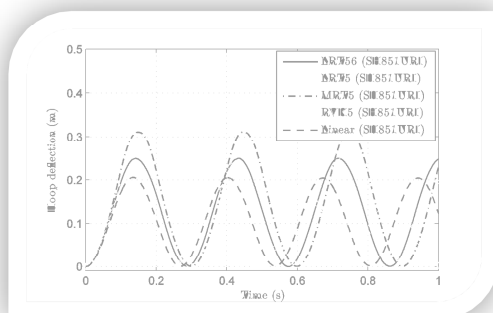
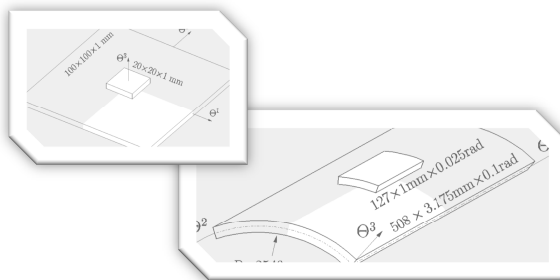


Nonlinear FE Simulation and Active Vibration Control of Piezoelectric Laminated Thin-Walled Smart Structures

A Ph.D. dissertation at the
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Abstract

This dissertation deals with nonlinear finite element modeling and active vibration control for piezoelectric integrated smart structures, and is presented in two parts.

In the first part, an electro-mechanically coupled large rotation finite element model is developed for static and dynamic analysis of thin-walled structures with piezoelectric sensor and actuator layers. The present large rotation theory is based on the first-order shear deformation hypothesis, which has six independent kinematic parameters but expressed by five nodal degrees of freedom. Unrestricted finite rotations are described by two rotations using the Euler angle representation method. Due to the assumption of small strains and weak electric potential, linear piezoelectric coupled constitutive equations and a linearly distributed electric potential through the thickness are considered. In order to show the necessity of the large rotation theory in the application of thin-walled composite or smart structures undergoing large rotations, several simplified nonlinear shell theories are implemented into finite elements for thin-walled structures as well.

The second part develops a disturbance rejection control with a Proportional-Integral (PI) observer which uses step functions to construct a fictitious model of disturbances for active vibration control of smart structures. To improve the dynamic behavior of the existing PI observer, a Generalized PI (GPI) observer is proposed and developed. Therefore, any unknown disturbances can be estimated and compensated by the present disturbance rejection control with either a PI or GPI observer. Additionally, PID, LQR and LQG control strategies are implemented to show the advantages of the proposed disturbance rejection control.

Zusammenfassung

Diese Dissertation behandelt die beiden Themenbereiche nichtlineare Finite Elemente Modellierung und Schwingungsdämpfung für intelligente Strukturen mit integrierten piezoelektrischen Schichten.

Im ersten Teil wird basierend auf einer Schalentheorie für kleine Dehnungen und finite Rotationen ein elektro-mechanisch gekoppeltes Finite Elemente Modell für die statische und dynamische Berechnung dünnwandiger Strukturen mit piezoelektrischen Sensor- und Aktorschichten entwickelt. Die hier verwendete nichtlineare Schalentheorie basiert auf der Schubdeformationstheorie erster Ordnung und hat sechs unabhängige kinematische Parameter, die durch fünf Knotenfreiheitsgrade ausgedrückt werden. Unter Verwendung von zwei Eulerwinkeln werden beliebig große Rotationen beschrieben. Aufgrund der Annahme kleiner Dehnungen und eines schwachen elektrischen Potentials werden lineare piezoelektrische konstitutive Gleichungen und ein linear in Dickenrichtung verlaufendes Potential verwendet. Auch mehrere vereinfachte nichtlineare Schalentheorien werden in Finite Elemente Modelle dünnwandiger Strukturen implementiert, um durch Vergleichsrechnungen zeigen zu können, in welchen Anwendungsfällen die Theorie finiter Rotationen für Strukturen aus Schichtverbundwerkstoffen oder intelligente Strukturen erforderlich ist.

Im zweiten Teil wird eine Störungsunterdrückungs-Regelung mit einem Proportional-Integral (PI) Beobachter entwickelt, der Sprungfunktionen nutzt, um ein fiktives Modell von Störungen für aktive Schwingungskontrolle von intelligenten Strukturen zu konstruieren. Um das dynamische Verhalten des bestehenden PI Beobachters zu verbessern, wird ein Generalized PI-(GPI) Beobachter vorgeschlagen und entwickelt. Daher lassen sich durch die vorliegende Störungsunterdrückungs-Regelung alle unbekannten Störungen entweder mit einem PI oder GPI Beobachter schätzen bzw. kompensieren. Um die Vorteile der vorgeschlagenen Störungsunterdrückungs-Regelung darzustellen, werden zusätzlich PID, LQR und LQG Steuerstrategien eingesetzt.

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Dedicated to my beloved parents and sisters.

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Nomenclature

Operators

$\dot{\square}$	first-order time derivative, or velocity
$\ddot{\square}$	second-order time derivative, or acceleration
$\square_{,\alpha}$	spatial derivative with respect to Θ^α
$\square_{ \alpha}$	covariant derivative with respect to Θ^α
$\delta\square$	variational operator
$\Delta\square$	incremental operator
$\exp$	exponential operator
rank	rank
$\square^T$	transposition
$\square^{-1}$	inverse
$\ \ \ \$	Euclidean norm
$ \ \$	absolute value
$\cdot$	scalar product, or dot product
$\times$	vector product
$\otimes$	tensor product
$[\]$	matrix
$\{ \}$	vector
$\bar{\square}$	quantities in the deformed configuration
$\hat{\square}$	normalized quantities in Chapter 2 and 3
$\check{\square}$	estimated values in Chapter 4
$\breve{\square}$	quantities in the material coordinate system

Symbols

X^i	CARTESIAN coordinate system
Θ^i	curvilinear coordinate system
$\mathbf{R}$	position vector for an arbitrary point in the shell space
$\mathbf{r}$	position vector for an arbitrary point at the mid-surface
$\mathbf{g}_i$	covariant base vectors in the shell space
$\mathbf{g}^i$	contravariant base vectors in the shell space
g_{ij}	covariant metric tensor in the shell space
g^{ij}	contravariant metric tensor in the shell space
$\mathbf{a}_i$	covariant base vectors at the mid-surface
$\mathbf{a}^i$	contravariant base vectors at the mid-surface
a_{ij}	covariant metric tensor at the mid-surface
a^{ij}	contravariant metric tensor at the mid-surface
δ_i^j	KRONECKER delta
$\Gamma_{\alpha\beta}^\lambda$	CHRISTOFFEL symbols of the second kind
$b_{\alpha\beta}$	covariant components of the curvature tensor
b_β^α	mixed components of the curvature tensor
$\mathbf{u}$	displacement vector for an arbitrary point in the shell space
$\mathbf{u}^0$	translational displacement vector at the mid-surface
$\mathbf{u}^1$	rotation vector of the Θ^3 -line
v_i	translational displacements in the shell space
v_i^0	translational displacements at the mid-surface
v_i^1	rotational displacements at the mid-surface
$\boldsymbol{\mu}$	shift tensor
μ_j^i	components of the shift tensor
H	mean curvature of the surface
K	GAUSSIAN curvature of the surface
$\mathbf{F}$	deformation gradient tensor in Chapter 2
$\mathbf{C}$	right CAUCHY-GREEN tensor in Chapter 2
	output matrix in Chapter 4
$\mathbf{G}$	RIEMANNIAN metric tensor in Chapter 2
ρ	mass density
E	YOUNG'S modulus
ν	POISSON'S ratio
G	shear modulus
$\boldsymbol{\varepsilon}$	GREEN-LAGRANGE strain vector
$\boldsymbol{\sigma}$	second PIOLA-KIRCHHOFF stress vector
$\mathbf{c}$	elasticity constant matrix

$\mathbf{e}, \mathbf{d}$	piezoelectric constant matrix
ϵ	dielectric constant matrix
$\mathbf{D}$	electric displacement vector
$\mathbf{E}$	electric field vector
T	kinetic energy
W_{int}	internal work
W_{ext}	external work
${}^m C$	configuration m , $m = 0, 1, 2$
$\mathbf{M}_{\text{uu}}$	mass matrix
$\mathbf{C}_{\text{uu}}$	damping matrix
$\mathbf{K}_{\text{uu}}$	stiffness matrix
$\mathbf{K}_{\text{u}\phi}$	piezoelectric coupled stiffness matrix
$\mathbf{K}_{\phi\text{u}}$	piezoelectric coupled capacity matrix
$\mathbf{K}_{\phi\phi}$	piezoelectric capacity matrix
$\mathbf{K}_{\text{ug}}$	geometrically induced stiffness due to mechanically induced stresses
$\mathbf{K}_{\phi\text{g}}$	geometrically induced stiffness due to electrically induced stresses
$\mathbf{S}_{\text{uu}}$	mechanically induced resultant stresses
$\mathbf{S}_{\text{u}\phi}$	electrically induced resultant stresses
$\mathbf{F}_{\text{ue}}$	total external force vector
$\mathbf{F}_{\text{ub}}$	element body force vector
$\mathbf{F}_{\text{us}}$	element surface force vector
$\mathbf{F}_{\text{uc}}$	element concentrated force vector
$\mathbf{f}_{\text{b}}$	body force vector of an arbitrary point in the shell space
$\mathbf{f}_{\text{s}}$	surface force vector of an arbitrary point at the mid-surface
$\mathbf{f}_{\text{c}}$	concentrated force vector of an arbitrary point at the mid-surface
$\mathbf{F}_{\text{ui}}$	total in-balance force vector
$\mathbf{F}_{\text{ut}}$	inertial in-balance force vector
$\mathbf{F}_{\text{uu}}$	mechanically induced in-balance force vector
$\mathbf{F}_{\text{u}\phi}$	electrically induced in-balance force vector
$\mathbf{G}_{\phi\text{e}}$	total external charge vector
$\mathbf{G}_{\phi\text{s}}$	surface charge vector
$\mathbf{G}_{\phi\text{c}}$	concentrated charge vector
$\mathbf{G}_{\phi\text{i}}$	total in-balance charge vector
$\mathbf{F}_{\phi\text{u}}$	mechanically induced in-balance charge vector
$\mathbf{F}_{\phi\phi}$	electrically induced in-balance charge vector
$\mathbf{q}$	nodal displacement vector
ϕ_{a}	actuation voltage vector applied on piezoelectric layer
ϕ_{s}	sensor voltage vector output from piezoelectric layer
$\mathbf{A}_0$	linear strain-displacement relation matrix

$\mathbf{A}_n$	nonlinear strain-displacement relation matrix
$\mathbf{A}$	system matrix
$\mathbf{B}$	control matrix
$\bar{\mathbf{B}}$	disturbance noise influence matrix in LQG control
$\mathbf{N}$	disturbance influence matrix in DR control
$\mathbf{V}$	system matrix of the fictitious model of disturbances
$\mathbf{H}$	output matrix of the fictitious model of disturbances
$\mathbf{x}$	state vector
$\mathbf{u}$	control input vector
$\mathbf{y}$	measured output vector
$\mathbf{z}$	controlled output vector
$\mathbf{d}$	disturbance noise vector
$\mathbf{n}$	measurement noise vector
$\mathbf{f}$	disturbance vector
$\mathbf{v}$	base vector for re-construction of disturbance fictitious model
$\mathbf{L}$	observer gain matrix
$\mathbf{e}_x$	estimation error of plant state vector $\mathbf{x}$
$\mathbf{e}_v$	estimation error of base function vector $\mathbf{v}$
$\bar{\mathbf{Q}}$	weighting matrix for controlled output
$\bar{\mathbf{R}}$	weighting matrix for plant input
$\mathbf{Q}_g$	weighting matrix for disturbance noise
$\mathbf{R}_g$	weighting matrix for measurement noise
$\mathbf{P}$	positive definite matrix solved by RACCATI equation
$\mathbf{K}_p$	proportional gain of PID control
$\mathbf{K}_i$	integral gain of PID control
$\mathbf{K}_d$	derivative gain of PID control
$\mathbf{K}_x$	free vibration control gain in DR control
$\mathbf{K}_v$	forced vibration control gain in DR control
$\mathbf{I}$	identity matrix
$\mathbf{0}$	zero matrix
$\mathbf{1}$	one matrix

Abbreviations

FE	Finite Element
DOF(s)	Degree(s) of Freedom
CT	Classical Theory
FOSD	First-Order Shear Deformation
SOSD	Second-Order Shear Deformation
TOSD	Third-Order Shear Deformation
HOSD	Higher-Order Shear Deformation
LRT56	Large Rotation Theory with six parameters expressed by five nodal DOFs
LRT5	Fully geometrically nonlinear shell theory with five parameters
MRT5	Moderate Rotation Theory with five parameters
RVK5	Refined VON KÁRMÁN type nonlinear shell theory with five parameters
LIN5	LINEar shell theory with five parameters
ANS	Assumed Natural Strain
EAS	Enhanced Assumed Strain
FI	Full Integration
URI	Uniformly Reduced Integration
SRI	Selectively Reduced Integration
TL	Total Lagrangian
PID	Proportional-Integral-Derivative
LQR	Linear Quadratic Regulator
LQG	Linear Quadratic Gaussian
DR	Disturbance Rejection
PI	Proportional-Integral
GPI	Generalized Proportional-Integral
ARE	Algebraic RICCATI Equation
PZT	Lead Zirconate Titanate
PVDF	Polyvinylidene Fluoride

Chapter 1

Introduction

1.1 Background

With the development of material science and light-weight design, laminated thin-walled structures made of isotropic or orthotropic materials are applied in many fields of technology, e.g. civil, automotive, aeronautical [1, 2] and aerospace [3–5] engineering. Thin-walled structures have a number of beneficial properties, e.g. reduction of weight, less raw material, etc., however they tend to be more unstable and sensitive to vibrations. In the recent decades, thin-walled structures with integrated layers or patches of smart materials, i.e. *piezoelectrics*, *electrostrictives*, *magnetostrictives* and *shape memory alloys*, which are so-called *smart structures*, have been proposed for vibration control [2, 3], shape control [4, 5], noise and acoustic control [6, 7], damage detection, energy harvesting [8–13] and health monitoring [1, 14, 15], among others.

Smart structures in this dissertation refer to those integrated with smart materials acting as sensors and actuators that can sense the changes of the environment and measure the system states itself, based on which a control action can be designed and carried out to make the structures perform in a desired way. There are many other related definitions, although somehow they have slight differences. Wada *et al.* [16] gave a general framework of the definition of intelligent structures. They started defining two basic categories, the *sensory structures* “which possess sensors that enable the determination or monitoring of the system states or characteristics”, and the *adaptive structures* which are “those which possess actuators that enable the alteration of system states or characteristics in a controlled manner”. A sensory structure may have sensors, but possesses no actuators. Conversely, an adaptive structure may have actuators, but

does not have sensors. Later they defined *controlled structures* as those with both sensors and actuators that are connected into a feedback architecture. The term *active structures* is also used frequently in the literature, which is defined as a subset of controlled structures with the sensors and actuators highly integrated into a host structure as one object [16]. Finally, they defined the *intelligent structures* as those where sensors and actuators are highly integrated into a feedback architecture which also includes control logic and electronics [16, 17]. From the biological engineering point of view, Rogers [18] defined *intelligent material system* as “those with intelligence and life features integrated in the micro-structure of the material system to reduce mass and energy and produce adaptive functionality”.

1.2 Literature review

There are lots of work that can be carried out in the field of thin-walled composite and smart structures. The dissertation will deal with two problems, on the one hand, how to predict the static and dynamic behavior of composite laminated and piezoelectric integrated thin-walled structures, and on the other hand, how to improve dynamic or static behavior by using piezoelectric materials as sensors and actuators. In light of this reason, the literature review in this section only introduces and analyzes the fields related to mathematical modeling and vibration suppression techniques.

1.2.1 Literature on modeling technique

In order to predict the static or dynamic behavior of piezoelectric integrated composite structures, an efficient and accurate electro-mechanically coupled modeling of both the mechanical and electrical responses such as mechanical displacement and electric potential are essential. A large amount of papers developed linear Finite Element (FE) models for static and dynamic analysis of smart structures.

The most accurate modeling techniques, 3-Dimensional (3-D) solid FE methods, can construct highly accurate FE models, but usually with large model size resulting in high costs of computation time, see Tzou and Tseng [19], Dube *et al.* [20], Kapuria and Dube [21], Ray *et al.* [22], Sze *et al.* [23–25] and He [26] for linear models. Recently, a solid finite element method was applied by Kapuria and Kumari [27] for piezoelectric fiber reinforced composite integrated smart structures.

Due to the small dimension of thickness compared to the other two dimensions, thin-walled structures can be treated as a 2-dimensional surface using various displacement distribution assumptions along the thickness direction. Additionally, if the width or the length is small enough, thin-walled structures can be shrunk to a line for consideration. The resulting FE solutions are called 2-Dimensional (2-D) and 1-Dimensional (1-D) FE methods, respectively. Concerning smart structures shaped like beams or curved beams, 1-D FE methods are favorably adopted using the assumptions of e.g. BERNOULLI beam theory [28, 29] or TIMOSHENKO beam theory [30, 31].

Compared to solid element theories, 2-D FE methods based on various hypotheses are most frequently used in smart plate and shell structures, due to relatively high accuracy and less computation time. Numerous papers can be found in the literature which have developed 2-D piezoelectric coupled FE models based on KIRCHHOFF-LOVE plate/shell theory, which is called Classical Theory (CT), see [32–40] among others. In addition, Shimpi [41] presented a model of two variable *refined plate theory*, which has a strong similarity with the classical plate theory.

Taking the transverse shear strains into account, which are neglected in KIRCHHOFF-LOVE theory, yields REISSNER-MINDLIN plate/shell theory known as First-Order Shear Deformation (FOSD) theory, see [42–53] among others, for modeling of smart structures using geometrically linear theory. With the assumption of linear displacement distribution through the thickness, FOSD hypothesis is applicable for thin to moderately thick plates and shells. In order to deal with thick structures, Third-Order Shear Deformation (TOSD) or Higher-Order Shear Deformation (HOSD) hypotheses were first proposed by Reddy [54, 55] for composite structures. Later the theory was extended and developed by Hanna and Leissa [56], Carrera and Demasi [57] for composite structures, by Correia *et al.* [58], Correia *et al.* [59], Moita *et al.* [60] for smart structures. Furthermore, Loja *et al.* [61] and Soares *et al.* [62] proposed *higher-order B-spline* FE strip models for laminated composite structures bonded with piezoelectric patches. Additionally, Ray and Reddy [63], Vasques and Rodrigues [64] applied *first-order zigzag shear deformation* (or layerwise first-order shear deformation) theory, and Kapuria *et al.* [65, 66] developed *third-order zigzag shear deformation* theory, for piezolaminated structures.

Since linear models presented above are only valid for problems at small strains and small rotations, some papers started taking geometric nonlinearity into account for modeling of thin-walled structures laminated with isotropic, orthotropic or anisotropic materials. A large amount of papers have developed FE models using VON KÁRMÁN

type geometrically nonlinear theory based on classical theory [67], FOSD [68] and TOSD [69–71] for isotropic and orthotropic layered structures. With considering more nonlinear strain-displacement terms, FOSD *moderate rotation* theories were proposed and developed by Librescu and Schmidt [72], Schmidt and Reddy [73], Schmidt and Weichert [74]. Later the theory was implemented and applied into FE analysis by Palmerio *et al.* [75, 76], Kreja *et al.* [77] for composite structures. Both the VON KÁRMÁN type nonlinear theory and the moderate rotation theory mentioned above are limited to the assumption of moderate rotations in structures. In order to consider large rotations in thin-walled structures, Habip [78], Habip and Ebcioğlu [79] first derived the *full geometrically nonlinear* strain-displacement relations for both static and dynamic equations of shells based on FOSD hypothesis, and Librescu [80] developed fully nonlinear plate and shell theory for composite laminated structures. The finite rotation theory based on FOSD hypothesis was further implemented into FE analysis by Gruttmann *et al.* [81], Basar *et al.* [82, 83], Sansour and Bufler [84], Wriggers and Gruttmann [85], Sansour and Bednarczyk [86], Brank *et al.* [87], Bischoff and Ramm [88], Kreja and Schmidt [89], Lentzen [90] and others. Additionally, the large rotation theories were simplified to use the *classical beam theory* by Saravia *et al.* [91], and KIRCHHOFF-LOVE theory with triangular element by Kuznetsov and Levyakov [92]. The theory was also applied to FE analysis of beams and arches by Miller and Palazotto [93]. In order to deal with thick plates and shells, the large rotation theory was extended to seven parameter theory based on TOSD hypothesis with the assumption of an *inextensible shell director* by Basar *et al.* [94, 95] for composite structures. Similar theories were developed by Bischoff and Ramm [88], Gummadi and Palazotto [96, 97] based on TOSD hypothesis. Recently, Arciniega and Reddy [98] proposed a large rotation theory based on Second-Order Shear Deformation (SOSD) hypothesis with displacement *quadratically distributed* along the thickness direction using *higher-order elements*, in which a solid constitutive equation was employed, meaning that an *extensible shell director* was considered. In order to analyze rubber-like soft materials, Basar and Ding [99] developed a large-strain shell finite element model with taking the transverse normal strain into account based on SOSD hypothesis. In addition, Dvorkin and Bathe [100], Stander *et al.* [101] developed a four-node assumed strain shell element for static analysis. Jiang and Chernuka [102] developed a four-node mixed interpolation shell element. Sze *et al.* [103] re-calculated several popular nonlinear benchmark problems using the shell element in ABAQUS. Moreover, Kožar and Ibrahimbegović [104], Masud *et al.* [105], Lopez and Sala [106] developed fully geometrically nonlinear solid FE models for static analysis of shell

structures.

A few papers in the literature named large or finite rotation theories consider fully geometrically nonlinear strain-displacement relations, but are based on a simplified kinematic hypothesis, which does not allow to treat arbitrarily large rotations properly. Large or finite rotation theories refer to those which take into account arbitrary unrestricted rotations in structures, as well as consider full geometric nonlinearities. Large rotation theory usually requires six independent kinematic parameters in FOSD hypothesis and even more parameters if it is based on HOSD hypothesis. Eliminating the drilling rotation, the reduction to two rotational DOFs represent arbitrary large rotations of the shell director. Using the assumption of an inextensible shell director, two typical ways have been proposed and developed, see [107] for the detailed classification. The formulation using two EULER angles to express the shell director rotation was developed and implemented by Gruttmann *et al.* [81], Bruechter and Ramm [108], Basar *et al.* [83], Wriggers and Gruttmann [85], Brank *et al.* [87], Kreja and Schmidt [89] and others. On the other hand, RODRIGUES rotation formulation was proposed earlier by Simo *et al.* [109, 110], and later implemented and applied by Sansour and Bufler [84], Betsch *et al.* [107, 111], Basar *et al.* [112], Wang and Thierauf [113], Lentzen [90].

It is well known that elements exhibit over stiffening due to *locking* phenomena, which very often occurs when using lower-order elements or coarse meshes. Locking problems arise from inconsistencies in discrete representation of the transverse shear energy and membrane energy in shell elements. Especially when the thickness tends to zero, the convergence of the element is seriously affected. Locking problems are well understood as *shear locking* and *membrane locking*. Shear locking is caused by the KIRCHHOFF constraints or shear constraints of vanishing transverse shear strains, while membrane locking results from hidden constraints in shell models. More detailed descriptions can be found e.g. in [98, 114, 115]. There are several numerical methods that can avoid locking problems like Assumed Natural Strain (ANS) [100, 116–118], Enhanced Assumed Strain (EAS) [119–122], Selectively Reduced Integration (SRI) [123] and Uniformly Reduced Integration (URI) [124–126]. Alternatively, locking effects can be reduced by using *high-order elements*. One famous high-order finite element method is called *h-p* finite element method, which was proposed and developed earlier by Pitkäranta *et al.* [114, 127], Leino and Pitkäranta [115] and later by [98, 128, 129].

Much fewer published papers have considered geometric nonlinearity in modeling of piezoelectric integrated thin-walled structures than for composite structures described

above. The majority of papers developed piezoelectric coupled FE models only considering VON KÁRMÁN type nonlinearity. The first implementation of VON KÁRMÁN type nonlinear theory into analysis of piezoelectric integrated structures was done by Im and Atluri [130]. Later on, Kapuria and Dumir [131] developed a VON KÁRMÁN type nonlinear FE static model based on the classical plate theory for *thermal buckling* analysis, and Varelis *et al.* [132] developed a model based on FOSD hypothesis for *buckling* and *post-buckling* analysis of piezoelectric laminated composite plates. Additionally, VON KÁRMÁN type nonlinear FE models were developed by Panda and Ray [133], Varelis and Saravanos [134] based on FOSD hypothesis, by Kapuria and Alam [135] based on first-order *zigzag* hypothesis with a global third-order variation, by Icardi [136] based on *third-order zigzag* hypothesis, and by Schmidt and Vu [137] based on both FOSD and TOSD hypotheses for plates and shells. For dynamic analysis, VON KÁRMÁN type nonlinear theories are mostly used in modeling of smart structures as well, based on FOSD [138], TOSD [137, 139], and first-order zigzag [140, 141] hypotheses. Compared to VON KÁRMÁN type nonlinearity, moderate rotation nonlinear theory considers more nonlinear effects, which was first proposed by Librescu and Schmidt [72], Schmidt and Reddy [73]. Later, the theory was further developed and implemented by Lentzen *et al.* [90, 142, 143] for both static and dynamic analysis of smart structures. Considering more nonlinear strain-displacement terms, fully geometrically nonlinear FE models were developed by Moita *et al.* [144] based on KIRCHHOFF-LOVE theory for static analysis, by Kundu *et al.* [145] based on FOSD hypothesis for *buckling* and *post-buckling* analysis, by Gao and Shen [125] based on FOSD hypothesis for dynamic analysis, and by Dash and Singh [146] based on TOSD hypothesis for dynamic analysis. However, the fully geometrically nonlinear FE models presented in [125, 144–146] are not real large rotation FE models, even though full geometric nonlinearities are included. This is because the rotations in these models cannot be arbitrary large, but are limited to the range of moderate rotations. In order to represent large or finite rotations in piezoelectric integrated structures, Chróścielewski *et al.* [147, 148] developed a 1-D FE model using fully geometrically nonlinear large rotation theory for shape and vibration control of curved beams.

Apart from the above 2-D or 1-D nonlinear FE models using various hypotheses, Marinković *et al.* [149, 150] developed a degenerated shell element for fully geometrically nonlinear analysis of thin-walled piezoelectric structures. Furthermore, 3-D fully geometrically nonlinear FE models were developed by Yi *et al.* [151], Klinkel and Wagner [152, 153] for static and dynamic analysis of piezoelectric smart structures. Besides,

a nonlinear FE model was developed by Lee *et al.* [154] based on TOSD hypothesis for active control of magnetostrictive laminated shells.

All the studies on modeling of piezoelectric coupled smart structures cited above assume linear electric potential distribution or constant electric field through the thickness. However, the electric potential can also be assumed in a higher-order distribution like the displacements. In this frame work, Marinković *et al.* proposed *quadratically distributed electric potential* through the thickness direction in the linear [53] and non-linear [149, 150] FE models for both static and dynamic analysis. Moreover, electric field nonlinear effects which refer to strong electric field are considered in a linear layer-wise FE model by Kapuria and Yasin [155]. More advanced nonlinear material constitutive laws including the hysteresis effect were developed by Klinkel [156] and Linnemann *et al.* [157] for piezoelectric materials.

1.2.2 Literature on vibration control

As mentioned before, smart structures are widely proposed for vibration control. Vibrations can be significantly suppressed by well designed controllers. On the contrary, the dynamic behavior of smart structures will become worse if improper control laws are applied. Many control schemes have been proposed and developed in the literature, among which most of them were designed based on linear FE models of smart structures.

The majority of papers in the literature presented *negative velocity proportional feedback* control using FE models based on various hypotheses, e.g. classical plate theory [36, 38, 39, 158–163], TIMOSHENKO beam theory [30], FOSD hypothesis [45, 46, 48, 164–168], TOSD hypothesis [60, 169], HOSD hypothesis [170], first-order zigzag hypothesis [63, 171] and by using commercial software [172]. In addition, Liu *et al.* [49] implemented the same control law but using a *mesh-free* model based the FOSD hypothesis. Furthermore, Kang [163] implemented a *negative velocity proportional feedback* control into a real smart beam experimentally for vibration control. Interestingly, Moita *et al.* [60] developed a genetic algorithm for optimizing the position of piezoelectric patches using *negative velocity proportional feedback* control based on a TOSD FE model. Apart from negative velocity proportional feedback control law, Narayanan and Balamurugan [30], Balamurugan and Narayanan [45], Tzou and Tseng [158–160] applied LYAPUNOV feedback control for active vibration control of smart structures based on the models obtained by FOSD and classical theories, respectively. Tzou and

Chai [28] applied *bang-bang* control method numerically based on EULER-BERNOULLI beam theory, and they also carried out the control schemes experimentally. The same control law was implemented by Zhang and Shen [40] based on the model obtained by classical plate theory but for *piezoelectric fiber reinforced* composite structures.

Optimal control laws are very popular in simulations of vibration suppression of smart structures. Numerous papers can be found in the literature that have developed Linear Quadratic Regulator (LQR) control with FE models based on e.g. classical plate/shell theory [163], TIMOSHENKO beam theory [30], FOSD hypothesis [45, 173, 174], layerwise theory [171], 3-D solution [175] and others [176]. Since LQR control is a *full state feedback* control, all state variables have to be measured. Therefore, LQR is an ideal method, which cannot be implemented into real systems in most of the cases. In light of this shortcoming, Linear Quadratic Gaussian (LQG) control was implemented by Vasques and Rodrigues [171] numerically, by Dong *et al.* [177] both numerically and experimentally, in which the state variables can be estimated using the measured signals. Additionally, Stavroulakis *et al.* [178] implemented LQR control and robust H_2 control with the model obtained based on EULER-BERNOULLI beam theory, and they compared the results with each other. Marinaki *et al.* [31] developed a particle swarm optimization based controller for vibration suppression of beams. Moreover, Roy and Chakraborty [179] proposed a *genetic algorithm* based LQR control for smart *fiber reinforced polymer* composite shell structures using the model derived by REISSNER-MINDLIN theory.

Some other advanced control schemes can also be found in the literature. Chen and Shen [180], Lin and Nien [181] presented an *independent modal space* control for vibration suppression of smart structures. Bhattacharya *et al.* [182] proposed an *independent modal space* based LQR control strategy for vibration control of laminated spherical shell with different fiber orientation and varying radius of curvature based on REISSNER'S hypothesis. Furthermore, Manjunath and Bandyopadhyay [183] proposed a *discrete sliding mode* control scheme with TIMOSHENKO beam theory for the vibration control of smart flexible beams. A *prediction* control algorithm was applied by Valliappan and Qi [175] for vibration control of beams with bonded piezoelectric patches. In addition, a *robust* control was considered by Li *et al.* [184], Hu and Vukovich [185, 186], Marinova *et al.* [187].

Most of the papers presented above developed controllers based on linear FE models. There are only a few papers in the literature considering geometrically nonlinear theory in vibration control simulation. To the author's knowledge till now, only some simple

control schemes, e.g. negative *velocity* or *displacement* proportional feedback control, were applied. Zhou and Wang [188] implemented negative velocity and displacement proportional feedback control based on the nonlinear model using BERNOULLI beam theory. The same control scheme was applied by Schmidt and Vu [137], Vu [189] using a TOSD VON KÁRMÁN nonlinear FE model, by Lentzen and Schmidt [142, 143] using FOSD moderate rotation nonlinear theory, and by Gao and Shen [125] based on a fully geometrically nonlinear FE plate model with FOSD hypothesis.

All the control schemes discussed previously require an accurate mathematical model, which are called conventional control schemes. However, some promising control strategies, e.g. *neural network* control, *fuzzy logic* control, etc., which are called *intelligent control*, may also be a good choice for vibration control of smart structures. Intelligent control has been well developed in the last three decades. Nevertheless, only a very limited number of papers developed *intelligent control* for vibration suppression of smart structures. *Neural network* based controllers for vibration suppression of smart structures were implemented numerically by Lee [190], Han and Acar [191], Valoor *et al.* [192], while Youn *et al.* [193], Kumar *et al.* [194], Qiu *et al.* [195] applied *neural network* controller experimentally. Furthermore, a *neural adaptive predictive* control was developed and applied by Jha and He [196]. Besides, a *fuzzy logic* control, was proposed by Shirazi *et al.* [197] for vibration suppression of *functionally graded* rectangular plate bonded with piezoelectric patches by using classical plate theory, as well as by Abreu and Ribeiro [198].

Most of those proposed control schemes above did not take disturbances into account as state variables fed back to the controller, since these disturbances are usually unknown and unmeasurable. However, the vibrations of smart structures in most of the cases are caused by disturbances. In order to estimate the unknown disturbances or inputs, a Proportional-Integral (PI) observer was first proposed and developed by Müller and Lückel [199, 200], and later applied and completed by [201–205]. Besides, *full-* and *reduced-order* observer [206, 207] and *sliding-mode* observer [208, 209] were developed as well for unknown disturbance estimation. Nevertheless, only a few of them compensated the estimated disturbances into the closed-loop system to improve the control effects.

1.3 Objectives and outline

In the previous sections, the state of the art of modeling and vibration control techniques for smart structures is presented. From the literature review, large rotation theory is well developed for static analysis of composite laminated thin-walled structures, as well as linear theory for piezoelectric coupled smart structures. Although a number of papers published recently developed nonlinear theories for modeling of piezoelectric coupled smart structures, the majority of them considered simplified nonlinear shell theories including VON KÁRMÁN type nonlinear theory, *moderate rotation* nonlinear theory and *fully geometrically nonlinear* theory with limited rotations. However, thin-walled shaped piezoelectric integrated smart structures are frequently undergoing large deflections and rotations. Therefore, the first objective of this dissertation is to develop piezoelectric coupled large rotation FE models for static and dynamic analysis of smart structures based on FOSD hypothesis. In order to emphasize the accuracy of the proposed large rotation theory, other simplified nonlinear shell theories are discussed and compared, which range from VON KÁRMÁN type nonlinearity to full geometric nonlinearity with large rotations.

By reviewing the control techniques for vibration suppression of smart structures published in the literature, it can be observed that most of the papers applied simple control laws, e.g. *negative velocity proportional feedback* control, LYAPUNOV control, *bang-bang* control, *optimal* control (LQR or LQG) and *robust* control, based on linear piezoelectric coupled FE models. Even though there are some papers taking geometric nonlinearity into account in dynamic analysis, only a few of them implemented very simple control laws (e.g. velocity feedback proportional control) based on the simplified nonlinear models. Additionally, very few papers took unknown disturbances into account, since they cannot be easily measured in most of the cases. However, disturbances are the main cause of vibrations. In light of this, the second part of this dissertation is to develop a Disturbance Rejection (DR) control with PI observer which uses step functions to build the fictitious model of disturbances. In order to improve the dynamic behavior, the PI observer is then extended to the Generalized PI (GPI) observer, in which nonlinear functions, e.g. sine, cosine, polynomial ones, are employed to represent disturbances. Therefore, the unknown disturbances can be estimated by either PI or GPI observer, and compensated by the proposed DR control.

In order to fulfill all these objectives, the dissertation is organized into five major chapters. In Chapter 2, we first introduce and compare the hypotheses that have been

already developed, which is followed by the definitions of base vectors and geometric quantities in curvilinear coordinate system. Afterwards, the strain-displacement relations for large rotation theories with six parameters based on FOSD hypothesis are derived, as well as those for various geometrically nonlinear shell theories ranging from VON KÁRMÁN nonlinearity to full geometric nonlinearity.

Chapter 3 develops large rotation electro-mechanically coupled FE models for static and dynamic analysis of composite and piezoelectric laminated thin-walled structures. The large rotation theory has six independent kinematic parameters expressed by five nodal DOFs using EULER angles to represent arbitrary rotations in structures. To demonstrate the effect of the proposed large rotation FE models, other simplified nonlinear FE models are developed as well. Those nonlinear models are linearized by Total-LAGRANGIAN formulations. In the last part of this chapter, several numerical algorithms are introduced for solving the coupled static and dynamic equations.

In Chapter 4, a DR control with PI observer is developed for vibration suppression of smart structures. Later, the PI observer is extended to the GPI observer, which has better dynamic characteristics. Thus, the unknown disturbances can be estimated either by the PI or GPI observer, and then the estimated disturbances are fed back to the controller as measured signals. In order to present the advantages of the proposed DR controller with PI or GPI observer, PID, LQR and LQG control algorithms are implemented as well.

In Chapter 5 and 6, the simulations of FE analysis and the results of active vibration control of smart structures are respectively presented. Chapter 5 first deals with the validation test of the present large rotation FE model by several static benchmark problems, buckling and post-buckling analysis. Later, the nonlinear FE models based on RVK5, MRT5, LRT5 and LRT56 shell theories are applied to static and dynamic analysis of piezoelectric integrated smart structures. Chapter 6 illustrates the active vibration control effects by various control schemes including PID, LQR, LQG and DR control. The results illustrate that the disturbances can be well re-constructed by using PI or GPI observer, which are considered in DR control.

The last chapter, Chapter 7, summarizes the present work and outlines the scope of the future work.

Chapter 2

Nonlinear shell theories

This chapter starts with discussing various hypotheses, and the differences between these hypotheses are outlined. Afterwards, the mathematical preliminaries, including the covariant and contravariant base vectors, the position vectors, the CHRISTOFFEL symbols, the shifter tensor, etc., will be defined and introduced. Based on FOSD hypothesis, the displacement vector and the six parameters for large rotation shell theory are discussed. Using these predefined quantities, the fully geometrically nonlinear strain-displacement relations in terms of six parameters based on FOSD hypothesis are obtained, which was first developed by Habip [78]. Representing the six parameters by the five nodal DOFs yields a large rotation shell theory, abbreviated as LRT56, which allows unrestricted finite rotations occurring in structures. The theory, LRT56, which was developed earlier by Kreja and Schmidt [89], Kreja [126] for static analysis of composite structures, will be extended to static and dynamic analysis of smart structures. Additionally, other simplified nonlinear shell theories are implemented as well to show the necessity of the LRT56 theory in the application of thin-walled structures undergoing large rotations.

2.1 Shear deformation hypotheses

The 3-D FE method is a very good choice for modeling of thin-walled composite and smart structures, which can obtain relatively high accuracy models but with large model size and high computation time. Due to small thickness of thin-walled plates and shells, 2-D FE methods are more frequently used in theoretical and numerical analysis using various hypotheses, as shown in Fig. 2.1. The main advantage of 2-D

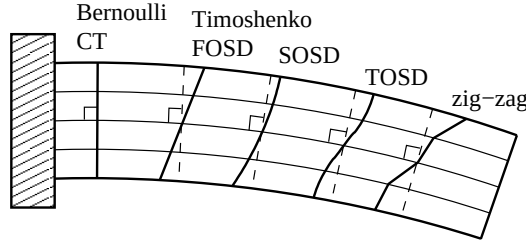


FIGURE 2.1: Various shear deformation hypotheses

FE models is that less computation time is needed due to small size of the models compared to 3-D ones, but they are still retaining a high accuracy. For thin-walled beam structures, even 1-D FE method can be employed using EULER-BERNOULLI or TIMOSHENKO or higher-order beam hypothesis.

The simplest hypothesis for plates and shells is the KIRCHHOFF-LOVE theory, which is also known as CT, see [32, 34, 36–40]. This theory can be understood as an extension of the EULER-BERNOULLI beam theory from 1-D case to 2-D case, which assumes that the straight lines normal to the mid-surface remain straight and normal after deformation. This assumption leads to zero transverse shear strains, which results in an inadequate prediction of the elastic behavior of layered composite and smart structures. In order to overcome this limitation, REISSNER-MINDLIN plate/shell theory [42] known as FOSD hypothesis was proposed, which takes transverse shear strains into account. Analogously, the FOSD plate/shell theory can be understood as an extension of the TIMOSHENKO beam theory with assuming that the straight lines normal to the mid-surface remain straight after deformation, but not necessarily normal. This theory assumes that a linear variation of displacement is distributed across the shell thickness.

In order to deal with moderately thick structures, SOSD, TOSD or any other HOSD hypotheses are much more preferred than FOSD hypothesis. Due to different hypotheses, the displacements can be distributed linearly, quadratically, cubically or according to other higher-order functions, see Fig. 2.1. Concerning laminated shell structures, *zigzag* theory can describe the through-thickness distribution of displacements better than other theories, since it satisfies the inter-layer shear stress continuity. If *first-order zigzag* hypothesis is not accurate enough, *second-* or *third-order zigzag* hypothesis can be employed.

2.2 Mathematical preliminaries

Two coordinate systems, the CARTESIAN coordinate system (X^1 , X^2 and X^3) and the curvilinear coordinate system (Θ^1 , Θ^2 and Θ^3), are used in this dissertation, as shown in Fig. 2.2. The fixed CARTESIAN coordinate system acts as a global coordinate system,

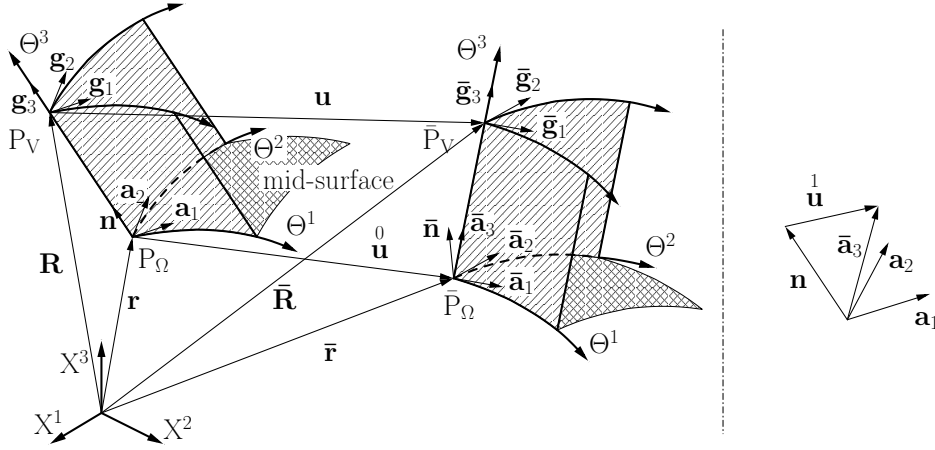


FIGURE 2.2: Definition of base vectors

while the convective curvilinear coordinate system acts as a local coordinate system, which can be plate, cylindrical, spherical or any other coordinates. The position vector of an arbitrary point in the shell space (V) is denoted by $\mathbf{R}(\Theta^1, \Theta^2, \Theta^3)$, while $\mathbf{r}(\Theta^1, \Theta^2)$ refers to that of an arbitrary point at the mid-surface (Ω). Two configurations are considered: the *undeformed configuration* which is shown in the left part of Fig. 2.2; and the *deformed configuration* shown in the middle part of the figure. Furthermore, the right part of the figure shows the rotation of the Θ^3 -line. An arbitrary point in the shell space and at the mid-surface is denoted by P_V and P_Ω , respectively.

The covariant and contravariant base vectors, $\mathbf{g}_i$, $\mathbf{g}^i$, with the corresponding metric tensors, g_{ij} , g^{ij} , as well as the CHRISTOFFEL symbols of the second kind, Γ_{ij}^k , at point P_V in the undeformed configuration are given by

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial \Theta^i} = \mathbf{R}_{,i}, \quad (2.1)$$

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j, \quad \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij} = g_{ji}, \quad \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij} = g^{ji}, \quad (2.2)$$

$$\Gamma_{ij}^k = \Gamma_{ji}^k = \mathbf{g}_{i,j} \cdot \mathbf{g}^k = -\mathbf{g}_i \cdot \mathbf{g}^k_{,j}, \quad (2.3)$$

where $\square_{,i}$ represents the spatial derivative with respect to Θ^i , the Latin indices vary from 1 to 3, and δ_i^j is the KRONECKER delta which is given by

$$\delta_i^j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j . \end{cases} \quad (2.4)$$

Analogously, the base vectors and geometric quantities at point P_Ω in the undeformed configuration are given by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \Theta^\alpha} = \mathbf{r}_{,\alpha} , \quad \mathbf{a}_3 = \mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} , \quad (2.5)$$

$$\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j , \quad \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\alpha\beta} = a_{\beta\alpha} , \quad \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta} = a^{\beta\alpha} , \quad (2.6)$$

$$\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma = -\mathbf{a}_\alpha \cdot \mathbf{a}^{\gamma}_{,\beta} , \quad (2.7)$$

where $\|\cdot\|$ represent the EUCLIDEAN norm, and the Greek indices vary only from 1 to 2. From the definition of the vector $\mathbf{n}$, it can be seen that $\mathbf{n}$ is a unit vector and perpendicular to the plane formed by $(\mathbf{a}_1, \mathbf{a}_2)$, which can be represented by the formulae as

$$\mathbf{a}_\alpha \cdot \mathbf{n} = 0 , \quad (2.8)$$

$$\mathbf{n} \cdot \mathbf{n} = 1 . \quad (2.9)$$

Taking the derivative of equations (2.8) and (2.9) with respect to Θ^β one obtains

$$\mathbf{a}_{\alpha,\beta} \cdot \mathbf{n} + \mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta} = 0 , \quad (2.10)$$

$$\mathbf{n} \cdot \mathbf{n}_{,\beta} = 0 . \quad (2.11)$$

The derivative of the covariant and contravariant base vectors at point P_Ω with respect to Θ^β can be obtained as

$$\mathbf{a}_{\alpha,\beta} = \Gamma_{\alpha\beta}^\delta \mathbf{a}_\delta + b_{\alpha\beta} \mathbf{n} , \quad (2.12)$$

$$\mathbf{a}^\alpha_{,\beta} = -\Gamma_{\delta\beta}^\alpha \mathbf{a}^\delta + b_\beta^\alpha \mathbf{n} , \quad (2.13)$$

$$\mathbf{n}_{,\beta} = -b_\beta^\delta \mathbf{a}_\delta = -b_{\lambda\beta} \mathbf{a}^\lambda . \quad (2.14)$$

Here, $b_{\alpha\beta}$ and b_{β}^{α} are the covariant and mixed components of the curvature tensor, respectively, which can be calculated by

$$b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n} = -\mathbf{a}_{\alpha} \cdot \mathbf{n}_{,\beta} , \quad (2.15)$$

$$b_{\beta}^{\alpha} = \mathbf{a}^{\alpha}_{,\beta} \cdot \mathbf{n} = -\mathbf{a}^{\alpha} \cdot \mathbf{n}_{,\beta} . \quad (2.16)$$

The relations between the covariant and mixed components of the curvature tensor can be obtained as

$$b_{\alpha}^{\lambda} = a^{\beta\lambda} b_{\alpha\beta} . \quad (2.17)$$

Additionally, the same notations, but with an overbar, are used for the base vectors and geometric quantities in the deformed configuration, which is shown in the middle part of Fig. 2.2. Thus, the base vectors in the undeformed and deformed configurations will be denoted as given in Table 2.1.

TABLE 2.1: Base vectors in the undeformed and deformed configurations

	Name	Undeformed	Deformed
Position vector in the shell space	$\mathbf{R}$	$\bar{\mathbf{R}}$	
Position vector at the mid-surface	$\mathbf{r}$	$\bar{\mathbf{r}}$	
Covariant base vectors in the shell space	$\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$	$\bar{\mathbf{g}}_1, \bar{\mathbf{g}}_2, \bar{\mathbf{g}}_3$	
Covariant base vectors at the mid-surface	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3(\mathbf{n})$	$\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3$	
Contravariant base vectors in the shell space	$\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$	$\bar{\mathbf{g}}^1, \bar{\mathbf{g}}^2, \bar{\mathbf{g}}^3$	
Contravariant base vectors at the mid-surface	$\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3(\mathbf{n})$	$\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3$	

2.3 Kinematics of shell structures

2.3.1 Through-thickness displacement distribution

According to the geometric relations in Fig. 2.2, the position vector of point P_V in the undeformed configuration can be expressed by the base vectors at the mid-surface as

$$\mathbf{R} = \mathbf{r} + \Theta^3 \mathbf{n} . \quad (2.18)$$

Due to the FOSD hypothesis that straight lines along the thickness direction remain straight, but not normal to the deformed mid-surface, the position vector at $\bar{P}_V$ in the

deformed configuration will be obtained as

$$\bar{\mathbf{R}} = \bar{\mathbf{r}} + \Theta^3 \bar{\mathbf{a}}_3 . \quad (2.19)$$

Furthermore, because of the FOSD hypothesis, the displacement vector $\mathbf{u}$ can be obtained, which is linearly distributed through the thickness direction, as

$$\mathbf{u} = \bar{\mathbf{R}} - \mathbf{R} = \overset{0}{\mathbf{u}} + \Theta^3 \overset{1}{\mathbf{u}} , \quad (2.20)$$

where $\overset{0}{\mathbf{u}}$ denotes the translational displacement vector at the mid-surface, and $\overset{1}{\mathbf{u}}$ is the rotational displacement vector, which describes the rotation of the Θ^3 -line from $\mathbf{n}$ to $\bar{\mathbf{a}}_3$. They can be respectively obtained as

$$\overset{0}{\mathbf{u}} = \bar{\mathbf{r}} - \mathbf{r} , \quad (2.21)$$

$$\overset{1}{\mathbf{u}} = \bar{\mathbf{a}}_3 - \mathbf{n} . \quad (2.22)$$

Taking the derivative of equations (2.21) and (2.22) with respect to Θ^α yields

$$\overset{0}{\mathbf{u}}_{,\alpha} = \bar{\mathbf{a}}_\alpha - \mathbf{a}_\alpha , \quad (2.23)$$

$$\overset{1}{\mathbf{u}}_{,\alpha} = \bar{\mathbf{a}}_{3,\alpha} - \mathbf{n}_{,\alpha} . \quad (2.24)$$

Further, the covariant and contravariant components of the translational displacement vector $\overset{0}{\mathbf{u}}$ and the rotational displacement vector $\overset{1}{\mathbf{u}}$ are defined as

$$\overset{0}{\mathbf{u}} = \overset{0}{v}_\alpha \mathbf{a}^\alpha + \overset{0}{v}_3 \mathbf{n} = \overset{0}{v}^\alpha \mathbf{a}_\alpha + \overset{0}{v}^3 \mathbf{n} , \quad (2.25)$$

$$\overset{1}{\mathbf{u}} = \overset{1}{v}_\alpha \mathbf{a}^\alpha + \overset{1}{v}_3 \mathbf{n} = \overset{1}{v}^\alpha \mathbf{a}_\alpha + \overset{1}{v}^3 \mathbf{n} . \quad (2.26)$$

Here, the six covariant components are considered as six independent kinematic parameters, among which the first three parameters, $\overset{0}{v}_1, \overset{0}{v}_2, \overset{0}{v}_3$, are the translational displacements at the mid-surface, and the last three parameters, $\overset{1}{v}_1, \overset{1}{v}_2, \overset{1}{v}_3$, are the generalized rotational displacements, i.e. the projections of $\overset{1}{\mathbf{u}}$ in the contravariant base vector triad of the undeformed configuration. The sixth parameter $\overset{1}{v}_3$ is usually neglected in linear or simplified nonlinear shell theories, due to the assumption of small or moderate rotations occurring in structures. However, when structures undergo large rotations and deflections, $\overset{1}{v}_3$ is no longer small. Therefore the sixth parameter $\overset{1}{v}_3$ must be considered in large rotation theory.

Using the defined covariant components of the vectors $\overset{0}{\mathbf{u}}$ and $\overset{1}{\mathbf{u}}$, equation (2.20) can also be re-written in scalar form as

$$v_\alpha(\Theta^1, \Theta^2, \Theta^3) = \overset{0}{v}_\alpha(\Theta^1, \Theta^2) + \Theta^3 \overset{1}{v}_\alpha(\Theta^1, \Theta^2), \quad (2.27)$$

$$v_3(\Theta^1, \Theta^2, \Theta^3) = \overset{0}{v}_3(\Theta^1, \Theta^2) + \Theta^3 \overset{1}{v}_3(\Theta^1, \Theta^2). \quad (2.28)$$

Taking the derivative of equations (2.25)-(2.26) with respect to Θ^β and considering the covariant components yields

$$\begin{aligned} \overset{n}{\mathbf{u}}_{,\beta} &= \overset{n}{v}_{\alpha,\beta} \mathbf{a}^\alpha + \overset{n}{v}_\alpha \mathbf{a}^{\alpha}_{,\beta} + \overset{n}{v}_{3,\beta} \mathbf{n} + \overset{n}{v}_3 \mathbf{n}_{,\beta} \\ &= \left(\overset{n}{v}_{\lambda,\beta} - \Gamma_{\lambda\beta}^\alpha \overset{n}{v}_\alpha - b_{\lambda\beta} \overset{n}{v}_3 \right) \mathbf{a}^\lambda + \left(\overset{n}{v}_{3,\beta} + b_\beta^\alpha \overset{n}{v}_\alpha \right) \mathbf{n}. \end{aligned} \quad (2.29)$$

Alternatively, taking the derivative of (2.25)-(2.26) with respect to Θ^β using the contravariant components one obtains

$$\begin{aligned} \overset{n}{\mathbf{u}}_{,\beta} &= \overset{n}{v}^{\alpha}_{,\beta} \mathbf{a}_\alpha + \overset{n}{v}^\alpha \mathbf{a}_{\alpha,\beta} + \overset{n}{v}^3_{,\beta} \mathbf{n} + \overset{n}{v}^3 \mathbf{n}_{,\beta} \\ &= \left(\overset{n}{v}^{\lambda}_{,\beta} + \Gamma_{\alpha\beta}^\lambda \overset{n}{v}^\alpha - b_\beta^\lambda \overset{n}{v}^3 \right) \mathbf{a}_\lambda + \left(\overset{n}{v}^3_{,\beta} + b_{\alpha\beta} \overset{n}{v}^\alpha \right) \mathbf{n}. \end{aligned} \quad (2.30)$$

Introducing the covariant derivative with respect to Θ^β represented by $\square_{|\beta}$

$$\overset{n}{v}_{\lambda|\beta} = \overset{n}{v}_{\lambda,\beta} - \Gamma_{\lambda\beta}^\alpha \overset{n}{v}_\alpha, \quad (2.31)$$

$$\overset{n}{v}^\lambda_{|\beta} = \overset{n}{v}^\lambda_{,\beta} + \Gamma_{\alpha\beta}^\lambda \overset{n}{v}^\alpha, \quad (2.32)$$

and using the following abbreviations

$$\overset{n}{\varphi}_{\lambda\beta} = \overset{n}{v}_{\lambda|\beta} - b_{\lambda\beta} \overset{n}{v}_3, \quad (2.33)$$

$$\overset{n}{\varphi}_\beta^\lambda = \overset{n}{v}^\lambda_{|\beta} - b_\beta^\lambda \overset{n}{v}^3, \quad (2.34)$$

$$\overset{n}{\varphi}_{3\beta} = \overset{n}{v}_{3,\beta} + b_\beta^\alpha \overset{n}{v}_\alpha, \quad (2.35)$$

$$\overset{n}{\varphi}_\beta^3 = \overset{n}{v}^3_{,\beta} + b_{\alpha\beta} \overset{n}{v}^\alpha, \quad (2.36)$$

equation (2.29) can be re-written as

$$\begin{aligned} \overset{n}{\mathbf{u}}_{,\beta} &= \overset{n}{\varphi}_{\lambda\beta} \mathbf{a}^\lambda + \overset{n}{\varphi}_{3\beta} \mathbf{n} \\ \text{or } \overset{n}{\mathbf{u}}_{,\beta} &= \overset{n}{\varphi}_\beta^\lambda \mathbf{a}_\lambda + \overset{n}{\varphi}_\beta^3 \mathbf{n}. \end{aligned} \quad (2.37)$$

Here, the overhead letter n assumes only the value 0 or 1. From equations (2.25)

and (2.26), it can be concluded that $\overset{n}{v}_3 = \overset{n}{v}^3$. Therefore, the relations between the abbreviations above can be obtained as

$$\overset{n}{\varphi}_\beta^\lambda = a^{\lambda\alpha} \overset{n}{\varphi}_{\alpha\beta}, \quad (2.38)$$

$$\overset{n}{\varphi}_\beta^3 = a^{33} \overset{n}{\varphi}_{3\beta} = \overset{n}{\varphi}_{3\beta}. \quad (2.39)$$

2.3.2 Shifter tensor

Taking the spatial derivative of (2.18) with respect to Θ^i and using (2.14) yields

$$\begin{aligned} \mathbf{g}_\alpha &= \mathbf{a}_\alpha + \Theta^3 \mathbf{n}_{,\alpha} = (\delta_\alpha^\delta - b_\alpha^\delta \Theta^3) \mathbf{a}_\delta = \mu_\alpha^\delta \mathbf{a}_\delta, \\ \mathbf{g}_3 &= \mathbf{a}_3 = \mu_3^3 \mathbf{a}_3, \end{aligned} \quad (2.40)$$

where μ_i^j denote the components of the shifter tensor $\boldsymbol{\mu}$, which can be obtained as

$$\mu_i^j = \begin{bmatrix} 1 - \Theta^3 b_1^1 & -\Theta^3 b_1^2 & 0 \\ -\Theta^3 b_2^1 & 1 - \Theta^3 b_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.41)$$

The shifter tensor can be expressed in terms of the covariant base vectors at the mid-surface as

$$\boldsymbol{\mu} = \mathbf{g}_i \otimes \mathbf{a}^i = \mu_\lambda^\delta \mathbf{a}_\delta \otimes \mathbf{a}^\lambda + \mathbf{a}_3 \otimes \mathbf{a}_3, \quad (2.42)$$

$$\boldsymbol{\mu}^T = \mathbf{a}^i \otimes \mathbf{g}_i = \mu_\lambda^\delta \mathbf{a}^\lambda \otimes \mathbf{a}_\delta + \mathbf{a}_3 \otimes \mathbf{a}_3. \quad (2.43)$$

Here $\otimes$ represents the tensor product. We further define the determinant of the shifter tensor, μ , as

$$\begin{aligned} \mu &= \det [\mu_i^j] = 1 - \Theta^3 (b_1^1 + b_2^2) + (\Theta^3)^2 (b_1^1 b_2^2 - b_1^2 b_2^1) \\ &= 1 - 2H\Theta^3 + K(\Theta^3)^2, \end{aligned} \quad (2.44)$$

where H and K denote respectively the mean and GAUSSIAN curvature of the surface.

Using the shifter tensor, the volume element

$$dV = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 d\Theta^1 d\Theta^2 d\Theta^3 = \sqrt{g} d\Theta^1 d\Theta^2 d\Theta^3 \quad (2.45)$$

can be related to the surface element as

$$dV = \mu d\Theta^3 d\Omega \quad (2.46)$$

where the surface area element is given by

$$d\Omega = |\mathbf{a}_1 \times \mathbf{a}_2| d\Theta^1 d\Theta^2 = \sqrt{a} d\Theta^1 d\Theta^2, \quad (2.47)$$

in which

$$g = \det[g_{ij}], \quad a = \det[a_{\alpha\beta}]. \quad (2.48)$$

2.4 Strain field

There are several strain measures available, among which the GREEN-LAGRANGE strain and the ALMANSI strain measures are the most popular two. They are respectively associated with the second PIOLA-KIRCHHOFF stresses and the CAUCHY stresses. The GREEN-LAGRANGE strains are referred to the undeformed configuration, while the ALMANSI strains are measured in the deformed configuration.

In problems of large rotations or deflections, the internal virtual work is given by (see e.g. [210, 211])

$$\delta W_{\text{int}} = \int_V \sigma^{ij} \delta \varepsilon_{ij} dV \quad (2.49)$$

where ε_{ij} and σ^{ij} denote the components of the GREEN-LAGRANGE strain tensor and the second PIOLA-KIRCHHOFF stress tensor, respectively. In such a way, the volume integral is referred to the undeformed configuration, which can be easily formulated. Due to this reason, the GREEN-LAGRANGE strains are mostly employed in large rotation theories.

The *deformation gradient* tensor $\mathbf{F}$ at point P_V in the shell space is defined as

$$\mathbf{F} = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i, \quad \mathbf{F}^T = \mathbf{g}^i \otimes \bar{\mathbf{g}}_i. \quad (2.50)$$

We now introduce the GREEN-LAGRANGE strain tensor which is defined as (see books e.g. [212])

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G}), \quad (2.51)$$

where $\mathbf{C}$ and $\mathbf{G}$ are the right CAUCHY-GREEN tensor and RIEMANNIAN metric tensor, respectively. Using equation (2.50), the right CAUCHY-GREEN tensor can be

calculated as

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = (\mathbf{g}^i \otimes \bar{\mathbf{g}}_i)(\bar{\mathbf{g}}_j \otimes \mathbf{g}^j) = \bar{g}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \quad (2.52)$$

and the RIEMANNIAN metric tensor in the undeformed configuration is

$$\mathbf{G} = \mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{g}^i = g_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = g^{ij} \mathbf{g}_i \otimes \mathbf{g}_j. \quad (2.53)$$

Substitution of equations (2.52) and (2.53) into (2.51), the GREEN-LAGRANGE strain tensor becomes

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\bar{g}_{ij} - g_{ij}) \mathbf{g}^i \otimes \mathbf{g}^j = \varepsilon_{ij} \mathbf{g}^i \otimes \mathbf{g}^j. \quad (2.54)$$

According to (2.18), the components of the covariant metric tensor for an arbitrary point in the shell space in the undeformed configuration can be derived in terms of the covariant base vectors at the mid-surface and their derivatives as

$$\begin{aligned} g_{\alpha\beta} &= \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta + \Theta^3(\mathbf{n}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}) + (\Theta^3)^2 \mathbf{n}_{,\alpha} \cdot \mathbf{n}_{,\beta}, \\ g_{\alpha 3} &= \mathbf{g}_\alpha \cdot \mathbf{g}_3 = \mathbf{a}_\alpha \cdot \mathbf{n} + \Theta^3 \mathbf{n}_{,\alpha} \cdot \mathbf{n} = 0, \\ g_{33} &= \mathbf{g}_3 \cdot \mathbf{g}_3 = \mathbf{n} \cdot \mathbf{n} = 1. \end{aligned} \quad (2.55)$$

Similarly, the components of the covariant metric tensor at $\bar{\mathbf{P}}_V$ can be obtained in terms of covariant base vectors at the mid-surface and their derivatives in the deformed configuration as

$$\begin{aligned} \bar{g}_{\alpha\beta} &= \bar{\mathbf{g}}_\alpha \cdot \bar{\mathbf{g}}_\beta = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta + \Theta^3(\bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_{3,\beta}) + (\Theta^3)^2 \bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_{3,\beta}, \\ \bar{g}_{\alpha 3} &= \bar{\mathbf{g}}_\alpha \cdot \bar{\mathbf{g}}_3 = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_3 + \Theta^3 \bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_3, \\ \bar{g}_{33} &= \bar{\mathbf{g}}_3 \cdot \bar{\mathbf{g}}_3 = \bar{\mathbf{a}}_3 \cdot \bar{\mathbf{a}}_3. \end{aligned} \quad (2.56)$$

Substituting the components of the covariant metric tensor in the shell space, given in (2.55) and (2.56), into the GREEN-LAGRANGE strain tensor, shown in (2.54), one obtains the in-plane, the transverse shear and the transverse normal components of the GREEN-LAGRANGE strain tensor in terms of the covariant base vectors at the mid-surface as (see Habip [78], who first developed the fully geometrically nonlinear

strain-displacement relations based on FOSD hypothesis)

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^0 + \Theta^3 \varepsilon_{\alpha\beta}^1 + (\Theta^3)^2 \varepsilon_{\alpha\beta}^2, \quad (2.57)$$

$$\varepsilon_{\alpha 3} = \varepsilon_{\alpha 3}^0 + \Theta^3 \varepsilon_{\alpha 3}^1, \quad (2.58)$$

$$\varepsilon_{33} = \varepsilon_{33}^0, \quad (2.59)$$

where the strain terms in the above equations are

$$2\varepsilon_{\alpha\beta}^0 = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta - \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \quad (2.60)$$

$$2\varepsilon_{\alpha\beta}^1 = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_{3,\beta} + \bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_\beta - \mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} - \mathbf{a}_{3,\alpha} \cdot \mathbf{a}_\beta, \quad (2.61)$$

$$2\varepsilon_{\alpha\beta}^2 = \bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_{3,\beta} - \mathbf{a}_{3,\alpha} \cdot \mathbf{a}_{3,\beta}, \quad (2.62)$$

$$2\varepsilon_{\alpha 3}^0 = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_3, \quad (2.63)$$

$$2\varepsilon_{\alpha 3}^1 = \bar{\mathbf{a}}_{3,\alpha} \cdot \bar{\mathbf{a}}_3, \quad (2.64)$$

$$2\varepsilon_{33}^0 = \bar{\mathbf{a}}_3 \cdot \bar{\mathbf{a}}_3 - 1. \quad (2.65)$$

Here, the strain terms have their own physical meanings, e.g. the in-plane longitudinal strains $(\varepsilon_{11}^0, \varepsilon_{22}^0)$, the in-plane shear strains $(\varepsilon_{12}^0, \varepsilon_{21}^0)$, the bending strains $(\varepsilon_{11}^1, \varepsilon_{22}^1)$, the torsional strains $(\varepsilon_{12}^1, \varepsilon_{21}^1)$, the transverse shear strains $(\varepsilon_{13}^0, \varepsilon_{23}^0)$, and the transverse normal strain (ε_{33}^0) .

Accordingly, the resultant internal forces and moments per unit length can be defined as [80]

$$\begin{aligned} \bar{L}^{\alpha\beta} &= \int_{\mathbf{h}} \mu (\Theta^3)^n \sigma^{\alpha\beta} d\Theta^3 & (n = 0, 1, 2), \\ \bar{L}^{\alpha 3} &= \int_{\mathbf{h}} \mu (\Theta^3)^n \sigma^{\alpha 3} d\Theta^3 & (n = 0, 1), \\ \bar{L}^{33} &= \int_{\mathbf{h}} \mu (\Theta^3)^n \sigma^{33} d\Theta^3 & (n = 0). \end{aligned} \quad (2.66)$$

Analogously, the physical meaning of the resultant internal forces and moments are the in-plane longitudinal forces $(\bar{L}^{11}, \bar{L}^{22})$, the in-plane shear forces $(\bar{L}^{12}, \bar{L}^{21})$, the bending moments $(\bar{L}^{11}, \bar{L}^{22})$, the torsional moments $(\bar{L}^{12}, \bar{L}^{21})$, the transverse shear forces $(\bar{L}^{13}, \bar{L}^{23})$, and the transverse normal force $(\bar{L}^{33})$, as shown in Fig. 2.3.

Considering the relations given in (2.23) and (2.24), the GREEN-LAGRANGE strain components in terms of the base vectors and displacement vectors in the undeformed

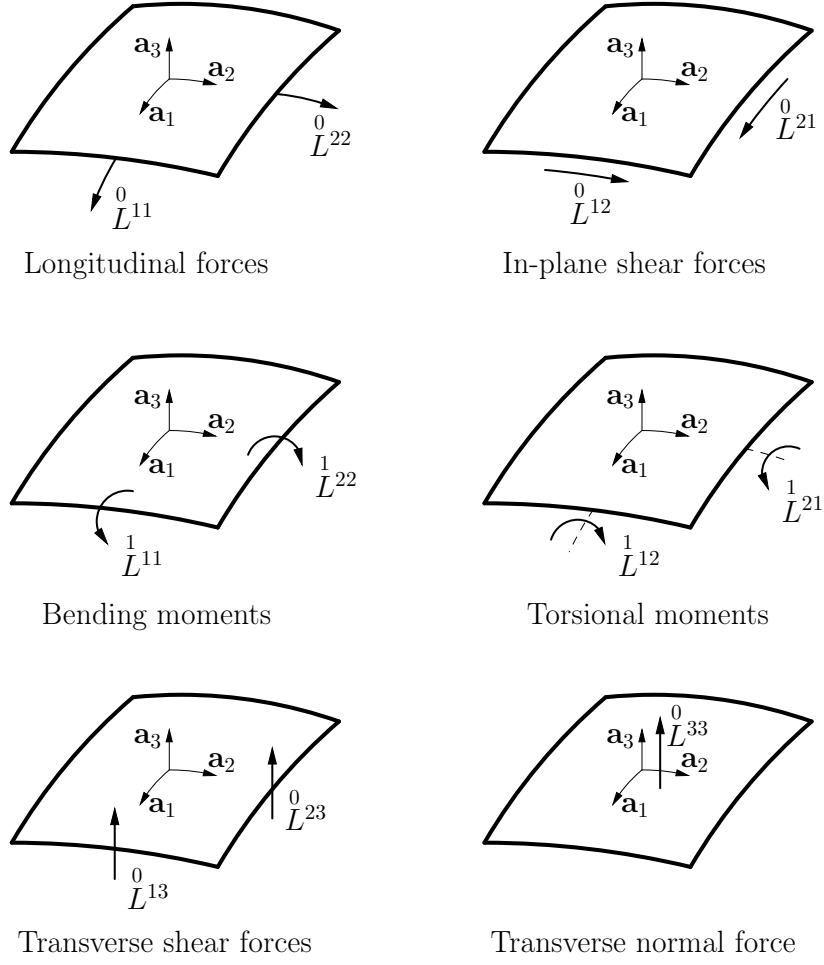


FIGURE 2.3: Physical meaning of the resultant internal forces and moments

configuration can be obtained as

$$2\varepsilon_{\alpha\beta}^0 = \mathbf{a}_\alpha \cdot \dot{\mathbf{u}}_{,\beta} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{a}_\beta + \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta}, \quad (2.67)$$

$$2\varepsilon_{\alpha\beta}^1 = \mathbf{a}_\alpha \cdot \dot{\mathbf{u}}_{,\beta} + \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{n}_{,\beta} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{a}_\beta + \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta} + \mathbf{n}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta}, \quad (2.68)$$

$$2\varepsilon_{\alpha\beta}^2 = \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{n}_{,\beta} + \mathbf{n}_{,\alpha} \cdot \dot{\mathbf{u}}_{,\beta}, \quad (2.69)$$

$$2\varepsilon_{\alpha 3}^0 = \mathbf{a}_\alpha \cdot \dot{\mathbf{u}} + \mathbf{a}_\alpha \cdot \mathbf{n} + \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{n}, \quad (2.70)$$

$$2\varepsilon_{\alpha 3}^1 = \dot{\mathbf{u}}_{,\alpha} \cdot \dot{\mathbf{u}} + \dot{\mathbf{u}}_{,\alpha} \cdot \mathbf{n} + \mathbf{n}_{,\alpha} \cdot \dot{\mathbf{u}} + \mathbf{n}_{,\alpha} \cdot \mathbf{n}, \quad (2.71)$$

$$2\varepsilon_{33}^0 = \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \dot{\mathbf{u}} \cdot \mathbf{n} + \mathbf{n} \cdot \dot{\mathbf{u}} + \mathbf{n} \cdot \mathbf{n} - 1. \quad (2.72)$$

Substituting equations (2.25), (2.26) and (2.37) into (2.67)-(2.72) yields the strain-displacement relations in terms of six parameters as

$$2\varepsilon_{\alpha\beta}^0 = \varphi_{\alpha\beta}^0 + \varphi_{\beta\alpha}^0 + \underline{\varphi_{3\alpha}^0 \varphi_{3\beta}^0} + \underline{\varphi_{\alpha}^0 \varphi_{\delta\beta}^0}, \quad (2.73)$$

$$2\varepsilon_{\alpha\beta}^1 = \varphi_{\alpha\beta}^1 - b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^0 + \varphi_{\beta\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^0 \\ + \underline{\varphi_{3\alpha}^1 \varphi_{3\beta}^1} + \underline{\varphi_{3\alpha}^0 \varphi_{3\beta}^0} + \underline{\varphi_{\alpha}^0 \varphi_{\delta\beta}^1} + \underline{\varphi_{\alpha}^1 \varphi_{\delta\beta}^0}, \quad (2.74)$$

$$2\varepsilon_{\alpha\beta}^2 = -b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^1 + \underline{\varphi_{3\alpha}^1 \varphi_{3\beta}^1} + \underline{\varphi_{\alpha}^1 \varphi_{\delta\beta}^1}, \quad (2.75)$$

$$2\varepsilon_{\alpha 3}^0 = v_{\alpha}^1 + \varphi_{3\alpha}^0 + \underline{\varphi_{\alpha}^0 v_{\delta}^1} + \underline{\varphi_{3\alpha}^1 v_3^1}, \quad (2.76)$$

$$2\varepsilon_{\alpha 3}^1 = \varphi_{3\alpha}^1 - b_{\alpha}^{\delta} v_{\delta}^1 + \underline{\varphi_{\alpha}^1 v_{\delta}^1} + \underline{\varphi_{3\alpha}^1 v_3^1}, \quad (2.77)$$

$$2\varepsilon_{33}^0 = 2 v_3^1 + a^{\lambda\delta} v_{\lambda}^1 v_{\delta}^1 + (v_3^1)^2. \quad (2.78)$$

2.5 Shell theories

In Section 2.4, the fully geometrically nonlinear strain-displacement relations in terms of six parameters based on FOSD hypothesis have been derived. Using the assumption of an *inextensible* shell director in thin-walled structures leads to $\varepsilon_{33}^0 = 0$. Based on this assumption, the linear and nonlinear shell theories listed in Table 2.2 have been developed and implemented in this dissertation.

TABLE 2.2: List of nonlinear shell theories based on FOSD hypothesis

Theory	Specification	Parameters
LRT56	Large rotation shell theory with six parameters expressed by five nodal DOFs	$v_{\alpha}^0, v_3^0, v_{\alpha}^1, v_3^1$
LRT5	Fully geometrically nonlinear shell theory with five parameters	$v_{\alpha}^0, v_3^0, v_{\alpha}^1$
MRT5	Moderate rotation shell theory with five parameters	
RVK5	Refined von Kármán type nonlinear shell theory with five parameters	
LIN5	Geometrically linear shell theory with five parameters	$v_{\alpha}^0, v_3^0, v_{\alpha}^1$

If the six independent kinematic parameters ($v_{\alpha}^0, v_3^0, v_{\alpha}^1, v_3^1$) in the fully geometrically nonlinear relations are expressed by five nodal DOFs (see Chapter 3), the resulting theory is abbreviated as LRT56 theory (see [89, 126, 213, 214]). Neglecting the sixth

TABLE 2.3: Strain-displacement relations for various shell theories

Strain	Strain-displacement relation	Theory
$2\varepsilon_{\alpha\beta}^0$	$\varphi_{\alpha\beta}^0 + \varphi_{\beta\alpha}^0 + \varphi_{3\alpha}^0 \varphi_{3\beta}^0 + \varphi_{\alpha}^{0\delta} \varphi_{\delta\beta}^0$	LRT56
		LRT5
	$\varphi_{\alpha\beta}^0 + \varphi_{\beta\alpha}^0 + \varphi_{3\alpha}^0 \varphi_{3\beta}^0$	MRT5
	$\varphi_{\alpha\beta}^0 + \varphi_{\beta\alpha}^0 + v_{3,\alpha}^0 v_{3,\beta}^0$	RVK5
	$\varphi_{\alpha\beta}^0 + \varphi_{\beta\alpha}^0$	LIN5
$2\varepsilon_{\alpha\beta}^1$	$\varphi_{\alpha\beta}^1 - b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^0 + \varphi_{\beta\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^0 + \varphi_{3\alpha}^0 \varphi_{3\beta}^1 + \varphi_{3\alpha}^1 \varphi_{3\beta}^0 + \varphi_{\alpha}^{0\delta} \varphi_{\delta\beta}^1 + \varphi_{\alpha}^{1\delta} \varphi_{\delta\beta}^0$	LRT56
		LRT5
	$\varphi_{\alpha\beta}^1 - b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^0 + \varphi_{\beta\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^0 + \varphi_{3\alpha}^0 \varphi_{3\beta}^1 + \varphi_{3\alpha}^1 \varphi_{3\beta}^0$	MRT5
	$\varphi_{\alpha\beta}^1 - b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^0 + \varphi_{\beta\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^0$	RVK5
		LIN5
$2\varepsilon_{\alpha\beta}^2$	$-b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^1 + \varphi_{3\alpha}^1 \varphi_{3\beta}^1 + \varphi_{\alpha}^{1\delta} \varphi_{\delta\beta}^1$	LRT56
		LRT5
	$-b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^1 + \varphi_{3\alpha}^1 \varphi_{3\beta}^1$	MRT5
	$-b_{\beta}^{\lambda} \varphi_{\lambda\alpha}^1 - b_{\alpha}^{\delta} \varphi_{\delta\beta}^1$	RVK5
		LIN5
$2\varepsilon_{\alpha 3}^0$	$v_{\alpha}^1 + \varphi_{3\alpha}^0 + \varphi_{\alpha}^{0\delta} v_{\delta}^1 + \varphi_{3\alpha}^1 v_3^0$	LRT56
	$v_{\alpha}^1 + \varphi_{3\alpha}^0 + \varphi_{\alpha}^{0\delta} v_{\delta}^1$	LRT5
		MRT5
	$v_{\alpha}^1 + \varphi_{3\alpha}^0$	RVK5
		LIN5
$2\varepsilon_{\alpha 3}^1$	$\varphi_{3\alpha}^1 - b_{\alpha}^{\delta} v_{\delta}^1 + \varphi_{\alpha}^{1\delta} v_{\delta}^1 + \varphi_{3\alpha}^1 v_3^1$	LRT56
	$\varphi_{3\alpha}^1 - b_{\alpha}^{\delta} v_{\delta}^1 + \varphi_{\alpha}^{1\delta} v_{\delta}^1$	LRT5
		MRT5
	$\varphi_{3\alpha}^1 - b_{\alpha}^{\delta} v_{\delta}^1$	RVK5
		LIN5
$2\varepsilon_{33}^0$	0	LRT56
		LRT5
		MRT5
		RVK5
		LIN5

parameter $\overset{1}{v}_3$ is permitted only for small or moderate rotations, see [89]. In case of full geometric nonlinearities being considered, this would yield a theory abbreviated as LRT5 [89, 126, 213, 214]. Further dropping the nonlinear strain-displacement terms marked by double lines in (2.73)-(2.78) yields the moderate rotation theory (MRT5) by Schmidt and Reddy [73] (see also [72, 74–77, 126, 213, 214]). Retaining only the nonlinear terms which contain the squares and products of derivatives of the transverse deflection in the in-plane longitudinal and shear strain components, yields the refined VON KÁRMÁN type nonlinear theory, abbreviated as RVK5 [89, 126]. Dropping the nonlinear terms marked by both single and double lines results in linear theory with five parameters, which is shorted as LIN5.

The strain-displacement relations for various shell theories mentioned above can be obtained as shown in Table 2.3, by using the abbreviations listed in Table 2.4.

TABLE 2.4: The expressions of the abbreviations for various shell theories

Theory	$\overset{n}{\varphi}_{\lambda\alpha} =$	$\overset{n}{\varphi}_{3\alpha} =$
LRT56	$\overset{0}{v}_{\lambda,\alpha} - \Gamma_{\lambda\alpha}^{\delta} \overset{0}{v}_{\delta} - b_{\lambda\alpha} \overset{0}{v}_3$	$\overset{0}{v}_{3,\alpha} + b_{\alpha}^{\delta} \overset{0}{v}_{\delta}$
	$\overset{1}{v}_{\lambda,\alpha} - \Gamma_{\lambda\alpha}^{\delta} \overset{1}{v}_{\delta} - b_{\lambda\alpha} \overset{1}{v}_3$	$\overset{1}{v}_{3,\alpha} + b_{\alpha}^{\delta} \overset{1}{v}_{\delta}$
LRT5, MRT5, RVK5, LIN5	$\overset{0}{v}_{\lambda,\alpha} - \Gamma_{\lambda\alpha}^{\delta} \overset{0}{v}_{\delta} - b_{\lambda\alpha} \overset{0}{v}_3$	$\overset{0}{v}_{3,\alpha} + b_{\alpha}^{\delta} \overset{0}{v}_{\delta}$
	$\overset{1}{v}_{\lambda,\alpha} - \Gamma_{\lambda\alpha}^{\delta} \overset{1}{v}_{\delta}$	$b_{\alpha}^{\delta} \overset{1}{v}_{\delta}$

2.6 Normalization

From equations (2.25), (2.26) and (2.54), it can be seen that the components of the displacement and strain tensors are referred to the base vectors which are not unit vectors. Therefore, *physical* (or *normalized*) components of the displacement and strain vectors are introduced, which are obtained by *normalization*. The displacement vector is defined with respect to the mid-surface contravariant basis as

$$\overset{n}{\mathbf{u}} = \overset{n}{v}_i \mathbf{a}^i = \overset{n}{v}_1 \mathbf{a}^1 + \overset{n}{v}_2 \mathbf{a}^2 + \overset{n}{v}_3 \mathbf{a}^3. \quad (2.79)$$

It can also be expressed in the corresponding contravariant basis, but with unit EUCLIDEAN length, as

$$\overset{n}{\mathbf{u}} = \hat{\overset{n}{v}}_1 \hat{\mathbf{a}}^1 + \hat{\overset{n}{v}}_2 \hat{\mathbf{a}}^2 + \hat{\overset{n}{v}}_3 \hat{\mathbf{a}}^3, \quad (2.80)$$

where, $\hat{v}_i$ denote the *physical* quantity of v_i , and $\hat{\mathbf{a}}^i = \frac{\mathbf{a}^i}{\|\mathbf{a}^i\|}$ represents the *normalized* vectors of $\mathbf{a}^i$. From equations (2.79) and (2.80), one can easily obtain

$$v_i = \frac{\hat{v}_i}{\|\mathbf{a}^i\|} . \quad (2.81)$$

Analogously, the *physical* components of the GREEN-LAGRANGE strain tensor, which is a second-order tensor expressed by the contravariant basis $\mathbf{g}^i \otimes \mathbf{g}^j$ in the shell space, can be calculated by the same procedure as

$$\boldsymbol{\varepsilon} = \varepsilon_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = \hat{\varepsilon}_{ij} \hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j . \quad (2.82)$$

Here again $\hat{\mathbf{g}}^i = \frac{\mathbf{g}^i}{\|\mathbf{g}^i\|}$ represents the *normalized* vector of $\mathbf{g}^i$, such that the *physical* components of the strain tensor are

$$\hat{\varepsilon}_{ij} = \|\mathbf{g}^i\| \|\mathbf{g}^j\| \varepsilon_{ij} . \quad (2.83)$$

2.7 Summary

This chapter deduced fully and simplified geometrically nonlinear strain-displacement relations based on FOSD hypothesis for various nonlinear shell theories. The differences between each nonlinear shell theory were analyzed and strengthened.

Chapter 3

Finite element formulation

In this chapter, a brief introduction of piezoelectric material is first presented. Considering the assumptions of *small strains* and *weak electric potential*, linear electro-mechanically coupled constitutive equations will be employed, as well as the linear distribution of electric potential through the thickness. In order to describe the unrestricted finite rotations in thin-walled smart structures, five mechanical nodal DOFs are defined to represent the six kinematic parameters in strain-displacement relations by using EULER angles. Furthermore, various eight-node elements with five mechanical nodal DOFs or additionally integrated with one electrical DOF using FI or URI integration scheme are developed for both composite and smart structures. Applying HAMILTON'S principle and the principle of virtual work will respectively yield the dynamic and static piezoelectric coupled FE models with considering various geometric nonlinearities discussed in Chapter 2 for smart structures. In the last part of this chapter, several numerical algorithms will be discussed for solving the equilibrium equation and the equation of motion.

3.1 Piezoelectric material

Piezoelectricity means electricity resulting from pressure, which describes the electric charge that accumulates in certain solid materials in response to applied mechanical stresses. The *direct piezoelectric effect* was first demonstrated by the brothers Pierre Curie and Jacques Curie in 1880. One year later, the *converse effect* was mathematically deduced from fundamental thermo-dynamic principles by Gabriel Lippmann. Shortly after, the complete reversibility of electro-elasto-mechanical deformations in piezoelectric crystals was proved by the Curies.

In most cases the piezoelectric materials are also *ferroelectric*, the piezoelectric phase can be transformed to a symmetric *non-piezoelectric* state at a certain high temperature, which here refers to the CURIE temperature, as shown in Fig. 3.1. The

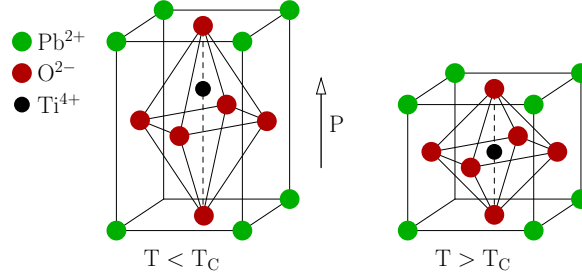


FIGURE 3.1: The configurations of PbTiO_3 crystalline structure

ion Ti^{4+} is shifted to one side of the crystalline structure when the temperature is below the CURIE point. As a consequence, the center of the positive electric charges of the unit cell is different from that of the negative charges. The crystal is then called *polarized*.

The piezoelectric material has two effects, namely the *direct* and *converse* effects, which are shown in Fig. 3.2. Applying a stress in direction x_1 will decrease the distance

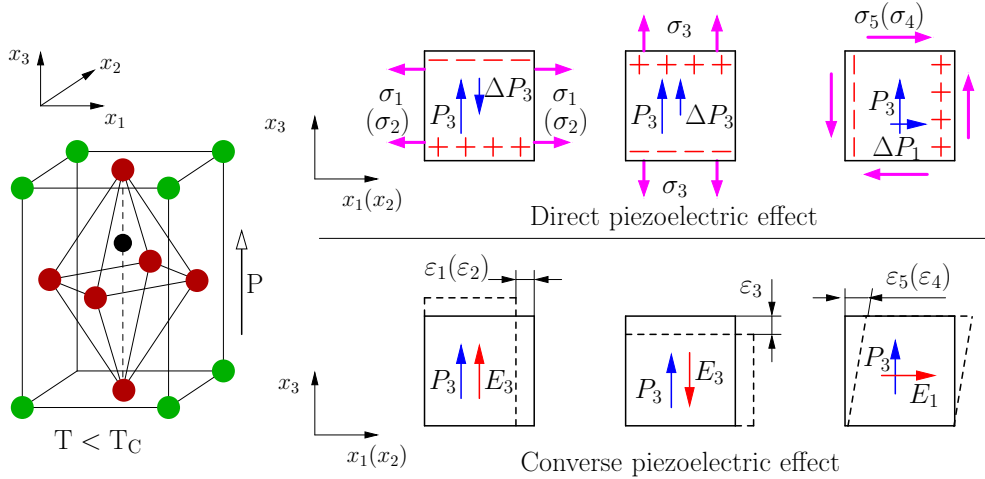


FIGURE 3.2: The direct and converse effects of piezoelectric material

between the ion of titanium and the geometric center of the unit cell. This can be understood as an additionally generated *polarization*, which results in extra electric charges due to the stresses. Similarly, applying a normal stress σ_{33} or shear stress σ_{13} will produce electric charges as well. Those phenomena are called *direct piezoelectric*

effect, which can be expressed separately as

$$\begin{aligned}\Delta P_1 &= d_{15}\sigma_5, \\ \Delta P_2 &= d_{24}\sigma_4, \\ \Delta P_3 &= d_{31}\sigma_1 + d_{32}\sigma_2 + d_{33}\sigma_3,\end{aligned}\tag{3.1}$$

where ΔP_i denotes the extra polarization in x_i direction.

In an analogous way, the physical meaning of the *converse* piezoelectric effect can be observed. Applying an electric field along the polarization direction will move the ion of titanium off the center in x_3 direction. This will result in stretching the cell along direction x_3 and squeezing along direction x_1 and x_2 , which yields additional strains given as

$$\begin{aligned}\varepsilon_1 &= d_{13}E_3, \\ \varepsilon_2 &= d_{23}E_3, \\ \varepsilon_3 &= d_{33}E_3.\end{aligned}\tag{3.2}$$

In the same way, applying an electric field along x_1 or x_2 direction yields additional shear strains as

$$\begin{aligned}\varepsilon_4 &= d_{42}E_2, \\ \varepsilon_5 &= d_{51}E_1.\end{aligned}\tag{3.3}$$

Here, $d_{13} = d_{31}$, $d_{23} = d_{32}$, $d_{42} = d_{24}$ and $d_{51} = d_{15}$ for isotropic piezoelectric material. More detailed information can be found e.g. in [17, 215].

3.2 Constitutive equations

Due to the kinematics of *small strains* and *weak electric potential*, linear piezoelectric constitutive equations are employed, which are expressed in material axes as

$$\check{\sigma}^{ij} = \check{c}^{ijkl}\check{\varepsilon}_{kl} - \check{e}^{mij}\check{E}_m,\tag{3.4}$$

$$\check{D}^m = \check{e}^{mkl}\check{\varepsilon}_{kl} + \check{\epsilon}^{mn}\check{E}_n,\tag{3.5}$$

in which $\check{\square}$ represents the components in the material coordinate system, the LATIN indices vary from 1, 2 to 3, and the indices ij or kl take the values only 11, 22, 33, 12 or 21, 13 or 31, 23 or 32. Furthermore, $\check{\sigma}^{ij}$, $\check{\varepsilon}_{kl}$, $\check{D}^m$, $\check{E}_n$, $\check{c}^{ijkl}$, $\check{e}^{mij}$ and $\check{\epsilon}^{mn}$ are the components of the second PIOLA-KIRCHHOFF stress and GREEN-LAGRANGE strain

tensors, the electric displacement and the electric field vectors, and the elasticity constant, piezoelectric constant and dielectric constant tensors, respectively, in material coordinates. The components of the stress and strain tensors are

$$[\check{\sigma}^{ij}] = \begin{bmatrix} \check{\sigma}^{11} & \check{\sigma}^{12} & \check{\sigma}^{13} \\ \check{\sigma}^{21} & \check{\sigma}^{22} & \check{\sigma}^{23} \\ \check{\sigma}^{31} & \check{\sigma}^{32} & \check{\sigma}^{33} \end{bmatrix}, \quad [\check{\epsilon}_{ij}] = \begin{bmatrix} \check{\epsilon}_{11} & \check{\epsilon}_{12} & \check{\epsilon}_{13} \\ \check{\epsilon}_{21} & \check{\epsilon}_{22} & \check{\epsilon}_{23} \\ \check{\epsilon}_{31} & \check{\epsilon}_{32} & \check{\epsilon}_{33} \end{bmatrix}. \quad (3.6)$$

Due to the symmetry of the stress and strain tensors, $\check{\sigma}^{ij} = \check{\sigma}^{ji}$ and $\check{\epsilon}_{ij} = \check{\epsilon}_{ji}$, the VOIGT notations are introduced to describe the second-order strain and stress tensors in vector form, which are defined as listed in Table 3.1. In such a way, the strains and

TABLE 3.1: VOIGT notation

ij or kl	p or q
11	1
22	2
33	3
23 or 32	4
13 or 31	5
12 or 21	6

stresses can be arranged in vector form as

$$\check{\sigma} = \begin{Bmatrix} \check{\sigma}^{11} \\ \check{\sigma}^{22} \\ \check{\sigma}^{33} \\ \check{\sigma}^{23} \\ \check{\sigma}^{13} \\ \check{\sigma}^{12} \end{Bmatrix} = \begin{Bmatrix} \check{\sigma}_1 \\ \check{\sigma}_2 \\ \check{\sigma}_3 \\ \check{\sigma}_4 \\ \check{\sigma}_5 \\ \check{\sigma}_6 \end{Bmatrix}, \quad \check{\epsilon} = \begin{Bmatrix} \check{\epsilon}_{11} \\ \check{\epsilon}_{22} \\ \check{\epsilon}_{33} \\ 2\check{\epsilon}_{23} \\ 2\check{\epsilon}_{13} \\ 2\check{\epsilon}_{12} \end{Bmatrix} = \begin{Bmatrix} \check{\epsilon}_1 \\ \check{\epsilon}_2 \\ \check{\epsilon}_3 \\ 2\check{\epsilon}_4 \\ 2\check{\epsilon}_5 \\ 2\check{\epsilon}_6 \end{Bmatrix}. \quad (3.7)$$

Using the VOIGT notations, the components of the fourth-order elasticity constant tensor in (3.4) can be arranged in matrix form as

$$[\check{c}^{ijkl}] = \begin{bmatrix} \check{c}^{1111} & \check{c}^{1122} & \check{c}^{1133} & 0 & 0 & 0 \\ \check{c}^{1122} & \check{c}^{2222} & \check{c}^{2233} & 0 & 0 & 0 \\ \check{c}^{1133} & \check{c}^{2233} & \check{c}^{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \check{c}^{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{c}^{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & \check{c}^{1212} \end{bmatrix} \quad (3.8)$$

with

$$\begin{aligned} \check{c}^{1111} &= E_1 \frac{1-\nu_{23}\nu_{32}}{\Delta}, & \check{c}^{2222} &= E_2 \frac{1-\nu_{31}\nu_{13}}{\Delta}, & \check{c}^{3333} &= E_3 \frac{1-\nu_{12}\nu_{21}}{\Delta}, \\ \check{c}^{1122} &= E_1 \frac{\nu_{21}-\nu_{31}\nu_{23}}{\Delta}, & \check{c}^{1133} &= E_3 \frac{\nu_{13}-\nu_{12}\nu_{23}}{\Delta}, & \check{c}^{2233} &= E_2 \frac{\nu_{32}-\nu_{12}\nu_{31}}{\Delta}, \\ \check{c}^{2323} &= G_{23}, & \check{c}^{1313} &= G_{13}, & \check{c}^{1212} &= G_{12}, \end{aligned} \quad (3.9)$$

where $\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13}$ and $\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$, where E_i denotes the YOUNG'S moduli, G_{ij} the shear moduli, and ν_{ij} the POISSON'S ratios.

The components of the third-order piezoelectric constant tensor and the second-order dielectric constant tensor in (3.5) can be arranged respectively in matrix form as

$$[\check{e}^{mkl}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \check{e}^{113} & 0 \\ 0 & 0 & 0 & \check{e}^{223} & 0 & 0 \\ \check{e}^{311} & \check{e}^{322} & \check{e}^{333} & 0 & 0 & 0 \end{bmatrix}, \quad [\check{\epsilon}^{mn}] = \begin{bmatrix} \check{\epsilon}^{11} & 0 & 0 \\ 0 & \check{\epsilon}^{22} & 0 \\ 0 & 0 & \check{\epsilon}^{33} \end{bmatrix}. \quad (3.10)$$

Using the assumption of constant electric field through the thickness, the components of the electric field vector, $\check{E}_i$, in material axes can be obtained as

$$\check{E}_i = -\frac{\partial \check{\phi}_i}{\partial \check{\Theta}^i}, \quad (3.11)$$

where $\check{\phi}_i$ is the electric potential in $\check{\Theta}^i$ direction. It can be expressed in matrix form for structures with multi-piezoelectric layers and various directions of electric field as

$$\check{\mathbf{E}} = -\nabla \check{\phi} = \mathbf{B}_\phi \check{\phi}, \quad (3.12)$$

where ∇ represents the gradient operator, $\check{\mathbf{E}}$ is the electric field vector, and $\mathbf{B}_\phi$ denotes the electric field matrix.

For isotropic materials, the material coordinate axes, $\check{\Theta}^1$ and $\check{\Theta}^2$, can be set the same as the curvilinear coordinate axes, Θ^1 and Θ^2 . However, in case that the fiber reinforcement direction of orthotropic material is not parallel to the curvilinear coordinate axes, like in the case shown in Fig. 3.3, a transformation matrix is necessary for converting the constitutive equations from the material coordinate axes to the curvilinear coordinate axes.

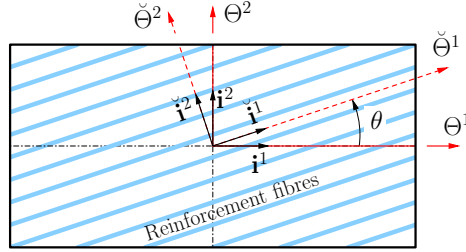


FIGURE 3.3: Orientation of reinforcement fibers

The components of the elasticity constant tensor used in (3.4) are referred to the unit covariant base vectors $\check{\mathbf{i}}_a$ in the material coordinate system, which must be transformed to the covariant shell base vectors $\mathbf{g}_i$ or the normalized ones $\hat{\mathbf{g}}_i = \frac{\mathbf{g}_i}{\|\mathbf{g}_i\|}$, since the formulations are developed in the curvilinear coordinate system. The transformation matrix can be obtained by means of the following equations

$$\begin{aligned} \mathbf{c} &= \check{c}^{abcd} \check{\mathbf{i}}_a \otimes \check{\mathbf{i}}_b \otimes \check{\mathbf{i}}_c \otimes \check{\mathbf{i}}_d \\ &= \hat{c}^{ijkl} \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j \otimes \hat{\mathbf{g}}_k \otimes \hat{\mathbf{g}}_l \\ &= c^{ijkl} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l, \end{aligned} \quad (3.13)$$

which leads to

$$\hat{c}^{ijkl} = \left(\hat{\mathbf{g}}^i \cdot \check{\mathbf{i}}_a \right) \left(\hat{\mathbf{g}}^j \cdot \check{\mathbf{i}}_b \right) \left(\hat{\mathbf{g}}^k \cdot \check{\mathbf{i}}_c \right) \left(\hat{\mathbf{g}}^l \cdot \check{\mathbf{i}}_d \right) \check{c}^{abcd}. \quad (3.14)$$

Here, the indices, a , b , c and d , have the same function as i , j , k and l , but they are used for the components in material coordinate system. Using the same rule one

obtains

$$\check{\varepsilon}_{ab} = \left(\hat{\mathbf{g}}^i \cdot \check{\mathbf{i}}_a \right) \left(\hat{\mathbf{g}}^j \cdot \check{\mathbf{i}}_b \right) \hat{\varepsilon}_{ij}, \quad (3.15)$$

$$\hat{\sigma}^{ij} = \left(\hat{\mathbf{g}}^i \cdot \check{\mathbf{i}}_a \right) \left(\hat{\mathbf{g}}^j \cdot \check{\mathbf{i}}_b \right) \check{\sigma}^{ab}, \quad (3.16)$$

$$\check{E}_a = \left(\hat{\mathbf{g}}^i \cdot \check{\mathbf{i}}_a \right) \hat{E}_i, \quad (3.17)$$

$$\hat{D}^i = \left(\hat{\mathbf{g}}^i \cdot \check{\mathbf{i}}_a \right) \check{D}^a, \quad (3.18)$$

which can be expressed in matrix form as

$$\check{\boldsymbol{\varepsilon}} = \mathbf{T} \hat{\boldsymbol{\varepsilon}}, \quad \hat{\boldsymbol{\sigma}} = \mathbf{T}^T \check{\boldsymbol{\sigma}}, \quad (3.19)$$

$$\check{\mathbf{E}} = \mathbf{Q} \hat{\mathbf{E}}, \quad \hat{\mathbf{D}} = \mathbf{Q}^T \check{\mathbf{D}}. \quad (3.20)$$

Due to the neglect of the transverse normal strain $\check{\varepsilon}_{33}$, the constitutive equations given in (3.4)-(3.5) are simplified to contain only five components, which can be expressed in matrix form as

$$\check{\boldsymbol{\sigma}} = \check{\mathbf{c}} \check{\boldsymbol{\varepsilon}} - \check{\mathbf{e}}^T \check{\mathbf{E}}, \quad (3.21)$$

$$\check{\mathbf{D}} = \check{\mathbf{e}} \check{\boldsymbol{\varepsilon}} + \check{\boldsymbol{\epsilon}} \check{\mathbf{E}}, \quad (3.22)$$

in which the second PIOLA-KIRCHHOFF stress vector $\check{\boldsymbol{\sigma}}$, the GREEN strain vector $\check{\boldsymbol{\varepsilon}}$, the electric displacement vector $\check{\mathbf{D}}$, and the electric field vector $\check{\mathbf{E}}$ are

$$\check{\boldsymbol{\sigma}} = \begin{Bmatrix} \check{\sigma}^{11} \\ \check{\sigma}^{22} \\ \check{\sigma}^{12} \\ \check{\sigma}^{23} \\ \check{\sigma}^{13} \end{Bmatrix}, \quad \check{\boldsymbol{\varepsilon}} = \begin{Bmatrix} \check{\varepsilon}_{11} \\ \check{\varepsilon}_{22} \\ 2\check{\varepsilon}_{12} \\ 2\check{\varepsilon}_{23} \\ 2\check{\varepsilon}_{13} \end{Bmatrix}, \quad \check{\mathbf{D}} = \begin{Bmatrix} \check{D}^1 \\ \check{D}^2 \\ \check{D}^3 \end{Bmatrix}, \quad \check{\mathbf{E}} = \begin{Bmatrix} \check{E}_1 \\ \check{E}_2 \\ \check{E}_3 \end{Bmatrix}. \quad (3.23)$$

In equations (3.21) and (3.22), $\check{\mathbf{c}}$ denotes the elasticity constant matrix, $\check{\mathbf{d}}$ and $\check{\mathbf{e}}$ are the piezoelectric constant matrices, with $\check{\mathbf{e}} = \check{\mathbf{d}}^T \check{\mathbf{c}}$, and $\check{\boldsymbol{\epsilon}}$ the dielectric constant matrix.

The elasticity constant matrix is given by

$$\check{\mathbf{c}} = \begin{bmatrix} \check{c}_{11} & \check{c}_{12} & 0 & 0 & 0 \\ \check{c}_{12} & \check{c}_{22} & 0 & 0 & 0 \\ 0 & 0 & \check{c}_{66} & 0 & 0 \\ 0 & 0 & 0 & \check{c}_{44} & 0 \\ 0 & 0 & 0 & 0 & \check{c}_{55} \end{bmatrix} \quad (3.24)$$

with

$$\begin{aligned} \check{c}_{11} &= \frac{E_1}{1-\nu_{12}\nu_{21}}, & \check{c}_{22} &= \frac{E_2}{1-\nu_{12}\nu_{21}}, & \check{c}_{12} &= \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \\ \check{c}_{66} &= G_{12}, & \check{c}_{55} &= \kappa G_{13}, & \check{c}_{44} &= \kappa G_{23}. \end{aligned} \quad (3.25)$$

Here, κ is the shear correction factor, which is usually given as $\frac{5}{6}$ or $\frac{\pi}{12}$. The piezoelectric constant matrix $\check{\mathbf{d}}$ and the dielectric constant matrix $\check{\boldsymbol{\epsilon}}$ can be written as

$$\check{\mathbf{d}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \check{d}_{15} \\ 0 & 0 & 0 & \check{d}_{24} & 0 \\ \check{d}_{31} & \check{d}_{32} & 0 & 0 & 0 \end{bmatrix}, \quad \check{\boldsymbol{\epsilon}} = \begin{bmatrix} \check{\epsilon}_{11} & 0 & 0 \\ 0 & \check{\epsilon}_{22} & 0 \\ 0 & 0 & \check{\epsilon}_{33} \end{bmatrix}. \quad (3.26)$$

With the help of the transformation matrix given in (3.19) and (3.20), one obtains the constitutive equations described in a curvilinear coordinate system as

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{c}}\hat{\boldsymbol{\epsilon}} - \hat{\mathbf{e}}^T\hat{\mathbf{E}}, \quad (3.27)$$

$$\hat{\mathbf{D}} = \hat{\mathbf{e}}\hat{\boldsymbol{\epsilon}} + \hat{\boldsymbol{\epsilon}}\hat{\mathbf{E}}, \quad (3.28)$$

where

$$\hat{\mathbf{c}} = \mathbf{T}^T\check{\mathbf{c}}\mathbf{T}, \quad \hat{\mathbf{e}} = \mathbf{Q}^T\check{\mathbf{e}}\mathbf{T}, \quad \hat{\boldsymbol{\epsilon}} = \mathbf{Q}^T\check{\boldsymbol{\epsilon}}\mathbf{Q}. \quad (3.29)$$

The transformation matrix $\mathbf{Q}$ is an identity matrix if the electrical coordinate axes are parallel to the curvilinear coordinate lines, and $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} t_{11}^2 & t_{21}^2 & t_{11}t_{21} & 0 & 0 \\ t_{12}^2 & t_{22}^2 & t_{12}t_{22} & 0 & 0 \\ 2t_{11}t_{12} & 2t_{21}t_{22} & t_{11}t_{22} + t_{12}t_{21} & 0 & 0 \\ 0 & 0 & 0 & t_{22} & t_{12} \\ 0 & 0 & 0 & t_{21} & t_{11} \end{bmatrix} \quad (3.30)$$

with

$$\begin{aligned} t_{11} &= \hat{\mathbf{g}}^1 \cdot \check{\mathbf{i}}_1 = \cos \theta, & t_{12} &= \hat{\mathbf{g}}^1 \cdot \check{\mathbf{i}}_2 = -\sin \theta, \\ t_{21} &= \hat{\mathbf{g}}^2 \cdot \check{\mathbf{i}}_1 = \sin \theta, & t_{22} &= \hat{\mathbf{g}}^2 \cdot \check{\mathbf{i}}_2 = \cos \theta. \end{aligned} \quad (3.31)$$

3.3 Resultant vectors

In order to reduce the volume integral in the variational formulation to a surface integral, we define the following two vectors respectively containing the strain components and the internal forces and moments as

$$\mathbf{L} = \left\{ \overset{0}{L}_{11}, \overset{0}{L}_{22}, \overset{0}{L}_{12}, \overset{1}{L}_{11}, \overset{1}{L}_{22}, \overset{1}{L}_{12}, \overset{2}{L}_{11}, \overset{2}{L}_{22}, \overset{2}{L}_{12}, \overset{0}{L}_{23}, \overset{0}{L}_{13}, \overset{1}{L}_{23}, \overset{1}{L}_{13} \right\}^T, \quad (3.32)$$

$$\mathbf{S} = \left\{ \overset{0}{\varepsilon}_{11}, \overset{0}{\varepsilon}_{22}, 2\overset{0}{\varepsilon}_{12}, \overset{1}{\varepsilon}_{11}, \overset{1}{\varepsilon}_{22}, 2\overset{1}{\varepsilon}_{12}, \overset{2}{\varepsilon}_{11}, \overset{2}{\varepsilon}_{22}, 2\overset{2}{\varepsilon}_{12}, 2\overset{0}{\varepsilon}_{23}, 2\overset{0}{\varepsilon}_{13}, 2\overset{1}{\varepsilon}_{23}, 2\overset{1}{\varepsilon}_{13} \right\}^T. \quad (3.33)$$

Here $\mathbf{S}$ is the *resultant* strain vector and $\mathbf{L}$ will be called *resultant* stress vector for simplicity.

According to the GREEN-LAGRANGE strain components given in (2.57)-(2.59), the strain can be expressed in terms of the *resultant* strain vector $\mathbf{S}$ as

$$\boldsymbol{\varepsilon} = \mathbf{H}_s \mathbf{S}, \quad (3.34)$$

with

$$\mathbf{H}_s = \begin{bmatrix} 1 & 0 & 0 & \Theta^3 & 0 & 0 & (\Theta^3)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Theta^3 & 0 & 0 & (\Theta^3)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \Theta^3 & 0 & 0 & (\Theta^3)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \Theta^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \Theta^3 \end{bmatrix}.$$

Using equations (3.32)-(3.33), the volume integral of the internal virtual work can be transformed to a surface integral as

$$\int_V \boldsymbol{\sigma}^T \delta \boldsymbol{\varepsilon} \, dV = \int_\Omega \mathbf{L}^T \delta \mathbf{S} \, d\Omega. \quad (3.35)$$

In order to express the nonlinear strain-displacement relations in matrix form, we introduce several *resultant* vectors containing the variables and their derivatives for displacement $\boldsymbol{\theta}$, normalized displacement $\hat{\boldsymbol{\theta}}$ and nodal DOFs $\hat{\boldsymbol{\theta}}_u$, which are respectively

defined as

$$\boldsymbol{\theta} = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{matrix} \right\}^T, \quad (3.36)$$

$$\hat{\boldsymbol{\theta}} = \left\{ \begin{matrix} \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{0} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{1} & \hat{0} & \hat{0} & \hat{0} & \hat{1} & \hat{1} & \hat{1} \end{matrix} \right\}^T, \quad (3.37)$$

$$\hat{\boldsymbol{\theta}}_u = \left\{ u_{,1} \ u_{,2} \ v_{,1} \ v_{,2} \ w_{,1} \ w_{,2} \ \varphi_{1,1} \ \varphi_{1,2} \ \varphi_{2,1} \ \varphi_{2,2} \ u \ v \ w \ \varphi_1 \ \varphi_2 \right\}^T. \quad (3.38)$$

Here, u , v , w , φ_1 and φ_2 are the five nodal DOFs, which will be further discussed in Section 3.4. Due to the normalization procedure, one obtains

$$\boldsymbol{\theta} = \mathbf{K}_{th} \hat{\boldsymbol{\theta}}. \quad (3.39)$$

Analogously, for later use, we define the vectors $\mathbf{v}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{v}}_u$ that only contain the displacements, the normalized displacements and the DOFs, respectively, as

$$\mathbf{v} = \left\{ \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 \end{matrix} \right\}^T, \quad (3.40)$$

$$\hat{\mathbf{v}} = \left\{ \begin{matrix} \hat{0} & \hat{0} & \hat{0} & \hat{1} & \hat{1} & \hat{1} \end{matrix} \right\}^T, \quad (3.41)$$

$$\hat{\mathbf{v}}_u = \left\{ u \ v \ w \ \varphi_1 \ \varphi_2 \right\}^T. \quad (3.42)$$

Similarly, applying normalization process yields

$$\mathbf{v} = \mathbf{K}_v \hat{\mathbf{v}}. \quad (3.43)$$

3.4 Rotation description

The parameters of the linear and nonlinear shell theories (LIN5, RVK5, MRT5, LRT5 and LRT56) mentioned above are expressed by five predefined nodal DOFs, namely three translational DOFs, u , v , w , and two rotational DOFs, φ_1 , φ_2 , as shown in Fig. 3.4. Here, u , v , w are the translational displacement along the Θ^1 -, Θ^2 - and Θ^3 -axis, respectively, and φ_1 , φ_2 are the rotations about the Θ^2 - and Θ^1 -line, respectively. The first three parameters, $\hat{v}_1$, $\hat{v}_2$, $\hat{v}_3$, for all shell theories can be expressed linearly by the three translational DOFs as

$$\hat{v}_1 = \|\mathbf{a}^1\| v_1 = u, \quad \hat{v}_2 = \|\mathbf{a}^2\| v_2 = v, \quad \hat{v}_3 = \|\mathbf{a}^3\| v_3 = w. \quad (3.44)$$

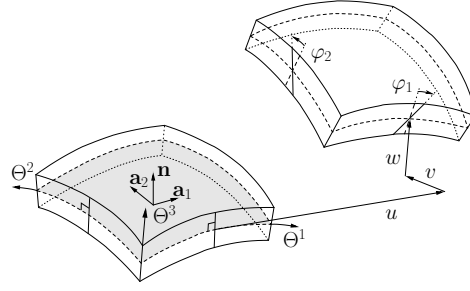
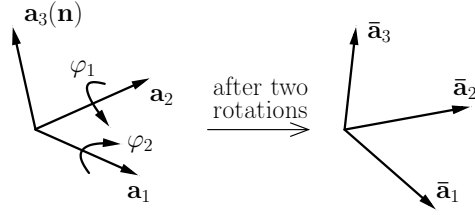


FIGURE 3.4: Degrees of freedom at any point on the mid-surface

The last three parameters, $\hat{v}_1, \hat{v}_2, \hat{v}_3$, in LRT56 theory, can be expressed nonlinearly by using the EULER angle representation, see [89, 213, 214, 216]. Rotating the curvilinear coordinate system sequentially by φ_1 about the Θ^2 -axis and φ_2 about the Θ^1 -axis, as shown in Fig. 3.5, yields the shell director being transformed from $\mathbf{n}$ in the undeformed configuration to $\bar{\mathbf{a}}_3$ in the deformed configuration. With these two

FIGURE 3.5: Rotation of the base vector triad by Euler angles φ_1 and φ_2

rotations, the transformation matrices will be respectively obtained as

$$\mathbf{Rot}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_2) & \sin(\varphi_2) \\ 0 & -\sin(\varphi_2) & \cos(\varphi_2) \end{bmatrix}, \quad \mathbf{Rot}_2 = \begin{bmatrix} \cos(\varphi_1) & 0 & \sin(\varphi_1) \\ 0 & 1 & 0 \\ -\sin(\varphi_1) & 0 & \cos(\varphi_1) \end{bmatrix}. \quad (3.45)$$

Here, the matrix $\mathbf{Rot}_1$ is produced by rotating φ_2 about the Θ^1 -axis, and the matrix $\mathbf{Rot}_2$ by rotating φ_1 about the Θ^2 -axis. After the two rotations, the total transformation matrix can be obtained to transfer the coordinates from the undeformed configuration to the deformed configuration as

$$\begin{Bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{Bmatrix} = \mathbf{Rot} \begin{Bmatrix} \bar{\Theta}^1 \\ \bar{\Theta}^2 \\ \bar{\Theta}^3 \end{Bmatrix} \quad (3.46)$$

with

$$\mathbf{Rot} = \mathbf{Rot}_2 \mathbf{Rot}_1 = \begin{bmatrix} \cos(\varphi_1) & -\sin(\varphi_1) \sin(\varphi_2) & \sin(\varphi_1) \cos(\varphi_2) \\ 0 & \cos(\varphi_2) & \sin(\varphi_2) \\ -\sin(\varphi_1) & -\cos(\varphi_1) \sin(\varphi_2) & \cos(\varphi_1) \cos(\varphi_2) \end{bmatrix}, \quad (3.47)$$

$$\mathbf{Rot}^{-1} = \begin{bmatrix} \cos(\varphi_1) & 0 & -\sin(\varphi_1) \\ -\sin(\varphi_1) \sin(\varphi_2) & \cos(\varphi_2) & -\cos(\varphi_1) \sin(\varphi_2) \\ \sin(\varphi_1) \cos(\varphi_2) & \sin(\varphi_2) & \cos(\varphi_1) \cos(\varphi_2) \end{bmatrix}. \quad (3.48)$$

Using this transformation matrix, the covariant base vector of the thickness direction in the deformed configuration can be expressed as [89]

$$\bar{\mathbf{a}}_3 = \sin(\varphi_1) \cos(\varphi_2) \hat{\mathbf{a}}^1 + \sin(\varphi_2) \hat{\mathbf{a}}^2 + \cos(\varphi_1) \cos(\varphi_2) \hat{\mathbf{a}}^3. \quad (3.49)$$

From the definition of the rotational displacement vector, $\hat{\mathbf{u}}^1 = \bar{\mathbf{a}}_3 - \mathbf{n}$, one obtains

$$\begin{aligned} \hat{\mathbf{u}}^1 &= \bar{\mathbf{a}}_3 - \mathbf{n} \\ &= \sin(\varphi_1) \cos(\varphi_2) \hat{\mathbf{a}}^1 + \sin(\varphi_2) \hat{\mathbf{a}}^2 + (\cos(\varphi_1) \cos(\varphi_2) - 1) \hat{\mathbf{a}}^3. \end{aligned} \quad (3.50)$$

Thus, the normalized rotational displacements are given by

$$\begin{aligned} \hat{v}_1^1 &= \sin(\varphi_1) \cos(\varphi_2), \\ \hat{v}_2^1 &= \sin(\varphi_2), \\ \hat{v}_3^1 &= \cos(\varphi_1) \cos(\varphi_2) - 1. \end{aligned} \quad (3.51)$$

In the linear (LIN5) or the simplified nonlinear shell theories (RVK5, MRT5, LRT5), small or moderate rotations are respectively assumed in structures. For small rotations ($\varphi_\alpha \ll 1$) and moderate rotations ($\varphi_\alpha^2 \ll 1$), it follows that $\sin(\varphi_\alpha) = \varphi_\alpha$ and $\cos(\varphi_\alpha) = 1$. Therefore, the normalized rotational displacements for the linear and simplified nonlinear shell theories are approximated as

$$\hat{v}_1^1 = \varphi_1, \quad \hat{v}_2^1 = \varphi_2, \quad \hat{v}_3^1 = 0. \quad (3.52)$$

Taking the spatial derivative of equation (3.51) with respect to Θ^α one obtains

$$\begin{aligned}\hat{v}_{1,\alpha} &= \frac{\partial \hat{v}_1}{\partial \Theta^\alpha} = \cos(\varphi_1) \cos(\varphi_2) \varphi_{1,\alpha} - \sin(\varphi_1) \sin(\varphi_2) \varphi_{2,\alpha}, \\ \hat{v}_{2,\alpha} &= \frac{\partial \hat{v}_2}{\partial \Theta^\alpha} = \cos(\varphi_2) \varphi_{2,\alpha}, \\ \hat{v}_{3,\alpha} &= \frac{\partial \hat{v}_3}{\partial \Theta^\alpha} = -\sin(\varphi_1) \cos(\varphi_2) \varphi_{1,\alpha} - \cos(\varphi_1) \sin(\varphi_2) \varphi_{2,\alpha}.\end{aligned}\tag{3.53}$$

For FE implementation, the nonlinear expressions for the rotational displacements given in (3.51) have to be linearized by means of the TAYLOR series expansion, with the higher-order terms neglected, as [89]

$$\Delta \hat{v}_i = \left. \frac{\partial \hat{v}_i}{\partial \varphi_1} \right|_t \Delta \varphi_1 + \left. \frac{\partial \hat{v}_i}{\partial \varphi_2} \right|_t \Delta \varphi_2,\tag{3.54}$$

where Δ represents the incremental operator. Therefore, the increment of the normalized displacements can be organized in matrix form as

$$\begin{Bmatrix} \Delta \hat{v}_1 \\ \Delta \hat{v}_2 \\ \Delta \hat{v}_3 \\ \Delta \hat{v}_1 \\ \Delta \hat{v}_2 \\ \Delta \hat{v}_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(\varphi_1) \cos(\varphi_2) & -\sin(\varphi_1) \sin(\varphi_2) \\ 0 & 0 & 0 & 0 & \cos(\varphi_2) \\ 0 & 0 & 0 & -\sin(\varphi_1) \cos(\varphi_2) & -\cos(\varphi_1) \sin(\varphi_2) \end{bmatrix}}_{\mathbf{T}_v} \begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta \varphi_1 \\ \Delta \varphi_2 \end{Bmatrix}.\tag{3.55}$$

Here, $\mathbf{T}_v$ is a transformation matrix produced by linearization. Thus, the incremental displacement vector $\Delta \mathbf{v}$ can be obtained as

$$\Delta \mathbf{v} = \mathbf{K}_v \Delta \hat{\mathbf{v}} = \mathbf{K}_v \mathbf{T}_v \Delta \hat{\mathbf{v}}_u = \mathbf{G}_v \Delta \hat{\mathbf{v}}_u,\tag{3.56}$$

where $\mathbf{G}_v = \mathbf{K}_v \mathbf{T}_v$. Similarly, a transformation matrix $\mathbf{T}_{th}$ can be obtained for $\Delta \boldsymbol{\theta}$ as

$$\Delta \boldsymbol{\theta} = \mathbf{K}_{th} \Delta \hat{\boldsymbol{\theta}} = \mathbf{K}_{th} \mathbf{T}_{th} \Delta \hat{\boldsymbol{\theta}}_u = \mathbf{G}_t \Delta \hat{\boldsymbol{\theta}}_u,\tag{3.57}$$

in which $\mathbf{G}_t = \mathbf{K}_{th}\mathbf{T}_{th}$. According to the FE method, the incremental displacements at any point can be interpolated by nodal DOFs using shape functions as

$$\Delta \hat{\mathbf{v}}_u = \mathbf{N}_u \Delta \mathbf{q}, \quad (3.58)$$

$$\Delta \hat{\boldsymbol{\theta}}_u = \mathbf{N}_t \Delta \mathbf{q}, \quad (3.59)$$

where $\mathbf{N}_u$ and $\mathbf{N}_t$ are the matrices of shape functions, and $\mathbf{q}$ represents the vector of nodal DOFs (or nodal displacements) at all specified nodes.

3.5 Shell element design

The finite element method is realized by discretization of the whole structure using specific defined elements. The most popular quadrilateral shell elements can be classified into LAGRANGE or *Serendipity* interpolation schemes, which are shown in Fig. 3.6. More detailed description of those two shell element families can be found in most FE books, e.g. Bathe [217], Zienkiewicz *et al.* [218], Kreja [126]. The elements with quadratic shape functions of both LAGRANGE and *Serendipity* families perform similarly, as well as the two types of elements with cubic shape functions. However, *Serendipity* elements have less nodes that will save computation time.

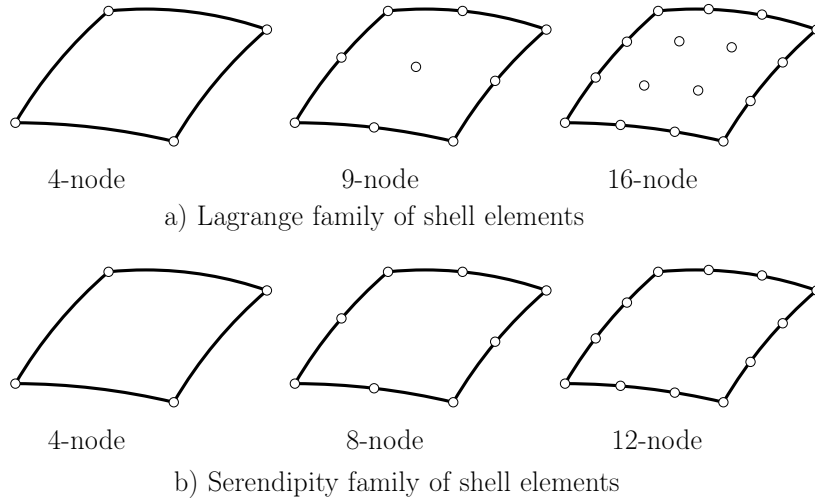


FIGURE 3.6: Lagrange and Serendipity families of shell elements

In the present study, only the eight-node *Serendipity* shell element is considered. The element can be transferred from the curvilinear coordinate system to the natural coordinate system (ξ, η) by using the JACOBIAN matrix, as shown in Fig. 3.7. The

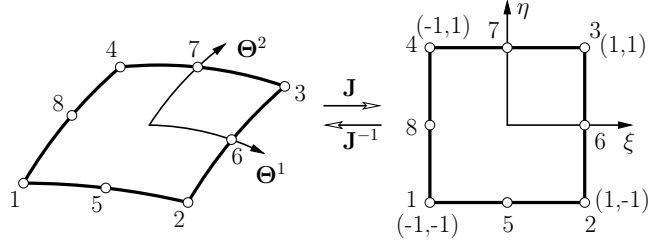


FIGURE 3.7: Eight-node shell element in natural coordinate system

interpolation functions in the natural coordinate system, which are usually called shape functions, can be expressed at each node as

$$\begin{aligned}
 N_I &= \frac{1}{4}(1 + \xi_I \xi)(1 + \eta_I \eta)(\xi_I \xi + \eta_I \eta - 1) & \text{for } I \in 1, 2, 3, 4, \\
 N_I &= \frac{1}{2}(1 - \xi^2)(1 + \eta_I \eta) & \text{for } I \in 5, 7, \\
 N_I &= \frac{1}{2}(1 - \eta^2)(1 + \xi_I \xi) & \text{for } I \in 6, 8.
 \end{aligned} \tag{3.60}$$

In such a way the matrices $\mathbf{N}_u$ and $\mathbf{N}_t$ given in equations (3.58) and (3.59) can be constructed by the shape functions in equation (3.60) and their derivatives.

Concerning the *membrane* and *shear locking* problems, there are several numerical methods, e.g. ANS, EAS, SRI or URI, see Chapter 1, which have been developed to avoid *locking* effects. In the present model, only the URI scheme is employed. To illustrate the *locking* effects, the FI scheme is used in some examples as well.

Two types of elements abbreviated as SH85FI and SH85URI, which mean eight-node isoparametric shell elements with five mechanical nodal DOFs respectively using FI and URI integration schemes, are developed for composite laminated structures. Additionally, two piezoelectric coupled elements denoted as SH851FI and SH851URI, which stands for eight-node isoparametric shell elements with five mechanical nodal DOFs and one electrical DOF per piezoelectric material layer respectively using FI and URI integration schemes, are proposed for piezoelectric integrated elements. All the shell elements used in the later simulations are listed in Table 3.2.

TABLE 3.2: Shell element types

Element	Mechanical DOFs	Electrical DOFs	Integration scheme
SH85FI	5	0	FI
SH85URI	5	0	URI
SH851FI	5	1	FI
SH851URI	5	1	URI

3.6 Variational formulations

In order to derive the equation of motion for composite or smart thin-walled structures, HAMILTON'S principle is employed, which is defined by

$$\int_{t_1}^{t_2} (\delta T - \delta W_{\text{int}} + \delta W_{\text{ext}}) dt = 0, \quad (3.61)$$

where δ denotes the variational operator, T , W_{int} and W_{ext} are the kinetic energy, the internal work and the external work, respectively. For the derivation of the static equilibrium equation, the principle of virtual work is used, which is given by

$$\delta W_{\text{int}} = \delta W_{\text{ext}}. \quad (3.62)$$

The variation of the kinetic energy, δT , can be calculated by [215]

$$\delta T = \int_V \rho \delta \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV = - \int_V \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} dV, \quad (3.63)$$

where ρ is the material density, $\dot{\square}$ and $\ddot{\square}$ represent the first- and second-order time derivative, respectively. Furthermore, $\mathbf{\hat{u}}$ denotes the vector of the normalized translational displacements in the shell space, which is given by

$$\mathbf{\hat{u}} = \begin{bmatrix} 1 & 0 & 0 & \Theta^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & \Theta^3 & 0 \\ 0 & 0 & 1 & 0 & 0 & \Theta^3 \end{bmatrix} \left\{ \begin{matrix} \hat{0} \\ v_1 \\ \hat{0} \\ v_2 \\ \hat{0} \\ v_3 \\ \hat{1} \\ v_1 \\ \hat{1} \\ v_2 \\ \hat{1} \\ v_3 \end{matrix} \right\} = \mathbf{Z}_u \hat{\mathbf{v}}, \quad (3.64)$$

where $\hat{\mathbf{v}}$ is the physical quantity of the generalized displacement vector.

According to (3.63) and (3.64), δT can be written as

$$\delta T = - \int_V \rho \delta \hat{\mathbf{v}}^T \mathbf{Z}_u^T \mathbf{Z}_u \ddot{\mathbf{v}} \, dV = - \int_\Omega \delta \hat{\mathbf{v}}^T \mathbf{H}_u \ddot{\mathbf{v}} \, d\Omega, \quad (3.65)$$

in which

$$\mathbf{H}_u = \int_h \rho \mathbf{Z}_u^T \mathbf{Z}_u \mu \, d\Theta^3. \quad (3.66)$$

The variation of the potential energy or internal virtual work, δW_{int} , is given by

$$\delta W_{\text{int}} = \int_V \left(\delta \hat{\boldsymbol{\varepsilon}}^T \hat{\boldsymbol{\sigma}} - \delta \hat{\mathbf{E}}^T \hat{\mathbf{D}} \right) \, dV. \quad (3.67)$$

Substituting equations (3.27) and (3.28) into (3.67) yields

$$\begin{aligned} \delta W_{\text{int}} &= \int_V \left(\delta \hat{\boldsymbol{\varepsilon}}^T \mathbf{c} \hat{\boldsymbol{\varepsilon}} - \delta \hat{\boldsymbol{\varepsilon}}^T \mathbf{e}^T \hat{\mathbf{E}} - \delta \hat{\mathbf{E}}^T \mathbf{e} \hat{\boldsymbol{\varepsilon}} - \delta \hat{\mathbf{E}}^T \boldsymbol{\epsilon} \hat{\mathbf{E}} \right) \, dV \\ &= \delta W_{\text{int}}^{(1)} + \delta W_{\text{int}}^{(2)} + \delta W_{\text{int}}^{(3)} + \delta W_{\text{int}}^{(4)}, \end{aligned} \quad (3.68)$$

where $\delta W_{\text{int}}^{(1)}$ and $\delta W_{\text{int}}^{(2)}$ are the pure and piezoelectric coupled mechanical internal virtual work, while $\delta W_{\text{int}}^{(3)}$ and $\delta W_{\text{int}}^{(4)}$ represent the coupled and pure electrical internal virtual work, respectively.

From equation (2.83), the transformation matrix $\mathbf{K}_e$ for the strain vector can be constructed as

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{K}_e \boldsymbol{\varepsilon} = \mathbf{K}_{\text{en}} \mathbf{K}_{\text{et}} \boldsymbol{\varepsilon}. \quad (3.69)$$

Here, the diagonal matrix $\mathbf{K}_e$ produced by normalization is a function of $(\Theta^1, \Theta^2, \Theta^3)$, which can be decomposed into $\mathbf{K}_{\text{en}}$ containing only Θ^3 and $\mathbf{K}_{\text{et}}$ depending on Θ^1 and Θ^2 . The components of $\mathbf{K}_{\text{et}}$ can be integrated into the resultant strain vector, which generates the partially normalized resultant strain vector $\hat{\mathbf{S}}$ as

$$\mathbf{K}_{\text{et}} \boldsymbol{\varepsilon} = \mathbf{K}_{\text{et}} \mathbf{H}_s \mathbf{S} = \mathbf{H}_s \mathbf{N}_{\text{ms}} \mathbf{S} = \mathbf{H}_s \hat{\mathbf{S}}. \quad (3.70)$$

By using the resultant strain and stress vectors, $\delta W_{\text{int}}^{(1)}$, $\delta W_{\text{int}}^{(2)}$, $\delta W_{\text{int}}^{(3)}$, $\delta W_{\text{int}}^{(4)}$, given in (3.68), can be obtained as

$$\delta W_{\text{int}}^{(1)} = \int_V \delta \hat{\boldsymbol{\varepsilon}}^T \mathbf{c} \hat{\boldsymbol{\varepsilon}} \, dV = \int_{\Omega} \delta \hat{\mathbf{S}}^T \mathbf{H}_c \hat{\mathbf{S}} \, d\Omega, \quad (3.71)$$

$$\delta W_{\text{int}}^{(2)} = - \int_V \delta \hat{\boldsymbol{\varepsilon}}^T \mathbf{e}^T \hat{\mathbf{E}} \, dV = \int_{\Omega} \delta \hat{\mathbf{S}}^T \mathbf{H}_e^T \mathbf{E} \, d\Omega, \quad (3.72)$$

$$\delta W_{\text{int}}^{(3)} = - \int_V \delta \hat{\mathbf{E}}^T \mathbf{e} \hat{\boldsymbol{\varepsilon}} \, dV = \int_{\Omega} \delta \mathbf{E}^T \mathbf{H}_e \hat{\mathbf{S}} \, d\Omega, \quad (3.73)$$

$$\delta W_{\text{int}}^{(4)} = - \int_V \delta \hat{\mathbf{E}}^T \boldsymbol{\epsilon} \hat{\mathbf{E}} \, dV = \int_{\Omega} \delta \mathbf{E}^T \mathbf{H}_g \mathbf{E} \, d\Omega, \quad (3.74)$$

with

$$\mathbf{H}_c = \int_h \mathbf{H}_s^T \mathbf{K}_{\text{en}}^T \mathbf{c} \mathbf{K}_{\text{en}} \mathbf{H}_s \, \mu \, d\Theta^3, \quad (3.75)$$

$$\mathbf{H}_e = - \int_h \mathbf{e} \mathbf{K}_{\text{en}} \mathbf{H}_s \, \mu \, d\Theta^3, \quad (3.76)$$

$$\mathbf{H}_g = - \int_h \boldsymbol{\epsilon} \, \mu \, d\Theta^3. \quad (3.77)$$

Furthermore, the external virtual work, δW_{ext} , can be derived as [189, 215]

$$\delta W_{\text{ext}} = \int_V \delta \hat{\mathbf{u}}^T \mathbf{f}_b \, dV + \int_{\Omega} \delta \hat{\mathbf{u}}^T \mathbf{f}_s \, d\Omega + \delta \hat{\mathbf{u}}^T \mathbf{f}_c - \int_{\Omega} \delta \boldsymbol{\phi}^T \boldsymbol{\varrho} \, d\Omega - \delta \boldsymbol{\phi}^T \mathbf{Q}_c, \quad (3.78)$$

where $\mathbf{f}_b$, $\mathbf{f}_s$ and $\mathbf{f}_c$ denote the body force, the surface distributed force and the concentrated force vectors. Additionally, $\boldsymbol{\varrho}$ is the surface charge vector and $\mathbf{Q}_c$ the applied concentrated electric charge vector.

3.7 Total Lagrangian formulation

According to the Total LAGRANGIAN (TL) incremental formulation [89, 90, 189, 219], three configurations are considered for structures, which are listed in Table 3.3. The configurations are characterized by the left superscripts 0, 1 or 2, the reference configurations are denoted by the left subscripts. Using the TL method, the stress vector, the strain vector, the displacement vector, etc. in the virtual configuration can be expressed by those in the current configuration and the incremental values as

$${}^2_0\mathbf{X} = {}^1_0\mathbf{X} + \Delta\mathbf{X}, \quad (\mathbf{X} = \mathbf{L}, \mathbf{S}, \mathbf{D}, \mathbf{E}, \mathbf{v}, \boldsymbol{\phi}). \quad (3.79)$$

TABLE 3.3: Notations for different configurations

Notation	Meaning
${}^0\text{C}$	Initial configuration, referring to the undeformed configuration
${}^1\text{C}$	Current configuration, referring to the deformed configuration
${}^2\text{C}$	Virtual configuration, which is called searched configuration
${}^m\text{C}$	Configuration m , $m = 0, 1, 2$,

The partially normalized resultant strain vector, ${}^m\hat{\mathbf{S}}$, in configuration ${}^m\text{C}$, referred to the undeformed configuration, can be expressed by a linear part $\mathbf{A}_0$ and a nonlinear part $\mathbf{A}_n({}^m\boldsymbol{\varrho})$ as

$${}^m\hat{\mathbf{S}} = \mathbf{N}_{\text{ms}} \left(\mathbf{A}_0 + \frac{1}{2} \mathbf{A}_n({}^m\boldsymbol{\varrho}) \right) {}^m\boldsymbol{\theta}. \quad (3.80)$$

Therefore, the increment and variation of $\hat{\mathbf{S}}$ can be obtained as

$$\Delta \hat{\mathbf{S}} = (\mathbf{B}_1 + \mathbf{B}_{\text{nl}}) \Delta \mathbf{q}, \quad (3.81)$$

$${}^2_0\delta \hat{\mathbf{S}} = \delta \Delta \hat{\mathbf{S}} = (\mathbf{B}_1 + 2 \mathbf{B}_{\text{nl}}) \delta \Delta \mathbf{q}. \quad (3.82)$$

Here, $\mathbf{B}_1$ and $\mathbf{B}_{\text{nl}}$ are

$$\mathbf{B}_1 = \mathbf{N}_{\text{ms}} \left(\mathbf{A}_0 + \mathbf{A}_n({}^1_0\boldsymbol{\varrho}) \right) \mathbf{G}_{\text{t}} \mathbf{N}_{\text{t}}, \quad (3.83)$$

$$\mathbf{B}_{\text{nl}} = \frac{1}{2} \mathbf{N}_{\text{ms}} \mathbf{A}_n(\Delta \boldsymbol{\varrho}) \mathbf{G}_{\text{t}} \mathbf{N}_{\text{t}}. \quad (3.84)$$

From equation (3.65), the variation of the kinetic energy in the virtual configuration, ${}^2_0\delta T$, can be obtained as

$$\begin{aligned} {}^2_0\delta T &= - \int_{\Omega} {}^2_0\delta \hat{\mathbf{v}}^{\text{T}} \mathbf{H}_{\text{u}0} {}^2_0\ddot{\hat{\mathbf{v}}} \, \text{d}\Omega \\ &= - \delta \Delta \mathbf{q}^{\text{T}} \left(\int_{\Omega} \mathbf{N}_{\text{u}}^{\text{T}} \mathbf{T}_{\text{v}}^{\text{T}} \mathbf{H}_{\text{u}0} {}^1\ddot{\hat{\mathbf{v}}} \, \text{d}\Omega + \int_{\Omega} \mathbf{N}_{\text{u}}^{\text{T}} \mathbf{T}_{\text{v}}^{\text{T}} \mathbf{H}_{\text{u}} \mathbf{T}_{\text{v}} \mathbf{N}_{\text{u}} \, \text{d}\Omega \Delta \ddot{\mathbf{q}} \right) \\ &= - \delta \Delta \mathbf{q}^{\text{T}} \left({}^1\mathbf{F}_{\text{ut}} + {}^1\mathbf{M}_{\text{uu}} \Delta \ddot{\mathbf{q}} \right), \end{aligned} \quad (3.85)$$

where ${}^1\mathbf{F}_{ut}$ and ${}^1\mathbf{M}_{uu}$ represent the inertial in-balance force and mass matrix, which are respectively calculated by

$${}^1\mathbf{F}_{ut} = \int_{\Omega} \mathbf{N}_u^T \mathbf{T}_v^T \mathbf{H}_{u0} {}^1\ddot{\mathbf{v}} \, d\Omega, \quad (3.86)$$

$${}^1\mathbf{M}_{uu} = \int_{\Omega} \mathbf{N}_u^T \mathbf{T}_v^T \mathbf{H}_u \mathbf{T}_v \mathbf{N}_u \, d\Omega. \quad (3.87)$$

From equation (3.71), the pure mechanical induced virtual work in the virtual configuration, ${}^2_0\delta W_{int}^{(1)}$, can be expressed as

$$\begin{aligned} {}^2_0\delta W_{int}^{(1)} &= \int_{\Omega} {}^2_0\delta \hat{\mathbf{S}}^T \mathbf{H}_{c0} {}^2\hat{\mathbf{S}} \, d\Omega \\ &= \delta \Delta \mathbf{q}^T \left(\int_{\Omega} (\mathbf{B}_l^T + 2 \mathbf{B}_{nl}^T) \mathbf{H}_{c0} {}^1\hat{\mathbf{S}} \, d\Omega \right. \\ &\quad \left. + \int_{\Omega} (\mathbf{B}_l^T + 2 \mathbf{B}_{nl}^T) \mathbf{H}_c (\mathbf{B}_l + \mathbf{B}_{nl}) \, d\Omega \Delta \mathbf{q} \right) \\ &= \delta \Delta \mathbf{q}^T \left({}^1\mathbf{F}_{uu} + ({}^1\mathbf{K}_{uu} + {}^1\mathbf{K}_{ug} + \mathbf{K}_{nl1}) \Delta \mathbf{q} \right), \end{aligned} \quad (3.88)$$

where ${}^1\mathbf{F}_{uu}$, ${}^1\mathbf{K}_{uu}$, ${}^1\mathbf{K}_{ug}$ and $\mathbf{K}_{nl1}$ denote the mechanically induced in-balance force vector, the linearized stiffness matrix, the geometrically induced stiffness matrix and the higher-order nonlinear stiffness matrix, respectively. The linearized and geometrically nonlinear stiffness matrices will be updated after every iteration, but the higher-order nonlinear stiffness matrix will be neglected due to small contribution.

The mechanically induced in-balance force vector ${}^1\mathbf{F}_{uu}$ and the linearized stiffness matrix ${}^1\mathbf{K}_{uu}$ can be respectively obtained as

$${}^1\mathbf{F}_{uu} = \int_{\Omega} \mathbf{B}_l^T \mathbf{H}_{c0} {}^1\hat{\mathbf{S}} \, d\Omega, \quad (3.89)$$

$${}^1\mathbf{K}_{uu} = \int_{\Omega} \mathbf{B}_l^T \mathbf{H}_c \mathbf{B}_l \, d\Omega. \quad (3.90)$$

Furthermore, the term $2\mathbf{B}_{nl}^T \mathbf{H}_{c0} {}^1\hat{\mathbf{S}}$ in equation (3.88) can be expanded to

$$2\mathbf{B}_{nl}^T \mathbf{H}_{c0} {}^1\hat{\mathbf{S}} = \mathbf{N}_t^T \mathbf{G}_t^T \mathbf{A}_n (\Delta \boldsymbol{\theta})^T \mathbf{N}_{ms}^T \mathbf{H}_{c0} {}^1\hat{\mathbf{S}}. \quad (3.91)$$

Here, we define $\hat{\mathbf{L}} = \mathbf{N}_{\text{ms}}^T \mathbf{H}_{\text{c0}}^1 \hat{\mathbf{S}}$, so that the components of the vector $\Delta \boldsymbol{\theta}$ can be extracted from the term $\mathbf{A}_n(\Delta \boldsymbol{\theta})^T \hat{\mathbf{L}}$ as

$$\mathbf{A}_n(\Delta \boldsymbol{\theta})^T \hat{\mathbf{L}} = {}^1\mathbf{S}_{\text{uu}} \Delta \boldsymbol{\theta} = {}^1\mathbf{S}_{\text{uu}} \mathbf{G}_t \mathbf{N}_t \Delta \mathbf{q} . \quad (3.92)$$

Consequently, an additional stiffness matrix ${}^1\mathbf{K}_{\text{ug}}$, which is called the geometrically induced stiffness due to the mechanically induced stresses, can be calculated as

$${}^1\mathbf{K}_{\text{ug}} = \int_{\Omega} \mathbf{N}_t^T \mathbf{G}_t^T {}^1\mathbf{S}_{\text{uu}} \mathbf{G}_t \mathbf{N}_t \, d\Omega , \quad (3.93)$$

where ${}^1\mathbf{S}_{\text{uu}}$ will be denoted as the mechanically induced resultant stresses.

From equation (3.72), the coupled mechanical internal virtual work in the virtual configuration, ${}_0^2\delta W_{\text{int}}^{(2)}$, can be expressed as

$$\begin{aligned} {}_0^2\delta W_{\text{int}}^{(2)} &= \int_{\Omega} {}_0^2\delta \hat{\mathbf{S}}^T \mathbf{H}_{\text{e}0}^T \mathbf{E} \, d\Omega \\ &= \delta \Delta \mathbf{q}^T \left(\int_{\Omega} (\mathbf{B}_l^T + 2 \mathbf{B}_{\text{nl}}^T) \mathbf{H}_{\text{e}0}^T \mathbf{E} \, d\Omega + \int_{\Omega} (\mathbf{B}_l^T + 2 \mathbf{B}_{\text{nl}}^T) \mathbf{H}_{\text{e}}^T \mathbf{B}_{\phi} \, d\Omega \Delta \boldsymbol{\phi} \right) \\ &= \delta \Delta \mathbf{q}^T \left({}^1\mathbf{F}_{\text{u}\phi} + ({}^1\mathbf{K}_{\text{u}\phi} + \mathbf{K}_{\text{nl2}}) \Delta \boldsymbol{\phi} + {}^1\mathbf{K}_{\phi\text{g}} \Delta \mathbf{q} \right) , \end{aligned} \quad (3.94)$$

where ${}^1\mathbf{F}_{\text{u}\phi}$, ${}^1\mathbf{K}_{\text{u}\phi}$, ${}^1\mathbf{K}_{\phi\text{g}}$ and $\mathbf{K}_{\text{nl2}}$ are the electrically induced in-balance force vector, the coupled stiffness matrix, the geometrically induced stiffness matrix due to the electrical displacement, and the higher-order nonlinear stiffness matrix which will be neglected as well. They can be calculated by

$${}^1\mathbf{F}_{\text{u}\phi} = \int_{\Omega} \mathbf{B}_l^T \mathbf{H}_{\text{e}0}^T \mathbf{E} \, d\Omega , \quad (3.95)$$

$${}^1\mathbf{K}_{\text{u}\phi} = \int_{\Omega} \mathbf{B}_l^T \mathbf{H}_{\text{e}}^T \mathbf{B}_{\phi} \, d\Omega , \quad (3.96)$$

$${}^1\mathbf{K}_{\phi\text{g}} = \int_{\Omega} \mathbf{N}_t^T \mathbf{G}_t^T {}^1\mathbf{S}_{\text{u}\phi} \mathbf{G}_t \mathbf{N}_t \, d\Omega . \quad (3.97)$$

in which ${}^1\mathbf{S}_{\text{u}\phi}$ is the electrically induced resultant stresses.

From equation (3.73), the coupled electrical internal virtual work in the virtual configuration, ${}^2_0\delta W_{\text{int}}^{(3)}$, can be calculated as

$$\begin{aligned} {}^2_0\delta W_{\text{int}}^{(3)} &= \int_{\Omega} {}^2_0\delta \mathbf{E}^T \mathbf{H}_{e0} {}^2\hat{\mathbf{S}} \, d\Omega \\ &= \delta \Delta \phi^T \left(\int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_{e0} {}^1\hat{\mathbf{S}} \, d\Omega + \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_e (\mathbf{B}_l + \mathbf{B}_{nl}) \, d\Omega \Delta \mathbf{q} \right) \\ &= \delta \Delta \phi^T \left({}^1\mathbf{F}_{\phi u} + ({}^1\mathbf{K}_{\phi u} + \mathbf{K}_{nl3}) \Delta \mathbf{q} \right), \end{aligned} \quad (3.98)$$

Here, ${}^1\mathbf{F}_{\phi u}$ and ${}^1\mathbf{K}_{\phi u}$ denote the mechanically induced in-balance charge vector and the piezoelectric coupled capacity matrix, which are respectively given by

$${}^1\mathbf{F}_{\phi u} = \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_{e0} {}^1\hat{\mathbf{S}} \, d\Omega, \quad (3.99)$$

$${}^1\mathbf{K}_{\phi u} = \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_e \mathbf{B}_l \, d\Omega. \quad (3.100)$$

From equation (3.74), the pure electric internal virtual work in the virtual configuration, ${}^2_0\delta W_{\text{int}}^{(4)}$, can be expressed as

$$\begin{aligned} {}^2_0\delta W_{\text{int}}^{(4)} &= \int_{\Omega} {}^2_0\delta \mathbf{E}^T \mathbf{H}_{g0} {}^2\mathbf{E} \, d\Omega \\ &= \delta \Delta \phi^T \left(\int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_{g0} {}^1\mathbf{E} \, d\Omega + \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_g \mathbf{B}_{\phi} \, d\Omega \Delta \phi \right) \\ &= \delta \Delta \phi^T \left({}^1\mathbf{F}_{\phi\phi} + {}^1\mathbf{K}_{\phi\phi} \Delta \phi \right), \end{aligned} \quad (3.101)$$

in which the electrically induced in-balance charge vector ${}^1\mathbf{F}_{\phi\phi}$ and the piezoelectric capacity matrix ${}^1\mathbf{K}_{\phi\phi}$ are calculated as

$${}^1\mathbf{F}_{\phi\phi} = \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_{g0} {}^1\mathbf{E} \, d\Omega, \quad (3.102)$$

$${}^1\mathbf{K}_{\phi\phi} = \int_{\Omega} \mathbf{B}_{\phi}^T \mathbf{H}_g \mathbf{B}_{\phi} \, d\Omega. \quad (3.103)$$

The variation of the external work in the virtual configuration, ${}^2_0\delta W_{\text{ext}}$, including the mechanical force and electric charge loads, are expressed as

$$\begin{aligned} {}^2_0\delta W_{\text{ext}} &= \int_V {}^2_0\delta \hat{\mathbf{u}}^T \mathbf{f}_b \, dV + \int_{\Omega} {}^2_0\delta \hat{\mathbf{u}}^T \mathbf{f}_s \, d\Omega + {}^2_0\delta \hat{\mathbf{u}}^T \mathbf{f}_c - \int_{\Omega} {}^2_0\delta \phi^T \boldsymbol{\varrho} \, d\Omega - {}^2_0\delta \phi^T \mathbf{Q}_c \\ &= \delta \Delta \mathbf{q}^T \left(\mathbf{F}_{ub} + \mathbf{F}_{us} + \mathbf{F}_{uc} \right) + \delta \Delta \phi^T \left(\mathbf{G}_{\phi s} + \mathbf{G}_{\phi c} \right), \end{aligned} \quad (3.104)$$

with

$$\mathbf{F}_{ub} = \int_V \mathbf{N}_u^T \mathbf{T}_v^T \mathbf{Z}_u^T \mathbf{f}_b dV, \quad (3.105)$$

$$\mathbf{F}_{us} = \int_{\Omega} \mathbf{N}_u^T \mathbf{T}_v^T \mathbf{Z}_u^T \mathbf{f}_s d\Omega, \quad (3.106)$$

$$\mathbf{F}_{uc} = \mathbf{N}_u^T \mathbf{T}_v^T \mathbf{Z}_u^T \mathbf{f}_c, \quad (3.107)$$

$$\mathbf{G}_{\phi s} = - \int_{\Omega} \boldsymbol{\rho} d\Omega, \quad (3.108)$$

$$\mathbf{G}_{\phi c} = -\mathbf{Q}_c, \quad (3.109)$$

where $\mathbf{F}_{ub}$, $\mathbf{F}_{us}$, $\mathbf{F}_{uc}$ are the element body force, surface force and concentrated force vectors, respectively, while $\mathbf{G}_{\phi s}$ and $\mathbf{G}_{\phi c}$ denote the element surface and concentrated electric charge vectors that are applied on piezoelectric material layers.

3.8 FE models

3.8.1 Dynamic FE model

Substituting equations (3.85), (3.88), (3.94), (3.98), (3.101) and (3.104) into HAMILTON'S principle given in (3.61) yields

$$\begin{aligned} 0 = & \delta \Delta \mathbf{q}^T \left({}^1\mathbf{F}_{ut} + {}^1\mathbf{M}_{uu} \Delta \ddot{\mathbf{q}} \right) \\ & + \delta \Delta \mathbf{q}^T \left({}^1\mathbf{F}_{uu} + ({}^1\mathbf{K}_{uu} + {}^1\mathbf{K}_{ug} + \mathbf{K}_{nl1}) \Delta \mathbf{q} \right) \\ & + \delta \Delta \mathbf{q}^T \left({}^1\mathbf{F}_{u\phi} + ({}^1\mathbf{K}_{u\phi} + \mathbf{K}_{nl2}) \Delta \phi + {}^1\mathbf{K}_{\phi g} \Delta \mathbf{q} \right) \\ & + \delta \Delta \phi^T \left({}^1\mathbf{F}_{\phi u} + {}^1\mathbf{K}_{\phi u} \Delta \mathbf{q} \right) \\ & + \delta \Delta \phi^T \left({}^1\mathbf{F}_{\phi\phi} + {}^1\mathbf{K}_{\phi\phi} \Delta \phi \right) \\ & - \delta \Delta \mathbf{q}^T \left(\mathbf{F}_{ub} + \mathbf{F}_{us} + \mathbf{F}_{uc} \right) \\ & - \delta \Delta \phi^T \left(\mathbf{G}_{\phi s} + \mathbf{G}_{\phi c} \right). \end{aligned} \quad (3.110)$$

In order to satisfy equation (3.110) unconditionally, the coefficient terms of $\delta \Delta \mathbf{q}^T$

and $\delta\Delta\phi^T$ must be set to zero, respectively, which yields a piezoelectric coupled dynamic FE model including an equation of motion and a sensor equation as

$${}^1\mathbf{M}_{uu0}\ddot{\mathbf{q}} + {}^1\bar{\mathbf{K}}_{uu}\Delta\mathbf{q} + {}^1\mathbf{K}_{u\phi}\Delta\phi_a = \mathbf{F}_{ue} - {}^1\mathbf{F}_{ui}, \quad (3.111)$$

$${}^1\mathbf{K}_{\phi u}\Delta\mathbf{q} + {}^1\mathbf{K}_{\phi\phi}\Delta\phi_s = \mathbf{G}_{\phi e} - {}^1\mathbf{G}_{\phi i}, \quad (3.112)$$

where ${}^1\mathbf{M}_{uu}$, ${}^1\bar{\mathbf{K}}_{uu}$, ${}^1\mathbf{K}_{u\phi}$, ${}^1\mathbf{K}_{\phi u}$ and ${}^1\mathbf{K}_{\phi\phi}$ represent the mass, the total stiffness, the coupled stiffness, the coupled capacity and the piezoelectric capacity matrices, respectively. In the right-hand side of the above equations, $\mathbf{F}_{ue}$, ${}^1\mathbf{F}_{ui}$, $\mathbf{G}_{\phi e}$ and ${}^1\mathbf{G}_{\phi i}$ denote the external force, the in-balance force, the external charge and the in-balance charge vectors, respectively. Additionally, $\ddot{\mathbf{q}}$ is the acceleration of the nodal DOF vector, $\mathbf{q}$ the nodal DOF vector, ϕ_a the vector of the electric potential applied on piezoelectric material layers, and ϕ_s the vector of the electric potential output from piezoelectric material layers. The total stiffness matrix, the in-balance force and charge vectors, the external force and charge vectors are calculated by

$${}^1\bar{\mathbf{K}}_{uu} = {}^1\mathbf{K}_{uu} + {}^1\mathbf{K}_{ug} + {}^1\mathbf{K}_{\phi g}, \quad (3.113)$$

$${}^1\mathbf{F}_{ui} = {}^1\mathbf{F}_{uu} + {}^1\mathbf{F}_{u\phi}, \quad (3.114)$$

$${}^1\mathbf{G}_{\phi i} = {}^1\mathbf{F}_{\phi u} + {}^1\mathbf{F}_{\phi\phi}, \quad (3.115)$$

$$\mathbf{F}_{ue} = \mathbf{F}_{ub} + \mathbf{F}_{us} + \mathbf{F}_{uc}, \quad (3.116)$$

$$\mathbf{G}_{\phi e} = \mathbf{G}_{\phi s} + \mathbf{G}_{\phi c}. \quad (3.117)$$

Due to the complexity and uncertainty, it is difficult to model the damping effect. Usually, the damping coefficient is assumed to be linear with respect to mass and stiffness. In this dissertation, the RAYLEIGH damping coefficients computation method [220] is employed, which is given by

$${}^1\mathbf{C}_{uu} = \frac{\alpha_1 + \alpha_2}{2} {}^1\mathbf{M}_{uu} + \frac{\beta_1 + \beta_2}{2} {}^1\bar{\mathbf{K}}_{uu}, \quad (3.118)$$

in which the coefficients α_1 , α_2 , β_1 and β_2 can be calculated as

$$\begin{aligned} \beta_1 &= \frac{2(\varsigma_1\omega_1 - \varsigma_m\omega_m)}{\omega_1^2 - \omega_m^2}, & \alpha_1 &= 2\varsigma_1\omega_1 - \beta_1\omega_1^2, \\ \beta_2 &= \frac{2(\varsigma_1\omega_1 - \varsigma_{2.5m}\omega_{2.5m})}{\omega_1^2 - \omega_{2.5m}^2}, & \alpha_2 &= 2\varsigma_1\omega_1 - \beta_2\omega_1^2. \end{aligned} \quad (3.119)$$

Here, ς_1 , ς_m and $\varsigma_{2.5m}$ ($m = 2, 4, 6, \dots$) refer to the damping ratio at 1, m and $2.5m$

modes, respectively. Similarly, ω_1 , ω_m and $\omega_{2.5m}$ are the angular frequencies at 1, m and $2.5m$ modes. The damping ratio at i^{th} mode can be assumed as

$$\varsigma_i = \begin{cases} \frac{\varsigma_m - \varsigma_1}{\omega_m - \omega_1}(\omega_i - \omega_1) + \varsigma_1 & 1 < i < m \\ \frac{\varsigma_m - \varsigma_1}{\omega_m - \omega_1}(\omega_{m+i} - \omega_m) + \varsigma_1 & m < i < 2.5m . \end{cases} \quad (3.120)$$

Adding the damping coefficient matrix yields the equation of motion with considering the damping effects as

$${}^1\mathbf{M}_{uu0} {}^2\ddot{\mathbf{q}} + {}^1\mathbf{C}_{uu0} {}^2\dot{\mathbf{q}} + {}^1\bar{\mathbf{K}}_{uu}\Delta\mathbf{q} + {}^1\mathbf{K}_{u\phi}\Delta\phi_a = \mathbf{F}_{ue} - {}^1\mathbf{F}_{ui} , \quad (3.121)$$

where ${}^2_0\dot{\mathbf{q}}$ represents the velocity of the nodal DOF vector in the virtual configuration.

3.8.2 Static FE model

By applying the FE method and the *principle of virtual work*, an electro-mechanically coupled static FE model including an equilibrium equation and a sensor equation for smart structures can be obtained as

$${}^1\bar{\mathbf{K}}_{uu}\Delta\mathbf{q} + {}^1\mathbf{K}_{u\phi}\Delta\phi_a = \mathbf{F}_{ue} - {}^1\mathbf{F}_{ui} , \quad (3.122)$$

$${}^1\mathbf{K}_{\phi u}\Delta\mathbf{q} + {}^1\mathbf{K}_{\phi\phi}\Delta\phi_s = \mathbf{G}_{\phi e} - {}^1\mathbf{G}_{\phi i} . \quad (3.123)$$

Here, the coefficient matrices have the same meanings as those given in Section 3.8.1.

3.9 Numerical algorithms

In the previous sections, we have derived the equations of motion and the equilibrium equations for smart structures. The former ones are second-order differential equations with respect to the time. The NEWMARK method (implicit method) and the Central Difference Algorithm (CDA, explicit method) are employed for solving the second-order differential equation of motion. NEWMARK method is used much more frequently in dynamic analysis due to less computation time costs resulting from larger time step compared to CDA. Concerning the first-order differential equation, the load control method, NEWTON-RAPHSON method, or the RIKS-WEMPNER method usually called

the arc-length control method, are employed to trace the static behavior of thin-walled structures. The load control method is only applicable for structures which have simple static response, while RIKS-WEMPNER method should be adopted for computations of structures with buckling and complex post-buckling behavior, including e.g. snap-through and snap-back phenomena or loops. The details of these numerical algorithms can be found in many books or thesis, see e.g. [89, 90, 126, 189, 219] among many others.

3.9.1 Newmark method

We start from the dynamic equation at time $t + \Delta t$, which is given by

$$\mathbf{M}_{\text{uu}}^{(t)} \ddot{\mathbf{q}}^{(t+\Delta t)} + \mathbf{C}_{\text{uu}}^{(t)} \dot{\mathbf{q}}^{(t+\Delta t)} + \mathbf{K}_{\text{uu}}^{(t)} \Delta \mathbf{q}^{(t)} = \mathbf{F}_{\text{ue}}^{(t)} - \mathbf{F}_{\text{ui}}^{(t)} \quad (3.124)$$

with the assumptions of $\mathbf{q}$ and $\dot{\mathbf{q}}$ at time $t + \Delta t$ as [90, 189]

$$\mathbf{q}^{(t+\Delta t)} = \mathbf{q}^{(t)} + (\Delta t) \dot{\mathbf{q}}^{(t)} + (\Delta t)^2 ((0.5 - \beta) \ddot{\mathbf{q}}^{(t)} + \beta \ddot{\mathbf{q}}^{(t+\Delta t)}) , \quad (3.125)$$

$$\dot{\mathbf{q}}^{(t+\Delta t)} = \dot{\mathbf{q}}^{(t)} + (\Delta t) ((1 - \gamma) \ddot{\mathbf{q}}^{(t)} + \gamma \ddot{\mathbf{q}}^{(t+\Delta t)}) . \quad (3.126)$$

Here superscript t refers to the time in the current configuration, and Δt is a small increment of time.

If the parameters $\gamma \geq 0.5$ and $\beta \geq (2\gamma + 1)^2/16$, the NEWMARK method is unconditionally stable [217], meaning that the size of the time step will not affect the stability of the solution, but it may affect the accuracy. The commonly used values are $\beta = 0.25$ and $\gamma = 0.5$, with which it is called linear acceleration method. For simplicity, some constants will be introduced for calculation as

$$\begin{aligned} a_0 &= \frac{1}{\beta(\Delta t)^2}, & a_1 &= \frac{\gamma}{\beta(\Delta t)}, & a_2 &= \frac{1}{\beta(\Delta t)}, & a_3 &= \frac{\gamma}{\beta}, \\ a_4 &= \frac{1}{2\beta}, & a_5 &= \left(1 - \frac{\gamma}{2\beta}\right)(\Delta t), & a_6 &= 1 - \frac{1}{2\beta}, & a_7 &= 1 - \frac{\gamma}{\beta}. \end{aligned} \quad (3.127)$$

Based on the assumptions given in (3.125)-(3.126), the incremental acceleration and velocity of the nodal displacement vector can be obtained as

$$\Delta \ddot{\mathbf{q}}^{(t)} = a_0 \Delta \mathbf{q}^{(t)} - a_2 \dot{\mathbf{q}}^{(t)} - a_4 \ddot{\mathbf{q}}^{(t)}, \quad (3.128)$$

$$\Delta \dot{\mathbf{q}}^{(t)} = a_1 \Delta \mathbf{q}^{(t)} - a_3 \dot{\mathbf{q}}^{(t)} + a_5 \ddot{\mathbf{q}}^{(t)}. \quad (3.129)$$

Substituting equations (3.128)-(3.129) into (3.124) yields

$$\Delta \mathbf{q}^{(t)} = \frac{\mathbf{F}_{\text{ue}}^{(t)} - \mathbf{F}_{\text{ui}}^{(t)} - (a_6 \mathbf{M}_{\text{uu}}^{(t)} + a_5 \mathbf{C}_{\text{uu}}^{(t)}) \ddot{\mathbf{q}}^{(t)} - (a_7 \mathbf{C}_{\text{uu}}^{(t)} - a_2 \mathbf{M}_{\text{uu}}^{(t)}) \dot{\mathbf{q}}^{(t)}}{a_0 \mathbf{M}_{\text{uu}}^{(t)} + a_1 \mathbf{C}_{\text{uu}}^{(t)} + \mathbf{K}_{\text{uu}}^{(t)}}. \quad (3.130)$$

3.9.2 Central difference algorithm

We consider the dynamic equation at time t expressed as

$$\mathbf{M}_{\text{uu}}^{(t)} \ddot{\mathbf{q}}^{(t)} + \mathbf{C}_{\text{uu}}^{(t)} \dot{\mathbf{q}}^{(t)} + \mathbf{K}_{\text{uu}}^{(t)} \Delta \mathbf{q}^{(t)} = \mathbf{F}_{\text{ue}}^{(t)} - \mathbf{F}_{\text{ui}}^{(t)}. \quad (3.131)$$

The central difference algorithm is based on the approximations of the acceleration $\ddot{\mathbf{q}}^{(t)}$ and the velocity $\dot{\mathbf{q}}^{(t)}$ at time t as [90, 189]

$$\ddot{\mathbf{q}}^{(t)} = \frac{1}{(\Delta t)^2} \left(\mathbf{q}^{(t+\Delta t)} - 2\mathbf{q}^{(t)} + \mathbf{q}^{(t-\Delta t)} \right), \quad (3.132)$$

$$\dot{\mathbf{q}}^{(t)} = \frac{1}{2(\Delta t)} \left(\mathbf{q}^{(t+\Delta t)} - \mathbf{q}^{(t-\Delta t)} \right). \quad (3.133)$$

Substituting the above assumptions into the dynamic equation yields the displacements at time $t + \Delta t$ as

$$\mathbf{q}^{(t+\Delta t)} = \left(\frac{1}{(\Delta t)^2} \mathbf{M}_{\text{uu}}^{(t)} + \frac{1}{2(\Delta t)} \mathbf{C}_{\text{uu}}^{(t)} \right)^{-1} \mathbf{F}_{\text{Residual}}^{(t)} \quad (3.134)$$

where

$$\begin{aligned} \mathbf{F}_{\text{Residual}}^{(t)} = & \mathbf{F}_{\text{ue}}^{(t)} - \mathbf{F}_{\text{ui}}^{(t)} + \frac{1}{(\Delta t)^2} \mathbf{M}_{\text{uu}}^{(t)} (2\mathbf{q}^{(t)} - \mathbf{q}^{(t-\Delta t)}) \\ & + \frac{1}{2(\Delta t)} \mathbf{C}_{\text{uu}}^{(t)} \mathbf{q}^{(t-\Delta t)} + \mathbf{K}_{\text{uu}}^{(t)} (\mathbf{q}^{(t-\Delta t)} - \mathbf{q}^{(t)}). \end{aligned} \quad (3.135)$$

For the first step, the displacement at time $t - \Delta t$ can be derived as

$$\mathbf{q}^{(t-\Delta t)} = \mathbf{q}^{(t)} + (\Delta t) \dot{\mathbf{q}}^{(t)} + \frac{(\Delta t)^2}{2} \ddot{\mathbf{q}}^{(t)}, \quad (3.136)$$

where $\mathbf{q}^{(t)}$ and $\dot{\mathbf{q}}^{(t)}$ are prescribed, and $\ddot{\mathbf{q}}^{(t)}$ can be calculated by

$$\ddot{\mathbf{q}}^{(t)} = (\mathbf{M}_{\text{uu}}^{(t)})^{-1} \left(\mathbf{F}_{\text{ue}}^{(t)} - \mathbf{F}_{\text{ui}}^{(t)} - \mathbf{C}_{\text{uu}}^{(t)} \dot{\mathbf{q}}^{(t)} - \mathbf{K}_{\text{uu}}^{(t)} \Delta \mathbf{q}^{(t)} \right). \quad (3.137)$$

3.9.3 Newton-Raphson method

The static equilibrium equation in the k^{th} iteration is given as

$$\mathbf{K}_{uu}^{(k)} \Delta \mathbf{q}^{(k)} = \mathbf{F}_{ue}^{(k)} - \mathbf{F}_{ui}^{(k)} . \quad (3.138)$$

Therefore, the incremental displacement vector in the k^{th} iteration can be calculated as

$$\Delta \mathbf{q}^{(k)} = (\mathbf{K}_{uu}^{(k)})^{-1} (\mathbf{F}_{ue}^{(k)} - \mathbf{F}_{ui}^{(k)}) . \quad (3.139)$$

Consequently, the displacement vector in iteration $k + 1$ can be obtained as

$$\mathbf{q}^{(k+1)} = \mathbf{q}^{(k)} + \Delta \mathbf{q}^{(k)} . \quad (3.140)$$

Updating the system matrices and vectors to $\mathbf{K}_{uu}^{(k+1)}$, $\mathbf{F}_{ue}^{(k+1)}$ and $\mathbf{F}_{ui}^{(k+1)}$ using $\mathbf{q}^{(k+1)}$, $\Delta \mathbf{q}$ can be calculated again until it converges within an accepted error ϵ as

$$\frac{\|\Delta \mathbf{q}^{(k)}\|}{\|\mathbf{q}^{(k+1)}\|} < \epsilon . \quad (3.141)$$

3.9.4 Riks-Wempner method

The RIKS-WEMPNER algorithm is one of the arc-length control methods for solving nonlinear equilibrium equations. It can be found in many books e.g. [126, 221]. There are two major strategies for searching the equilibrium point, namely along the normal plane or the spherical surface. Here, the iteration procedure of the RIKS-WEMPNER algorithm goes along the normal plane with stiffness matrices updated in every iteration, as shown in Fig. 3.8.

By introducing a proportional loading factor, the nonlinear equilibrium equation in the i^{th} iteration can be re-written as

$${}^1\bar{\mathbf{K}}_{uu}^{(i)} \Delta \mathbf{q}^{(i)} = \lambda^{(i)} \mathbf{F}_{ue}^{(i)} - {}^1\mathbf{F}_{ui}^{(i)} . \quad (3.142)$$

Here, the proportional loading factor $\lambda^{(i)}$ varies between 0 and 1. The vector $\mathbf{t}^{(i)}$, which is tangent to the equilibrium path at i^{th} iteration, is defined as

$$\mathbf{t}^{(0)} = \begin{Bmatrix} \Delta \mathbf{q}^{(0)} \\ \Delta \lambda^{(0)} \end{Bmatrix} , \quad \mathbf{t}^{(i)} = \begin{Bmatrix} \Delta \lambda^{(i)} \Delta \mathbf{q}_I^{(i)} \\ \Delta \lambda^{(i)} \end{Bmatrix} \quad (i \geq 1) . \quad (3.143)$$

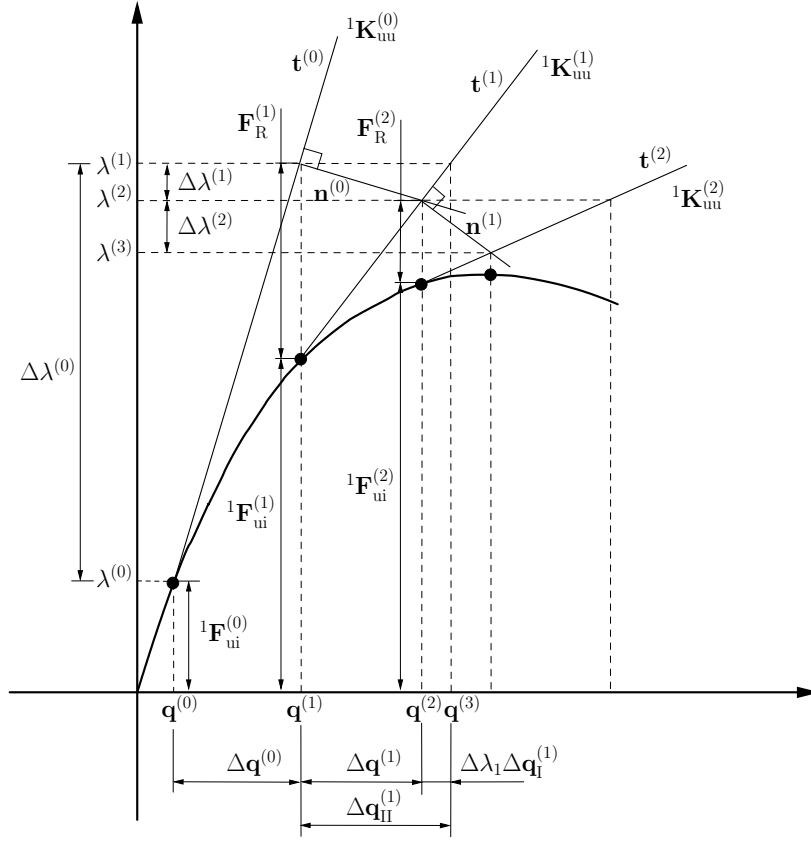


FIGURE 3.8: Schematic procedure for the RIKS-WEMPNER method

The incremental displacement vector $\Delta \mathbf{q}^{(i)}$ can be calculated by the linearized equilibrium equation as

$${}^1\bar{\mathbf{K}}_{uu}^{(i)} \Delta \mathbf{q}^{(i)} = \Delta \lambda^{(i)} \mathbf{F}_{ue}^{(i)}. \quad (3.144)$$

The searching orientation vector $\mathbf{n}^{(i)}$, which is normal to the tangent vector $\mathbf{t}^{(i)}$, can be defined as

$$\mathbf{n}^{(i)} = \begin{Bmatrix} \Delta \mathbf{q}^{(i+1)} \\ -\Delta \lambda^{(i+1)} \end{Bmatrix}. \quad (3.145)$$

The initial increment of the loading factor $\Delta \lambda^{(0)}$ is prescribed, and the next incremental loading factor can be obtained by using the constraints of $\mathbf{n}^{(i)} \cdot \mathbf{t}^{(i)} = 0$ as

$$\Delta \lambda^{(i)} = \begin{cases} \frac{(\Delta \mathbf{q}^{(0)})^T \Delta \mathbf{q}_{II}^{(1)}}{(\Delta \mathbf{q}^{(0)})^T \Delta \mathbf{q}_I^{(1)} + \Delta \lambda^{(0)}} & i = 1 \\ \frac{(\Delta \mathbf{q}_I^{(i-1)})^T \Delta \mathbf{q}_{II}^{(i)}}{(\Delta \mathbf{q}_I^{(i-1)})^T \Delta \mathbf{q}_I^{(i)} + 1} & i \geq 2 \end{cases} \quad (3.146)$$

with

$$\begin{aligned}\Delta \mathbf{q}_I^{(i)} &= ({}^1\mathbf{K}_{uu}^{(i)})^{-1} \mathbf{F}_{ue}^{(i)}, \\ \Delta \mathbf{q}_{II}^{(i)} &= ({}^1\mathbf{K}_{uu}^{(i)})^{-1} \left({}^1\mathbf{K}_{uu}^{(i-1)} \Delta \mathbf{q}^{(i-1)} + {}^1\mathbf{F}_{ui}^{(i-1)} - {}^1\mathbf{F}_{ui}^{(i)} \right).\end{aligned}\quad (3.147)$$

The arc length of the first loading case can be calculated by

$$\Delta S_0 = \|\mathbf{t}^{(0)} \cdot \mathbf{t}^{(0)}\| = \sqrt{(\Delta \lambda^{(0)})^2 + (\Delta \mathbf{q}^{(0)})^T \cdot \Delta \mathbf{q}^{(0)}}. \quad (3.148)$$

The arc length can be fixed during all the loading cases, but it can be updated as well according to the desired and the actual number of iterations by using the updating equation given as

$$\Delta S_i = \Delta S_{i-1} \sqrt{\frac{I_{des}}{I_{i-1}}}, \quad (3.149)$$

where ΔS_{i-1} is the current arc length, and ΔS_i represents the updated one. Furthermore, I_{des} and I_{i-1} are the desired and the current number of iteration, respectively. Therefore, the first incremental loading factor for the next loading step can be obtained as

$$\Delta \lambda_i^{(0)} = \frac{\pm \Delta S_i}{\sqrt{1 + (\Delta \mathbf{q}_I^{(0)})_i^T \cdot (\Delta \mathbf{q}_I^{(0)})_i}}. \quad (3.150)$$

Here, the sign of ΔS_i can be determined by the stiffness matrix.

3.10 Summary

This chapter developed static and dynamic geometrically nonlinear FE models for piezoelectric integrated smart structures. In the FE models, linear piezoelectric constitutive equations and constant electric field through the thickness were considered. Four types of shell elements were developed for composite or piezoelectric laminated thin-walled structures. In LRT56 FE models, EULER angles were used to represent unrestricted finite rotations in shell structures. In the end of this chapter, several numerical algorithms for solving static and dynamic equations were discussed.

Chapter 4

Active vibration control

In this chapter, several control strategies are discussed and developed based on linear piezoelectric coupled dynamic FE models of smart structures. The linear FE models are developed based on the FOSD hypothesis in consideration of linear strain-displacement relations which have been discussed in Chapter 2. Due to the assumptions of *small strains* and *weak electric potential*, linear constitutive equations and constant electric field through the thickness, which have been discussed in Chapter 3, are adopted in the present linear FE models. Since FE models usually contain a large number of DOFs, a decomposition and reduction technique is employed to reduce the size of FE models. Based on the decomposed and reduced model, a state space model can be constructed for control design.

From the literature, it can be noticed that the majority of papers applied very simple control schemes, like, e.g. negative velocity proportional feedback or LYAPUNOV feedback control, for vibration suppression of smart structures. Most of them did not take disturbances into account in their models, which are the main cause of vibrations. However, in control engineering, disturbances can be estimated by various observers, e.g. PI observer [199, 200], *full-* and *reduced-order* observer [206, 207] and *sliding-mode* observer [208, 209]. It can be observed that very few papers have considered the estimated disturbances being fed back to the controller as measured signals.

The aim of this part is to develop a DR control with PI observer, based on the work done by Müller *et al.* [199–202, 205], for vibration suppression of smart structures. In order to improve the dynamic behavior of the observer, the PI observer is then extended to the GPI observer, in which the fictitious model of disturbances can be constructed by any nonlinear function. For the purpose of comparison, PID, LQR and LQG control schemes are implemented and discussed as well. All the developed and

implemented controllers will be validated by applying different disturbances and by several examples of smart structures in Chapter 6.

4.1 Linear FE dynamic model

Geometrically linear FE model is the simplest special case of the nonlinear model presented in Chapter 2 and Chapter 3. Due to the assumptions of small strains and small deflections, linear FE formulations can be obtained much easier and simpler. In order to get quick understanding of modeling of smart structures in control problems, a brief description of the linear FE model is presented in the first part of this chapter.

4.1.1 Linear strain-displacement relations

From the geometry of plate and shell structures, the covariant components of the displacement vector in the shell space based on the FOSD hypothesis can be expressed by only five parameters as

$$\begin{aligned} v_\alpha(\Theta^1, \Theta^2, \Theta^3) &= v_\alpha^0(\Theta^1, \Theta^2) + \Theta^3 v_\alpha^1(\Theta^1, \Theta^2), \\ v_3(\Theta^1, \Theta^2, \Theta^3) &= v_3^0(\Theta^1, \Theta^2), \end{aligned} \quad (4.1)$$

in which v_1^0, v_2^0, v_3^0 are the translational displacements at the mid-surface, and v_1^1, v_2^1 are the two rotational displacements about the Θ^2 - and Θ^1 -axis, respectively. Defining the displacement vector of an arbitrary point in the shell space as

$$\mathbf{u} = \left\{ v_1 \quad v_2 \quad v_3 \right\}^T. \quad (4.2)$$

equation (4.1) can be expressed in matrix form as

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 & \Theta^3 & 0 \\ 0 & 1 & 0 & 0 & \Theta^3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1^0 \\ v_2^0 \\ v_3^0 \\ v_1^1 \\ v_2^1 \end{Bmatrix} = \mathbf{Z}_u \mathbf{v}. \quad (4.3)$$

In Chapter 2, the in-plane and transverse shear components of the GREEN-LAGRANGE strain tensor have been obtained based on the FOSD hypothesis with neglecting the

transverse normal strain in thin-walled structures, which are given by

$$\begin{aligned}\varepsilon_{\alpha\beta} &= \overset{0}{\varepsilon}_{\alpha\beta} + \Theta^3 \overset{1}{\varepsilon}_{\alpha\beta} + (\Theta^3)^2 \overset{2}{\varepsilon}_{\alpha\beta} , \\ \varepsilon_{\alpha 3} &= \overset{0}{\varepsilon}_{\alpha 3} + \Theta^3 \overset{1}{\varepsilon}_{\alpha 3} .\end{aligned}\tag{4.4}$$

The longitudinal/shear strains, the bending/torsion strains, the correction strains and the transverse shear strains are respectively given by

$$\begin{aligned}2\overset{0}{\varepsilon}_{\alpha\beta} &= \overset{0}{\varphi}_{\alpha\beta} + \overset{0}{\varphi}_{\beta\alpha} , \\ 2\overset{1}{\varepsilon}_{\alpha\beta} &= \overset{1}{v}_{\alpha|\beta} - b_{\beta}^{\lambda} \overset{0}{\varphi}_{\lambda\alpha} + \overset{1}{v}_{\beta|\alpha} - b_{\alpha}^{\delta} \overset{0}{\varphi}_{\delta\beta} , \\ 2\overset{2}{\varepsilon}_{\alpha\beta} &= -b_{\beta}^{\lambda} \overset{1}{v}_{\lambda|\alpha} - b_{\alpha}^{\delta} \overset{1}{v}_{\delta|\beta} , \\ 2\overset{0}{\varepsilon}_{\alpha 3} &= \overset{1}{v}_{\alpha} + \overset{0}{v}_{3,\alpha} + b_{\alpha}^{\delta} \overset{0}{v}_{\delta} , \\ 2\overset{1}{\varepsilon}_{\alpha 3} &= 0 ,\end{aligned}\tag{4.5}$$

with

$$\overset{0}{\varphi}_{\lambda\alpha} = \overset{0}{v}_{\lambda|\alpha} - b_{\lambda\alpha} \overset{0}{v}_3 ,\tag{4.6}$$

$$\overset{n}{v}_{\lambda|\alpha} = \overset{n}{v}_{\lambda,\alpha} - \Gamma_{\lambda\alpha}^{\delta} \overset{n}{v}_{\delta} ,\tag{4.7}$$

where $b_{\lambda\alpha}$ and b_{α}^{λ} are the covariant and mixed components of the curvature tensor, $\Gamma_{\lambda\alpha}^{\delta}$ denote the CHRISTOFFEL symbols of the second kind, and $\square_{|\alpha}$ represents the covariant derivative with respect to Θ^{α} . Additionally, the Greek indices represent the numbers 1 or 2, the header n assumes 0 or 1.

The strain vector can be transformed to the resultant strain vector as (see equation (3.34))

$$\boldsymbol{\varepsilon} = \mathbf{H}_s \mathbf{S} .\tag{4.8}$$

Further, the strain-displacement relations can be expressed in matrix form as

$$\mathbf{S} = \mathbf{A}_0 \boldsymbol{\theta}\tag{4.9}$$

with the resultant displacement vector

$$\boldsymbol{\theta} = \left\{ \overset{0}{v}_{1,1} \ \overset{0}{v}_{1,2} \ \overset{0}{v}_{2,1} \ \overset{0}{v}_{2,2} \ \overset{0}{v}_{3,1} \ \overset{0}{v}_{3,2} \ \overset{1}{v}_{1,1} \ \overset{1}{v}_{1,2} \ \overset{1}{v}_{2,1} \ \overset{1}{v}_{2,2} \ \overset{0}{v}_1 \ \overset{0}{v}_2 \ \overset{0}{v}_3 \ \overset{1}{v}_1 \ \overset{1}{v}_2 \right\}^T ,\tag{4.10}$$

where $\mathbf{A}_0$ is the linear strain-displacement coefficient matrix. The resultant displacement vector $\boldsymbol{\theta}$ can be expressed by nodal DOFs using normalization as

$$\boldsymbol{\theta} = \mathbf{K}_{\text{th}} \hat{\boldsymbol{\theta}}_{\text{u}} , \quad (4.11)$$

in which

$$\hat{\boldsymbol{\theta}}_{\text{u}} = \left\{ u_{,1} \ u_{,2} \ v_{,1} \ v_{,2} \ w_{,1} \ w_{,2} \ \varphi_{1,1} \ \varphi_{1,2} \ \varphi_{2,1} \ \varphi_{2,2} \ u \ v \ w \ \varphi_1 \ \varphi_2 \right\} . \quad (4.12)$$

Again using the normalization, the physical quantity of the strain vector can be obtained as

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{K}_{\text{e}} \boldsymbol{\varepsilon} . \quad (4.13)$$

Here the transformation matrix $\mathbf{K}_{\text{e}}$ is produced by normalization. With the help of equations (3.59) and (4.11), the normalized strain vector will become

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{K}_{\text{e}} \mathbf{H}_{\text{s}} \mathbf{A}_0 \mathbf{K}_{\text{th}} \mathbf{N}_{\text{t}} \mathbf{q} = \mathbf{B}_{\text{u}} \mathbf{q} , \quad (4.14)$$

where $\mathbf{B}_{\text{u}}$ is the strain field matrix.

4.1.2 Dynamic FE model

In order to obtain a dynamic FE model of smart structures, Hamilton's principle is applied, which is given by

$$\int_{t_1}^{t_2} (\delta T - \delta W_{\text{int}} + \delta W_{\text{ext}}) dt = 0 , \quad (4.15)$$

where the variation of the kinetic energy, δT , the internal virtual work, δW_{int} , and the external virtual work, δW_{ext} , are defined as

$$\delta T = - \int_V \rho \delta \hat{\mathbf{u}}^T \ddot{\hat{\mathbf{u}}} dV , \quad (4.16)$$

$$\delta W_{\text{int}} = \int_V (\delta \hat{\boldsymbol{\varepsilon}}^T \hat{\boldsymbol{\sigma}} - \delta \hat{\mathbf{E}}^T \hat{\mathbf{D}}) dV , \quad (4.17)$$

$$\delta W_{\text{ext}} = \int_V \delta \hat{\mathbf{u}}^T \mathbf{f}_{\text{b}} dV + \int_{\Omega} \delta \hat{\mathbf{u}}^T \mathbf{f}_{\text{s}} d\Omega + \delta \hat{\mathbf{u}}^T \mathbf{f}_{\text{c}} - \int_{\Omega} \delta \boldsymbol{\phi}^T \boldsymbol{\varrho} d\Omega - \delta \boldsymbol{\phi}^T \mathbf{Q}_{\text{c}} . \quad (4.18)$$

Substituting the constitutive equations given in (3.27)-(3.28) and the displacement formulation in (4.3) into (4.16)-(4.18), one obtains a linear electro-mechanically coupled

dynamic FE model, including an equation of motion and a sensor equation, which are respectively given as

$$\mathbf{M}_{uu}\ddot{\mathbf{q}} + \mathbf{C}_{uu}\dot{\mathbf{q}} + \mathbf{K}_{uu}\mathbf{q} + \mathbf{K}_{u\phi}\phi_a = \mathbf{F}_{ue} , \quad (4.19)$$

$$\mathbf{K}_{\phi u}\mathbf{q} + \mathbf{K}_{\phi\phi}\phi_s = \mathbf{G}_{\phi e} . \quad (4.20)$$

Here, $\mathbf{M}_{uu}$, $\mathbf{C}_{uu}$, $\mathbf{K}_{uu}$, $\mathbf{K}_{u\phi}$, $\mathbf{K}_{\phi u}$, $\mathbf{K}_{\phi\phi}$, $\mathbf{F}_{ue}$, $\mathbf{G}_{\phi e}$, $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$, ϕ_a and ϕ_s have the same physical meanings as described in Section 3.8.1, but the matrices and vectors are obtained by the linear shell theory.

The mass matrix, the stiffness matrix, the piezoelectric coupled stiffness matrix, the coupled capacity matrix, the piezoelectric capacity matrix, the external force vector and the external electric charge vector are respectively calculated as

$$\mathbf{M}_{uu} = \int_V \rho \mathbf{N}_v^T \mathbf{Z}_u^T \mathbf{Z}_u \mathbf{N}_v dV , \quad (4.21)$$

$$\mathbf{K}_{uu} = \int_V \mathbf{B}_u^T \mathbf{c} \mathbf{B}_u dV , \quad (4.22)$$

$$\mathbf{K}_{u\phi} = \mathbf{K}_{\phi u}^T = - \int_V \mathbf{B}_u^T \mathbf{e}^T \mathbf{B}_\phi dV , \quad (4.23)$$

$$\mathbf{K}_{\phi\phi} = - \int_V \mathbf{B}_\phi^T \boldsymbol{\epsilon} \mathbf{B}_\phi dV , \quad (4.24)$$

$$\mathbf{F}_{ue} = \int_V \mathbf{N}_v^T \mathbf{Z}_u^T \mathbf{f}_b dV + \int_\Omega \mathbf{N}_v^T \mathbf{Z}_u^T \mathbf{f}_s d\Omega + \mathbf{N}_v^T \mathbf{Z}_u^T \mathbf{f}_c , \quad (4.25)$$

$$\mathbf{G}_{\phi e} = - \int_\Omega \boldsymbol{\rho} d\Omega - \mathbf{Q}_c , \quad (4.26)$$

where ρ is the density, $\mathbf{N}_v$ the shape function matrix, $\mathbf{B}_u$ the strain field matrix, and $\mathbf{B}_\phi$ the electric field matrix. Furthermore, $\mathbf{f}_b$, $\mathbf{f}_s$ and $\mathbf{f}_c$ are respectively the vectors of body, surface and concentrated force, $\boldsymbol{\rho}$ and $\mathbf{Q}_c$ denote the surface and concentrated electric charge vectors, respectively.

The damping matrix $\mathbf{C}_{uu}$ in (4.19) is calculated using RAYLEIGH damping coefficients computation method described in Section 3.8.1, which is linear with respect to mass and stiffness matrices.

4.2 State space model

4.2.1 Model decomposition and reduction

Due to the discretization by elements, one always obtains large size dynamic FE models for smart structures, resulting in high costs of computation time. In order to retain the main dynamic characteristics, a truncated modal matrix [215] $\mathbf{S}_r$ including the first r modes, which is usually known as mode shape matrix, is introduced to decompose and reduce the modes of FE models. Using the truncated modal matrix, the nodal displacement vector $\mathbf{q}$ can be transformed to the reduced modal displacement vector $\mathbf{z}_r$ as

$$\mathbf{q} = \mathbf{S}_r \mathbf{z}_r . \quad (4.27)$$

Substituting (4.27) into (4.19) and left-multiplying by the transposed modal matrix one obtains the decomposed and reduced equation of motion as

$$\tilde{\mathbf{M}}_{uu} \ddot{\mathbf{z}}_r + \tilde{\mathbf{C}}_{uu} \dot{\mathbf{z}}_r + \tilde{\mathbf{K}}_{uu} \mathbf{z}_r = \mathbf{S}_r^T \mathbf{F}_{ue} - \mathbf{S}_r^T \mathbf{K}_{u\phi} \boldsymbol{\phi}_a , \quad (4.28)$$

where $\tilde{\mathbf{M}}_{uu}$, $\tilde{\mathbf{C}}_{uu}$ and $\tilde{\mathbf{K}}_{uu}$ are the modal mass, damping and stiffness matrices, respectively, which are diagonal. They are given as

$$\tilde{\mathbf{M}}_{uu} = \mathbf{S}_r^T \mathbf{M}_{uu} \mathbf{S}_r , \quad (4.29)$$

$$\tilde{\mathbf{C}}_{uu} = \mathbf{S}_r^T \mathbf{C}_{uu} \mathbf{S}_r , \quad (4.30)$$

$$\tilde{\mathbf{K}}_{uu} = \mathbf{S}_r^T \mathbf{K}_{uu} \mathbf{S}_r . \quad (4.31)$$

Further we assume that no extra external electric charges are applied on piezoelectric patches acting as sensors, which means $\mathbf{G}_{\phi e} = \mathbf{0}$. Again using equation (4.27), the sensor equation given in (4.20) can be expressed by modal coordinates as

$$\boldsymbol{\phi}_s = -\mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \mathbf{S}_r \mathbf{z}_r . \quad (4.32)$$

Modal truncation method is used for example. But other more advanced methods could be applied like those in [222–224].

4.2.2 State space description

In control engineering, two kinds of models are frequently used for controller design: transfer function and state space model. The former one can be only used for Single-Input-Single-Output (SISO) systems, but the latter one is a general description, which is not restricted by the number of inputs and outputs. According to the decomposed and reduced dynamic FE model given in equations (4.28) and (4.32), a state space model can be constructed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) , \quad (4.33)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) , \quad (4.34)$$

$$\mathbf{z}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) . \quad (4.35)$$

Here, equations (4.33), (4.34) and (4.35) represent the state equation, the measured output equation and the controlled output equation, respectively. In the state space model, $\mathbf{u}(t)$, $\mathbf{y}(t)$, $\mathbf{z}(t)$ and $\mathbf{x}(t)$ denote the system input vector (manipulated vector or control input vector), the measured output vector, the controlled output vector, and the state vector. Additionally, $\mathbf{A}$ denotes the system matrix, $\mathbf{B}$ the control matrix and $\mathbf{C}$ the system output matrix. Concerning smart structures, the system output can be displacements or sensor voltages, and the system input can be actuation voltages or forces, as shown in Table 4.1. They can be determined by system output and control matrices, respectively. The system input and output can be chosen from Table 4.1,

TABLE 4.1: Definition of system input/output and the corresponding matrices

Input or output		Definition	
input	force	$\mathbf{u} = \mathbf{F}_{\text{ue}}$	$\mathbf{B} = \mathbf{B}_{\text{f}}$
	voltage	$\mathbf{u} = \boldsymbol{\phi}_{\text{a}}$	$\mathbf{B} = \mathbf{B}_{\text{v}}$
output	displacement	$\mathbf{y} = \mathbf{q}$	$\mathbf{C} = \mathbf{C}_{\text{d}}$
	voltage	$\mathbf{y} = \boldsymbol{\phi}_{\text{s}}$	$\mathbf{C} = \mathbf{C}_{\text{v}}$

in which $\mathbf{B}_{\text{f}}$, $\mathbf{B}_{\text{v}}$, $\mathbf{C}_{\text{d}}$ and $\mathbf{C}_{\text{v}}$ are the force control matrix, the actuation voltage control matrix, the displacement output matrix and the sensor output voltage matrix, respectively.

Further we define the state vector as

$$\mathbf{x} = \begin{Bmatrix} \mathbf{z}_{\text{r}} \\ \dot{\mathbf{z}}_{\text{r}} \end{Bmatrix} . \quad (4.36)$$

Therefore, the system matrix can be derived as

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\tilde{\mathbf{M}}_{uu}^{-1} \tilde{\mathbf{K}}_{uu} & -\tilde{\mathbf{M}}_{uu}^{-1} \tilde{\mathbf{C}}_{uu} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Omega}_r^2 & -2\boldsymbol{\Lambda}_r \boldsymbol{\Omega}_r \end{bmatrix}, \quad (4.37)$$

in which $\boldsymbol{\Omega}_r$ and $\boldsymbol{\Lambda}_r$ are diagonal matrices composed of the first r eigen-frequencies and damping ratios, respectively. Additionally, the control matrices and the output matrices can be obtained as

$$\mathbf{B}_v = \begin{bmatrix} \mathbf{0} \\ -\tilde{\mathbf{M}}_{uu}^{-1} \mathbf{S}_r^T \mathbf{K}_{u\phi} \end{bmatrix}, \quad (4.38)$$

$$\mathbf{B}_f = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{M}}_{uu}^{-1} \mathbf{S}_r^T \end{bmatrix}, \quad (4.39)$$

$$\mathbf{C}_v = \begin{bmatrix} -\mathbf{K}_{\phi\phi}^{-1} \mathbf{K}_{\phi u} \mathbf{S}_r & \mathbf{0} \end{bmatrix}, \quad (4.40)$$

$$\mathbf{C}_d = \begin{bmatrix} \mathbf{S}_r & \mathbf{0} \end{bmatrix}. \quad (4.41)$$

In most of the cases, the controlled output is set to $\mathbf{z}(t) = \mathbf{y}(t)$, which leads to $\mathbf{F} = \mathbf{C}$ and $\mathbf{G} = \mathbf{0}$. The controller should be designed to make the error vector as small as possible in the shortest possible amount of time. In vibration suppression, usually the desired output signal should be zero.

4.3 PID control

The theoretical analysis of a Proportional-Integral-Derivative (PID) controller was first published by the Russian-American engineer Nicolas Minorsky in 1922. Now, it is one of the most popular controllers applied in industries. In this dissertation, the PID control algorithm is implemented into vibration suppression for smart structures in order to compare with other controllers. Therefore, the theory will be briefly introduced, which also can be found in many books, e.g. [225, 226].

From the previous section, a mathematical model of smart structures in state space form can be constructed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (4.42)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \quad (4.43)$$

Here $\mathbf{z}(t) = \mathbf{y}(t)$ is considered for simplification. PID control can be realized by applying voltages on actuators, which depend on the output error $\mathbf{e}(t)$, its integral and its derivative, as shown in Fig. 4.1. The vectors $\mathbf{r}(t)$, $\mathbf{y}(t)$, $\mathbf{e}(t)$ and $\mathbf{u}(t)$ are defined as

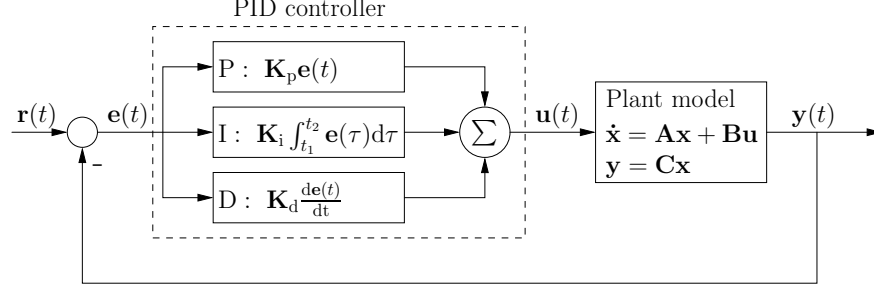


FIGURE 4.1: The sketch of PID closed-loop control system

the reference signal, the measured output, the output error, and the system input (or manipulated variable), respectively. Additionally, the output error is defined as

$$\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t) . \quad (4.44)$$

According to the strategy of PID algorithm, the manipulated variable can be defined as the sum of terms proportional to the output error, its integral and its derivative as

$$\mathbf{u}(t) = \mathbf{K}_p \mathbf{e}(t) + \mathbf{K}_i \int_0^t \mathbf{e}(\tau) d\tau + \mathbf{K}_d \frac{d\mathbf{e}(t)}{dt} . \quad (4.45)$$

Here, $\mathbf{K}_p$, $\mathbf{K}_i$ and $\mathbf{K}_d$ denote the proportional gain, the integral gain and the derivative gain, respectively. The proportional term produces a control action that is proportional to the output error. A high proportional gain results in a strong control action for a given output error. However, high proportional gain will make the system become unstable. The integral term is proportional to the sum of the error, which gives the accumulated offset that should have been corrected previously. The integral term can eliminate the residual steady-state error that occurs with a pure proportional controller. However, the integral part can also cause a larger overshoot. The derivative term is calculated by determining the rate of the error over time and multiplying this rate of change by the derivative gain. Derivative action predicts system behavior and thus improves settling time and stability of the system.

In the field of vibration control of smart structures, the reference signal is usually set to be zero, meaning that the control aim is to eliminate the vibration. Therefore, equation (4.44) becomes

$$\mathbf{e}(t) = -\mathbf{y}(t) = -\mathbf{C}\mathbf{x} . \quad (4.46)$$

Due to the integral part, a new state variable has to be introduced to deduce the time-continuous closed-loop state space model with PID controller. Assuming the new state variable as

$$\boldsymbol{\vartheta}(t) = \int_0^t \mathbf{y}(\tau) \, d\tau, \quad (4.47)$$

leads to

$$\dot{\boldsymbol{\vartheta}}(t) = \mathbf{y}(t). \quad (4.48)$$

Substitution of (4.46) and (4.47) into (4.45) yields

$$\mathbf{u} = -\mathbf{K}_p \mathbf{C} \mathbf{x} - \mathbf{K}_i \boldsymbol{\vartheta} - \mathbf{K}_d \mathbf{C} \dot{\mathbf{x}}. \quad (4.49)$$

Extending the state vector to

$$\tilde{\mathbf{x}} = \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\vartheta} \end{Bmatrix}, \quad (4.50)$$

the closed-loop state space model with PID controller can be obtained in terms of the extended state vector $\tilde{\mathbf{x}}$ as

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix} \tilde{\mathbf{x}} = \tilde{\mathbf{A}} \tilde{\mathbf{x}}, \quad (4.51)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}}, \quad (4.52)$$

with

$$\tilde{\mathbf{A}}_{11} = (\mathbf{I} + \mathbf{B} \mathbf{K}_d \mathbf{C})^{-1} (\mathbf{A} - \mathbf{B} \mathbf{K}_p \mathbf{C}), \quad (4.53)$$

$$\tilde{\mathbf{A}}_{12} = (\mathbf{I} + \mathbf{B} \mathbf{K}_d \mathbf{C})^{-1} (-\mathbf{B} \mathbf{K}_i), \quad (4.54)$$

$$\tilde{\mathbf{A}}_{21} = \mathbf{C}, \quad (4.55)$$

$$\tilde{\mathbf{A}}_{22} = \mathbf{0}. \quad (4.56)$$

The plant system input signal, which is the output of the PID controller, can be calculated as

$$\mathbf{u} = (\mathbf{I} + \mathbf{K}_d \mathbf{C} \mathbf{B})^{-1} \begin{bmatrix} -(\mathbf{K}_p \mathbf{C} + \mathbf{K}_d \mathbf{C} \mathbf{A}) & -\mathbf{K}_i \end{bmatrix} \tilde{\mathbf{x}}. \quad (4.57)$$

4.4 Optimal control

Optimal control is concerned with operating a dynamic system at minimum cost. The main resulting theory is Linear Quadratic Regulator (LQR) control, which is a full state feedback control. Because all state variables in the system usually cannot be measured, the implementation of LQR control is limited in many cases. In order to overcome this fatal disadvantage, Linear Quadratic Gaussian (LQG) observer is proposed to estimate the state variables according to the measured signals. The combined control strategy, LQR/LQG control (which is also called LQG control) uses LQG as an observer to estimate the state variables and LQR as an optimal solution to produce a control gain by minimizing the cost function. These two control strategies can be found in [227–229], among many others.

4.4.1 LQR control

An LQR control strategy can be developed from a state space model of smart structures, which is described as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) , \quad (4.58)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) , \quad (4.59)$$

$$\mathbf{z}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) . \quad (4.60)$$

Further assuming all state variables can be measured, an optimized control gain can be obtained by minimizing the cost function. Therefore, the manipulated vector can be calculated and applied to the system. The process of an LQR controlled system is shown in Fig. 4.2.

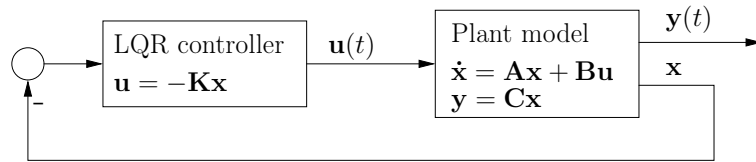


FIGURE 4.2: The sketch of LQR closed-loop control system

The cost function can be defined as the sum of the energy of the controlled output and the system input, which is expressed as

$$J_{LQR} = \int_{t_0}^{\infty} (\mathbf{z}(t)^T \bar{\mathbf{Q}} \mathbf{z}(t) + \rho \mathbf{u}(t)^T \bar{\mathbf{R}} \mathbf{u}(t)) dt . \quad (4.61)$$

Here, $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ are the weighting matrices for the controlled output energy and the plant input energy, respectively, which are symmetric positive definite matrices. Furthermore, the positive constant coefficient ρ is used for trading-off between the energy of the controlled output and the plant input. Actually, there is no method to calculate the exact weighting matrices, but they can be approximated by BRYSON'S rule [229] as

$$\bar{Q}_{ii} = \frac{1}{\max(|z_i|^2)}, \quad \bar{R}_{ii} = \frac{1}{\max(|u_i|^2)}, \quad (4.62)$$

where $|\square|$ denotes the absolute value. Substituting $\mathbf{z}(t)$ given in (4.60) into (4.61) one obtains a general form of the cost function expressed in terms of the state vector $\mathbf{x}(t)$ and the plant input vector $\mathbf{u}(t)$ as

$$J_{LQR} = \int_{t_0}^{\infty} (\mathbf{x}(t)^T \mathbf{Q}_r \mathbf{x}(t) + \mathbf{u}(t)^T \mathbf{R}_r \mathbf{u}(t) + 2\mathbf{x}(t)^T \mathbf{N}_r \mathbf{u}(t)) dt, \quad (4.63)$$

with

$$\mathbf{Q}_r = \mathbf{F}^T \bar{\mathbf{Q}} \mathbf{F}, \quad \mathbf{R}_r = \mathbf{G}^T \bar{\mathbf{Q}} \mathbf{G} + \rho \bar{\mathbf{R}}, \quad \mathbf{N}_r = \mathbf{F}^T \bar{\mathbf{Q}} \mathbf{G}. \quad (4.64)$$

In the state-feedback version of LQR problem, all the state variables are assumed to be measured, such that the plant can be controlled. The manipulated vector of the full state feedback LQR optimal control can be designed as

$$\mathbf{u} = -\mathbf{K}\mathbf{x}, \quad (4.65)$$

where the control gain is given by

$$\mathbf{K} = \mathbf{R}_r^{-1} (\mathbf{B}^T \mathbf{P} + \mathbf{N}_r^T). \quad (4.66)$$

Here, the symmetric positive definite matrix $\mathbf{P}$ is the solution of the following Algebraic RICCATI Equation (ARE) as

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} + \mathbf{Q}_r - (\mathbf{P} \mathbf{B} + \mathbf{N}_r) \mathbf{R}_r^{-1} (\mathbf{B}^T \mathbf{P} + \mathbf{N}_r^T) = \mathbf{0}. \quad (4.67)$$

Substituting equation (4.65) into the state space model in (4.58)-(4.59) yields the closed-loop system with consideration of an LQR controller as

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}, \quad (4.68)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}. \quad (4.69)$$

4.4.2 LQG control

LQG observer

As mentioned in Section 4.4.1, the LQR solution is a full state feedback control scheme, which requires all state variables to be measurable. However, in most of the cases, state variables cannot be completely measured in reality, since some of them are difficult to be observed or even impossible to be detected. Therefore, one possibility is using an observer to re-construct the state variables by using the measured system output signals, as shown in Fig. 4.3. Instead of the LQR control law given in equation (4.65),

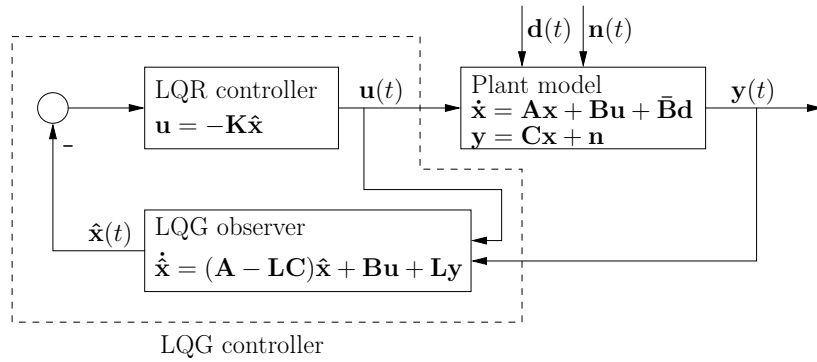


FIGURE 4.3: The sketch of LQG closed-loop control system

the manipulated vector of an LQG control can be defined as

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t) , \quad (4.70)$$

where the estimated state vector $\hat{\mathbf{x}}(t)$ can be achieved by using LUENBERGER observer architecture as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}}) , \quad (4.71)$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} . \quad (4.72)$$

Here, $\mathbf{L}$ denotes the observer gain matrix. Theoretically, the larger the observer gain is, the faster the error converges to zero. However, large observer gain will magnify the measurement noises. In light of this, the plant model for design of LQG controller

should consider both disturbances and measurement noises, which becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \bar{\mathbf{B}}\mathbf{d}(t) , \quad (4.73)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{n}(t) , \quad (4.74)$$

$$\mathbf{z}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) , \quad (4.75)$$

where $\mathbf{d}(t)$ is the disturbance noise vector applied on the plant, $\bar{\mathbf{B}}$ the disturbance influence matrix, and $\mathbf{n}(t)$ denotes the measurement noise vector.

Subsequently, the estimation error between the plant state vector and the estimated state vector is introduced as

$$\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}} . \quad (4.76)$$

Therefore, the time rate of the estimation error, or the error dynamic model, can be obtained as

$$\dot{\mathbf{e}}_x = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}_x + \bar{\mathbf{B}}\mathbf{d} - \mathbf{L}\mathbf{n} , \quad (4.77)$$

from which it can be seen that the observer gain matrix $\mathbf{L}$ not only influences the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$, but also magnifies the measurement noises. In order to balance between the convergence speed of the error dynamics and the magnification of noises, an optimal observer gain can be determined by LQG method. The resulting observer is called KALMAN-BUCY filter or *linear quadratic estimator* [229] in the literature.

The observer gain is designed to minimize the asymptotic expectation value of the estimation error given as

$$J_{LQG} = \lim_{t \rightarrow \infty} E(\mathbf{e}_x(t)^T \mathbf{e}_x(t)) , \quad (4.78)$$

where $E(\square)$ indicates the expectation of $\square$. Furthermore, $\mathbf{d}(t)$ and $\mathbf{n}(t)$ are the zero-mean GAUSSIAN noise with the covariances:

$$E(\mathbf{d}(t_1), \mathbf{d}(t_2)) = \mathbf{Q}_g \delta(t_1 - t_2) , \quad (4.79)$$

$$E(\mathbf{n}(t_1), \mathbf{n}(t_2)) = \mathbf{R}_g \delta(t_1 - t_2) . \quad (4.80)$$

Based on the criteria given in (4.78) one obtains the optimized observer gain as

$$\mathbf{L} = \mathbf{P}\mathbf{C}^T \mathbf{R}_g^{-1} , \quad (4.81)$$

where $\mathbf{P}$ is a positive definite matrix obtained from the observer algebraic RICCATI equation:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \bar{\mathbf{B}}\mathbf{Q}_g\bar{\mathbf{B}}^T - \mathbf{P}\mathbf{C}^T\mathbf{R}_g^{-1}\mathbf{C}\mathbf{P} = \mathbf{0}. \quad (4.82)$$

If a positive definite $\mathbf{P}$ exists, the error dynamic model defined in (4.77) is asymptotically stable.

LQR/LQG feedback control

Since the plant state variables are difficult or even impossible to be measured, the estimated ones are fed back to the LQG controller instead of the plant state variables. Extending the state vector to

$$\tilde{\mathbf{x}} = \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \end{Bmatrix}, \quad (4.83)$$

the manipulated vector $\mathbf{u}$ can be re-written in terms of the extended state vector as

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{e}_x) = \begin{bmatrix} -\mathbf{K} & \mathbf{K} \end{bmatrix} \tilde{\mathbf{x}}. \quad (4.84)$$

Substituting equation (4.84) into the state equation defined in (4.73) yields

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}_x + \bar{\mathbf{B}}\mathbf{d}. \quad (4.85)$$

Therefore, the closed-loop system with LQR/LQG controller can be obtained as

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}}_x \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \end{Bmatrix} + \begin{bmatrix} \bar{\mathbf{B}} & \mathbf{0} \\ \bar{\mathbf{B}} & -\mathbf{L} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{n} \end{Bmatrix}, \quad (4.86)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \end{Bmatrix} + \mathbf{n}, \quad (4.87)$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{F} - \mathbf{G}\mathbf{K} & \mathbf{G}\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \end{Bmatrix}. \quad (4.88)$$

4.5 Disturbance rejection control

In the previous sections, it can be noticed that the PID and LQR control methods do not take any disturbance into account, both of which are designed based on ideal plant models. In LQG control scheme, plant disturbances and measurement noises are considered, which refer to random signals with relatively small amplitudes. However,

if the structures are subjected to random or periodic disturbances (forces or voltages), which, in contrast to noises, are varying smoothly, they will be forced to vibrate. The control schemes mentioned above can suppress the free vibration very well. Unfortunately, the forced vibrations are slightly damped by those controllers, since none of them considers this kind of disturbances. In this section, a DR (Disturbance Rejection) control [205, 230] will be proposed and developed for vibration suppression of smart structures by using PI observer. Additionally, the PI observer is extended to the GPI observer in order to obtain better dynamic properties. The schematic flow of the proposed DR control system is presented in Fig. 4.4.

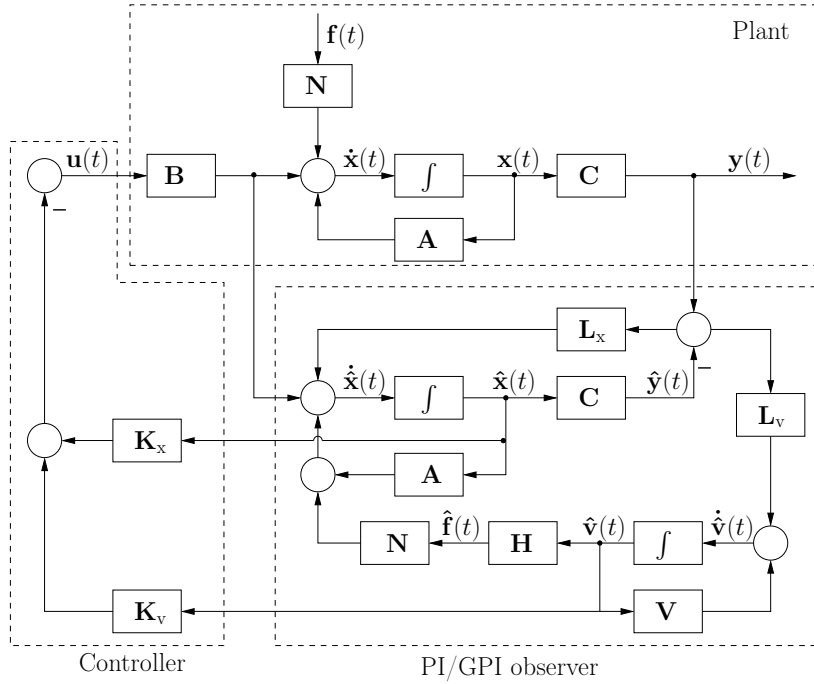


FIGURE 4.4: The sketch of DR closed-loop control system with PI/GPI observer

4.5.1 Problem statement

Considering a smart structure which is under unknown disturbance forces at specific positions or unknown disturbance voltages at specific piezoelectric patches, the plant state space model can be constructed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}\mathbf{f}(t), \quad (4.89)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad (4.90)$$

$$\mathbf{z}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t), \quad (4.91)$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$ and $\mathbf{y}(t)$ denote the n -dimensional state vector, the r -dimensional control input vector, and the m -dimensional measurement vector, respectively. The vector $\mathbf{f}(t)$ represents the p -dimensional unknown disturbance vector, $\mathbf{N}$ is the disturbance influence matrix. Additionally, the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are the system matrix, the control input matrix and the system output matrix, respectively. The rank of these matrices are

$$\text{rank } \mathbf{A} = n, \quad \text{rank } \mathbf{B} = r, \quad \text{rank } \mathbf{C} = m, \quad \text{rank } \mathbf{N} = p. \quad (4.92)$$

4.5.2 Fictitious model of disturbances

Basically, any nonlinear disturbance $\mathbf{f}(t)$ can be exactly expressed as

$$\mathbf{f}(t) = \mathbf{H}\mathbf{v}(t) + \mathbf{\Delta}(t), \quad (4.93)$$

where $\mathbf{H}$ is a coefficient matrix, $\mathbf{v}(t)$ a finite dimensional vector of base functions, and $\mathbf{\Delta}(t)$ the residual error. Here, $\mathbf{\Delta}(t)$ is assumed to be small, such that it can be neglected in most cases. In such a way, the dynamic fictitious model of unknown disturbances will be linearly approximated as

$$\mathbf{f}(t) \approx \mathbf{H}\mathbf{v}(t), \quad (4.94)$$

$$\dot{\mathbf{v}}(t) = \mathbf{V}\mathbf{v}(t), \quad (4.95)$$

where the coefficient matrices $\mathbf{H}$ and $\mathbf{V}$ denote the output and system matrices of the fictitious model of disturbances. A suitable choice of the matrices $\mathbf{H}$ and $\mathbf{V}$ requires usually a good understanding of the system behavior. One simple way to construct the linear formulation of unknown disturbances is to use the FOURIER series, which is an expansion of a periodic function $\mathbf{f}(t)$ in terms of an infinite sum of sines and cosines, such that the components of an unknown disturbance vector will be expressed as [231]

$$f_i = a_{i0} + \sum_{j=1}^{\infty} \left(a_{ij} \cos(\omega_{ij}t) + b_{ij} \sin(\omega_{ij}t) \right). \quad (4.96)$$

The simplest choice for approximation of f_i is to retain only the constant terms expressed as

$$f_i \approx a_{i0}, \quad (4.97)$$

which leads to the disturbance vector $\mathbf{f}(t)$ being composed of step functions only.

Defining the following base functions given as

$$\mathbf{v}_1 = a_{10}, \quad \mathbf{v}_2 = a_{20}, \quad \dots, \quad \mathbf{v}_n = a_{p0}, \quad (4.98)$$

where p denotes the number of disturbances, yields $\mathbf{H}$ and $\mathbf{V}$ being an identity matrix and a zero matrix, respectively, as

$$\mathbf{H} = \mathbf{I}, \quad \mathbf{V} = \mathbf{0}. \quad (4.99)$$

The resulting observer is defined as PI observer [205]. It is also proved by Müller [205] that if the observer is fast enough, it will follow the disturbances even by step functions.

Moreover, the i^{th} disturbance can also be approximated by constant and cosine terms as

$$f_i \approx a_{i0} + a_{i1} \cos(\omega_{i1}t). \quad (4.100)$$

Here, ω_{ij} is the angular frequency of the cosine base functions, which is usually given as $\omega_{ij} = j$, ($j = 1, 2, \dots$). If the disturbances applied on smart structures are periodic signals with the frequencies known, ω_{ij} can be given as the disturbance frequencies, which leads to a better dynamic behavior of the observer. If only one disturbance is considered, introducing the base functions as

$$\mathbf{v}_1 = a_{10}, \quad \mathbf{v}_2 = a_{11} \cos(\omega_{11}t), \quad \mathbf{v}_3 = a_{11} \sin(\omega_{11}t), \quad (4.101)$$

one obtains the $\mathbf{H}$ and $\mathbf{V}$ matrices as

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}, \quad (4.102)$$

in which $\omega = \omega_{11}$. The fictitious model of disturbances is constructed based on nonlinear functions composed of step and cosine terms as shown in equation (4.100), which has a better dynamic behavior than that approximated only by step functions. However, it also can be constructed by other nonlinear functions, e.g. sine or polynomial ones, which will lead to $\mathbf{H} \neq \mathbf{I}$ and $\mathbf{V} \neq \mathbf{0}$ analogously.

4.5.3 Extended observer

Using the disturbance fictitious model described in (4.93), the state equation of the plant model can be re-written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{N}\mathbf{H}\mathbf{v} + \mathbf{N}\Delta. \quad (4.103)$$

Extending the state vector to

$$\tilde{\mathbf{x}} = \begin{Bmatrix} \mathbf{x} \\ \mathbf{v} \end{Bmatrix}, \quad (4.104)$$

an extended state space model for the plant system will be obtained as

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{NH} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{v} \end{Bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{N} \\ \mathbf{0} \end{bmatrix} \Delta, \quad (4.105)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{v} \end{Bmatrix}. \quad (4.106)$$

According to the classical LUENBERGER observer structure, the extended observer system can be obtained as

$$\begin{aligned} \begin{Bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{v}}} \end{Bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{NH} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_v \end{bmatrix} (\mathbf{y} - \hat{\mathbf{y}}) \\ &= \begin{bmatrix} \mathbf{A} - \mathbf{L}_x \mathbf{C} & \mathbf{NH} \\ -\mathbf{L}_v \mathbf{C} & \mathbf{V} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_v \end{bmatrix} \mathbf{y}, \end{aligned} \quad (4.107)$$

with $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$. Here, the observer gains $\mathbf{L}_x$ and $\mathbf{L}_v$ can be calculated by using classical ways, like the pole placement design method, to make the extended system asymptotically stable, if the extended system is detectable. Arbitrary eigenvalues can be realized if the system is completely observable [230]:

$$\text{rank} \begin{bmatrix} \lambda \mathbf{I}_n - \mathbf{A} & -\mathbf{NH} \\ \mathbf{0} & \lambda \mathbf{I}_s - \mathbf{V} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = n + s \quad \text{for all } \lambda \in \mathbb{C}. \quad (4.108)$$

The observability condition given in (4.108) includes complete observability of the linear part of the plant state space model in (4.89)-(4.91). The complete observability

depends on the fictitious model (4.94)-(4.95). Independent on $(\mathbf{H}, \mathbf{V})$ robust observability is obtained for

$$\text{rank} \begin{bmatrix} \lambda \mathbf{I}_n - \mathbf{A} & -\mathbf{N} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = n + p \quad \text{for all } \lambda \in \mathbb{C}. \quad (4.109)$$

If (4.109) is satisfied, then (4.108) is also satisfied. From (4.109) it is shown that the number of measurements must not be less than the number of unknown disturbances [205, 230]:

$$m \geq p. \quad (4.110)$$

From the extended observer model according to equation (4.107), the estimated state variable vectors $\dot{\hat{\mathbf{x}}}$ and $\dot{\hat{\mathbf{v}}}$ can be expressed separately as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{N}\mathbf{H}\hat{\mathbf{v}} + \mathbf{B}\mathbf{u} + \mathbf{L}_x(\mathbf{y} - \hat{\mathbf{y}}), \quad (4.111)$$

$$\dot{\hat{\mathbf{v}}} = \mathbf{V}\hat{\mathbf{v}} + \mathbf{L}_v(\mathbf{y} - \hat{\mathbf{y}}). \quad (4.112)$$

Solving the first-order linear ordinary differential equation given in (4.112) one obtains $\hat{\mathbf{v}}$ as

$$\hat{\mathbf{v}} = \int_0^t \left(\exp(\mathbf{V}(t - \tau)) \mathbf{L}_v(\mathbf{y}(\tau) - \hat{\mathbf{y}}(\tau)) \right) d\tau + \exp(\mathbf{V}t) \hat{\mathbf{v}}_0, \quad (4.113)$$

in which $\hat{\mathbf{v}}_0$ is the initial condition of the base functions at time $t = 0$, which is usually treated as zero, and $\exp(\square)$ denotes the exponential operator. Considering $\hat{\mathbf{v}}_0 = \mathbf{0}$ and substituting (4.113) into (4.111) yields

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{N}\mathbf{H} \int_0^t \left(\exp(\mathbf{V}(t - \tau)) \mathbf{L}_v(\mathbf{y}(\tau) - \hat{\mathbf{y}}(\tau)) \right) d\tau + \mathbf{L}_x(\mathbf{y} - \hat{\mathbf{y}}). \quad (4.114)$$

Here, the estimating procedure for the state variable $\mathbf{x}$ with taking the proportional and the weighted integral of the measurement error $(\mathbf{y} - \hat{\mathbf{y}})$ into account, which is a general expression of the PI observer stated in [199, 200], is defined as Generalized Proportional-Integral (GPI) observer. PI observer is a special case of GPI observer, which is available only when $\mathbf{H} = \mathbf{I}$ and $\mathbf{V} = \mathbf{0}$.

4.5.4 Estimation error dynamic analysis

In order to guarantee the stability and convergence of the observer system, the estimation error dynamic model has to be constructed and analyzed. In smart structure

systems, the estimation error vectors of the plant state variables and the disturbances are of importance, which are respectively defined as

$$\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}} , \quad (4.115)$$

$$\mathbf{e}_f = \mathbf{f} - \mathbf{H}\hat{\mathbf{v}} . \quad (4.116)$$

For simplicity, eliminating the residual error $\Delta(t)$ in (4.93), the estimation error vector of disturbance will be approximated as

$$\mathbf{e}_f = \mathbf{H}(\mathbf{v} - \hat{\mathbf{v}}) + \Delta \approx \mathbf{H}(\mathbf{v} - \hat{\mathbf{v}}) . \quad (4.117)$$

Therefore, a new estimation error vector can be introduced to represent $\mathbf{e}_f$, which is defined as

$$\mathbf{e}_v = \mathbf{v} - \hat{\mathbf{v}} . \quad (4.118)$$

So the estimation error dynamic model of the extended observer system can be obtained as

$$\begin{aligned} \begin{Bmatrix} \dot{\mathbf{e}}_x \\ \dot{\mathbf{e}}_v \end{Bmatrix} &= \begin{bmatrix} \mathbf{A} - \mathbf{L}_x\mathbf{C} & \mathbf{NH} \\ -\mathbf{L}_v\mathbf{C} & \mathbf{V} \end{bmatrix} \begin{Bmatrix} \mathbf{e}_x \\ \mathbf{e}_v \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}\Delta \\ \mathbf{0} \end{Bmatrix} \\ &= \mathbf{A}_b \begin{Bmatrix} \mathbf{e}_x \\ \mathbf{e}_v \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}\Delta \\ \mathbf{0} \end{Bmatrix} , \end{aligned} \quad (4.119)$$

where

$$\mathbf{A}_b = \mathbf{A}_e - \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_v \end{bmatrix} \mathbf{C}_e , \quad (4.120)$$

with the following abbreviations:

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{NH} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} , \quad (4.121)$$

$$\mathbf{C}_e = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} . \quad (4.122)$$

One of our control aims is to make the estimation error in (4.119) convergent to zero as fast as possible. The system matrix $\mathbf{A}_b$ of the error dynamic model determines the error dynamic behavior, which is dependent not only on the system matrices of the plant, but also on the observer gains. Due to the unchangeable properties of the plant matrices, the only way that can improve the error dynamic behavior is to design

the observer gains.

4.5.5 Observer gain design

In order to successfully estimate the state variables, the error dynamic model has to be stabilized and the estimation errors must converge to zero as soon as possible. According to the LYAPUNOV stability criterion for linear model, the estimation error model is asymptotically stable, if the LYAPUNOV algebraic equation is satisfied

$$\mathbf{A}_b^T \mathbf{P} + \mathbf{P} \mathbf{A}_b = -\mathbf{Q}, \quad (4.123)$$

where $\mathbf{Q}$ is an arbitrary symmetric positive definite matrix. The estimation error dynamic model is asymptotically stable if and only if for any $\mathbf{Q} = \mathbf{Q}^T > 0$ there exists a unique $\mathbf{P} = \mathbf{P}^T > 0$, such that the LYAPUNOV algebraic equation, (4.123), is satisfied. Further, the observer gains are assumed to be calculated as

$$\begin{bmatrix} \mathbf{L}_x^T & \mathbf{L}_v^T \end{bmatrix} = \mathbf{C}_e \mathbf{P}^{-1}. \quad (4.124)$$

Substituting equations (4.120) and (4.124) into (4.123) yields

$$\mathbf{A}_e^T \mathbf{P} + \mathbf{P} \mathbf{A}_e = 2\mathbf{C}_e^T \mathbf{C}_e - \mathbf{Q}. \quad (4.125)$$

Now equation (4.125) can be solved by specific manners, where only the symmetric positive definite matrix $\mathbf{P}$ is unknown. On the one hand, the equation can be refined to a standard LYAPUNOV equation, which is recommended by Müller [205, 230]. On the other hand, the equation can be modified to a standard RICCATI equation. From the later simulations, it can be found that the matrix $\mathbf{P}$ obtained by the LYAPUNOV approach cannot guarantee the observer to be robust and stable, while the RICCATI approach produces excellent observer gains.

Lyapunov approach

It is known from the LYAPUNOV stability criterion that $\mathbf{Q}$ can be any symmetric positive definite matrix. Thus, $\mathbf{Q}$ can be assumed as

$$\mathbf{Q} = 2a\mathbf{P} + b\mathbf{I} \quad (a > 0, b > 0). \quad (4.126)$$

Further substituting (4.126) into (4.125) one obtains a standard LYAPUNOV algebraic equation as

$$\underbrace{(\mathbf{A}_e + a\mathbf{I})^T}_{\tilde{\mathbf{A}}^T} \mathbf{P} + \mathbf{P} \underbrace{(\mathbf{A}_e + a\mathbf{I})}_{\tilde{\mathbf{A}}} = \underbrace{2\mathbf{C}_e^T \mathbf{C}_e - b\mathbf{I}}_{\tilde{\mathbf{Q}}} . \quad (4.127)$$

Although the positive definite matrix $\mathbf{P}$ can be obtained by inserting different a and b , the solution is only valid when all the eigenvalues of the system matrix $\mathbf{A}_b$ of the error model are placed in the left half plane. The LYAPUNOV algebraic equation can be numerically solved by using the available Matlab function “lyap”.

Riccati approach

Alternatively, equation (4.125) can be re-written as

$$\mathbf{P}\mathbf{A}_e + \mathbf{A}_e^T \mathbf{P} - 2\mathbf{C}_e^T \mathbf{C}_e + \mathbf{Q} = \mathbf{0} . \quad (4.128)$$

Multiplying by $\mathbf{P}^{-1}$ on the left- and right-hand side of equation (4.128) yields an algebraic RICCATI equation as [231]

$$\mathbf{A}_e \mathbf{P}^{-1} + \mathbf{P}^{-1} \mathbf{A}_e^T - 2\mathbf{P}^{-1} \mathbf{C}_e^T \mathbf{C}_e \mathbf{P}^{-1} + \mathbf{P}^{-1} \mathbf{Q} \mathbf{P}^{-1} = \mathbf{0} . \quad (4.129)$$

Assuming the symmetric positive definite matrix $\mathbf{Q}$ as

$$\mathbf{Q} = b\mathbf{P}^2 \quad (b > 0) , \quad (4.130)$$

equation (4.129) becomes a standard form of RICCATI equation as

$$\mathbf{A}_e \mathbf{P}^{-1} + \mathbf{P}^{-1} \mathbf{A}_e^T - 2\mathbf{P}^{-1} \mathbf{C}_e^T \mathbf{C}_e \mathbf{P}^{-1} + b\mathbf{I} = \mathbf{0} . \quad (4.131)$$

Analogously, the solution is only valid for a given b when $\mathbf{A}_b$ is asymptotically stable. Usually, a larger b produces larger observer gains, which shortens the rise time of the estimated signal, but with larger overshoot. However, a large b will amplify the noises in the system as well, and additionally the RICCATI equation may become unsolvable if b is extremely large.

4.5.6 Control gain design

As described before, the estimated signals from the extended observer will be fed back to the controller, by which the free vibrations and the forced vibrations will be respectively counteracted. Therefore, the control action should consist of two parts, one is $-\mathbf{K}_x \hat{\mathbf{x}}(t)$ for counteracting the free vibrations, and the other term is $-\mathbf{K}_v \hat{\mathbf{v}}(t)$ for the forced vibrations, which is defined as

$$\mathbf{u}(t) = -\mathbf{K}_x \hat{\mathbf{x}}(t) - \mathbf{K}_v \hat{\mathbf{v}}(t) , \quad (4.132)$$

in which $\mathbf{K}_x$ can be designed by any ordinary method like pole placement, linear quadratic regulator, etc. In the later simulations, the control gain $\mathbf{K}_x$ is derived by using linear quadratic regulator optimization method given in (4.66) of Section 4.4.1.

However, the control gain $\mathbf{K}_v$, which compensates the unknown disturbance effects, can be obtained in a specific manner under the assumption of a linear mapping $\mathbf{X}$ existing between $\mathbf{x}$ and $\mathbf{v}$ as [232]

$$\mathbf{x} = \mathbf{X} \mathbf{v} , \quad (4.133)$$

such that

$$\dot{\mathbf{x}} = \mathbf{X} \dot{\mathbf{v}} = \mathbf{X} \mathbf{V} \mathbf{v} . \quad (4.134)$$

Further substituting the linear mapping into the plant state space model yields the equilibrium equation for computation of control gain $\mathbf{K}_v$ as

$$(\mathbf{A} - \mathbf{B} \mathbf{K}_x) \mathbf{X} - \mathbf{X} \mathbf{V} - \mathbf{B} \mathbf{K}_v + \mathbf{N} \mathbf{H} = \mathbf{0} , \quad (4.135)$$

$$(\mathbf{F} - \mathbf{G} \mathbf{K}_x) \mathbf{X} - \mathbf{G} \mathbf{K}_v = \mathbf{0} . \quad (4.136)$$

Exact solution

Since the unknown matrix $\mathbf{X}$ respectively appears on the left and right of the first two terms in (4.135), it is difficult to solve the above equations. Using the fictitious model given in (4.102) and assuming that the unknown matrices $\mathbf{X}$ and $\mathbf{K}_v$ are composed of three parts as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix} , \quad (4.137)$$

$$\mathbf{K}_v = \begin{bmatrix} \mathbf{K}_{v1} & \mathbf{K}_{v2} & \mathbf{K}_{v3} \end{bmatrix} , \quad (4.138)$$

yields six equilibrium equations from (4.135)-(4.136) as [231]

$$\begin{aligned}
 (\mathbf{A} - \mathbf{BK}_x)\mathbf{X}_1 - \mathbf{BK}_{v1} &= -\mathbf{NH}_1, \\
 (\mathbf{A} - \mathbf{BK}_x)\mathbf{X}_2 - \omega\mathbf{X}_3 - \mathbf{BK}_{v2} &= -\mathbf{NH}_2, \\
 (\mathbf{A} - \mathbf{BK}_x)\mathbf{X}_3 + \omega\mathbf{X}_2 - \mathbf{BK}_{v3} &= -\mathbf{NH}_3, \\
 (\mathbf{F} - \mathbf{GK}_x)\mathbf{X}_1 - \mathbf{GK}_{v1} &= \mathbf{0}, \\
 (\mathbf{F} - \mathbf{GK}_x)\mathbf{X}_2 - \mathbf{GK}_{v2} &= \mathbf{0}, \\
 (\mathbf{F} - \mathbf{GK}_x)\mathbf{X}_3 - \mathbf{GK}_{v3} &= \mathbf{0},
 \end{aligned} \tag{4.139}$$

where $\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 \end{bmatrix}$. Re-arranging equation (4.139) one obtains a linear equation in matrix form as

$$\begin{bmatrix} \mathbf{A} - \mathbf{BK}_x & \mathbf{0} & \mathbf{0} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} - \mathbf{BK}_x & -\omega\mathbf{I} & \mathbf{0} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \omega\mathbf{I} & \mathbf{A} - \mathbf{BK}_x & \mathbf{0} & \mathbf{0} & -\mathbf{B} \\ \mathbf{F} - \mathbf{GK}_x & \mathbf{0} & \mathbf{0} & -\mathbf{G} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} - \mathbf{GK}_x & \mathbf{0} & \mathbf{0} & -\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} - \mathbf{GK}_x & \mathbf{0} & \mathbf{0} & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{K}_{v1} \\ \mathbf{K}_{v2} \\ \mathbf{K}_{v3} \end{bmatrix} = \begin{bmatrix} -\mathbf{NH}_1 \\ -\mathbf{NH}_2 \\ -\mathbf{NH}_3 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{4.140}$$

Consequently, $\mathbf{K}_v$ can be easily derived by solving the above linear equation.

Approximate solution

Alternatively, the control gain $\mathbf{K}_v$ can be calculated by assuming $\dot{\mathbf{x}} = \mathbf{0}$, due to the idea that the gain is only used for counteracting the forced vibrations. This assumption neglects the disturbance dynamics and considers only the steady-state behavior. As a consequence, the observer may fail to correctly estimate fast varying disturbances. Further substituting the linear mapping into the plant state space model, yields the equilibrium equation for statically compensating as

$$(\mathbf{A} - \mathbf{BK}_x)\mathbf{X} - \mathbf{BK}_v + \mathbf{NH} = \mathbf{0}, \tag{4.141}$$

$$(\mathbf{F} - \mathbf{GK}_x)\mathbf{X} - \mathbf{GK}_v = \mathbf{0}. \tag{4.142}$$

In such a way, $\mathbf{K}_v$ and $\mathbf{X}$ can be solved by

$$\begin{bmatrix} \mathbf{A} - \mathbf{BK}_x & -\mathbf{B} \\ \mathbf{F} - \mathbf{GK}_x & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{K}_v \end{bmatrix} = - \begin{bmatrix} \mathbf{NH} \\ \mathbf{0} \end{bmatrix}, \quad (4.143)$$

which is also used for PI observer.

4.5.7 Closed-loop system

Substituting equation (4.132) into (4.89), yields the closed-loop control system with the estimation of unknown disturbances using PI or GPI observer as

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}}_x \\ \dot{\hat{\mathbf{v}}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK}_x & \mathbf{BK}_x & -\mathbf{BK}_v \\ 0 & \mathbf{A} - \mathbf{L}_x\mathbf{C} & -\mathbf{NH} \\ 0 & \mathbf{L}_v\mathbf{C} & \mathbf{V} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \\ \hat{\mathbf{v}} \end{Bmatrix} + \begin{bmatrix} \mathbf{N} \\ \mathbf{N} \\ \mathbf{0} \end{bmatrix} \mathbf{f}, \quad (4.144)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \\ \hat{\mathbf{v}} \end{Bmatrix}, \quad (4.145)$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{F} - \mathbf{GK}_x & \mathbf{GK}_x & -\mathbf{GK}_v \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \\ \hat{\mathbf{v}} \end{Bmatrix}. \quad (4.146)$$

Additionally, the control input can be calculated as

$$\mathbf{u} = \begin{bmatrix} -\mathbf{K}_x & \mathbf{K}_x & -\mathbf{K}_v \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{e}_x \\ \hat{\mathbf{v}} \end{Bmatrix}. \quad (4.147)$$

4.6 Summary

This chapter mainly developed a disturbance rejection control with PI or GPI observer for vibration suppression of piezoelectric integrated smart structures. In order to show the advantages of the proposed disturbance rejection control, PID, LQR and LQG control strategies were discussed and implemented as well.

Chapter 5

Numerical FE analysis*

This chapter first deals with the validation tests of nonlinear FE models obtained based on RVK5, MRT5, LRT5 and LRT56 shell theories by composite laminated thin-walled structures. Later the nonlinear FE models are implemented into the simulation of buckling and post-buckling problems. Finally, the electro-mechanically coupled FE models based on the FOSD large rotation shell theory and the simplified nonlinear shell theories are applied to the static and dynamic analysis of piezoelectric laminated plates and shells.

5.1 Benchmark problems

5.1.1 Asymmetric cross-ply laminated plate

In the first validation test, an asymmetric cross-ply laminated plate shown in Fig. 5.1 is considered, which was first proposed by Sun *et al.* [233], and used as benchmark problem by Reddy [69], Basar *et al.* [94], Kreja and Schmidt [89]. The plate is made of two substrate layers with a stacking sequence of $[90^\circ/0^\circ]$. The fiber reinforcement along the Θ^1 -axis defines the material angle 0° . The thickness for each substrate layer is 0.02 in. The dimensions of the plate are displayed in the figure, and the material properties are listed in Table 5.1. The composite structure is meshed by

*Part of the numerical results reported in this chapter were published in the articles “Large rotation FE transient analysis of piezolaminated thin-walled smart structures” by S. Q. Zhang and R. Schmidt, *Smart Mater. Struct.*, 22(10): 105025, 2013 [213], Copyright (2013) IOP Publishing Ltd.; and “Large rotation theory for static analysis of composite and piezoelectric laminated thin-walled structures” by S. Q. Zhang and R. Schmidt, *Thin-Walled Struct.*, 78: 16-25, 2014 [214], Copyright (2013) Elsevier Ltd.

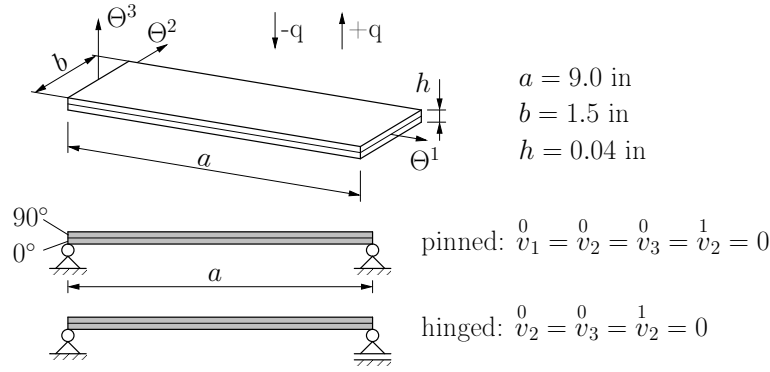


FIGURE 5.1: Asymmetric cross-ply laminated plate

TABLE 5.1: Material properties of the composite plate

Orthotropic material
$E_1 = 2.0 \times 10^7 \text{ lb/in}^2$
$E_2 = 1.4 \times 10^6 \text{ lb/in}^2$
$\nu_{12} = \nu_{23} = 0.3$
$G_{12} = G_{23} = G_{13} = 0.7 \times 10^7 \text{ lb/in}^2$

9×2 SH85URI or SH851FI elements along the Θ^1 - and Θ^2 -axis, respectively. Two boundary conditions, pinned and hinged, are considered. The difference between these two boundary conditions is that the translational movement along the Θ^1 -direction in the pinned case is fixed compared to the case of the hinged boundary condition. The rotations around the Θ^2 -axis are not restricted in both boundary conditions. A uniformly distributed force is applied on the top surface, which is measured in the unit lb/in^2 . The static response of the mid-point displacement is calculated and measured using the non-dimensionalized deflection $|w|/h$.

For the case of pinned edges, the numerical results of both loading cases $\pm \mathbf{q}$ obtained by LRT56 theory are given in Fig. 5.2 (a) for small loading and (b) for large loading, which indicate that the present results have excellent agreement with those obtained by Reddy [69] using TOSD RVK5 theory, since the deformation in this case does not exceed the range of moderate rotations. Our results agree also very well with those obtained by Basar *et al.* [94] based on TOSD large rotation theory, except for the fact that the latter authors do not properly account for the load-deflection response at low loads for the $+\mathbf{q}$ loading case. Our result in this range confirms that of Reddy [69]. It reveals a complex load-deflection response showing first a softening tendency, then turning into a stress stiffening behavior due to the VON KÁRMÁN effect.

The results for the case of hinged edges, as shown in Fig. 5.3, illustrate that those

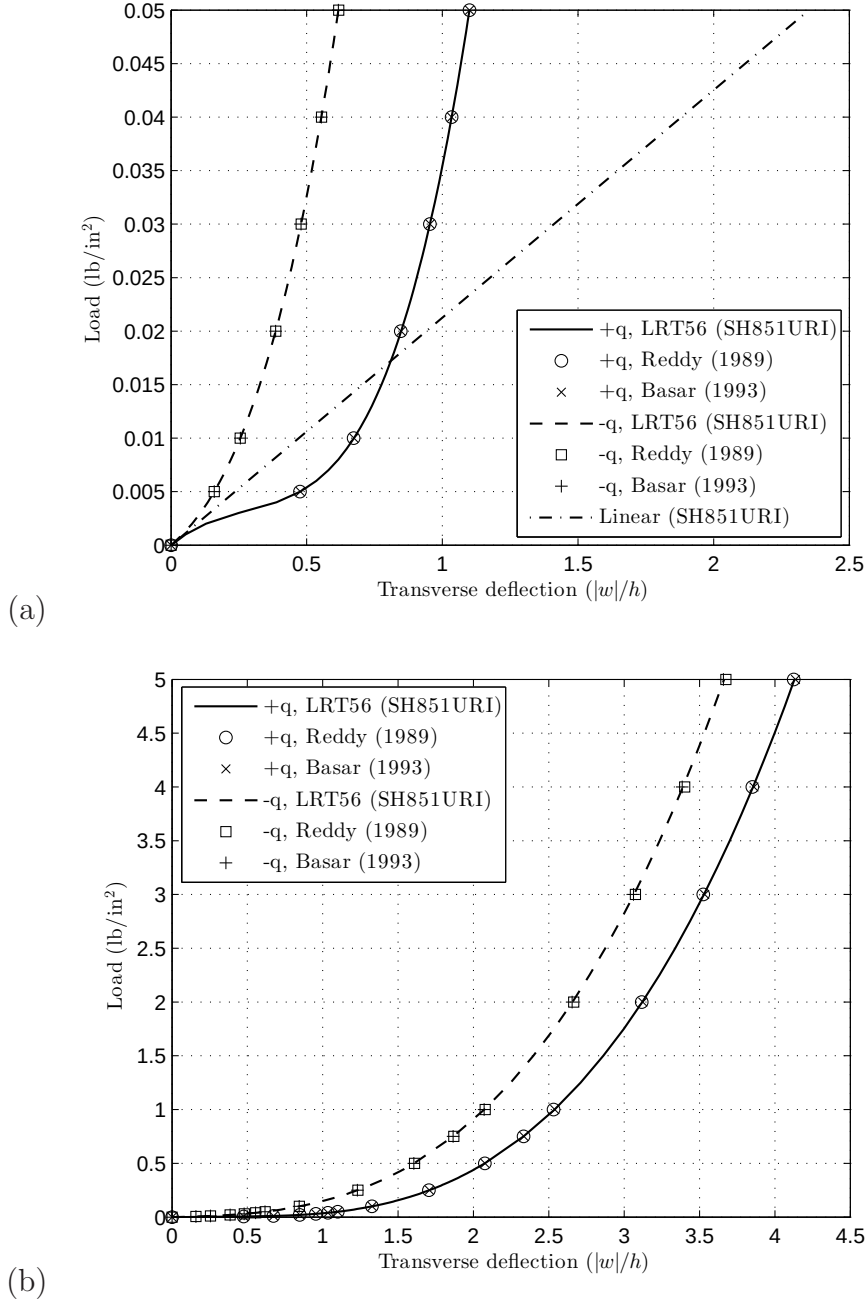


FIGURE 5.2: Static response of the mid-point displacement for the pinned case under a uniform pressure: (a) small pressure, (b) large pressure

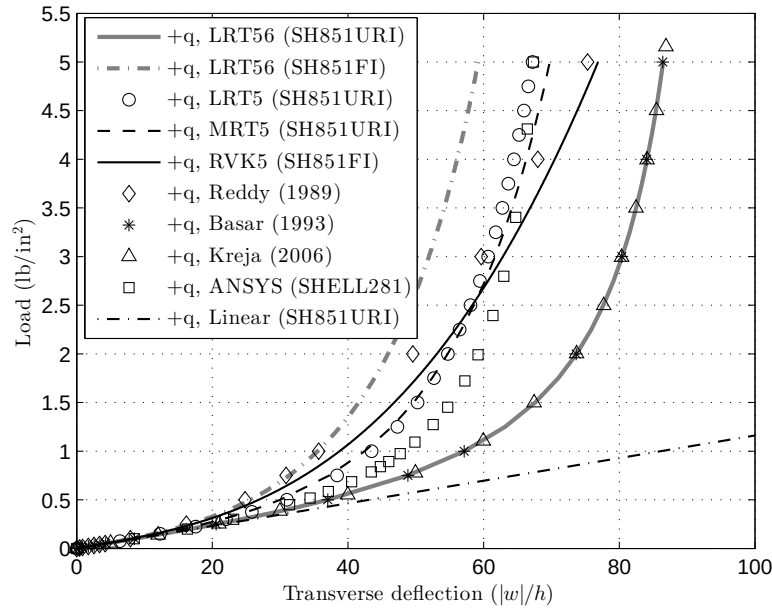


FIGURE 5.3: Static response of the mid-point displacement for the hinged case under a uniform pressure

TABLE 5.2: Mid-point displacement of the asymmetric cross-ply laminated plate by LRT56 theory using SH85URI elements

Load (+q lb/in ²)	Present (LRT56, $ w /h$)	Reddy [69]	Basar <i>et al.</i> [94]
0.005	0.4303	0.429	0.429
0.01	0.8606	0.858	0.858
0.02	1.7208	1.71	1.717
0.03	2.5803	2.55	2.574
0.04	3.4385	3.37	3.430
0.05	4.2951	4.19	4.285
0.10	8.5413	7.92	8.525
0.25	20.5799	16.17	20.57
0.50	37.0014	24.82	37.03
0.75	48.8280	30.87	48.83
1.0	57.1631	35.69	57.14
2.0	73.7032	49.56	73.64
3.0	80.4214	59.65	80.36
4.0	84.0963	68.00	84.03
5.0	86.4767	75.33	86.41

calculated by LRT56 using SH851URI element agrees quite well with the results reported in [89, 94], since both of them applied large rotation theory in their FOSD [89] and TOSD [94] models. Large discrepancies occur in the results obtained by RVK5, MRT5 and LRT5, due to the fact that these theories are not suitable for the analysis of deformed structures at large rotations. The first two, RVK5 and MRT5, use simplified nonlinear strain-displacement relations, which restrict their applicability to the range of moderate rotations. The LRT5 theory fails despite fully geometrically nonlinear strain-displacement relations are employed, because it does not properly describe large rotations due to putting $\frac{1}{v_3} = 0$ in the kinematic hypothesis (2.28). It is interesting to note that, due to the lack of proper updating of the rotations, LRT5 theory cannot yield better results than MRT5. Big differences are observed between the results obtained by the present LRT56 theory using the SH851FI and SH851URI elements, respectively. This is due to the fact that the full integration converges to the exact value slower than the uniformly reduced integration, meaning that the difference can be decreased by increasing the number of elements. It can be observed additionally that the load-deflection curve calculated by RVK5 using the SH851FI elements based on the FOSD hypothesis agrees quite well with Reddy's [69] result, who implemented the same type of nonlinear theory in the model but based on the TOSD hypothesis.

5.1.2 Asymmetrically loaded thin arch

The second simulation is an asymmetrically loaded thin arch as shown in Fig. 5.4, which is a very typical large rotation problem. The dimensions and material properties are given in the figure. Two edges are hinged with free rotations about the Θ^1 -axis. The arch is meshed by 1×25 SH85URI elements along the Θ^1 - and Θ^2 -axis, respectively. A concentrated force is applied on the point located 40° from the left hinged edge, where also the radial displacements w are calculated. The static response obtained by various nonlinear shell theories is presented in Fig. 5.5.

The results illustrate that the load-deflection curves differ considerably from each other, which results from the different strain-displacement relations considered in the shell theories and different assumptions for the magnitudes of rotations in the structure. Since the arch undergoes large rotations and deflections, the simplified nonlinear shell theories fail to predict the response precisely, rather than LRT56. It can be seen that the present result calculated by LRT56 theory agrees very well with that analyzed using ANSYS.

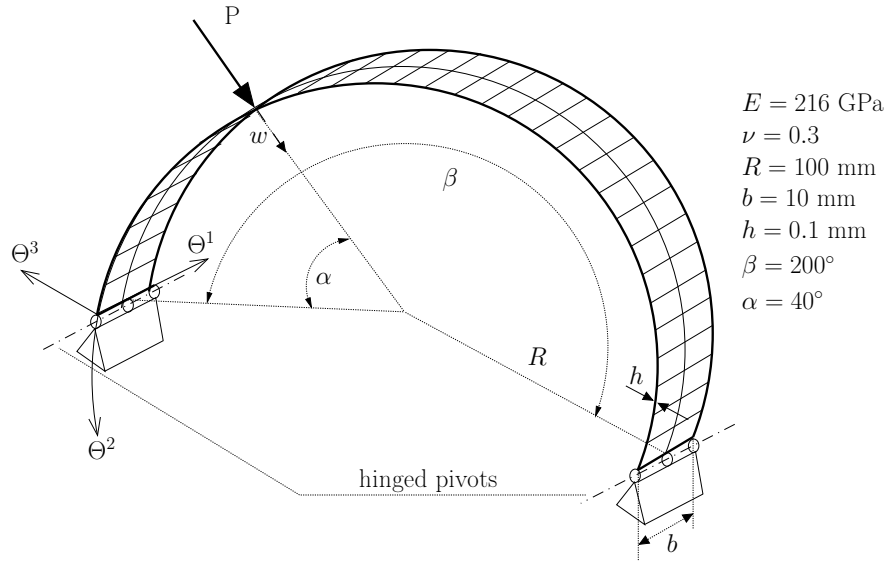


FIGURE 5.4: Asymmetrically loaded hinged thin arch

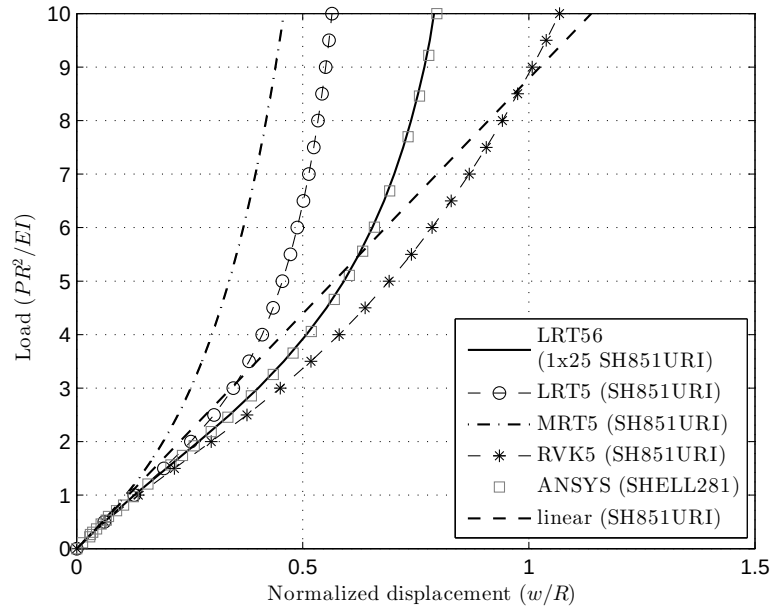


FIGURE 5.5: Static response of the asymmetrically loaded hinged thin arch

5.1.3 Spherical shell with a hole

In the next example, a spherical shell with an 18° hole is considered, as shown in Fig. 5.6, which is a very popular benchmark problem for large rotation analysis of isotropic shells, see [84, 89, 98, 101–103, 105, 234] among others. The radius of the

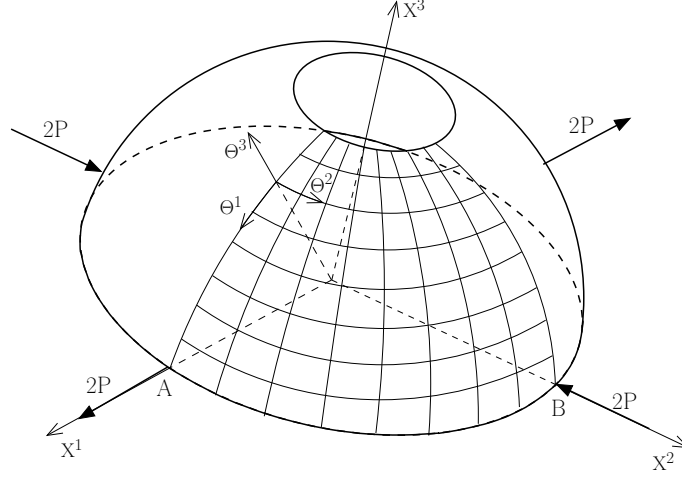


FIGURE 5.6: Spherical shell with an 18° hole under a pair of stretching and compressing forces

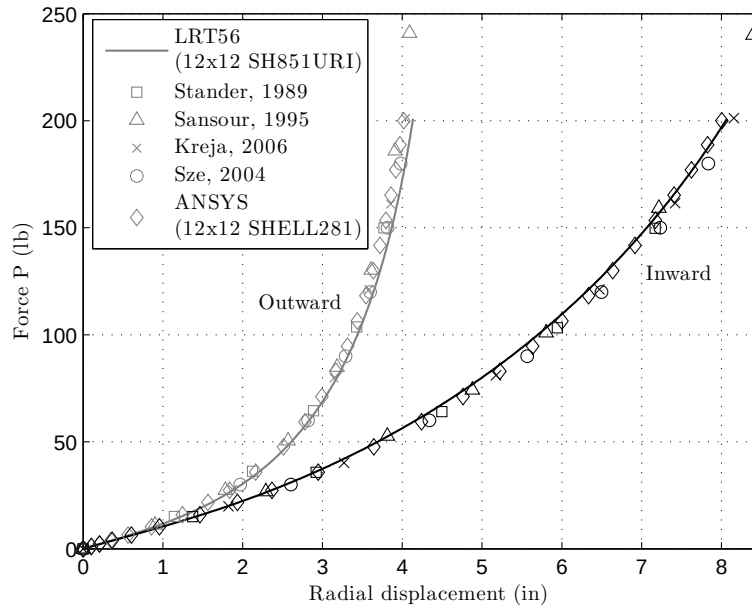


FIGURE 5.7: Outward and inward displacement response of the spherical shell

spherical shell is $R = 10$ in and the thickness is $h = 0.04$ in. The material properties are $E = 6.825 \times 10^7$ psi and $\nu = 0.3$. Due to the symmetry, only a quarter of the shell is considered with appropriate symmetry boundary conditions. The quarter shell

is meshed by 12×12 SH85URI elements. A pair of stretching and compressing forces are perpendicularly applied on the shell as shown in Fig. 5.6. The outward displacement at point A and the inward displacement at point B are respectively calculated by LRT56 theory, and displayed in Fig. 5.7. From the results, one can see that the present static response of the inward and outward displacements obtained by LRT56 theory agrees quite well with that published in the literature, as well as with that computed by ANSYS using SHELL281 elements.

5.2 Buckling and post-buckling analysis

5.2.1 Hinged panel with cross-ply laminates

We consider a cross-ply laminated cylindrical shell with different thicknesses and lay-ups, depicted in Fig. 5.8. This structure was analyzed earlier by Saigal *et al.* [235],

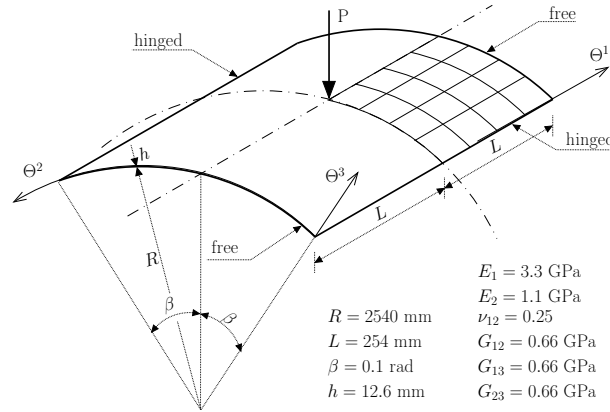


FIGURE 5.8: Cylindrical panel with layered orthotropic materials under a concentrated force applied at the mid-point

and later by Laschet and Jeusette [236], Brank *et al.* [87], respectively. The cylindrical panel is subjected to a concentrated force at the mid-point. The dimensions and material properties can be found in Fig. 5.8.

Concerning the boundary conditions, the two straight edges are hinged and the two curved ones are free. The hinged boundary condition implies that only the rotations about the Θ^1 -axis are free. The panel with the stacking sequence of $[90^\circ/0^\circ/90^\circ]$ and $[0^\circ/90^\circ/0^\circ]$ has not only the geometric symmetry, but also the material symmetry. Therefore, only a quarter of the panel can be calculated using the symmetry boundary conditions. The panel with these two stacking sequences is meshed by 4×4 SH85URI elements. Firstly, the panel with a total thickness of 12.6 mm comprised of three

substrate layers of 4.3 mm each is analyzed. The static response of the mid-point displacement is presented in Fig. 5.9 for the stacking sequence of $[0^\circ/90^\circ/0^\circ]$ and in Fig. 5.10 for $[90^\circ/0^\circ/90^\circ]$.

As the figures illustrate, the results obtained by RVK5 and MRT5 theories agree excellently with those presented by Laschet [236] and Brank [87]. However, LRT56 and LRT5 predict a stiffer response in the post-buckling range, which results from the different number of nonlinear strain terms that are considered. Additionally, the curves calculated by LRT5 and LRT56 are almost the same, implying that only moderate rotations occur in the panel. It can also be observed that changing the material sequence will lead to a different limit load, meaning that the stiffness of structures can be improved by optimizing the sequence.

With the same lay-ups of the panel as in the previous calculation, reducing the total thickness to 6.3 mm yields a significantly softer static response than in the previous two cases, which is shown in Fig. 5.11 and Fig. 5.12, respectively. The results of the panel with the lay-up of $[0^\circ/90^\circ/0^\circ]$ illustrate a significant, complicated snap-back behavior. Additionally, changing the lay-up to $[90^\circ/0^\circ/90^\circ]$ yields a higher limit load, which improves the stability behavior of the panel, and leads to a simpler load-deflection path.

5.2.2 Hinged panel with angle-ply laminates

The same geometry and material properties of the cylindrical panel are considered in this computation. The orthotropic material layered shell consists of two substrate layers with the angle-ply stacking sequence $[45^\circ/-45^\circ]$ or $[-45^\circ/45^\circ]$. Even though the material of the panel with these two stacking sequences is unsymmetrical, a quarter of the panel is still considered in order to compare with the results published in the literature. Two meshes are considered in these two stacking sequences, 1×1 and 4×4 SH85URI elements. Firstly, the panel with a total thickness of 12.6 mm and 6.3 mm for each sub-layer is analyzed. The results are shown in Fig. 5.13. For the panel with the stacking sequence $[45^\circ/-45^\circ]$, the result obtained by MRT5 theory with the mesh of 1×1 agrees quite well with that reported in [235], where the same number of elements is considered. Refining the mesh to 4×4 elements leads to the panel being softer in the pre-buckling range. Similar with the cross-ply laminated panel, LRT56 theory predicts stiffer behavior in the post-buckling range. The structural stiffness changes by re-arranging the stacking sequence.

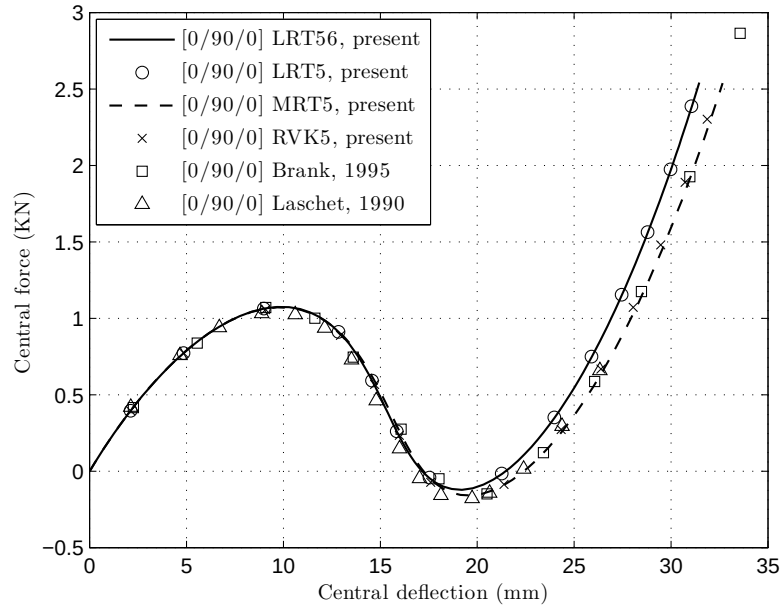


FIGURE 5.9: Static response of the mid-point displacement for the cross-ply laminated panel with a thickness of 12.6 mm and a stacking sequence $[0^\circ/90^\circ/0^\circ]$

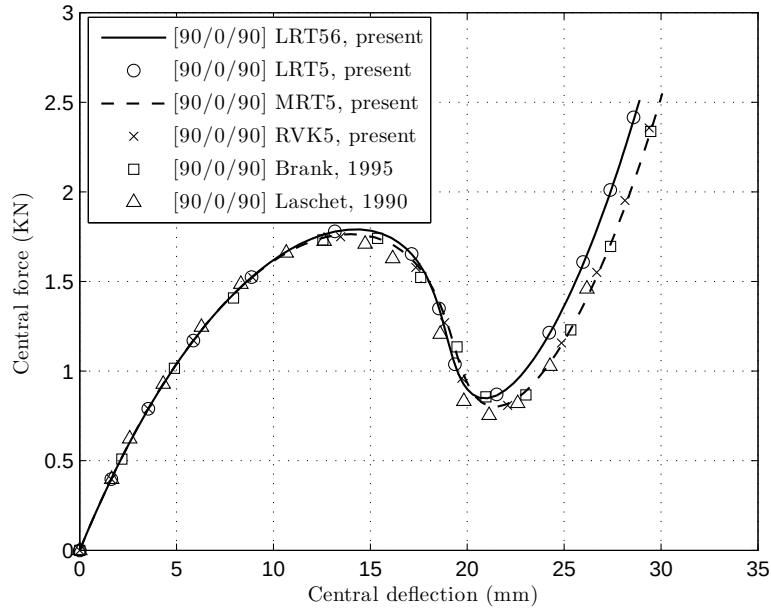


FIGURE 5.10: Static response of the mid-point displacement for the cross-ply laminated panel with a thickness of 12.6 mm and a stacking sequence $[90^\circ/0^\circ/90^\circ]$

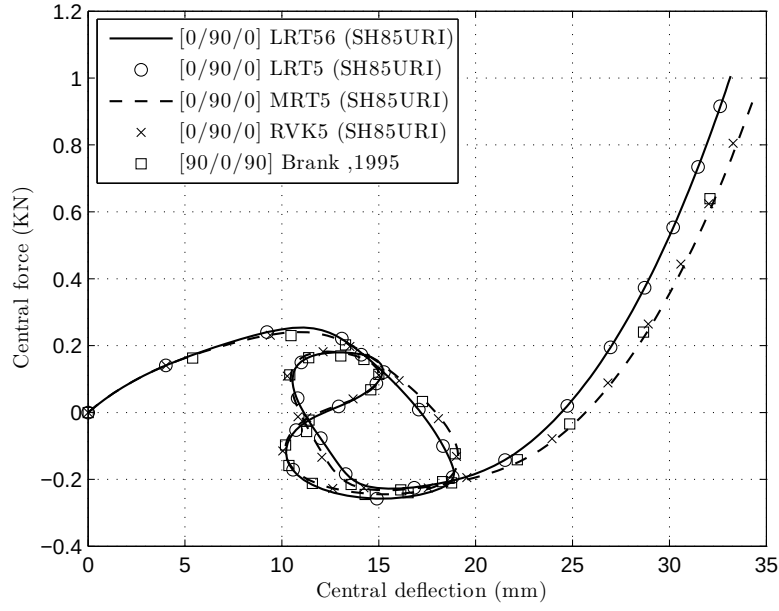


FIGURE 5.11: Static response of the mid-point displacement for the cross-ply laminated panel with a thickness of 6.3 mm and a stacking sequence $[0^\circ/90^\circ/0^\circ]$

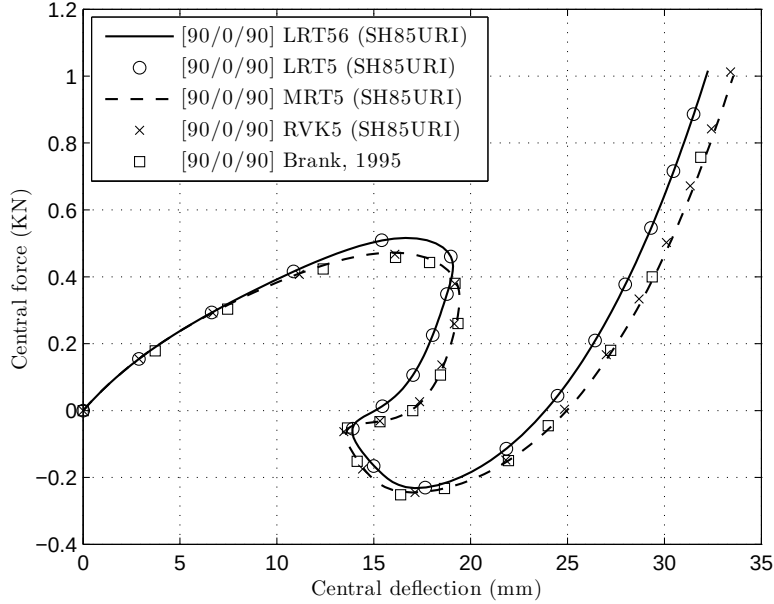


FIGURE 5.12: Static response of the mid-point displacement for the cross-ply laminated panel with a thickness of 6.3 mm and a stacking sequence $[90^\circ/0^\circ/90^\circ]$

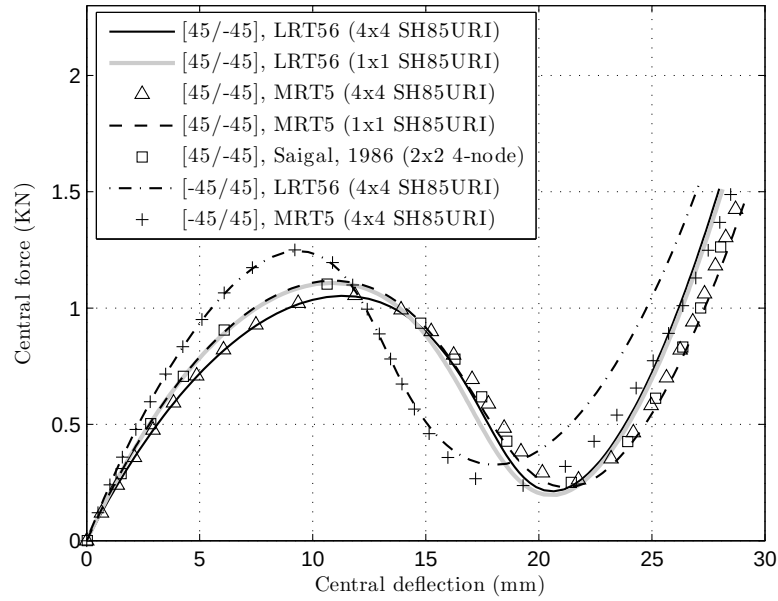


FIGURE 5.13: Static response of the mid-point displacement for the angle-ply laminated panel with a thickness of 12.6 mm and stacking sequences $[45^\circ / -45^\circ]$ and $[-45^\circ / 45^\circ]$

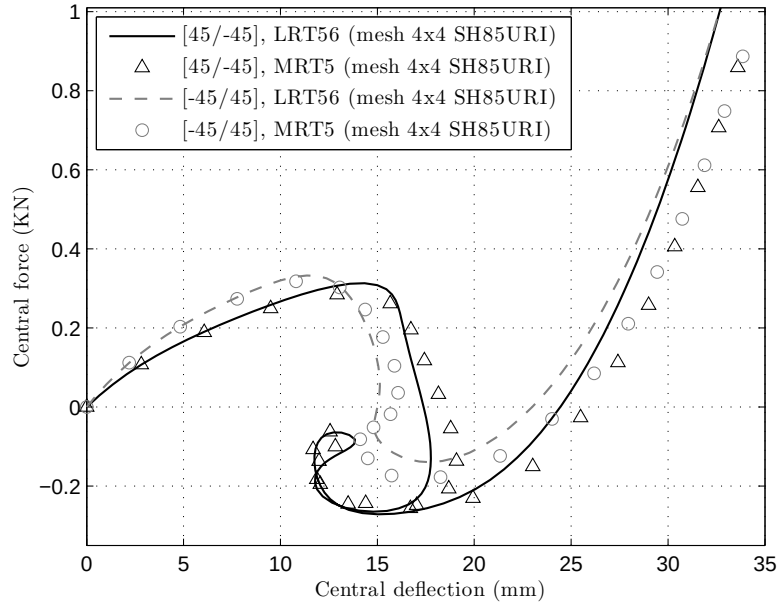


FIGURE 5.14: Static response of the mid-point displacement for the angle-ply laminated panel with a thickness of 6.3 mm and stacking sequences $[45^\circ / -45^\circ]$ and $[-45^\circ / 45^\circ]$

Analogously, reducing the total thickness to 6.3 mm, the cylindrical shell performs much softer than the thick one, as shown in Fig. 5.14. Again, the shell with the stacking sequence $[-45^\circ/45^\circ]$ is stiffer than that of $[45^\circ/-45^\circ]$ which exhibits a complex snap-through and snap-back load-deflection path.

5.3 FE analysis of smart structures

5.3.1 Cantilevered smart beam

A cantilevered beam bonded with one piezoelectric patch shown in Fig. 5.15, which was first calculated by Yi *et al.* [151] using solid finite element method, is analyzed here both statically and dynamically. The dimensions of the host structure and the

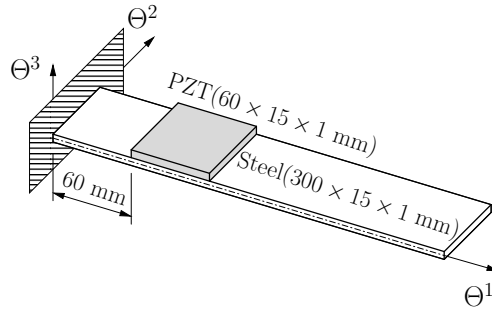


FIGURE 5.15: Cantilevered beam with one piezoelectric patch bonded

PZT patch, as well as the position of the PZT patch are displayed in Fig. 5.15. The material properties are given in Table 5.3. Here the piezoelectric coupling coefficients

TABLE 5.3: Material properties of the cantilevered smart beam

Steel	PZT
$E = 197 \text{ GPa}$	$E = 67 \text{ GPa}$
$\nu = 0.33$	$\nu = 0.33$
$\rho = 7900 \text{ kg/m}^3$	$\rho = 7800 \text{ kg/m}^3$
	$d_{31} = d_{32} = -1.7119 \times 10^{-10} \text{ C/N}$
	$\epsilon_{33} = 2.03 \times 10^{-8} \text{ F/m}$

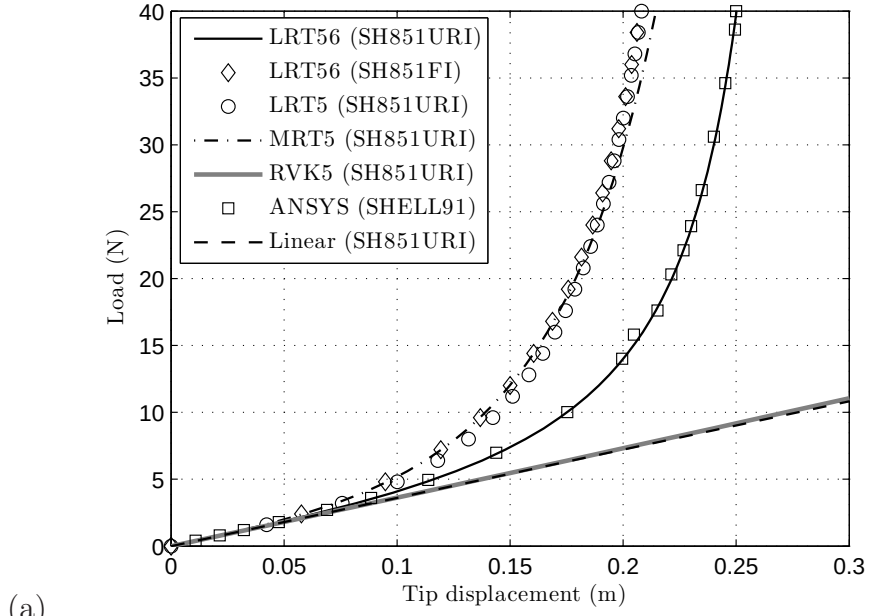
d_{31} and d_{32} are different from those in [151], but the same as those in [142]. The piezoelectric potential of the bonded surface is $\phi = 0$, while at the upper surface of the piezoelectric patch the physical equipotential condition is enforced.

Static analysis

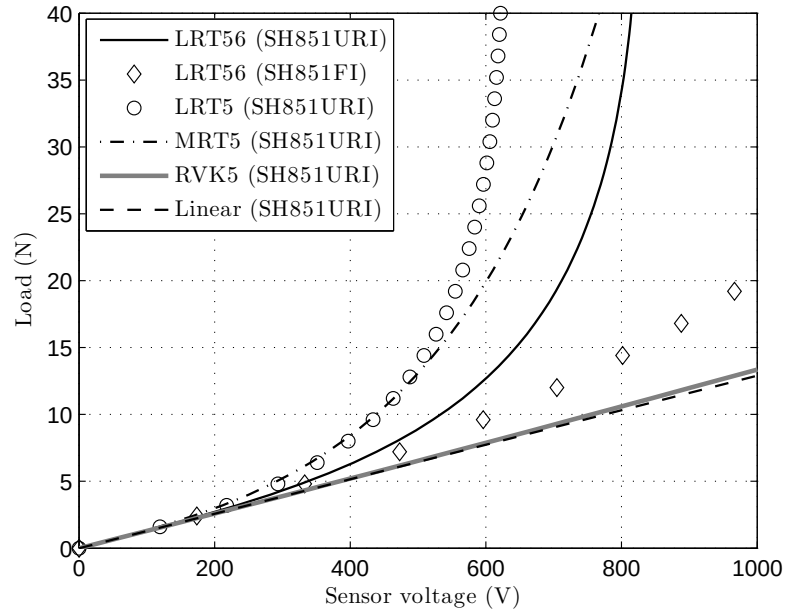
The smart beam is meshed by 10×1 SH851URI or SH851FI elements along the Θ^1 - and Θ^2 -line, respectively, for static analysis. A concentrated force varying from 0 N to 40 N is applied on the tip point at the free end. The static response of the tip displacement is displayed in Fig. 5.16 (a), which is obtained by using various nonlinear shell theories with consideration of short circuit electrodes of the sensor patch. Using the same range of the tip load, one obtains a static load-voltage response, which is presented in Fig. 5.16 (b). It can be seen that the tip displacement response calculated by LRT56 theory using SH851URI elements agrees quite well with that obtained by ANSYS. However, large differences of both displacements and sensor voltages are existing between those obtained by LRT56 and the simplified nonlinear theories, e.g. RVK5, MRT5 and LRT5, in the range of large rotations. Due to the reasons explained in Section 5.1.1, the simplified nonlinear theories fail to predict the static behavior of smart structures undergoing large rotations. The maximum value of the rotational DOFs, φ_1 and φ_2 , at the centerline nodes are shown in Figs. 5.17 (a) and (b), respectively. It illustrates that the rotations φ_1 about the Θ^2 -axis are very large, reaching over 80° when the force is around 35 N. However, the rotations φ_2 are almost zero, since the beam is bent only in one direction without any torsion effect. The small deviation of φ_2 from 0° is caused by the error margin of the numerical method. Furthermore, it can be observed that the static response calculated by LRT56 using SH851FI and SH851URI elements has big differences. This is because the SH851FI elements exhibit *membrane* and *shear locking* phenomena, while SH851URI elements avoid these locking effects.

Dynamic analysis

Two meshes of 5×1 and 10×1 eight-node shell elements are used in the following simulations. A concentrated step force of 10 N is applied on the tip point of the free end. The linear dynamic response is calculated by the NEWMARK method with a time step of 1×10^{-3} s. The nonlinear dynamic response using the SH851FI elements is achieved by CDA method with a time step of 1×10^{-7} s, while the NEWMARK method with a time step of 5×10^{-6} s is applied for the model using SH851URI elements. The tip displacement and sensor voltage transient response obtained by various nonlinear theories using SH851URI element is presented in Fig. 5.18 (a) and (b), respectively. It can be seen from Fig. 5.18 (a) that LRT56 yields a stiffer response than RVK5, but a softer one than MRT5 and LRT5. These simplified nonlinear theories



(a)



(b)

FIGURE 5.16: Static response of the cantilevered smart beam: (a) tip displacement, (b) sensor output voltage

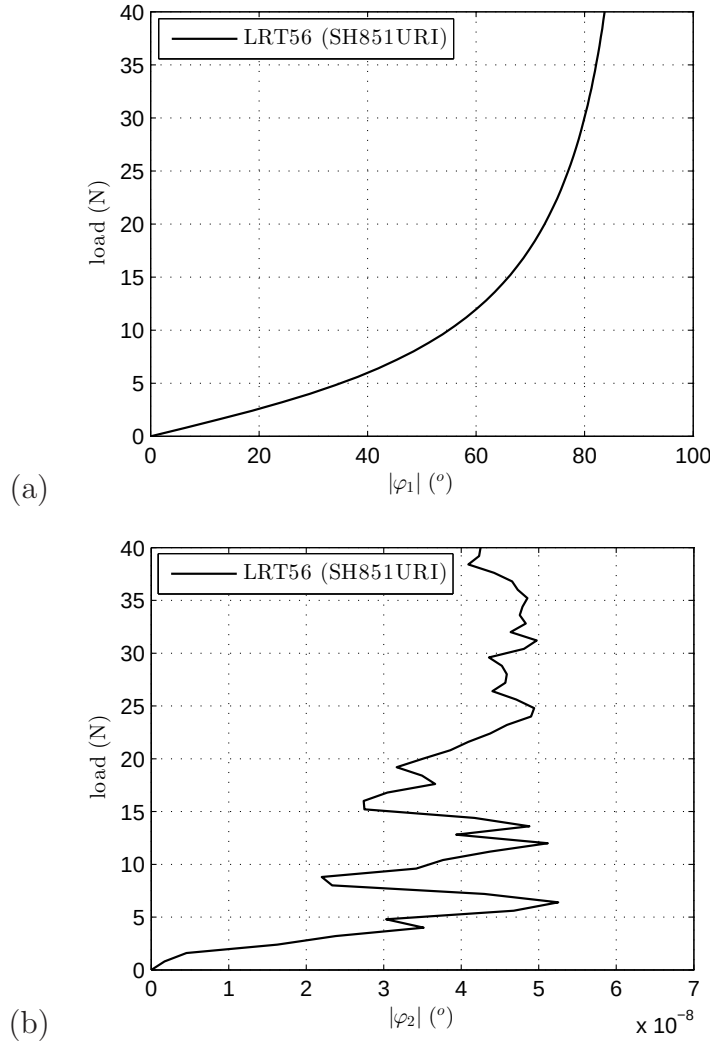


FIGURE 5.17: Maximum value of rotations at the centerline nodes of the cantilevered smart beam: (a) rotations φ_1 about Θ^2 -axis, (b) rotations φ_2 about Θ^1 -axis

fail because in this problem really large rotations occur. A deeper static investigation of the structure shows that under a quasi-statically applied tip force 10 N the structure undergoes maximum rotations of more than 50° , see Fig. 5.17 (a). Furthermore, the static deflections predicted by LRT56 are larger than those by MRT5 and LRT5, and lower than those predicted by RVK5. In Fig. 5.18 (b), the linear theory overpredicts the sensor output voltage by far, because it does not account for the stress stiffening effects. It can be also recognized that the larger displacement amplitudes of the LRT56 transient response lead to larger amplitudes of the sensor output voltage than those predicted by MRT5 and LRT5.

The next group of figures show the transient response obtained by LRT56 theory

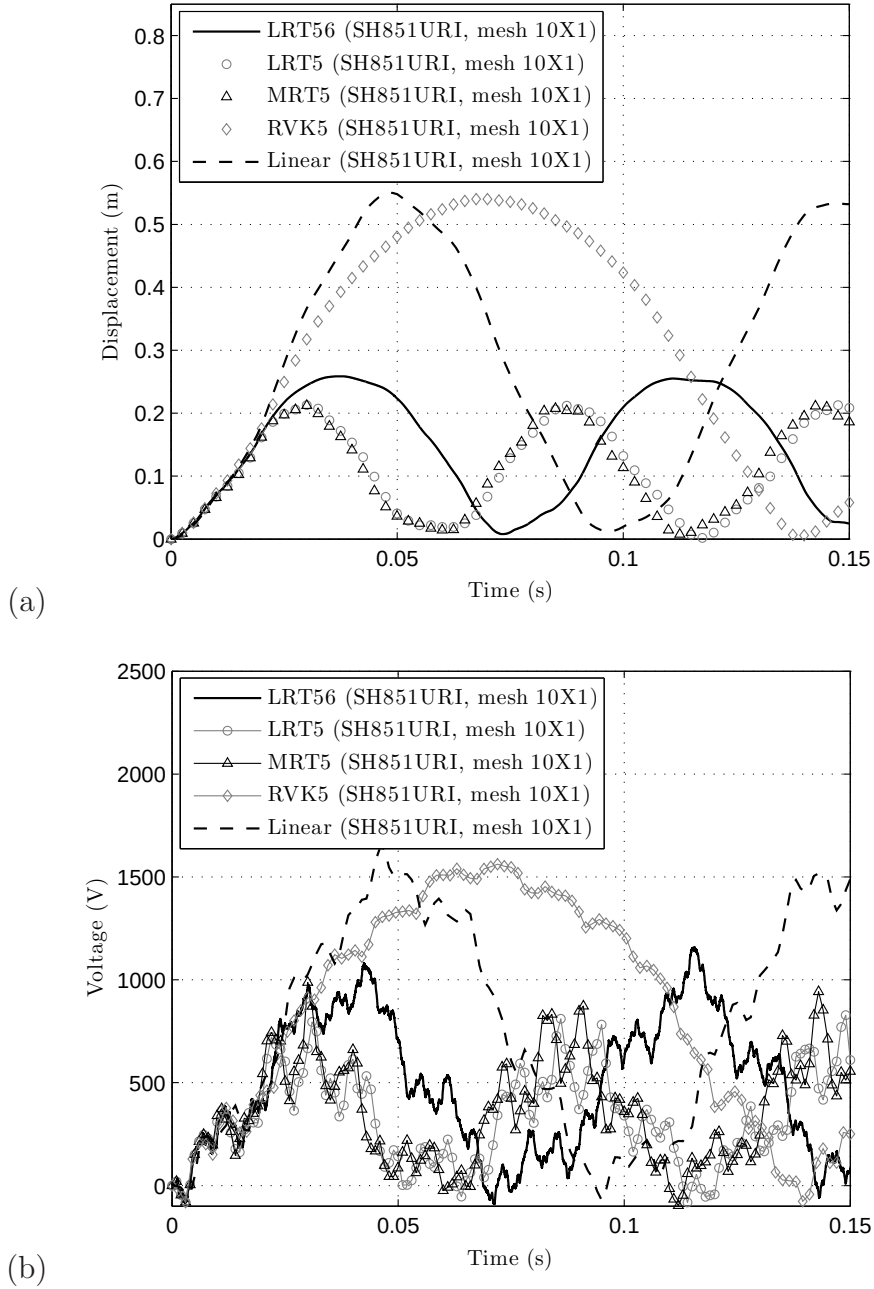


FIGURE 5.18: Dynamic response of the cantilevered beam under a step tip force of 10 N using various shell theories: (a) tip displacement, (b) sensor output voltage

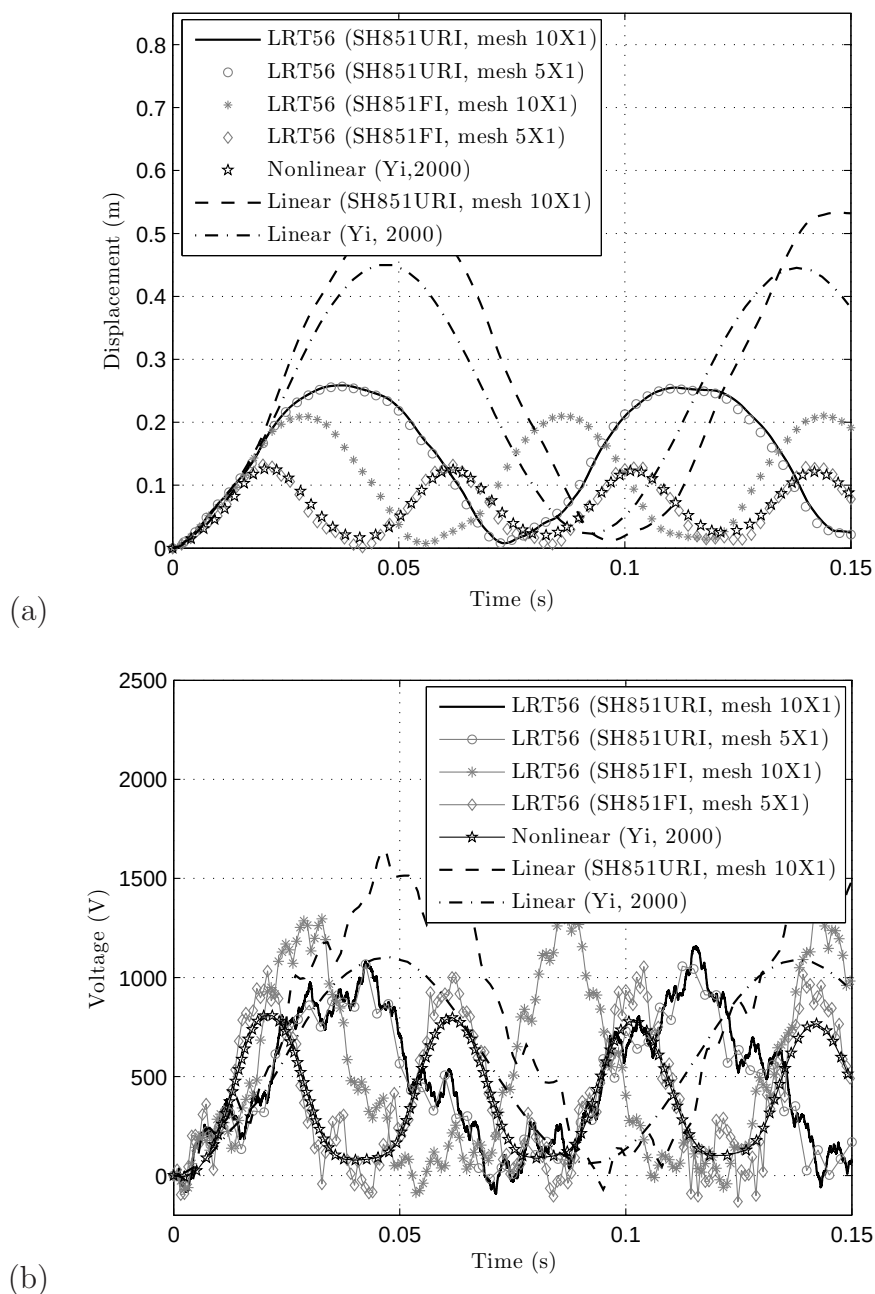


FIGURE 5.19: Dynamic response of the cantilevered beam under a step tip force of 10 N using various meshes and integration schemes: (a) tip displacement, (b) sensor output voltage

using both SH851URI and SH851FI elements with two different meshes, which is displayed in Fig. 5.19 (a) for the displacements and (b) for the sensor output voltages along with a comparison to the literature. From Fig. 5.19 (a), it can be seen that the transient response obtained by LRT56 using the discretization of 5×1 or 10×1 SH851URI elements is almost identical. This indicates that the 5×1 mesh yields already the converged solution. In Fig. 5.19 (a), we have also added a result obtained by a 5×1 mesh of SH851FI elements. This was done to compare the results with those given by Yi *et al.* [151], who used the fully geometrically nonlinear 3-D theory applying a mesh of 5×1 20-node solid elements for the master structure and 1×1 for the piezoelectric patch without avoiding the locking effects. That is equivalent to using the present mesh of 5×1 SH851FI elements. It can be seen that these solutions are indeed in very good agreement. However, due to the locking effects in the models obtained using SH851FI elements, the solutions are not well converged. This is shown by increasing the number of elements from 5×1 to 10×1 with the same element type SH851FI which leads to a totally different dynamic response compared to the one obtained by 5×1 elements.

5.3.2 Fully clamped smart plate

The second example of smart structures is a fully clamped plate with one piezoelectric centrally bonded patch, as shown in Fig. 5.20, which was first proposed by Yi *et al.* [151] like the previous one. The structure is made up of the same materials as

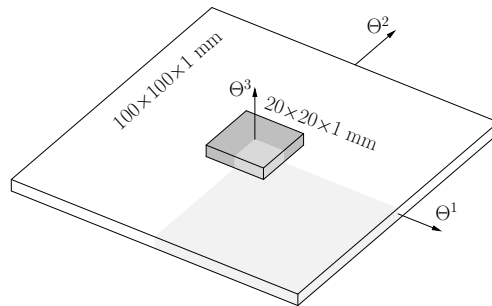


FIGURE 5.20: Fully clamped plate with one piezoelectric patch centrally bonded

the cantilevered smart beam, whose material properties are shown in Table 5.3. Due to the symmetry of the structure, only a quarter of the plate is considered, which is meshed with 5×5 SH851URI shell elements for both static and dynamic analysis.

Static analysis

Firstly, the structural response to a uniformly distributed pressure load up to 2×10^7 Pa is simulated with short circuit electrodes of the sensor patch. The mid-point deflection is calculated using various nonlinear theories as shown in Fig. 5.21 (a). The load deflection curves obtained by different theories including the commercial code ANSYS are almost the same. This can be explained by the fact that due to the clamped boundary conditions the plate undergoes only moderate rotations. The rotations, φ_1 and φ_2 , calculated by LRT56 theory using SH851URI elements under a pressure of 2×10^7 Pa, which are shown in Fig. 5.22 (a) and (b), respectively, are smaller than about 10° . It can be seen additionally that the rotations sharply jump to a maximum value from the fixed edges and smoothly decrease to zero at the mid-point of the plate. In Fig. 5.21 (b), the results for the sensor output voltage are displayed. It can be seen that the linear theory overpredicts the sensor output voltage, because it does not account for the stress stiffening effects. Since the plate undergoes only moderate rotations, the results obtained by all nonlinear theories are in very good agreement.

Dynamic analysis

The dynamic response is calculated by applying a uniformly distributed step pressure with the amplitude of 2×10^4 N/m². The linear and nonlinear dynamic response is calculated by NEWMARK method with a time step of 1×10^{-5} s for the linear case and 1×10^{-7} s for the nonlinear case. The dynamic response of the mid-point displacement and sensor output voltage both for linear and nonlinear simulations is presented in Fig. 5.23 (a) and (b), respectively. It can be observed that the linear and nonlinear transient response is almost identical, due to relatively small deflections resulting in weak nonlinear effects. The linear theory overpredicts slightly both the deflections and the sensor voltages. This is due to the fact that the linear theory does not account for the VON KÁRMÁN stress stiffening effect. The dynamic behavior obtained by LRT56 theory is in good accordance with that obtained by RVK5 theory. This indicates that the plate is undergoing only moderate rotations and deflections. The rotations of the plate can be seen from the static rotation analysis given in Fig. 5.22. The present results of both displacement and sensor output voltage agree quite well with those calculated by Lentzen and Schmidt [142], in which FOSD MRT5 theory was applied.

Increasing the pressure to 2×10^5 N/m², the results obtained by linear, RVK5 and LRT56 theories are displayed in Fig. 5.24. Now the amplitudes of the displacement

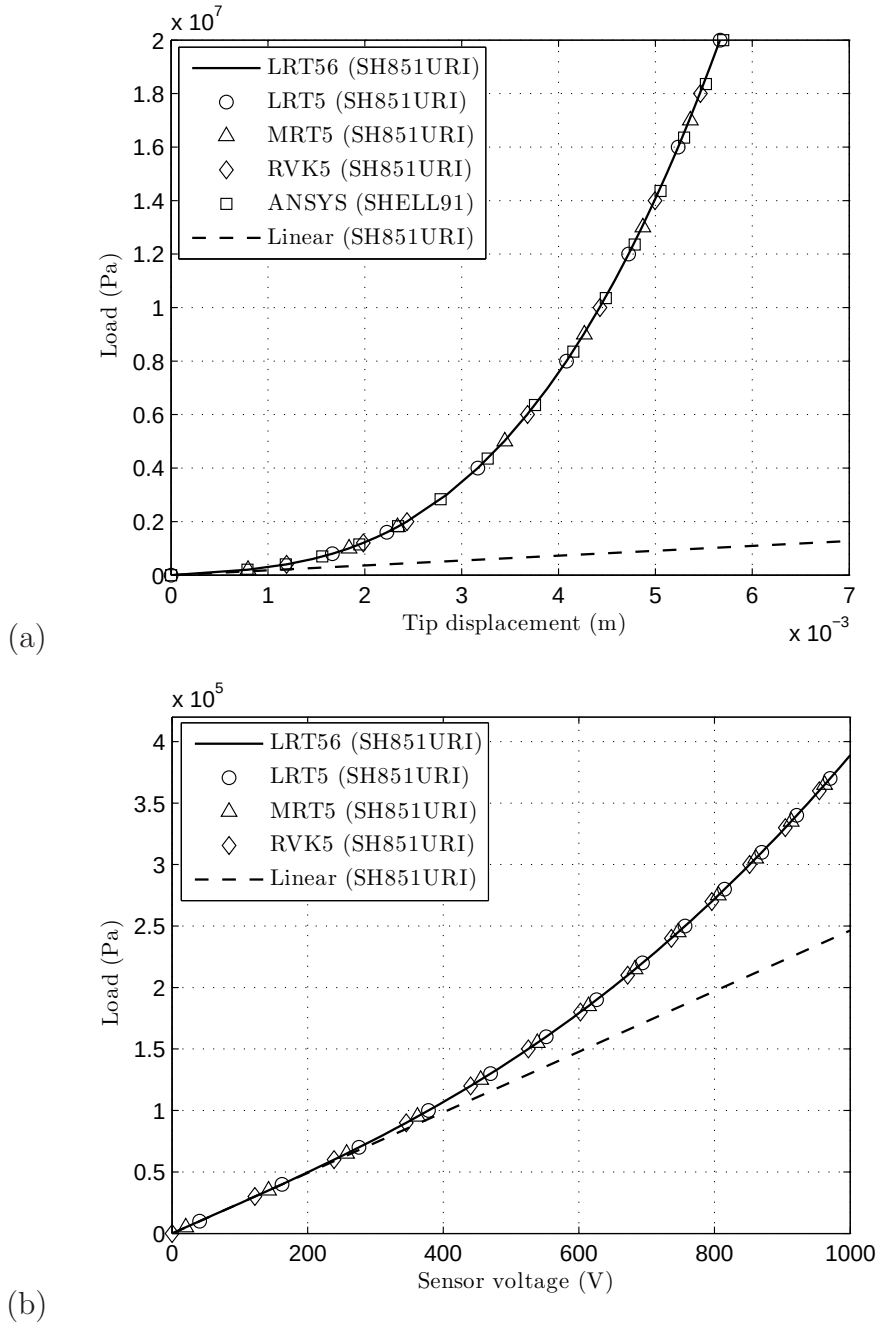


FIGURE 5.21: Static response of the fully clamped plate: (a) mid-point displacement, (b) sensor output voltage

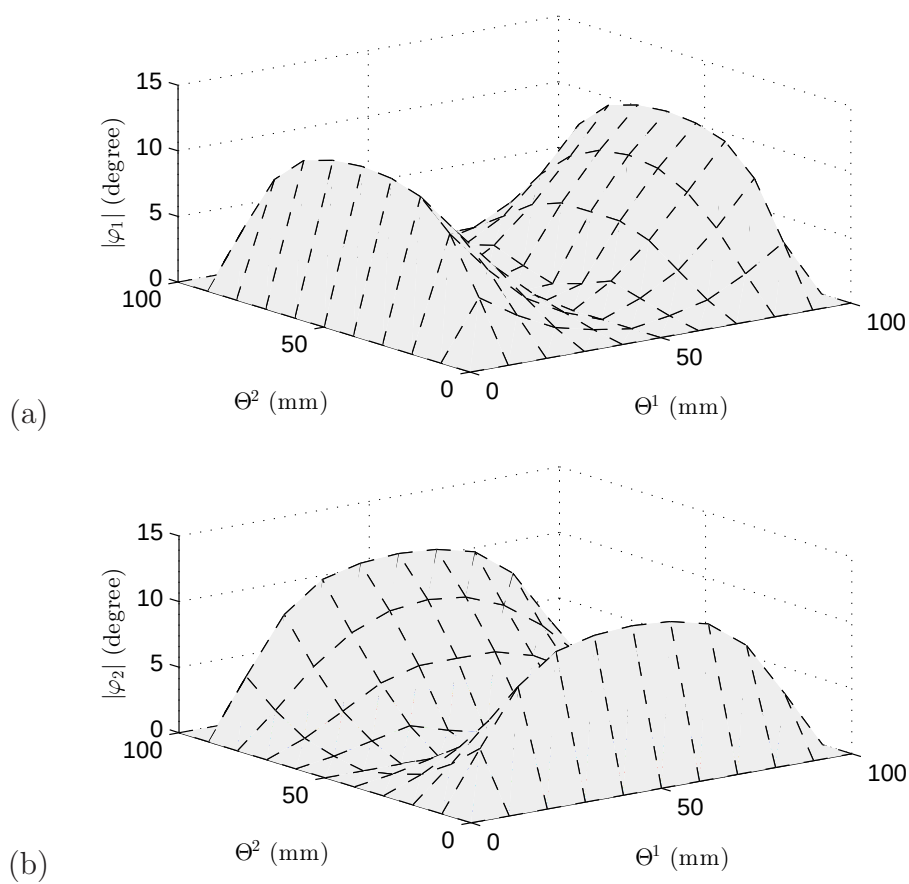


FIGURE 5.22: Rotations at each node of the plate under a pressure of 2×10^7 Pa:
 (a) rotations φ_1 about Θ^2 -axis, (b) rotations φ_2 about Θ^1 -axis

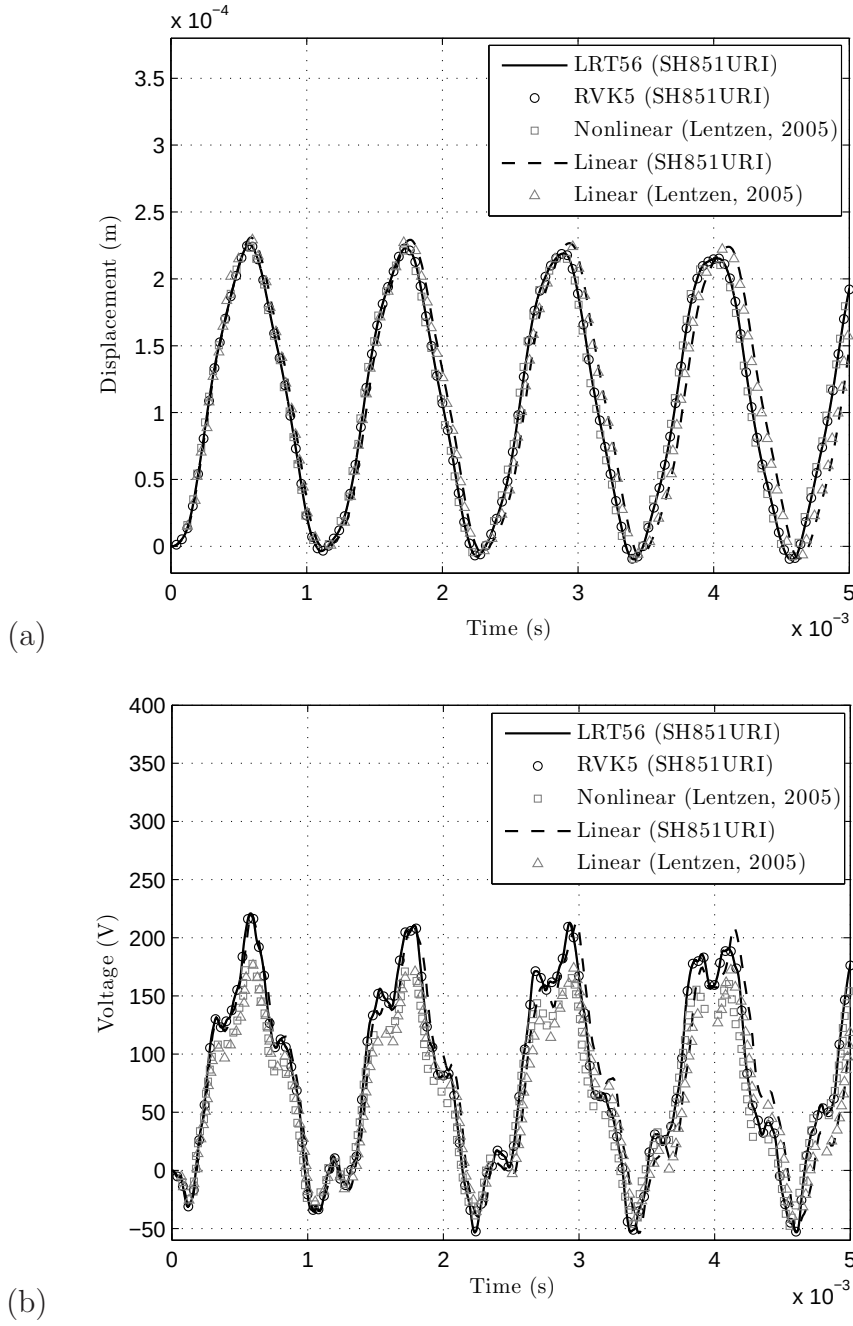


FIGURE 5.23: Dynamic response of the fully clamped plate under a step pressure of 2×10^4 Pa: (a) mid-point displacement, (b) sensor output voltage

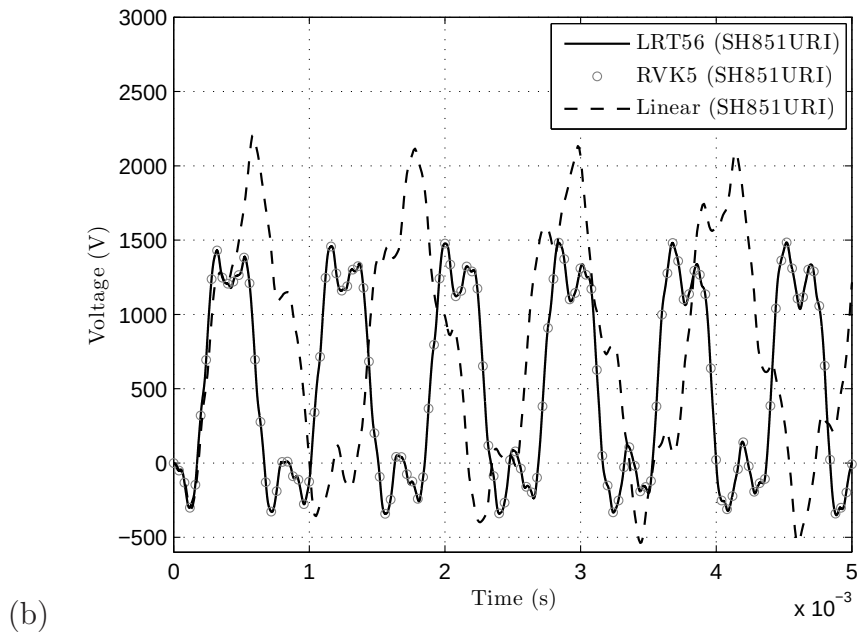
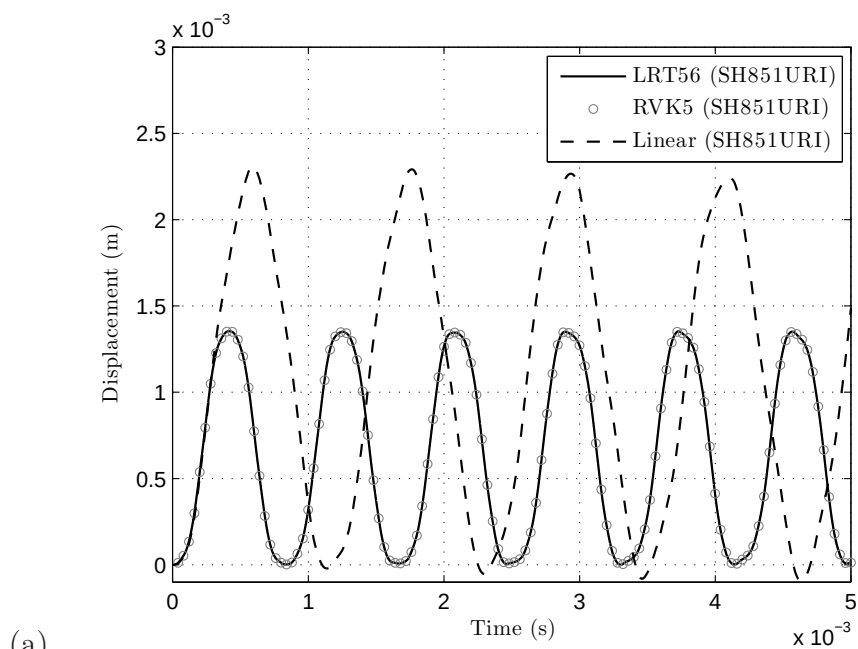


FIGURE 5.24: Dynamic response of the fully clamped plate under a step pressure of 2×10^5 Pa: (a) mid-point displacement, (b) sensor output voltage

are in the order of the magnitude of the plate thickness. Here a big difference can be observed between the amplitudes of the linear and nonlinear vibrations. This is due to the stress stiffening effect. Since the linear theory does not account for this effect, it overpredicts the amplitudes of both mid-point displacement and sensor output voltage. The RVK5 and LRT56 theories predict the same dynamic response, which shows that due to the clamped boundary conditions still only moderate rotations occur in the plate.

5.3.3 Fully clamped cylindrical smart shell

The following simulations consider a fully clamped cylindrical shell with one piezoelectric centrally integrated patch, as shown in Fig. 5.25, which was first proposed and calculated by Yi *et al.* [151] as well. The piezoelectric integrated cylindrical shell

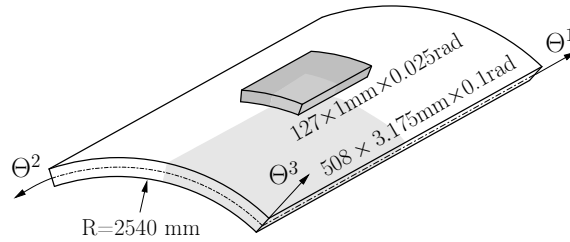


FIGURE 5.25: Fully clamped cylindrical shell with one piezoelectric patch centrally bonded

consists of a host shell made up of orthotropic material and a piezoelectric patch as actuator or sensor, whose material properties can be found in Table 5.4. The direction

TABLE 5.4: Material properties of the fully clamped smart cylindrical shell

Host shell	PZT
$E_1 = 124 \text{ GPa}$	$E = 67 \text{ GPa}$
$E_2 = 96.53 \text{ GPa}$	$\nu = 0.33$
$\nu_{12} = \nu_{23} = 0.34$	$\rho = 7800 \text{ kg/m}^3$
$G_{12} = G_{13} = G_{23} = 6.205 \text{ GPa}$	$d_{31} = d_{32} = -1.7119 \times 10^{-10} \text{ C/N}$
$\rho = 1520 \text{ kg/m}^3$	$\epsilon_{33} = 2.03 \times 10^{-8} \text{ F/m}$

of the fiber reinforcement of the orthotropic material is along the Θ^1 -axis. The structure is symmetric as the previous plate. Therefore, only a quarter of the cylindrical shell is simulated with a mesh of 8×4 SH851URI elements along the Θ^1 - and Θ^2 -axis, respectively.

Static analysis

The structural response to a uniformly distributed inner pressure up to 2×10^7 Pa is analyzed with short circuit electrodes of the sensor patch. The plot of the mid-point displacements using various geometrically nonlinear theories is shown in Fig. 5.26 (a). It can be seen that the results obtained by RVK5, MRT5, LRT5 and LRT56, as well as those calculated by ANSYS, are almost identical to each other, due to the fact that the shell undergoes only moderate rotations. This can be confirmed by the rotations, φ_1 and φ_2 , at each node, which are smaller than about 10° , as shown in Fig. 5.27 (a) and (b), respectively. It can also be seen that there are very slight discrepancies of displacement response between the curves obtained by LRT56/LRT5 and those by MRT5/RVK5. This is due to the fact that both MRT5 and RVK5 take less terms in the nonlinear strain-displacement relations into account than LRT56 and LRT5. Since the shell structure undergoes only moderate rotations, the sensor output voltage obtained by all nonlinear theories are nearly identical, as shown in Fig. 5.26 (b).

Dynamic analysis

In the dynamic analysis, the NEWMARK method is applied with a time step 1×10^{-5} s for the linear case and 1×10^{-7} s for the nonlinear case. The transient response of the mid-point displacement and the sensor voltage of the shell structure subjected to a uniformly distributed step load of 6×10^4 Pa is calculated and displayed in Fig. 5.28 (a) and (b), respectively. It can be seen that there is only a small difference between the results of both displacement and sensor voltage obtained by LRT56 and RVK5 theories, implying that the cylindrical shell undergoes deformations only in the range of moderate rotations. The dynamic response of the mid-point displacement agrees quite well with that in [142], where the linear and MRT5 theory were used, as shown in Fig. 5.28 (a).

Further, increasing the pressure to 6×10^5 Pa one obtains the dynamic response of displacement and sensor voltage shown in Fig. 5.29 (a) and (b), respectively. It can be seen that even at this load only small discrepancies exist between the results predicted by LRT56 and RVK5 theories. This indicates that due to the clamped boundary conditions the cylindrical shell is still undergoing only moderate rotations, although the sensor voltage output would be beyond the range of applicability of the linear piezoelectric constitutive relations.

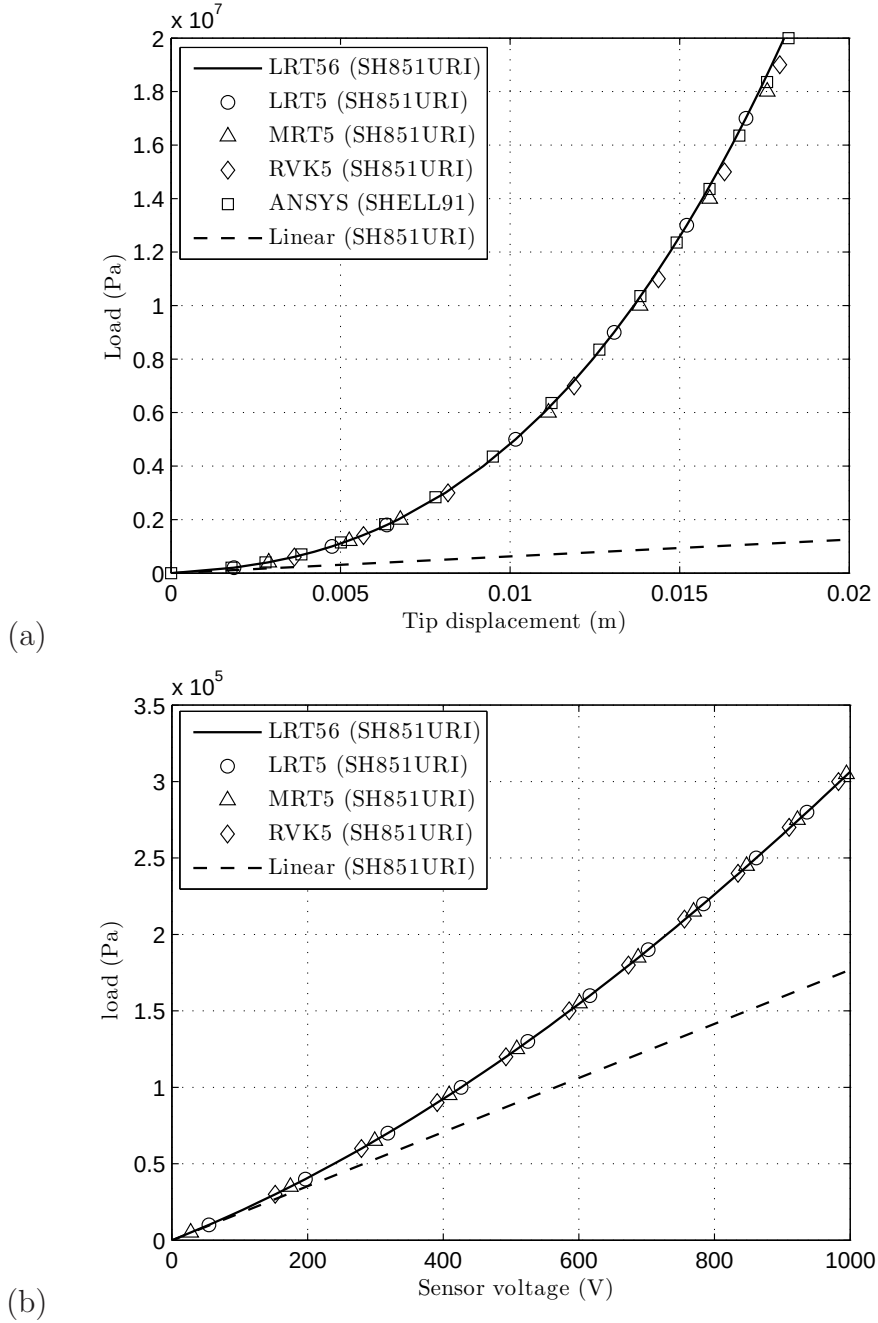


FIGURE 5.26: Static response of the fully clamped smart cylindrical shell: (a) mid-point displacement, (b) sensor output voltage

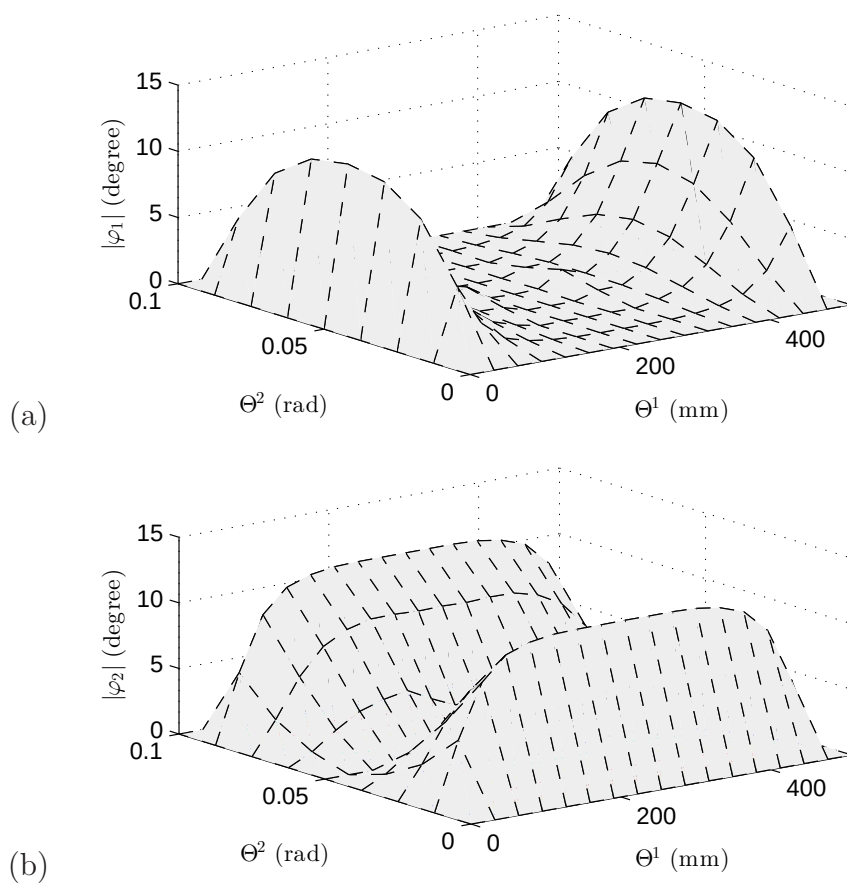


FIGURE 5.27: Rotations at each node of the cylindrical shell under a pressure of 2×10^7 Pa: (a) rotations φ_1 about Θ^2 -axis, (b) rotations φ_2 about Θ^1 -axis

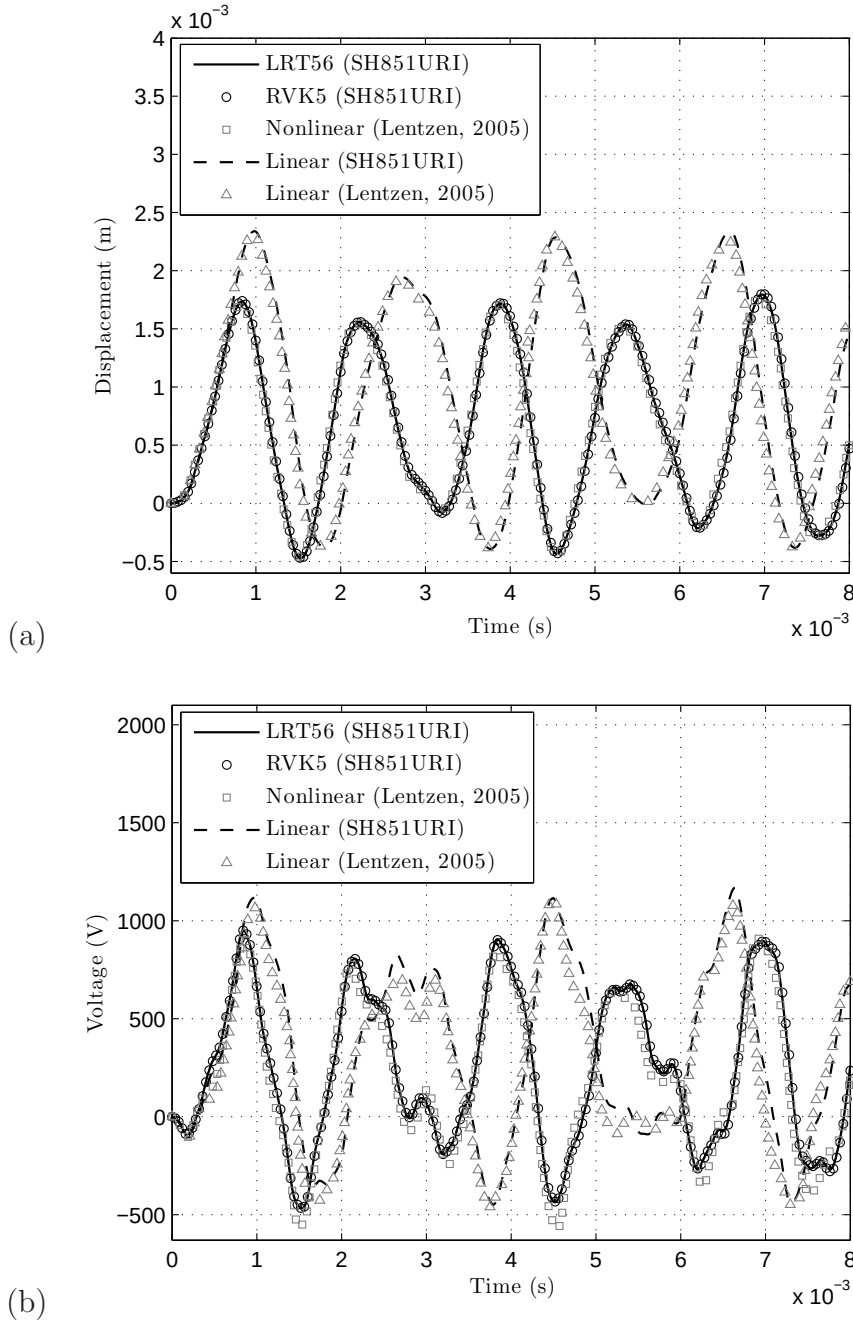


FIGURE 5.28: Dynamic response of the fully clamped cylindrical shell under a step pressure of 6×10^4 Pa: (a) mid-point displacement, (b) sensor output voltage

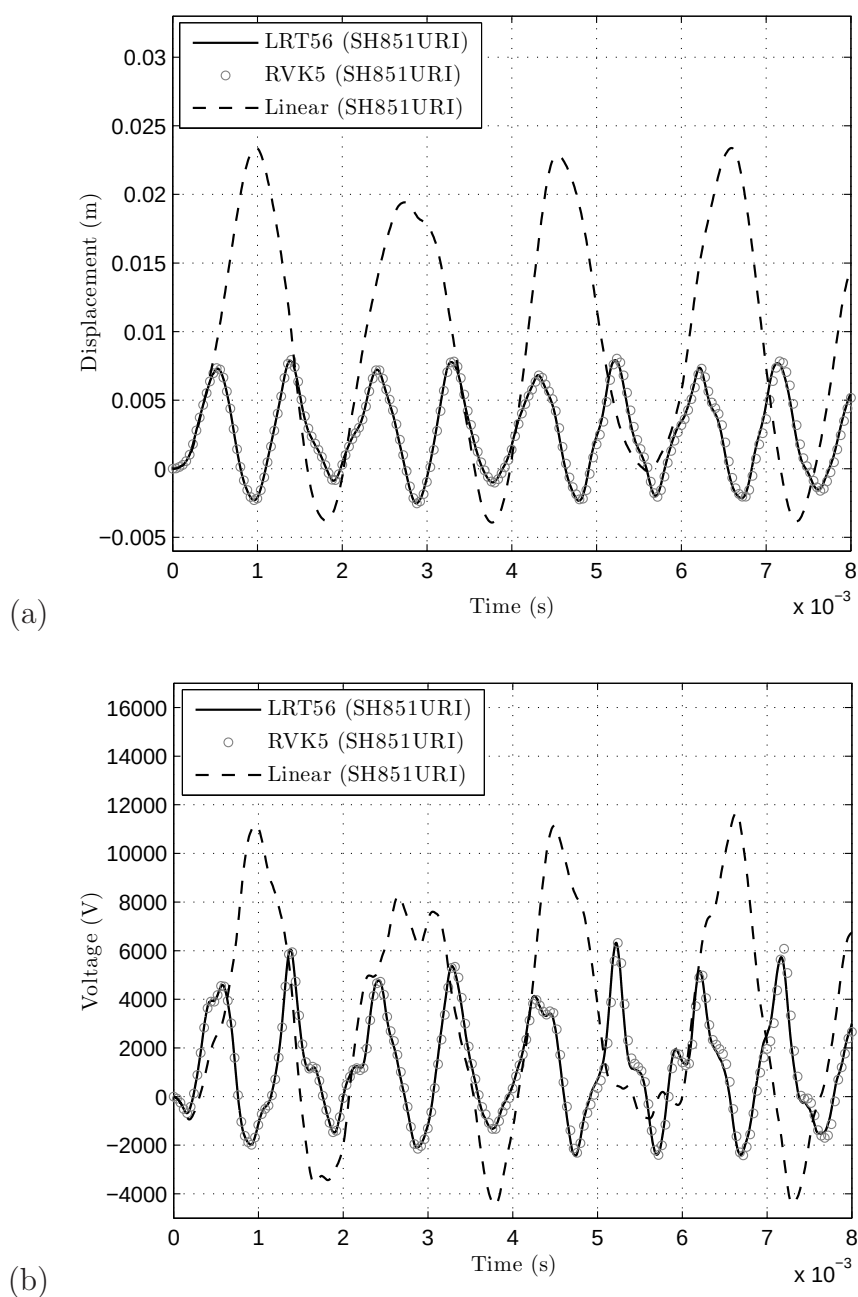


FIGURE 5.29: Dynamic response of the fully clamped cylindrical shell under a step pressure of 6×10^5 Pa: (a) mid-point displacement, (b) sensor output voltage

5.3.4 PZT laminated semicircular cylindrical shell

The last example for the nonlinear simulation of smart structures is a PZT laminated semicircular cylindrical shell, as shown in Fig. 5.30, which was first proposed and calculated by Tzou and Ye [237]. The cylindrical shell consists of one metallic layer

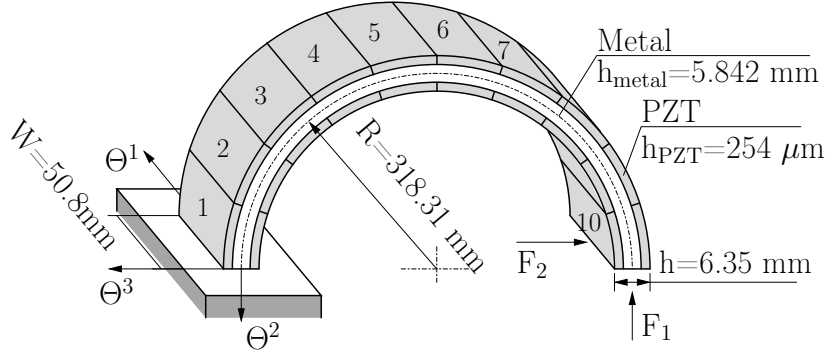


FIGURE 5.30: PZT laminated semicircular cylindrical shell

in the middle as the host structure and two PZT layers bonded on the both surfaces. The material properties are shown in Table 5.5. The cylindrical shell is clamped at

TABLE 5.5: Material properties of the PZT laminated semicircular cylindrical shell

Host shell	PZT
$E = 68.95 \text{ GPa}$	$E = 63 \text{ GPa}$
$\nu = 0.3$	$\nu = 0.3$
$\rho = 7750 \text{ kg/m}^3$	$\rho = 7600 \text{ kg/m}^3$
	$d_{31} = d_{32} = -1.79 \times 10^{-10} \text{ C/N}$
	$\epsilon_{33} = 1.65 \times 10^{-8} \text{ F/m}$

one straight edge. The structure is discretized by 1×10 SH851URI elements along the Θ^1 - and Θ^2 -direction, respectively.

Linear analysis

Firstly, the linear dynamic behavior of the semicircular cylindrical shell is analyzed. The first five eigen-frequencies are calculated by linear theory and compared with those reported in the literature, as shown in Table 5.6. From the table, it can be seen that the present five eigen-frequencies agree quite well with those calculated by Sze and Yao [24] using both ABAQUS and their own code. But there is a big difference between the present results and Tzou's results [237], because probably a different YOUNG'S modulus of the metal has been used in the calculations.

TABLE 5.6: First five eigen-frequencies of the PZT laminated semicircular cylindrical shell (Hz)

	mesh	mode 1	mode 2	mode 3	mode 4	mode 5
present	1×10 SH851URI	3.7199	5.8530	11.7782	33.4356	40.3605
Sze and Yao [24]	1×10 S9R5	3.7475	5.8971	11.856	33.634	40.626
Sze and Yao [24]	2×20 4-node	3.6810	5.8041	11.691	33.231	40.450
Tzou and Ye [237]	2×10 Triangular	8.17	25.66	86.93	194.14	346.08

Nonlinear static analysis

In the next simulation, the nonlinear static response of the tip point displacements and the sensor output voltages of the inner layer is calculated with applying a load in hoop direction. Considering the concentrated force F_1 varying from 0 to 200 N applied on the tip point in hoop direction, one obtains the load-deflection curves in both hoop and radial directions, and the load-voltage curves of the inner PZT layer, as displayed in Fig. 5.31. The results show that RVK5 theory predicts stiffer displacements and sensor output voltages of the inner PZT layer than linear theory does. It can also be observed that MRT5 and LRT5 theories predict very similar static response of the hoop displacement and the sensor output voltage, which confirm our statement in Section 2 that the LRT5 theory is restricted to the range of moderate rotations even though full geometrically nonlinear relations are used. By comparing the results predicted by RVK5, MRT5, LRT5 and LRT56 in Fig. 5.31 (a), it can be recognized that they deviate at a common point from the LRT56 load-deflection curve which indicates that large rotations occur. The hoop deflections obtained by MRT5, LRT5 and LRT56 show first a softening tendency which turns at large loads to a stiffening behavior. The plot of the sensor output voltage follows this tendency. Interestingly, the radial deflections obtained by LRT56 first increase but later decrease with increasing load. The figures also illustrate that big differences are existing among the results obtained by linear and various nonlinear shell theories. It can be concluded that the results obtained by LRT56 are the most accurate ones, since the full geometric nonlinearities and large rotations are considered in the theory.

Nonlinear dynamic analysis

In the last part, the semicircular cylindrical shell is used for nonlinear vibration simulation using various shell theories, which have been applied in the previous section

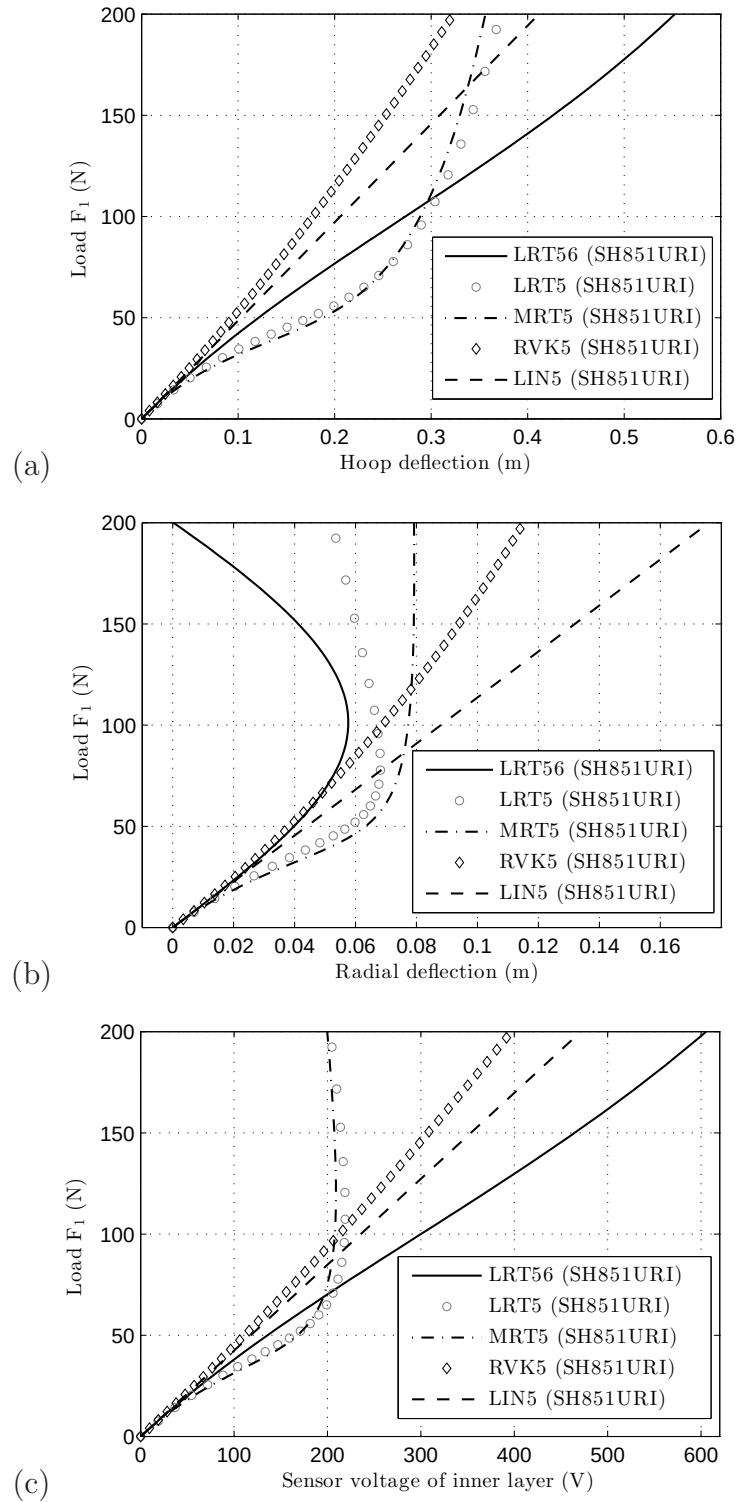


FIGURE 5.31: Static response of the PZT laminated semicircular cylindrical shell under a concentrated force F_1 in the hoop direction: (a) hoop deflection, (b) radial deflection, (c) sensor output voltage of the inner PZT layer

for static analysis. A concentrated step force $F_1 = 50$ N is applied on the tip point at the free end in hoop direction along positive Θ^2 -line. The dynamic response of the tip displacement in the hoop and radial directions, as well as the output voltage of the inner PZT layer is calculated using NEWMARK method with a time step 1×10^{-3} s for the linear case and 1×10^{-4} s for the nonlinear case, which is respectively shown in Figs. 5.32 (a), (b) and (c).

Since MRT5 and LRT5 theories predict similar static response, it can be seen that there is not much difference between the transient behavior obtained by MRT5 and LRT5 theories. Interestingly, the dynamic response of the hoop deflection obtained by the nonlinear shell theories LRT56 and MRT5 has larger amplitudes than those predicted by linear theory, while those obtained by RVK5 have smaller amplitudes. This corresponds to the static results displayed in Fig. 5.31 (a) which shows that LRT56 and MRT5 predict a softer, but RVK5 a stiffer response than the linear simulations. Furthermore, comparing the simulations obtained by LRT56 and MRT5 in Fig. 5.31 and Fig. 5.32, it can be observed that in a wide range of load (including the step load $F_1 = 50$ N) MRT5 overpredicts the static hoop and radial deflections, the amplitudes of the vibrations in hoop and radial directions, as well as the static sensor output voltage. These observations are also confirmed by a comparison of the frequencies of the transient response of the tip deflection and the sensor output voltage in Fig. 5.32, which shows that the stiffest response is predicted by RVK5, followed, in this sequence, by the linear theory, LRT56 and MRT5. Additionally, MRT5, LRT5 and LRT56 predict a superimposed transient response in the graph of the radial deflection. This yields smaller amplitudes than those predicted by the linear theory, which is in contrast to the tendency in the static analysis. A similar phenomenon occurs in the graph of the inner layer sensor output voltage predicted by MRT5 and LRT5.

Comparing all these nonlinear dynamic results, it can be observed that big differences exist between those obtained by various nonlinear shell theories. We may conclude that those calculated by LRT56 theory are the best simulations and yield the time response closest to the real one, because full geometric nonlinearities and arbitrary rotations are considered in the theory, in contrast to the simplified nonlinear shell theories RVK5, MRT5 and LRT5.

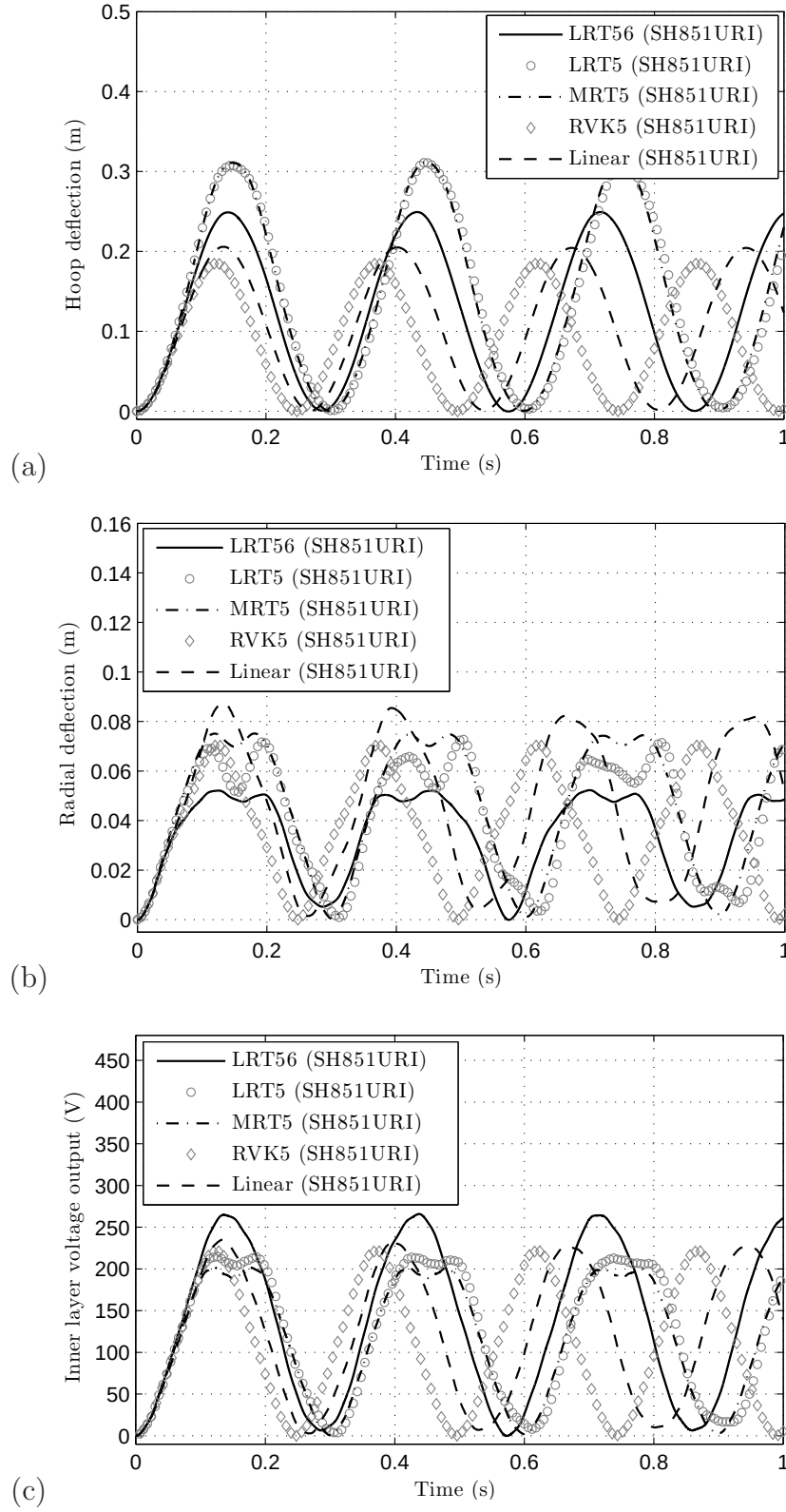


FIGURE 5.32: Dynamic response of the PZT laminated semicircular cylindrical shell under a step tip force of 50 N: (a) hoop deflection, (b) radial deflection, (c) sensor output voltage of the inner PZT layer

Chapter 6

Simulation of active vibration control*

This chapter presents the results of active vibration control by various control strategies, including PID, LQR and LQG control, as well as DR control with PI or GPI observer. In order to show the advantages of each control strategy, several kinds of disturbances, e.g. step disturbance, periodic disturbance and random disturbance, are applied to smart structures. Additionally, these controllers are also tested in different piezoelectric bonded smart systems.

6.1 Active vibration control of a smart beam

6.1.1 Cantilevered beam with collocated piezoelectric patches

A cantilevered beam with collocated piezoelectric patches bonded on both surfaces at a distance of 50 mm from the cantilevered end (two-PZT-patch beam), as shown in Figure 6.1, is used for the validation test of the PID, LQR, LQG and DR control. The upper piezoelectric patch acts as a sensor and the lower one as an actuator. The patches have opposite polarizations in the direction of the outward normal vectors of the upper and lower surfaces, respectively. A concentrated force disturbance will be applied at the tip point of the free end. The dimensions of the master structure are $350 \times 25 \times 0.8$ mm, and those of the piezoelectric patches are $75 \times 25 \times 0.25$ mm.

*Part of the numerical results reported in this chapter were published in the article “Disturbance rejection control for vibration suppression of piezoelectric laminated thin-walled structures” by S. Q. Zhang *et al.*, *J. Sound Vib.*, 333: 1209-1223, 2014 [231], Copyright (2013) Elsevier Ltd.

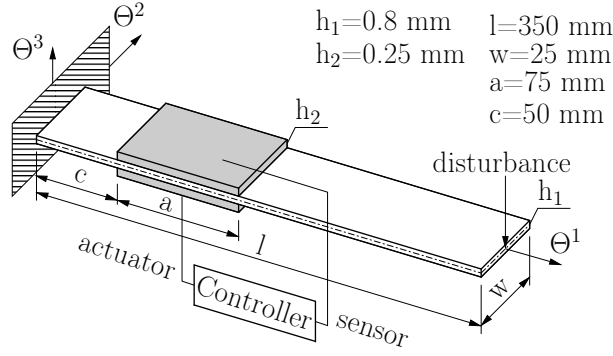


FIGURE 6.1: Cantilevered beam with collocated piezoelectric patches

The smart structure is made up of spring steel as the host structure and the two PZT patches as sensors and actuators. The material properties are shown in Table 6.1.

TABLE 6.1: Material properties of the two-PZT-patch beam

Steel	PZT
$E = 210 \text{ GPa}$	$E = 67 \text{ GPa}$
$\nu = 0.3$	$\nu = 0.3$
$\rho = 7900 \text{ kg/m}^3$	$\rho = 7800 \text{ kg/m}^3$
	$d_{31} = d_{32} = -2.1 \times 10^{-10} \text{ C/N}$
	$\epsilon_{33} = 2.13 \times 10^{-8} \text{ F/m}$

A linear piezoelectric coupled dynamic FE model is built based on the linear variant of FOSD shell theory described in the previous chapters. Furthermore, the damping matrix is obtained by the RAYLEIGH damping coefficient computation method with a damping ratio of 0.8% for the first six modes. In order to validate the FE model, two different meshes (5×1 and 14×1 along the Θ^1 - and Θ^2 -axis, respectively), as well as two element types (SH851FI and SH851URI), are considered. The first five eigen-frequencies are calculated and compared in Table 6.2. From the table, it can be

TABLE 6.2: First five eigen-frequencies of the two-PZT-patch beam (Hz)

Element	mesh	mode 1	mode 2	mode 3	mode 4	mode 5
SH851FI	5×1	6.2144	35.0444	110.1355	176.5464	178.0647
SH851FI	14×1	6.1507	33.7150	97.3652	166.0701	175.0620
SH851URI	5×1	6.1300	33.5314	97.0450	162.4192	175.9936
SH851URI	14×1	6.0672	33.2706	95.0806	163.4342	175.0233

seen that only slight locking effects occur in the SH851FI elements. In order to avoid

a large size of the FE model, a discretization by 5×1 SH851FI elements will be used and the system is reduced to the first 12 modes.

6.1.2 Results of LQR and LQG control

Two weighting matrices $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ are required in LQR control, which can be approximately calculated by BRYSON's rule given in (4.62). If the maximum output and input are set to be 10 V and 100 V for example, the weighting matrices $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ are 1×10^{-2} and 1×10^{-4} , respectively. Other cases can be found in Table 6.3.

TABLE 6.3: LQR control parameters for the two-PZT-patch beam

Case	$\bar{\mathbf{Q}}$	$\bar{\mathbf{R}}$	ρ	Max. output	Max. input
LQR case 1	1×10^{-2}	1×10^{-4}	1	10 V	100 V
LQR case 2	4×10^{-2}	1×10^{-4}	1	5 V	100 V
LQR case 3	4×10^{-2}	2.5×10^{-5}	1	5 V	200 V

Firstly, the free vibrations are considered to test the controllers. A concentrated force of 0.2 N is applied quasi-statically at the tip point, and then the beam is released. Using the parameters of LQR control given in Table 6.3, the free vibrations can be actively suppressed by LQR controller, which are presented in Fig. 6.2. The results show that increasing $\bar{\mathbf{Q}}$ or decreasing $\bar{\mathbf{R}}$ will generate a larger amplitude of the control input voltage, which results in smaller amplitude of vibrations.

Since LQR control is a full state feedback control, it needs all state variables being measured, which is not applicable in most of the cases. Compared with LQR control scheme, LQG control is more practical, which contains an observer for estimation of the state variables using the measured signals. It has been discussed before that the LQG control needs two additional weighting matrices, $\mathbf{Q}_g$ and $\mathbf{R}_g$, for computation of the observer gains, which can be determined by the noises existing in the system. All the parameters of LQG control are given in Table 6.4.

TABLE 6.4: LQG control parameters for the two-PZT-patch beam

Case	$\bar{\mathbf{Q}}$	$\bar{\mathbf{R}}$	ρ	$\mathbf{Q}_g$	$\mathbf{R}_g$
LQG case 1				1×10^{-10}	1×10^{-6}
LQG case 2	4×10^{-2}	2.5×10^{-5}	1	1×10^{-10}	1×10^{-8}
LQG case 3				1×10^{-8}	1×10^{-12}

Large $\mathbf{Q}_g$ means large disturbance noises are considered, and similarly large measurement noises are added if $\mathbf{R}_g$ is large. Considering $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$ values in LQR control of case 3, the results can be derived using three different pairs of weighting matrices for observer gains, which are shown in Fig. 6.3. From the figure, it can be noticed that the weighting matrices $\mathbf{Q}_g$ and $\mathbf{R}_g$ do not influence the damping ratio in the free vibration case, since all results obtained by LQR and LQG control using the parameters in Table 6.4 are same.

Applying a step disturbance force with the amplitude of 0.1 N starting from 0.2 s one obtains the dynamic behavior with LQG control, as shown in Fig. 6.4. It can be seen that $\mathbf{Q}_g$ and $\mathbf{R}_g$ affect the steady-state error. However, the steady-state error cannot be counteracted as good as by LQR control. Consequently, LQG control will not be better than LQR control.

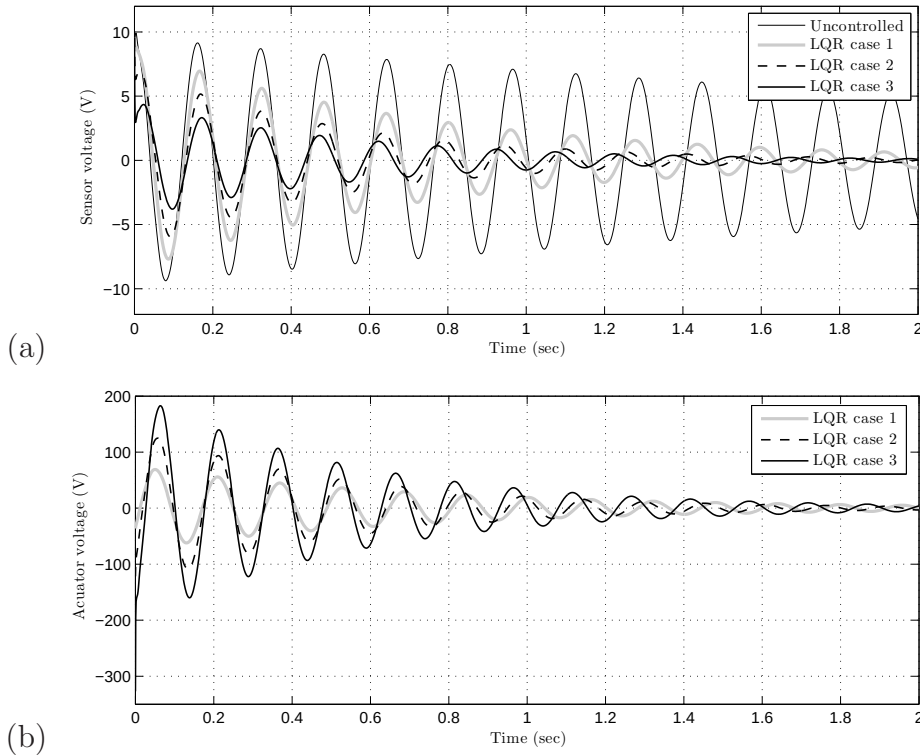


FIGURE 6.2: The dynamic behavior of the two-PZT-patch beam by LQR control for free vibration: (a) sensor output, (b) control input

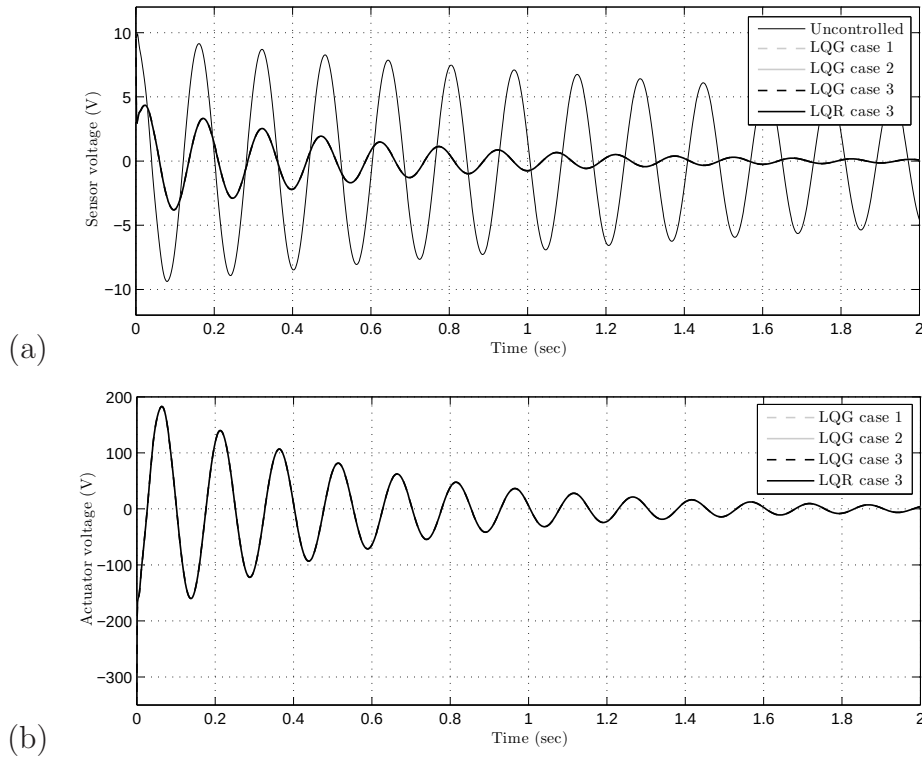


FIGURE 6.3: The dynamic behavior of the two-PZT-patch beam by LQR and LQG control for free vibration: (a) sensor output, (b) control input

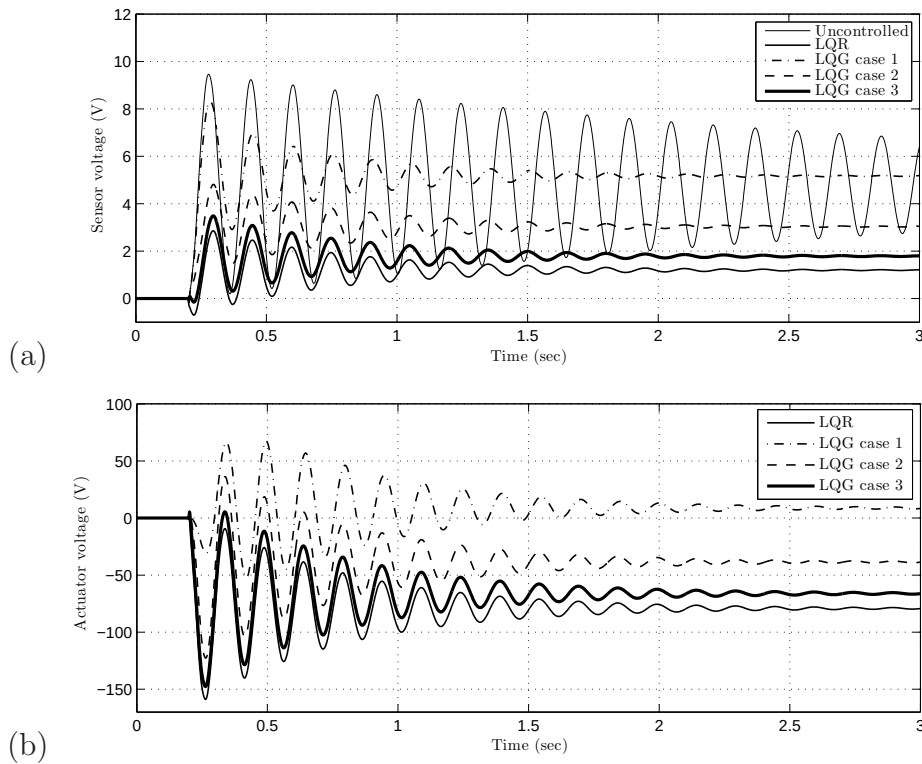


FIGURE 6.4: The dynamic behavior of the two-PZT-patch beam by LQR and LQG control under a step disturbance force: (a) sensor output, (b) control input

6.1.3 Results of PID control

From the theoretical approach of PID algorithm discussed in Chapter 4, three control gains, $\mathbf{K}_p$, $\mathbf{K}_i$ and $\mathbf{K}_d$, respectively counteract the output error, the integral of the output error and the error dynamics. Unfortunately, there is no mathematical way to calculate these gains. Usually, the gains can be tuned by empirical formulae, which can be found in many control books, e.g. [225]. Here, the gains are tuned manually, which is also very often used in practical projects, rather than using empirical formulae. The gains considered in the following simulations are listed in Table 6.5, which are tuned in order to make the amplitudes of the control input as large as those in LQR control.

TABLE 6.5: PID control parameters for the two-PZT-patch beam

Case	Control type	$\mathbf{K}_p$	$\mathbf{K}_i$	$\mathbf{K}_d$
PID case 1	D control	0	0	0.198
PID case 2		0	0	0.424
PID case 3		0	0	0.759
PID case 4	PD control	4.5	0	0.198
PID case 5		17.2	0	0.424
PID case 6		66	0	0.759
PID case 7	PID control	4.5	200	0.198
PID case 8		17.2	200	0.424
PID case 9		66	200	0.759

Considering the free vibration of the system, the measured signals are oscillating around zero, thus no steady-state error occurs. Therefore, $\mathbf{K}_p$ and $\mathbf{K}_i$ have no effect on the damping ratio. For the free vibration of the two-PZT-patch beam, only D control (PID case 1, 2, 3) schemes are simulated, whose results are shown in Fig. 6.5. From the results, it can be concluded that larger derivative gains $\mathbf{K}_d$ lead to larger damping ratios in the free vibration case. However the system will become unstable if $\mathbf{K}_d$ is too large. As with LQR and LQG control, a similar vibration suppression effect can be achieved by tuning the derivative gain of D control, as shown in Fig. 6.6.

Next, the same step force as applied in the LQR control simulation is considered to excite the two-PZT-patch beam. The dynamic response regulated by PID and LQR with various parameters is shown in Fig. 6.7. The results show that D control (PID case 1) only attenuates the dynamic vibration, rather than the steady-state error.

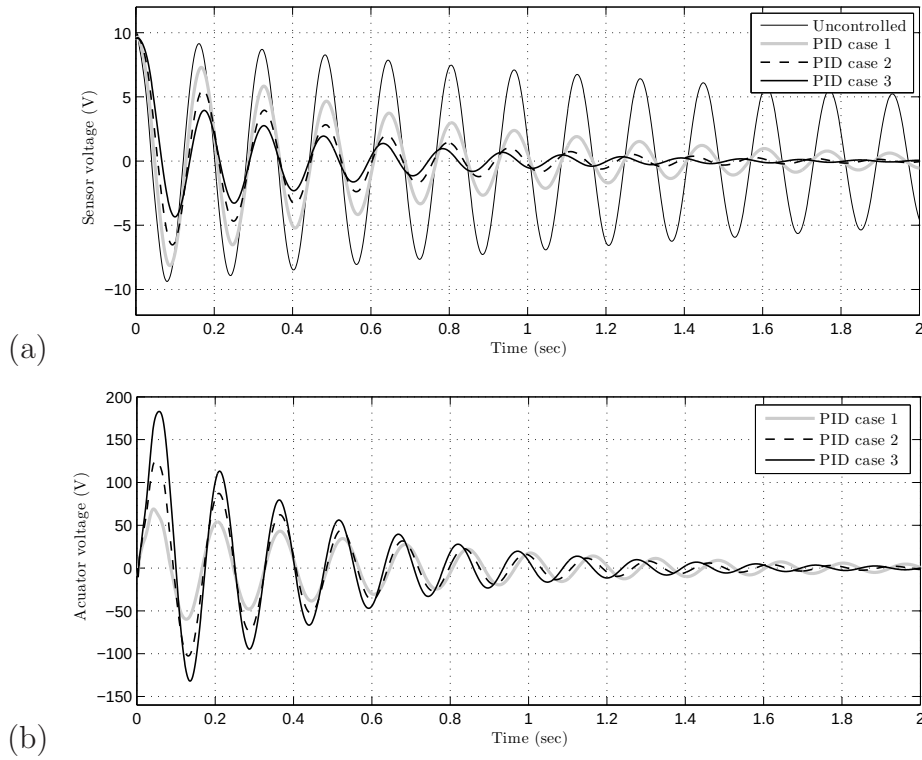


FIGURE 6.5: The dynamic behavior of the two-PZT-patch beam by PID control for free vibration: (a) sensor output, (b) control input

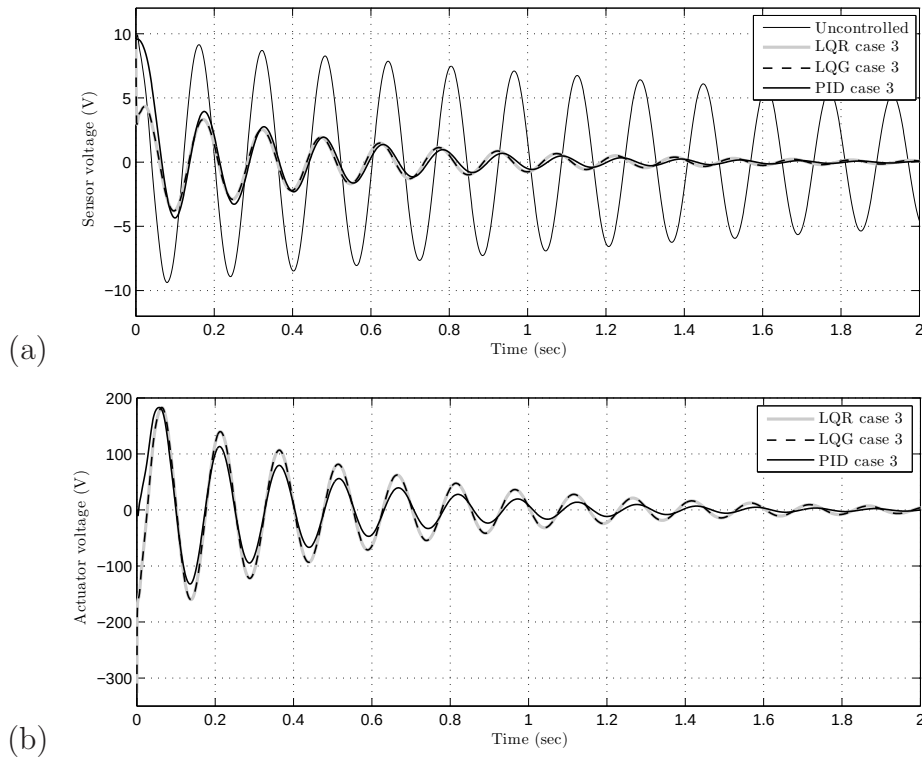


FIGURE 6.6: The dynamic behavior of the two-PZT-patch beam by PID, LQR and LQG control for free vibration: (a) sensor output, (b) control input

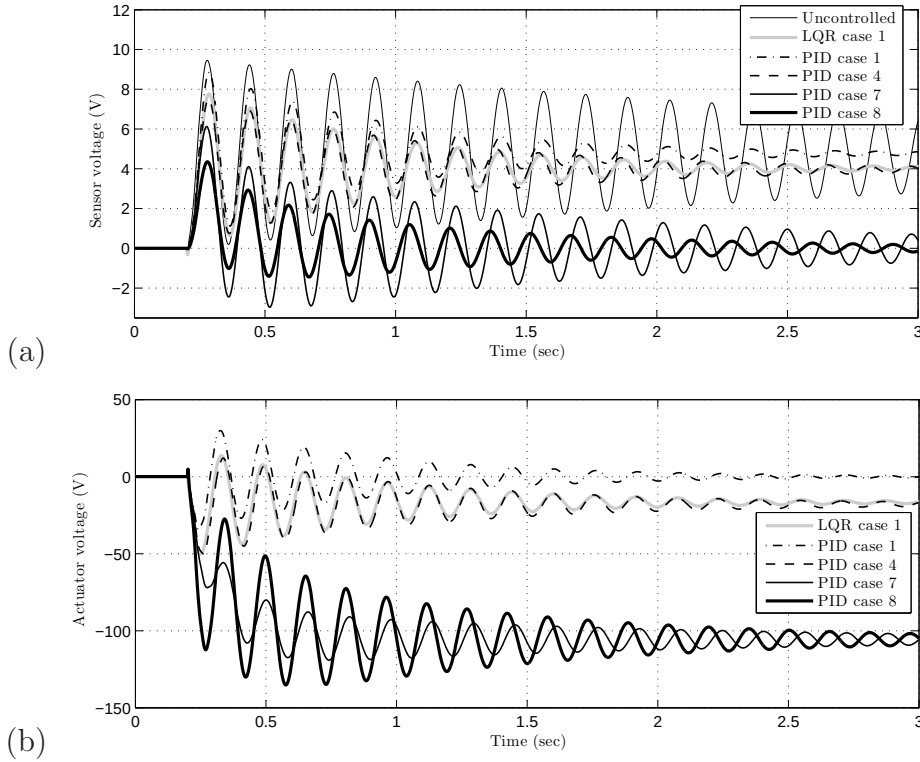


FIGURE 6.7: The dynamic behavior of the two-PZT-patch beam by PID and LQR control under a step disturbance force: (a) sensor output, (b) control input

Adding $\mathbf{K}_p$ results in PD control (PID case 4), which not only attenuates the dynamic part, but also the steady-state error. If the gains $\mathbf{K}_p$ and $\mathbf{K}_d$ (PID case 4) are tuned to generate the same amplitude of the control input as LQR control (LQR case 1), it will yield similar control effects. Further considering the integral of the steady-state error by adding $\mathbf{K}_i$, the resulting PID (case 7) control law eliminates the steady-state error completely. However the integral part will deteriorate the dynamic behavior. Again, we can tune $\mathbf{K}_p$ and $\mathbf{K}_d$ to improve the dynamic behavior, as PID (case 8) predicts.

6.1.4 Results of disturbance rejection control

The unknown disturbances are assumed to be concentrated forces applied at the tip point of the beam given in Section 6.1.1. The DR control with PI or GPI observer is tested on the smart beam under various kinds of external force disturbances, namely step disturbance, harmonic disturbance, triangle wave disturbance, square wave disturbance and random disturbance.

Parameters configuration

The control gain matrix $\mathbf{K}_x$ for DR control is derived based on the weighting matrices $\bar{\mathbf{Q}} = 1/(10)^2$ and $\bar{\mathbf{R}} = 1/(200)^2$ through all the cases in this section, as well as that for the simulation of LQR control. The observer gains are obtained by solving the algebraic RICCATI equation given in (4.131) using $b = 100$ in all simulations if it is not stated.

As mentioned before, the GPI observer can be realized by using the matrices $\mathbf{H}$ and $\mathbf{V}$ given in (4.102), in which the angular frequency can be either known or unknown. From the theory, it is known that the same fictitious model of disturbances described by $\mathbf{H}$ and $\mathbf{V}$ matrices will lead to the same performance of the GPI observer, which produces the same estimated disturbances. In Section 4.5.6, it has been discussed that there are two ways for calculating the control gain matrix $\mathbf{K}_v$, which are respectively given in (4.140) for the exact solution, and in (4.143) for the approximate solution. Therefore, for a GPI observer four possibilities exist, the notations of which are explained in Table 6.6, where ω_0 is the frequency of the periodic disturbance.

TABLE 6.6: Notations for the GPI observers

Case	ω [rad/s]	Solution of $\mathbf{K}_v$
GPI case 1	$\omega = 1$	approximate solution, (4.143)
GPI case 2	$\omega = 1$	exact solution, (4.140)
GPI case 3	$\omega = \omega_0$	approximate solution, (4.143)
GPI case 4	$\omega = \omega_0$	exact solution, (4.140)

Two kinds of periodic disturbances are considered in the simulation, including low frequency disturbances which have $\omega = \pi$ rad/s and high frequency ones with $\omega = 10\pi$ rad/s. Based on the considered frequencies, the control gain matrix $\mathbf{K}_v$ is calculated both approximately and exactly, as shown in Table 6.7. The data given in Table 6.7, illustrate that the approximate and exact solution will give similar control gain if the frequency of the periodic disturbance is small enough. Otherwise, large disturbance frequency will affect the control gains significantly, which influences the control effects of vibration suppression.

TABLE 6.7: Control gains $\mathbf{K}_v$ solved by different solutions

Case	ω [rad/s]	$\mathbf{K}_v$	Applicable example
GPI case 1	$\omega = 1$	[815.4122, 815.4122, 0]	all disturbances
GPI case 2	$\omega = 1$	[815.4122, 815.8548, 12.6937]	
GPI case 3	$\omega = \pi$	[815.4122, 815.4122, 0]	low frequency disturbances
GPI case 4	$\omega = \pi$	[815.4122, 819.8013, 40.0779]	
GPI case 3	$\omega = 10\pi$	[815.4122, 815.4122, 0]	high frequency disturbances
GPI case 4	$\omega = 10\pi$	[815.4122, 1740.4256, 890.6405]	

Step disturbance

In the first DR control simulation, the two-PZT-patch beam is excited by a step disturbance force, which occurs at 0.5 s with the amplitude of 0.1 N. As described above, there are two approaches for obtaining the observer gains, LYAPUNOV approach given in (4.127) and RICCATI approach in (4.131). Using the PI observer with the gains calculated by these two approaches, one obtains the estimated step disturbance forces, as shown in Fig. 6.8, from which it can be seen that the PI observer calculated by LYAPUNOV approach behaves not as good as that by RICCATI approach. The former one estimates the step disturbance with larger overshoot, while the latter one performs excellently. But this depends very much on the choice of the parameters a and b .

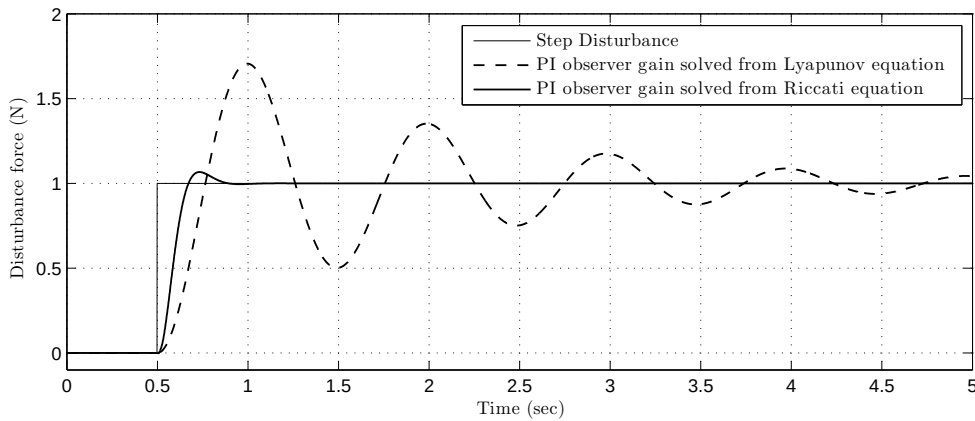
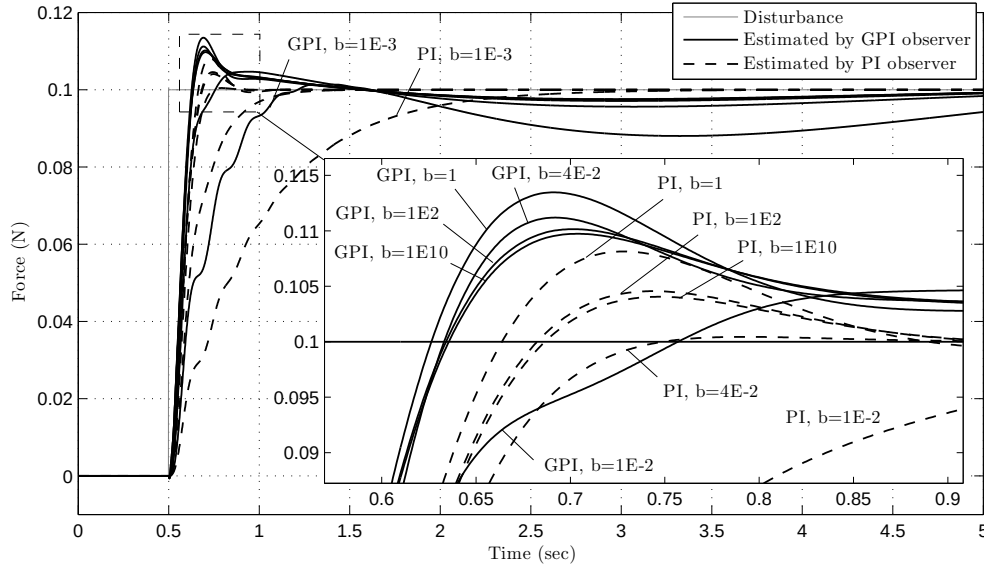


FIGURE 6.8: Estimated step disturbances using PI observer with observer gains obtained by LYAPUNOV and RICCATI approaches

As discussed before, the observer gains are affected by the parameter b in (4.131), which determines the observer dynamic behavior. The estimated signals of the step disturbance by the PI and GPI observers using different b , varying from $b = 1 \times 10^{-3}$ to $b = 1 \times 10^{10}$, are calculated and shown in Fig. 6.9. Generally, the signals estimated

FIGURE 6.9: Estimated step disturbances using various b

by the GPI observer rise much faster than those by the PI observer with the same b , but they have larger overshoot than those estimated by the PI observer, implying that the GPI observer has better dynamic behavior. Interestingly, the rise time of the disturbance estimated by the PI observer decreases as b increases from $b = 1 \times 10^{-3}$ to about $b = 1$, and then the rise time increases, but slowly. A similar phenomenon occurs in the results derived by the GPI observer.

The vibrations are suppressed by LQR control and DR control with PI or GPI observer, respectively, which are shown in Fig. 6.10. The uncontrolled and controlled sensor signals are displayed in Fig. 6.10 (a), and the corresponding control input voltages applied on the actuator are shown in Fig. 6.10 (b). The figures illustrate that the DR control methods which take the unknown disturbances into account by using PI or GPI observer lead to better vibration suppression than LQR control. Additionally, the disturbances can be estimated by PI or GPI observer, as shown in Fig. 6.10 (c), from which it can be seen that the signal estimated by the GPI observer has shorter rise time but larger overshoot than that estimated by the PI observer.

Harmonic disturbance

In the second validation test, a harmonic disturbance force with an angular frequency of π rad/s is applied on the tip point of the smart beam. The harmonic disturbance is produced by the function $\mathbf{f}(t) = 0.1 \times \cos(\pi t)$ N. The sensor signals and the control

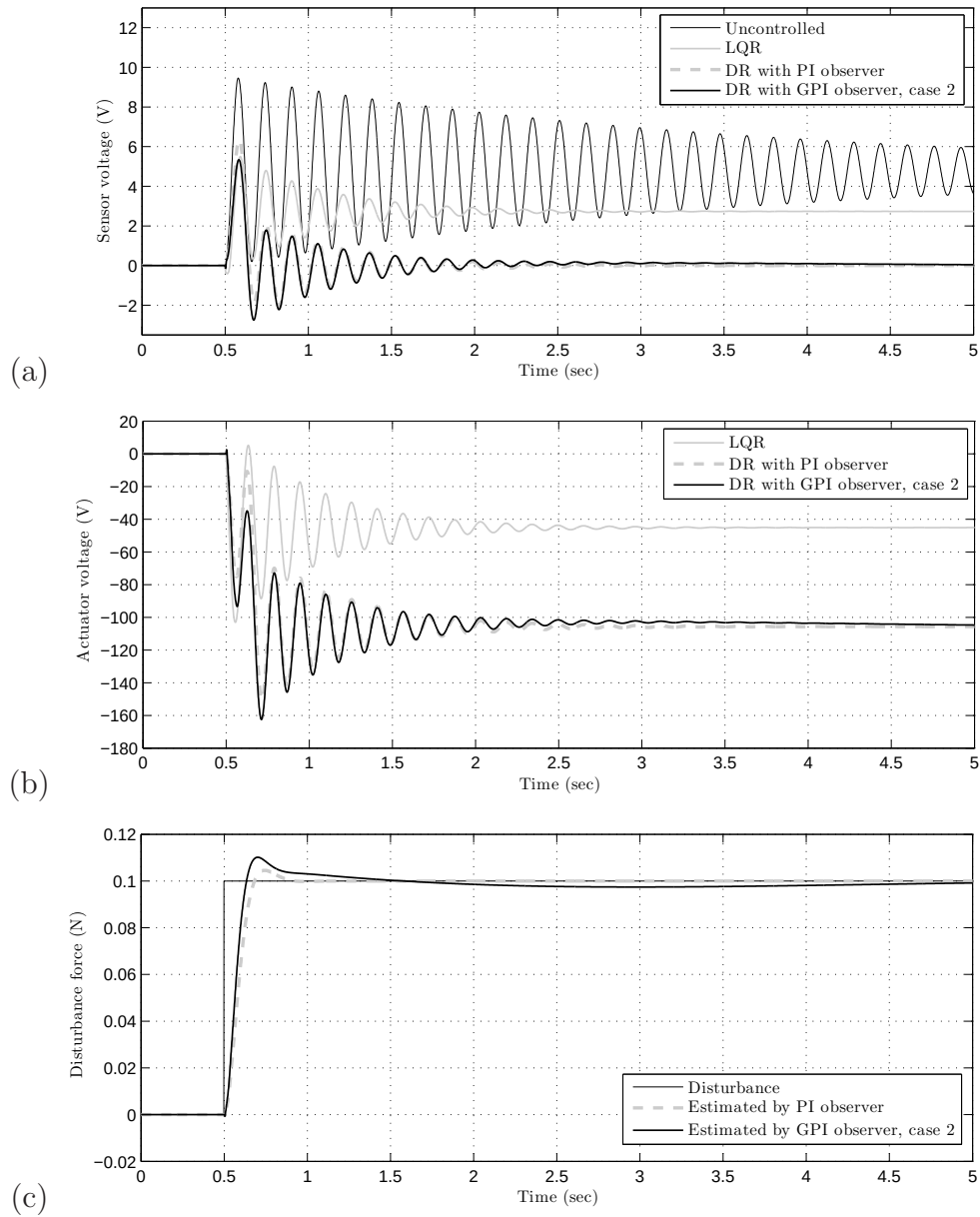


FIGURE 6.10: The dynamic behavior of the two-PZT-patch beam under a step disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

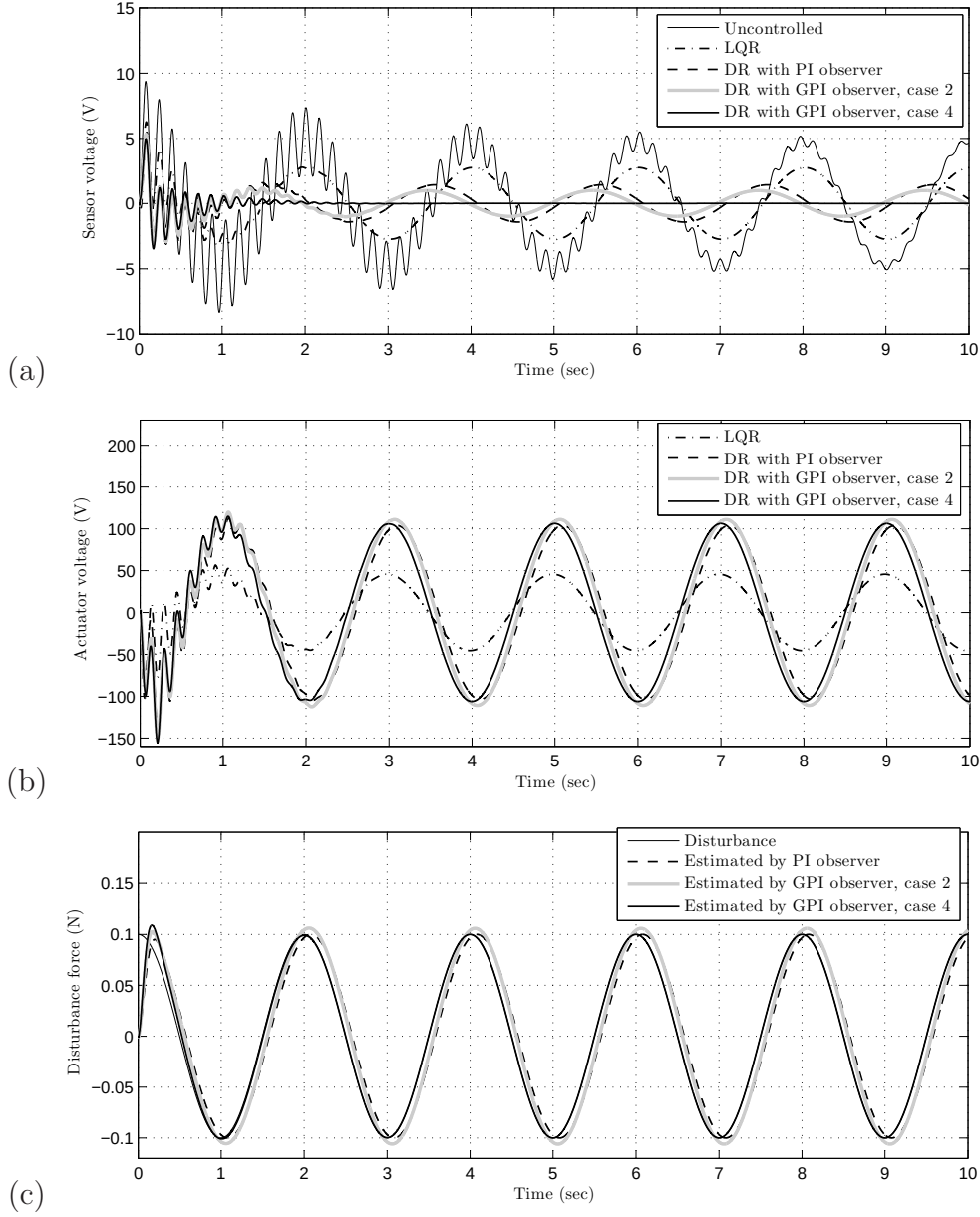


FIGURE 6.11: The dynamic behavior of the two-PZT-patch beam under a harmonic disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

input signals are presented in Fig. 6.11 (a) and (b), respectively. If the disturbance is absolutely unknown, $\omega = 1$ rad/s is considered, otherwise $\omega = \pi$ rad/s is employed if the disturbance frequency is known.

The controlled sensor signals in Fig. 6.11 (a) illustrate that the vibrations suppressed by DR control with PI or GPI observer have smaller amplitudes than those obtained by LQR control. Furthermore, the DR control with GPI observer (case 2, $\omega = 1$ rad/s) has a better capability for vibration suppression than that with PI observer. If the angular frequency of the disturbance is considered (DR control with GPI observer, case 4), the beam is perfectly damped, as shown in Fig. 6.11 (a). The estimates of the disturbance are displayed in Fig. 6.11 (c), indicating that the GPI observer with angular frequency known predicts the disturbance almost exactly, while the GPI observer with angular frequency unknown and the PI observer estimate the disturbance almost the same, but with a slight time delay, which results in bigger sensor amplitudes than that of GPI observer with angular frequency known in Fig. 6.11 (a). Due to the better dynamic behavior of the GPI observer, even if the angular frequency is unknown, the disturbance estimated by the GPI observer with $\omega = 1$ rad/s has a smaller time delay than that by the PI observer, which leads to better vibration suppression. This can be explained by the fact that the first eigen-frequency of the beam is very large compared to the excitation frequency. Therefore, even just a slight time delay may significantly and negatively impact the vibration suppression.

Triangle wave disturbance

A triangle periodic wave disturbance force with the angular frequency of $\omega = \pi$ rad/s and the amplitude of 0.1 N is simulated in the third validation test. The controlled vibration and the control input are displayed in Fig. 6.12 (a) and (b), respectively. Similar effects on vibration suppression using various control schemes can be observed as in the previous simulation. The vibration response is better suppressed by DR control with either PI or GPI observer than that by LQR control. The best result is obtained by the GPI observer especially if the angular frequency of the triangle wave disturbance is known (GPI case 4). The estimated disturbances are displayed in Fig. 6.12 (c), which illustrates that all the proposed observers can estimate the disturbances very well. The GPI observer with the disturbance frequency known predicts the closest signal to the original disturbance among those three.

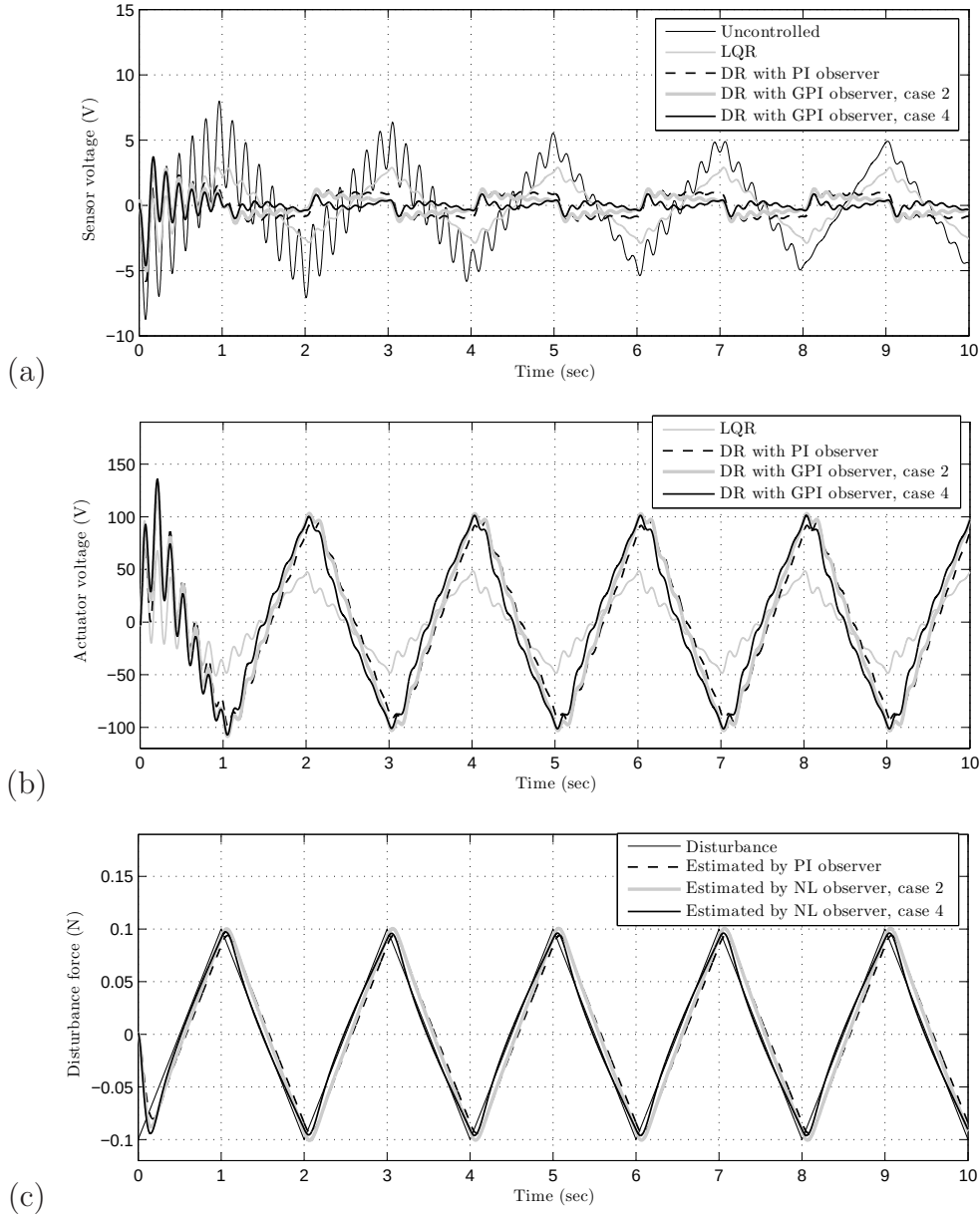


FIGURE 6.12: The dynamic behavior of the two-PZT-patch beam under a triangle wave disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

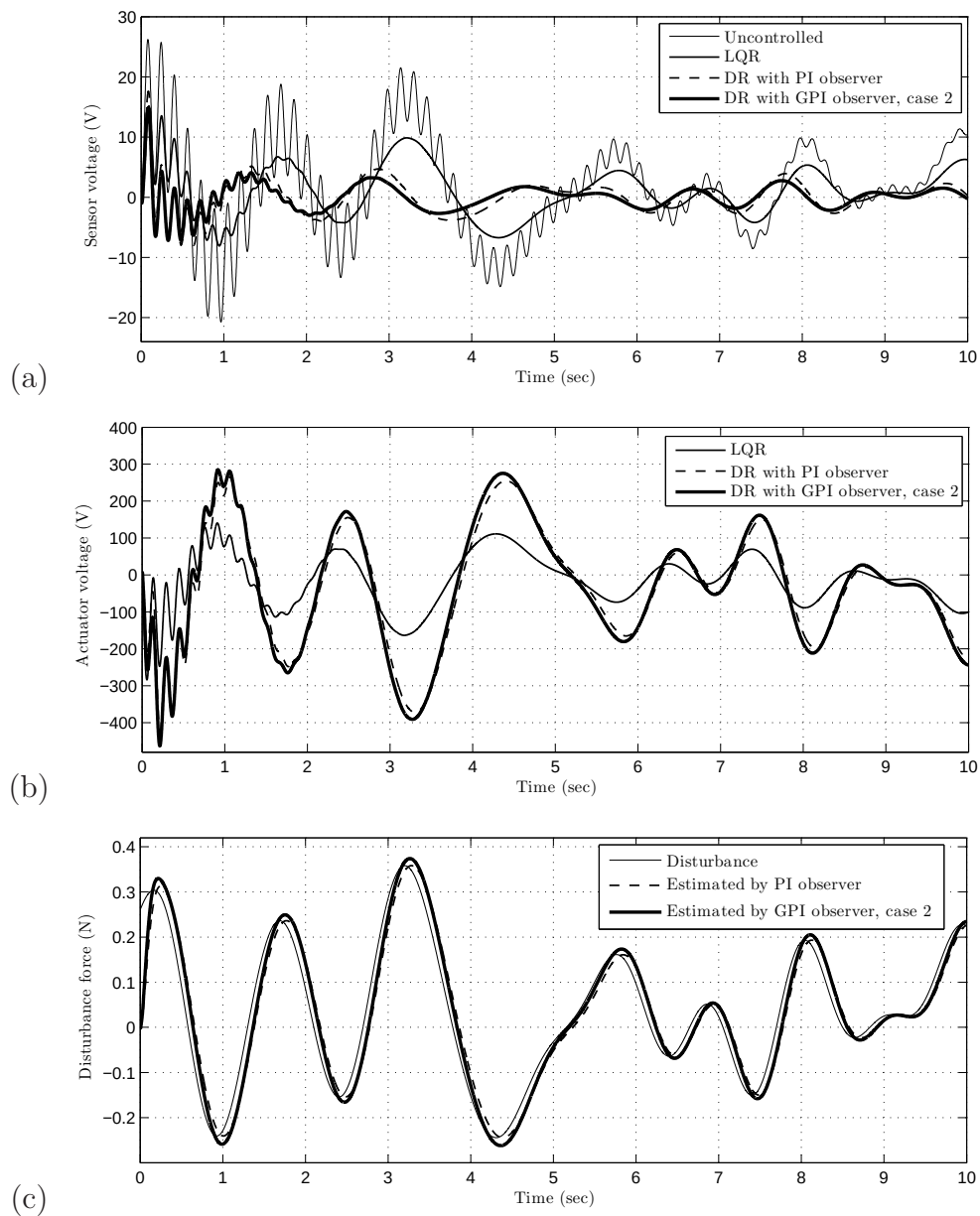


FIGURE 6.13: The dynamic behavior of the two-PZT-patch beam under a random disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

Random disturbance

Next, a random disturbance force is considered for the validation test. The dynamic response of the sensor output and the control input is respectively displayed in Fig. 6.13 (a) and (b). The time history of the disturbance forces and the estimated signals are shown in Fig. 6.13 (c). DR control with either PI or GPI observer provides a significant damping effect. Again the best results are obtained by DR control with GPI observer (case 2).

High frequency periodic disturbance

From the results presented above, it can be seen that DR control perfectly suppresses the beam vibration caused by a periodic disturbance, whose frequency is low compared to the first eigen-frequency of 6.21 Hz. If the beam is under a high frequency periodic disturbance, which is near the first eigen-frequency, the vibration may not be successfully suppressed by DR control with all PI and GPI observers. Considering a harmonic disturbance with the frequency of 5 Hz and the amplitude of 0.1 N, the uncontrolled/controlled sensor signals, the control input signals and estimated disturbances are obtained and displayed in Fig. 6.14 (a), (b) and (c), respectively. The vibrations damped by DR control with PI and GPI (case 1) observers are very similar, and both of them are worse than those regulated by LQR control. Because of the existence of the rising time in the estimation process, the observer may fail to re-construct the unknown disturbances which vary rapidly. Due to the failure of the estimator (PI and GPI case 1), uncorrected estimated signals are considered in DR control, which probably leads to negative effect in vibration suppression. However, the GPI observers of case 3 and 4 consider the disturbance frequency, which estimate excellent signals, as shown in Fig. 6.14 (c). Since the approximate solution is used in GPI (case 3) observer, the vibrations are suppressed not as good as those by GPI (case 4) where the control gain is calculated in the exact way. However, both case 3 and 4 perform better than LQR control, the latter one performs the best among all the presented control schemes.

Changing the harmonic disturbance to a periodic square wave disturbance with the same frequency and amplitude, the sensor signals, the control input signals and the estimated disturbances can be obtained by LQR control, DR control with PI and GPI observers, which are shown in Fig. 6.15. Similarly, the PI and GPI (case 1) observers predict small amplitude disturbances, as well as shifted phases, as shown

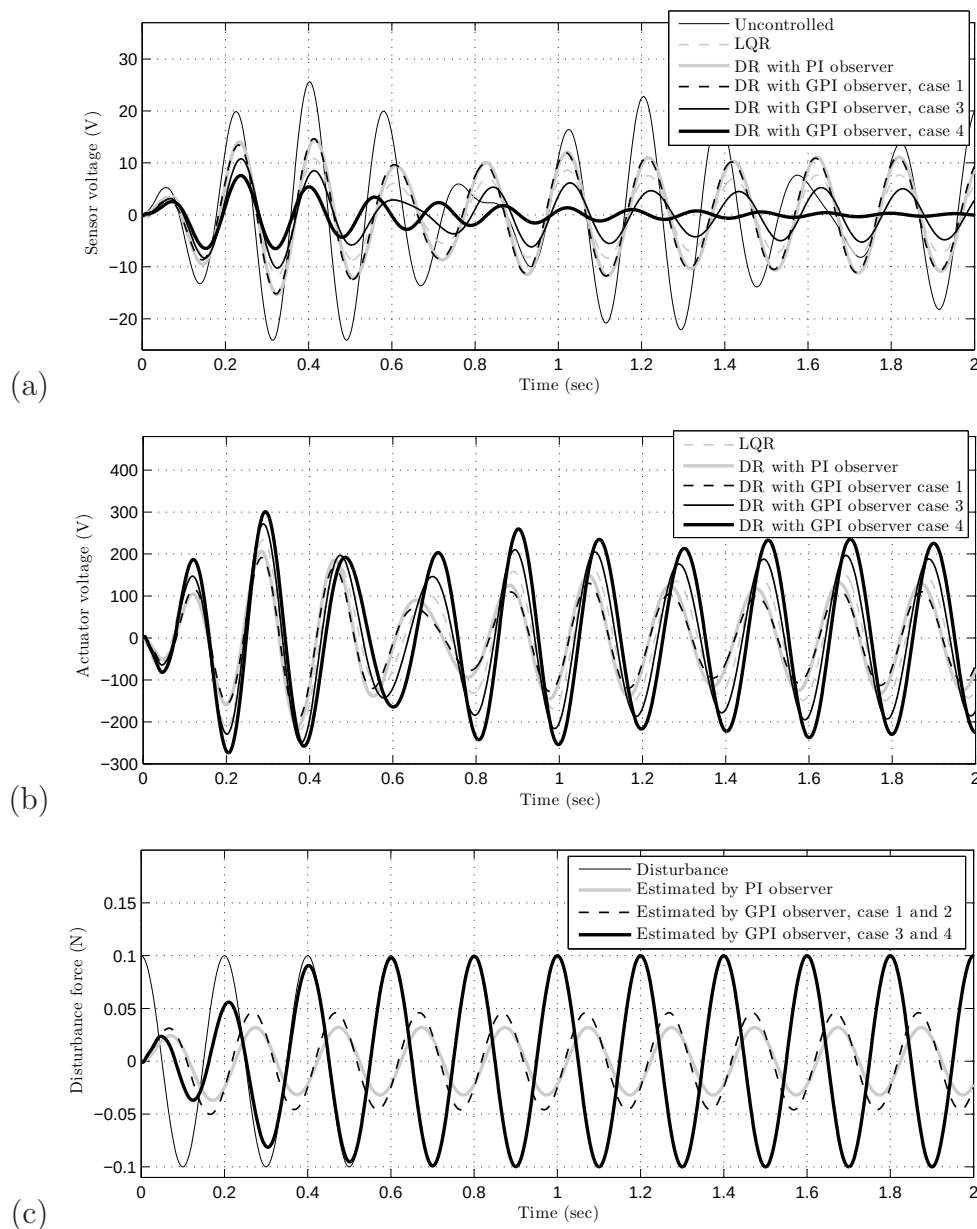


FIGURE 6.14: The dynamic behavior of the two-PZT-patch beam under a high frequency harmonic disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

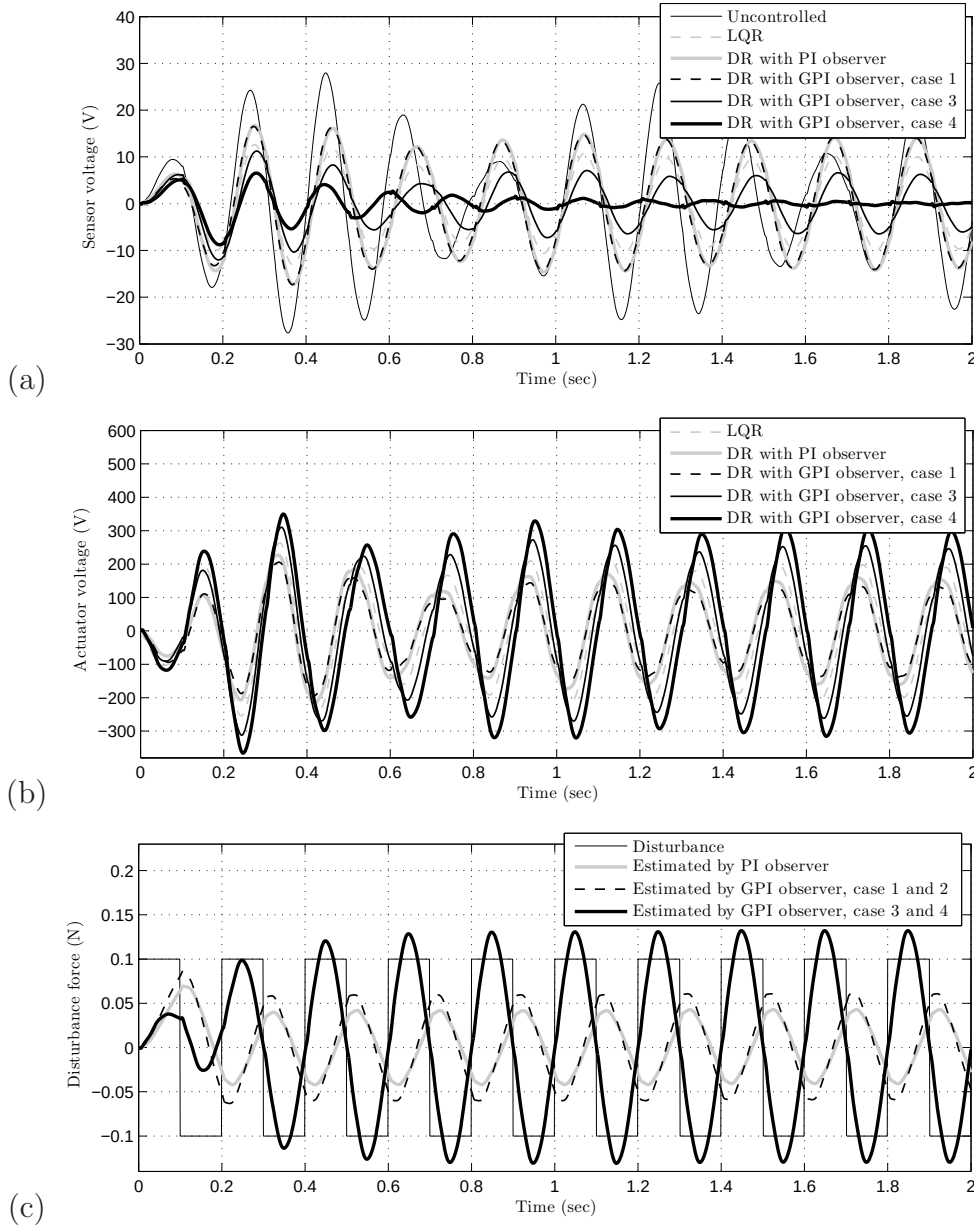


FIGURE 6.15: The dynamic behavior of the two-PZT-patch beam under a high frequency square disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

in Fig. 6.15 (c), which results in negative effect on vibration suppression. However, GPI observers of case 3 and 4 estimate the disturbance with relatively high accuracy, and have similar amplitude and the same phase compared to the original disturbance. Due to this reason, the vibrations suppressed by DR control with GPI (case 3 or 4) observer are better than those using LQR control. Analogously, the DR control with GPI observer (case 4) using the exact solution and considering the disturbance frequency gives the best result.

6.2 Active vibration control of a smart plate

6.2.1 Piezolaminated composite plate

The second example of control problems is a cantilevered PZT composite plate proposed by Lam *et al.* [34], which is shown in Fig. 6.16. The cantilevered piezolaminated

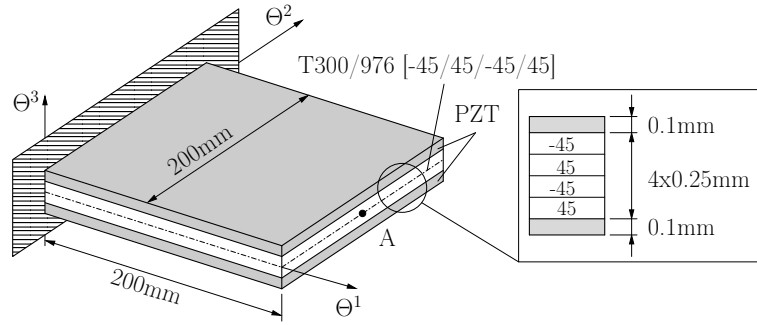


FIGURE 6.16: Piezolaminated composite plate

composite plate consists of one composite master layer, on which two PZT layers with opposite polarizations pointing outward are bonded at the top and bottom surfaces. The host structure is made of T300/976 graphite-epoxy composite material with four substrate layers, whose stacking sequence consists of antisymmetric angle plies $[-45^\circ/45^\circ/-45^\circ/45^\circ]$. The composite structure is 200 mm square. The total thickness of the master layer is 1 mm comprised of four substrate layers of 0.25 mm each, and the thickness of each PZT layer is 0.1 mm. The material properties are given in Table 6.8. A dynamic FE model is derived by a discretization of 5×5 SH851URI elements for the plate. Again, the damping matrix is obtained by the RAYLEIGH damping coefficient computation method in consideration of the damping ratio of 0.8% for the first six modes.

TABLE 6.8: Material properties of the piezolaminated composite plate

T300/976	PZT
$E_1 = 150$ GPa	$E = 63$ GPa
$E_2 = 9$ GPa	$\nu = 0.3$
$G_{12} = G_{13} = 7.1$ GPa	$\rho = 7600$ kg/m ³
$G_{23} = 2.5$ GPa	$d_{31} = d_{32} = -2.54 \times 10^{-10}$ C/N
$\nu = 0.3$	$\epsilon_{33} = 1.5 \times 10^{-8}$ F/m
$\rho = 1600$ kg/m ³	

6.2.2 Validation test

Firstly, the FE model is validated by static analysis, all the PZT layers on the upper and lower surfaces are assumed to be actuators, and the displacements of the centerline are computed. The plate is subjected to a uniformly distributed load of 100 N/m². Equal amplitude voltages with an opposite sign are applied across the thickness of the two PZT layers, respectively. The voltages of 0, 30 and 50 V for the actuators are applied to flatten the plate. The shapes of the centerline are presented in Fig. 6.17, which show that the present curves are in good agreement with the results calculated by Lam *et al.* [34] based on the classical laminated plate theory.

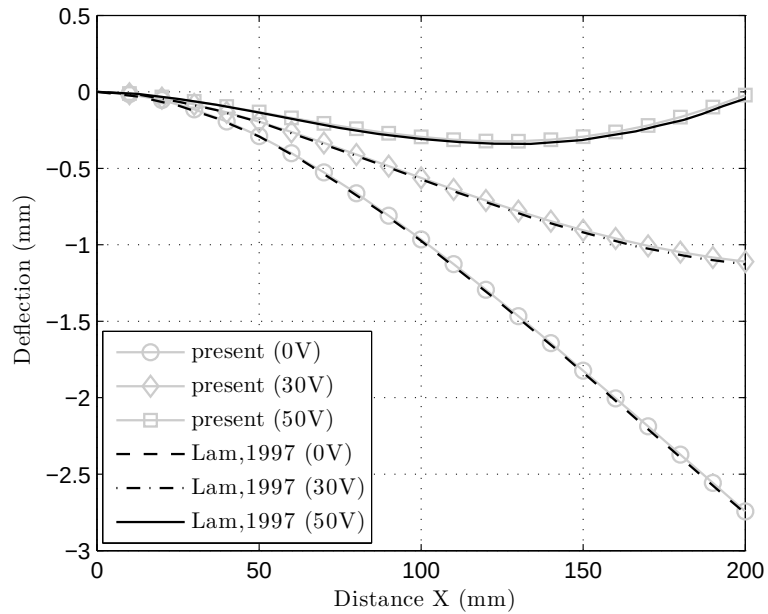


FIGURE 6.17: The centerline deflection of the piezolaminated plate under a uniformly distributed load and different input actuation voltages

In addition, the FE model is tested by dynamic analysis. The first five eigen-frequencies of the plate are calculated, as shown in Table 6.9, which indicates that good agreement with those reported in [34] has been achieved.

TABLE 6.9: The first five eigen-frequencies of the piezolaminated plate (Hz)

Mode	Lam <i>et al.</i> [34]	Present	Discrepancy
1	21.4657	21.5083	0.20%
2	63.3491	63.2409	0.17%
3	130.8221	129.9076	0.79%
4	182.4224	183.4276	0.55%
5	218.2750	217.8606	0.19%

6.2.3 Control simulation of the plate

Due to the large size of the system matrices, the FE model is reduced to retain the first 12 modes for the simulation of vibration control. The disturbances are considered as concentrated forces applied on point A at the free end. Several control strategies are applied with the parameters listed in Table 6.10, where ω_0 refers to the angular frequency of the disturbance signal. The parameters for PID control are tuned man-

TABLE 6.10: Parameters of various control schemes for piezolaminated plate

Controller	Parameters
PID:	$\mathbf{K}_p = 2, \mathbf{K}_i = 400, \mathbf{K}_d = 0.3$
LQR:	$\bar{\mathbf{Q}} = 0.04, \bar{\mathbf{R}} = 0.0025$
DR with PI observer:	$\bar{\mathbf{Q}} = 0.04, \bar{\mathbf{R}} = 0.0025, b = 1 \times 10^5$
DR with GPI observer, case 1:	$\bar{\mathbf{Q}} = 0.04, \bar{\mathbf{R}} = 0.0025, b = 1 \times 10^5$ $\omega = 1 \text{ rad/s, approximate solution}$
DR with GPI observer, case 3:	$\bar{\mathbf{Q}} = 0.04, \bar{\mathbf{R}} = 0.0025, b = 1 \times 10^5$ $\omega = \omega_0 \text{ rad/s, approximate solution}$
DR with GPI observer, case 4:	$\bar{\mathbf{Q}} = 0.04, \bar{\mathbf{R}} = 0.0025, b = 1 \times 10^5$ $\omega = \omega_0 \text{ rad/s, exact solution}$

ually, meaning that the present values are not the best. For the other four kinds of control strategies, we use the same weighting matrices $\bar{\mathbf{Q}}$ and $\bar{\mathbf{R}}$.

In the first vibration control simulation, a harmonic disturbance with the angular frequency of $2\pi \text{ rad/s}$ and the amplitude of 1 N, which can be described by the function

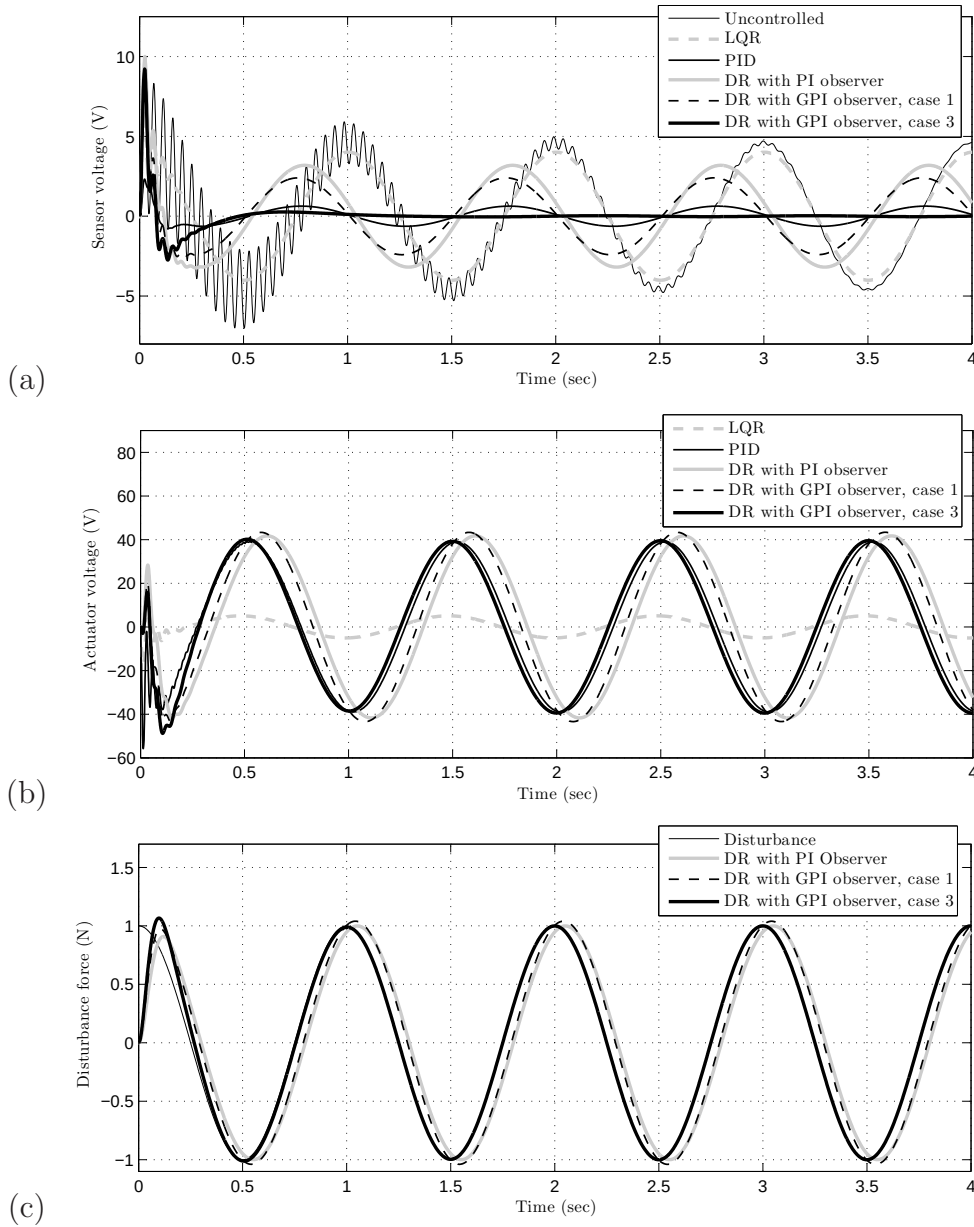


FIGURE 6.18: The dynamic behavior of the piezolaminated plate by various control strategies under a low frequency harmonic disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

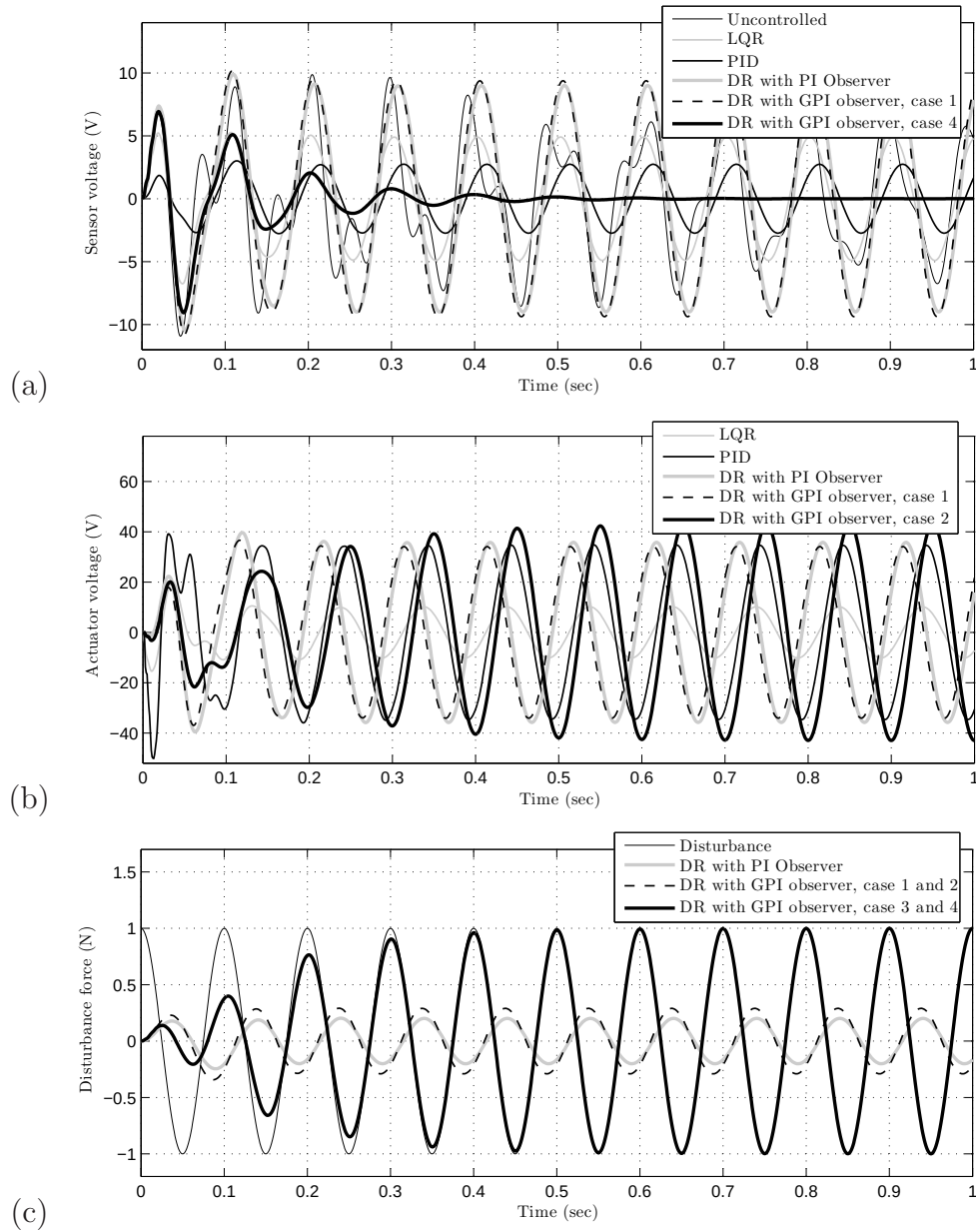


FIGURE 6.19: The dynamic behavior of the piezolaminated plate by various control strategies under a high frequency harmonic disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

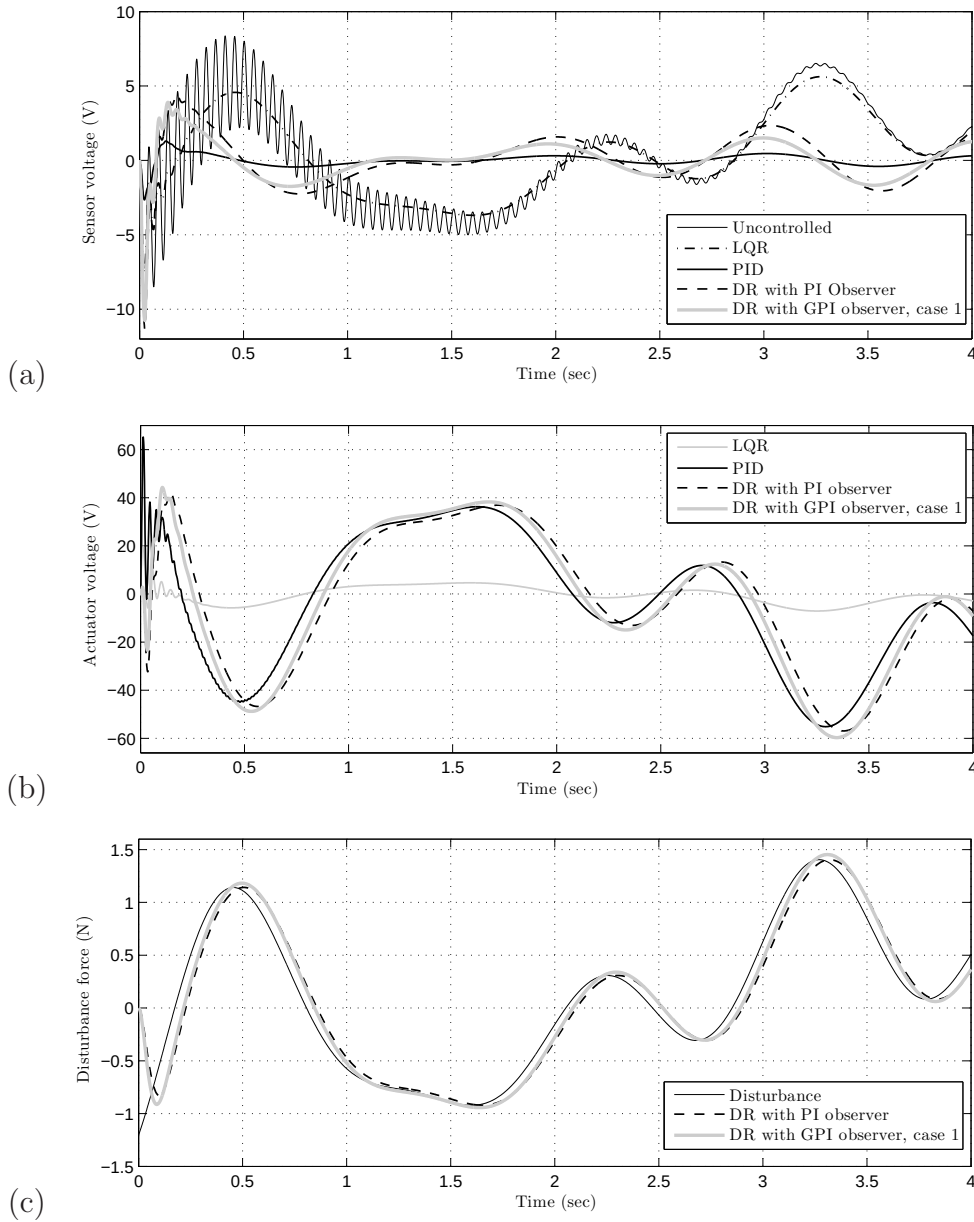


FIGURE 6.20: The dynamic behavior of the piezolaminated plate by various control strategies under a random disturbance force: (a) sensor output, (b) control input, (c) estimated disturbance

$\mathbf{f}(t) = \cos(2\pi t)$ N, is applied on the smart plate. The uncontrolled/controlled vibrations regulated by various controllers and the corresponding control input voltages are shown in Fig. 6.18 (a) and (b), respectively. As the results show, the vibrations are much more significantly damped by PID control and DR control with GPI observer (case 3) than those suppressed by other controllers. As mentioned before, the parameters for PID control are not the best, which means that the control effect may be improved at a certain extent. Furthermore, it can be recognized that the DR control with PI observer and GPI observer (case 1) perform very similar, and both of them are better than LQR control. The biggest advantage of DR control with PI or GPI observer is that it can estimate the unknown disturbances, which are displayed in Fig. 6.18 (c).

Increasing the angular frequency of the periodic disturbance from 2π rad/s to 20π rad/s one obtains the control effects as displayed in Fig. 6.19. Again, the vibrations suppressed by DR control with GPI observer (case 4) perform the best. Unlike the previous simulation, PID control scheme is unable to follow the disturbances varying rapidly, resulting in larger amplitudes of the vibrations compared to the case with low frequency ones presented in Fig. 6.18.

In the next simulation, the plate is excited by a random disturbance and regulated by various control strategies with the parameters shown in Table 6.10. The dynamic behavior of the sensor output and control input voltages is obtained and shown in Fig. 6.20 (a) and (b), respectively. Since the random disturbance is not a periodic signal, we can only apply DR control with PI observer or GPI observer (case 1). The results show that the vibrations suppressed by PID control have the smallest amplitudes among all the applied control strategies. Again, the DR control with GPI observer (case 1) yields slightly better results than those with PI observer. Both of them generate better dynamic behavior than LQR does. Additionally, PI and GPI observers can estimate the unknown disturbances, which are shown in Fig. 6.20 (c).

Chapter 7

Conclusion

7.1 Summary and concluding remarks

This dissertation dealt first with geometrically nonlinear modeling techniques for both composite laminated and piezoelectric integrated thin-walled structures, which are presented in Chapters 2, 3 and 5. Second, the active vibration control of smart structures was discussed in Chapters 4 and 6.

In the first part of this dissertation, linear and nonlinear electro-mechanically coupled FE models based on LIN5, RVK5, MRT5, LRT5 and LRT56 shell theories have been developed for static and dynamic analysis of piezoelectric integrated thin-walled smart structures. The large rotation shell theory, LRT56, has six independent kinematic parameters but expressed by five nodal DOFs using the EULER angle representation method. The theory considers not only the fully geometrically nonlinear strain-displacement relations, but also the unrestricted finite rotations. Other simplified nonlinear shell theories, RVK5, MRT5, LRT5, as well as linear shell theory, LIN5, have only five independent kinematic parameters which are linearly represented by the five nodal DOFs, respectively, implying that only small or moderate rotations are considered in these theories. The former two theories, RVK5 and MRT5, consider the simplified nonlinear strain-displacement relations, while the third one, LRT5, includes the full geometric nonlinearities.

Due to the assumption of *small strains* and *weak electric potential*, linear piezoelectric coupled constitutive equations and linearly distributed electric potential through the thickness direction are considered. Four types of shell elements, SH85FI and

SH85URI for *composite* structures, SH851FI and SH851URI for *piezoelectric* laminated smart structures, have been developed, in which FI refers to full integration and URI represents uniformly reduced integration. Using HAMILTON's principle, dynamic FE models including equations of motion and sensor equations have been developed based on those nonlinear shell theories. Analogously, static FE models including equilibrium equations and sensor equations have been derived based on various nonlinear shell theories by using the principle of virtual work.

From the results presented in Chapter 5, it can be concluded that simplified nonlinear shell theories, RVK5, MRT5, LRT5, will fail to predict both static and dynamic response for composite and piezoelectric laminated thin-walled structures in the range of large rotations. This is because only simplified nonlinear strain-displacement relations are considered in the RVK5 and MRT5 theories, and in addition, no proper rotation updating is possible in all these simplified nonlinear shell theories. In the case of smart structures undergoing large deflections and rotations, large rotation theory (LRT56) has to be considered. Concerning the severe *shear locking* phenomena existing in thin-walled smart structures, a proper method, e.g. ANS, EAS or URI algorithms, has to be considered to avoid these locking effects in both static and dynamic analysis.

The second part of this dissertation firstly has developed a linear piezoelectric coupled FE model based on LIN5 shell theory for active vibration control of smart structures. Afterwards, a state space model has been constructed for control design, based on which PID, LQR and LQG control strategies have been implemented for vibration suppression. In order to consider the effects from disturbances, a DR control with PI observer which uses step functions as a fictitious model of disturbances has been developed for active vibration control. To improve the dynamic behavior of the existing PI observer, a GPI observer, which can employ sine, cosine or polynomial functions to represent the disturbances, has been proposed and developed. Using either the PI or GPI observer, any unknown disturbances then can be estimated and re-constructed. The estimated disturbances can be fed back to the controller as measured signals and compensated in a specific manner.

From the results discussed in Chapter 6, it can be concluded that PID is a powerful controller, which can excellently damp the vibrations but at a high cost of input energy. Compared to PID, LQR is an ideal control strategy, which produces the results at the lowest cost of energy. However, the vibrations suppressed by LQR control may not be as good as those by PID control, especially when the structures vibrate under disturbances. Since LQR needs all state variables to be measured, which may be

impossible in real applications, LQG control has to be implemented, such that the state variables will be estimated by LQG observer. The same optimal control law is adopted in LQR and LQG control, but LQG will not be better than LQR control. The proposed DR control is based on LQR optimization, which on the one hand can suppress the free vibrations at an optimized consumption of input energy, on the other hand can counteract the forced vibrations. At the meantime, the disturbances can be additionally re-constructed by PI or GPI observer. For the disturbances with relatively high frequency, DR control with GPI observer considering the disturbance frequency gives the best results among all other implemented controllers.

7.2 Future research

Even though this dissertation has taken a certain amount of research on nonlinear FE modeling in which the geometric nonlinearity ranges from the simplest VON KÁRMÁN type nonlinearity to the large rotation nonlinearity, and on active vibration control of smart structures using PID, LQR, LQG and DR control strategies, there still exists much interesting work that can be further considered and developed.

In the first part of the dissertation, the electro-mechanically coupled problems have been considered. However many other coupled problems can be further developed, for example, electro-thermo-mechanically coupled problems, magneto-mechanically coupled structures. Secondly, a higher-order shear deformation hypothesis can be adopted for large rotation FE models. Concerning the *locking* problems, higher-order elements can be implemented in FE models, which will not only eliminate the locking phenomena, but also provide many unmeasurable beneficial properties [98].

Concerning the present large rotation FE code, some static computations for shell structures cannot converge at large loads, due to the problems that are not known yet. But the numerical results can be improved by changing the element type, discretizing the structure with fine elements or using more robust rotation updating method (e.g. RODRIGUES rotation formulation).

Considering the active vibration control part, only linear FE piezoelectric integrated smart structures are considered in the present dissertation. However, more advanced mathematical models with explicit nonlinear expressions, based on e.g. VON KÁRMÁN type nonlinear theory, moderate rotation theory, large rotation theory, or other advanced nonlinear theories, can be developed for control design. Moreover, many other

intelligent control schemes like *neural network*, *fuzzy logic* based control, or combination with other conventional control laws can be developed and implemented in vibration suppression of smart structures. Additionally, those control schemes may be applied to real smart structures.

In the development of DR control in Chapter 4, two algorithms have been proposed for the design of observer gains, namely the LYAPUNOV approach given in equation (4.127) and the RICCATI approach in equation (4.131). The latter one obtains excellent observer gains which meet the stability requirement. But the former one produces the observer gains that deteriorate the system. Therefore, LYAPUNOV approach can be improved in our future work as well. In addition, DR control strategy can also be applied in estimating nonlinearity of smart structures, like material hysteresis or geometric nonlinearities.

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Appendix A

Geometric quantities

A.1 Plate structure

The CARTESIAN coordinate system (X^1, X^2, X^3) and the curvilinear coordinate system $(\Theta^1, \Theta^2, \Theta^3)$ of a plate structure are shown in Fig. A.1.

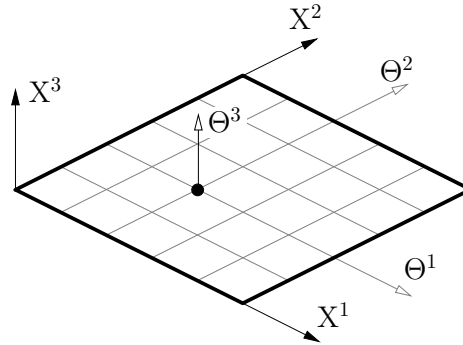


FIGURE A.1: Curvilinear coordinates for a plate structure

The curvilinear coordinates are defined as

$$\Theta^1 = X^1, \quad \Theta^2 = X^2, \quad \Theta^3 = X^3. \quad (\text{A.1})$$

The position vectors of an arbitrary point in the shell space and at the mid-surface are respectively expressed as

$$\mathbf{R} = \begin{Bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{Bmatrix}, \quad \mathbf{r} = \begin{Bmatrix} \Theta^1 \\ \Theta^2 \\ 0 \end{Bmatrix}. \quad (\text{A.2})$$

The covariant base vectors for an arbitrary point in the shell space are

$$\mathbf{g}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{g}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{g}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (\text{A.3})$$

The covariant and contravariant metric tensors in the shell space are

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = [g_{ij}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.4})$$

Using the formulation $\mathbf{g}^i = g_{ij}\mathbf{g}_j$ one obtains the contravariant base vectors in the shell space

$$\mathbf{g}^1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{g}^2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{g}^3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (\text{A.5})$$

The covariant base vectors of the point at the mid-surface are

$$\mathbf{a}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{a}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{a}_3 = \mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (\text{A.6})$$

The covariant and contravariant metric tensors at the mid-surface will be

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a^{ij} = [a_{ij}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.7})$$

The contravariant base vectors at the mid-surface are

$$\mathbf{a}^1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{a}^2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{a}^3 = \mathbf{a}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (\text{A.8})$$

The partial derivatives of the covariant base vectors at the mid-surface are

$$\mathbf{a}_{i,j} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (\text{A.9})$$

The covariant and mixed components of the curvature tensor are

$$b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad b_{\alpha}^{\beta} = a^{\beta\gamma} \cdot b_{\alpha\gamma} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{A.10})$$

The components of the shifter tensor are

$$\mu_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} - \Theta^3 b_{\alpha}^{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{A.11})$$

The CHRISTOFFEL symbols of the second kind for the point at the mid-surface are

$$\Gamma_{\alpha\beta}^1 = \Gamma_{\alpha\beta}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{A.12})$$

Therefore, the covariant derivatives and the abbreviations ${}^n\varphi_{\alpha\beta}$ can be obtained as

$${}^n v_{\alpha|\beta} = {}^n v_{\alpha,\beta}, \quad {}^n \varphi_{\alpha\beta} = {}^n v_{\alpha,\beta}, \quad {}^n \varphi_{3\alpha} = {}^n v_{3,\alpha}. \quad (\text{A.13})$$

A.2 Cylindrical structure

The CARTESIAN coordinate system (X^1, X^2, X^3) and the curvilinear coordinate system $(\Theta^1, \Theta^2, \Theta^3)$ of a cylindrical structure are shown in Fig. [A.2](#).

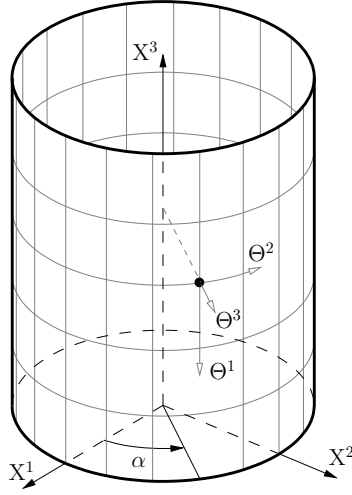


FIGURE A.2: Curvilinear coordinates for a cylindrical structure

The curvilinear coordinates are defined as

$$\Theta^1 = -z, \quad \Theta^2 = \alpha, \quad \Theta^3 = r - R. \quad (\text{A.14})$$

Here, R denotes the the radius of the mid-surface, and r represents the radius of an arbitrary large cylindrical surface. The position vectors of an arbitrary point in the shell space and at the mid-surface are respectively expressed as

$$\mathbf{R} = \begin{Bmatrix} (R + \Theta^3) \cos(\Theta^2) \\ (R + \Theta^3) \sin(\Theta^2) \\ -\Theta^1 \end{Bmatrix}, \quad \mathbf{r} = \begin{Bmatrix} R \cos(\Theta^2) \\ R \sin(\Theta^2) \\ -\Theta^1 \end{Bmatrix}. \quad (\text{A.15})$$

The covariant base vectors for an arbitrary point in the shell space are

$$\mathbf{g}_1 = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{g}_2 = \begin{Bmatrix} -(R + \Theta^3) \sin(\Theta^2) \\ (R + \Theta^3) \cos(\Theta^2) \\ 0 \end{Bmatrix}, \quad \mathbf{g}_3 = \begin{Bmatrix} \cos(\Theta^2) \\ \sin(\Theta^2) \\ 0 \end{Bmatrix}. \quad (\text{A.16})$$

The covariant and contravariant metric tensors in the shell space are

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (R + \Theta^3)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = [g_{ij}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(R + \Theta^3)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.17})$$

The contravariant base vectors in the shell space are

$$\mathbf{g}^1 = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{g}^2 = \begin{Bmatrix} -\frac{\sin(\Theta^2)}{(R + \Theta^3)} \\ \frac{\cos(\Theta^2)}{(R + \Theta^3)} \\ 0 \end{Bmatrix}, \quad \mathbf{g}^3 = \begin{Bmatrix} \cos(\Theta^2) \\ \sin(\Theta^2) \\ 0 \end{Bmatrix}. \quad (\text{A.18})$$

The covariant base vectors at the mid-surface are

$$\mathbf{a}_1 = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{a}_2 = \begin{Bmatrix} -R \sin(\Theta^2) \\ R \cos(\Theta^2) \\ 0 \end{Bmatrix}, \quad \mathbf{a}_3 = \mathbf{n} = \begin{Bmatrix} \cos(\Theta^2) \\ \sin(\Theta^2) \\ 0 \end{Bmatrix}. \quad (\text{A.19})$$

The covariant and contravariant metric tensors at the mid-surface are

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a^{ij} = [a_{ij}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.20})$$

The contravariant base vectors at the mid-surface are

$$\mathbf{a}^1 = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{a}^2 = \begin{Bmatrix} -\frac{\sin(\Theta^2)}{R} \\ \frac{\cos(\Theta^2)}{R} \\ 0 \end{Bmatrix}, \quad \mathbf{a}^3 = \mathbf{a}_3 = \begin{Bmatrix} \cos(\Theta^2) \\ \sin(\Theta^2) \\ 0 \end{Bmatrix}. \quad (\text{A.21})$$

The partial derivatives of the covariant base vectors at the mid-surface are

$$\mathbf{a}_{1,1} = \mathbf{a}_{1,2} = \mathbf{a}_{1,3} = \mathbf{a}_{2,3} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{a}_{2,2} = \begin{Bmatrix} -R \cos(\Theta^2) \\ -R \sin(\Theta^2) \\ 0 \end{Bmatrix}, \quad \mathbf{a}_{3,2} = \begin{Bmatrix} -\sin(\Theta^2) \\ \cos(\Theta^2) \\ 0 \end{Bmatrix}. \quad (\text{A.22})$$

The covariant and mixed components of the curvature tensor are

$$b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 = \begin{bmatrix} 0 & 0 \\ 0 & -R \end{bmatrix}, \quad b_{\alpha}^{\beta} = a^{\beta\gamma} \cdot b_{\alpha\gamma} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}. \quad (\text{A.23})$$

The components of the shifter tensor are

$$\mu_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} - \Theta^3 b_{\alpha}^{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{\Theta^3}{R} \end{bmatrix}. \quad (\text{A.24})$$

The CHRISTOFFEL symbols of the second kind for the point at the mid-surface are

$$\Gamma_{\alpha\beta}^1 = \Gamma_{\alpha\beta}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{A.25})$$

Therefore, the covariant derivatives and the abbreviations $\overset{n}{\varphi}_{\alpha\beta}$ can be obtained as

$$\left\{ \begin{array}{l} \overset{n}{v}_{1|1} = \overset{n}{v}_{1,1} \\ \overset{n}{v}_{1|2} = \overset{n}{v}_{1,2} \\ \overset{n}{v}_{2|1} = \overset{n}{v}_{2,1} \\ \overset{n}{v}_{2|2} = \overset{n}{v}_{2,2} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \overset{n}{\varphi}_{11} = \overset{n}{v}_{1,1} \\ \overset{n}{\varphi}_{12} = \overset{n}{v}_{1,2} \\ \overset{n}{\varphi}_{21} = \overset{n}{v}_{2,1} \\ \overset{n}{\varphi}_{22} = \overset{n}{v}_{2,2} + R\overset{n}{v}_3 \\ \overset{n}{\varphi}_{31} = \overset{n}{v}_{3,1} \\ \overset{n}{\varphi}_{32} = \overset{n}{v}_{3,2} - \frac{1}{R}\overset{n}{v}_2 \end{array} \right. \quad (\text{A.26})$$

A.3 Spherical structure

The CARTESIAN coordinate system (X^1, X^2, X^3) and the curvilinear coordinate system $(\Theta^1, \Theta^2, \Theta^3)$ of a spherical structure are shown in Fig. A.3.

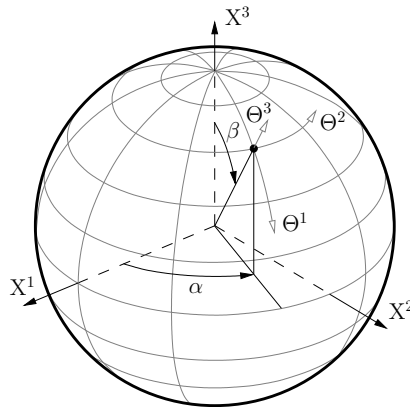


FIGURE A.3: Curvilinear coordinates for a spherical structure

Here, R denotes the radius of the mid-surface, and r is the radius of an arbitrary large spherical surface. The curvilinear coordinates are defined as

$$\Theta^1 = R\beta, \quad \Theta^2 = \alpha, \quad \Theta^3 = r - R. \quad (\text{A.27})$$

The position vectors of an arbitrary point in the shell space and at the mid-surface are respectively expressed as

$$\mathbf{R} = (R + \Theta^3) \begin{Bmatrix} \sin\left(\frac{\Theta^1}{R}\right) \cos(\Theta^2) \\ \sin\left(\frac{\Theta^1}{R}\right) \sin(\Theta^2) \\ \cos\left(\frac{\Theta^1}{R}\right) \end{Bmatrix}, \quad \mathbf{r} = R \begin{Bmatrix} \sin\left(\frac{\Theta^1}{R}\right) \cos(\Theta^2) \\ \sin\left(\frac{\Theta^1}{R}\right) \sin(\Theta^2) \\ \cos\left(\frac{\Theta^1}{R}\right) \end{Bmatrix}. \quad (\text{A.28})$$

The covariant base vectors for an arbitrary point in the shell space are

$$\begin{aligned} \mathbf{g}_1 &= \left(1 + \frac{\Theta^3}{R}\right) \begin{Bmatrix} \cos\left(\frac{\Theta^1}{R}\right) \cos(\Theta^2) \\ \cos\left(\frac{\Theta^1}{R}\right) \sin(\Theta^2) \\ -\sin\left(\frac{\Theta^1}{R}\right) \end{Bmatrix}, \\ \mathbf{g}_2 &= (R + \Theta^3) \begin{Bmatrix} -\sin\left(\frac{\Theta^1}{R}\right) \sin(\Theta^2) \\ \sin\left(\frac{\Theta^1}{R}\right) \cos(\Theta^2) \\ 0 \end{Bmatrix}, \quad \mathbf{g}_3 = \begin{Bmatrix} \sin\left(\frac{\Theta^1}{R}\right) \cos(\Theta^2) \\ \sin\left(\frac{\Theta^1}{R}\right) \sin(\Theta^2) \\ \cos\left(\frac{\Theta^1}{R}\right) \end{Bmatrix}. \end{aligned} \quad (\text{A.29})$$

The covariant metric tensor in the shell space is

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{bmatrix} \left(1 + \frac{\Theta^3}{R}\right)^2 & 0 & 0 \\ 0 & (R + \Theta^3)^2 \sin^2\left(\frac{\Theta^1}{R}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.30})$$

The contravariant metric tensor in the shell space is

$$g^{ij} = [g_{ij}]^{-1} = \begin{bmatrix} \frac{R^2}{(R + \Theta^3)^2} & 0 & 0 \\ 0 & \frac{1}{(R + \Theta^3)^2 \sin^2 \left(\frac{\Theta^1}{R} \right)} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.31})$$

The contravariant base vectors in the shell space are

$$\begin{aligned} \mathbf{g}^1 &= \frac{R}{R + \Theta^3} \begin{bmatrix} \cos \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ -\sin \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \\ \mathbf{g}^2 &= \frac{1}{R + \Theta^3} \begin{bmatrix} -\frac{\sin (\Theta^2)}{\sin \left(\frac{\Theta^1}{R} \right)} \\ \frac{\cos (\Theta^2)}{\sin \left(\frac{\Theta^1}{R} \right)} \\ 0 \end{bmatrix}, \quad \mathbf{g}^3 = \begin{bmatrix} \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \end{bmatrix}. \end{aligned} \quad (\text{A.32})$$

The covariant base vectors at the mid-surface are

$$\begin{aligned} \mathbf{a}_1 &= \begin{bmatrix} \cos \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ -\sin \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \quad \mathbf{a}_2 = R \begin{bmatrix} -\sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ 0 \end{bmatrix}, \\ \mathbf{a}_3 = \mathbf{n} &= \begin{bmatrix} \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \end{bmatrix}. \end{aligned} \quad (\text{A.33})$$

The covariant and contravariant metric tensors at the mid-surface are

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sin^2 \left(\frac{\Theta^1}{R} \right) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a^{ij} = a_{ij}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2 \sin^2 \left(\frac{\Theta^1}{R} \right)} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (\text{A.34})$$

The contravariant base vectors at the mid-surface are

$$\mathbf{a}^1 = \begin{bmatrix} \cos \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ -\sin \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \quad \mathbf{a}^2 = \begin{bmatrix} -\frac{\sin (\Theta^2)}{\sin \left(\frac{\Theta^1}{R} \right)} \\ \frac{\cos (\Theta^2)}{\sin \left(\frac{\Theta^1}{R} \right)} \\ 0 \end{bmatrix}, \quad \mathbf{a}^3 = \begin{bmatrix} \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \end{bmatrix}. \quad (\text{A.35})$$

The partial derivatives of the covariant base vectors at the mid-surface are

$$\begin{aligned} \mathbf{a}_{1,1} &= -\frac{1}{R} \begin{bmatrix} \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, & \mathbf{a}_{1,2} = \mathbf{a}_{2,1} &= \begin{bmatrix} -\cos \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ 0 \end{bmatrix}, \\ \mathbf{a}_{2,2} &= \begin{bmatrix} -\sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ -\sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ 0 \end{bmatrix}, & \mathbf{a}_{3,1} = \frac{1}{R} \begin{bmatrix} \cos \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ \cos \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ -\sin \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \\ \mathbf{a}_{3,2} &= \begin{bmatrix} -\sin \left(\frac{\Theta^1}{R} \right) \sin (\Theta^2) \\ \sin \left(\frac{\Theta^1}{R} \right) \cos (\Theta^2) \\ 0 \end{bmatrix}, & \mathbf{a}_{1,3} = \mathbf{a}_{2,3} = \mathbf{a}_{3,3} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (\text{A.36})$$

The covariant and mixed components of the curvature tensor are

$$b_{\alpha\beta} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -R \sin^2 \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \quad b_{\alpha}^{\beta} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}. \quad (\text{A.37})$$

The components of the shifter tensor are

$$\mu_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} - \Theta^3 \cdot b_{\alpha}^{\beta} = \begin{bmatrix} 1 + \frac{\Theta^3}{R} & 0 \\ 0 & 1 + \frac{\Theta^3}{R} \end{bmatrix}. \quad (\text{A.38})$$

The CHRISTOFFEL symbols of the second kind for the point at the mid-surface are

$$\Gamma_{\alpha\beta}^1 = \begin{bmatrix} 0 & 0 \\ 0 & -R \sin \left(\frac{\Theta^1}{R} \right) \cos \left(\frac{\Theta^1}{R} \right) \end{bmatrix}, \quad \Gamma_{\alpha\beta}^2 = \begin{bmatrix} 0 & \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} \\ \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} & 0 \end{bmatrix}. \quad (\text{A.39})$$

Therefore, the covariant derivatives and the abbreviations $\overset{n}{\varphi}_{\alpha\beta}$ can be obtained as

$$\left\{ \begin{array}{l} \overset{n}{v}_{1|1} = \overset{n}{v}_{1,1} \\ \overset{n}{v}_{1|2} = \overset{n}{v}_{1,2} - \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} \overset{n}{v}_2 \\ \overset{n}{v}_{2|1} = \overset{n}{v}_{2,1} - \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} \overset{n}{v}_2 \\ \overset{n}{v}_{2|2} = \overset{n}{v}_{2,2} + R \sin \left(\frac{\Theta^1}{R} \right) \cos \left(\frac{\Theta^1}{R} \right) \overset{n}{v}_1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \overset{n}{\varphi}_{11} = \overset{n}{v}_{1,1} + \frac{1}{R} \overset{n}{v}_3 \\ \overset{n}{\varphi}_{12} = \overset{n}{v}_{1,2} - \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} \overset{n}{v}_2 \\ \overset{n}{\varphi}_{21} = \overset{n}{v}_{2,1} - \frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)} \overset{n}{v}_2 \\ \overset{n}{\varphi}_{22} = \overset{n}{v}_{2,2} + R \sin \left(\frac{\Theta^1}{R} \right) \cos \left(\frac{\Theta^1}{R} \right) \overset{n}{v}_1 \\ \quad + R \sin^2 \left(\frac{\Theta^1}{R} \right) \overset{n}{v}_3 \\ \overset{n}{\varphi}_{31} = \overset{n}{v}_{3,1} - \frac{1}{R} \overset{n}{v}_1 \\ \overset{n}{\varphi}_{32} = \overset{n}{v}_{3,2} - \frac{1}{R} \overset{n}{v}_2 \end{array} \right. \quad (\text{A.40})$$

We introduce several variables that are frequently used in the strain-displacement expressions, as shown in Table A.1.

TABLE A.1: Notations of frequently used geometric quantities

Notation	Sphere	Cylinder	Plate
$a_1 = a^{11}$	1	1	1
$a_2 = a^{22}$	$\frac{1}{R^2 \sin^2 \left(\frac{\Theta^1}{R} \right)}$	$\frac{1}{R^2}$	1
$b_1 = b_{11}$	$-\frac{1}{R}$	0	0
$b_2 = b_{22}$	$-R \sin^2 \left(\frac{\Theta^1}{R} \right)$	$-R$	0
$t_1 = -\Gamma_{12}^2 = -\Gamma_{21}^2$	$-\frac{\cos \left(\frac{\Theta^1}{R} \right)}{R \sin \left(\frac{\Theta^1}{R} \right)}$	0	0
$t_2 = -\Gamma_{22}^1$	$R \sin \left(\frac{\Theta^1}{R} \right) \cos \left(\frac{\Theta^1}{R} \right)$	0	0
$c_1 = b_{\cdot 1}^1$	$c_1 = a_1 b_1$		
$c_2 = b_{\cdot 2}^2$	$c_2 = a_2 b_2$		

Using the notations introduced in Table A.1, a general expression of ${}^n\varphi_{\alpha\beta}$ for the three curvilinear coordinate systems can be obtained as

$$\left\{ \begin{array}{lll} {}^n\varphi_{11} = & {}^nv_{1,1} - b_{11}{}^nv_3 & = {}^nv_{1,1} - b_1{}^nv_3 \\ {}^n\varphi_{12} = & {}^nv_{1,2} - \Gamma_{12}^2{}^nv_2 & = {}^nv_{1,2} + t_1{}^nv_2 \\ {}^n\varphi_{21} = & {}^nv_{2,1} - \Gamma_{21}^2{}^nv_2 & = {}^nv_{2,1} + t_1{}^nv_2 \\ {}^n\varphi_{22} = & {}^nv_{2,2} - \Gamma_{22}^1{}^nv_1 - b_{22}{}^nv_3 & = {}^nv_{2,2} + t_2{}^nv_1 - b_2{}^nv_3 \\ {}^n\varphi_{31} = & {}^nv_{3,1} + b_{\cdot 1}^1{}^nv_1 & = {}^nv_{3,1} + c_1{}^nv_1 \\ {}^n\varphi_{32} = & {}^nv_{3,2} + b_{\cdot 2}^2{}^nv_2 & = {}^nv_{3,2} + c_2{}^nv_2 \end{array} \right. \quad (\text{A.41})$$

Appendix B

Strain fields of LRT56 theory

B.1 Strain-displacement relations

The GREEN-LAGRANGE strain terms ($\overset{n}{\varepsilon}_{\alpha\beta}$, $\overset{n}{\varepsilon}_{\alpha 3}$ and $\overset{0}{\varepsilon}_{33} = 0$) can be derived as

$$\begin{aligned}\overset{0}{\varepsilon}_{11} &= \overset{0}{v}_{1,1} - b_1 \overset{0}{v}_3 \quad (\mathbf{A}_0) \\ (\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 (\overset{0}{v}_{1,1})^2 + a_1 (b_1)^2 (\overset{0}{v}_3)^2 + a_2 (\overset{0}{v}_{2,1})^2 + a_2 (t_1)^2 (\overset{0}{v}_2)^2 + (\overset{0}{v}_{3,1})^2 + (c_1)^2 (\overset{0}{v}_1)^2 \right] \\ (\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[-a_1 b_1 \overset{0}{v}_{1,1} \overset{0}{v}_3 - a_1 b_1 \overset{0}{v}_3 \overset{0}{v}_{1,1} + a_2 t_1 \overset{0}{v}_{2,1} \overset{0}{v}_2 + a_2 t_1 \overset{0}{v}_2 \overset{0}{v}_{2,1} + c_1 \overset{0}{v}_{3,1} \overset{0}{v}_1 + c_1 \overset{0}{v}_1 \overset{0}{v}_{3,1} \right] \\ &\hspace{15em} (\text{B.1})\end{aligned}$$

$$\begin{aligned}\overset{0}{\varepsilon}_{22} &= \overset{0}{v}_{2,2} + t_2 \overset{0}{v}_1 - b_2 \overset{0}{v}_3 \quad (\mathbf{A}_0) \\ (\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 (\overset{0}{v}_{1,2})^2 + (a_1 (t_1)^2 + (c_2)^2) (\overset{0}{v}_2)^2 + a_2 (\overset{0}{v}_{2,2})^2 + a_2 (t_2)^2 (\overset{0}{v}_1)^2 \right. \\ &\quad \left. + a_2 (b_2)^2 (\overset{0}{v}_3)^2 + (\overset{0}{v}_{3,2})^2 \right] \\ (\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_1 t_1 \overset{0}{v}_{1,2} \overset{0}{v}_2 + a_1 t_1 \overset{0}{v}_2 \overset{0}{v}_{1,2} + a_2 t_2 \overset{0}{v}_{2,2} \overset{0}{v}_1 + a_2 t_2 \overset{0}{v}_1 \overset{0}{v}_{2,2} \right] \\ (\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[-a_2 b_2 \overset{0}{v}_{2,2} \overset{0}{v}_3 - a_2 b_2 \overset{0}{v}_3 \overset{0}{v}_{2,2} + c_2 \overset{0}{v}_{3,2} \overset{0}{v}_2 + c_2 \overset{0}{v}_2 \overset{0}{v}_{3,2} \right] \\ (\mathbf{A}_{n_4}) &+ \frac{1}{2} \left[-a_2 t_2 b_2 \overset{0}{v}_1 \overset{0}{v}_3 - a_2 t_2 b_2 \overset{0}{v}_3 \overset{0}{v}_1 \right] \\ &\hspace{15em} (\text{B.2})\end{aligned}$$

$$\begin{aligned}
2\varepsilon_{12}^0 &= v_{1,2}^0 + v_{2,1}^0 + 2t_1 v_2^0 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{\mathbf{n}_1}) &+ \frac{1}{2} \left[a_1 v_{1,2}^0 v_{1,1}^0 + a_1 v_{1,1}^0 v_{1,2}^0 - (a_1 t_1 b_1 + a_2 b_2 t_1) v_2^0 v_3^0 - (a_1 t_1 b_1 + a_2 b_2 t_1) v_3^0 v_2^0 \right. \\
&\quad \left. + a_2 v_{2,2}^0 v_{2,1}^0 + a_2 v_{2,1}^0 v_{2,2}^0 + v_{3,2}^0 v_{3,1}^0 + v_{3,1}^0 v_{3,2}^0 \right] \\
(\mathbf{A}_{\mathbf{n}_2}) &+ \frac{1}{2} \left[-a_1 b_1 v_{1,2}^0 v_3^0 - a_1 b_1 v_3^0 v_{1,2}^0 + a_1 t_1 v_2^0 v_{1,1}^0 + a_1 t_1 v_{1,1}^0 v_2^0 + a_2 t_2 v_1^0 v_{2,1}^0 + a_2 t_2 v_{2,1}^0 v_1^0 \right] \\
(\mathbf{A}_{\mathbf{n}_3}) &+ \frac{1}{2} \left[a_2 t_1 v_{2,2}^0 v_2^0 + a_2 t_1 v_2^0 v_{2,2}^0 - a_2 b_2 v_3^0 v_{2,1}^0 - a_2 b_2 v_{2,1}^0 v_3^0 + c_1 v_{3,2}^0 v_1^0 + c_1 v_1^0 v_{3,2}^0 \right] \\
(\mathbf{A}_{\mathbf{n}_4}) &+ \frac{1}{2} \left[(a_2 t_2 t_1 + c_1 c_2) v_1^0 v_2^0 + (a_2 t_2 t_1 + c_1 c_2) v_2^0 v_1^0 \right] \\
(\mathbf{A}_{\mathbf{n}_5}) &+ \frac{1}{2} \left[c_2 v_2^0 v_{3,1}^0 + c_2 v_{3,1}^0 v_2^0 \right]
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\varepsilon_{11}^1 &= v_{1,1}^1 - a_1 b_1 v_{1,1}^0 + a_1 (b_1)^2 v_3^0 - b_1 v_3^1 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{\mathbf{n}_1}) &+ \frac{1}{2} \left[a_1 v_{1,1}^0 v_{1,1}^1 + a_1 v_{1,1}^1 v_{1,1}^0 + a_2 v_{2,1}^0 v_{2,1}^1 + a_2 v_{2,1}^1 v_{2,1}^0 + a_2 (t_1)^2 v_2^0 v_2^1 + a_2 (t_1)^2 v_2^1 v_2^0 \right. \\
&\quad \left. + c_1 v_{3,1}^0 v_1^1 + c_1 v_1^0 v_{3,1}^1 + a_1 (b_1)^2 v_3^0 v_3^1 + a_1 (b_1)^2 v_3^1 v_3^0 + c_1 v_1^0 v_{3,1}^1 + c_1 v_{3,1}^0 v_1^1 \right] \\
(\mathbf{A}_{\mathbf{n}_2}) &+ \frac{1}{2} \left[-a_1 b_1 v_3^0 v_{1,1}^1 - a_1 b_1 v_{1,1}^0 v_3^1 + a_2 t_1 v_{2,1}^0 v_2^1 + a_2 t_1 v_2^0 v_{2,1}^1 + a_2 t_1 v_2^1 v_{2,1}^0 + a_2 t_1 v_{2,1}^1 v_2^0 \right. \\
&\quad \left. + (c_1)^2 v_1^0 v_1^1 + (c_1)^2 v_1^1 v_1^0 - a_1 b_1 v_{1,1}^0 v_3^1 - a_1 b_1 v_3^0 v_{1,1}^1 + v_{3,1}^0 v_{3,1}^1 + v_{3,1}^1 v_{3,1}^0 \right]
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\varepsilon_{22}^1 &= v_{2,2}^1 + t_2 v_1^1 - a_2 b_2 v_{2,2}^0 - a_2 b_2 t_2 v_1^0 + a_2 (b_2)^2 v_3^0 - b_2 v_3^1 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{\mathbf{n}_1}) &+ \frac{1}{2} \left[a_1 v_{1,2}^0 v_{1,2}^1 + a_1 v_{1,2}^1 v_{1,2}^0 + (a_1 (t_1)^2 + (c_2)^2) v_2^0 v_2^1 + (a_1 (t_1)^2 + (c_2)^2) v_2^1 v_2^0 \right. \\
&\quad \left. + a_2 v_{2,2}^0 v_{2,2}^1 + a_2 v_{2,2}^1 v_{2,2}^0 + a_2 (t_2)^2 v_1^0 v_1^1 + a_2 (t_2)^2 v_1^1 v_1^0 + a_2 (b_2)^2 v_3^0 v_3^1 \right. \\
&\quad \left. + a_2 (b_2)^2 v_3^1 v_3^0 + v_{3,2}^0 v_{3,2}^1 + v_{3,2}^1 v_{3,2}^0 \right] \\
(\mathbf{A}_{\mathbf{n}_2}) &+ \frac{1}{2} \left[a_1 t_1 v_{1,2}^0 v_2^1 + a_1 t_1 v_2^0 v_{1,2}^1 + a_1 t_1 v_2^1 v_{1,2}^0 + a_1 t_1 v_{1,2}^1 v_2^0 + a_2 t_2 v_{2,2}^0 v_1^1 + a_2 t_2 v_1^0 v_{2,2}^1 \right. \\
&\quad \left. + a_2 t_2 v_1^1 v_{2,2}^0 + a_2 t_2 v_{2,2}^1 v_1^0 \right] \\
(\mathbf{A}_{\mathbf{n}_3}) &+ \frac{1}{2} \left[-a_2 b_2 v_3^0 v_{2,2}^1 - a_2 b_2 v_{2,2}^0 v_3^1 + c_2 v_{3,2}^0 v_2^1 + c_2 v_2^0 v_{3,2}^1 - a_2 b_2 v_{2,2}^1 v_3^0 - a_2 b_2 v_3^0 v_{2,2}^1 \right. \\
&\quad \left. + c_2 v_2^1 v_{3,2}^0 + c_2 v_{3,2}^0 v_2^1 \right] \\
(\mathbf{A}_{\mathbf{n}_4}) &+ \frac{1}{2} \left[-a_2 b_2 t_2 v_3^0 v_1^1 - a_2 b_2 t_2 v_1^0 v_3^1 - a_2 b_2 t_2 v_3^1 v_1^0 - a_2 b_2 t_2 v_1^1 v_3^0 \right]
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
2\varepsilon_{12}^1 &= \overset{1}{v}_{1,2} + \overset{1}{v}_{2,1} + 2t_1 \overset{1}{v}_2 - a_2 b_2 \overset{0}{v}_{2,1} - a_1 b_1 \overset{0}{v}_{1,2} - (a_2 b_2 t_1 + a_1 b_1 t_1) \overset{0}{v}_2 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 \overset{0}{v}_{1,1} \overset{1}{v}_{1,2} + a_1 \overset{1}{v}_{1,2} \overset{0}{v}_{1,1} - a_1 b_1 t_1 \overset{0}{v}_3 \overset{1}{v}_2 - a_1 b_1 t_1 \overset{1}{v}_2 \overset{0}{v}_3 + a_2 \overset{0}{v}_{2,1} \overset{1}{v}_{2,2} + a_2 \overset{1}{v}_{2,2} \overset{0}{v}_{2,1} \right. \\
&\quad + (a_2 t_1 t_2 + c_1 c_2) \overset{1}{v}_1 \overset{0}{v}_2 + (a_2 t_1 t_2 + c_1 c_2) \overset{0}{v}_2 \overset{1}{v}_1 + a_1 \overset{1}{v}_{1,1} \overset{0}{v}_{1,2} + a_1 \overset{0}{v}_{1,2} \overset{1}{v}_{1,1} \\
&\quad \left. + a_2 t_2 \overset{1}{v}_{2,1} \overset{0}{v}_1 + a_2 t_2 \overset{0}{v}_1 \overset{1}{v}_{2,1} + \overset{0}{v}_{3,1} \overset{1}{v}_{3,2} + \overset{1}{v}_{3,2} \overset{0}{v}_{3,1} + \overset{1}{v}_{3,1} \overset{0}{v}_{3,2} + \overset{0}{v}_{3,2} \overset{1}{v}_{3,1} \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_1 t_1 \overset{0}{v}_{1,1} \overset{1}{v}_2 + a_1 t_1 \overset{1}{v}_2 \overset{0}{v}_{1,1} - a_1 b_1 \overset{0}{v}_3 \overset{1}{v}_{1,2} - a_1 b_1 \overset{1}{v}_{1,2} \overset{0}{v}_3 + a_2 t_2 \overset{0}{v}_{2,1} \overset{1}{v}_1 + a_2 t_2 \overset{1}{v}_1 \overset{0}{v}_{2,1} \right. \\
&\quad \left. + a_2 t_1 \overset{0}{v}_2 \overset{1}{v}_{2,2} + a_2 t_1 \overset{1}{v}_{2,2} \overset{0}{v}_2 + c_1 \overset{0}{v}_1 \overset{1}{v}_{3,2} + c_1 \overset{1}{v}_{3,2} \overset{0}{v}_1 - a_1 b_1 \overset{1}{v}_3 \overset{0}{v}_{1,2} - a_1 b_1 \overset{0}{v}_{1,2} \overset{1}{v}_3 \right] \\
(\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[c_2 \overset{0}{v}_{3,1} \overset{1}{v}_2 + c_2 \overset{1}{v}_2 \overset{0}{v}_{3,1} + a_1 t_1 \overset{1}{v}_{1,1} \overset{0}{v}_2 + a_1 t_1 \overset{0}{v}_2 \overset{1}{v}_{1,1} + a_2 \overset{1}{v}_{2,1} \overset{0}{v}_{2,2} + a_2 \overset{0}{v}_{2,2} \overset{1}{v}_{2,1} \right. \\
&\quad \left. + c_1 \overset{1}{v}_1 \overset{0}{v}_{3,2} + c_1 \overset{0}{v}_{3,2} \overset{1}{v}_1 - a_2 b_2 \overset{0}{v}_{2,1} \overset{1}{v}_3 - a_2 b_2 \overset{1}{v}_3 \overset{0}{v}_{2,1} \right] \\
(\mathbf{A}_{n_4}) &+ \frac{1}{2} \left[(c_1 c_2 + a_2 t_1 t_2) \overset{0}{v}_1 \overset{1}{v}_2 + (c_1 c_2 + a_2 t_1 t_2) \overset{1}{v}_2 \overset{0}{v}_1 - a_2 b_2 \overset{1}{v}_{2,1} \overset{0}{v}_3 - a_2 b_2 \overset{0}{v}_3 \overset{1}{v}_{2,1} \right. \\
&\quad \left. - (a_2 b_2 t_1 + a_1 b_1 t_1) \overset{0}{v}_2 \overset{1}{v}_3 - (a_2 b_2 t_1 + a_1 b_1 t_1) \overset{1}{v}_3 \overset{0}{v}_2 \right] \\
(\mathbf{A}_{n_5}) &+ \frac{1}{2} \left[-a_2 t_1 b_2 \overset{1}{v}_2 \overset{0}{v}_3 - a_2 t_1 b_2 \overset{0}{v}_3 \overset{1}{v}_2 + c_2 \overset{1}{v}_{3,1} \overset{0}{v}_2 + c_2 \overset{0}{v}_2 \overset{1}{v}_{3,1} \right] \\
(\mathbf{A}_{n_6}) &+ \frac{1}{2} \left[a_2 t_1 \overset{1}{v}_2 \overset{0}{v}_{2,2} + a_2 t_1 \overset{0}{v}_{2,2} \overset{1}{v}_2 \right]
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\varepsilon_{11}^2 &= -a_1 b_1 \overset{1}{v}_{1,1} + a_1 (b_1)^2 \overset{1}{v}_3 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 (\overset{1}{v}_{1,1})^2 + a_2 (\overset{1}{v}_{2,1})^2 + a_2 (t_1)^2 (\overset{1}{v}_2)^2 + (c_1)^2 (\overset{1}{v}_1)^2 + a_1 (b_1)^2 (\overset{1}{v}_3)^2 + (\overset{1}{v}_{3,1})^2 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_2 t_1 \overset{1}{v}_{2,1} \overset{1}{v}_2 + a_2 t_1 \overset{1}{v}_2 \overset{1}{v}_{2,1} - a_1 b_1 \overset{1}{v}_{1,1} \overset{1}{v}_3 - a_1 b_1 \overset{1}{v}_3 \overset{1}{v}_{1,1} + c_1 \overset{1}{v}_{3,1} \overset{1}{v}_1 + c_1 \overset{1}{v}_1 \overset{1}{v}_{3,1} \right]
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\varepsilon_{22}^2 &= -a_2 b_2 \overset{1}{v}_{2,2} - a_2 b_2 t_2 \overset{1}{v}_1 + a_2 (b_2)^2 \overset{1}{v}_3 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 (\overset{1}{v}_{1,2})^2 + (a_1 (t_1)^2 + (c_2)^2) (\overset{1}{v}_2)^2 + a_2 (\overset{1}{v}_{2,2})^2 + a_2 (t_2)^2 (\overset{1}{v}_1)^2 \right. \\
&\quad \left. + a_2 (b_2)^2 (\overset{1}{v}_3)^2 + (\overset{1}{v}_{3,2})^2 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_1 t_1 \overset{1}{v}_{1,2} \overset{1}{v}_2 + a_1 t_1 \overset{1}{v}_2 \overset{1}{v}_{1,2} + a_2 t_2 \overset{1}{v}_{2,2} \overset{1}{v}_1 + a_2 t_2 \overset{1}{v}_1 \overset{1}{v}_{2,2} \right] \\
(\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[-a_2 b_2 \overset{1}{v}_{2,2} \overset{1}{v}_3 - a_2 b_2 \overset{1}{v}_3 \overset{1}{v}_{2,2} - c_2 \overset{1}{v}_{3,2} \overset{1}{v}_2 - c_2 \overset{1}{v}_2 \overset{1}{v}_{3,2} \right] \\
(\mathbf{A}_{n_4}) &+ \frac{1}{2} \left[-a_2 b_2 t_2 \overset{1}{v}_1 \overset{1}{v}_3 - a_2 b_2 t_2 \overset{1}{v}_3 \overset{1}{v}_1 \right]
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
2\varepsilon_{12}^2 &= -a_2 b_2 v_{2,1}^1 - (a_2 b_2 t_1 + a_1 b_1 t_1) v_2^1 - a_1 b_1 v_{1,2}^1 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 v_{1,1}^1 v_{1,2}^1 + a_1 v_{1,2}^1 v_{1,1}^1 + a_2 v_{2,1}^1 v_{2,2}^1 + a_2 v_{2,2}^1 v_{2,1}^1 + (a_2 t_1 t_2 + c_1 c_2) v_2^1 v_1^1 \right. \\
&\quad \left. + (a_2 t_1 t_2 + c_1 c_2) v_1^1 v_2^1 + v_{3,1}^1 v_{3,2}^1 + v_{3,2}^1 v_{3,1}^1 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_1 t_1 v_{1,1}^1 v_2^1 + a_1 t_1 v_2^1 v_{1,1}^1 + a_2 t_2 v_{2,1}^1 v_1^1 + a_2 t_2 v_1^1 v_{2,1}^1 - a_1 b_1 v_3^1 v_{1,2}^1 - a_1 b_1 v_{1,2}^1 v_3^1 \right] \\
(\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[a_2 t_1 v_2^1 v_{2,2}^1 + a_2 t_1 v_{2,2}^1 v_2^1 - a_2 b_2 v_{2,1}^1 v_3^1 - a_2 b_2 v_3^1 v_{2,1}^1 + c_1 v_1^1 v_{3,2}^1 + c_1 v_{3,2}^1 v_1^1 \right] \\
(\mathbf{A}_{n_4}) &+ \frac{1}{2} \left[- (a_1 b_1 t_1 + a_2 b_2 t_1) v_3^1 v_2^1 - (a_1 b_1 t_1 + a_2 b_2 t_1) v_2^1 v_3^1 \right] \\
(\mathbf{A}_{n_5}) &+ \frac{1}{2} \left[c_2 v_{3,1}^1 v_2^1 + c_2 v_2^1 v_{3,1}^1 \right]
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
2\varepsilon_{23}^0 &= v_2^0 + v_{3,2}^0 + c_2 v_2^0 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 v_{1,2}^0 v_1^1 + a_1 v_1^0 v_{1,2}^1 + a_2 v_{2,2}^0 v_2^1 + a_2 v_2^0 v_{2,2}^1 + v_{3,2}^0 v_3^1 + v_3^0 v_{3,2}^1 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[a_1 t_1 v_2^0 v_1^1 + a_1 t_1 v_1^0 v_2^1 + a_2 t_2 v_1^0 v_2^1 + a_2 t_2 v_2^0 v_1^1 \right] \\
(\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[-a_2 b_2 v_3^0 v_2^1 - a_2 b_2 v_2^0 v_3^1 + c_2 v_2^0 v_3^1 + c_2 v_3^0 v_2^1 \right]
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
2\varepsilon_{13}^0 &= v_1^0 + v_{3,1}^0 + c_1 v_1^0 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 v_{1,1}^0 v_1^1 + a_1 v_1^0 v_{1,1}^1 + a_2 v_{2,1}^0 v_2^1 + a_2 v_2^0 v_{2,1}^1 + v_{3,1}^0 v_3^1 + v_3^0 v_{3,1}^1 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[-a_1 b_1 v_3^0 v_1^1 - a_1 b_1 v_1^0 v_3^1 + a_2 t_1 v_2^0 v_2^1 + a_2 t_1 v_2^1 v_2^0 + c_1 v_1^0 v_3^1 + c_1 v_3^0 v_1^1 \right]
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
2\varepsilon_{23}^1 &= (c_2 - a_2 b_2) v_2^1 + v_{3,2}^1 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n_1}) &+ \frac{1}{2} \left[a_1 v_{1,2}^1 v_1^1 + a_1 v_1^1 v_{1,2}^1 + a_2 v_{2,2}^1 v_2^1 + a_2 v_2^1 v_{2,2}^1 + v_{3,2}^1 v_3^1 + v_3^1 v_{3,2}^1 \right] \\
(\mathbf{A}_{n_2}) &+ \frac{1}{2} \left[(a_1 t_1 + a_2 t_2) v_1^1 v_2^1 + (a_1 t_1 + a_2 t_2) v_2^1 v_1^1 \right] \\
(\mathbf{A}_{n_3}) &+ \frac{1}{2} \left[(c_2 - a_2 b_2) v_2^1 v_3^1 + (c_2 - a_2 b_2) v_3^1 v_2^1 \right]
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
2\varepsilon_{13}^1 &= (c_1 - a_1 b_1) v_1^1 + v_{3,1}^1 \quad (\mathbf{A}_0) \\
(\mathbf{A}_{n1}) &+ \frac{1}{2} \left[a_1 v_{1,1}^1 v_1^1 + a_1 v_1^1 v_{1,1}^1 + a_2 v_{2,1}^1 v_2^1 + a_2 v_2^1 v_{2,1}^1 + v_{3,1}^1 v_3^1 + v_3^1 v_{3,1}^1 \right] \\
(\mathbf{A}_{n2}) &+ \frac{1}{2} \left[2a_2 t_1 (v_2^1)^2 + (c_1 - a_1 b_1) v_1^1 v_3^1 + (c_1 - a_1 b_1) v_3^1 v_1^1 \right]
\end{aligned} \tag{B.13}$$

B.2 Strain-displacement relations in matrix form

In order to express the nonlinear strain-displacement terms in matrix form, we arrange the nonlinear relations in multi-matrices using the following equation

$$\mathbf{S} = \left(\mathbf{A}_0 + \frac{1}{2} (\mathbf{A}_{n1} + \mathbf{A}_{n2} + \cdots + \mathbf{A}_{n6}) \right) \boldsymbol{\theta}, \tag{B.14}$$

with

$$\mathbf{S} = \left\{ \varepsilon_{11}^0, \varepsilon_{22}^0, 2\varepsilon_{12}^0, \varepsilon_{11}^1, \varepsilon_{22}^1, 2\varepsilon_{12}^1, \varepsilon_{11}^2, \varepsilon_{22}^2, 2\varepsilon_{12}^2, 2\varepsilon_{23}^0, 2\varepsilon_{13}^0, 2\varepsilon_{23}^1, 2\varepsilon_{13}^1 \right\}^T, \tag{B.15}$$

$$\boldsymbol{\theta} = \left\{ v_{1,1}^0, v_{1,2}^0, v_{2,1}^0, v_{2,2}^0, v_{3,1}^0, v_{3,2}^0, v_{1,1}^1, v_{1,2}^1, v_{2,1}^1, v_{2,2}^1, v_{3,1}^1, v_{3,2}^1, v_1^0, v_2^0, v_3^0, v_1^1, v_2^1, v_3^1 \right\}^T. \tag{B.16}$$

To shorten the strain expressions, several new variables are introduced as

$$d_1 = a_1 (t_1)^2 + (c_2)^2, \tag{B.17}$$

$$d_2 = -(a_1 t_1 b_1 + a_2 b_2 t_1), \tag{B.18}$$

$$d_3 = a_2 t_1 t_2 + c_1 c_2, \tag{B.19}$$

$$d_4 = a_1 t_1 + a_2 t_2, \tag{B.20}$$

$$d_5 = c_1 - a_1 b_1, \tag{B.21}$$

$$d_6 = c_2 - a_2 b_2. \tag{B.22}$$

The linear and nonlinear matrices are respectively obtained as

Linear matrix $\mathbf{A}_0$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1	1														$-b_1$				1
2				1									t_2		$-b_2$				2
3		1	1											$2t_1$					3
4	$-a_1b_1$						1								$a_1(b_1)^2$			$-b_1$	4
5				$-a_2b_2$						1			$-a_2b_2t_2$		$a_2(b_2)^2$	t_2		$-b_2$	5
6		$-a_1b_1$	$-a_2b_2$					1	1					$-a_2b_2t_1$ $-a_1b_1t_1$			$2t_1$		6
7							$-a_1b_1$											$a_1(b_1)^2$	7
8										$-a_2b_2$						$-a_2b_2t_2$		$a_2(b_2)^2$	8
9								$-a_1b_1$	$-a_2b_2$								$-a_2b_2t_1$ $-a_1b_1t_1$		9
10						1								c_2			1		10
11					1								c_1			1			11
12												1					$c_2 - a_2b_2$		12
13											1					$c_1 - a_1b_1$			13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{n1}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1	$a_1^0 v_{1,1}$		$a_2^0 v_{2,1}$		$v_{3,1}^0$								$(c_1)^2 v_1^0$	$a_2(t_2)^2 v_2^0$	$a_1(b_2)^2 v_3^0$				1
2		$a_1^0 v_{1,2}$		$a_2^0 v_{2,2}$		$v_{3,2}^0$							$a_2(t_2)^2 v_1^0$	$d_1^0 v_2$	$a_2(b_2)^2 v_3^0$				2
3	$a_1^0 v_{1,2}$	$a_1^0 v_{1,1}$	$a_2^0 v_{2,2}$	$a_2^0 v_{2,1}$	$v_{3,2}^0$	$v_{3,1}^0$								$d_2^0 v_3$	$d_2^0 v_2$				3
4	$a_1^1 v_{1,1}$		$a_2^1 v_{2,1}$		$c_1^1 v_1$		$a_1^0 v_{1,1}$		$a_2^0 v_{2,1}$		$c_1^0 v_1$		$c_1^1 v_{3,1}$	$a_2(t_1)^2 v_2^1$	$a_1(b_1)^2 v_3^1$	$c_1^0 v_{3,1}$	$a_2(t_1)^2 v_2^0$	$a_1(b_1)^2 v_3^0$	4
5		$a_1^1 v_{1,2}$		$a_2^1 v_{2,2}$		$v_{3,2}^1$		$a_1^0 v_{1,2}$		$a_2^0 v_{2,2}$		$v_{3,2}^0$	$a_2(t_2)^2 v_1^1$	$d_1^1 v_2$	$a_2(b_2)^2 v_3^1$	$a_2(t_2)^2 v_1^0$	$d_1^0 v_2$	$a_2(b_2)^2 v_3^0$	5
6	$a_1^1 v_{1,2}$	$a_1^1 v_{1,1}$	$a_2^1 v_{2,2}$		$v_{3,2}^1$	$v_{3,1}^1$	$a_1^0 v_{1,2}$	$a_1^0 v_{1,1}$	$a_2^0 t_2 v_1$	$a_2^0 v_{2,1}$	$v_{3,2}^0$	$v_{3,1}^0$	$a_2 t_2 v_{2,1}^1$	$d_3^1 v_1$	$-a_1 b_1 t_1 v_2^1$	$d_3^0 v_2$	$-a_1 b_1 t_1 v_3^0$		6
7							$a_1^1 v_{1,1}$		$a_2^1 v_{2,1}$		$v_{3,1}^1$					$(c_1)^2 v_1^1$	$a_2(t_1)^2 v_2^1$	$a_1(b_1)^2 v_3^1$	7
8							$a_1^1 v_{1,2}$		$a_2^1 v_{2,2}$		$v_{3,2}^1$					$a_2(t_2)^2 v_1^1$	$d_1^1 v_2$	$a_2(b_2)^2 v_3^1$	8
9							$a_1^1 v_{1,2}$	$a_1^1 v_{1,1}$	$a_2^1 v_{2,2}$	$a_2^1 v_{2,1}$	$v_{3,2}^1$	$v_{3,1}^1$				$d_3^1 v_2$	$d_3^1 v_1$		9
10		$a_1^1 v_1$		$a_2^1 v_2$		v_3^1										$a_1^0 v_{1,2}$	$a_2^0 v_{2,2}$	$v_{3,2}^0$	10
11	$a_1^1 v_1$		$a_2^1 v_2$		v_3^1											$a_1^0 v_{1,1}$	$a_2^0 v_{2,1}$	$v_{3,1}^0$	11
12							$a_1^1 v_1$		$a_2^1 v_2$		v_3^1					$a_1^1 v_{1,2}$	$a_2^1 v_{2,2}$	$v_{3,2}^1$	12
13							$a_1^1 v_1$		$a_2^1 v_2$		v_3^1					$a_1^1 v_{1,1}$	$a_2^1 v_{2,1}$	$v_{3,1}^1$	13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{n2}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1	$-a_1 b_1^0 v_3$		$a_2 t_1^0 v_2$		$c_1^0 v_1$								$c_1^0 v_{3,1}$	$a_2 t_1^0 v_{2,1}$	$-a_1 b_1^0 v_{1,1}$				1
2		$a_1 t_1^0 v_2$		$a_2 t_2^0 v_1$									$a_2 t_2^0 v_{2,2}$	$a_1 t_1^0 v_{1,2}$					2
3	$a_1 t_1^0 v_2$	$-a_1 b_1^0 v_3$	$a_2 t_2^0 v_1$										$a_2 t_2^0 v_{2,1}$	$a_1 t_1^0 v_{1,1}$	$-a_1 b_1^0 v_{1,2}$				3
4	$-a_1 b_1^1 v_3$		$a_2 t_1^1 v_2$		$v_{3,1}^1$		$-a_1 b_1^0 v_3$		$a_2 t_1^0 v_2$		$v_{3,1}^0$		$(c_1)^2 v_1^1$	$a_2 t_1^1 v_{2,1}$	$-a_1 b_1^1 v_{1,1}$	$(c_1)^2 v_1^0$	$a_2 t_1^0 v_{2,1}$	$-a_1 b_1^0 v_{1,1}$	4
5		$a_1 t_1^1 v_2$		$a_2 t_2^1 v_1$				$a_1 t_1^0 v_2$		$a_2 t_2^0 v_1$			$a_2 t_2^1 v_{2,2}$	$a_1 t_1^1 v_{1,2}$		$a_2 t_2^0 v_{2,2}$	$a_1 t_1^0 v_{1,2}$		5
6	$a_1 t_1^1 v_2$	$-a_1 b_1^1 v_3$	$a_2 t_2^1 v_1$					$-a_1 b_1^0 v_3$		$a_2 t_1^0 v_2$		$c_1^0 v_1$	$c_1^1 v_{3,2}$	$a_2 t_1^1 v_{2,2}$	$-a_1 b_1^1 v_{1,2}$	$a_2 t_2^0 v_{2,1}$	$a_1 t_1^0 v_{1,1}$	$-a_1 b_1^0 v_{1,2}$	6
7							$-a_1 b_1^1 v_3$		$a_2 t_1^1 v_2$		$c_1^1 v_1$					$c_1^1 v_{3,1}$	$a_2 t_1^1 v_{2,1}$	$-a_1 b_1^1 v_{1,1}$	7
8								$a_1 t_1^1 v_2$		$a_2 t_2^1 v_1$						$a_2 t_2^1 v_{2,2}$	$a_1 t_1^1 v_{1,2}$		8
9							$a_1 t_1^1 v_2$	$-a_1 b_1^1 v_3$	$a_2 t_2^1 v_1$							$a_2 t_2^1 v_{2,1}$	$a_1 t_1^1 v_{1,1}$	$-a_1 b_1^1 v_{1,2}$	9
10													$a_2 t_2^1 v_2$	$a_1 t_1^1 v_1$		$a_1 t_1^0 v_2$	$a_2 t_2^0 v_1$		10
11													$c_1^1 v_3$	$a_2 t_1^1 v_2$	$-a_1 b_1^1 v_1$	$-a_1 b_1^0 v_3$	$a_2 t_1^0 v_2$	$c_1^0 v_1$	11
12																$d_4^1 v_2$	$d_4^1 v_1$		12
13																$d_5^1 v_3$	$2a_2 t_1^1 v_2$	$d_5^1 v_1$	13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{n3}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2				$-a_2 b_2^0 v_3$		$c_2^0 v_2$								$c_2^0 v_{3,2}$	$-a_2 b_2^0 v_{2,2}$				2
3			$-a_2 b_2^0 v_3$	$a_2 t_1^0 v_2$		$c_1^0 v_1$							$c_1^0 v_{3,2}$	$a_2 t_1^0 v_{2,2}$	$-a_2 b_2^0 v_{2,1}$				3
4																			4
5				$-a_2 b_2^1 v_3$		$c_2^1 v_2$				$-a_2 b_2^0 v_3$		$c_2^0 v_2$		$c_2^1 v_{3,2}$	$-a_2 b_2^1 v_{2,2}$		$c_2^0 v_{3,2}$	$-a_2 b_2^0 v_{2,2}$	5
6			$-a_2 b_2^1 v_3$	$a_2^1 v_{2,1}$	$c_2^1 v_2$	$c_1^1 v_1$	$a_1 t_1^0 v_2$		$a_2^0 v_{2,2}$					$a_1 t_1^1 v_{1,1}$		$c_1^0 v_{3,2}$	$c_2^0 v_{3,1}$	$-a_2 b_2^0 v_{2,1}$	6
7																			7
8										$-a_2 b_2^1 v_3$		$-c_2^1 v_2$					$-c_2^1 v_{3,2}$	$-a_2 b_2^1 v_{2,2}$	8
9									$-a_2 b_2^1 v_3$	$a_2 t_1^1 v_2$		$c_1^1 v_1$				$c_1^1 v_{3,2}$	$a_2 t_1^1 v_{2,2}$	$-a_2 b_2^1 v_{2,1}$	9
10														$c_2^1 v_3$	$-a_2 b_2^1 v_2$		$-a_2 b_2^0 v_3$	$c_2^0 v_2$	10
11																			11
12																	$d_6^1 v_3$	$d_6^1 v_2$	12
13																			13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{n4}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2													$-a_2 t_2 b_2 v_3^0$		$-a_2 t_2 b_2 v_1^0$				2
3													$d_3 v_2^0$	$d_3 v_1^0$					3
4																			4
5													$-a_2 b_2 t_2 v_3^1$		$-a_2 b_2 t_2 v_1^1$	$-a_2 b_2 t_2 v_3^0$		$-a_2 b_2 t_2 v_1^0$	5
6									$-a_2 b_2 v_3^0$				$d_3 v_2^1$	$d_2 v_3^1$	$-a_2 b_2 v_{2,1}^1$		$d_3 v_1^0$	$d_2 v_2^0$	6
7																			7
8																$-a_2 b_2 t_2 v_3^1$		$-a_2 b_2 t_2 v_1^1$	8
9																	$d_2 v_3^1$	$d_2 v_2^1$	9
10																			10
11																			11
12																			12
13																			13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{n5}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3					$c_2^0 v_2$									$c_2^0 v_{3,1}$					3
4																			4
5																			5
6											$c_2^0 v_2$			$c_2^1 v_{3,1}$	$-a_2 t_1 b_2^1 v_2$		$-a_2 t_1 b_2^0 v_3$		6
7																			7
8																			8
9											$c_2^1 v_2$						$c_2^1 v_{3,1}$		9
10																			10
11																			11
12																			12
13																			13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Nonlinear matrix $\mathbf{A}_{\mathbf{n}_6}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3																			3
4																			4
5																			5
6				$a_2 t_1^1 v_2$													$a_2 t_1^0 v_{2,2}$		6
7																			7
8																			8
9																			9
10																			10
11																			11
12																			12
13																			13
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

B.3 Mechanically or electrically induced stresses

Assuming a known vector

$$\mathbf{L} = \left\{ L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}, L_{13} \right\}^T,$$

such that the term $\mathbf{A}_{n1}^T \mathbf{L}$ can be re-written as

1. $a_1^0 v_{1,1} L_1 + a_1^0 v_{1,2} L_3 + a_1^1 v_{1,1} L_4 + a_1^1 v_{1,2} L_6 + a_1^1 v_1 L_{11}$
2. $a_1^0 v_{1,2} L_2 + a_1^0 v_{1,1} L_3 + a_1^1 v_{1,2} L_5 + a_1^1 v_{1,1} L_6 + a_1^1 v_1 L_{10}$
3. $a_2^0 v_{2,1} L_1 + a_2^0 v_{2,2} L_3 + a_2^1 v_{2,1} L_4 + a_2^1 v_{2,2} L_6 + a_2^1 v_2 L_{11}$
4. $a_2^0 v_{2,2} L_2 + a_2^0 v_{2,1} L_3 + a_2^1 v_{2,2} L_5 + a_2^1 v_2 L_{10}$
5. $v_{3,1}^0 L_1 + v_{3,2}^0 L_3 + c_1^1 v_1 L_4 + v_{3,2}^1 L_6 + v_3^1 L_{11}$
6. $v_{3,2}^0 L_2 + v_{3,1}^0 L_3 + v_{3,2}^1 L_5 + v_{3,1}^1 L_6 + v_3^1 L_{10}$
7. $a_1^0 v_{1,1} L_4 + a_1^0 v_{1,2} L_6 + a_1^1 v_{1,1} L_7 + a_1^1 v_{1,2} L_9 + a_1^1 v_1 L_{13}$
8. $a_1^0 v_{1,2} L_5 + a_1^0 v_{1,1} L_6 + a_1^1 v_{1,2} L_8 + a_1^1 v_{1,1} L_9 + a_1^1 v_1 L_{12}$
9. $a_2^0 v_{2,1} L_4 + a_2^0 v_{2,2} L_6 + a_2^1 v_{2,1} L_7 + a_2^1 v_{2,2} L_9 + a_2^1 v_2 L_{13}$
10. $a_2^0 v_{2,2} L_5 + a_2^0 v_{2,1} L_6 + a_2^1 v_{2,2} L_8 + a_2^1 v_{2,1} L_9 + a_2^1 v_2 L_{12}$
11. $c_1^0 v_1 L_4 + v_{3,2}^0 L_6 + v_{3,1}^1 L_7 + v_{3,2}^1 L_9 + v_3^1 L_{13}$
12. $v_{3,2}^0 L_5 + v_{3,1}^0 L_6 + v_{3,2}^1 L_8 + v_{3,1}^1 L_9 + v_3^1 L_{12}$
13. $(c_1)^2 v_1^0 L_1 + a_2(t_2)^2 v_1^0 L_2 + c_1^1 v_{3,1}^1 L_4 + a_2(t_2)^2 v_1^1 L_5 + a_2 t_2^1 v_{2,1}^1 L_6$
14. $a_2(t_1)^2 v_2^0 L_1 + d_1^0 v_2^0 L_2 + d_2^0 v_3^0 L_3 + a_2(t_1)^2 v_2^1 L_4 + d_1^1 v_2^1 L_5 + d_3^1 v_1^1 L_6$
15. $a_1(b_1)^2 v_3^0 L_1 + a_2(b_2)^2 v_3^0 L_2 + d_2^0 v_2^0 L_3 + a_1(b_1)^2 v_3^1 L_4 + a_2(b_2)^2 v_3^1 L_5 - a_1 b_1 t_1^1 v_2^1 L_6$
16. $c_1^0 v_{3,1}^0 L_4 + a_2(t_2)^2 v_1^0 L_5 + d_3^0 v_2^0 L_6 + (c_1)^2 v_1^1 L_7 + a_2(t_2)^2 v_1^1 L_8 + d_3^1 v_2^1 L_9$
 $+ a_1^0 v_{1,2}^0 L_{10} + a_1^0 v_{1,1}^0 L_{11} + a_1^1 v_{1,2}^1 L_{12} + a_1^1 v_{1,1}^1 L_{13}$
17. $a_2(t_1)^2 v_2^0 L_4 + d_1^0 v_2^0 L_5 - a_1 b_1 t_1^0 v_3^0 L_6 + a_2(t_1)^2 v_2^1 L_7 + d_1^1 v_2^1 L_8 + d_3^1 v_1^1 L_9$
 $+ a_2^0 v_{2,2}^0 L_{10} + a_2^0 v_{2,1}^0 L_{11} + a_2^1 v_{2,2}^1 L_{12} + a_2^1 v_{2,1}^1 L_{13}$
18. $a_1(b_1)^2 v_3^0 L_4 + a_2(b_2)^2 v_3^0 L_5 + a_1(b_1)^2 v_3^1 L_7 + a_2(b_2)^2 v_3^1 L_8 + v_{3,2}^0 L_{10} + v_{3,1}^0 L_{11}$
 $+ v_{3,2}^1 L_{12} + v_{3,1}^1 L_{13}$

The term $\mathbf{A}_{n2}^T \mathbf{L}$ will be

1. $-a_1 b_1^0 v_3 L_1 + a_1 t_1^0 v_2 L_3 - a_1 b_1^1 v_3 L_4 + a_1 t_1^1 v_2 L_6$
2. $a_1 t_1^0 v_2 L_2 - a_1 b_1^0 v_3 L_3 + a_1 t_1^1 v_2 L_5 - a_1 b_1^1 v_3 L_6$
3. $a_2 t_1^0 v_2 L_1 + a_2 t_2^0 v_1 L_3 + a_2 t_1^1 v_2 L_4 + a_2 t_2^1 v_1 L_6$
4. $a_2 t_2^0 v_1 L_2 + a_2 t_2^1 v_1 L_5$
5. $c_1^0 v_1 L_1 + v_{3,1}^1 L_4$
6. ---
7. $-a_1 b_1^0 v_3 L_4 - a_1 b_1^1 v_3 L_7 + a_1 t_1^1 v_2 L_9$
8. $a_1 t_1^0 v_2 L_5 - a_1 b_1^0 v_3 L_6 + a_1 t_1^1 v_2 L_8 - a_1 b_1^1 v_3 L_9$
9. $a_2 t_1^0 v_2 L_4 + a_2 t_1^1 v_2 L_7 + a_2 t_2^1 v_1 L_9$
10. $a_2 t_2^0 v_1 L_5 + a_2 t_1^0 v_2 L_6 + a_2 t_2^1 v_1 L_8$
11. $v_{3,1}^0 L_4 + c_1^1 v_1 L_7$
12. $c_1^0 v_1 L_6$
13. $c_1^0 v_{3,1} L_1 + a_2 t_2^0 v_{2,2} L_2 + a_2 t_2^0 v_{2,1} L_3 + (c_1)^2 v_1^1 L_4 + a_2 t_2^1 v_{2,2} L_5 + c_1^1 v_{3,2} L_6$
 $+ a_2 t_2^1 v_2 L_{10} + c_1^1 v_3 L_{11}$
14. $a_2 t_1^0 v_{2,1} L_1 + a_1 t_1^0 v_{1,2} L_2 + a_1 t_1^0 v_{1,1} L_3 + a_2 t_1^1 v_{2,1} L_4 + a_1 t_1^1 v_{1,2} L_5 + a_2 t_1^1 v_{2,2} L_6$
 $+ a_1 t_1^1 v_1 L_{10} + a_2 t_1^1 v_2 L_{11}$
15. $-a_1 b_1^0 v_{1,1} L_1 - a_1 b_1^0 v_{1,2} L_3 - a_1 b_1^1 v_{1,1} L_4 - a_1 b_1^1 v_{1,2} L_6 - a_1 b_1^1 v_1 L_{11}$
16. $(c_1)^2 v_1^0 L_4 + a_2 t_2^0 v_{2,2} L_5 + a_2 t_2^0 v_{2,1} L_6 + c_1^1 v_{3,1} L_7 + a_2 t_2^1 v_{2,2} L_8 + a_2 t_2^1 v_{2,1} L_9$
 $+ a_1 t_1^0 v_2 L_{10} - a_1 b_1^0 v_3 L_{11} + d_4^1 v_2 L_{12} + d_5^1 v_3 L_{13}$
17. $a_2 t_1^0 v_{2,1} L_4 + a_1 t_1^0 v_{1,2} L_5 + a_1 t_1^0 v_{1,1} L_6 + a_2 t_1^1 v_{2,1} L_7 + a_1 t_1^1 v_{1,2} L_8 + a_1 t_1^1 v_{1,1} L_9$
 $+ a_2 t_2^0 v_1 L_{10} + a_2 t_1^0 v_2 L_{11} + d_4^1 v_1 L_{12} + 2a_2 t_1^1 v_2 L_{13}$
18. $-a_1 b_1^0 v_{1,1} L_4 - a_1 b_1^0 v_{1,2} L_6 - a_1 b_1^1 v_{1,1} L_7 - a_1 b_1^1 v_{1,2} L_9 + c_1^0 v_1 L_{11} + d_5^1 v_1 L_{13}$

The term $\mathbf{A}_{n3}^T \mathbf{L}$ can be re-written as

3. $-a_2 b_2^0 v_3 L_3 - a_2 b_2^1 v_3 L_3$
4. $-a_2 b_2^0 v_3 L_2 + a_2 t_1^0 v_2 L_3 - a_2 b_2^1 v_3 L_5 + a_2^1 v_{2,1} L_6$
5. $c_2^1 v_2 L_6$
6. $c_2^0 v_2 L_2 + c_1^0 v_1 L_3 + c_2^1 v_2 L_5 + c_1^1 v_1 L_6$
7. $a_1 t_1^0 v_2 L_6$
8. _____
9. $a_2 v_{2,2}^0 L_6 - a_2 b_2^1 v_3 L_9$
10. $-a_2 b_2^0 v_3 L_5 - a_2 b_2^1 v_3 L_8 + a_2 t_1^1 v_2 L_9$
11. _____
12. $c_2^0 v_2 L_5 - c_2^1 v_2 L_8 + c_1^1 v_1 L_9$
13. $c_1^0 v_{3,2} L_3$
14. $c_2^0 v_{3,2} L_2 + a_2 t_1^0 v_{2,2} L_3 + c_2^1 v_{3,2} L_5 + a_1 t_1^1 v_{1,1} L_6 + c_2^1 v_3 L_{10}$
15. $-a_2 b_2^0 v_{2,2} L_2 - a_2 b_2^0 v_{2,1} L_3 - a_2 b_2^1 v_{2,2} L_5 - a_2 b_2^1 v_2 L_{10}$
16. $c_1^0 v_{3,2} L_6 + c_1^1 v_{3,2} L_9$
17. $c_2^0 v_{3,2} L_5 + c_2^0 v_{3,1} L_6 - c_2^1 v_{3,2} L_8 + a_2 t_1^1 v_{2,2} L_9 - a_2 b_2^0 v_3 L_{10} + d_6^1 v_3 L_{12}$
18. $-a_2 b_2^0 v_{2,2} L_5 - a_2 b_2^0 v_{2,1} L_6 - a_2 b_2^1 v_{2,2} L_8 - a_2 b_2^1 v_{2,1} L_9 + c_2^0 v_2 L_{10} + d_6^1 v_2 L_{12}$

The term $\mathbf{A}_{n4}^T \mathbf{L}$ can be expanded to

9. $-a_2 b_2^0 v_3 L_6$
13. $-a_2 t_2 b_2^0 v_3 L_2 + d_3^0 v_2 L_3 - a_2 b_2 t_2^1 v_3 L_5 + d_3^1 v_2 L_6$
14. $d_3^0 v_1 L_3 + d_2^1 v_3 L_6$
15. $-a_2 t_2 b_2^0 v_1 L_2 - a_2 b_2 t_2^1 v_1 L_5 - a_2 b_2^1 v_{2,1} L_6$
16. $-a_2 b_2 t_2^0 v_3 L_5 - a_2 b_2 t_2^1 v_3 L_8$
17. $d_3^0 v_1 L_6 + d_2^1 v_3 L_9$
18. $-a_2 b_2 t_2^0 v_1 L_5 + d_2^0 v_2 L_6 - a_2 b_2 t_2^1 v_1 L_8 + d_2^1 v_2 L_9$

The term $\mathbf{A}_{n5}^T \mathbf{L}$ will be

- 5. $c_2^0 v_2 L_3$
- 11. $c_2^0 v_2 L_6 + c_2^1 v_2 L_9$
- 14. $c_2^0 v_{3,1} L_3 + c_2^1 v_{3,1} L_6$
- 15. $-a_2 t_1 b_2^1 v_2 L_6$
- 17. $-a_2 t_1 b_2^0 v_3 L_6 + c_2^1 v_{3,1} L_9$

The term $\mathbf{A}_{n6}^T \mathbf{L}$ will be

- 4. $a_2 t_1^1 v_2 L_6$
- 17. $a_2 t_1^0 v_{2,2} L_6$

Re-arranging the expanded expressions, the corresponding mechanically or electrically induced resultant stresses can be obtained as

$$\mathbf{A}_{nx}^T \mathbf{L} = \mathbf{S}_{uux} \boldsymbol{\theta}, \quad \text{where } x = 1, 2, \dots, 6. \quad (\text{B.23})$$

To shorten the expressions in matrix form, we introduce the following coefficient expressions

$$f_1 = (c_1)^2 L_1 + a_2 (t_2)^2 L_2, \quad (\text{B.24})$$

$$f_2 = a_2 (t_1)^2 L_1 + d_1 L_2, \quad (\text{B.25})$$

$$f_3 = a_2 (t_1)^2 L_4 + d_1 L_5, \quad (\text{B.26})$$

$$f_4 = a_1 (b_1)^2 L_1 + a_2 (b_2)^2 L_2, \quad (\text{B.27})$$

$$f_5 = (c_1)^2 L_7 + a_2 (t_2)^2 L_8, \quad (\text{B.28})$$

$$f_6 = a_2 (t_1)^2 L_4 + d_1 L_5, \quad (\text{B.29})$$

$$f_7 = a_2 (t_1)^2 L_7 + d_1 L_8, \quad (\text{B.30})$$

$$f_8 = a_1 (b_1)^2 L_4 + a_2 (b_2)^2 L_5, \quad (\text{B.31})$$

$$f_9 = a_1 (b_1)^2 L_7 + a_2 (b_2)^2 L_8. \quad (\text{B.32})$$

From $\mathbf{A}_{\text{nl}}^T \mathbf{L}$, we will obtain $\mathbf{S}_{\text{uu1}}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1	$a_1 L_1$	$a_1 L_3$					$a_1 L_4$	$a_1 L_6$								$a_1 L_{11}$			1
2	$a_1 L_3$	$a_1 L_2$					$a_1 L_6$	$a_1 L_5$								$a_1 L_{10}$			2
3			$a_2 L_1$	$a_2 L_3$					$a_2 L_4$	$a_2 L_6$							$a_2 L_{11}$		3
4			$a_2 L_3$	$a_2 L_2$						$a_2 L_5$							$a_2 L_{10}$		4
5					L_1	L_3						L_6				$c_1 L_4$		L_{11}	5
6					L_3	L_2					L_6	L_5						L_{10}	6
7	$a_1 L_4$	$a_1 L_6$					$a_1 L_7$	$a_1 L_9$								$a_1 L_{13}$			7
8	$a_1 L_6$	$a_1 L_5$					$a_1 L_9$	$a_1 L_8$								$a_1 L_{12}$			8
9			$a_2 L_4$						$a_2 L_7$	$a_2 L_9$			$a_2 t_2 L_6$				$a_2 L_{13}$		9
10			$a_2 L_6$	$a_2 L_5$					$a_2 L_9$	$a_2 L_8$							$a_2 L_{12}$		10
11						L_6					L_7	L_9	$c_1 L_4$					L_{13}	11
12					L_6	L_5					L_9	L_8						L_{12}	12
13									$a_2 t_2 L_6$		$c_1 L_4$		f_1			$a_2 (t_2)^2 L_5$			13
14														f_2	$d_2 L_3$	$d_3 L_6$	f_3		14
15														$d_2 L_3$	f_4		$-a_1 b_1 t_1 L_6$	f_8	15
16	$a_1 L_{11}$	$a_1 L_{10}$			$c_1 L_4$		$a_1 L_{13}$	$a_1 L_{12}$					$a_2 (t_2)^2 L_5$	$d_3 L_6$		f_5	$d_3 L_9$		16
17			$a_2 L_{11}$	$a_2 L_{10}$					$a_2 L_{13}$	$a_2 L_{12}$				f_6	$-a_1 b_1 t_1 L_6$	$d_3 L_9$	f_7		17
18					L_{11}	L_{10}					L_{13}	L_{12}			f_8			f_9	18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

From $\mathbf{A}_{n2}^T \mathbf{L}$, we will obtain $\mathbf{S}_{uu2}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1														$a_1 t_1 L_3$	$-a_1 b_1 L_1$		$a_1 t_1 L_6$	$-a_1 b_1 L_4$	1
2														$a_1 t_1 L_2$	$-a_1 b_1 L_3$		$a_1 t_1 L_5$	$-a_1 b_1 L_6$	2
3													$a_2 t_2 L_3$	$a_2 t_1 L_1$		$a_2 t_2 L_6$	$a_2 t_1 L_4$		3
4													$a_2 t_2 L_2$			$a_2 t_2 L_5$			4
5											L_4		$c_1 L_1$						5
6																			6
7															$-a_1 b_1 L_4$		$a_1 t_1 L_9$	$-a_1 b_1 L_7$	7
8														$a_1 t_1 L_5$	$-a_1 b_1 L_6$		$a_1 t_1 L_8$	$-a_1 b_1 L_9$	8
9														$a_2 t_1 L_4$		$a_2 t_2 L_9$	$a_2 t_1 L_7$		9
10													$a_2 t_2 L_5$	$a_2 t_1 L_6$		$a_2 t_2 L_8$			10
11					L_4											$c_1 L_7$			11
12													$c_1 L_6$						12
13			$a_2 t_2 L_3$	$a_2 t_2 L_2$	$c_1 L_1$					$a_2 t_2 L_5$		$c_1 L_6$				$(c_1)^2 L_4$	$a_2 t_2 L_{10}$	$c_1 L_{11}$	13
14	$a_1 t_1 L_3$	$a_1 t_1 L_2$	$a_2 t_1 L_1$					$a_1 t_1 L_5$	$a_2 t_1 L_4$	$a_2 t_1 L_6$						$a_1 t_1 L_{10}$	$a_2 t_1 L_{11}$		14
15	$-a_1 b_1 L_1$	$-a_1 b_1 L_3$					$-a_1 b_1 L_4$	$-a_1 b_1 L_6$								$-a_1 b_1 L_{11}$			15
16			$a_2 t_2 L_6$	$a_2 t_2 L_5$					$a_2 t_2 L_9$	$a_2 t_2 L_8$	$c_1 L_7$		$(c_1)^2 L_4$	$a_1 t_1 L_{10}$	$-a_1 b_1 L_{11}$		$d_4 L_{12}$	$d_5 L_{13}$	16
17	$a_1 t_1 L_6$	$a_1 t_1 L_5$	$a_2 t_1 L_4$				$a_1 t_1 L_9$	$a_1 t_1 L_8$	$a_2 t_1 L_7$				$a_2 t_2 L_{10}$	$a_2 t_1 L_{11}$		$d_4 L_{12}$	$2a_2 t_1 L_{13}$		17
18	$-a_1 b_1 L_4$	$-a_1 b_1 L_6$					$-a_1 b_1 L_7$	$-a_1 b_1 L_9$					$c_1 L_{11}$			$d_5 L_{13}$			18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

From $\mathbf{A}_{n3}^T \mathbf{L}$, we will obtain $\mathbf{S}_{uu3}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3															$-a_2 b_2 L_3$			$-a_2 b_2 L_3$	3
4								$a_2 L_6$					$a_2 t_1 L_3$	$-a_2 b_2 L_2$				$-a_2 b_2 L_5$	4
5																	$c_2 L_6$		5
6												$c_1 L_3$	$c_2 L_2$			$c_1 L_6$	$c_2 L_5$		6
7													$a_1 t_1 L_6$						7
8																			8
9				$a_2 L_6$														$-a_2 b_2 L_9$	9
10															$-a_2 b_2 L_5$		$a_2 t_1 L_9$	$-a_2 b_2 L_8$	10
11																			11
12													$c_2 L_5$			$c_1 L_9$	$-c_2 L_8$		12
13						$c_1 L_3$													13
14				$a_2 t_1 L_3$		$c_2 L_2$	$a_1 t_1 L_6$					$c_2 L_5$						$c_2 L_{10}$	14
15			$-a_2 b_2 L_3$	$-a_2 b_2 L_2$						$-a_2 b_2 L_5$							$-a_2 b_2 L_{10}$		15
16						$c_1 L_6$						$c_1 L_9$							16
17					$c_2 L_6$	$c_2 L_5$				$a_2 t_1 L_9$		$-c_2 L_8$			$-a_2 b_2 L_{10}$			$d_6 L_{12}$	17
18			$-a_2 b_2 L_6$	$-a_2 b_2 L_5$					$-a_2 b_2 L_9$	$-a_2 b_2 L_8$				$c_2 L_{10}$			$d_6 L_{12}$		18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

From $\mathbf{A}_{n4}^T \mathbf{L}$, we will obtain $\mathbf{S}_{uu4}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3																			3
4																			4
5																			5
6																			6
7																			7
8																			8
9															$-a_2 b_2 L_6$				9
10																			10
11																			11
12																			12
13														$d_3 L_3$	$-a_2 t_2 b_2 L_2$		$d_3 L_6$	$-a_2 b_2 t_2 L_5$	13
14												$d_3 L_3$						$d_2 L_6$	14
15								$-a_2 b_2 L_6$				$-a_2 t_2 b_2 L_2$				$-a_2 b_2 t_2 L_5$			15
16															$-a_2 b_2 t_2 L_5$			$-a_2 b_2 t_2 L_8$	16
17												$d_3 L_6$						$d_2 L_9$	17
18												$-a_2 b_2 t_2 L_5$	$d_2 L_6$			$-a_2 b_2 t_2 L_8$	$d_2 L_9$		18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

From $\mathbf{A}_{n5}^T \mathbf{L}$, we will obtain $\mathbf{S}_{uu5}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3																			3
4																			4
5													$c_2 L_3$						5
6																			6
7																			7
8																			8
9																			9
10																			10
11													$c_2 L_6$				$c_2 L_9$		11
12																			12
13																			13
14				$c_2 L_3$						$c_2 L_6$									14
15																	$-a_2 t_1 b_2 L_6$		15
16																			16
17										$c_2 L_9$				$-a_2 t_1 b_2 L_6$					17
18																			18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

From $\mathbf{A}_{n6}^T \mathbf{L}$, we will obtain $\mathbf{S}_{uu6}$ as

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
1																			1
2																			2
3																			3
4																	$a_2 t_1 L_6$		4
5																			5
6																			6
7																			7
8																			8
9																			9
10																			10
11																			11
12																			12
13																			13
14																			14
15																			15
16																			16
17				$a_2 t_1 L_6$															17
18																			18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Appendix C

Normalization

C.1 Physical components of the strains

The relations between the original and normalized components of the GREEN strain tensor in the three defined curvilinear coordinate systems are shown in Table C.1, with

TABLE C.1: Physical quantities of the GREEN strains

	Sphere	Cylinder	Plate	General
$\hat{n}_{\varepsilon_{11}}$	$\frac{R^2}{(R + \Theta^3)^2} {}^n\varepsilon_{11}$	${}^n\varepsilon_{11}$		$s_1^2 {}^n\varepsilon_{11}$
$\hat{n}_{\varepsilon_{22}}$	$\frac{1}{(R + \Theta^3)^2 \sin^2(\frac{\Theta^1}{R})} {}^n\varepsilon_{22}$	$\frac{1}{(R + \Theta^3)^2} {}^n\varepsilon_{22}$	${}^n\varepsilon_{22}$	$s_2^2 {}^n\varepsilon_{22}$
$\hat{n}_{\varepsilon_{12}}$	$\frac{R}{(R + \Theta^3)^2 \sin(\frac{\Theta^1}{R})} {}^n\varepsilon_{12}$	$\frac{1}{(R + \Theta^3)} {}^n\varepsilon_{12}$	${}^n\varepsilon_{12}$	$s_1 s_2 {}^n\varepsilon_{12}$
$\hat{n}_{\varepsilon_{13}}$	$\frac{R}{R + \Theta^3} {}^n\varepsilon_{13}$	${}^n\varepsilon_{13}$		$s_1 {}^n\varepsilon_{13}$
$\hat{n}_{\varepsilon_{23}}$	$\frac{1}{(R + \Theta^3) \sin(\frac{\Theta^1}{R})} {}^n\varepsilon_{23}$	$\frac{1}{(R + \Theta^3)} {}^n\varepsilon_{23}$	${}^n\varepsilon_{23}$	$s_2 {}^n\varepsilon_{23}$

the general coefficients in Table C.2

TABLE C.2: Coefficients for the normalized strains

	Sphere	Cylinder	Plate	General
$s_1 = \ \mathbf{g}^1\ $	$\frac{R}{R + \Theta^3}$	1	1	$\sqrt{g^{11}}$
$s_2 = \ \mathbf{g}^2\ $	$\frac{1}{(R + \Theta^3) \sin(\frac{\Theta^1}{R})}$	$\frac{1}{R + \Theta^3}$	1	$\sqrt{g^{22}}$

C.2 Physical components of the displacements

The relations between the original and normalized components of the displacement vector in the three defined curvilinear coordinate systems are shown in Table C.3, with the general coefficients in Table C.4.

TABLE C.3: Physical quantities of the displacements

	Sphere	Cylinder	Plate	General
$\overset{0}{v}_{1,1}$	$\overset{0}{v}_{1,1}$			$\overset{0}{v}_{1,1}$
$\overset{0}{v}_{1,2}$	$\overset{0}{v}_{1,2}$			$\overset{0}{v}_{1,2}$
$\overset{0}{v}_{2,1}$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{0}{v}_{2,1} + \cos\left(\frac{\Theta^1}{R}\right) \overset{0}{v}_2$	$R \overset{0}{v}_{2,1}$	$\overset{0}{v}_{2,1}$	$k_1 \overset{0}{v}_{2,1} + k_2 \overset{0}{v}_2$
$\overset{0}{v}_{2,2}$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{0}{v}_{2,2}$	$R \overset{0}{v}_{2,2}$	$\overset{0}{v}_{2,2}$	$k_1 \overset{0}{v}_{2,2}$
$\overset{0}{v}_{3,1}$	$\overset{0}{v}_{3,1}$			$\overset{0}{v}_{3,1}$
$\overset{0}{v}_{3,2}$	$\overset{0}{v}_{3,2}$			$\overset{0}{v}_{3,2}$
$\overset{1}{v}_{1,1}$	$\overset{1}{v}_{1,1}$			$\overset{1}{v}_{1,1}$
$\overset{1}{v}_{1,2}$	$\overset{1}{v}_{1,2}$			$\overset{1}{v}_{1,2}$
$\overset{1}{v}_{2,1}$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{1}{v}_{2,1} + \cos\left(\frac{\Theta^1}{R}\right) \overset{1}{v}_2$	$R \overset{1}{v}_{2,1}$	$\overset{1}{v}_{2,1}$	$k_1 \overset{1}{v}_{2,1} + k_2 \overset{1}{v}_2$
$\overset{1}{v}_{2,2}$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{1}{v}_{2,2}$	$R \overset{1}{v}_{2,2}$	$\overset{1}{v}_{2,2}$	$k_1 \overset{1}{v}_{2,2}$
$\overset{1}{v}_{3,1}$	$\overset{1}{v}_{3,1}$			$\overset{1}{v}_{3,1}$
$\overset{1}{v}_{3,2}$	$\overset{1}{v}_{3,2}$			$\overset{1}{v}_{3,2}$
$\overset{0}{v}_1$	$\overset{0}{v}_1$			$\overset{0}{v}_1$
$\overset{0}{v}_2$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{0}{v}_2$	$R \overset{0}{v}_2$	$\overset{0}{v}_2$	$k_1 \overset{0}{v}_2$
$\overset{0}{v}_3$	$\overset{0}{v}_3$			$\overset{0}{v}_3$
$\overset{1}{v}_1$	$\overset{1}{v}_1$			$\overset{1}{v}_1$
$\overset{1}{v}_2$	$R \sin\left(\frac{\Theta^1}{R}\right) \overset{1}{v}_2$	$R \overset{1}{v}_2$	$\overset{1}{v}_2$	$k_1 \overset{1}{v}_2$
$\overset{1}{v}_3$	$\overset{1}{v}_3$			$\overset{1}{v}_3$

TABLE C.4: Coefficients for the normalized displacements

	Sphere	Cylinder	Plate
k_1	$R \sin\left(\frac{\Theta^1}{R}\right)$	R	1
k_2	$\cos\left(\frac{\Theta^1}{R}\right)$	0	0