(P) = 1, \chi_{1,2}(-I) = 1\),
- \(\chi_P(J_0) = 1, \chi_P(T_g) = 1\) (where \(g \in \mathbb{Z}^3\)), \(\chi_P(P) = -1, \chi_P(-I) = 1\),
- \(\chi_{\det}(J_0) = 1, \chi_{\det}(T_g) = 1\) (where \(g \in \mathbb{Z}^3\)), \(\chi_{\det}(P) = -1, \chi_{\det}(-I) = -1\).

Of course, (6.32) gives the answer how to identify paramodular forms of level \(p\) with orthogonal modular forms with respect to \(\Gamma_{(2p)}\). Note that the additional abelian character \(\chi_{\det}\) on the orthogonal side is in a way “redundant” and does not interfer with the identification of the modular forms: If \(f\) is an orthogonal modular form of weight \(k\) with respect to the abelian character \(\chi\), then \(\chi(-I) = (-1)^k\) has to hold or otherwise \(f\) vanishes identically (cf. (5.17)). Here, \(\chi_{\det}\) assumes this role of allowing odd weight. Again, a summary of chapter 3 in [Mar04] gives (where the same would hold for \(p \in \{2,3\}\), but we omit that since we did not introduce the corresponding abelian characters):

(6.34) Theorem. Let \(p \geq 5\) be a prime number, \(k \in \mathbb{N}_0\) and \(l, n \in \{0,1\}\). If \(f \in [\Gamma_p, k, \kappa_1 \chi_p^n]\) with Fourier-expansion

\[
f(Z) = \sum_{T = (\frac{t}{2}, \frac{z}{2}) \in \Gamma_p, T \geq 0} \alpha_f(T) e^{2\pi i (n t_1 + p m t_2 + l z)} , \quad Z = (\frac{t}{2}, \frac{z}{2}) \in \mathcal{H}(\mathbb{R}) \, ,
\]

then \(g := f \circ \Phi_p \in [\Gamma_{(2p)}, k, \chi_1 \chi_p^{l+n+k} \chi_{\det}]\). Moreover, the Fourier-expansion of \(g\) is given by

\[
g(w) = \sum_{m, n \in \mathbb{N}_0} \sum_{t \in \mathbb{Z}} \sum_{\substack{\tau_1, \tau_2 \in \mathbb{Z} \cap \frac{t}{2} \leq 4pnm}} \alpha_g(m, t, n) e^{2\pi i (n t_1 + m t_2 + l z)} , \quad w = (\tau_1, \tau_2) \in \mathcal{H}_{(2p)} \, ,
\]

where

\[
\alpha_g(m, t, n) = \alpha_f \left( \begin{array}{c} n \\ t/2 \\ pm \end{array} \right)
\]

Furthermore, the map

\[
[\Gamma_p, k, \kappa_1 \chi_p^n] \to [\Gamma_{(2p)}, k, \chi_1 \chi_p^{l+n+k} \chi_{\det}] , \quad f \mapsto f \circ \Phi_p
\]

is an isomorphism.

So since we can identify paramodular forms with orthogonal modular forms, we can again construct Borcherds products. Note that for \(S = 2p \in \mathbb{Z}^{1 \times 1}\), where \(p\) is a prime number,

\[
\Lambda^2 = \{ \pm \frac{n}{2p} ; \ n = \ldots , p \} = \{ \frac{1}{2p} \} \simeq C_{2p}
\]
while this immediately leads to

In virtue of (6.5) the product expansion of the corresponding Borcherds product

and

such that \( k \in \mathbb{Z} \) exists \( M \) rational quadratic divisors of fixed discriminant, i.e. if \( \lambda \in \mathbb{Z}^2 \), then there exists \( M \in \Gamma(2p) \) such that \( \lambda_1^+ = M\lambda_2^+ \).

Let us get back to the issue of determining the abelian character of a Borcherds product:

(6.35) Proposition. Let \( p \) be a prime number and \( S = 2p \). Then \( \Gamma(2p) \) acts transitively on the set of rational quadratic divisors of fixed discriminant, i.e. if \( \lambda_1, \lambda_2 \in \mathbb{Z}^2 \) such that \( \delta(\lambda_1^+) = \delta(\lambda_2^+) \) then there exists \( M \in \Gamma(2p) \) such that \( \lambda_1^+ = M\lambda_2^+ \).

(6.36) Proposition. Let \( S = 2p \) and suppose \( f \in [\text{Mp}_2(\mathbb{Z}), -1/2, \rho_2^\pm]_\infty \) with Fourier-expansion

\[ f(\tau) = \sum_{t \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}} \sum_{n \in -pt^2 + \mathbb{Z}} c_t(n) e^{2\pi int} e_t, \quad \tau \in \mathcal{H}, \]

such that \( k = c_0(0) \in 2\mathbb{Z} \) and \( c_t(n) \in \mathbb{Z} \) whenever \( n < 0 \). Then for the Borcherds product \( \psi_k \) attached to \( f \) we get

\[ \psi_k \in [\Gamma(2p), k, \chi_{2,1}^d, \chi_P^{\beta+k}, \chi_{\text{det}}], \]

where

\[ \chi = \frac{1}{12} \sum_{t \in \mathbb{Z}} c_{t/2p}(-t^2/(4p)) \]

and

\[ \beta = \sum_{n \in \mathbb{N}, t \in \mathbb{Z}} \sigma_1(n)c_{t/2p}(-n - t^2/(4p)) \].

Proof: In virtue of (6.5) the product expansion of the corresponding Borcherds product \( \psi \) is given by

\[ \psi(w) = e^{2\pi i (\varphi,w)_0} \prod_{\lambda_0 \in \mathbb{Z} \times \frac{1}{2p}\mathbb{Z} \times \mathbb{Z}} \left( 1 - e^{2\pi i (\lambda_0,w)_0} \right)^{c_{\lambda_0}(\varphi(\lambda_0))}. \]

This immediately leads to

\[ \psi_k|_k T_{(0,0,1)}(w) = e^{2\pi i (\varphi,w+(0,0,1))_0} \prod_{\lambda_0 \in \mathbb{Z} \times \frac{1}{2p}\mathbb{Z} \times \mathbb{Z}} \left( 1 - e^{2\pi i (\lambda_0,w+(0,0,1))_0} \right)^{c_{\lambda_0}(\varphi(\lambda_0))} = e^{2\pi i (\varphi,(0,0,1))_0} \psi_k(w), \]
because \((\lambda_0, (0,0,1)) \in \mathbb{Z}\) for all \(\lambda_0 \in \mathbb{Z} \times \frac{1}{2p} \mathbb{Z} \times \mathbb{Z}\). But then
\[
(e_f, (0,0,1)) = e_{z_0} = \frac{1}{24} \sum_{t \in \frac{1}{2p} \mathbb{Z}} c_t(-pt^2) = \frac{1}{24} \sum_{t \in \mathbb{Z}} c_{t/2p}(-t^2 / (4p)),
\]
hence the first part of the assertion follows. Now, we consider the transformation behavior under \(P\). For appropriate \(w = (\tau_1, z, \tau_2) \in \mathcal{H}(2p)\) (which means those for which the quotient is well-defined) we get
\[
\frac{\psi_k(\tau_1, z, \tau_2)}{\psi_k|_P(\tau_1, z, \tau_2)} = e^{2\pi i (e_f(\tau_1, z, \tau_2))_0} \prod_{(m, l, n) \in \mathbb{Z} \times \frac{1}{2p} \mathbb{Z} \times \mathbb{N}} \left(1 - e^{2\pi i (nt_1 + mt_2 - pt_z)}\right)^{c_t(mn - pt^2)} .
\]

So as long as \(m, n \geq 0\) in the products, the terms cancel out. Hence
\[
\frac{\psi_k(\tau_1, z, \tau_2)}{\psi_k|_P(\tau_1, z, \tau_2)} = e^{2\pi i (e_f(\tau_1, z, \tau_2))_0} \prod_{(m, l, n) \in \mathbb{Z} \times \frac{1}{2p} \mathbb{Z} \times \mathbb{N}} \left(1 - e^{2\pi i (nt_1 - mt_2 - pt_z)}\right)^{c_t(-mn - pt^2)} .
\]

remains, since \(c_t(\cdot) = c_{-t}(\cdot)\) has to hold (see (6.9)). Furthermore, if \((m, t, n)\) occurs in the product, then so does \((m, -t, n)\). Hence all terms in \(t\) cancel out, and plugging in the formulas from (6.4) for \((e_f, (\tau_1 - \tau_2, 0, \tau_2 - \tau_1))_0 = (\tau_1 - \tau_2)e_{z_0} - (\tau_1 - \tau_2)e_{z_0}\) we get
\[
\frac{\psi_k(\tau_1, z, \tau_2)}{\psi_k|_P(\tau_1, z, \tau_2)} = \prod_{n \in \mathbb{N}, t \in \frac{1}{2p} \mathbb{Z}} \left(e^{-2\pi i (\tau_1 - \tau_2)n_1(n)}\right)^{c_t(-n - pt^2)} \prod_{m, n \in \mathbb{N}, t \in \frac{1}{2p} \mathbb{Z}} \left(-e^{2\pi i (nt_1 - mt_2)}\right)^{c_t(-mn - pt^2)} = \prod_{n \in \mathbb{N}, t \in \frac{1}{2p} \mathbb{Z}} (-1)^{c_t(n)c_t(-n - pt^2)},
\]

where the second identity is a simple consideration taking into account that \(c_t(\cdot)\) is the sum over all divisors, so in the left product we collect all \(m, n\) with the same value of \(mn\) (hence \(\bar{n}\) and \(\bar{m}\) run through the divisors of \(mn\)). The assertion concerning \(\chi_{\text{det}}\) is clear due to the considerations above (6.35). \(\square\)

So we have all tools we need at hand and can begin with constructing Borcherds products. But this time, we will not stay in the orthogonal world, since the symplectic world seems to be more beneficial this time. So we should find out how the rational quadratic divisors transfer to \(\mathcal{H}(\mathbb{R})\). So let \(\lambda = (r, m, t/2p, n, 0) \in \Lambda^1_p\) be primitive (we will only have to consider rational quadratic divisors coming from such a \(\lambda\)). Then for \(w = (\tau_1, z, \tau_2) \in \mathcal{H}(2p)\) we have
\[
w \in \lambda^\perp \iff r + nt_1 + pm(t_2/p) - zt = 0 .
\]
Using the identification via $\Phi_p$ yields the corresponding divisor in $\mathcal{H}(\mathbb{R})$.

We will now analyze the special case $p = 7$, since we have seen in section 6.2 that this might be the key to start the reduction process. So from now on, let $S = 14 \in \mathbb{Z}^{1 \times 1}$. We need to determine the obstruction space $[\text{Mp}_2(\mathbb{Z}), 2 + 1/2, \rho_S]$. Unfortunately, we cannot draw upon the vector-valued theta-series from (5.46) since $S$ is scalar, so we cannot choose “$r = 2$”, only $r \in \{0, 1\}$. But there is a way around this. From [Mar04, le.6.2.4] we cite a special case:

**(6.37) Lemma.** Let $k \in \frac{1}{2} \mathbb{N}_0$ and $f \in [\text{Mp}_2(\mathbb{Z}), k, \rho_S]$. Then

$$\Delta^{-1}[f, \Delta] := 12f' - kf\Delta^{-1}\Delta' \in [\text{Mp}_2(\mathbb{Z}), k + 2, \rho_S]$$

holds, where $\Delta \in [\text{SL}_2(\mathbb{Z}), 12]_0$ is the Modular Discriminant (cf. [KK07, pp.162]).

This means we can increase the weight. Of course, $\Delta^{-1}[f, \Delta]$ could be vanishing identically, but fortunately this is not the case for $\Theta(\cdot ; S, 1)$ (where $p_0 = 1$ is the constant polynomial). In the next proposition about the obstruction space we choose a different way of presenting the possible principal parts since there are too many summands that are involved.

**(6.38) Proposition.** We have $\dim[\text{Mp}_2(\mathbb{Z}), 5/2, \rho_S] = 2$. A basis is given by $E_{5/2}(\cdot ; -S_1)$ and $\Theta^x_{5/2}(\cdot ; S) := \Delta^{-1}[\Theta(\cdot ; S, 1), \Delta]$. If we use the common abbreviation $q = e^{2\pi i \tau}$, then we get

$$E_{5/2,0}(\tau; -S_1) = 2 - \frac{192}{5} q^{3/4} + O(q^{7/4})$$
$$E_{5/2,1/2}(\tau; -S_1) = -\frac{96}{5} q^{3/4} - \frac{272}{5} q^{7/4} + O(q^{11/4})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -\frac{2}{5} q^{1/28} + O(q^{29/28})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -\frac{14}{5} q^{1/7} + O(q^{8/7})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -30q^{9/28} + O(q^{37/28})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -22q^{1/7} + O(q^{11/7})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -\frac{242}{5} q^{25/28} + O(q^{53/28})$$
$$E_{5/2,\pm \frac{1}{4}}(\tau; -S_1) = -\frac{24}{5} q^{37/28} - 70q^{9/7} + O(q^{16/7})$$

and

$$\Theta^x_{5/2,0}(\tau; S) = -\frac{1}{2} + 12q^1 + O(q^2)$$
$$\Theta^x_{5/2,1/2}(\tau; S) = 0q^{3/4} + 41q^{7/4} + O(q^{11/4})$$
$$\Theta^x_{5/2,\pm \frac{1}{4}}(\tau; S) = -\frac{1}{14} q^{1/28} + O(q^{29/28})$$
$$\Theta^x_{5/2,\pm \frac{1}{4}}(\tau; S) = \frac{17}{14} q^{1/7} + O(q^{8/7})$$
$$\Theta^x_{5/2,\pm \frac{1}{4}}(\tau; S) = \frac{47}{14} q^{9/28} + O(q^{37/28})$$
$$\Theta^x_{5/2,\pm \frac{1}{4}}(\tau; S) = \frac{89}{14} q^{4/7} + O(q^{11/7})$$
$$\Theta^x_{5/2,\pm \frac{1}{4}}(\tau; S) = \frac{143}{14} q^{25/28} + O(q^{53/28})$$
\[ \Theta_{5/2, \pm \frac{1}{2}}(\tau; S) = 0q^{2/7} + \frac{299}{14}q^{9/7} + O(q^{16/7}) \, . \]

Therefore, a basis for possible principal parts and constant terms of weakly holomorphic vector-valued modular forms satisfying \( c_\mu(n) = 0 \) for those \( n < 0 \) such that \( -n \) does not occur as an exponent above can be read off the following table:

<table>
<thead>
<tr>
<th>( e_0 )</th>
<th>( e_{1/2} )</th>
<th>( e_{1/14} )</th>
<th>( e_{1/7} )</th>
<th>( e_{3/14} )</th>
<th>( e_{2/7} )</th>
<th>( e_{3/14} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>3q^{-1/28}</td>
<td>( q^{-1/7} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>2q^{-1/28}</td>
<td>3q^{-1/7}</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>48 + 2q^{-1}</td>
<td>( q^{-3/7} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>82</td>
<td>0q^{-3/7} + q^{-7/7}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90 + q^{-1}</td>
<td>0</td>
<td>q^{-1/28}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>102 + q^{-1}</td>
<td>0</td>
<td>q^{-1/28}</td>
<td>( q^{-9/28} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>106 + q^{-1}</td>
<td>( q^{-3/7} + q^{-7/7} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>112</td>
<td>0</td>
<td>0</td>
<td>2q^{-9/28}</td>
<td>( q^{-3/7} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>114 + q^{-1}</td>
<td>0</td>
<td>0</td>
<td>( q^{-3/7} )</td>
<td>0</td>
<td>( q^{-3/7} )</td>
<td>0</td>
</tr>
</tbody>
</table>

where we use the following abbreviations for the components:

\[
\begin{align*}
    e_0 &= e_0, \\
    e_{1/2} &= e_{\frac{1}{2}}, \\
    e_{1/14} &= e_{\frac{1}{14}} + e_{-\frac{1}{14}}, \\
    e_{1/7} &= e_{\frac{1}{7}} + e_{-\frac{1}{7}}, \\
    e_{3/14} &= e_{\frac{3}{14}} + e_{-\frac{3}{14}}, \\
    e_{2/7} &= e_{\frac{2}{7}} + e_{-\frac{2}{7}}, \\
    e_{5/14} &= e_{\frac{5}{14}} + e_{-\frac{5}{14}}, \\
    e_{3/7} &= e_{\frac{3}{7}} + e_{-\frac{3}{7}}.
\end{align*}
\]

Again, we want to apply (6.5) to these principal parts to obtain Borcherds products \( \psi_{5,k} \) which have zeros along rational quadratic divisors with discriminant \( \leq 28 \) plus the discriminants 36 and 49 (since this might be helpful for future work). According to (6.35) the orthogonal modular group \( \Gamma_{14} \) acts transitively on the set of rational quadratic divisors of fixed discriminant. But like mentioned before, we are getting back to the symplectic world, now. So we will write down the divisors in terms of \( \mathcal{H}(\mathbb{R}) \), directly. Once more, it suffices to consider the following representatives \( \lambda_{\delta} \) of discriminant \( \delta \), where \( Z \in \mathcal{H}(\mathbb{R}) \) shall always be given by \( Z = (\tau_1, z, 7\tau_2) \) (hence the corresponding element in \( \mathcal{H}_{14} \) is \( w = (\tau_1, z, 7\tau_2) \)):

\[
\begin{align*}
    \lambda_{1} &= (0, 0, \frac{1}{14}, 0, 0) = \{ Z \in \mathcal{H}(\mathbb{R}); z = 0 \}, \\
    \lambda_{4} &= (1, 0, \frac{1}{7}, 0, 0) = \{ Z \in \mathcal{H}(\mathbb{R}); z = \frac{1}{2} \}, \\
    \lambda_{8} &= (0, 1, \frac{3}{7}, 1, 0) = \{ Z \in \mathcal{H}(\mathbb{R}); \tau_1 - 6z + 7\tau_2 = 0 \}.
\end{align*}
\]
Next, making use of (6.36) and (6.34) regarding the attached abelian characters of the Borcherds products and their counterparts in the symplectic world, we obtain the following theorem. Note that the basis above is just some random and not quite optimal basis in terms of appearing weights and divisors. Therefore, we are going to list the Borcherds products associated to this basis, first. But afterwards, we will present some further, more interesting Borcherds products. We will not give any details on how to obtain them, but each one is a product and quotient of the first nine Borcherds products. And since there is more information involved this time, we are going to choose some other way to write it down, again. Finally, note that obviously all Borcherds products vanishing along $\lambda_\perp^1$ have to be cusp forms.

(6.39) **Theorem.** There exist the following Borcherds products $\psi_{5,\lambda}$, which are all paramodular forms of weight $k$ with respect to $\Gamma_{\text{max}}^7$ and some abelian character $\nu$. Their zeros are all along rational quadratic divisors. For $\delta$ one of the discriminants appearing above denote

$$D_\delta = \bigcup_{M \in \Gamma_{\text{max}}^7} M(\lambda_\perp^1).$$

The Borcherds products can be read off the following tables, giving their weight, the orders along the $D_\delta$ and their attached abelian characters.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\nu$</th>
<th>$D_1$</th>
<th>$D_4$</th>
<th>$D_8$</th>
<th>$D_{10}$</th>
<th>$D_{16}$</th>
<th>$D_{21}$</th>
<th>$D_{25}$</th>
<th>$D_{28}$</th>
<th>$D_{36}$</th>
<th>$D_{49}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\kappa_{1,2}$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>$\lambda_7$</td>
<td>5</td>
<td>3</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
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</tr>
<tr>
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<td>0</td>
<td>1</td>
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<td>0</td>
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<td>51</td>
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</tr>
<tr>
<td>53</td>
<td>$\kappa_{1,2}$</td>
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<td>1</td>
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<td>0</td>
<td>1</td>
<td>0</td>
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and
Now, we can come to a possible reduction process, one that seems most promising. But first, note that a reduction process including $\lambda_1^b$ does not seem to be possible due to the arrangement of the orders. But we will see now that a reduction process with respect to $\lambda_9^b$ might be possible.

Let $\psi_{5,2}$, $\psi_{5,5}$, $\psi_{5,7}$ and $\psi_{5,9}$ be the unique Borcherds products from above of weights 2, 5, 7 and 9. Furthermore, let $\psi_{5,12}$ denote the first Borcherds product of weight 12 from above (the one with abelian character $\chi_7^2$).

First, assume $f \in \Gamma_{\max}^7, k_{1,2}^\chi_7^m$ (with $m \in \mathbb{Z}$). Of course, we can multiply $f$ with $\psi_{5,5}$. By construction (which we did not execute explicitly) $\psi_{5,5}$ is given by

$$\psi_{5,5} = \frac{\psi_{5,7}}{\psi_{5,2}},$$

which could also be read off the tables from the theorem by regarding the orders of the divisors.

Hence we obtain

$$\psi_{5,5} f = \frac{\psi_{5,7} f}{\psi_{5,2}} \in [\Gamma_{\max}^7, k + 5, \chi_7^{m+1}] .$$

Next, assume $f \in [\Gamma_{\max}^7, k_7^{k+1}]$. Then by definition we have

$$f(V_7(Z)) = (-1)^k f|_{V_7(Z)} = (-1)^{k+1} f(Z) = -f(Z).$$

A straightforward computation yields that $V_7$ acts trivially on $\lambda_{28}^b$. Hence $f$ vanishes along $\lambda_{28}^b$. But then having a look at the tables yields that $\psi_{5,7} f$ vanishes whenever $\psi_{5,12}$, and $\psi_{5,13}$ always vanishes of first order. So due to the considerations in (6.6) we get

$$\frac{\psi_{5,7} f}{\psi_{5,12}} \in [\Gamma_{\max}^7, k - 5, \chi_7^{k+1}] = [\Gamma_{\max}^7, k - 5, \chi_7^{k-5}] .$$

And finally, since we are multiplying with $\psi_{5,7}$ anyways, if $f \in [\Gamma_{\max}^7, 2k + 1, \chi_7]$, then note

$$\psi_{5,7} f \in [\Gamma_{\max}^7, 2k + 8, 1] ,$$

hence we only have to consider $f \in [\Gamma_{\max}^7, 2k, 1]$ from now on. Again, suppose that we found paramodular forms $g_1, \ldots, g_n$ of even weights and trivial character with respect to $\Gamma_{7}^{\max}$, such that given $f \in [\Gamma_{7}^{\max}, 2k, 1]$ there is a polynomial $p$ (homogeneous in the weights) with $p(g_1, \ldots, g_n) \in [\Gamma_{7}^{\max}, 2k, 1]$ such that $f - p(g_1, \ldots, g_n)$ vanishes on $\lambda_{28}^b$. Then deriving from the tables in (6.39)
(and taking (6.6) into account) this leads to
\[ \frac{\psi_{5,7}(f - p(g_1, \ldots, g_n))}{\psi_{5,9}} \in [\Gamma_7^{max}, 2k - 2, 1] . \]

Therefore, by induction we would obtain that for every \( f \in [\Gamma_7^{max}, k, v] \), where \( v \) is an abelian character for \( \Gamma_7^{max} \), there exists \( j \in \mathbb{N}_0 \) such that
\[ \psi_{5,7}^{j}f \in \mathbb{C}[\psi_{5,2}, \psi_{5,7}, \psi_{5,9}, \psi_{5,12}, g_1, \ldots, g_n] . \]

The rest of the reduction process might again work in analogy to the considerations done in [De01, pp.128] (or in [DK04]) concerning Hermitian modular forms for \( \mathbb{Q}(\sqrt{-2}) \). Moreover, note that in each step of the reduction process we had to consider \( \psi_{5,7}^{j}f \), where \( l \in \{2, 4, 5\} \) - while by applying the same methods we used for the other cases one also obtains \( \psi_{4,7}|_{\mathcal{H}(l)} \circ \Phi^{-1} \in \mathbb{C}^* \psi_{5,7} \).

So the issue of getting rid of the factor \( \psi_{5,7}^{j} \) might even be delayed until the considerations concerning the graded ring for \( A_{2,1} \), without the need to actually determine the graded rings for lesser dimensions.

Therefore, it seems necessary to have a closer look at the restriction to \( \lambda_9^\perp \). Of course, regarding that we have to find the paramodular forms \( g_1, \ldots, g_n \), we have to analyze what “kind of modular form” the restriction of \( f \in [\Gamma_7^{max}, 2k, 1] \) is. Obviously, \( f|_{\lambda_9^\perp} \) possesses an appropriate transformation behavior with respect to \( \text{Stab}_{\Gamma_7^{max}}(\lambda_9^\perp) \). Therefore, let us determine this stabilizer to find out what kind of modular forms we obtain. Furthermore, note that
\[ \left\{ \begin{pmatrix} \tau_1 & 1 \\ \frac{1}{3} & \tau_2 \end{pmatrix} ; \tau_1, \tau_2 \in \mathcal{H} \right\} \simeq \mathcal{H} \times \mathcal{H} . \]

The proceeding which will follow now is taken from the work of Marschner. Citing [Mar04, pp.29] for our special setting yields:

(6.40) **Lemma.** a) All matrices in \( \text{Sp}_2(\mathbb{Q}) \) preserving \( \lambda_1^\perp \) are given by
\[
M_1 \times M_2 = \begin{pmatrix}
    a_1 & 0 & b_1 & 0 \\
    0 & a_2 & b_2 & 0 \\
    c_1 & 0 & d_1 & 0 \\
    0 & c_2 & 0 & d_2
\end{pmatrix}
\] and \( M_1 \times \text{tr} M_2 := \begin{pmatrix}
    0 & a_1 & 0 & b_1 \\
    a_2 & 0 & b_2 & 0 \\
    0 & c_1 & 0 & d_1 \\
    c_2 & 0 & d_2 & 0
\end{pmatrix} \),

where \( M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \), \( M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \) \( \in \text{SL}_2(\mathbb{Q}) \).

b) We have
\[ \lambda_9^\perp = M_{1/3}(\lambda_1^\perp) , \quad \text{where } M_{1/3} = \text{Trans}(\sqrt{2}(0 \ 1 \ \ 1 \ 0)) \),

and thus
\[ \text{Stab}_{\Gamma_7}(\lambda_9^\perp) = M_{1/3}(\text{Stab}_{\text{Sp}_2(\mathbb{Q})}(\lambda_1^\perp)) M_{1/3}^{-1} \cap \Gamma_7 . \]

Note that one easily verifies that \( V_7(\lambda_9^\perp) = \lambda_9^\perp \), hence \( \text{Stab}_{\Gamma_7^{max}}(\lambda_9^\perp) = \langle \text{Stab}_{\Gamma_7}(\lambda_9^\perp), V_7 \rangle \). So let us determine the stabilizer in \( \Gamma_7 \) with the help of the preceding lemma. For \( M \in \text{SL}_2(\mathbb{R}) \) use the
abbreviation
\[ \text{SL}_2(\mathbb{Z})^M = M \text{SL}_2(\mathbb{Z})M^{-1}. \]

(6.41) Proposition. Define \( P_3 = \text{diag}(1,3) \) and \( P_{21} = \text{diag}(1,21) \). We have
\[ \text{Stab}_{\Gamma_7}(\lambda_9^\perp) \simeq \left\{ \begin{pmatrix} a_1 & b_1/3 \\
3c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2/21 \\
21c_2 & d_2 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})^{P_3} \times \text{SL}_2(\mathbb{Z})^{P_{21}} ; \right. \]
\[ \left( \begin{array}{c}
\begin{pmatrix} a_1 & b_1 \\
c_1 & d_1 \end{pmatrix} \\
\begin{pmatrix} b_2 & c_2 \\
d_2 & a_2 \end{pmatrix} \end{array} \equiv \begin{array}{c}
\begin{pmatrix} 1 \\
1 
end{pmatrix}
\end{array} \mod 3 \right). \]

Proof: Let \( M_1 = \begin{pmatrix} a_1 & b_1 \\
c_1 & d_1 \end{pmatrix}, \ M_2 = \begin{pmatrix} a_2 & b_2 \\
c_2 & d_2 \end{pmatrix} \in \text{SL}_2(\mathbb{Q}). \) A straightforward calculation gives
\[ M := M_{1/3}(M_1 \times M_2)M_{1/3}^{-1} = \begin{pmatrix}
a_1 & \frac{1}{3}c_2 & -\frac{1}{3}c_2 + b_1 & -\frac{1}{3}a_1 + \frac{1}{3}d_2 \\
\frac{1}{3}c_1 & a_2 & -\frac{1}{3}a_2 + \frac{1}{3}d_1 & -\frac{1}{3}c_1 + b_2 \\
c_1 & 0 & d_1 & -\frac{1}{3}c_2 \\
0 & c_2 & -\frac{1}{3}c_2 & d_2
\end{pmatrix}. \]

But then \( M \in \text{Sp}_2(\mathbb{Q}) \) yields that \( M \in \Gamma_7 \) holds if and only if \( M \) is of the shape given in 6.10. For those entries where no sum occurs this immediately leads to
\[ a_1 = \tilde{a}_1 \in \mathbb{Z}, \ a_2 = \tilde{a}_2 \in \mathbb{Z}, \ c_1 = 3\tilde{c}_1 \in 3\mathbb{Z}, \ c_2 = 21\tilde{c}_2 \in 21\mathbb{Z}, \ d_1 = \tilde{d}_1 \in \mathbb{Z}, \ d_2 = \tilde{d}_2 \in \mathbb{Z}. \]

So the upper right block equals
\[ \frac{1}{3} \begin{pmatrix}
-7\tilde{c}_2 + 3b_1 & -\tilde{a}_1 + \tilde{d}_2 \\
-\tilde{a}_2 + \tilde{d}_1 & -\tilde{c}_1 + 3b_2
\end{pmatrix}. \]

Again, 6.10 yields
\[ -7\tilde{c}_2 + 3b_1 \in 3\mathbb{Z}, \ -\tilde{a}_1 + \tilde{d}_2 \in 3\mathbb{Z}, \ -\tilde{a}_2 + \tilde{d}_1 \in 3\mathbb{Z}, \ -7\tilde{c}_1 + 21b_2 \in 3\mathbb{Z}, \]
and this implies
\[ a_1 \equiv d_2 \mod 3, \ d_1 \equiv a_2 \mod 3 \]
and
\[ b_1 = \frac{1}{3}b_1 \in \frac{1}{3}\mathbb{Z}, \ b_2 = \frac{1}{21}b_2 \in \frac{1}{21}\mathbb{Z}, \]

since we have \(-7\tilde{c}_2 + 3b_1 \in \mathbb{Z} \) and \(-7\tilde{c}_1 + 21b_2 \in \mathbb{Z}, \) in particular. But then the remaining conditions are given by
\[ -7\tilde{c}_2 + \tilde{b}_1 \in 3\mathbb{Z}, \ -7\tilde{c}_1 + \tilde{b}_2 \in 3\mathbb{Z}, \]

or equivalently
\[ b_1 \equiv c_2 \mod 3, \ c_1 \equiv b_2 \mod 3. \]

Therefore, if we can show that \( M_{1/3}(M_1 \times M_2)M_{1/3}^{-1} \notin \Gamma_7 \) for all \( M_1 \) and \( M_2, \) then the assertion
follows. We compute

\[ M := M_{1/3}(M_1 \times M_2)M^{-1}_{1/3} = \begin{pmatrix} \frac{1}{3}c_1 & a_1 & -\frac{1}{3}a_1 + \frac{1}{3}d_2 & -\frac{1}{3}c_2 + b_1 \\ a_2 & \frac{1}{3}c_1 & -\frac{1}{3}c_1 + b_2 & -\frac{1}{3}a_2 + \frac{1}{3}d_1 \\ 0 & c_1 & -\frac{1}{3}c_1 & d_1 \\ c_2 & 0 & d_2 & -\frac{1}{3}c_2 \end{pmatrix}. \]

Making use of 6.10 once more yields

\[ a_1 = 7\tilde{a}_1 \in 7\mathbb{Z}, \quad a_2 = \tilde{a}_2 \in \mathbb{Z}, \quad c_1 = 3\tilde{c}_1 \in 3\mathbb{Z}, \quad c_2 = 21\tilde{c}_2 \in 21\mathbb{Z}, \quad d_1 = \tilde{d}_1 \in \mathbb{Z}, \quad d_2 = 7\tilde{d}_2 \in 7\mathbb{Z}, \]

and thus also

\[ b_1 = \frac{1}{3}b_1 \in \frac{1}{3}\mathbb{Z}, \quad b_2 = \frac{1}{3}b_2 \in \frac{1}{3}\mathbb{Z}. \]

But this would imply

\[ \det(M_2) = \det \begin{pmatrix} \frac{1}{3}b_2 \\ 21\tilde{c}_2 \\ 7\tilde{d}_2 \end{pmatrix} = 7\tilde{a}_2 \tilde{d}_2 - 7\tilde{b}_2 \tilde{c}_2 \in 7\mathbb{Z}, \]

which contradicts \( M_2 \in \text{SL}_2(\mathbb{Q}). \)

Now, let \( M \in \text{Stab}_{G_7}(\lambda_0^+) \) be given like in the preceding proposition, hence \( M = M_{1/3}(M_1 \times M_2)M^{-1}_{1/3} \), with \( M_1 \) and \( M_2 \) like above. A computation shows

\[ M(Z) = \begin{pmatrix} a_1 \tau_1 + \frac{1}{3}b_1 \\ 3a_1 \tau_1 + b_1 \\ a_2 \tau_2 + \frac{1}{3}b_2 \\ 3a_2 \tau_2 + b_2 \end{pmatrix} \]

for \( Z = \left( \frac{\tau_1}{3}, \frac{1}{3} \right) \), which means that \( \text{Stab}_{G_7}(\lambda_0^+) \) acts on \( Z \in \lambda_0^+ \simeq \mathcal{H} \times \mathcal{H} \) like \( \text{SL}_2(\mathbb{Z})_{P_1} \times \text{SL}_2(\mathbb{Z})_{P_2} \) does on \( (\tau_1, \tau_2) \in \mathcal{H} \times \mathcal{H} \). Furthermore, define

\[ \mathcal{G} := \left\{ \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \in \text{SL}_2(\mathbb{Z})^2 \mid \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right) \equiv \left( \begin{pmatrix} d_1 & c_2 \\ a_2 & d_2 \end{pmatrix} \right) \right\}. \quad (6.12) \]

One easily verifies that \( \mathcal{G} \) is a group, and view of (6.41)

\[ \text{Stab}_{G_7}(\lambda_0^+) \simeq \mathcal{G} \]

holds. Let the corresponding isomorphism be denoted by \( \Psi_\mathcal{G} : \text{Stab}_{G_7}(\lambda_0^+) \to \mathcal{G} \), where \( M \in \text{Stab}_{G_7}(\lambda_0^+) \) is first mapped to \( M_{1/3}MM^{-1}_{1/3} = M_1 \times M_2 \in \text{SL}_2(\mathbb{Z})_{P_1} \times \text{SL}_2(\mathbb{Z})_{P_2} \), and this again is canonically mapped to \( \mathcal{G} \). Moreover, let the biholomorphic map \( \Phi_\mathcal{G} \) be given by

\[ \Phi_\mathcal{G} : \lambda_0^+ \to \mathcal{H} \times \mathcal{H} , \quad \left( \begin{pmatrix} \tau_1 \\ 1/3 \end{pmatrix}, \frac{1}{3} \right) \mapsto \left\{ (3\tau_1, 21\tau_2) \right\}. \quad (6.13) \]

A straightforward calculation shows

\[ M(Z) = \Phi_\mathcal{G}^{-1}(\Psi_\mathcal{G}(M)\langle \Phi_\mathcal{G}(Z) \rangle), \]
where
\[(M_1, M_2)(\tau_1, \tau_2) := (M_1\langle \tau_1 \rangle, M_2\langle \tau_2 \rangle)\]
for \((M_1, M_2) \in \text{SL}_2(\mathbb{R})^2\) and \((\tau_1, \tau_2) \in \mathcal{H} \times \mathcal{H}\). Furthermore, given \(M_1 = \left(\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right)\), \(M_2 = \left(\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right) \in \text{SL}_2(\mathbb{Z})\) and \(Z = \left(\begin{array}{c} \tau_1 \\ \tau_2 \end{array}\right) \in \delta_\tau^\perp\), one computes
\[
(\Psi_G^{-1}(M_1, M_2))\{Z\} = (3c_1 \tau_1 + d_1)(21c_2 \tau_2 + d_2) = M_1\{3 \tau_1\} \cdot M_2\{21 \tau_2\}.
\]
Therefore, define
\[f|_k(M_1, M_2)(\tau_1, \tau_2) := (M_1\{\tau_1\} \cdot M_2\{\tau_2\})^{-k}f(M_1\langle \tau_1 \rangle, M_2\langle \tau_2 \rangle)
\]
for functions \(f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}\). Then due to the considerations above we get
\[f \circ \Phi^{-1}_G|_k(\Psi_G(M)) = (f|_kM) \circ \Phi^{-1}_G
\]
holds for all \(f : \delta^\perp_\tau \rightarrow \mathbb{C}\) and all \(M \in \text{Stab}_{\mathcal{H}}(\lambda_\tau^\perp)\). So by this correspondence, analyzing \(f|_k\), where \(f \in [\tau_7, k, \nu]\) actually means considering holomorphic functions \(g : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}\) which possess some proper transformation behavior under \(g \mapsto g|_k(M_1, M_2)\), where \((M_1, M_2) \in \mathcal{G}\). Furthermore, due to the Koecher’s principal, paramodular forms are holomorphic at all cusps (cf. [Mar04, p.9]). Therefore, we only have to consider those forms \(g\) that are also holomorphic at all cusps. And finally, note that
\[V_7(Z) = \left(\begin{array}{cc} 7\tau_2 \\ 1/3 \\ \tau_1/7 \end{array}\right)
\]
for all \(Z = \left(\begin{array}{c} \tau_1 \\ 1/3 \tau_2 \end{array}\right) \in \delta_\tau^\perp\), and thus
\[V_7(Z) = \Phi^{-1}_G(21\tau_2, 3\tau_1).
\]
Hence we get to the following

**6.42 Definition.** Let \(N \in \mathbb{N}\), \(\Gamma\) a group with \(\text{SL}_2(\mathbb{Z})/|N|^2 \leq \Gamma \leq \text{SL}_2(\mathbb{Z})^2\) and \(\nu\) an abelian character of \(\Gamma\). Then a function \(f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}\) is called a (two-dimensional) modular form of weight \(k \in \mathbb{Z}\) with respect to \(\Gamma\) and \(\nu\) if \(f\) satisfies:

\[(M2.1)\] \(f|_kM = \nu(M) \cdot f\) for all \(M \in \Gamma\),

\[(M2.2)\] \(f\) is holomorphic on \(\mathcal{H} \times \mathcal{H}\),

\[(M2.3)\] For all \(M \in \text{SL}_2(\mathbb{Z})^2\), \(f|_kM\) is bounded on \(\{(\tau_1, \tau_2) \in \mathcal{H} : \text{Im}(\tau_1), \text{Im}(\tau_2) > y_0\}\) for all \(y_0 > 0\).

If in addition
\[f(\tau_1, \tau_2) = \epsilon f(\tau_2, \tau_1)
\]
for all \((\tau_1, \tau_2) \in \mathcal{H} \times \mathcal{H}\), where \(\epsilon \in \{\pm 1\}\), then \(f\) is called symmetric if \(\epsilon = 1\), and skew-symmetric if \(\epsilon = -1\).
The space of all (two-dimensional) modular forms of weight \( k \) with respect to \( \Gamma \) and \( \nu \) is denoted by \( [\Gamma, k, \nu] \), as usual. The subspaces of symmetric and skew-symmetric forms are denoted by \( [\Gamma, k, \nu]^1 \) and \( [\Gamma, k, \nu]^{-1} \), respectively.

Note that (M2.2) is fulfilled if and only if \( f \) is holomorphic in both variables separately. Furthermore, due to the cocycle relations it suffices to verify (M2.1) for generators of \( \Gamma \) only, like in all other cases of modular forms. And finally, again like in all other cases we considered so far, (M2.3) is equivalent so demand \( f \) to possess an appropriate Fourier-expansion. Since \( \text{SL}_2(\mathbb{Z})[N]^2 \leq \Gamma \) has to hold (or also confer [Mar04, p.41]), this condition is given by

\[(6.43) \text{ Proposition.}\]

- For all \( M \in \text{SL}_2(\mathbb{Z})^2 \), \( f|_k M \) possesses a Fourier-expansion of the shape
  \[ f|_k M(\tau_1, \tau_2) = \sum_{n_1,n_2 \in \mathbb{N}_0} \alpha_f(n_1, n_2; M)e^{2\pi i(n_1 \tau_1 + n_2 \tau_2)/N} \]

And this immediately gives rise to the definition of cusp forms, where we demand these Fourier-expansions to run over \( \mathbb{N}^2 \) instead of \( \mathbb{N}_0^2 \). The subspace of cusp forms is again denoted by \( [\Gamma, k, \nu]_0 \).

Obviously, we have to determine \( [\mathcal{G}, k, \nu] \) for certain abelian characters \( \nu \). By definition we have \( \mathcal{G} \leq \text{SL}_2(\mathbb{Z})^2 \), and also \( \text{SL}_2(\mathbb{Z})[3]^2 \leq \mathcal{G} \) since the congruence conditions are fulfilled for all \( M \in \text{SL}_2(\mathbb{Z})[3]^2 \), of course. Furthermore, \( \text{SL}_2(\mathbb{Z})[3]^2 \leq \text{SL}_2(\mathbb{Z})^2 \) and

\[ \text{SL}_2(\mathbb{Z})^2 / \text{SL}_2(\mathbb{Z})[3]^2 \simeq \text{SL}_2(\mathbb{F}_3)^2 \]

hold in virtue of [KK07, pp.133]. Hence \( \mathcal{G} / \text{SL}_2(\mathbb{Z})[3]^2 \) can be identified as a subgroup of \( \text{SL}_2(\mathbb{F}_3)^2 \). Of course, \( \mathcal{G} \) is generated by \( \text{SL}_2(\mathbb{Z})[3]^2 \) and any pre-images of the generators of \( \mathcal{G} / \text{SL}_2(\mathbb{Z})[3]^2 \). Concerning generators of \( \mathcal{G} / \text{SL}_2(\mathbb{Z})[3]^2 \) one can use some computer algebra system since \( \text{SL}_2(\mathbb{F}_3)^2 \) is finite. Doing so and also taking the results from [De02] concerning generators of \( \text{SL}_2(\mathbb{Z})[3] \) into account we obtain in analogy to [Mar04, le.3.6.7] (without any further proof):

\[(6.43) \text{ Proposition.}\]

a) \( \text{SL}_2(\mathbb{Z})[3] \) is generated by

\[ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}, \quad \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}. \]

b) \( \text{SL}_2(\mathbb{Z})[3]^2 \leq \mathcal{G} \leq \text{SL}_2(\mathbb{Z})^2 \).

c) \( \mathcal{G} / \text{SL}_2(\mathbb{Z})[3]^2 \simeq \text{SL}_2(\mathbb{F}_3) \)

d) \( \mathcal{G} \) is given by

\[ \mathcal{G} = \left\langle \text{SL}_2(\mathbb{Z})[3]^2, \; T_G := \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \; J_G := \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \right\rangle \]

So deriving from the preceding proposition we have the following setting: Let \( \nu \) be any abelian character for \( \mathcal{G} \). Then \( \nu \) is also an abelian character for \( \text{SL}_2(\mathbb{Z})[3]^2 \), and \( f \in [\mathcal{G}, k, \nu] \) holds if and
only if
\[ f \in [\text{SL}_2(\mathbb{Z})|^3, k, \nu] \quad \text{and} \quad f|_k T_G = \nu(T_G) \cdot f \quad \text{and} \quad f|_k J_G = \nu(J_G) \cdot f. \quad (6.16) \]

Now, note that due to our reduction process concerning paramodular forms of level 7 we only need to consider the trivial character and even weight. But for now, let us assume we have not multiplied with \( \psi_{5,7} \) to get rid of the character \( \chi_7 \). Then we have to deal with paramodular forms \( f \in [\Gamma_7^{\text{max}}, k, \chi^3] \). Thus we have
\[ f|_k M = f \quad \text{for all } M \in \Gamma_7, \quad f(V_7(Z)) = (-1)^k f|_k V_7(Z) = f(Z). \]

Hence we have to determine all spaces \([G, k, 1]^1\). Like we have seen, this means that we first need to determine \([\text{SL}_2(\mathbb{Z})|^3, k, 1]\). This was already done in [Mar04], indeed. But before we can present the result, we need to define some special modular forms for \( \text{SL}_2(\mathbb{Z})[^3] \), namely special Eisenstein-series.

Since we will not need any other Eisenstein-series, we only consider the following one (cf. [Mar04, pp.37]): Let \( \chi \) and \( \psi \) be Dirichlet characters, then for \( \tau \in \mathcal{H} \) we define
\[ E_1(\tau, s; \chi, \psi) := \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus (m,n) \neq (0,0)}} \psi(m) \psi(n)(m\tau + n)^{-1}|m\tau + n|^{-2s}. \quad (6.17) \]

This series only converges for \( s > \frac{1}{2} \), but according to [Mi89, thm.7.2.9], \( E_1(\tau, s; \chi, \psi) \) is meromorphically continuable in \( s \) to \( \mathbb{C} \), and this continuation is holomorphic in \( 0 - \) given \( \chi \psi(-1) = -1 \) holds. Now, let \( \chi^{(1)}_0 \) be the trivial Dirichlet character mod 1 (i.e. \( \chi^{(1)}_0 \equiv 1 \)), \( \chi^{(3)}_0 \) be the trivial Dirichlet character mod 3 (i.e. \( \chi^{(3)}_0(n) = 1 \) for \( n \in \mathbb{Z} \setminus 3\mathbb{Z} \), \( \chi^{(3)}_0(n) = 0 \) for \( n \in 3\mathbb{Z} \), and \( \chi_3 \) the unique non-trivial Dirichlet character mod 3 (i.e. \( \chi_3(3n) = 0 \), \( \chi_3(3n + 1) = 1 \), \( \chi_3(3n - 1) = -1 \) for \( n \in \mathbb{Z} \). Then we define two special Eisenstein-series for \( \tau \in \mathcal{H} \):
\[ E_{1,3}^*(\tau) := \frac{9}{2\pi^3} E_1(\tau, 0; \chi^{(1)}_0, \chi_3), \]
\[ E_{3,3}^*(\tau) := -\frac{9}{4\pi^3} E_1(\tau, 0; \chi_3, \chi^{(3)}_0), \quad (6.18) \]

where the prefactors are chosen such that the Fourier-expansions are normalized in the sense that the constant term equals 1 (cf. [Mar04, pp.37]). Note that according to [Mar04, thm.4.2.7] we have \( E_{1,3}^*, E_{3,3}^* \in [\text{SL}_2(\mathbb{Z})[^3, 1, 1] \), and every \( f \in [\text{SL}_2(\mathbb{Z})[^3, k, 1] \) (where \( k \in \mathbb{N}_0 \)) is given by \( f = p_{k,f}(E_{1,3}^*, E_{3,3}^*) \), where \( p_{k,f} \) is a homogeneous polynomial of degree \( k \) in two variables. Moreover, these Eisenstein-series suffice to generate the graded ring of modular forms with respect to \( \text{SL}_2(\mathbb{Z})[^3 \mathbb{Z}^2 \). The following theorem is taken from [Mar04, thm.4.3.3], [Mar04, cor.4.3.4] and [Mar04, p.44]:
6.3 Paramodular forms of level 7 and degree 2

(6.44) Theorem. Let \( A \) be the graded ring of all modular forms with respect to \( \text{SL}_2(\mathbb{Z})[3]^2 \) and the trivial character, i.e.

\[
A = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})[3]^2, k, 1],
\]

\( A^1 \) the subring of the symmetric forms, i.e.

\[
A^1 = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})[3]^2, k, 1]^1,
\]

and \( A^{-1} \) the \( A^1 \)-module of all skew-symmetric forms, i.e.

\[
A^{-1} = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})[3]^2, k, 1]^{-1}.
\]

a) We have

\[
A \simeq \mathbb{C}[X_1, X_2, X_3, X_4]/(X_1X_4 - X_2X_3),
\]

\[
A^1 \simeq \mathbb{C}[X_1, X_4, X_2 + X_3],
\]

\[
A^{-1} \simeq (X_2 - X_3) \cdot \mathbb{C}[X_1, X_4, X_2 + X_3],
\]

where the isomorphisms are induced by

\[
X_1 \mapsto F_1(\tau_1, \tau_2) := E_{3,3}^1(\tau_1) \cdot E_{3,3}(\tau_2),
\]

\[
X_2 \mapsto F_2(\tau_1, \tau_2) := E_{3,3}^1(\tau_1) \cdot E_{3,3}(\tau_2),
\]

\[
X_3 \mapsto F_3(\tau_1, \tau_2) := E_{1,3}^1(\tau_1) \cdot E_{3,3}(\tau_2),
\]

\[
X_4 \mapsto F_4(\tau_1, \tau_2) := E_{1,3}^1(\tau_1) \cdot E_{1,3}(\tau_2).
\]

b) The following identities hold:

\[
E_{3,3}^*|J_1 = -\frac{1}{\sqrt{3}}(E_{1,3}^* - E_{3,3}^*),
\]

\[
E_{1,3}^*|J_1 = -\frac{1}{\sqrt{3}}(2E_{3,3}^* + E_{1,3}^*),
\]

\[
E_{3,3}^*|T_1 = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right)E_{3,3}^* + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right)E_{1,3}^*,
\]

\[
E_{1,3}^*|T_1 = (1 - \frac{1}{\sqrt{3}})E_{3,3}^* + \frac{1}{\sqrt{3}}E_{1,3}^*.
\]

Deriving from the preceding theorem, the next steps are obvious (and already described in [Mar04, pp.44]): Every (symmetric) modular form \( f \in [G, k, 1] \) is contained in \( [\text{SL}_2(\mathbb{Z})[3]^2, k, 1] \) (and is symmetric). Therefore, there exists a homogeneous polynomial \( p_{k,f} \) of degree \( k \) in four (or three) variables, such that \( f = p_{k,f}(F_1, F_2, F_3, F_4) \) (or \( f = p_{k,f}(F_1, F_4, F_2 + F_3) \)). On the other hand, if such a polynomial in the \( F_j \) is given, then in virtue of (6.43) it is an element in \([G, k, 1]\) if and only if it is also invariant under \( T_{G} = (T_1, J_1^{-1}T_1^{-1}J_1) \) and \( J_{G} = (J_1, J_1^{-1}) \). Note that in view of the preceding theorem the behavior of the \( F_j \) under these two elements can be determined easily by regarding the transformation behavior of \( E_{3,3}^* \) and \( E_{1,3}^* \). Therefore, define

\[
M_f = \frac{1}{3} \begin{pmatrix}
1 & -1 & -1 & 1 \\
-2 & -1 & 2 & 1 \\
-2 & 2 & -1 & 1 \\
4 & 2 & 2 & 1
\end{pmatrix}
\]

(6.19)
Using a computer algebra system (here, [MAGMA] was used) one can verify that \( M_J \) is of order 2, while \( M_T \) is of order 3, and

\[
F_j \mid_1 \Sigma = 4 \sum_{k=1}^4 (M_J)_{j,k} F_k, \quad F_j \mid_1 T_{\Sigma} = 4 \sum_{k=1}^4 (M_T)_{j,k} F_k
\]

for \( j \in \{1, \ldots, 4\} \). And finally, if we want to obtain symmetric forms, we also need

\[
M_{\text{sym}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

since \( M_J \) and \( M_T \) do not preserve the subring \( C[X_1, X_4, X_2 + X_3] \).

Defining \( G = \langle M_J, M_T \rangle \) and \( \hat{G} = \langle M_J, M_T, M_{\text{sym}} \rangle \), we thus have to determine the invariant rings

\[
\left( C[X_1, X_2, X_3, X_4] / (X_1 X_4 - X_2 X_3) \right)^H
\]

for \( H \in \{G, \hat{G}\} \) (where one has to note that both groups are finite), since these will be isomorphic to \( \bigoplus_{k \in \mathbb{N}_0} [G, k, 1] \) and \( \bigoplus_{k \in \mathbb{N}_0} [\hat{G}, k, 1] \), respectively. The algebraic methods to do so are described in chapter 5 of [Mar04]. This can be implemented in [MAGMA], which has been done for this current thesis, indeed.

In the appendix, the results of these calculations are written down. For now, we just give the following result:

**(6.45) Corollary.** \( \bigoplus_{k \in \mathbb{N}_0} [G, k, 1] \) is generated by some modular forms (found in the appendix) \( g_1, g_2, g_3 \) and \( g_4 \) of weights 1, 2, 3 and 4, respectively, where \( g_2 \) can be omitted due to the relation \( F_1 F_4 = F_2 F_3 \).

\( \bigoplus_{k \in \mathbb{N}_0} [\hat{G}, k, 1] \) is generated by the same elements plus a further modular form \( g_6 \) of weight 6 (which is neither symmetric nor skew-symmetric, whereas \( g_6 \) could be replaced by the skew-symmetric form \( \tilde{g}_6(\tau_1, \tau_2) = g_6(\tau_1, \tau_2) - g_6(\tau_2, \tau_1) \)).

So this is where the obvious problem arises: So far, we have not found any paramodular forms of level 7 with respect to \( \Gamma_7 \) and the trivial character of weights 1, 2 or 3 – and we will see further below that there actually exist none. Hence there exist no paramodular forms whose restrictions to \( \delta_\perp \) coincide with the modular forms of weights 1 and 3 from the preceding corollary. So the question remains if we can find any paramodular forms with the property that for every paramodular form \( f \) there exists a polynomial in these such that the difference between \( f \) and this polynomial vanishes along \( \delta_\perp \). Thus, one maybe has to consider subgroups of \( \Gamma_7 \), first, in
order to obtain paramodular forms of lower weight, and then go back up to $\Gamma_7$ – similar to what was done in [Kr05] concerning quaternionic modular forms for the Hurwitz order. But for now, we have to leave it at this state that we are not able to determine the graded ring of paramodular forms of level 7. Further approaches that might yield graded rings of paramodular forms can be found in [Mar04]. However, note again that analyzing the structure of the spaces of paramodular forms of level 7 might be the key to initiate a successful reduction process that may result in being able to determine the structure of the graded ring of quaternionic modular forms.

But still, there is one last thing left to do here. In order to determine the graded ring of paramodular forms of level 7 in some future work, it might be helpful to know the dimension of the space of paramodular forms with respect to $\Gamma_7$ and the trivial character for a fixed weight $k$. Indeed, it is possible to determine this dimension for all $k$, just like it was done for $\Gamma_5$ in [Mar04, prop.7.3.3]. To actually achieve an analog of that proposition, we need some special paramodular forms, namely Maass lifts for the paramodular setting. We will keep it very brief since it does not seem necessary to present the whole theory here. We just need some special paramodular forms such that we can prove the analog of said proposition. The exact details can be found in chapter 6 of [Mar04].

We start by recalling the definition of the famous Dedekind $\eta$-function:

$$\eta : \mathcal{H} \to \mathbb{C}, \quad \tau \mapsto e^{\frac{1}{12} \pi i \tau} \prod_{n \in \mathbb{N}} \left( 1 - e^{2 \pi i n \tau} \right).$$

(6.22)

It is well known (for example, confer [KK07, pp.187]) that

$$\eta|_{\frac{1}{2}} T_{Mp} = e^{\frac{1}{12} \pi i \eta}, \quad \eta|_{\frac{1}{2}} J_{Mp} = e^{-\frac{1}{4} \pi i \eta}$$

(6.23)

holds. This induces an abelian character $\nu_\eta$ of $Mp_2(\mathbb{Z})$, and $\eta \in [Mp_2(\mathbb{Z}), \frac{1}{2}, \nu_\eta]$. (Note that this also means $\eta^2 \in [SL_2(\mathbb{Z}), 1, \nu^2_\eta]$. Furthermore, $\nu^{12}_\eta \equiv 1$ holds.)

Next, we need half-integral Jacobi forms with abelian characters. We will not give any details here. These can be found in chapter 6 of [Mar04]. We already introduced the metaplectic group $Mp_2(\mathbb{Z})$ (see (5.35)). Now, we also need the so-called integral Heisenberg group defined by

$$H(\mathbb{Z}) := \left\{ [\lambda, \mu; \kappa] := \text{Rot}(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \text{ Trans}(\begin{smallmatrix} 0 & \mu \\ \mu & \lambda \end{smallmatrix}) ; \lambda, \mu, \kappa \in \mathbb{Z} \right\}. \quad (6.24)$$

$Mp_2(\mathbb{Z})$ acts on $H(\mathbb{Z})$, but we omit the details here (cf. [Mar04, p.72]). The metaplectic Jacobi group is then defined by

$$Mj_2(\mathbb{Z}) := Mp_2(\mathbb{Z}) \ltimes H(\mathbb{Z}). \quad (6.25)$$

Given $k \in \frac{1}{2} \mathbb{Z}$, $Mj_2(\mathbb{Z})$ acts on the set of all vector-valued functions $f : \mathcal{H}(\mathbb{R}) \to V$ (where $V$ is a $\mathbb{C}$-vector space) via

$$f|_k((M, \varphi), [\lambda, \mu; \kappa])(Z) := \varphi(\tau_1)^{-2k} f \left( (M \times I) \text{ Rot}(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \text{ Trans}(\begin{smallmatrix} 0 & \mu \\ \mu & \lambda \end{smallmatrix}) (Z) \right), \quad (6.26)$$
with \( Z = \left( \frac{\tau}{2}, \frac{z}{2} \right) \in \mathcal{H}(\mathbb{R}) \) and \( ((M, \varphi), [\lambda, \mu; \kappa]) \in \text{M}_{1,2}(\mathbb{Z}) \). According to [GN98], all characters of \( \text{M}_{1,2}(\mathbb{Z}) \) are given by

\[
v_{a,b} := v_{\eta}^{a} \times v_{H}^{b},
\]

where \( a, b \in \mathbb{Z} \) and

\[
v_{H}(\lambda, \mu; \kappa) := (-1)^{\lambda+\mu+\lambda\mu+\kappa}.
\]

Note that \( v_{242} \equiv 1 \). Now, we can define Jacobi forms (cf. [Mar04, def.6.1.1]):

**6.46 Definition.** Let \( v_{a,b} \in \text{M}_{1,2}(\mathbb{Z})^{ab} \). A holomorphic function \( \Phi : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \) is called a Jacobi form of weight \( k \in \frac{1}{2} \mathbb{Z} \) and index \( m \in \frac{1}{2} \mathbb{Z} \) with respect to \( v_{a,b} \) if the function \( \tilde{\Phi}_{m} : \mathcal{H}(\mathbb{R}) \rightarrow \mathbb{C} \),

\[
(\frac{\tau}{2}, z) \mapsto \Phi(\tau, z)e^{2\pi i mz},
\]

satisfies

\[
\tilde{\Phi}_{m}|_{k}M = v_{a,b}(M) \cdot \tilde{\Phi}_{m} \quad \text{for all } M \in \text{M}_{1,2}(\mathbb{Z})
\]

and if \( \Phi \) possesses a Fourier-expansion of the shape

\[
\Phi(\tau, z) = \sum_{n, t \in \mathbb{Z}, n \geq 0, 4m n - t^2 \geq 0} c(n, t)e^{2\pi i (nt + tz)}, \quad (\tau, z) \in \mathcal{H} \times \mathbb{C},
\]

where \( n \) and \( t \) have bounded denominators depending on \( v_{a,b} \). Moreover, \( \Phi \) is called a cusp form if \( c(n, t) \neq 0 \) implies \( 4mn - t^2 > 0 \). The space of Jacobi forms of weight \( k \) and index \( m \) with respect to \( v_{a,b} \) is denoted by \( \text{M}_{1,2}(\mathbb{Z}), k, m, v_{a,b} \), and the subspace of cusp forms by \( \text{M}_{1,2}(\mathbb{Z}), k, m, v_{a,b}\)0.

According to [Mar04, le.6.1.2] \( \text{M}_{1,2}(\mathbb{Z}), k, m, v_{a,b} = \{0\} \) holds if \( 2m \equiv b \pmod{2} \), hence we assume \( 2m \equiv b \pmod{2} \) from now on. From [Mar04, le.6.1.3] we cite the following important lemma:

**6.47 Lemma.** a) If \( \Phi \in \text{M}_{1,2}(\mathbb{Z}), k, m, 1 \)0, then

\[
\eta^{-j} \Phi \in \text{M}_{1,2}(\mathbb{Z}), k - \frac{j}{2}, m, v_{-j,0}
\]

holds for all \( j \in \mathbb{N} \) with \( jm \leq 18 \).

b) Let \( \Phi_{j} \in \text{M}_{1,2}(\mathbb{Z}), k_{j}, m_{j}, v_{j} \), where \( j \in \{1, 2\} \), \( k_{j}, m_{j} \in \frac{1}{2} \mathbb{Z} \) and \( v_{j} \in \text{M}_{1,2}(\mathbb{Z})^{ab} \). We define

\[
[\Phi_{1}, \Phi_{2}] := \frac{1}{2\pi i}(m_{2}\Phi_{1}'(\Phi_{2} - m_{1} \Phi_{1}'),
\]

where

\[
\Phi_{1}'(\tau, z) = \frac{\partial \Phi_{1}(\tau, z)}{\partial z}.
\]

Then we get

\[
[\Phi_{1}, \Phi_{2}] \in \text{M}_{1,2}(\mathbb{Z}), k_{1} + k_{2} + 1, m_{1} + m_{2}, v_{1} v_{2}
\].

Next, we need some special Jacobi forms:

- \( \eta \in \text{M}_{1,2}(\mathbb{Z}), \frac{1}{2}, 0, v_{1,0} \),
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• \( E_{k,m} \in [MJ_2(\mathbb{Z}), k, m, 1] \) (for \( m \in \mathbb{N}, k \in 2\mathbb{N}, k \geq 4 \)),
• \( \Phi_{10,1} \in [MJ_2(\mathbb{Z}), 10, 1, 1]_0 \), \( \Phi_{12,1} \in [MJ_2(\mathbb{Z}), 12, 1, 1]_0 \),
• \( \vartheta_{1/2} \in [MJ_2(\mathbb{Z}), 1/2, 1/2, 1/2, 1] \), \( \vartheta_{3/2} \in [MJ_2(\mathbb{Z}), 3/2, 3/2, 1] \),
• \( \Phi_{6,7/2} = [\vartheta_{1/2} \vartheta_{3/2}, \eta^7] \in [MJ_2(\mathbb{Z}), 6, 7/2, v_{12,1}] \).

Here, \( E_{k,m} \) is the \( m \)-th Fourier-Jacobi coefficient of the Siegel Eisenstein-series of weight \( k \) (defined analogous to our quaternionic Eisenstein-series), while \( \Phi_{10,1} \) and \( \Phi_{12,1} \) are the first Fourier-Jacobi coefficients of the unique Siegel cusp forms of weight 10 and 12. (We will not need them for further progress, but they are needed to obtain further Maaß lifts, which might be interesting for future work – cf. [Mar04, pp.82].) \( \vartheta_{1/2} \) and \( \vartheta_{3/2} \) are certain Theta series. We omit all details here. Except for \( \Phi_{6,7/2} \), which we just defined for the current thesis, said details can be found in [Mar04, pp.74] and [EZ85]. The important thing is that the Fourier-expansions of these Jacobi forms can be computed explicitly, which was done for this thesis using [MAGMA] and [SAGE], indeed.

Now, we come to the Maaß lifts. Again, we refer to [Mar04, thm.6.3.1, thm.6.3.2] and omit all details, here. Just note that the lift works similar to the quaternionic Maaß lifts: Given a Jacobi form of index \( m \), a Hecke operator is applied to gain a Jacobi form of higher index. Then one sums over these newly created Jacobi forms (multiplied by \( e^{2\pi i \tau^2} \) to the power of the index) and obtains a paramodular form. Given a Jacobi form \( \Phi \), this lift is denoted by \( \text{Lift}_\Phi \) (in order to use the notation of Marschner). We obtain the following important paramodular forms:

• \( F_4 = \text{Lift}_{E_{4,7}} \in [\Gamma_7^{\text{max}}, 4, 1] \),
• \( F_6 = \text{Lift}_{E_{6,7}} \in [\Gamma_7^{\text{max}}, 6, 1] \),
• \( H_6 = \text{Lift}_{\Phi_{6,7/2}} \in [\Gamma_7^{\text{max}}, 6, \kappa_{1,2}] \),
• \( F_{12} = \text{Lift}_{G_6, E_{6,7}} \in [\Gamma_7^{\text{max}}, 12, 1] \) (with \( G_6 \) the elliptic Eisenstein-series of weight six, see (4.3)),
• \( \psi_{5,12} \in [\Gamma_7^{\text{max}}, 12, \chi_7] \) (the Borcherds product from (6.39)).

The importance of these paramodular forms will become clear in a moment. From [Mar04, thm.8.6.1, rem.8.6.2] we cite the following theorem in analogy to (6.44):

(6.48) Theorem. Let \( \mathcal{A} \) be the graded ring of all modular forms with respect to \( \text{SL}_2(\mathbb{Z})^2 \) and the trivial character, i.e.

\[
\mathcal{A} = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})^2, k, 1],
\]

\( \mathcal{A}^1 \) the subring of the symmetric forms, i.e.

\[
\mathcal{A}^1 = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})^2, k, 1]^1,
\]

and \( \mathcal{A}^{-1} \) the \( \mathcal{A}^1 \)-module of all skew-symmetric forms, i.e.

\[
\mathcal{A}^{-1} = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})^2, k, 1]^{-1}.
\]
Furthermore, let \( \mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12}) \) and \( \mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12})^{-1} \) be the \( \mathcal{A}^1 \)-modules given by
\[
\mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12}) = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})^2, k, v_{\eta}^{12} \times v_{\eta}^{12}]^1
\]
and
\[
\mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12})^{-1} = \bigoplus_{k \in \mathbb{N}_0} [\text{SL}_2(\mathbb{Z})^2, k, v_{\eta}^{12} \times v_{\eta}^{12}]^{-1}.
\]
We have
\[
\mathcal{A}^1 = \mathbb{C}[G_4(\tau_1)G_4(\tau_2), G_6(\tau_1)G_6(\tau_2), G_4^3(\tau_1)G_6^2(\tau_2) + G_6^2(\tau_1)G_4^3(\tau_2)],
\]
\[
\mathcal{A}^{-1} = (G_4^2(\tau_1)G_6^2(\tau_2) - G_6^2(\tau_1)G_4^2(\tau_2)) \cdot \mathcal{A}^1,
\]
\[
\mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12}) = \eta^{12}(\tau_1)\eta^{12}(\tau_2) \cdot \mathcal{A}^1,
\]
\[
\mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12})^{-1} = (G_4^2(\tau_1)G_6^2(\tau_2) - G_6^2(\tau_1)G_4^2(\tau_2)) \cdot \mathcal{A}(v_{\eta}^{12} \times v_{\eta}^{12})^{-1}.
\]

Next, we have the following setting: Define the Witt operator \( W_7 \) on functions \( f : \mathcal{H}(\mathbb{R}) \to \mathbb{C} \) by
\[
W_7(f)(\tau_1, \tau_2) := f\left(\begin{array}{c} \tau_1 \\ 0 \\ \tau_2/7 \end{array}\right), \quad (\tau_1, \tau_2) \in \mathcal{H} \times \mathcal{H}.
\]
In virtue of [Mar04, pp.94] we obtain for \( f \in [\Gamma_7^{max}, k, \kappa_0^m \chi_{12}^2] \) (with \( m, n \in \mathbb{Z} \)):
\[
W_7(f) \in [\text{SL}_2(\mathbb{Z})^2, k, v_{\eta}^{12m} \times v_{\eta}^{12n}]^{\epsilon}
\]
in analogy to the situation we had for \( \delta_{\eta}^1 \), where \( \epsilon = 1 \) if \( k + n \equiv 0 \mod 2 \), and \( \epsilon = -1 \) if \( k + n \equiv 1 \mod 2 \).

Note that we already mentioned that we calculated explicit Fourier-coefficients for the Jacobi forms which are lifted to \( F_4, F_6, H_6 \) and \( F_{12} \). Using these for the formulas presented in [Mar04, thm.6.3.1, thm.6.3.2] it is possible to calculate explicit Fourier coefficients of these lifts, hence also for \( W_7(\cdot) \). This can be done using some computer algebra system, and was done for this thesis, indeed, as well as for the Fourier coefficients of the generators in (6.48). Moreover, according to that theorem, \( W_7(\cdot) \) can always be expressed in these generators, and solving some linear equation systems concerning sufficiently many Fourier-coefficients yields the exact expression. Furthermore, in view of (6.39) and the remark above, \( W_7(\psi_{5,12}) \in [\text{SL}_2(\mathbb{Z}), 12, 1]^{-1} \) does not vanish identically, while \( \dim[\text{SL}_2(\mathbb{Z}), 12, 1]^{-1} = 1 \) in virtue of (6.48). This and the results from the linear equation systems yield

(6.49) Proposition.

- \( W_7(F_4)(\tau_1, \tau_2) = \frac{1}{250} G_4(\tau_1)G_4(\tau_2), \)
- \( W_7(F_6)(\tau_1, \tau_2) = -\frac{1}{504} G_6(\tau_1)G_6(\tau_2), \)
- \( W_7(H_6)(\tau_1, \tau_2) = 8\eta^{12}(\tau_1)\eta^{12}(\tau_2), \)
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- \( W_7\left( -\frac{7}{2185792}, F_4^2 - \frac{53}{412898008}, F_6^2 - \frac{13}{389847}, F_{12} \right) = G_4^5(\tau_1)G_6^5(\tau_2) + G_6^2(\tau_1)G_4^3(\tau_2), \)
- \( W_7(\psi_{5,12})(\tau_1, \tau_2) = c(G_4^3(\tau_1)G_6^2(\tau_2) - G_6^2(\tau_1)G_4^3(\tau_2)) \) for some \( c \in \mathbb{C}_*. \)

Because of the preceding proposition we obtain the following corollary in analogy to [Mar04, cor.7.3.2] for \( \Gamma_5 \). We omit the proof since it is exactly the same for \( \Gamma_7 \) when taking the preceding proposition into account.

(6.50) Corollary. Let \( k \in \mathbb{N}_0 \).

- a) \( W_7 : [\Gamma_7, k, 1] \to [\text{SL}_2(\mathbb{Z}), k, 1] \) and \( W_7 : [\Gamma_7, k, \kappa_{1,2}] \to [\text{SL}_2(\mathbb{Z}), k, \nu^2_7 \times \nu^{12}_7] \) are surjective homomorphisms.
- b) The \( \Phi \)-operator \( \Phi : [\Gamma_7, k, 1] \to [\text{SL}_2(\mathbb{Z}), k, 1] \) is a surjective homomorphism.

Thus, all conditions that were needed to prove proposition 7.3.3 in [Mar04], where \( \Gamma_5 \) was considered, are met for \( \Gamma_7 \), too. Therefore, we cite that proposition (adapted to \( \Gamma_7 \)) and omit the proof since it would be literally the same.

(6.51) Proposition. Let \( k \in \mathbb{N}_0 \). Then the following holds true:

\[
\dim[\Gamma_7, k, 1] - \dim[\Gamma_7, k, 1]_0 = \begin{cases} 
\dim[\text{SL}_2(\mathbb{Z}), k, 1] + \dim[\text{SL}_2(\mathbb{Z}), k, 1]_0, & \text{if } k \text{ is even,} \\
0, & \text{if } k \text{ is odd.}
\end{cases}
\]

For \( k \geq 5 \) Ibukiyama determined a dimension formula for \( [\Gamma_7, k, 1]_0 \) in [Ib85]. Furthermore, there also exists an explicit dimension formula for \( [\text{SL}_2(\mathbb{Z}), k, 1] \) and \( [\text{SL}_2(\mathbb{Z}), k, 1]_0 \) (see [KK07, pp.173]). Therefore, we can determine the dimensions of \( [\Gamma_7, k, 1] \) and \( [\Gamma_7, k, 1]_0 \) for all \( k \geq 5 \). For the remaining cases there are other explanations below the table.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \dim[\Gamma_7, k, 1] )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>5</td>
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<tr>
<td>( \dim[\Gamma_7, k, 1]_0 )</td>
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<td>4</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>( \dim[\Gamma_7, k + 12, 1] )</td>
<td>14</td>
<td>11</td>
<td>17</td>
<td>16</td>
<td>27</td>
<td>24</td>
<td>36</td>
<td>33</td>
<td>48</td>
<td>45</td>
<td>63</td>
<td>60</td>
</tr>
<tr>
<td>( \dim[\Gamma_7, k + 12, 1]_0 )</td>
<td>11</td>
<td>11</td>
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<td>33</td>
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<td>60</td>
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</tr>
<tr>
<td>( \dim[\Gamma_7, k + 24, 1] )</td>
<td>83</td>
<td>76</td>
<td>100</td>
<td>97</td>
<td>128</td>
<td>121</td>
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<td>148</td>
<td>187</td>
<td>180</td>
<td>224</td>
<td>217</td>
</tr>
<tr>
<td>( \dim[\Gamma_7, k + 24, 1]_0 )</td>
<td>78</td>
<td>76</td>
<td>97</td>
<td>97</td>
<td>123</td>
<td>121</td>
<td>150</td>
<td>148</td>
<td>182</td>
<td>180</td>
<td>219</td>
<td>217</td>
</tr>
</tbody>
</table>

Note that the dimension for \( k = 3 \) and \( k = 4 \) are due to a recent result of Ibukiyama in [Ib07], where dimension formulas were determined for weights 3 and 4, too (and also confer [PY09, thm.3.1]). For \( k = 4 \), the cusp form is given by \( \psi_{3,2}^2 \) and a non-cusp form by \( F_4 \). Furthermore, there cannot exist a paramodular form of weight 1, or otherwise it would be impossible that the dimension goes down when passing from weight 2k to weight 2k + 1. And finally, there cannot exist a paramodular form of weight 2: Concerning that issue, confer [PY09], saying that according to a private communication to Poor, Ibukiyama has proven that there exist no
non-trivial paramodular cusp forms of weight 2 and level $p$ for primes $p \leq 23$.

Like mentioned before, we cannot go any further from this point. But after all we pointed out where the critical points are concerning a reduction process like we described it in the current chapter. It was shown how one might be able to determine the needed graded rings of modular forms (and certain modules regarding characters) once the structure of those for lower dimensions has been analyzed. And regarding quaternionic modular forms, we supposedly have all the needed tools at hand, meaning Eisenstein-series, Maass lifts, theta-series and Borcherds products with “good-natured” divisors. This was not done for the cases of lower dimension, since the general setting concerning these lower dimensional cases has already been worked out before: [Kl06] concerning the five-dimensional case of orthogonal modular forms, [De01] concerning the Hermitian case and [Mar04] regarding the paramodular setting (although we already mentioned the basics here). So actually, the next step in some future work would be to explicitly determine the graded ring of paramodular forms of level 7. From there, one can hopefully climb all the way back up to quaternionic modular forms utilizing the reduction process we described here – noting that of course there might occur several problems similar to those concerning paramodular forms of level 7, but nonetheless the basic approach has been worked out here, here.

6.4 A maximal set of algebraically independent forms for $\Gamma(\mathcal{O})$

In this final section we will determine a maximal set (and hopefully minimal concerning the weights) of algebraically independent quaternionic modular forms with respect to $\Gamma(\mathcal{O})$. Note that quaternionic modular forms $f_1, \ldots, f_n$ are algebraically independent if the only polynomial $p \in \mathbb{C}[X_1, \ldots, X_n]$ that is homogenous with respect to the weights of $f_1, \ldots, f_n$ and fulfills $p(f_1, \ldots, f_n) \equiv 0$ (i.e. the function $p(f_1, \ldots, f_n)$ vanishes identically) is the zero polynomial $p = 0$.

First, recall some facts: Let $g \in M_3^3(\mathbb{Z})$ be a generator of the one-dimensional space $M_3^3(\mathbb{Z})$. We defined $f_{7,i} = M_3^{(i)}(\mathbb{Z}) \in [\text{Sp}_2(\mathcal{O}), 7, v_i]$ and $f_{7,-i} = M_3^{(-i)}(\mathbb{Z}) \in [\text{Sp}_2(\mathcal{O}), 7, v_{-i}]$. Recall that these spaces are one-dimensional (see (3.36)). In virtue of (3.22) $f_{7,i} |_{k}(i^1 I) = -f_{7,-i}$ holds, and we have $f_{7,i}(Z') = -f_{7,i}(Z)$ and $f_{7,-i}(Z') = -f_{7,-i}(Z)$ in view of (3.20) and (3.21). The transformation behavior with respect to $\text{Sp}_2(\mathcal{O})$ can be read off the values of the multiplier systems (see (1.57)). So one easily verifies

$$F_{14} := f_{7,i} f_{7,-i} \in [\Gamma(\mathcal{O}), 14, 1].$$

(6.31)

We have seen that restricting quaternionic modular forms for $\Gamma(\mathcal{O})$ and the trivial character to $\{(\frac{z_1}{z_2}, \frac{z_3}{z_2}) \in \mathcal{H}(\mathbb{H}) : z = z_0 + z_2 i_2, \ z_0, z_2 \in \mathbb{C}\}$ yields symmetric Hermitian modular forms for $\mathcal{O}(i)$ with respect to the full Hermitian modular group $\text{Sp}_2(\mathcal{Z}[i])$ and the abelian character $\det^{k/2}$ (confer (5.5), (5.34) and as well (5.24), (5.27) and (5.32)). We abbreviate this restriction by writing “$|_{\mathcal{O}(i)}$”. Thus $[\Gamma(\mathcal{O}), k, 1]|_{\mathcal{O}(i)} \subset [\text{Sp}_2(\mathcal{Z}[i]), k, \det^{k/2}]$. In view of (3.26) we get

$$F_{14}|_{\mathcal{O}(i)} \equiv 0.$$  

(6.32)

On the other hand, we already mentioned before that computing some Fourier-coefficients
yields that $f_{7,i}|_{\mathcal{H}_{A_{2,1}}} \text{ and } f_{7,-i}|_{\mathcal{H}_{A_{2,1}}}$ do not vanish identically. Note that "$|_{\mathcal{H}_{A_{2,1}}}$" is an abuse of notation here, since $f_{7,i}$ and $f_{7,-i}$ are living on $\mathcal{H}($$\mathbb{H})$, not on $\mathcal{H}_{A_{2}^{(2)}}$. So actually, we mean the restriction to $\left\{ \left( \begin{smallmatrix} z_i \\ z_j \end{smallmatrix} \right) \in \mathcal{H}($$\mathbb{H}) \right\}; z = z_0 + z_1 \frac{1+i\sqrt{3}}{2} + z_2 i, z_0, z_1, z_2 \in \mathbb{C} \}$). Nevertheless, we will stick to this notation in order to not getting confused by introducing even more notation. And recall that we already mentioned $f_{7,i}|_{\mathcal{H}_{A_{2,1}}} = -f_{7,-i}|_{\mathcal{H}_{A_{2,1}}}$. Anyways, we obtain

$$F_{14}|_{\mathcal{H}_{A_{2,1}}} \not\equiv 0 \ . \quad (6.33)$$

Furthermore, deriving from (5.32) and (6.12) we define

$$F_{12} := \phi_{1,6}^2 \circ \phi_{H}^{-1} \in [\Gamma(\mathcal{O}), 12, 1] . \quad (6.34)$$

According to (6.12) $\psi_{1,6}$ vanishes along $\mathcal{H}_{A_{2,1}}$ (seen as a submanifold of $\mathcal{H}_{A_{2}^{(2)}}$). Hence we also get

$$F_{12}|_{\mathcal{H}_{A_{2,1}}} \equiv 0 , \quad F_{12}|_{Q(\mathbb{i})} \equiv 0 , \quad (6.35)$$

since $\mathcal{H}_{A_{2}^{(2)}}$ is a submanifold of $\mathcal{H}_{A_{2,1}}$ (see (5.5) and 5.24).

We are now going to show that $\{ E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14} \}$ is a set of algebraically independent quaternion modular forms with respect to $\Gamma(\mathcal{O})$. (Recall that $E_4 \in [\Gamma(\mathcal{O}), k, 1]$ holds for all $k \in 2\mathbb{N}, k \geq 4$ in virtue of (4.69).) Note that this set is a maximal set of algebraically independent forms: Quaternionic modular forms are functions in six complex variables and can be identified with orthogonal modular forms for $O(2, 6)$, so a result from algebraic geometry is that every set of more than seven forms is algebraically dependend (also confer [FH00] or [Kr85, ch.III, thm.2.12] – the proof would be completely analogous for our setting). Moreover, this set hopefully is minimal in the sense of involved weights. Concerning the quaternionic modular forms we already discovered (meaning Eisenstein-series, Maaß lifts, theta series and Borcherds products) it is minimal. But to actually prove that it is minimal concerning all (maybe yet unknown) quaternionic modular forms, one would need more information on the dimensions of the spaces of quaternionic modular forms of small weights. But the estimates coming from the trace (see (4.9)) are not good enough to give an answer in that direction. At least it seems plausible that the weights are minimal since they are already small compared to the results in [Kr05] concerning the Hurwitz order.

But before we get to prove the algebraic independency, we need a preparatory proposition concerning the Eisenstein-series that are involved. According to [DK03, cor.9] the graded ring of symmetric Hermitian modular forms for $Q(i)$ and character $\det^{k/2}$ (i.e. $\bigoplus_{k \in 2\mathbb{Z}}[\text{Sp}_2(\mathbb{Z}[i]), k, \det^{k/2}])$ is generated by

$$E_{Q(i),4}, E_{Q(i),6}, \phi_4^2, E_{Q(i),10}, E_{Q(i),12} ,$$

where $E_{Q(i),k}$ denotes the Hermitian Eisenstein-series of weight $k$ with respect to $Q(i)$, and $\phi_4$ is some Borcherds product (cf. [DK03]). Moreover, these five forms are algebraically independent. Note again that $f|_{Q(i)} \in [\text{Sp}_2(\mathbb{Z}[i]), k, \det^{k/2}]$ holds for all $f \in [\Gamma(\mathcal{O}), k, 1]$ (see above).
(6.52) Proposition. Let \( \phi_i \) denote the restriction of quaternionic modular forms to the Hermitian half-space like described above, hence such that one obtains Hermitian modular forms for \( \mathbb{Q}(i) \). Then the following identities hold:

\[
\begin{align*}
E_{Q(i),4} & = E_4|_{Q(i)}, \\
E_{Q(i),6} & = E_6|_{Q(i)}, \\
\phi_4^2 & = \frac{1}{57600} (13E_4^2|_{Q(i)} - 13E_8|_{Q(i)}), \\
E_{Q(i),10} & = \frac{1}{831} (11E_4|_{Q(i)}E_6|_{Q(i)} + 820E_{10}|_{Q(i)}), \\
E_{Q(i),12} & = \frac{1}{1012990319} (6191640E_4^2|_{Q(i)} - 1095250E_6^2|_{Q(i)} - 8123661E_4|_{Q(i)}E_8|_{Q(i)} + 101547590E_{12}|_{Q(i)}).
\end{align*}
\]

Proof: Since the graded ring of symmetric Hermitian modular forms for \( \mathbb{Q}(i) \) and character \( \det^{k/2} \) is generated by the forms above, the restrictions of the quaternionic Eisenstein-series have to be polynomials in these. In (4.68) we determined a formula for the Fourier-coefficients of the quaternionic Eisenstein-series. Hence we can also compute the Fourier-coefficients of their restrictions to the Hermitian half-space, like described in (1.73). Furthermore, there also exists a formula for the Fourier-coefficients of the Hermitian Eisenstein-series (cf. [Kr91]). And finally, the Borcherds product \( \phi_i \) is also a Hermitian Maass lift (cf. [De01, Satz 6.11]), hence its Fourier-expansion is easily computable (cf. [De01] for details).

So in other words, the Fourier-expansions of all involved forms are easily computable. Since we already know that a representation has to exist, if suffices to consider sufficiently enough Fourier-coefficients to decide how such a representation looks like – one simply has to solve some linear equation system. This was done for this thesis using SAGE, indeed, hence a computation yielded an expression (in form of a polynomial) of the restricted Eisenstein-series in terms of the generators of the graded ring. Of course, this also works the other way round, hence a computation yields the expressions above. The explicit calculations can be omitted here, of course.

(6.53) Corollary. \( [\Gamma(\mathcal{O}), k, 1]|_{Q(i)} = [\text{Sp}_2(\mathbb{Z}[i]), k, \det^{k/2}] \) holds for all \( k \in 2\mathbb{N}_0 \). And furthermore, \( \{E_4|_{Q(i)}, E_6|_{Q(i)}, E_8|_{Q(i)}, E_{10}|_{Q(i)}, E_{12}|_{Q(i)}\} \) is an algebraically independent set.

Proof: The assertions are obvious in the light of (6.52) and the generators of the graded ring of symmetric Hermitian modular forms for \( \mathbb{Q}(i) \) being algebraically independent, since every algebraic relation of the \( E_k|_{Q(i)} \) would automatically yield a relation for these generators.

So we finally come to the announced theorem:

(6.54) Theorem. \( \{E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}\} \) is a set of seven algebraically independent quaternionic modular forms with respect to \( \Gamma(\mathcal{O}) \) and the trivial character.

Proof: So suppose we have \( p \in \mathbb{C}[X_1, \ldots, X_7] \) (which is homogeneous regarding the weights) such that \( p(E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}) \equiv 0 \). We have to prove that \( p \) is the zero polynomial.

Let \( n_{12} \) be the maximal degree in \( X_6 \) (hence in \( F_{12} \)) and \( n_{14} \) the maximal degree in \( X_7 \) (hence in
6.4 A maximal set of algebraically independent forms for $\Gamma(\mathcal{O})$

First, we decompose $p$.

$$p(E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}) = p_E(E_4, E_6, E_8, E_{10}, E_{12}) + \sum_{j=1}^{n_{14}} F_{14j}^i (E_4, E_6, E_8, E_{10}, E_{12}) + \sum_{j=1}^{n_{12}} F_{12j}^i (E_4, E_6, E_8, E_{10}, E_{12}, F_{14}).$$

Of course, such a decomposition is always possible by sorting the monomials accordingly. Now, $p(E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}) \equiv 0$ yields that also the restriction to the Hermitian half-space vanishes identically, of course. And since $F_{12}|_{\mathcal{Q}(i)} = F_{14}|_{\mathcal{Q}(i)} \equiv 0$, this gives

$$0 \equiv p_E(E_4|_{\mathcal{Q}(i)}, E_6|_{\mathcal{Q}(i)}, E_8|_{\mathcal{Q}(i)}, E_{10}|_{\mathcal{Q}(i)}, E_{12}|_{\mathcal{Q}(i)}).$$

But then $p_E$ has to be the zero polynomial in view of (6.53), because the restrictions of the Eisenstein-series are algebraically independent, hence

$$p(E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}) = \sum_{j=1}^{n_{14}} F_{14j}^i (E_4, E_6, E_8, E_{10}, E_{12}) + \sum_{j=1}^{n_{12}} F_{12j}^i (E_4, E_6, E_8, E_{10}, E_{12}, F_{14}).$$

Next, we restrict to $\mathcal{H}_{A_{2,1}}$, and $F_{12}|_{\mathcal{H}_{A_{2,1}}} \equiv 0$ yields

$$0 \equiv \sum_{j=1}^{n_{14}} F_{14}^j|_{\mathcal{H}_{A_{2,1}}} p_{14j} (E_4|_{\mathcal{H}_{A_{2,1}}}, E_6|_{\mathcal{H}_{A_{2,1}}}, E_8|_{\mathcal{H}_{A_{2,1}}}, E_{10}|_{\mathcal{H}_{A_{2,1}}}, E_{12}|_{\mathcal{H}_{A_{2,1}}})$$

$$\Leftrightarrow 0 \equiv \sum_{j=1}^{n_{14}} F_{14}^{j-1}|_{\mathcal{H}_{A_{2,1}}} p_{14j} (E_4|_{\mathcal{H}_{A_{2,1}}}, E_6|_{\mathcal{H}_{A_{2,1}}}, E_8|_{\mathcal{H}_{A_{2,1}}}, E_{10}|_{\mathcal{H}_{A_{2,1}}}, E_{12}|_{\mathcal{H}_{A_{2,1}}}),$$

since $F_{14}|_{\mathcal{H}_{A_{2,1}}} \neq 0$. But then, restricting to the Hermitian half-space again implies that $p_{14,1}$ has to be the zero polynomial, just like we had for $p_E$. Hence we can even devide the equation above by $F_{14}^2|_{\mathcal{H}_{A_{2,1}}}$. Therefore, by induction we obtain that all $p_{14,j}$ have to be the zero polynomial, which leads to

$$0 \equiv p(E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}) = \sum_{j=1}^{n_{12}} F_{12j}^i (E_4, E_6, E_8, E_{10}, E_{12}, F_{14}).$$

$F_{14} \neq 0$ allows to devide this equation by $F_{14}$ and we obtain the initial situation from above – but the degree of the polynomial decreased by 14. Or in other words, $p_{12,1}$ has to be the zero polynomial due to the considerations above. So by another induction, all $p_{12,j}$ have to be the zero polynomial. Thus $p = 0$ and the assertion follows.

Note that all quaternionic modular forms in $\{E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}\}$ are constructed by Maaß lifts: In (4.68) we saw that the quaternionic Eisenstein-series $E_k$ are Maaß lifts of even weight and trivial character (for $k \geq 8$, while $E_4$ and $E_6$ were even initially defined as Maaß lifts). Furthermore, by definition $F_{14} = f_{ij} f_{j,-i}$ is the product of two Maaß lifts of odd weight for $v_i$ and $v_{-i}$, respectively. And finally, in (3.37) we already mentioned that $\psi_{1,6} \circ \varphi^{-1}_{\mathbb{H}}$ coincides (up to some constant factor) with the unique Maaß lift of weight six with respect to $v_{\det}$. This
is because the counterpart of this Maaß lift in the orthogonal setting has abelian character $\chi_{\pi_1}^S$, hence it vanishes along $H_{A_2}$ and we can divide by $\psi_{1,6}$ (cf. (5.24), (5.32) and (6.12)). Hence $F_{12}$ is the square of a Maaß lift of even weight for $\nu_{\det}$.

At this final point, one can make a conjecture. \{ $E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}$ \} is a set of seven algebraically independent quaternionic modular forms with respect to $\Gamma(O)$ and the trivial character. And it is quite reasonable that this set is also minimal with respect to the occurring weights – although we cannot prove it at this point. In [Kr05] Krieg proved that the graded ring of quaternionic modular forms for the Hurwitz order (with respect to the extended modular group) is generated by seven algebraically independent modular forms of weights 4, 6, 10, 12, 16, 18 and 24 (namely the quaternionic Eisenstein-series for the Hurwitz order of said weights). Therefore, since all aspects concerning the Hurwitz order and our order $O$ are so similar, one can conjecture that the situation is similar here, too – which means that the graded ring for our order might be isomorphic to $\mathbb{C}[X_1, \ldots, X_7]$, too, without any algebraic relations. And since we assume the set above to be minimal regarding the occurring weights, our graded ring has to be given by $\mathbb{C}[E_4, E_6, E_8, E_{10}, E_{12}, F_{12}, F_{14}]$. Of course, this is just a conjecture and cannot be answered as of yet. Nevertheless, it is not without any reason.
7 Appendix

Throughout this work, some results were omitted since they would not have yielded any further information at that point. So we make up for these omissions, now.

7.1 Transformation behavior of quaternionic theta-constants

In (2.16) we had: Let $\theta = (\theta_1, \ldots, \theta_{21})'$. There is a homomorphism $\Psi : \text{Sp}_2(O) \rightarrow \text{GL}_{21}(\mathbb{C})$ such that

$$\theta|_2 M := (\theta_1|_2 M, \ldots, \theta_{21}|_2 M)' = \Psi(M) \cdot \theta$$

holds for all $M \in \text{Sp}_2(O)$.

Note that we also determined the behavior under $Z \mapsto Z'$ and $Z \mapsto Z[i_1]$, hence we even obtain a homomorphism for $\Gamma(O)$. The explicit images of the generators of $\Gamma(O)$ under $\Psi$ are given as follows. Here, $\rho = e^{\frac{i\pi}{2}}$. Furthermore, we write permutation matrices as their image in the permutation group $S_{21}$ in order to keep it well-arranged. A cycle $(n_1, \ldots, n_m)$ denotes the permutation $n_j \rightarrow n_{j+1}, j \in \{1, \ldots, m-1\}$, and $n_m \rightarrow n_1$, and if $\pi_1$ and $\pi_2$ are two permutation cycles, then $\pi_1 \circ \pi_2$ denotes the composition $\pi_1 \circ \pi_2$.

$$\Psi(J) = \frac{1}{9}$$

$$\Psi(\text{Trans}(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) = \text{diag}(1,1,\rho,\overline{\rho},\rho,\rho,\rho,\overline{\rho},\rho,\rho,\rho^2,\overline{\rho}^2,\rho^2,\rho^2,\rho^2,\rho^2)$$

$$\Psi(\text{Trans}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) = \text{diag}(1,\rho,\rho^2,1,\overline{\rho},\rho,\rho,\rho,\rho,\rho^2,\overline{\rho}^2,\rho^2,\rho^2,\rho^2,\rho^2)$$

$$\Psi(\text{Trans}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) = \text{diag}(1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho^2,\rho^2,\rho^2,\rho^2)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,\rho,\rho^2,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,1,\rho)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,1,\rho)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,\rho)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,\rho)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,\rho)$$

$$\Psi(\text{Trans}(\frac{0}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqrt{3})) = \text{diag}(1,1,1,1,1,1,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,\rho,1,\rho)$$
\[ \Psi(\text{Rot}(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})) = (2,4)(3,13)(7,8)(9,14)(10,16)(11,17)(12,15)(19,20) \]
\[ \Psi(\text{Rot}(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})) = (2,5,6)(3,18,21)(7,16,15)(8,14,17)(9,19,11)(10,20,12) \]
\[ \Psi(\text{Rot}(\begin{smallmatrix} 1+i \cdot \sqrt{3} & 0 \\ 0 & 1 \end{smallmatrix})) = (5,6)(7,8)(9,12)(10,11)(14,15)(16,17)(18,21)(19,20) \]
\[ \Psi(\text{Rot}(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})) = (5,8,6,7)(9,10,12,11)(14,17,15,16)(18,19,21,20) \]
\[ \Psi(\tau) = \Psi(\text{Rot}(i_1 I)) = (7,8)(9,10)(11,12)(14,16)(15,17)(19,20) \]

Note that the matrix group spanned by these matrices has order 13063680 = 2^93^65^17^1, while omitting \( \Psi(\tau) \) (hence only considering \( \Psi(\text{Sp}_2(O)) \)) yields a subgroup of index 2. Again, this means that the group and the dimension are too large to be able to compute the ring of invariants, which would yield quaternionic modular forms with respect to \( \text{Sp}_2(O) \) or \( \Gamma(2) \) with trivial character. At least the Molien or Hilbert series are easily computable with \([\text{MAGMA}]\), but the computation on the primary and secondary invariants does not determine.

### 7.2 Invariants for \([\mathcal{G}, k, 1]\)

Let
\[ \mathcal{G} = \left\langle \text{SL}_2(\mathbb{Z})|3|, T_G := \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), I_G := \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right\rangle \]
and
\[ F_1(\tau_1, \tau_2) = E_{3,3}(\tau_1) \cdot E_{3,3}(\tau_2) , \quad F_2(\tau_1, \tau_2) = E_{1,3}(\tau_1) \cdot E_{1,3}(\tau_2) , \]
\[ F_3(\tau_1, \tau_2) = E_{1,3}(\tau_1) \cdot E_{1,3}(\tau_2) , \quad F_4(\tau_1, \tau_2) := E_{1,3}(\tau_1) \cdot E_{1,3}(\tau_2) . \]

Then we had the following result in (6.45): \( \bigoplus_{k \in \mathbb{N}} [\mathcal{G}, k, 1] \) is generated by some modular forms \( g_1, g_2, g_3 \) and \( g_4 \) of weights 1, 2, 3 and 4, respectively, where \( g_2 \) can be omitted due to the relation \( F_1 F_4 = F_2 F_3 \). \( \bigoplus_{k \in \mathbb{N}} [\mathcal{G}, k, 1] \) is generated by the same elements plus a further modular form \( g_6 \) of weight 6 (which is neither symmetric nor skew-symmetric, whereas \( g_6 \) could be replaced by the skew-symmetric form \( g_6(\tau_1, \tau_2) = g_6(\tau_1, \tau_2) - g_6(\tau_2, \tau_1) \)).

Like described in that situation, these forms are obtained by computing the invariant rings
\[ \left( \mathbb{C}[X_1, X_2, X_3, X_4]/(X_1 X_4 - X_2 X_3) \right)^H \]
for \( H \in \{ G, \hat{G} \} \), where \( G = \langle M_I, M_T \rangle \) and \( \hat{G} = \langle M_I, M_T, M_{\text{sym}} \rangle \). Like described in chapter 5 of [Mar04], one first has to compute the invariant ring with respect to \( \mathbb{C}[X_1, X_2, X_3, X_4] \) and then take the algebraic relations into account. Doing so, we obtain using \([\text{MAGMA}]\):

- \( g_1 = F_1 - F_2 - F_3 - \frac{1}{2} F_4 \)
- \( g_2 = F_1^2 - 2 F_1 F_2 - 2 F_1 F_3 + \frac{1}{2} F_1 F_4 + F_2^2 + \frac{1}{2} F_2 F_3 + F_2 F_4 + F_3^2 + \frac{1}{4} F_4^2 \)
- \( g_3 = F_1^3 + \frac{3}{2} F_1^2 F_2 + \frac{3}{2} F_1 F_3^2 - \frac{3}{2} F_1 F_4 + \frac{3}{2} F_2 F_3 - F_2^2 F_3 - \frac{3}{4} F_2 F_4 - \frac{3}{4} F_3 F_4 - \frac{3}{4} F_4^2 \)
- \( g_4 = F_1^3 + \frac{3}{2} F_1^2 F_2 + \frac{3}{2} F_1 F_3^2 - \frac{3}{2} F_1 F_4 + \frac{3}{2} F_2 F_3 - F_2^2 F_3 - \frac{3}{4} F_2 F_4 - \frac{3}{4} F_3 F_4 - \frac{3}{4} F_4^2 \)
- \( g_5 = F_1^3 + \frac{3}{2} F_1^2 F_2 + \frac{3}{2} F_1 F_3^2 - \frac{3}{2} F_1 F_4 + \frac{3}{2} F_2 F_3 - F_2^2 F_3 - \frac{3}{4} F_2 F_4 - \frac{3}{4} F_3 F_4 - \frac{3}{4} F_4^2 \)
- \( g_6 = F_1^3 + \frac{3}{2} F_1^2 F_2 + \frac{3}{2} F_1 F_3^2 - \frac{3}{2} F_1 F_4 + \frac{3}{2} F_2 F_3 - F_2^2 F_3 - \frac{3}{4} F_2 F_4 - \frac{3}{4} F_3 F_4 - \frac{3}{4} F_4^2 \)
7.2 Invariants for $[G,k,1]$

Here, we have the relation $g_2 - g_1^2 = \frac{1}{2}(F_1F_4 - F_2F_3) \equiv 0$, hence $g_2$ can be omitted. But according to the proceedings in chapter 5 of [Mar04] one can compute that $g_1$, $g_3$ and $g_4$ are algebraically independent. Furthermore, we have the following unique relation concerning the secondary invariant $g_6$ (meaning that $G_{k \in \mathbb{N}_0}, [G,k,1]$ is isomorphic to $\mathbb{C}[g_1,g_3,g_4,g_6]$ modulo the ideal generated by this single polynomial):

$0 \equiv -564457g_1^{12} + 9548740g_1^9g_3 - 15649389g_1^6g_4 - 25449225g_1^3g_5 + 158837490g_1^2g_6 - 148053582g_1g_7 + 121252250g_8 - 214962525g_9 + 1050853500g_{10} + 19223750g_{11} - 723921300g_{12} + (-5280912g_1^6 + 47941920g_1^3g_3 - 71292312g_1^2g_4 + 51409800g_1)g_6 - 11389248g_7^2$
Bibliography


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