Minimal immersions in Finsler spaces

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Zusammenfassung

Die vorliegende Arbeit verallgemeinert Resultate von [ST03], [SST04] und [CS09] bezüglich Finslerminimalflächen in Finslermannigfaltigkeiten, welche mit dem Busemann-Hausdorff-Volumen versehen sind, wovon sich der Finslersche Flächeninhalt ableiten lässt. Es werden eine Finslermetrik F als auf $\mathbb{R}^{m+1}$ definiert angenommen und Bedingungen bezüglich der m-Symmetrisierung von F formuliert, die die Elliptizität des Finslerschen Flächeninhalts garantieren. In [ST03] und [SST04] wurden spezielle Randersräume betrachtet und in [CS09] Finslermannigfaltigkeiten mit sogenannten (α, β)-Metriken, welche Randersmetriken als Spezialfall umfassen. Verallgemeinert werden in der vorliegenden Arbeit die Sätze vom Bernsteintyp, welche in [SST04] und [CS09] bewiesen wurden, allerdings umfasst diese Arbeit auch etliche neue Ergebnisse. Darunter unter anderem Existenz- und Regularitätssätze für Minimierer, Hebbarkeit von Singularitäten, Einschlusssätze, isoperimetrische Ungleichungen und Krümmungsabschätzungen für Finslerminimalflächen.

Abstract

The present thesis generalizes results established in [ST03], [SST04] and [CS09] on Finsler-minimal hypersurfaces in Finsler manifolds equipped with the Busemann-Hausdorff volume wherefrom the Finsler area derives. Therefore, a Finsler structure F on $\mathbb{R}^{m+1}$ is assumed and some conditions involving the m-symmetrization $F_{(m)}$ of F are formulated to guarantee the ellipticity of the Finsler area. The present thesis generalizes especially Bernstein-type theorems of [SST04] and [CS09] beyond the class of (α, β)-metrics. There are even many new results some of which with no direct counterpart in the aforementioned references. These results include existence and regularity of minimizers, removability of singularities, enclosure theorems, isoperimetric inequalities and curvature estimates for Finsler-minimal hypersurfaces.

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Contents

Introduction

The present thesis covers an investigation of Finsler-minimal immersions into Finsler manifolds and their properties. Finsler-minimal immersions generalize minimal immersions into Euclidean space, i.e., Finsler-minimal immersions are critical immersions to the Finsler area functional. In other words, the first variation of such an immersion w.r.t. Finsler area vanishes. Finsler area in this context means the Busemann-Hausdorff definition of area in a Finsler manifold (see [Bus47]). There were only very few results known regarding Finsler-minimal immersions. Especially, Souza, Spruck and Tenenblat [SST04] established results for Finsler-minimal graphs in a Randers-Minkowski space, i.e. a Minkowski space where the Finsler metric is a Randers metric. A Randers metric is a Riemannian metric plus an additive linear perturbation term. In [SST04], some results regarding (local) uniqueness, a Bernstein-type theorem in dimension 3, and removability of singularities are given for such graphs. To the author's knowledge, most of the formerly established results including [SST04] were restricted to graphs and target Finsler manifolds of a very specific structure and there were essentially no results regarding existence, isoperimetric inequalities or curvature estimates for Finsler-minimal immersions.

In the beginning, we name some of the basic notions necessary to speak of Finsler-minimal immersions. The Busemann-Hausdorff volume, an extension of the notion of volume on Riemannian manifolds, is one of two main choices of volume on Finsler manifolds. Here, the most influential work was that of Busemann [Bus47] from 1947. Therein, he not only defined this Finsler volume, but also showed that it coincides on reversible Finsler manifolds with the Hausdorff measure induced by the Finsler structure. This property gives it a basic geometric interpretation and makes it a natural choice of volume.[1] Most of the founding definitions and basic results on that type of Finsler volume are due to Busemann. Busemann especially derived from his volume a *Finsler area* for immersions, which leads to Finsler-minimal immersions in Finsler manifolds (see [She98]). Shen gave in [She98] a Finsler version of mean curvature, stemming from the Euler-Lagrange equation and the first variation of the Finsler area. Then, Finsler-minimal immersions are immersions of vanishing Finsler mean curvature. So, Finsler-minimal immersions can be characterized by a differential equation and they are critical points of Finsler area. Further investigations on the topic have been carried out by [ST03], [SST04], where these characterizing differential equations are computed for Finsler-minimal graphs in a Randers-Minkowski space setting, i.e. in a Minkowski space with Finsler metric $F = \alpha + \beta$ with a Riemannian metric α and a linear 1-form β. These differential equations are identified as of *mean curvature type* (cf. [GT01]) and being elliptic as such up to a threshold $1/\sqrt{3}$ on the Riemannian norm of the linear perturbation term β. An example in [SST04] shows even more that this threshold is sharp. It is a cone, which is a solution to the characterizing differential equations of Finsler-minimal graphs for β of norm equal to

[1] Notice that the alternative Holmes-Thompson volume (see [ÁPB06]) leads to a different notion of Finsler-minimal immersions that we do not address here.

$1/\sqrt{3}$. For elliptic differential equations of mean curvature type Jenkins and Serrin [JS63] as well as Simon [Sim77a] proved Bernstein-type theorems, which where then applied by Souza et al. to the Finsler-minimal graphs as the solution of such an equation. Thereby, [SST04] obtained, among other results, Bernstein-type theorems for critical graphs to Finsler area in Randers-Minkowski space. Cui and Shen [CS09] applied this method to more general (α, β)-metrics (see Definition 1.4.13). Again, they classify the characterizing differential equation of critical graphs of Finsler area as being elliptic and of mean curvature type. Notice that a Randers metric is a special type of (α, β)-metric. The purpose of this thesis is to investigate minimizers and critical immersions of Busemann's Finsler area, generalize results established in [SST04] and [CS09] as well as present some entirely new results not restricted to graphs. In contrast to those sources, we use a variational approach as the problem naturally arises by a variation of Finsler area.

The drawback of the approach in [SST04] and [CS09] is that the characterizing differential equations are highly non-linear in nature. This is the reason why their coefficients can only be computed in an involved manner. Therefore, classifying the differential equations as elliptic becomes also quite complicated, which restricts their approach to very specific Finsler spaces. So, the main idea of the present thesis is to choose a variational approach. In fact, we apply the theory of so called *parametric* or *Cartan functionals* to Finsler area. The Finsler area, reinterpreted as an elliptic Cartan functional, leads, for instance, automatically to a characterization of Finsler-minimal graphs by elliptic differential equations of mean curvature type but allows also to treat parametric surfaces beyond graphs. Notice that Cartan functionals are defined by their invariance under reparametrization of the immersion under consideration. Assume a smooth immersion $X : \mathscr{M} \to \mathbb{R}^n$ with an oriented smooth m-manifold $\mathscr{M}$ and a Lagrangian $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$, $N = \binom{n}{m}$, and define the m-form

$$i \quad := \quad I(X, \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m})\mathrm{d}u$$

in local coordinates $u = (u^\alpha)_{\alpha=1}^m$ on $\mathscr{M}$. This is globally well-defined, since we assume

$$I(x, tz) \quad = \quad tI(x, z)$$

for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^N$ and $t > 0$. Then,

$$\mathcal{I}(X) := \int_{\mathscr{M}} i$$

is called *parametric functional* or *Cartan functional*. The most commonly known example is the Euclidean area functional $\mathcal{A}(X)$ with the Euclidean area integrand $A(z) := |z|$ as a choice of $I(x, z)$, where $|\cdot|$ is the Euclidean metric. There has been established a wide range of definitions and results regarding minimizers and critical immersions of Cartan functionals. Namely, there are existence and regularity results (cf. [HvdM03b], [HvdM03c] and [Whi91]), results regarding removability of singularities ([HS09] and [Sim77b]), enclosure theorems (cf. [BF09] and [Cla02]), isoperimetric inequalities (cf. [CvdM02] and [Win02]), curvature estimates and Bernstein-type theorems (cf. [Sim77c] and [Win07]). Most of these results assume $n = m + 1$ as a choice of dimension. So, the Finsler area $\mathcal{A}^F(X)$ of a smooth immersion X from an m-manifold into $\mathbb{R}^{m+1}$ equipped with a Finsler metric F is the main scope of our investigation and is identified as a Cartan functional. Most results regarding Cartan functionals assume that the Lagrangian I is positive for non-zero second argument and *elliptic*, i.e. for all $R > 0$, there is a constant $\Lambda = \Lambda(R) > 0$ such that

$$I(x, \cdot) - \Lambda|\cdot| \quad \text{is convex}$$

for all $x \in \mathbb{R}^{m+1}$ with $|x| \leq R$. Hence, we compute the respective Lagrangian $A^F = A^F(x, z)$ for the Finsler area (see Proposition 2.1.9) and then establish assumptions on F to ensure that the Finsler area integrand A^F is indeed elliptic. There are two valuable new observations useful to achieve this goal. First, the Finsler area is symmetrizing in the sense that F and the m-symmetrization $F_{(m)}$ generate the same Finsler area, i.e. $A^F = A^{F_{(m)}}$, where

$$F_{(m)}(x,y) \quad := \quad 2^{\frac{1}{m}}(F^{-m}(x,y)+F^{-m}(x,-y))^{-\frac{1}{m}}$$

for all $(x,y) \in T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$ (see Definition 2.1.10 and Corollary 2.1.13). $F_{(m)}$ is indeed the unique reversible real-valued function on $T\mathbb{R}^{m+1}$ with that property (see Theorem 2.3.2). Second, the Finsler area integrand can be represented in terms of the so called *spherical Radon transform* $\mathcal{R}$, namely

$$A^F(x,z) \quad = \quad \frac{1}{\mathcal{R}(F^{-m}(x,\cdot))(z)}$$

for all $(x,z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$ (see Corollary 2.3.1). The spherical Radon transform is a mapping from the space of continuous functions on $\mathbb{R}^{m+1}\backslash\{0\}$ to the space of (-1)-homogeneous functions on $\mathbb{R}^{m+1}\backslash\{0\}$ (see Definition 2.2.1). The theory of the spherical Radon transform presented here relies mostly on [Hel00] and [Gro96] as well as partially on [BEGM03]. An overview to the topic is given in [Gar94]. Of special use is the continuity property of $\mathcal{R}$ and its inverse $\mathcal{T}$ if we restrict the domain of $\mathcal{R}$ to the Fréchet space of smooth even $(-m)$-homogeneous functions on $\mathbb{R}^{m+1}\backslash\{0\}$ (cf. Theorem 2.2.20 and Theorem 2.2.28). Using the representation in terms of the spherical Radon transform as well as the spherical Radon transforms' inverse, we construct for reversible Finsler metrics F, G on $\mathbb{R}^{m+1}$ and a given $x \in \mathbb{R}^{m+1}$ a reversible 1-homogeneous function $F_{x,\delta}(\cdot)$ such that

$$A^F(x,\cdot) - \delta A^G(x,\cdot) \quad = \quad A^{F_{x,\delta}}(\cdot)$$

for $\delta \le \delta_0$ with $\delta_0 = \delta_0(x,F,G) > 0$ sufficiently small. We show for δ_0 eventually chosen even smaller that there is an integer $l \ge 2$ such that

$$\begin{aligned} \widehat{\rho}_2(F(x,\cdot) - F_{x,\delta}(\cdot)) \quad &\le \quad C\widehat{\rho}_l(A^F(x,\cdot) - A^{F_{x,\delta}}(\cdot)) \\ &\le \quad \delta C\widehat{\rho}_l(A^G(x,\cdot)), \end{aligned}$$

where we exploit the continuity of the spherical Radon transform as an operator in terms of the seminorms $\widehat{\rho}_j(f) := \max\{|\mathrm{D}^\alpha f(\zeta)| : \zeta \in \mathbb{S}^m, \ |\alpha| \le j\}$, for $j \in \mathbb{N}_0$. Hence if we choose δ small enough in function of F, G and x, $F_{x,\delta}$ is close enough to $F(x,\cdot)$ to be positive away from zero and convex, i.e. satisfies **(F2)**. Consequently, Busemann's convexity theorem (cf. [Bus49]) implies for fixed $x \in \mathbb{R}^{m+1}$ that $A^{F_{x,\delta}}$ is convex in the second argument. This implies the ellipticity comparison in Lemma 2.3.7 since $A^F(x,\cdot) - \delta A^G(x,\cdot)$ is then convex. Ultimately this leads in combination with the observation regarding $F_{(m)}$ to Corollary 2.3.8, which states essentially the following new result:

If $(\mathbb{R}^{m+1}, F)$ and $(\mathbb{R}^{m+1}, F_{(m)})$ are both Finsler manifolds, then is A^F an elliptic Cartan integrand.

We call the geometric assumption that $(\mathbb{R}^{m+1}, F)$ and $(\mathbb{R}^{m+1}, F_{(m)})$ are both Finsler manifolds **(GA3)**. So, it is of special interest to know when $F_{(m)}$ is a Finsler metric. A sufficient condition on F is

$$||\mathrm{d}_y F_a||_{\widehat{F_s}}\big|_{(x,y)} \quad < \quad \frac{1}{m+1} \quad \text{and} \quad F_a(x,y)(F_a)_{yy}|_{(x,y)}(\xi,\xi) \quad \le \quad 0$$

for all $x \in \mathbb{R}^{m+1}$ and $y, \xi \in \mathbb{R}^{m+1}\backslash\{0\}$, if we split $F = F_s + F_a$ in an even part F_s and an odd part F_a with respect to the second argument (see Theorem 2.4.13). These conditions are exemplarily verified for the Randers, two order and Matsumoto metric, which are the three

(α, β)-metrics presented in [CS09]. We even reproduce their sharp bounds on the Riemannian norm of β. For the Randers, two order and Matsumoto metric this is done in Example 2.4.3, Example 2.4.4 and Example 2.4.5 by computing directly $F_{(m)}$. For a Minkowski metric with linear perturbation term this is done in Example 2.4.16 and again for the two order metric in Example 2.4.17 via the sufficient condition. The sufficient condition can not be applied to the Matsumoto metric, as shown in Example 2.4.18), which were shown in [CS09] to be sharp.

Ellipticity of the Finsler area integrand allows us to apply the aforementioned results on Cartan functionals to Finsler area and we thereby get existence, regularity and removability results, enclosure theorems, isoperimetric inequalities, curvature estimates and Bernstein-type theorems for Finsler-minimal immersions or Finsler area minimizers. Most of these results seem to be completely new except for the removability results and the Bernstein-type theorems some of which have counterparts in some very constrained Finsler settings.

The existence results encompass existence for minimizers of $\mathcal{A}^F$ for a given boundary configuration with suitable properties in a Sobolev setting (see Corollary 3.2.1) and in a smooth setting (see Theorem 3.2.4 and Corollary 3.2.6), both for the choice $m = 2$. Notice that finding a minimizer of some area functional spanning a given boundary configuration is generally referred to as *Plateau problem*. Higher regularity for minimizers in the sense of Corollary 3.2.1 can be achieved if the Finsler metric F is close enough to the Euclidean metric $|\cdot|$ up to second order derivatives. This exploits the fact that the Finsler area integrand as a Cartan integrand then possesses a perfect dominance function by Corollary 1.6.21. These existence results in a Sobolev setting have already been published jointly by Heiko von der Mosel and the author in [OvdM13]. Theorem 3.2.4 gives existence of embedded Finsler-minimal surfaces for $m = 2$ and a given boundary curve Γ in a smooth setting, where F satisfies (**GA3**) and some additional assumptions on F and Γ are set.

Removability results can be found in Theorem 3.2.9 and Corollary 3.2.8, of which the latter is derived from an energy estimate in Theorem 3.2.7. These removability results are all carried out in a Minkowski setting and generalize a removability result regarding isolated singularities already shown in [SST04] for $m = 2$ in a Randers-Minkowski setting.

The enclosure theorems include a convex hull property (see Theorem 3.3.1) and an enclosure result for Finsler admissible domains (see Theorem 3.3.2), both in a Minkowski setting. The convex hull property basically states that the image of a Finsler-minimal immersion is contained in the convex hull of the image of the boundary of the immersion's domain. Theorem 3.3.2 essentially states that if the boundary is contained in a Finsler admissible domain so is the whole Finsler-minimal hypersurface.

Isoperimetric inequalities are given in Theorem 3.4.1 for the minimizers of Corollary 3.2.1 and in Theorem 3.4.3 for Finsler-minimal immersions in a Minkowski setting. Notice that Theorem 3.4.3 imposes no conditions on $F_{(m)}$ for most of its statements.

Curvature estimates for locally Finsler area minimizing C^2-hypersurfaces and Finsler-minimal graphs in a Minkowski space $(\mathbb{R}^{m+1}, F)$ are given in Theorem 3.5.12 assuming $m \in \{2, 3\}$ and (**GA3**) as well as in Theorem 3.5.13 assuming $m \leq 7$ and the even stronger condition that the Finsler metric F in question is close enough to the Euclidean metric up to third order derivatives. Thereby, we can directly derive Bernstein theorems for entire Finsler-minimal graphs in $\mathbb{R}^{m+1}$ (see Theorem 3.5.15 and Corollary 3.5.17). In Theorem 3.5.14, a Bernstein-type theorem for arbitrary $m \geq 2$ independent of the former curvature estimates is then presented. It states that an entire Finsler-minimal graph subject to a certain growth condition in a Minkowski space $(\mathbb{R}^{m+1}, F)$ with F close enough to the Euclidean metric up to third order derivatives, is a plane. These Bernstein results generalize Bernstein-type theorems established in [SST04] for $m = 2$ in a Randers-Minkowski space setting and in [CS09] for some Minkowski spaces with (α, β)-metric and $m \leq 7$.

The thesis is structured into three chapters. Chapter 1 contains basic definitions and notions as well as basic results regarding linear algebra, real analysis, functional analysis, manifolds, Finsler metrics, immersions and Cartan functionals. These results are grouped in sections according to the respective topics and in particular the section on Cartan functionals goes deep into detail and hence contains several subsections. Chapter 2 defines in detail the Finsler area similar to [She98] and in accordance with [Bus47]. This is part of section 2.1, where will be also presented the identification of the Finsler area integrand as a Cartan integrand. In section 2.2, we present the spherical Radon transform in a way adapted to our needs. Continuity properties as an operator and behaviour with respect to differentiation are investigated which will prove useful in the proof of ellipticity of the Finsler area integrand. In section 2.3, we relate the Finsler area integrand to the spherical Radon transform and show ellipticity of the Finsler area integrand under suitable assumptions on F, i.e. that the m-symmetrization $F_{(m)}$ is a Finsler metric on $\mathbb{R}^{m+1}$. Therefore, the relation of the Finsler area integrand to the spherical Radon transform are exploited as well as the latter's specific properties shown in section 2.2. In section 2.4, we investigate the ellipticity of $F_{(m)}$ for some special (α, β)-metrics, among others. We establish a sufficient condition and discuss its merits and drawbacks again exemplarily on the previously mentioned (α, β)-metrics. Section 2.5 introduces a Finsler mean curvature and relates it to similar notions for Cartan integrands. Chapter 3 collects the results attained for the Finsler-minimal immersions and Finsler area minimizers resulting from theory on Cartan functionals. Existence and regularity results, enclosure theorems, isoperimetric inequalities, curvature estimates and Bernstein-type theorems can be found in separate sections of chapter 3.

Preliminaries of linear Algebra, Analysis and Finsler manifolds

In this chapter, we aim to give some basics on linear algebra, real analysis, measure theory, functional analysis, manifolds, Finsler geometry, immersions as well as parametric or Cartan functionals. The selection of these topics is driven by the needs on the application side in the following chapters. The respective sections will by no means be exhaustive, but present most of the background needed.

1.1 Linear Algebra

This section contains basic definitions and theorems of matrix calculus and linear algebra. All results presented here are either standard theory or quite easily derived from it by direct calculations. In our presentation of the topic, we mostly rely on the books of Jänich [Jän94] and [Jän01] as well as Lee [Lee03]. Only the part on projections on finite dimensional vector spaces is non-standard in its presentation, even though it also involves only straight forward computations. We chose this approach due to needs on application side. Nevertheless, for a general overview on projections, we refer to a book of Meyer (see [Mey00, Section 5.13]).

1.1.1 Matrices

Some words to clarify the notation. $\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$ is the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of non-negative integer numbers, $\mathbb{Z}$ is the set of integer numbers and $\mathbb{R}$ is the set of real numbers. For $k, l \in \mathbb{N}$ let $M = (M^i_j) = (M_{ij}) \in \mathbb{R}^{k\times l}$ be a matrix with elements $M^i_j = M_{ij}$, row-index i and column-index j. $M^{(i)}$ is the matrix obtained by dropping the i-th row of M and $M_{(j)}$ is the matrix obtained by dropping the j-th column of M. For $s \in \mathbb{N}$, $1 \le s \le k$ and $1 \le i_1 < \ldots < i_s \le k$, the matrix $M^{i_1\ldots i_s} \in \mathbb{R}^{k\times s}$ is the matrix formed by the rows $i_1 \ldots i_s$ of M. $M_{j_1\ldots j_t}$ is defined analogously w.r.t. the columns of M. Sometimes these notations will be combined. By δ^i_j, δ_{ij} or δ^{ij} we denote the *Kronecker delta* defined by

$$\delta^i_j \quad := \quad \delta_{ij} \quad := \quad \delta^{ij} \quad := \quad \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j \in \mathbb{N}$. The *identity matrix* $\mathrm{Id} \in \mathbb{R}^{k\times k}$ is defined componentwise by

$$\mathrm{Id}^i_j \quad := \quad \delta^i_j.$$

The components of the product $A \cdot B$ of two matrices $A \in \mathbb{R}^{k\times l}$ and $B \in \mathbb{R}^{l\times m}$ are given by

$$(A \cdot B)^i_j = \sum_{s=1}^{l} A^i_s B^s_j$$

for $i, j \in \mathbb{N}$, $1 \leq i \leq k$ and $1 \leq j \leq m$. By $M^T \in \mathbb{R}^{l\times k}$ we denote the *transpose* of the matrix $M \in \mathbb{R}^{k\times l}$, i.e. $(M^T)^j_i = M^i_j$ for $i, j \in \mathbb{N}$, $1 \leq i \leq k$ and $1 \leq j \leq l$. M^{-1} is the *inverse* of a matrix M. We write shortly M^{-T} if we refer to the inverse of the transpose of the matrix M. M is said to be *symmetric* if M equals its transpose M^T. For a quadratic matrix M, i.e. k=l, the *determinant* will be denoted by $\det M$. We denote by $\mathrm{GL}(n)$ the set of matrices $M \in \mathbb{R}^{n\times n}$ of non-zero determinant $\det M \neq 0$ and by $\mathrm{SL}(n)$ the set of matrices $M \in \mathbb{R}^{n\times n}$ of determinant $\det M = 1$. The *trace* of a matrix $M \in \mathbb{R}^{k\times k}$ is the sum of all the diagonal elements of M, $\mathrm{trace}(M) = \sum_{i=1}^{k} M^i_i$. A matrix $M = (M_{ij}) \in \mathbb{R}^{n\times n}$ is said to be *positive semidefinite* if

$$\xi^T M \xi = \sum_{i,j=1}^{n} \xi^i M_{ij} \xi^j \geq 0$$

for all $\xi = (\xi^i) \in \mathbb{R}^n$. M is said to be *positive definite* if

$$\xi^T M \xi = \sum_{i,j=1}^{n} \xi^i M_{ij} \xi^j > 0$$

for all $\xi = (\xi^i) \in \mathbb{R}^n \backslash \{0\}$. *Negative semidefinite* and *negative definite* are defined in an analogous way up to the inequalities in the opposite direction in the defining relation. We will use the Einstein summation convention, which means that we sum over repeated indices if the indices are not set explicitly.

Theorem 1.1.1 (Leibniz formula for determinats [Jän94, p. 116]**).** *Let $M = (M_{ij}) \in \mathbb{R}^{k\times k}$ be a quadratic matrix. Then the determinant of M is given by*

$$\det M = \sum_{\sigma \in S_k} \mathrm{sgn}\sigma \prod_{i=1}^{n} M_{\sigma(i)i}, \tag{1.1.1}$$

where S_k is the set of all permutations *of $1, \ldots, k$, i.e. the set of all bijections on the set $\{1, \ldots, k\}$, and* $\mathrm{sgn}\sigma$ *is the sign of the permutation σ.*

Theorem 1.1.2 (Laplace expansion of determinants [Jän94, pp. 107, 110]**).** *Let $M = (M^i_j) \in \mathbb{R}^{k\times k}$ be a matrix. Then*

$$\begin{aligned}\det M &= \sum_{j,s=1}^{k} (-1)^{s+j} M^s_j \det M^{(s)}_{(j)} \\ &= \sum_{i,s=1}^{k} (-1)^{i+s} M^i_s \det M^{(i)}_{(s)}.\end{aligned}$$

for $i, j \in \mathbb{N}$, $1 \leq i, j \leq k$.

Theorem 1.1.3 (Inverse matrix [Jän94, p. 110]**).** *Let $M = (M^i_j) \in \mathbb{R}^{k\times k}$ with $\det M \neq 0$. Then*

$$(M^{-1})^i_j = \frac{(-1)^{i+j} \det M^{(j)}_{(i)}}{\det M}$$

for $i, j \in \mathbb{N}$, $1 \leq i, j \leq k$.

The following theorem is a consequence of the Laplace expansion Theorem 1.1.2.

Theorem 1.1.4 (Derivative of the determinant). *Let* $M = (M^i_j) \in \mathbb{R}^{k\times k}$. *Then*

$$\frac{\partial}{\partial M^i_j} \det M = (-1)^{i+j} \det M^{(i)}_{(j)} \tag{1.1.2}$$

for $i, j \in \mathbb{N}$, $1 \leq i, j \leq k$. *Especially if* $\det M \neq 0$, (1.1.2) *can be written as*

$$\frac{\partial}{\partial M^i_j} \det M = \det M (M^{-1})^j_i \tag{1.1.3}$$

for $i, j \in \mathbb{N}$, $1 \leq i, j \leq k$.

Proof. By using Theorem 1.1.2 and Theorem 1.1.3 we get

$$\begin{aligned}
\frac{\partial}{\partial M^i_j} \det M &= \frac{\partial}{\partial M^i_j} \sum_{s=1}^{k} (-1)^{i+s} M^i_s \det M^{(i)}_{(s)} \\
&= \sum_{s=1}^{k} (-1)^{i+s} \delta_{sj} \det M^{(i)}_{(s)} \\
&= (-1)^{i+j} \det M^{(i)}_{(j)} \\
&= \det M \frac{(-1)^{i+j} \det M^{(i)}_{(j)}}{\det M} \\
&= \det M (M^{-1})^j_i .
\end{aligned}$$

□

The following theorem on the componentwise derivative of the inverse of a quadratic matrix is formulated using the aforementioned Einstein summation convention.

Theorem 1.1.5 (Derivative of the inverse). *Let* $M = (M^i_j) \in \mathbb{R}^{k\times k}$ *with* $\det M \neq 0$ *and* $M^{-1} = (N^i_j) \in \mathbb{R}^{k\times k}$. *Then*

$$\frac{\partial N^s_t}{\partial M^i_j} = -N^s_q \frac{\partial M^q_r}{\partial M^i_j} N^r_t = -N^s_i N^j_t$$

for $i, j, s, t \in \mathbb{N}$, $1 \leq i, j, s, t \leq k$ *or in matrix notation*

$$\frac{\partial M^{-1}}{\partial M^i_j} = -M^{-1} \frac{\partial M}{\partial M^i_j} M^{-1},$$

where the partial derivative $\frac{\partial}{\partial M^i_j}$ *is applied componentwise.*

Proof. The matrix M and the inverse matrix M^{-1} obey the following relation

$$M^q_r N^r_t = \delta^s_t \tag{1.1.4}$$

for $q, t \in \mathbb{N}$, $1 \leq q, t \leq k$. If we differentiate this relation, we get

$$\frac{\partial M^q_r}{\partial M^i_j} N^r_t + M^q_r \frac{\partial N^r_t}{\partial M^i_j} = 0$$

and therefore

$$M_r^q \frac{\partial N_t^r}{\partial M_j^i} \quad = \quad -\frac{\partial M_r^q}{\partial M_j^i} N_t^r .$$

If we multiply with N_q^s for $1 \leq s, q \leq k$ and contract index q, we get

$$\frac{\partial N_t^s}{\partial M_j^i} \quad = \quad -N_q^s \frac{\partial M_r^q}{\partial M_j^i} N_t^r \quad = \quad -N_i^s N_t^j .$$

□

Theorem 1.1.6 (**Binet-Cauchy formula** [EG92, Section 3.2., Theorem 4])**.** *Let $M \in \mathbb{R}^{k\times l}$ and $N \in \mathbb{R}^{l\times k}$ be matrices with $k, l \in \mathbb{N}$, $l \geq k$. Then*

$$\det MN \quad = \sum_{1\leq i_1<\ldots<i_k\leq l} \det M_{i_1\ldots i_k} \det N^{i_1\ldots i_k} .$$

The following theorem can be verified using Theorem 1.1.2.

Theorem 1.1.7 (**Trace and determinant**)**.** *Let $L \in \mathbb{R}^{n\times n}$ and $\xi := (\xi_1|\ldots|\xi_n) \in \mathbb{R}^{n\times n}$ be matrices. Then*

$$\operatorname{trace}(L)\det(\xi_1, \cdots, \xi_n) \quad = \quad \sum_{i=1}^n \det(\xi_1, \cdots \xi_{i-1}, L\xi_i, \xi_{i+1} \cdots, \xi_n) \tag{1.1.5}$$

Proof. With Theorem 1.1.2 we get

$$\begin{aligned}
&\sum_{i=1}^n \det(\xi_1, \cdots \xi_{i-1}, L\xi_i, \xi_{i+1} \cdots, \xi_n) \\
\overset{\text{Thm. 1.1.2}}{=} &\sum_{i=1}^n \sum_{j=1}^n (-1)^{i+j} (L\xi_i)^j \det(\xi_1, \cdots \xi_{i-1}, \xi_{i+1} \cdots, \xi_n)^j \\
= \quad &\sum_{i=1}^n \sum_{j=1}^n (-1)^{i+j} \sum_{k=1}^n L_k^j \xi_i^k \det(\xi_1, \cdots \xi_{i-1}, \xi_{i+1} \cdots, \xi_n)^j \\
= \quad &\sum_{j=1}^n \sum_{k=1}^n (-1)^{k+j} L_k^j \sum_{i=1}^n (-1)^{i+k} \xi_i^k \det(\xi_1, \cdots \xi_{i-1}, \xi_{i+1} \cdots, \xi_n)^j \\
\overset{\text{Thm. 1.1.2}}{=} &\sum_{j=1}^n \sum_{k=1}^n (-1)^{k+j} L_k^j \det \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{j-1} \\ \xi^k \\ \xi^{j+1} \\ \vdots \\ \xi^n \end{pmatrix} \\
= \quad &\sum_{j=1}^n \sum_{k=1}^n (-1)^{k+j} L_k^j \delta_j^k \det \xi \\
= \quad &\sum_{j=1}^n L_j^j \det \xi \\
= \quad &\operatorname{trace}(L) \det \xi ,
\end{aligned}$$

where we used the identity

$$\det \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{j-1} \\ \xi^k \\ \xi^{j+1} \\ \vdots \\ \xi^n \end{pmatrix} = \delta_j^k \det \xi.$$

Remember that ξ^i denotes the i-th row and ξ_i denotes the i-th column of the matrix ξ. □

The following theorem of Matsumoto is especially useful in situations, when we deal with an invertible matrix to which we add a dyadic perturbation.

Theorem 1.1.8 (**Theorem of Matsumoto** [Mat86, Theorem 30.2, p. 206])**.** *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric and invertible matrix, $C \in \mathbb{R}^n$ be a vector and $\varepsilon \in \{-1, 1\}$. Then the determinant of the matrix $Q + \varepsilon CC^T$ is given by*

$$\det(Q + \varepsilon CC^T) = (1 + \varepsilon C^T Q^{-1} C) \det Q.$$

If $1 + \varepsilon C^T Q^{-1} C \neq 0$ then the inverse of $Q + \varepsilon CC^T$ can be written as

$$(Q + \varepsilon CC^T)^{-1} = Q^{-1} - \frac{\varepsilon}{1 + \varepsilon C^T Q^{-1} C} (Q^{-1}C)(Q^{-1}C)^T.$$

Proof. Due to the fact that Q is a symmetric matrix, there exists a hermetian matrix $A \in \mathbb{C}^{n\times n}$ s.t. $Q = A^2$, i.e. through the Eigendecomposition of the matrix Q. Therein $\mathbb{C}$ is the set of complex numbers, hermetian means that $A = \overline{A}^T$ with $\overline{A}$ the componentwise complex conjugation of A. Thereby, we get

$$\begin{aligned} \det(Q + \varepsilon CC^T) &= \det(A^2 + \varepsilon A(A^{-1}C)\overline{(A^{-1}C)}^T A) \\ &= \det A \det(\mathrm{Id} + \varepsilon (A^{-1}C)\overline{(A^{-1}C)}^T) \det A \\ &= \det(\mathrm{Id} + \varepsilon (A^{-1}C)\overline{(A^{-1}C)}^T) \det Q. \end{aligned}$$

On the other hand, the matrix $\mathrm{Id} + \varepsilon (A^{-1}C)\overline{(A^{-1}C)}^T$ is hermetian and has the real Eigenvalues 1 of multiplicity $n - 1$ and $1 + \varepsilon C^T Q^{-1} C$ of multiplicity 1. So, we get

$$\det(Q + \varepsilon CC^T) = (1 + \varepsilon C^T Q^{-1} C) \det Q.$$

The remaining assertion regarding the inverse matrix of $Q + \varepsilon CC^T$ follows by verifying that

$$(Q + \varepsilon CC^T)(Q^{-1} - \frac{\varepsilon}{1 + \varepsilon C^T Q^{-1} C} (Q^{-1}C)(Q^{-1}C)^T) = \mathrm{Id}.$$

In this case, we assumed $1 + \varepsilon C^T Q^{-1} C \neq 0$. □

The following corollary is a direct consequence of Theorem 1.1.8.

Corollary 1.1.9. *Let $C \in \mathbb{R}$ and $a, b \in \mathbb{R}^n$ for $n \geq 2$. Then, there holds*

$$\det\left(C\mathrm{Id} + aa^T - bb^T\right) = C^{n-2}\left((C + a^T a)(C - bb^T) + (a^T b)^2\right).$$

Proof. Assume $C \neq 0$ and $C + a^T a \neq 0$. With Theorem 1.1.8 we compute

$$\begin{aligned} \det\left(C\mathrm{Id} + aa^T\right) &= C^n(1 + \frac{1}{C}a^T a) \\ &= C^{n-1}(C + a^T a) \\ \left(C\mathrm{Id} + aa^T\right)^{-1} &= \frac{1}{C}\mathrm{Id} - \frac{1}{1 + \frac{1}{C}a^T a}\frac{1}{C^2}aa^T \\ &= \frac{1}{C}\left(\mathrm{Id} - \frac{1}{C + a^T a}aa^T\right). \end{aligned}$$

By using once again Theorem 1.1.8 we get

$$\begin{aligned} \det\left(C\mathrm{Id} + aa^T - bb^T\right) &= \det\left(C\mathrm{Id} + aa^T\right)\left(1 - \frac{1}{C}b^T\left(\mathrm{Id} - \frac{1}{C + a^T a}aa^T\right)b\right) \\ &= C^{n-1}(C + a^T a)\left(1 - \frac{1}{C}\left(b^T b - \frac{(a^T b)^2}{C + a^T a}\right)\right) \\ &= C^{n-2}\left((C + a^T a)(C - b^T b) + (a^T b)^2\right). \end{aligned}$$

By the continuity of the determinant we can extend this result to the cases of $C = 0$ or $C + a^T a = 0$, respectively. □

1.1.2 Finite dimensional vector spaces and related topics

For a (real) finite-dimensional *vector space* V we denote by V^* the *dual space*, i.e. the space of all linear real valued mappings on V. If V is of *dimension* n, i.e. $\dim V = n$, there exists a basis $\{b_i\}_{i=1}^n$, s.t. each vector $v \in V$ can be expressed by $v = \sum_{i=1}^n v^j b_j$ with $v^j \in \mathbb{R}$ for $j = 1, \ldots, n$. For such a basis, there exists a so called *dual basis* $\{\theta^j\}_{j=1}^n$ of the dual space V^* defined by

$$\theta^j(b_i) = \delta_i^j$$

for $i, j \in \{1, \ldots, n\}$. Actually, a vector space V and the dual space of the dual space of V are isomorphic, i.e. $V \cong V^{**} := (V^*)^*$, by means of the canonical linear isomorphism $\Phi_{\mathrm{d}} : V \to V^{**}$, which is given by

$$\Phi_{\mathrm{d}}(v)(\omega) := \omega(v) \tag{1.1.6}$$

for $v \in V$ and $\omega \in V^*$. Notice that an *isomorphism* between to finite-dimensional vector spaces is a linear and bijective mapping.

A *k-linear real-valued mapping* T on V is a mapping

$$T : \underbrace{V \times \ldots \times V}_{k\text{-times}} \to \mathbb{R},$$

which is linear in each of its components that is

$$\begin{aligned} & T(v_1, \ldots, v_{i-1}, v_i + \alpha w, v_{i+1}, \ldots, v_k) \\ = \; & T(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k) + \alpha T(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_k) \end{aligned}$$

for all $\alpha \in \mathbb{R}$, $w \in V$, $v_1, \ldots v_k \in V$ and $i = 1, \ldots, k$. Such a mapping T will be called *covariant k-tensor*. The set of all covariant k-tensors on V will be denoted by $T^k(V)$. $T^k(V)$ is a real vector space, where addition of two elements and multiplication with a scalar are defined pointwise.

For a choice of $\omega^1, \ldots, \omega^k \in V^*$ we define the k-linear real-valued mapping $\omega^1 \otimes \ldots \otimes \omega^k$ on V by

$$\omega^1 \otimes \ldots \otimes \omega^k(v_1, \ldots, v_k) \quad := \quad \omega^1(v_1) \cdot \ldots \cdot \omega^k(v_k)$$

for all $v_1, \ldots, v_k \in V$. We call $\omega^1 \otimes \ldots \otimes \omega^k$ the *tensor product* of $\omega^1, \ldots, \omega^k$. In fact $T^k(V)$ is the linear span of such tensor products. Notice, the *linear span* of a subset S of a vector space is the set of all linear combinations of elements in S and will be denoted by span$S$. A basis of $T^k(V)$ is given by

$$\{\theta^{i_1} \otimes \ldots \otimes \theta^{i_k}\}_{i_1,\ldots,i_k=1}^n,$$

so $T^k(V)$ is a n^k-dimensional real vector space. With respect to this basis, each element T of $T^k(V)$ can be expressed as

$$T \quad = \quad T_{i_1 \ldots i_k} \theta^{i_1} \otimes \ldots \otimes \theta^{i_k}.$$

Further, we define the real vector space of all *contravariant k-tensors* $T_k(V)$ in a similar way. We define $T_k(V)$ as the real vector space of real-valued k-linear mappings on V^*. For a choice of $v_1, \ldots, v_k \in V$ we define the k-linear real-valued mapping $v_1 \otimes \ldots \otimes v_k$ on V^* by

$$v_1 \otimes \ldots \otimes v_k(\omega^1, \ldots, \omega^k) \quad := \quad \omega^1(v_1) \cdot \ldots \cdot \omega^k(v_k)$$

for all $\omega^1, \ldots, \omega^k \in V^*$ (cf. (1.1.6)). We call $v_1 \otimes \ldots \otimes v_k$ the *tensor product* of $v_1, \ldots, v_k$. Again, $T_k(V)$ is the linear span of such tensor products. A basis of $T_k(V)$ is given by

$$\{b_{i_1} \otimes \ldots \otimes b_{i_k}\}_{i_1,\ldots,i_k=1}^n,$$

so $T_k(V)$ is a n^k-dimensional real vector space. With respect to this basis, each element $\tilde{T}$ of $T_k(V)$ can be expressed as

$$\tilde{T} = \tilde{T}^{i_1 \ldots i_k} b_{i_1} \otimes \ldots \otimes b_{i_k}.$$

By means of (1.1.6) we see that $T_k(V) \cong T^k(V^*)$.

For a choice $v_1, \ldots, v_k \in V$ we define the alternating k-linear real-valued mapping $v_1 \wedge \ldots \wedge v_k$ on V^* by

$$v_1 \wedge \ldots \wedge v_k(\omega^1, \ldots, \omega^k) \quad = \quad \det \begin{pmatrix} \omega^1(v_1) & \ldots & \omega^1(v_k) \\ \vdots & & \vdots \\ \omega^k(v_1) & \ldots & \omega^k(v_k) \end{pmatrix}$$

for all $\omega^1, \ldots, \omega^k \in V^*$. Such a mapping $v_1 \wedge \ldots \wedge v_k$ will be called *simple k-vector*. The set of all simple k-vectors is a subset of the vector space of all k-linear real valued mappings on V^*, namely $T_k(V)$. Especially,

$$v_1 \wedge \ldots \wedge v_k \quad = \quad \sum_{\sigma \in S_k} \mathrm{sgn}\sigma \, v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}$$

for $v_1, \ldots, v_k \in V$.

The linear span of the set of all simple k-vectors will be denoted by $\Lambda^k(V^*)$ and will be called *k-th exterior power* of V^*. An element of $\Lambda^k(V^*)$ is referred to as a *k-vector*. For a basis $\{b_i\}_{i=1}^n$ of V a basis of $\Lambda^k(V^*)$ is given by

$$\{b_{i_1} \wedge \ldots \wedge b_{i_k}\}_{1 \leq i_1 < \ldots < i_k \leq n}.$$

Therefore, the dimension of $\Lambda^k(V^*)$ equals the *binomial coefficient* $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. Further, by using (1.1.6) we can identify $\omega^1 \wedge \ldots \wedge \omega^k \in \Lambda^k(V)$ for $\omega^1, \ldots, \omega^k \in V^*$ with the alternating k-linear real valued mapping on V defined by

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) \quad := \quad \det \begin{pmatrix} \omega^1(v_1) & \ldots & \omega^1(v_k) \\ \vdots & & \vdots \\ \omega^k(v_1) & \ldots & \omega^k(v_k) \end{pmatrix}$$

for all $v_1, \ldots, v_k \in V$. $\Lambda^k(V)$ is a subset of $T^k(V)$. Elements of $\Lambda^k(V)$ will be referred to as k-*covectors*. We now concentrate on $V = \mathbb{R}^n$ the n-dimensional real vector space, equipped with the *Euclidean scalar product* $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the standard basis $\{e_i\}_{i=1}^n$ and the dual standard basis $\{\delta^j\}_{j=1}^n$. The standard *Euclidean norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ derives from the Euclidean scalar product through $|y| := \sqrt{\langle y, y \rangle}$ for all $y \in \mathbb{R}^n$. We can construct an induced scalar product on $\Lambda^k(\mathbb{R}^n)$ and $\Lambda^k((\mathbb{R}^n)^*)$ by defining it for the basis elements

$$\langle e_{i_1} \wedge \ldots \wedge e_{i_k}, e_{j_1} \wedge \ldots \wedge e_{j_k} \rangle \quad := \quad \prod_{l=1}^k \delta_{i_l j_l},$$
$$\langle \delta^{i_1} \wedge \ldots \wedge \delta^{i_k}, \delta^{j_1} \wedge \ldots \wedge \delta^{j_k} \rangle \quad := \quad \prod_{l=1}^k \delta^{i_l j_l},$$

for $1 \le i_1 < \ldots < i_k \le n$ and $1 \le j_1 < \ldots < j_k \le n$. Notice that the presentation of tensors and k-vectors here is essentially based on [Jän01] and [Lee03, pp. 260-268, 294-302].

From now on, for indices running from 1 to m we will write greek letters and for indices running from 1 to n or $m+1$ we will write latin letters. $\Lambda^m(\mathbb{R}^{m+1})$ can be identified with $\mathbb{R}^{m+1}$ by means of the isometric linear isomorphism $\Phi_\mathrm{n} : \Lambda^m(\mathbb{R}^{m+1}) \to \mathbb{R}^{m+1}$ by defining it for the canonical basis vectors by

$$\Phi_\mathrm{n}(e_1 \wedge \ldots \wedge e_{k-1} \wedge e_{k+1} \wedge \ldots \wedge e_{m+1}) \quad := \quad (-1)^{k-1} e_k,$$

for $1 \le k \le m+1$ and extending it linearly to common m-vectors. Notice that every m-vector in $\Lambda^m(\mathbb{R}^{m+1})$ is a simple m-vector due to codimension 1. This justifies the following definition of the m-vector in codimension 1, which is compatible with the precedent construction.

Definition 1.1.10. For $\xi_\alpha \in \mathbb{R}^{m+1}$ with $\xi_\alpha = \xi_\alpha^j e_j$ and $\alpha \in \{1, \ldots, m\}$, set $\xi := (\xi_1 | \ldots | \xi_m) \in \mathbb{R}^{(m+1)\times m}$. Define the *m-vector* $\xi_1 \wedge \ldots \wedge \xi_m$ by

$$\xi_1 \wedge \ldots \wedge \xi_m \quad := \quad (\xi_1 \wedge \ldots \wedge \xi_m)^j e_j$$
$$(\xi_1 \wedge \ldots \wedge \xi_m)^j \quad := \quad (-1)^{j-1} \det \xi^{(j)}.$$

Theorem 1.1.11. *(cf. [Fla63, Ch. 2.6, p. 14])Let $\xi := (\xi_1 | \ldots | \xi_m) \in \mathbb{R}^{(m+1)\times m}$ and $\eta := (\eta_1 | \ldots | \eta_m) \in \mathbb{R}^{(m+1)\times m}$ be the matrices. Then*

$$\langle \xi_1 \wedge \ldots \wedge \xi_m, \eta_1 \wedge \ldots \wedge \eta_m \rangle \quad = \quad \det(\xi^T \eta), \tag{1.1.7}$$
$$|\xi_1 \wedge \ldots \wedge \xi_m| \quad = \quad \sqrt{\det \xi^T \xi}. \tag{1.1.8}$$

Proof. We start by showing (1.1.7).

$$\langle \xi_1 \wedge \ldots \wedge \xi_m, \eta_1 \wedge \ldots \wedge \eta_m \rangle \overset{\text{Dfn. 1.1.10}}{=} (-1)^{i-1} \det \xi^{(i)} \delta_{ij} (-1)^{j-1} \det \eta^{(j)}$$
$$= \sum_{i=1}^{m+1} \det \xi^{(i)} \det \eta^{(i)}$$

$$\begin{aligned} &= \sum_{i=1}^{m+1} \det(\xi^T)_{(i)} \det \eta^{(i)} \\ &\overset{\text{Thm. 1.1.6}}{=} \det(\xi^T \eta). \end{aligned}$$

(1.1.8) follows directly from (1.1.7) and the definition of the Euclidean norm. □

Theorem 1.1.12. *Let* $\xi := (\xi_1|\dots|\xi_m) \in \mathbb{R}^{(m+1)\times m}$ *and* $\eta \in \mathbb{R}^{m+1}$. *Then*

$$\langle \eta, \xi_1 \wedge \dots \wedge \xi_m \rangle = \det(\eta, \xi_1, \dots, \xi_m). \tag{1.1.9}$$

Proof.

$$\begin{aligned} \langle \eta, \xi_1 \wedge \dots \wedge \xi_m \rangle &\overset{\text{Dfn. 1.1.10}}{=} \eta^i \delta_{ij} (-1)^{j-1} \det \xi^{(j)} \\ &= \eta^i \delta_{ij} (-1)^{1+j} \det \xi^{(j)} \\ &\overset{\text{Thm. 1.1.2}}{=} \det(\eta, \xi_1, \dots, \xi_m), \end{aligned}$$

where $\eta = \eta^i e_i \in \mathbb{R}^{m+1}$. □

Definition 1.1.13 (Orthogonal complement). For a set of vectors $S \subset \mathbb{R}^{m+1}$ we define the *orthogonal complement to* S by

$$S^\perp := \{\zeta \in \mathbb{R}^{m+1} : \langle \eta, \zeta \rangle = 0 \text{ for all } \eta \in S\}.$$

For a single vector $\eta \in \mathbb{R}^{m+1}$ we define the *orthogonal complement to* η as the set $\{\eta\}^\perp$ and denote it by $\eta^\perp$.

Corollary 1.1.14 (Normal). *Let* $\xi := (\xi_1|\dots|\xi_m) \in \mathbb{R}^{(m+1)\times m}$. *Then*

$$\langle \xi_i, \xi_1 \wedge \dots \wedge \xi_m \rangle = 0$$

for all $i = 1, \dots, m$. *If additionally* $\xi_1 \wedge \dots \wedge \xi_m \neq 0$, *there holds*

$$(\xi_1 \wedge \dots \wedge \xi_m)^\perp = \operatorname{span}\{\xi_1, \dots, \xi_m\},$$

where the linear span of a set of vectors is the set of all linear combinations of the vectors.

Proof. Corollary 1.1.14 is a direct consequence of Theorem 1.1.12 and the properties of the determinant. □

Theorem 1.1.15. *Let* $\xi = (\xi_1|\dots|\xi_m) \in \mathbb{R}^n$. *Then*

$$|\xi_1 \wedge \dots \wedge \xi_m| \leq \prod_{i=1}^m |\xi_i|, \tag{1.1.10}$$

If $\xi_1, \dots, \xi_m$ *are linearly independent, equality in* (1.1.10) *holds if and only if* $\xi_1, \dots, \xi_m$ *are pairwise orthogonal vectors.*

Proof. Assume that $\xi_1, \dots, \xi_m$ are linearly independent. We will exploit the alternating property of the wedge product. To do so, we define by means of the Gram-Schmidt process

$$\begin{aligned} \tilde{\xi}_1 &:= \xi_1, \quad \eta_1 := \frac{\tilde{\xi}_1}{|\tilde{\xi}_1|}, \\ \tilde{\xi}_i &:= \xi_i - \sum_{j=1}^{i-1} \langle \xi_i, \eta_j \rangle \eta_j, \quad \eta_i := \frac{\tilde{\xi}_i}{|\tilde{\xi}_i|} \end{aligned}$$

for $i = 2, \ldots, m$. Notice that $\eta_1, \ldots, \eta_m$ and $\tilde{\xi}_1, \ldots, \tilde{\xi}_m$ are orthogonal by construction and $|\eta_i| = 1$ for $i = 1, \ldots, m$ by construction. Using the fact that the wedge product is alternating and multilinear, we get

$$\xi_1 \wedge \ldots \wedge \xi_m = \left(\prod_{i=1}^m |\tilde{\xi}_i| \right) (\eta_1 \wedge \ldots \wedge \eta_m)$$

Further, $\{\eta_i\}_{i=1}^m$ is a set of orthonormal vectors and therefore, due to the definition of the scalar product on $\Lambda^m(\mathbb{R}^n)$, the vector $\eta_1 \wedge \ldots \wedge \eta_m$ has norm 1. On the other hand, we can estimate the norm of $\tilde{\xi}_i$ in the following way

$$\begin{aligned} |\tilde{\xi}_i|^2 = \langle \tilde{\xi}_i, \tilde{\xi}_i \rangle &= |\xi_i|^2 - 2 \sum_{j=1}^{i-1} \langle \xi_i, \eta_j \rangle \langle \xi_i, \eta_j \rangle + \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} \langle \xi_i, \eta_j \rangle \langle \xi_i, \eta_k \rangle \langle \eta_j, \eta_k \rangle \\ &= |\xi_i|^2 - \sum_{j=1}^{i-1} \langle \xi_j, \eta_j \rangle^2 \leq |\xi_i|^2. \end{aligned} \tag{1.1.11}$$

Thereby, we get

$$|\xi_1 \wedge \ldots \wedge \xi_m| = \left(\prod_{i=1}^m |\tilde{\xi}_i| \right) |\eta_1 \wedge \ldots \wedge \eta_m| = \prod_{i=1}^m |\tilde{\xi}_i| \leq \prod_{i=1}^m |\xi_i|.$$

If the equality holds in (1.1.10), then by the latter estimate follows that

$$\prod_{i=1}^m |\tilde{\xi}_i| = \prod_{i=1}^m |\xi_i|$$

and this together with (1.1.11) yields $|\xi_i| = |\tilde{\xi}_i|$ for all $i = 1, \ldots, m$. This identity can be rewritten to

$$|\xi_i|^2 = |\xi_i|^2 - \sum_{j=1}^{i-1} \langle \xi_i, \eta_j \rangle^2$$

and thereby

$$\sum_{j=1}^{i-1} \langle \xi_i, \eta_j \rangle^2 = 0$$

for all $i = 1, \ldots, m$. So, we conclude that $\xi_i = \tilde{\xi}_i$ for $i = 1, \ldots, m$ and since $\tilde{\xi}_i$ are orthogonal by construction, we get the second statement of the theorem.

On the other hand if we start by assuming that $\xi_1, \ldots, \xi_m$ are linearly independent orthogonal vectors, the norm of the m-vector $\xi_1 \wedge \ldots \wedge \xi_m$ computes to

$$\begin{aligned} |\xi_1 \wedge \ldots \wedge \xi_m| &= \sqrt{\det \xi^T \xi} \\ &= \sqrt{\det(\langle \xi_i, \xi_j \rangle)} \\ &= \sqrt{\det(|\xi_i|^2 \delta_{ij})} = \prod_{i=1}^m |\xi_i| \end{aligned}$$

by means of (1.1.8).

Finally, (1.1.10) can be extended to linearly dependent $\xi_1, \ldots, \xi_m$ as the equality is continuous with respect to approximation of the mentioned set of vectors by linearly independent ones. □

Theorem 1.1.16 (Normal transformation). *Let $L : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be an isomorphism, i.e. a linear and bijective mapping. Then the following is true:*

$$(L\xi_1) \wedge \ldots \wedge (L\xi_m) \quad = \quad (\det L)L^{-T}(\xi_1 \wedge \ldots \wedge \xi_m)$$

for all vectors ξ_i in $\mathbb{R}^{m+1}$ for $i = 1, \ldots, m$.

Remark 1.1.17. In a slight abuse of notation, the linear mapping L will be identified with its transformation matrix $L = (L_i^j)$, wherein L_j^i are defined by $Le_i = L_i^j e_j$, where $\{e_i\}_{i=1,\ldots,m+1}$ is the canonical basis of $\mathbb{R}^{m+1}$.

Proof of Theorem 1.1.16. Express the linear mapping L in the canonical basis $\{e_i\}_{i=1,\ldots,m+1}$ by

$$Le_i \quad = \quad L_i^j e_j$$

for $i = 1, \ldots, m+1$. By Definition 1.1.10 and Theorem 1.1.6 the components of the m-vector $(L\xi_1) \wedge \ldots \wedge (L\xi_m)$ can be written as

$$\begin{aligned}
((L\xi_1) \wedge \ldots \wedge (L\xi_m))^j &\overset{\text{Dfn. 1.1.10}}{=} (-1)^{j-1} \det\left((L\xi)^{(j)}\right) \\
&= (-1)^{j-1} \det((L_k^l \xi_\beta^k)^{(j)}) \\
&\overset{\text{Thm. 1.1.6}}{=} (-1)^{j-1} \sum_{l=1}^{m+1} \det L_{(l)}^{(j)} \det \xi^{(l)} \\
&= \det L \sum_{l=1}^{m+1} \frac{(-1)^{j+l} \det L_{(l)}^{(j)}}{\det L} (-1)^{l-1} \det \xi^{(l)} \\
&\overset{\text{Thm. 1.1.3}}{=} \det L \sum_{l=1}^{m+1} (L^{-T})_l^j (\xi_1 \wedge \ldots \wedge \xi_m)^l \\
&= ((\det L)L^{-T}\xi_1 \wedge \ldots \wedge \xi_m)^j
\end{aligned}$$

□

In the rest of the section, we introduce some concept of linear projections on finite dimensional vector spaces. We present it in a probably non-standard manner due to needs on application side. We refer to [Mey00, Section 5.13] for an overview on the topic, even though Meyer uses there another approach.

In the following definition, we introduce scalar products on $\mathbb{R}^n$ issued from a positive definite quadratic matrix. Notice that a *scalar product* p on a real vector space V is a symmetric, positive-definite and bilinear mapping $p : V \times V \to \mathbb{R}$.

Definition 1.1.18 (A-scalar product). Let $A = (A_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. We define the *A-scalar product* $\langle\cdot,\cdot\rangle_A$ on $\mathbb{R}^n$ by

$$\langle v, w\rangle_A \quad = \quad v^T A w \quad = \quad v^i A_{ij} w^j$$

for $v = (v^i), w = (w^i) \in \mathbb{R}^n$. A matrix $M \in \mathbb{R}^{n\times n}$ is called *A-symmetric* if

$$\langle Mv, w\rangle_A \quad = \quad \langle v, Mw\rangle_A$$

which means in matrix notation

$$M^T A \quad = \quad AM.$$

A matrix P is called a *projection* if $P^2 = P$. Given two positive integers m and n s.t. $n \geq m$ and a matrix $C \in \mathbb{R}^{n\times m}$ with rank$C = m$, then we define the *C-projection w.r.t. to A*, $P_{A,C}$, by

$$P_{A,C} \quad := \quad \mathrm{Id} - C(C^TAC)^{-1}C^TA.$$

$P_{A,C}$ is an A-symmetric projection.

Remark 1.1.19. Let V, W be two finite dimensional vector spaces. The *rank* of a linear mapping $f : V \to W$ is the dimension of the image of V w.r.t. f. We write shortly rank(f). We say that f is of *full rank* if rank$(f) = \min\{\dim V, \dim W\}$. The rank of a matrix $M \in \mathbb{R}^{n\times m}$ is then defined as the rank of its induced linear mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ with $f(x) := M \cdot x$ for all $x \in \mathbb{R}^m$.

Theorem 1.1.20 (**Kernel of projections**). *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $C = (c_1, \dots, c_m) \in \mathbb{R}^{n\times m}$ a matrix with $n \geq m$. Then the Kernel of the C-projection w.r.t. A equals the span of the columns of C, i.e.*

$$\ker P_{A,C} \quad = \quad \mathrm{span}\{c_\alpha : \alpha = 1, \dots, m\}.$$

Remark 1.1.21. Let V and W be two vector spaces. The *kernel* $\ker(f)$ of a linear mapping $f : V \to W$ is defined by $\ker(f) := \{v \in V : f(v) = 0 \in W\}$.

Proof of Theorem 1.1.20. Assume $\xi \in \mathbb{R}^n$ with $P_{A,C}\xi = 0$, then there holds

$$\xi \quad = \quad C(C^TAC)^{-1}C^TA\xi \quad = \quad \sum_{\alpha=1}^{m} c_\alpha w^\alpha,$$

wherein $w := (C^TAC)^{-1}C^TA\xi \in \mathbb{R}^m$. So, $\xi \in \mathrm{span}\{c_\alpha, \alpha = 1, \dots, m\}$. On the other hand if we assume that $y \in \mathrm{span}\{c_\alpha, \alpha = 1, \dots, m\}$, we express y as

$$y \quad = \quad \sum_{\alpha=1}^{m} c_\alpha w^\alpha$$

for $w \in \mathbb{R}^m$. Further,

$$P_{A,C}C \quad = \quad C - C(C^TAC)^{-1}C^TAC \quad = \quad C - C \quad = \quad 0$$

such that for each column of C holds

$$P_{A,C}c_\alpha \quad = \quad 0$$

for $\alpha = 1, \dots, m$. Therefore, we deduce that $P_{A,C}y = \sum_{\alpha=0}^{m} w^\alpha P_{A,C}c_\alpha = 0$ such that $y \in \ker P_{A,C}$. □

Theorem 1.1.22 (**Comparative estimate of projections**). *Given two symmetric positive definite matrices $A, B \in \mathbb{R}^{n\times n}$ and a matrix $C = (c_1, \dots, c_m) \in \mathbb{R}^{n\times m}$ of full rank, where $n \geq m$. There exist positive constants $\Lambda_1 = \Lambda_1(A, B)$, $\Lambda_2 = \Lambda_2(A, B) > 0$ such that*

$$\Lambda_1\langle P_{B,C}\xi, \xi\rangle_B \quad \leq \quad \langle P_{A,C}\xi, \xi\rangle_A \quad \leq \quad \Lambda_2\langle P_{B,C}\xi, \xi\rangle_B \tag{1.1.12}$$

for all $\xi \in \mathbb{R}^n$. The constants Λ_1 and Λ_2 depend continuously on A and B. Further, the transformation behaviour of the constants Λ_1 and Λ_2 for an invertible matrix $N \in \mathbb{R}^{n\times n}$ is given by

$$\Lambda_i(A, B) \quad = \quad \Lambda_i(N^TAN, N^TBN) \tag{1.1.13}$$

for $i = 1, 2$.

Proof. Define the constant

$$\begin{aligned}\tilde{\Lambda}_1 \quad &:= \quad \tilde{\Lambda}_1(A,B,C) \quad := \quad \min_{\eta\in S_{B,C}} \langle P_{A,C}\eta,\eta\rangle_A \\ &\qquad\qquad\qquad\qquad = \quad \min_{\eta\in S_{B,C}} \langle P_{A,C}\eta, P_{A,C}\eta\rangle_A\end{aligned}$$

with the compact set

$$S_{B,C} \quad := \quad \{\eta\in\mathbb{R}^n : |\eta|_B = 1,\ \langle \eta, c_\alpha\rangle_B = 0 \text{ for } \alpha = 1,\dots,m\}.$$

By definition it holds that $\tilde{\Lambda}_1 \geq 0$.

Assume that $\tilde{\Lambda}_1 = 0$. Then there exists $\eta_0 \in S_{B,C}$ such that $\langle P_{A,C}\eta_0, P_{A,C}\eta_0\rangle_A = 0$ due to the compactness of $S_{B,C}$. Therewith and by Theorem 1.1.20, we get $\eta_0 \in \ker P_{A,C} \cap S_{B,C} = \operatorname{span}\{c_\alpha : \alpha = 1,\dots m\} \cap S_{B,C} = \emptyset$, which is a contradiction. Hence, $\tilde{\Lambda}_1 > 0$.

Now we decompose $\xi\in\mathbb{R}^n$ in the following way

$$\begin{aligned}\xi \quad &= \quad P_{B,C}\xi + (\mathrm{Id} - P_{B,C})\xi \\ &=: \quad \tau + \sigma,\end{aligned}$$

where obviously $\tau = P_{B,C}\xi \in \{\eta : \langle \eta, c_\alpha\rangle_B = 0 \text{ for } \alpha = 1,\dots m\}$ and $\sigma = (\mathrm{Id} - P_{B,C})\xi \in \operatorname{span}\{c_\alpha : \alpha = 1,\dots m\} = \ker P_{A,C} = \ker P_{B,C}$ by Theorem 1.1.20. Putting this into the expression $\langle P_{A,C}\xi,\xi\rangle_A$ yields

$$\begin{aligned}\langle P_{A,C}\xi,\xi\rangle_A \quad &= \quad \langle P_{A,C}\xi, P_{A,C}\xi\rangle_A \\ &= \quad \langle P_{A,C}(\tau+\sigma), P_{A,C}(\tau+\sigma)\rangle_A \\ &= \quad \langle P_{A,C}\tau, P_{A,C}\tau\rangle_A + 2\langle P_{A,C}\sigma, P_{A,C}\tau\rangle_A + \langle P_{A,C}\sigma, P_{A,C}\sigma\rangle_A \\ &= \quad \langle P_{A,C}\tau, P_{A,C}\tau\rangle_A \\ &\geq \quad \tilde{\Lambda}_1\langle\tau,\tau\rangle_B \\ &= \quad \tilde{\Lambda}_1\langle P_{B,C}\xi, P_{B,C}\xi\rangle_B \\ &= \quad \tilde{\Lambda}_1\langle P_{B,C}\xi,\xi\rangle_B. \qquad (1.1.14)\end{aligned}$$

Further, it can easily be shown that the following transformation behaviour of the respective expressions is true for matrices $N\in\mathbb{R}^{n\times n}$ and $M\in\mathbb{R}^{m\times m}$, both of full rank:

$$\langle P_{N^TAN,N^{-1}CM}\eta,\eta\rangle_{N^TAN} \quad = \quad \langle P_{A,C}N\eta, N\eta\rangle_A, \qquad (1.1.15)$$

$$S_{N^TBN,N^{-1}CM} \quad = \quad N^{-1}S_{B,C} \qquad (1.1.16)$$

for all $\eta\in\mathbb{R}^n$. Therefore, the transformation behaviour of $\tilde{\Lambda}_1$ is given by

$$\begin{aligned}\tilde{\Lambda}_1(N^TAN, N^TBN, N^{-1}CM) &\overset{(1.1.15)\&(1.1.16)}{=} \min_{\eta\in N^{-1}S_{B,C}} \langle P_{A,C}N\eta, N\eta\rangle_A \\ &= \quad \min_{N\eta\in S_{B,C}} \langle P_{A,C}N\eta, N\eta\rangle_A \\ &= \quad \min_{\zeta\in S_{B,C}} \langle P_{A,C}\zeta,\zeta\rangle_A \\ &= \quad \tilde{\Lambda}_1(A,B,C). \qquad (1.1.17)\end{aligned}$$

Define the *orthogonal group* to the Euclidean scalar product

$$O(n,m) \quad := \quad \{(w_1|\cdots|w_m)\in\mathbb{R}^{n\times m} : \langle w_\alpha, w_\beta\rangle = \delta_{\alpha\beta} \text{ for } \alpha,\beta = 1,\dots,m\}.$$

$O(n,m)$ is a closed subset of the compact set $\mathbb{S}^{n-1}\times\ldots\times\mathbb{S}^{n-1}\subset\mathbb{R}^{n\times m}$ and as such compact by itself. Further, it is a commonly known fact that for every $C\in\mathbb{R}^{n\times m}$ there is an invertible matrix $M \in \mathbb{R}^{m\times m}$ s.t. $CM \in O(n,m)$ (i.e. by means of the Gram-Schmidt process). This motivates the definition

$$\Lambda_1(A,B) \quad := \quad \min_{D\in O(n,m)} \tilde{\Lambda}_1(A,B,D).$$

$\Lambda_1(A,B)$ is positive being the minimum of a continuous and positive function over a compact set. We have

$$\begin{aligned}\tilde{\Lambda}_1(A,B,C) &\overset{(1.1.17)}{=} \tilde{\Lambda}_1(A,B,CM)\\ &\geq \ \Lambda_1(A,B)\end{aligned}$$

for a properly chosen invertible matrix $M \in \mathbb{R}^{m\times m}$. So, we get

$$\Lambda_1(A,B) \quad = \quad \min_{D\in O(n,m)} \tilde{\Lambda}_1(A,B,D) = \inf_{D\in\mathbb{R}^{n\times m} \text{ of full rank}} \tilde{\Lambda}_1(A,B,D). \tag{1.1.18}$$

The transformation behaviour of $\Lambda_1(A,B)$ follows directly from the transformation behaviour of $\tilde{\Lambda}_1(A,B,C)$ in the following way

$$\begin{aligned}\Lambda_1(N^TAN,N^TBN) &\overset{(1.1.18)}{=} \inf_{D\in\mathbb{R}^{n\times m} \text{ of full rank}} \tilde{\Lambda}_1(N^TAN,N^TBN,D)\\ &\overset{\tilde{D}=ND}{=} \inf_{\tilde{D}\in\mathbb{R}^{n\times m} \text{ of full rank}} \tilde{\Lambda}_1(N^TAN,N^TBN,N^{-1}\tilde{D})\\ &\overset{(1.1.17)}{=} \inf_{\tilde{D}\in\mathbb{R}^{n\times m} \text{ of full rank}} \tilde{\Lambda}_1(A,B,\tilde{D})\\ &\overset{(1.1.18)}{=} \Lambda_1(A,B),\end{aligned}$$

which proves (1.1.13). Further, (1.1.14) and (1.1.18) lead to the estimate

$$\langle P_{A,C}\xi,\xi\rangle_A \quad \geq \quad \Lambda_1\langle P_{B,C}\xi,\xi\rangle_B.$$

By exchanging the role of A and B we get the constant $\Lambda_2 := \Lambda_2(A,B) := \frac{1}{\Lambda_1(B,A)} > 0$. The continuity of the constants Λ_1 and Λ_2 w.r.t A and B follows by construction. □

1.2 Real and Functional Analysis

1.2.1 Real Analysis and Measure theory

In this section, we will introduce some function spaces on subsets of the real vector space $\mathbb{R}^m$ and state some related theorems. We will also present shortly some useful measure theoretical theorems. The section is mostly based on the books of Evans & Gariepy [EG92] and Gilbarg & Trudinger [GT01]. For an open set $\Omega \subset \mathbb{R}^m$ denote its *topological boundary* or *boundary* by $\partial\Omega$ and its *closure* by $\overline{\Omega}$. $\Omega \subset \mathbb{R}^m$ is said to be a *domain* if $\Omega \subset \mathbb{R}^m$ is an open and connected set. We will write $\Omega \subset\subset \mathbb{R}^m$ for an open set Ω, whose closure $\overline{\Omega}$ is a compact subset of $\mathbb{R}^m$. A *compact* set in $\mathbb{R}^m$ is a closed bounded set. Denote by $C^k(\Omega)$ the set of k-times continuously differentiable functions on an open set $\Omega \subset \mathbb{R}^m$, where $k \in \mathbb{N}\cup\{0,\infty\}$, and by $C^k(\overline{\Omega})$ the set of k-times continuously differentiable functions on an open set $\Omega \subset \mathbb{R}^m$, whose derivatives can be extended continuously up to the boundary of Ω. For $\Omega \subset\subset \mathbb{R}^m$, the vector space $C^0(\overline{\Omega})$ becomes a *Banach space*, i.e. a complete normed vector space, by means of the *supremum norm*

$$||f||_{C^0(\overline{\Omega})} \quad := \quad \sup_{\Omega}|f(x)|$$

for $f \in C^0(\overline{\Omega})$. Notice that a *norm* $||\cdot||$ on a vector space V is a mapping $||\cdot|| : V \to [0,\infty)$ satisfying the following conditions:

(i) if $||v|| = 0$ holds true for some $v \in V$, then $v = 0 \in V$;

(ii) $||tv|| = |t|||v||$ for all $t \in \mathbb{R}$ and $v \in V$;

(iii) $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in V$.

Therein, (iii) is called triangle inequality. If $|| \cdot ||$ only satisfies (ii) and (iii) then $|| \cdot ||$ is called *seminorm*. A *normed vector space* is a pair $(V, || \cdot ||)$ where V is a vector space and $|| \cdot ||$ is a norm on V. For $k \in \mathbb{N}$ the vector space $C^k(\overline{\Omega})$ becomes a Banach space by means of the norm

$$||f||_{C^k(\overline{\Omega})} \quad := \quad \sum_{l=0}^{k} \sum_{|\alpha|=l} ||\mathrm{D}^\alpha f||_{C^0(\overline{\Omega})},$$

where $f \in C^k(\overline{\Omega})$ and α are multi-indices. Notice that for a function $f \in C^k(\overline{\Omega})$ or $C^k(\Omega)$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$ we set $|\alpha| := \sum_{i=1}^m \alpha_i$ and

$$\mathrm{D}^\alpha f(x) \quad := \quad \frac{\partial^{|\alpha|}}{\partial(x^1)^{\alpha_1} \dots \partial(x^m)^{\alpha_m}} f(x)$$

for $x = (x^i)_{i=1,\dots,m} \in \Omega$.

Definition 1.2.1 (Hölder continuity). Let Ω be an open set (or the closure of an open set). A function $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ is said to be *Hölder continuous with exponent* $\sigma \in (0, 1]$ on Ω if there exists a constant $C \geq 0$ s.t.

$$|f(x) - f(y)| \quad \leq \quad C|x - y|^\sigma \tag{1.2.1}$$

for $x, y \in \Omega$. The real vector space of all Hölder continuous functions on Ω to a given exponent $\sigma \in (0, 1]$ will be denoted by $C^{0,\sigma}(\Omega)$. A function $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ is said to be *locally Hölder continuous* on Ω if $f|_K$ is Hölder continuous for every compact set $K \subset \Omega$. The real vector space of all locally Hölder continuous functions to a given exponent $\sigma \in (0, 1]$ on Ω will be denoted by $C^{0,\sigma}_{\mathrm{loc}}(\Omega)$. Further, we define the *Hölder norm* of f on Ω by

$$\mathrm{Höl}_{\sigma,\Omega}(f) \quad := \quad \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\sigma}, \tag{1.2.2}$$

which is the smallest constant C s.t. f satisfies (1.2.1) on Ω for a given fixed constant $\sigma \in (0, 1]$.

Remark 1.2.2.

- For every $\sigma \in (0, 1]$ holds $C^{0,\sigma}(\Omega) \subset C^0(\Omega)$.
- Let k be a non-negative integer and $\sigma \in (0, 1]$. We define the spaces $C^{k,\sigma}(\Omega)$, $C^{k,\sigma}(\overline{\Omega})$ and $C^{k,\sigma}_{\mathrm{loc}}(\overline{\Omega})$ as the spaces of the respective C^k functions, whose differentials of order k are all (locally) Hölder continuous functions to the Hölder exponent σ. If $\Omega \subset\subset \mathbb{R}^m$, then $C^{k,\sigma}(\overline{\Omega})$ becomes a Banach space by means of the norm

$$|| \cdot ||_{C^{k,\sigma}(\Omega)} \quad := \quad || \cdot ||_{C^k(\Omega)} + \mathrm{Höl}_{\sigma,\Omega}(\cdot).$$

- A (locally) Hölder continuous function with exponent $\sigma = 1$ on an open set Ω will also be referred to as a *(locally) Lipschitz continuous* function or *(locally) Lipschitz function*. The so called Lipschitz constant Lip_Ω coincides with the corresponding Hölder norm $\mathrm{Höl}_{1,\Omega}$.

In the following, we will present some theorems of measure theory. Before doing so, some notational conventions will be given. Denote by $\mathscr{H}^m$ the *m-dimensional Hausdorff measure* on $\mathbb{R}^n$, where we choose $m \leq n$. For a $\mathscr{H}^m$-integrable function $f : \mathbb{R}^n \to \mathbb{R}$, we write $\int_A f(x)\mathrm{d}\mathscr{H}^m(x)$ for the integral of f over the measurable set $A \subset \mathbb{R}^n$. In the case of $n = m$, we will also write $\int_A f\mathrm{d}x = \int_A f(x)\mathrm{d}x = \int_A f(x)\mathrm{d}\mathscr{H}^n(x)$ what is reasonable as the n-dimensional Hausdorff measure coincides then with the n-dimensional Lebesgue measure. The notion $\mathscr{H}^m$ *almost every* x or shortly $\mathscr{H}^m$-a.e. x refers to all $x \in \mathbb{R}^n$ up to a $\mathscr{H}^m$ null set. $\mathscr{H}^m$ *almost everywhere* or shortly $\mathscr{H}^m$-a.e. means for $\mathscr{H}^m$ almost every $x \in \mathbb{R}^n$.

Theorem 1.2.3 (Rademacher's theorem [EG92, Section 3.1.2, Theorem 2]**).** *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function. Then f is differentiable $\mathscr{H}^m$ almost everywhere.*

Definition 1.2.4 (Jacobian). Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function. Then the *Jacobian determinant* of f is

$$Jf(x) \quad := \quad \sqrt{\det(\mathrm{D}f(x))^T(\mathrm{D}f(x))}$$

for $\mathscr{H}^m$-a.e. x. Therein, $\mathrm{D}f(x) \in \mathbb{R}^{n\times m}$ is the *Jacobian matrix* of f given by

$$\mathrm{D}f(x) \quad := \quad \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \dots & \frac{\partial f^1}{\partial x^m}(x) \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1}(x) & \dots & \frac{\partial f^n}{\partial x^m}(x) \end{pmatrix}$$

for all $x \in \mathbb{R}^m$, where f is differentiable.

Remark 1.2.5.

- Notice that if not said otherwise, differentiable means continuously differentiable.
- A generalization of the Jacobian for mappings on manifolds in special choices of local coordinates is given in (2.1.4).

Theorem 1.2.6 (Area formula [EG92, Section 3.3.2 Theorem 1]**).** *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function with $m \leq n$. Then for each $\mathscr{H}^m$-measurable subset $A \subset \mathbb{R}^m$, there holds*

$$\int_A Jf\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n} \mathscr{H}^0(A \cap f^{-1}(\{y\}))\mathrm{d}\mathscr{H}^m(y).$$

Theorem 1.2.7 (Change of variables [EG92, Section 3.3.3, Theorem 2]**).** *For $m \leq n$, let Ω be a $\mathscr{H}^m$-measurable subset of $\mathbb{R}^m$ and $f : \Omega \to \mathbb{R}^n$ be Lipschitz continuous. Then for each $\mathscr{H}^m$-integrable function $g : \Omega \to \mathbb{R}$ with $\int_\Omega |g(x)|\mathrm{d}x < \infty$, there holds*

$$\int_\Omega g(x)Jf(x)\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n} \sum_{x\in f^{-1}(\{y\})} g(x)\mathrm{d}\mathscr{H}^m(y).$$

Remark 1.2.8.

- This theorem is a slightly adapted version of that in [EG92] due to the choice of a $\mathscr{H}^m$-measurable subset Ω of $\mathbb{R}^m$.
- For a set $A \in \mathbb{R}^n$ we define $f^{-1}(A) := \{x \in \Omega : f(x) \in A\}$.
- Due to the area formula, Theorem 1.2.6, the set $f^{-1}(\{y\})$ is at most countable for $\mathscr{H}^m$-a.e. $y \in \mathbb{R}^n$.

Definition 1.2.9 (**L^p-space and Sobolev space**)**.** Let $\Omega \subset \mathbb{R}^m$ be an open set and $p \geq 1$. Denote by $\mathscr{L}^p(\Omega)$ the *space of p-integrable functions on* Ω, i.e. $\mathscr{H}^m$-measurable functions $u : \Omega \to \mathbb{R}$ s.t.

$$\int_\Omega |u(x)|^p \mathrm{d}\mathscr{H}^m(x) \quad < \quad \infty.$$

Further, let $L^p(\Omega)$ be the modulo space derived from the equivalence relation $\sim_{\mathscr{H}^m}$, where for two $\mathscr{H}^m$-measurable functions $u, v : \Omega \to \mathbb{R}$ holds $u \sim_{\mathscr{H}^m} v$ if and only if $u = v$ $\mathscr{H}^m$ almost everywhere. $L^p(\Omega)$ will be referred to as *L^p-space* and becomes a Banach space by means of the *L^p-norm*

$$||u||_{L^p(\Omega)} \quad := \quad \left(\int_\Omega |u(x)|^p \mathrm{d}\mathscr{H}^m(x)\right)^{\frac{1}{p}}.$$

If for a function $u \in L^p(\Omega)$ and a multi-index α exists a $h \in L^p(\Omega)$ s.t.

$$\int_\Omega h(x)\varphi(x)\mathrm{d}\mathscr{H}^m(x) \quad = \quad (-1)^{|\alpha|}\int_\Omega u(x)\mathrm{D}^\alpha\varphi(x)\mathrm{d}\mathscr{H}^m(x)$$

for all $\varphi \in C_0^\infty(\Omega)$, we call h a *weak derivative* of u and denote it by $\mathrm{D}^\alpha u$. Therein, $C_0^\infty(\Omega)$ is the set of functions in $C^\infty(\Omega)$ with compact support in Ω. Notice that the *support* supp$g$ of a function $g : A \subset \mathbb{R}^m \to \mathbb{R}^n$ is the closure of the set where the function is non-zero, i.e.

$$\mathrm{supp}g := \overline{\{a \in A : g(a) \neq 0\}}.$$

We define the *Sobolev space* for a non-negative integer k and integrability $p \geq 1$ by

$$W^{k,p}(\Omega) \quad := \quad \{u \in L^p(\Omega) : \mathrm{D}^\alpha u \in L^p(\Omega) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \leq k\}.$$

$W^{k,p}(\Omega)$ becomes a Banach space by means of the *$W^{k,p}$-norm*

$$||u||_{W^{k,p}(\Omega)} \quad := \quad \sum_{\substack{\alpha \text{ multi-index,} \\ |\alpha| \leq k}} ||\mathrm{D}^\alpha u||_{L^p(\Omega)}. \tag{1.2.3}$$

Remark 1.2.10.

- The basic definitions of Lebesgue and Sobolev spaces can be found in [GT01, pp. 145, 153]. A more detailed treatise is given in [Ada75].
- Notice that Definition 1.2.9 can easily extended to functions $u = (u^1, \ldots, u^n)^T : \Omega \to \mathbb{R}^n$ with an open set $\Omega \subset \mathbb{R}^m$. We denote the Sobolev space of such functions by $W^{k,p}(\Omega, \mathbb{R}^n)$ and set the norm of u to

  $$||u||_{W^{k,p}(\Omega,\mathbb{R}^n)} \quad := \quad \sum_{i=1}^n \sum_{\substack{\alpha \text{ multi-index,} \\ |\alpha| \leq k}} ||\mathrm{D}^\alpha u^i||_{L^p(\Omega)}. \tag{1.2.4}$$

  Further, the local Sobolev space $W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is defined by

  $$W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \quad := \quad \left\{u : \Omega \to \mathbb{R}^n \ : \ u \in W^{k,p}(\Omega', \mathbb{R}^n) \text{ for all open sets } \Omega' \subset\subset \Omega\right\}$$

  (cf. [GT01, p. 154]). For an open set $\Omega' \subset \Omega$, where the closure $\overline{\Omega'} \subset \Omega$ is a compact set contained in Ω, we write $\Omega' \subset\subset \Omega$.

1.2.2 Fréchet spaces and some examples

We based this section on a book of Rudin (see [Rud91]) regarding the functional analysis of Fréchet spaces. The theory on Fréchet spaces is especially useful in the investigation of the spherical Radon transform (see section 2.2). We adapted some of the notation to this setting, but it essentially remains similar to that of [Rud91]. This section is structured as follows: We start by introducing the Fréchet space with corresponding notations and then we discuss function spaces of k-homogeneous functions on $\mathbb{R}^n\backslash\{0\}$.

Definition 1.2.11 (**Fréchet space** (cf. [Rud91])). Let V be a vector space. A *seminorm* ρ on V is a mapping $\rho : V \to [0, \infty)$ such that

$$\begin{aligned} \rho(tv) &= |t|\rho(v) \quad \text{for all } v \in V \text{ and } t \in \mathbb{R}, \\ \rho(v+w) &\leq \rho(v) + \rho(w) \quad \text{for all } v, w \in V. \end{aligned}$$

A family of seminorms $\mathfrak{P}$ on V is said to be *separating* if for every $v \in V\backslash\{0\}$ there exists an $\rho \in \mathfrak{P}$ with $\rho(v) \neq 0$. A family of seminorms $\mathfrak{P}$ on V generates a topology on V with local basis

$$\mathfrak{B} \quad := \quad \left\{\cap_{l=1}^k V_l : \{V_1, \dots, V_k\} \subset \mathfrak{V}\right\},$$

where

$$\begin{aligned} \mathfrak{V} &:= \{V_{i,j} : i, j \in \mathbb{N}\}, \\ V_{i,j} &:= \left\{v \in V : \rho_i(v) < \frac{1}{j}\right\}. \end{aligned}$$

A set $O \subset V$ is *open* if for every $v \in O$ and $\rho \in \mathfrak{P}$, there exists an $\varepsilon = \varepsilon(v, \rho) > 0$ such that the set $\{w : \rho(v-w) < \varepsilon\}$ is a subset of O. A sequence $\{v_k\}_{k\in\mathbb{N}} \subset V$ is a *Cauchy-sequence w.r.t. a seminorm* $\rho \in \mathfrak{P}$ if to every $\varepsilon > 0$ there corresponds an integer $k_0 = k_0(\varepsilon)$ such that $\rho(v_k - v_l) < \varepsilon$ for all $k, l \geq k_0$. A sequence $\{v_k\}_{k\in\mathbb{N}} \subset V$ is a *Cauchy-sequence* in V if to every $\varepsilon > 0$ and $\rho \in \mathfrak{P}$ there corresponds an integer $k_0 = k_0(\varepsilon, \rho)$ such that $\rho(v_k - v_l) < \varepsilon$ for all $k, l \geq k_0$. A sequence $\{v_k\}_{k\in\mathbb{N}} \subset V$ *converges to* v *w.r.t. a seminorm* $\rho \in \mathfrak{P}$ if to every $\varepsilon > 0$ there corresponds an integer $k_0 = k_0(\varepsilon)$ such that $\rho(v_k - v) < \varepsilon$ for all $k \geq k_0$. We will abbreviate this by $v_k \xrightarrow[k\to\infty]{\rho} v$. A sequence $\{v_k\}_{k\in\mathbb{N}}$ *converges* to v in V if to every $\varepsilon > 0$ and $\rho \in \mathfrak{P}$ there corresponds an integer $k_0 = k_0(\varepsilon, \rho)$ such that $\rho(v_k - v) < \varepsilon$ for all $k \geq k_0$. We will abbreviate this by $v_k \xrightarrow[k\to\infty]{} v$. V is *complete* if for every Cauchy-sequence $\{v_k\}_{k\in\mathbb{N}}$ in V, there exists a vector $v \in V$ such that v_k converges to v. A vector space V with a countable separating family of seminorms $\mathfrak{P}$, which is complete, will be referred to as a *Fréchet space*.

Remark 1.2.12. The ingredients to this definition can be found in [Rud91]. There, another definition of the Fréchet spaces is used in [Rud91, 1.8, p. 9], but the more suitable characterizations we used in Definition 1.2.11 can be found in the remarks [Rud91, 1.38 (b), (c), p. 29].

In the following, we define some special function spaces (see Definition 1.2.13), which are vector spaces subset to $C^\infty(\mathbb{R}^n\backslash\{0\})$. On these function spaces, we define two families of seminorms, one also used in [Rud91] on $C^\infty(\mathbb{R}^n\backslash\{0\})$ (see Theorem 1.2.15) and another one is defined in Definition 1.2.18 equivalent to the former one (see Lemma 1.2.19).

Definition 1.2.13 (**Homogeneity**). A function $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ is called *even* if $f(y) = f(-y)$ for all $y \in \mathbb{R}^n$. A function $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ is called *odd* if $f(y) = -f(-y)$ for all $y \in \mathbb{R}^n$. For $k \in \mathbb{Z}$, a function $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ is called *k-homogeneous* or *homogeneous of degree* k if f satisfies the following condition:

$$f(ty) \quad = \quad t^k f(y) \quad \text{for all } t > 0 \text{ and } y \in \mathbb{R}^n.$$

For $k \in \mathbb{Z}$ and $l \in \mathbb{N} \cup \{0, \infty\}$, $\mathrm{HF}^l_k(\mathbb{R}^n \backslash \{0\})$ denotes the space of l-times differentiable functions on $\mathbb{R}^n \backslash \{0\}$, which are homogeneous of degree k. Further, $\mathrm{HF}^l_{k,e}(\mathbb{R}^n \backslash \{0\})$ denotes the space of even l-times differentiable functions on $\mathbb{R}^n \backslash \{0\}$, which are homogeneous of degree k and $\mathrm{HF}^l_{k,o}(\mathbb{R}^n \backslash \{0\})$ denotes the space of odd l-times differentiable functions on $\mathbb{R}^n \backslash \{0\}$, which are homogeneous of degree k.

Remark 1.2.14. The vector spaces $\mathrm{HF}^l_k(\mathbb{R}^n \backslash \{0\})$, $\mathrm{HF}^l_{k,e}(\mathbb{R}^n \backslash \{0\})$ and $\mathrm{HF}^l_{k,o}(\mathbb{R}^n \backslash \{0\})$ are subspaces of $C^l(\mathbb{R}^n \backslash \{0\})$.

In the next theorem, we construct a countable family of seminorms on $C^\infty(\mathbb{R}^n \backslash \{0\})$ using Euclidean balls. The open Euclidean *ball* of radius $R > 0$ and center $p \in \mathbb{R}^n$ is defined by

$$B_R(p) \quad := \quad B^n_R(p) \quad := \quad \{x \in \mathbb{R}^n : |x - a| < R\},$$

the closed ball of radius $R > 0$ an center $p \in \mathbb{R}^n$ is defined by

$$\overline{B_R(p)} \quad := \quad \overline{B^n_R(p)} \quad := \quad \{x \in \mathbb{R}^n : |x - p| \leq R\}$$

and the *unit sphere* is defined by

$$\mathbb{S}^{n-1} \quad := \quad \{x \in \mathbb{R}^n : |x| = 1\}.$$

The *unit ball* in $\mathbb{R}^n$, i.e. the ball of radius 1 around 0, will sometimes be referred to by $\mathbb{B}^n$.

Theorem 1.2.15 ([Rud91, Section 1.46])**.** *$C^\infty(\mathbb{R}^n \backslash \{0\})$ together with the countable family of seminorms $\{\rho_j\}_{j \in \mathbb{N}_0}$, where*

$$\rho_j(f) \quad = \quad \max\{|\mathrm{D}^\alpha f(y)| : y \in K_j,\ |\alpha| \leq j\} \tag{1.2.5}$$

for $j \in \mathbb{N}_0$ and $f \in C^\infty(\mathbb{R}^n \backslash \{0\})$, is a Fréchet space. Therein, we choose a sequence of nested compact sets

$$K_j \quad := \quad \overline{B_{2^{j+1}}(0) \backslash B_{2^{-(j+1)}}(0)}$$

for $j \in \mathbb{N}_0$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^n_0$ with $|\alpha| = \sum_{i=1}^n \alpha_i$.

Remark 1.2.16. The seminorms $\rho_j(\cdot)$ are even well-defined on $C^l(\mathbb{R}^n \backslash \{0\})$ for $j \leq l$.

Theorem 1.2.17. *$\mathrm{HF}^\infty_k(\mathbb{R}^n \backslash \{0\})$, $\mathrm{HF}^\infty_{k,e}(\mathbb{R}^n \backslash \{0\})$ and $\mathrm{HF}^\infty_{k,o}(\mathbb{R}^n \backslash \{0\})$ are Fréchet spaces induced by the family of seminorms $\{\rho_j\}_{j \in \mathbb{N}_0}$.*

Proof. Due to Remark 1.2.14, $\mathrm{HF}^\infty_k(\mathbb{R}^n \backslash \{0\})$, $\mathrm{HF}^\infty_{k,e}(\mathbb{R}^n \backslash \{0\})$ and $\mathrm{HF}^\infty_{k,o}(\mathbb{R}^n \backslash \{0\})$ are subspaces of $C^\infty(\mathbb{R}^n \backslash \{0\})$. So, the only thing left to show is that they are both closed w.r.t. the topology introduced in Theorem 1.2.15. For a sequence of converging functions $f_l \to f$ with f_l an infinitely differentiable function homogeneous of degree k for each l, the convergence in C^∞ leads to pointwise convergence of the sequence. This way the properties of being homogeneous, even or odd carry over to the limit f. □

Definition 1.2.18. For an l-times differentiable function f on $\mathbb{R}^n \backslash \{0\}$, i.e. $f \in C^l(\mathbb{R}^n \backslash \{0\})$, we define the seminorms

$$\widehat{\rho}_j(f) \quad := \quad \max\{|\mathrm{D}^\alpha f(\zeta)| : \zeta \in \mathbb{S}^{n-1},\ |\alpha| \leq j\} \tag{1.2.6}$$

for all $j \in \mathbb{N}_0$.

Lemma 1.2.19. *For an l-times differentiable and k-homogeneous function f on $\mathbb{R}^n \backslash \{0\}$, i.e. $f \in \mathrm{HF}^l_k(\mathbb{R}^n \backslash \{0\})$, holds*

$$\widehat{\rho}_l(f) \quad \leq \quad \rho_l(f) \quad \leq \quad 2^{(l+1)(|k|+l)} \widehat{\rho}_l(f) \tag{1.2.7}$$

for all $l \in \mathbb{N}_0$.

Proof. The first estimate in (1.2.7) is trivial since $\mathbb{S}^{n-1} \subset K_l$ for all $l \in \mathbb{N}_0$. To show the second estimate, we will make use of the $(k-|\alpha|)$-homogeneity of $\mathrm{D}^\alpha f$, i.e.

$$\mathrm{D}^\alpha f(y) \quad = \quad |y|^{k-|\alpha|}\mathrm{D}^\alpha f(\frac{y}{|y|}) \ \text{ for all } \ y \in \mathbb{R}^n\backslash\{0\},$$

so that we can deduce directly

$$|\mathrm{D}^\alpha f(y)| \quad \leq \quad |y|^{k-|\alpha|}\max\{|\mathrm{D}^\alpha f(\zeta)| : \zeta \in \mathbb{S}^{n-1},\ |\alpha| \leq l\}$$

for all $y \in \mathbb{R}^n\backslash\{0\}$, multi-indices α with $|\alpha| \leq l$. Further, there holds that

$$|y|^{k-|\alpha|} \quad \leq \quad \begin{cases} (2^{l+1})^{k-\alpha}, & \text{for } k-\alpha \geq 0 \\ (2^{-(l+1)})^{k-\alpha}, & \text{for } k-\alpha < 0 \end{cases}$$

since $2^{-(l+1)} \leq |y| \leq 2^{l+1}$ for $y \in K_l$. So, we get

$$|y|^{k-|\alpha|} \quad \leq \quad 2^{(l+1)|k-|\alpha||} \quad \leq \quad 2^{(l+1)(|k|+|\alpha|)} \quad \leq \quad 2^{(l+1)(|k|+l)}.$$

Consequently, we deduce that

$$\begin{aligned} |\mathrm{D}^\alpha f(y)| \quad &\leq \quad 2^{(l+1)(k+|\alpha|)}\max\{|\mathrm{D}^\alpha f(\zeta)| : \zeta \in \mathbb{S}^{n-1},\ |\alpha| \leq l\} \\ &\leq \quad 2^{(l+1)(k+l)}\rho_l(f) \end{aligned}$$

for all $y \in \mathbb{R}^n\backslash\{0\}$ and multi-indices α with $|\alpha| \leq l$. Taking the maximum over the left-hand side yields the desired estimate. □

Remark 1.2.20.

- The seminorm-equivalence (1.2.7) implies that the families of seminorms $\{\rho_j\}_{j\in\mathbb{N}_0}$ and $\{\widehat{\rho}_j\}_{j\in\mathbb{N}_0}$ induce the same Fréchet topology on $\mathrm{HF}^\infty_k(\mathbb{R}^n\backslash\{0\})$. This is also the case for the subspaces $\mathrm{HF}^\infty_{k,e}(\mathbb{R}^n\backslash\{0\})$ and $\mathrm{HF}^\infty_{k,o}(\mathbb{R}^n\backslash\{0\})$.
- Each vector space $\mathrm{HF}^l_k(\mathbb{R}^n\backslash\{0\})$, $\mathrm{HF}^l_{k,e}(\mathbb{R}^n\backslash\{0\})$ and $\mathrm{HF}^l_{k,o}(\mathbb{R}^n\backslash\{0\})$ for $l \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ becomes a Banach spaces by means of the norm $\widehat{\rho}_l(\cdot)$. A *Banach space* is a complete normed vector space.

Definition 1.2.21. For a k-homogeneous function $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ define the restriction of f to the unit sphere $\Phi(f) := \Phi_k(f)$ by

$$\Phi_k(f)(\cdot) \quad := \quad f|_{\mathbb{S}^{n-1}}(\cdot). \tag{1.2.8}$$

For a spherical function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ define its k-homogeneous extension $\Psi_k(g)$ by

$$\Psi_k(g)(\cdot) \quad := \quad |\cdot|^k g\left(\frac{\cdot}{|\cdot|}\right). \tag{1.2.9}$$

A function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ is said to be *continuous* (or l*-times differentiable*) if its 0-homogeneous extension is continuous (or l-times differentiable), i.e. $\Psi_0(g) \in C^0(\mathbb{R}^n\backslash\{0\})$ (or $\Psi_0(g) \in C^l(\mathbb{R}^n\backslash\{0\})$) and we denote the set of such functions by $C^0(\mathbb{S}^{n-1})$ (or $C^l(\mathbb{S}^{n-1})$). For $g \in C^1(\mathbb{S}^{n-1})$, the *spherical gradient* $\nabla_o g$ is defined by

$$\nabla_o g \quad := \quad \Phi_0(\frac{\partial}{\partial z^i}\Psi_0(g))\delta^{ij}e_i,$$

where $\Psi_0(g) = \Psi_0(g)(z)$ for $z = (z^i) \in \mathbb{R}^n\backslash\{0\}$ and $\{e_i\}_{i=1}^n$ is the canonical basis of the Euclidean vector space $\mathbb{R}^n$. For $g \in C^2(\mathbb{S}^{n-1})$, the *spherical Laplacian* $\Delta_o g$ is defined by

$$\Delta_o g \quad := \quad \Phi_0(\Delta\Psi_0(g)),$$

where $\Delta = \delta^{ij}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^j}$ is the Euclidean Laplacian.

Remark 1.2.22.

- The mappings Φ_k and Ψ_k are inverse one to another. If we choose functions $f : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ k-homogeneous and $g : \mathbb{S}^{n-1} \to \mathbb{R}$, then

$$\begin{aligned} f &= \Psi_k \circ \Phi_k(f), \\ g &= \Phi_k \circ \Psi_k(g). \end{aligned}$$

  This is the reason, we will sometimes write Φ_k^{-1} instead of Ψ_k.

- It can be easily shown that in the definition of the continuity and l-times differentiability on $\mathbb{S}^{n-1}$, the k-homogeneous extension can be used in an analogous way instead of the 0-homogeneous extension. Further, the notions of continuity and differentiability on $\mathbb{S}^{n-1}$ are compatible with those given for manifolds (as $\mathbb{S}^{n-1}$ is one) in Definition 1.3.7 (cf. [Bär10, Proposition 3.1.11]).

- The spherical Laplacian coincides (locally) with the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ as we define it in Definition 1.5.5.

Theorem 1.2.23. *For $l \in \{0, 1\}$, there is a bijective linear mapping from $\mathrm{HF}_k^l(\mathbb{R}^n\backslash\{0\})$ onto $C^l(\mathbb{S}^{n-1})$ and from $\mathrm{HF}_{k,e}^l(\mathbb{R}^n\backslash\{0\})$ onto $C_e^l(\mathbb{S}^{n-1})$. The bijection is given by Φ_k as in Definition 1.2.21 and its inverse is given by $\Phi_k^{-1} = \Psi_k$.*

Proof. Define for $f \in \mathrm{HF}_k^l(\mathbb{R}^n\backslash\{0\})$ the mapping

$$\Phi(f)(\zeta) \quad := \quad \Phi_k(f)(\zeta) \quad := \quad f|_{\mathbb{S}^{n-1}}(\zeta),\ \zeta \in \mathbb{S}^{n-1}$$

and for $g \in C^l(\mathbb{S}^{n-1})$

$$\Psi_k(g)(y) \quad := \quad |y|^k g\left(\frac{y}{|y|}\right),\ y \in \mathbb{R}^n\backslash\{0\}.$$

These mappings are both linear in f and g, respectively. Looking at their composition yields

$$\begin{aligned} \Psi_k \circ \Phi_k(f)(y) &= |y|^k \Phi_k(f)\left(\frac{y}{|y|}\right) \\ &= |y|^k f\left(\frac{y}{|y|}\right) \\ &= f(y) \end{aligned}$$

and

$$\begin{aligned} \Phi_k \circ \Psi_k(g)(\zeta) &= \Psi_k(g)|_{\mathbb{S}^{n-1}}(\zeta) \\ &= |\zeta|^k g\left(\frac{\zeta}{|\zeta|}\right) \\ &= g(\zeta). \end{aligned}$$

So, both mappings are inverse one to another and therefore bijective. Moreover if f or g is even, $\Phi_k(f)$ or $\Psi_k(g)$ is even, respectively. Also the differentiability in the case $l = 1$ carries over due to Definition 1.2.21 and Remark 1.2.22. □

1.3 Manifolds

The scope of this section is to present some basic differential geometry concepts like the notions manifold and tangent vector, among others. The section is mainly based on [Lee03] and in parts on [Jän01]. Remember that the Einstein summation convention will be used, i.e. the summation over repeated indices. Further, where it seems useful, the Ricci calculus will be a notational tool.

Definition 1.3.1 (**Topoligical manifold**). An *m-dimensional topological manifold* is a second countable Hausdorff space $\mathscr{M}$ s.t. for each point p in $\mathscr{M}$ there exists an open neighbourhood U of p and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$, with $\varphi(U)$ an open neighbourhood of $\varphi(p)$ in $\mathbb{R}^m$. We call such a homeomorphism φ a *(local) coordinate chart* on $\mathscr{M}$ and denote it shortly by (U, φ).

Definition 1.3.2 (**Manifold with boundary**). The closed m-dimensional *upper halfspace* $\mathbb{H}^m \subset \mathbb{R}^m$ is defined as

$$\mathbb{H}^m \quad := \quad \{x = (x^1, \dots, x^m) \in \mathbb{R}^m : x^m \geq 0\}.$$

We denote by $\mathrm{int}\mathbb{H}^m$ and $\partial\mathbb{H}^m$ the interior and boundary of $\mathbb{H}^m$, respectively, as subsets of $\mathbb{R}^m$, i.e.

$$\begin{aligned} \mathrm{int}\mathbb{H}^m \quad &:= \quad \{x = (x^1, \dots, x^m) \in \mathbb{R}^m : x^m > 0\}, \\ \partial\mathbb{H}^m \quad &:= \quad \{x = (x^1, \dots, x^m) \in \mathbb{R}^m : x^m = 0\}. \end{aligned}$$

An *m-dimensional topological manifold with boundary* is a second countable Hausdorff space $\mathscr{M}$ s.t. for each point p in $\mathscr{M}$ there exists an open neighbourhood U of p and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{H}^m$ with $\varphi(U)$ a (relatively) open neighbourhood of $\varphi(p)$ in $\mathbb{H}^m$. We call such a homeomorphism φ a *(local) coordinate chart* of $\mathscr{M}$ and denote it shortly by (U, φ). A coordinate chart (U, φ) for which $\varphi(U) \subset \mathrm{int}\mathbb{H}^m$ holds, is an *interior coordinate chart*. Conversely if $\varphi(U) \cap \partial\mathbb{H}^m \neq \emptyset$, (U, φ) is a *boundary coordinate chart*. A *C^k manifold with boundary* is a topological manifold whose transition maps are all k-times continuously differentiable for $k \in \mathbb{N} \cup \{0, \infty\}$ fixed. A *smooth manifold with boundary* or *differentiable manifold with boundary* is a C^∞-manifold with boundary.

Remark 1.3.3.

- An m-dimensional topological manifold $\mathscr{M}$ with boundary, whose charts (U, φ) do all have no intersection of their images $\varphi(U)$ with $\partial\mathbb{H}^m$, is an m-dimensional topological manifold. Similarly, we say C^k *manifold* or *smooth manifold* if all the images of its charts do have no intersection with $\partial\mathbb{H}^m$.
- If we intend to name the dimension of an m-dimensional topological, C^k or smooth manifold $\mathscr{M}$, we shortly write topological, C^k or smooth *m-manifold*. Another way to name the dimension is to write it as an upper index, i.e. $\mathscr{M}^m$. Further, $\dim\mathscr{M}$ denotes the *dimension of a manifold*.
- Given a coordinate chart (U, φ) on an m-dimensional topological manifold $\mathscr{M}$ (with boundary), we call U the *coordinate domain* of each of its points. The map φ is called the *(local) coordinate map*, and the component functions $(x^1, \dots, x^m)$ of φ, defined by $\varphi(p) = (x^1(p), \dots, x^m(p))$ for $p \in U$, are called *local coordinates* (x^α) on U. Therefore, a local coordinate chart (U, φ) on a manifold $\mathscr{M}$ (with boundary) will be sometimes referred to by $(U, (x^\alpha))$, where the x^α for $\alpha = 1, \dots, m$ stand for the coordinate functions of φ. Further, we will say that a coordinate chart (U, φ) contains p as shorthand for a coordinate chart (U, φ) whose coordinate domain contains p and local coordinates (x^α) at p if their coordinate domain contains p.

Definition 1.3.4 (Regularity structures on manifolds). Let $\mathscr{M}$ be a topological m-manifold (with boundary). A *transition map* from φ to ψ for two coordinate charts (U, φ), (V, ψ) with $U \cap V \neq \emptyset$ is the mapping $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. A *smooth atlas* $\mathscr{A}$ is a set of coordinate charts, where the union of the coordinate domains covers $\mathscr{M}$ and where all of the transition maps are smooth, i.e.

- $\mathscr{M} = \bigcup_{(U,\varphi)\in\mathscr{A}} U$;
- $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is of class C^∞ for all $(U, \varphi), (V, \psi) \in \mathscr{A}$ with $U \cap V \neq \emptyset$.

For every atlas $\mathscr{A}$, there exists a unique *maximal smooth atlas*; that is an atlas containing $\mathscr{A}$ and which is not contained in any other atlas different from the maximal smooth atlas itself. A *smooth structure* on an m-dimensional topological manifold $\mathscr{M}$ is a maximal smooth atlas. A *smooth manifold* is an m-dimensional topological manifold endowed with a smooth structure on $\mathscr{M}$.

Remark 1.3.5.

- We can define notions as C^k *atlas*, C^k *structure* and C^k *manifold* with boundary for $k \in \mathbb{N} \cup \{\infty\}$ in a similar and straightforward manner by adapting Definition 1.3.4 accurately. In this context, a C^∞ manifold is nothing else but a smooth manifold.
- For two coordinate charts on $\mathscr{M}$, namely (U, φ) with local coordinates (x^α) and (V, ψ) with local coordinates $(\tilde{x}^\alpha)$, in some cases, we write the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ in the shorthand notation

$$\psi \circ \varphi^{-1}(x) \quad = \quad (\tilde{x}^1(x), \ldots, \tilde{x}^m(x))^T \quad = \quad (\tilde{x}^\alpha(x))_{\alpha=1,\ldots,m} \quad =: \quad (\tilde{x}^\alpha(x))$$

  for $x \in \varphi(U \cap V)$.

Definition 1.3.6 (Boundary of a manifold). Let $\mathscr{M}$ be an m-dimensional smooth manifold with boundary. A *boundary point* is a point on $\mathscr{M}$, whose image under a smooth coordinate chart lies in $\partial\mathbb{H}^m$, and an *interior point* is a point on $\mathscr{M}$, whose image under a smooth coordinate chart lies in $\mathrm{int}\mathbb{H}^m$. The set of all boundary points of $\mathscr{M}$ is called the *boundary* of $\mathscr{M}$ and will be denoted by $\partial\mathscr{M}$; the set of all interior points of $\mathscr{M}$ is called the *interior* of $\mathscr{M}$ and will be denoted by $\mathrm{int}\mathscr{M}$.

Definition 1.3.7 (Differentiable functions). Assume $l \in \mathbb{N} \cup \{\infty\}$. Let $\mathscr{M}$ and $\mathscr{N}$ be smooth manifolds. A function $f : \mathscr{M} \to \mathscr{N}$ is said to be *l-times continuously differentiable* in p if and only if there exist coordinate charts (U, φ) on $\mathscr{M}$ and (V, ψ) on $\mathscr{N}$ with U containing p and V containing $f(p)$ such that

$$\psi \circ f \circ \varphi^{-1}$$

is l-times continuously differentiable at $\varphi(p)$. f is said to be l-times continuously differentiable if it is l-times continuously differentiable in every point $p \in \mathscr{M}$. In the case of a smooth manifold with boundary, the differentials have to exist in the interior and have to be continuously extendable up to the boundary in each boundary chart. In any case, $C^l(\mathscr{M}, \mathscr{N})$ will denote the set of all l-times continuously differentiable functions from $\mathscr{M}$ to $\mathscr{N}$. Especially, we set

$$C^l(\mathscr{M}) \quad := \quad C^l(\mathscr{M}, \mathbb{R}).$$

Further, $C^l_0(\mathscr{M})$ denotes the subset of functions in $C^l(\mathscr{M})$ of compact support (cf. subsection 1.2.1). The *support* of a function $g : \mathscr{M} \to \mathbb{R}$ is the set

$$\mathrm{supp} g \quad := \quad \overline{\{p \in \mathscr{M} : g(p) \neq 0\}}.$$

Remark 1.3.8. Sometimes we will say smooth instead of l-times differentiable.

Definition 1.3.9 (Tangential vectors). Let $\mathscr{M}$ be an m-dimensional smooth manifold. A *tangent vector* X to $\mathscr{M}$ at p is a linear mapping $X : \mathcal{E}_p(\mathscr{M}) \to \mathbb{R}$, which is a *derivation*, i.e. for $f, g \in \mathcal{E}_p(\mathscr{M})$

$$X(f \cdot g)|_p \quad = \quad X(f)|_p \cdot g(p) + f(p) \cdot X(g)|_p.$$

Therein,

$$\mathcal{E}_p(\mathscr{M}) \quad := \quad \{[f]_p : f \in C^\infty(\mathscr{M})\}$$

wherein $[f]_p$ is the equivalence class of $f \in C^\infty(\mathscr{M})$ w.r.t. the equivalence relation $\sim_p$. $f \sim_p g$ if and only if there exists a neighbourhood U_p of p s.t. the restrictions of f and g to U_p coincide. This is the *algebraic definition of a tangent vector* to $\mathscr{M}$ in p. The *tangent space* $T_p\mathscr{M}$ to $\mathscr{M}$ at p is the set of all tangent vectors to $\mathscr{M}$ at p.

Remark 1.3.10.

- In Definition 1.3.9, the set $T_p\mathscr{M}$ for $p \in \mathscr{M}$ can be shown to be an m-dimensional real vector space. For a coordinate chart $(U, \varphi) = (U, (x^\alpha))$ with $p \in U$, we get a basis of the vector space $T_p\mathscr{M}$ by defining the derivations

$$\left(\frac{\partial}{\partial x^\alpha}(f \circ \varphi^{-1})\right)|_{\varphi(p)} \;\text{ for } \alpha = 1, \ldots, m$$

  and $f \in \mathcal{E}_p(\mathscr{M})$. These vectors will be referred to, in a slight abuse of notation, as $\frac{\partial}{\partial x^\alpha}|_p$. Each $\frac{\partial}{\partial x^\alpha}|_p$ is called *coordinate vector*.

- There are two other characterizing definitions of tangent vectors, the geometric and physical one, giving essentially the same vector space up to an isomorphism. The *geometric definition of a tangent vector* is given by the equivalence class

$$v \quad := \quad [c]_p^{\text{geo}} \quad := \quad \{\tilde{c} : \tilde{c} \in C^1((-\varepsilon, \varepsilon), \mathscr{M}) \text{ for some } \varepsilon > 0, \tilde{c} \sim_p^{\text{geo}} c\}$$

  for a given curve $c : (-\varepsilon, \varepsilon) \to \mathscr{M}$ for some $\varepsilon > 0$ and with $c(0) = p$. Two curves $c, \tilde{c} : (-\varepsilon, \varepsilon) \to \mathscr{M}$ are said to be geometrically equivalent at $p \in \mathscr{M}$, namely $\tilde{c} \sim_p^{\text{geo}} c$ if and only if for a coordinate chart (U, φ) with $p \in U$ holds $\varphi(c(p)) = \varphi(\tilde{c}(p))$ and $\frac{\partial}{\partial t}|_{t=0}\varphi(c(t)) = \frac{\partial}{\partial t}|_{t=0}\varphi(\tilde{c}(t))$. For a coordinate chart $(U, \varphi) = (U, (x^\alpha))$ with $p \in U$ and $\beta \in \{1, \ldots, m\}$, the basis vector $\frac{\partial}{\partial x^\beta}|_p$ w.r.t. (U, φ) can be expressed in the geometrical interpretation of the tangent vector to $\mathscr{M}$ at p as the equivalence class of the curve $\varphi^{-1}(p + te_\beta)$ for $t \in (-\varepsilon, \varepsilon)$ with ε sufficiently small, wherein e_γ for $\gamma = 1, \ldots, m$ are the standard basis vectors of $\mathbb{R}^m$. The *physical definition of a tangent vector* is the following. Let $\mathcal{D}_p(\mathscr{M}) := \{(U, \varphi) : p \in U \text{ and } (U, \varphi) \text{ is a coordinate chart of } \mathscr{M}\}$. A physically defined tangent vector to $\mathscr{M}$ in p is a mapping $v : \mathcal{D}_p(\mathscr{M}) \to \mathbb{R}^m$ which behaves well under coordinate change, i.e.

$$v^\beta(U, \varphi) \quad = \quad \frac{\partial}{\partial \tilde{x}^\alpha}(x^\beta \circ \psi^{-1})|_{\psi(p)} v^\alpha(V, \psi)$$

  for $(U, \varphi) = (U, (x^\alpha))$ and $(V, \psi) = (V, (\tilde{x}^\alpha))$ in $\mathcal{D}_p(\mathscr{M})$.

- The dependencies, i.e. on a point p of the manifold $\mathscr{M}$, will be omitted if they can be deduced from the context. Especially, we will write shortly $\frac{\partial x^\beta}{\partial \tilde{x}^\alpha}$ for the components of the differential of the transition map, so namely for

$$\frac{\partial}{\partial \tilde{x}^\alpha}(x^\beta \circ \psi^{-1})|_{\psi(p)}$$

  for coordinate charts (U, φ) and (V, ψ) with local coordinates (x^α) and $(\tilde{x}^\alpha)$, respectively and $p \in U \cap V$.

Definition 1.3.11 (Cotangent vectors). A *cotangent vector* θ to $\mathscr{M}$ at p is a linear mapping $\theta : T_p\mathscr{M} \to \mathbb{R}$. The *cotangent space* $(T_p\mathscr{M})^*$ to $\mathscr{M}$ at p is the space of all cotangent vectors to $\mathscr{M}$ at p and as such the dual space to $T_p\mathscr{M}$. For a local coordinate chart (U, x) on $\mathscr{M}$ with $p \in U$, a basis of the cotangent space to $\mathscr{M}$ at p is given by the dual basis $\mathrm{d}x^1, \dots, \mathrm{d}x^m$ to the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}$ of the tangential space $T_p\mathscr{M}$ to $\mathscr{M}$ at p.

Definition 1.3.12 (Tangent bundle and Cotangent bundle). Let $\mathscr{M}$ be an m-dimensional smooth manifold. The *tangent bundle* $T\mathscr{M}$ is defined as the disjoint union $\bigcup_{p\in\mathscr{M}} T_p\mathscr{M}$ of all the tangent spaces $T_p\mathscr{M}$ to $\mathscr{M}$. The *cotangent bundle* $T^*\mathscr{M}$ is defined as the disjoint union $\bigcup_{p\in\mathscr{M}} (T_p\mathscr{M})^*$ of all the cotangent spaces $(T_p\mathscr{M})^*$ to $\mathscr{M}$.

Example 1.3.13. Let $\mathbb{R}^m$ be the m-dimensional real vector space. In standard coordinates, we identify the coordinate vectors with the Euclidean standard basis $\{e_\alpha\}_{\alpha=1}^m$. Hence, each tangent space $T_p\mathbb{R}^m$ can be identified with $\mathbb{R}^m$. Further, the tangent bundle $T\mathbb{R}^m$ can be identified with $\mathbb{R}^m \times \mathbb{R}^m$, i.e. $T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m$.

Definition 1.3.14 (Differential). Let $\mathscr{M}$ be an m-dimensional smooth manifold, $\mathscr{N}$ an n-dimensional smooth manifold and $f : \mathscr{M} \to \mathscr{N}$ a smooth mapping. The *differential* $\mathrm{d}f : T\mathscr{M} \to T\mathscr{N}$ of f is defined pointwise in the following way: Let $v \in T_p\mathscr{M}$, then we get a tangent vector on $\mathscr{N}$ at $f(p)$ by

$$v(g \circ f)|_p \in T_p\mathscr{M}$$

for a function $g \in \mathcal{E}_p(\mathscr{N})$. By using a coordinate chart (U, φ) with local coordinates (u^α) on $\mathscr{M}$, $p \in U$ and a coordinate chart $(V, (x^i))$ on $\mathscr{N}$ with $f(p) \in V$, the differential of f can be locally expressed by

$$\left(\frac{\partial}{\partial u^\alpha}(x^i \circ f \circ \varphi^{-1})|_{\varphi(p)}\right) \frac{\partial}{\partial x^i}|_{f(p)} \otimes \mathrm{d}u^\alpha|_p.$$

Remark 1.3.15.

- Definition 1.3.6-1.3.9, Definition 1.3.11-1.3.12, and Definition 1.3.14 also make sense if we assume $\mathscr{M}$ to be a C^k manifold with boundary with $k \geq 1$ or a smooth manifold with boundary. In this case, the physical definition of a tangent vector in Remark 1.3.10 has to be adapted to contain outward-pointing vectors (cf. Definition 1.3.18). In Definition 1.3.14, we then require, similar to Definition 1.3.7, the differential of a function to be extended continuously up to the boundary. The regularity assumption of f being smooth in Definition 1.3.14 can then be replaced by f being of class C^l for $1 \leq l \leq k$.

- In the spirit of the last point of Remark 1.3.10, the notion $\frac{\partial f^\beta}{\partial u^\alpha}|_p$ or $f^\beta_{u^\alpha}|_p$ will be a shorthand notation for the expression

$$\frac{\partial}{\partial u^\alpha}(x^i \circ f \circ \varphi^{-1})|_{\varphi(p)}$$

as found in Definition 1.3.14.

Definition 1.3.16 (Oriented manifold). Let $\mathscr{M}$ be an m-dimensional smooth manifold with smooth structure $\mathscr{A}$. A smooth atlas $\tilde{\mathscr{A}}$ subset to $\mathscr{A}$ is said to be *consistently oriented* if all of the transition maps of $\tilde{\mathscr{A}}$ have differentials of positive determinant, i.e.

$$\det\left(\frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\right) \quad > \quad 0,$$

for all coordinate charts $(U, (x^\alpha))$, $(V, (\tilde{x}^\beta))$ in $\tilde{\mathscr{A}}$. $\mathscr{M}$ is said to be an *orientable smooth manifold* if there exists an consistently oriented smooth atlas $\tilde{\mathscr{A}}$. Further, due to [Lee03, p.

327], the smooth structure $\mathscr{A}$ can then be partitioned into two disjoint consistently oriented atlases, i.e.

$$\begin{aligned}\mathscr{A} &= \mathscr{A}_1 \cup \mathscr{A}_2\\ \mathscr{A}_1 \cap \mathscr{A}_2 &= \emptyset\end{aligned}$$

with $\mathscr{A}_1$, $\mathscr{A}_2$ consistently oriented atlases. $\tilde{\mathscr{A}}$ is then either a subset of $\mathscr{A}_1$ or $\mathscr{A}_2$. $\mathscr{A}_1$ and $\mathscr{A}_2$ are the *orientations* of $\mathscr{M}$. $\mathscr{M}$ together with an orientation $\mathscr{O} \in \{\mathscr{A}_1, \mathscr{A}_2\}$ is said to be an *oriented smooth manifold*. A coordinate chart $(U, \varphi) \in \mathscr{A}$ such that $(U, \varphi) \in \mathscr{O}$ is said to be positively oriented and is said to be negatively oriented otherwise.

Remark 1.3.17.

- An analogous construction as in Definition 1.3.16 can be carried out for C^k manifold for an integer $k \geq 1$.
- Sometimes if a smooth manifold $\mathscr{M}$ is endowed in a somehow natural or commonly known way with an orientation, we will shortly write $-\mathscr{M}$ to refer to the smooth manifold endowed with the other orientation.

Definition 1.3.18 (Inward-pointing vector). Let $\mathscr{M}$ be a smooth manifold with boundary $\partial\mathscr{M}$ and $q \in \partial\mathscr{M}$. A vector $v \in T_q\mathscr{M}$ is said to be *inward-pointing* if $v \notin T_q\partial\mathscr{M}$ and if there exists for some $\varepsilon > 0$ a smooth curve $c : [0, \varepsilon] \to \mathscr{M}$ s.t. $c(0) = q$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} c(t) = v$. A vector in $T_q\mathscr{M}$ is said to be *outward-pointing* if its multiplicative inverse is inward-pointing.

Definition 1.3.19 (Compact manifold). A smooth manifold $\mathscr{M}$ (with boundary) is said to be compact if for every family of open sets $\{V_i\}_{i\in I}$ with $V_i \subset \mathscr{M}$ for all $i \in I$, $\mathscr{M} \subset \bigcup_{i\in I} V_i$ and index set I, there exists a finite subset $J \subset I$ s.t. $\mathscr{M} \subset \bigcup_{j\in J} V_j$.

Definition 1.3.20 (Immersion). Let $\mathscr{M}$ be an m-dimensional smooth manifolds with boundary and $\mathscr{N}$ an n-dimensional smooth manifolds with boundary. An *immersion* $X : \mathscr{M} \to \mathscr{N}$ is a C^1 mapping, whose differential $\mathrm{d}X|_p : T_p\mathscr{M} \to T_{X(p)}\mathscr{N}$ is of full rank m for all $p \in \mathscr{M}$.

Definition 1.3.21 (Homeomorphism and Embedding). Let $\mathscr{M}$ be an m-dimensional smooth manifold (with boundary) and $\mathscr{N}$ an n-dimensional smooth manifold (with boundary). A *homeomorphism* $X : \mathscr{M} \to \mathscr{N}$ is a mapping such that for each open set $\mathscr{O} \subset \mathscr{N}$ w.r.t. to the topology of $\mathscr{N}$, the set $X^{-1}(\mathscr{O}) = \{u \in \mathscr{M} : X(u) \in \mathscr{O}\}$ is open w.r.t. the topology of $\mathscr{M}$. A *smooth embedding* $X : \mathscr{M} \to \mathscr{N}$ is an injective immersion, which is a homeomorphism onto its image $X(\mathscr{M})$. $X(\mathscr{M})$ is therein thought to be equipped with the subspace topology inherited from $\mathscr{N}$.

Definition 1.3.22 (Immersed and embedded manifold). Let $\mathscr{N}$ be an n-dimensional smooth manifold (with boundary). A subset $\mathscr{M} \subset \mathscr{N}$, which is the image of a smooth immersion, is called *m-dimensional immersed submanifold* or *immersed m-submanifold*, where $\mathscr{M}$ is seen as a smooth m-manifold with the smooth structure derived from an immersion, whose image is $\mathscr{M}$. A subset $\mathscr{M} \subset \mathscr{N}$ is called an *m-dimensional embedded submanifold* or *embedded m-submanifold* if $\mathscr{M}$ is a smooth m-manifold, whose *inclusion map* $\iota : \mathscr{M} \hookrightarrow \mathscr{N}$ is a smooth embedding. If $n = m + 1$ an immersed m-submanifold is called *immersed hypersurface* and an embedded m-submanifold is called *embedded hypersurface*. If $m = 2$ we call them *immersed surface* or *embedded surface*, respectively.

Remark 1.3.23.

- Further information on the concept of smooth embeddings and embedded submanifolds presented in Definition 1.3.21 and Definition 1.3.22 can be found in [Lee03, pp. 156, 174-180].

- Every embedded submanifold is also an immersed submanifold by definition (see [Lee03, pp. 174, 187]).
- A submanifold or hypersurface can also be a manifold with boundary. We assume this depending on the context, where the respective notions are used.
- The notions defined in Definition 1.3.22 can be adapted to versions of regularity C^1, as this regularity is necessary to the definition of an immersion. Hence, we will speak of a submanifold or hypersurface of class C^k for $k \geq 1$ in that sense.

Definition 1.3.24 (Vector bundle). Let $\mathscr{E}$, $\mathscr{M}$ be smooth manifolds (with boundary) and $\pi \in C^\infty(\mathscr{E}, \mathscr{M})$. Set $\mathscr{E}_p := \pi^{-1}(\{p\})$ for all $p \in \mathscr{M}$. Then $(\mathscr{E}, \mathscr{M}, \pi) = \bigcup_{p \in \mathscr{M}} \mathscr{E}_p$ is called a C^∞ *or smooth vector bundle of rank* n over $\mathscr{M}$ if and only if

- each fibre $\mathscr{E}_p := \pi^{-1}(\{p\})$ has an n-dimensional vector space structure.
- for every point p on $\mathscr{M}$ exists an open neighbourhood $\mathscr{U}$ of $\mathscr{M}$ and a diffeomorphism $\psi : \pi^{-1}(\mathscr{U}) \to \mathscr{U} \times \mathbb{R}^n$ s.t. $\psi|_{\mathscr{E}_p} \to \{p\} \times \mathbb{R}^n$ is an isomorphism between vector spaces. Such a map ψ is said to be a *smooth local trivialization* of $\mathscr{E}$ and $(\mathscr{U}, \psi)$ is called *smooth bundle chart* on $\mathscr{E}$.

A smooth map $S : \mathscr{M} \to \mathscr{E}$ is called a smooth *section* of $\mathscr{E}$ if the composition $\pi \circ S = \mathrm{id}_{\mathscr{M}}$, where $\mathrm{id}_{\mathscr{M}}$ is the identity map on $\mathscr{M}$. A local smooth section of $\mathscr{E}$ is a smooth section $S : \mathscr{U} \to \mathscr{E}$, where $\mathscr{U}$ is some open subset of $\mathscr{M}$. Let $\mathscr{V}$ be an open subset of $\mathscr{M}$. Local sections $\sigma_1, \dots, \sigma_n$ of $\mathscr{E}$ over $\mathscr{V}$ are said to be *independent* if $\sigma_1(p), \dots, \sigma_n(p)$ are linearly independent vectors in $\mathscr{E}_p$ for every $p \in \mathscr{M}$. Analogously, $\sigma_1, \dots, \sigma_n$ are said to *span* $\mathscr{E}$ if the linear span of $\sigma_1(p), \dots, \sigma_n(p)$ equals $\mathscr{E}_p$ for every $p \in \mathscr{M}$. A *local smooth frame* for $\mathscr{E}$ over $\mathscr{V}$ is an ordered n-tuple $(\sigma_1, \dots, \sigma_n)$ of independent local smooth sections over $\mathscr{V}$, which span E, i.e. $\sigma_1(p), \dots, \sigma_n(p)$ is a basis of the fibre E_p for every $p \in \mathscr{V}$.

Remark 1.3.25.

- $(\mathscr{E}, \mathscr{M}, \pi)$ will be abbreviated by $\bigcup_{p \in \mathscr{M}} \mathscr{E}_p$ to refer to a smooth vector bundle $\mathscr{E}$ of rank n over a smooth manifold $\mathscr{M}$ and we write $(u, v) \in \bigcup_{p \in \mathscr{M}} \mathscr{E}_p$ to refer to its elements, wherein $u \in \mathscr{M}$ and $v \in \mathscr{E}_u$. The map π is then given implicitly via $\pi(u, v) := u$ for $(u, v) \in \bigcup_{p \in \mathscr{M}} \mathscr{E}_p$.
- A C^k vector bundle of rank n can be defined in a similar way by replacing smooth in Definition 1.3.24 by k-times differentiable.
- Similar to the manifold case, we can define such notions as *smooth bundle atlas*, based on the definition of a smooth bundle chart.

Example 1.3.26. The tangent bundle of an m-dimensional smooth manifold $\mathscr{M}$ is a smooth vector bundle of rank m. A trivial smooth section of the tangent bundle is the zero section $o = \{(p, 0) \in T\mathscr{M}\}$. Let $(U, (u^\alpha))$ be a smooth coordinate chart on $\mathscr{M}$. A local smooth frame of $T\mathscr{M}$ over U is given by $(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^m})$ (cf. induced bundle coordinates in Definition 1.4.1). A section of $T\mathscr{M}$ is called *vector field*.

The cotangent bundle of a smooth m-manifold $\mathscr{M}$ is a smooth vector bundle of rank m. A trivial smooth section of the tangent bundle is the zero section $o = \{(p, 0) \in T^*\mathscr{M}\}$. Let $(U, (u^\alpha))$ be a smooth coordinate chart on $\mathscr{M}$. A local smooth frame of $T\mathscr{M}$ over U is given by $(\mathrm{d}u^1, \dots, \mathrm{d}u^m)$. A section of $T^*\mathscr{M}$ is called *covector field*.

Definition 1.3.27 (Tensor). Let $\mathscr{M}$ be a smooth manifold. The *bundle of covariant k-tensors* on $\mathscr{M}$ will be referred to as $T^k(\mathscr{M})$ and is defined by

$$T^k(\mathscr{M}) \quad := \quad \bigcup_{p \in \mathscr{M}} T^k(T_p\mathscr{M}).$$

A smooth section of $T^k(\mathscr{M})$ is called *covariant k-tensor field*. The *bundle of contravariant k-tensors* on $\mathscr{M}$ will be referred to as $T_k(\mathscr{M})$ and is defined by

$$T_k(\mathscr{M}) \quad := \quad \bigcup_{p\in\mathscr{M}} T_k(T_p\mathscr{M}).$$

A smooth section of $T_k(\mathscr{M})$ is called *contravariant k-tensor field*.

Definition 1.3.28 (Partition of unity). Let $\mathscr{M}$ be a smooth manifold with boundary $\partial\mathscr{M}$. A smooth *partition of unity* on $\mathscr{M}$ is a set of continuous functions $\{\varphi_i\}_{i\in I}$ for an index set I s.t. for every $p \in \mathscr{M}$ holds

- $0 \le \varphi_i(p) \le 1$ for all $i \in I$;
- there exists a neighbourhood where all except a finite number of φ_i vanish;
- the well-defined sum $\sum_{i\in I} \varphi_i(p) = 1$.

Let (V_i, x_i) be a covering set of coordinate charts for $i \in I$, then there exists a subordinate partition of unity $\{\varphi_i\}_{i\in I}$ with $\text{supp}\varphi_i \subset V_i$ for all $i \in I$. A partition of unity $\{\varphi_i\}_{i\in I}$ for which all φ_i are smooth or C^k will be, respectively called a smooth or C^k partition of unity.

Remark 1.3.29. Smooth partitions of unity like in Definition 1.3.28 exist on every smooth manifold $\mathscr{M}$ with boundary (see [Lee03, Theorem 2.25, p. 54]).

Definition 1.3.30 (Alternating form). Let $\mathscr{M}$ be a smooth manifold with boundary $\partial\mathscr{M}$. A *smooth (alternating) k-form* ω is a function s.t. $\omega|_p \in \Lambda^k((T_p\mathscr{M})^*)$ for all $p \in \mathscr{M}$ and which can be expressed in local coordinates (x^α) at $p \in \mathscr{M}$ by

$$\omega|_p \quad = \sum_{1\le\alpha_1<\ldots<\alpha_k\le m} \omega_{\alpha_1,\ldots,\alpha_k}|_p \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_k} \tag{1.3.1}$$

wherein the coefficients $\omega_{\alpha_1,\ldots,\alpha_k}|_p$ depend smoothly on p and (1.3.1) is independent of the actual choice of local coordinates, i.e.

$$\omega_{\alpha_1,\ldots,\alpha_k}|_p \quad = \sum_{1\le\beta_1<\ldots<\beta_k\le m} \tilde{\omega}_{\beta_1,\ldots,\beta_k}|_p \det\left(\frac{\partial\tilde{x}^{\beta_t}}{\partial x^{\alpha_s}}\right)_{s,t=1,\ldots,k} \tag{1.3.2}$$

for local coordinates (x^α) and $(\tilde{x}^\beta)$ at p. The vector space of all k-forms will be denoted by $\Omega^k(\mathscr{M})$.

Remark 1.3.31.

- The set of alternating k-tensors $\Omega^k(\mathscr{M}) := \bigcup_{p\in\mathscr{M}} \Lambda^k(T_p\mathscr{M})$ is a subset of $T^k(\mathscr{M})$. A section ω of $\Omega^k(\mathscr{M})$, i.e. $\omega \in \Gamma(\Omega^k(\mathscr{M}))$, is a smooth alternating k-form and vice versa.
- A smooth partitions of unity is useful to construct a global smooth k-form from well-transforming local representations as in (1.3.1).

Definition 1.3.32 (Bundle maps). For two smooth vector bundles $\pi : \mathscr{E} \to \mathscr{M}$ and $\tilde{\pi} : \tilde{\mathscr{E}} \to \tilde{\mathscr{M}}$, a smooth *bundle map* from $\mathscr{E}$ to $\tilde{\mathscr{E}}$ is a pair of smooth maps $F : \mathscr{E} \to \tilde{\mathscr{E}}$ and $f : \mathscr{M} \to \tilde{\mathscr{M}}$ s.t. $\tilde{\pi} \circ F = f \circ \pi$ and where the restriction $F|_{\mathscr{E}_p} : \mathscr{E}_p \to \tilde{\mathscr{E}}_{f(p)}$ is linear for every $p \in \mathscr{M}$.

Remark 1.3.33. Definition 1.3.32 can easily be adapted to define C^k bundle maps.

Example 1.3.34. Let $\mathscr{M}$, $\mathscr{N}$ be smooth manifolds and $X : \mathscr{M} \to \mathscr{N}$ a smooth mapping. The differential $\mathrm{d}X : T\mathscr{M} \to T\mathscr{N}$ is a smooth bundle map from $T\mathscr{M}$ to $T\mathscr{N}$.

1.4 Finsler metrics and Finsler manifolds

This section introduces some basics on Finsler geometry. Our main sources regarding these fundamentals are the books of Shen [She01] and Bao et al. [BCS00]. Nevertheless, we will present to some extend weakened notions of Finsler metric and Finsler manifold, similar to [MRTZ09] or [PT09]. Therefore, we will present this theory in a slightly different way than usual. Notice further that in its properties, the Finsler metric is very similar if not equivalent to the so-called elliptic Cartan integrand, which will be presented in section 1.6. At the end of this section, we give a set of useful assumptions for real-valued maps defined on the tangent bundle. Most of the definitions in this section assume a smooth n-manifold, but can be generalized in a straightforward manner to a smooth n-manifold with boundary.

Definition 1.4.1. Let $\mathscr{N}$ be a smooth n-manifold and $T\mathscr{N} = \bigcup_{x\in\mathscr{N}} T_x\mathscr{N}$ be its tangent bundle. The subset $o := \{(x,0) \in T\mathscr{N}\}$ of $T\mathscr{N}$ is the *zero section* of $T\mathscr{N}$. Let $F : T\mathscr{N} \to \mathbb{R}$ be a real-valued map on the tangent bundle. F is a continuous real-valued mapping on the tangent bundle if we write

$$F \in C^0(T\mathscr{N}). \tag{\textbf{b}}$$

F is called *homogeneous* or *homogeneous of degree one* if

$$F(x,ty) \quad = \quad tF(x,y) \tag{\textbf{h}}$$

for all $t > 0$, $(x,y) \in T\mathscr{N}$. F is called *k-homogeneous* or *homogeneous of degree k* if

$$F(x,ty) \quad = \quad t^kF(x,y) \tag{\textbf{k}}$$

for all $t > 0$, $(x,y) \in T\mathscr{N}$. F is called *positive* if

$$F(x,y) \quad > \quad 0 \tag{\textbf{p}}$$

for all $(x,y) \in T\mathscr{N}\backslash o$. F is called *reversible* or *even* if

$$F(x,y) \quad = \quad F(x,-y) \tag{\textbf{s}}$$

for all $(x,y) \in T\mathscr{N}$. F is called *odd* if

$$F(x,y) \quad = \quad -F(x,-y) \tag{\textbf{a}}$$

for all $(x,y) \in T\mathscr{N}$. F is called *convex* if

$$F(x,t_1y_1 + t_2y_2) \quad \le \quad t_1F(x,y_1) + t_2F(x,y_2) \tag{\textbf{c}}$$

for all $t_1, t_2 \in (0,1)$ with $t_1 + t_2 = 1$ and $y_k \in T_x\mathscr{N}$ for $k = 1,2$. F is called *elliptic* if $F \in C^2(T\mathscr{N}\backslash o)$ and

$$g^F_{ij}(x,y) \quad := \quad (\frac{F^2}{2})_{y^iy^j}(x,y) \quad > \quad 0 \tag{\textbf{e}}$$

for all $(x,y) \in T\mathscr{N}\backslash o$, this means that the matrix $(g^F_{ij}(x,y))_{ij}$ is positive-definite for every choice of local coordinates $x^1,\cdots,x^n$ together with their *induced bundle coordinates* $y^1,\cdots,y^n$ via $y = y^i\frac{\partial}{\partial x^i}|_x \in T_x\mathscr{N}$ and $F(x,y) = F(x^1,\cdots,x^n,y^1,\cdots,y^n)$. The expression $(g^F_{ij}(x,y))_{ij}$ is named the *fundamental tensor* and (locally) defines the *fundamental form* by

$$g^F|_{(x,y)}(u,v) \quad := \quad \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\{F^2(x,y+su+tv)\}|_{s=t=0} \quad = \quad g^F_{ij}(x,y)u^iv^j,$$

for $u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i} \in T_x\mathscr{N}$.

Remark 1.4.2.

- Local coordinates (x^i) on an open set $U \subset \mathscr{N}$ induce local coordinates (y^i) on $T_xU \subset T_x\mathscr{N}$ for each $x \in U$ by means of $y = y^i \frac{\partial}{\partial x^i}|_x$ for each $y \in T_xU$ (see Definition 1.4.1 (**e**)). Hence, we get local coordinates (x^i, y^i) on an open set $TU \subset T\mathscr{N}$. These local coordinates are sometimes called *local bundle coordinates on* U, *local bundle coordinates at* x or *local bundle coordinates*. For a function $F : T\mathscr{N} \to \mathbb{R}$, which is differentiable in some point $(x, y) \in T\mathscr{N}$, we set

$$F_{x^i}(x,y) \quad := \quad \frac{\partial}{\partial x^i}F(x,y) \quad \text{and} \quad F_{y^i}(x,y) \quad := \quad \frac{\partial}{\partial y^i}F(x,y)$$

  for $i = 1, \ldots, m$ to ease notation of derivatives w.r.t. local bundle coordinates. The vectors $\frac{\partial}{\partial x^i}|_{(x,y)}$ and $\frac{\partial}{\partial y^i}|_{(x,y)}$ are elements of $T_{(x,y)}(T\mathscr{N})$ for $i = 1, \ldots, n$. Notice that the tangent bundle is by itself a manifold (cf. [BCS00]).

- For a homogeneous function $F : T\mathscr{N} \to [0, \infty)$ with $F \in C^2(T\mathscr{N}\backslash o)$, the condition (**e**) implies (**p**) and (**c**) (see [BCS00, pp. 7-9]).

- For a homogeneous function $F : T\mathscr{N} \to [0, \infty)$ with $F \in C^2(T\mathscr{N}\backslash o)$, the condition (**c**) is equivalent to

$$\xi^i F_{y^i y^j}(x,y)\xi^j \quad \geq \quad 0$$

  in local bundle coordinates (x^i, y^i) at $x \in \mathscr{N}$ with $y, \xi \in T_x\mathscr{N}$, where $\xi = \xi^i \frac{\partial}{\partial x^i}|_x$.

- A function $F : T\mathscr{N} \to \mathbb{R}$ is said to be *irreversible* if the condition (**s**) does not apply to F (cf. [CS05, p. 1]).

The following theorem is useful for some computations involving k-homogeneous functions. We state the theorem for k-homogeneous functions $f : \mathbb{R}^{m+1} \to \mathbb{R}$ as defined in Definition 1.2.13. Nevertheless, it can be directly applied to k-homogeneous functions $F : T\mathscr{N} \to \mathbb{R}$ as defined in Definition 1.4.1, since $F(x, \cdot)$ is k-homogeneous in the sense of Definition 1.2.13 for every $x \in \mathscr{N}$. Therein, we identify the vector space $T_x\mathscr{N}$ with $\mathbb{R}^n$.

Theorem 1.4.3 (Euler's theorem [BCS00, Theorem 1.2.1]**).** *Let $H : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ be a function which is differentiable away from the origin. Then the following two statements are equivalent*

- *H is homogeneous of degree k, i.e.*

$$H(ty) \quad = \quad t^k H(y) \quad \text{for all } t > 0,\ y \in \mathbb{R}^n\backslash\{0\}.$$

- *The radial directional derivative of H equals k-times H. Namely,*

$$y^i H_{y^i}(y) \quad = \quad kH(y) \quad \text{for all } t > 0,\ y \in \mathbb{R}^n\backslash\{0\}.$$

Remark 1.4.4. For a function H as in Theorem 1.4.3, which is homogeneous of degree k, its directional derivatives of first order are all homogeneous of degree $k - 1$.

Example 1.4.5. Let $\mathscr{N}$ be a smooth manifold and $F : T\mathscr{N} \to \mathbb{R}$ be a homogeneous function of class at least C^2 on $T\mathscr{N}\backslash o$. In local bundle coordinates (x^i, y^i), there holds

$$\begin{aligned} F_{y^i}(x,y)y^i \quad &= \quad F(x,y), \\ F_{y^i y^j}(x,y)y^j \quad &= \quad 0, \end{aligned}$$

as a direct consequence of Theorem 1.4.3 and since

$$g^F_{ij}(x,y) \quad = \quad F_{y^i}(x,y)F_{y^i}(x,y) + F(x,y)F_{y^iy^j}(x,y)$$

by differentation, we finally deduce that

$$g^F_{ij}(x,y)y^iy^j \quad = \quad F^2(x,y).$$

Definition 1.4.6 (**Finsler metric** [BCS00, pp. 2-3]). Let $\mathscr{N}$ be a smooth n-manifold and $T\mathscr{N}$ its tangent bundle. Then, a function $F : T\mathscr{N} \to [0,\infty)$ is called *weak Finsler structure* or *weak Finsler metric* on $\mathscr{N}$ if $F \in C^\infty(T\mathscr{N}\backslash o)$ and F is a homogeneous, positive definite and convex function on the tangent bundle, which means that (**h**), (**p**), (**c**) and (**b**) hold. $F : T\mathscr{N} \to [0,\infty)$ is called *Finsler structure* or *Finsler metric* on $\mathscr{N}$ if $F \in C^\infty(T\mathscr{N}\backslash o)$ and F is a homogeneous and elliptic function on the tangent bundle, which means that (**h**), (**e**) and (**b**) hold. A smooth manifold $\mathscr{N}$ with a (weak) Finsler structure F is called *(weak) Finsler manifold* or *(weak) Finsler space* and will be denoted by $(\mathscr{N}, F)$ or $(\mathscr{N}^n, F)$.

Remark 1.4.7.

- The notion of a weak Finsler structure has been introduced by Papadopoulos and Troyanov [PT09, Definition 5.1, p. 6] and there is a similar concept of Finsler metric presented by Matveev et al. [MRTZ09, p. 937].
- In Definition 1.4.6, we do not necessarily need as much regularity to guarantee that the conditions (**b**), (**h**), (**p**), (**c**) or (**e**) are well-defined. So, we will speak of F as a (weak) C^k-Finsler structure on a smooth n-manifold $\mathscr{N}$ meaning that $F \in C^k(T\mathscr{N}\backslash o)$ and then $(\mathscr{N}, F)$ will be referred to as a *(weak) C^k-Finsler manifold* for $(k \geq 0)$ $k \geq 2$.
- Every Finsler metric is also a weak Finsler metric due to Remark 1.4.2.
- For a reversible weak Finsler manifold $(\mathscr{N}, F)$ and $x \in \mathscr{N}$ each tangent space $T_x\mathscr{N}$ becomes a normed vector space with norm $F(x,\cdot)$.
- The *Finslerian length* of a smooth curve $\gamma : [a,b] \to \mathscr{N}$ is defined by

$$\mathcal{L}_F(\gamma) \quad := \quad \int_{[a,b]} F(\gamma(t), \dot{\gamma}(t))\mathrm{d}t \tag{1.4.1}$$

where $\dot{\gamma}(t) := \frac{\partial\gamma}{\partial t}(t)$ for all $t \in [a,b]$. The main idea of defining a Finsler metric F on an smooth manifold $\mathscr{N}$ is to measure the distance between two points p, q on $\mathscr{N}$. This distance can be measured by computing the infimum of the Finslerian length of smooth curves joining p and q, i.e.

$$d_F(p,q) \quad := \quad \inf_{\gamma \in C_{p,q}} \mathcal{L}_F(\gamma), \tag{1.4.2}$$

where $C_{p,q} := \{\gamma \in C^\infty([0,1], \mathscr{N}) : \gamma(0) = p, \gamma(1) = q\}$. We refer to $d_F(p,q)$ as the *Finslerian distance* from p to q for all $p, q \in \mathscr{N}$. The *Finslerian distance* $\mathrm{dist}_F(A,B)$ of a set $A \subset \mathscr{N}$ to a set $B \subset \mathscr{N}$ is defined by

$$\mathrm{dist}_F(A,B) \quad := \quad \inf_{p\in A,\ q \in B} d_F(p,q). \tag{1.4.3}$$

We will sometimes speak of the *Finslerian distance* $d_F(p,B)$ of a point $p \in \mathscr{N}$ to a set $B \subset \mathscr{N}$ where we set $d_F(p,B) := d_F(\{p\}, B)$. We can even define the *Finslerian length* of a continuous curve $\gamma : [a,b] \to \mathscr{N}$ by setting

$$\mathcal{L}_F(\gamma) \quad := \quad \sup\left\{\sum_{i=1}^k d_F(\gamma(t_{i-1}), \gamma(t_i)) : a \leq t_0 < t_1 \leq \ldots \leq t_k = b\right\}. \tag{1.4.4}$$

The *Euclidean length* $\mathcal{L}(\gamma)$ of a continuous curve γ in $\mathbb{R}^n$ is defined by $\mathcal{L}(\gamma) := \mathcal{L}_{|\cdot|}(\gamma)$. The *Euclidean distance* $d(p,q)$ from p to q is defined by $d(p,q) = d_{|\cdot|}(p,q)$ and can be computed to be $d(p,q) = |p-q|$ for all $p,q \in \mathbb{R}^n$. The *Euclidean distance* $\mathrm{dist}(A,B)$ of a set $A \subset \mathbb{R}^n$ to a set $B \subset \mathbb{R}^n$ is defined by $\mathrm{dist}(A,B) := \mathrm{dist}_{|\cdot|}(A,B)$. The *Euclidean distance* $\mathrm{dist}(A,B)$ of a point $p \in \mathbb{R}^n$ to a set $B \subset \mathbb{R}^n$ is defined by $\mathrm{dist}(p,B) := \mathrm{dist}_{|\cdot|}(p,B)$.

Definition 1.4.8 (**Dual Finsler metric** (cf. [BCS00, pp. 406-411] & [She98, p. 35])**.**
Let $(\mathscr{N},F)$ be a weak Finsler manifold. The *dual Finsler structure* or *dual Finsler metric* $F^*: T^*\mathscr{N} \to [0,\infty)$ is defined by

$$F^*(x,\theta) \quad = \quad \sup_{y\in T_x\mathscr{N}} \frac{\theta(y)}{F(x,y)} \tag{1.4.5}$$

for $(x,\theta) \in T^*\mathscr{N}$.

Remark 1.4.9.

- In the setting of Definition 1.4.8 assume that F is a Finsler metric so that the coefficients g^F_{ij} as in Definition 1.4.1 are well-defined in local bundle coordinates (x^i,y^i) on $\mathscr{N}$. Denote by $(g^F)^{ij}$ the coefficients of the inverse matrix to the matrix with coefficients g^F_{ij}. Then

$$F^*(x,\theta) \quad = \quad \sup_{y\in T_x\mathscr{N}} \sqrt{(g^F)^{ij}(x,y)\theta_i|_x\theta_j|_x} \tag{1.4.6}$$

  for $x \in \mathscr{N}$, $y = y^i\frac{\partial}{\partial x^i}|_x \in T_x\mathscr{N}$ and $\theta = \theta_i \mathrm{d}x^i|_x \in (T_x\mathscr{N})^*$ (see [GH96, p. 16]). Sometimes we write $||\theta||_{\widehat{g^F}}|_x$ instead of $F^*(x,\theta)$.

- If F is a reversible weak Finsler metric, $F^*(x,\cdot)$ is the *dual norm* on $(T_x\mathscr{N})^*$ to the norm $F(x,\cdot)$ on $T_x\mathscr{N}$ at $x \in \mathscr{N}$ (cf. [Rud91, p. 92]).

Example 1.4.10. Let $\mathscr{N}$ be a smooth n-manifold together with a family of scalar products $a = \{a|_x\}_{x\in\mathscr{N}}$ on $T\mathscr{N} = \bigcup_{x\in\mathscr{N}} T_x\mathscr{N}$ such that the quantities

$$a_{ij}(x) \quad := \quad a|_x(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) \quad \text{for} \quad x \in \mathscr{N}$$

are smooth in local coordinates (x^i). Such a family of scalar products a is called a *Riemannian metric*. The pair $(\mathscr{N},a)$ is then called a *Riemannian manifold*. Remember that a scalar product p on a real vector space V is a symmetric, positive-definite and bilinear mapping $p: V\times V \to \mathbb{R}$ (see section 1.1.2). A Riemannian manifold is automatically a Finsler manifold with Finsler metric $\alpha(x,y)$ given in local coordinates by

$$\alpha(x,y) \quad = \quad \sqrt{a_{ij}(x)y^iy^j}$$

for $x \in \mathscr{N}$ and $y = y^i\frac{\partial}{\partial x^i}|_x \in T_x\mathscr{N}$. In fact, the fundamental tensor $(g^\alpha)_{ij}(x,y)$ - given in local bundle coordinates (x^i,y^i) - coincides with $a_{ij}(x)$. In this work, the Finsler metric issued from a Riemannian metric will also be called a *Riemannian metric* or *Riemannian Finsler metric*, if we have to make a distinction. Let β be a 1-form that is an element of $\Omega^1(\mathscr{N}) = \Gamma(T^*\mathscr{N})$. Notice that a 1-form can be seen as a family $\{b|_x\}_{x\in\mathscr{N}}$, wherein the quantities

$$b_i(x) \quad := \quad b|_x(\frac{\partial}{\partial x^i}) \quad \text{for} \quad x \in \mathscr{N}$$

are smooth in local coordinates (x^i). The *(dual) norm* of such a 1-form β w.r.t. the Riemannian Finsler metric α is given by

$$||\beta||_{\widehat{\alpha}}\big|_x \quad := \quad \alpha^*(x, b|_x) \quad = \quad \sup_{y \in T_x\mathscr{N}} \frac{\beta(x,y)}{\alpha(x,y)}. \tag{1.4.7}$$

Expressed in local coordinates (x^i) this gives rise to

$$||\beta||_{\widehat{\alpha}}\big|_x \quad := \quad \alpha^*(x, b|_x) \quad = \quad \sqrt{b^i(x)b_i(x)} \quad = \quad \sqrt{a^{ij}(x)b_i(x)b_j(x)},$$

wherein a^{ij} denote the coefficients of the inverse matrix to the matrix with coefficients a_{ij} and we raise or lower indices by multiplying and contracting with a^{ij} or a_{ij}, respectively. A general overview on Riemannian metrics can be found in [dC92] or [Jos02]. An investigation from the Finslerian point of view can be found in [BCS00, p. 351].

Definition 1.4.11 (**Minkowski space** [BCS00, pp. 6-7, 275])**.** Let F be a non-negative real-valued function on $\mathbb{R}^n$ with the following properties:

- $F \in C^\infty(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$;
- $F(ty) = tF(y)$ for all $t > 0$ and $y \in \mathbb{R}^n$;
- the matrix $(g^F_{ij}(y)) \in \mathbb{R}^{n \times n}$ is positive definite for all $y \neq 0$, where $g^F_{ij}(y) := (\frac{F^2}{2})_{y^i y^j}(y)$.

Such a function is called a *Minkowski metric* and a finite-dimensional vector space $\mathbb{R}^n$ together with a Minkowski metric F is called a *Minkowski space*. A Minkowski metric induces a Finsler structure on $\mathbb{R}^n$ by means of translation. So, a Minkowski space is also a Finsler manifold. A Finsler manifold $(\mathscr{N}, F)$ is called *locally Minkowski space* if for all $x \in \mathscr{N}$ exists an open set $\Omega = \Omega_x \subset \mathscr{N}$ containing x, with a coordinate chart $x^1, \cdots, x^n$ and corresponding bundle coordinates $y^1, \cdots, y^n$ (see Definition 1.4.1) such that

$$F(x,y) = F(y^1, \cdots, y^n) \quad \text{for all } (x,y) \in T\Omega.$$

Remark 1.4.12.

- Every n-dimensional real vector space can be identified by means of an appropriate isomorphism with the Euclidean real vector space $\mathbb{R}^n$, which is equipped with the Euclidean metric. Having this identification in mind, we can also speak of a Minkowski space (V, F) for an arbitrary finite-dimensional vector space.
- When talking about a Minkowski metric F on $\mathbb{R}^n$, we will simply write in a slight abuse of notation $F : \mathbb{R}^n \to \mathbb{R}$ and $F(y)$ for all $y \in \mathbb{R}^n$, inducing a Finsler structure $\tilde{F}$ on $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ by translation, i.e. $\tilde{F}(x,y) := F(y)$ for all $(x,y) \in T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.
- The second and third property Definition 1.4.11 can be formulated even if we only assume $F \in C^k(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ for $k \geq 2$. Hence, we will speak of a C^k*-Minkowski metric* and a C^k*-Minkowski space* in that sense.

Some useful examples of Finsler metrics are the so called (α, β)-metrics, which are to some extend of a simple structure.

Definition 1.4.13 (**(α, β)-Finsler space** [AIM93, p. 49])**.** Let $\mathscr{N}$ be a smooth n-manifold. Let α be a Riemannian Finsler metric and $\beta(x, \cdot)$ a 1-form on $T\mathscr{N}$ (cf. Example 1.4.10). An (α, β)*-metric* is a scalar function on $T\mathscr{N}$ defined by

$$F \quad := \quad \alpha\phi(\frac{\beta}{\alpha}), \tag{1.4.8}$$

where ϕ is a C^∞-function on $(-b_0, b_0)$. For any Riemannian metric α and any 1-form β on $\mathscr{N}$ with $||\beta||_{\widehat{\alpha}}\big|_x < b_0$ for all $x \in \mathscr{N}$ the function $F := \alpha\phi(\frac{\beta}{\alpha})$ is a Finsler metric if and only if

$$\phi(s) \quad > \quad 0 \quad \text{and} \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) \quad > \quad 0 \tag{1.4.9}$$

for all $|s| \leq b < b_0$. Such a manifold will be denoted by (α, β)-*Finsler space*.

Remark 1.4.14. A proof that the condition (1.4.9) characterizes an (α, β)-metric as a Finsler metric can be found in [She09, Lemma 2.1].

Example 1.4.15 (**Randers metric**). The choice $\phi(s) = 1 + s$ leads to the (α, β)-metric $F = \alpha + \beta$, a so called *Randers metric*, which is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < 1$ for all $x \in \mathscr{N}$. In this case the Finsler manifold $(\mathscr{N}, F)$ is also called *Randers manifold*. The Randers metric was introduced by Randers in [Ran41]. For more details on this topic, we refer to [AIM93], [BCS00] or [She01].

Example 1.4.16 (**Two order metric**). The choice $\phi(s) = (1 + s)^2$ leads to the (α, β)-metric $F := \alpha\phi(\frac{\beta}{\alpha})$, which is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < 1$ for all $x \in \mathscr{N}$, since

$$\phi(s) \quad = \quad (1 + s)^2, \quad \phi'(s) \quad = \quad 2(1 + s), \quad \phi''(s) \quad = \quad 2,$$

where $\phi(s)$ is well-defined and positive for all $|s| \leq b < 1$, and

$$\begin{aligned} \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) &= (1 + s)^2 - 2s(1 + s) + 2(b^2 - s^2) \\ &= 1 + 2s + s^2 - 2s - 2s^2 + 2b^2 - 2s^2 \\ &= 1 + 2b^2 - 3s^2 \\ &\geq 1 - b^2 \quad > \quad 0 \end{aligned}$$

for all $|s| \leq b < 1$. This (α, β)-metric is called *two order metric* and it seems to have been introduced by Cui and Schen [CS09].

Example 1.4.17 (**Matsumoto metric**). The choice $\phi(s) = \frac{1}{1-s}$ leads to the (α, β)-metric $F := \alpha\phi(\frac{\beta}{\alpha})$, which is well-defined if $||\beta||_{\widehat{\alpha}}\big|_x < 1$ for all $x \in \mathscr{N}$. F is a Finsler metric if $||\beta||_{\widehat{\alpha}}\big|_x < \frac{1}{2}$ for all $x \in \mathscr{N}$, since

$$\phi(s) \quad = \quad \frac{1}{1 - s}, \quad \phi'(s) \quad = \quad \frac{1}{(1 - s)^2}, \quad \phi''(s) \quad = \quad \frac{2}{(1 - s)^3},$$

where $\phi(s)$ is well-defined and positive for all $|s| \leq b < 1$, and

$$\begin{aligned} \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) &= \frac{(1 - s)^2 - s(1 - s) + 2(b^2 - s^2)}{(1 - s)^3} \\ &= \frac{1 - 2s + s^2 - s + s^2 + 2b^2 - 2s^2}{(1 - s)^3} \\ &= \frac{1 + 2b^2 - 3s}{(1 - s)^3} \\ &\geq \frac{2b^2 - 3b + 1}{(1 - s)^3} \quad > \quad 0 \end{aligned}$$

for all $|s| \leq b < \frac{1}{2}$, since $2b^2 - 3b + 1 = 2(b - 1)(b - \frac{1}{2})$ is positive for $b \in [0, \frac{1}{2})$. This (α, β)-metric is called *Matsumoto metric*. It was first mentioned by Matsumoto in [Mat89].

We conclude this section with the introduction of a set of properties of functions on the tangent bundle of a manifold, to ease reference on such a kind of function. Let $\mathscr{N}$ be an n-manifold for some $n \geq 2$. A function $F : T\mathscr{N} \to \mathbb{R}$ will sometimes be subject to the following assumption:

(M) *$F : T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function on the tangent space of $\mathscr{N} = \mathbb{R}^n$ independent of its first argument, so $F(x,y) \equiv F(y)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.*

In case F is subject to **(M)**, we sometimes simply write $F : \mathbb{R}^n \to \mathbb{R}$ and $F = F(y)$. We then get a structure on $T\mathbb{R}^n \cong \mathbb{R}^n$ by means of translation (cf. Remark 1.4.12).

For a function $F : T\mathscr{N} \to [0,\infty)$ we formulate the following list of assumptions:

(F1) *F is a 1-homogeneous, positive and continuous function on the tangent bundle of $\mathscr{N}$, i.e.* (**b**)*,* (**p**) *and* (**h**) *hold,*

(F2) *F is a weak C^0-Finsler metric, a 1-homogeneous, positive, continuous and convex function on the tangent bundle of $\mathscr{N}$, i.e.* (**b**)*,* (**p**)*,* (**h**) *and* (**c**) *hold,*

(F3) *F is a C^2-Finsler metric, a 1-homogeneous and elliptic function on the tangent bundle of $\mathscr{N}$ with $F \in C^2(T\mathscr{N}\backslash o)$, i.e.* (**b**) *and* (**e**) *hold.*

Notice that the following implications hold:

$$\textbf{(F3)} \Rightarrow \textbf{(F2)} \Rightarrow \textbf{(F1)}.$$

Assuming that the properties **(F3)** and **(M)** hold for F is equivalent to assuming $(\mathscr{N}, F)$ to be a C^2-Minkowski space. When the property **(M)** holds for F, then we will simply write in a slight abuse of notation $F : \mathbb{R}^n \to \mathbb{R}$ and $F(y)$ for all $y \in \mathbb{R}^n$, thereby inducing a structure on $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ by translation.

1.5 Immersions into Euclidean space

This section is dedicated to the fundamental differential geometric quantities regarding immersions of codimension 1 which map into Euclidean space. Notice that an immersion of *codimension* 1 is an immersion mapping from a smooth m-manifold with boundary to a smooth $(m+1)$-mainfold. We already gave a more general definition of immersions in Definition 1.3.20. The definitions and theorems of this section are a synthesis of standard theory to the topic and can be found, for example, in the books by Forster [For09], do Carmo [dC93] and Bär [Bär10] in a $\mathbb{R}^3$ setting and further in the one by Kühnel [Küh99] for hypersurfaces in $\mathbb{R}^{m+1}$. We additionally compute some basic identities, which are useful in later computations regarding the comparison of Finsler mean curvature to other curvature notions. Remember also that we label indices running from 1 to m with greek letters, indices running from 1 to $m+1$ with latin letters and we use the Einstein summation convention, i.e. we sum over repeated indices. Further, we drop some times the dependencies on points of the manifold - if they are clear given the context - and suppress the actual coordinates chart in which a local quantity is expressed.

Definition 1.5.1. Let $\mathscr{M}$ be an oriented smooth m-manifold (with boundary) and an immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$ of class C^1 that is a C^1-mapping whose differential is of full rank everywhere on $\mathscr{M}$ (see Definition 1.3.20). Its *first fundamental form* g is defined in local coordinates (u^α) at $p \in \mathscr{M}$ by

$$g|_p(v,w) \quad := \quad g_{\alpha\beta} \mathrm{d}u^\alpha \otimes \mathrm{d}u^\beta(v,w) \quad := \quad g_{\alpha\beta} v^\alpha w^\beta \tag{1.5.1}$$

for $v = v^\alpha \frac{\partial}{\partial u^\alpha}, w = w^\alpha \frac{\partial}{\partial u^\alpha} \in T_p\mathscr{M}$, $T_p\mathscr{M}$ the tangent space to $\mathscr{M}$ at p and

$$g_{\alpha\beta} \quad := \quad \left\langle \frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} \right\rangle. \tag{1.5.2}$$

This defines a Riemannian metric on $\mathscr{M}$ if X is smooth. We define the *Gauss map* $N : \mathscr{M} \to \mathbb{S}^m$ of X in a positively oriented coordinate chart (V, u) on $\mathscr{M}$ by

$$N(p) = \frac{\frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m}}{\left|\frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m}\right|}$$

for $p \in V$. The Gauss map is a well-defined global continuous map on $\mathscr{M}$ as can be shown by using the definition. If X is of class C^2 then we can define the *second fundamental form* in local coordinates (u^α) at $p \in \mathscr{M}$ as

$$h|_p(v, w) := h_{\alpha\beta} \mathrm{d}u^\alpha \otimes \mathrm{d}u^\beta (v, w) := h_{\alpha\beta} v^\alpha w^\beta \tag{1.5.3}$$

for $p \in \mathscr{M}$, $v = v^\alpha \frac{\partial}{\partial u^\alpha}$, $w = w^\alpha \frac{\partial}{\partial u^\alpha} \in T_p\mathscr{M}$, $T_p\mathscr{M}$ the tangent space to $\mathscr{M}$ at p and

$$h_{\alpha\beta} := -\left\langle \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} \right\rangle \tag{1.5.4}$$

$$= \left\langle N, \frac{\partial^2 X}{\partial u^\alpha \partial u^\beta} \right\rangle. \tag{1.5.5}$$

Further, the so called *Christoffel symbols* are then given by

$$\frac{\partial^2 X}{\partial u^\alpha \partial u^\beta} = \mathit{\Gamma}^\gamma_{\alpha\beta} \frac{\partial X}{\partial u^\gamma} + h_{\alpha\beta} N \tag{1.5.6}$$

or more concretely by

$$\mathit{\Gamma}^\gamma_{\alpha\beta} := g^{\gamma\tau} \left\langle \frac{\partial^2 X}{\partial u^\alpha \partial u^\beta}, \frac{\partial X}{\partial u^\tau} \right\rangle \tag{1.5.7}$$

$$= \frac{g^{\gamma\tau}}{2} \left(\frac{\partial g_{\alpha\tau}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\tau} + \frac{\partial g_{\tau\beta}}{\partial u^\alpha} \right), \tag{1.5.8}$$

wherein $g^{\gamma\tau}$ are the coefficients of the inverse matrix to the matrix $(g_{\alpha\beta}) \in \mathbb{R}^{m\times m}$. The *(Euclidean) mean curvature* H of the C^2-immersion X is given in local coordinates (u^α) of $\mathscr{M}$ by

$$H = g^{\alpha\beta} h_{\alpha\beta}. \tag{1.5.9}$$

Remark 1.5.2.

- The first and second fundamental form are smooth covariant 2-tensor fields on $\mathscr{M}$ for a smooth immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$. The first fundamental form g defines even a Riemannian metric on $\mathscr{M}$ and hence $(\mathscr{M}, g)$ is a Riemannian manifold. The corresponding Riemannian Finsler metric is given in local coordinates u^α on $\mathscr{M}$ by $\sqrt{g_{\alpha\beta} u^\alpha v^\beta}$ for $u = u^\alpha \frac{\partial}{\partial u^\alpha}$ and $v = v^\alpha \frac{\partial}{\partial u^\alpha}$ in $T\mathscr{M}$.
- For a smooth immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$ with an oriented smooth m-manifold $\mathscr{M}$, we define the vector bundle $TX := \mathrm{d}X(T\mathscr{M}) = \bigcup_{p\in\mathscr{M}} \mathrm{d}X(T_p\mathscr{M}) \subset T\mathbb{R}^{m+1}$ and call it the *tangent bundle to the immersion* X.

Theorem 1.5.3 (**Normal derivative**). *Let $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be an immersion of class C^2. Then we have in local coordinates (u^α) on $\mathscr{M}$*

$$\frac{\partial N}{\partial u^\alpha} = -h_{\alpha\tau} g^{\tau\gamma} \frac{\partial X}{\partial u^\gamma}, \tag{1.5.10}$$

where the respective quantities are those of Definition 1.5.1.

Proof. By the definition of the Gauss map N for an immersion X (see Definition 1.5.1) $\{\frac{\partial X}{\partial u^1}, \ldots, \frac{\partial X}{\partial u^m}, N\}$ is a basis of $\mathbb{R}^{m+1}$. So, we can express $\frac{\partial N}{\partial u^\alpha}$ for $\alpha = 1, \ldots, m$ as a linear combination of those basis vectors, i.e.

$$\frac{\partial N}{\partial u^\alpha} = W^\gamma \frac{\partial X}{\partial u^\gamma} + W^{m+1} N,$$

with unique coefficients W^i for $i = 1, \ldots, m+1$ and therewith

$$\begin{aligned} -h_{\alpha\tau} &= \left\langle \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\tau} \right\rangle \\ &= \left\langle W^\gamma \frac{\partial X}{\partial u^\gamma} + W^{m+1} N, \frac{\partial X}{\partial u^\tau} \right\rangle \\ &= W^\gamma \left\langle \frac{\partial X}{\partial u^\gamma}, \frac{\partial X}{\partial u^\tau} \right\rangle \\ &= W^\gamma g_{\gamma\tau}. \end{aligned}$$

Contracting with $g^{\tau\gamma}$ yields

$$W^\gamma = -h_{\alpha\tau} g^{\tau\gamma}$$

and further we get

$$\begin{aligned} W^{m+1} &= \left\langle \frac{\partial N}{\partial u^\alpha}, N \right\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial u^\alpha} (\langle N, N \rangle) \\ &= \frac{1}{2} \frac{\partial}{\partial u^\alpha} (1) = 0. \end{aligned}$$

Putting all together concludes the proof. □

Lemma 1.5.4. *Let $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be an immersion of class C^2. Then we have in local coordinates (u^α) on $\mathscr{M}$ the following identity*

$$-\left| \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right| g^{\alpha\beta} \frac{\partial X}{\partial u^\beta} = \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}, \quad (1.5.11)$$

where the respective quantities are those found in Definition 1.5.1.

Proof. Start by setting

$$W := \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}.$$

The vector W can be represented in terms of the basis $\left\{\frac{\partial X}{\partial u^1}, \ldots, \frac{\partial X}{\partial u^m}, N\right\}$ as

$$W = W^\gamma \frac{\partial X}{\partial u^\gamma} + W^{m+1} N,$$

where we have to compute the coefficients W^i for $i = 1, \ldots, m+1$. By exploiting $\langle \frac{\partial X}{\partial u^\gamma}, N \rangle = 0$ for all $\gamma = 1, \ldots, m$ and Theorem 1.1.12, we get

$$\begin{aligned} W^{m+1} &= \langle W, N \rangle \\ &= \left\langle \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}, N \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \det\left(\frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}} \cdots, \frac{\partial X}{\partial u^m}, N\right) \\
&= 0 \\
W^\gamma g_{\gamma\beta} &= W^\gamma \left\langle \frac{\partial X}{\partial u^\gamma}, \frac{\partial X}{\partial u^\beta} \right\rangle \\
&= \left\langle W^\gamma \frac{\partial X}{\partial u^\gamma}, \frac{\partial X}{\partial u^\beta} \right\rangle \\
&= \left\langle W, \frac{\partial X}{\partial u^\beta} \right\rangle \\
&= \left\langle \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}, \frac{\partial X}{\partial u^\beta} \right\rangle \\
&= \left\langle \frac{\partial X}{\partial u^\beta}, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right\rangle \\
&= (-1)^{\alpha-1} \left\langle \frac{\partial X}{\partial u^\beta}, N \wedge \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right\rangle \\
&= (-1)^{\alpha-1} \det\left(\frac{\partial X}{\partial u^\beta}, N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, \frac{\partial X}{\partial u^{\alpha+1}}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&= (-1)^{2\alpha-1} \det\left(N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, \frac{\partial X}{\partial u^\beta}, \frac{\partial X}{\partial u^{\alpha+1}}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&= -\,\delta_{\alpha\beta} \det\left(N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&= -\,\delta_{\alpha\beta} \left\langle N, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right\rangle \\
&= -\,\delta_{\alpha\beta} \left\langle \frac{\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}}{\left|\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}\right|}, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right\rangle \\
&= -\,\delta_{\alpha\beta} \left|\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}\right|,
\end{aligned}
$$

wherein $\delta_{\alpha\beta}$ is the Kronecker delta. By multiplying with the inverse matrix of $g_{\gamma\beta}$ we get

$$
W^\gamma = -\,g^{\gamma\alpha} \left|\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}\right|
$$

thereby yielding (1.5.11). □

Definition 1.5.5 (Laplace-Beltrami operator). Let $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be an immersion of class C^2. The *Laplace-Beltrami Operator* for a function $f : \mathscr{M} \to \mathbb{R}^{m+1}$ is defined in local coordinates (u^α) on $\mathscr{M}$ by

$$
\Delta_g f \quad := \quad g^{\alpha\beta}\left(\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} - \Gamma^\tau_{\alpha\beta} \frac{\partial f}{\partial u^\tau}\right). \tag{1.5.12}
$$

This representation is invariant under coordinate change and therefore the Laplace-Beltrami Operator is globally defined on $\mathscr{M}$. The mentioned quantities are those found in Definition 1.5.1.

Theorem 1.5.6. *Assume the setting of Definition 1.5.5. The Laplace-Beltrami operator can be written in the form*

$$
\Delta_g f \quad = \quad \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial u^\alpha} \left(\sqrt{\det g} g^{\alpha\beta} \frac{\partial f}{\partial u^\beta}\right), \tag{1.5.13}
$$

wherein $\det g$ *is set to be the determinant of the matrix* $(g_{\alpha\beta}) \in \mathbb{R}^{m \times m}$.

Proof. By using (1.5.8) and Theorem 1.1.4 we get in local coordinates (u^α) on $\mathscr{M}$

$$\begin{aligned}
& \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial u^\alpha}\left(\sqrt{\det g}g^{\alpha\beta}\frac{\partial f}{\partial u^\beta}\right) \\
= & \frac{1}{2}g^{\gamma\tau}\frac{\partial g_{\gamma\tau}}{\partial u^\alpha}g^{\alpha\beta}\frac{\partial f}{\partial u^\beta}+\frac{\partial g^{\alpha\beta}}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta}+g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha\partial u^\beta} \\
= & \frac{1}{2}g^{\gamma\alpha}\frac{\partial g_{\gamma\alpha}}{\partial u^\tau}g^{\tau\beta}\frac{\partial f}{\partial u^\beta}-g^{\alpha\gamma}\frac{\partial g_{\gamma\tau}}{\partial u^\alpha}g^{\tau\beta}\frac{\partial f}{\partial u^\beta}+g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha\partial u^\beta} \\
= & -g^{\alpha\gamma}\left(\frac{\partial g_{\gamma\tau}}{\partial u^\alpha}-\frac{\partial g_{\gamma\alpha}}{\partial u^\tau}+\frac{\partial g_{\tau\alpha}}{\partial u^\gamma}\right)\frac{g^{\tau\beta}}{2}\frac{\partial f}{\partial u^\beta}+g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha\partial u^\beta} \\
= & -g^{\alpha\gamma}\Gamma^\beta_{\alpha\gamma}\frac{\partial f}{\partial u^\beta}+g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha\partial u^\beta} \\
= & g^{\alpha\beta}\left(\frac{\partial f}{\partial u^\alpha\partial u^\beta}-\Gamma^\gamma_{\alpha\beta}\frac{\partial f}{\partial u^\gamma}\right) \\
= & \Delta_g f.
\end{aligned}$$

□

Theorem 1.5.7 (Laplace-Beltami and mean curvature). *Let $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be an immersion of class C^2. Then it holds that*

$$\Delta_g X = HN, \tag{1.5.14}$$

where the Laplace-Beltrami operator is taken component wise. Therein, N is the Gauss map and H is the mean curvature of X.

Proof. By (1.5.6) we directly get in local coordinates $u = (u^\alpha)$ on $\mathscr{M}$

$$\begin{aligned}
\Delta_g X &= g^{\alpha\beta}\left(\frac{\partial^2 X}{\partial u^\alpha\partial u^\beta}-\Gamma^\tau_{\alpha\beta}\frac{\partial X}{\partial u^\tau}\right) \\
&= g^{\alpha\beta}h_{\alpha\beta}N \\
&= HN.
\end{aligned}$$

□

In the following, we define the shape operator. The shape operator will be useful to define the mean curvature notions adapted to Cartan integrands as presented in subsection 1.6.2 (cf. [Cla99] and [Cla02]).

Definition 1.5.8 (Shape operator). Let $\mathscr{M}$ be an oriented smooth m-manifold, $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be an immersion of class C^2 and $N : \mathscr{M} \to \mathbb{S}^m$ the Gauss map. We can identify $T_{N(p)}\mathbb{S}^m \subset \mathbb{R}^{m+1}$ with $(N(p))^\perp$ due to the properties of $\mathbb{S}^m$. Further, $(N(p))^\perp = \mathrm{d}X|_p(T_p\mathscr{M}) \subset T_{X(p)}\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1}$, as we regard $\mathrm{d}X|_p(T_p\mathscr{M})$ as embedded in $T_{X(p)}\mathbb{R}^{m+1}$ by means of the inclusion mapping. So, we can identify $T_{N(p)}\mathbb{S}^m$ with $\mathrm{d}X|_p(T_p\mathscr{M})$. Hence, the mapping

$$S := -(\mathrm{d}X)^{-1}\circ \mathrm{d}N : T\mathscr{M} \to T\mathscr{M}$$

is well defined and is called *shape operator*, since $\mathrm{d}N : T\mathscr{M} \to \mathrm{d}X(T\mathscr{M})$ and $(\mathrm{d}X)^{-1} : \mathrm{d}X(T\mathscr{M}) \to T\mathscr{M}$.

Remark 1.5.9.

- The shape operator is given by $-(\mathrm{d}X)^{-1}\circ \mathrm{d}N : T\mathscr{M} \to T\mathscr{M}$. Choose $p \in \mathscr{M}$ and $v \in T_p\mathscr{M}$ and local coordinates (u^α) at p. Then holds

$$\mathrm{d}N|_p(v) = v^\alpha\frac{\partial N}{\partial u^\alpha}(p)$$

$$\overset{\text{Thm. 1.5.3}}{=} - v^\gamma h_{\gamma\alpha} g^{\alpha\beta} \frac{\partial X}{\partial u^\beta}(p)$$
$$= \mathrm{d}X|_p(-v^\gamma h_{\gamma\alpha} g^{\alpha\beta} \frac{\partial}{\partial u^\beta}|_p) \tag{1.5.15}$$

s.t. we deduce $-(\mathrm{d}X)^{-1} \circ \mathrm{d}N(v) = v^\gamma h_{\gamma\alpha} g^{\alpha\beta} \frac{\partial}{\partial u^\beta}|_p$.

- It can be seen easily by using a local coordinate representation that

$$g|_p(S(v), w) \quad = \quad g|_p(v, S(w))$$

holds for $p \in \mathscr{M}$ and $v, w \in T_p\mathscr{M}$. So, the shape operator S is at any point $p \in \mathscr{M}$ an self-adjoint operator, and can be diagonalized as such. The resulting ordered Eigenvalues $\kappa_1(p) \geq \ldots \geq \kappa_m(p)$ are called *principal curvatures* at p and a corresponding choice of Eigenvectors $v_1|_p, \ldots, v_m|_p$ are called *principal curvature directions*, which are chosen to form an g-orthonormal base of $T_p\mathscr{M}$. The principal curvatures can also be seen as the Eigenvalues of the second fundamental form as we easily deduce by (1.5.15) that

$$h|_p(v, w) \quad = \quad g|_p(S(v), w)$$

for all $p \in \mathscr{M}$ and $v, w \in T_p\mathscr{M}$. Notice that the principal curvatures of the boundary of a convex set are non-negative if we choose as normal the inward-pointing normal.

1.6 Cartan functionals

In the present section, we introduce some basic definitions regarding Cartan functionals, i.e. functionals invariant under reparametrization of the involved immersion. We start by giving a short introduction to the topic and continue by some mean curvature notions issuing from the first variation of a Cartan functional. Notice that we also investigate the non-parametric situation in this context. Then, some existence, regularity and removability results are given, some of which use the curvature notions mentioned before. Especially, geometric boundary value problems for Cartan functionals on 2-dimensional surfaces are considered, either in a Sobolev as well as in a smooth setting. The existence results in particular are not restricted to a Minkowski setting. Further, enclosure results are presented followed by a presentation of isoperimetric inequalities. The listing of theorems on Cartan functionals is concluded by curvature estimates for Finsler-minimal immersions, which imply Bernstein-type theorems. The enclosure theorems, isoperimetric inequalities and curvature estimates do all assume a Minkowski setting and involve some of the Cartan mean curvature notions mentioned in this section.

1.6.1 Review of variational results of Cartan functionals

In the following, we discuss some basics about parametric variational integrals or Cartan functionals. The definitions in this section are essentially based on a paper of Hildebrandt and von der Mosel [HvdM05]. The name Cartan functional is therein a tribute to a paper of Cartan [Car33], wherein he develops a theory of anisotopic surface area in arbitrary codimension similar to the Finslerian length of curves. Especially, Cartan also requires the 1-homogeneity of his Lagrangians in order to have notions of area invariant under reparametrization.

Similar to [HvdM05], we identify $\Lambda^m(\mathbb{R}^n) \cong \mathbb{R}^N$ by using the canonical isometry w.r.t. to the Euclidean standard norm $|\cdot|$ on $\mathbb{R}^N$. Further, if we choose $N := \dim \Lambda^m(\mathbb{R}^n) = \binom{n}{m}$, we identify $\mathbb{R}^n \times \Lambda^m(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^N$. Especially, if we choose $n = m+1$, N computes to $\binom{m+1}{m} = m+1$. Notice that [HvdM05] actually only deals with Cartan functionals, where N is set to $\binom{n}{2}$ and in most other sources, it is chosen to be $\binom{n}{m}$ for some integer m.

Definition 1.6.1 (**Cartan integrand**). Let N, n be two integers with $N \geq n$. A *Lagrangian* $I(x, z)$ is a function of class $C^0(\mathbb{R}^n \times \mathbb{R}^N)$. A Lagrangian I is said to be *parametric* if I satisfies the condition

$$I(x, tz) \quad = \quad tI(x, z) \quad \text{for all} \quad t > 0,\ (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{H}$$

A *Cartan integrand* is a parametric Lagrangian (cf. [HvdM05, p. 2]). A Lagrangian I is said to be *positive definite* if there are two constants M_1 and M_2 with $0 < M_1 \leq M_2$ such that

$$M_1|z| \quad \leq \quad I(x, z) \quad \leq \quad M_2|z| \quad \text{for all} \quad (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{D}$$

Remark 1.6.2. Let $n \geq m$, $N := \binom{n}{m}$ be integers and a Cartan integrand I on $\mathbb{R}^m \times \mathbb{R}^N$, i.e. (**H**) holds, which is strictly positive on $\mathbb{R}^m \times (\mathbb{R}^N \backslash \{0\})$. Further, let $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^n$ be a C^1-immersion. Then we can define the *parametric functional* or *Cartan functional* $\mathcal{I}$ of X in the following way

$$\mathcal{I}(X) \quad := \quad \int_{\mathscr{M}} i,$$

where the m-form i on $\mathscr{M}$ is given in positively oriented local coordinates $(u^1, \ldots, u^m) : \mathscr{U} \to \Omega$ for $\mathscr{U} \subset \mathscr{M}$, $\Omega \subset \mathbb{R}^{m+1}$ by

$$i|_p \quad = \quad I(X(x), \frac{\partial X}{\partial u^1}|_p \wedge \ldots \wedge \frac{\partial X}{\partial u^m}|_p) \mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^m$$

for $p \in \mathscr{U}$. This form is well-defined globally on $\mathscr{M}$ due to its invariance under a coordinate change and $\mathcal{I}_{\mathscr{M}}(X) \in [0, \infty) \cup \{\infty\}$. Notice that we simply write $\mathcal{I}(\mathscr{M}) := \mathcal{I}(\iota|_{\mathscr{M}})$ for an embedded smooth m-manifold $\mathscr{M}$ with embedding $\iota : \mathscr{M} \hookrightarrow \mathbb{R}^n$. Thereby, we can even define $\mathcal{I}(\mathscr{S})$ for $\mathcal{H}^m$-measurable subsets $\mathscr{S} \subset \mathscr{M}$ by an approximation argument. If we choose $n = m + 1$, $\mathscr{M} = \Omega$ for an open subset $\Omega \subset \mathbb{R}^m$ and (u^α) to be the standard coordinates on $\mathbb{R}^m$, $\mathcal{I}(X)$ computes to

$$\mathcal{I}(X) \quad = \quad \int_\Omega I(X, \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m}) \mathrm{d}u^1 \ldots \mathrm{d}u^m.$$

The value $\mathcal{I}(X)$ can be guaranteed to be finite by assuming $X \in W^{1,m}(\Omega, \mathbb{R}^{m+1})$ and I to satisfy the positive definiteness relation (**D**). This can be seen by using (**D**) and Theorem 1.1.15 to get the simple estimate

$$\begin{aligned} I(X, \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m}) \quad &\leq \quad M_2 \left| \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m} \right| \\ &\leq \quad M_2 \prod_{\alpha=1}^m \left| \frac{\partial X}{\partial u^\alpha} \right| \\ &\leq \quad M_2 \left| \mathrm{D}X \right|^m. \end{aligned}$$

Therewith follows directly

$$\begin{aligned} \int_\Omega I(X, \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m}) \mathrm{d}u^1 \ldots \mathrm{d}u^m \quad &\leq \quad M_2 \int_\Omega \left| \mathrm{D}X \right|^m \mathrm{d}u^1 \ldots \mathrm{d}u^m \\ &< \quad \infty. \end{aligned}$$

Example 1.6.3 (**Euclidean area**). A special parametric Lagrangian or Cartan integrand is the *Euclidean area integrand* $A(z) := |z|$. It leads to the classical *Euclidean area*

$$\mathcal{A}(X) \quad := \quad \int_\Omega \left| \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m} \right| \mathrm{d}u^1 \ldots \mathrm{d}u^m$$

for an open set $\Omega \subset \mathbb{R}^m$, an immersion $X \in C^1(\Omega, \mathbb{R}^{m+1})$ and standard coordinates (u^α) on $\mathbb{R}^m$.

Now we say some words on the notation of gradients and Hessians. Let $f : \Omega \times \Theta \to \mathbb{R}$ be a mapping with open sets $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^N$ and $(x, z) \in \Omega \times \Theta$. If f is differentiable in z at (x, z), we write

$$\nabla_z f(x, z) \quad = \quad f_z(x, z) \quad = \quad (f_{z^i}(x, z)) \in \mathbb{R}^N$$

for the gradient of f w.r.t. z at (x, z). Notice that f_{z^i} is the i-th partial derivative of f w.r.t. z. If f is two times differentiable in z at (x, z), we write

$$\nabla_z^2 f(x, z) \quad = \quad f_{zz}(x, z) \quad = \quad (f_{z^i z^j}(x, z)) \in \mathbb{R}^{N \times N}$$

for the Hessian matrix of f w.r.t. z at (x, z). We define f_x, f_{xz}, f_{zx} and f_{xx} in a similar way.

Again, in accordance with [HvdM05], we define ellipticity of Cartan integrands.

Definition 1.6.4 (Ellipticity). Let N, n be two integers with $N \geq n$. A Cartan integrand I is said to be *semi-elliptic* on $\Omega \times \mathbb{R}^N$ for $\Omega \subset \mathbb{R}^n$ if it is convex in the second argument, i.e. if

$$I(x, tz_1 + (1 - t)z_2) \leq tI(x, z_1) + (1 - t)I(x, z_2), \tag{\textbf{C}}$$

for all $x \in \Omega$, $z_1, z_2 \in \mathbb{R}^N$ and $t \in (0, 1)$. Moreover, a Cartan integrand I is called *elliptic* if there exists for every $R > 0$ a constant $\Lambda_1(R) > 0$ such that

$$I - \Lambda_1(R)A \text{ is semi-elliptic} \tag{\textbf{E}}$$

on $\overline{B_R^n(0)} \times \mathbb{R}^N$. Remember that $\overline{B_R^n(p)}$ denotes the closed Euclidean ball with radius $R > 0$ and center $p \in \mathbb{R}^n$, i.e. $\overline{B_R^n(p)} := \{v \in \mathbb{R}^n : |v - p| \leq 0\}$.

Remark 1.6.5.

- Assume that I is a Cartan integrand of class $C^2(\mathbb{R}^n \times (\mathbb{R}^N \backslash \{0\}))$. Then I being semi-elliptic, i.e. (**C**) holds, is equivalent to the assumption that the Hessian matrix $I_{zz}(x, z)$ is positive semidefinite for $(x, z) \in \Omega \times (\mathbb{R}^N \backslash \{0\})$. Hence, I elliptic means that for every $R > 0$ there is a positive constant $\Lambda_1(R) > 0$ such that $I_{zz} - \Lambda_1(R)A_{zz}$ is positive semidefinite on $\overline{B_R^n(0)} \times (\mathbb{R}^N \backslash \{0\})$ (cf. [GH96, p. 72]).

- Notice that the conditions (**H**), (**C**) and (**E**) for $n = m + 1$ and $N = \binom{m+1}{m} = m + 1$ are equivalent to (**h**), (**c**) and (**e**) of Definition 1.4.1, if we choose therein $\mathscr{N} = \mathbb{R}^{m+1}$. Especially, the equivalence of (**E**) and (**e**) follows by an argument of [GH96, p. 72].

In the following lemma, we choose $n = m + 1$ and $N = \binom{m+1}{m} = m + 1$. Therein, we give a useful quantification of ellipticity.

Lemma 1.6.6 (Ellipticity quantification (cf. [GH96])**).** *Let $I \in C^2(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \backslash \{0\}))$ be a Cartan integrand. Then the following two statements are equivalent:*

(i) I is elliptic;

(ii) For any $R > 0$ there are constants $\Lambda_i = \Lambda_i(R) > 0$ such that

$$\frac{\Lambda_1(R)}{|z|} \left[|\xi|^2 - \left\langle \frac{z}{|z|}, \xi \right\rangle^2 \right] \quad \leq \quad \xi^T \cdot I_{zz}(x, z) \cdot \xi \quad \leq \quad \frac{\Lambda_2(R)}{|z|} \left[|\xi|^2 - \left\langle \frac{z}{|z|}, \xi \right\rangle^2 \right] \tag{\textbf{EQ}}$$

for all $(x, z) \in \overline{B_R^{m+1}(0)} \times (\mathbb{R}^{m+1} \backslash \{0\})$.

Proof. By Remark 1.6.5 we already know that I elliptic is equivalent to the following statement: *For any $R > 0$ there is a positive constant $\Lambda_1 = \Lambda_1(R) > 0$ such that $I_{zz} - \Lambda_1(R)A_{zz}$ is positive semidefinite on $\overline{B_R^{m+1}(0)} \times (\mathbb{R}^N \backslash \{0\})$.*
Since the Hessian matrix of the Euclidean area integrand computes to

$$A_{zz}(z) \quad = \quad \frac{1}{|z|}\left[\mathrm{Id} - \frac{z}{|z|}\left(\frac{z}{|z|}\right)^T\right] \tag{1.6.1}$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$, the condition of $I_{zz} - \Lambda_1(R)A_{zz}$ being positive semidefinite is equivalent to

$$\frac{\Lambda_1(R)}{|z|}\xi^T \cdot \left[\mathrm{Id} - \frac{z}{|z|}\left(\frac{z}{|z|}\right)^T\right] \cdot \xi \quad \leq \quad \xi^T \cdot I_{zz}(x,z) \cdot \xi \tag{1.6.2}$$

for $(x, z) \in \overline{B_R^{m+1}(0)} \times (\mathbb{R}^N \backslash \{0\})$ and $\xi \in \mathbb{R}^{m+1}$. On the other hand, the expression

$$\Pi_{z^\perp} \quad := \quad \mathrm{Id} - \frac{z}{|z|}\left(\frac{z}{|z|}\right)^T$$

can be seen as the *orthogonal projection* on the linear subspace spanned by $\frac{z}{|z|}$. $\Pi_{z^\perp}$ especially fulfills $\Pi_{z^\perp}^T \cdot \Pi_{z^\perp} = \Pi_{z^\perp}^2 = \mathrm{Id}$. Using Theorem 1.4.3 and the 0-homogeneity of the gradient $I_z(x, z) \in \mathbb{R}^{m+1}$ w.r.t z, we deduce that $I_{zz}(x, z) \cdot z = 0$. So, we can write

$$I_{zz}(x,z) \quad = \quad \Pi_{z^\perp}^T \cdot I_{zz}(x,z) \cdot \Pi_{z^\perp},$$

what is a continuous expression in $(x, z) \in \overline{B_R^{m+1}(0)} \times (\mathbb{R}^{m+1} \backslash \{0\})$. This together with the (-1)-homogeneity of I_{zz} implies

$$\xi^T \cdot I_{zz}(x,z) \cdot \xi \quad \leq \quad \frac{\Lambda_2(R)}{|z|}\xi^T \cdot \left[\mathrm{Id} - \frac{z}{|z|}\left(\frac{z}{|z|}\right)^T\right] \cdot \xi, \tag{1.6.3}$$

for $R > 0$ and $(x, z) \in \overline{B_R^{m+1}(0)} \times (\mathbb{R}^N \backslash \{0\})$, where we choose

$$\Lambda_2(R) \quad := \quad \max_{(x,z)\in \overline{B_R^{m+1}(0)}\times \mathbb{S}^m} \max_{\xi \in \mathbb{S}^m} \xi^T \cdot I_{zz}(x,z) \cdot \xi$$

as the largest eigenvalue of the Hessian matrix I_{zz} on $\overline{B_R^{m+1}(0)}$. Notice that by (1.6.2), we can deduce that $\Lambda_2(R) \geq \Lambda_1(R)$. (1.6.2) and (1.6.3) together conclude the proof, since

$$\xi^T \cdot \left[\mathrm{Id} - \frac{z}{|z|}\left(\frac{z}{|z|}\right)^T\right] \cdot \xi \quad = \quad |\xi|^2 - \left\langle \frac{z}{|z|}, \xi \right\rangle^2.$$

□

1.6.2 Notions of mean curvature

In the following definition, we will give notions of mean curvature for a Cartan integrand I. We introduce the I-mean curvature based on the one proposed by Clarenz in [Cla99], which can also be found in [CvdM02]. This mean curvature is related to other such constructions, i.e. that of Bergner and Dittrich in [BD08], which they developed in the more general setting of weighted mean curvature. Further, we introduce the I-mean curvature used by White in [Whi91] and name it full I-mean curvature here. The full I-mean curvature issues directly from the first variation of the Cartan functional. Again, this mean curvature notion is related to the one of Clarenz. In the end of the section, we introduce, among others, I-mean convex domains, which rely on a definition proposed by White in [Whi91] and by Bergner and Fröhlich in [BF09]. Notice that these definitions apply only to the case of codimension 1, since we assume $n = m+1$ and $N = \binom{m+1}{m} = m+1$.

Definition 1.6.7 (***I*-mean curvature** [Cla99, Definition 1.2, p. 14])**.** Let I be a Cartan integrand in $C^2(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$, $\mathscr{M}$ be an oriented smooth m-manifold and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ a C^2-immersion with normal $N : \mathscr{M} \to \mathbb{S}^m$. The *$I$-shape operator* is defined by

$$S_I \quad := \quad (\mathrm{d}X)^{-1}\, I_{zz}(X,N)\, \mathrm{d}X \circ S,$$

wherein $S := -(\mathrm{d}X)^{-1} \circ \mathrm{d}N : T\mathscr{M} \to T\mathscr{M}$ denotes the standard (Euclidean) *shape operator* (see Definition 1.5.8). Further, we define the *I-mean curvature* as

$$H_I \quad := \quad \mathrm{trace}_g S_I.$$

Therein, trace_g is the *trace* of a bundle map $f : T\mathscr{M} \to T\mathscr{M}$ w.r.t. the first fundamental form g of X, i.e. $\mathrm{trace}_g f := g^{\alpha\beta} g(f(\frac{\partial}{\partial x^\alpha}), \frac{\partial}{\partial x^\beta})$ with $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ in local coordinates (u^α) on $\mathscr{M}$ (see (1.5.1)).

The following theorem gives a coordinate representation of the I-mean curvature. This result can be found in a work of Bergner and Fröhlich [BF09].

Theorem 1.6.8 (cf. [BF09, Remark 1, p. 365])**.** *Let $\mathscr{M}$ be an oriented smooth m-manifold with boundary, $I \in C^2(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ be a Cartan integrand and $X \in C^2(\mathscr{M}, \mathbb{R}^{m+1})$ be an immersion with normal $N : \mathscr{M} \to \mathbb{S}^m$. The I-mean curvature can be given in local coordinates (u^α) on $\mathscr{M}$ by*

$$H_I \quad = \quad -\, g^{\alpha\beta} \left\langle I_{zz}(X,N) \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} \right\rangle. \tag{1.6.4}$$

wherein $g^{\alpha\beta}$ are the coefficients of the inverse of the matrix with coefficients $g_{\alpha\beta}$ given by

$$g_{\alpha\beta} \quad := \quad \left\langle \frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} \right\rangle.$$

The latter are the coefficients of the first fundamental form in local coordinates.

Proof. The differential of X is a mapping

$$\mathrm{d}X : T\mathscr{M} \to TX,$$

wherein $TX = \mathrm{d}X(T\mathscr{M}) = \bigcup_{p\in\mathscr{M}} \mathrm{d}X(T_p\mathscr{M}) \subset T\mathbb{R}^{m+1}$ is the tangent bundle to X (see Remark 1.5.2). In local coordinates (u^α) on $\mathscr{M}$ and with the standard basis (e_i) of $\mathbb{R}^{m+1}$ one has

$$\mathrm{d}X(v) \quad = \quad \mathrm{d}X(v^\alpha \frac{\partial}{\partial u^\alpha}) \quad = \quad v^\alpha \frac{\partial X}{\partial u^\alpha} \quad = \quad v^\alpha \frac{\partial X^i}{\partial u^\alpha} e_i.$$

We denote the inverse of $\mathrm{d}X$ by

$$(\mathrm{d}X)^{-1} : TX \to T\mathscr{M}.$$

Let $V = V^i e_i$ be a vector field in TX, i.e. $\langle V, N\rangle = 0$. Then there exists for a suitable choice of local coordinates (u^α) a vector field in (or section) $v = v^\alpha \frac{\partial}{\partial u^\alpha}$ s.t.

$$\mathrm{d}X(v) \quad = \quad V.$$

In local coordinates we get

$$(\mathrm{d}X)^{-1}(V) \quad = \quad V^i \delta_{ij} \frac{\partial X^j}{\partial u^\alpha} g^{\alpha\beta} \frac{\partial}{\partial u^\beta}$$

for all $V \in TX$. Further, the differential of the normal vector $\mathrm{d}N$ maps into the tangent bundle TX.

$$\begin{aligned}
S_I(\frac{\partial}{\partial u^\alpha}) &= -(\mathrm{d}X)^{-1} I_{zz}(X,N)\mathrm{d}X \circ S(\frac{\partial}{\partial u^\alpha}) \\
&= -(\mathrm{d}X)^{-1} I_{zz}(X,N)\mathrm{d}N(\frac{\partial}{\partial u^\alpha}) \\
&= -(\mathrm{d}X)^{-1} I_{zz}(X,N)\frac{\partial N}{\partial u^\alpha} \\
&= -I_{z_i z_k}(X,N)\frac{\partial N^k}{\partial u^\alpha}\frac{\partial X^i}{\partial u^\beta} g^{\beta\gamma}\frac{\partial}{\partial u^\gamma}
\end{aligned}$$

Thereby, we get

$$\begin{aligned}
H_I(X,N) &= \mathrm{trace}_g S_I \\
&= g^{\tau\upsilon} g(S_I(\frac{\partial}{\partial u^\tau}), \frac{\partial}{\partial u^\upsilon}) \\
&= -g^{\tau\upsilon} g\left(I_{z_i z_k}(X,N)\frac{\partial N^k}{\partial u^\tau}\frac{\partial X^i}{\partial u^\beta} g^{\beta\gamma}\frac{\partial}{\partial u^\gamma}, \frac{\partial}{\partial u^\upsilon}\right) \\
&= -g^{\tau\upsilon} g^{\beta\gamma} g_{\gamma\upsilon} I_{z_i z_k}(X,N)\frac{\partial N^k}{\partial u^\tau}\frac{\partial X^i}{\partial u^\beta} \\
&= -g^{\tau\upsilon} \delta^\beta_\upsilon I_{z_i z_k}(X,N)\frac{\partial N^k}{\partial u^\tau}\frac{\partial X^i}{\partial u^\beta} \\
&= -g^{\tau\beta} I_{z_i z_k}(X,N)\frac{\partial N^k}{\partial u^\tau}\frac{\partial X^i}{\partial u^\beta} \\
&= -g^{\tau\beta}\left\langle I_{zz}(X,N)\frac{\partial N}{\partial u^\tau}, \frac{\partial X}{\partial u^\beta}\right\rangle.
\end{aligned}$$

□

The following proposition expresses the I-mean curvature as a weighted sum of the principal curvatures. This result can be found in a work of Bergner and Fröhlich [BF09].

Proposition 1.6.9 (cf. [BF09, Proposition 1])**.** *Let $\mathscr{M}$ be an oriented smooth m-manifold with boundary, $I \in C^2(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ be a Cartan integrand and $X \in C^2(\mathscr{M}, \mathbb{R}^{m+1})$ be an immersion with normal N as well as principal curvatures $\kappa_\alpha \in \mathbb{R}$, $\alpha = 1, \ldots, m$. Then the I-mean curvature of X can be expressed as the weighted sum of its principal curvatures, i.e.*

$$H_I(X,N) \quad = \quad \sum_{\alpha=1}^m \varrho_\alpha(X,N)\kappa_\alpha.$$

The weight factors are further given by

$$\varrho_\alpha(X,N) \quad = \quad \langle Izz(X,N)\mathrm{d}X(v_\alpha), \mathrm{d}X(v_\alpha)\rangle,$$

where $v_\alpha \in \mathbb{R}^{m+1}$ for $i = 1, \ldots, m$ are the principal curvature directions. If I is additionally elliptic, then the weights are all positive, i.e. $\varrho_\alpha(X,N) > 0$ for $\alpha = 1, \ldots, m$.

The I-mean curvature can be interpreted as being derived from the first variation of the Cartan functional corresponding to I. In [Cla99], the first variation has been carried out for normal variations constructed in a certain way. Nevertheless, this is no big obstruction, as it can be shown that only the normal part of the variation directions contributes to the first variation, as by arguments of Jost in [Jos02, Lemma 8.2]. Variations in general directions of Cartan functionals are considered by White in [Whi91]. The following lemma is a synthesis of these sources.

Lemma 1.6.10 (First variation (cf. [Cla99])**.** *Let $\mathscr{M}$ be an oriented smooth m-manifold with boundary $\partial\mathscr{M}$, $I \in C^2(\mathbb{R}^{m+1}\times(\mathbb{R}^{m+1}\backslash\{0\}))$ be a Cartan integrand and $X \in C^2(\mathscr{M}, \mathbb{R}^{m+1})$ be an immersion with normal N. Let $\tilde{X} \in C^2((-\varepsilon,\varepsilon)\times\mathscr{M}, \mathbb{R}^{m+1})$ be a mapping for some $\varepsilon > 0$, where each function $X_t(\cdot) := X(t,\cdot)$ is an immersion with normal N_t, s.t. $X_0 = X$, $N_0 = N$ and $X_t = X$ outside a compact set $\mathscr{K} \subset \mathscr{M}\backslash\partial\mathscr{M}$. We call $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$ a* variation *of X. We define $V := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}X_t : \mathscr{M} \to \mathbb{R}^{m+1}$ as the* variational vector field*. Notice that the support of V is subset to $\mathscr{K}$ and hence is compact in $\mathscr{M}\backslash\partial\mathscr{M}$. Then holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{I}(X_t) \quad = \quad -\int_{\mathscr{M}} \langle h_I, V\rangle \mathrm{d}A,$$

where $h_I = h_I(X,N) : \mathscr{M} \to \mathbb{R}^{m+1}$ is vector valued. Especially, h_I computes to

$$h_I(X,N) \quad = \quad (H_I(X,N) - \mathrm{trace}\, I_{xz}(X,N)) \cdot N. \tag{1.6.5}$$

We call $h_I(X,N)$ the full I-mean curvature*. We call an immersion X* I-minimal *if*

$$h_I(X,N) \quad = \quad 0. \tag{1.6.6}$$

The property of X being I-minimal is equivalent to X being a critical immersion of $\mathcal{I}$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{I}(X_t) = 0$ for all variations $\{X_t\}_{t\in(-\varepsilon,\varepsilon)}$ of X.

Remark 1.6.11.

- The I-mean curvature as defined in [Cla99] is a scalar value and does not fully reflect the first variation of the Cartan functional $\mathcal{I}$ in the case that I indeed depends on the first argument. The I-mean curvature defined in [Whi91] for $m = 2$ does fully reflect the first variation and is in fact a vector field. Nevertheless, both definitions are related and coincide for I independent on the first argument in the sense that $H_I(X,N) = \langle h_I(X,N), N\rangle$. We distinguish them especially by the simple additive word "full" to refer to the definition based on [Whi91] but extended to arbitrary dimension $m \geq 2$.

- An immersed submanifold is called *I-minimal* if it is immersed by an I-minimal immersion.

Proof of Lemma 1.6.10. A proof of this has been done in the proof of [Cla99, Satz 1.1] for a special type of variation $X_t = X + t\varphi N$, where $\varphi \in C_0^\infty(\mathscr{M})$. Therein, $C_0^\infty(\mathscr{M})$ is the set of all real-valued C^∞ functions on $\mathscr{M}$ of compact support. The proof generalizes straightforwardly to the type of variation, we chose in the theorem's statement. □

Now we present an example, where we compute the various I-mean curvature notions for graphs and relate them to the so-called non-parametric functional corresponding to the Cartan functional $\mathcal{I}$. We especially look at the first variation of the non-parametric functional, i.e. its Euler-Lagrange equation.

Example 1.6.12 (**Non-parametric functional**). Let $I : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a Cartan integrand (i.e. $I = I(x,z)$) and an embedded hypersurface $\mathscr{M}$, which is the graph of a C^2-function $f : \Omega \to \mathbb{R}$ for an open subset $\Omega \subset \mathbb{R}^m$. The manifold $\mathscr{M}$ can be represented by the immersion $X : \Omega \to \mathbb{R}^{m+1}$ with

$$X(u) \quad := \quad \begin{pmatrix} u \\ f(u) \end{pmatrix}$$

for $u \in \Omega$. We set $X_{u^\alpha} := \frac{\partial X}{\partial u^\alpha}$ and so forth. We intend to compute

$$H_I(X,N) \quad = \quad -\, g^{\alpha\beta}\langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle,$$

where w.l.o.g. the normal map $N : \Omega \to \mathbb{S}^m$ is given by

$$N \quad = \quad \frac{1}{\sqrt{1+|\nabla f|^2}} \begin{pmatrix} -\nabla f(u) \\ 1 \end{pmatrix}.$$

Notice that ∇ denotes here the Euclidean gradient on $\mathbb{R}^m$. We see directly that the normal map is well-defined on Ω since X is an immersion. The components of the normal map to $\mathscr{M}$ write down to

$$N^i \quad = \quad \frac{1}{\sqrt{1+|\nabla f|^2}}(-\delta^{i\gamma} f_{u^\gamma} + \delta^i_{m+1}).$$

Notice that we can write $|\nabla f|^2 = \delta^{\tau\sigma} f_{u^\tau} f_{u^\sigma}$. The derivatives of the components of the normal map, i.e. $N^i_{u^\alpha}$, compute to

$$\begin{aligned} N^i_{u^\alpha} \quad &= \quad -\frac{f_{u^\tau}\delta^{\tau\sigma} f_{u^\sigma u^\alpha}}{\sqrt{1+|\nabla f|^2}^3}(-\delta^{i\gamma} f_{u^\gamma} + \delta^i_{m+1}) + \frac{1}{\sqrt{1+|\nabla f|^2}}(-\delta^{i\gamma} f_{u^\gamma u^\alpha}) \\ &= \quad -\frac{f_{u^\tau}\delta^{\tau\sigma} f_{u^\sigma u^\alpha}}{1+|\nabla f|^2} N^i + \frac{1}{\sqrt{1+|\nabla f|^2}}(-\delta^{i\gamma} f_{u^\gamma u^\alpha}). \end{aligned} \tag{1.6.7}$$

On the other hand, we have

$$X^i_{u^\alpha} \quad = \quad \delta^i_\alpha + \delta^i_{m+1} f_{u^\alpha}$$

and

$$X^i_{u^\alpha u^\beta} \quad = \quad \delta^i_{m+1} f_{u^\alpha u^\beta}.$$

Thereby follows

$$g_{\alpha\beta} \quad = \quad \langle X_{u^\alpha}, X_{u^\beta}\rangle \quad = \quad X^i_{u^\alpha}\delta_{ij}X^j_{u^\beta} \quad = \quad \delta_{\alpha\beta} + f_{u^\alpha} f_{u^\beta}.$$

We deduce by Theorem 1.1.8 that

$$g^{\alpha\beta} \quad = \quad \delta^{\alpha\beta} - \frac{\delta^{\alpha\gamma} f_{u^\gamma} f_{u^\lambda}\delta^{\lambda\beta}}{1+|\nabla f|^2}.$$

Since I satisfies (**H**) together with Theorem 1.4.3 and (1.6.7), we have $I_{z^i}(x,tz) = I_{z^i}(x,z)$, $I_{z^i}(x,z)z^i = I(x,z)$ and $I_{z^i z^j}(x,z)z^j = 0$ for all $t > 0$ and $(x,z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. Therefore holds

$$\begin{aligned} \langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle \quad &= \quad I_{z^i z^j}(X,N)N^i_{u^\alpha}X^j_{u^\beta} \\ &= \quad \Big(\frac{\partial}{\partial u^\alpha}(I_{z^j}(X,N))\Big)X^j_{u^\beta} - I_{x^i z^j}(X,N)X^j_{u^\beta}X^i_{u^\alpha} \\ &= \quad \Big(\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + f_{u^\beta}\frac{\partial}{\partial u^\alpha}(I_{z^{m+1}}(X,N)\Big) - I_{x^i z^j}(X,N)X^j_{u^\beta}X^i_{u^\alpha}. \end{aligned}$$

Combining the former identities with the following computation

$$g^{\alpha\beta} f_{u^\alpha} \quad = \quad \frac{1}{1+|\nabla f|^2}\delta^{\alpha\beta} f_{u^\beta}$$

as well as regrouping and inverse application of the chain rule lead to

$$\begin{aligned}
& g^{\alpha\beta}\langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle + I_{x^iz^j}(X,N)X^i_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} \\
= \ & g^{\alpha\beta}\big(\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + f_{u^\beta}\frac{\partial}{\partial u^\alpha}(I_{z^{m+1}}(X,N)\big) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) \\
& + f_{u^\gamma} g^{\gamma\alpha}\big(- f_{u^\lambda}\delta^{\lambda\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \frac{\partial}{\partial u^\alpha}(I_{z^{m+1}}(X,N))\big) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} f_{u^\gamma} g^{\gamma\alpha} N^i \frac{\partial}{\partial u^\alpha}(I_{z^i}(X,N)) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} f_{u^\gamma} g^{\gamma\alpha}(\frac{\partial}{\partial u^\alpha}(N^i I_{z^i}(X,N)) - N^i_{u^\alpha} I_{z^i}(X,N) \\
& - X^i_{u^\alpha} I_{x^i}(X,N) + X^i_{u^\alpha} I_{x^i}(X,N)) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} f_{u^\gamma} g^{\gamma\alpha}(\frac{\partial}{\partial u^\alpha}(I(X,N)) - \frac{\partial}{\partial u^\alpha}(I(X,N)) \\
& + X^i_{u^\alpha} I_{x^i}(X,N)) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} f_{u^\gamma} g^{\gamma\alpha} X^i_{u^\alpha} I_{x^i}(X,N) \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} X^{m+1}_{u^\gamma} g^{\gamma\alpha} X^i_{u^\alpha} I_{x^i}(X,N).
\end{aligned}$$

By simple linear algebra we deduce that

$$\delta^{ij} \quad = \quad X^i_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} + N^i N^j$$

and thereby

$$\delta^{(m+1)j} \quad = \quad X^{m+1}_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} + N^{m+1} N^j.$$

Hence, we conclude

$$\begin{aligned}
X^{m+1}_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} \quad &= \quad \delta^{(m+1)j} - N^{m+1}N^j \\
&= \quad \delta^{(m+1)j} - \frac{1}{\sqrt{1+|\nabla f|^2}} N^j.
\end{aligned}$$

This together with $I_{x^i}(x,z) = I_{x^iz^j}(x,z)z^j$ by Theorem 1.4.3 leads to

$$\begin{aligned}
& g^{\alpha\beta}\langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle + \delta^{ij} I_{x^iz^j}(X,N) \\
= \ & g^{\alpha\beta}\langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle + I_{x^iz^j}(X,N)X^i_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} + I_{x^iz^j}(X,N)N^iN^j \\
= \ & g^{\alpha\beta}\langle I_{zz}(X,N)N_{u^\alpha}, X_{u^\beta}\rangle + I_{x^iz^j}(X,N)X^i_{u^\alpha} g^{\alpha\beta} X^j_{u^\beta} + I_{x^i}(X,N)N^i \\
= \ & \delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}(I_{z^\beta}(X,N)) + \sqrt{1+|\nabla f|^2} I_{x^{m+1}}(X,N).
\end{aligned}$$

So, the (full) I-mean curvature can be written in the simple form

$$\begin{aligned}
& \langle h_I(X,N), N\rangle \\
= \ & H_I(X,N) - \mathrm{trace} I_{xz}(X,N)
\end{aligned}$$

$$= \; -\delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\left[I_{z^\beta}(X,N)\right] - \sqrt{1+|\nabla f|^2} I_{x^{m+1}}(X,N)$$
$$= \; -\delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\left[I_{z^\beta}\left(\begin{pmatrix} u \\ f(u)\end{pmatrix}, \begin{pmatrix} -\nabla f(u) \\ 1\end{pmatrix}\right)\right] - I_{x^{m+1}}\left(\begin{pmatrix} u \\ f(u)\end{pmatrix}, \begin{pmatrix} -\nabla f(u) \\ 1\end{pmatrix}\right),$$

where we used again (**H**). Indeed, when we look at the following functional

$$\mathcal{J}(f) \;:=\; \int_\Omega j(u, f(u), \nabla f(u))\mathrm{d}u \tag{1.6.8}$$

with

$$j(u,w,p) \;:=\; I(\begin{pmatrix} u \\ w\end{pmatrix}, \begin{pmatrix} -p \\ 1\end{pmatrix}), \tag{1.6.9}$$

for all $w \in \mathbb{R}$ and $u, p \in \mathbb{R}^m$, the Euler-Lagrange equation for critical graphs of $\mathcal{J}$ can be written as

$$0 \;=\; -\delta^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\left[j_{p^\beta}(u,f(u),\nabla f(u))\right] + j_w(u,f(u),\nabla f(u)) \;=\; -\langle h_I(X,N), N\rangle, \tag{1.6.10}$$

what corresponds - as we have seen - to vanishing I-mean curvature. $\mathcal{J}$ is the *non-parametric functional* corresponding to the parametric or Cartan functional $\mathcal{I}$.

The following definition is motivated by the works of White [Whi91] as well as Bergner and Fröhlich [BF09]. White defines in [Whi91] the notion I-mean convex for a given Cartan integrand I by means of the full I-mean curvature, which he also defined similar to the classical mean curvature via a variational approach. Bergner and Fröhlich define in [BF09] weighted mean convex sets. This definition is applicable to Cartan integrands and is then based on the I-mean curvature. Their approach coincides with the one of White in the case that the underlying Cartan integrand is independent of the first argument. They also give examples for weighted mean convex sets, such as convex sets of smooth boundary and hyperboloids of a certain structure (see [BF09, Remark 3, p. 368]). So, the following definition is synthesis of both sources.

Definition 1.6.13 (***I*-mean convex**)**.** Let Ω be a domain with C^2-boundary $\partial\Omega$ in $\mathbb{R}^{m+1}$. Assume that $\partial\Omega$ has the full I-mean curvature h_I. Ω is called

- *star-shaped* (w.r.t. some $q_0 \in \Omega$) if for every $q \in \Omega$ holds that $q_0 + \lambda(q - q_0) \in \Omega$ for any $0 < \lambda < 1$;
- *I-mean convex* if $\langle h_I, N\rangle \geq 0$ at every $q \in \partial\Omega$, wherein $N : \partial\Omega \to \mathbb{S}^m$ is the inward-pointing normal;
- *strictly I-mean convex* if $\langle h_I, N\rangle > 0$ at every $q \in \partial\Omega$, wherein $N : \partial\Omega \to \mathbb{S}^m$ is the inward-pointing normal;
- *admissible* if there exists an index set J, which is non-empty, and a family of star-shaped I-mean convex C^2-domains $\{\Omega^j\}_{j\in J}$ such that

$$\Omega \;=\; \bigcap_{j\in J}\Omega^j \quad \text{and} \quad \overline{\Omega} \;=\; \bigcap_{j\in J}\overline{\Omega^j}.$$

Remark 1.6.14.

- Let $k \geq 1$. An open set $\Omega \subset \mathbb{R}^{m+1}$ with *C^k-boundary* $\partial\Omega$ means that the topological boundary $\partial\Omega$ can be locally represented as the graph of a C^k function. In other words, Ω is an embedded m-submanifold of class C^k.

- The notions I-mean convex and strictly I-mean convex are motivated by [Whi91] and [BF09].
- Sometimes we say that $\partial\Omega$ is (strictly) I-mean convex if $\Omega \subset \mathbb{R}^{m+1}$ is (strictly) I-mean convex.
- Instead of evaluating ($\langle h_I, N\rangle > 0$) $\langle h_I, N\rangle \geq 0$ to characterize Ω as (strictly) I-mean convex, we can assume that ($\langle h_I, V\rangle > 0$) $\langle h_I, V\rangle \geq 0$ for all inward-pointing vectors V on $\partial\Omega$.
- In the case that the Cartan integrand I is elliptic and independent of its first argument, convex sets with C^2-boundary are also I-mean convex and convex sets in general are admissible. This is due to the fact that a convex set can be written as the intersection of half spaces (see [BF09, Remark 3]).

1.6.3 Existence of minimizers and regularity

This section contains some existence and regularity results regarding minimizer of the Cartan functional under some assumptions.

We present existence results in a Sobolev and a smooth setting. The results in a Sobolev setting are geometric boundary value problems for Cartan functionals on two-dimensional surfaces, which have first been treated by Sigalov [Sig50, Sig53, Sig58], Cesari [Ces52], and Danskin [Dan52] establishing the existence of continuous minimizers, before Morrey [Mor61, Mor08], and Rešetnjak [Reš62] gave a simplification of their methods and obtained minimizers with interior Hölder continuity. However, non of these contributions produced conformally parametrized minimizing surfaces, i.e. satisfying (1.6.12) a.e. on the domain, which is crucial for higher regularity. In the work of Hildebrandt and von der Mosel [HvdM99, HvdM02, HvdM03a, HvdM03b, HvdM03c, HvdM05, HvdM09] the existence of conformal minimizers was established, and higher regularity investigated. Notice that they restricted their results to mappings from a 2-dimensional unit ball to n-dimensional ambient space for $n \geq 3$. Hence, we choose $N = \binom{n}{2} = \frac{n(n-1)}{2}$, even though we will choose $n = 3$ later on application side (see section 3.2). In this section, we present the results of [HvdM03b], especially with a result on existence in a certain class of functions (see [HvdM03b, Theorem 1.4]) and a result on regularity for such minimizer (see [HvdM03b, Theorem 1.5]). Higher regularity results for such minimizers are then presented for Cartan integrands with perfect dominance function (see [HvdM03b, Theorem 1.9], [HvdM03c, Theorem 1.1]). In particular, if the Cartan integrand is close enough to the Euclidean area integrand in a C^2-topology (see [HvdM03a, Theorem 1.3]). Afterwards, existence in a smooth setting will be presented. Therefore, White [Whi91] and Winklmann [Win03] are cited giving existence of 2-dimensional I-minimal immersed submanifolds of suitable properties in a 3-dimensional ambient space.

Finally, some results regarding removability of singularities for critical immersions to Cartan functionals are presented. We start by an energy estimate due to Hildebrandt and Sauvigny [HS09] leading to a removability result. Further, we present a removability result due to Simon [Sim77b].

We start with some basic definitions necessary to formulate the existence results in a smooth setting. Remember that we choose $N = \binom{n}{2} = \frac{n(n-1)}{2}$.

Definition 1.6.15 (Curves in vector spaces). A *curve* c in $\mathbb{R}^n$ is a continuous mapping of an interval into $\mathbb{R}$. Such a curve can be represented as a mapping $c : [a, b] \to \mathbb{R}^n$ for an interval $[a, b]$. A curve c is said to be a *simple curve* if it is injective, i.e. for all $x, y \in [a, b]$ with $x \neq y$ holds $c(x) \neq c(y)$. A curve c is said to be a *closed curve* if $c(a) = c(b)$. A closed curve can be reparametrized to be a continuous mapping $c : \mathbb{S}^1 \to \mathbb{R}^n$. A simple closed curve is called a *Jordan curve*. A *rectifiable curve* is a curve of finite Euclidean length.

Definition 1.6.16 ([DHKW92, vol. I., p. 231])**.** Let Γ be a closed Jordan curve in $\mathbb{R}^n$ and let $\varphi : \mathbb{S}^1 \to \Gamma$ be a homeomorphism from $\mathbb{S}^1$ onto Γ. Then a continuous mapping $\psi : \mathbb{S}^1 \to \Gamma$ of $\mathbb{S}^1$ onto Γ is said to be *weakly monotonic* if there is a non-decreasing continuous function $\tau : [0, 2\pi] \to \mathbb{R}$ with $\tau(0) = 0,\ \tau(2\pi) = 2\pi$ such that

$$\psi((\cos\theta, \sin\theta)) \quad = \quad \psi((\cos\tau(\theta), \sin\tau(\theta)))$$

for $0 \le \theta \le 2\pi$.

First, the formulation of the *Plateau problem* in a weak setting regarding regularity of the minimizer. This is necessary to give the results of [HvdM03b]. Let Γ be a rectifiable Jordan curve in $\mathbb{R}^n$ and $B := \{(u^1, u^2) : (u^1)^2 + (u^2)^2 < 1\}$ the *unit disc*, which is chosen to be the parameter domain of the competing surfaces $X : B \to \mathbb{R}^n$. $\mathcal{C}(\Gamma)$ denotes the class of surfaces $X \in W^{1,2}(B, \mathbb{R}^n)$ whose trace $X|_{\partial B}$ on ∂B is a continuous, weakly monotonic mapping of ∂B onto Γ. $\mathcal{C}(\Gamma)$ is non-empty.

Theorem 1.6.17 ([HvdM03b, Theorem 1.4])**.** *Suppose that $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ satisfies* (**H**)*,* (**D**) *and* (**C**) *and denote by $\mathcal{I}$ the corresponding Cartan functional. Then there exists a minimizer X of $\mathcal{I}$ in $\mathcal{C}(\Gamma)$, i.e.*

$$\mathcal{I}(X) \quad = \quad \inf_{\mathcal{C}(\Gamma)} \mathcal{I}(\cdot). \tag{1.6.11}$$

Therein, X is (Euclidean) conformally parametrized *almost everywhere on B by construction, i.e.*

$$|X_{u^1}|^2 \quad = \quad |X_{u^2}|^2 \quad \text{and} \quad \langle X_{u^1}, X_{u^2}\rangle \quad = \quad 0 \quad \mathscr{H}^2\text{-a.e. on } B. \tag{1.6.12}$$

Theorem 1.6.18 ([HvdM03b, Theorem 1.5])**.** *Suppose that $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ satisfies* (**H**)*,* (**D**) *and* (**C**) *and denote by $\mathcal{I}$ the corresponding Cartan functional. Then every minimizer X of $\mathcal{I}$ in $\mathcal{C}(\Gamma)$, which is (Euclidean) conformally parametrized almost everywhere on B satisfies*

$$X \in C^0(\bar{B}, \mathbb{R}^n) \cap C^{0,\sigma}(B, \mathbb{R}^n) \cap W^{1,q}(B, \mathbb{R}^n) \tag{1.6.13}$$

for $\sigma = M_1/M_2 \in (0, 1]$ and $q > 2$, where M_i for $i = 1, 2$ are given by (**D**)*.*

To gain higher regularity for these minimizers in a weak setting, we need a so-called dominance function for the Cartan integrand. This dominance function plays then the role, which is played by the Lagrangian of the Dirichlet energy for the Euclidean area.

Definition 1.6.19 (**Perfect dominance function** [HvdM03a, Definition 1.2])**.** Let $I = I(x, z) \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ be a Cartan integrand. The *associated Lagrangian* to a Cartan integrand $I = I(x, z)$ is set to be

$$i(x, Z) \quad = \quad I(x, z_1 \wedge z_2)$$

for $Z = (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$. A *perfect dominance function* for I is a function $D^I \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} \backslash \{0\}))$ such that the following conditions are satisfied for all $(x, Z) = (x, (z_1, z_2)) \in \mathbb{R}^n \times \mathbb{R}^{2n}$:

(**D1**) $i(x, Z) \le D^I(x, Z)$;

(**D2**) $i(x, Z) = D^I(x, Z)$ for all (x, Z) with $|z_1| = |z_2|$ and $\langle z_1, z_2\rangle = 0$;

(**D3**) $D^I(x, tZ) = t^2 D^I(x, Z)$ for all $t > 0$;

(**D4**) there are universal constants $0 < \mu_1 \le \mu_2$ such that $\mu_1|Z| \le D^I(x, Z) \le \mu_2|Z|$;

(D5) for any $R_0 > 0$ there is a constant $\lambda_G(R_0) > 0$ such that

$$\xi^T \cdot D^I_{zz}(x, Z) \cdot \xi \quad \geq \quad \lambda_G(R_0)|\xi|^2$$

for $|x| \leq R_0$, $Z \neq 0$ and $\xi \in \mathbb{R}^{2n}$.

The following theorem established by Hildebrandt and von der Mosel in [HvdM03a] guarantees the existence of a perfect dominance function for some Cartan integrands.

Theorem 1.6.20 (**Perfect dominance function** [HvdM03a, Theorem 1.3])**.** *Let $I^* \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times \mathbb{R}^N \backslash \{0\})$ be a Cartan integrand satisfying the conditions* (**H**), (**D**) *(see page 47) with constants $M_1(I^*), M_2(I^*)$. In addition, let I^* be elliptic in the sense of Definition 1.6.4 with*

$$\Lambda(I^*) \quad := \quad \inf_{R_0 \in (0,\infty]} \Lambda_{I^*}(R_0) > 0. \tag{1.6.14}$$

Then the Cartan integrand I defined by

$$I(x, z) \quad := \quad k|z| + I^*(x, z) \tag{1.6.15}$$

possesses a perfect dominance function for all fixed

$$k \quad > \quad k_0(I^*) \quad := \quad 2[M_2(I^*) - \min\{\Lambda(I^*), \frac{M_1(I^*)}{2}\}]. \tag{1.6.16}$$

This corollary is a direct consequence of Theorem 1.6.20 and guarantees the existence of a perfect dominance function for Cartan integrands close enough to the Euclidean area integrand in a C^2-topology.

Corollary 1.6.21. *Let $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times \mathbb{R}^N \backslash \{0\})$ be a Cartan integrand. If*

$$\delta \quad := \quad \sup_{x \in \mathbb{R}^n} \{\widehat{\rho}_2(I(x, \cdot) - A(\cdot))\} \quad < \quad \frac{1}{5},$$

then the Cartan integrand I possesses a perfect dominance function. I then especially satisfies (**H**), (**D**) *and* (**E**).

Proof. We intend to apply Theorem 1.6.20 to the present situation and therefore we define the scaled Cartan integrand $I_R(x, z) := RI(x, z)$. We represent $I_R(x, z)$ as the sum

$$I_R \quad = \quad A + I^*_R$$

with $I^*_R(x, z) := RI(x, z) - A(z) = RI(x, z) - |z|$ for some $R > 0$ and apply Theorem 1.6.20 with $k = 1$. We start by computing constants regarding the positive definiteness and ellipticity of the involved Cartan-integrands. We estimate

$$\begin{aligned} I(x, z) \quad &= \quad |z|I(x, \frac{z}{|z|}) \\ &\geq \quad |z|(A(\frac{z}{|z|}) + I(x, \frac{z}{|z|}) - A(\frac{z}{|z|})) \\ &\geq \quad |z|(1 - \widehat{\rho}_0(I(x, \cdot) - A(\cdot))) \\ &\geq \quad |z|(1 - \delta) > 0 \end{aligned}$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^N \backslash \{0\}$ and $\delta < 1$. In a similar way, we compute

$$I(x, z) \quad \leq \quad |z|(1 + \delta)$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^N \backslash \{0\}$. Further, regarding the ellipticity constant of I, we get the following

$$\begin{aligned}
\xi^T |z| I_{zz}(x, \frac{z}{|z|})\xi &= \xi^T(A_{zz}(\frac{z}{|z|}) + I_{zz}(x, \frac{z}{|z|}) - A_{zz}(\frac{z}{|z|}))\xi \\
&= (\Pi_{z^\perp}\xi)^T(A_{zz}(\frac{z}{|z|}) + I_{zz}(x, \frac{z}{|z|}) - A_{zz}(\frac{z}{|z|}))(\Pi_{z^\perp}\xi) \\
&= \xi^T A_{zz}(\frac{z}{|z|})\xi + (\Pi_{z^\perp}\xi)^T(I_{zz}(x, \frac{z}{|z|}) - A_{zz}(\frac{z}{|z|}))(\Pi_{z^\perp}\xi) \\
&= \xi^T A_{zz}(\frac{z}{|z|})\xi + (\Pi_{z^\perp}\xi)^i(I_{z^i z^j}(x, \frac{z}{|z|}) - A_{z^i z^j}(\frac{z}{|z|}))(\Pi_{z^\perp}\xi)^j \\
&= \xi^T A_{zz}(\frac{z}{|z|})\xi - \widehat{\rho}_2(I(x,\cdot) - A(\cdot))|\Pi_{z^\perp}\xi|^2 \\
&= \xi^T A_{zz}(\frac{z}{|z|})\xi - \delta|\Pi_{z^\perp}\xi|^2 \\
&= (1-\delta)\xi^T A_{zz}(\frac{z}{|z|})\xi \\
&=: \Lambda(C)\xi^T A_{zz}(\frac{z}{|z|})\xi
\end{aligned}$$

where we exploited the fact that $A_{zz}(z)z = 0 = I_{zz}(x, z)z$ by Theorem 1.4.3 and $0 < \delta < 1$. Therein, $\Pi_{z^\perp}$ denotes the orthogonal projection onto $z^\perp$ and can be expressed by $\Pi_{z^\perp}\eta = \eta - \langle \eta, z/|z|\rangle z/|z| = A_{zz}(\frac{z}{|z|}) \cdot \eta$ for all $\eta \in \mathbb{R}^n$.

So, we get as positive definiteness constants for I_R^* (see (**D**))

$$M_1(I_R^*)|z| \quad := \quad (R(1-\delta)-1)|z| \quad \leq \quad I_R^*(x,z) \quad \leq \quad (R(1+\delta)-1)|z| \quad =: \quad M_2(I_R^*)|z|,$$

wherein we assume $R > (1-\delta)^{-1}$. In a next step, we compute the ellipticity constant $\Lambda(I_R^*)$. Hence, we compute the Hessian of I_R^*, i.e.

$$\begin{aligned}
\xi^T(I_R^*)_{zz}(x,z)\xi &= R\xi^T I_{zz}(x,z)\xi - \xi^T A_{zz}(z)\xi \\
&\geq (R\Lambda(C) - 1)\xi^T A_{zz}(z)\xi \\
&= (R(1-\delta)-1)\frac{1}{|z|}|\Pi_{z^\perp}\xi|^2 \\
&=: \frac{\Lambda(I_R^*)}{|z|}|\Pi_{z^\perp}\xi|^2
\end{aligned}$$

where $z \in \mathbb{R}^N \backslash \{0\}$. Now we can write

$$I_R(x,z) \quad = \quad A(z) + I_R^*(x,z)$$

whereon we apply Theorem 1.6.20. So, $k = 1$ in Theorem 1.6.20 and we compute

$$\begin{aligned}
k_0(I_R^*) &= 2[M_2(I_R^*) - \min\{\Lambda(I_R^*), \frac{M_1(I_R^*)}{2}\}] \\
&= 2[R(1+\delta) - 1 - (R(1-\delta)-1)/2] \\
&= R + 3\delta R - 1
\end{aligned}$$

by exploiting that $\Lambda(I_R^*) = R(1-\delta) - 1 > (R(1-\delta)-1)/2 = M_1(I_R^*)/2$. If R tends to $(1-\delta)^{-1}$ from above, we get as limit $4\delta/(1-\delta)$. The limit is strictly less than 1 for $\delta < 1/5$ such that we can assure that $1 > k_0(I_R^*)$ for R sufficiently close to $(1-\delta)^{-1}$. So, the conditions of Theorem 1.6.20 are fulfilled and thereby we know that I_R possesses a perfect dominance function. As I_R is just a positive scalar multiple of I, and the fact that the property of a Cartan integrand to have a perfect dominance function is invariant w.r.t. positive scaling, we get that I possesses a perfect dominance function by itself. □

In the following two theorems, we present the regularity improving results for minimizers of the Plateau problem in a weak setting (see Theorem 1.6.17). Notice that none of the two theorems excludes branching points. Branching points are points, where the minimizer's differential fails to be of full rank.

Theorem 1.6.22 (**Higher regularity in the interior** [HvdM03b, Theorem 1.9]). *Let $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times \mathbb{R}^N \backslash \{0\})$ be a positive definite elliptic Cartan integrand (i.e. I satisfies the conditions* (**H**)*,* (**D**) *and* (**E**)*) and assume that I possesses a perfect dominance function. Then any Euclidean conformally parametrized minimizer X of $\mathcal{I}$ in $\mathcal{C}(\Gamma)$ is of class $W^{2,2}_{\text{loc}}(B,\mathbb{R}^n) \cap C^{1,\sigma}(B,\mathbb{R}^n)$ for some $\sigma > 0$.*

Theorem 1.6.23 (**Higher regularity** [HvdM03c, Theorem 1.1]). *Let $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times \mathbb{R}^N \backslash \{0\})$ be a positive definite elliptic Cartan integrand (i.e. I satisfies the conditions* (**H**)*,* (**D**) *and* (**E**)*) and assume that I possesses a perfect dominance function and that Γ is of class C^4. Then there is some $\alpha \in (0,1)$ such that any Euclidean conformally parametrized minimizer X of $\mathcal{I}$ in $\mathcal{C}(\Gamma)$ is of class $W^{2,2}(B,\mathbb{R}^n) \cap C^{1,\alpha}(\bar{B},\mathbb{R}^n)$ and satisfies*

$$||X||_{W^{2,2}(B,\mathbb{R}^n)} + ||X||_{C^{1,\alpha}(\bar{B},\mathbb{R}^n)} \quad \leq \quad c(\Gamma, I)$$

where the constant $c(\Gamma, I)$ depends only on Γ and I.

The next result gives existence of an embedded surface in $\mathbb{R}^3$ minimizing the Cartan functional $\mathcal{I}$ among all embedded surfaces of a certain topological genus, which are spanned into a given boundary curve contained in an I-mean convex surface. This result was established by White in [Whi91].

Theorem 1.6.24 (**Existence in a smooth setting** [Whi91, Theorem 3.4]). *Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary $\partial\Omega$. Further, let $I \in C^0(\Omega \times \mathbb{R}^3) \cap C^2(\Omega \times (\mathbb{R}^3 \backslash \{0\}))$ be an even (i.e. $I(x,z) = I(x,-z)$), positive definite and elliptic Cartan integrand and assume that Ω is strictly I-mean convex. Then for a smooth simple closed curve c on $\partial\Omega$ and each $g \geq 0$ exists a smooth embedded surface that minimizes $\mathcal{I}(\mathscr{M})$ among all embedded surfaces $\mathscr{M}$ with boundary $\partial\mathscr{M} = c$ and $\text{genus}(\mathscr{M}) \leq g$.*

Remark 1.6.25. Some words on the *genus* of an oriented smooth manifold can be found in [HvdM03b, p. 39]. We denote the genus of an oriented smooth manifold $\mathscr{M}$ by $\text{genus}(\mathscr{M})$.
.

The following existence result in a smooth setting is due to Winkelmann [Win03]. It is restricted to Cartan functionals with even positive definite elliptic Cartan integrand independent of the first argument.

Theorem 1.6.26 (**Existence in a smooth setting** [Win03, Thm. 1.2 and Cor. 1.3]). *Let $I \in C^0(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \backslash \{0\})$ be an elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**D**)*,* (**H**) *and* (**E**)*). Further, let Γ be a graph of bounded slope over the boundary $\partial\Omega$ of some plane bounded convex domain $\Omega \subset \mathbb{R}^2$, then there is an (up to reparametrizations) unique Finsler-minimal immersions $X \in C^\infty(\Omega,\mathbb{R}^3) \cap C^0(\Omega,\mathbb{R}^3)$ spanning Γ. Notice that the surface X parametrizes a graph.*

Remark 1.6.27.

- In the context of Theorem 1.6.26, an immersion $X \in C^\infty(\Omega,\mathbb{R}^3) \cap C^0(\Omega,\mathbb{R}^3)$ is said to span Γ if $X(\partial\Omega) = \Gamma$.
- The *bounded slope* condition in the last part of the theorem means that we find a constant $R > 0$, so that we can write Γ as a graph,

$$\Gamma \quad = \quad \{(u,\gamma(u)) \in \mathbb{R}^3 : u = (u^1,u^2) \in \partial\Omega\}$$

for some function $\gamma : \partial\Omega \to \mathbb{R}$, such that for any curve point $(u_0, \gamma(u_0)) \in \Gamma$ there exist two vectors $p_0^+, p_0^- \in \overline{B_R^2(0)} \subset \mathbb{R}^2$ such that the two affine linear functions

$$\ell_0^+(u) \quad := \quad p_0^+ \cdot (u - u_0) + \gamma(u_0) \quad \text{and} \quad \ell_0^-(u) \quad := \quad p_0^- \cdot (u - u_0) + \gamma(u_0)$$

satisfy $\ell_0^-(u) \leq \gamma(u) \leq \ell_0^+(u)$ for all $u \in \partial\Omega$. In particular, if Ω is strictly convex and Γ is a C^2-graph over $\partial\Omega$ then Γ satisfies the bounded slope condition (see [GT01, pp. 309, 310]).

The following energy estimate for I-minimal graphs in codimension 1 is due to Hildebrandt and Sauvigny [HS09]. Therein, we choose $n = m+1$ and $N = m+1$ for a positive integer m.

Theorem 1.6.28 (**Energy estimate for graphs** [HS09, Theorem 3.1])**.** *Let $I \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^2(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}\backslash\{0\})$ be a positive definite, elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**D**)*,* (**H**) *and* (**E**)*). Let Ω be an open set in $\mathbb{R}^m$ with compact closure, K a compact subset of Ω with $\mathcal{H}^{m-1}(K) = 0$. Let $\Lambda_1 > 0$ be the lower ellipticity constant of I (see* (**E**)*). Assume two critical graphs of the non-parametric problem (i.e. both satisfy* (1.6.10)*) $f_1, f_2 \in C^0(\overline{\Omega}\times\mathbb{R}) \cap C^2(\Omega\times\mathbb{R})$, then we have the following weighted energy estimate*

$$\int_{\Omega\backslash K} \mu(f_1, f_2)|\nabla f_1 - \nabla f_2|^2 \mathrm{d}x \quad \leq \quad \frac{2}{\Lambda_1} \int_{\partial\Omega} |f_1 - f_2| \mathrm{d}\mathcal{H}^{m-1}$$

with

$$\mu(f_1, f_2)(x) \quad := \quad (\max\{W_1(x), W_2(x)\})^{-3} \quad \text{for } \ x \in \Omega\backslash K.$$

Therein are

$$W_i(x) \quad := \quad \sqrt{1 + |\nabla f_i|^2} \quad \text{for } \ i \ = \ 1, 2.$$

Proof. A combination of (1.6.10) and [HS09, Theorem 3.1] proves the stated assertion. Notice that the ellipticity of the characterizing differential equation (1.6.10) derives directly from the ellipticity of I. □

The following corollary is an uniqueness result, which is a direct consequence of the energy estimate in Theorem 1.6.28.

Corollary 1.6.29 (**Uniqueness** [HS09, Theorem 4.1])**.** *Assume that I, Ω, K and f_1, f_2 are satisfying the conditions of Theorem 1.6.28. Assume further that $\Omega\backslash K$ is connected and that*

$$f_1(x) \quad = \quad f_2(x) \quad \text{for all } \ x \in \partial\Omega.$$

Then it follows that

$$f_1 \quad \equiv \quad f_2 \quad \text{on } \ \overline{\Omega}\backslash K.$$

Proof. The assertion follows directly of Theorem 1.6.28 in a similar way as [HS09, Theorem 4.1] is based on [HS09, Theorem 2.1] in the proof given in [HS09]. □

The following removability theorem for I-minimal graphs was established by Simon in [Sim77b].

Theorem 1.6.30 (**Removable singularities** [Sim77b, Theorem 1])**.** *Let $I \in C^0(\mathbb{R}^n \times \mathbb{R}^N) \cap C^2(\mathbb{R}^n \times \mathbb{R}^N\backslash\{0\})$ be a positive definite, elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**D**)*,* (**H**) *and* (**E**)*). Let $f \in C^2(\Omega\backslash K)$ be a critical graph of the non-parametric problem on $\Omega\backslash K$ (i.e. f satisfies* (1.6.10) *on $\Omega\backslash K$), wherein K is a locally compact subset of Ω with $\mathcal{H}^{m-1}(K) = 0$. Then f can be extended as critical graph of class C^2 on Ω. In other words, there exists a function $\tilde{f} \in C^2(\Omega)$ which is a solution to* (1.6.10) *even on Ω such that $\tilde{f} = f$ on $\Omega\backslash K$.*

Remark 1.6.31. A set $K \subset \mathbb{R}^n$ is *locally compact* if for each $x_0 \in K$ there is $r = r(x_0) > 0$ such that $A \cap \overline{B_r^n(x_0)}$ is compact (see [Sim96, Section 4.1, p. 91]).

1.6.4 Enclosure results

In this section, we present some enclosure results. Notice that all the enclosure results presented here assume that the Cartan integrand in consideration is independent of the first argument.

The following result, established Clarenz in [Cla02], is a convex hull property for critical immersions of a Cartan functional.

Theorem 1.6.32 (**Convex hull property** [Cla02, Theorem 2.3]). *Let $I \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^3(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \backslash \{0\})$ be an elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**H**) *and* (**E**)*). Let $\mathscr{M}$ be a compact oriented m-manifold with boundary $\partial\mathscr{M}$, $\mathscr{M} = \mathrm{int}\mathscr{M} \cup \partial\mathscr{M}$ and $X \in C^2(\mathrm{int}\mathscr{M}, \mathbb{R}^{m+1}) \cap C^0(\mathscr{M}, \mathbb{R}^{m+1})$ be an I-minimal immersion. Then we have*

$$\overline{X(\mathscr{M})} \subset \mathrm{conv}X(\partial\mathscr{M}),$$

wherein $\mathrm{conv}\Sigma$ *denotes the* convex hull *of the subset $\Sigma \subset \mathbb{R}^{m+1}$.*

Remark 1.6.33. In contrast to [Cla02, Theorem 2.3], we require $\mathscr{M}$ to be compact as it seems necessary to apply the maximum principle [GT01, p. 32] on each coordinate domain of $\mathscr{M}$ in the proof of [Cla02, Theorem 2.3] properly and then use a finite covering of the manifold with such coordinate domains. By using the fact that $\mathrm{conv}A \subset \mathrm{conv}(A \cup B)$ holds true for each $A, B \subset \mathbb{R}^{m+1}$, we can then conclude the proof correctly.

If the boundary configuration of an I-minimal immersed hypersurface is contained in a set of special characteristics then the whole I-minimal surface that spans that boundary is contained in this set. This is essentially what the following theorem established by Bergner and Fröhlich [BF09] is about. Most of these results were made in the more general setting of weighted mean curvature whereof the I-mean curvature is a special case.

Theorem 1.6.34 (**Enclosure result** [BF09, Lemma 2]). *Let $I \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^2(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \backslash \{0\})$ be a positive definite, elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**D**), (**H**) *and* (**E**)*). For some bounded open set $\Omega \subset \mathbb{R}^m$, let $X \in C^0(\overline{\Omega}, \mathbb{R}^{m+1}) \cap C^2(\Omega, \mathbb{R}^{m+1})$ be an I-minimal immersed hypersurface. Furthermore, let $\Sigma \subset \mathbb{R}^{m+1}$ be an admissible domain. Then, the following two implications hold*

$$X(\partial\Omega) \subset \overline{\Sigma} \Rightarrow X(\Omega) \subset \overline{\Sigma} \tag{1.6.17}$$

and

$$X(\partial\Omega) \subset \overline{\Sigma} \ \wedge \ X(\Omega) \cap \Sigma \neq \emptyset \Rightarrow X(\Omega) \subset \Sigma. \tag{1.6.18}$$

As mentioned in Remark 1.6.14, we know by [BF09, Remark 3] that every convex domain is admissible. Thereby, we deduce the following corollary.

Corollary 1.6.35 (**Enclosure result for convex domains**). *Let $I \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^2(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \backslash \{0\})$ be a positive definite, elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies* (**D**), (**H**) *and* (**E**)*). For some bounded open set $\Omega \subset \mathbb{R}^m$, let $X \in C^0(\overline{\Omega}, \mathbb{R}^{m+1}) \cap C^2(\Omega, \mathbb{R}^{m+1})$ be an I-minimal immersed hypersurface. Furthermore, let $\Sigma \subset \mathbb{R}^{m+1}$ be a convex domain. Then, the following two implications hold*

$$X(\partial\Omega) \subset \overline{\Sigma} \Rightarrow X(\Omega) \subset \overline{\Sigma} \tag{1.6.19}$$

and

$$X(\partial\Omega) \subset \overline{\Sigma} \ \wedge \ X(\Omega) \cap \Sigma \neq \emptyset \Rightarrow X(\Omega) \subset \Sigma. \tag{1.6.20}$$

Remark 1.6.36. Such a result was shown for $m = 2$ and Euclidean balls of arbitrary radius as convex domains by Sauvigny in [Sau88] and can also be found in [Frö08, p. 57].

1.6.5 Isoperimetric inequalities

In this section, we will present some isoperimetric inequalities for immersions in terms of the Cartan functional. Let $\mathscr{M}$ be a compact oriented smooth m-manifold with boundary $\partial\mathscr{M}$ and a smooth immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$. Define $\varphi := X|_{\partial\mathscr{M}}$ as the restriction of X to $\partial\mathscr{M}$. Notice that φ is a smooth immersion on $\partial\mathscr{M}$ by construction. We define in this section

$$\mathrm{d}A \quad := \quad \mathrm{d}V_{X^*|\cdot|}$$

and

$$\mathrm{d}S \quad := \quad \mathrm{d}V_{\varphi^*|\cdot|}.$$

For the definitions of $\mathrm{d}V_{X^*|\cdot|}$ and $\mathrm{d}V_{\varphi^*|\cdot|}$, we refer to (2.1.1) for proper choices of dimension. Hence, $\int_{\mathscr{M}} \mathrm{d}A = \mathcal{A}_m^{|\cdot|}(X) = \mathcal{A}(X)$ is the m-dimensional *Euclidean area* of $X(\mathscr{M})$ and $\int_{\partial\mathscr{M}} \mathrm{d}S = \mathcal{A}_{m-1}^{|\cdot|}(X|_{\partial\mathscr{M}})$ is the $(m-1)$-dimensional Euclidean area of $X(\partial\mathscr{M})$.

Theorem 1.6.37 ([Win02, Theorem 2.2]). *Let $I \in C^0(\mathbb{R}^{m+1}\times\mathbb{R}^{m+1})\cap C^2(\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}\backslash\{0\})$ be a Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies (**H**)). Let $\mathscr{M}$ be a compact oriented smooth m-manifold with boundary $\partial\mathscr{M}$, $\mathscr{M} = \mathrm{int}\mathscr{M} \cup \partial\mathscr{M}$, $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be a smooth immersion with normal map $N : \mathscr{M} \to \mathbb{S}^m$ and $X(\mathscr{M}) \subset \overline{B_R^{m+1}(a)}$. Then we have*

$$\mathcal{I}(X) \quad \le \quad \frac{R}{m}\left(\int_{\mathscr{M}} |H_I(X,N)|\mathrm{d}A + ||I_z||_{\infty,\mathbb{S}^m}\int_{\partial\mathscr{M}} \mathrm{d}S\right).$$

If X is an I-minimal immersion (i.e. $H_I \equiv 0$), we get

$$\mathcal{I}(X) \quad \le \quad \frac{R}{m}||I_z||_{\infty,\mathbb{S}^m}\int_{\partial\mathscr{M}} \mathrm{d}S.$$

If X is closed (i.e. $\partial\mathscr{M} = \emptyset$), we get

$$\mathcal{I}(X) \quad \le \quad \frac{R}{m}\int_{\mathscr{M}} |H_I(X,N)|\mathrm{d}A.$$

Now we state some more specialized result for I-minimal immersions and $m = 2$.

Theorem 1.6.38 ([Win02, Theorem 3.2]). *Let $I \in C^0(\mathbb{R}^3\times\mathbb{R}^3)\cap C^2(\mathbb{R}^3\times\mathbb{R}^3\backslash\{0\})$ be a Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies (**H**)). Let $\mathscr{M}$ be a compact oriented 2-manifold with boundary $\partial\mathscr{M}$, where $\partial\mathscr{M}$ is assumed to consist of exactly $k \ge 1$ closed curves $\gamma_1,\dots,\gamma_k$. Let $X : \mathscr{M} \to \mathbb{R}^3$ be an I-minimal smooth immersion with boundary curves $\Gamma_i = X(\gamma_i)$ for $i = 1,\dots,k$. Then we have*

$$\mathcal{I}(X) \quad \le \quad ||I_z||_{\infty,\mathbb{S}^2}\sum_{i=1}^k\left(\frac{L_i^2}{4\pi} + \frac{L_i}{2}\mathrm{dist}(a,\Gamma_i)\right).$$

for every point $a \in \mathbb{R}^3$. Therein, we let $L_i := \int_{\gamma_i} \mathrm{d}S$ stand for the length of Γ_i for $i = 1,\dots,k$.

Theorem 1.6.39 ([Win02, Corollary 3.3]). *Let $I \in C^0(\mathbb{R}^3\times\mathbb{R}^3)\cap C^2(\mathbb{R}^3\times\mathbb{R}^3\backslash\{0\})$ be a positive definite elliptic Cartan integrand independent of the first argument (i.e. $I = I(z)$ satisfies (**H**), (**D**) and (**E**)). Let $\Lambda_i > 0$ for $i = 1,2$ be the ellipticity constants as in . Let $\mathscr{M}$ be a compact oriented 2-manifold with boundary $\partial\mathscr{M}$, where $\partial\mathscr{M}$ is assumed to consist of exactly two closed curves γ_1,γ_2. Let $X : \mathscr{M} \to \mathbb{R}^3$ be an I-minimal smooth immersion with boundary curves $\Gamma_i = X(\gamma_i)$ for $i = 1,2$ such that $X(\mathscr{M})$ is connected. Then we have*

$$\mathcal{I}(X) \quad \le \quad ||I_z||_{\infty,\mathbb{S}^2}\sum_{i=1}^k\left(\frac{L_i^2}{4\pi} + \frac{L_i}{2}\mathrm{dist}(a,\Gamma_i)\right).$$

for every point $a \in \mathbb{R}^3$. Therein, we let $L_i := \int_{\gamma_i} \mathrm{d}S$ stand for the length of Γ_i for $i = 1, \ldots, k$. If $k = 2$ and $X(\mathscr{M})$ is connected, then

$$\mathcal{I}(X) \quad \leq \quad ||I_z||_{\infty,\mathbb{S}^2} \sum_{i=1}^{k} \left(\frac{L_i^2}{4\pi} + \frac{L_i}{2} \mathrm{dist}(a, \Gamma_i) \right).$$

1.6.6 Curvature estimates and Bernstein results

In this section, we introduce some concepts and sets necessary to formulate curvature estimates of Simon [Sim77c] for a special class of hypersurfaces embedded in $\mathbb{R}^{m+1}$, which minimize locally an elliptic Cartan functional I. These curvature estimates imply Bernstein theorems for $m \leq 7$, i.e. for an elliptic Cartan functional on $\mathbb{R}^{m+1}$ independent of the first argument, an entire I-minimal graph is an m-hyperplane. Therein, an *entire graph* is a graph

$$G(u) \quad := \quad \begin{pmatrix} u \\ f(u) \end{pmatrix}$$

for $u \in \mathbb{R}^m$ and a function $f : \mathbb{R}^m \to \mathbb{R}$. Simon developed the curvature estimates even for currents but we restrict these results to a more regular setting. Afterwards, we present a Bernstein theorem established by Winklmann in [Win07] for I-minimal graphs in general dimension assuming a growth condition on the gradient of the graph.

We start by defining some basic notions.

Definition 1.6.40 ([Sim77c])**.** Let $\mathcal{U}$ be the set of all complete oriented C^2 m-manifolds $\mathscr{M}$ with boundary $\partial\mathscr{M}$ embedded in $\mathbb{R}^{m+1}$ such that $\mathcal{H}^m(\mathscr{M} \cap K) < \infty$ for all compact subsets K of $\mathbb{R}^{m+1}$. $\mathcal{U}(x_0, R)$ will denote the set of $\mathscr{M} \in \mathcal{U}$, which contain $x_0 \in \mathscr{M}$ and $\partial\mathscr{M} \subset \mathbb{R}^{m+1} \backslash B_R(x_0)$.

We say that $\mathscr{M} \in \mathcal{U}$ is *I-minimizing in A*, where A is chosen to be an open subset of $\mathbb{R}^{m+1}$, if for each bounded open V with $\bar{V} \subset A$ holds

$$\mathcal{I}(\mathscr{M} \cap V) \quad \leq \quad \mathcal{I}(\mathscr{N}),$$

whenever $\mathscr{N} \in \mathcal{U}$ satisfies $\bar{\mathscr{N}} \subset A$ and $\partial(\mathscr{M} \cap V) = \partial\mathscr{N}$.

$\mathcal{M}_I(x_0, R)$ will denote the collection of $\mathscr{M} \in \mathcal{U}(x_0, R)$ such that $\mathscr{M} \cap B_R(x_0)$ is I-minimizing in $B_R(x_0)$ and such that $\bar{\mathscr{M}} \cap B_R(x_0) = \partial U_{\mathscr{M}} \cap B_R(x_0)$ for some open set $U_{\mathscr{M}} \subset \mathbb{R}^{m+1}$. $\mathcal{M}_I^{'}(x_0, R)$ will denote the collection of $\mathscr{M} \in \mathcal{U}(x_0, R)$ of vanishing I-mean curvature, which can be represented as a graph of a C^2-function f on $B_R^m(x_0^{'})$, i.e.

$$\begin{aligned} \mathscr{M} \cap B_R^{m+1}(x_0) \quad = \quad & \{x = (x^1, \ldots, x^{m+1})^T \in \mathbb{R}^{m+1} : x^{'} = (x^1, \ldots, x^m)^T \in B_R^m(x_0^{'}) \\ & \text{and } x^{m+1} = f(x^{'})\} \end{aligned}$$

and f satisfies (1.6.10). Therein, we define $x^{'} := (x^1, \ldots, x^m)^T$ for $x := (x^1, \ldots, x^{m+1}) \in \mathbb{R}^{m+1}$. Notice that each $\mathscr{M} \in \mathcal{M}_I^{'}(x_0, R)$ is I-minimizing in $B_R^m(x_0^{'}) \times \mathbb{R}$ and hence $\mathcal{M}_I^{'}(x_0, R) \subset \mathcal{M}_I(x_0, R)$ by an argument in [Sim77c].

Remark 1.6.41.

- A complete embedded manifold with boundary is automatically a closed set as a subset of $\mathbb{R}^{m+1}$. Complete is meant with respect to the manifolds topology. The definition of manifold with boundary used here includes the boundary.
- The set $\mathcal{U}$ corresponds to the set of similar name in [Sim77c, p. 267] as conditions (3) and (4) of [Sim77c, p. 267] are fulfilled automatically for each complete oriented embedded C^2 m-manifold $\mathscr{M}$ with boundary with $\mathscr{M} \in \mathcal{U}$.

- Notice, we used that in the notation of [Sim77c] $\mathrm{reg}\mathscr{M} = \mathrm{int}\mathscr{M}$, $\mathrm{sing}\mathscr{M} = \mathrm{spt}\partial[\![\mathscr{M}]\!] = \partial\mathscr{M}$ for such manifolds $\mathscr{M} \in \mathcal{U}$ by an application of Stokes' theorem an the special properties of $\mathscr{M}$. So, the definition of $\mathcal{U}$ here is a bit more restrictive than the one in [Sim77c]. The definitions of $\mathrm{reg}\mathscr{M}$, $\mathrm{sing}\mathscr{M}$ and $\mathrm{spt}\partial[\![\mathscr{M}]\!]$ are given in [Sim77c, pp. 265-268].

The following two corollaries are results of [Sim77c]. Essentially, they give for some $m \leq 7$ and a Cartan integrand I with additional properties constants $c = c(m, I)$ such that the curvature estimate

$$\sum_{i=1}^{m} \kappa_i^2(x_0) \quad \leq \quad \frac{c}{R^2}, \tag{1.6.21}$$

holds, where $\kappa_1(x_0), \dots, \kappa_m(x_0)$ are the principal curvatures of $\mathscr{M} \in \mathcal{M}_I(x_0, R)$ at x_0. The first corollary gives a result for dimensions $m = 2$ and $m = 3$.

Corollary 1.6.42 ([Sim77c, Corollary 1]). *Let I be a positive definite, elliptic Cartan integrand of class $C^{2,\alpha}_{\mathrm{loc}}$ on $\mathbb{R}^{m+1}\backslash\{0\}$, which is independent of the first argument, i.e. $I = I(z) : \mathbb{R}^{m+1} \to \mathbb{R}$ satisfies* (**D**), (**H**), (**E**) *and $I \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$. If $m = 2$ then there is a constant $c = c(2, I)$ such that* (1.6.21) *holds for $\mathscr{M} \in \mathcal{M}_I(x_0, R)$. If $m = 3$ there is a constant $c = c(3, I)$ such that* (1.6.21) *holds for any $\mathscr{M} \in \mathcal{M}'_I(x_0, R)$. Especially if f is a function in $C^2(\mathbb{R}^3)$, whose graph has vanishing I-mean curvature (i.e.* (1.6.10) *holds on $\mathbb{R}^3$) then f is affine linear.*

Similar results for higher dimension (i.e. $m \leq 7$) in [Sim77c] make use of an additional C^3-closeness of the Cartan integrand I in consideration to the standard Euclidean area integrand $|\cdot|$. This closeness will be expressed by the seminorm $\widehat{\rho}_3$, defined by

$$\widehat{\rho}_3(f) \quad := \quad \max\{|\mathrm{D}^\alpha f(\zeta)| : \zeta \in \mathbb{S}^m,\ |\alpha| \leq 3\}$$

for $f \in C^3(\mathbb{R}^{m+1}\backslash\{0\})$ as given in Definition 1.2.18.

Corollary 1.6.43 ([Sim77c, Corollary 2]). *Let $I : \mathbb{R}^{m+1} \to \mathbb{R}$ be a positive definite, elliptic Cartan integrand of class $C^3(\mathbb{R}^{m+1}\backslash\{0\})$ independent of the first argument on $\mathbb{R}^{m+1}$. In case $m \leq 6$ there is an $\eta > 0$ with the following property: If I satisfies $\widehat{\rho}_3(I(\cdot) - |\cdot|) < \eta$ then there is a constant $c = c(m, I)$ such that* (1.6.21) *holds for each $\mathscr{M} \in \mathcal{M}_I(x_0, R)$. In case $m \leq 7$ there is an $\eta > 0$ with the following property: If I satisfies $\widehat{\rho}_3(I(\cdot) - |\cdot|) < \eta$ then there is a constant $c = c(m, I)$ such that* (1.6.21) *holds for each $\mathscr{M} \in \mathcal{M}'_I(x_0, R)$. Especially if f is a function in $C^2(\mathbb{R}^m)$, whose graph has vanishing I-mean curvature (i.e.* (1.6.10) *holds on $\mathbb{R}^m$) then f is affine linear.*

The following Bernstein theorem for general $m \geq 2$ is due to Winklmann (see [Win07]). Crucial to this theorem is an additional growth condition on the gradient of the considered graph. Without such additional conditions, one can not expect Bernstein theorems for general $m \geq 2$.

Theorem 1.6.44 ([Win07, Theorem 4.1]). *Let $m \geq 2$. Let $I : \mathbb{R}^{m+1} \to \mathbb{R}$ be a positive definite, elliptic Cartan integrand of class $C^3(\mathbb{R}^{m+1}\backslash\{0\})$ independent of the first argument on $\mathbb{R}^{m+1}$. There exists $\delta(m, \gamma) > 0$ and $\gamma \in (0, 1)$ such that the following holds: Assume*

$$\widehat{\rho}_3(F(\cdot) - |\cdot|) \quad \leq \quad \delta$$

and that f is a Finsler-minimal graph with

$$|\mathrm{D}f(x)| = O(|(x, f(x))|^\gamma) \text{ für } |x| \to \infty, \tag{1.6.22}$$

wherein $|(x, f(x))| = \sqrt{|x|^2 + f^2(x)}$. Then f is an affine linear function.

Finsler Area

2.1 Finsler Area as a Cartan functional

In this section, we define the Busemann-Hausdorff area or Finsler area for immersed submanifolds in Finsler manifolds. For this, we follow the definition of Busemann [Bus47] concerning the volume on a Finsler manifold and the resulting area notion. But we use an adapted version of the more modern notation Shen used in [She01] and [She98] to express the notions of volume and area. His main aim in [She98] was to define an analogon of the mean curvature for Finsler manifolds, which will be discussed in the section 2.5. The Busemann-Hausdorff area is based on the Busemann-Hausdorff volume. Notice that this volume is one of many possible choices of volume on a manifold.[1]

Definition 2.1.1 (**Finsler area** [Bus47])**.** Let $\mathscr{M}^m$ be an oriented compact smooth m-manifold with boundary $\partial\mathscr{M}$, $(\mathscr{N}^n, F)$ an n-dimensional Finsler manifold and $X : \mathscr{M}^m \to \mathscr{N}^n$ be an immersion of class C^2. For any point $q \in \mathscr{N}$ there exist local coordinates $x^1, \ldots, x^n$ on a suitable open neighborhood $W_q \subset \mathscr{N}$ and local coordinates $u^1, \ldots, u^m$ on a suitable open neighborhood $V_q \subset \mathscr{M}$ such that the given immersion $X : \mathscr{M} \to \mathscr{N}$ satisfies $X(V_q) \subset W_q$. We denote the local coordinate chart (W_q, x) on $\mathscr{N}$ by $\mathfrak{X}$. F induces through the immersion X a pullback Finsler metric X^*F on $\mathscr{M}$, where

$$X^*F(u,v) \quad := \quad F(X(u), \mathrm{d}X|_u(v)) \quad \text{ for } \ (u,v) \in T\mathscr{M}.$$

We define the *m-dimensional Busemann-Hausdorff area form* or *m-dimensional Finsler area form* as the Busemann-Hausdorff volume form (see [Bus47], [She01] and [She98]) of X^*F on $\mathscr{M}$ given in local coordinates by

$$\begin{aligned} \mathrm{d}V_{X^*F} \quad &:= \quad \sigma_{X^*F}(u)\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^m, \\ \sigma_{X^*F}(u) \quad &:= \quad \frac{\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{(y^\alpha) \in \mathbb{R}^m : X^*F(u, y^\alpha \frac{\partial}{\partial u^\alpha}|_u) \le 1\})}, \end{aligned} \tag{2.1.1}$$

wherein $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure. $\mathrm{d}V_{X^*F}$ is a global m-form on $\mathscr{M}$ due to the invariance under coordinate change of the expression (2.1.1) (see [She98, Lemma 5.1]).

The *m-dimensional Busemann-Hausdorff area* or *m-dimensional Finsler area* of the immersion X is defined by

$$\mathcal{A}_m^F(X) \quad := \quad \int_{\mathscr{M}} \mathrm{d}V_{X^*F}. \tag{2.1.2}$$

[1] Notice that the alternative Holmes-Thompson volume (see [ÁPB06]) leads to different notions of area and Finsler-minimal immersions that we do not address here.

A rewritten version of the local expression $\sigma_{X^*F}(u)$ gives

$$\sigma_{X^*F}(u) = \frac{\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{(y^\alpha)\in\mathbb{R}^m : F(X(u), y^\alpha X^i_\alpha(u)\frac{\partial}{\partial u^i}|_u)\leq 1\})}$$

wherein coefficients $X^i_\alpha(u)$ are defined by the representation of the differential of X in local coordinates

$$\mathrm{d}X = X^i_\alpha(u)\mathrm{d}u^\alpha \otimes \frac{\partial}{\partial x^i} \tag{2.1.3}$$

and we set for such a choice of local coordinates the matrix

$$\mathrm{D}X(u) := (X^i_\alpha(u)). \tag{2.1.4}$$

So, we define for the chosen local coordinate chart $\mathfrak{X}$ on $\mathscr{N}$ the expression

$$a^F_{m,\mathfrak{X}}(x,P) := \begin{cases} \frac{\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{(y^\alpha)\in\mathbb{R}^m : F(x, y^\alpha P^i_\alpha\frac{\partial}{\partial x^i}|_x)\leq 1\})} & \text{if } \operatorname{rank}P = m,\\ 0 & \text{if } \operatorname{rank}P < m. \end{cases} \tag{2.1.5}$$

Therein, $P = (P^i_\alpha) \in \mathbb{R}^{n\times m}$, $P_\alpha = (P^i_\alpha)_{i=1\dots n} \in \mathbb{R}^n$, $P^i = (P^i_\alpha)_{\alpha=1\dots m} \in \mathbb{R}^m$. We call $a^F_{m,\mathfrak{X}}(x,P)$ the *m-dimensional Finsler area integrand* for a given choice of local coordinates $\mathfrak{X}$ on $\mathscr{N}$. Thereby, we get in local coordinates on $\mathscr{M}$

$$\sigma_{X^*F}(u) = a^F_{m,\mathfrak{X}}(X(u), \mathrm{D}X(u)). \tag{2.1.6}$$

In the case when $n = m+1$ we call $\mathcal{A}^F(X) := \mathcal{A}^F_m(X)$ simply the *Busemann-Hausdorff area* or *Finsler area* and $a^F_{\mathfrak{X}}(x,P) := a^F_{m,\mathfrak{X}}(x,P)$ the *Finsler area integrand* for a given choice of local coordinates $\mathfrak{X}$ on $\mathscr{N}$.

Remark 2.1.2.

- Assuming the setting of Definition 2.1.1, we have that the manifold $\mathscr{M}$ is compact and can therefore be covered by a family of local coordinates $u^1_i, \dots, u^m_i$ on suitable neighbourhoods U_i for $i = 1, \dots, N$, whose image under X is a subset of a coordinate chart (W_i, x_i) on $\mathscr{N}$. Further, we choose a subordinate *partition of unity* $\{\varphi_j\}_j$. This means that each $\varphi_j : \mathscr{N} \to [0,1]$ is a smooth function on $\mathscr{N}$ with $\operatorname{supp}\varphi_j \subset U_{q_j}$ for some q_j and such that for each point p on $\mathscr{N}$ there are only finitely many $\varphi_j(p)$ differing from zero. So, the sum $\sum_j \varphi_j(p) = 1$ is well-defined (see [Lee03, p. 54]).

$$\begin{aligned}\mathcal{A}^F(X) &:= \int_{\mathscr{M}} \mathrm{d}V_{X^*F}\\ &= \int_{\mathscr{M}} \sum_j \varphi_j \mathrm{d}V_{X^*F}\\ &= \sum_j \int_{U_{q_j}} \varphi_j \mathrm{d}V_{X^*F}\\ &= \sum_j \int_{u_{q_j}(U_{q_j})} \varphi_j(u_{q_j})\sigma_{X^*F}(u_{q_j})\mathrm{d}u^1_{q_j}\wedge\cdots\wedge\mathrm{d}u^m_{q_j}\\ &= \sum_i \int_{u_{q_j}(U_{q_j})} \varphi_i(u_{q_j})\sigma_{X^*F}(u_{q_j})\mathrm{d}u^1_{q_j}\wedge\cdots\wedge\mathrm{d}u^m_{q_j}\\ &= \sum_i \int_{u_{q_j}(U_{q_j})} \varphi_i(u_{q_j})a^F_{m,(W_i,x_i)}(X,\nabla X)\mathrm{d}u^1_{q_j}\wedge\cdots\wedge\mathrm{d}u^m_{q_j}\end{aligned}$$

- In Definition 2.1.1, it is sufficient to assume a smooth n-manifold $\mathscr{N}$ and a function $F : \mathrm{T}\mathscr{N} \to [0,\infty)$ satisfying **(F1)** to assure that the expressions in Definition 2.1.1 are well-defined. Remember that **(F1)** means that F is a continuous and homogeneous function on $\mathrm{T}\mathscr{N}$, which is positive on $\mathrm{T}\mathscr{N}\backslash o$. We will use this generalized assumption **(F1)** in the rest of this section.

- The compactness condition on $\mathscr{M}$ in Definition 2.1.1 is not a nessecary obstruction, it just guarantees the finiteness of the integral
$$\int_{\mathscr{M}} \mathrm{d}V_{X^*F}.$$

- Indeed, the Finsler area as in Definition 2.1.1 is invariant under reparametrization of the immersion X. This is the reason, we can rewrite the Finsler area integrand for $n = m+1$ to be dependent of $P_1 \wedge \cdots \wedge P_m$ instead of the whole matrix $P = (P_1|\cdots|P_m) \in \mathbb{R}^{(m+1)\times m}$, due to a result of Morrey [Mor08, chapter 9] reinvestigated by von der Mosel in [vdM92]. We do this explicitly in Proposition 2.1.9.

Before we investigate the Finsler area, we give some motivating examples.

Example 2.1.3 (**Riemannian area on** $\mathbb{R}^3$). We look at $\mathbb{R}^3$ endowed with the Riemannian metric
$$F(x,y) \quad := \quad \sqrt{a_{ij}(x)y^iy^j}$$
for $x \in \mathbb{R}^3$, $y = (y^i) \in \mathbb{R}^3$ and $A(x) = (a_{ij}(x))_{ij} \in \mathbb{R}^{3\times3}$, $A(x)$ a symmetric matrix, depending smoothly on x, everything expressed in cartesian coordinates. It is easy to show by using integration theory and the properties of ellipsoids that
$$\begin{aligned} &\mathscr{H}^2\left(\{w = (w^\alpha) \in \mathbb{R}^2 : F(x, w^\alpha P_\alpha) \le 1\}\right) \\ = \quad &(\det(P^i_\alpha a_{ij}(x)P^j_\beta)_{\alpha\beta})^{-\frac12}\mathscr{H}^2(\mathbb{B}^2)\end{aligned}$$
with $P = (P^i_\alpha) \in \mathbb{R}^{3\times2}$, P of full rank, $P_\alpha = (P^i_\alpha)_{i=1\ldots3} \in \mathbb{R}^3$ for $\alpha = 1,2$. By means of the Cauchy-Binet formula we get
$$\begin{aligned} &(\det(PA(x)P^T)^{-1} \\ = \quad &(\det(P^i_\alpha a_{ij}(x)P^j_\beta)_{\alpha\beta})^{-1} \\ \overset{\text{Thm. 1.1.6}}{=} \quad &\sum_{1\le i_1<i_2\le3}\sum_{1\le j_1<j_2\le3} \det P^{i_1i_2}\det(A(x))^{j_1j_2}_{i_1i_2}\det(P^T)_{j_1j_2} \\ = \quad &\sum_{i=1}^3\sum_{j=1}^3(-1)^{i+1}\det P^{(i)}(-1)^{i+j}\det(A(x))^{(j)}_{(i)}(-1)^{j+1}\det P^{(j)} \\ = \quad &\det A(x)\sum_{i=1}^3\sum_{j=1}^3(-1)^{i+1}\det P^{(i)}\frac{(-1)^{i+j}}{\det A(x)}\det(A(x))^{(j)}_{(i)}(-1)^{j+1}\det P^{(j)} \\ \overset{\text{Thm. 1.1.3 \& Dfn. 1.1.10}}{=} &(P_1\wedge P_2)^TA^{-1}(x)(P_1\wedge P_2)\det A(x).\end{aligned}$$

This way we get as Finsler area integrand in standard coordinates on $\mathbb{R}^{m+1}$
$$a^F(x,P) \quad = \quad \left((P_1\wedge P_2)^TA^{-1}(x)(P_1\wedge P_2)\right)^{\frac12}(\det A(x))^{\frac12},$$
where we omitted the dependence of a^F of the actual choice of coordinates as we chose a specific one in the beginning.

Example 2.1.4 (**Euclidean area**). Choose $(\mathscr{N}, F)$ to be the Euclidean space $(\mathbb{R}^{m+1}, |\cdot|)$. Then, we can easily compute in a similar way as in Example 2.1.3 that

$$a^F(x,P) \quad = \quad |P_1 \wedge \ldots \wedge P_m|$$

for all $x \in \mathbb{R}^{m+1}$ and $P = (P_1|\cdots|P_m) \in \mathbb{R}^{(m+1)\times m}$. Hence, the Euclidean area $\mathcal{A}$ defined in Example 1.6.3 coincides with the Finsler area $\mathcal{A}^{|\cdot|}$ issued from the Euclidean metric.

The following theorem illustrates the transformation behaviour of the Finsler area integrand and is a result established by Shen in [She98].

Theorem 2.1.5 (**Transformation behaviour** [She98, Lemma 5.1]). *Let $n \geq m > 0$, $\mathscr{N}$ be a smooth n-manifold and $F : \mathrm{T}\mathscr{N} \to [0,\infty)$ satisfy* (**F1**)*. Choose $\mathfrak{X} = (W, \varphi) = (W, (x^i))$ to be a local coordinate chart on $\mathscr{N}$. Further, let $P = (P^i_\alpha)$ and $Q = (Q^i_\alpha)$ be matrices in $\mathbb{R}^{n\times m}$, $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ span the same subspace of $\mathbb{R}^n$ and $C = (C^\beta_\alpha)$ be the matrix in $\mathbb{R}^{m\times m}$ s.t. $P = QC$. Then we have*

$$a^F_{m,\mathfrak{X}}(x,P) \quad = \quad (\det C) a^F_{m,\mathfrak{X}}(x,Q) \tag{2.1.7}$$

$$\quad = \quad \pm\sqrt{\frac{\det P^T P}{\det Q^T Q}}\, a^F_{m,\mathfrak{X}}(x,Q) \tag{2.1.8}$$

for all $x \in \varphi(W) \subset \mathbb{R}^n$. The sign in (2.1.8) *depends on whether the bases $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ are oriented in the same way or not.*

Proof. For the case that $\mathrm{rank} P = \mathrm{rank} Q < m$ the claim follows directly, so we assume P and Q to be of full rank. For the first identity (2.1.7) we refer to [She98, Lemma 5.1] and the second comes from

$$\det P^T P \quad = \quad \det (QC)^T QC \quad = \quad \det C^T Q^T QC \quad = \quad (\det C)^2 \det Q^T Q$$

by using the multiplicativity of the determinant. □

Theorem 2.1.6 (**Spherical integral representation of Finsler area**). *Let $n \geq m > 0$, $\mathscr{N}$ be a smooth n-manifold and $F : \mathrm{T}\mathscr{N} \to [0,\infty)$ satisfy* (**F1**)*. Choose $\mathfrak{X} = (W, \varphi) = (W, (x^i))$ to be a local coordinate chart on $\mathscr{N}$. Further, let $P = (P^i_\alpha)$ be a matrix in $\mathbb{R}^{n\times m}$ of full rank and $P_1, \ldots, P_m$ be the columns of P. Then we have*

$$a^F_{m,\mathfrak{X}}(x,P) \quad = \quad \left(\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} F^{-m}\left(x, \theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}\Big|_x\right) \mathrm{d}S(\theta)\right)^{-1} \tag{2.1.9}$$

for all $x \in \varphi(W) \subset \mathbb{R}^n$. Therein $\mathrm{d}S$ *is the* spherical measure.

Remark 2.1.7.

- The representation (2.1.9) shows especially why it is sufficient to assume that $F : T\mathscr{N} \to [0,\infty)$ satisfies (**F1**) (i.e. (**b**), (**h**) and (**p**) hold) in order to assure that $a^F_{m,\mathfrak{X}}$ is well-defined for every choice of local coordinates $\mathfrak{X}$ on a smooth manifold $\mathscr{N}$. Further, the expression $F\left(x, \theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)$ is positive for every θ in $\mathbb{S}^{m-1}$ because the linear combination of the columns of P, namely $\theta^\alpha P_\alpha$, differs from zero and (**p**) holds for F. Actually, $a^F_{m,\mathfrak{X}}(x,P)$ is positive for every $x \in W$ and matrix $P \in \mathbb{R}^{(m+1)\times m}$ of full rank. Further, it is easy to see that the homogeneity of F and (2.1.9) imply $a^F_{m,\mathfrak{X}}(x,tP) = t^m a^F_{m,\mathfrak{X}}(x,P)$ for $t > 0$.
- It seems to the author that Theorem 2.1.6 gives a new type of representation of the m-dimensional Finsler area integrand.

Proof of Theorem 2.1.6. We essentially need only to rewrite the Hausdorff measure term in expression (2.1.5) by a change of variables using [For09, Satz 8, p. 144], i.e.

$$\begin{aligned}
&\mathscr{H}^m(\{(y^\alpha) \in \mathbb{R}^m : F(x, y^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x) \leq 1\}) \\
=& \int_{\mathbb{S}^{m-1}} \int_0^\infty \chi_{\left\{(r\theta^i)_i \in \mathbb{R}^m : rF(\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x) \leq 1\right\}}(s\theta) s^{m-1} \mathrm{d}s\mathrm{dS}(\theta) \\
=& \int_{\mathbb{S}^{m-1}} \int_0^{\frac{1}{F(\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x)}} s^{m-1} \mathrm{d}s\mathrm{dS}(\theta) \\
=& \int_{\mathbb{S}^{m-1}} \frac{1}{mF^m(\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x)} \mathrm{dS}(\theta).
\end{aligned}$$

Further, the following identity holds

$$\mathscr{H}^{m-1}(\mathbb{S}^{m-1}) \overset{\text{[For09]}}{=} m\mathscr{H}^m(\mathbb{B}^m).$$

Putting all together concludes the proof. □

The following corollary allows for a comparative quantification of Finsler areas issued from two different Finsler metrics. The corollary is a direct consequence of Theorem 2.1.6.

Corollary 2.1.8 (Positivity quantification). *Let $n \geq m > 0$, $\mathscr{N}$ be a smooth n-manifold and $F, G : \mathrm{T}\mathscr{N} \to [0, \infty)$ both satisfy* (**F1**)*. Choose $\mathfrak{X} = (W, \varphi) = (W, (x^i))$ to be a local coordinate chart on $\mathscr{N}$. Let there be a fixed $x \in W \subset \mathscr{N}$ with $\varphi(x) = \tilde{x}$ such that there exist positive constants $c_2(x) \geq c_1(x) > 0$ with*

$$c_1(x)G(x, y) \quad \leq \quad F(x, y) \quad \leq \quad c_2(x)G(x, y) \tag{2.1.10}$$

for all $y \in \mathrm{T}_x\mathscr{N}$. Then follows

$$m_1(\tilde{x})a^G_{m,\mathfrak{X}}(\tilde{x}, P) \quad \leq \quad a^F_{m,\mathfrak{X}}(\tilde{x}, P) \quad \leq \quad m_2(\tilde{x})a^G_{m,\mathfrak{X}}(\tilde{x}, P) \tag{2.1.11}$$

for all $P = (P^i_\alpha) \in \mathbb{R}^{n\times m}$ of full rank, where we set $m_i(\tilde{x}) := c_i^m(x)$ for $i = 1, 2$.

Proof. By (2.1.10) follows

$$c_2^{-m}(x)G^{-m}(x, y) \quad \leq \quad F^{-m}(x, y) \quad \leq \quad c_1^{-m}(x)G^{-m}(x, y)$$

for all $y \in \mathrm{T}_x\mathscr{N}$. Combining this with the representation of $a^G_{m,\mathfrak{X}}$ and $a^F_{m,\mathfrak{X}}$ in (2.1.9) yields the stated (2.1.11). □

In the following proposition, we give a representation of the Finsler area that will become useful to reinterpret the Finsler area as a Cartan functional.

Proposition 2.1.9 (Normal representation of Finsler area). *Let m be an integer, $\mathscr{N}$ a smooth $(m+1)$manifold and $F : T\mathscr{N} \to [0, \infty)$ satisfy* (**F1**)*. Further, let $\mathfrak{X} = (W, \varphi) = (W, (x^i))$ be a local coordinate chart on $\mathscr{N}$. We define*

$$A^F_{\mathfrak{X}}(x, z) \quad := \quad \begin{cases} \frac{|z|\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{y \in \mathbb{R}^{m+1} : \langle y, z\rangle = 0\} \cap \{y \in \mathbb{R}^n : F(x, y^i \frac{\partial}{\partial x^i}|_x) \leq 1\})} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0 \end{cases}$$

for $x \in \varphi(W) \subset \mathscr{N}$, $z \in \mathbb{R}^{m+1}$. Then

$$a^F_{m,\mathfrak{X}}(x, P) \quad = \quad A^F_{\mathfrak{X}}(x, P_1 \wedge \ldots \wedge P_m) \tag{2.1.12}$$

for all $P \in \mathbb{R}^{n\times m}$. $A^F_{\mathfrak{X}}$ is homogeneous of degree 1 in z and is positive on $\varphi(W) \times \mathbb{R}^{m+1} \backslash \{0\} \cong \mathrm{T}(\varphi(W))$.

Proof of Proposition 2.1.9. We start by showing the identity (2.1.12). Let $P = (P^i_\alpha) \in \mathbb{R}^{(m+1)\times m}$, P be of full rank and $P_\alpha = (P^i_\alpha)_{i=1\dots m+1} \in \mathbb{R}^{m+1}$. We define

$$M \quad := \quad \{(y^\alpha) \in \mathbb{R}^m : F(x, y^\alpha P^i_\alpha \frac{\partial}{\partial x^i}) \le 1\}, \tag{2.1.13}$$

$$\begin{aligned} \Omega \quad := \quad & \{(w^i) \in \mathbb{R}^{m+1} : F(x, w^i \frac{\partial}{\partial x^i}) \le 1\} \\ & \cap \{w = (w^i) \in \mathbb{R}^{m+1} : \langle w, P_1 \wedge P_2 \wedge \ldots \wedge P_m \rangle = 0\}. \end{aligned} \tag{2.1.14}$$

The set Ω can be seen as the intersection of the Finsler unit ball with the hyperplane spanned by $P_1, \ldots, P_m$ in tangent space at the point X given in local coordinates. We define a linear mapping by $\varphi : \mathbb{R}^m \to \mathbb{R}^{m+1}$ by $\varphi(y) = y^\alpha P_\alpha$, wherein $y = (y^\alpha) \in \mathbb{R}^m$. This mapping has due to the assumptions on P full rank. As a first step we show that

$$\varphi(M) \quad = \quad \Omega. \tag{2.1.15}$$

Indeed, for $w = (w^i) \in \varphi(M)$ there exists $v = (v^\alpha) \in \mathbb{R}^m$, such that $w = \varphi(v) = v^\alpha P_\alpha$ and $F(x, v^\alpha P^i_\alpha \frac{\partial}{\partial x^i}) \le 1$. For such w holds $\langle w, P_1 \wedge P_2 \wedge \ldots \wedge P_m \rangle = y^\alpha \langle P_\alpha, P_1 \wedge P_2 \wedge \ldots \wedge P_m \rangle = 0$ and $F(x, w^i \frac{\partial}{\partial x^i}) \le 1$. Therefore, $w \in \Omega$ and thereby follows

$$\varphi(M) \quad \subset \quad \Omega.$$

On the other hand if we assume $w \in \Omega$, then holds $\det(w, P_1, P_2, \ldots, P_m) = \langle w, P_1 \wedge P_2 \wedge \ldots \wedge P_m \rangle = 0$ and $F(x, w^i \frac{\partial}{\partial x^i}) \le 1$. Since $\det(w, P_1, P_2, \ldots P_m) = 0$, we deduce that the columns of the matrix $(w, P_1, P_2, \ldots P_m)$ are linearly dependent. We assumed further that P_1, …, P_m are linearly independent, and this way it follows that w is a linear combination of P_1, …, P_m. So, there exists $v = (v^\alpha) \in \mathbb{R}^m$ such that $w = v^\alpha P_\alpha$ and $F(x, v^\alpha P^i_\alpha \frac{\partial}{\partial x^i}) = G(x, w^i \frac{\partial}{\partial x^i}) \le 1$. Therefore, $w \in \varphi(M)$ and consequently

$$\varphi(M) \quad \supset \quad \Omega.$$

This proves (2.1.15).

In a next step, we need to show that

$$\mathscr{H}^0(M \cap \varphi^{-1}(\cdot)) \quad = \quad \chi_\Omega(\cdot) \tag{2.1.16}$$

holds, wherein χ_Ω is the *characteristic function* of the set Ω, i.e.

$$\chi_\Omega(z) \quad := \quad \begin{cases} 1 & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{R}^m \backslash \Omega \end{cases}$$

for all $z \in \mathbb{R}^m$. We already know that φ is injective, given the fact that P_1, …, P_m are linearly independent and $\varphi(\mathbb{R}^m) = \operatorname{span}\{P_1, P_2, \ldots, P_m\}$. There are two different cases to discuss.

Case 1:
If $w \notin \operatorname{span}\{P_1, P_2, \ldots, P_m\} = \varphi(\mathbb{R}^m)$ it follows that $\varphi^{-1}(w) = \emptyset$ and then $M \cap \varphi^{-1}(w) = \emptyset$. We get $\mathscr{H}^0(M \cap \varphi^{-1}(w)) = 0$.

Case 2:
Now let $w \in \operatorname{span}\{P_1, P_2, \ldots, P_m\}$. Since φ is injective and $\varphi(M) = \Omega$, we conclude

$$\operatorname{span}\{P_1, P_2, \ldots, P_m\} \quad = \quad \varphi(\mathbb{R}^m \backslash M) \dot{\cup} \varphi(M) \quad = \quad \varphi(\mathbb{R}^2 \backslash M) \dot{\cup} \Omega.$$

Thereby, if $w \in \Omega$ we have $\sharp \left(M \cap \varphi^{-1}(w)\right) = 1$ due to the injectivity of φ and consequently $\mathscr{H}^0(M \cap \varphi^{-1}(w)) = 1$. Therein $\sharp S$ denotes the *cardinality* of the set S. On the other side the injectivity of φ yields $\sharp\left(M\cap\varphi^{-1}(w)\right)=0$ and therefore $\mathscr{H}^0(M\cap\varphi^{-1}(w))=0$ if $w\in\varphi(\mathbb{R}^m\backslash M)$.

Both cases together yield $\mathscr{H}^0(M\cap\varphi^{-1}(\cdot))=\chi_\Omega(\cdot)$ as claimed in (2.1.16).

Using the area formula (see Theorem 1.2.6), we get

$$\begin{aligned}
&\mathscr{H}^m(M)\\
=\;&\int_M 1\mathrm{d}y^1\dots\mathrm{d}y^m\\
=\;&\frac{1}{|P_1\wedge\ldots\wedge P_m|}\int_M |P_1\wedge\ldots\wedge P_m|\mathrm{d}y^1\dots\mathrm{d}y^m\\
=\;&\frac{1}{|P_1\wedge\ldots\wedge P_m|}\mathcal{A}_M(\varphi)\\
\overset{\text{Thm. 1.2.6}}{=}\;&\frac{1}{|P_1\wedge\ldots\wedge P_m|}\int_{\mathbb{R}^{m+1}}\mathscr{H}^0(M\cap\varphi^{-1}(w))\mathrm{d}\mathscr{H}^m(w)\\
=\;&\frac{1}{|P_1\wedge\ldots\wedge P_m|}\int_{\mathbb{R}^{n+1}}\chi_\Omega(w)\mathrm{d}\mathscr{H}^m(w)\\
=\;&\frac{\mathscr{H}^m(\{w\in\mathbb{R}^{m+1}:<P_1\wedge\ldots\wedge P_m,w>=0\}\cap\{w\in\mathbb{R}^{m+1}:F(x,w^i\frac{\partial}{\partial x^i})\le 1\})}{|P_1\wedge\ldots\wedge P_m|}.
\end{aligned}$$

By substitution we finally get

$$\begin{aligned}
&a^F_{m,\mathfrak{X}}(x,P)\\
=\;&\frac{|P_1\wedge\ldots\wedge P_m|\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{w\in\mathbb{R}^{m+1}:w\wedge P_1\wedge\ldots\wedge P_m=0\}\cap\{w\in\mathbb{R}^n:F(x,w^i\frac{\partial}{\partial x^i})\le 1\})}.
\end{aligned}$$

So, we showed (2.1.12).

By exploiting (2.1.12) and the fact that every $z\in\mathbb{R}^{m+1}\cong\Lambda^m(\mathbb{R}^{m+1})$ can be represented as $z=P_1\wedge\ldots\wedge P_m$ for properly chosen $P_1,\dots,P_m$, we can identify

$$A^F_{\mathfrak{X}}(x,z)\quad=\quad a^F_{m,\mathfrak{X}}(x,P) \tag{2.1.17}$$

for $P:=(P_1|\cdots|P_m)\in\mathbb{R}^{(m+1)\times m}$. This expression is obviously homogeneous of degree 1 in z. So, we get by using (2.1.17) and Remark 2.1.7

$$\begin{aligned}
A^F_{\mathfrak{X}}(x,tz)\;&=\;A^F_{\mathfrak{X}}(x,(t^{\frac{1}{m}}P_1)\wedge\ldots\wedge(t^{\frac{1}{m}}P_m))\\
&\overset{(2.1.17)}{=}a^F_{m,\mathfrak{X}}(x,t^{\frac{1}{m}}P)\\
&\overset{\text{Rmk. 2.1.7}}{=}ta^F_{m,\mathfrak{X}}(x,P)\\
&\overset{(2.1.17)}{=}tA^F_{\mathfrak{X}}(x,z)
\end{aligned}$$

the stated homogeneity of degree 1 in the second argument of $A^F_{\mathfrak{X}}$. Further, combination of (2.1.17) and Remark 2.1.7 yields also the positivity of $A^F_{\mathfrak{X}}$ on $W\times\mathbb{R}^{m+1}\backslash\{0\}$. □

In the following, we define a symmetrization of a Finsler metric, which generates the same Finsler area (see Theorem 2.1.12).

Definition 2.1.10 (Symmetrization). Let $n\ge m>0$ be integers, $\mathscr{N}$ be a smooth n-manifold and $F:T\mathscr{N}\to[0,\infty)$ be a function satisfying **(F1)**. We define the *m-symmetrization* of F as

$$F_{(m)}(x,y)\quad:=\quad\begin{cases}2^{\frac{1}{m}}(F^{-m}(x,y))+F^{-m}(x,-y))^{-\frac{1}{m}} & \text{for } (x,y)\in T\mathscr{N}\backslash o,\\ 0 & \text{for } (x,y)\in o.\end{cases} \tag{2.1.18}$$

Remark 2.1.11.

- As F is a homogeneous and positive function on the tangent bundle $T\mathscr{N}$ of class C^0, i.e. satisfies **(F1)**, it can easily be shown that $F_{(m)}$ is also a homogeneous and positive function on the tangent bundle $T\mathscr{N}$ of class C^0, i.e. satifies **(F1)**. Since $F_{(m)}$ is homogeneous, it can be continuously extended by zero to points on the zero section. Further, $F_{(m)}$ is at least as regular as F on $T\mathscr{N}$ or $T\mathscr{N}\backslash o$, respectively.

- The name m-symmetrization for $F_{(m)}$ is justified by the fact that $F_{(m)}$ is obviously even and that F equals $F_{(m)}$ if and only if F is by itself even. $F_{(m)}(x,y)$ is even more the m-harmonic mean of $F(x,y)$ and $F(x,-y)$ for all $(x,y) \in T\mathscr{N}$.

- The property of F being elliptic, i.e. (**e**), does in general not carry over to its m-symmetrization $F_{(m)}$.

The following theorem especially shows that the m-symmetrization of a Finsler metric generates the same Finsler area.

Theorem 2.1.12. *Let $0 < m \leq n$, $\mathscr{N}$ be an n-manifold, $F : \mathrm{T}\mathscr{N} \to [0,\infty)$ be a function satisfying* **(F1)** *and choose local coordinates $\mathfrak{X} = (W,\varphi) = (W,(x^i))$ on $\mathscr{N}$. Further, let P be a matrix in $\mathbb{R}^{n\times m}$ of full rank. Then we have*

$$a^F_{m,\mathfrak{X}}(x,P) \quad = \quad a^{F_{(m)}}_{m,\mathfrak{X}}(x,P) \tag{2.1.19}$$

for all $x \in \varphi(W) \subset \mathbb{R}^n$. Further, let $\mathscr{M}$ be a smooth oriented m-manifold with boundary and $X : \mathscr{M} \to \mathscr{N}$ an immersion. Then the m-dimensional Finsler area w.r.t. F of X coincides with the m-dimensional Finsler area w.r.t. $F_{(m)}$ of X, i.e.

$$\mathcal{A}^F(X) \quad = \quad \mathcal{A}^{F_{(m)}}(X).$$

Proof. Assume the setting of Theorem 2.1.12. Let $P = (P^i_\alpha)$ be a matrix in $\mathbb{R}^{n\times m}$ of full rank. By applying Theorem 2.1.6 we get

$$\begin{aligned}
&a^{F_{(m)}}_{m,\mathfrak{X}}(x,P)\\
\overset{\text{Thm. 2.1.6}}{=}\ & \left(\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} F^{-m}_{(m)}\left(x,\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta)\right)^{-1}\\
=\ & \left(\frac{1}{2\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} F^{-m}\left(x,\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right) + F^{-m}\left(x,-\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta)\right)^{-1}\\
=\ & \left(\frac{1}{2\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\left[\int_{\mathbb{S}^{m-1}} F^{-m}\left(x,\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta)\right.\right.\\
&\qquad\left.\left. + \int_{\mathbb{S}^{m-1}} F^{-m}\left(x,-\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta)\right]\right)^{-1}\\
=\ & \left(\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} F^{-m}\left(x,\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta)\right)^{-1}\\
\overset{\text{Thm. 2.1.6}}{=}\ & a^F_{m,\mathfrak{X}}(x,P)
\end{aligned}$$

wherein we exploited the fact that

$$\int_{\mathbb{S}^{m-1}} F^{-m}\left(x,\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta) \quad = \quad \int_{\mathbb{S}^{m-1}} F^{-m}\left(x,-\theta^\alpha P^i_\alpha \frac{\partial}{\partial x^i}|_x\right)\mathrm{d}S(\theta).$$

The rest follows then by definition of the m-dimensional Finsler area (see Definition 2.1.1). ☐

Corollary 2.1.13. *Let m be an integer, $\mathscr{N}$ a smooth $(m+1)$-manifold and $F : T\mathscr{N} \to [0,\infty)$ satisfy* (**F1**)*. Further, let $\mathfrak{X} = (W,\varphi) = (W,(x^i))$ be a local coordinate chart on $\mathscr{N}$. We define*

$$A^F_{\mathfrak{X}}(x,z) \quad = \quad A^{F_{(m)}}_{\mathfrak{X}}(x,z)$$

for $x \in W \subset \mathscr{N}$, $z \in \mathbb{R}^{m+1}$.

Proof. Corollary 2.1.13 is a direct conclusion of Theorem 2.1.12 by using the representation (2.1.12). □

The following lemma is a collection of all the results in this section regarding the Finsler area for codimension 1 and an ambient space, which is a real vector space.

Lemma 2.1.14 (Collection of results for $\mathbb{R}^{m+1}$). *Let $\mathscr{M}$ be a compact oriented smooth m-manifold, $\Omega \subset \mathbb{R}^{m+1}$ an open subset of the standard Euclidean space equipped with a function $F : T\Omega \cong \Omega\times\mathbb{R}^{m+1} \to [0,\infty)$ satisfying* (**F1**)*. The Finsler area for an immersion $X : \mathscr{M} \to \Omega$ of class C^1 is defined by*

$$\mathcal{A}^F(X) \quad := \quad \int_{\mathscr{M}} \mathrm{d}V_{X^*F},$$

with

$$\mathrm{d}V_{X^*F} \quad = \quad A^F(X, \frac{\partial X}{\partial u^1} \wedge \ldots \wedge \frac{\partial X}{\partial u^m})\mathrm{d}u, \tag{2.1.20}$$

$$A^F(x,z) \quad := \quad \begin{cases} \frac{|z|\mathscr{H}^m(\mathbb{B}^m)}{\mathscr{H}^m(\{w\in\mathbb{R}^{m+1}: <w,z>=0,\ F(x,w)\le 1\})} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0 \end{cases} \tag{2.1.21}$$

for local coordinates (u^α) on $\mathscr{M}$ and $x \in \Omega$. The function $A^F : T\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0,\infty)$ is homogeneous of degree 1 and positive on $\Omega \times \mathbb{R}^{m+1}\backslash\{0\}$, i.e. especially satisfies (**H**)*.*

If we choose $(f_i)_{i=1,\ldots,m}$ as a basis of the orthogonal complement $z^\perp$ and we then set $f := |f_1 \wedge \ldots \wedge f_m|$, we can represent the area integrand as

$$A^F(x,z) \quad = \quad \begin{cases} \frac{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}{f\int_{\mathbb{S}^{m-1}} F^{-m}(x,\theta^i f_i)\mathrm{dS}(\theta)} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases} \tag{2.1.22}$$

which gives us a mean to relate the Finsler area integrand to integral expressions of F.

Further, F and its m-symmetrization $F_{(m)}$ lead both to the same Finsler area and Finsler area integrand, i.e.

$$\mathcal{A}^F(X) \quad = \quad \mathcal{A}^{F_{(m)}}(X) \tag{2.1.23}$$

and

$$A^F(x,z) \quad = \quad A^{F_{(m)}}(x,z) \tag{2.1.24}$$

for all $(x,z) \in T\Omega \cong \Omega \times \mathbb{R}^{m+1}$.

Remark 2.1.15.

- In Lemma 2.1.14 we computed the expressions $\mathrm{d}V_{X^*F}$ and A^F w.r.t. standard cartesian coordinates.
- If we choose the basis (f_i), $i = 1,\ldots,m$ such that $f = |z|$ we get

$$A^F(x,z) \quad = \quad \frac{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}{\int_{\mathbb{S}^{m-1}} F^{-m}(x,\theta^i f_i)\mathrm{dS}(\theta)}.$$

This way, the dependence on z is only given in the integral expression of the denominator. On the other side if we choose $f = 1$ and show that the integral expression in the denominator is invariant under transformations of $(f_i), i = 1, \ldots, m$ such that they span the same hyperplane and $f = 1$ rests untouched, we get another proof for the homogeneity of degree 1 of $A^F(x, z)$ in z.

- Given the assumptions in Lemma 2.1.14, the Finsler area integrand is homogeneous and positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \backslash \{0\})$ and it will be shown in the section 2.3 that it is also continuous. Therefore, A^F is a Cartan integrand and fulfills by itself **(F1)**.

Proof of Lemma 2.1.14. Lemma 2.1.14 except for (2.1.22) directly results from Proposition 2.1.9, Theorem 2.1.12 and Corollary 2.1.13 in the special setting of (global) cartesian coordinates on $\mathbb{R}^{m+1}$. (2.1.22) can be shown by means of Theorem 2.1.6 in the following way:

$$\begin{aligned} A^F(x,z) &\overset{(\mathbf{H})}{=} \frac{|z|}{f} A^F(x, f\frac{z}{|z|}) \\ &= \frac{|z|}{f} A^F(x, f_1 \wedge \ldots \wedge f_m) \\ &\overset{\text{Thm. 2.1.9}}{=} \frac{|z|}{f} a^F(x, (f_1|\cdots|f_m)) \\ &\overset{\text{Thm. 2.1.6}}{=} \frac{|z| \mathscr{H}^{m-1}(\mathbb{S}^{m-1})}{f \int_{\mathbb{S}^{m-1}} F^{-m}(x, \theta^i f_i) \mathrm{dS}(\theta)} \end{aligned}$$

□

We conclude this section with the introduction of a set of properties of functions on the tangent bundle of a manifold, to ease reference on such a kind of function. Let $m,\ n \in \mathbb{N}$ with $m \le n$. For an n-manifold $\mathscr{N}$ and a function $F : T\mathscr{N} \to [0, \infty)$, we formulate the following general assumptions:

(GA1) *F satisfies $F \in C^0(T\mathscr{N}) \cap C^\infty(T\mathscr{N}\backslash o)$ and* **(F1)***; $F_{(m)}$ satisfies* **(F2)***,*

(GA2) *F satisfies $F \in C^0(T\mathscr{N}) \cap C^\infty(T\mathscr{N}\backslash o)$ and* **(F1)***; $F_{(m)}$ satisfies* **(F3)***,*

(GA3) *F satisfies $F \in C^0(T\mathscr{N}) \cap C^\infty(T\mathscr{N}\backslash o)$ and* **(F3)***; $F_{(m)}$ satisfies* **(F3)***,*

(GAM) *F satisfies $F \in C^0(T\mathbb{R}^n) \cap C^\infty(T\mathbb{R}^n\backslash o)$ and* **(F1)***; $F_{(m)}$ satisfies* **(F3)** *and* **(M)***.*

The following chain of implications holds:

(GAM) ⇒ **(GA3)** ⇒ **(GA2)** ⇒ **(GA1)**.

Notice further that if F satisfies **(F1)** or **(M)**, $F_{(m)}$ does automatically also satisfy **(F1)** or **(M)**. If F satisfies **(F1)**, $F_{(m)}$ does also satisfy **(F1)**. On the other hand that **(F2)** or **(F3)** is satisfied by F implies not that **(F2)** or **(F3)** is satisfied by $F_{(m)}$. So, essentially ellipticity does not carry over from F to its m-symmetrization $F_{(m)}$.

2.2 The spherical Radon transform

This section is dedicated to the spherical Radon transform. As will be shown in section 2.3, the spherical Radon transform is related to the Finsler area integrand (see Corollary 2.3.1). From this point of view, an investigation of its topological properties as a linear operator will be crucial to a better understanding of the Finsler area. The convenient framework for such an investigation is the setting of Fréchet spaces, which has been introduced to some extend in subsection 1.2.2 based on [Rud91]. In this context, we are especially interested in the continuity and inversion properties of the spherical Radon transform. This section is based on the books of Helgason [Hel00] and Groemer [Gro96] as well as an overview article of Gardner [Gar94]. For the investigation of the transformation and differentiation behaviour of the spherical Radon transform, we reproduce results of Bailey et al. [BEGM03] in a higher dimensional setting.

This section is structured as follows. First we define the spherical Radon transform. We then develop some representations for the spherical Radon transform, which will be useful to relate the spherical Radon transform to the Finsler area integrand. Additionally, we classify how regular the transformed functions are. Further, some invariance properties of the spherical Radon transform are given, which lead to a statement on how differentiation commutes with the operator itself. This leads to a quantification of the continuity of the spherical Radon transform seen as an operator on special Fréchet spaces.

For a continuous function on the sphere, the Radon transform computes the mean of this function w.r.t. lower dimensional subspheres.

Definition 2.2.1 (**Spherical Radon transform**)**.** The spherical Radon transform $\widehat{\mathcal{R}}$ defined on the function space $C^0(\mathbb{S}^m)$ of continuous functions on the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ is given as

$$\widehat{\mathcal{R}}(f)(\zeta) \quad := \quad \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^m \cap \zeta^\perp} f(w) \mathrm{d}\mathscr{H}^{m-1}(w) \tag{2.2.1}$$

for $f \in C^0(\mathbb{S}^m)$ and $\zeta \in \mathbb{S}^m$. The spherical Radon transform $\mathcal{R}$ on the space of continuous functions on $\mathbb{R}^{m+1}\backslash\{0\}$ is defined as

$$\mathcal{R}(f)(z) \quad := \quad \frac{1}{|z|} \widehat{\mathcal{R}}(f|_{\mathbb{S}^m}) \left(\frac{z}{|z|} \right) \tag{2.2.2}$$

for $f \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ and $z \in \mathbb{R}^{m+1}\backslash\{0\}$.

Remark 2.2.2.

- Restricting the domain of definition of $\mathcal{R}$ to $\mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ leads to the following relation between both types of spherical Radon transforms, namely

$$\mathcal{R} \quad = \quad \Phi_{-1}^{-1} \circ \widehat{\mathcal{R}} \circ \Phi_{-m}, \quad \widehat{\mathcal{R}} \quad = \quad \Phi_{-1} \circ \mathcal{R} \circ \Phi_{-m}^{-1},$$

  where Φ_{-1}^{-1}, Φ_{-m}^{-1} are the inverse mappings to Φ_{-1}, Φ_{-m}, respectively, which where defined in Definition 1.2.21.

- The spherical Radon transforms $\widehat{\mathcal{R}}(f)(\cdot)$ and $\mathcal{R}(f)(\cdot)$ are even functions by definition.

- In (2.2.1) the spherical Radon transform operates on spherical functions, which justifies the name spherical Radon transform. The classical Radon transform derives from the integration of functions on $\mathbb{R}^{m+1}$ along lines, while the spherical Radon transform derives form the integration over great circles in $\mathbb{S}^m$. Lines and circles are the (locally) geodesic curves in Euclidean and spherical geometry, respectively. Therefore, the sphere is to some extent the natural setting for the spherical Radon transform. For more information on this subject (see [Hel00]). (2.2.2) is, as we will see, the representation of the spherical Radon transform on the function spaces which is the most suitable to the investigation of the Finsler Area. This is due to the choice of homogeneities.

Lemma 2.2.3 (Symmetry of $\mathcal{R}(f)$). *Let $f \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$. $\mathcal{R}(f)$ is an even function, i.e.*

$$\mathcal{R}(f)(z) \quad = \quad \mathcal{R}(f)(-z) \quad \textit{for all } z \in \mathbb{R}^{m+1}\backslash\{0\}. \tag{2.2.3}$$

Proof. We will simply exploit the definition of the spherical Radon transform:

$$\begin{aligned}\mathcal{R}(f)(z) \quad &= \quad \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}\cap z^\perp} f(w)\mathrm{d}\mathscr{H}^{m-1}(w)\\ &= \quad \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}\cap (-z)^\perp} f(w)\mathrm{d}\mathscr{H}^{m-1}(w)\\ &= \quad \mathcal{R}(f)(-z)\end{aligned}$$

for all $z \in \mathbb{R}^{m+1}\backslash\{0\}$. □

The following lemma gives a more analytical representation of the spherical Radon transform of a continuous function.

Lemma 2.2.4 (Orthogonal basis representation). *For $g \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ one has the identity*

$$\mathcal{R}(g)(z) \quad = \quad \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta) \tag{2.2.4}$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$, where $\{o_1,\dots,o_m\} \subset \mathbb{R}^{m+1}$ is an orthonormal basis of the subspace $z^\perp \in \mathbb{R}^{m+1}$.

Proof. By means of local coordinate charts $\mathbb{S}^{m-1} \subset \bigcup_{t=1}^M V_t \subset \mathbb{R}^m$ and respective coordinates $y_t = (y_t^1,\dots,y_t^{m-1}) : V_t \to \Omega_t \subset \mathbb{R}^{m-1}$, we define the disjoint sets $A_t := V_t - \bigcup_{s=1}^{t-1} V_s$ for $t = 1,\dots,M$ and use the characteristic functions χ_{A_t} of the sets A_t to partition the integrand

$$g(\theta^\kappa o_\kappa) \quad = \quad \sum_{t=1}^M \chi_{A_t}(\theta)g(\theta^\kappa o_\kappa) \quad =: \quad \sum_{t=1}^M g_t(\theta)$$

to find (cf. [Bär10, Definition 3.7.4, p. 143])

$$\int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta) \quad = \quad \sum_{t=1}^M \int_{\mathbb{S}^{m-1}} g_t(\theta)\mathrm{d}\mathscr{H}^{m-1}(\theta).$$

In each term on the right-hand side we apply the area formula Theorem 1.2.7 with respect to the (injective) transformation $T_t : \Omega_t \to \mathbb{R}^{m+1}$ given by $T_t(y_t) := \theta_t^\kappa(y_t)o_\kappa$ for $t = 1,\dots,M$ to obtain

$$\begin{aligned}\int_{\mathbb{S}^{m-1}} g_t(\theta)\mathrm{d}\mathscr{H}^{m-1}(\theta) \quad &= \quad \int_{V_t} g_t(\theta)\mathrm{d}\mathscr{H}^{m-1}(\theta)\\ &= \quad \int_{\Omega_t} g_t(\theta_t(y_t))\sqrt{\det(D\theta_t^T(y_t)D\theta_t(y_t))}\mathrm{d}y_t^1\dots\mathrm{d}y_t^{m-1}\\ &= \quad \int_{\mathbb{R}^{m+1}} g_t(\theta_t(y_t))|_{y_t\in T_t^{-1}(\zeta)}\mathrm{d}\mathscr{H}^{m-1}(\zeta)\\ &= \quad \int_{\mathbb{R}^{m+1}} (\chi_{A_t}(\theta_t(y_t))g(\theta_t^\kappa(y_t)o_\kappa)|_{y_t\in T_t^{-1}(\zeta)}\mathrm{d}\mathscr{H}^{m-1}(\zeta),\end{aligned}$$

since $\theta_t^\kappa(y_t)o_k = T_t(y_t) = \zeta$ for $y_t \in T_t^{-1}(\zeta)$, and because $T_t(y_t) \in z^\perp$ and $|T_t(y_t)| = 1$ by definition of T_t. Recall that the system $\{o_1,\dots,o_m\}$ is an orthonormal basis of $z^\perp$.

Now for $y_t \in T_t^{-1}(\zeta)$ one has $T_t(y_t) = \theta_t^\kappa(y_t)o_\kappa = \zeta \in \mathbb{R}^{m+1}$, and therefore

$$\theta_t(y_t) \quad = \quad (\theta_t^1(y_t), \ldots, \theta_t^m(y_t))^T \quad = \quad (o_1 \cdot \zeta, \ldots, o_m \cdot \zeta)^T \quad = \quad \Phi^T \zeta$$

and

$$\zeta \quad = \quad \theta_t^\kappa(y_t) o_k \quad = \quad \Phi\Phi^T\zeta$$

for the matrix $\Phi := (o_1|\ldots|o_m) \in \mathbb{R}^{(m+1)\times m}$ with the orthonormal basis vectors o_α, $\alpha = 1, \ldots, m$, as column vectors. This implies for any set $A \subset \mathbb{R}^m$ that $\theta = (\theta^1, \ldots, \theta^m)^T = \Phi^T\zeta \in A$ with $\zeta = \Phi\Phi^T\zeta$ if and only if $\zeta \in \Phi A := \{\xi \in \mathbb{R}^{m+1} : \xi = \Phi a \text{ for some } a \in A\}$ since $\Phi^T\Phi = \mathrm{Id}_{\mathbb{R}^m}$. Hence the characteristic functions satisfy $\chi_A(\Phi^T\zeta) = \chi_{\Phi A}(\zeta)$ for all $\zeta \in \mathbb{R}^{m+1}$ with $\zeta = \Phi\Phi^T\zeta$, in particular we find

$$\chi_{A_t}(\theta_t(y_t))\big|_{y_t \in T_t^{-1}(\zeta)} \quad = \quad \chi_{\Phi A_t}(\zeta),$$

where the sets ΦA_t are also disjoint, since any $\xi \in \Phi A_s \cap \Phi A_t$ for $1 \leq s < t \leq M$ has the representations $\xi = \Phi a_s = \Phi a_t$ for some $a_s \in A_s$ and $a_t \in A_t$, which implies $\Phi a_s = a_s^\kappa o_\kappa = a_t^\kappa o_\kappa = \Phi a_t$, i.e. $a_s = a_t$ as the o_κ are linearly independent. But then $a_s = a_t \in A_s \cap A_t = \emptyset$, which is a contradiction. So, $\mathbb{S}^m \cap z^\perp$ is the disjoint union of the sets ΦA_t for $t = 1, \ldots, M$, i.e. $\mathbb{S}^m \cap z^\perp = \dot{\bigcup}_{t=1}^M \Phi A_t$. Summarizing we conclude

$$\begin{aligned}
\int_{\mathbb{S}^{m-1}} g(\theta^k o_k)\,\mathrm{d}\mathscr{H}^{m-1}(\theta) \quad &= \quad \sum_{t=1}^M \int_{\mathbb{R}^{m+1}} \chi_{\Phi A_t}(\zeta) g(\zeta)\,\mathrm{d}\mathscr{H}^{m-1}(\zeta) \\
&= \quad \sum_{t=1}^M \int_{\Phi A_t} g(\zeta)\,\mathrm{d}\mathscr{H}^{m-1}(\zeta) \\
&= \quad \int_{\mathbb{S}^m \cap z^\perp} g(\zeta)\,\mathrm{d}\mathscr{H}^{m-1}(\zeta) \\
&= \quad \int_{\mathbb{S}^m \cap (z/|z|)^\perp} g(\zeta)\,\mathrm{d}\mathscr{H}^{m-1}(\zeta)
\end{aligned}$$

and therefore, by Definition 2.2.1 of the spherical Radon transform,

$$\frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\,\mathrm{d}\mathscr{H}^{m-1}(\theta) \quad = \quad \mathcal{R}(g)(z)$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$. □

As a direct consequence of Lemma 2.2.4, we can classify the spherical Radon transform as a bounded operator.

Corollary 2.2.5 (**C^0-bounded operator**). *For $f, g \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ holds*

$$\widehat{\rho}_0(\mathcal{R}(f) - \mathcal{R}(g)) \quad \leq \quad \widehat{\rho}_0(f - g). \tag{2.2.5}$$

Especially, if we choose $g \equiv 0$, we get

$$\widehat{\rho}_0(\mathcal{R}(f)) \quad \leq \quad \widehat{\rho}_0(f). \tag{2.2.6}$$

So, the mapping

$$\mathcal{R} : \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) \to \mathrm{HF}^0_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$$

is a bounded linear operator. Therein, we think of $\mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ and $\mathrm{HF}^0_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$ as Banach spaces with norm $\widehat{\rho}_0$.

Proof. For $z \in \mathbb{R}^{m+1}\backslash\{0\}$ choose an orthonormal basis $\{o_1,\dots,o_m\}$ of $z^\perp$. Exploiting reprasentation (2.2.4) yields

$$\begin{aligned} |\mathcal{R}(f)(z) - \mathcal{R}(g)(z)| &= \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\left|\int_{\mathbb{S}^{m-1}} f(\theta^\kappa o_\kappa) - g(\theta^\kappa o_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)\right| \\ &\leq \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} |f(\theta^\kappa o_\kappa) - g(\theta^\kappa o_\kappa)|\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &\leq \frac{1}{|z|}\widehat{\rho}_0(f-g). \end{aligned}$$

Notice that $\theta^\kappa o_\kappa \in \mathbb{S}^m$ for all $\theta \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. Taking the supremum over $z \in \mathbb{S}^m$ on both sides of the estimate leads to the desired inequality (2.2.5) and, by setting $g \equiv 0$, also (2.2.6). By (2.2.5) (or (2.2.6)) $\mathcal{R} : \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) \to \mathrm{HF}^0_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$ is a bounded linear operator, i.e. a continuous linear operator w.r.t. the involved normed vector spaces. □

The following lemma is an improvement of Lemma 2.2.4 for $(-m)$-homogeneous functions. It will be useful to relate the spherical Radon transform to the Finsler area in Corollary 2.3.1.

Lemma 2.2.6 (General basis representation). *For $g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ one has the identity*

$$\mathcal{R}(g)(z) = \frac{|\xi_1 \wedge \dots \wedge \xi_m|}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa \xi_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta) \tag{2.2.7}$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$, where $\{\xi_1,\dots,\xi_m\} \subset \mathbb{R}^{m+1}$ is a basis of the subspace $z^\perp \in \mathbb{R}^{m+1}$.

Proof. By continuity of $\mathcal{R}$ (see Corollary 2.2.5) it is sufficient to prove the lemma for C^1-functions. For an orthonormal basis $\{o_1,\dots,o_m\} \subset \mathbb{R}^{m+1}$ of an m-dimensional subspace of $\mathbb{R}^{m+1}$ we can form the m-vector

$$o_1 \wedge \dots \wedge o_m = \sum_{i=1}^{m+1} \det(e_i|o_1|\dots|o_m)e_i \in \mathbb{R}^{m+1},$$

where the e_i denote the standard basis vectors of $\mathbb{R}^{m+1}$, $i = 1,\dots,m$, and we have by Theorem 1.1.11

$$|o_1 \wedge \dots \wedge o_m|^2 = \langle o_1 \wedge \dots \wedge o_m, o_1 \wedge \dots \wedge o_m\rangle = \det(\langle o_i, o_j\rangle) = 1.$$

Lemma 2.2.4 applied to the m-vector $z := o_1 \wedge \dots \wedge o_m$ (so that $\mathrm{span}\{o_1,\dots,o_m\} = z^\perp$) yields

$$\mathcal{R}(g)(o_1 \wedge \dots \wedge o_m) = \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)$$

for any $g \in C^1(\mathbb{R}^{m+1}\backslash\{0\})$. By means of the Gauß map $\nu : \mathbb{S}^{m-1} \to \mathbb{R}^m$, which coincides with the position vector at every point on $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ we can apply [For09, Satz 3, p. 245] to rewrite the Radon transform in terms of differential forms:

$$\begin{aligned} \mathcal{R}(g)(o_1 \wedge \dots \wedge o_m) &= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\theta^\sigma\nu_\sigma(\theta)\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa o_\kappa)\theta^\sigma(-1)^{\sigma-1}\mathrm{d}\theta^1 \wedge \dots \wedge \widehat{\mathrm{d}\theta^\sigma} \wedge \dots \wedge \mathrm{d}\theta^m \\ &=: \mathcal{I}(g)(O), \end{aligned} \tag{2.2.8}$$

where $O = (o_1|\dots|o_m) \in \mathbb{R}^{(m+1)\times m}$ assembles the orthonormal basis vectors $o_1,\dots,o_m$ as columns. Now we claim that

$$\mathcal{I}(g)(\Xi B) = \frac{1}{\det B}\mathcal{I}(g)(\Xi) \tag{2.2.9}$$

for any $B = (b^\alpha_\beta) \in \mathbb{R}^{m\times m}$ with positive determinant, and $\Xi := (\xi_1|\ldots|\xi_m) \in \mathbb{R}^{(m+1)\times m}$, where $\{\xi_1,\ldots,\xi_m\}$ is an arbitrary set of m linearly independent vectors in $\mathbb{R}^{m+1}$ replacing the $o_\alpha, \alpha = 1,\ldots,m$, in the defining integral for $\mathcal{I}(g)(\cdot)$ in (2.2.8). Indeed, B represents the linear map $\beta : \mathbb{R}^m \to \mathbb{R}^m$ with $\beta^\alpha(x) = b^\alpha_\beta x^\beta$ for $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$ with $(\beta^{-1})^\alpha(y) = a^\alpha_\beta y^\beta$ for $(y_1,\ldots,y_m) \in \mathbb{R}^m$, where $A = (a^\alpha_\beta) := B^{-1} \in \mathbb{R}^{m\times m}$, and we have $\mathrm{d}\beta^\alpha = b^\alpha_\beta \mathrm{d}x^\beta$ for $i = 1,\ldots,m$, so that $\mathrm{d}\theta^\alpha = a^\alpha_\tau b^\tau_\sigma \mathrm{d}\theta^\sigma$, and $\theta^\sigma = a^\sigma_\tau b^\tau_\eta \theta^\eta = b^\tau_\eta \theta^\eta a^\sigma_\tau = \beta^\tau(\theta) a^\sigma_\tau$. By means of the matrix

$$\Xi B \quad = \quad (\xi_1,\ldots,\xi_m)B \quad = \quad (b^\tau_1\xi_\tau|\ldots|b^\tau_m\xi_\tau) \in \mathbb{R}^{(m+1)\times m},$$

we can write the left-hand side of (2.2.9) as

$$\begin{aligned} &\mathcal{I}(g)(\Xi B) \\ = &\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa b^\tau_\kappa \xi_\tau)\theta^\sigma(-1)^{\sigma-1}\mathrm{d}\theta^1\wedge\ldots\wedge\widehat{\mathrm{d}\theta^\sigma}\wedge\ldots\wedge\mathrm{d}\theta^m \\ = &\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\beta^\tau(\theta)\xi_\tau)\beta^\kappa(\theta)a^\sigma_\kappa(-1)^{\sigma-1}a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}\wedge\ldots\wedge\widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}}\wedge\ldots\wedge a^m_{\tau_m}\mathrm{d}\beta^{\tau_m}. \end{aligned}$$

Now it is a routine matter in computations with determinants to verify that the last integrand on the right-hand side equals

$$\frac{1}{\det B} g(\beta^\tau(\theta)\xi_\tau)\beta^\sigma(\theta)(-1)^{\sigma-1}\mathrm{d}\beta^1\wedge\ldots\wedge\widehat{\mathrm{d}\beta^\sigma}\wedge\ldots\wedge\mathrm{d}\beta^m. \tag{2.2.10}$$

To prove this equality, it is sufficient to show

$$\begin{aligned} &\beta^\kappa(\theta)a^\sigma_\kappa(-1)^{\sigma-1}a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}\wedge\ldots\wedge\widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}}\wedge\ldots\wedge a^m_{\tau_m}\mathrm{d}\beta^{\tau_m} \\ = &\frac{1}{\det B}\beta^\sigma(\theta)(-1)^{\sigma-1}\mathrm{d}\beta^1\wedge\ldots\wedge\widehat{\mathrm{d}\beta^\sigma}\wedge\ldots\wedge\mathrm{d}\beta^m \end{aligned} \tag{2.2.11}$$

For this purpose, we start to rewrite the following form

$$a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}\wedge\ldots\wedge\widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}}\wedge\ldots\wedge a^m_{\tau_m}\mathrm{d}\beta^{\tau_m}.$$

We evaluate the form at the standard basis vectors as we can deduce the values of the form for arbitrary input vectors by linear extension. So, for a choice of indices $\gamma_1,\ldots,\gamma_{m-1} \in \{1,\ldots,m\}$ and following the definitions at the beginning of subsection 1.1.2 we identify

$$a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}\wedge\ldots\wedge\widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}}\wedge\ldots\wedge a^m_{\tau_m}\mathrm{d}\beta^{\tau_m}\Big(\frac{\partial}{\partial x^{\gamma_1}},\ldots,\frac{\partial}{\partial x^{\gamma_{m-1}}}\Big)$$

with

$$\det\begin{pmatrix} a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}(\frac{\partial}{\partial x^{\gamma_1}}) & \cdots & a^1_{\tau_1}\mathrm{d}\beta^{\tau_1}(\frac{\partial}{\partial x^{\gamma_{m-1}}}) \\ \vdots & & \vdots \\ \widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}(\frac{\partial}{\partial x^{\gamma_1}})} & \cdots & \widehat{a^\sigma_{\tau_\sigma}\mathrm{d}\beta^{\tau_\sigma}(\frac{\partial}{\partial x^{\gamma_{m-1}}})} \\ \vdots & & \vdots \\ a^m_{\tau_m}\mathrm{d}\beta^{\tau_m}(\frac{\partial}{\partial x^{\gamma_1}}) & \cdots & a^m_{\tau_m}\mathrm{d}\beta^{\tau_m}(\frac{\partial}{\partial x^{\gamma_{m-1}}}) \end{pmatrix} \quad = \quad \det\begin{pmatrix} a^1_{\tau_1}b^{\tau_1}_{\gamma_1} & \cdots & a^1_{\tau_1}b^{\tau_1}_{\gamma_{m-1}} \\ \vdots & & \vdots \\ \widehat{a^\sigma_{\tau_\sigma}b^{\tau_\sigma}_{\gamma_1}} & \cdots & \widehat{a^\sigma_{\tau_\sigma}b^{\tau_\sigma}_{\gamma_{m-1}}} \\ \vdots & & \vdots \\ a^m_{\tau_m}b^{\tau_m}_{\gamma_1} & \cdots & a^m_{\tau_m}b^{\tau_m}_{\gamma_{m-1}} \end{pmatrix}.$$

Further, we can write the latter matrix in the former equality as a matrix product. So

$$\det\left(\begin{pmatrix} a^1_1 & \cdots & a^1_m \\ \vdots & & \vdots \\ \widehat{a^\sigma_1} & \cdots & \widehat{a^\sigma_m} \\ \vdots & & \vdots \\ a^m_1 & \cdots & a^m_m \end{pmatrix} \cdot \begin{pmatrix} b^1_{\gamma_1} & \cdots & b^m_{\gamma_{m-1}} \\ \vdots & & \vdots \\ b^1_{\gamma_1} & \cdots & b^m_{\gamma_{m-1}} \end{pmatrix}\right) \quad = \quad \det(A^{(\sigma)}\cdot B_{\gamma_1\cdots\gamma_{m-1}})$$

and applying the Binet-Cauchy formula (see Theorem 1.1.6) leads to

$$\det(A^{(\sigma)} \cdot B_{\gamma_1\cdots\gamma_{m-1}}) = \sum_{\tau=1}^{m} \det A^{(\sigma)}_{(\tau)} \det B^{(\tau)}_{\gamma_1\cdots\gamma_{m-1}}.$$

Hence, together with Theorem 1.1.3, we get

$$\begin{aligned}
& a^1_{\tau_1} \mathrm{d}\beta^{\tau_1} \wedge \ldots \wedge \widehat{a^\sigma_{\tau_\sigma} \mathrm{d}\beta^{\tau_\sigma}} \wedge \ldots \wedge a^m_{\tau_m} \mathrm{d}\beta^{\tau_m} \Big(\frac{\partial}{\partial x^{\gamma_1}}, \ldots, \frac{\partial}{\partial x^{\gamma_{m-1}}}\Big) \\
= & \sum_{\tau=1}^{m} \det A^{(\sigma)}_{(\tau)} \det B^{(\tau)}_{\gamma_1\cdots\gamma_{m-1}} \\
= & \sum_{\tau=1}^{m} \frac{(-1)^{\sigma+\tau} \det A^{(\sigma)}_{(\tau)}}{\det A} \det B^{(\tau)}_{\gamma_1\cdots\gamma_{m-1}} (-1)^{\sigma+\tau} \det A \\
= & \sum_{\tau=1}^{m} b^\tau_\sigma \det B^{(\tau)}_{\gamma_1\cdots\gamma_{m-1}} (-1)^{\sigma+\tau} (\det B)^{-1} \\
= & \sum_{\tau=1}^{m} b^\tau_\sigma (-1)^{\sigma+\tau} (\det B)^{-1} \mathrm{d}\beta^1 \wedge \ldots \wedge \widehat{\mathrm{d}\beta^\tau} \wedge \ldots \wedge \mathrm{d}\beta^m \Big(\frac{\partial}{\partial x^{\gamma_1}}, \ldots, \frac{\partial}{\partial x^{\gamma_{m-1}}}\Big)
\end{aligned}$$

and so

$$\begin{aligned}
& \beta^\kappa(\theta) a^\sigma_\kappa (-1)^{\sigma-1} a^1_{\tau_1} \mathrm{d}\beta^{\tau_1} \wedge \ldots \wedge \widehat{a^\sigma_{\tau_\sigma} \mathrm{d}\beta^{\tau_\sigma}} \wedge \ldots \wedge a^m_{\tau_m} \mathrm{d}\beta^{\tau_m} \\
= & \beta^\kappa(\theta) a^\sigma_\kappa (-1)^{\sigma-1} \sum_{\tau=1}^{m} b^\tau_\sigma (-1)^{\sigma+\tau} (\det B)^{-1} \mathrm{d}\beta^1 \wedge \ldots \wedge \widehat{\mathrm{d}\beta^\tau} \wedge \ldots \wedge \mathrm{d}\beta^m \\
= & \beta^\kappa(\theta) (-1)^{\kappa-1} (\det B)^{-1} \mathrm{d}\beta^1 \wedge \ldots \wedge \widehat{\mathrm{d}\beta^\kappa} \wedge \ldots \wedge \mathrm{d}\beta^m
\end{aligned}$$

proving (2.2.11).

Now, remember the form mentioned in (2.2.10), which is in fact the so-called pullback $\beta^*\omega$ of the form

$$\omega(\theta) = \frac{1}{\det B} g(\theta^\tau \xi_\tau) \theta^\sigma (-1)^{\sigma-1} \mathrm{d}\theta^1 \wedge \ldots \wedge \widehat{\mathrm{d}\theta^\sigma} \wedge \ldots \wedge \mathrm{d}\theta^m$$

under the linear mapping β. Since $\det B = \det \mathrm{D}\beta > 0$ by assumption we obtain by the transformation formula for differential forms (see, e.g. [For09, Satz 1, p. 235])

$$\begin{aligned}
\mathcal{I}(g)(\Xi B) &= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \beta^*\omega = \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\beta(\mathbb{S}^{m-1})} \omega \\
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \omega = \frac{1}{\det B} \mathcal{I}(g)(\Xi)
\end{aligned}$$

where we have used the fact that ω is a closed form and that the closed surface $\beta(\mathbb{S}^{m-1})$ contains the origin as the only singularity of the differential form ω in its interior, since β as a linear map maps 0 to 0 (see, e.g. [For09, Corollar, p. 257]). Recall that g was assumed to be of class $C^1(\mathbb{R}^{m+1}\backslash\{0\})$. Hence the claim (2.2.9) is proved. With arguments analogous to [Mor08, pp. 349,350] (or in more detail [vdM92, pp. 7-11]) one can use relation (2.2.9) for fixed $g \in C^1(\mathbb{R}^{m+1}\backslash\{0\})$ to show that there is a (-1)-homogeneous function $\mathcal{J}(g)(\cdot) : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ such that

$$\mathcal{I}(g)(\Xi) = \mathcal{J}(g)(\xi_1 \wedge \ldots \wedge \xi_m) \quad \text{for} \quad \Xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^{(m+1)\times m}, \tag{2.2.12}$$

whenever $\xi_1, \ldots, \xi_m \in \mathbb{R}^{m+1}$ are linearly independent.

For a hyperplane $(\xi_1 \wedge \ldots \wedge \xi_m)^\perp \subset \mathbb{R}^{m+1}$, where $\xi_1, \ldots, \xi_m \in \mathbb{R}^{m+1}$ are linearly independent vectors, we can now choose an appropriately oriented orthonormal basis $\{o_1, \ldots, o_m\} \subset \mathbb{R}^{m+1}$ such that

$$o_1 \wedge \ldots \wedge o_m \quad = \quad \frac{\xi_1 \wedge \ldots \wedge \xi_m}{|\xi_1 \wedge \ldots \wedge \xi_m|}.$$

For the matrix $F = (o_1|\ldots|o_m) \in \mathbb{R}^{(m+1)\times m}$ we consequently obtain by (-1)-homogeneity of $\mathcal{R}(g)(\cdot)$ and of $\mathcal{J}(g)(\cdot)$

$$\begin{aligned}
\mathcal{R}(g)(\xi_1 \wedge \ldots \wedge \xi_m) &= \mathcal{R}(g)(o_1 \wedge \ldots \wedge o_m)\frac{1}{|\xi_1 \wedge \ldots \wedge \xi_m|} \\
&\overset{(2.2.8)}{=} \mathcal{I}(g)(F)\frac{1}{|\xi_1 \wedge \ldots \wedge \xi_m|} \\
&\overset{(2.2.12)}{=} \mathcal{J}(g)(o_1 \wedge \ldots \wedge o_m)\frac{1}{|\xi_1 \wedge \ldots \wedge \xi_m|} \\
&= \mathcal{J}(g)(\xi_1 \wedge \ldots \wedge \xi_m) \\
&\overset{(2.2.12)}{=} \mathcal{I}(g)(\Xi) \qquad (2.2.13)
\end{aligned}$$

which is relation (2.2.8) even for matrices $\Xi = (\xi_1|\ldots|\xi_m) \in \mathbb{R}^{(m+1)\times m}$, whose column vectors $\xi_i,\ i = i, \ldots, m$, are merely linearly independent. So, for arbitrary $z \in \mathbb{R}^{m+1}\backslash\{0\}$ and basis vectors $\xi_1, \ldots, \xi_m \in \mathbb{R}^{m+1}$ of $z^\perp$ holds

$$z \quad = \quad \pm \frac{|z|}{|\xi_1 \wedge \ldots \wedge \xi_m|}\xi_1 \wedge \ldots \wedge \xi_m$$

depending on the orientation of $\xi_1, \ldots, \xi_m$. By exploiting the fact that the spherical Radon transform of g is an even (-1)-homogeneous function (see (2.2.3)) and by (2.2.13), we deduce

$$\begin{aligned}
\mathcal{R}(g)(z) &= \mathcal{R}(g)(\pm\frac{|z|}{|\xi_1 \wedge \ldots \wedge \xi_m|}\xi_1 \wedge \ldots \wedge \xi_m) \\
&= \mathcal{R}(g)(\frac{|z|}{|\xi_1 \wedge \ldots \wedge \xi_m|}\xi_1 \wedge \ldots \wedge \xi_m) \\
&= \frac{|\xi_1 \wedge \ldots \wedge \xi_m|}{|z|}\mathcal{R}(g)(\xi_1 \wedge \ldots \wedge \xi_m) \\
&= \frac{|\xi_1 \wedge \ldots \wedge \xi_m|}{|z|}\mathcal{I}(g)(\Xi),
\end{aligned}$$

which proves (2.2.7) for any $g \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ and therefore also for any function $g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ by approximation. □

A consequence of Lemma 2.2.6 is the following corollary regarding the continuity and differentiability of transformed functions. Notice that the regularity of the transformed functions depends on the regularity of the input functions themselves.

Corollary 2.2.7 (**Continuity and Differentiability**). *The spherical Radon transform of a function*

$$g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$$

satisfies

$$\mathcal{R}(g) \in \mathrm{HF}^0_{-1}(\mathbb{R}^{m+1}\backslash\{0\}).$$

Further, if we assume

$$g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$$

then
$$\mathcal{R}(g) \in \mathrm{HF}^0_{-1}(\mathbb{R}^{m+1}\backslash\{0\}) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$$
and if we assume even more
$$g \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$$
then
$$\mathcal{R}(g) \in \mathrm{HF}^1_{-1}(\mathbb{R}^{m+1}\backslash\{0\}).$$

Proof. We begin with some basic constructions, which we will need throughout the proof. For two arbitrary linearly independent vectors $\zeta, \eta \in \mathbb{R}^{m+1}\backslash\{0\}$ and an orthonormal basis $o_2, \ldots, o_m$ of $\mathrm{span}\{\zeta, \eta\}^\perp$ we define for every $\tau \in \mathrm{span}\{\zeta, \eta\}$

$$\begin{aligned} \xi_1^\tau &:= -\tau \wedge o_2 \wedge \ldots \wedge o_m, \\ \xi_\kappa^\tau &:= o_\kappa \text{ for } \kappa = 2, \ldots, m. \end{aligned}$$

By simple linear algebra holds

$$|\xi_1^\tau| \overset{(1.1.8)}{=} \sqrt{\det \begin{pmatrix} \langle \tau, \tau\rangle & \langle \tau, o_2\rangle & \ldots & \langle \tau, o_m\rangle \\ \langle o_2, \tau\rangle & \langle o_2, o_2\rangle & \ldots & \langle o_2, o_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle o_m, \tau\rangle & \langle o_m, o_2\rangle & \ldots & \langle o_m, o_m\rangle \end{pmatrix}} = |\tau|, \tag{2.2.14}$$

$$\tau = \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau. \tag{2.2.15}$$

Equality (2.2.15) can be deduced as follows: By Theorem 1.1.15, (2.2.14) and the fact that $\xi_1^\tau, \ldots, \xi_m^\tau$ are orthogonal vectors, we deduce $|\xi_1^\tau \wedge \ldots \wedge \xi_m^\tau| = \prod_{i=1}^m |\xi_i^\tau| = |\tau|$. Therefore, to prove $\tau = \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau$, it is sufficient to show that $\langle \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau, \tau\rangle = |\xi_1^\tau \wedge \ldots \wedge \xi_m^\tau||\tau|$, because if we assume $\tau \neq \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau$ with $|\xi_1^\tau \wedge \ldots \wedge \xi_m^\tau| = |\tau|$ and $\langle \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau, \tau\rangle = |\xi_1^\tau \wedge \ldots \wedge \xi_m^\tau||\tau|$, we get

$$\begin{aligned} 0 &\neq \langle \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau - \tau, \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau - \tau\rangle \\ &= |\xi_1^\tau \wedge \ldots \wedge \xi_m^\tau|^2 - 2\langle \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau, \tau\rangle + |\tau|^2 \quad = \quad 0, \end{aligned}$$

what is a contradiction. Hence, by calculation, we get

$$\begin{aligned} \langle \xi_1^\tau \wedge \ldots \wedge \xi_m^\tau, \tau\rangle &\overset{(1.1.9)}{=} \det(\tau, \xi_1^\tau, \xi_2^\tau, \ldots, \xi_m^\tau) \\ &= -\det(\xi_1^\tau, \tau, o_2, \ldots, o_m) \\ &\overset{(1.1.9)}{=} \langle \xi_1^\tau, -\tau \wedge o_2 \wedge \ldots \wedge o_m\rangle \\ &= \langle \xi_1^\tau, \xi_1^\tau\rangle \\ &= |\xi_1^\tau|^2 \quad = \quad |\xi_1^\tau||\tau|, \end{aligned}$$

thereby showing (2.2.15).

Define the compact set $K := \overline{B_R(0)\backslash B_r(0)} \subset \mathbb{R}^{m+1}\backslash\{0\}$ with $R > 1 > r > 0$, and choose $\zeta, \eta \in K$. For such ζ, η holds by construction that $\xi_\kappa^\zeta, \xi_\kappa^\eta \in K$ for $\kappa = 1, \ldots, m$ and further that

$$\theta^\kappa \xi_\kappa^\zeta, \theta^\kappa \xi_\kappa^\eta \in K \tag{2.2.16}$$

for every $\theta \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$.

Assume now that g is a continuous, $(-m)$-homogeneous function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$, and therefore uniformly continuous on the compact set K. So, for every $\varepsilon > 0$ there is a $\delta = \delta(K, \varepsilon) > 0$ s.t.

$$|g(\xi) - g(\tau)| \quad < \quad \varepsilon \tag{2.2.17}$$

for all $\xi,\tau \in K$ with $|\xi-\tau|<\delta$. We will exploit the representation of the spherical Radon transform of Lemma 2.2.6 resulting in

$$\begin{aligned}|\mathcal{R}(g)(\zeta)-\mathcal{R}(g)(\eta)| &= \left|\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa\xi_\kappa^\zeta)-g(\theta^\kappa\xi_\kappa^\eta)\mathrm{d}\mathscr{H}^{m-1}(\theta)\right| \\ &\le \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} |g(\theta^\kappa\xi_\kappa^\zeta)-g(\theta^\kappa\xi_\kappa^\eta)|\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &\le \frac{\varepsilon}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &= \varepsilon\end{aligned}$$

for $\zeta,\eta\in K$ with $|\zeta-\eta|<\delta$, due to the choice of δ and the following relations:

$$\begin{aligned}|\xi_\kappa^\zeta-\xi_\kappa^\eta| &= 0 \quad \text{for} \quad \kappa=2,\dots,m, \\ |\xi_1^\zeta-\xi_1^\eta| &= |(\zeta-\eta)\wedge o_2\wedge\dots\wedge o_m| \quad \le \quad |\zeta-\eta| \quad < \quad \delta,\end{aligned}$$

and thereby

$$|\theta^\kappa\xi_\kappa^\zeta-\theta^\kappa\xi_\kappa^\eta| \quad < \quad \delta \tag{2.2.18}$$

for every $\theta\in\mathbb{S}^{m-1}\subset\mathbb{R}^m$. The case when $\zeta,\eta\in\mathbb{R}^{m+1}\backslash\{0\}$ are linearly dependent can be treated in an analogous way, by choosing an orthonormal basis $o_1,\dots,o_m$ of $\operatorname{span}\{\zeta\}^\perp$ with $\frac{\zeta}{|\zeta|}=o_1\wedge\dots\wedge o_m$ and setting for every $\tau\in\operatorname{span}\{\zeta\}$

$$\begin{aligned}\xi_1^\tau &:= \left\langle \tau,\frac{\zeta}{|\zeta|}\right\rangle o_1, \\ \xi_\kappa^\tau &= o_\kappa \quad \text{for} \quad \kappa=2,\dots,m.\end{aligned}$$

We can proceed as in the linearly independent case and for $\zeta,\eta\in K$ the equations (2.2.16) and (2.2.18) hold true. Summarizing, we conclude that $\mathcal{R}(g)(\cdot)$ is a continuous function.

Regarding the carry-over of local Hölder-continuity, we can apply the same method. Assume $g\in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})\cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$. Then, for the compact set K there is a constant $C>0$ s.t.

$$|g(\xi)-g(\tau)| \quad \le \quad C|\xi-\tau|^\alpha \tag{2.2.19}$$

for all $\xi,\tau\in K\subset\mathbb{R}^{m+1}\backslash\{0\}$. Choose again $\zeta,\eta\in K$. The computation (2.2.16) for $\zeta,\eta\in K$ implies $\theta^\kappa\xi_\kappa^\zeta,\theta^\kappa\xi_\kappa^\tau\in K$ for all $\theta\in\mathbb{S}^{m-1}\subset\mathbb{R}^m$. Thereby, we get

$$\begin{aligned}|\mathcal{R}(g)(\zeta)-\mathcal{R}(g)(\eta)| &= \left|\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa\xi_\kappa^\zeta)-g(\theta^\kappa\xi_\kappa^\eta)\mathrm{d}\mathscr{H}^{m-1}(\theta)\right| \\ &\le \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} |g(\theta^\kappa\xi_\kappa^\zeta)-g(\theta^\kappa\xi_\kappa^\eta)|\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &\le \frac{C}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} |\theta^\kappa(\xi_\kappa^\zeta-\xi_\kappa^\eta)|^\alpha\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &\le C|\xi_1^\zeta-\xi_1^\eta|^\alpha \\ &\le C|\zeta-\eta|^\alpha.\end{aligned} \tag{2.2.20}$$

So, we get that $\mathcal{R}(g)(\cdot)\in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$.

In the last part of the proof, we will assume that $g\in\mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$. Let further $\zeta,\omega\in\mathbb{R}^{m+1}\backslash\{0\}$ be two orthogonal vectors and set $\eta_t:=\zeta+t\omega$ for small $t>0$. Then the directional derivative of $\mathcal{R}(g)(\cdot)$ at ζ in direction ω computes to

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)(\eta_t) \quad = \quad \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\int_{\mathbb{S}^{m-1}} g(\theta^\kappa\xi_\kappa^{\eta_t})\mathrm{d}\mathscr{H}^{m-1}(\theta)$$

$$
\begin{aligned}
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} g(\theta^\kappa \xi_\kappa^{\eta_t}) \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g_{y^i}(\theta^\kappa \xi_\kappa^\zeta)\theta^\kappa \left(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\xi_\kappa^{\eta_t}\right)^i \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g_{y^i}(\theta^\kappa \xi_\kappa^\zeta)\theta^1 \left(\xi_1^\omega\right)^i \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
&= \frac{1}{|\zeta|^2\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g_{y^i}(\theta^\kappa \xi_\kappa^\zeta)\langle\theta^\kappa \xi_\kappa^\zeta, \xi_1^\zeta\rangle \left(\xi_1^\omega\right)^i \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
&= \frac{1}{|\zeta|^2\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \langle \xi_1^\omega, (\nabla_y g)(\theta^\kappa \xi_\kappa^\zeta)\rangle\langle\theta^\kappa \xi_\kappa^\zeta, \xi_1^\zeta\rangle \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
&= \frac{1}{|\zeta|^2}\mathcal{R}(\langle \xi_1^\omega, \nabla_y g\rangle\langle y, \xi_1^\zeta\rangle)(\zeta),
\end{aligned}
$$

where we applied [For09, §11 Satz 2, p. 99], which is applicable in local coordinates and by covering the $(m-1)$-dimensional sphere with a countable number of local coordinate charts also in the present situation (cf. proof of Lemma 2.2.6). This yields the differentiability of $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)(\eta_t)$ w.r.t. t. Besides, due to $\omega \perp \zeta$, we get

$$
\begin{aligned}
\xi_1^\omega &= -\,\omega \wedge o_2 \wedge \ldots o_m &= \mp|\omega|\frac{\zeta}{|\zeta|}, \\
\xi_1^\zeta &= -\,\zeta \wedge o_2 \wedge \ldots o_m &= \pm|\zeta|\frac{\omega}{|\omega|},
\end{aligned}
$$

and thereby

$$
\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)(\eta_t) = -\frac{1}{|\zeta|}\mathcal{R}\left(\left\langle \frac{\zeta}{|\zeta|}, \nabla_y g\right\rangle \langle y, \omega\rangle\right)(\zeta). \tag{2.2.21}
$$

For $\zeta, \omega \in \mathbb{R}^{m+1}\backslash\{0\}$ two linearly dependent vectors we can write

$$
\omega = \langle \omega, \frac{\zeta}{|\zeta|}\rangle \frac{\zeta}{|\zeta|}.
$$

By using this representation and the (-1)-homogeneity of $\mathcal{R}(g)$ we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)(\eta_t) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)((1+t\langle\omega, \frac{\zeta}{|\zeta|^2}\rangle)\zeta) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\left(\frac{1}{1+t\langle\omega, \frac{\zeta}{|\zeta|^2}\rangle}\right)\right)\mathcal{R}(g)(\zeta) \\
&= -\,\langle\omega, \frac{\zeta}{|\zeta|^2}\rangle\mathcal{R}(g)(\zeta).
\end{aligned} \tag{2.2.22}
$$

So, $\mathcal{R}(g)$ is at $\zeta \in \mathbb{R}^{m+1}\backslash\{0\}$ continuously differentiable in $m+1$ orthogonal directions and as such a function continuously differentiable in ζ (see [For08, §6, Satz 2, p. 65]). Further, putting together (2.2.21) and (2.2.22), we get for arbitrary $\zeta, \omega \in \mathbb{R}^{m+1}\backslash\{0\}$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{R}(g)(\eta_t) &= -\,\langle\omega, \frac{\zeta}{|\zeta|^2}\rangle\mathcal{R}(g)(\zeta) - \frac{1}{|\zeta|^2}\mathcal{R}(\langle\zeta, \nabla_y g\rangle\langle y, \omega\rangle)(\zeta) \\
&= -\,\frac{1}{|\zeta|^2}\mathcal{R}\left(\langle\omega, \zeta\rangle g + \langle\zeta, \nabla_y g\rangle\langle y, \omega\rangle\right)(\zeta).
\end{aligned}
$$

□

Corollary 2.2.8 (Continuity and Differentiability). *The spherical Radon transform of a continuous function f on the sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$, namely $\widehat{\mathcal{R}}(f)$ is a continuous function on the sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. Further, if f is additionally differentiable of degree one, i.e. $f \in C^1(\mathbb{S}^m)$, then its spherical Radon transform is differentiable of degree one.*

Proof. The corollary is a direct consequence of Corollary 2.2.7 by remembering the Definition 1.2.21. □

The following lemma gives some useful estimates, which are a byproduct of the proof to Corollary 2.2.7.

Lemma 2.2.9. *Let $f, g \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ with $f \leq g$. Then holds*

$$\mathcal{R}(f) \leq \mathcal{R}(g). \tag{2.2.23}$$

Even more, if $f < g$, then is

$$\mathcal{R}(f) < \mathcal{R}(g). \tag{2.2.24}$$

Let $f \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ and $g \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$. Then holds

$$\begin{aligned} |\mathcal{R}(fg)(z)| &\leq \left(\max_{z^\perp \cap \mathbb{S}^m} |f|\right) \mathcal{R}(|g|)(z) \\ &\leq \widehat{\rho}_0(f)\mathcal{R}(|g|)(z), \end{aligned} \tag{2.2.25}$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$. If we choose $f \in C^0(\mathbb{R}^{m+1}\backslash\{0\})$ and $g(\cdot) := |\cdot|^{-m}$, we get

$$|\mathcal{R}(fg)(z)| \leq \left(\max_{z^\perp \cap \mathbb{S}^m} |f|\right) |z|^{-1} \leq \widehat{\rho}_0(f)|z|^{-1}.$$

Further, if $f \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^{m+1}\backslash\{0\})$, we get

$$\mathrm{Höl}_{\alpha,K}(\mathcal{R}(f)) \leq \mathrm{Höl}_{\alpha,K}(f), \tag{2.2.26}$$

$$\mathrm{Höl}_{\alpha,\mathbb{S}^m}(\mathcal{R}(f)) \leq \mathrm{Höl}_{\alpha,\mathbb{S}^m}(f), \tag{2.2.27}$$

where $K := \overline{B_R(0)\backslash B_r(0)}$ for a choice $0 < r < 1 < R < \infty$.

Proof. (2.2.23) and (2.2.25) follow directly by exploiting the monotonicity of the integration operator and Definition 2.2.1. On the other hand, (2.2.26) is a direct consequence of (2.2.20) as the same constant C appears in (2.2.19) and (2.2.20). Further, (2.2.27) can be done in a quite similar way and is a limiting case of (2.2.26). □

In the following, we will present some algebraic invariance properties of the spherical Radon transform. These can be written in an infinitesimal version (see Theorem 2.2.12). These results and methods have been done first to our knowledge by Bailey et al. [BEGM03] for the case $m = 2$ and matrices of determinant equal to 1. It can be generalized in a straight-forward manner.

Lemma 2.2.10 (Transformations (cf. [BEGM03, p. 579])**).** *For any $g \in \mathrm{HF}^0_{-m}\left(\mathbb{R}^{m+1}\backslash\{0\}\right)$ one has*

$$\mathcal{R}(g) \circ L = \mathcal{R}(g \circ (\det L)^{\frac{1}{m}} L^{-T}) \tag{2.2.28}$$

for every invertible matrix $L \in \mathrm{GL}(m+1) \subset \mathbb{R}^{(m+1)\times(m+1)}$.

The following corollary is just the restriction to matrices of determinant equal to 1.

Corollary 2.2.11 (**Transformations** (cf. [BEGM03, p. 579])). *For any* $g \in \mathrm{HF}^0_{-m}\left(\mathbb{R}^{m+1}\backslash\{0\}\right)$ *and* $L \in \mathrm{SL}(m+1)$ *one obtains*

$$\mathcal{R}(g) \circ L \quad = \quad \mathcal{R}(g \circ L^{-T}). \tag{2.2.29}$$

Proof of Lemma 2.2.10. Let $g \in \mathrm{HF}^0_{-m}\left(\mathbb{R}^{m+1}\backslash\{0\}\right)$. According to the well-known formula (see Theorem 1.1.16)

$$\begin{aligned} L(\xi_1 \wedge \ldots \wedge \xi_m) &= (\det L)(L^{-T}\xi_1) \wedge \ldots \wedge (L^{-T}\xi_m) \\ &= ((\det L)^{\frac{1}{m}} L^{-T}\xi_1) \wedge \ldots \wedge ((\det L)^{\frac{1}{m}} L^{-T}\xi_m) \end{aligned}$$

for any invertible matrix $L \in \mathbb{R}^{(m+1)\times(m+1)}$, we can now conclude with (2.2.7) for matrices $\Xi = (\xi_1|\ldots|\xi_m) \in \mathbb{R}^{(m+1)\times m}$ of maximal rank m,

$$\begin{aligned} \mathcal{R}(g)(L(\xi_1 \wedge \ldots \wedge \xi_m)) &= \mathcal{R}(g)(((\det L)^{\frac{1}{m}} L^{-T}\xi_1) \wedge \ldots \wedge ((\det L)^{\frac{1}{m}} L^{-T}\xi_m)) \\ &\overset{(2.2.7)}{=} \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g(\theta^\kappa (\det L)^{\frac{1}{m}} L^{-T}\xi_\kappa)\,\mathrm{d}\mathscr{H}^{m-1}(\theta) \\ &= \mathcal{R}(g \circ (\det L)^{\frac{1}{m}} L^{-T})(\xi_1 \wedge \ldots \wedge \xi_m), \end{aligned}$$

which proves the lemma, since for $z \in \mathbb{R}^{m+1}\backslash\{0\}$ and any appropriately oriented basis $\{\xi_1, \ldots, \xi_m\} \subset \mathbb{R}^{m+1}$ of the subspace $z^\perp \subset \mathbb{R}^{m+1}$, we have

$$z \quad = \quad |z| \frac{\xi_1 \wedge \ldots \wedge \xi_m}{|\xi_1 \wedge \ldots \wedge \xi_m|}$$

and therefore by (-1)-homogeneity

$$\begin{aligned} \mathcal{R}(g)(Lz) &= \mathcal{R}(g)(L(\xi_1 \wedge \ldots \wedge \xi_m)) \frac{|\xi_1 \wedge \ldots \wedge \xi_m|}{|z|} \\ &= \mathcal{R}(g \circ (\det L)^{\frac{1}{m}} L^{-T})(\xi_1 \wedge \ldots \wedge \xi_m) \frac{|\xi_1 \wedge \ldots \wedge \xi_m|}{|z|} \\ &= \mathcal{R}(g \circ (\det L)^{\frac{1}{m}} L^{-T})\left(\frac{|z|\xi_1 \wedge \ldots \wedge \xi_m}{|\xi_1 \wedge \ldots \wedge \xi_m|}\right) \\ &= \mathcal{R}(g \circ (\det L)^{\frac{1}{m}} L^{-T})(z) \end{aligned}$$

for any $g \in \mathrm{HF}^0_{-m}\left(\mathbb{R}^{m+1}\backslash\{0\}\right)$. □

The algebraic invariance of Corollary 2.2.11 leads to an infinitesimal property of the spherical Radon transform.

Theorem 2.2.12 (cf. [BEGM03, p. 580]). *For the spherical Radon transform holds the following identity*

$$P^j_i z_j \frac{\partial}{\partial z_i} \mathcal{R}(f)(z) \quad = \quad -\mathcal{R}\left(P^j_i y^i \frac{\partial}{\partial y^j} f\right)(z) \quad \textit{for all } z \in \mathbb{R}^{m+1}\backslash\{0\}, \tag{2.2.30}$$

where $P = (P^j_i) \in \mathbb{R}^{(m+1)\times(m+1)}$ *is a trace-free matrix and* $f \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$.

Remark 2.2.13.

- The expression $P^i_j y^j \frac{\partial f}{\partial y^i}$ in (2.2.30) is meant in the following way: the spherical Radon transform $\mathcal{R}$ is applied to the $(-m)$-homogeneous and continuous function $g : \mathbb{R}^{m+1}\backslash\{0\} \to \mathbb{R}$ defined by $g(y) := P^i_j y^j \frac{\partial f}{\partial y^i}(y)$ for all $y \in \mathbb{R}^{m+1}\backslash\{0\}$.

- The relation (2.2.30) has been shown in [BEGM03] for $m = 2$.

Proof. The proof of Theorem 2.2.12 will exploit the transformation behaviour of Corollary 2.2.11, which leads to an infinitesimal property of the spherical Radon transform by differentiating it along the manifold $\mathrm{SL}(m+1) := \{M \in \mathbb{R}^{(m+1)\times(m+1)} : \det M = 1\}$ at the identity matrix. Accordingly, we will at first investigate the manifold structure of $\mathrm{SL}(m+1)$. $\mathrm{SL}(m+1) \subset \mathbb{R}^{(m+1)\times(m+1)} \cong \mathbb{R}^{(m+1)^2}$ can be represented implicitly as the set $\mathrm{SL}(m+1) = \{M \in \mathbb{R}^{(m+1)\times(m+1)} : f(M) = 0\}$, where the mapping $f : \mathbb{R}^{(m+1)\times(m+1)} \cong \mathbb{R}^{(m+1)^2} \to \mathbb{R}$ is defined by $f(M) := \det M - 1$ for every $M \in \mathbb{R}^{(m+1)\times(m+1)}$. f is, as a polynomial function, a smooth function on $\mathbb{R}^{(m+1)\times(m+1)}$. Hence, the differential of f can be computed by Theorem 1.1.4 as follows

$$\frac{\partial}{\partial m^i_j} f(M) \quad = \quad \frac{\partial}{\partial m^i_j}(\det M - 1) \quad = \quad \det M (M^{-1})^j_i$$

for every $M = (m^i_j) \in \mathbb{R}^{(m+1)\times(m+1)}$ of full rank. Especially if $\det M = 1$, M is of full rank and thereby the differential of f is of rank one on $\mathrm{SL}(m+1)$ such that $\mathrm{SL}(m+1)$ is a submanifold of $\mathbb{R}^{(m+1)\times(m+1)}$ of dimension $(m+1)^2 - 1 = m(m+2)$ (see [For08, §9, Satz 2, pp. 104-105]). Further, the tangent space $\mathrm{T}_{\mathrm{Id}}\mathrm{SL}(m+1)$ at the identity matrix $\mathrm{Id} = (\delta^i_j) \in \mathbb{R}^{(m+1)\times(m+1)}$ can be identified with the hyperplane orthogonal to the gradient of the implicit function f at the identity matrix (see [For08, §9, Satz 3, pp. 107-108]), i.e.

$$\begin{aligned}
\mathrm{T}_{\mathrm{Id}}\mathrm{SL}(m+1) \quad &= \quad \{P = (P^i_j) \in \mathbb{R}^{(m+1)\times(m+1)} : P^i_j \frac{\partial}{\partial m^i_j} f(M)|_{M=\mathrm{Id}} = 0\} \\
&= \quad \{P = (P^i_j) \in \mathbb{R}^{(m+1)\times(m+1)} : P^i_j \det \mathrm{Id}(\mathrm{Id}^{-1})^j_i = 0\} \\
&= \quad \{P = (P^i_j) \in \mathbb{R}^{(m+1)\times(m+1)} : P^i_i = P^i_j \delta^j_i = 0\}.
\end{aligned}$$

So, the tangent space $\mathrm{T}_{\mathrm{Id}}\mathrm{SL}(m+1)$ is the vector space of all trace-free matrices. Choose now an arbitrary differentiable curve $M : (-\varepsilon, \varepsilon) \to \mathrm{SL}(m+1)$ such that

$$\begin{aligned}
M(0) \quad &= \quad \mathrm{Id}, \\
M^{'}(0) \quad &:= \quad \frac{\partial}{\partial t}|_{t=0} M(t) \quad = \quad P
\end{aligned}$$

with $P = (P^i_j) \in \mathbb{R}^{(m+1)\times(m+1)}$ a trace-free matrix, i.e. $\mathrm{trace} P = P^i_i = 0$. By applying Corollary 2.2.11 to the curve M and differentiation w.r.t. t, we get

$$\frac{\partial}{\partial t}|_{t=0} \mathcal{R}(g)(M(t)z) \quad = \quad \frac{\partial}{\partial t}|_{t=0} \mathcal{R}(g \circ (M(t))^{-T})(z), \tag{2.2.31}$$

where $g \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ and $z = (z_i) \in \mathbb{R}^{m+1}\backslash\{0\}$. The left-hand side of (2.2.31) is well-defined, since $\mathcal{R}(g)(\cdot)$ is a differentiable function according to Corollary 2.2.8. After application of the chain rule, the left-hand side becomes

$$\frac{\partial}{\partial t}|_{t=0} \mathcal{R}(g)(M(t)z) \quad = \quad \frac{\partial}{\partial z_i} \mathcal{R}(g)(z) P^j_i z_j.$$

On the other hand, the right-hand side of (2.2.31) computes to

$$\begin{aligned}
&\frac{\partial}{\partial t}|_{t=0} \mathcal{R}(g \circ (M(t))^{-T})(z) \\
= \quad &\frac{\partial}{\partial t}|_{t=0} \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} g((M(t))^{-T} \theta^\kappa \xi_\kappa) \mathrm{d}\mathscr{H}^{m-1}(\theta) \\
= \quad &\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})} \int_{\mathbb{S}^{m-1}} \frac{\partial}{\partial t}|_{t=0} g((M(t))^{-T} \theta^\kappa \xi_\kappa) \mathrm{d}\mathscr{H}^{m-1}(\theta)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \frac{\partial g}{\partial y^i}(\theta^\kappa \xi_\kappa)\frac{\partial}{\partial t}|_{t=0}\left((M(t))^{-1}\right)^i_j(\theta^\kappa\xi^j_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)\\
&\overset{\text{Thm. 1.1.5 + chain rule}}{=} \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \frac{\partial g}{\partial y^i}(\theta^\kappa \xi_\kappa)\\
&\quad\cdot\left(-\left((M(t))^{-1}\right)^i_q(\frac{\partial}{\partial t}(M^q_r(t))^T)\left((M(t))^{-1}\right)^r_j\right)|_{t=0}(\theta^\kappa\xi^j_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)\\
&= \frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \frac{\partial g}{\partial y^i}(\theta^\kappa \xi_\kappa)\left(-P^i_j\right)(\theta^\kappa\xi^j_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)\\
&= -P^i_j\frac{1}{\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}} \frac{\partial g}{\partial y^i}(\theta^\kappa \xi_\kappa)(\theta^\kappa\xi^j_\kappa)\mathrm{d}\mathscr{H}^{m-1}(\theta)\\
&= -P^i_j\mathcal{R}\left(y^j\frac{\partial g}{\partial y^i}\right)(z),
\end{aligned}
$$

wherein the second equality is admissible, because analogous to the proof of Lemma 2.2.4, we can express the Hausdorff integral as a sum of Lebesgue integrals over domains in $\mathbb{R}^{m-1}$ using the area formula of Theorem 1.2.7, a covering family of coordinate charts and apply on each of these Lebesgue integrals [For09, §11 Satz 2, p. 99]. □

The following theorem was established by Bailey et al. (see [BEGM03, p. 581]) for $m = 2$.

Theorem 2.2.14 (cf. [BEGM03, p. 581]). *Let g be a $(-m)$-homogeneous and continuously differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $g \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$. For the Radon-transform $\mathcal{R}$ holds*

$$
z_i\frac{\partial}{\partial z_j}\mathcal{R}(g)(z) \quad = \quad \mathcal{R}\left(-\frac{\partial}{\partial y^i}(y^j g)\right)(z), \tag{2.2.32}
$$

wherein $z = (z_j) \in \mathbb{R}^{m+1}\backslash\{0\}$.

Proof. Let g be a $(-m)$-homogeneous and continuously differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $g \in \mathrm{HF}^1_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$. By Corollary 2.2.7 we know that $\mathcal{R}(g)$ is one times differentiable. Due to Euler's theorem (see Theorem 1.4.3), the $(-m)$-homogeneity of g and the (-1)-homogeneity of $\mathcal{R}(g)$, we get

$$
\begin{aligned}
y^i\frac{\partial}{\partial y^i}g &= -mg,\\
z_i\frac{\partial}{\partial z_i}\mathcal{R}(g) &= -\mathcal{R}(g).
\end{aligned}
$$

Choose the trace-free matrix

$$
P^l_k = \delta^k_i\delta^j_l - \frac{1}{m+1}\delta^k_l\delta^j_i,
$$

for $i, j, k, l \in \mathbb{N}$ with $1 \le i, j, k, l \le m+1$. Therewith, we get

$$
\begin{aligned}
P^l_k z_l\frac{\partial}{\partial z_k}\mathcal{R}(g) &= z_j\frac{\partial}{\partial z_i}\mathcal{R}(g) - \frac{1}{m+1}z_l\frac{\partial}{\partial z_l}\mathcal{R}(g)\delta^j_i\\
&= z_j\frac{\partial}{\partial z_i}\mathcal{R}(g) + \frac{1}{m+1}\mathcal{R}(g)\delta^j_i,\\
\mathcal{R}(-P^l_k y^k\frac{\partial}{\partial y^l}g) &= \mathcal{R}\left(-y^i\frac{\partial}{\partial y^j}g + \frac{1}{m+1}y^l\frac{\partial}{\partial y^l}g\delta^j_i\right)\\
&= \mathcal{R}\left(-y^i\frac{\partial}{\partial y^j}g - \frac{m}{m+1}g\delta^j_i\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{R}\left(-\frac{\partial}{\partial y^j}(y^i g) + \frac{1}{m+1} g \delta_i^j\right) \\
&= \mathcal{R}\left(-\frac{\partial}{\partial y^j}(y^i g)\right) + \frac{1}{m+1}\mathcal{R}(g)\,\delta_i^j,
\end{aligned}
$$

which together with Theorem 2.2.12 concludes the proof. □

The following theorem results from iterated use of Theorem 2.2.14.

Theorem 2.2.15. *For a $(-m)$-homogeneous and k-times differentiable function g on $\mathbb{R}^{m+1}\backslash\{0\}$ the (-1)-homogeneous function $\mathcal{R}(g)$ is k-times differentiable on $\mathbb{R}^{m+1}\backslash\{0\}$ and for their derivatives holds*

$$
z_{i_1} \dots z_{i_k} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(g) \quad = \quad (-1)^k \mathcal{R}\left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{j_1} \dots y^{j_k} g)\right).
$$

Remark 2.2.16. By means of Theorem 2.2.15 it is possible to compute the tangential derivatives of $\widehat{\mathcal{R}}(f)$ on $\mathbb{S}^m$ from the tangential derivatives of f on $\mathbb{S}^m$.

Proof of Theorem 2.2.15. We prove the assertion by induction. For $k = 1$ it is true due to Corollary 2.2.8 and Theorem 2.2.14. Now we make the step $k \to k+1$. Assume that the claim holds for a chosen k. Thereby, we can rewrite the following expression

$$
\begin{aligned}
& z_{i_1} \dots z_{i_{k+1}} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_{k+1}}} \mathcal{R}(g) + \sum_{l=1}^{k} \delta_{j_{k+1}}^{i_l} z_{i_1} \dots \widehat{z_{i_l}} \dots z_{i_{k+1}} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(g) \\
= \;& z_{i_{k+1}} \frac{\partial}{\partial z_{j_{k+1}}} \left(z_{i_1} \dots z_{i_k} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(g)\right) \\
= \;& z_{i_{k+1}} \frac{\partial}{\partial z_{j_{k+1}}} \left((-1)^k \mathcal{R}\left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{j_1} \dots y^{j_k} g)\right)\right)
\end{aligned}
$$

Again we use Corollary 2.2.8, the fact that $\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{j_1} \dots y^{j_k} g)$ is continuously differentiable on $\mathbb{R}^{m+1}\backslash\{0\}$ for $g \in \mathrm{HF}_{-m}^{k+1}(\mathbb{R}^{m+1}\backslash\{0\})$ and Theorem 2.2.14 to deduce

$$
\begin{aligned}
& z_{i_1} \dots z_{i_{k+1}} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_{k+1}}} \mathcal{R}(g) + \sum_{l=1}^{k} \delta_{j_{k+1}}^{i_l} z_{i_1} \dots \widehat{z_{i_l}} \dots z_{i_{k+1}} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(g) \\
&= (-1)^{k+1} \mathcal{R}\left(\frac{\partial}{\partial y^{i_{k+1}}} \left(y^{j_{k+1}} \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (y^{j_1} \dots y^{j_k} g)\right)\right) \\
&\overset{\text{Lem. 2.2.19}}{=} (-1)^{k+1} \mathcal{R}\left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \frac{\partial}{\partial y^{i_{k+1}}} (y^{j_1} \dots y^{j_k} y^{j_{k+1}} g)\right) \\
&+ \sum_{l=1}^{k} (-1)^k \delta_{j_{k+1}}^{i_l} \mathcal{R}\left(\frac{\partial}{\partial y^{i_1}} \cdots \widehat{\frac{\partial}{\partial y^{i_l}}} \cdots \frac{\partial}{\partial y^{i_{k+1}}} (y^{j_1} \dots y^{j_k} g)\right) \\
&= (-1)^{k+1} \mathcal{R}\left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \frac{\partial}{\partial y^{i_{k+1}}} (y^{j_1} \dots y^{j_k} y^{j_{k+1}} g)\right) \\
&+ \sum_{l=1}^{k} \delta_{j_{k+1}}^{i_l} z_{i_1} \dots \widehat{z_{i_l}} \dots z_{i_{k+1}} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(g).
\end{aligned}
$$

So, the assertion follows. □

The following lemma gives a rule for repeated differentiation of products of functions. For this result, we refer to [Ada75, p. 2].

Lemma 2.2.17 (**Leibniz rule** [Ada75, p. 2])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f, g : \Omega \to \mathbb{R}$ k-times differentiable at $y \in \Omega$. Then there holds*

$$D^\alpha (f \cdot g) \quad = \quad \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g.$$

for any multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

Remark 2.2.18.

- For two multi-indices $\alpha = (\alpha_1, \ldots \alpha_m)$, $\beta = (\beta_1, \ldots \beta_m) \in \mathbb{N}_0^n$ the inequality $\beta \leq \alpha$ is meant componentwise, i.e. $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.
- For two multi-indices $\alpha = (\alpha_1, \ldots \alpha_m)$, $\beta = (\beta_1, \ldots \beta_m) \in \mathbb{N}_0^n$ with $\beta \leq \alpha$, the binomial coefficient is defined by

$$\binom{\alpha}{\beta} \quad := \quad \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

The following lemma is an alternative non-standard representation of the result in Lemma 2.2.17.

Lemma 2.2.19 (**Leibniz rule**)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f, g : \Omega \to \mathbb{R}$ k-times differentiable at $y \in \Omega$. Then there holds*

$$\begin{aligned} & \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} (f \cdot g)(y) \\ = \quad & \sum_{t=0}^{k} \sum_{\sigma \in S_k} \frac{1}{t!(k-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right)(y) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} g \right)(y) \end{aligned}$$

where S_k is the symmetric group of order $k \in \mathbb{N}$.

Proof. We will prove the assertion by induction. We omit the dependency on the actual point $y \in \Omega$ and think of it as being implicitly set. Further, we set

$$\frac{\partial}{\partial y^{i_{s_1}}} \cdots \frac{\partial}{\partial y^{i_{s_t}}} f \quad := \quad f$$

every time when there is some i s.t. $s_i \notin \{1, \ldots, k\}$. A permutation $\sigma \in S_k$ is extended to the set of integers $\mathbb{Z}$ by defining $\sigma(j) := j$ for all $j \in \mathbb{Z} \backslash \{1, \ldots, k\}$. We start the proof by verifying the assertion for $\underline{k = 1}$:

$$\begin{aligned} \frac{\partial}{\partial y^{i_1}} f \quad &= \quad \left(\frac{\partial}{\partial y^{i_1}} f \right) g + f \left(\frac{\partial}{\partial y^{i_1}} g \right) \\ &= \quad \sum_{t=0}^{1} \sum_{\sigma \in S_1} \frac{1}{t!(1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(1)}}} g \right) \end{aligned}$$

In the next step, we assume that the assertion holds true for a distinct $k \in \mathbb{N}$ and we show that thereby the assertion holds true for $\underline{k+1}$:

$$\begin{aligned} & \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \frac{\partial}{\partial y^{i_{k+1}}} (f \cdot g) \\ = \quad & \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} \left(\left(\frac{\partial}{\partial y^{i_{k+1}}} f \right) g + f \left(\frac{\partial}{\partial y^{i_{k+1}}} g \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=0}^{k} \sum_{\sigma \in S_k} \frac{1}{t!(k-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} \frac{\partial}{\partial y^{i_{k+1}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} g \right) \\
&\quad + \sum_{t=0}^{k} \sum_{\sigma \in S_k} \frac{1}{t!(k-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} \frac{\partial}{\partial y^{i_{k+1}}} g \right) \\
&= \sum_{t=1}^{k+1} \sum_{\sigma \in S_k} \frac{1}{(t-1)!(k+1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t-1)}}} \frac{\partial}{\partial y^{i_{k+1}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} g \right) \\
&\quad + \sum_{t=0}^{k} \sum_{\sigma \in S_k} \frac{1}{t!(k-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} \frac{\partial}{\partial y^{i_{k+1}}} g \right) \\
&= \sum_{t=1}^{k} \sum_{\sigma \in S_k} \frac{1}{(t-1)!(k+1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t-1)}}} \frac{\partial}{\partial y^{i_{k+1}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} g \right) \\
&\quad + \sum_{t=1}^{k} \sum_{\sigma \in S_k} \frac{1}{t!(k-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} \frac{\partial}{\partial y^{i_{k+1}}} g \right) \\
&\quad + f \left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_{k+1}}} g \right) + \left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_{k+1}}} f \right) g \\
&= \sum_{t=1}^{k} \sum_{\substack{\sigma \in S_{k+1} \\ \sigma^{-1}(k+1) \leq t}} \frac{1}{t!(k+1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k+1)}}} g \right) \\
&\quad + \sum_{t=1}^{k} \sum_{\substack{\sigma \in S_{k+1} \\ \sigma^{-1}(k+1) > t}} \frac{1}{t!(k+1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k+1)}}} g \right) \\
&\quad + f \left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_{k+1}}} g \right) + \left(\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_{k+1}}} f \right) g \\
&= \sum_{t=0}^{k+1} \sum_{\sigma \in S_{k+1}} \frac{1}{t!(k+1-t)!} \left(\frac{\partial}{\partial y^{i_{\sigma(1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(t)}}} f \right) \left(\frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k+1)}}} g \right)
\end{aligned}$$

So, the assertion follows for all $k \in \mathbb{N}$ by induction. □

The following theorem shows boundedness of the spherical Radon transform on spaces of functions with higher order derivatives. It is a consequence of Theorem 2.2.15.

Theorem 2.2.20 (**Boundedness of the spherical Radon transform**). *Let f be a $(-m)$-homogeneous and k-times differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $f \in \mathrm{HF}^k_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$. Then the spherical Radon transform of f is a (-1)-homogeneous and k-times differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $\mathcal{R}(f) \in \mathrm{HF}^k_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$, and satisfies*

$$\widehat{\rho}_k(\mathcal{R}(f)) \quad \leq \quad \widehat{C}(m,k)\widehat{\rho}_k(f), \tag{2.2.33}$$

wherein $\widehat{C}(m,k) := (m+1)^k \sum_{t=0}^{k} \frac{k!}{(k-t)!}\binom{k}{t}$ is a constant depending on k and m only. Especially, by (2.2.33), the spherical Radon transform is a bounded linear operator from the Fréchet space $\mathrm{HF}^\infty_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ to the Fréchet space $\mathrm{HF}^\infty_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$. Therein, the Fréchet spaces $\mathrm{HF}^\infty_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ and $\mathrm{HF}^\infty_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$ are equipped with the family of seminorms $\{\widehat{\rho}_k\}_{k\in\mathbb{N}_0}$.

Corollary 2.2.21. *Let f be a $(-m)$-homogeneous and k-times differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $f \in \mathrm{HF}^k_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$. Then the spherical Radon transform of f is a (-1)-homogeneous and k-times differentiable function on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e. $\mathcal{R}(f) \in \mathrm{HF}^k_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$,* *and satisfies*

$$\rho_k(\mathcal{R}(f)) \quad \leq \quad C(m,k)\rho_k(f), \tag{2.2.34}$$

wherein $C(m,k) := 2^{(k+1)^2}\widehat{C}(m,k)$ *is a constant depending on* k *and* m *only and* $\widehat{C}(m,k)$ *is the constant given in Theorem 2.2.20. Especially, by (2.2.34), the spherical Radon transform is a bounded linear operator from the Fréchet space* $\mathrm{HF}^{\infty}_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ *to the Fréchet space* $\mathrm{HF}^{\infty}_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$*, where both of these function spaces can be interpreted as closed subspaces of the Fréchet space* $\mathrm{C}^{\infty}(\mathbb{R}^{m+1}\backslash\{0\})$ *equipped with the Fréchet topology induced by the family of seminorms* $\{\rho_k\}_{k\in\mathbb{N}_0}$*.*

Proof. The differentiability of $\mathcal{R}(f)$ is part of Theorem 2.2.20. Further, by (1.2.7) and (2.2.34), we get the following chain of inequalities

$$\begin{aligned}\rho_k(\mathcal{R}(f)) \quad &\leq \quad 2^{(k+1)^2}\widehat{\rho}_k(\mathcal{R}(f))\\ &\leq \quad 2^{(k+1)^2}\widehat{C}(m,k)\widehat{\rho}_k(f) \quad \leq \quad 2^{(k+1)^2}\widehat{C}(m,k)\rho_k(f).\end{aligned}$$

The rest follows by the definition of Fréchet spaces (see Definition 1.2.11). □

Proof of Theorem 2.2.20. The differentiability of $\mathcal{R}(f)$ is part of Theorem 2.2.15. To estimate the seminorms of $\mathcal{R}(f)$ by seminorms of $f \in \mathrm{HF}^{k}_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ we use again Theorem 2.2.15 and the following inequality derived from Definition 2.2.1 (cf. proof of Corollary 2.2.5):

$$\begin{aligned}|\mathcal{R}(f)(z)| \quad &= \quad \left|\frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^{m-1}\cap\frac{z}{|z|}^{\perp}} f(w)\mathrm{d}\mathscr{H}^{m-1}(w)\right|\\ &\leq \quad \frac{1}{|z|}\max_{y\in\mathbb{S}^m}|f(y)|\end{aligned} \tag{2.2.35}$$

for $z \in \mathbb{R}^{m+1}\backslash\{0\}$. So, Theorem 2.2.15 and (2.2.35) together yield

$$\begin{aligned}&\left|\frac{\partial}{\partial z_{j_1}}\dots\frac{\partial}{\partial z_{j_k}}\mathcal{R}(f)(z)\right|\\ =\ &\left|(-1)^k\frac{z_{i_1}}{|z|^2}\dots\frac{z_{i_k}}{|z|^2}\mathcal{R}\left(\frac{\partial}{\partial y^{i_1}}\dots\frac{\partial}{\partial y^{i_k}}(y^{j_1}\dots y^{j_k}f)\right)(z)\right|\\ \leq\ &\frac{(m+1)^k}{|z|^k}\max_{1\leq i_1,\dots,i_k\leq m+1}\left|\mathcal{R}\left(\frac{\partial}{\partial y^{i_1}}\dots\frac{\partial}{\partial y^{i_k}}(y^{j_1}\dots y^{j_k}f)\right)(z)\right|\\ \leq\ &\frac{(m+1)^k}{|z|^{k+1}}\max_{1\leq i_1,\dots,i_k\leq m+1}\max_{\zeta\in\mathbb{S}^m}\left|\frac{\partial}{\partial y^{i_1}}\dots\frac{\partial}{\partial y^{i_k}}(y^{j_1}\dots y^{j_k}f(y))|_{y=\zeta}\right|\\ =\ &\frac{(m+1)^k}{|z|^{k+1}}\max_{1\leq i_1\leq\dots\leq i_k\leq m+1}\max_{\zeta\in\mathbb{S}^m}\left|\left(\frac{\partial}{\partial y^{i_1}}\dots\frac{\partial}{\partial y^{i_k}}(y^{j_1}\dots y^{j_k}f(y))\right)|_{y=\zeta}\right|\end{aligned} \tag{2.2.36}$$

On the other hand, we compute

$$\begin{aligned}&\left|\frac{\partial}{\partial y^{i_1}}\dots\frac{\partial}{\partial y^{i_k}}(y^{j_1}\dots y^{j_k}f(y))\right|\\ \overset{\text{Th. 2.2.19}}{=}\ &\left|\sum_{t=0}^{k}\sum_{\sigma\in S_k}\frac{1}{t!(k-t)!}\frac{\partial}{\partial y^{i_{\sigma(1)}}}\dots\frac{\partial}{\partial y^{i_{\sigma(t)}}}(y^{j_1}\dots y^{j_k})\frac{\partial}{\partial y^{i_{\sigma(t+1)}}}\dots\frac{\partial}{\partial y^{i_{\sigma(k)}}}(f(y))\right|\\ \leq\ &\sum_{t=0}^{k}\sum_{\sigma\in S_k}\frac{1}{t!(k-t)!}\frac{k!}{(k-t)!}|y|^{k-t}\left|\frac{\partial}{\partial y^{i_{\sigma(t+1)}}}\dots\frac{\partial}{\partial y^{i_{\sigma(k)}}}(f(y))\right|\\ =\ &\sum_{t=0}^{k}\frac{k!}{(k-t)!}|y|^{k-t}\sum_{\sigma\in S_k}\frac{1}{t!(k-t)!}\left|\frac{\partial}{\partial y^{i_{\sigma(t+1)}}}\dots\frac{\partial}{\partial y^{i_{\sigma(k)}}}(f(y))\right|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{k} \frac{k!}{(k-t)!}|y|^{k-t}|y|^{-m-k+t} \sum_{\sigma\in S_k} \frac{1}{t!(k-t)!} \left| \frac{\partial}{\partial y^{i_{\sigma(t+1)}}} \cdots \frac{\partial}{\partial y^{i_{\sigma(k)}}} (f(\frac{y}{|y|})) \right| \\
&\leq \sum_{t=0}^{k} \frac{k!}{(k-t)!} \binom{k}{t} |y|^{-m} \max_{1\leq s_1\leq \ldots \leq s_{k-t}\leq m+1} \max_{\zeta\in\mathbb{S}^m} \left| \left(\frac{\partial}{\partial y^{s_1}} \cdots \frac{\partial}{\partial y^{s_{k-t}}} f(y) \right) |_{y=\zeta} \right| \\
&\leq |y|^{-m} \sum_{t=0}^{k} \frac{k!}{(k-t)!} \binom{k}{t} \widehat{\rho}_{k-t}(f)
\end{aligned}
$$

for $y = (y^i) \in \mathbb{R}^{m+1}\backslash\{0\}$. Notice that we used therein

$$
\left| \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_t}} (y^{j_1} \cdot \ldots \cdot y^{j_k}) \right| \quad \leq \quad \frac{k!}{(k-t)!} |y|^{k-t}
$$

for every choice $y = (y^i) \in \mathbb{R}^{m+1}\backslash\{0\}$, $t, k \in \mathbb{N}_0$ with $t \leq k$ and indices $i_r, j_q \in \{0, \ldots, m+1\}$ for $r = 1, \ldots, t$ and $q = 1, \ldots, k$. This can be shown by induction. Thereby, we get

$$
\left| \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_k}} \mathcal{R}(f)(z) \right| \quad \leq \quad \frac{(m+1)^k}{|z|^{k+1}} \sum_{t=0}^{k} \frac{k!}{(k-t)!} \binom{k}{t} \widehat{\rho}_{k-t}(f) \tag{2.2.37}
$$

and this leads to

$$
\begin{aligned}
& \widehat{\rho}_k(\mathcal{R}(f)) \\
= \; & \max_{l\in\{0,\ldots,k\}} \max_{0\leq j_1\leq\ldots\leq j_l\leq m+1} \max_{z\in\mathbb{S}^m} \left| \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_l}} \mathcal{R}(f)(z) \right| \\
\overset{(2.2.37)}{\leq} \; & \max_{l\in\{0,\ldots,k\}} (m+1)^l \sum_{t=0}^{l} \frac{l!}{(l-t)!} \binom{l}{t} \widehat{\rho}_{l-t}(f) \\
\leq \; & \left(\max_{l\in\{0,\ldots,k\}} (m+1)^l \sum_{t=0}^{l} \frac{l!}{(l-t)!} \binom{l}{t} \right) \widehat{\rho}_k(f) \\
= \; & \left((m+1)^k \sum_{t=0}^{k} \frac{k!}{(k-t)!} \binom{k}{t} \right) \widehat{\rho}_k(f) \\
=: \; & \widehat{C}(m,k) \widehat{\rho}_k(f).
\end{aligned}
$$

Notice that the equality

$$
\max_{l\in\{0,\ldots,k\}} \left((m+1)^l \sum_{t=0}^{l} \frac{l!}{(l-t)!} \binom{l}{t} \right) \quad = \quad (m+1)^k \sum_{t=0}^{k} \frac{k!}{(k-t)!} \binom{k}{t}
$$

can be shown easily, as the expression $(m+1)^l \sum_{t=0}^{l} \frac{l!}{(l-t)!} \binom{l}{t}$ is increasing in $l \in \{0, \ldots, k\}$. □

The following lemma basically states that odd functions are annihilated by the spherical Radon transform.

Lemma 2.2.22 (Kernel [Gro96, Prop. 3.4.12]**).** *The Kernel of the spherical Radon transform $\widehat{\mathcal{R}}$ is made up by the continuous odd functions on $\mathbb{S}^m$, i.e.*

$$
\ker(\widehat{\mathcal{R}}) \quad = \quad \{g \in C^0(\mathbb{S}^m) : g(\zeta) = -g(-\zeta) \text{ for all } \zeta \in \mathbb{S}^m\}. \tag{2.2.38}
$$

Corollary 2.2.23 (Kernel). *The Kernel of the spherical Radon transform $\mathcal{R}$ is made up by the continuous $(-m)$-homogeneous odd functions on $\mathbb{R}^{m+1}\backslash\{0\}$, i.e.*

$$\ker(\mathcal{R}) \quad = \quad \{g \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) : g(z) = -g(-z) \text{ for all } z \in \mathbb{R}^{m+1}\backslash\{0\}\}. \tag{2.2.39}$$

Proof. The corollary is a direct consequence of Lemma 2.2.22. □

Corollary 2.2.24 (Injectivity). *Let $k \in \mathbb{N}_0$. The spherical Radon transform*

$$\mathcal{R} : \mathrm{HF}^k_{-m}(\mathbb{R}^{m+1}\backslash\{0\}) \to \mathrm{HF}^k_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$$

restricted to the subspace of even functions, i.e.

$$\mathcal{R}|_{\mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})} : \mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\}) \to \mathrm{HF}^k_{-1,e}(\mathbb{R}^{m+1}\backslash\{0\}),$$

is injective.

Proof. We start by recapitulating the fact that the image of the spherical Radon transform is a subspace of the space of continuous (-1)-homogeneous and even functions on $\mathbb{R}^{m+1}\backslash\{0\}$ (see Lemma 2.2.3). As the intersection $\ker(\mathcal{R}) \cap \mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\}) = \{0\}$ only contains the zero function, we conclude that

$$\ker(\mathcal{R}|_{\mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})}) \quad = \quad \{0\}.$$

Choose now $f_1, f_2 \in \mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})$ and assume $\mathcal{R}(f_1) = \mathcal{R}(f_2)$. By the linearity of $\mathcal{R}$ follows that $\mathcal{R}(f_1 - f_2) = 0$. Hence,

$$f_1 - f_2 \in \ker(\mathcal{R}|_{\mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})}) \quad = \quad \{0\}.$$

So, $f_1 - f_2 = 0$ and therefore $f_1 = f_2$. This proves the injectivity of $\mathcal{R}$ restricted to $\mathrm{HF}^k_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})$. □

The following proposition, which can be found in a book by Groemer [Gro96], can be seen as a first step towards surjectivity of the spherical Radon transform.

Proposition 2.2.25 ([Gro96, Prop. 3.6.4]). *Let $m \geq 1$ and f be a real valued even function on $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. If f is $2\lfloor(m+2)/2\rfloor$-times continuously differentiable (i.e. $f \in C^{2\lfloor(m+2)/2\rfloor}(\mathbb{S}^m)$ with $f(\zeta) = f(-\zeta)$ for all $\zeta \in \mathbb{S}^m$), there is an even continuous function g on $\mathbb{S}^m$ (i.e. $g \in C^0(\mathbb{S}^m)$ with $g(\zeta) = g(-\zeta)$ for all $\zeta \in \mathbb{S}^m$) such that*

$$f \quad = \quad \widehat{\mathcal{R}}(g).$$

Notice that $\lfloor\cdot\rfloor$ is the floor function.

An adaption of the proof of [Gro96, Prop. 3.6.4] yields an improved result in form of the following proposition.

Proposition 2.2.26. *Let $m \geq 1$ and f be a real valued even function on $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. If f is $2\lfloor(m+3)/2\rfloor$-times continuously differentiable (i.e. $f \in C^{2\lfloor(m+3)/2\rfloor}(\mathbb{S}^m)$ with $f(\zeta) = f(-\zeta)$ for all $\zeta \in \mathbb{S}^m$), there is an even one times continuously differentiable function g on $\mathbb{S}^m$ (i.e. $g \in C^1(\mathbb{S}^m)$ with $g(\zeta) = g(-\zeta)$ for all $\zeta \in \mathbb{S}^m$) such that*

$$f \quad = \quad \widehat{\mathcal{R}}(g).$$

Proof. This result follows by adapting the proof given by Groemer in [Gro96, Prop 3.6.4]. There, he starts with a so called condensed expansion of the sperical function f, namely

$$f \quad \sim \quad \sum_{n=0}^{\infty} Q_n$$

where the series converges in $L^2(\mathbb{S}^m)$ to f and the Q_n are spherical harmonics of degree n, which means restrictions to $\mathbb{S}^m$ of harmonic polynoms on $\mathbb{R}^{m+1}$ of degree n.[1] Further, he showed in [Gro96, Proposition 3.6.4] that if the series

$$g \quad \sim \quad \sum_{n=0}^{\infty} \frac{1}{\nu_{m+1,n}} Q_n \tag{2.2.40}$$

converges uniformly to g, the limit g is a continuous function, whose image under $\mathcal{R}$ yields f. The coefficients $\nu_{m+1,n}$ are defined in [Gro96, Lemma 3.4.7] as follows

$$\nu_{m+1,n} \quad := \quad \begin{cases} (-1)^{n/2} \frac{1 \cdot 3 \cdot \ldots \cdot (n-1)}{m \cdot (m+2) \cdot \ldots \cdot (m+n-2)} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

That the expressions in (2.2.40) converge properly is shown by computing an estimate, which uses the degree of differentiability of f (denoted by $2k_1 \in \mathbb{N}_0$) to gain control over the convergence of the series:

$$\frac{1}{|\nu_{m+1,n}|} |Q_n| \quad \leq \quad C_{1,m+1} ||\Delta_o^{k_1} f||_{L^2} n^{m-2k_1-1}.$$

Herein is $\Delta_o f := \Phi_0(\Delta \Phi_0^{-1}(f))$ the Laplace-Beltrami operator on $\mathbb{S}^m$ and $C_{1,m+1}$ a constant depending on $m+1$. So, we have uniform convergence for $m - 2k_1 - 1 < -1$, which is fulfilled for $k_1 > m/2$. In a next step, we intend to do the same kind of procedure for

$$\frac{1}{|\nu_{m+1,n}|} |\nabla_o Q_n|,$$

wherein $\nabla_o f := (\Phi_0(\nabla_{y^i} \Phi_0^{-1}(f))) e_i$. We choose a L^2-orthonormal basis $(H_j)_{j=1,\ldots,N}$ for the finite dimensional subspace $\mathcal{H}_n^{m+1}$ of spherical harmonics of degree n on $\mathbb{S}^m$. Its dimension will be denoted by $N = N(m,n)$ (see [Gro96, Theorem 3.1.4]). It follows that

$$Q_n(u) \quad = \quad \sum_{j=1,\ldots,N} a_j H_j(u)$$

with $a_j := \langle Q_n, H_j \rangle_{L^2}$. Then by using [Gro96, Theorem 3.3.3], [Gro96, Theorem 3.2.11] and the proof of [Gro96, Proposition 3.6.2] we get

$$\begin{aligned} |\nabla_o Q_n(u)|^2 \quad &= \quad |\sum_{j=1,\ldots,N} a_j \nabla_o H_j(u)|^2 \\ &\leq \quad \sum_{j=1,\ldots,N} a_j^2 \sum_{j=1,\ldots,N} |\nabla_o H_j(u)|^2 \\ &\leq \quad \sum_{j=1,\ldots,N} a_j^2 \sum_{j=1,\ldots,N} -\langle \Delta_o H_j(u), H_j(u) \rangle \\ &\leq \quad n(n+m-1) \sum_{j=1,\ldots,N} a_j^2 \sum_{j=1,\ldots,N} \langle H_j(u), H_j(u) \rangle \end{aligned}$$

[1] A function $f : \mathbb{S}^m \to \mathbb{R}$ is an element of $L^2(\mathbb{S}^m)$ if and only if $||f||_{\mathbb{S}^m} := \int_{\mathbb{S}^m} |f|^2 \mathrm{d}A < \infty$. $L^2(\mathbb{S}^m)$ together with the norm $|| \cdot ||_{\mathbb{S}^m}$ is then a Banach space.

$$\leq \quad n(n+m-1)C_{2,m+1}||\Delta_o^{k_1} f||_{L^2}^2 n^{m-4k_1-1}$$
$$\leq \quad \tilde{C}_{2,m+1}||\Delta_o^{k_1} f||_{L^2}^2 n^{m-4k_1+1}$$

By [Gro96, Lemma 3.4.8] one has $|1/\nu_{m+1,n}| \leq C_{3,m+1} n^{(m-1)/2}$ where $C_{3,m+1}$ depends on m only. This finally gives the desired estimate

$$\frac{1}{|\nu_{m+1,n}|}|\nabla_o Q_n| \quad \leq \quad C_{4,m+1}||\Delta_o^{k_1} f||_{L^2} n^{m-2k_1} \tag{2.2.41}$$

where $C_{4,m+1}$ depends on m only. So, the series in (2.2.40) converges uniformly if $m-2k_1 < -1$, which means that $k_1 > (m+2)/2$. Therefore, we choose $k_1 = [(m+1)/2]$. Then due to [For11, Satz5, §21, p. 258] adapted to spherical functions (i.e. by means of local coordinate charts and compactness of $\mathbb{S}^m$), the uniform convergence of

$$g \quad = \quad \sum_{n\geq 0, n \text{ even}} \frac{1}{\nu_{m+1,n}} Q_n \quad \text{and} \quad h \quad = \quad \sum_{n\geq 0, n \text{ even}} \frac{1}{\nu_{m+1,n}} \nabla_o Q_n,$$

we deduce that g is continuously differentiable with $\nabla_o g = h$. Further, $f = \widehat{\mathcal{R}}(g)$. □

Theorem 2.2.27. *Let $m \geq 1$ and f be a real valued even and (-1)-homogeneous function on $\mathbb{R}^{m+1}\backslash\{0\}$. If $l \in \mathbb{N} \cup \{\infty\}$ and f is $2[(m+2)/2]+l$-times continuously differentiable (i.e. $f \in \mathrm{HF}_{e,-1}^{2[(m+2)/2]+l}(\mathbb{R}^{m+1}\backslash\{0\})$), then there exists an even $(-m)$-homogeneous and l-times continuously differentiable function g on $\mathbb{R}^{m+1}\backslash\{0\}$ (i.e. $g \in \mathrm{HF}_{e,-m}^{l}(\mathbb{R}^{m+1}\backslash\{0\})$) such that*

$$f \quad = \quad \mathcal{R}(g).$$

Proof. Lets start with the case $l = 1$. By Proposition 2.2.26 for the even spherical function $\widehat{f} := \Phi_{-1}(f) = f|_{\mathbb{S}^m}$, there exists a continuously differentiable even spherical function $\widehat{g}$ s.t.

$$\widehat{f} \quad = \quad \widehat{\mathcal{R}}(\widehat{g}),$$

where we used that $2[(m+2)/2]+1 \geq 2[(m+3)/2]$. After defining the continuously differentiable even $(-m)$-homogeneous function $g := \Phi_{-m}^{-1}(\widehat{g})$ we get

$$\begin{aligned} f \quad &= \quad \Phi_{-1}^{-1}(\widehat{f}) \\ &= \quad \Phi_{-1}^{-1} \circ \widehat{\mathcal{R}} \circ \Phi_{-m} \circ \Phi_{-m}^{-1}(\widehat{g}) \\ &= \quad \mathcal{R}(\Phi_{-m}^{-1}(\widehat{g})) \\ &= \quad \mathcal{R}(g). \end{aligned}$$

Now we conclude the induction by assuming that the statement of Theorem 2.2.27 holds true for l and we will prove that the assertion holds for the next integer $l+1$. Then for an $2[(m+2)/2]+l+1$-times continuously differentiable even (-1)-homogeneous function f the even (-1)-homogeneous function $z_i \frac{\partial}{\partial z_j} f$ is continuously differentiable of degree $2[(m+2)/2]+l$ for all $i, j \in \{1, \ldots, m+1\}$. Then by induction assumption there exists an l-times differentiable h_i^j such that

$$z_i \frac{\partial}{\partial z_j} f \quad = \quad \mathcal{R}(h_i^j)$$

and further, due to Theorem 2.2.14

$$z_i \frac{\partial}{\partial z_j} f \quad = \quad \mathcal{R}(\frac{\partial}{\partial y^i}(y^j g))$$

for $f = \mathcal{R}(g)$, where g is continuously differentiable (see the first step of the induction). Further, $\mathcal{R}$ is injective on $\mathrm{HF}^0_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})$ such that $\frac{\partial}{\partial y^i}(y^j g)$ and h^j_i have to coincide. Further, g is l-times differentiable also by the induction hypothesis. So, we get

$$y^j \frac{\partial g}{\partial y^i} + \delta^j_i g \quad = \quad \frac{\partial}{\partial y^i}(y^j g) \quad = \quad h^j_i$$

and by contracting it with y^j yields

$$\frac{\partial g}{\partial y^i} \quad = \quad |y|^{-2}(y_j h^j_i - y_i g),$$

where on the right hand side of the equation are only terms present, which are at least l-times differentiable on $\mathbb{R}^{m+1}\backslash\{0\}$. Therefore, g is continuously differentiable of degree $l+1$. □

In the following theorem, we collect some of the attained results about the spherical Radon transform.

Theorem 2.2.28 (Bijective and bounded in Fréchet topology). *The spherical Radon transform $\mathcal{R}$ is a bijective bounded linear operator from the Fréchet space $\mathrm{HF}^\infty_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})$ onto the Fréchet space $\mathrm{HF}^\infty_{-1,e}(\mathbb{R}^{m+1}\backslash\{0\})$. The inverse of the spherical Radon transform will be denoted by $\mathcal{T}$ and is also a bounded linear mapping.*

Proof. The injectivity is due to Corollary 2.2.23. The surjectivity comes from Theorem 2.2.27 by setting $l = \infty$ there. The bijectivity induces the existence of the inverse transform $\mathcal{T}$. The boundedness of the spherical Radon transform has been shown in Theorem 2.2.20. The boundedness of the inverse operator $\mathcal{T}$ can be shown through the application of the open mapping theorem for Fréchet spaces as can be found in [Rud91, 2.12 Corollaries, p. 46]. □

The following theorem is stated just for the sake of completeness.

Theorem 2.2.29. *For a (-1)-homogeneous, even and infinitely times differentiable function f on $\mathbb{R}^{m+1}\backslash\{0\}$ (i.e. $f \in \mathrm{HF}^\infty_{-1,e}(\mathbb{R}^{m+1}\backslash\{0\})$) the $(-m)$-homogeneous and even function $\mathcal{T}(f)$ is infinitely times differentiable on $\mathbb{R}^{m+1}\backslash\{0\}$ (i.e. $\mathcal{T}(f) \in \mathrm{HF}^\infty_{-m,e}(\mathbb{R}^{m+1}\backslash\{0\})$) and for their derivatives holds*

$$y^{i_1}\dots y^{i_{k+1}} \frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_{k+1}}} \mathcal{T}(f) \quad = \quad (-1)^k \mathcal{T}\left(\frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_{k+1}}}(z_{j_1}\dots z_{j_{k+1}} f)\right).$$

Proof. The proof will be carried out by induction with respect to k. For a function $f \in \mathrm{HF}^\infty_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$, there is by Theorem 2.2.28 an unique function $g \in \mathrm{HF}^\infty_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ such that $f = \mathcal{R}(g)$. By Theorem 2.2.15 we get

$$z_i \frac{\partial}{\partial z_j} f \quad = \quad z_i \frac{\partial}{\partial z_j}\mathcal{R}(g) \quad = \quad -\mathcal{R}\left(\frac{\partial}{\partial y^i}(y^j g)\right) \quad = \quad -\mathcal{R}\left(\frac{\partial}{\partial y^i}(y^j \mathcal{T}(f))\right).$$

Applying the inverse spherical Radon transform on the former equality leads to

$$\frac{\partial}{\partial y^i}\left(y^j \mathcal{T}(f)\right) \quad = \quad -\mathcal{T}\left(z_i \frac{\partial}{\partial z_j} f\right),$$

and carrying out differentiation yields

$$\begin{aligned} y^j \frac{\partial}{\partial y^i}\mathcal{T}(f) \quad &= \quad -\mathcal{T}\left(z_i \frac{\partial}{\partial z_j} f\right) - \delta^i_j \mathcal{T}(f) \\ &= \quad -\mathcal{T}\left(\delta^i_j f + z_i \frac{\partial}{\partial z_j} f\right) \end{aligned}$$

$$= \quad -\mathcal{T}\left(\frac{\partial}{\partial z_j}(z_i f)\right). \tag{2.2.42}$$

So, we proved the assertion for $k = 1$. Now we assume that the assertion holds for k, what we call the induction hypothesis **(IH)**, and show the assertion for $k+1$. So, we look at

$$\begin{aligned}
& y^{i_1}\dots y^{i_{k+1}}\frac{\partial}{\partial y^{j_1}}\cdots\frac{\partial}{\partial y^{j_{k+1}}}\mathcal{T}(f) \\
= \quad & y^{i_1}\frac{\partial}{\partial y^{j_1}}\left(y^{i_2}\dots y^{i_{k+1}}\frac{\partial}{\partial y^{j_2}}\cdots\frac{\partial}{\partial y^{j_{k+1}}}\mathcal{T}(f)\right) \\
& - y^{i_1}\frac{\partial}{\partial y^{j_1}}\left(y^{i_2}\dots y^{i_{k+1}}\right)\frac{\partial}{\partial y^{j_2}}\cdots\frac{\partial}{\partial y^{j_{k+1}}}\mathcal{T}(f) \\
= \quad & y^{i_1}\frac{\partial}{\partial y^{j_1}}\left(y^{i_2}\dots y^{i_{k+1}}\frac{\partial}{\partial y^{j_2}}\cdots\frac{\partial}{\partial y^{j_{k+1}}}\mathcal{T}(f)\right) \\
& - y^{i_1}\sum_{s=2}^{k+1}(y^{i_2}\dots y^{i_{s-1}}\cdot\delta^{i_s}_{j_1}\cdot y^{i_{s+1}}\dots y^{i_{k+1}})\frac{\partial}{\partial y^{j_2}}\cdots\frac{\partial}{\partial y^{j_{k+1}}}\mathcal{T}(f) \\
\overset{\textbf{(IH)}}{=} \quad & y^{i_1}\frac{\partial}{\partial y^{j_1}}\left((-1)^k\mathcal{T}\left(\frac{\partial}{\partial z_{i_2}}\cdots\frac{\partial}{\partial z_{i_{k+1}}}(z_{j_2}\dots z_{j_{k+1}}f)\right)\right) \\
& + (-1)^{k+1}\mathcal{T}\left(\sum_{s=2}^{k+1}\delta^{i_s}_{j_1}\frac{\partial}{\partial z_{i_1}}\frac{\partial}{\partial z_{i_2}}\cdots\frac{\partial}{\partial z_{i_{s-1}}}\frac{\partial}{\partial z_{i_{s+1}}}\cdots\frac{\partial}{\partial z_{i_{k+1}}}(z_{j_2}\dots z_{j_{k+1}}f)\right) \\
\overset{(2.2.42)}{=} & (-1)^{k+1}\mathcal{T}\left(\frac{\partial}{\partial z_{i_1}}\left(z_{j_1}\frac{\partial}{\partial z_{i_2}}\cdots\frac{\partial}{\partial z_{i_{k+1}}}(z_{j_2}\dots z_{j_{k+1}}f)\right)\right) \\
& + (-1)^{k+1}\mathcal{T}\left(\sum_{s=2}^{k+1}\delta^{i_s}_{j_1}\frac{\partial}{\partial z_{i_1}}\frac{\partial}{\partial z_{i_2}}\cdots\frac{\partial}{\partial z_{i_{s-1}}}\frac{\partial}{\partial z_{i_{s+1}}}\cdots\frac{\partial}{\partial z_{i_{k+1}}}(z_{j_2}\dots z_{j_{k+1}}f)\right) \\
\overset{\text{Lem. 2.2.19}}{=} & (-1)^{k+1}\mathcal{T}\left(\frac{\partial}{\partial z_{i_1}}\cdots\frac{\partial}{\partial z_{i_{k+1}}}(z_{j_1}\dots z_{j_{k+1}}f)\right).
\end{aligned}$$

This concludes the proof. □

Remark 2.2.30.

- Theorem 2.2.29 leads not to a similar boundedness statement like the one for the spherical Radon transform due to the lack of a simple C^0-estimate for the inverse spherical Radon transform. Nevertheless, to the author's opinion it should be possible to extract some estimate from the proof of Proposition 2.2.26 and use this estimate to show the boundedness without using the open mapping theorem (cf. proof of Theorem 2.2.28).
- The transformation behaviour of $\mathcal{T}$ under $\mathrm{SL}(m+1)$ can be shown to be of the very same form as the one of the spherical Radon transform:

  $$\mathcal{T}(f)\circ L \quad = \quad \mathcal{T}(f\circ L^{-T})$$

  for all $L \in \mathrm{SL}(m+1)$ and $f \in \mathrm{HF}^{\infty}_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$. This is a direct consequence of Corollary 2.2.11.

2.3 Finsler area in terms of the spherical Radon transform

In this section, we will identify the Finsler area integrand as a Cartan integrand in the sense of section 1.6, investigate it regarding convexity in the second argument and establish some closeness results. A starting point to the convexity discussion is a theorem by Busemann [Bus49], which he established in the context of his investigation of convex bodies of the Brunn-Minkowski type. It will be one of the core tools to our argumentation as well as the representation of the Finsler area integrand in terms of the spherical Radon transform (see Corollary 2.3.1). Especially the boundedness of the spherical Radon transform and its inverse will become useful. Both facts will lead to a ellipticity result for the Finsler area integrand seen as a Cartan integrand as well as the aforementioned closeness result. The closeness result essentially states that if two Finsler metrics are close in some C^k topology, then the corresponding Finsler area integrands are close in such a topology.

The following corollary is a direct consequence of the basis representation of the spherical Radon transform (see Lemma 2.2.6) and a similar construction for the Finsler area that we stated in (2.1.22). This connection between Finsler area integrand and spherical Radon transoform is the reason for us to investigate the latter. The spherical Radon transform will be especially useful to investigate how the Finsler area depends on its underlying Finsler metric.

Corollary 2.3.1 (**Spherical Radon transform and Finsler area**). *Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set and $F : \mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfy* (**F1**)*. Then*

$$A^F(x, z) \quad = \quad \frac{1}{\mathcal{R}(F^{-m}(x, \cdot))(z)} \tag{2.3.1}$$

for $(x, z) \in \Omega \times \big(\mathbb{R}^{m+1}\backslash\{0\}\big)$. Remember, if we choose an orthonormal basis (f_i) of the orthogonal complement $z^\perp$ to $z \in \mathbb{R}^{m+1}\backslash\{0\}$, then holds

$$A^F(x, z) \quad = \quad \frac{|z|\mathcal{H}^m(\mathbb{S}^{m-1})}{\int_{\mathbb{S}^{m-1}} F^{-m}(x, \theta^i f_i)\mathrm{dS}(\theta)}$$

by Lemma 2.1.14.

Proof. The assertion follows by combining the identities (2.1.22) and (2.2.7). □

The following theorem establishes uniqueness of the m-symmetrization $F_{(m)}$ as a reversible function generating the Finsler area.

Theorem 2.3.2 (**Uniqueness of the symmetrization**). *Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set and $F : \mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfy* (**F1**)*. The m-symmetrization $F_{(m)}$ is the unique reversible function generating the same Finsler area as F, i.e. for all reversible $G : \mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfying* (**F1**) *and $A^G \equiv A^F$ holds $G \equiv F_{(m)}$ on $\mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1}$.*

Proof. This follows directly from the injectivity property of the spherical Radon transform established in Corollary 2.2.24 and the representation of the Finsler area integrand in terms of the spherical Radon transform (see Corollary 2.3.1). □

The following corollary deals especially with the regularity of the Finsler area A^F given some regularity of F.

Corollary 2.3.3 (**Cartan integrand and regularity**). *Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set and $F : \mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfy* (**F1**)*. Then is A^F a Cartan integrand, i.e. $A^F \in C^0(\Omega \times \mathbb{R}^{m+1})$ and* (**H**) *holds that is*

$$A^F(x, tz) \quad = \quad tA^F(x, z) \quad \text{for all } t > 0 \text{ and } (x, z) \in \Omega \times \mathbb{R}^{m+1}.$$

Further, since (**F1**) *implies that*

$$F(x, y) \quad > \quad 0 \quad \text{for all } (x, y) \in \Omega \times \mathbb{R}^{m+1},$$

then is

$$A^F(x,z) \quad > \quad 0 \quad \text{for all } (x,z) \in \Omega \times \mathbb{R}^{m+1}.$$

If for some $k \in \mathbb{N}_0$ holds

$$F \in C^k(\Omega \times (\mathbb{R}^{m+1} \backslash \{0\})$$

then is

$$A^F \in C^k(\Omega \times (\mathbb{R}^{m+1} \backslash \{0\})$$

and even more if for some $k \in \mathbb{N}_0$ and $\alpha > 0$ holds

$$F \in C^{k,\alpha}_{\text{loc}}(\Omega \times (\mathbb{R}^{m+1} \backslash \{0\})$$

then is

$$A^F \in C^{k,\alpha}_{\text{loc}}(\Omega \times (\mathbb{R}^{m+1} \backslash \{0\}).$$

Proof. This follows directly by combining Corollary 2.3.1 and **(F1)** with Corollary 2.2.7, Lemma 2.2.9 and Theorem 2.2.20. □

The following theorem was established by Busemann in [Bus49] for $m = 2$ and straightforwardly generalized to $m \geq 2$ by Milman and Pajor in [MP89]. We also refer for the proof to a book of Thompson [Tho96, Theorem 7.1.1]. Notice that Busemann originally investigated geometric quantities as appear in the fractional expression in (2.1.21). He studied these in the context of convex bodies of Brunn-Minkowski type.

Theorem 2.3.4 (**Busemann's convexity theorem** [Tho96, Theorem 7.1.1])**.** *Let $\mathbb{R}^{m+1}$ be equipped with a reversible weak C^0-Minkowski metric $F : \mathbb{R}^{m+1} \to [0,\infty)$ (i.e. $F(y) = F(-y)$ for all $y \in \mathbb{R}^{m+1}$ like in (**s**)), which satisfies **(F2)** and **(M)**. Then the Finsler area integrand $A^F(z)$ is a semi-elliptic Cartan integrand, hence (**H**) and (**C**) hold.*

Remark 2.3.5. The former theorem is of interest, because if the weak C^0-Minkowski metric F is reversible, i.e. $F(y) = F(-y)$ for all $y \in \mathbb{R}^{m+1}$, then we automatically get convexity of the Finsler area integrand. We will need some kind of improvement of this theorem to gain ellipticity, what is needed on the application side.

The following corollary is a direct consequance of Theorem 2.3.4.

Corollary 2.3.6 (**Semi-ellipticity**)**.** *Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set and $F : \mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1} \to [0,\infty)$ be a function satisfying **(GA1)**. Then the Finsler area integrand $A^F(x,z)$ is a semi-elliptic Cartan integrand, hence (**H**) and (**C**) hold.*

Proof. We start by identifying $\mathrm{T}\Omega \cong \Omega \times \mathbb{R}^{m+1}$. A^F is homogeneous as a direct consequence of the representation as 0-homogeneous quotient of volumes times the Euclidean area integrand in (2.1.21). As $A^F = A^{F_{(m)}}$ holds by means of (2.1.24) and F satisfies **(GA1)**, we can restrict our investigation to $F_{(m)}$ as the Finsler area generating function $F_{(m)}$ is a weak Finsler metric on Ω, i.e. $F_{(m)}$ is even (see (**s**)) and satisfies **(F2)**. Thus, for a fixed $x_0 \in \Omega$ the function $H : \mathbb{R}^{m+1} \to [0,\infty)$ defined by

$$H(y) \quad := \quad F_{(m)}(x_0, y)$$

for all $y \in \mathbb{R}^{m+1}$, can be seen as a reversible weak C^0-Minkowski metric. Hence, we can apply Theorem 2.3.4 to the present situation. This yields the convexity of A^F in the second argument. So, A^F is semi-elliptic. □

The following lemma quantifies the convexity or more precisely computes the ellipticity constants in comparison of two Finsler area integrands. We prove it after we stated some corollaries.

Lemma 2.3.7 (**Ellipticity comparison of Finsler area integrands**). *Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set endowed with two reversible Finsler metrics F and G. Then for every $x \in \Omega$ exist constants $\Lambda_i(x) = \Lambda_i(F, G, x)$ for $i = 1, 2$ so that*

$$\Lambda_1(x)\xi^i A^G_{z^i z^j}(x,z)\xi^j \quad \leq \quad \xi^i A^F_{z^i z^j}(x,z)\xi^j \quad \leq \quad \Lambda_2(x)\xi^i A^G_{z^i z^j}(x,z)\xi^j, \tag{2.3.2}$$

holds for all $\xi \in \mathbb{R}^{m+1}$ and all $z \in \mathbb{R}^{m+1}\backslash\{0\}$.

The following corollary is a consequence of Lemma 2.3.7. If we set therein $G(x, y) = |y|$, we get as a result ellipticity constants from below and above as in Lemma 1.6.6.

Corollary 2.3.8 (**Ellipticity of the Finsler area integrands**). *Let $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfy* (**GA2**)*. The Finsler area integrand A^F is an elliptic Cartan integrand of class $C^\infty(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}\backslash\{0\})$, which is positive on $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}\backslash\{0\}$.*

Notice especially that F and $F_{(m)}$ being both Finsler metrics on $T\mathbb{R}^{m+1}$, i.e. F satisfies (**GA3**), directly implies (**GA2**). Hence, the following corollary is a special case of Corollary 2.3.8.

Corollary 2.3.9. *Let $(\mathbb{R}^{m+1}, F)$ and $(\mathbb{R}^{m+1}, F_{(m)})$ be Finsler manifolds, i.e. F satisfies* (**GA3**)*. The corresponding Finsler area integrand $A^F \equiv A^{F_{(m)}}$ is then elliptic.*

Proof of Corollary 2.3.8. F satisfies (**F1**) by (**GA2**). So, the m-symmetrization $F_{(m)}$ is well-defined, satisfies (**F1**) and is thereby at least as regular as F. Further, (**GA2**) implies that $F_{(m)}$ satisfies (**F3**). The positivity is a direct consequence of F satisfying (**F1**) and Corollary 2.3.3, the parametric property (**H**) of A^F follows by (2.1.21) and A^F is least as regular as F by Corollary 2.3.3. The ellipticity is due to Lemma 2.3.7 by setting $G(x, y) = |y|$ the Euclidean metric and the fact that the expression

$$\Lambda_1(R) \quad := \quad \inf_{x \in B^{m+1}_R(0)} \inf_{\xi \in \mathbb{S}^m \cap z^\perp} \xi^i A^F_{z^i z^j}(x,z)\xi^j$$

depends continuously on R as the coefficients $A^F_{z^i z^j}(x, z)$ depend continuously on x, z and ξ (parametric integrals). Further, as the infima of $A^F_{z^i z^j}(x, z)$, the infima are attained for a special choice of points x_0, z_0 and ξ_0 such that

$$\Lambda_1(R) \quad = \quad \xi_0^i A^F_{z^i z^j}(x_0, z_0)\xi_0^j.$$

So, by Lemma 2.3.7 holds

$$\Lambda_1(R) \quad = \quad \xi_0^i A^F_{z^i z^j}(x_0,z_0)\xi_0^j \quad \geq \quad \Lambda_1(x_0)\xi_0^i A^E_{z^i z^j}(x_0,z_0)\xi_0^j \quad = \quad \Lambda_1(x_0) \quad > \quad 0.$$

On the other hand, we have

$$\begin{aligned}
\xi^i A^F_{z^i z^j}(x,z)\xi^j \quad &= \quad \xi_1^i A^F_{z^i z^j}(x,z)\xi_1^j + 2\xi_1^i A^F_{z^i z^j}(x,z)\xi_2^j + \xi_2^i A^F_{z^i z^j}(x,z)\xi_2^j \\
&= \quad \xi_1^i A^F_{z^i z^j}(x,z)\xi_1^j \\
&= \quad \frac{|\xi_1|^2}{|z|}\frac{\xi_1^i}{|\xi_1|} A^F_{z^i z^j}(x, \frac{z}{|z|})\frac{\xi_1^j}{|\xi_1|} \\
&\geq \quad \frac{\Lambda_1(R)}{|z|}|\xi_1|^2 \\
&= \quad \frac{\Lambda_1(R)}{|z|}(|\xi|^2 - \langle \frac{z}{|z|}, \xi\rangle^2) \\
&= \quad \xi^i A^E_{z^i z^j}(x,z)\xi^j
\end{aligned}$$

for arbitrary $\xi = \xi_1 + \xi_2$ with $\xi_1 \perp z$ and $\xi_2 \in \operatorname{span}\{z\}$ and $(x, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}\backslash\{0\}$. ☐

Proof of Lemma 2.3.7. We start by choosing a fixed $x \in \mathbb{R}^{m+1}$. As F and G are reversible Finsler metrics, they are both especially positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. So, we can assume by Corollary 2.1.8 that there are constants $m_i = m_i(x) > 0$ for $i = 1, 2$ such that

$$m_1(x)G(x,y) \quad \leq \quad F(x,y) \quad \leq \quad m_2(x)G(x,y)$$

for all $y \in \mathbb{R}^{m+1}$. By Corollary 2.3.1 we have a representation of the Finsler area integrand as a fractional expression in the spherical Radon transform. So, we need to show that there is a positive constant $\Lambda = \Lambda(x, F, G, m) > 0$ such that $A^F(x,\cdot) - \Lambda A^G(x,\cdot)$ is convex in the second argument. Choose some $z \in \mathbb{R}^{m+1}\backslash\{0\}$. To prove existence of such a constant, we look at the expressions

$$\begin{aligned} A^F(x,z) - \delta A^G(x,z) &= \frac{1}{\mathcal{R}(F^{-m}(x,\cdot))(x,z)} - \delta\frac{1}{\mathcal{R}(G^{-m}(x,\cdot))(x,z)} \\ &= \frac{\mathcal{R}(G^{-m}(x,\cdot))(x,z) - \delta\mathcal{R}(F^{-m}(x,\cdot))(x,z)}{\mathcal{R}(F^{-m}(x,\cdot))(x,z)\mathcal{R}(G^{-m}(x,\cdot))(x,z)}. \end{aligned}$$

Especially, it holds that

$$\begin{aligned} A^F(x,z) - \delta A^G(x,z) &= \frac{1}{\mathcal{R}(F^{-m}(x,\cdot))(x,z)} - \delta\frac{1}{\mathcal{R}(G^{-m}(x,\cdot))(x,z)} \\ &\geq (m_1^m(x) - \delta)A^G(x,z) \quad > \quad 0 \end{aligned}$$

for $\delta < m_1^m(x)$. On each element of the sequences the inverse Radon transform will be applied to obtain a new sequence of positive even $(-m)$-homogeneous functions on $\mathbb{R}^{m+1}\backslash\{0\}$, namely

$$\varphi_{x,\delta}(y) \quad := \quad \mathcal{T}(\phi_{x,\delta}(\cdot))(y),$$

for all $y \in \mathbb{R}^{m+1}\backslash\{0\}$, where we define the even, (-1)-homogeneous function

$$\begin{aligned} \phi_{x,\delta}(\cdot) &:= \frac{1}{A^F(x,\cdot) - \delta A^G(x,\cdot)} \\ &= \frac{\mathcal{R}(F^{-m}(x,\cdot))\mathcal{R}(G^{-m}(x,\cdot))}{\mathcal{R}(G^{-m}(x,\cdot)) - \delta\mathcal{R}(F^{-m}(x,\cdot))} \end{aligned}$$

for fixed $x \in \mathbb{R}^{m+1}$. Notice that $\phi_{x,\delta}(\cdot)$ is well-defined on $\mathbb{R}^{m+1}\backslash\{0\}$ for $\delta < 1/m_1^m(x)$ as the nominator is positive. If the parameter $\delta \in [0, 1/m_1^m(x))$ tends to zero, the sequence of functions $\phi_{x,\delta}(\cdot)$ tends with respect to the Fréchet topology on $\mathrm{HF}^{\infty}_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ to

$$\mathcal{R}(F^{-m}(x,\cdot))$$

for every $x \in \mathbb{R}^{m+1}$. Remember that the inverse Radon transform has been identified in Theorem 2.2.28 to be a bounded operator on suitable Fréchet spaces of homogeneous functions. Therefrom, we know that for every non-negative integer $l \in \mathbb{N}_0$ there exists a non-negative integer $N = N(m,l) \in \mathbb{N}_0$ and a positive constant $C = C(m,l) > 0$ such that

$$\widehat{\rho}_l(\mathcal{T}(f)) \quad \leq \quad C\widehat{\rho}_N(f)$$

for all $f \in \mathrm{HF}^{\infty}_{-1}(\mathbb{R}^{m+1}\backslash\{0\})$. Therefore and due to the convergence of $\phi_{x,\delta}(\cdot)$, we deduce that $\varphi_{x,\delta}(\cdot)$ tends with respect to the Fréchet topology on $\mathrm{HF}^{\infty}_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$ to

$$\varphi_{x,0}(\cdot) \quad = \quad \mathcal{T}\Big(\mathcal{R}(F^{-m}(x,\cdot))\Big)(\cdot) \quad = \quad \frac{1}{F^m(x,\cdot)}$$

for δ tending to zero and for $x \in \mathbb{R}^{m+1}$ fixed. Since

$$\frac{1}{F^m(x,y)} \quad \geq \quad \frac{1}{m_2^m(x)}\frac{1}{G^m(x,y)} \quad > \quad 0$$

and $\varphi_{x,\delta}(\cdot)$ converges to $F^{-m}(x,\cdot)$ as δ converges to zero, there is some positive constant $\delta_0 = \delta_0(x, F, G, m) > 0$ such that $\varphi_{x,\delta}(y) > 0$ for $y \in \times\mathbb{R}^{m+1}\backslash\{0\}$. As $\varphi_{x,\delta}$ are positive valued for $\delta < \delta_0$, we are able to define

$$F_{x,\delta}(\cdot) \quad := \quad \varphi_{x,\delta}^{-\frac{1}{m}}(\cdot)$$

for $x \in \mathbb{R}^{m+1}$. By construction holds the following

$$\begin{aligned} A^{F_\delta}(x,\cdot) &= \mathcal{R}^{-1}(F_{x,\delta}^{-m}(\cdot)) \\ &= \mathcal{R}^{-1}(\varphi_{x,\delta}(\cdot)) \\ &= \mathcal{R}^{-1}(\mathcal{T}(\phi_{x,\delta}(\cdot))) \\ &= \phi_{x,\delta}^{-1}(\cdot) \\ &= A^F(x,\cdot) - \delta A^G(x,\cdot). \end{aligned}$$

So, the Finsler area integrand of $F_{x,\delta}$ is actually the expression, we intend to investigate. By choosing δ_0 again sufficiently small we can even assure that $F_{x,\delta}(\cdot)$ is sufficiently close to $F(x,\cdot)$ up to second order such that $g_{ij}^{F_{x,\delta}}(\cdot)$ is close enough to $g_{ij}^F(x,\cdot)$ to be positive definite on its own. Since $F_{x,\delta}(\cdot)$ is then especially convex, we can apply Theorem 2.3.4 to the present situation and get that $A^{F_{x,\delta}}(\cdot) = A^F(x,\cdot) - \delta A^G(x,\cdot)$ is convex for $\delta < \delta_0 = \delta_0(x, F, G, m)$. □

The following lemma is a closeness estimate for Finsler area integrands. It essentially states that two Finsler area integrands are close in a C^k topology, if the corresponding Finsler metrics are C^k close.

Lemma 2.3.10 (Finsler area closeness estimates). *Let $\mathbb{R}^{m+1}$ be equipped with two functions $F, G : \mathrm{T}\mathbb{R}^{m+1} \equiv \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0,\infty)$ with $F, G \in C^k(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ and both satisfying* (**F1**)*. There is a constant $C = C(m,k)$ such that for every $x \in \mathbb{R}^{m+1}$ holds*

$$\widehat{\rho}_k(A^G(x,\cdot) - A^F(x,\cdot)) \leq C\widehat{\rho}_k(G(x,\cdot) - F(x,\cdot))\left(\frac{\tilde{\rho}_k(F(x,\cdot))}{F_0(x)}\frac{\tilde{\rho}_k(G(x,\cdot))}{G_0(x)}\right)^{k^2+(m+1)k+m}$$

where we set $F_0(x) := \min\{1, \min_{\mathbb{S}^m} F(x,\cdot)\}$, $G_0(x) := \min\{1, \min_{\mathbb{S}^m} G(x,\cdot)\}$ and $\tilde{\rho}_k(f) := \max\{1, \widehat{\rho}_k(f)\}$ for a function $f \in C^k(\mathbb{R}^{m+1}\backslash\{0\})$.

Proof. We start by mentioning some general observations for functions $f, g, h \in C^k(\mathbb{R}^{m+1}\backslash\{0\})$ with $f, g > 0$ on $\mathbb{R}^{m+1}\backslash\{0\}$. In the following, we will denote with $C(m,k)$ generic constants depending on m and k that may change from line to line. By the product rule (see Lemma 2.2.17) we have

$$\mathrm{D}^\alpha(fg) \quad = \quad \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta f \mathrm{D}^{\alpha-\beta} g \tag{2.3.3}$$

for a multi-index α with $|\alpha| \leq k$, so that we deduce

$$|\mathrm{D}^\alpha(fg)| \quad \leq \quad \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\widehat{\rho}_{|\beta|}(f)\widehat{\rho}_{|\alpha-\beta|}(g).$$

Hence, we get

$$|\mathrm{D}^\alpha(fg)| \quad \leq \quad (\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta})\widehat{\rho}_k(f)\widehat{\rho}_k(g) \quad =: \quad C(m,k)\widehat{\rho}_k(f)\widehat{\rho}_k(g)$$

and consequently

$$\widehat{\rho}_k(fg) \quad \leq \quad C(m,k)\widehat{\rho}_k(f)\widehat{\rho}_k(g). \tag{2.3.4}$$

Iterating the former inequalities leads to

$$|\mathrm{D}^\alpha(f^l)| \quad \leq \quad C^l(m,k)\widehat{\rho}_k^l(f)$$

for $l \in \mathbb{N}$, what gives

$$\widehat{\rho}_k(f^l) \quad \leq \quad C^l(m,k)\widehat{\rho}_k^l(f).$$

Claim: For all $k,l \in \mathbb{N}$, there is a constant $C = C(m,k,l)$ such that

$$|\mathrm{D}^\alpha(\frac{1}{f^l})| \quad \leq \quad C(m,k,l)f_0^{-(l+k)}\tilde{\rho}_k^k(f) \tag{2.3.5}$$

for a multi-index α with $|\alpha| \leq k$. Therein, we set $f_0 := \min\{1, \min_{\mathbb{S}^m} f\}$ and recall $\tilde{\rho}_k(f) := \max\{1, \widehat{\rho}_k(f)\}$. We show this claim by induction over k. For $k = 0$ we do have

$$|\frac{1}{f^l}| \quad \leq \quad \frac{1}{(\min\{1, \min_{\mathbb{S}^m} f\})^l} = \frac{1}{f_0^l},$$

so that the claim holds for this special choice of k. Now, say that the claim holds for $0, \dots, k$, what we will call the induction hypothesis, i.e. (IH). We will show that it then holds also for $k+1$. Therefore, we choose a multi-index α with $|\alpha| \leq k+1$ and represent it as $\alpha = \bar{\alpha} + e_j$ for $\bar{\alpha} \leq k$ and a conveniently chosen standard basis vector e_j with $j \in \{1, \dots, m+1\}$. Thereby, we get

$$\begin{aligned}
|\mathrm{D}^\alpha(\frac{1}{f^l})| &= |\mathrm{D}^{\bar{\alpha}}(\partial_j(\frac{1}{f^l}))| \\
&= |\mathrm{D}^{\bar{\alpha}}(-\frac{1}{f^{l+1}}\partial_j f)| \\
&\overset{(2.3.3)}{=} |-\sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} \mathrm{D}^\beta(f^{-(l+1)})\mathrm{D}^{\bar{\alpha}-\beta}(\partial_j f)| \\
&= |-\sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} \mathrm{D}^\beta(f^{-(l+1)})\mathrm{D}^{\bar{\alpha}+e_j-\beta}(f)| \\
&\leq \sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} |\mathrm{D}^\beta(f^{-(l+1)})||\mathrm{D}^{\alpha-\beta}(f)| \\
&\overset{(\mathrm{IH})}{\leq} \sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} C(m,|\beta|,l+1)f_0^{-(l+1+|\beta|)}\tilde{\rho}_{|\beta|}^{|\beta|}(f)\tilde{\rho}_{|\alpha-\beta|}(f) \\
&\leq \left(\sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} C(m,|\beta|,l+1)\right) f_0^{-(l+k+1)}\tilde{\rho}_k^k(f)\tilde{\rho}_{k+1}(f) \\
&\leq \left(\sum_{0\leq\beta\leq\bar{\alpha}} \binom{\bar{\alpha}}{\beta} C(m,|\beta|,l+1)\right) f_0^{-(l+k+1)}\tilde{\rho}_{k+1}^{k+1}(f) \\
&=: C(m,k+1,l)f_0^{-(l+k+1)}\tilde{\rho}_{k+1}^{k+1}(f),
\end{aligned}$$

what concludes the proof of the claim.

As a consequence of (2.3.3) and (2.3.5), we estimate

$$|\mathrm{D}^\alpha(\frac{h}{fg})| \overset{(2.3.3)}{=} |\sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta(h)\mathrm{D}^{\alpha-\beta}(\frac{1}{fg})|$$

$$
\begin{aligned}
&\overset{(2.3.3)}{=} |\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta(h)\sum_{0\le\gamma\le\alpha-\beta}\binom{\alpha-\beta}{\gamma}\mathrm{D}^\gamma(\frac{1}{f})\mathrm{D}^{\alpha-\beta-\gamma}(\frac{1}{g})| \\
&\overset{(2.3.5)}{\le} \sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}|\mathrm{D}^\beta(h)|\sum_{0\le\gamma\le\alpha-\beta}\binom{\alpha-\beta}{\gamma}C(m,|\gamma|,1)f_0^{-(k+1)}\tilde{\rho}_{|\gamma|}^{|\gamma|}(f) \\
&\quad \cdot C(m,|\alpha-\beta-\gamma|,1)g_0^{-(k+1)}\tilde{\rho}_{|\alpha-\beta-\gamma|}^{|\alpha-\beta-\gamma|}(g) \\
&\le \quad C(m,k)\widehat{\rho}_k(h)\frac{\tilde{\rho}_k^k(f)}{f_0^{k+1}}\frac{\tilde{\rho}_k^k(g)}{g_0^{k+1}} \qquad (2.3.6)
\end{aligned}
$$

for a multi-index α with $|\alpha|\le k$. Even more, we have

$$
\begin{aligned}
& |D^\alpha(f^{-m}-g^{-m})| \\
= \;& |-D^\alpha((f-g)\sum_{t=0}^{m-1}f^{t-m}g^{-1-t})| \\
= \;& |\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta(f-g)\mathrm{D}^{\alpha-\beta}(\sum_{t=0}^{m-1}f^{t-m}g^{-1-t}))| \\
= \;& |\sum_{t=0}^{m-1}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta(f-g)\mathrm{D}^{\alpha-\beta}(f^{t-m}g^{-1-t}))| \\
\overset{(2.3.3)}{=} & |\sum_{t=0}^{m-1}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta(f-g)\sum_{0\le\gamma\le\alpha-\beta}\binom{\alpha-\beta}{\gamma}(\mathrm{D}^\gamma f^{t-m}\mathrm{D}^{\alpha-\beta-\gamma}g^{-1-t}))| \\
\overset{(2.3.5)}{=} & \sum_{t=0}^{m-1}\sum_{0\le\beta\le\alpha}\sum_{0\le\gamma\le\alpha-\beta}\binom{\alpha}{\beta}\binom{\alpha-\beta}{\gamma}|\mathrm{D}^\beta(f-g)| \\
& \cdot C(m,|\gamma|,m-t)C(m,|\alpha-\beta-\gamma,1+t)\frac{\tilde{\rho}_{|\gamma|}^{|\gamma|}(f)}{f_0^{m-t+|\gamma|}}\frac{\tilde{\rho}_{|\alpha-\beta-\gamma|}^{|\alpha-\beta-\gamma|}(g)}{g_0^{t+1+|\alpha-\beta-\gamma|}} \\
\le \;& C(m,k)\widehat{\rho}_k(f-g)\frac{\tilde{\rho}_k^k(f)}{f_0^{m+k}}\frac{\tilde{\rho}_k^k(g)}{g_0^{m+k}} \qquad (2.3.7)
\end{aligned}
$$

for a multi-index α with $|\alpha|\le k$. By using the definition of the spherical Radon transform (see Definition 2.2.1) we get

$$
\begin{aligned}
\mathcal{R}(F^{-m}(x,\cdot))(\zeta) &= \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^m\cap\zeta^\perp}F^{-m}(x,\theta)\mathrm{d}\mathcal{H}^{m-1}(\theta) \\
&\ge \frac{1}{\mathcal{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^m\cap\zeta^\perp}\min_{\xi\in\mathbb{S}^m}F^{-m}(x,\xi)\mathrm{d}\mathcal{H}^{m-1}(\theta) \\
&= \min_{\xi\in\mathbb{S}^m}F^{-m}(x,\xi) \\
&= (\max_{\xi\in\mathbb{S}^m}F(x,\xi))^{-m} \\
&= (\widehat{\rho}_0(F(x,\cdot))^{-m}
\end{aligned}
$$

for $\zeta\in\mathbb{S}^m$ and taking the minimum over the unit sphere of the left-hand side yields

$$
\min_{\xi\in\mathbb{S}^m}\mathcal{R}(F^{-m}(x,\cdot))(\xi) \quad = \quad (\widehat{\rho}_0(F(x,\cdot))^{-m}.
$$

Thereby, we get

$$
\min\{1,\min_{\mathbb{S}^m}\mathcal{R}(F^{-m}(x,\cdot))\} \quad \ge \quad (\max\{1,\max_{\zeta\in\mathbb{S}^m}F(x,\zeta)\})^{-m} \quad = \quad \tilde{\rho}_0^{-m}(F(x,\cdot)) \qquad (2.3.8)
$$

and in the same way $\min\{1, \min_{\mathbb{S}^m} \mathcal{R}(G^{-m}(x,\cdot))\} \geq \tilde{\rho}_0^{-m}(G(x,\cdot))$. By (2.2.25), where we choose $g \equiv 1$, we get $|\mathcal{R}(f)(\zeta)| \leq \widehat{\rho}_0(|f|)$ for $\zeta \in \mathbb{S}^m$ and so

$$\widehat{\rho}_0(\mathcal{R}(f)) \quad \leq \quad \widehat{\rho}_0(f). \tag{2.3.9}$$

Now we do have the required tools for showing the thesis of the theorem. First we rewrite by using (2.3.1):

$$\begin{aligned}\widehat{\rho}_k(A^F(x,\cdot) - A^G(x,\cdot)) &= \widehat{\rho}_k(\mathcal{R}^{-1}(F^{-m}(x,\cdot)) - \mathcal{R}^{-1}(G^{-m}(x,\cdot))) \\ &= \widehat{\rho}_k(\frac{\mathcal{R}(G^{-m}(x,\cdot)) - \mathcal{R}(F^{-m}(x,\cdot))}{\mathcal{R}(F^{-m}(x,\cdot))\mathcal{R}(G^{-m}(x,\cdot))})\end{aligned}$$

Afterwards applying (2.3.6) and (2.3.8) yields

$$\begin{aligned}\widehat{\rho}_k(A^F(x,\cdot) - A^G(x,\cdot)) \quad &\leq \quad C(m,k)\widehat{\rho}_k(\mathcal{R}(G^{-m}(x,\cdot)) - \mathcal{R}(F^{-m}(x,\cdot))) \\ &\quad \cdot \frac{\tilde{\rho}_k^k(\mathcal{R}(F^{-m}(x,\cdot)))}{\tilde{\rho}_0^{-m(k+1)}(F(x,\cdot))} \frac{\tilde{\rho}_k^k(\mathcal{R}(G^{-m}(x,\cdot)))}{\tilde{\rho}_0^{-m(k+1)}(G(x,\cdot))}.\end{aligned}$$

Using further (2.3.9), (2.3.7) and (2.3.5) yields

$$\begin{aligned}&\widehat{\rho}_k(A^F(x,\cdot) - A^G(x,\cdot)) \\ \overset{(2.3.9)}{\leq}\ & C(m,k)\widehat{\rho}_k(G^{-m}(x,\cdot) - F^{-m}(x,\cdot)) \\ &\cdot \tilde{\rho}_k^k(F^{-m}(x,\cdot))\tilde{\rho}_0^{m(k+1)}(F(x,\cdot))\tilde{\rho}_k^k(G^{-m}(x,\cdot))\tilde{\rho}_0^{m(k+1)}(G(x,\cdot)) \\ \overset{(2.3.7)}{\leq}\ & C(m,k)\widehat{\rho}_k(G(x,\cdot) - F(x,\cdot))\frac{\tilde{\rho}_k^k(F(x,\cdot))}{F_0^{m+k}(x)} \frac{\tilde{\rho}_k^k(G(x,\cdot))}{G_0^{m+k}(x)} \\ &\cdot \tilde{\rho}_k^k(F^{-m}(x,\cdot))\tilde{\rho}_0^{m(k+1)}(F(x,\cdot))\tilde{\rho}_k^k(G^{-m}(x,\cdot))\tilde{\rho}_0^{m(k+1)}(G(x,\cdot)) \\ \overset{(2.3.5)}{\leq}\ & C(m,k)\widehat{\rho}_k(G(x,\cdot) - F(x,\cdot))\frac{\tilde{\rho}_k^k(F(x,\cdot))}{F_0^{m+k}(x)} \frac{\tilde{\rho}_k^k(G(x,\cdot))}{G_0^{m+k}(x)} \\ &\cdot \left(\frac{\tilde{\rho}_k^k(F(x,\cdot))}{F_0^{m+k}}\right)^k \tilde{\rho}_0^{m(k+1)}(F(x,\cdot)) \left(\frac{\tilde{\rho}_k^k(G(x,\cdot))}{G_0^{m+k}}\right)^k \tilde{\rho}_0^{m(k+1)}(G(x,\cdot)) \\ \leq\ & C(m,k)\widehat{\rho}_k(G(x,\cdot) - F(x,\cdot))\left(\frac{\tilde{\rho}_k(F(x,\cdot))}{F_0(x)} \frac{\tilde{\rho}_k(G(x,\cdot))}{G_0(x)}\right)^{k^2+(m+1)k+m}.\end{aligned}$$

□

2.4 Finsler area and ellipticity of the symmetrization

As we have seen in section 2.3 (i.e. Theorem 2.3.4, Corollary 2.3.6 and Corollary 2.3.8), the ellipticity of the m-symmetrization of a Finsler metric is crucial to the investigation of ellipticity of the corresponding Finsler area integrand. In this section, we will establish some theorems implying ellipticity of the m-symmetrization of a Finsler metric. We start with Example 2.4.1, where we consider a Finsler metric of a structure such that its m-symmetrization is automatically a Riemannian metric. This construction is motivated by a similar example of Cui, Shen in [CS09], which they carried out in an (α, β)-setting. In Corollary 2.4.2, we consider (α, β)-metrics. A characterizing ellipticity condition for the m-symmetrization is given in this corollary. Afterwards, we will present in Theorem 2.4.13 a sufficient condition on the Finsler metric guaranteeing the ellipticity of the m-symmetrization in a general setting. Further, we will discuss the merits of the former results by applying it on different sorts of so called (α, β)-metrics, namely the Randers, two order and Matsumoto metric (see section 1.4).

The following example exploits the structure of the m-symmetrization of a Finsler metric. Therein, we present Finsler metrics of a special structure, whose m-symmetrization is automatically a Riemannian metric. Cui and Shen [CS09] investigated such metrics in an (α,β)-setting.

Example 2.4.1 (m-**perturbed Riemannian metric**). Let $(\mathscr{N},\alpha)$ be a Finsler manifold with Riemannian metric α. Let $F: T\mathscr{N}\to[0,\infty)$ be a Finsler metric of the following form:

$$F(x,y) \quad := \quad \begin{cases}((\alpha(x,y))^{-m}+h(x,y))^{\frac{1}{-m}} & \text{for all } (x,y)\in T\mathscr{N}\backslash o,\\ 0 & \text{for all } (x,y)\in o,\end{cases}$$

where $h: T\mathscr{N}\backslash o\to\mathbb{R}$ is continuous, $(-m)$-homogeneous and odd, i.e. h satisfies (**b**), (**k**) with $k=-m$ and (**a**). Therein, o is the zero section. In this case, the m-symmetrization $F_{(m)}$ equals the Riemannian metric α, i.e.

$$\begin{aligned}F_{(m)}(x,y) &= 2^{\frac{1}{m}}(F^{-m}(x,y)+F^{-m}(x,-y))^{-\frac{1}{m}}\\ &= 2^{\frac{1}{m}}((\alpha(x,y))^{-m}+h(x,y)+(\alpha(x,-y))^{-m}+h(x,-y))^{-\frac{1}{m}}\\ &= 2^{\frac{1}{m}}((\alpha(x,y))^{-m}+h(x,y)+(\alpha(x,y))^{-m}-h(x,y))^{-\frac{1}{m}}\\ &= 2^{\frac{1}{m}}(2(\alpha(x,y))^{-m})^{-\frac{1}{m}}\\ &= \alpha(x,y)\end{aligned}$$

for all $(x,y)\in T\mathscr{N}\backslash o$. So, F and $F_{(m)}$ are Finsler metrics and therefore F satisfies (**GA3**). Corollary 2.3.9 then guarantees that $A^F\equiv A^\alpha$ is an elliptic Cartan integrand for a choice $\mathscr{N}=\mathbb{R}^{m+1}$. Notice that an (α,β)-metric of this structure was mentioned in [CS09, Theorem 1.1].

The following corollary identifies the m-symmetrization of an (α,β)-metric as an (α,β)-metric by itself and gives a characterizing condition for its ellipticity.

Corollary 2.4.2 (**Ellipticity of the Finsler area integrand for** (α,β)**-Finsler space**). *Let $(\mathscr{N},F)$ be a Finsler manifold with (α,β)-metric $F=\alpha\phi(\frac{\beta}{\alpha})$ (as in Definition 1.4.13). The m-symmetrization $F_{(m)}$ is then an (α,β)-metric given by*

$$F_{(m)} \quad = \quad \alpha\phi_{(m)}(\frac{\beta}{\alpha}) \quad \text{with} \quad \phi_{(m)}(s) \quad := \quad 2^{\frac{1}{m}}(\phi^{-m}(s)+\phi^{-m}(-s))^{-\frac{1}{m}}, \tag{2.4.1}$$

what is well-defined for $||\beta||_{\widehat{\alpha}}\big|_x<b_0$ for all $x\in\mathscr{N}$ and for all $|s|<b_0$, where $\phi\in C^\infty((-b_0,b_0))$ as in Definition 1.4.13. $F_{(m)}$ is a Finsler metric if and only if $\phi_{(m)}$ satisfies

$$\phi_{(m)}(s)-s\phi_{(m)}^{'}(s)+(b^2-s^2)\phi_{(m)}^{''}(s) \quad > \quad 0 \tag{2.4.2}$$

for all $|s|\le b<b_0$. Especially, if we additionally assume $\mathscr{N}=\mathbb{R}^{m+1}$, then the Finsler area integrand A^F is an elliptic Cartan integrand.

Proof. Definition 2.1.10 leads to

$$\begin{aligned}F_{(m)} &= 2^{\frac{1}{m}}(F^{-m}(x,y)+F^{-m}(x,-y))^{-\frac{1}{m}}\\ &= 2^{\frac{1}{m}}\left((\alpha(x,y)\phi(\frac{\beta(x,y)}{\alpha(x,y)}))^{-m}+(\alpha(x,-y)\phi(\frac{\beta(x,-y)}{\alpha(x,-y)}))^{-m}\right)^{-\frac{1}{m}}\\ &= \alpha(x,y)2^{\frac{1}{m}}\left(\phi^{-m}(\frac{\beta(x,y)}{\alpha(x,y)})+\phi^{-m}(-\frac{\beta(x,y)}{\alpha(x,y)})\right)^{-\frac{1}{m}}\end{aligned}$$

$$= \quad \alpha(x,y)\phi_{(m)}(\frac{\beta(x,y)}{\alpha(x,y)})$$

for $(x,y) \in T\mathscr{N}$. By Definition 1.4.13 we deduce that condition (2.4.2) is equivalent to $F_{(m)}$ being an elliptic Finsler metric. Notice that

$$\phi_{(m)}(s) \quad > \quad 0$$

holds automatically, since $\phi_{(m)}(s)$ is the harmonic mean of the positive values $\phi(s)$ and $\phi(-s)$ for $|s| \leq b < 1$. Finally, if F and $F_{(m)}$ are both Finsler metrics on $\mathbb{R}^{m+1}$ (i.e. F satisfies (**GA3**)), then is A^F an elliptic Cartan integrand by Corollary 2.3.9. □

The following examples illustrate Corollary 2.4.2.

Example 2.4.3 (**Randers metric**). Let F and ϕ be as in Example 1.4.15 defined on a smooth 3-manifold $\mathscr{N}$. We can then write $F_{(2)} = \alpha\phi_{(2)}(\beta/\alpha)$ with $\phi_{(2)}(s) = 2^{\frac{1}{2}}(\phi^{-2}(s)+\phi^{-2}(-s))^{-\frac{1}{2}}$. Corollary 2.4.2 implies that $F_{(2)}$ is a Finsler metric if and only if

$$\phi_{(2)}(s) - s\phi_{(2)}^{'}(s) + (b^2 - s^2)\phi_{(2)}^{''}(s) \quad > \quad 0.$$

We compute

$$\begin{aligned}
\phi_{(2)}(s) &= \frac{1-s^2}{\sqrt{1+s^2}}, \\
\phi_{(2)}^{'}(s) &= \frac{-2s}{\sqrt{1+s^2}} - \frac{(1-s^2)s}{\sqrt{1+s^2}^3} \\
&= \frac{-2s(1+s^2)-(1-s^2)s}{\sqrt{1+s^2}^3} \\
&= \frac{-s(3+s^2)}{\sqrt{1+s^2}^3}, \\
\phi_{(2)}^{''}(s) &= \frac{-3-9s^2}{\sqrt{1+s^2}^3} + \frac{3s^2(3+s^2)}{\sqrt{1+s^2}^5} \\
&= \frac{-3(1+3s^2)(1+s^2)+3s^2(3+s^2)}{\sqrt{1+s^2}^5} \\
&= \frac{-3(1+s^2)+6s^2}{\sqrt{1+s^2}^5} \\
&= \frac{-3(1-s^2)}{\sqrt{1+s^2}^5}.
\end{aligned}$$

Thereby, we get

$$\begin{aligned}
&\phi_{(2)}(s) - s\phi_{(2)}^{'}(s) + (b^2 - s^2)\phi_{(2)}^{''}(s) \\
= &\frac{1-s^2}{\sqrt{1+s^2}} - s\frac{-s(3+s^2)}{\sqrt{1+s^2}^3} + (b^2-s^2)\frac{-3(1-s^2)}{\sqrt{1+s^2}^5} \\
= &\frac{(1-s^2)(1+s^2)^2 + s^2(3+s^2)(1+s^2) - 3(b^2-s^2)(1-s^2)}{\sqrt{1+s^2}^5} \\
= &\frac{(1+3s^2)(1+s^2) - 3(b^2-s^2)(1-s^2)}{\sqrt{1+s^2}^5} \\
= &\frac{1+4s^2+3s^4-3b^2+3b^2s^2+3s^2-3s^4}{\sqrt{1+s^2}^5}
\end{aligned}$$

$$= \frac{1 - 3b^2 + 7s^2 + 3b^2s^2}{\sqrt{1+s^2}^5},$$

where the last expression is positive if and only if $|s| \le b < b_0 := \frac{1}{\sqrt{3}}$, especially in the case $s = 0$. Hence, $F_{(2)}$ is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < \frac{1}{\sqrt{3}}$ for all $x \in \mathcal{N}$. In the case that we choose $\mathcal{N} = \mathbb{R}^3$, then A^F is an elliptic Cartan integrand and we reproduced the bound on $||\beta||_{\widehat{\alpha}}\big|_x$ of [SST04] and [CS09, Example 6.1].

Example 2.4.4 (**Two order metric**). Let F and ϕ be as in Example 1.4.16 defined on a smooth 3-manifold $\mathcal{N}$. We can then write $F_{(2)} = \alpha\phi_{(2)}(\beta/\alpha)$ with $\phi_{(2)}(s) = 2^{\frac{1}{2}}(\phi^{-2}(s)+\phi^{-2}(-s))^{-\frac{1}{2}}$. Corollary 2.4.2 implies that $F_{(2)}$ is a Finsler metric if and only if

$$\phi_{(2)}(s) - s\phi_{(2)}^{'}(s) + (b^2 - s^2)\phi_{(2)}^{''}(s) \quad > \quad 0.$$

We compute

$$\begin{aligned}
\phi_{(2)}(s) &= 2^{\frac{1}{2}}((1+s)^{-4} + (1-s)^{-4})^{-\frac{1}{2}} \\
&= \frac{(1-s^2)^2}{\sqrt{1+6s^2+s^4}}, \\
\phi_{(2)}^{'}(s) &= \frac{-4s(1-s^2)}{\sqrt{1+6s^2+s^4}} - \frac{(1-s^2)^2(6s+2s^3)}{\sqrt{1+6s^2+s^4}^3} \\
&= \frac{-4s(1+6s^2+s^4) - (1-s^2)(6s+2s^3)}{\sqrt{1+6s^2+s^4}^3} \\
&= \frac{-4s - 24s^3 - 4s^5 - 6s - 2s^3 + 6s^3 + 2s^5}{\sqrt{1+s^2}^3} \\
&= \frac{-10s - 20s^3 - 2s^5}{\sqrt{1+6s^2+s^4}^3}, \\
\phi_{(2)}^{''}(s) &= \frac{-10 - 60s^2 - 10s^4}{\sqrt{1+6s^2+s^4}^3} + \frac{3(5s+10s^3+s^5)(12s+4s^3)}{\sqrt{1+6s^2+s^4}^5} \\
&= \frac{-(10+60s^2+10s^4)(1+6s^2+s^4) + 3(5s+10s^3+s^5)(12s+4s^3)}{\sqrt{1+6s^2+s^4}^5} \\
&= \frac{-10+60s^2+40s^4+36s^6+2s^8}{\sqrt{1+6s^2+s^4}^5}.
\end{aligned}$$

Thereby, we get

$$\begin{aligned}
&\phi_{(2)}(s) - s\phi_{(2)}^{'}(s) + (b^2-s^2)\phi_{(2)}^{''}(s) \\
= \; &\frac{(1-s^2)^2}{\sqrt{1+6s^2+s^4}} - s\frac{-10s-20s^3-2s^5}{\sqrt{1+6s^2+s^4}^3} + (b^2-s^2)\frac{-10+60s^2+40s^4+36s^6+2s^8}{\sqrt{1+6s^2+s^4}^5} \\
= \; &\Big(1 - 10(b^2-s^2) + 6s^2 + s^4 + (14s^2+10s^4+6s^6+s^8)(1+6s^2+s^4) \\
&+ (b^2-s^2)(60s^2+40s^4+36s^6+2s^8)\Big)\Big/\Big(\sqrt{1+6s^2+s^4}^5\Big),
\end{aligned}$$

where the last expression is positive if and only if $s \le b < b_0 := \frac{1}{\sqrt{10}}$, especially in the case $s = 0$. Hence, $F_{(2)}$ is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < \frac{1}{\sqrt{10}}$ for all $x \in \mathcal{N}$. In the case that we choose $\mathcal{N} = \mathbb{R}^3$, then A^F is an elliptic Cartan integrand and we reproduced the bound on $||\beta||_{\widehat{\alpha}}\big|_x$ of [CS09, Example 6.3].

Example 2.4.5 (**Matsumoto metric**). Let F and ϕ be as in Example 1.4.17 defined on a smooth 3-manifold $\mathscr{N}$. We can then write $F_{(2)} = \alpha\phi_{(2)}(\beta/\alpha)$ with $\phi_{(2)}(s) = 2^{\frac{1}{2}}(\phi^{-2}(s) + \phi^{-2}(-s))^{-\frac{1}{2}}$. Corollary 2.4.2 implies that $F_{(2)}$ is a Finsler metric if and only if

$$\phi_{(2)}(s) - s\phi'_{(2)}(s) + (b^2 - s^2)\phi''_{(2)}(s) \quad > \quad 0.$$

We compute

$$\begin{aligned}\phi_{(2)}(s) &= 2^{\frac{1}{2}}((1-s)^2 + (1+s)^2)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{1+s^2}}, \\ \phi'_{(2)}(s) &= \frac{-s}{\sqrt{1+s^2}^3} \\ \phi''_{(2)}(s) &= \frac{-1}{\sqrt{1+s^2}^3} + \frac{3s^2}{\sqrt{1+s^2}^5} \\ &= \frac{-1+2s^2}{\sqrt{1+s^2}^5}.\end{aligned}$$

Thereby, we get

$$\begin{aligned}&\phi_{(2)}(s) - s\phi'_{(2)}(s) + (b^2 - s^2)\phi''_{(2)}(s) \\ =\ &\frac{1}{\sqrt{1+s^2}} - s\frac{-s}{\sqrt{1+s^2}^3} + (b^2 - s^2)\frac{-1+2s^2}{\sqrt{1+s^2}^5} \\ =\ &\frac{(1+s^2)^2 + s^2(1+s^2) + (b^2-s^2)(-1+2s^2)}{\sqrt{1+s^2}^5} \\ =\ &\frac{1-(b^2-s^2) + 2s^2 + s^4 + s^2(1+s^2) + 2s^2(b^2-s^2)}{\sqrt{1+s^2}^5},\end{aligned}$$

where the last expression is positive if and only if $|s| \leq b < 1$, especially in the case $s = 0$. Hence, $F_{(2)}$ is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < 1$ for all $x \in \mathbb{R}^3$. F is a Finsler metric for $||\beta||_{\widehat{\alpha}}\big|_x < 1/2$ for all $x \in \mathbb{R}^3$ by Example 1.4.17 and so is $F_{(2)}$, what implies **(GA3)** for such a bound on $||\beta||_{\widehat{\alpha}}\big|_x$ for all $x \in \mathscr{N}$. In the case that we choose $\mathscr{N} = \mathbb{R}^3$, then A^F is an elliptic Cartan integrand and we reproduced the bound on $||\beta||_{\widehat{\alpha}}\big|_x$ of [CS09, Example 6.2].

In the following, we present a sufficient condition guaranteeing that the m-symmetrization $F_{(m)}$ of a Finsler metric F is indeed a Finsler metric by itself. Before we present this result, we give some basic definitions and combinatorial theorems, which are useful to prove the implications of the aforementioned sufficient condition.

Definition 2.4.6 (**Even and odd part of a function on the tangent space**). Let $\mathscr{N}$ be a smooth n-manifold (with boundary) and $F : T\mathscr{N} \to \mathbb{R}$ be a continuous function on the tangent space. We define the *even part* (or *symmetric part*) $F_s : T\mathscr{N} \to \mathbb{R}$ of F as

$$F_s(x,y) \quad := \quad \frac{F(x,y) + F(x,-y)}{2} \quad \text{for all} \quad (x,y) \in T\mathscr{N}, \tag{2.4.3}$$

and the *odd part* (or *antisymmetric part*) $F_a : T\mathscr{N} \to \mathbb{R}$ of F as

$$F_a(x,y) \quad := \quad \frac{F(x,y) - F(x,-y)}{2} \quad \text{for all} \quad (x,y) \in T\mathscr{N}. \tag{2.4.4}$$

Thereby holds especially that

$$F(x,y) \quad = \quad F_s(x,y) + F_a(x,y) \tag{2.4.5}$$

for all $(x,y) \in T\mathscr{N}$.

Remark 2.4.7.

- F_s and F_a are at least as regular as F, as they are just linear combinations of instances of F.
- $F : T\mathscr{N} \to [0, \infty)$ satisfying **(F1)** implies directly that the even part F_s is non-negative, positive away from zero and homogeneous of degree 1, so satisfies **(F1)** on its own. Further, if even more F satisfies **(F3)**, it can be easily shown that the fundamental tensor $(g_s)_{ij} := g_{ij}^{F_s}$ is positive definite. So, F_s also satisfies **(F3)**. All together means especially that F_s is a Finsler metric if F is one. Notice that F_s is reversible as a Finsler metric by construction.
- In the following, we state some simple relations worth mentioning:

$$\begin{aligned} (F_s)_{y_i}(x,-y) &= -(F_s)_{y_i}(x,y), & (F_a)_{y_i}(x,-y) &= (F_a)_{y_i}(x,y),\\ F_{y_i}(x,y)\xi^i &= (F_a)_{y_i}(x,y)\xi^i, & F_{y_i}(x,-y)\xi^i &= (F_a)_{y_i}(x,y)\xi^i, \end{aligned}$$

  for all $(x,y) \in \mathrm{T}\mathscr{N}$ and $\xi = (\xi^i) \in \mathbb{R}^n$ with $(F_s)_{y_i}(x,-y)\xi^i = 0$. Therein, we assumed $\mathscr{N}$ to be a smooth n-manifold and $F \in C^1(T\mathscr{N})$.

In the following definition, we extend the definition of binomial coefficients to negative integers in the lower argument.

Definition 2.4.8 (Binomial coefficient). Let $m \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. The *binomial coefficient* indexed by m and k is defined by

$$\binom{m}{k} := \begin{cases} \frac{m!}{k!(m-k)!} & \text{for } k \in \{0,1,\dots,m\},\\ 0 & \text{else.} \end{cases} \tag{2.4.6}$$

with $0! := 1$ and $m! := m \cdot (m-1) \cdot \ldots \cdot 2 \cdot 1$ for $m \in \mathbb{N}$.

Theorem 2.4.9 (Pascal's rule (cf. [Mer03])**).** *Let $m \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. Then holds*

$$\binom{m}{k} = \binom{m}{m-k}. \tag{2.4.7}$$

If we assume $m \in \mathbb{N}$ and $k \le m$, then holds

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}. \tag{2.4.8}$$

Proof. We start by showing (2.4.7). Let $m \in \mathbb{N}_0$ and $k \in \{0,\dots,m\}$. Then holds

$$\begin{aligned} \binom{m}{k} &= \frac{m!}{k!(m-k)!}\\ &= \frac{m!}{(m-(m-k))!(m-k)!}\\ &= \binom{m}{m-k}. \end{aligned}$$

On the other hand if $k \notin \{0,\dots,m\}$ then is $m-k \notin \{0,\dots,m\}$ and it follows

$$\binom{m}{k} = 0 = \binom{m}{m-k}.$$

Now we come to show (2.4.8). So, let $m \in \mathbb{N}$ and $k \in \{1,\dots,m\}$. We then compute

$$\binom{m-1}{k} + \binom{m-1}{k-1} = \frac{(m-1)!}{k!(m-1-k)!} + \frac{(m-1)!}{(k-1)!(m-k)!}$$

$$
\begin{aligned}
&= \frac{(m-1)!}{(k-1)!(m-1-k)!}\left(\frac{1}{k}+\frac{1}{m-k}\right)\\
&= \frac{(m-1)!}{(k-1)!(m-1-k)!}\frac{m}{k(m-k)}\\
&= \frac{m!}{k!(m-k)!}\\
&= \binom{m}{k}.
\end{aligned}
$$

For $k = 0$ holds

$$
\binom{m}{0} = 1 = \binom{m-1}{0} + \binom{m-1}{-1}
$$

and for $k < 0$ holds

$$
\binom{m}{k} = 0 = \binom{m-1}{k} + \binom{m-1}{k-1}.
$$

This concludes the proof. □

Theorem 2.4.10 (**Binomial theorem** (cf. [Mer03]))**.** *Let $m \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then holds*

$$
(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k. \tag{2.4.9}
$$

Proof. We show the claimed first equality by induction with respect to m. The initial step $m = 1$ follows directly. Then we assume as induction hypothesis **(IH)** that the claimed relation holds for some $m \in \mathbb{N}$. We show now that claim then follows for $m + 1$:

$$
\begin{aligned}
(x+y)^{m+1} &= (x+y)\cdot(x+y)^m\\
&\overset{\textbf{(IH)}}{=} (x+y)\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k}\\
&= \sum_{k=0}^{m}\binom{m}{k}x^{k+1}y^{m-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k}\\
&\overset{l=k+1}{=} \sum_{l=1}^{m+1}\binom{m}{l-1}x^l y^{m+1-l} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k}\\
&\overset{l=k+1}{=} y^{m+1} + x^{m+1} + \sum_{k=1}^{m}\binom{m}{k-1}x^k y^{m+1-k} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m+1-k}\\
&= y^{m+1} + x^{m+1} + \sum_{k=1}^{m}\left(\binom{m}{k-1} + \binom{m}{k}\right)x^k y^{m+1-k}\\
&\overset{(2.4.8)}{=} y^{m+1} + x^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^k y^{m+1-k}\\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k},
\end{aligned}
$$

what shows the claimed first equality for $m+1$ and thereby concludes the proof by induction. The second stated equality stems from an exchange of the role of x and y. □

Lemma 2.4.11. *For $m \in \mathbb{N}$, $m \geq 2$ holds*

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} \quad \geq \quad 0, \tag{2.4.10}$$

for $x \in \mathbb{R}$ with $x^2 \leq \frac{1}{m-1}$. For $m = 2$ or m odd it is sufficient to assume $x^2 \leq 1$.

Proof. The proof will be split in four distinct cases, namely $m = 2$, $m = 2q+1$, $m = 2(2q+1)$ and $m = 4q$. Before discussing each case, we give a set of useful and simple identities for binomial coefficients:

$$\binom{m}{l} \quad = \quad \binom{m}{m-l} \tag{2.4.11}$$

$$\binom{m}{l} - \binom{m}{l-1} \quad = \quad \frac{m+1-2l}{m+1} \binom{m+1}{l} \tag{2.4.12}$$

for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$,

$$\binom{m}{l} - \binom{m}{l-1} \quad = \quad \frac{m}{m-l} \binom{m-1}{l} - \frac{m}{m-l+1} \binom{m-1}{l-1} \tag{2.4.13}$$

$$\geq \quad \frac{m}{m+1-l} \left(\binom{m-1}{l} - \binom{m-1}{l-1} \right) \tag{2.4.14}$$

for $m \in \mathbb{N}$ and $l \in \mathbb{Z} \backslash \{m, m+1\}$.

Case 1: Let $m = 2$ and $x^2 \leq 1$. Then holds

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{2}{2} \rfloor} \left(\binom{2}{2k+1} - \binom{2}{2k} \right) x^{2k} \quad &= \quad (\binom{2}{1} - \binom{2}{0})x^0 + (\binom{2}{3} - \binom{2}{2})x^2 \\ &= \quad (2-1)x^0 + (0-1)x^2 \\ &= \quad 1 - x^2 \\ &\geq \quad 0 \end{aligned}$$

for $x^2 \leq 1$ as stated.

Case 2: Let $m = 2q+1$ for $q \in \mathbb{N}$ and $x^2 \leq 1$. By (2.4.11) and multiple change of the summation index we get

$$\begin{aligned} & 2\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} \\ = \quad & 2\sum_{k=0}^{q} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} \\ \overset{(2.4.11)}{=} & \sum_{k=0}^{q} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} + \sum_{k=0}^{q} \left(\binom{m}{m-(2k+1)} - \binom{m}{m-2k} \right) x^{2k} \\ \overset{m=2q+1}{=} & \sum_{k=0}^{q} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} + \sum_{k=0}^{q} \left(\binom{m}{2(q-k)} - \binom{m}{2(q-k)+1} \right) x^{2k} \\ \overset{l=q-k}{=} & \sum_{k=0}^{q} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) x^{2k} + \sum_{l=0}^{q} \left(\binom{m}{2l} - \binom{m}{2l+1} \right) x^{2(q-l)} \\ = \quad & \sum_{k=0}^{q} \left(\binom{m}{2k+1} - \binom{m}{2k} \right) (x^{2k} - x^{2(q-k)}). \end{aligned}$$

By (2.4.12) we know that $\left(\binom{m}{2k+1}-\binom{m}{2k}\right)$ is non-negative for $2(2k+1)\leq m+1=2q+2$ or more precisely $k\leq q/2$ and is negative otherwise. Since x^2 is less than one the term $x^{2k}-x^{2(q-k)}$ is non-negative for $x^2\neq 0$ if and only if $2k\leq 2(q-k)$ or more precisely if $k\leq q/2$. Both estimates combined yield that the product $\left(\binom{m}{2k+1}-\binom{m}{2k}\right)(x^{2k}-x^{2(q-k)})$ is non-negative for $k=0,\ldots,q$. So, we showed the theorem's statement for m odd.

Case 3: Let $m=2(2q+1)$ for $q\in\mathbb{N}$ and $x^2\leq\frac{1}{m-1}$.

$$\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
=&\sum_{k=0}^{2q+1}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
=&\sum_{k=0}^{q}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}+\sum_{k=q+1}^{2q}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}-x^m\\
\overset{(2.4.14)\&(2.4.15)}{\geq}&\ 1-x^m+\sum_{k=0}^{q}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&+\sum_{k=q+1}^{2q}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
\overset{x^2\leq\frac{1}{m-1}}{\geq}&\sum_{k=0}^{q}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&+\sum_{k=q+1}^{2q}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k},
\end{aligned}$$

where we used

$$\begin{aligned}
\left(\binom{m}{2\cdot 0+1}-\binom{m}{2\cdot 0}\right)x^{2\cdot 0}&=m-1\\
&=1+\frac{m}{m-2\cdot 0}\left(\binom{m-1}{2\cdot 0+1}-\binom{m-1}{2\cdot 0}\right)x^{2\cdot 0}. \quad (2.4.15)
\end{aligned}$$

Again, we know that $\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)$ is non-negative if and only if $2(2k+1)\leq m=2(2q+1)$, what is equivalent to $k\leq q$. On the other hand, we know that $1\leq\frac{m}{m-2k}\leq\frac{m}{2}\leq m-1$ for $0\leq k\leq 2q$ and $m\geq 2$. So, we get

$$\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
\geq&\sum_{k=0}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{k=q+1}^{2q}(m-1)\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
\overset{x^2\leq\frac{1}{m-1}}{\geq}&\sum_{k=0}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{k=q+1}^{2q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2(k-1)}\\
\overset{(2.4.11)}{=}&\sum_{k=0}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=q+1}^{2q}\left(\binom{m-1}{m-1-(2k+1)}-\binom{m-1}{m-1-2k}\right)x^{2(k-1)}\\
\overset{m=2(2q+1)}{=}\;&\sum_{k=0}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&+\sum_{k=q+1}^{2q}\left(\binom{m-1}{2(2q-k)}-\binom{m-1}{2(2q-k)+1}\right)x^{2(k-1)}\\
\overset{l=2q-k}{=}\;&\sum_{k=0}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{l=0}^{q-1}\left(\binom{m-1}{2l}-\binom{m-1}{2l+1}\right)x^{2(2q-l-1)}\\
=\;&\left(\binom{m-1}{2q+1}-\binom{m-1}{2q}\right)x^{2q}+\sum_{k=0}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)(x^{2k}-x^{2(2q-k-1)})
\end{aligned}
$$

Remember that $\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)$ is non-negative for $k\leq q$. Further, given the fact that $x^2\leq\frac{1}{m-1}\leq 1$ we have $x^{2k}-x^{2(2q-k-1)}\geq 0$ for $2k\leq 2(2q-k-1)$, which means for $k\leq q-\frac{1}{2}$, what is fulfilled. So, the assertion follows for $m=2(2q+1)$.

<u>Case 4:</u> Let $m=4q$ for $q\in\mathbb{N}$ and $x^2\leq\frac{1}{m-1}$.

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
=\;&\sum_{k=0}^{2q}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
=\;&m-1+\sum_{k=1}^{q-1}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}+\sum_{k=q+1}^{2q-2}\left(\binom{m}{2k}-\binom{m}{2k}\right)x^{2k}\\
&+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}-x^m\\
\geq\;&m-1+\sum_{k=1}^{q-1}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&+\sum_{k=q}^{2q-2}\frac{m}{m-2k}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}-x^m
\end{aligned}
$$

Again, we know that $\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)$ is non-negative if and only if $2(2k+1)\leq m=4q$, what is for $k\leq q-\frac{1}{2}$. On the other hand, we know that $1\leq\frac{m}{m-2k}\leq\frac{m}{4}\leq m-1$ for $0\leq k\leq 2q-2$ and $m\geq 2$. So, we get

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
\geq\;&m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&+\sum_{k=1}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{k=q}^{2q-2}(m-1)\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}
\end{aligned}
$$

$$\begin{aligned}
&\overset{x^2\leq\frac{1}{m-1}}{\geq} m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&\quad+\sum_{k=1}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{k=q}^{2q-2}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2(k-1)}\\
&\overset{(2.4.11)}{=} m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&\quad+\sum_{k=1}^{q}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&\quad+\sum_{k=q}^{2q+1}\left(\binom{m-1}{m-1-(2k+1)}-\binom{m-1}{m-1-2k}\right)x^{2(k-1)}\\
&\overset{m=4q}{=} m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&\quad+\sum_{k=1}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}\\
&\quad+\sum_{k=q}^{2q-2}\left(\binom{m-1}{2(2q-k-1)}-\binom{m-1}{2(2q-k-1)+1}\right)x^{2(k-1)}\\
&\overset{l=2q-k-1}{=} m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&\quad+\sum_{k=1}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)x^{2k}+\sum_{k=1}^{q-1}\left(\binom{m-1}{2l}-\binom{m-1}{2l+1}\right)x^{2(2q-l-2)}\\
&= m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&\quad+\sum_{k=1}^{q-1}\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)(x^{2k}-x^{2(2q-k-2)})
\end{aligned}$$

The term $\left(\binom{m-1}{2k+1}-\binom{m-1}{2k}\right)$ is non-negative for $2(2k+1)\leq m=4q$ or more precisely $k\leq q-\frac{1}{2}$. Further, given the fact that $x^2\leq\frac{1}{m-1}\leq 1$ we have $x^{2k}-x^{2(2q-k-2)}\geq 0$ for $2k\leq 2(2q-k-2)$, which means for $k\leq q-1$, what is fulfilled. So, this leads to

$$\begin{aligned}
&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\left(\binom{m}{2k+1}-\binom{m}{2k}\right)x^{2k}\\
&\geq m-1-x^m+\left(\binom{m}{m-1}-\binom{m}{m-2}\right)x^{m-2}\\
&= m-1-x^m+\left(m-\frac{m(m-1)}{2}\right)x^{m-2}\\
&= m-1-x^m-\frac{m(m-3)}{2}x^{m-2}\\
&= m-1-x^{m-2}\left(x^2+\frac{m(m-3)}{2}\right)\\
&\overset{x^2\leq\frac{1}{m-1}}{\geq} m-1-\left(\frac{1}{m-1}\right)^{\frac{m-2}{2}}\left(\frac{1}{m-1}+\frac{m(m-3)}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{m\geq 4}{\geq} \quad m-1-\frac{1}{m-1}\left(\frac{1}{3}+\frac{m(m-3)}{2}\right) \\
&\overset{\frac{m}{m-1}\leq 2}{\geq} \quad m-1-\frac{1}{3(m-1)}-\frac{2(m-3)}{2} \\
&\overset{m\geq 4}{\geq} \quad 2-\frac{1}{9} \\
&= \quad \frac{17}{9} > 0.
\end{aligned}
$$

So, the assertion follows for $m = 4q$. □

Lemma 2.4.12. *Let $m \in \mathbb{N}$. Then holds*

$$
\Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} x^{2k+1}\Big)^2 \quad < \quad \Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} x^{2k}\Big)^2 \tag{2.4.16}
$$

for $|x| < 1$.

Proof. First of all, we assume $x \in [0,1] := \{x \in \mathbb{R} : x \geq 0 \text{ and } x \leq 1\}$, since the left hand side of (2.4.16) can be rewritten to

$$
\begin{aligned}
\Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} x^{2k+1}\Big)^2 \quad &= \quad x^2\Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} x^{2k}\Big)^2 \\
&= \quad |x|^2\Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} |x|^{2k}\Big)^2 \\
&= \quad \Big(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} |x|^{2k+1}\Big)^2
\end{aligned}
$$

and on the right hand side do only appear even powers of x. By means of (2.4.9) we can rewrite

$$
f(x) \quad := \quad \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} x^{2k+1} \quad = \quad \frac{1}{2}((1+x)^m - (1-x)^m)
$$

and

$$
g(x) \quad := \quad \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} x^{2k} \quad = \quad \frac{1}{2}((1+x)^m + (1-x)^m)
$$

for $x \in [0,1]$. We intend to show that $f(x) < g(x)$ on $[0,1) := [0,1]\backslash\{1\}$, what would prove the claim by taking the square of the non-negative values on both sides. Hence, it is sufficient to show that

$$
h(x) \quad := \quad \frac{f(x)}{g(x)} \quad < \quad 1
$$

on $[0,1)$. Notice that $h(x)$ is differentiable and well-defined on $[0,1]$ since $g(x) > 0$ for all $x \in [0,1]$. The derivative of first order of $h(x)$ yields

$$
\frac{\mathrm{d}}{\mathrm{d}x} h(x) \quad = \quad m\big((1+x)^{m-1} + (1-x)^{m-1}\big)(2g(x))^{-1}
$$

$$
\begin{aligned}
&\quad - m\big(((1+x)^m - (1-x)^m)((1+x)^{m-1} - (1-x)^{m-1})\big)(2g(x))^{-2} \\
&= \quad m\big[((1+x)^{m-1} + (1-x)^{m-1})((1+x)^m + (1-x)^m) \\
&\quad - ((1+x)^m - (1-x)^m)((1+x)^{m-1} - (1-x)^{m-1})\big](2g(x))^{-2} \quad \geq \quad 0,
\end{aligned}
$$

since

$$
\begin{aligned}
(1+x)^{m-1} &\geq (1-x)^{m-1}, \\
(1+x)^m &\geq (1-x)^m, \\
(1+x)^{m-1} + (1-x)^{m-1} &\geq (1+x)^{m-1} - (1-x)^{m-1}
\end{aligned}
$$

and

$$
(1+x)^m + (1-x)^m \geq (1+x)^m - (1-x)^m
$$

for $x \in [0,1]$. Notice that all the inequalities are strict for $x \neq 0$ and hence $\frac{\mathrm{d}}{\mathrm{d}x}h(x)$ is positive on $(0,1] := [0,1]\backslash\{0\}$. So, as $h(x)$ is monotonically increasing on $[0,1]$, strictly monotonically increasing on $(0,1]$ and since $h(0) = 0$ and $h(1) = 1$, we have shown that $h(x) < 1$ on $[0,1)$. As $h(x) > 0$ is positive for all $x \in (0,1]$, we even get $h(x) < 1$ on $x \in [0,1)$. This concludes the proof. □

Now we have all the ingredients to show the following theorem, which gives a sufficient condition imposed on the Finsler metric F to guarantee that the m-symmetrization $F_{(m)}$ is a Finsler metric.

Theorem 2.4.13 (**Sufficient condition**). *Let $(\mathscr{N}, F)$ be a Finsler manifold such that the Finsler metric F satisfies the following two conditions:*

$$
\sup_{\xi \in T_x\mathscr{N}\backslash\{0\}} \frac{(\mathrm{d}_y F_a|_{(x,y)}(\xi))^2}{g^{F_s}|_{(x,y)}(\xi,\xi)} < \frac{1}{m+1} \quad \text{for all} \quad (x,y) \in T\mathscr{N}, \tag{2.4.17}
$$

$$
\sup_{\xi \in T_x\mathscr{N}} F_a(x,y)(F_a)_{yy}|_{(x,y)}(\xi,\xi) \leq 0 \quad \text{for all} \quad (x,y) \in T\mathscr{N}. \tag{2.4.18}
$$

Then the m-symmetrization $F_{(m)}$ of F is a Finsler metric on $\mathscr{N}$, i.e. F satisfies (**GA3**)*. Notice that F_s and F_a are the even and odd part of F, respectively, and we define*

$$
\begin{aligned}
\mathrm{d}_y F_a|_{(x,y)}(\xi) &:= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F_a(x, y+t\xi) = (F_a)_{y^i}(x,y)\xi^i, \\
(F_a)_{yy}|_{(x,y)}(\xi,\omega) &:= \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F_a(x, y+t\xi+s\omega) = (F_a)_{y^iy^j}(x,y)\xi^i\omega^j, \\
g^{F_s}|_{(x,y)}(\xi,\omega) &:= \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\left(\frac{F_s^2}{2}\right)(x, y+t\xi+s\omega) = g^{F_s}_{ij}(x,y)\xi^i\omega^j
\end{aligned}
$$

for $(x,y) \in T\mathscr{N}$, bundle coordinates (x^i, y^i) on $T\mathscr{N}$ and $\xi = \xi^i\frac{\partial}{\partial y^i}$, $\omega = \omega^i\frac{\partial}{\partial y^i} \in T_x\mathscr{N}$. Corollary 2.3.9 then guarantees that $A^F \equiv A^{F_{(m)}}$ is an elliptic Cartan integrand for a choice $\mathscr{N} = \mathbb{R}^{m+1}$.

Remark 2.4.14.

- The condition (2.4.18) is especially fulfilled if the odd part F_a of F is linear or constantly zero.
- By standard linear algebra we know that

$$
\sup_{\xi \in T_x\mathscr{N}\backslash\{0\}} \frac{(\mathrm{d}_y F_a|_{(x,y)}(\xi))^2}{g^{F_s}|_{(x,y)}(\xi,\xi)} = (g^{F_s})^{ij}(x,y)(F_a)_{y^i}(x,y)(F_a)_{y^j}(x,y)
$$

$$
\begin{aligned}
&=: \quad (g^{F_s})^*|_{(x,y)}(\mathrm{d}_y F_a|_{(x,y)}, \mathrm{d}_y F_a|_{(x,y)}) \\
&=: \quad ||\mathrm{d}_y F_a||^2_{\widehat{F_s}}|_{(x,y)},
\end{aligned}
$$

where the latter two quantities are globally well-defined scalars due to the right transformation behaviour of the defining local coordinates expressions. So, relation (2.4.17) is equivalent to

$$
||\mathrm{d}_y F_a||_{\widehat{F_s}} \quad < \quad \frac{1}{\sqrt{m+1}} \quad \text{on} \quad T\mathcal{N}. \tag{2.4.19}
$$

This gives the first condition a very geometric interpretation in the following way: F is the sum of a reversible Finsler metric F_s and an odd perturbation F_a. If the differential w.r.t. the second argument of the perturbation term F_a, measured in the dual norm issued from F_s, is smaller than a dimension dependent constant, then is the m-symmetrization $F_{(m)}$ a Finsler metric.

- Let $(\mathbb{R}^{m+1}, F)$ be a Finsler manifold such that (2.4.17) and (2.4.18) hold. F then satisfies **(GA3)** by Theorem 2.4.13 and consequently A^F is an elliptic Cartan integrand by Corollary 2.3.8.

Proof of Theorem 2.4.13. First of all, we start with some relations that hold for every homogeneous function $f : T\mathcal{N} \to \mathbb{R}$ of class $C^2(T\mathcal{N}\backslash\{0\})$. These relations follow quite directly from Euler's theorem (see Theorem 1.4.3, Example 1.4.5). Namely, such an f satisfies

$$
\begin{aligned}
f_{y^i}(x,y)y^i &= f(x,y), \\
f_{y^i y^j}(x,y)y^j &= 0, \\
g^f|_{(x,y)}(y,w) &= y^i(f_{y^i}(x,y)f_{y^j}(x,y) + f(x,y)f_{y^i y^j}(x,y))w^j \\
&= f(x,y)f_{y^j}(x,y)w^j, \\
g^f|_{(x,y)}(y,y) &= f(x,y)f_{y^j}(x,y)y^j \\
&= f^2(x,y)
\end{aligned}
$$

for all $x = (x^i) \in \mathcal{N}$ and $y = y^i \frac{\partial}{\partial x^i}$, $w = w^i \frac{\partial}{\partial x^i} \in T\mathcal{N}$ with $y \neq 0$. We will use them throughout the proof without mentioning them especially.

Let throughout the proof $(x, y) \in T\mathcal{N}\backslash o$ with bundle coordinates (x^i, y^i). To show the claim, we look at the m-symmetrization $F_{(m)}$ of the Finsler metric F. We intend to show that $F_{(m)}$ is a Finsler metric, i.e. that $F_{(m)}$ satisfies **(F3)**. Remember that we defined in Definition 2.1.10 the m-symmetrization $F_{(m)}$ as

$$
F_{(m)}(x,y) \quad := \quad 2^{\frac{1}{m}}(F^{-m}(x,y) + F^{-m}(x,-y))^{-\frac{1}{m}}.
$$

So, the m-symmetrization $F_{(m)}$ is nothing else but the so called m-harmonic mean of $F(x,y)$ and $F(x,-y)$. It is easy to see that $F_{(m)} \in C^0(T\mathcal{N}) \cap C^\infty(T\mathcal{N}\backslash o)$, since $F_{(m)}$ depends in a relatively direct manner from $F \in C^0(T\mathcal{N}) \cap C^\infty(T\mathcal{N}\backslash o)$. We also deduce directly that $F_{(m)}$ is homogeneous and $F_{(m)} > 0$ on $T\mathcal{N}\backslash o$ by construction, i.e. $F_{(m)}$ satisfies **(F1)** (see Remark 2.1.11). So, the last thing, we need to discuss, is the fundamental form $g^{F_{(m)}}$ of $F_{(m)}$. In other words, we need to show the ellipticity of $F_{(m)}$.

We start by decomposing F in its even and odd part, namely $F = F_s + F_a$ with

$$
F_s(x,y) \quad = \quad F_s(x,-y) \quad \text{and} \quad F_a(x,y) \quad = \quad -F_a(x,-y).
$$

Reversely, we can write

$$
F_s(x,y) \quad = \quad (F(x,y) + F(x,-y))/2 \quad \text{and} \quad F_a(x,y) \quad = \quad (F(x,y) - F(x,-y))/2.
$$

Remark 2.4.7 implies that F_s is a reversible Finsler metric, since F is a Finsler metric. Hence, F_s is especially homogeneous and $F_s > 0$ on $T\mathscr{M}\backslash o$. Some arbitrary tangent vector $w = w^j \frac{\partial}{\partial x^j} \in T_x\mathscr{N}$ can be decomposed into $w = \alpha y + \xi$, where $\alpha \in \mathbb{R}$ and $\xi = \xi^j \frac{\partial}{\partial x^j} \in T_x\mathscr{N}$ with

$$g^{F_s}|_{(x,y)}(y,\xi) \quad = \quad y^i g_{ij}^{F_s}(x,y)\xi^j \quad = \quad F_s(x,y)(F_s)_{y^j}(x,y)\xi^j \quad = \quad 0. \tag{2.4.20}$$

Notice that the latter condition poses a real one, since we deduce that $g^{F_s}|_{(x,y)}(y,y) = ((F_s)_{y^i}(x,y)y^i)^2 = F_s^2(x,y) > 0$ by Euler's theorem (see Theorem 1.4.3). The decomposition of w can be constructed in the following way:

$$w \quad = \quad \frac{g^{F_s}|_{(x,y)}(y,w)}{g^{F_s}|_{(x,y)}(y,y)} y + (w - \frac{g^{F_s}|_{(x,y)}(y,w)}{g^{F_s}|_{(x,y)}(y,y)} y) \quad =: \quad \alpha y + \xi.$$

Now we start to compute the fundamental tensor of $F_{(m)}$. Choose some arbitrary $w \in T_x\mathscr{N}$, and decompose it as mentioned above to $w = \alpha y + \xi$ with $\alpha \in \mathbb{R}$ and $\xi \in T_x\mathscr{N}$ such that $g^{F_s}|_{(x,y)}(\xi,\xi) = y^i g_{ij}^{F_s}(x,y)\xi^j = F_s(y)(F_s)_{y^j}(x,y)\xi^j = 0$. Hence, we can write

$$g^{F_{(m)}}|_{(x,y)}(w,w) \quad = \quad \alpha^2 g^{F_{(m)}}|_{(x,y)}(y,y) + 2\alpha g^{F_{(m)}}|_{(x,y)}(y,\xi) + g^{F_{(m)}}|_{(x,y)}(\xi,\xi). \tag{2.4.21}$$

Again, using Euler's theorem (see Theorem 1.4.3) and the homogeneity of $F_{(m)}$, we compute the expression $g^{F_{(m)}}|_{(x,y)}(y,y)$ in the first term on the right-hand side of (2.4.21) to be

$$\begin{aligned} g^{F_{(m)}}|_{(x,y)}(y,y) \quad &= \quad y^i g_{ij}^{F_{(m)}}(x,y)y^j \\ &= \quad y^i((F_{(m)})_{y^i}(x,y)(F_{(m)})_{y^i}(x,y) + (F_{(m)})(x,y)(F_{(m)})_{y^i y^j})(x,y)y^j \\ &= \quad F_{(m)}^2(y). \end{aligned} \tag{2.4.22}$$

For the next steps, we compute explicitly the componentwise derivatives of $F_{(m)}$, where we make use of the representation of $F_{(m)}$ in terms of F :

$$\begin{aligned} (F_{(m)})_{y^i}(x,y) \quad &= \quad \frac{1}{2}F_{(m)}^{m+1}(x,y)(F^{-(m+1)}(x,y)F_{y^i}(x,y) - F^{-(m+1)}(x,-y)F_{y^i}(x,-y)), \\ (F_{(m)})_{y^i y^i}(x,y) \quad &= \quad \left\{\frac{1}{2}F_{(m)}^{m+1}(x,y)(F^{-(m+1)}(x,y)F_{y^i}(x,y) - F^{-(m+1)}(x,-y)F_{y^i}(x,-y))\right\}_{y^j} \\ &= \quad \frac{m+1}{4}F_{(m)}^{2m+1}(x,y)\Big(F^{-(m+1)}(x,y)F_{y^i}(x,y) - F^{-(m+1)}(x,-y)F_{y^i}(x,-y)\Big) \\ &\quad \cdot \Big(F^{-(m+1)}(x,y)F_{y^j}(x,y) - F^{-(m+1)}(x,-y)F_{y^j}(x,-y)\Big) \\ &\quad + \frac{1}{2}F_{(m)}^{m+1}(x,y)\Big(- (m+1)F^{-(m+2)}(x,y)F_{y^i}(x,y)F_{y^j}(x,y) \\ &\quad - (m+1)F^{-(m+2)}(x,-y)F_{y^i}(x,-y)F_{y^j}(x,-y) \\ &\quad + F^{-(m+1)}(x,y)F_{y^i y^j}(x,y) + F^{-(m+1)}(x,-y)F_{y^i y^j}(x,-y)\Big). \end{aligned}$$

Thereby, we compute the expression $g^{F_{(m)}}|_{(x,y)}(y,\xi)$ in the second term on the right-hand side of (2.4.21) to be

$$\begin{aligned} g^{F_{(m)}}|_{(x,y)}(y,\xi) \quad &= \quad y^i g_{ij}^{F_{(m)}}(x,y)\xi^j \\ &= \quad F_{(m)}(x,y)\left(F_{(m)}\right)_{y^j}(x,y)\xi^j \\ &= \quad \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{F_{y^j}(x,y)}{F^{m+1}(x,y)} - \frac{F_{y^j}(x,-y)}{F^{m+1}(x,-y)}\Big)\xi^j \end{aligned}$$

$$= \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\Big)(F_a)_{y^j}(x,y)\xi^j, \qquad (2.4.23)$$

where we exploited in the last step that

$$0 = (F_s)_{y^i}(x,y)\xi^i = \frac{1}{2}(F_{y^i}(x,y) - F_{y^i}(x,-y))\xi^i$$

and therefore

$$\begin{aligned}(F_a)_{y^i}(x,-y)\xi^i &= (F_a)_{y^i}(x,y)\xi^i \\ &= \frac{1}{2}(F_{y^i}(x,y) + F_{y^i}(x,-y))\xi^i \\ &= F_{y^i}(x,y)\xi^i.\end{aligned}$$

In the same way, we compute the expression $g^{F_{(m)}}|_{(x,y)}(\xi,\xi)$ in the last term on the right-hand side of (2.4.21) to be

$$\begin{aligned} g^{F_{(m)}}|_{(x,y)}(\xi,\xi) &= \xi^i g_{ij}^{F_{(m)}}(x,y)\xi^j \\ &= \xi^i((F_{(m)})_{y^i}(x,y)(F_{(m)})_{y^j}(x,y) + F_{(m)}(x,y)(F_{(m)})_{y^iy^j}(x,y))\xi^j \\ &= \frac{m+2}{4}F_{(m)}^{2m+2}(x,y)\Big(\Big(\frac{F_{y^j}(x,y)}{F^{m+1}(x,y)} - \frac{F_{y^j}(x,-y)}{F^{m+1}(x,-y)}\Big)\xi^j\Big)^2 \\ &\quad + \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j}{F^{m+2}(x,y)} - \frac{m+1}{F^{m+2}(x,y)}\big(F_{y^i}(x,y)\xi^i\big)^2 \\ &\quad + \frac{\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j}{F^{m+2}(x,-y)} - \frac{m+1}{F^{m+2}(x,-y)}\big(F_{y^i}(x,-y)\xi^i\big)^2\Big) \\ &= \frac{m+2}{4}F_{(m)}^{2m+2}(x,y)\Big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\Big)^2\big((F_a)_{y^j}(x,y)\xi^j\big)^2 \\ &\quad + \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,y)} \\ &\quad + \frac{\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,-y)}\Big). \qquad (2.4.24)\end{aligned}$$

In the following, we replace the respective terms in (2.4.21) by (2.4.22), (2.4.23) and (2.4.24). So, we get

$$\begin{aligned} & g^{F_{(m)}}|_{(x,y)}(w,w) \\ = \; & \alpha^2 F_{(m)}^2(x,y) \\ & + 2\alpha\frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\Big)(F_a)_{y^j}(x,y)\xi^j \\ & + \frac{m+2}{4}F_{(m)}^{2m+2}(x,y)\Big(\Big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\Big)(F_a)_{y^j}(x,y)\xi^j\Big)^2 \\ & + \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,y)} \\ & + \frac{\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,-y)}\Big).\end{aligned}$$

Choose $\varepsilon \in [\frac{1}{\sqrt{m+2}}, 1]$ and by completing the square it follows that

$$g^{F_{(m)}}|_{(x,y)}(w,w)$$

$$
\begin{aligned}
= \quad & \Big(\alpha\varepsilon F_{(m)}(x,y) + \frac{1}{2\varepsilon}F_{(m)}^{m+1}(x,y)\Big(\big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\big)(F_a)_{y^j}(x,y)\xi^j\Big)\Big)^2 \\
& + (1-\varepsilon^2)\alpha^2 F_{(m)}^2(x,y) \\
& + \big(\frac{m+2}{4} - \frac{1}{4\varepsilon^2}\big)F_{(m)}^{2m+2}(x,y)\Big(\big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\big)(F_a)_{y^j}(x,y)\xi^j\Big)^2 \\
& + \frac{F_{(m)}^{m+2}(x,y)}{2}\Big(\frac{\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,y)} \\
& + \frac{\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j - (m+1)\big((F_a)_{y^i}(x,y)\xi^i\big)^2}{F^{m+2}(x,-y)}\Big).
\end{aligned}
$$

Now we look at a part of the former expression. Set $\delta_\varepsilon = \frac{m+2}{4} - \frac{1}{4\varepsilon^2}$, whereby $\delta_\varepsilon \in [0, \frac{m+1}{4}]$ due to the choice of $\varepsilon \in [\frac{1}{\sqrt{m+2}}, 1]$. We define

$$
\begin{aligned}
& P(x,y,\delta_\varepsilon,m,\xi) \\
:= \quad & 2\delta_\varepsilon F_{(m)}^m(x,y)\big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\big)^2\big((F_a)_{y^i}(x,y)\xi^i\big)^2 \\
& + \frac{1}{F^{m+2}(x,y)}\big(\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big) \\
& + \frac{1}{F^{m+2}(x,-y)}\big(\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big) \\
= \quad & \frac{1}{F^{m+2}(x,y)F^{m+2}(x,-y)}\Bigg[2\delta_\varepsilon\frac{(F^{m+1}(x,y) - F^{m+1}(x,-y))^2}{\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}(x,y)F_a^{2k}(x,y)}\big((F_a)_{y^i}(x,y)\xi^i\big)^2 \\
& + F^{m+2}(x,-y)\big(\xi^i F(x,y)F_{y^iy^j}(x,y)\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big) \\
& + F^{m+2}(x,y)\big(\xi^i F(x,-y)F_{y^iy^j}(x,-y)\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big)\Bigg],
\end{aligned}
$$

where we used the binomial theorem (see Theorem 2.4.10) and $F = F_s + F_a$ to write

$$
\begin{aligned}
F_{(m)}^m(x,y) &= 2\big(\frac{1}{F^m(x,y)} + \frac{1}{F^m(x,-y)}\big)^{-1} \\
&= \frac{2F^m(x,y)F^m(x,-y)}{F^m(x,y) + F^m(x,-y)} \\
&= \frac{2F^m(x,y)F^m(x,-y)}{(F_s(x,y)+F_a(x,y))^m + (F_s(x,-y)+F_a(x,-y))^m} \\
&= \frac{2F^m(x,y)F^m(x,-y)}{(F_s(x,y)+F_a(x,y))^m + (F_s(x,y)-F_a(x,y))^m} \\
&= \frac{2F^m(x,y)F^m(x,-y)}{\sum_{k=0}^m F_s^{m-k}(x,y)F_a^k(x,y) + \sum_{k=0}^m F_s^{m-k}(x,y)(-1)^k F_a^k(x,y)} \\
&= \frac{F^m(x,y)F^m(x,-y)}{\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} F_s^{m-2k}(x,y)F_a^{2k}(x,y)}. \qquad (2.4.25)
\end{aligned}
$$

Remember that for $k, m \in \mathbb{N}_0$, $\binom{m}{k}$ denotes the binomial coefficient, for $k > m$ and $k < 0$ we set $\binom{m}{k} := 0$ (see Definition 2.4.8). By multiplying $P(y,\delta_\varepsilon,m)$ with a positive factor we obtain

$$
\begin{aligned}
& Q(x,y,\delta_\varepsilon,m,\xi) \\
:= \quad & \left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}(x,y)F_a^{2k}(x,y)\right)F^{m+2}(x,y)F^{m+2}(x,-y)P(x,y,\delta_\varepsilon,m,\xi)
\end{aligned}
$$

Hence, we can express $g^{F_s}|_{(x,y)}(w,w)$ in the following way

$$
\begin{aligned}
& g^{F_{(m)}}|_{(x,y)}(w,w) \\
= \; & \Big(\alpha\varepsilon F_{(m)}(x,y) + \frac{1}{2\varepsilon}F_{(m)}^{m+1}(x,y)\Big(\big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\big)(F_a)_{y^j}(x,y)\xi^j\Big)\Big)^2 \\
& + (1-\varepsilon^2)\alpha^2 F_{(m)}^2(x,y) \\
& + P(x,y,\delta_\varepsilon,m,\xi) \\
= \; & \Big(\alpha\varepsilon F_{(m)}(x,y) + \frac{1}{2\varepsilon}F_{(m)}^{m+1}(x,y)\Big(\big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\big)(F_a)_{y^j}(x,y)\xi^j\Big)\Big)^2 \\
& + (1-\varepsilon^2)\alpha^2 F_{(m)}^2(x,y) \\
& + \frac{Q(x,y,\delta_\varepsilon,m,\xi)}{\Big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}(x,y)F_a^{2k}(x,y)\Big)F^{m+2}(x,y)F^{m+2}(x,-y)}.
\end{aligned} \tag{2.4.26}
$$

We see in (2.4.26) that it is useful to investigate the expression Q regarding its positivity, what we will proceed with. As in (2.4.25), using $F = F_s + F_a$, the symmetries of F_s, F_a and multiple use of the binomial theorem yields

$$
\begin{aligned}
& Q(x,y,\delta_\varepsilon,m,\xi) \\
= \; & \Big[8\delta_\varepsilon\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_s^{m-2k}(x,y)F_a^{2k+1}(x,y)\big)^2((F_a)_{y^i}(x,y)\xi^i)^2 \\
& + \big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}(x,y)F_a^{2k}(x,y)\big)\big(\sum_{k=0}^{m+2}\binom{m+2}{k}(-1)^k F_s^{m+2-k}(x,y)F_a^k(x,y)\big) \\
& \cdot \big(\xi^i(F_s(x,y)+F_a(x,y))((F_s)_{y^iy^j}(x,y)+(F_a)_{y^iy^j}(x,y))\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big) \\
& + \big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}(x,y)F_a^{2k}(x,y)\big)\big(\sum_{k=0}^{m+2}\binom{m+2}{k}F_s^{m+2-k}(x,y)F_a^k(x,y)\big) \\
& \cdot \big(\xi^i(F_s(x,y)-F_a(x,y))((F_s)_{y^iy^j}(x,y)-(F_a)_{y^iy^j}(x,y))\xi^j - (m+1)(\xi^i(F_a)_{y^i}(x,y))^2\big)\Big].
\end{aligned}
$$

As we do have dependencies on (x,y) only and none on $(x,-y)$ any more, we will omit from now on the dependencies on (x,y), where possible. Regrouping yields

$$
\begin{aligned}
& Q(x,y,\delta_\varepsilon,m,\xi) \\
= \; & \Big[\Big(8\delta_\varepsilon\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_s^{m-2k}F_a^{2k+1}\big)^2 \\
& - 2(m+1)\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\big)\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_s^{m+2-2k}F_a^{2k}\big)\Big)((F_a)_{y^i}\xi^i)^2 \\
& + 2\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\big)\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_s^{m+2-2k}F_a^{2k}\big)\xi^i(F_s(F_s)_{y^iy^j}+F_a(F_a)_{y^iy^j})\xi^j \\
& - 2\big(\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\big)\big(\sum_{k=0}^{\lfloor \frac{m-1}{2}\rfloor+1}\binom{m+2}{2k+1}F_s^{m+1-2k}F_a^{2k+1}\big) \\
& \cdot \xi^i(F_a(F_s)_{y^iy^j}+F_s(F_a)_{y^iy^j})\xi^j\Big].
\end{aligned}
$$

Further regrouping and an index shift lead to

$$\begin{aligned}
&Q(x,y,\delta_\varepsilon,m,\xi)\\
= \quad &\Bigg[\Bigg(8\delta_\varepsilon\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_s^{m-2k}F_a^{2k+1}\Big)^2\\
&-2(m+1)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_s^{m+2-2k}F_a^{2k}\Big)\Bigg)((F_a)_{y^i}\xi^i)^2\\
&+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k-1}\Big)F_s^{m+2-2k}F_a^{2k}\Big)\\
&\cdot\xi^i(F_s(F_s)_{y^iy^j})\xi^j\\
&+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_s^{m-2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k+1}\Big)F_s^{m+2-2k}F_a^{2k}\Big)\\
&\cdot\xi^i(F_a(F_a)_{y^iy^j})\xi^j\Bigg].
\end{aligned}$$

Further simplification of the computation can be performed by assuming additionally w.l.o.g.

$$F_s(x,y) \quad = \quad 1 \tag{2.4.27}$$

due to a scaling argument exploiting the 0-homogeneity of $g^{F_{(m)}}|_{(x,y)}$ in its reference direction y, i.e. $g^{F_{(m)}}|_{(x,y)} = g^{F_{(m)}}|_{(x,y/F_s)}$. So, we get

$$\begin{aligned}
&Q(x,y,\delta_\varepsilon,m,\xi)\\
= \quad &\Bigg[\Bigg(8\delta_\varepsilon\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+1}\Big)^2\\
&-2(m+1)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_a^{2k}\Big)\Bigg)((F_a)_{y^i}\xi^i)^2\\
&+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k-1}\Big)F_a^{2k}\Big)\xi^i(F_s(F_s)_{y^iy^j})\xi^j\\
&+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k+1}\Big)F_a^{2k}\Big)\xi^i(F_a(F_a)_{y^iy^j})\xi^j\Bigg].
\end{aligned}$$

In a next step, we set specifically $\delta_1 = \frac{m+1}{4}$ and hence $\varepsilon = 1$. We assume even more that $\xi \neq 0$ with (2.4.20). So, we get

$$\begin{aligned}
&Q(x,y,\delta_1,m,\xi)\\
= \quad &\Bigg[2(m+1)\Bigg(\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+1}\Big)^2\\
&-\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_a^{2k}\Big)\Bigg)((F_a)_{y^i}\xi^i)^2\\
&+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k-1}\Big)F_a^{2k}\Big)\xi^i(F_s(F_s)_{y^iy^j})\xi^j
\end{aligned}$$

$$+2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k+1}\Big)F_a^{2k}\Big)\xi^i(F_a(F_a)_{y^iy^j})\xi^j\Big]. \quad (2.4.28)$$

Now we consider the coefficient of $\xi^i(F_a(F_a)_{y^iy^j})\xi^j$ in (2.4.28). By Lemma 2.4.11 we see that

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k+1}\Big)F_a^{2k} \quad \leq \quad 0,$$

where the assumption of Lemma 2.4.11 is fulfilled, since by means of (2.4.17) we deduce not only that

$$(\mathrm{d}_y F_a|_{(x,y)}(\eta))^2 \quad = \quad ((F_a)_{y^i}(x,y)\eta^i)^2 \quad \leq \quad \frac{1}{m+1}g^{F_s}|_{(x,y)}(\eta,\eta) \quad (2.4.29)$$

for all $\eta \in T_x\mathscr{N}$ but also by means of Euler's theorem (see Theorem 1.4.3) and choosing $\eta = y$ that

$$\begin{aligned}
F_a^2(x,y) \quad &= \quad ((F_a)_{y^i}(x,y)y^i)^2 \\
&< \quad \frac{1}{m+1}g^{F_s}|_{(x,y)}(y,y) \\
&= \quad \frac{1}{m+1}y^i((F_s)_{y^i}(x,y)(F_s)_{y^j}(x,y)+F_s(x,y)(F_s)_{y^iy^j}(x,y))y^j \\
&= \quad \frac{1}{m+1}F_s^2(x,y) \\
&\overset{F_s(y)=1}{=} \quad \frac{1}{m+1}<1.
\end{aligned} \quad (2.4.30)$$

Therefore and with condition (2.4.18) the expression

$$2\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+2}{2k}-\binom{m+2}{2k+1}\Big)F_a^{2k}\Big)\xi^i(F_a(F_a)_{y^iy^j})\xi^j \quad \geq \quad 0 \quad (2.4.31)$$

is non-negative. Now we look at the coefficient of $((F_a)_{y^i}\xi^i)^2$ in (2.4.28). By regrouping and (2.4.8) we get

$$\begin{aligned}
&\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+1}\Big)^2-\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\binom{m+2}{2k}F_a^{2k}\Big) \\
\overset{(2.4.8)}{=}\ &\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\Big(\binom{m}{2k+1}+\binom{m}{2k}\Big)F_a^{2k+1}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+1}\Big) \\
&-\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m+1}{2k}+\binom{m+1}{2k-1}\Big)F_a^{2k}\Big) \\
\overset{(2.4.8)}{=}\ &\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\Big(\binom{m}{2k+1}+\binom{m}{2k}\Big)F_a^{2k+1}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+1}\Big) \\
&-\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}+\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor+1}\Big(\binom{m}{2k-1}+\binom{m+1}{2k-1}\Big)F_a^{2k}\Big).
\end{aligned}$$

An index shift (i.e. $l = k+1$), multiple regrouping and application of (2.4.8) lead to

$$\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+2}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k}\Big)+\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2k+1}F_a^{2k+2}\Big)$$

$$
\begin{aligned}
& -(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}+\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(\binom{m}{2l+1}+\binom{m+1}{2l+1})F_a^{2l+2}) \\
= \;& (\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+2})(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(\binom{m+1}{2k+1}-\binom{m}{2k})F_a^{2k})-(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})^2 \\
= \;& (\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+1})^2-(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})^2 \quad < \quad 0,
\end{aligned}
\tag{2.4.32}
$$

where the last expression is negative by Lemma 2.4.12, since $F_a^2 < 1$. The rewritten coefficient of $((F_a)_{y^i}\xi^i)^2$ together with (2.4.29), (2.4.31), (2.4.32) and $\xi \neq 0$ lead to the estimate

$$
\begin{aligned}
& Q(x,y,\delta_1,m,\xi) \\
= \;& 2\Big[(m+1)\Big((\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+1})^2-(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})^2\Big)((F_a)_{y^i}\xi^i)^2 \\
& +(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}(\binom{m+2}{2k}-\binom{m+2}{2k-1})F_a^{2k})g^{F_s}(\xi,\xi) \\
& +(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor+1}(\binom{m+2}{2k}-\binom{m+2}{2k+1})F_a^{2k})\xi^i(F_a(F_a)_{y^iy^j})\xi^j\Big] \\
\overset{(2.4.31)}{\geq} \;& 2\Big[(m+1)\Big((\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+1})^2-(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})^2\Big)((F_a)_{y^i}\xi^i)^2 \\
& +(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}(\binom{m+2}{2k}-\binom{m+2}{2k-1})F_a^{2k})g^{F_s}(\xi,\xi)\Big] \\
\overset{(2.4.29)\&(2.4.32)}{>} \;& 2\Big((\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+1})^2-(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})^2 \\
& +(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}(\binom{m+2}{2k}-\binom{m+2}{2k-1})F_a^{2k})\Big)g^{F_s}(\xi,\xi) \\
= \;& 2\Big((\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k+1}F_a^{2k+1})^2 \\
& +(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k})(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor+1}(\binom{m+2}{2k}-\binom{m+2}{2k-1}-\binom{m}{2k})F_a^{2k})\Big)g^{F_s}(\xi,\xi),
\end{aligned}
$$

where we used also the identity

$$
\begin{aligned}
g^{F_s}|_{(x,y)}(\xi,\xi) &= \xi^i((F_s)_{y^i}(x,y)(F_s)_{y^j}(x,y)+F_s(x,y)(F_s)_{y^iy^j}(x,y))\xi^j \\
&= ((F_s)_{y^i}(x,y)\xi^i)^2+\xi^i(F_s(x,y)(F_s)_{y^iy^j}(x,y))\xi^j \\
&= \xi^i(F_s(x,y)(F_s)_{y^iy^j}(x,y))\xi^j
\end{aligned}
$$

for the special choice of $\xi \neq 0$ in (2.4.20). Notice the following identity, where we made multiple use of (2.4.8):

$$
\binom{m+2}{2k}-\binom{m+2}{2k-1}-\binom{m}{2k} \quad = \quad \binom{m+1}{2k}+\binom{m+1}{2k-1}-\binom{m+2}{2k-1}-\binom{m}{2k}
$$

$$
\begin{aligned}
&= \binom{m}{2k-1} + \binom{m+1}{2k-1} - \binom{m+2}{2k-1} \\
&= \binom{m}{2k-1} - \binom{m+1}{2k-2}.
\end{aligned}
$$

This in conjunction with some index shift and multiple use of (2.4.8) leads to

$$
\begin{aligned}
&Q(x,y,\delta_1,m,\xi) \\
> \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+1})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+1}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+1})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k+1}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor+1} (\binom{m}{2k-1} - \binom{m+1}{2k-2}) F_a^{2k}) \Big) g^{F_s}(\xi,\xi) \\
\overset{k=l+1}{=} \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+1})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+1}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+2})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k})(\sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} (\binom{m}{2l+1} - \binom{m+1}{2l}) F_a^{2l+2}) \Big) g^{F_s}(\xi,\xi) \\
= \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+1})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+1}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (\binom{m}{2k+1} + \binom{m}{2k} - \binom{m+1}{2k} \\
&+ \binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+2}) \Big) g^{F_s}(\xi,\xi) \\
\overset{(2.4.8)}{=} \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k+1} F_a^{2k+1})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+1}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (\binom{m+1}{2k+1} - \binom{m+1}{2k} \\
&+ \binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+2}) \Big) g^{F_s}(\xi,\xi) \\
= \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} + \binom{m}{2k}) F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+2}) \\
&+ (\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (\binom{m+1}{2k+1} - \binom{m+1}{2k}) F_a^{2k+2}) \Big) g^{F_s}(\xi,\xi) \\
\overset{(2.4.8)}{=} \; & 2\Big((\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m+1}{2k+1} F_a^{2k})(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (\binom{m}{2k+1} - \binom{m}{2k}) F_a^{2k+2})
\end{aligned}
$$

$$+\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}\Big)\Big(\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor}\Big(\binom{m+1}{2k+1}-\binom{m+1}{2k}\Big)F_a^{2k+2}\Big)\Big)g^{F_s}(\xi,\xi)$$

for $\xi \neq 0$ with (2.4.20). Therein, the last expression is non-negative, since $g^{F_s}|_{(x,y)}(\xi,\xi)$ is non-negative as well as

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\Big(\binom{m}{2k+1}-\binom{m}{2k}\Big)F_a^{2k+2} \quad \geq \quad 0$$

and

$$\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor}\Big(\binom{m+1}{2k+1}-\binom{m+1}{2k}\Big)F_a^{2k+2} \quad \geq \quad 0$$

hold by Lemma 2.4.11, where we used also that $F_a^2 < \frac{1}{m+1}$ by (2.4.30). So, we conclude that

$$Q(x,y,\delta_1,m,\xi) \quad > \quad 0 \tag{2.4.33}$$

for $\xi \neq 0$ with (2.4.20).

Now back to the general problem. Again, choose an arbitrary $w \in T_x\mathscr{N}\setminus\{0\}$. Represent $w = \alpha y + \xi$ with $\alpha \in \mathbb{R}$ and $\xi \in T_x\mathscr{N}$ such that $g^{F_s}|_{(x,y)}(y,\xi) = 0$. If $\xi = 0$, we deduce that $\alpha \neq 0$ as $w \neq 0$ and by (2.4.26) we get

$$\begin{aligned} g^{F_{(m)}}(w,w) &= (\alpha\varepsilon F_{(m)} + 0)^2 + (1-\varepsilon^2)\alpha^2 F_{(m)}^2 + 0 \\ &= \alpha^2 F_{(m)}^2 > 0, \end{aligned}$$

where we used that $Q(x,y,\delta_\varepsilon,m,0) = 0$ for all $\varepsilon \in [\frac{1}{\sqrt{m+2}},1]$. On the other hand, when $\xi \neq 0$, we get by chosing $\varepsilon = 1$ in (2.4.26) and combining it with (2.4.33) that

$$\begin{aligned} & g^{F_{(m)}}|_{(x,y)}(w,w) \\ = \;& g^{F_{(m)}}|_{(x,\frac{y}{F_s(x,y)})}(w,w) \\ \overset{(2.4.26)}{=} \;& \Big(\alpha F_{(m)}(x,\tilde{y}) + \frac{1}{2}F_{(m)}^{m+1}(x,\tilde{y})\Big(\Big(\frac{1}{F^{m+1}(x,\tilde{y})} - \frac{1}{F^{m+1}(x,-\tilde{y})}\Big)(F_a)_{\tilde{y}^j}(x,\tilde{y})\xi^j\Big)\Big)^2 \\ & + \frac{Q(x,\tilde{y},\delta_1,m,\xi)}{\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}(x,\tilde{y})\Big)F^{m+2}(x,\tilde{y})F^{m+2}(x,-\tilde{y})} \\ \geq \;& \frac{Q(x,\tilde{y},\delta_1,m,\xi)}{\Big(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}(x,\tilde{y})\Big)F^{m+2}(x,\tilde{y})F^{m+2}(x,-\tilde{y})} \overset{(2.4.33)}{>} 0, \end{aligned}$$

where we used that $g^{F_s}|_{(x,y)}(y,\xi) = 0$ implies $g^{F_s}|_{(x,\tilde{y})}(\tilde{y},\xi) = 0$ and $F_s(x,\tilde{y}) = 1$, since we set $\tilde{y} := y/F_s(x,y)$. This shows the claim. □

Remark 2.4.15 (**Sharpness of** (2.4.17)). In the following, some words on the sharpness of the condition (2.4.17) by looking at the positivity of the expression $Q(x,y,\delta_1,m,\xi)$ in the proof of Theorem 2.4.13. We choose $(x,y),(x,w) \in T_x\mathscr{N}$ with $F_a(x,y) = 0$, $F_s(x,y) = 1$, $w = \alpha y + \xi \in T_x\mathscr{N}\setminus\{0\}$ such that $\alpha = 0$, $\xi \in T_x\mathscr{N}$ and $g^{F_s}|_{(x,y)}(y,\xi) = 0$. Notice that in this case, we can write $F(x,y) = F_s(x,y) + F_a(x,y) = F_s(x,y)$ and $F(x,-y) = F_s(x,y) - F_a(x,y) = F_s(x,y)$. We then deduce once again by choosing $\varepsilon = 1$ in (2.4.26) and combining it with (2.4.33) that

$$\begin{aligned} & g^{F_{(m)}}|_{(x,y)}(w,w) \\ \overset{(2.4.26)}{=} \;& \Big(\alpha F_{(m)}(x,y) + \frac{1}{2}F_{(m)}^{m+1}(x,y)\Big(\Big(\frac{1}{F^{m+1}(x,y)} - \frac{1}{F^{m+1}(x,-y)}\Big)(F_a)_{y^j}(x,y)\xi^j\Big)\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{Q(x,y,\delta_1,m,\xi)}{\left(\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2k}F_a^{2k}(x,y)\right)F^{m+2}(x,y)F^{m+2}(x,-y)} \\
\overset{\alpha=0}{=}\ & \left(\frac{1}{2}F_{(m)}^{m+1}(x,y)\left(\left(\frac{1}{F_s^{m+1}(x,y)}-\frac{1}{F_s^{m+1}(x,y)}\right)(F_a)_{y^j}(x,y)\xi^j\right)\right)^2 \\
&+\frac{Q(x,y,\delta_1,m,\xi)}{F_s^{2(m+2)}(x,y)} \\
\overset{F_s=1}{=}\ & Q(x,y,\delta_1,m,\xi) \\
=\ & 2\left(\xi^i((F_s)_{y^iy^j}(x,y))\xi^j-(m+1)((F_a)_{y^i}(x,y)\xi^i)^2\right) \\
\overset{(2.4.33)}{=} & 2\left(g^{F_s}|_{(x,y)}(\xi,\xi)-(m+1)(\mathrm{d}_yF_a|_{(x,y)}(\xi))^2\right).
\end{aligned}
$$

For such special choices of y and ξ the condition

$$
\frac{(\mathrm{d}_yF_a|_{(x,y)}(\xi))^2}{g^{F_s}|_{(x,y)}(\xi,\xi)} \quad < \quad \frac{1}{m+1}
$$

is indeed sharp and necessary to ensure the positivity of the term $Q(x,y,\delta_1,m,\xi)$. Notice that for $n:=\dim\mathscr{N}>1$, such tangent vectors y,ξ can always be constructed in the following way: Choose some tangent vectors $\tau,\eta\in T_x\mathscr{N}\backslash\{0\}$ such that $\eta\notin\operatorname{span}\{\tau\}$, what is possible since $\dim T_x\mathscr{N}=n>1$. Then define

$$
\tilde{\zeta}_\theta \quad := \quad \cos\theta\tau+\sin\theta\eta\in T_x\mathscr{N}\backslash\{0\}
$$

for $\theta\in[0,\pi]$. As $F_s>0$ on $T_x\mathscr{N}\backslash\{0\}$, we can set even further $\zeta_\theta:=\tilde{\zeta}_\theta/F_s(x,\tilde{\zeta}_\theta)$. So, we ensured that $F_s(x,\zeta_\theta)=1$. Now we set $f(\theta):=F_a(\zeta_\theta)$. f is continuous on $[0,\pi]$ and

$$
\begin{aligned}
f(0) \quad &= \quad F_a(x,\zeta_0) \\
&= \quad F_a(x,\frac{\tilde{\zeta}_0}{F_s(x,\tilde{\zeta}_0)}) \\
&= \quad F_a(x,\frac{\tau}{F_s(x,\tau)}) \\
&= \quad -F_a(x,\frac{-\tau}{F_s(x,-\tau)}) \\
&= \quad -F_a(x,\frac{\tilde{\zeta}_\pi}{F_s(x,\tilde{\zeta}_\pi)}) \\
&= \quad -F_a(x,\zeta_\pi) \quad = \quad -f(\pi).
\end{aligned}
$$

By the intermediate value theorem follows then that there is some $\tilde{\theta}\in[0,\pi]$ such that $f(\tilde{\theta})=0$ and thereby we find some

$$
y \quad := \quad \zeta_{\tilde{\theta}}\in T_x\mathscr{N}
$$

with $F_s(x,y)=1$ and $F_a(x,y)=0$. Then choose $\tilde{\xi}\in\{\tau,\eta\}$ such that $\tilde{\xi}\notin\operatorname{span}\{y\}$ and set

$$
\xi \quad := \quad \tilde{\xi}-\left(g^{F_s}|_{(x,y)}(y,\tilde{\xi})/g^{F_s}|_{(x,y)}(y,\tilde{\xi})\right)\tilde{\xi}.
$$

The following three examples illustrate some merits and drawbacks of Theorem 2.4.13. We especially see in the first two examples that Theorem 2.4.13 can be applied to the Randers and two order metric but fails to be applicable to the Matsumoto metric, in contrast to Corollary 2.4.2. Nevertheless, Theorem 2.4.13 can be applied to Finsler metrics which are no (α,β)-metrics, as we show in the first of the following examples.

Example 2.4.16 (**Reversible Finsler metric plus linear perturbation**). Let $(\mathscr{N}, F)$ be an $(m+1)$-dimensional Finsler manifold with

$$F(x,y) \quad = \quad F_s(x,y) + \beta(x,y) \quad \text{for all } \ (x,y) \in T\mathscr{N},$$

where F_s is a reversible Finsler metric and β a smooth 1-form on $\mathscr{N}$ such that the condition (2.4.17) is fulfilled. Then Theorem 2.4.13 is applicable, what implies that $F_{(m)}$ is a Finsler metric. The condition (2.4.18) is automatically fulfilled, since the odd part $F_a(x,y) = \beta(x,y)$ is linear in y. Notice that this class of Finsler metrics contains the Randers metric investigated in [SST04] as a special case. For Randers metrics, we even reproduce their bound on $||\beta||_{\widehat{\alpha}}$. The class of Finsler metrics considered in this example is indeed a broader class of Finsler metrics as we can choose, for instance, F_s to be the regularized quartic metric (cf. [BCS00, p. 15]) on $\mathscr{N} = \mathbb{R}^{m+1}$, i.e.

$$F_s(x,y) \quad := \quad \sqrt{\sqrt{\sum_{i=1}^{m+1}(y^i)^4} + \varepsilon \sum_{i=1}^{m+1}(y^i)^2}$$

for all $(x,y) \in T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ with $y = (y^i)$ and some fixed $\varepsilon > 0$.

Example 2.4.17 (**Two order metric**). Herein, we study the application of Theorem 2.4.13 to the two order metric (see Example 1.4.16) on a smooth 3-manifold $\mathscr{N}$. Notice that we choose $(x,y) \in T\mathscr{N}\backslash o$ and we omit the dependence on (x,y), where possible. Remember that the two order metric has the following structure:

$$F(x,y) \quad = \quad \alpha(x,y)\phi(\frac{\beta(x,y)}{\alpha(x,y)}) \quad \text{with} \quad \phi(s) \quad = \quad (1+s)^2$$

for all $|s| < 1$ and Riemannian Finsler metric α and 1-form β such that $||\beta||_{\widehat{\alpha}} < 1$. We start by computing the even and odd part of the two order metric:

$$\begin{aligned}
F_s(x,y) \quad &= \quad \alpha(x,y)\phi_s(\frac{\beta(x,y)}{\alpha(x,y)}), \\
\phi_s(s) \quad &= \quad \frac{1}{2}\left((1+s)^2 + (1-s)^2\right) = 1 + s^2, \\
F_a(x,y) \quad &= \quad \alpha(x,y)\phi_a(\frac{\beta(x,y)}{\alpha(x,y)}), \\
\phi_a(s) \quad &= \quad \frac{1}{2}\left((1+s)^2 - (1-s)^2\right) = 2s.
\end{aligned}$$

The odd part F_a is in fact a linear perturbation term such that the condition 2.4.18 of Theorem 2.4.13 is already fulfilled. Choose local bundle coordinates (x^i, y^i) on $T\mathscr{N}$ as well as a_{kl}, a^{kl} and b_i as in Example 1.4.10. Hence, $\alpha(x,y) = \sqrt{a_{kl}(x)y^k y^l}$ and $\beta(x,y) = b_i(x)y^i$. Concretely, we get

$$\begin{aligned}
F_a(x,y) \quad &= \quad 2\beta(x,y), \\
(F_a)_{y^i}(x,y) \quad &= \quad 2b_i, \\
(F_a)_{y^i y^j}(x,y) \quad &= \quad 0.
\end{aligned}$$

So, the only thing left to check is the condition (2.4.17). Therefore, we compute the expression $(g^{F_s})^{kl}(F_a)_{y^k}(F_a)_{y^l} = 4(g^{F_s})^{kl}b_k b_l$ (see Remark 2.4.14) and we search an optimal bounding value to $||\beta||_{\widehat{\alpha}}^2 = a^{kl}b_k b_l$ such that the former expression is bounded by $\frac{1}{3}$.

$$(g^{F_s})_{kl} \quad = \quad \left(F_s^2/2\right)_{y^k y^l}$$

$$\begin{aligned}
&= a_{kl} + 2b_kb_l + 6\frac{\beta^2}{\alpha^2}b_kb_l - 4\frac{\beta^3}{\alpha^4}b_ka_{lj}y^j - 4\frac{\beta^3}{\alpha^4}b_la_{kj}y^j \\
&\quad - 4\frac{\beta^4}{\alpha^6}a_{kj}y^ja_{lj}y^j - \frac{\beta^4}{\alpha^4}a_{kl} \\
&= (1-\frac{\beta^4}{\alpha^4})a_{kl} + 2(1+\frac{\beta^2}{\alpha^2})b_kb_l + 4(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)(\frac{\beta}{\alpha}b_l - \frac{\beta^2}{\alpha^3}a_{lj}y^j).
\end{aligned}$$

We define as auxiliary quantities

$$\begin{aligned}
g_{kl} &:= (1-\frac{\beta^4}{\alpha^4})a_{kl} + 2(1+\frac{\beta^2}{\alpha^2})b_kb_l, \\
C_k &:= 2(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)
\end{aligned}$$

and we get by applying the Theorem of Matsumoto (see Theorem 1.1.8)

$$\begin{aligned}
\det g_{kl} &= (1-\frac{\beta^4}{\alpha^4})(1+2\frac{1+\frac{\beta^2}{\alpha^2}}{(1-\frac{\beta^4}{\alpha^4})}a^{kl}b_kb_l)\det a_{kl} \\
&= (1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)\det a_{kl}, \\
g^{kl} &= \frac{a^{kl}}{(1-\frac{\beta^4}{\alpha^4})} - \frac{(1-\frac{\beta^4}{\alpha^4})\det a_{kl}}{\det g_{kl}}2\frac{(1+\frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^4}{\alpha^4})^2}a^{ki}b_ia^{lj}b_j \\
&= \frac{a^{kl}}{(1-\frac{\beta^4}{\alpha^4})} - 2\frac{\det a_{kl}}{\det g_{kl}}\frac{(1+\frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^4}{\alpha^4})}a^{ki}b_ia^{lj}b_j.
\end{aligned}$$

Now we compute some terms we will need later on.

$$\begin{aligned}
g^{kl}b_l &= \left(1 - \frac{2(1+\frac{\beta^2}{\alpha^2})a^{ij}b_ib_j}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\right)\frac{a^{kl}b_l}{(1-\frac{\beta^4}{\alpha^4})} \\
&= \frac{(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l-2a^{ij}b_ib_j)a^{kl}b_l}{(1-\frac{\beta^4}{\alpha^4})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)} \\
&= \frac{a^{kl}b_l}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}, \\
g^{kl}b_kb_l &= \frac{a^{kl}b_kb_l}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}, \\
a^{kl}C_kC_l &= 4a^{kl}(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)(\frac{\beta}{\alpha}b_l - \frac{\beta^2}{\alpha^3}a_{lj}y^j) \\
&= 4(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)(\frac{\beta}{\alpha}a^{kl}b_l - \frac{\beta^2}{\alpha^3}y^k) \\
&= 4(\frac{\beta^2}{\alpha^2}a^{kl}b_kb_l - \frac{\beta^4}{\alpha^4} - \frac{\beta^4}{\alpha^4} + \frac{\beta^4}{\alpha^4}) \\
&= 4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2}), \\
a^{kl}C_kb_l &= 2(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)a^{kl}b_l \\
&= 2\frac{\beta}{\alpha}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2}), \\
g^{kl}C_kb_l &= \frac{2(\frac{\beta}{\alpha}b_k - \frac{\beta^2}{\alpha^3}a_{kj}y^j)a^{kl}b_l}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\frac{\beta}{\alpha}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}, \\
g^{kl}C_kC_l &= \frac{a^{kl}C_kC_l}{(1-\frac{\beta^4}{\alpha^4})} - 2\frac{\det a_{kl}}{\det g_{kl}}\frac{(1+\frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^4}{\alpha^4})}(C_ka^{ki}b_i)^2 \\
&= \frac{1}{(1-\frac{\beta^4}{\alpha^4})}\Big(a^{kl}C_kC_l - 2\frac{\det a_{kl}}{\det g_{kl}}(1+\frac{\beta^2}{\alpha^2})(C_ka^{ki}b_i)^2\Big) \\
&= \frac{1}{(1-\frac{\beta^4}{\alpha^4})}\Big(4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2}) - 8\frac{\det a_{kl}}{\det g_{kl}}(1+\frac{\beta^2}{\alpha^2})\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})^2\Big) \\
&= \frac{4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^4}{\alpha^4})}\Big(1 - 2\frac{\det a_{kl}}{\det g_{kl}}(1+\frac{\beta^2}{\alpha^2})(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})\Big) \\
&= \frac{4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^4}{\alpha^4})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\Big(1 - \frac{\beta^2}{\alpha^2} + 2a^{kl}b_kb_l - 2a^{kl}b_kb_l + 2\frac{\beta^2}{\alpha^2})\Big) \\
&= \frac{4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}.
\end{aligned}
$$

Using the former computations and applying the Theorem of Matsumoto once again, we get

$$
\begin{aligned}
\det(g^{F_s})_{kl} &= (1+g^{kl}C_kC_l)\det g_{kl} \\
&= \big(1 + \frac{4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\big)(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)\det a_{kl} \\
&= \big((1-\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l) + 4\frac{\beta^2}{\alpha^2}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})\big)\frac{(1+\frac{\beta^2}{\alpha^2})}{(1-\frac{\beta^2}{\alpha^2})}\det a_{kl} \\
&= \big(1 - 3\frac{\beta^2}{\alpha^2} + 2a^{kl}b_kb_l\big)\frac{(1+\frac{\beta^2}{\alpha^2})^2}{(1-\frac{\beta^2}{\alpha^2})}\det a_{kl}, \\
(g^{F_s})^{kl}b_kb_l &= g^{kl}b_kb_l - \frac{\det g_{kl}}{\det(g^{F_s})_{kl}}(g^{kl}b_kC_l)^2 \\
&= \frac{a^{kl}b_kb_l}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)} - \frac{(1-\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}{(1+\frac{\beta^2}{\alpha^2})(1-3\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)} \\
&\quad \cdot \Big(\frac{2\frac{\beta}{\alpha}(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\Big)^2 \\
&= \frac{1}{(1+\frac{\beta^2}{\alpha^2})(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\Big(a^{kl}b_kb_l - \frac{4\frac{\beta^2}{\alpha^2}(1-\frac{\beta^2}{\alpha^2})(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})^2}{(1+\frac{\beta^2}{\alpha^2})^2(1-3\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}\Big) \\
&= \frac{\Big((1+\frac{\beta^2}{\alpha^2})^2(1-3\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)a^{kl}b_kb_l - 4\frac{\beta^2}{\alpha^2}(1-\frac{\beta^2}{\alpha^2})(a^{kl}b_kb_l - \frac{\beta^2}{\alpha^2})^2\Big)}{(1+\frac{\beta^2}{\alpha^2})^3(1-\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)(1-3\frac{\beta^2}{\alpha^2}+2a^{kl}b_kb_l)}.
\end{aligned}
$$

It holds that

$$
\frac{\beta}{\alpha} \quad \leq \quad \sqrt{a^{kl}b_kb_l} \quad =: \quad K.
$$

Therein, we already know that $0 \leq K < 1$. So, we define now a function $f : [0, K] \to \mathbb{R}$ by

$$
f(t) \quad := \quad \frac{4\big((1+t)^2(1+2K-3t)K - 4t(1-t)(K-t)^2\big)}{(1+t)^3(1+2K-t)(1+2K-3t)}
$$

and for this choice of function holds

$$(g^{F_s})^{kl}(F_a)_{y^k}(F_a)_{y^l} \quad = \quad 4(g^{F_s})^{kl}b_kb_l \quad = \quad f(\frac{\beta}{\alpha}).$$

We will now make a discussion of the critical points of f.

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \quad = \quad -24\frac{(1-t)^2(K-2t)(K-t)}{(1+2K-3t)^2(1+t)^4}.$$

Therefore and due to the choice of the functions domain, the candidates for critical points are $t=0,\ K/2,\ K$. We want the function to be bounded by $\frac{1}{3}$. So, we look now at

$$\begin{aligned} f(0)-\frac{1}{3} \quad &= \quad -\frac{1-10K}{3(1+K)^2} \quad < \quad 0 \text{ for all } K<\frac{1}{10}, \\ f(K/2)-\frac{1}{3} \quad &= \quad -\frac{K^4-40K^3+24K^2-160K+16}{3(2+K)^4} \quad < \quad 0 \quad \text{for all} \quad K<\frac{1}{10}, \\ f(K)-\frac{1}{3} \quad &= \quad -\frac{K^2-10K+1}{3(1+K)^2} \quad < \quad 0 \quad \text{for all} \quad K<\frac{1}{10}. \end{aligned}$$

So, we see that we get the sharp bound $\frac{1}{10}$ for K and by reinsertion of $t=\frac{\beta}{\alpha}$ also

$$(g^{F_s})^{kl}(F_a)_{y^k}(F_a)_{y^l} \quad < \quad \frac{1}{3}$$

for $||\beta||_{\widehat{\alpha}}=\sqrt{a^{kl}b_kb_l}<\frac{1}{\sqrt{10}}$. Thus, we found again the same threshold for $||\beta||_{\widehat{\alpha}}$ as in [CS09] and Example 2.4.4.

Example 2.4.18 (**Matsumoto metric**). Herein, we will show that the Matsumoto metric known from Example 1.4.17 does not fit the assumptions of Theorem 2.4.13. Explicitly, the Matsumoto metric violates the condition (2.4.18). Notice that we choose $(x,y)\in T\mathscr{N}\backslash o$ and we omit the dependence on (x,y), where possible. Remember that the Matsumoto metric has the following structure:

$$F(x,y) \quad = \quad \alpha(x,y)\phi(\frac{\beta(x,y)}{\alpha(x,y)}) \quad \text{with} \quad \phi(s) \quad = \quad \frac{1}{1-s}$$

for all $|s|<1$ and Riemannian Finsler metric α and 1-form β such that $||\beta||_{\widehat{\alpha}}<1$. We start by computing the odd part of the Matsumoto metric. The odd part F_a of the Mathsumoto metric computes to

$$\begin{aligned} F_a(x,y) \quad &:= \quad \frac{F(x,y)-F(-y)}{2} \\ &= \quad \frac{\alpha(x,y)}{2}(\phi(\frac{\beta(x,y)}{\alpha(x,y)})-\phi(-\frac{\beta(x,y)}{\alpha(x,y)})) \\ &= \quad \alpha(x,y)\phi_a(\frac{\beta(x,y)}{\alpha(x,y)}), \end{aligned}$$

wherein we set

$$\begin{aligned} \phi_a(s) \quad &:= \quad \frac{\phi(s)-\phi(-s)}{2} \\ &= \quad \frac{1}{2}\left(\frac{1}{1-s}-\frac{1}{1+s}\right) \\ &= \quad \frac{1}{2}\left(\frac{1+s-1+s}{1-s^2}\right) \end{aligned}$$

$$= \frac{s}{1-s^2}.$$

Further, the derivatives of ϕ_a compute to be

$$\begin{aligned}
\phi_a'(s) &:= \frac{\partial}{\partial s}\phi_a(s) = \frac{1}{1-s^2} - \frac{s}{(1-s^2)^2}(-2s) \\
&= \frac{1-s^2}{(1-s^2)^2} + \frac{2s^2}{(1-s^2)^2} \\
&= \frac{1+s^2}{(1-s^2)^2}, \\
\phi_a''(s) &= := \frac{\partial^2}{\partial s^2}\phi_a(s) = \frac{\partial}{\partial s}\left(\frac{1+s^2}{(1-s^2)^2}\right) \\
&= \frac{2s}{(1-s^2)^2} - 2\frac{1+s^2}{(1-s^2)^3}(-2s) \\
&= \frac{2s-2s^3}{(1-s^2)^3} + \frac{4s+4s^3}{(1-s^2)^3} \\
&= \frac{6s+2s^3}{(1-s^2)^3}
\end{aligned}$$

Choose local bundle coordinates (x^i, y^i) on $T\mathcal{N}$ as well as a_{kl}, a^{kl} and b_i as in Example 1.4.10. Hence, $\alpha(x,y) = \sqrt{a_{kl}(x)y^k y^l}$ and $\beta(x,y) = b_i(x)y^i$. With these settings, we compute the Hessian of F_a:

$$\begin{aligned}
(F_a)_{y^i} &= \alpha_{y^i}\phi_a(\frac{\beta}{\alpha}) + \alpha\phi_a'(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\right), \\
(F_a)_{y^i y^j} &= \alpha_{y^i y^j}\phi_a(\frac{\beta}{\alpha}) + \alpha_{y^i}\phi_a'(\frac{\beta}{\alpha})\left(\frac{\beta_{y^j}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right) \\
&\quad + \alpha_{y^j}\phi_a'(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\right) + \alpha\phi_a''(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\right)\left(\frac{\beta_{y^j}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right) \\
&\quad + \alpha\phi_a'(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i y^j}}{\alpha} - \frac{\beta_{y^i}}{\alpha}\frac{\alpha_{y^j}}{\alpha} - \left(\frac{\beta_{y^j}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right)\frac{\alpha_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i y^j}}{\alpha} + \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right) \\
&= \left(\phi_a(\frac{\beta}{\alpha}) - \frac{\beta}{\alpha}\phi_a'(\frac{\beta}{\alpha})\right)\alpha_{y^i y^j} + \alpha\phi_a'(\frac{\beta}{\alpha})\frac{\beta_{y^i y^j}}{\alpha} \\
&\quad + \alpha\phi_a''(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\right)\left(\frac{\beta_{y^j}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right).
\end{aligned}$$

Exploiting that $\beta_{y^i y^j} = 0$ yields

$$(F_a)_{y^i y^j} = \left(\phi_a(\frac{\beta}{\alpha}) - \frac{\beta}{\alpha}\phi_a'(\frac{\beta}{\alpha})\right)\alpha_{y^i y^j} + \alpha\phi_a''(\frac{\beta}{\alpha})\left(\frac{\beta_{y^i}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^i}}{\alpha}\right)\left(\frac{\beta_{y^j}}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha_{y^j}}{\alpha}\right).$$

So, we get

$$F_a\,(F_a)_{y^i y^j} = \left(\phi_a^2(\frac{\beta}{\alpha}) - \frac{\beta}{\alpha}\phi_a(\frac{\beta}{\alpha})\phi_a'(\frac{\beta}{\alpha})\right)\alpha\alpha_{y^i y^j} + \phi_a(\frac{\beta}{\alpha})\phi_a''(\frac{\beta}{\alpha})\left(\beta_{y^i} - \frac{\beta}{\alpha}\alpha_{y^i}\right)\left(\beta_{y^j} - \frac{\beta}{\alpha}\alpha_{y^j}\right).$$

Further,

$$\begin{aligned}
\beta(x,y) &= b_k y^k \\
\beta_{y^i}(x,y)b^i &= b_i b^i = b_i a^{ij} b_j = b^i a_{ij} b^j = ||\beta||^2_{\hat{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
\alpha(x,y) &= \sqrt{y^k a_{kl} y^l} \\
\alpha_{y^i}(x,y) &= \frac{a_{il}y^l}{\sqrt{y^k a_{kl} y^l}} \\
\alpha_{y^i}(x,y)b^i &= \frac{b^i a_{il}y^l}{\sqrt{y^k a_{kl} y^l}} = \frac{b_l y^l}{\sqrt{y^k a_{kl} y^l}} = \frac{\beta(x,y)}{\alpha(x,y)} \\
\alpha_{y^i y^j}(x,y) &= \frac{a_{ij}}{\sqrt{y^k a_{kl} y^l}} - \frac{a_{il}y^l a_{jk} y^k}{\sqrt{y^k a_{kl} y^l}^3} = \frac{a_{ij}}{\alpha(x,y)} - \frac{a_{il}y^l a_{jk}y^k}{\alpha^3(x,y)} \\
\alpha_{y^i y^j}(x,y)b^i b^j &= \frac{a_{ij}b^i b^j}{\alpha(x,y)} - \frac{b^i a_{il} y^l b^j a_{jk} y^k}{\alpha^3(x,y)} = \frac{1}{\alpha(x,y)}\left(||\beta||^2_{\hat{\alpha}} - \left(\frac{\beta(x,y)}{\alpha(x,y)}\right)^2\right),
\end{aligned}
$$

where we raise and lower indices by contracting with a_{kl} and a^{kl}, respectively, in accordance with the Ricci calculus. Hence, we set $b^i = a^{ij} b_j$. The former computations imply

$$
\begin{aligned}
F_a\,(F_a)_{y^i y^j}\, b^i b^j &= \left(\phi_a^2(\frac{\beta}{\alpha}) - \frac{\beta}{\alpha}\phi_a(\frac{\beta}{\alpha})\phi_a^{'}(\frac{\beta}{\alpha})\right)\left(||\beta||^2_{\hat{\alpha}} - \left(\frac{\beta}{\alpha}\right)^2\right) \\
&\quad + \phi_a(\frac{\beta}{\alpha})\phi_a^{''}(\frac{\beta}{\alpha})\left(||\beta||^2_{\hat{\alpha}} - \left(\frac{\beta}{\alpha}\right)^2\right)^2 .
\end{aligned}
$$

If $0 \neq \frac{\beta}{\alpha} = ||\beta||_{\hat{\alpha}}/\sqrt{2} =: t < ||\beta||_{\hat{\alpha}}$, which can be ensured by choosing y appropriately, we get

$$
\begin{aligned}
F_a\,(F_a)_{y^i y^j}\, b^i b^j &= \left(\phi_a^2(t) - t\phi_a(t)\phi_a^{'}(t)\right) t^2 + \phi_a(t)\phi_a^{''}(t) t^4 \\
&= \frac{t^4}{(1-t^2)^2} - \frac{t^4+t^6}{(1-t^2)^3} + \frac{6t^6+2t^8}{(1-t^2)^4} \\
&= \frac{t^4+t^8-2t^6-(t^4+t^6-t^6-t^8)+6t^6+2t^8}{(1-t^2)^4} \\
&= \frac{4t^6(1+t^2)}{(1-t^2)^4} > 0.
\end{aligned}
$$

So, condition (2.4.18) is not fulfilled. Therefore, we can not use Theorem 2.4.13 to reproduce the bound on $||\beta||_{\hat{\alpha}}$ of Example 2.4.5.

2.5 Finsler mean curvature and Cartan integrand mean curvatures

In the following section, we give the definition of the Finsler mean curvature as proposed by Shen in [She98]. He introduces the Finsler mean curvature by means of the first variation of the Finsler area, namely the Busemann-Hausdorff area. After defining the Finsler mean curvature in this context, he shows that this notion, which he gives a representation in local coordinates for, is independent of the actual choice of local coordinates. The local coordinate representation derives from the Euler-Lagrange equation of the first variation of the Finsler area. Further, we showed in Lemma 2.1.14 that in some circumstances the Finsler area can be interpreted as a Cartan functional. Therefore, we assume as target space a real vector space together with cartesian coordinates, which is endowed with a Finsler metric. So, the question arises, in which way the Finsler mean curvature is related to the notions of mean curvature for Cartan functionals. Based on the local coordinate representation of the Finsler mean curvature, we compute here these relating equations.

At the beginning of the section, we give some definitions and results regarding the Finsler mean curvature as defined in [She98]. Then the computations regarding a comparison of Finsler mean curvature to Cartan mean curvatures is given. We conclude the section by defining some notions related to Finsler mean curvature, namely Finsler mean convex and admissible. These definitions are motivated by the application of the theory of Cartan functionals to Finsler area (cf. Definition 1.6.13).

We start by giving the definition of a Finsler mean curvature in a general setting. We further define Finsler-minimal immersions to be immersions of vanishing mean curvature.

Definition 2.5.1 (**Finsler mean curvature** [She98])**.** Let $\mathscr{M}$ be an oriented m-manifold and $(\mathscr{N}, F)$ an n-dimensional Finsler manifold. The *Finsler mean curvature* $\mathcal{H}_X^F$ for a C^2-immersion $X : \mathscr{M} \to \mathscr{N}$ is given in local coordinates (u^α) on $\mathscr{M}$ and a coordinate chart $\mathfrak{X} = (W, (x^i))$ on $\mathscr{N}$ by the formula (see [She98, p. 563])

$$\begin{aligned} \mathcal{H}_X^F|_u(V) \quad &= \quad \frac{1}{a_{\mathfrak{X}}^F(x,P)} \Bigg\{ \frac{\partial}{\partial x^i} a_{\mathfrak{X}}^F(x,P) - \frac{\partial^2}{\partial x^j \partial p_\beta^i} a_{\mathfrak{X}}^F(x,P) \frac{\partial X^j}{\partial u^\beta}(u) \\ &\quad - \frac{\partial^2}{\partial p_\alpha^i \partial p_\beta^j} a_{\mathfrak{X}}^F(x,P) \frac{\partial^2 X^j}{\partial u^\alpha \partial u^\beta}(u) \Bigg\} \Bigg|_{(x,P)=(X(u),\mathrm{D}X(u))} V^i \end{aligned} \tag{2.5.1}$$

wherein $P = (p_\alpha^i) \in \mathbb{R}^{n\times m}$, $V = V^i \frac{\partial}{\partial x^i}|_{X(u)} \in \mathrm{T}_{X(u)}\mathscr{N}$ and $\mathrm{D}X(u) = \left(\frac{\partial X^i}{\partial u^\alpha}\right) \in \mathbb{R}^{n\times m}$. If the immersion X is smooth, then $\mathcal{H}_X^F \in \Gamma(X^*T^*\mathscr{N})$ is a smooth section of the pullback bundle of the cotangent bundle to $\mathscr{N}$ by X, namely $X^*T^*\mathscr{N} = \bigcup_{u\in\mathscr{M}} T^*_{X(u)}\mathscr{N}$. X is said to be a *Finsler-minimal immersion* if X has vanishing Finsler mean curvature, i.e. $\mathcal{H}_X^F \equiv 0$.

Remark 2.5.2. Definition 2.5.1 makes even sense in the case we weaken the assumptions on $F : \mathrm{T}\mathscr{N} \to [0,\infty)$. Instead of choosing F to be a Finsler metric, we assume F to be smooth away from the zero section and that it satisfies **(F1)**. Further, the smoothness assumptions can be weakened to $F \in C^0(\mathrm{T}\mathscr{M}) \cap C^2(\mathrm{T}\mathscr{M}\backslash o)$.

The following theorem shows how the Finsler mean curvature relates to critical immersions of Finsler area. Concretely, Finsler-minimal immersions are critical immersions of the Finsler area and vice versa.

Theorem 2.5.3 (**First variation** [She98, Theorem 1.2])**.** *Let $\mathscr{M}$ be a smooth oriented m-manifold and $(\mathscr{N}, F)$ an n-dimensional Finsler manifold. The smooth immersion $X : \mathscr{M} \to \mathscr{N}$ has vanishing Finsler mean curvature w.r.t. the Finsler metric F (i.e. X is a Finsler-minimal immersion) if and only if it is a critical immersion to the Finsler area functional*

$$\mathcal{A}^F(X) \quad = \quad \int_{\mathscr{M}} \mathrm{d}V_{X^*F}.$$

A smooth variation *of X is a mapping $\tilde{X} \in C^\infty((-\varepsilon,\varepsilon)\times\mathscr{M},\mathscr{N})$ such that for every $t \in (-\varepsilon,\varepsilon)$ the mapping $X_t(\cdot) := \tilde{X}(t,\cdot)$ is a smooth immersion, $X_t = X$ on a compact set $\mathscr{K} \subset \mathscr{M}$ and $X_0 = X$. X is an* critical immersion *of the functional $\mathcal{A}^F$ if*

$$\int_{\mathscr{M}} \mathcal{H}_X^F(V)\mathrm{d}V_{X^*F} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{A}^F(X_t) = 0,$$

for every smooth variation $\tilde{X}$ of X and variational vector field *$V := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}X_t \in \Gamma(X^*\mathrm{T}\mathscr{N})$ with $X^*\mathrm{T}\mathscr{N} := \bigcup_{u\in\mathscr{M}} \mathrm{T}_{X(u)}\mathscr{N}$.*

Proof. [She98, Theorem 1.2] together with the fundamental lemma of calculus of variations. □

In the following, we restrict to codimension 1, namely we choose the Finsler manifold $(\mathscr{N}, F) = (\mathbb{R}^{m+1}, F)$ of dimension $n = m + 1$ as the immersion's target manifold, and we refer to the Finsler area integrand a^F in its Cartan integrand form A^F (see Definition 1.6.13). As shown in [She98, Lemma 6.3], the Finsler mean curvature $\mathcal{H}_X^F(V)$ vanishes in directions V tangential to the immersion X (cf. [EJ07, Lemma 8.1.1] for the Euclidean case). Hence, $\mathcal{H}_X^F$ is determined by its value in directions transversal to the tangent space at X. Transversal to the tangent space means that the vector in consideration is not contained in the linear span of the tangent space. As the codimension is 1 and the normal vector field N to X is transversal to the tangent space of X at every point, it is sufficient to compute $\mathcal{H}_X^F(N)$ to determine $\mathcal{H}_X^F$ completely. $\mathcal{H}_X^F(N)$ is related to A^F-mean curvature in Theorem 2.5.4. Afterwards, a representation of $\mathcal{H}_X^F$ in terms of the (full) A^F-mean curvature is given in Corollary 2.5.6.

Theorem 2.5.4 (Mean curvature comparison in normal direction). *Let $\mathscr{M}$ be a smooth oriented m-manifold and $(\mathbb{R}^{m+1}, F)$ a Finsler manifold. For a C^2-immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$ with normal $N : \mathscr{M} \to \mathbb{S}^m$ holds*

$$\mathcal{H}_X^F(N) \quad = \quad \frac{1}{A^F(X,N)}\Big\{\operatorname{trace}\left(A^F_{xz}(X,N)\right) - H_{A^F}(X,N)\Big\}. \tag{2.5.2}$$

Therein, $\operatorname{trace}\left(A^F_{xz}(x,z)\right) := \delta^{ij} A^F_{x^i z^j}(x,z) = \delta^{ij}\frac{\partial^2 A^F}{\partial x^i \partial z^j}(x,z)$.

Remark 2.5.5. Theorem 2.5.4 gives a direct relation between the Finsler mean curvature of Definition 2.5.1 and the A^F-mean curvature in Definition 1.6.7.

Proof. Choose standard cartesian coordinates (x^i) on the target space $\mathbb{R}^{m+1}$. Remember that we define in such coordinates for a function $f : \omega \times \Omega \to \mathbb{R}$ with $\omega, \Omega \subset \mathbb{R}^{m+1}$ open subsets, which is sufficiently differentiable, the matrices

$$\begin{aligned} f_x(x,z) \quad &:= \quad \left(\frac{\partial f}{\partial x^i}(x,z)\right) \in \mathbb{R}^{m+1}, \\ f_{xz}(x,z) \quad &:= \quad \left(\frac{\partial^2 f}{\partial x^i \partial z^i}(x,z)\right) \in \mathbb{R}^{(m+1)\times(m+1)}, \\ f_{zz}(x,z) \quad &:= \quad \left(\frac{\partial^2 f}{\partial z^i \partial z^i}(x,z)\right) \in \mathbb{R}^{(m+1)\times(m+1)}. \end{aligned}$$

Therein, f_{x^i} denotes f differentiated by $\frac{\partial}{\partial x^i}$ and f_{z^i} denotes f differentiated by $\frac{\partial}{\partial z^i}$ and so forth. Choose local coordinates (u^α) on $\mathscr{M}$, w.l.o.g. such that

$$N \quad = \quad \frac{\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}}{|\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}|}. \tag{2.5.3}$$

Notice that if $-N = \frac{\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}}{|\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}|}$ replacing u^1 by $-u^1$ results in new local coordinates, which fulfill (2.5.3). Let $g_{\alpha\beta}$ be the coefficients of the first fundamental form of X, $g^{\alpha\beta}$ the coefficients of the inverse to the first fundamental form and $\sqrt{\det g} := \sqrt{\det(g_{\alpha\beta})} = |\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}|$. We denote by $h_{\alpha\beta}$ the coefficients of the second fundamental form. $\mathrm{D}X \in \mathbb{R}^{(m+1)\times m}$ is the differential matrix of X, whose components are $\frac{\partial X^i}{\partial u^\alpha}$.

For further investigation, we use Definition 2.5.1 and compute the terms, which appear therein. Thereby, we replace a^F by A^F and try to express all the derivetives of a^F in terms of A^F. We choose a vector $z \in \mathbb{R}^{m+1}\backslash\{0\}$ and a matrix $P = (p_1|\cdots|p_m) = (p^i_\alpha) \in \mathbb{R}^{(m+1)\times m}$ of full rank such that $z = p_1 \wedge \ldots \wedge p_m$. Using the linearity of the wedge product one gets

$$\frac{\partial}{\partial p^i_\alpha} p_1 \wedge \cdots \wedge p_m \quad = \quad \frac{\partial}{\partial p^i_\alpha} \sum_{j=1}^{m+1} p^j_\alpha \, p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_j \wedge p_{\alpha+1} \cdots \wedge p_m$$

$$= \sum_{j=1}^{m+1} \delta^{ij} p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_j \wedge p_{\alpha+1} \cdots \wedge p_m$$
$$= p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_i \wedge p_{\alpha+1} \wedge \cdots \wedge p_m \tag{2.5.4}$$

Let us now rewrite the Finsler mean curvature by using Definition 1.1.10, Theorem 1.1.12, Euler's Theorem and the chain rule.

$$\begin{aligned}
\frac{\partial^2 A^F}{\partial x^b \partial p^i_\alpha}(x,z) &= \frac{\partial^2 A^F}{\partial x^b \partial z_c}(x,z)\frac{\partial z_c}{\partial p^i_\alpha} \\
&= \frac{\partial^2 A^F}{\partial x^b \partial z_c}(x,z)(p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_i \wedge p_{\alpha+1} \cdots \wedge p_n)_c, \qquad (2.5.5)\\
\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z) z_d &= 0, \qquad (2.5.6)\\
\frac{\partial}{\partial p^i_\alpha}\left(\frac{z_d}{|z|}\right) &= z_d \frac{\partial}{\partial p^i_\alpha}\left(\frac{1}{|z|}\right) + \frac{\partial z_d}{\partial p^i_\alpha}\frac{1}{|z|}. \qquad (2.5.7)
\end{aligned}$$

Using (2.5.6) and (2.5.7) together gives

$$\begin{aligned}
\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,\frac{z}{|z|})\frac{\partial}{\partial p^i_\alpha}\left(\frac{z_d}{|z|}\right) &= |z|\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z)\frac{\partial}{\partial p^i_\alpha}\left(\frac{z_d}{|z|}\right) \\
&= \frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z)\frac{\partial z_d}{\partial p^i_\alpha}, \qquad (2.5.8)\\
\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,\frac{z}{|z|})\frac{\partial}{\partial p^i_\alpha}\left(\frac{z_c}{|z|}\right)\frac{\partial}{\partial p^j_\beta}\left(\frac{z_d}{|z|}\right) &= |z|\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z)\frac{\partial}{\partial p^i_\alpha}\left(\frac{z_c}{|z|}\right)\frac{\partial}{\partial p^j_\beta}\left(\frac{z_d}{|z|}\right) \\
&= \frac{1}{|z|}\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z)\frac{\partial z_c}{\partial p^i_\alpha}\frac{\partial z_d}{\partial p^j_\beta}. \qquad (2.5.9)
\end{aligned}$$

Using (2.5.5), the 0-homogeneity of A^F_{zx} and Theorem 1.1.12 we get

$$\begin{aligned}
&\frac{\partial^2 A^F}{\partial x^b \partial p^i_\alpha}(x,z)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial X^b}{\partial u^\alpha}N_i \\
&= \sum_{\alpha=1}^m \left(\frac{\partial^2 A^F}{\partial x^b \partial z_c}(x,z)(p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_i \wedge p_{\alpha+1} \cdots \wedge p_n)_c\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial X^b}{\partial u^\alpha}N_i \\
&= \sum_{\alpha=1}^m \left(\frac{\partial^2 A^F}{\partial x^b \partial z_c}(x,\frac{z}{|z|})(p_1 \wedge \cdots \wedge p_{\alpha-1} \wedge e_i \wedge p_{\alpha+1} \cdots \wedge p_n)_c\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial X^b}{\partial u^\alpha}N_i \\
&= \sum_{\alpha=1}^m \frac{\partial^2 A^F}{\partial x^b \partial z_c}(X,N)\frac{\partial X^b}{\partial u^\alpha}(\frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \cdots \wedge \frac{\partial X}{\partial u^m})_c \\
&= \sum_{\alpha=1}^m \left\langle A^F_{zx}(X,N)\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \cdots \wedge \frac{\partial X}{\partial u^m}\right\rangle \\
&= \sum_{\alpha=1}^m \det\left(A^F_{zx}(X,N)\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}} \cdots, \frac{\partial X}{\partial u^m}\right). \qquad (2.5.10)
\end{aligned}$$

In the following computation, we used the 1-homogeneity of A^F_x and Theorem 1.1.12

$$\begin{aligned}
&\frac{\partial A^F}{\partial x^i}(x,z)|_{(x,P)=(X,\mathrm{D}X)}N_i \\
&= \left\langle A^F_x(X, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}), N\right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \left\langle A_x^F(X,N)\left|\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right|, N\right\rangle \\
&= \left\langle A_x^F(X,N), \frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right\rangle \\
&= \det\left(A_x^F(X,N), \frac{\partial X}{\partial u^1},\cdots,\frac{\partial X}{\partial u^m}\right). \qquad (2.5.11)
\end{aligned}$$

We also compute with the chain rule, (2.5.4), (1.5.10), (1.5.11) and (1.5.14)

$$\begin{aligned}
&\left(\frac{\partial^2 z_d}{\partial p_\alpha^i \partial p_\beta^b}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial z_d}{\partial p_\alpha^i}|_{(x,P)=(X,\mathrm{D}X)}\right)N_i \\
&= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial z_d}{\partial p_\alpha^i}|_{(x,P)=(X,\mathrm{D}X)}N_i\right) - \frac{\partial z_d}{\partial p_\alpha^i}|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial N_i}{\partial u^\alpha} \\
&= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^{\alpha-1}}\wedge N\wedge\frac{\partial X}{\partial u^{\alpha+1}}\cdots\wedge\frac{\partial X}{\partial u^m}\right)_d \\
&\quad -\left(\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^{\alpha-1}}\wedge\frac{\partial N}{\partial u^\alpha}\wedge\frac{\partial X}{\partial u^{\alpha+1}}\cdots\wedge\frac{\partial X}{\partial u^m}\right)_d \\
&= \frac{\partial}{\partial u^\alpha}\left(-\left|\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right|g^{\alpha\beta}\frac{\partial X}{\partial u^\beta}\right) \\
&\quad -(-h_{\alpha\beta})g^{\beta\gamma}\left(\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^{\alpha-1}}\wedge\frac{\partial X}{\partial u^\gamma}\wedge\frac{\partial X}{\partial u^{\alpha+1}}\cdots\wedge\frac{\partial X}{\partial u^m}\right)_d \\
&= -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial u^\alpha}\left(\sqrt{\det g}g^{\alpha\beta}\frac{\partial X^d}{\partial u^\beta}\right)\left|\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right| \\
&\quad + h_{\alpha\beta}g^{\beta\alpha}N_d\left|\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right| \\
&= -(\Delta_g X - HN)_d\left|\frac{\partial X}{\partial u^1}\wedge\cdots\wedge\frac{\partial X}{\partial u^m}\right| \\
&= 0. \qquad (2.5.12)
\end{aligned}$$

The 0-homogeneity of $\frac{\partial A^F}{\partial z_d}$ and (2.5.12) imply

$$\begin{aligned}
&\left(\frac{\partial A^F}{\partial z_d}(x,z)\frac{\partial^2 z_d}{\partial p_\alpha^i \partial p_\beta^b}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= \left(\frac{\partial A^F}{\partial z_d}(x,\frac{z}{|z|})\frac{\partial^2 z_d}{\partial p_\alpha^i \partial p_\beta^b}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= \frac{\partial A^F}{\partial z_d}(X,N)\left(\frac{\partial^2 z_d}{\partial p_\alpha^i \partial p_\beta^b}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= 0 \qquad (2.5.13)
\end{aligned}$$

Multiple use of the chain rule, the -1-homogeneity of A_{zz}^F, (2.5.13), (2.5.8), (2.5.5) and Theorem 1.1.12 lead to

$$\begin{aligned}
&\frac{\partial^2 A^F}{\partial p_\alpha^i \partial p_\beta^b}(x,z)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= \left(\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x,z)\frac{\partial z_c}{\partial z_\alpha^i}\frac{\partial z_d}{\partial z_\beta^b} + \frac{\partial A^F}{\partial z_d}(x,z)\frac{\partial^2 z_d}{\partial p_\alpha^i \partial p_\beta^b}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x, \frac{z}{|z|})\frac{\partial z_c}{\partial p^i_\alpha}\frac{\partial}{\partial p^b_\beta}\left(\frac{z_d}{|z|}\right)\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial^2 X^b}{\partial u^\alpha \partial u^\beta}N_i \\
&= \left(\frac{\partial^2 A^F}{\partial z_c \partial z_d}(x, \frac{z}{|z|})\frac{\partial z_c}{\partial p^i_\alpha}\right)|_{(x,P)=(X,\mathrm{D}X)}\frac{\partial N_d}{\partial u^\alpha}N_i \\
&= \sum_{\alpha=1}^m \frac{\partial^2 A^F}{\partial z_c \partial z_d}(X,N)\frac{\partial N_d}{\partial u^\alpha}(\frac{\partial X}{\partial u^1}\wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}}\wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}}\cdots \wedge \frac{\partial X}{\partial u^m})_c \\
&= \sum_{\alpha=1}^m \left\langle A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}\wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}}\wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}}\cdots \wedge \frac{\partial X}{\partial u^m}\right\rangle \\
&= \sum_{\alpha=1}^m \det\left(A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right). \qquad (2.5.14)
\end{aligned}
$$

Putting together (2.5.1), (2.5.10), (2.5.11) and (2.5.13) we compute another representation for the Finsler mean curvature, i.e.

$$
\begin{aligned}
\mathcal{H}^F_X(N) = \frac{1}{A^F(X, \frac{\partial X}{\partial u^1}\wedge \ldots \wedge \frac{\partial X}{\partial u^m})}\Bigg\{ &\det\left(A^F_x(X,N), \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&- \sum_{\alpha=1}^m \det\left(A^F_{zx}(X,N)\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right) \\
&- \sum_{\alpha=1}^m \det\left(A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right)\Bigg\}.
\end{aligned}
\qquad (2.5.15)
$$

This further simplifies by exploiting the 1-homogeneity of A^F_x, exchanging columns of some of the involved matrices and applying Theorem 1.1.7, i.e.

$$
\begin{aligned}
\mathcal{H}^F_X(N) &= \frac{|\frac{\partial X}{\partial u^1}\wedge \ldots \wedge \frac{\partial X}{\partial u^m}|}{A^F(X,N)}\Bigg\{ \det\left(A^F_{xz}(X,N)N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&\quad + \sum_{\alpha=1}^m \det\left(N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, A^F_{zx}(X,N)\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right) \\
&\quad - \sum_{\alpha=1}^m \det\left(A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right)\Bigg\} \\
&= \frac{|\frac{\partial X}{\partial u^1}\wedge \ldots \wedge \frac{\partial X}{\partial u^m}|}{A^F(X,N)}\Bigg\{ \det\left(A^F_{zx}(X,N)N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&\quad + \sum_{\alpha=1}^m \det\left(N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, A^F_{zx}(X,N)\frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right) \\
&\quad - \sum_{\alpha=1}^m \det\left(A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right)\Bigg\} \\
&= \frac{1}{A^F(X,N)}\Bigg\{\mathrm{tr}\left(A^F_{zx}(X,N)\right)\det\left(N, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^m}\right) \\
&\quad - \sum_{\alpha=1}^m \det\left(A^F_{zz}(X,N)\frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}\cdots, \frac{\partial X}{\partial u^m}\right)\Bigg\}.
\end{aligned}
$$

Hence, we get

$$
\mathcal{H}^F_X(N) = \frac{1}{|\frac{\partial X}{\partial u^1}\wedge \ldots \wedge \frac{\partial X}{\partial u^m}|A^F(X,N)}\Bigg\{\left|\frac{\partial X}{\partial u^1}\wedge \ldots \wedge \frac{\partial X}{\partial u^m}\right| \mathrm{trace}\left(A^F_{xz}(X,N)\right)
$$

$$- \sum_{\alpha=1}^{m} \det \left(A^F_{zz}(X,N) \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}} \cdots, \frac{\partial X}{\partial u^m} \right) \Bigg\}. \tag{2.5.16}$$

Theorem 1.6.8 implies that the A^F-mean curvature is given in local coordinates as follows

$$H_{A^F}(X,N) = -g^{\alpha\beta} \left\langle A^F_{zz}(X,N) \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} \right\rangle. \tag{2.5.17}$$

Remember further the identity (1.5.11), i.e.

$$- \left| \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right| g^{\alpha\beta} \frac{\partial X}{\partial u^\beta} = \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m}. \tag{2.5.18}$$

This together with (2.5.17) and (1.1.9) leads to a rewritten form of the A^F-mean curvature, namely

$$\begin{aligned} & \left| \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right| H_{A^F}(X,N) \\ = & \sum_{\alpha=1}^{m} \left\langle A^F_{zz}(X,N) \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1} \wedge \cdots \wedge \frac{\partial X}{\partial u^{\alpha-1}} \wedge N \wedge \frac{\partial X}{\partial u^{\alpha+1}} \wedge \cdots \wedge \frac{\partial X}{\partial u^m} \right\rangle \\ = & \sum_{\alpha=1}^{m} \det \left(A^F_{zz}(X,N) \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^1}, \cdots, \frac{\partial X}{\partial u^{\alpha-1}}, N, \frac{\partial X}{\partial u^{\alpha+1}}, \cdots, \frac{\partial X}{\partial u^m} \right\rangle. \end{aligned}$$

If we put this into the representation (2.5.16) of the Finsler mean curvature, we get

$$\mathcal{H}^F_X(N) = \frac{1}{A^F(X,N)} \Big\{ \operatorname{trace} \left(A^F_{xz}(X,N) \right) - H_{A^F}(X,N) \Big\},$$

what concludes the proof. □

Corollary 2.5.6 (Mean curvature comparison). *Let $\mathscr{M}$ be a smooth oriented m-manifold and $(\mathbb{R}^{m+1}, F)$ a Finsler manifold. For a C^2-immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$ with Gauss map $N : \mathscr{M} \to \mathbb{S}^m$ holds*

$$\begin{aligned} \mathcal{H}^F_X(V) &= \frac{1}{A^F(X,N)} \Big\{ \operatorname{trace} \left(A^F_{xz}(X,N) \right) - H_{A^F}(X,N) \Big\} \langle N, V \rangle \\ &= - \frac{h_{A^F}(X,N)}{A^F(X,N)} \langle N, V \rangle. \end{aligned} \tag{2.5.19}$$

Remark 2.5.7.

- Due to the fact that $A^F(x,-z) = A^F(x,z)$ for all $(x,z) \in T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ and the characterizing identity (2.5.19), the Finsler mean curvature $\mathcal{H}^F_X$ is independent of the actual choice of orientation of the immersion X.
- Assume $(\mathbb{R}^{m+1}, |\cdot|)$ with a choice of Finsler metric $F = |\cdot|$. The Euclidean area integrand then becomes $A^{|\cdot|} = |\cdot|$, which is a simple example for a Cartan integrand. The mean curvature of a smooth immersion $X : \mathscr{M} \to \mathbb{R}^{m+1}$ with smooth oriented m-manifold $\mathscr{M}$ (see Definition 1.5.1) equals by Theorem 1.6.8 the multiplicative inverse of the (full) $|\cdot|$-mean curvature $-H_{|\cdot|}$ and $-h_{|\cdot|}$. The latter expression is by Corollary 2.5.6 directly related to the Finsler mean curvature $\mathcal{H}^F$ and we thereby get

$$H = -H_{|\cdot|}(N,X) = -h_{|\cdot|}(X,N) = \mathcal{H}^{|\cdot|}_X(N).$$

We exploited also the fact that $(\mathbb{R}^{m+1}, |\cdot|)$ is a Minkowski space.

In the following definition, we introduce among others the notion Finsler area mean convex. The definition is motivated by similar notions for Cartan integrands, which can be found in [Whi91] and [BF09] (see Definition 1.6.13). In chapter 3, we apply theorems of those sources to a Finsler setting, whereby their mean convex notions translate directly to those in Definition 2.5.8. We exploit especially the fact that the (full) A^F-mean curvature is related to the Finsler mean curvature by means of Corollary 2.5.6.

Definition 2.5.8. Let $(\mathbb{R}^{m+1}, F)$ be a Finsler manifold, i.e. F satisfies **(F3)**. Let Ω be a domain with C^2-boundary $\partial\Omega$ in $\mathbb{R}^{m+1}$. Assume that $\partial\Omega$ has the Finsler mean curvature vector $\mathcal{H}_X^F$. Ω is called

- *star-shaped* (w.r.t. some $q_0 \in \Omega$) if for every $q \in \Omega$ holds that $q_0 + \lambda(q - q_0) \in \Omega$ for any $0 < \lambda < 1$;
- *Finsler mean convex* if $\mathcal{H}_X^F(V) \leq 0$ at every $q \in \partial\Omega$, wherein $V \in \mathrm{T}_q\mathbb{R}^{m+1}$ is an inward-pointing vector and $X : \mathscr{M} \to \partial\Omega$ is a parametrization of a neighbourhood of q;
- *strictly Finsler mean convex* if $\mathcal{H}_X^F(V) < 0$ at every $q \in \partial\Omega$, wherein $V \in \mathrm{T}_q\mathbb{R}^{m+1}$ is an inward-pointing vector and $X : \mathscr{M} \to \partial\Omega$ is a parametrization of a neighbourhood of q;
- *Finsler admissible* if there exists an index set I, which is non-empty, and a family of star-shaped Finsler mean convex C^2-domains $\{\Omega^i\}_{i \in I}$ such that

$$\Omega = \bigcap_{i \in I} \Omega^i \text{ and } \overline{\Omega} = \bigcap_{i \in I} \overline{\Omega^i}.$$

Remark 2.5.9.

- In case the Finsler area integrand A^F is elliptic and independent of the first argument, what is especially fulfilled if F is a Minkowski metric, i.e. **(F3)** and **(M)** hold. Then, every convex set is admissible as it can be written as the intersection of all half spaces containing the set. Especially, if we assume that the boundary of the convex set is of class C^2, the boundary is Finsler mean convex.
- The notions defined in Definition 2.5.8 equal the ones of Definition 1.6.13, respectively, when we choose therein A^G as the Cartan integrand in consideration.

Application of Cartan functional theory on Finsler area

The major aim of this chapter is the application of commonly known results for Cartan functionals (see section 1.6) to Finsler area. There is a wide range of theorems concerning existence and regularity of minimizers of Cartan functionals, inclusion theorems for critical immersions and Bernstein type theorems issuing from curvature estimates for locally minimizing immersions. We will give some results without being exhaustive.

3.1 Introduction

In this introductory section, we recapitulate some the implications that properties of F have on the Finsler area integrand A^F.

Let $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ be a function satisfying (**F1**), i.e. F is a continuous homogeneous function, which is positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. Then Corollary 2.3.3 implies that A^F is a continuous Cartan integrand, which is positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$, i.e.

$$A^F \in C^0(\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}),$$

A^F satisfies (**H**) or in other words $A^F(x, tz) = tA^F(x, z)$ for all $t > 0$ and $(x, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$, and

$$A^F \quad > \quad 0$$

on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. The latter property can be quantified, i.e. for each $R > 0$ holds

$$m_1^m|z| \quad \leq \quad A^F(x, z) \quad \leq \quad m_2^m|z| \tag{3.1.1}$$

for all $(x, z) \in \overline{B_R^{m+1}(0)} \times \mathbb{R}^{m+1}$ with positive constants m_1, m_2 defined by

$$m_1 \quad := \quad m_1(R) \quad := \quad \min_{\overline{B_R^{m+1}(0)} \times \mathbb{S}^m} F(\cdot, \cdot),$$
$$m_2 \quad := \quad m_2(R) \quad := \quad \max_{\overline{B_R^{m+1}(0)} \times \mathbb{S}^m} F(\cdot, \cdot).$$

Notice that for such positive constants m_1, m_2 holds

$$m_1|y| \quad \leq \quad F(x, y) \quad \leq \quad m_2|y|$$

for all $(x,y) \in \overline{B_R^{m+1}(0)} \times \mathbb{R}^{m+1}$. If m_1 and m_2 can be chosen to be positive constants independent of R, we say in correspondence to (**D**) that F is *positive definite*, i.e.

$$0 \quad < \quad m_1 \quad := \quad \inf_{\mathbb{R}^{m+1}\times\mathbb{S}^m} F(\cdot,\cdot) \quad \leq \quad \sup_{\mathbb{R}^{m+1}\times\mathbb{S}^m} F(\cdot,\cdot) \quad =: \quad m_2 \quad < \quad \infty. \qquad (\mathbf{d})$$

In this case, A^F is also positive definite as a Cartan integrand, i.e. A^F satisfies (**D**) with constants $M_1 := m_1^m$ and $M_2 := m_2^m$. If $F \in C^k(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ for $k \in \mathbb{N} \cup \{0,\infty\}$, then

$$A^F \in C^k(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$$

by Corollary 2.3.3. So, in that sense A^F is at least as regular as F. Notice even more that A^F is even, i.e. $A^F(x,z) = A^F(x,-z)$ for all $(x,z) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$ (see (2.1.21)). If F satisfies (**GA1**), i.e. $F \in C^\infty(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ satisfies (**F1**) and its m-symmetrization $F_{(m)}$ satisfies (**F2**), then Corollary 2.3.6 guarantees that A^F is semi-elliptic. If F satisfies (**GA2**) that is $F \in C^\infty(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ satisfies (**F1**) and $F_{(m)}$ satisfies (**F3**) ($F_{(m)}$ is especially elliptic), then Corollary 2.3.8 guarantees that A^F is an elliptic Cartan integrand, which is positive on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. Notice that we will assume in most theorems of this chapter that F satisfies the stronger condition (**GA3**) even though (**GA2**) would be sufficient. That F satisfies (**GA3**) is equivalent to $(\mathbb{R}^{m+1},F)$ and $(\mathbb{R}^{m+1},F_{(m)}$ being Finsler manifolds. The assumption of $(\mathbb{R}^{m+1},F)$ and $(\mathbb{R}^{m+1},F_{(m)})$ to be Minkowski spaces is equivalent to F satisfying (**GAM**). In the following, if $A^F \in C^2(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\}))$ is an elliptic Cartan integrand, we will denote its lower and upper ellipticity constants by $\Lambda_1 = \Lambda_1(R)$ and $\Lambda_2 = \Lambda_2(R)$ for all $R > 0$. Then for each $R > 0$ is

$$\frac{\Lambda_1(R)}{|z|}\left[|\xi|^2 - \left\langle \xi, \frac{z}{|z|}\right\rangle^2\right] \quad \leq \quad A^F_{z^iz^j}(x,z)\xi^i\xi^j \quad \leq \quad \frac{\Lambda_2(R)}{|z|}\left[|\xi|^2 - \left\langle \xi, \frac{z}{|z|}\right\rangle^2\right]$$

for all $(x,z) \in \overline{B_R^{m+1}(0)} \times (\mathbb{R}^{m+1}\backslash\{0\})$ and $\xi \in \mathbb{R}^{m+1}$. This stems from the ellipticity quantification (**EQ**) presented in Lemma 1.6.6.

3.2 Existence and regularity for Finsler surfaces

What follows is the formulation of the *Plateau problem* (cf. subsection 1.6.3). Let Γ be a closed, rectifiable Jordan curve in $\mathbb{R}^3$ with $B := \{(u^1,u^2) : (u^1)^2 + (u^2)^2 < 1\}$ the unit disc, which is chosen to be the parameter domain of the competing surfaces $X : B \to \mathbb{R}^3$. $\mathcal{C}(\Gamma)$ denotes the class of surfaces $X \in H^{1,2}(B,\mathbb{R}^3)$ whose trace $X|_{\partial B}$ on ∂B is a continuous, weakly monotonic mapping of ∂B onto Γ. $\mathcal{C}(\Gamma)$ is non-empty.

Corollary 3.2.1. *Let $(\mathbb{R}^3,F)$ be a weak Finsler manifold such that $(\mathbb{R}^3,F_{(2)})$ is also a weak Finsler manifold. Especially, $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0,\infty)$ then satisfies* (**GA1**). *Assume further that F is positive definite, i.e.*

$$0 \quad < \quad m_1 \quad := \quad \inf_{\mathbb{R}^3\times\mathbb{S}^2} F(\cdot,\cdot) \quad \leq \quad \sup_{\mathbb{R}^3\times\mathbb{S}^2} F(\cdot,\cdot) \quad =: \quad m_2 < \infty$$

(see (**d**)*). Then for any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ there exists a minimizer $X \in \mathcal{C}(\Gamma)$ of the Finsler area, i.e.*

$$\mathcal{A}^F(X) \quad = \quad \inf_{\mathcal{C}(\Gamma)} \mathcal{A}^F(\cdot).$$

Furthermore, the minimizer X is Euclidean conformally parametrized almost everywhere on B, i.e.

$$|X_{u^1}|^2 \quad = \quad |X_{u^2}|^2 \quad and \quad \langle X_{u^1}, X_{u^2}\rangle \quad = \quad 0 \quad \mathscr{H}^2\text{-a.e. on } B,$$

and

$$X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^{0,\gamma}(B, \mathbb{R}^3) \cap W^{1,2}(B, \mathbb{R}^3)$$

for $\gamma := (m_1/m_2)^2 \in (0,1]$ *and some* $q > 2$*.*

Remark 3.2.2. The notion rectifiable for a Jordan curve can be formulated in terms of an positive definite Finsler metric F on $\mathbb{R}^{m+1}$ (cf. Corollary 3.2.1). F then satisfies

$$m_1|y| \quad \leq \quad F(x,y) \quad \leq \quad m_2|y|$$

for all $(x,y) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \backslash \{0\})$ with constants m_i for $i = 1,2$ as in (**d**). Then follows directly that

$$m_1 \mathcal{L}(c) \quad \leq \quad \mathcal{L}_F(c) \quad \leq \quad m_2 \mathcal{L}(c) \tag{3.2.1}$$

for any curve $c : I \to \mathbb{R}^{m+1}$ of class C^1 on an interval I, since $\mathcal{L}(c) = \mathcal{L}_{|\cdot|}(c) = \int_I |\dot{c}| \mathrm{d}t$ and $\mathcal{L}_F(c) = \int_I F(c, \dot{c}) \mathrm{d}t$ by (1.4.1).

Proof of Corollary 3.2.1. As discussed in section 3.1, the assumptions on F and $F_{(2)}$ directly imply that A^F is a positive definite semi-elliptic Cartan integrand, i.e. (**H**), (**D**) and (**C**) hold. Thus, we can apply Theorem 1.6.17 to the present situation what yields the stated result. □

The following theorem is a result of a combination of Lemma 2.3.10, Corollary 1.6.21, Theorem 1.6.22 and Theorem 1.6.23.

Theorem 3.2.3 (Higher regularity of minimizers). *There is a universal constant* $\delta_0 \in (0,1)$ *such that any Finsler-area minimizing and conformally parametrized (see* (1.6.12)*) surface* $X \in \mathcal{C}(\Gamma)$ *is of class* $W^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^3) \cap C^{1,\sigma}(B, \mathbb{R}^3)$ *for some* $\sigma \in (0,1)$ *if the Finsler structure* $F = F(x,y)$ *satisfies*

$$\widehat{\rho}_2(F(x,\cdot) - |\cdot|) \quad < \quad \delta_0$$

for all $x \in \mathbb{R}^3$*. Moreover if in addition the boundary contour* Γ *is assumed to be of class* C^4 *one obtains* $X \in X^{2,2}(B, \mathbb{R}^3) \cap C^{1,\alpha}(\bar{B}, \mathbb{R}^3)$ *and a constant* $c = \mathcal{C}(\Gamma, F)$ *depending on* Γ *and* F *such that*

$$||X||_{W^{2,2}(B,\mathbb{R}^3)} + ||X||_{C^{1,\alpha}(\bar{B},\mathbb{R}^3)} \quad \leq \quad \mathcal{C}(\Gamma, F).$$

Proof. We will use Lemma 2.3.10 wherein we set F, $G(x,\cdot) := |\cdot|$, $m = 2$, $k = 2$. Thereby, the estimate becomes

$$\widehat{\rho}_2(|\cdot| - A^F(x,\cdot)) \quad \leq \quad C \widehat{\rho}_2(|\cdot| - F(x,\cdot)) \left(\frac{\tilde{\rho}_2(F(x,\cdot))}{F_0(x)} \frac{\tilde{\rho}_2(|\cdot|)}{|\cdot|_0} \right)^{12}, \tag{3.2.2}$$

where $\tilde{\rho}_2(f(\cdot)) := \max\{1, \widehat{\rho}_2(f(\cdot))\}$ for some function $f \in C^2(\mathbb{S}^2)$ and $f_0 := \min\{1, \min_{\mathbb{S}^m} f(\cdot)\}$ for some function $f \in C^0(\mathbb{S}^2)$. Assume that

$$\widehat{\rho}_2(F(x,\cdot) - |\cdot|) \quad < \quad \delta_0 \tag{3.2.3}$$

holds, where we will set the constant δ_0 later on in the proof. Therewith, we compute by using the triangle inequality

$$\begin{aligned} \widehat{\rho}_2(F(x,\cdot)) \quad &\leq \quad \widehat{\rho}_2(|\cdot|) + \widehat{\rho}_2(F(x,\cdot) - |\cdot|) \\ &\leq \quad \widehat{\rho}_2(|\cdot|) + \delta_0 \end{aligned}$$

and therefore

$$\begin{aligned} \tilde{\rho}_2(F(x,\cdot)) &= \max\{1, \widehat{\rho}_2(F(x,\cdot))\} \\ &\leq \max\{1, \widehat{\rho}_2(|\cdot|) + \delta_0\} \\ &\leq \tilde{\rho}_2(|\cdot|) + \delta_0. \end{aligned} \tag{3.2.4}$$

Even more, direct computation yields

$$\begin{aligned} |\cdot|_0 &= \min\{1, \min_{\mathbb{S}^m} |\cdot|\} \\ &= 1. \end{aligned} \tag{3.2.5}$$

Using the triangle inequality again together with (3.2.3) and (3.2.5), we compute

$$\begin{aligned} \min_{\mathbb{S}^m} F(x,\cdot) &= \min_{\mathbb{S}^m}(|\cdot| + F(x,\cdot) - |\cdot|) \\ &\geq \min_{\mathbb{S}^m}(|\cdot| - |F(x,\cdot) - |\cdot||) \\ &\geq \min_{\mathbb{S}^m} |\cdot| - \max_{\mathbb{S}^m}(|F(x,\cdot) - |\cdot||) \\ &= 1 - \widehat{\rho}_0(F(x,\cdot)) \\ &\geq 1 - \delta_0 \end{aligned}$$

and consequently

$$\begin{aligned} F_0(x,\cdot) &= \min\{1, \min_{\mathbb{S}^m} F_0(\cdot)\} \\ &\geq 1 - \delta_0 \end{aligned} \tag{3.2.6}$$

for $\delta_0 > 0$. Putting (3.2.4), (3.2.5) and (3.2.6) into (3.2.2) yields

$$\begin{aligned} \widehat{\rho}_2(|\cdot| - A^F(x,\cdot)) &\leq C\widehat{\rho}_2(|\cdot| - F(x,\cdot)) \left(\frac{\tilde{\rho}_2(|\cdot|)(\tilde{\rho}_2(|\cdot|) + \delta_0)}{1 - \delta_0} \right)^{12} \\ &\leq C\widehat{\rho}_2(|\cdot| - F(x,\cdot)) \left(\tilde{\rho}_2(|\cdot|)(2\tilde{\rho}_2(|\cdot|) + 1)\right)^{12} \\ &\leq \tilde{C}\delta_0 \end{aligned} \tag{3.2.7}$$

for $0 < \delta_0 < 1/2$ and with $\tilde{C} = \tilde{C}(|\cdot|) := (\tilde{\rho}_2(|\cdot|)(2\tilde{\rho}_2(|\cdot|) + 1))^{12} C > 0$. We chose now $\delta_0 < \min\{\frac{1}{2}, \frac{1}{5\tilde{C}}\}$. Combining this choice of δ_0 with (3.2.7) implies

$$\widehat{\rho}_2(|\cdot| - A^F(x,\cdot)) < \frac{1}{5}.$$

So, the Finsler area integrand A^F - as a Cartan integrand (see Definition 1.6.1) - fulfills the assumptions of Theorem 1.6.20 and thereby A^F possesses a perfect dominance function. Hence, Theorem 1.6.22 and Theorem 1.6.23 can be applied to the present situation and yield the stated higher regularity of the minimizer. □

The following theorem establishes existence in a smooth setting. Remember that the notion Finsler mean convex is essentially another name for A^F-mean convex (see section 2.5).

Theorem 3.2.4 (Existence in a smooth setting). *Let $(\mathbb{R}^3, F)$ be a Finsler manifold such that $(\mathbb{R}^3, F_{(2)})$ is also a Finsler manifold, i.e. $F : T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ satisfies* **(GA3)***. Then, the following to statements are true:*

(i) *If $\Gamma \subset \mathbb{R}^3$ is a smooth closed Jordan curve contained in the boundary of a strictly Finsler mean convex body, then for each $g \geq 0$ there exists a smooth embedded surface spanning Γ, which minimizes the Finsler area among all smooth embedded surfaces M with $\operatorname{genus}(M) \leq g$ spanning Γ.*

(ii) *If $(\mathbb{R}^3, F)$ is a Minkowski space, i.e. $F = F(y)$, and Γ is a graph of bounded slope over the boundary $\partial\Omega$ of some plane bounded convex domain $\Omega \subset \mathbb{R}^2$, then there is an (up to reparametrizations) unique Finsler-minimal immersions $X \in C^\infty(\Omega, \mathbb{R}^3) \cap C^0(\Omega, \mathbb{R}^3)$ spanning Γ. Notice that the surface X parametrizes a graph.*

Remark 3.2.5.

- In (i) a smooth embedded surface M spans Γ if $\partial M = \Gamma$. In (ii) an immersion $X \in C^\infty(\Omega, \mathbb{R}^3) \cap C^0(\Omega, \mathbb{R}^3)$ is said to span Γ if $X(\partial\Omega) = \Gamma$;
- $\operatorname{genus}(M)$ denotes the topological genus of the smooth oriented manifold M (see Remark 1.6.25);
- The *bounded slope* condition in the last part of the theorem means that we find a constant $R > 0$, so that we can write Γ as a graph,

$$\Gamma = \{(u, \gamma(u)) \in \mathbb{R}^3 : u = (u^1, u^2) \in \partial\Omega\}$$

  for some function $\gamma : \partial\Omega \to \mathbb{R}$, such that for any curve point $(u_0, \gamma(u_0)) \in \Gamma$ there exist two vectors $p_0^+, p_0^- \in \overline{B_R^2(0)} \subset \mathbb{R}^2$ such that the two affine linear functions

$$\ell_0^+(u) := p_0^+ \cdot (u - u_0) + \gamma(u_0) \quad \text{and} \quad \ell_0^-(u) := p_0^- \cdot (u - u_0) + \gamma(u_0)$$

  satisfy $\ell_0^-(u) \leq \gamma(u) \leq \ell_0^+(u)$ for all $u \in \partial\Omega$. In particular, if Ω is strictly convex and Γ is a C^2-graph over $\partial\Omega$ then Γ satisfies the bounded slope condition (see Remark 1.6.27 or [GT01, pp. 309, 310]).

Proof of Theorem 3.2.4. As discussed in section 3.1, the assumptions on F and $F_{(2)}$ directly imply that A^F is a positive definite elliptic Cartan integrand, i.e. (**H**), (**D**) and (**E**) hold. Thus, we can apply Theorem 1.6.24 to the present situation what yields the stated result (i). Notice that A^G is even in the second argument and the notion Finsler mean convex of Definition 2.5.8 translates directly to A^F-mean convex given in Definition 1.6.13. The statement of (ii) follows in the same way by Theorem 1.6.26 to the Cartan integrand A^F, which is independent of the first argument for a Minkowski metric F. □

Corollary 3.2.6 (Existence in a smooth setting). *Let $(\mathbb{R}^3, F)$ be a Minkowski space such that $(\mathbb{R}^3, F_{(2)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ then satisfies* (**GAM**). *If $\Gamma \subset \mathbb{R}^3$ is a smooth closed Jordan curve contained in the boundary of a strictly convex body, then there exists a smooth embedded Finsler-minimal surface spanning Γ.*

Proof. The statement of the theorem follows directly by Theorem 3.2.4 (i), if we take into account that in a Minkowski space every strictly convex set with smooth boundary is Finsler mean convex (see Remark 2.5.9). □

The following theorem incorporates an energy estimate for Finsler-minimal graphs in a Minkowski setting, which leads ultimately to an uniqueness result for such graphs.

Theorem 3.2.7 (Energy estimate for graphs). *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfies* (**GAM**). *Let Ω be an open set in $\mathbb{R}^m$ with compact closure, K a compact subset of Ω with $\mathscr{H}^{m-1}(K) = 0$. Let $\Lambda_1 > 0$ be the lower ellipticity constant of the Finsler area integrand, whose existence is ensured by Lemma 2.3.7. Assume two critical graphs of the non-parametric* *problem (cf. Example 1.6.12) $f_1, f_2 \in C^0(\overline{\Omega} \times \mathbb{R}) \cap C^2(\Omega \times \mathbb{R})$ then we have the following weighted energy estimate*

$$\int_{\Omega\backslash K} \mu(f_1,f_2)|\nabla f_1 - \nabla f_2|^2 \mathrm{d}x \quad \leq \quad \frac{2}{\Lambda_1}\int_{\partial\Omega} |f_1 - f_2| \mathrm{d}\mathscr{H}^{m-1}$$

with

$$\mu(f_1,f_2)(x) \quad := \quad (\max\{W_1(x), W_2(x)\})^{-3} \quad \text{for } x \in \Omega\backslash K.$$

Therein are

$$W_i(x) \quad := \quad \sqrt{1+|\nabla f_i|^2} \quad \text{for } i = 1,2.$$

Proof. The assumptions on F and $F_{(m)}$ imply as mentioned in section 3.1 that $A^F \in C^0(\mathrm{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^\infty(\mathrm{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$ is a positive definite elliptic Cartan integrand independent of the first argument., i.e. $A^F = A^F(z)$ satisfies (**H**), (**D**) and (**E**). Thus, we can apply Theorem 1.6.28 to the present situation what yields the stated estimate. □

Corollary 3.2.8 (Uniqueness). *Assume that F, Ω, K and f_1, f_2 are satisfying the conditions of Theorem 3.2.7. Assume further that $\Omega\backslash K$ is connected and that*

$$f_1(x) \quad = \quad f_2(x) \quad \text{for all } x \in \partial\Omega.$$

Then it follows that

$$f_1 \quad \equiv \quad f_2 \quad \text{on } \overline{\Omega}\backslash K.$$

Proof. We can either apply Corollary 1.6.29 to the present situation analogous to the proof of Theorem 3.2.7 or deduce the statement directly from Theorem 3.2.7. □

The following theorem is a result regarding the removability of singularities of Finsler-minimal graphs.

Theorem 3.2.9 (Removable singularities). *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0,\infty)$ satisfies* (**GAM**). *Let $f \in C^2(\Omega\backslash K)$ be a critical graph of the Finsler area integrand on $\Omega\backslash K$ (cf. Example 1.6.12), wherein K is a locally compact subset of Ω with $\mathscr{H}^{m-1}(K) = 0$. Then f is extendable as critical graph on Ω in C^2.*

Proof. The assumptions on F and $F_{(m)}$ imply as mentioned in section 3.1 that $A^F \in C^0(\mathrm{R}^{m+1} \times \mathbb{R}^{m+1}) \cap C^\infty(\mathrm{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$ is a positive definite elliptic Cartan integrand independent of the first argument., i.e. $A^F = A^F(z)$ satisfies (**H**), (**D**) and (**E**). Thus, we can apply Theorem 1.6.30 to the present situation what yields the stated result. □

3.3 Enclosure theorems

In this section, we present some enclosure results. We start by giving a convex hull property that a Finsler-minimal immersion's image is contained in the convex hull of its boundary configuration. Afterwards we present some enclosure results for Finsler-minimal immersions, whose image is contained in an Finsler admissible domain (see Definition 2.5.8). All enclosure results are restricted to a Minkowski setting, i.e. we assume (**GAM**).

The following convex hull property derives from Theorem 1.6.32. Again, the assumptions on the Finsler metric F and its m-symmetrization $F_{(m)}$ are set to ensure that A^F fulfills the theorem's conditions (see section 3.1).

Theorem 3.3.1 (Convex hull property). *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfies* **(GAM)**. *Let $\mathscr{M}$ be a compact oriented m-manifold with boundary $\partial\mathscr{M}$, $\mathscr{M} = \mathrm{int}\mathscr{M} \cup \partial\mathscr{M}$ and $X \in C^2(\mathrm{int}\mathscr{M}, \mathbb{R}^{m+1}) \cap C^0(\mathscr{M}, \mathbb{R}^{m+1})$ be a Finsler-minimal immersion. Then we have*

$$X(\mathscr{M}) \subset \mathrm{conv}X(\partial\mathscr{M}),$$

wherein $\mathrm{conv}\Sigma$ *denotes the convex hull of the subset* Σ *of* $\mathbb{R}^{m+1}$.

As an application of Theorem 1.6.34 together with Definition 2.5.8 and section 3.1, we get

Theorem 3.3.2 (Enclosure in Finsler admissible domains (cf. [BF09, Lemma 2])**.** *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfies* **(GAM)**. *For some bounded open set $\Omega \subset \mathbb{R}^m$, let $X \in C^0(\overline{\Omega}, \mathbb{R}^{m+1}) \cap C^2(\Omega, \mathbb{R}^{m+1})$ be a Finsler-minimal immersion. Furthermore, let $\Sigma \subset \mathbb{R}^{m+1}$ be an Finsler admissible domain (see Definition 2.5.8). Then, the following two implications hold*

$$X(\partial\Omega) \subset \overline{\Sigma} \quad \Rightarrow \quad X(\Omega) \subset \overline{\Sigma} \tag{3.3.1}$$

and

$$X(\partial\Omega) \subset \overline{\Sigma} \ \wedge \ X(\Omega) \cap \Sigma \neq \emptyset \quad \Rightarrow \quad X(\Omega) \subset \Sigma. \tag{3.3.2}$$

As a special case to Theorem 3.3.2, we present the following corollary, which can also be derived by Corollary 1.6.35.

Corollary 3.3.3 (Enclosure in convex domains). *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfies* **(GAM)**. *For some bounded open set $\Omega \subset \mathbb{R}^m$, let $X \in C^0(\overline{\Omega}, \mathbb{R}^{m+1}) \cap C^2(\Omega, \mathbb{R}^{m+1})$ be a Finsler-minimal immersion. Furthermore, let $\Sigma \subset \mathbb{R}^{m+1}$ be a convex domain. Then, the following two implications hold*

$$X(\partial\Omega) \subset \overline{\Sigma} \quad \Rightarrow \quad X(\Omega) \subset \overline{\Sigma} \tag{3.3.3}$$

and

$$X(\partial\Omega) \subset \overline{\Sigma} \ \wedge \ X(\Omega) \cap \Sigma \neq \emptyset \quad \Rightarrow \quad X(\Omega) \subset \Sigma. \tag{3.3.4}$$

Remark 3.3.4.

- (3.3.3) can also be derived directly from Theorem 3.3.1.
- Σ in Corollary 3.3.3 can be chosen to be a *Finslerian ball* of radius $R > 0$ and center x_0, i.e.

$$B_R^F(x_0) \quad := \quad \{y \in \mathbb{R}^{m+1} : F(y - x_0) < R\},$$

  since $B_R^F(x_0)$ is a convex set.

3.4 Isoperimetric inequalities

First, we start by giving an isoperimetric inequality in the setting of Corollary 3.2.1. Especially, we restrict to $m = 2$ and to minimizers $X \in \mathcal{C}(\varGamma)$.

Theorem 3.4.1 (Isoperimetric inequality). *Let $(\mathbb{R}^3, F)$ be a weak Finsler manifold such that $(\mathbb{R}^3, F_{(2)})$ is also a weak Finsler manifold. Especially, $F : T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ then satisfies (**GA1**). Assume further that F is positive definite, i.e.*

$$0 \quad < \quad m_1 \quad := \quad \inf_{\mathbb{R}^3 \times \mathbb{S}^2} F(\cdot, \cdot) \quad \leq \quad \sup_{\mathbb{R}^3 \times \mathbb{S}^2} F(\cdot, \cdot) \quad =: \quad m_2 \quad < \quad \infty$$

*(see (**d**)). Then any minimizer $X \in \mathcal{C}(\Gamma)$ of the Finsler area $\mathcal{A}^F$ satisfies the simple isoperimetric inequality*

$$\mathcal{A}^F(X) \quad \leq \quad \frac{m_2^2}{4\pi m_1^2} \left(\mathcal{L}_F(\Gamma)\right)^2 .$$

Proof. The isoperimetric inequality can be proven in a similar way as in the proof of [CvdM02, Theorem 3, p. 628]. Let $Y \in \mathcal{C}(Y)$ be a minimizer of the Euclidean area $\mathcal{A} = \mathcal{A}^{|\cdot|}$. The isoperimetric inequality for classical minimal surfaces [DHS10, Theorem 1, p. 330] can be written as

$$\mathcal{A}(Y) \quad \leq \quad \frac{1}{4\pi} \left(\mathcal{L}(\Gamma)\right)^2 ,$$

where $\mathcal{L}(\Gamma)$ is the Euclidean length of Γ (see (1.4.1)). Hence, we conclude

$$\mathcal{A}^F(X) \overset{X \text{ min.}}{\leq} \mathcal{A}^F(Y) \overset{(3.1.1)}{\leq} m_2^2 \mathcal{A}(Y) \quad \leq \quad \frac{m_2^2}{4\pi} \left(\mathcal{L}(\Gamma)\right)^2 \quad \leq \quad \frac{m_2^2}{4\pi m_1^2} \left(\mathcal{L}_F(\Gamma)\right)^2 ,$$

where we made use of the growth condition

$$m_1|y| \quad \leq \quad F(x, y) \quad \leq \quad m_2|y|$$

for all $(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \backslash \{0\}$ (see (**d**)). Notice that the growth condition directly implies

$$m_1 \mathcal{L}(\Gamma) \quad \leq \quad \mathcal{L}_F(\Gamma) \quad \leq \quad m_2 \mathcal{L}(\Gamma),$$

when we exploit the defining relation (1.4.1) of the Finslerian length. □

Lemma 3.4.2 (Estimates of the Finsler area integrand gradient). *Let $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfy (**F1**) and $F \in C^1(\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \backslash \{0\}))$. Denote by $|\cdot|$ the Euclidean metric on $\mathbb{R}^{m+1}$ and by A^F the Finsler area integrand corresponding to F. Then the following estimates hold true:*

(i) For all $(x, z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \backslash \{0\}$ holds

$$|\nabla_z A^F(x, z)| \quad \leq \quad A^F\left(x, \frac{z}{|z|}\right) \left(1 + (m+1)m^2 \max_{y \in z^\perp \cap \mathbb{S}^m} \left(\left\langle z, \frac{\nabla_y F(x, y)}{F(x, y)} \right\rangle\right)^2\right)^{\frac{1}{2}} .$$

(ii) For a ball $\overline{B_R^{m+1}(0)} \subset \mathbb{R}^{m+1}$ holds

$$\begin{aligned}
& ||\nabla_z A^F(\cdot, \cdot)||_{\infty, \overline{B_R^{m+1}(0)} \times \mathbb{S}^m} \\
:= \quad & \max_{(x,z) \in \overline{B_R^{m+1}(0)} \times \mathbb{S}^m} |\nabla_z A(x, z)| \\
\leq \quad & m_2^m \left(1 + (m+1)m^2 \max_{(x,z) \in \overline{B_R^{m+1}(0)} \times \mathbb{S}^m} \max_{y \in z^\perp \cap \mathbb{S}^m} \left(\left\langle z, \frac{\nabla_y F(x, y)}{F(x, y)} \right\rangle\right)^2\right)^{\frac{1}{2}} \\
\leq \quad & m_2^m \left(1 + (m+1)\frac{m^2}{m_1^2} \max_{(x,z) \in \overline{B_R^{m+1}(0)} \times \mathbb{S}^m} \max_{y \in z^\perp \cap \mathbb{S}^m} \langle z, \nabla_y F(x, y) \rangle^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \quad m_2^m \left(1 + (m+1)\frac{m^2}{m_1^2}||\nabla_y F(x,\cdot)||^2_{\infty,\overline{B_R^{m+1}(0)\times\mathbb{S}^m}}\right)^{\frac{1}{2}}.$$

wherein $m_1 = m_1(R)$ *and* $m_2 = m_2(R)$ *are the constants defined in* (3.1.1)*. If* F *satisfies additionally* **(F3)***, then we estimate*

$$||\nabla_z A^F(x,\cdot)||_{\infty,\overline{B_R^{m+1}(0)\times\mathbb{S}^m}} \quad\leq\quad m_2^m \left(1 + \lambda_2(m+1)\left(\frac{m}{m_1}\right)^2\right)^{\frac{1}{2}},$$

wherein

$$\lambda_2 \quad := \quad \lambda_2(R) \quad := \quad \max_{(x,y),(x,\xi)\in\overline{B_R^{m+1}(0)\times\mathbb{S}^m}} g^F|_{(x,y)}(\xi,\xi) \quad > \quad 0$$

is an upper ellipticity bound of F *measured in terms of the fundamental form (see Definition 1.4.1).*

Proof. In a first step, we compute the derivative of A^F with respect to the second argument. Therefore, we write the Finsler area integrand A^F in terms of the spherical Radon transform (see Corollary 2.3.1) and exploit the differentiation rule of Theorem 2.2.14. Notice that we write the argument functions to the Radon transform in dependence of y, whereby y should be the argument whereon the spherical Radon transform is actually applied. Further, we drop the arguments of terms involving $F = F(x,y)$, $A^F = A^F(x,z)$ and $\mathcal{R}(f) = \mathcal{R}(f)(z)$ for all $f \in \mathrm{HF}^0_{-m}(\mathbb{R}^{m+1}\backslash\{0\})$, $x \in \mathbb{R}^{m+1}$ and $y, z \in \mathbb{R}^{m+1}\backslash\{0\}$ to simplify notations. Hence, we get

$$\begin{aligned} & A^F_{z_i}(x,z) \\ = \;& \left(\frac{1}{\mathcal{R}(F^{-m}(x,y))(z)}\right)_{z_i} \\ = \;& -\frac{1}{\mathcal{R}^2(F^{-m})}[\mathcal{R}(F^{-m})]_{z_i} \\ \overset{(2.2.32)}{=} \;& \frac{1}{|z|\mathcal{R}^2(F^{-m})}\mathcal{R}(\frac{z^k}{|z|}\frac{\partial}{\partial y^k}(y^i F^{-m})) \\ = \;& \frac{z^i}{|z|^2\mathcal{R}(F^{-m})} - m\frac{1}{|z|[\mathcal{R}(F^{-m})]^2}\mathcal{R}(\frac{z^k}{|z|}y^i F^{-m-1}F_{y^k}) \end{aligned}$$

for all $(x,z) \in \mathbb{R}^{m+1}\times(\mathbb{R}^{m+1}\backslash\{0\}$. Remember that $f_{z_i} := \frac{\partial f}{\partial z_i}$ for all $f = f(z) \in C^1(\mathbb{R}^{m+1}\backslash\{0\})$ and $f_{y^i} := \frac{\partial f}{\partial y^i}$ for all $f = f(y) \in C^1(\mathbb{R}^{m+1}\backslash\{0\})$. Raising and lowering of indices is carried out here with the Euclidean fundemental tensor and its inverse, namely $w_i = \delta_{ij}w^j$ and $w^i = \delta^{ij}w_j$ for all $w = (w^i) \in \mathbb{R}^{m+1}$. Therewith, we are able to compute the Euclidean norm of the Finsler area integrand's gradient. So, we get

$$\begin{aligned} & |\nabla_z A^F(x,z)|^2 \\ = \;& A^F_{z_i}\delta_{ij}A^F_{z_j} \\ = \;& \frac{|z|^2}{|z|^4(\mathcal{R}(F^{-m}))^2} - 2m\frac{1}{|z|^2(\mathcal{R}(F^{-m}))^3}\mathcal{R}(\frac{z^k}{|z|}\frac{z_i}{|z|}y^i F^{-m-1}F_{y^k}) \\ & + m^2\frac{1}{|z|^2(\mathcal{R}(F^{-m}))^4}\mathcal{R}(\frac{z^k}{|z|}y^i F^{-m-1}F_{y^k})\delta_{ij}\mathcal{R}(\frac{z^l}{|z|}y^j F^{-m-1}F_{y^l}) \end{aligned} \tag{3.4.1}$$

By Definition 2.2.1 we compute

$$\begin{aligned} & \mathcal{R}(\frac{z^k}{|z|}\frac{z_i}{|z|}y^i F^{-m-1}F_{y^k}) \\ = \;& \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^m\cap(\frac{z}{|z|})^\perp}\frac{z^k}{|z|}\frac{z_i}{|z|}y^i F^{-m-1}(x,y)F_{y^k}(x,y)\mathrm{d}\mathscr{H}^{m-1}(y) \end{aligned}$$

$$= \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^m\cap(\frac{z}{|z|})^\perp} F^{-m-1}(x,y)\left\langle \frac{z}{|z|}, y\right\rangle \left\langle \frac{z}{|z|}, \nabla_y F(x,y)\right\rangle \mathrm{d}\mathscr{H}^{m-1}(y)$$

So, we thereof derive that

$$\mathcal{R}(\frac{z^k}{|z|}\frac{z_i}{|z|}y^i F^{-m-1}F_{y^k}) \quad = \quad 0, \tag{3.4.2}$$

since $\left\langle \frac{z}{|z|}, y\right\rangle = 0$ for all $y \in (\frac{z}{|z|})^\perp$. Similarly, we derive by exploiting Definition 2.2.1 again that

$$\begin{aligned}
&|\mathcal{R}(\frac{z^k}{|z|}y^j F^{-m-1}F_{y^k})|\\
= \;& \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\left|\int_{\mathbb{S}^m\cap(\frac{z}{|z|})^\perp}\frac{z^k}{|z|}y^j F^{-m-1}(x,y)F_{y^k}(x,y)\mathrm{d}\mathscr{H}^{m-1}(y)\right|\\
= \;& \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\left|\int_{\mathbb{S}^m\cap(\frac{z}{|z|})^\perp} y^j F^{-m}(x,y)\left\langle \frac{z}{|z|}, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle \mathrm{d}\mathscr{H}^{m-1}(y)\right|\\
\leq \;& \frac{1}{|z|\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}\int_{\mathbb{S}^m\cap(\frac{z}{|z|})^\perp} F^{-m}(x,y)\mathrm{d}\mathscr{H}^{m-1}(y)\max_{y\in\mathbb{S}^m\cap(\frac{z}{|z|})^\perp}\left|\left\langle \frac{z}{|z|}, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle\right|\\
= \;& \mathcal{R}(F^{-m})\max_{y\in\mathbb{S}^m\cap z^\perp}\left|\left\langle \frac{z}{|z|}, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle\right|
\end{aligned} \tag{3.4.3}$$

Combining (3.4.2) and (3.4.3) with (3.4.1) leads to

$$\begin{aligned}
&|\nabla_z A^F(x,z)|^2\\
\overset{(3.4.2)}{=}\;& \frac{1}{|z|^2(\mathcal{R}(F^{-m}))^2}\\
&+m^2\frac{1}{|z|^2(\mathcal{R}(F^{-m}))^4}\mathcal{R}(\frac{z^k}{|z|}y^i F^{-m-1}F_{y^k})\delta_{ij}\mathcal{R}(\frac{z_l}{|z|}y^j F^{-m-1}F_{y^l})\\
\overset{(3.4.3)}{\leq}\;& \frac{1}{|z|^2(\mathcal{R}(F^{-m}))^2}+m^2\frac{1}{|z|^2(\mathcal{R}(F^{-m}))^4}\sum_{i=1}^{m+1}\left[(\mathcal{R}(F^{-m}))^2\max_{y\in z^\perp\cap\mathbb{S}^m}\left(\left\langle \frac{z}{|z|}, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle\right)^2\right]\\
=\;& \frac{1}{|z|^2(\mathcal{R}(F^{-m}))^2}\left(1+(m+1)m^2\max_{y\in z^\perp\cap\mathbb{S}^m}\left(\left\langle \frac{z}{|z|}, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle\right)^2\right)\\
\overset{(2.3.1)}{=}\;& (A^F(x,\frac{z}{|z|}))^2\left(1+(m+1)m^2\max_{y\in z^\perp\cap\mathbb{S}^m}\left(\left\langle z, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle\right)^2\right).
\end{aligned}$$

So, we have shown the first estimate Lemma 3.4.2 (i). Taking the maximum on both sides w.r.t. $\overline{B_R^{m+1}(0)}\times\mathbb{S}^m$ yields directly the first estimate in Lemma 3.4.2 (ii). As noted in section 3.1, we have

$$m_1^m(R) \quad = \quad \min_{\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} A^F(\cdot,\cdot) \quad \leq \quad \max_{\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} A^F(\cdot,\cdot) \quad = \quad m_2^m(R).$$

Thereby, we get

$$\begin{aligned}
&||\nabla_z A^F(\cdot,\cdot)||_{\infty,\overline{B_R^{m+1}(0)}\times\mathbb{S}^m}\\
\leq \;& \max_{\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} A^F(\cdot,\cdot)\left(1+(m+1)m^2\max_{(x,z)\in\overline{B_R^{m+1}(0)}\times\mathbb{S}^m}\max_{y\in z^\perp\cap\mathbb{S}^m}\left\langle z, \frac{\nabla_y F(x,y)}{F(x,y)}\right\rangle^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \quad m_2^m \left(1 + (m+1)\left(\frac{m}{m_1}\right)^2 \max_{(x,z)\in\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} \max_{y\in z^\perp\cap\mathbb{S}^m} \langle z, \nabla_y F(x,y)\rangle^2\right)^{\frac{1}{2}}$$

$$\leq \quad m_2^m \left(1 + (m+1)\left(\frac{m}{m_1}\right)^2 ||\nabla_y F(\cdot,\cdot)||^2_{\infty,\overline{B_R^{m+1}(0)}\times\mathbb{S}^m}\right)^{\frac{1}{2}}$$

If F satisfies additionally **(F3)**, we especially know that F is convex (see Remark 1.4.2) and hence the Hessian matrix $F_{yy}(x,y) := (F_{y^iy^j}(x,y)) \in \mathbb{R}^{(m+1)\times(m+1)}$ is positive semi-definite. Knowing this and that $F > 0$ on $\mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$, we conclude

$$\begin{aligned}
|\nabla_y F(x,y)|^2 &= \max_{v\in\mathbb{S}^m} \langle \nabla_y F(x,y), v\rangle^2 \\
&\leq \max_{v=(v^i)\in\mathbb{S}^m} v^i\{F_{y^i}(x,y)F_{y^i}(x,y) + F(x,y)F_{y^iy^j}(x,y)\}v^j \\
&= \max_{v=(v^i)\in\mathbb{S}^m} v^i g^F_{ij}(x,y)v^j \\
&= \max_{v\in\mathbb{S}^m} g^F|_{(x,y)}(v,v)
\end{aligned}$$

for all $(x,y) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1}\backslash\{0\})$. If we take now the maximum over $\overline{B_R^{m+1}(0)} \times \mathbb{S}^m$ on both sides, we get

$$\begin{aligned}
||\nabla_y F(\cdot,\cdot)||^2_{\infty,\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} &\leq \max_{(x,y),(x,v)\in\overline{B_R^{m+1}(0)}\times\mathbb{S}^m} g^F|_{(x,y)}(v,v) \\
&= \lambda_2(R) \quad = \quad \lambda_2.
\end{aligned}$$

This concludes the proof as thereby follows the last inequality in Lemma 3.4.2 (ii). □

The next result is based on Theorem 1.6.37 and Theorem 1.6.38. In order to apply those theorems to the Finsler area integrand and reinterpret the resulting inequalites in a Finsler setting, we need some additional definitions. Let $\mathscr{M}$ be a compact oriented smooth m-manifold with boundary $\partial\mathscr{M}$, $F : T\mathscr{M} \to [0,\infty)$ satisfy **(F1)** and $X : \mathscr{M} \to \mathbb{R}^{m+1}$ be a smooth immersion. Similar to section 1.6.5, we define $\mathrm{d}A_F := \mathrm{d}V_{X^*F}$ and $\mathrm{d}S_F := \mathrm{d}V_{X|^*_{\partial\mathscr{M}}F}$, where the Busemann-Haudorff area form is given in (2.1.1). Hence, $\int_{\mathscr{M}} \mathrm{d}A_F = \mathcal{A}^F_m(X)$ is the m-dimensional Finsler area of $X(\mathscr{M})$ and $\int_{\partial\mathscr{M}} \mathrm{d}S_F = \mathcal{A}^F_{m-1}(X|_{\partial\mathscr{M}})$ is the $(m-1)$-dimensional Finsler area of $X(\partial\mathscr{M})$.

Theorem 3.4.3. *Let $(\mathbb{R}^{m+1}, F)$ be a weak Minkowski space, i.e. $F = F(y) \in C^0(\mathbb{R}^{m+1}\backslash\{0\}) \cap C^\infty(\mathbb{R}^{m+1}\backslash\{0\})$ satisfies* **(F2)***. Let $\mathscr{M}$ be an oriented smooth m-manifold with boundary $\partial\mathscr{M}$.*

(i) Any immersion $X \in C^2(\mathscr{M},\mathbb{R}^{m+1}) \cap C^1(\bar{\mathscr{M}},\mathbb{R}^{m+1})$ with $X(\partial\mathscr{M}) \subset \overline{B_R^F(a)} \subset \mathbb{R}^{m+1}$ satisfies the following inequality

$$\mathcal{A}_F(X) \quad \leq \quad \frac{R}{m}\frac{m_2}{m_1}\left(\int_{\mathscr{M}} F^*(\mathcal{H}_F)\mathrm{d}A_F + c\left(\frac{m_2}{m_1}\right)^{m-1}\int_{\partial\mathscr{M}} \mathrm{d}S_F\right),$$

with a constant

$$c \quad := \quad c(m,F,\nabla_y F) \quad := \quad \left(1 + (m+1)m^2 \max_{z\in\mathbb{S}^m}\max_{y\in z^\perp\cap\mathbb{S}^m}\left(\left\langle z, \frac{\nabla_y F(y)}{F(y)}\right\rangle\right)^2\right)^{\frac{1}{2}}.$$

If additionally X is a Finsler-minimal immersion an isoperimetric inequality holds as follows

$$\mathcal{A}_F(X) \quad \leq \quad c\frac{R}{m}\left(\frac{m_2}{m_1}\right)^m \int_{\partial\mathscr{M}} \mathrm{d}S_F.$$

(ii) If $m = 2$ and $\partial\mathscr{M}$ consists of exactly $k \geq 1$ closed curves $\gamma_1, \ldots, \gamma_k$ whose image curve $\Gamma_i := X(\gamma_i)$ have finite Finslerian lengths $L_i^F := \mathcal{L}_F(\Gamma_i)$ for $i = 1, \ldots, k$, then any Finsler-minimal immersion $X \in C^2(\mathscr{M}, \mathbb{R}^{m+1}) \cap C^1(\bar{\mathscr{M}}, \mathbb{R}^{m+1})$ satisfies the isoperimetric inequality

$$\mathcal{A}_F(X) \quad \leq \quad c_1 \left(\frac{m_2}{m_1}\right)^2 \left(\sum_{i=1}^{k} \frac{\left(L_i^F\right)^2}{4\pi} + L_i^F \frac{\mathrm{dist}_F(a, \Gamma_i)}{2}\right)$$

with a constant

$$c_1 \quad := \quad c(2, F, \nabla_y F).$$

If $(\mathbb{R}^3, F)$ and $(\mathbb{R}^3, F_{(m)})$ are both Minkowski spaces, i.e. F satisfies **(GAM)***, $k = 2$ and $X(\mathscr{M})$ is connected, then there is even a constant $c_2 = c_2(\frac{\Lambda_2}{\Lambda_1})$ such that*

$$\mathcal{A}_F(X) \quad \leq \quad c_1 c_2 \left(\frac{m_2}{m_1}\right) \frac{\left((L_1^F)^2 + (L_2^F)^2\right)}{4\pi}.$$

Remark 3.4.4.

- Let F in Theorem 3.4.3 (i) additionally satisfy **(F3)** and denote by λ_2 the upper ellipticity constant of F as mentioned in Lemma 3.4.2 (ii). In this situation, the proof of Lemma 3.4.2 (ii) gives an estimate of the constant $c = c(m, F, \nabla_y F)$, namely

$$c \quad = \quad c(m, F, \nabla_y F) \quad \leq \quad \left(1 + \lambda_2(m+1)\left(\frac{m}{m_1}\right)^2\right)^{\frac{1}{2}}.$$

- Lemma 3.4.2 (i) implies that there is no closed smooth Finsler-minimal hypersurface in $(\mathbb{R}^{m+1}, F)$, since the second estimate would then imply that its Finsler area vanishes. So, in our special setting, we have a alternative proof of [She98, Theorem 1.3]. Notice that a *closed manifold* in this context is a complete manifold without boundary. Further, the Finsler area of an embedded submanifold or hypersurface $\mathscr{M} \subset \mathbb{R}^{m+1}$ is defined by $\mathcal{A}_F(\mathscr{M}) := \mathcal{A}_F(i|_{\mathscr{M}})$, where $i : \mathscr{M} \hookrightarrow \mathbb{R}^{m+1}$ is the inclusion map.

Proof of Theorem 3.4.3. We start by giving some estimates, which will become useful throughout the proof. The constants m_1 and m_2 are given by

$$0 \quad < \quad m_1 \quad = \quad \min_{\mathbb{S}^m} F(\cdot) \quad \leq \quad \max_{\mathbb{S}^m} F(\cdot) \quad = \quad m_2 \quad < \quad \infty.$$

Thereby, the following estimate holds

$$m_1|y| \quad \leq \quad F(y) \quad \leq \quad m_2|y| \tag{3.4.4}$$

and for the dual metrics holds

$$\frac{1}{m_2}|\lambda|^* \quad \leq \quad F^*(\lambda) \quad \leq \quad \frac{1}{m_1}|\lambda|^*, \tag{3.4.5}$$

where λ is a linear real-valued mapping on $\mathbb{R}^{m+1}$. Using (3.1.1), we directly conclude

$$m_1^m|z| \quad \leq \quad A^F(z) \quad \leq \quad m_2^m|z|.$$

Further, we have that

$$\int_{\partial\mathscr{M}} \mathrm{d}S \quad = \quad \mathcal{A}_{m-1}^{|\cdot|}(X|_{\partial\mathscr{M}}) \overset{\text{Cor. 2.1.8}}{\leq} \frac{1}{m_1^{m-1}} \mathcal{A}_{m-1}^F(X|_{\partial\mathscr{M}}) \quad = \quad \frac{1}{m_1^{m-1}} \int_{\partial\mathscr{M}} \mathrm{d}S_F. \tag{3.4.6}$$

Lemma 3.4.2 leads to

$$||A_z^F||_{\infty,\mathbb{S}^m} \quad \leq \quad m_2^m\left(1+(m+1)m^2 \max_{z\in\mathbb{S}^m}\max_{y\in z^\perp\cap\mathbb{S}^m}\left\langle z, \frac{\nabla_y F(y)}{F(y)}\right\rangle^2\right)^{\frac{1}{2}} \quad = \quad m_2^m c \tag{3.4.7}$$

Even more, by Theorem 2.5.4, the following identity holds

$$-H_{A^F}(X,N) \quad = \quad \mathcal{H}_X^F(N)\cdot A^F(X,N),$$

where $N:\mathscr{M}\to\mathbb{S}^m$ is the normal map of X. Hence,

$$\begin{aligned}\int_{\mathscr{M}}|H_{A^F}(X,N)|\mathrm{d}A \quad &= \quad \int_{\mathscr{M}}|\mathcal{H}_X^F(N)|A^F(X,N)\mathrm{d}A\\ &= \quad \int_{\mathscr{M}}|\mathcal{H}_X^F(N)|A^F(X,N)\mathrm{d}A\\ &= \quad \int_{\mathscr{M}}|\mathcal{H}_X^F(N)|\mathrm{d}A_F\\ &= \quad \int_{\mathscr{M}}|\mathcal{H}_X^F|^*\mathrm{d}A_F\\ &\overset{(3.4.5)}{\leq} \quad m_2\int_{\mathscr{M}}F^*(\mathcal{H}_X^F)\mathrm{d}A_F.\end{aligned} \tag{3.4.8}$$

Notice that $\overline{B_R^F(a)}\subset\overline{B_{R/m_1}^{|\cdot|}(a)}=\overline{B_{R/m_1}^{m+1}(a)}$ and then Theorem 1.6.37 under the assumptions of (i) gives

$$\mathcal{A}^F(X) \quad \leq \quad \frac{1}{m}\frac{R}{m_1}\left(\int_{\mathscr{M}}|H_{A^F}(X,N)|\mathrm{d}A+||A_z^F||_{\infty,\mathbb{S}^m}\int_{\partial\mathscr{M}}\mathrm{d}S\right).$$

Combining this inequality with (3.4.8), (3.4.7) and (3.4.6) lead to

$$\begin{aligned}\mathcal{A}^F(X) \quad &\leq \quad \frac{1}{m}\frac{R}{m_1}\left(m_2\int_{\mathscr{M}}F^*(\mathcal{H}_X^F)\mathrm{d}A_F+cm_2\left(\frac{m_2}{m_1}\right)^{m-1}\int_{\partial\mathscr{M}}\mathrm{d}S_F\right)\\ &= \quad \frac{R}{m}\frac{m_2}{m_1}\left(\int_{\mathscr{M}}F^*(\mathcal{H}_X^F)\mathrm{d}A_F+c\left(\frac{m_2}{m_1}\right)^{m-1}\int_{\partial\mathscr{M}}\mathrm{d}S_F\right),\end{aligned}$$

what proves (i). Next, we assume the setting of (ii). We will give some additional estimates. (3.4.4) implies directly a similar comparison estimate for the related lengths and distances, i.e.

$$m_1\mathcal{L}(\Gamma) \quad \leq \quad \mathcal{L}_F(\Gamma) \quad \leq \quad m_2\mathcal{L}(\Gamma) \tag{3.4.9}$$

for any continuous curve Γ in $\mathbb{R}^{m+1}$ and

$$m_1|p-q| \quad \leq \quad d_F(p,q) \quad \leq \quad m_2|p-q|. \tag{3.4.10}$$

for all $p,q\in\mathbb{R}^{m+1}$. This can be directly derived from the definitions of the Finslerian length and distance in Remark 1.4.7. Similarly, using (3.4.10) and Remark 1.4.7 leads to

$$m_1\mathrm{dist}(a,B) \quad \leq \quad \mathrm{dist}_F(A,B) \quad \leq \quad m_2\mathrm{dist}(a,B) \tag{3.4.11}$$

for all $A,B\subset\mathbb{R}^{m+1}$. Then Theorem 1.6.38 leads to

$$\mathcal{A}^F(X) \quad \leq \quad ||A_z^F||_{\infty,\mathbb{S}^2}\sum_{i=1}^k\left(\frac{\left(\int_{\gamma_i}\mathrm{d}S\right)^2}{4\pi}+\left(\int_{\gamma_i}\mathrm{d}S\right)\frac{\mathrm{dist}(a,\Gamma_i)}{2}\right),$$

where $\int_{\gamma_i} \mathrm{d}S$ equals the Euclidean length $\mathcal{L}(\Gamma_i)$ of Γ_i for $i = 1, \ldots, k$. Combining this estimate with (3.4.9) and (3.4.11) for $m = 2$ yields

$$\mathcal{A}^F(X) \quad \leq \quad c_1 \left(\frac{m_2}{m_1}\right)^2 \sum_{i=1}^{k} \left(\frac{(\mathcal{L}_F(\Gamma_i))^2}{4\pi} + \mathcal{L}_F(\Gamma_i) \frac{\mathrm{dist}_F(a, \Gamma_i)}{2} \right).$$

Notice that for $i = 1, \ldots, k$ the expressions $\int_{\gamma_i} \mathrm{d}S_F$ and $\mathcal{L}_F(\Gamma_i)$ can differ from each other if F is irreversible, but we could nevertheless replace $\mathcal{L}_F(\Gamma_i)$ by $\int_{\gamma_i} \mathrm{d}S_F$ if needed. The last statement of (ii) for $k = 2$ can be derived analogously from Theorem 1.6.39. □

3.5 Curvature estimates and Bernstein-type theorems for Finsler-minimal immersions

3.5.1 Curvature estimates for Finsler-minimal immersions

The subject of this section is to establish normal curvature estimates for Finsler-minimal immersions. These estimates ultimately lead to Bernstein-type theorems and are an application of the classical theory of Berstein-type theorems for elliptic Cartan functionals. We intend to apply Corollary 1.6.42 and Corollary 1.6.43 on the Finsler area as a Cartan functional. These theorems were established by Simon in [Sim77c] for currents. The resulting theorems are applicable to $(m+1)$-dimensional Minkowski spaces, where the Minkowski metric satisfies the assumption **(GA3)**, where $m \leq 7$. Notice that it would be sufficienten to only assume that the metric $F = F(y) : \mathbb{R}^{m+1} \to [0, \infty)$ satisfies **(GA2)**.

In the following, we recall the definition of the Finslerian length for smooth curves (see also Remark 1.4.7) and define the so-called geodesic spray coefficients. The geodesic spray coefficients determine the differential equations characterizing critical curves of the Finslerian length functional; critical curves meant in the sense that these are curves of vanishing first variation of Finslerian length, where only variations are allowed that leave the endpoints of the curve unchanged (see [BCS00, §3.8 & §5.3], [CS05, §3.2] and [She01, §5.1]). Nevertheless, we need them to compute the related normal curvature.

Definition 3.5.1 (**Geodesics** [BCS00, §3.8]). Let $(\mathscr{N}, F)$ be an n-dimensional smooth Finsler manifold. Choose a local coordinate chart $(U, (x^\alpha))$ on $\mathscr{N}$. The *Finslerian length* of a smooth curve $\gamma : [a, b] \to \mathscr{N}$ is defined by

$$\mathcal{L}_F(\gamma) \quad := \quad \int_{[a,b]} F(\gamma(t), \dot{\gamma}(t))\mathrm{d}t,$$

where $\dot{\gamma}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$. The *geodesic spray coefficients* are defined in local coordinates by

$$G^j_F(x, y) \quad = \quad -(g^F)^{jk}(x, y)\left\{ \left(\frac{F^2}{2}\right)_{x^k}(x, y) - \left(\frac{F^2}{2}\right)_{y^k x^l}(x, y)y^l \right\}$$

for $x \in U \subset \mathscr{N}$, $y = y^i \frac{\partial}{\partial x^i} \in TU$ and $j \in \{1, \ldots, n\}$. A *geodesic* or *Finslerian geodesic of constant speed* $\gamma : [a, b] \to \mathscr{M}$ can be characterized by the set of equations

$$\ddot{\gamma}^j(t) + G^j_F(\gamma(t), \dot{\gamma}(t)) \quad = \quad 0 \qquad (3.5.1)$$

for all $t \in [a, b]$ and $j = 1, \ldots, n$.

Remark 3.5.2.

- The Finslerian length coincides with the 1-dimensional Busemann-Hausdorff volume, as defined in Definition 2.1.1 for reversible Finsler metrics. If the Finsler metric is irreversible, the 1-dimensional Busemann-Hausdorff volume coincides with the Finslerian length to the 1-symmetrization $F_{(1)}$ of F (see Theorem 2.1.12).
- In the setting of Definition 3.5.1, (3.5.1) implies that the smooth curve γ satisfies

$$F(\gamma(t),\dot\gamma(t)) \quad = \quad \text{const} \quad \text{for all} \quad t\in[a,b].$$

  Hence, γ is of constant speed measured in terms of the Finsler metric. (3.5.1) implies even more that γ is a critical curve to $\mathcal{L}_F(\gamma)$.
- In the case that $(\mathscr{N},F)$ is a Minkowski space, the Finsler metric $F(x,y)\cong F(y)$ for all $(x,y)\in T\mathscr{N}$ has a Minkowski structure and thereby the geodesic spray coefficients obviously do all equal zero. Let $p,q\in\mathscr{N}$ and identify $\mathscr{N}\equiv\mathbb{R}^n$ as a vector space. The Finslerian geodesic of constant speed from p to q is given by

$$\gamma(t) \quad := \quad p+t\frac{q-p}{F(q-p)}$$

  for all $t\in[0,F(q-p)]$. Hence, in a Minkowski space Finslerian geodesics of constant speed are line segments.
- The expressions in Definition 3.5.1 are even well-defined if we only assume $F\in C^0(T\mathscr{N})\cap C^2(T\mathscr{N})$ and γ to be of class C^2.

The following definition introduces the Finslerian normal curvature for immersions.

Definition 3.5.3 (**Finslerian normal curvature** (cf. [She01, section 14.2])). Let $\mathscr{M}$ be a smooth m-manifold, $(\mathscr{N},F)$ be an n-dimensional Finsler manifold and $X:\mathscr{M}\to\mathscr{N}$ a smooth immersion for an integer $k\geq 2$. We define X^*F as the pullback of the Finsler metric F under X, i.e.

$$X^*F(u,v) \quad := \quad F(X(u),\mathrm{d}X(v))$$

for all $(u,v)\in T\mathscr{M}$. The *normal curvature* or *Finslerian normal curvature* $\mathcal{K}^F_X$ is in local coordinates (u^α) on $\mathscr{M}$ and (x^i) on $\mathscr{N}$ defined by

$$\mathcal{K}^F_X|_{(u,v)} \quad = \quad -\left\{X^j_{u^\alpha u^\beta}(u)v^\alpha v^\beta - X^j_{u^\alpha}(u)G^\alpha_{X^*F}(u,v)+G^j_F(X(u),\mathrm{d}X|_u(v))\right\}\frac{\partial}{\partial x^j}|_{X(u)} \tag{3.5.2}$$

for all $(u,v)\in T\mathscr{M}\backslash o$ with $v=v^\alpha\frac{\partial}{\partial u^\alpha}\in T_u\mathscr{M}\backslash\{0\}$. Therein, the geodesic spray coefficients G^j_F and $G^\alpha_{X^*F}$ are given in Definition 3.5.1 with respect to the respective local coordinates. This local representation leads directly to the (global) smooth section $\mathcal{K}^F_X$ of $\widehat{X^*T\mathscr{N}}\subset T\mathscr{N}$, where $\widehat{X^*T\mathscr{N}}:=\bigcup_{(u,v)\in T\mathscr{M}\backslash o}T_{X(u)}\mathscr{N}$ is at least a C^k vector bundle over $T\mathscr{M}$. The immersion X is said to be *totally geodesic* if $\mathcal{K}^F_X\equiv 0$.

Remark 3.5.4.

- The normal curvature notion $\mathcal{K}^F_X|_{(u,v)}$ given in Definition 3.5.3 equals the normal curvature $\mathcal{A}|_u(v)$ in [She01, §14.2]. A similar definition of normal curvature by the same author can be found in [She98, (96), p. 571]. In both sources, the normal curvature is defined as a vector field along the immersed surface. The definition in [She01] is slightly different from the one in [She98] at least for irreversible Finsler metrics (cf. Theorem 2.1.12), as it is based on the Finslerian length instead of the 1-dimensional Busemann-Hausdorff volume. Some further theoretical results on the normal curvature can be found as well in [Bej87].

- It is quite easy to show that the geodesic spray coefficients $G^{\alpha}_{X^*F}$ are well-defined even for immersions X of regularity C^k. Then, X^*F is a C^{k-1} Finsler metric on $\mathscr{M}$ and $\mathcal{K}^F_X$ is a C^{k-2} section of $\widehat{X^*T\mathscr{N}}$.
- For $v \in T\mathscr{M}$ and $\lambda > 0$ holds $\mathcal{K}^F_X|_{(u,\lambda v)} = \lambda^2 \mathcal{K}^F_X|_{(u,v)}$, what follows directly from the definition of $\mathcal{K}^F_X|_{(u,v)}$.
- $\widehat{X^*T\mathscr{N}} := \bigcup_{(u,v)\in T\mathscr{M}\setminus o} T_{X(u)}\mathscr{N}$ can be equipped in a natural fashion with a scalar product induced from the Finsler metric F on $T\mathscr{N}\setminus o$, namely $\{g^F|_{(X(u),\mathrm{d}X(v))}(\cdot,\cdot)\}_{(u,v)\in T\mathscr{M}\setminus o}$.

The following theorem was established by Shen in [She98] and characterizes to some extend totally geodesic immersions. Remember that in a Minkowski space geodesics are line segments (see Remark 3.5.2).

Proposition 3.5.5 (**Totally geodesic immersions** (cf. [She98, Proposition 8.3]))**.** *Let $(\mathscr{N}, F)$ be an $(m+1)$-dimensional Minkowski space and $\mathscr{M}$ a smooth m-manifold with boundary. The image $X(\mathscr{M})$ of a totally geodesic immersion $X : \mathscr{M} \to \mathscr{N}$ is contained in a hyperplane of $\mathscr{N}$.*

The representation of normal curvature in a projective way is useful to establish estimates and is given in Lemma 3.5.6.

Lemma 3.5.6 (**Projective normal curvature representation**)**.** *Assume the situation in Definition 3.5.3. Then, the normal curvature can be locally expressed as*

$$\begin{aligned} \mathcal{K}^F_X|_{(u,v)} &= -\Big(\delta^j_l - X^j_{u^\alpha}(u)(g^{X^*F})^{\alpha\beta}(u,v)X^k_{u^\beta}(u)g^F_{kl}(X(u),\mathrm{d}X|_u(v))\Big) \\ &\quad \Big(X^l_{u^\sigma u^\tau}(u)v^\sigma v^\tau + G^l_F(X(u),\mathrm{d}X|_u(v))\Big)\frac{\partial}{\partial x^j}|_{X(u)}. \end{aligned} \tag{3.5.3}$$

Remark 3.5.7. We call (3.5.3) a projective representation of the normal curvature, due to the coefficients

$$\delta^j_l - X^j_{u^\alpha}(u)(g^{X^*F})^{\alpha\beta}(u,v)X^k_{u^\beta}(u)g^F_{kl}(X(u),\mathrm{d}X|_u(v))$$

being coefficients of a C-projection w.r.t. A, where we choose $C = (X^j_{u^\alpha}(u)) \in \mathbb{R}^{n\times m}$ and $A = \big(g^F_{ij}(X(u),\mathrm{d}X|_u(v))\big) \in \mathbb{R}^{n\times n}$. These projectional coefficients could also locally define a projection on $\widehat{X^*T\mathscr{N}}$. It seems to the author that a similar approach has been presented by [Bej87] in a coordinate-free manner.

Proof of Lemma 3.5.6. Let F be a Finsler metrics. During this proof, the dependencies of points on the manifold of the mentioned expression will be suppressed if not explicitly needed. The expressions are evaluated at $(u,v) \in T\mathscr{M}$ or $(X(u),\mathrm{d}X(v)) \in T_{X(u)}\mathscr{M}$ depending on the respective domain of them.

$$\begin{aligned} \mathcal{F}(u,v) &:= X^*F(u,v) = F(X(u),\mathrm{d}X|_u(v)), \\ g^{\mathcal{F}}_{\alpha\beta} &= \left(\frac{\mathcal{F}^2}{2}\right)_{v^\alpha v^\beta} = X^i_{u^\alpha} g^F_{ij} X^j_{u^\beta}, \\ \left(\frac{\mathcal{F}^2}{2}\right)_{u^\beta} &= \left(\frac{F^2}{2}\right)_{x^k} X^k_{u^\beta} + \left(\frac{F^2}{2}\right)_{y^k} X^k_{u^\beta u^\tau} v^\tau, \\ \left(\frac{\mathcal{F}^2}{2}\right)_{v^\beta u^\tau} v^\tau &= \left(\frac{F^2}{2}\right)_{y^k} X^k_{u^\beta u^\tau} v^\tau + \left(\frac{F^2}{2}\right)_{y^k x^l} X^k_{u^\beta} X^l_{u^\tau} v^\tau \\ &\quad + \left(\frac{F^2}{2}\right)_{y^k y^l} X^k_{u^\beta} X^l_{u^\sigma u^\tau} v^\sigma v^\tau, \end{aligned}$$

$$\begin{aligned}
G_F^\alpha(u,v) &= -(g^F)^{\alpha\beta}\Big\{\Big(\frac{F^2}{2}\Big)_{u^\beta} - \Big(\frac{F^2}{2}\Big)_{v^\beta u^\tau} v^\tau\Big\} \\
&= -(g^F)^{\alpha\beta}\Big\{\Big(\frac{F^2}{2}\Big)_{x^k} X^k_{u^\beta} - \Big(\frac{F^2}{2}\Big)_{y^k x^l} X^k_{u^\beta} X^l_{u^\tau} v^\tau \\
&\quad - \Big(\frac{F^2}{2}\Big)_{y^k y^l} X^k_{u^\beta} X^l_{u^\sigma u^\tau} v^\sigma v^\tau\Big\} \\
&= (g^F)^{\alpha\beta}\Big\{G^i_F g^F_{ik} X^k_{u^\beta} + g^F_{kl} X^k_{u^\beta} X^l_{u^\sigma u^\tau} v^\sigma v^\tau\Big\} \\
&= (g^F)^{\alpha\beta} X^k_{u^\beta} g^F_{kl}\Big\{X^l_{u^\sigma u^\tau} v^\sigma v^\tau + G^l_F\Big\}.
\end{aligned}$$

Putting all together in the defining local representation of the normal curvature yields

$$\begin{aligned}
\mathcal{K}^F_X|_{(u,v)} &= -\Big\{X^j_{u^\alpha u^\beta} v^\alpha v^\beta - X^j_{u^\alpha} F^\alpha_{X^*G} + G^j_F\Big\}\frac{\partial}{\partial x^j}|_X \\
&= -\Big\{X^j_{u^\sigma u^\tau} v^\sigma v^\tau + G^j_F - X^j_{u^\alpha}(g^F)^{\alpha\beta} X^k_{u^\beta} g^F_{kl}\{X^l_{u^\sigma u^\tau} v^\sigma v^\tau + G^l_F\}\Big\}\frac{\partial}{\partial x^j}|_X \\
&= -\Big\{\delta^j_l - X^j_{u^\alpha}(g^F)^{\alpha\beta} X^k_{u^\beta} g^F_{kl}\Big\}\Big\{X^l_{u^\sigma u^\tau} v^\sigma v^\tau + G^l_F\Big\}\frac{\partial}{\partial x^j}|_X
\end{aligned}$$

This proves the assertion. □

Example 3.5.8 (Euclidean normal curvature). Assume the setting of Definition 3.5.3. Further, let $(\mathscr{N}, F)$ be the Euclidean space. So, we can identify $(\mathscr{N}, F)$ with $(\mathbb{R}^{m+1}, |\cdot|)$. Then by Remark 3.5.2 the geodesic spray coefficients of F vanish. Lemma 3.5.6 implies that the normal curvature can be written in local coordinates as

$$\begin{aligned}
\mathcal{K}^F_X|_{(u,v)} &= -\Big(\delta^j_l - X^j_{u^\alpha}(u)(g^{X^*F})^{\alpha\beta}(u,v) X^k_{u^\beta}(u) g^F_{kl}(X(u), \mathrm{d}X|_u(v))\Big) \\
&\quad \Big(X^l_{u^\sigma u^\tau}(u) v^\sigma v^\tau + G^l_F(X(u), \mathrm{d}X|_u(v))\Big)\frac{\partial}{\partial x^j}|_{X(u)} \\
&= -\Big(\delta^j_l - X^j_{u^\alpha}(u)(g^{X^*F})^{\alpha\beta}(u,v) X^k_{u^\beta}(u) g^F_{kl}(X(u), \mathrm{d}X|_u(v))\Big) \\
&\quad \Big(X^l_{u^\sigma u^\tau}(u) v^\sigma v^\tau\Big)\frac{\partial}{\partial x^j}|_{X(u)}
\end{aligned}$$

Further, we deduce $g^{X^*F}_{\alpha\beta} = X^k_{u^\alpha}\delta_{kl}X^l_\beta = \langle \frac{\partial X}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta}\rangle =: g_{\alpha\beta}$ are the coefficients of the first fundamental form of X and $g^F_{kl}(X(u), \mathrm{d}X|_u(v)) = \delta_{kl}$ due to the choice of metric on the target space. Thereby, the normal cuvature becomes

$$\begin{aligned}
\mathcal{K}^F_X|_{(u,v)} &= -\Big(\delta^j_l - X^j_{u^\alpha}(u) g^{\alpha\beta}(u,v) X^k_{u^\beta}(u)\delta_{kl}\Big)\Big(X^l_{u^\sigma u^\tau}(u) v^\sigma v^\tau\Big)\frac{\partial}{\partial x^j}|_{X(u)} \\
&= -v^\sigma v^\tau\Big(X^j_{u^\sigma u^\tau}(u) - \langle X_{u^\sigma u^\tau}(u), \frac{\partial X}{\partial u^\beta}\rangle g^{\alpha\beta}\frac{\partial X^j}{\partial u^\alpha}\Big)\frac{\partial}{\partial x^j}|_{X(u)} \\
&= -v^\sigma v^\tau \langle X_{u^\sigma u^\tau}(u), N\rangle N^j \frac{\partial}{\partial x^j}|_{X(u)} \\
&= -v^\sigma v^\tau h_{\sigma\tau} N^j \frac{\partial}{\partial x^j}|_{X(u)},
\end{aligned} \tag{3.5.4}$$

where N is the normal map of X and $h_{\sigma\tau}$ are the coefficients of the second fundamental form of X. So, the normal curvature points only in normal direction. If we intend to measure the (squared) length of this section in some way, we compute

$$g^F|_{(X(u),\mathrm{d}X(v))}(\mathcal{K}^F_X|_{(u,v)}, \mathcal{K}^F_X|_{(u,v)}) = v^\alpha v^\beta h_{\alpha\beta} N^i \delta_{ij} v^\sigma v^\tau h_{\sigma\tau} N^j$$

$$= \left(v^\alpha v^\beta h_{\alpha\beta}\right)^2. \tag{3.5.5}$$

So, the normal curvature is determined in the Euclidean setting by the second fundamental form. Choosing the vector v appropriately, we get the squared principal curvatures of the surface. These observations justifies at least in the Euclidean setting the designation as a normal curvature.

For later interpretation of the Cartan functional curvature estimates in a Finsler-Minkowski setting, we need some estimates to compare Finslerian normal curvatures issued from different Minkowski metrics. This is given in the following Theorem 3.5.9.

Theorem 3.5.9 (Normal curvature comparison). *Let $(\mathscr{N}, F)$ and $(\mathscr{N}, E)$ be Minkowski spaces. Let $X : \mathscr{M} \to \mathscr{N}$ be a C^2-immersion. Then there exist constants $C_i = C_i(F,E) > 0$ for $i = 1,2$ such that*

$$C_1 g^E\left(\mathcal{K}_X^E, \mathcal{K}_X^E\right) \quad \le \quad g^F\left(\mathcal{K}_X^F, \mathcal{K}_X^F\right) \quad \le \quad C_2 g^E\left(\mathcal{K}_X^E, \mathcal{K}_X^E\right) \tag{3.5.6}$$

for all $(u,v) \in T\mathscr{M}\backslash o$, where $g^F\left(\mathcal{K}_X^F, \mathcal{K}_X^F\right)$ is an abbreviation for

$$g^F|_{(X(u),\mathrm{d}X(v))}\left(\mathcal{K}_X^F|_{(u,v)}, \mathcal{K}_X^F|_{(u,v)}\right).$$

Proof. Let E and F be Minkowski metrics. During this proof, the dependencies of the mentioned expressions evaluated at points on manifolds are suppressed if not explicitly needed. The expressions are evaluated at $(u,v) \in T\mathscr{M}\backslash o$ or $(X(u), \mathrm{d}X(v)) \in T_{X(u)}\mathscr{M}\backslash\{0\}$ depending on the respective domain of them. In local coordinates the geodesic spray coefficients G_E^q and G_F^q vanish due to the Minkowski structure of E and F. So, we get

$$\begin{aligned}
& g^F|_{(X(u),\mathrm{d}X(v))}\left(\mathcal{K}_X^F|_{(u,v)}, \mathcal{K}_X^F|_{(u,v)}\right) \\
\overset{\text{Lem. 3.5.6}}{=} & g_{qr}^F\left(\delta_l^r - \delta_j^r X_{u^\alpha}^j (g^F)^{\alpha\beta} X_{u^\beta}^k g_{kl}^F\right)\left(X_{u^\sigma u^\tau}^l v^\sigma v^\tau\right)\left(\delta_t^q - \delta_p^q X_{u^\mu}^p (g^F)^{\mu\nu} X_{u^\nu}^s g_{st}^F\right)\left(X_{u^\rho u^\mu}^t v^\rho v^\mu\right) \\
= & g_{qr}^F\left(\delta_l^r - \delta_j^r X_{u^\alpha}^j (g^F)^{\alpha\beta} X_{u^\beta}^k g_{kl}^F\right)\left(X_{u^\sigma u^\tau}^l v^\sigma v^\tau\right)\left(X_{u^\rho u^\mu}^q v^\rho v^\mu\right)
\end{aligned} \tag{3.5.7}$$

and analogously for the Minkowski metric E. Notice that we can assume $\mathscr{N} \cong \mathbb{R}^n$, $F(x,y) \equiv F(y)$, $E(x,y) \equiv E(y)$, $g^F(x,y) \equiv g^F(y)$ and $g^E(x,y) \equiv g^E(y)$ for all $(x,y) \in T\mathscr{N}\backslash o \cong \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$, since $(\mathscr{N}, F)$ and $(\mathscr{N}, E)$ are Minkowski spaces. We can apply Theorem 1.1.22 to (3.5.7) by setting therein $A = (g_{kl}^F(\mathrm{d}X(v))) \in \mathbb{R}^{n\times n}$ and $B = (g_{kl}^E) \in \mathbb{R}^{n\times n}$ and define positive constants $C_1 := \min_{\mathbb{R}^{m+1}} \Lambda_1(A,B)$ and $C_2 := \max_{\mathbb{R}^{m+1}} \Lambda_2(A,B)$ s.t. the normal curvature w.r.t. F of X relates to the normal curvature w.r.t. E of X at least locally as in (3.5.6) for $C_2 = C_2((g_{ij}^F), (g_{ij}^E))$ as in Theorem 1.1.22. These constants C_1, C_2 are by construction (see Theorem 1.1.22) invariant under coordinate change on $\mathscr{M}$ and $\mathscr{N}$. So, we can write $C_2 = C_2(g^F, g^E)$. □

Motivated by theorems presented in subsection 1.6.6 based on [Sim77c], we adapt Definition 1.6.40 to the a Finsler-Minkowski setting.

Definition 3.5.10. Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space. Define the sets

$$\begin{aligned}
\mathcal{U}_F(x_0,R) &:= \{M : M \text{ is a } C^2\text{-hypersurface with } C^2 \text{ boundary } \partial M \subset \mathbb{R}^{m+1}\backslash B_R^F(x_0)\}, \\
\mathcal{N}_F(x_0,R) &:= \{M : M \in \mathcal{U}^F(x_0,R), M \cap B_R^F(x_0) \text{ is } A^F\text{-minimizing in } B_R^F(x_0) \\
&\qquad \text{and there exists an open set } U_M \subset \mathbb{R}^{m+1} \\
&\qquad \text{such that } \overline{M} \cap B_R^F(x_0) = \partial U_M \cap B_R^F(x_0)\},
\end{aligned}$$

where A^F-minimizing in $B_R^F(x_0)$ is meant as defined in Definition 1.6.40. $\mathcal{N}_F'(x_0,R)$ is the set of $M \in \mathcal{U}_F(x_0,R)$ representable in as a graph $x^{m+1} = f(x^1,\ldots,x^m)$, $(x^1,\ldots,x^m) \in$

$D_R^F(x_0^{'}) := \{x : x^{m+1} = 0\} \cap B_R^F((x_0^1, \ldots, x_0^m, 0))$, where f is a C^2 function whose graph on $D_R^F(x_0^{'})$ is Finsler-minimal. Analogous to the argument in Definition 1.6.40 holds that $\mathcal{N}_F^{'}(x_0, R) \subset \mathcal{N}_F(x_0, R)$.

Remark 3.5.11. As mentioned in section 3.1, the following growth condition holds for a Minkowski space $(\mathbb{R}^{m+1}, F)$:

$$m_1|\cdot| \quad \leq \quad F(\cdot) \quad \leq \quad m_2|\cdot|$$

for constants $m_1 = \min_{y\in\mathbb{S}^{m+1}} F(\cdot)$ and $m_2 = \max_{y\in\mathbb{S}^{m+1}} F(\cdot)$. Thereby follows

$$\overline{B^{|\cdot|}_{\frac{R}{m_2}}(x_0)} \subset \overline{B^F_R(x_0)} \subset \overline{B^{|\cdot|}_{\frac{R}{m_1}}(x_0)}$$

and consequently

$$\mathbb{R}^{m+1}\backslash\overline{B^{|\cdot|}_{\frac{R}{m_1}}(x_0)} \subset \mathbb{R}^{m+1}\backslash\overline{B^F_R(x_0)} \subset \mathbb{R}^{m+1}\backslash\overline{B^{|\cdot|}_{\frac{R}{m_2}}(x_0)}.$$

Notice that it is easy to show that A^F-minimizing in an open set A implies A^F-minimizing in an open set B if $B \subset A$. So, we get the following inclusions

$$\begin{aligned}
\mathcal{U}(x_0, R/m_1) &\subset \mathcal{U}_F(x_0, R) \subset \mathcal{U}(x_0, R/m_2),\\
\mathcal{M}(x_0, R/m_1) &\subset \mathcal{N}_F(x_0, R) \subset \mathcal{M}(x_0, R/m_2),\\
\mathcal{M}^{'}(x_0, R/m_1) &\subset \mathcal{N}_F^{'}(x_0, R) \subset \mathcal{M}^{'}(x_0, R/m_2).
\end{aligned}$$

Therein, $\mathcal{U}$, $\mathcal{M}$ and $\mathcal{M}^{'}$ are defined as in Definition 1.6.40.

The following Theorem 3.5.12 and Theorem 3.5.13 establish the announced curvature estimates. Therein, the curvature estimate for some C^2-hypersurface M in a Minkowski space $(\mathbb{R}^{m+1}, F)$ takes the following form: *There exists a constant $C = C(F) > 0$ such that*

$$\left.\frac{g^F|_{(X,\mathrm{d}X)}(\mathcal{K}_X^F, \mathcal{K}_X^F)}{X^*G^2}\right|_{(u_0,v_0)} \leq \frac{C}{R^2}, \tag{3.5.8}$$

where the C^2 immersion $X : \mathscr{M} \to M$ is a parametrization of the hypersurface M with $X(\mathscr{M}) = M$ and $u_0 \in \mathscr{M}$ with $X(u_0) = x_0$ and $v_0 \in T_{u_0}\mathscr{M}$.

Theorem 3.5.12. *Let $(\mathbb{R}^{m+1}, F)$ and $(\mathbb{R}^{m+1}, F_{(m)})$ be Minkowski spaces. Let $M \in \mathcal{N}_F(x_0, R)$ with $m = 2$ or $M \in \mathcal{N}_F^{'}(x_0, R)$ with $m = 3$, then there exists a constant $C = C(F) > 0$ such that (3.5.8) holds.*

Theorem 3.5.13. *Let $(\mathbb{R}^{m+1}, F)$ be Minkowski spaces and choose $|\cdot|$ to be the Euclidean metric. In the case $m \leq 6$ there is a $\delta = \delta(m) > 0$ such that for all Minkowski spaces $(\mathbb{R}^{m+1}, F)$ whose Finsler metric F satisfies*

$$\widehat{\rho}_3(F(\cdot) - |\cdot|) \quad \leq \quad \delta, \tag{3.5.9}$$

there exists a constant $C = C(F) > 0$ such that (3.5.8) holds for every $M \in \mathcal{N}_F(x_0, R)$. In the case $m \leq 7$ there is a $\delta > 0$ such that for all Minkowski spaces $(\mathbb{R}^{m+1}, F)$ whose Finsler metric F satisfies (3.5.9) w.r.t δ, there exists a constant $C = C(F) > 0$ such that (3.5.8) holds for every $M \in \mathcal{N}_F^{'}(x_0, R)$.

Proof of Theorem 3.5.12 and Theorem 3.5.13. Similar theorems for elliptic Cartan integrands independent of the first argument are Corollary 1.6.42 and Corollary 1.6.43, which are based on [Sim77c, Theorem 1, Corollary 1 & Corollary 2]. We intend to apply those theorems to the present situation. There, the assumptions are all made on the elliptic Cartan integrand, whose role is played here by A^F. The sets $\mathcal{N}_F(x_0,R)$ and $\mathcal{N}_F'(x_0,R)$ can be replaced by $\mathcal{M}(x_0,R/m_2)$ and $\mathcal{M}'(x_0,R/m_2)$ due to the inclusions presented in Remark 3.5.11. Further, (3.5.8) can be written in the context of Corollary 1.6.42 and Corollary 1.6.43 as

$$\sum_{\alpha=1}^{m}\kappa_\alpha^2(u_0)\le\frac{Cm_2^2}{R^2}$$

for a hypersurface M with principal (Euclidean) curvatures k_α for $i=1,\dots,m$, which is A^F-minimizing in $B_R^{|\cdot|}(x_0)$. We represent M by a C^2 immersion $X:\mathscr{M}\to\mathbb{R}^{m+1}$ with a smooth oriented m-manifold $\mathscr{M}$ as in (3.5.8). The C^3 closeness of F to the Euclidean metric $|\cdot|$ implies C^3 closeness of A^F to the Euclidean area integrand $A^{|\cdot|}=|\cdot|$ (see Lemma 2.3.10). If we choose $\mathscr{N}$ to be the $(m+1)$-dimensional real vector space $\mathbb{R}^{m+1}$ and E to be the standard Euclidean metric in Theorem 3.5.9 the estimate becomes

$$\begin{aligned}
&\frac{1}{C_2}g^F|_{(X(u),\mathrm{d}X|_u(v))}\left(\mathcal{K}_X^F|_{(u,v)},\mathcal{K}_X^F|_{(u,v)}\right)\\
\overset{\text{Th. 3.5.9}}{\le}\;& g^E|_{(X(u),\mathrm{d}X|_u(v))}\left(\mathcal{K}_X^E|_{(u,v)},\mathcal{K}_X^E|_{(u,v)}\right)\\
\overset{(3.5.5)}{=}\;&(h_{\rho\mu}v^\rho v^\mu)^2\\
\le\;&\max_\alpha\kappa_\alpha^2 g_{\rho\mu}v^\rho v^\mu\\
\le\;&\left(\sum_\alpha\kappa_\alpha^2\right)|\mathrm{d}X(v)|^2\\
\le\;&\frac{1}{m_1^2}\sum_\alpha\kappa_\alpha^2F^2(\mathrm{d}X(v))\\
\le\;&\frac{(X^*F)^2(v)}{m_1^2}\sum_\alpha\kappa_\alpha^2
\end{aligned}$$

The curvature estimates of Corollary 1.6.42 and Corollary 1.6.43 imply then estimates of the following type:

$$\frac{g^F|_{(X(u_0),\mathrm{d}X|_{u_0}(v))}\left(\mathcal{K}_X^F|_{(u_0,v)},\mathcal{K}_X^F|_{(u_0,v)}\right)}{(X^*F)^2(v)}\quad\le\quad CC_2\left(\frac{m_2}{m_1}\right)^2R^{-2},$$

where $v\in T_{u_0}\mathscr{M}$ and $X(\partial\mathcal{M})\subset\mathbb{R}^{m+1}\backslash\overline{B_R^F(X(u_0))}\subset\mathbb{R}^{m+1}\backslash\overline{B_{\frac{R}{m_2}}^{|\cdot|}(X(u_0))}$. So, we get a Minkowski version of the curvature estimate. Ultimately, Theorem 3.5.12 and Theorem 3.5.13 follow then directly from Corollary 1.6.42 and Corollary 1.6.43. □

3.5.2 Bernstein-type theorems for Finsler-minimal immersions

Herein, we intend to state some Bernstein type theorems, which either derive from the curvature estimates of the previous subsection 3.5.1 or the theory of Cartan integrands presented in subsection 1.6.6. We start with a result in general dimension as a consequence of Theorem 1.6.44 and section 3.1.

Theorem 3.5.14. *Let $m\ge2$. Let $(\mathbb{R}^{m+1},F)$ be a Minkowski space such that $(\mathbb{R}^{m+1},F_{(m)})$ is also a Minkowski space, i.e. $F:T\mathbb{R}^{m+1}\cong\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}\to[0,\infty)$ satisfies* **(GAM)**. *There exists $\delta(m,\gamma)>0$ and $\gamma\in(0,1)$ such that the following holds: Assume*

$$\widehat{\rho}_3(F(\cdot)-|\cdot|)\quad\le\quad\delta$$

and that f is a Finsler-minimal graph with

$$|\mathrm{D}f(x)| = O(|(x, f(x))|^\gamma) \text{ für } |x| \to \infty, \tag{3.5.10}$$

wherein $|(x, f(x))| = \sqrt{|x|^2 + f^2(x)}$. *Then f is an affine linear function.*

Now we present Bernstein theorems stemming from the curvature estimates established in subsection 3.5.1. Assume that a curvature estimate as in Theorem 3.5.12 and Theorem 3.5.13 holds at every point of an entire Finsler-minimal graph. We deduce that the Finslerian normal curvature vanishes at every point by letting R tend to infinity for such a graph. So, the graph as a hypersurface is totally geodesic. Finally, Proposition 3.5.5 implies that the entire Finsler-minimal graph has to be a plane (cf. [Bej87] and [She98, Proposition 8.3]). This follows also directly from Corollary 1.6.42 and Corollary 1.6.43.

Theorem 3.5.15. *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space such that $(\mathbb{R}^{m+1}, F_{(m)})$ is also a Minkowski space, i.e. $F : T\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to [0, \infty)$ satisfies* **(GAM)**. *For $m = 2$ or $m = 3$ every entire Finsler-minimal graph is a plane.*

Remark 3.5.16. An *entire graph* in $\mathbb{R}^{m+1}$ is the graph of a function, which is defined on $\mathbb{R}^m$.

Corollary 3.5.17. *Let $(\mathbb{R}^{m+1}, F)$ be a Minkowski space and $|\cdot|$ be the Euclidean metric. In the case $m \leq 7$ there is an $\delta = \delta(m) > 0$ such that for all Minkowski spaces $(\mathbb{R}^{m+1}, F)$, whose Finsler metric F satisfies*

$$\widehat{\rho}_3(F(\cdot) - |\cdot|) \quad \leq \quad \delta, \tag{3.5.11}$$

every entire Finsler-minimal graph of class C^2 is a plane.

Remark 3.5.18. In section 2.4, we computed for some exemplary Finsler metrics conditions, which imply that the m-symmetrization is a Finsler metric by itself. Hence, those Finsler metrics satisfy **(GA3)** when they fulfill these conditions. We especially reproduced the sharp bounds on $||\beta||_{\widehat{\alpha}}$ for the Minkowski Randers metric presented by Souza et al. [SST04] and the special Minkowski (α, β)-metrics of Cui, Shen [CS09]. So, these metrics satisfy **(GAM)**. Thereby, Theorem 3.5.15 and Corollary 3.5.17 include the Bernstein-type theorems presented in [SST04] and [CS09] for these distinct Finsler metrics as special cases.

Nomenclature

A Vector and Set Notation

$\binom{\alpha}{\beta}$	binomial coefficient for multi-indices	92
$\binom{n}{k}$	binomial coefficient	14
$\det M$	determinant of the quadratic matrix $M \in \mathbb{R}^{k\times k}$	8
$\langle \cdot, \cdot \rangle_A$	A-scalar product on $\mathbb{R}^n$	17
$\mathbb{B}^n$	ball of radius 1 around 0 in $\mathbb{R}^n$	25
$\mathbb{C}$	set of complex numbers	11
$\mathbb{H}^m$	closed m-dimensional upper halfspace	28
$\mathbb{N}$	set of natural numbers	7
$\mathbb{N}_0$	set of non-negative integer numbers	7
$\mathbb{R}$	set of real numbers	7
$\mathbb{S}^{n-1}$	sphere of radius 1 around 0 in $\mathbb{R}^n$	25
$\mathbb{Z}$	set of integer numbers	7
$\mathrm{dim}V$	dimension of the vector space V	12
$\mathrm{GL}(n)$	general linear group	8
$\mathrm{int}\mathbb{H}^m$	boundary of the m-dimensional upper halfspace	28
$\mathrm{int}\mathbb{H}^m$	interior of the m-dimensional upper halfspace	28
$\ker(f)$	kernel of a linear mapping $f : V \to W$	18
$\mathrm{rank}(f)$	rank of the linear mapping $f : V \to W$	18
$\mathrm{sgn}\sigma$	sign of the permutation $\sigma \in S_k$	8
$\mathrm{SL}(n)$	special linear group	8
$\mathrm{span}S$	set of all linear combinations of elements in S	13

$\operatorname{trace}(M)$	trace of a matrix $M \in \mathbb{R}^{k\times l}$ 8
$\omega^1 \otimes \ldots \otimes \omega^k$	tensor product of $\omega^1, \ldots, \omega^k$ 13
$\overline{\Omega}$	closure of an open set $\Omega \subset \mathbb{R}^m$ 20
$\overline{B_R(p)}$, $\overline{B^n_R(p)}$	closed ball of radius R and center p in $\mathbb{R}^n$ 25
$\partial\Omega$	topological boundary of an open set $\Omega \subset \mathbb{R}^m$ 20
$\sharp S$	cardinality of a set S 72
B	unit disc 57
$B_R(p)$, $B^n_R(p)$	open ball of radius R around p in $\mathbb{R}^n$ 25
$O(n,m)$	orthogonal group to the Euclidean scalar product 19
$P_{A,C}$	C-projection w.r.t. to A 18
$S^\perp$	orthogonal complement of the set $S \subset \mathbb{R}^{m+1}$ 15
S_k	set of all permutations of $1, \ldots, k$ 8
$T^k(V)$	set of all covariant k-tensors on V 12
$T_k(V)$	set of all contravariant k-tensors on V 13
V^*	dual space to the vector space V 12
$v_1 \otimes \ldots \otimes v_k$	tensor product of $v_1, \ldots, v_k$ 13

B Functional Notation

$(\tilde{x}^\alpha(x))$	shorthand for a transition map 29
(U, φ)	coordinate chart $\varphi : U \to \varphi(U) \subset \mathbb{H}^m$ 28
(x^α)	local coordinates 28
χ_Ω	characteristic function of the set Ω 72
Δ	Euclidean Laplacian 26
δ^i_j	Kronecker delta 7
δ^{ij}	Kronecker delta 7
Δ_o	spherical Laplacian 26
δ_{ij}	Kronecker delta 7
$\frac{\partial f^\beta}{\partial u^\alpha}$	components of the differential of f in local coordinates 31
$\frac{\partial x^\beta}{\partial \tilde{x}^\alpha}$	components of the differential of the transition map $(x^\beta(\tilde{x}))$ 30
$\frac{\partial}{\partial x^\alpha}$	coordinate vector to local coordinates (x^α) 30
$\mathrm{d}f$	differential of $f : \mathscr{M} \to \mathscr{N}$ 31
$\mathrm{D}f(x)$	Jacobian matrix of f at x 22

$\mathrm{D}^\alpha f(x)$	differential of f at x w.r.t. the multi-index α	21
$\operatorname{supp} g$	support of a function g	23
trace_g	trace of a bundle map $f : T\mathscr{M} \to T\mathscr{M}$ w.r.t. g	50
∇	Euclidean gradient	53
∇_o	spherical gradient	26
$\nabla_z f$	gradient of f w.r.t. z	48
$\Gamma^\gamma_{\alpha\beta}$	Christoffel symbols	42
$f^\beta_{u^\alpha}$	components of the differential of f in local coordinates	31
f_{z^i}	i-th partial derivative of f w.r.t. z	48
f_{zz}	Hessian matrix of f w.r.t. z	48
f_z	gradient of f w.r.t. z	48
$g_{\alpha\beta}$	coefficients of the first fundamental form	42
H	Euclidean mean curvature	42
H_I	I-mean curvature	50
h_I	full I-mean curvature	52
$h_{\alpha\beta}$	coefficients of the second fundamental form	42
$Jf(x)$	Jacobian determinant of f at x	22
S_I	I-shape operator	50

C Function Spaces

$\|\cdot\|_{C^0(\overline{\Omega})}$	supremum norm on $C^0(\overline{\Omega})$	20
$\|\cdot\|_{C^k(\overline{\Omega})}$	norm on $C^k(\overline{\Omega})$	21
$\|\cdot\|_{C^{k,\sigma}(\overline{\Omega})}$	norm on $C^{k,\sigma}(\overline{\Omega})$	21
$\|\cdot\|_{L^p(\Omega)}$	L^p-norm on Ω	23
$\|\cdot\|_{W^{k,p}(\Omega)}$	$W^{k,p}$-norm on Ω	23
$\text{Höl}_{\sigma,\Omega}(\cdot)$	Hölder norm	21
$\mathrm{HF}^l_k(\mathbb{R}^n\backslash\{0\})$	vector space of l-times differentiable k-homogeneous functions	25
$\mathrm{HF}^l_{k,e}(\mathbb{R}^n\backslash\{0\})$	vector space of even l-times differentiable k-homogeneous functions	25
$\mathrm{HF}^l_{k,o}(\mathbb{R}^n\backslash\{0\})$	vector space of odd l-times differentiable k-homogeneous functions	25
$\mathscr{L}^p(\Omega)$	space of p-integrable functions on Ω	23
$\Phi_k(\cdot)$	restriction of a function to the unit sphere	26
$\Psi_k(\cdot)$	k-homogeneous extension of a spherical function	26

$\rho_j(\cdot)$	seminorm of order j on $C^\infty(\mathbb{R}^n\backslash\{0\})$	25
$\rho_j(\cdot)$	spherical seminorm of order j on $C^l(\mathbb{R}^n\backslash\{0\})$ for $j \leq l$	25
$C_0^\infty(\Omega)$	set of functions in $C^\infty(\Omega)$ with compact support in Ω	23
$C^k(\Omega)$	set of k-times continuously differentiable functions on an open set $\Omega \subset \mathbb{R}^m$	20
$C^k(\overline{\Omega})$	set of k-times continuously differentiable functions on the closure of an open set $\Omega \subset \mathbb{R}^m$	20
$C^l(\mathscr{M})$	set of l-times differentiable real-valued functions on $\mathscr{M}$	29
$C^l(\mathscr{M}, \mathscr{N})$	set of l-times differentiable functions from $\mathscr{M}$ to $\mathscr{N}$	29
$C_0^l(\mathscr{M})$	set of l-times differentiable real-valued functions on $\mathscr{M}$ of compact support	29
$C^{0,\sigma}(\Omega)$	vector space of all Hölder continuous functions on Ω to a given exponent $\sigma \in (0,1]$	21
$C^{0,\sigma}_{\mathrm{loc}}(\Omega)$	vector space of all locally Hölder continuous functions on Ω to a given exponent $\sigma \in (0,1]$	21
$L^p(\Omega)$	L^p-space on Ω	23
$W^{k,p}(\Omega)$	Sobolev space of order k and integrability p	23
$W^{k,p}(\Omega, \mathbb{R}^n)$	Sobolev space of order k and integrability p	23
$W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$	local Sobolev space of order k and integrability p	23

D Measures

$\mathscr{H}^m$	m-dimensional Hausdorff measure	22
$\mathscr{H}^m$-a.e.	$\mathscr{H}^m$ almost everywhere	22

E Finsler Manifolds

$(\mathscr{N}, F)$	Finsler manifold	37
$(g^F)^{ij}$	inverse to the fundamental tensor	38
$\lVert \mathrm{d}_y F_a \rVert_{\widehat{F_s}}$	dual norm of the odd part w.r.t. the even part	121
$\mathcal{A}(\cdot)$	Euclidean area	47
$\mathcal{A}^F(\cdot)$	Finsler area	68
$\mathcal{A}^F_m(\cdot)$	m-dimensional Finsler area	68
$\mathcal{K}^F_X$	Finslerian normal curvature of the immersion $X : \mathscr{M} \to (\mathscr{N}, F)$	159
$\mathcal{L}(\gamma)$	Euclidean length of a curve $\gamma : [a,b] \to \mathbb{R}^n$	38
$\mathcal{L}_F(\gamma)$	Finslerian length of a curve $\gamma : [a,b] \to (\mathscr{N}, F)$	37

$\dim \mathscr{M}$	dimension of a manifold 28
$\mathrm{dist}_F(A, B)$	Finslerian distance of a set A to a set B 37
$\mathrm{dist}_F(p, B)$	Finslerian distance of a point p to a set B 37
$\mathrm{d}A$	m-dimensional Euclidean area element 63
$\mathrm{d}S$	$(m-1)$-dimensional Euclidean area element 63
$\mathrm{d}V_{X^*F}$	m-dimensional Finsler area form 67
$\mathrm{genus}(\mathscr{M})$	genus of an oriented smooth manifold $\mathscr{M}$ 60
$\mathrm{int}\mathscr{M}$	interior of a manifold 29
$\mathscr{M}^m$	manifold of dimension m 28
$\Omega^k(\mathscr{M})$	vector space of all k-forms on $\mathscr{M}$ 34
$\partial\mathscr{M}$	boundary of a manifold 29
$A(\cdot)$	Euclidean area integrand 47
$a^F_{m,\mathfrak{X}}(\cdot,\cdot)$	m-dimensional Finsler area integrand 68
$A^F(\cdot,\cdot)$	m-dimensional Finsler area integrand on $\mathbb{R}^{m+1}$ 75
$B^F_R(x_0)$	Finslerian ball of radius $R > 0$ and center x_0 151
$d(p, q)$	Euclidean distance between two points $p, q \in \mathbb{R}^n$ 38
$d_F(p, q)$	Finslerian distance between two points $p, q \in \mathscr{N}$ 37
F^*	dual Finsler structure to the Finsler structure F 38
F_a	odd or antisymmetric part of a function F 112
F_s	even or symmetric part of a function F 112
g^F_{ij}	fundamental tensor 35
$G^j_F(x, y)$	geodesic spray coefficient 158
o	zero section 35
$T\mathscr{M}$	tangent bundle to $\mathscr{M}$ 31
$T^*\mathscr{M}$	cotangent bundle to $\mathscr{M}$ 31
$T^k(\mathscr{M})$	bundle of covariant k-tensors on $\mathscr{M}$ 33
$T_k(\mathscr{M})$	bundle of contravariant k-tensors on $\mathscr{M}$ 34
$T_p\mathscr{M}$	tangent space to $\mathscr{M}$ at p 30
TX	tangent bundle to the immersion X 42
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Erklärung

Hiermit erkläre ich eidesstattlich, dass ich die vorliegende Arbeit selbständig verfasst und dabei nur die angegebenen Quellen verwendet habe. Ferner erkläre ich, dass durch die Veröffentlichung der Arbeit keine Urheberschutzrechte verletzt werden und dass ich bisher keine Promotionsanträge gestellt habe.

Aachen, den den 19. Juni 2014