# Independence and $k$-Independence in Graphs in Terms of Degrees

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Michael Hoschek

# Preface

The independence number of a graph is the cardinality of a largest set of vertices such that no two vertices in the set are connected by an edge. The problem of determining a maximum independent set is a fundamental problem in graph theory. Since there is no efficient algorithm for most classes of graphs, approximations in form of lower and upper bounds on the independence number are of particular interest.

In addition to the strictly mathematical point of view, this graphical invariant appears in several economic and scientific applications. With an appropriate reformulation, some practical problems can be interpreted as graphs or networks. The approximation of the decision variables can be sufficient to decide whether a process is feasible or not. Let us have a look at two examples.

The number of bonded atoms of a given molecule can be modeled by a graph. The independence number of the graph is a measure of chemical stability: the larger the independent set, the lower the stability of the molecule. An approximation of the stability indicates whether a chemical compound is possible or not. Another application is the following problem. Given a set of $n$ computer processes which consume a certain resource, for example, hard disk space or random access memory. How many processes should ideally be running to make the system as efficient as possible without getting a resource conflict? The solution is to model a graph with $n$ vertices representing the processes. If two vertices are connected they use the same resource. An approximation of the independence number yields the quantity of processes which can run simultaneously.

In 1985, the independence number was generalized. For a positive integer $k$, the $k$-independence number of a graph is the cardinality of a largest set of vertices such that each vertex in the set has less than $k$ connections. Thus, a 1-independent set is independent in the classic sense. Considering the resource conflict of processes again, the $k$-independent set describes a relaxation of the problem. Instead of running processes having unique access to one resource, we look at $k$ processes using the same resource. An approximation of the $k$-independence number would help to solve this modified problem.

The advantage of approximation algorithms is the quality of not requiring all information of a given graph or network. The results often derive from parameters such as order or size of the graph. In this thesis our focus lies on the relation between the $k$-independence number and the degree sequence for any positive integer $k$. The degree sequence of a graph is an ordered sequence of its vertex degrees, i.e. the number of edges incident to the vertices. Our main objective is to improve and construct lower bounds on the $k$-independence number.

This thesis is structured as follows:

The first chapter will offer a concise introduction to the definitions and notions used in this work. Most of the definitions are widely used.

In Chapter 2 we will present a detailed survey of related work regarding lower bounds on the independence number. We will take a closer look at the bound found by Murphy [30] and the residue of a graph. The concept of the residue was introduced by Fajtlowicz [13]. Motivated by this investigation we will present an optimization of Murphy's algorithm in Chapter 3 using additional information of the degree sequence. This leads to an improvement for graphs with certain properties and still guarantees a lower bound on the independence number.

The next chapter will deal with the generalized invariant, the $k$-independence number, which was introduced by Fink and Jacobson [16, 17]. Again, we will provide a survey of known results and investigate further properties and characteristics.

Based on the Murphy algorithm we will develop a new algorithm in Chapter 5, which computes a lower bound on the $k$-independence number for all graphs. We will implement the algorithm in Matlab and perform a detailed analysis of the new result followed by a comparison of well-known bounds.

In Chapter 6 we will follow an idea by Turán [34] and investigate an extremal problem in terms of $k$-independent sets. We will study the minimum size of a graph with given $k$-independence number and offer a proof of this result. As a consequence, we will construct another new lower bound on the $k$-independence number for certain graphs. Again, we will implement the algorithm in Matlab and compare our results with well-known bounds.

Finally, we will present concluding remarks on the contributions of the thesis and discuss future research.

# Contents

# 1 Definitions and Notations

We will introduce the basic definitions and terms by presenting a survey of standard notations used in this work. For notation and graph theory terminology, we will generally follow Berge [5] and Diestel [11].

## 1.1 Graphs

Let $V$ and $E$ be finite sets. A *graph* $G$ is an ordered pair $G = (V, E)$, where $V$ is the set of *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges*, formed by pairs of vertices. The cardinality of $V$ is its *order*, written as $|V| = n$, and the number of edges is called the *size* of a graph denoted by $|E| = m$. A *simple* graph is an unweighted, undirected graph containing no loops or multiple edges. In the following, all considered graphs are simple.

Two vertices $v$ and $w$ are *adjacent* or *neighbors* if there is an edge $e = \{v, w\}$ in $G$. The *neighborhood* $N(v)$ of the vertex $v$ consists of the set of vertices adjacent to $v$. The number $d_G(v) = d(v) = |N(v)|$ is called the *degree* of $v \in V$. If the vertex set is numerated, i.e. $V = \{v_1, v_2, \ldots, v_n\}$, we write $d_i = d_G(v_i)$. Furthermore, the *maximum degree* of a graph $G$ is the largest vertex degree denoted by $\Delta(G)$. For the minimum degree of $G$ we write $\delta(G)$. The number

$$\overline{d}(G) = \frac{1}{n}\sum_{i=1}^{n} d_i$$

is the *average degree* of $G$. If we sum up all the vertex degrees in $G$, we count every edge twice. Thus, $\overline{d}(G) = \frac{2m}{n}$.

A *complete* graph $K_n$ is a graph in which each pair of vertices is connected by an edge. If all vertices of $G$ have the same degree $d$, then $G$ is *d-regular*, or simply *regular*.

The graph $G' = (V', E')$ is an *induced subgraph* of $G$ if $V' \subseteq V$ and for any pair of vertices $v$ and $w$ of $V'$, $\{v, w\}$ is an edge of $G'$ if and only if $\{v, w\}$ is an edge of G. We say that $V'$ induces $G'$ in $G$ and write $G[V']$. The *complement* of $G$ is the graph $\overline{G} = (V, \overline{E})$, where the edges in $\overline{E}$ are exactly the edges that are not in $G$. The *adjacency matrix* $A = (a_{ij})_{n \times n}$ of $G$ is defined by

$$a_{ij} := \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $A$ are called the eigenvalues of the graph $G$.

## 1.2 Partitions and Degree Sequences

A set $\rho = (a_1, a_2, \ldots, a_n)$ of non-negative integers with sum $a$ is a *partition* of $a$. Usually, the terms are ordered and we write $\rho = (a_1 \geq a_2 \geq \ldots \geq a_n)$ or $\rho = (a_1 \leq a_2 \leq \ldots \leq a_n)$. The partition $\rho$ has *length* $n$. We use superscripts to denote multiple terms with the same value; for example, $(4,4,4,3,3,2,2,2)$ will be written as $(4^3,3^2,2^3)$.

The *degree sequence* $\pi(G) = \pi = (d_1,d_2,\dots,d_n)$ of a graph $G = (V,E)$ is the ordered sequence of its vertex degrees and a partition of $2|E|$, i.e. $\sum_{i=1}^{n} d_i = 2|E|$. A partition $\pi = (d_1,d_2,\dots,d_n)$ is called *graphical* if there exists a graph $G$ having $\pi$ as degree sequence. We say $G$ is a *realization* of $\pi$. A partition $\pi = (d_1,d_2,\dots,d_n)$ *majorizes* a partition $\sigma = (e_1,e_2,\dots,e_n)$, denoted by $\pi \trianglerighteq \sigma$, if $d_i \ge e_i$ for $1 \le i \le n$. In this case we say the graph $G$ majorizes $H$ if $G$ is a realization of $\pi$ and $H$ is a realization of $\sigma$. Another important tool is the dominance order on partitions. A partition $\pi = (d_1,d_2,\dots,d_n)$ *dominates* a partition $\sigma = (e_1,e_2,\dots,e_m)$, denoted by $\pi \succeq \sigma$, if and only if

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{m} e_i \quad \text{and} \quad \sum_{i=1}^{k} d_i \ge \sum_{i=1}^{k} e_i, \quad \text{for } k \in \{1,2,\dots,n\}.$$

## 1.3 Independence and $k$-Independence

An *independent set* of a graph $G$ is a subset of $V$ such that no pair of vertices in the subset is adjacent. The cardinality of a maximum independent set is called the *independence number* of $G$ and is denoted by $\alpha(G)$. A generalization of independent sets was made by Fink and Jacobson [16, 17]. Given a positive integer $k$, a set $I_k \subseteq V$ is a *k-independent set* if the subgraph induced by $I_k$ has maximum degree at most $k-1$. The *k-independence number* of a graph $G$ is the cardinality of a largest $k$-independent set and is denoted by $\alpha_k(G)$ (see Figure 1.1). In particular, a 1-independent set is independent in the classic sense and $\alpha_1(G) = \alpha(G)$.

The graph theoretical pendant of an independent set is a so-called clique. A *clique* $W$ of a graph $G$ is a subset of $V$ such that the subgraph induced by $W$ is a complete graph of order $|W|$. The clique number $\omega(G)$ is the number of vertices in a maximum clique in $G$. Hence, a clique in $G$ is an independent set in the complement $\overline{G}$.

Finally, a *coloring* of a graph $G$ is a labeling of vertices with colors such that no two adjacent vertices have the same color. The smallest number of colors needed to color a graph G is called *chromatic number* and is denoted $\chi(G)$.

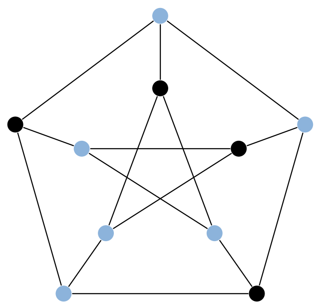


Figure 1.1: Petersen graph $G$ with $\alpha(G) = 4$ (black) and $\alpha_2(G) = 6$ (blue)

# 2 Independence in Graphs

The problem of computing the independence number is known to be $NP$-hard for most graphs. It was proven by Garey and Johnson [18] and Karp [26]. Thus, upper and lower bounds on the independence number are of great importance. In this chapter we will survey well-known bounds in terms of easily computable invariants. We will focus primarily on the residue of a graph introduced by Fajtlowicz [13] and a lower bound proven by Murphy [30]. Besides, we will start with bounds which provide a basis for improvements and new approximations in recent years. If not stated otherwise, we will consider a simple graph $G = (V, E)$ of order $n$ and size $m$.

## 2.1 Lower Bounds on the Independence Number

**Definition 2.1**
*A **lower bound** on the independence number is a graph invariant $l$ such that for every graph $G$*

$$l(G) \leq \alpha(G).$$

*The bound $l$ is called sharp for $G$ if equality holds.*

The first bound is based on the connection between coloring and independence and is called the chromatic number bound. For more details we refer to Berge [5] or Gould [19].

**Theorem 2.2**
*For every graph $G$ and chromatic number $\chi(G)$*

$$\frac{n}{\chi(G)} \leq \alpha(G).$$

**Proof:**
Let $G$ be a graph with a coloring using $\chi(G)$ colors. Each color class is an independent set because no two vertices with the same color are adjacent. Since $\alpha(G)$ is the maximum independent set, each color class has at most $\alpha(G)$ vertices and thus, $\alpha(G) \cdot \chi(G) \geq n$ which leads to

$$\frac{n}{\chi(G)} \leq \alpha(G).$$

■

**Theorem 2.3**
*If $G$ is a graph of maximum degree $\Delta$, then*

$$\frac{n}{1 + \Delta} \leq \alpha(G).$$

**Proof:**
Every graph $G$ satisfies $\chi(G) \leq \Delta + 1$ (see Diestel [11]). Together with the previous result, we get

$$\frac{n}{1+\Delta} \leq \frac{n}{\chi(G)} \leq \alpha(G).$$

■

A well-known bound on the independence number which uses the complete degree sequence of a graph is the following. It was found by Caro and Wei, independently.

**Theorem 2.4** (*Caro 1979, [7] and Wei 1981, [35]*)
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph $G$. Then*

$$CW(\pi) := \sum_{i=1}^{n} \frac{1}{1+d_i} \leq \alpha(G).$$

*The bound is called the Caro-Wei bound.*

**Proof:**
We show a probabilistic proof of their result. Let $G$ be a random graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ and $V = \{v_1, v_2, \ldots, v_n\}$ a random order of the vertices of $G$. We define a subset $U \subseteq V$ with

$$U := \{v_i \in V \mid \{v_i, v_j\} \in E \Rightarrow i < j\}.$$

Then $U$ is an independent set. The probability of $v_i \in V$ being included in the independent set $U$ is

$$\text{Prob}(v_i \in U) = \frac{d(v_i)!}{(d(v_i)+1)!} = \frac{1}{d(v_i)+1}$$

since there are $d(v_i)!$ permutations of $v_i$ and its neighbors, where $v_i$ is the leftmost vertex. The expected size of $U$ is

$$|U| = \sum_{i=1}^{n} \text{Prob}(v_i \in U) = \sum_{i=1}^{n} \frac{1}{d(v_i)+1}.$$

Since $|U| \leq \alpha(G)$, we conclude

$$\sum_{i=1}^{n} \frac{1}{d(v_i)+1} = \sum_{i=1}^{n} \frac{1}{1+d_i} \leq \alpha(G).$$

■

The next inequality is one of the oldest non-trivial bounds and a consequence of Turán's famous result. For detailed information, please refer to Chapter 6.1.

**Theorem 2.5** (*Turán 1941, [34] and Griggs 1983, [20]*)
*For every graph $G$ with average degree $\overline{d}$,*

$$\frac{n}{1+\overline{d}} \leq \alpha(G).$$

**Proof:**
We will present an alternative proof using the Caro-Wei bound. Let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Using the Cauchy-Schwarz inequality, we get

$$n^2 = \left( \sum_{i=1}^{n} \frac{1}{\sqrt{1+d_i}} \sqrt{1+d_i} \right)^2 \leq \left( \sum_{i=1}^{n} \frac{1}{1+d_i} \right) \left( \sum_{i=1}^{n} (1+d_i) \right).$$

Since the Caro-Wei bound is a lower bound on the independence number, we rearrange the above inequality and obtain

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{1+d_i} \geq \frac{n^2}{\sum_{i=1}^{n}(1+d_i)} = \frac{n^2}{n + \sum_{i=1}^{n} d_i} = \frac{n}{1 + \frac{1}{n}\sum_{i=1}^{n} d_i} = \frac{n}{1+\overline{d}}.$$

■

**Corollary 2.6**
*The following chain of inequalities shows the quality of the considered bounds in terms of degrees:*

$$\frac{n}{1+\Delta} \leq \frac{n}{1+\overline{d}} \leq \sum_{i=1}^{n} \frac{1}{1+d_i} \leq \alpha(G).$$

Hansen and Lorea investigated independent sets of hypergraphs. A special case is applicable to simple graphs.

**Theorem 2.7** (*Hansen and Lorea 1979, [23]*)
*Let $G$ be a graph with degree sequence $\pi$, maximum degree $\Delta$, and minimum degree $\delta$. Then*

$$HL(\pi) := \frac{n + \Delta - \delta}{1+\Delta} \leq \alpha(G).$$

**Remark 2.8**
If $G$ is a regular graph, we have $\Delta = \delta = \overline{d}$. It follows

$$\frac{n+\Delta-\delta}{1+\Delta} = \frac{n}{1+\overline{d}}.$$

Hence, the Hansen-Lorea bound is equal to Turán's bound, and to the Caro-Wei bound. There are also graphical partitions in which the Hansen-Lorea bound and the Caro-Wei bound improve one another. For example, the degree sequences $\pi = (3,2,2,1)$ and $\sigma = (3,1,1,1)$ yield

$$CW(\pi) = \frac{17}{12} < HL(\pi) = \frac{3}{2}$$
$$CW(\sigma) = \frac{7}{4} > HL(\sigma) = \frac{3}{2}.$$

We will close the section by presenting a lower bound including the eigenvalues of the corresponding adjacency matrix of a graph $G$. It is a consequence of a result found by Wilf.

**Theorem 2.9** (*Wilf 1967, [36]*)
*Let $G$ be a graph with chromatic number $\chi(G)$ and largest eigenvalue $\lambda_{\max}$ of the corresponding adjacency matrix of $G$. Then*

$$\chi(G) \leq \lambda_{\max} + 1.$$

**Corollary 2.10**
*For any graph $G$ with largest eigenvalue $\lambda_{\max}$,*

$$\frac{n}{1+\lambda_{\max}} \leq \alpha(G).$$

*The quotient is called Wilf's bound.*

**Proof:**
Using the chromatic number bound and Wilf's result, we get

$$\frac{n}{1+\lambda_{\max}} \leq \frac{n}{\chi(G)} \leq \alpha(G).$$

■

**Lemma 2.11**
*Suppose the largest eigenvalue of a graph is $\lambda_{\max} = d$ for a positive integer $d$. Further, the corresponding eigenvector is $v = (1, \ldots, 1)^\top$, i.e. a vector with value 1 in every entry. Then the graph is $d$-regular.*

**Proof:**
Suppose $A \in \mathbb{R}^{n \times n}$ is the corresponding adjacency matrix and $\mathbb{1} := (1, \ldots, 1)^\top$. Then, $A\mathbb{1} = d\mathbb{1}$ is equivalent to the fact that each row of $A$ adds up to $d$. This means every vertex of the graph has exactly $d$ neighbors. Hence, the graph is $d$-regular. ■

**Corollary 2.12**
*For $d$-regular graphs with $\pi = (d^n)$ we get*

$$\frac{n}{1+\lambda_{max}} = \frac{n}{1+d} = CW(\pi),$$

*so the Caro-Wei bound and Wilf's bound are equal for regular graphs.*

We will show that Wilf's bound is not better than the Caro-Wei bound for general graphs and use a spectral graph theory result, which can be found, for example, in Spielman [32].

**Lemma 2.13**
*For every graph $G$ with average degree $\overline{d}$, maximum degree $\Delta$, and largest eigenvalue $\lambda_{\max}$, we obtain*

$$\overline{d} \leq \lambda_{\max} \leq \Delta.$$

**Proof:**
Let $A \in \mathbb{R}^{n\times n}$ be the corresponding adjacency matrix of $G$. If $\lambda_{\max}$ is the largest eigenvalue, it follows

$$\lambda_{\max} = \max_{x\neq 0} \frac{x^\top Ax}{x^\top x}.$$

Set $x = (1,\dots,1)^\top = \mathbb{1}$, which leads to

$$\lambda_{\max} = \max_{x\neq 0} \frac{x^\top Ax}{x^\top x} \geq \frac{\mathbb{1}^\top A\mathbb{1}}{\mathbb{1}^\top \mathbb{1}}.$$

$$A\mathbb{1} \leq \lambda_{\max}\mathbb{1} \;\Leftrightarrow\; \sum_{i,j=1}^{n} a_{ij} \leq n\lambda_{\max} \;\Leftrightarrow\; \sum_{i=1}^{n} d_i \leq n\lambda_{\max} \;\Leftrightarrow\; \overline{d} \leq \lambda_{\max}$$

which implies the first inequality.

Let $x$ be an eigenvector of $\lambda_{\max}$ and let $v$ be the vertex with maximum value $x(v) \geq x(u)$ for all $u \in V$. Without loss of generality we assume $x(v) \neq 0$ and obtain

$$\lambda_{\max} = \frac{Ax(v)}{x(v)} = \frac{\sum_{\{u,v\}\in E} x(u)}{x(v)} = \sum_{\{u,v\}\in E} \frac{x(u)}{x(v)} \leq \sum_{\{u,v\}\in E} 1 = d(v) \leq \Delta.$$

■

**Corollary 2.14**
*The Caro-Wei bound and Turán's bound always provides a better approximation than Wilf's bound.*

**Proof:**
The statement follows immediately from the previous result:

$$\frac{n}{1+\lambda_{\max}} \leq \frac{n}{1+\overline{d}} \leq \sum_{i=1}^{n} \frac{1}{1+d_i}.$$

■

## 2.2 The Residue of a Graph

The residue of a graph is a parameter computed by successive reduction of its degree sequence. Favaron et al. [15] proved that the residue forms a lower bound on the independence number, which had been conjectured by Fajtlowicz [13]. The proof was simplified by Griggs and Kleitman [21] and Triesch [33].

The reduction process was introduced by Havel [24] and Hakimi [22] as a means of determining whether a partition is graphical or not.

**Definition 2.15**
*Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be a partition and $d_1 \leq n-1$. A* ***Havel-Hakimi reduction step*** *removes the largest element $d_1$, subtracts 1 from the next $d_1$ largest elements and reorders the terms, if necessary, in non-increasing order. We get a new sequence*

$$\mathcal{H}^1(\pi) := \mathcal{H}(\pi) := (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

*If $\pi$ is graphical, the operator $\mathcal{H}$ can be applied i times, for $i \in \{0,1,\ldots,n-1\}$, such that $\mathcal{H}^i(\pi)$ is a sequence of zeros. The procedure is called the* ***Havel-Hakimi algorithm****.*

**Example:**
We present the Havel-Hakimi algorithm for the partition $\pi = (4^2, 3^2, 2^2)$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $\pi$ | 4 | 4 | 3 | 3 | 2 | 2 |
| $\mathcal{H}(\pi)$ | | 3 | 2 | 2 | 2 | 1 |
| $\mathcal{H}^2(\pi)$ | | | 1 | 1 | 1 | 1 |
| $\mathcal{H}^3(\pi)$ | | | | 1 | 1 | 0 |
| $\mathcal{H}^4(\pi)$ | | | | | 0 | 0 |

**Definition 2.16**
*The* ***residue*** *$R(\pi)$ of a partition $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ is the number of zeros remaining at the end of the Havel-Hakimi algorithm. Alternatively,*

$$R(\pi) := n - s,$$

*where s is the number of Havel-Hakimi reduction steps to obtain a sequence full of zeros. If $\pi$ is the degree sequence of a graph G, we write $R(G)$.*

Havel and Hakimi independently provided necessary and sufficient conditions for a graphical partition. Both proved that a partition $\pi$ is graphical if and only if $\mathcal{H}(\pi)$ is graphical. For example, the partition $\pi = (2,1,0)$ does not belong to a graph because one reduction step leads to $\mathcal{H}(\pi) = (0,-1)$.

**Theorem 2.17** (*Havel 1955, [24] and Hakimi 1962, [22]*)
*A partition $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ is graphical if and only if the reduced partition $\mathcal{H}(\pi) = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphical.*

**Proof:**
*Sufficiency* Suppose $G'$ is a graph with degree sequence $\mathcal{H}(\pi)$. We generate a graph $G$ by adding a vertex to $G'$ that is adjacent to each one of $d_1$ vertices having degrees $d_2 - 1, \ldots, d_{d_1+1} - 1$. Then the degree sequence of $G$ is $\pi = (d_1, d_2, \ldots, d_n)$.

*Necessity* Suppose $G = (V, E)$ is a graph with degree sequence $\pi$. Label the vertices of $G$ with $v_1, v_2, \ldots, v_n$ such that $d(v_i) = d_i$ for $i = 1, \ldots, n$. If the neighborhood $N(v_1) = \{v_2, v_3, \ldots, v_{d_1+1}\}$, then the induced subgraph $G'$ obtained by deleting $v_1$ has degree sequence $\mathcal{H}(\pi)$. So suppose $N(v_1) \neq \{v_2, v_3, \ldots, v_{d_1+1}\}$. There must exist two vertices $v_j$ and $v_k$ with $d_j > d_k$ such that $\{v_1, v_k\} \in E$, but $\{v_1, v_j\} \notin E$. Since $d_j > d_k$, there exists a vertex $v_l$ with $\{v_j, v_l\} \in E$ but $\{v_k, v_l\} \notin E$. Now we will substitute the edges $\{v_1, v_k\}$ and

$\{v_j, v_l\}$ by the edges $\{v_1, v_j\}$ and $\{v_k, v_l\}$. This 2-switch transformation, illustrated in Figure 2.1, does not change the degree sequence of $G$, but it increases the cardinality of the set $N(v_1) \cap \{v_2, v_3, \ldots, v_{d_1+1}\}$. If there are still vertices $v_2, v_3, \ldots, v_{d_1+1}$ that are not adjacent to $v_1$, we can repeat the above process until we finally get $N(v_1) = \{v_2, v_3, \ldots, v_{d_1+1}\}$. At this point the graph $G' = G \setminus \{v_1\}$ will be a graph with degree sequence

$$\mathcal{H}(\pi) = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

■

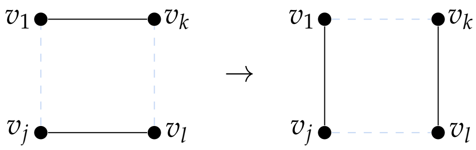



Figure 2.1: 2-switch transformation

For a realization $G$ of a partition, the residue provides a lower bound on the independence number of $G$.

**Theorem 2.18** (*Favaron et al. 1991, [15]*)
*For any graph $G$, the residue of $G$ is at most the independence number, that is*

$$R(G) \leq \alpha(G).$$

The original proof by Favaron, Mahéo and Saclé is based on the fact that the residue maintains the dominance order.

**Theorem 2.19** (*Favaron et al. 1991, [15]*)
*Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\sigma = (e_1, e_2, \ldots, e_n)$ are partitions with $\pi \succeq \sigma$. Then*

$$R(\pi) \geq R(\sigma).$$

Griggs and Kleitman [21] used a greedy algorithm in their proof which forms a maximal independent set by choosing, at each step, the maximum degree vertex in the graph $G$ and removing the vertex until the remaining graph has no more edges. We will denote the size of the resulting independent set by $\mathcal{A}(G)$. The size $\mathcal{A}(G)$ depends on the choice of the vertex if there are more vertices of maximum degree. The greedy algorithm is called $MAX$ and was introduced by Johnson [29] and again by Griggs [20].

| | |
|---|---|
| Algorithm: | $MAX$ |
| Input: | graph $G$ |
| | while $\Delta(G) \neq 0$ do |
| | $v \leftarrow$ any vertex of highest degree in $G$ |
| | $G \leftarrow G \setminus \{v\}$ |
| | endwhile |
| | $\mathcal{A}(G) \leftarrow$ number of vertices in the remaining graph $G$ |
| Output: | independent set in $G$ of size $\mathcal{A}(G)$ |

**Theorem 2.20** (*Griggs and Kleitman 1994, [21]*)
*For any graph $G$ and any possible result $\mathcal{A}(G)$ produced by the greedy algorithm MAX,*

$$R(G) \leq \mathcal{A}(G) \leq \alpha(G).$$

Triesch [33] generalized Theorem 2.19 and simplified the proof by introducing so-called elimination sequences. The main idea is to put the eliminated degree in a new partition at each step of the Havel-Hakimi algorithm.

**Definition 2.21**
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical partition and $\mathcal{H}^i(\pi)$ the partition after $i \in \{1, \ldots, n-1\}$ Havel-Hakimi reduction steps. The partition*

$$E(\pi) := (\max(\pi), \max(\mathcal{H}(\pi)), \max(\mathcal{H}^2(\pi)), \ldots, \max(\mathcal{H}^{s-1}(\pi)))$$

*is called* ***elimination sequence*** *of $\pi$, where $s = s(\pi)$ is the number of reduction steps to obtain a sequence full of zeros.*

In the previous example $\pi = (4^2, 3^2, 2^2)$, we have $E(\pi) = (4, 3, 1, 1)$. In fact, the elimination sequence is a partition of $\frac{1}{2}\sum_{i=1}^{n} d_i$.

**Theorem 2.22** (*Triesch 1996, [33]*)
*If $\pi$ and $\sigma$ are graphical partitions with $\pi \succeq \sigma$, then*

$$E(\pi) \succeq E(\sigma).$$

**Remark 2.23**
Using the definition of dominance order, $E(\pi) \succeq E(\sigma)$ implies that the number of positive terms in $E(\pi)$ is at most the number of positive terms in $E(\sigma)$. Hence, the number of Havel-Hakimi reduction steps in $\pi$ is at most the number of reduction steps in $\sigma$. This leads to

$$R(\pi) = n - s(\pi) \geq n - s(\sigma) = R(\sigma),$$

and Theorem 2.19 follows immediately.

It is interesting that the dominance order of the elimination sequences does not imply the dominance of the partitions themselves. For example, the sequences $\pi = (5^3, 3^2, 2^2, 1)$ and $\sigma = (5, 4^4, 3, 1^2)$ result in

$$\begin{aligned} E(\pi) = (5,4,3,1) &\succeq (5,3,2,2,1) = E(\sigma), \\ \pi = (5,5,5,3,3,2,2,1) &\not\succeq (5,4,4,4,4,3,1,1) = \sigma. \end{aligned}$$

Favaron et al. investigated the quality of the residue including a result related to Caro-Wei's bound.

**Theorem 2.24** (*Favaron et al. 1991, [15]*)
*Let $\pi$ be a graphical partition. Then*

$$CW(\pi) \leq R(\pi).$$

**Corollary 2.25**
*For any graph G on n vertices and largest eigenvalue $\lambda_{\max}$,*

$$\frac{n}{1+\lambda_{\max}} \leq R(G).$$

**Proof:**
We showed in Corollary 2.14 that the Caro-Wei bound always strengthens Wilf's bound, and thus, in combination with Theorem 2.24, the result follows immediately. ■

**Remark 2.26**
The previous result answers an open question by Willis [37], who conjectured that the residue is always better than Wilf's bound.

In the following part we will study some properties of the residue and offer explicit formulas for certain graph types.

**Lemma 2.27** (*Jelen 1996, [27]*)
*A graphical partition $\pi$ has residue $R(\pi) = 1$ if and only if $\pi$ is degree sequence of a complete graph.*

**Proof:**
*Sufficiency* Suppose $\pi = ((n-1)^n)$ is a degree sequence of a complete graph of order $n \in \mathbb{N}$. The Havel-Hakimi reduction steps yield

$$\begin{aligned}
\mathcal{H}(\pi) &= ((n-2)^{n-1}) \\
\mathcal{H}^2(\pi) &= ((n-3)^{n-2}) \\
&\vdots \\
\mathcal{H}^{n-2}(\pi) &= (1,1) \\
\mathcal{H}^{n-1}(\pi) &= (0).
\end{aligned}$$

The residue is $R(\pi) = 1$.

*Necessity* This will be done by induction on $n$. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a degree sequence with $R(\pi) = 1$. If $n = 1$, then $\pi = (0)$ is sequence of a complete graph of order 1. Assume that $R(\pi) = 1$ for $n \geq 2$, then $d_1 \neq 0$, and applying a Havel-Hakimi reduction step yields $R(\mathcal{H}(\pi)) = 1$. By the induction hypothesis, $\mathcal{H}(\pi)$ is degree sequence of a complete graph of order $n-1$. With $\mathcal{H}(\pi) = ((n-2)^{n-1})$ we obtain $\pi = ((n-1)^n)$. ■

**Remark 2.28**
From Lemma 2.27 it follows immediately that the residue of $r \geq 1$ disjoint complete graphs yields $r$.

For a semi-regular graph, i.e. its maximum degree and minimum degree differ by at most 1, there exists an explicit formula to compute the residue.

**Theorem 2.29** (*Favaron et al. 1991, [15]*)
*Let $\pi = (d^l, (d-1)^{n-l})$ be the degree sequence of a semi-regular graph of order $n$ with $d \geq 1$ and $1 \leq l \leq n$. Then*

$$R(\pi) = \left\lceil \frac{l}{d+1} + \frac{n-l}{d} \right\rceil.$$

**Corollary 2.30**
*If we set $n = l$ in Theorem 2.29, $\pi = (d^n)$ is the degree sequence of a $d$-regular graph. The residue yields*

$$R(\pi) = \left\lceil \frac{n}{d+1} \right\rceil.$$

Let us consider a couple of special regular graphs.

- Complete graphs are regular graphs. Using the formula for $\pi = ((n-1)^n)$ we get

$$R(\pi) = \left\lceil \frac{n}{(n-1)+1} \right\rceil = 1$$

  as we already know.

- Circles are 2-regular graphs $\pi = (2^n)$ with residue

$$R(\pi) = \left\lceil \frac{n}{3} \right\rceil.$$

- A $d$-regular graph of order $n$ with $d + 2 \leq n \leq 2d + 2$ has residue 2. A generalization of this result reads as follows.

**Theorem 2.31**
*Let $\pi = (d^n)$ be the degree sequence of a $d$-regular graph with $r(d+1) - d \leq n \leq r(d+1)$ for $r \geq 2$. Then $R(\pi) = r$.*

**Proof:**

$$r = \left\lceil \frac{r(d+1)-d}{d+1} \right\rceil \leq R(\pi) \leq \left\lceil \frac{r(d+1)}{d+1} \right\rceil = r.$$

■

Now we will investigate the quality of the residue. In fact, among functions of degree sequences, the residue is one of the best lower bounds on the independence number so far. We have seen that the residue bound is sharp for complete graphs. However, the bound can also be arbitrarily weak. For example, the degree sequence $\pi = (n^{2n})$ for $n \in \mathbb{N}$ has residue $R(\pi) = 2$, and a complete bipartite graph is a realization of $\pi$ with independence number $n$. Since there is no unique realization of a degree sequence, a detailed analysis of the quality is difficult. For example, for $\pi = (4^8)$ the residue is $R(\pi) = 2$. In this case, we have three different realizations and three different independence numbers (see Figure 2.2).

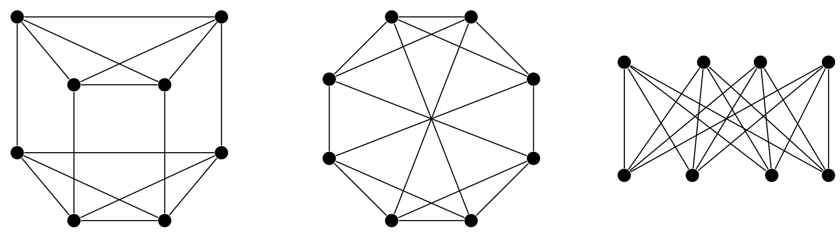



Figure 2.2: Realization problem $\pi = (4^8)$

It appears reasonable to consider the following parameter

$$\alpha_{\min}(\pi) = \min\{\alpha(G) \mid G \text{ is a realization of } \pi\}.$$

We will see graphical partitions $\pi$ for which the difference between $\alpha_{\min}(\pi)$ and the residue $R(\pi)$ can become arbitrarily large. First, we will present graph types where the residue provides good results. The independence number can exceed its residue by at most 1.

**Theorem 2.32** (*Nelson and Radcliffe 2004, [31]*)
*In the class of (semi-)regular graphs there always exists a graph G such that*

$$R(G) \leq \alpha(G) \leq R(G) + 1.$$

**Theorem 2.33** (*Barrus 2012, [3]*)
*If G is a unigraph, that is the unique realization of its degree sequence, then*

$$R(G) \leq \alpha(G) \leq R(G) + 1.$$

**Lemma 2.34** (*Jelen 1996, [27]*)
*Let $\pi = (d^n)$ be a graphical partition with $n = d + 1 + a$ and $0 < a \leq d$. Then there always exists a graph $G$ with degree sequence $\pi$ and $\alpha(G) \leq 3 = R(\pi) + 1$.*

**Sketch of the proof:**
We will present the idea of the proof. For more details please see the paper of Jelen [27] or Nelson and Radcliffe [31]. At first, we compute the residue of $\pi$ and obtain

$$2 = \left\lceil \frac{(d+1)+1}{d+1} \right\rceil \leq R(\pi) \leq \left\lceil \frac{(d+1)+d}{d+1} \right\rceil = 2.$$

If $n$ is an even number, we construct two complete graphs $K_{\frac{n}{2}}$ and connect the vertices between the graphs until every vertex has degree $d$. The result is a graph $G$ with $\alpha(G) = 2$. If $n$ is odd, we construct two complete graphs of order $\frac{d+a}{2}$ and one isolated vertex. We repeat the above process taking into account the isolated vertex. The resulting graph has degree sequence $\pi = (d^n)$ and independence number $\alpha(G) \leq 3$.

**Proposition 2.35**
*In the class of regular graphs, there always exists a graph G such that*

$$R(G) \leq \alpha(G) \leq R(G) + 1.$$

**Proof:**
Let $\pi = (d^n)$ be the degree sequence of a $d$-regular graph. We use induction on $n$. For $d+1 \leq n \leq 2d+1$ the statement is true by Lemma 2.34. Suppose $n > 2d+1$, then there exists a partition of $n$:

$$n = m(d+1) + a, \quad m \geq 1, \quad a \geq d+1$$

The residue of $\pi$ yields

$$R(\pi) = \left\lceil \frac{n}{d+1} \right\rceil = \left\lceil \frac{m(d+1)+a}{d+1} \right\rceil = m + \left\lceil \frac{a}{d+1} \right\rceil.$$

Since $a \geq d+1$, there is a graphical partition $\pi' = (d^a)$ with residue $R(\pi') = \lceil \frac{a}{d+1} \rceil$. By induction, $\pi'$ has a realization $G'$ with

$$R(G') \leq \alpha(G') \leq R(G') + 1.$$

Now we construct a $d$-regular graph $G$ with degree sequence $\pi = (d^n)$

$$G := \left( \bigcup_{i=1}^{m} K_d \right) \cup G'.$$

It follows

$$\alpha(G) = m + \alpha(G') \leq m + R(G') + 1 = m + \left\lceil \frac{a}{d+1} \right\rceil + 1 = R(G) + 1.$$

■

We will close the section with an example which illustrates that the residue of two disjoint graphs $G, H$ is larger when considering the residues separately than the residue of the union, that is

$$R(G \dot{\cup} H) \leq R(G) + R(H).$$

In Chapter 4 we will present a result by Amos et al. [2]. The authors proved that for any disconnected graph, the residue of the union is at least the sum of the residues componentwise.

**Example:**
Suppose $\sigma_{n+1} = (n, 1^n)$ for $n \in \mathbb{N}$ is the degree sequence of a star graph $S_{n+1}$. It is a graph such that exactly 1 vertex is adjacent to all other vertices. Further, we consider a complete graph $K_n$ and attach a degree-one-vertex to each vertex of $K_n$. The resulting graph has degree sequence $\kappa_n = (n^n, 1^n)$ and is denoted by $\overline{K}_n$.

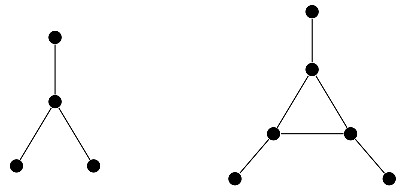


Figure 2.3: $S_4$ and $\overline{K}_3$

The residues of the considered graphs can be easily computed. One step in the Havel-Hakimi algorithm leads to

$$\sigma_{n+1} = (n, 1^n) \quad \Rightarrow \quad \mathcal{H}(\sigma_{n+1}) = (0^n).$$

Thus, the residue of the star graph is $R(S_{n+1}) = n$. The modified complete graph $\overline{K}_n$ is the unique realization of its degree sequence and has a maximum independent set of cardinality $n$. Using Theorem 2.33, we obtain

$$\alpha(\overline{K}_n) \leq R(\overline{K}_n) + 1 \quad \Leftrightarrow \quad R(\overline{K}_n) \geq n - 1.$$

Now we consider the degree sequence of a disjoint union of $S_{n+1}$ and $\overline{K}_n$:

$$\sigma_n \cup \kappa_n = (n^{n+1}, 1^{2n}).$$

Obviously, this is also the degree sequence of a complete graph $K_n$ and $n$ copies of complete graphs $K_2$. Since the residue of $n+1$ disjoint complete graphs is $n+1$, we conclude

$$R(\sigma_n \cup \kappa_n) = n + 1.$$

This leads to

$$R(S_{n+1}) + R(\overline{K}_n) - R(S_{n+1} \cup \overline{K}_n) \geq n - 2,$$

which grows arbitrarily large as $n$ approaches infinity. Since $\alpha(S_{n+1}) + \alpha(\overline{K}_n) = \alpha(S_{n+1} \cup \overline{K}_n)$, the residue has poor quality in this case.

## 2.3 Murphy's Bound

Murphy [30] developed an algorithm which yields another lower bound on the independence number in terms of the degree sequence. Instead of deleting vertices of high degree, just as in the Havel-Hakimi algorithm, Murphy considers vertices of low degree. In his proof, Murphy uses a greedy algorithm that computes a collection of pairwise non-adjacent vertices. These vertices represent an independent set. Since the procedure removes vertices of minimum degree, the algorithm will be denoted by $MIN$. A description of this algorithm will be given below.

| | |
|---|---|
| Algorithm: | *MIN* |
| Input: | graph $G$ |
| | $j \leftarrow 0$ |
| | while $G \neq \emptyset$ do |
| | $j \leftarrow j+1$ |
| | $v_j \leftarrow$ any vertex of smallest degree in $G$ |
| | $C_j \leftarrow \{v_j\} \cup \{w : w$ is adjacent to $v_j$ in $G\}$ |
| | $G \leftarrow G \setminus C_j$ |
| | endwhile |
| | $r \leftarrow j$ |
| Output: | independent set in $G$ of size $r$ |

Murphy shows inductively that the size of the independent set resulting from the greedy algorithm *MIN* is at least Murphy's bound. Before formally defining Murphy's bound, we will illustrate the procedure for the degree sequence $\pi = (1^3, 2^4, 3, 5^2, 6^3)$. Since we consider vertices of low degree, the sequence is sorted in increasing order. Mark the first term in $\pi$ (see Figure 2.4). If the marked vertex has degree $d$, move $d+1$ positions to the right and mark the next degree. We continue the process until we move beyond the last term of the sequence $\pi$. The sum of all marked terms is Murphy's lower bound on the independence number. In our example we obtain the result 5.

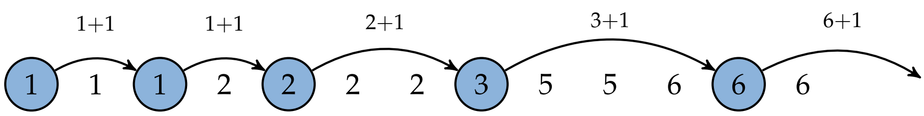



Figure 2.4: The principle of Murphy's bound for $\pi = (1^3, 2^4, 3, 5^2, 6^3)$

For the formal definition, we will follow Bauer et al. [4].

**Definition 2.36**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be a partition in increasing order. The iterative function $a : \mathbb{N} \mapsto \{d_1, d_2, \ldots, d_n, \infty\}$ has the rule: Set $a(1) = d_1$. If $a(j) = d_k$ for $1 \leq j \leq n$, then*

$$a(j+1) = \begin{cases} d_{k+a(j)+1}, & \text{if } k + a(j) + 1 \leq n, \\ \infty, & \text{otherwise.} \end{cases}$$

*If $a(j) = \infty$, then $a(j+1) = \infty$. The value $a(j)$ is the step length and the number*

$$M(\pi) := \max\{j \in \mathbb{N} \mid a(j) \neq \infty\}$$

*is called **Murphy's bound** of the partition $\pi$. If $\pi$ is the degree sequence of a graph $G$, we write $M(G)$.*

**Theorem 2.37** (*Murphy 1991, [30]*)
*Let $G$ be a graph with degree sequence $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$. Then*

$$M(G) \leq \alpha(G).$$

Alternatively, the Murphy bound $M(\pi)$ can be computed by the following algorithm.

**Algorithm 1** Murphy's algorithm

**Input:** $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$
**Output:** $M(\pi) \geq 1$

```
j = 0
m_j(π) = m_j
m_0 = 0
while m_j < n do
    i = m_j
    m_{j+1} = m_j + d_{i+1} + 1
    j = j + 1
end while
m_j = n
M(π) = j
```

**Theorem 2.38** (*Murphy 1991, [30]*)
*For any partition $\pi$ the Murphy bound strengthens the Caro-Wei bound, that is*

$$CW(\pi) \leq M(\pi).$$

Now we will consider certain graph types to compute Murphy's bound.

**Lemma 2.39**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be a graphical partition.*

*(i)* $M(\pi) = 1$ *if and only if $\pi$ is the degree sequence of a complete graph.*

*(ii)* *If $\pi$ is the degree sequence of a graph consisting of $r$ disjoint cliques, then* $M(\pi) = r$.

**Proof:**
*(i) Sufficiency* Suppose $\pi = ((n-1)^n)$ is a degree sequence of a complete graph. To compute Murphy's bound, we mark the first term $d_1 = n-1$. Now we move $d_1 + 1 = n$ positions to the right. Since $\pi$ has $n$ terms, we leave the sequence in the first step and obtain $M(\pi) = 1$.
*Necessity* Let $\pi = (d_1, d_2, \ldots, d_n)$ be a degree sequence with $M(\pi) = 1$. Using the definition of Murphy's bound, we have $a(1) = d_1$, $a(2) = \infty$ and $1 + a(1) + 1 > n$. This yields $d_1 = n-1$, and the result follows from the fact that $d_1$ is the minimum degree vertex.

*(ii)* Every clique is a complete graph with Murphy bound 1 by part one of the lemma. Since we have $r$ disjoint cliques, we obtain $M(\pi) = r$. ■

**Lemma 2.40**
*Let $\pi = (d^n)$ be the degree sequence of a $d$-regular graph of order $n \geq d+1$. Then*

$$M(\pi) = \left\lceil \frac{n}{d+1} \right\rceil.$$

**Proof:**
Let $r$ be a non-negative integer such that

$$r < \frac{n}{d+1} \quad \text{and} \quad r+1 \geq \frac{n}{d+1}.$$

Thus, $\lceil \frac{n}{d+1} \rceil = r+1$. Suppose $r$ is the number of iterations in the Murphy algorithm (see Algorithm 1):

$$\begin{aligned} m_1 &= d+1 \\ m_2 &= m_1 + d + 1 = 2(d+1) \\ m_3 &= m_2 + d + 1 = 3(d+1) \\ \vdots &= \vdots \\ m_r &= r(d+1) \\ m_{r+1} &= (r+1)(d+1) \end{aligned}$$

Since $r(d+1) < n$ and $m_{r+1} = (r+1)(d+1) \geq n$, the while-loop terminates at this step and we obtain $M(\pi) = r+1 = \lceil \frac{n}{d+1} \rceil$. ■

In case of regular graphs, the residue and the Murphy bound achieve the same value.

**Corollary 2.41**
*Suppose G is a regular graph of order n. Then*

$$R(G) = M(G).$$

In contrast to this result, there are graphical partitions in which either the residue or the Murphy bound improve the other one. For example, $\pi = (1, 2^2, 3^3)$ leads to $M(\pi) = 3$ and $R(\pi) = 2$ and $\sigma = (2^3, 4^3)$ yields $M(\sigma) = 2$ and $R(\sigma) = 3$. In the next section we will carry out a detailed comparison between both bounds.

**Corollary 2.42**
*In the class of regular graphs there always exists a graph G such that*

$$M(G) \leq \alpha(G) \leq M(G) + 1.$$

**Proof:**
The statement follows immediately from Proposition 2.35 and from the fact that the residue and Murphy yield the same bound for the independence number of regular graphs. ■

The next result will show that Murphy's bound is sharp for certain graphs.

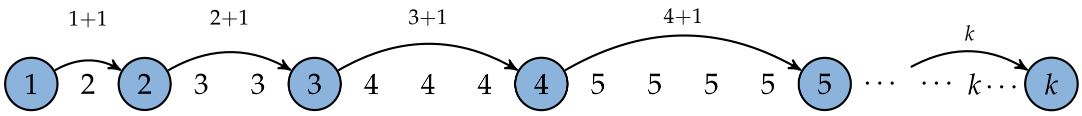
**Theorem 2.43**
*Let $\pi = (1^1, 2^2, 3^3, \ldots, k^k)$ be graphical for $k \in \mathbb{N}$. Then $M(\pi) = k$ and $\pi$ has a realization G such that*

$$M(G) = \alpha(G).$$

**Proof:**
We compute the Murphy bound for $\pi$. The first term in $\pi$ has value $d_1 = 1$, and we move 2 positions to the right and reach $d_3 = 2$. Thereafter, we move 3 positions in $\pi$ and reach $d_6 = 3$ and so on. In every iteration the step length increases by 1. Summing up all step lengths yields

$$1+2+3+\ldots+k = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

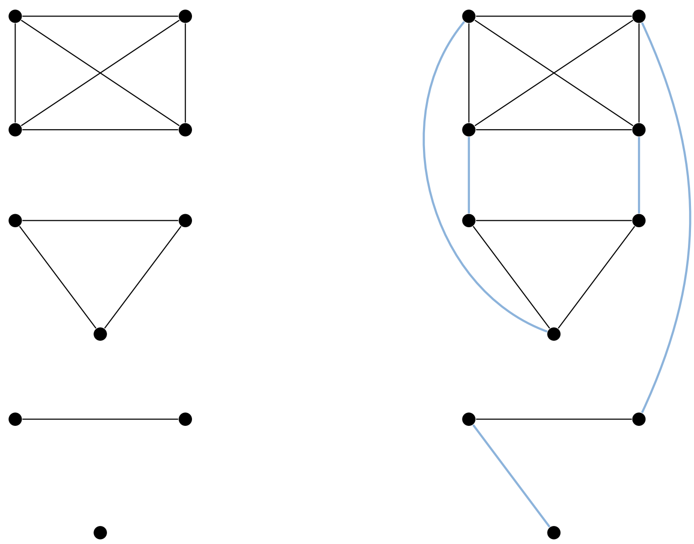
Since the partition $\pi$ consists of exactly $\frac{k(k+1)}{2}$ elements, the resulting number of marked degrees is $k$ until we move beyond the last degree. Thus, Murphy's bound yields $M(\pi) = k$.




Now we construct a graph $G$ with degree sequence $\pi = (1^1, 2^2, 3^3, \ldots, k^k)$ and $\alpha(G) = k$. Suppose $G'$ is a graph consisting of $k$ disjoint complete graphs such that

$$G' := K_1 \dot{\cup} K_2 \dot{\cup} \ldots \dot{\cup} K_k.$$

Hence, the graph $G'$ has $\frac{k(k+1)}{2}$ vertices and is a realization of the degree sequence $\pi' = (0, 1^2, 2^3, \ldots, (k-1)^k)$ with $\alpha(G') = k$. To obtain the graph $G$, we join $(k-1)$ vertices of $K_k$ to the vertices of $K_{k-1}$ such that each regarded vertex has one more neighbor. Further, we add an edge between the remaining vertex of $K_k$ and a vertex of $K_{k-2}$. We can repeat the above process (illustrated in Figure 2.5) with all complete graphs of $G'$ until the remaining vertex of $K_2$ will be joined with $K_1$. The resulting graph $G$ is a realization of $\pi = (1^1, 2^2, 3^3, \ldots, k^k)$. Since the Murphy bound is a lower bound on the independence number and $\alpha(G) \leq \alpha(G') = k$, we conclude $M(G) = \alpha(G)$. ■



$\pi' = (0,1,1,2,2,2,3,3,3,3)$ $\alpha(G') = 4$ $\pi = (1,2,2,3,3,3,4,4,4,4)$ $\alpha(G) = 4$

Figure 2.5: Graph $G'$ and modified graph $G$ for $k = 4$

We have seen that the residue maintains the dominance order of partitions. If we consider majorization of partitions instead of dominance, the Murphy bound has a similar property. We will present an alternative proof of the following result by Jelen [28].

**Theorem 2.44**
*Suppose $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ and $\sigma = (e_1 \leq e_2 \leq \ldots \leq e_n)$ are partition with $\pi \trianglerighteq \sigma$. Then*

$$M(\pi) \leq M(\sigma).$$

*The Murphy bound is monotonically decreasing.*

**Proof:**

Suppose $M(\sigma) = r$ for a positive integer $r$ and denote by $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_r$ the marked degrees such that

$$\sigma = (\overbrace{\bar{e}_1, e_2, \ldots}^{\bar{e}_1+1}, \overbrace{\bar{e}_2, \ldots, \ldots}^{\bar{e}_2+1}, \ldots, \overbrace{\bar{e}_{r-1}, \ldots, \ldots}^{\bar{e}_{r-1}+1}, \overbrace{\bar{e}_r, \ldots, e_n}^{\leq \bar{e}_r}).$$

Since $\sigma$ has length $n$, it follows

$$\bar{e}_1 + 1 + \bar{e}_2 + 1 + \ldots + \bar{e}_{r-1} + 1 + \bar{e}_r \geq n.$$

Now we compute the Murphy bound for $\pi$ and divide the sequence into $r$ parts as follows:

$$\pi = (\overbrace{d_1, d_2, \ldots, d_{\bar{e}_1+1}}^{\text{part 1}}, \overbrace{d_{\bar{e}_1+2}, \ldots, d_{\bar{e}_1+\bar{e}_2+2}}^{\text{part 2}}, \ldots, \overbrace{d_{\bar{e}_1+\bar{e}_2+\ldots+\bar{e}_{r-1}+r}, \ldots, d_n}^{\text{part } r})$$

Since $\pi$ majorizes $\sigma$, $d_i \geq e_i$ for $1 \leq i \leq n$, we obtain

$$m_1 = d_1 + 1 \geq \bar{e}_1 + 1$$

in the first step of the Murphy algorithm. We leave part 1 in the first step.

$$\pi = (d_1 \overset{\geq \bar{e}_1+1}{\curvearrowright} \cdots \quad \cdots \; d_{\bar{e}_1+1} \quad \cdots)$$

The next step yields

$$m_2 = m_1 + d_{m_1+1} + 1 \geq \bar{e}_1 + 1 + d_{\bar{e}_1+2} + 1 \geq \bar{e}_1 + \bar{e}_2 + 2,$$

and we leave the second part. Continuing in this way, we obtain

$$m_r \geq \bar{e}_1 + \bar{e}_2 + \ldots + \bar{e}_r + r \geq n.$$

The while-loop, if not before, is left in this step and we obtain $M(\pi) \leq r = M(\sigma)$. ■

**Remark 2.45**
The residue is not monotonically decreasing. For example, the partition $\pi = (1,1,1,3)$ majorizes the partition $\sigma = (1,1,1,1)$, but the residue bound yields $R(\pi) = 3$ and $R(\sigma) = 2$.

## 2.4 The Residue in Comparison with Murphy's Bound

The different constructions of both parameters complicate a comparison. The residue is the result of the Havel-Hakimi reduction steps while the Murphy bound relies on counting and cutting the degree sequence successively. Both bounds yield the same approximation for certain graph types, for example, for regular graphs. In the following section we will investigate graphical partitions for which the residue constitutes an improvement over Murphy's bound and vice versa.

**Example:**

$$\pi = (6,6,5,5,3,2,2,2,1) \quad \Rightarrow \quad R(\pi) = 5 > M(\pi) = 3$$
$$\sigma = (6,5,5,5,5,4,4,4,2) \quad \Rightarrow \quad R(\sigma) = 2 < M(\sigma) = 3$$
$$\tau = (6,5,4,4,4,3,2,1,1) \quad \Rightarrow \quad R(\tau) = 3 = M(\tau)$$

We will also see sequences where the difference between the bounds can become arbitrarily large. At first we will reflect on the behavior of Murphy's bound after applying the Havel-Hakimi algorithm.

**Theorem 2.46** (*Jelen 1996, [27]*)
*Suppose $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ is a graphical partition and $\mathcal{H}(\pi)$ the reduced partition when applying the Havel-Hakimi reduction step. With the notation of Algorithm 1, we obtain*

*(i)* $M(\pi) = M(\mathcal{H}(\pi)) + 1$, *if* $m_{M(\pi)-1}(\pi) = m_{M(\mathcal{H}(\pi))}(\mathcal{H}(\pi)) = n - 1$,

*(ii)* $M(\pi) \leq M(\mathcal{H}(\pi))$ *otherwise.*

**Proof:**
We define the partition $\sigma = (d_1, d_2, \ldots, d_{n-1})$, which can be obtained from $\pi$ by deleting the largest element $d_n$. Thus, $\sigma$ has length $n - 1$, and since the partition $\sigma$ majorizes the partition $\mathcal{H}(\pi)$, we conclude $M(\sigma) \leq M(\mathcal{H}(\pi))$ by means of Theorem 2.44. If $m_{M(\pi)-1} < n - 1$, we obtain

$$M(\pi) = M(\sigma) \leq M(\mathcal{H}(\pi)).$$

On the other hand $M(\pi) = M(\sigma) + 1$, if

$$m_{M(\pi)-1}(\pi) = m_{M(\sigma)}(\sigma) = n - 1,$$

and we can perform one more iteration for $\pi$:

$$m_{M(\pi)}(\pi) = m_{M(\pi)-1}(\pi) + d_n + 1.$$

Hence, $M(\pi) = M(\sigma) + 1 = M(\mathcal{H}(\pi)) + 1$ if and only if $m_{M(\mathcal{H}(\pi))}(\mathcal{H}(\pi)) = n - 1$. ■

With one exception the Murphy bound does not become smaller when applying the Havel-Hakimi algorithm. For a detailed quality comparison, we consider graph types for which computing their Murphy bounds or residues can be done easily. Besides, we need the following upper bound on the independence number, which can be found in Barrus [3].

**Lemma 2.47**
*Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be the degree sequence of a non-empty graph $G = (V, E)$. Further, we define a parameter $l \in \mathbb{N}$ such that $l := \max\{i \mid d_i \geq i,\ 1 \leq i \leq n\}$. Then*

$$\alpha(G) \leq n - l.$$

**Proof:**
We denote with $W \subseteq V$ a subset containing $l$ vertices of largest degree $d_1, d_2, \ldots, d_l$. Let $U \subseteq V$ be a maximum independent set in $G$. We distinguish two cases:

*Case 1:* Set $U$ does not contain any vertex of $W$. Then $V \setminus W$ has only $n - l$ vertices and

$$\alpha(G) = |U| \leq n - l.$$

*Case 2:* Set $U$ contains at least one vertex $w \in W$. Since the neighbors of $w$ are not in $U$, we get

$$\alpha(G) = |U| \leq n - d(w) \leq n - d_l \leq n - l,$$

where $l = \max\{i \mid d_i \geq i\}$ is used for the last inequality. ■

We use the above result to compute the independence number and the residue of so-called split graphs.

**Definition 2.48**
*A graph $G = (V, E)$ is called a* ***split graph*** *if its vertex set $V$ can be partitioned into disjoint sets $A$ and $B$ such that $A$ is an independent set and $B$ is a clique.*

**Example:**

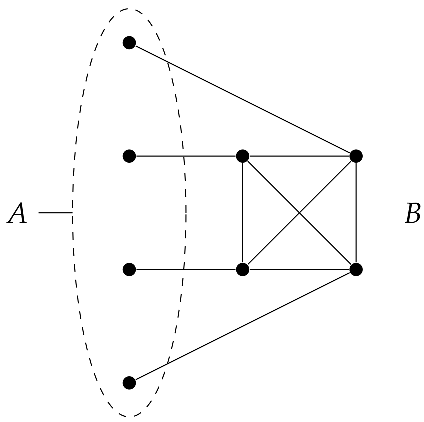



Figure 2.6: Split graph with independent set $|A| = 4$ and clique $|B| = 4$

**Proposition 2.49**
*Let $G$ be a split graph of order $n$. Then the residue bound is sharp, that is*

$$R(G) = \alpha(G).$$

**Proof:**
Suppose $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ is the degree sequence of $G$. Since the residue is always a lower bound on the independence number, we have

$$R(G) \leq \alpha(G) \leq n - l,$$

where $l = \max\{i \mid d_i \geq i\}$ and Lemma 2.47 are used for the last inequality. It suffices to show that $R(G) \geq n - l$. Suppose $V = A \dot{\cup} B$ with $|B| = m$ such that $A$ is an independent set and $B$ is a clique. If necessary, we reorder the degree sequence such that

$$\pi = (\underbrace{d_1 \geq \ldots \geq d_m}_{\in B}, \underbrace{d_{m+1} \geq \ldots \geq d_n}_{\in A}).$$

Since $d(v) \leq m$ for all $v \in A$, we obtain $d_{m+1} - m \leq 0$ when applying the Havel-Hakimi algorithm $m$ times. Hence, we can perform at most $m$ reduction steps. The residue $R(G) = n - s$, where $s \leq m$ is the number of reduction steps. We obtain

$$d_s \geq m \geq s \quad \overset{l=\max\{i \mid d_i \geq i\}}{\Rightarrow} \quad l \geq s$$

and finally,

$$n - l \leq n - s = R(G).$$

■

The Murphy bound does not reach the quality of the residue as the following example indicates: For a split graph $G$ with sequence $\pi = (5,5,2,2,2,2)$ the Murphy bound yields $M(G) = 2$, and the residue is $R(G) = 4 = \alpha(G)$.

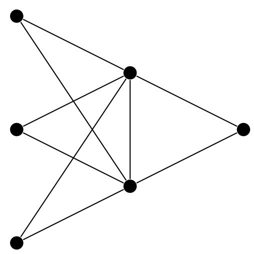


Figure 2.7: Split graph with $\pi = (5,5,2,2,2,2)$

From the example and the previous theorem we conclude:

**Corollary 2.50**
*In the class of split graphs, the residue always improves Murphy's bound. For any split graph G it holds*

$$M(G) \leq R(G).$$

We can proceed to the next step and formulate the following result.

**Theorem 2.51**
*There are graphical partitions $\pi$ for which the difference between*

$$\min\{\alpha(G) \mid G \textit{ is a realization of } \pi\}$$

*and Murphy's bound $M(\pi)$ can become arbitrarily large.*

**Proof:**
We consider the partition $((n-1)^n, 0^{n+2})$, which is the degree sequence of a split graph with $n+2$ isolated vertices and a complete graph of order $n$. Now we connect each vertex of the complete graph to all $n+2$ isolated vertices. The resulting graph is still a split graph $G$ with degree sequence $\pi = ((2n+1)^n, n^{n+2})$ and independence number $\alpha(G) = n+2$. The Murphy bound yields $M(\pi) = 2$ and since the residue is a sharp bound for split graphs, we conclude

$$\alpha_{\min}(\pi) - M(\pi) = R(\pi) - M(\pi) = n$$

for all $n \in \mathbb{N}$. ■

Jelen [27] proved with the help of Murphy's bound that the gap between the residue and the independence number can also become arbitrarily large. He creates graphical partitions for which Murphy bound computations can be carried out easily. The basis of the partitions is a sequence $(d_n)_{n\in\mathbb{N}_0}$ of non-negative integers with $d_n := \frac{1}{2}(3^{n+1}-1)$. The first values are

$$\begin{aligned} d_0 &= 1 \\ d_1 &= 4 \\ d_2 &= 13 \\ d_3 &= 40 \\ d_4 &= 121 \\ d_5 &= 364. \end{aligned}$$

Consider now the partitions

$$\phi_n := ((d_{n+1}-1)^{d_{n+1}-1}, (d_{n+1}-2)^{d_{n+1}-2}, \ldots, (2d_n)^{2d_n}, \ldots, d_n^{d_n})$$

and

$$\rho_n := ((d_{n+1}-1)^{d_{n+1}}, (d_{n+1}-2)^{d_{n+1}-1}, \ldots, (2d_n+1)^{2d_n+2}, (2d_n-1)^{2d_n}, \ldots, d_n^{d_n+1}).$$

The partition $\rho_n$ arises from $\phi_n$ by splitting the $2d_n$ terms of value $2d_n$ into two equal parts. For $i = 0, 1, \ldots, d_n - 1$ the $d_n - i$ largest terms of value $2d_n + i$ successively increase by one, and the $d_n - i$ smallest terms of value $2d_n - i$ decrease by one. Thus, the partitions have the same length.

**Example:**
For $n = 0$ we obtain

$$\phi_0 = (3^3, 2^2, 1^1) \quad \Rightarrow \quad \rho_0 = (3^4, 1^2).$$

For $n = 1$ we have $d_1 = 4$, and the transformation reads as follows:

$$
\begin{array}{lrcr}
i=0: & 8,8,8,8 & \nearrow & 9,9,9,9 \\
 & 8,8,8,8 & \searrow & 7,7,7,7 \\
i=1: & 9,9,9 & \nearrow & 10,10,10 \\
 & 7,7,7 & \searrow & 6,6,6 \\
i=2: & 10,10 & \nearrow & 11,11 \\
 & 6,6 & \searrow & 5,5 \\
i=3: & 11 & \nearrow & 12 \\
 & 5 & \searrow & 4
\end{array}
$$

$$\phi_1 = (12^{12},11^{11},10^{10},9^9,8^8,7^7,6^6,5^5,4^4) \quad \Rightarrow \quad \rho_1 = (12^{13},11^{12},10^{11},9^{10},7^8,6^7,5^6,4^5).$$

**Lemma 2.52** (*Jelen 1996, [27]*)
*The partitions $\phi_n$ and $\rho_n$ are graphical for all $n \in \mathbb{N}_0$ and satisfy $\rho_n \succeq \phi_n$.*

The partitions $\phi_n$ and $\rho_n$ form a base for the following partitions. For $n \in \mathbb{N}$ we define

$$\pi_n := (\phi_{n-1},\phi_{n-2},\dots,\phi_0) \quad \text{and} \quad \sigma_n := (\rho_{n-1},\rho_{n-2},\dots,\rho_0).$$

It follows immediately by Lemma 2.52 that $\pi_n$ and $\sigma_n$ are graphical with $\sigma_n \succeq \pi_n$. For example, for $n=2$ we have

$$\pi_2 = (\phi_1,\phi_0) = (12^{12},11^{11},10^{10},9^9,8^8,7^7,6^6,5^5,4^4,3^3,2^2,1^1)$$

and

$$\sigma_2 = (\rho_1,\rho_0) = (12^{13},11^{12},10^{11},9^{10},7^8,6^7,5^6,4^5,3^4,1^2).$$

**Theorem 2.53** (*Jelen 1996, [27]*)
*For the graphical partition $\pi_n$ and for all $n \in \mathbb{N}$,*

$$R(\pi_n) - M(\pi_n) \geq n.$$

**Proof:**
The partition $\pi_n$ contains each term $i$ exactly $i$ times for $i = 1,2,\dots,d_n - 1$. With Theorem 2.43 the Murphy bound can be easily computed, and we obtain $M(\pi_n) = d_n - 1$. The partition $\sigma_n$ is the degree sequence of $d_n - 1 - n$ disjunct complete graphs with $R(\sigma_n) = d_n - 1 - n$. Since $\sigma_n \succeq \pi_n$, we have $R(\sigma_n) \geq R(\pi_n)$. Thus,

$$R(\pi_n) \leq R(\sigma_n) = d_n - 1 - n = M(\pi_n) - n$$

$$\Leftrightarrow \quad M(\pi_n) - R(\pi_n) \geq n.$$

■

As a consequence, and combined with the fact that Murphy's bound is always a lower bound on the independence number of a graph, it follows:

**Corollary 2.54** (*Jelen 1996, [27]*)
*There are graphical partitions $\pi$ for which the difference between*

$$\min\{\alpha(G) \,|\, G \text{ is a realization of } \pi\}$$

*and the residue $R(\pi)$ can become arbitrarily large.*

# 3 A Refinement of Murphy's Algorithm for Certain Graphs

We have seen that the residue offers a genuine improvement on Murphy's bound for some graphs. The partition $\pi = (13^6, 6^8)$, for instance, yields $M(\pi) = 2$ whereas the residue is $R(\pi) = 8$. The differences in quality result from the distinct approaches. Murphy obtains a lower bound on the independence number by starting with vertices of low degree so essential information may be lost. This is particularly remarkable if the degree sequence consists of terms with significant increase of values (see example above). In this chapter we will characterize such graphical partitions and present an optimization of Murphys algorithm, using additional information on graphical sequences. This leads to improvements under certain conditions. Unless we use the Havel-Hakimi algorithm, all considered sequences are in increasing order.

## 3.1 Dual Partitions

**Definition 3.1**
*A partition $\pi$ of length $n \in \mathbb{N}$ is called **dual** if it is a sequence that contains only two distinct values. We write $\pi = (a^k, b^{n-k})$ for $a, b, k \in \mathbb{N}$.*

The following lemma shows that Murphy's bound is weak for some dual partitions.

**Lemma 3.2**
*Suppose $\pi = (n^{n+1}, (2n)^n)$ is a graphical dual partition for some $n \in \mathbb{N}$. Then $M(\pi) = 2$ and $R(\pi) = n + 1$.*

**Proof:**
Using Murphy's algorithm, we compute

$$\begin{aligned} m_1 &= n+1 \\ m_2 &= n+1+2n+1 = 3n+2. \end{aligned}$$

Since the partition $\pi$ has length $2n+1$, the algorithm stops and $M(\pi) = 2$. On the other hand we perform a Havel-Hakimi reduction step and obtain

$$\mathcal{H}(\pi) = ((2n-1)^{n-1}, (n-1)^{n+1}).$$

For $i = 1, 2, \dots, n$ we have

$$\mathcal{H}^i(\pi) = ((2n-i)^{n-i}, (n-i)^{n+1}).$$

This leads to $\mathcal{H}^n(\pi) = (0^{n+1})$, and the residue yields $R(\pi) = n + 1$. ■

Since Murphy's algorithm starts with vertices of low degree, the process excludes all those low degrees in the first steps. However, in case of dual partitions with significant difference between both values, the vertices of lower degree are not necessarily adjacent. Otherwise, the graph would not be realizable. Murphy's algorithm ignores this fact. Thus, we will present a refinement of Murphy's algorithm by eliminating this lack of information for certain degree sequences.

**Part 1.** Let $\pi = (a^k, b^{n-k})$, $2 < a < b$ be a graphical dual partition with the following conditions:

- $2a = b$ (refinement condition)
- $k = a + 1$ (graphical pre-condition)
- $a \leq n - k \leq a + 2$ (graphical pre-condition)

---

**Algorithm 2** Refined Murphy algorithm for dual partitions

---

**Input:** partition $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$
**Output:** refined Murphy bound $\overline{M}(\pi) \geq 1$
  $j = 0$
  $m_0 = 0$
  **while** $m_j < n$ **do**
    $i = m_j$
    $m_{j+1} = m_j + d_{i+1} + 1$
    **if** graphical pre-conditions of part 1 true **then**
      **if** $2d_{i+1} = d_{m_j+1}$ **then** (refinement condition)
        $m_{j+1} = m_{j+1} - d_{i+1} + 1$ (refinement)
      **else**
        $m_{j+1} = m_{j+1}$
      **end if**
    **end if**
    $j = j + 1$
  **end while**
  $\overline{M}(\pi) = j$

---

The *graphical pre-conditions* characterize the dual partitions and ensure their realizability. The *refinement condition* checks the difference between two marked degree values in Murphy's algorithm. If the difference is large enough, that is $2d_{i+1} = d_{m_j+1}$, we perform a *refinement step*. Here we add the information that not all vertices of low degree are adjacent. We reduce the step length

$$m_{j+1} = m_j + d_{i+1} + 1 - d_{i+1} + 1 = m_j + 2.$$

If one of the conditions is false, then $\overline{M}(\pi) = M(\pi)$.

**Example:**
The partition $\pi = (4^5, 8^5)$ satisfies the graphical pre-conditions and yields $M(\pi) = 2$ and $R(\pi) = 5$.

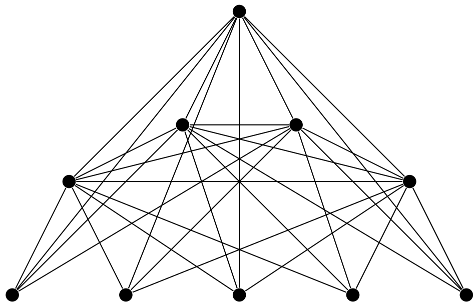


Figure 3.1: Realization of $\pi = (4^5, 8^5)$ with $\alpha_{\min}(\pi) = 5$

Murphy's algorithm without refinement:

$$\begin{aligned} m_0 &= 0 \\ m_1 &= 0+4+1=5 \\ m_2 &= 5+8+1=14 > n=10 \\ \Rightarrow \quad M(\pi) &= 2. \end{aligned}$$

The refined algorithm yields

$$\begin{aligned} m_0 &= 0 \\ m_1 &= 0+4+1=5 \\ \text{refinement:} \quad 2d_1 &= d_6 \quad \checkmark \\ \Rightarrow \quad m_1 &= m_1 - d_1 + 1 = 2 \\ m_2 &= 2+4+1=7 \\ \text{refinement:} \quad 2d_3 &= d_8 \quad \checkmark \\ \Rightarrow \quad m_2 &= m_2 - d_3 + 1 = 4 \\ m_3 &= 4+4+1=9 \\ \text{refinement:} \quad 2d_5 &= d_{10} \quad \checkmark \\ \Rightarrow \quad m_3 &= m_3 - d_5 + 1 \ = 6 \\ m_4 &= 5+8+1=14 > n=10 \\ \Rightarrow \quad \overline{M}(\pi) &= 4. \end{aligned}$$

**Lemma 3.3**
*Let $\pi = (a^k, (2a)^{n-k})$ be the degree sequence satisfying the graphical pre-conditions, i.e. $k = a+1$ and $a \leq n-k \leq a+2$. Then*

$$\left\lceil \frac{a+1}{2} \right\rceil \leq \overline{M}(\pi) \leq \left\lceil \frac{a+1}{2} \right\rceil + 1$$

*and for $n \in \mathbb{N}$*

$$\overline{M}(\pi) - M(\pi) \geq \frac{n}{4} - 3 = \mathcal{O}(n).$$

**Proof:**
Since the refinement condition is true, we have

$$m_{j+1} = m_j + d_{i+1} + 1 - d_{i+1} + 1 = m_j + 2, \quad 1 \leq i,j \leq n.$$

Thus, the step length is 2, and we perform at least $\lceil \frac{a+1}{2} \rceil$ steps until we mark the second but last or last term of value $a$. If $n - k \geq a + 1$, we can perform even one more step. We obtain $\lceil \frac{a+1}{2} \rceil \leq \overline{M}(\pi) \leq \lceil \frac{a+1}{2} \rceil + 1$.

$$\begin{aligned} 4\overline{M}(\pi) &\geq 2(a+1) \\ &\geq n - k + a \\ &= n - 1, \end{aligned}$$

where the graphical pre-conditions $n - k \leq a + 2$ and $k = a + 1$ are used for the estimation. Together with $M(\pi) = 2$ we conclude

$$\overline{M}(\pi) - M(\pi) \geq \frac{n}{4} - 3.$$

■

The result provides a genuine improvement on Murphy's bound for the partitions considered.

| partition $\pi$ | $M(\pi)$ | $\overline{M}(\pi)$ | $R(\pi)$ |
|---|---|---|---|
| $(3^4, 6^5)$ | 2 | 3 | 3 |
| $(4^5, 8^6)$ | 2 | 4 | 4 |
| $(6^7, 12^7)$ | 2 | 5 | 7 |
| $(8^9, 16^{10})$ | 2 | 6 | 8 |
| $(20^{21}, 40^{22})$ | 2 | 11 | 20 |
| $(55^{56}, 110^{56})$ | 2 | 29 | 56 |
| $(333^{334}, 666^{334})$ | 2 | 168 | 334 |
| $(1000^{1001}, 2000^{1002})$ | 2 | 502 | 1000 |
| $(12242^{12243}, 24484^{12243})$ | 2 | 6123 | 12243 |

**Theorem 3.4**
*The refined Murphy bound is still a lower bound on the independence number.*

**Proof:**
Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be the degree sequence of a graph $G$. If $\pi$ does not satisfy the refinement and the graphical pre-conditions, then $\overline{M}(G) = M(G) \leq \alpha(G)$. Otherwise,

$$\pi = (a^{a+1}, (2a)^{n-(a+1)})$$

with $2a + 1 \leq n \leq 2a + 3$. With Lemma 3.2 the residue yields $R(\pi) = a + 1$ if $n = 2a + 1$.

*Case:* $n = 2a + 2$

$$\begin{aligned} \pi &= ((2a)^{a+1}, a^{a+1}) \\ \mathcal{H}(\pi) &= ((2a-1)^{a}, a, (a-1)^{a}) \\ \mathcal{H}^{i}(\pi) &= ((2a-i)^{a+1-i}, (a+1-i)^{i}, (a-i)^{a+1-i}), \quad 1 \le i \le a+1. \end{aligned}$$

We obtain $\mathcal{H}^{a}(\pi) = (a, 1^{a}, 0)$ and $\mathcal{H}^{a+1}(\pi) = (0^{a+1})$.

*Case:* $n = 2a + 3$

$$\begin{aligned} \pi &= ((2a)^{a+2}, a^{a+1}) \\ \mathcal{H}(\pi) &= ((2a-1)^{a+1}, a^{2}, (a-1)^{a-1}) \end{aligned}$$

The Havel-Hakimi reduction steps can be carried out inductively, and we get

$$\begin{aligned} \mathcal{H}^{a+1}(\pi) &= ((a-1), 1^{a+1}) \\ \mathcal{H}^{a+2}(\pi) &= (1^{2}, 0^{a-1}) \\ \mathcal{H}^{a+3}(\pi) &= (0^{a}). \end{aligned}$$

Thus, the residue is $R(\pi) \ge a$ for $2a+1 \le n \le 2a+3$. Since the residue is a lower bound on the independence number and $a > 2$, we conclude

$$\overline{M}(G) \le \left\lceil \frac{a+1}{2} \right\rceil + 1 \le a \le R(G) \le \alpha(G).$$

■

Due to the special properties of the degree sequences, we present a weakening of the conditions.

**Part 2.** Let $\pi = (a^{k}, b^{n-k})$, $2 < a < b$, be a graphical dual partition with weakened conditions:

- $2a - 1 \le b$ (refinement condition)
- $a \le k \le a + 1$ (graphical pre-condition)
- $a \le n - k \le a + 2$ (graphical pre-condition)

This partition has more variability. We change the refinement condition in such a way that the if-statement reads as follows:

$$\text{If } 2d_{i+1} - 1 \le d_{m_j+1} \text{ true for } 0 \le j \le n \text{ and } i = m_j.$$

The following table shows test instances with an improvement on Murphy's bound. However, it is obvious that the residue yields still a better approximation.

| partition $\pi$ | $M(\pi)$ | $\overline{M}(\pi)$ | $R(\pi)$ |
|---|---|---|---|
| $(4^4,7^5)$ | 2 | 3 | 3 |
| $(5^5,9^6)$ | 2 | 4 | 4 |
| $(5^6,11^7)$ | 2 | 4 | 6 |
| $(8^8,15^9)$ | 2 | 5 | 7 |
| $(8^9,17^{10})$ | 2 | 6 | 9 |
| $(20^{20},39^{21})$ | 2 | 11 | 19 |
| $(20^{21},41^{22})$ | 2 | 12 | 21 |
| $(50^{50},99^{51})$ | 2 | 26 | 49 |
| $(50^{51},101^{52})$ | 2 | 27 | 51 |
| $(200^{201},401^{202})$ | 2 | 102 | 201 |
| $(555^{556},1111^{557})$ | 2 | 279 | 556 |
| $(1412^{1412},2823^{1413})$ | 2 | 707 | 1411 |
| $(23766^{23767},47533^{23768})$ | 2 | 11885 | 23767 |

## 3.2 Double Partitions

The refined Murphy algorithm has an impact on limited sequences and graphs. We present an extension of the algorithm for partitions with more than two different degree values.

**Definition 3.5**
*A partition $\pi = (d_1^{k_1}, d_2^{k_2}, \ldots, d_n^{k_n})$ with $d_i, k_i \in \mathbb{N}$ is called* ***double partition*** *if $2d_i - 1 \leq d_{i+1}$ for $i = 1, \ldots, n-1$.*

**Remark 3.6**
The name *double partition* arises from the fact that the degree values increase with almost twice the value of the previous value. Dual partitions are special double partitions.

**Part 3.** Let $\pi = (d_1^{k_1}, d_2^{k_2}, \ldots, d_n^{k_n})$, $n > 2$ be a graphical double partition with the following conditions:

- $2d_i - 1 \leq d_{i+1}$ for $i = 1, \ldots, n-1$ (refinement condition)
- $d_i \leq k_i \leq d_i + 1$ for $i = 1, \ldots, n-2$ (graphical pre-condition)
- $d_{n-1} \leq k_n \leq d_{n-1} + 2$ (graphical pre-condition)
- $k_{n-1} \leq \frac{2}{3} d_{n-1} + 2$ (graphical pre-condition)

Apart from the last property, the conditions are equal to the ones in part 2. The last property guarantees a high residue of the partition for a better comparison. The refined algorithm reads as follows:

**Algorithm 3** Refined Murphy algorithm

**Input:** partition $\pi = (d_1 \le d_2 \le \ldots \le d_n)$
**Output:** refined Murphy bound $\overline{M}(\pi) \ge 1$

```
j = 0
m_0 = 0
while m_j < n do
    i = m_j
    m_{j+1} = m_j + d_{i+1} + 1
    if graphical pre-conditions of part 1,2 or 3 true then
        if 2d_{i+1} - 1 <= d_{m_j+1} then    (refinement condition)
            m_{j+1} = m_{j+1} - d_{i+1} + 1  (refinement)
        else
            m_{j+1} = m_{j+1}
        end if
    end if
    j = j + 1
end while
M̄(π) = j
```

| partition $\pi$ | $M(\pi)$ | $\overline{M}(\pi)$ | $R(\pi)$ |
|---|---|---|---|
| $(4^5, 8^6, 16^9)$ | 3 | 7 | 11 |
| $(10^{11}, 20^{15}, 40^{21})$ | 3 | 14 | 26 |
| $(50^{51}, 100^{68}, 200^{100})$ | 3 | 61 | 121 |
| $(4^5, 8^8, 15^{16}, 30^{20}, 60^{31})$ | 5 | 26 | 49 |
| $(9^{10}, 17^{18}, 35^{36}, 70^{48}, 140^{71})$ | 5 | 27 | 89 |
| $(2500^{2501}, 5000^{3335}, 10000^{5000})$ | 3 | 2919 | 5836 |
| $(244^{245}, 488^{488}, 976^{652}, 1952^{976})$ | 4 | 693 | 1385 |
| $(100^{101}, 200^{201}, 400^{401}, 800^{536}, 1600^{800})$ | 5 | 620 | 1239 |
| $(6^7, 12^{12}, 23^{23}, 46^{47}, 92^{92}, 184^{124}, 368^{184})$ | 7 | 153 | 246 |
| $(7200^{7201}, 14400^{14401}, 28800^{19201}, 57600^{28800})$ | 4 | 20403 | 40804 |

The test instances show significant improvements. Yet, even then, we cannot reach the quality of the residue. On the other hand it is important to state that dual and double partitions are of a specific nature and Murphy's bound could be further improved.

# 4 *k*-Independence in Graphs

This chapter deals with the generalized concept of independence in graphs. We will survey some well-known results and investigate properties of $k$-independence. As described in the previous chapters, we will consider lower bounds and focus on the $k$-residue, which is a generalization of the residue.

## 4.1 Basic Properties

First, we will formally define $k$-independence and introduce a few properties and necessary tools.

**Definition 4.1**
*Let $G = (V, E)$ be a graph and $k$ a positive integer. A **k-independent set** $X \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by $X$ is at most $k-1$, that is*

$$\Delta(G[X]) \leq k - 1.$$

*The cardinality of a maximum k-independent set is denoted by $\alpha_k(G)$.*

The following general properties can be concluded immediately from the definition of $k$-independence.

**Remark 4.2**

- For $k = 1$ the 1-independent set is the classical independent set and we write $\alpha_1(G) = \alpha(G)$.
- The complete vertex set $V$ is $k$-independent if and only if the maximum degree of $G$ is less than $k$. Thus, it makes sense to consider only the cases $1 \leq k \leq \Delta(G)$.
- Since every induced subgraph of $k$ vertices has maximum degree at most $k-1$, we obtain $\alpha_k(G) \geq k$ for $1 \leq k \leq |V|$.
- Every induced subgraph of a complete graph is complete with $\Delta(K_k) = k-1$ and $\Delta(K_{k+1}) = k$. Thus, a complete graph $K_n$ on $n$ vertices satisfies $\alpha_k(K_n) = k$ for $1 \leq k \leq n$.
- Every $k$-independent set is also a $(k+1)$-independent set and so $\alpha_k(G) \leq \alpha_{k+1}(G)$. The $k$-independence number increases strictly, for example, for complete graphs we obtain

  $$\alpha(K_n) = 1 < \alpha_2(K_n) = 2 < \ldots < \alpha_{n-1}(K_n) = n-1 < \alpha_n(K_n) = n.$$

  On the other hand for the star graph $S_{n+1}$ with degree sequence $\pi = (n, 1^n)$ the $k$-independence number does not change for $1 \leq k \leq n$, i.e.

  $$\alpha(S_{n+1}) = \alpha_2(S_{n+1}) = \ldots = \alpha_n(S_{n+1}) = n.$$

A more general relationship between the parameters is produced by the following result.

**Theorem 4.3** (*Blidia et al. 2008, [6]*)
*For every graph G and integers i,k with $1 \leq i \leq k$,*

$$\alpha_{k+1}(G) \leq \alpha_i(G) + \alpha_{k-i+1}(G).$$

**Corollary 4.4**
*For every graph G and every positive integer k,*

*(i)* $\alpha_{k+1}(G) \leq \alpha_k(G) + \alpha(G)$,

*(ii)* $\alpha_{k+1}(G) \leq 2\alpha_{\lceil k+1/2 \rceil}(G)$,

*(iii)* $\alpha_{k+1}(G) \leq (k+1)\alpha(G)$.

Since the problem of determining the classic independence number of a graph is $NP$-complete, we state the following complexity result.

**Proposition 4.5** (*Jelen 1996, [27]*)
*The computation of maximal k-independent sets is $NP$-complete for all $k \in \mathbb{N}$.*

## 4.2 Lower Bounds on the *k*-Independence Number

A generalization of the independence number suggests to extend well-known bounds and results. Favaron succeeded in extending the Caro-Wei bound.

**Theorem 4.6** (*Favaron 1988, [14]*)
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph G and k a positive integer. Then*

$$F_k(\pi) := |\{i \,|\, d_i = 0\}| + \sum_{i:\, d_i \neq 0} \frac{k}{1 + kd_i} \leq \alpha_k(G).$$

*For $k = 1$, Favaron's bound is consistent with the Caro-Wei bound:*

$$F_1(\pi) = \sum_{i=1}^{n} \frac{1}{1 + d_i} = CW(\pi).$$

Caro and Tuza [9] investigated the size of $k$-independent sets in uniform hypergraphs. In case of simple graphs they improved Favaron's bound. The original proof had been published incorrectly and was corrected by Jelen [27].

**Theorem 4.7** (*Caro and Tuza 1991, [9]*)
*If G is a graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$ and k a positive integer, then*

$$CT_k(\pi) := \sum_{i=1}^{n} f_k(d_i) \leq \alpha_k(G),$$

*where*

$$f_k(x) = \begin{cases} 1 - \dfrac{x}{2k}, & \text{if } 0 \leq x \leq k, \\ \dfrac{k+1}{2x+2}, & \text{if } x > k. \end{cases}$$

**Corollary 4.8**
*For every graph G with average degree $\overline{d}$ and for every positive integer k,*

$$n \cdot f_k(\overline{d}) \leq \alpha_k(G).$$

**Proof:**
The second derivative of the real-valued function $f_k$ is

$$f_k''(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq k, \\ \dfrac{8(k+1)}{(2x+2)^3}, & \text{if } x > k. \end{cases}$$

Hence, $f_k''(x) \geq 0$ for all $x \in \mathbb{R}$ and $f_k$ is a convex function. Using the Caro-Tuza bound and Jensen's inequality, we obtain

$$\alpha_k(G) \geq \sum_{i=1}^{n} f_k(d_i) = n \sum_{i=1}^{n} \frac{1}{n} \cdot f_k(d_i) \geq n \cdot f_k\left(\frac{1}{n}\sum_{i=1}^{n} d_i\right) = n \cdot f_k(\overline{d}).$$

■

**Corollary 4.9**
*For every graph G with $\overline{d} \geq k$*

$$\frac{k+1}{2(\overline{d}+1)} n \leq \alpha_k(G).$$

**Theorem 4.10** (*Caro and Tuza 1991, [9]*)
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph G and k a positive integer. Then*

$$F_k(\pi) \leq CT_k(\pi).$$

Hopkins and Staton [25] investigated vertex partitions and so-called $k$-small subsets. The notion $k$-small is consistent with $k$-independent.

**Theorem 4.11** (*Hopkins and Staton 1986, [25]*)
*If G is a graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$, maximum degree $d_1 = \Delta$ and k a positive integer, then*

$$HS_k(\pi) := \frac{n}{1 + \lfloor \frac{\Delta}{k} \rfloor} \leq \alpha_k(G).$$

Note that, for $k = 1$, the Hopkins-Staton bound yields $\frac{n}{1+\Delta}$ from Theorem 2.3. A direct consequence of the Hopkins-Staton result is the following relationship between $\alpha_k(G)$ and $\alpha_j(G)$ for $1 \le j \le k$, which can also be found in Caro and Hansberg [8].

**Theorem 4.12**
*Let G be a graph and $1 \le j \le k$ two positive integers. Then*

$$\alpha_k(G) \le \left(1 + \left\lfloor \frac{k-1}{j} \right\rfloor\right) \alpha_j(G).$$

**Proof:**
Let $X$ be a maximum $k$-independent set of $G$. Then $\Delta(G[X]) \le k-1$ and we conclude

$$\alpha_j(G) \ge \alpha_j(G[X]) \ge \frac{|X|}{1 + \left\lfloor \frac{\Delta(G[X])}{j} \right\rfloor} \ge \frac{\alpha_k(G)}{1 + \left\lfloor \frac{k-1}{j} \right\rfloor},$$

where Theorem 4.11 is used for the second inequality. ■

Caro and Hansberg established a lower bound on the $k$-independence number in terms of the average degree of a graph.

**Theorem 4.13** (*Caro and Hansberg 2013, [8]*)
*If G is a graph with degree sequence $\pi = (d_1, d_2, \ldots, d_n)$, average degree $\overline{d}$ and k a positive integer, then*

$$CH_k(\pi) := \frac{kn}{k + \lceil \overline{d} \rceil} \le \alpha_k(G).$$

Favaron's bound was improved by Caro and Tuza while Caro and Hansberg improved the currently best general bound by Caro and Tuza. However, the lower bounds of Hopkins-Staton and Caro-Hansberg are mutually non-comparable as the following example indicates.

**Example:**
Let $\pi_n = (n, 1^n)$ be the degree sequence of a star graph on $n+1$ vertices and $n, k \in \mathbb{N}$. The average degree yields $\overline{d} = \frac{2n}{n+1}$ and hence, $\lceil \overline{d} \rceil = 2$. This leads to

$$HS_k(\pi_n) = \frac{n+1}{1 + \lfloor \frac{n}{k} \rfloor} \quad \text{and} \quad CH_k(\pi_n) = \frac{k(n+1)}{k+2}.$$

In case of $k = 2$,

$$HS_2(\pi_n) = \frac{n+1}{1 + \lfloor \frac{n}{2} \rfloor} \le 2 \quad \text{and} \quad CH_2(\pi_n) = \frac{n+1}{2}.$$

$$\Rightarrow \quad CH_2(\pi_n) - HS_2(\pi_n) \ge \frac{n+1}{2} - 2 = \mathcal{O}(n).$$

On the other hand let $\sigma_n = ((k+1)^n)$ be the degree sequence of $(k+1)$-regular graph on $n$ vertices. The bounds yield

$$HS_k(\sigma_n) = \frac{n}{1 + \left\lfloor \frac{k+1}{k} \right\rfloor} = \frac{n}{2} \quad \text{and} \quad CH_k(\sigma_n) = \frac{kn}{2k+1},$$

thus, $HS_k(\sigma_n) > CH_k(\sigma_n)$ for every $n \in \mathbb{N}$.

We will close the section with an upper bound on the $k$-independence number and present an extension of Lemma 2.47.

**Theorem 4.14**
*Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be the degree sequence of a non-empty graph $G$ and $k, l \in \mathbb{N}$ such that*

$$l = \max\{i \mid d_i \geq i,\, 1 \leq i \leq n\}.$$

*Then*

$$\alpha_k(G) \leq n + k - l - 1.$$

**Proof:**
For $k = 1$, $\alpha(G) \leq n - l$, which is precisely the result of Lemma 2.47. Suppose $X$ is a maximum $k$-independent set of $G$ for $k \geq 2$. Further, let $v_1, v_2, \ldots, v_l$ be the vertices of largest degrees $d_1, d_2, \ldots, d_l$.

*Case 1:* Set $X$ does not contain any vertex of $\{v_1, v_2, \ldots, v_l\}$. Then $X$ has at most $n - l$ vertices and

$$\alpha_k(G) = |X| \leq n - l \leq n + k - l - 1.$$

*Case 2:* Set $X$ contains at least one vertex $v_i$ of $\{v_1, v_2, \ldots, v_l\}$. Since $v_i$ has at most $k - 1$ neighbors in $X$,

$$d_i \leq n - \alpha_k(G) + (k - 1).$$

Rearranging the inequality leads to

$$\begin{aligned} \alpha_k(G) &\leq n + k - d_i - 1 \\ &\leq n + k - d_l - 1 \\ &\leq n + k - l - 1, \end{aligned}$$

where $l = \max\{i \mid d_i \geq i,\, 1 \leq i \leq n\}$ is used for the last inequality. ■

## 4.3 The *k*-Residue of a Graph

Since the residue of a graph is a lower bound on its independence number, it seems promising to consider a modified residue for a lower bound on the $k$-independence number. Jelen [28] was able to prove a lower bound by defining a generalization of the residue, the so-called $k$-residue of a graph.

Before formally defining the $k$-residue, we will introduce a necessary tool. Jelen modified the elimination sequence from Definition 2.21 by adding the resulting sequences of zeros obtained by the Havel-Hakimi algorithm.

**Definition 4.15**
*Let $\pi = (d_1 \geq d_2 \geq \ldots \geq d_n)$ be a graphical partition and $\mathcal{H}^i(\pi)$ the partition after $i \in \{1,\ldots,n-1\}$ Havel-Hakimi reduction steps. The partition*

$$\overline{E}(\pi) := (\max(\pi), \max(\mathcal{H}(\pi)), \max(\mathcal{H}^2(\pi)), \ldots, \max(\mathcal{H}^{s-1}(\pi)), \underbrace{0,\ldots,0}_{n-s})$$

*is called the **extended elimination sequence** of $\pi$, where $s = s(\pi)$ is the number of reduction steps to obtain a sequence full of zeros.*

**Definition 4.16**
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical partition and denote the number of terms with value $i$ in $\overline{E}(\pi)$ with $g_i(\pi)$. For a positive integer $k$,*

$$R_k(\pi) := \frac{1}{k}\sum_{i=0}^{k-1}(k-i)\cdot g_i(\pi)$$

*is called the **k-residue** of $\pi$. If $\pi$ is a degree sequence of a graph $G$, we write $R_k(G)$.*

For $k = 1$ we obtain $R_1(\pi) = g_0(\pi)$, which gives the number of zeros in the extended elimination sequence. Thus, the 1-residue equals the residue of a graph.

**Example:**
The graphical partition $\pi = (4^2, 3^2, 2^2)$ with reduction steps

| | | | | | | |
|---|---|---|---|---|---|---|
| $\pi$ | 4 | 4 | 3 | 3 | 2 | 2 |
| $\mathcal{H}(\pi)$ | | 3 | 2 | 2 | 2 | 1 |
| $\mathcal{H}^2(\pi)$ | | | 1 | 1 | 1 | 1 |
| $\mathcal{H}^3(\pi)$ | | | | 1 | 1 | 0 |
| $\mathcal{H}^4(\pi)$ | | | | | 0 | 0 |

has the extended elimination sequence $\overline{E}(\pi) = (4,3,1,1,0,0)$. For $1 \leq k \leq 4$ the $k$-residue yields

$$\begin{aligned}
R_1(\pi) &= 2\\
R_2(\pi) &= \frac{1}{2}(2\cdot g_0(\pi) + 1\cdot g_1(\pi)) = 3\\
R_3(\pi) &= \frac{1}{3}(3\cdot g_0(\pi) + 2\cdot g_1(\pi) + 1\cdot g_2(\pi)) = \frac{10}{3}\\
R_4(\pi) &= \frac{1}{4}(4\cdot g_0(\pi) + 3\cdot g_1(\pi) + 2\cdot g_2(\pi) + 1\cdot g_3(\pi)) = \frac{15}{4}.
\end{aligned}$$

We have seen that the dominance order of partitions is a useful tool in the study of the residue. Jelen generalized the result of Favaron et al. with the following statement.

**Lemma 4.17** (*Jelen 1999, [28]*)
*Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\sigma = (e_1, e_2, \ldots, e_n)$ are partitions with $\pi \succeq \sigma$. Then*

$$R_k(\pi) \geq R_k(\sigma).$$

In order to show that the $k$-residue is a lower bound on the $k$-independence number, Jelen follows an idea of Griggs and Kleitman [21] and considers a heuristic algorithm for finding large $k$-independent sets. The greedy algorithm *k-MAX* is an extension of the procedure *MAX* (see Chapter 2.2), which repeatedly deletes the vertex of highest degree until the remaining graph has a maximum degree of less than $k$. The size of the resulting $k$-independent set will be denoted by $\mathcal{A}_k$.

| | |
|---|---|
| Algorithm: | *k-MAX* |
| Input: | graph $G$ |
| | while $\Delta(G) \geq k$ do |
| | $v \leftarrow$ any vertex of highest degree in $G$ |
| | $G \leftarrow G \setminus \{v\}$ |
| | endwhile |
| | $\mathcal{A}_k(G) \leftarrow$ number of vertices in the remaining graph $G$ |
| Output: | $k$-independent set in $G$ of size $\mathcal{A}_k(G)$ |

**Theorem 4.18** (*Jelen 1999, [28]*)
*For every graph $G = (V, E)$ and every positive integer $k$, the $k$-residue of $G$ is at most the $k$-independence number,*

$$R_k(G) \leq \alpha_k(G).$$

**Proof:**
Since $\mathcal{A}_k(G) \leq \alpha_k(G)$ for any possible result produced by the algorithm *k-MAX*, it suffices to show that $R_k(G) \leq \mathcal{A}_k(G)$. This will be done by induction on $|V| = n$.

For $n = 1$, we obtain $R_k(G) = \mathcal{A}_k(G) = 1$ for every $k \in \mathbb{N}$. Let $G$ be a graph on $n+1$ vertices with degree sequence $\pi = (d_1 \geq d_2 \geq \ldots \geq d_{n+1})$. If the maximum degree $d_1$ is at most $k-1$, the *k-MAX* algorithm computes $\mathcal{A}_k(G) = n + 1 \geq R_k(G)$ since the $k$-residue does not exceed the length of $\pi$. From now on we assume $d_1 \geq k$ and let $\sigma := \mathcal{H}(\pi)$ be the partition after the first Havel-Hakimi reduction step. Then $R_k(G) = R_k(\pi) = R_k(\sigma)$ by definition. Further, we select a vertex $v \in V$ with $d(v) = d_1$ and consider the graph $G' := G \setminus \{v\}$ with $n$ vertices. Suppose $\rho$ is the degree sequence of $G'$, then $\rho \succeq \sigma$, and by Lemma 4.17 it follows

$$R_k(G) = R_k(\sigma) \leq R_k(\rho) = R_k(G').$$

Using the induction hypothesis, we obtain $R_k(G') \leq \mathcal{A}_k(G')$ and finally,

$$R_k(G) \leq R_k(G') \leq \mathcal{A}_k(G') = \mathcal{A}_k(G).$$

■

Next, we will show that the $k$-residue of any complete graph is easy to compute by an explicit formula.

**Proposition 4.19**
*Let $K_n$ be a complete graph on n vertices and $1 \leq k \leq n$. Then the k-residue yields*

$$R_k(K_n) = \frac{k+1}{2}.$$

**Proof:**
Let $\pi = ((n-1)^n)$ be the degree sequence of $K_n$. Since The Havel-Hakimi reduction steps yields

$$\mathcal{H}^i(\pi) = ((n-1-i)^{n-i}) \quad \text{for} \quad 1 \leq i \leq n-1,$$

the extended elimination sequence has the form

$$\overline{E}(\pi) = (n-1, n-2, \ldots, 1, 0).$$

Each term only occurs once and we obtain

$$\begin{aligned} R_k(\pi) &= \frac{1}{k}\sum_{i=0}^{k-1}(k-i)\cdot g_i(\pi) \\ &= \frac{1}{k}\sum_{i=0}^{k-1}\cdot(k-i) \\ &= \frac{1}{k}(k+(k-1)+\ldots+2+1) \\ &= \frac{1}{k}\left(\frac{k(k+1)}{2}\right) \\ &= \frac{k+1}{2}. \end{aligned}$$

■

We have seen that the classical residue can easily be computed for regular graphs. Unfortunately, there exists no explicit formula for the $k$-residue. With the help of the following fact, there is at least an estimation.

**Lemma 4.20** (*Jelen 1999, [28]*)
*Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical partition with extended elimination sequence $\overline{E}(\pi)$, and $g_i(\pi)$ is the frequency of value i in $\overline{E}(\pi)$. Then*

$$g_i(\pi) \leq g_0(\pi) = R(\pi) \quad \textit{for} \quad 1 \leq i \leq d_1.$$

**Corollary 4.21**
*For every graphical partition $\pi$ and every positive integer k,*

$$R_k(\pi) \leq \frac{k+1}{2} \cdot R(\pi).$$

**Proof:**

$$\begin{aligned} R_k(\pi) &= \frac{1}{k}\sum_{i=0}^{k-1}(k-i)\cdot g_i(\pi) \\ &\leq \frac{1}{k}\sum_{i=0}^{k-1}(k-i)\cdot g_0(\pi) \\ &= \frac{1}{k}(k+(k-1)+\ldots+2+1)\cdot R(\pi) \\ &= \frac{k+1}{2}\cdot R(\pi). \end{aligned}$$

■

For $d$-regular graphs of order $n$ the residue yields $R(\pi) = \lceil \frac{n}{d+1} \rceil$ and we conclude:

**Corollary 4.22**
*Suppose G is a d-regular graph and $k \leq d$. Then*

$$R_k(G) \leq \frac{k+1}{2}\left\lceil \frac{n}{d+1} \right\rceil.$$

The following result provides information regarding the quality of the Caro-Tuza bound.

**Theorem 4.23** (*Jelen 1999, [28]*)
*For every graphical partition $\pi$ and every positive integer $k$,*

$$CT_k(\pi) \leq R_k(\pi).$$

The $k$-residue always improves the Caro-Tuza bound. However, a comparison with other lower bounds from Chapter 4.2 is difficult. There are graphical partitions in which the $k$-residue and the bound of Caro-Hansberg or Hopkins-Staton improve each other respectively.

**Comparison of $R_k$ and $HS_k$**

The star graph $\pi_n = (n, 1^n)$ yields $R_k(\pi_n) = n$ for $k \leq n$ and

$$HS_k(\pi_n) = \frac{n+1}{1+\lfloor \frac{n}{k} \rfloor} \leq \frac{n+1}{2}.$$

It follows

$$R_k(\pi_n) - HS_k(\pi_n) \geq \frac{n}{2} - 1 = \mathcal{O}(n).$$

The partition $\sigma_n = ((2k-1)^{2kn})$ for $k, n \in \mathbb{N}$ shows that the $k$-residue can also be arbitrarily weak. Since $\sigma_n$ is a degree sequence of a $(2k-1)$-regular graph, we use Corollary 4.22 and obtain

$$R_k(\sigma_n) \leq \frac{k+1}{2}\left\lceil \frac{2kn}{2k} \right\rceil = \frac{(k+1)n}{2}.$$

The Hopkins-Staton bound yields

$$HS_k(\sigma_n) = \frac{2kn}{1+\lfloor\frac{2k-1}{k}\rfloor} = \frac{2kn}{1+1} = kn.$$

The difference between both values is

$$HS_k(\sigma_n) - R_k(\sigma_n) \geq kn - \frac{(k+1)n}{2} = \frac{n}{2}(k-1) = \mathcal{O}(n).$$

**Comparison of $R_k$ and $CH_k$**

We consider again the star graph $\pi_n = (n,1^n)$ with $R_k(\pi_n) = n$ for $k \leq n$ and

$$CH_k(\pi_n) = \frac{k(n+1)}{k+2}.$$

In case of $k = 2$ we obtain

$$CH_2(\pi_n) = \frac{n+1}{2}$$

and

$$R_2(\pi_n) - CH_2(\pi_n) > n - \frac{n}{2} - 1 = \frac{n}{2} - 1 = \mathcal{O}(n).$$

Otherwise, the partition $\tau_k = ((2k)^{2k+1})$ for every positive integer $k$ has the $k$-residue $R_k(\tau_k) = \frac{k+1}{2}$, and the Caro-Hansberg bound yields

$$CH_k(\tau_k) = \frac{k(2k+1)}{k+2k} = \frac{2k+1}{3}.$$

It follows

$$CH_k(\tau_k) - R_k(\tau_k) = \frac{2k+1}{3} - \frac{k+1}{2} = \frac{k-1}{6} = \mathcal{O}(k).$$

The comparison indicates that all considered bounds could be further improved. In 2014, Amos, Davilla and Pepper [2] proved that the $k$-residue of disjoint unions of graphs is at most the sum of the $k$-residues of the graphs considered separately. The result shows that for certain partitions the sum of the residues of its components grows arbitrarily larger than the residue of the union and thus, even improves all known tractable lower bounds on the $k$-independence number.

**Theorem 4.24** (*Amos et al. 2014, [2]*)
*For any disconnected graph $G$ with components $G_1, G_2, \ldots, G_p$ and positive integer $k$,*

$$R_k(G) \leq \sum_{i=1}^{p} R_k(G_i) \leq \sum_{i=1}^{p} \alpha_k(G_i) = \alpha_k(G).$$

# 5 A New Lower Bound on the $k$-Independence Number

The Murphy algorithm in Chapter 2.3 computes a lower bound on the independence number of a graph in the classic sense. A generalization of Murphy's result for $k$-independence has not yet been found. Motivated by this fact, we will present a new lower bound on the $k$-independence number based on Murphy's algorithm. The new bound improves all known bounds for some graphs.

## 5.1 The $M_k$-Bound

Let us recall the greedy algorithm $MIN$ which selects a vertex of minimum degree, deletes that vertex and its neighbors from the graph and repeats this process until the graph is empty. The result is a collection of pairwise independent vertices.

Following the idea, we will study the relation to a heuristic algorithm for constructing large $k$-independent sets in a graph and introduce the greedy algorithm $k$-$MIN$. It repeatedly removes $k$ vertices of minimum degree and all its neighbors until the remaining graph has less than $k$ vertices. Obviously, the $k$ chosen vertices form a $k$-independent set and thus, $k$-$MIN$ computes a collection of disjoint $k$-independent sets. The size of the resulting $k$-independent set in the graph is the sum of this collection plus the remaining vertex set of cardinality less than $k$ and will be denoted by $\mathcal{B}_k$. Of course, $\mathcal{B}_k$ depends on the chosen vertices if there is more than one possibility to choose $k$ vertices of smallest degree.
The $k$-$MIN$ algorithm reads as follows:

| | |
|---|---|
| Algorithm: | $k$-$MIN$ |
| Input: | graph $G=(V,E)$, $k \in \mathbb{N}$ |
| | $j \leftarrow 0$ |
| | while $\lvert V \rvert \geq k$ do |
| | $j \leftarrow j+1$ |
| | $\{v_1, v_2, \ldots, v_k\} \leftarrow$ any $k$ vertices of smallest degree in $G$ |
| | $C_j \leftarrow \{v_1, v_2, \ldots, v_k\} \cup N(v_1) \cup N(v_2) \cup \ldots \cup N(v_k)$ |
| | $G \leftarrow G \setminus C_j$ |
| | endwhile |
| | $r \leftarrow \lvert V \rvert$ |
| | $\mathcal{B}_k(G) \leftarrow j \cdot k + r$ |
| Output: | $k$-independent set in $G$ of size $\mathcal{B}_k(G)$ |

Note that for $k = 1$ the while-condition $|V| \geq 1$ is equivalent to a non-empty graph $G \neq \emptyset$. Further, the parameter $r$ is equal to zero, and the 1-$MIN$ algorithm is precisely the greedy algorithm $MIN$ used by Murphy.

We adapt the $k$-$MIN$ algorithm such that the input is the degree sequence of a graph $G$ exclusively. The algorithm reads as follows:

**Algorithm 4** $M_k$-algorithm

**Input:** $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$, $k \in \mathbb{N}$
**Output:** $M_k(\pi) \geq 1$

```
j = 0
m_0 = 0
while m_j + k − 1 < n do
    i = m_j
    m_{j+1} = m_j + ∑_{l=1}^{k} d_{i+l} + k
    j = j + 1
end while
s = n − m_j
if s < 0 then
    s = 0
end if
m_j + s = n
M_k(π) := j · k + s
```

Following the principle of Murphy's algorithm, mark the first $k$ terms in the sequence $\pi$. If the marked vertices have degree $d_1, d_2, \ldots, d_k$, move $d_1 + d_2 + \ldots + d_k + k$ positions to the right and mark the next $k$ degrees. We continue the process until we move beyond the last term of the sequence $\pi$ or mark the remaining part if it is smaller than $k$, which is denoted by $s$ in the algorithm. The difference

$$m_{j+1} - m_j = \sum_{l=1}^{k} d_{i+l} + k$$

is the step-size of the $(j+1)$-th iteration. The sum of all marked terms is the number $M_k(\pi)$.

**Definition 5.1**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be a partition and $k$ a positive integer. The number $M_k(\pi)$ determined by the Algorithm 4 is called the **$M_k$-bound** of the partition $\pi$. If $\pi$ is the degree sequence of a graph $G$, we write $M_k(G)$.*

**Example:**
We illustrate the procedure for $\pi = (1^3, 2^4, 3, 5^2, 6^3)$. The algorithm computes the number $M_2(\pi) = 6$.

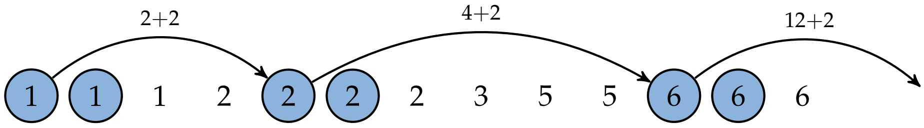



Figure 5.1: The principle of the $M_k$-algorithm for $k = 2$ and $\pi = (1^3, 2^4, 3, 5^2, 6^3)$

The graphical partition $\sigma = (1^9, 2^4, 3)$ yields $M_3(\sigma) = 8$ with the remaining part $s = 2$.

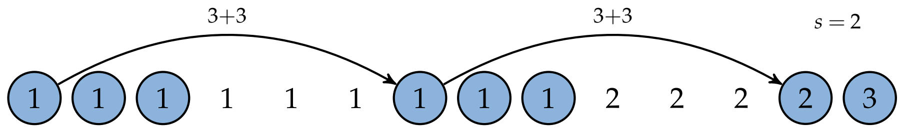



Figure 5.2: $\sigma = (1^9, 2^4, 3)$ with $M_3(\sigma) = 8$

**Theorem 5.2**
*For $k = 1$, the $M_1$-bound is identical with Murphy's bound.*

**Proof:**
For $k = 1$ the algorithm reads as follows:

**Algorithm 5** $M_1$-algorithm

**Input:** $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$
**Output:** $M_1(\pi) \geq 1$
$j = 0$
$m_0 = 0$
**while** $m_j < n$ **do**
  $i = m_j$
  $m_{j+1} = m_j + d_{i+1} + 1$
  $j = j + 1$
**end while**
$s = n - m_j$
**if** $s < 0$ **then**
  $s = 0$
**end if**
$m_j + s = n$
$M_1(\pi) = j \cdot 1 + s$

Since the difference $n - m_j$ is at most zero after the while-loop, we obtain $s = 0$ and $M_1(\pi) = j$. This is precisely Murphy's algorithm and thus, $M_1(\pi) = M(\pi)$. ■

We proceed to our main result of the chapter and to the proof that the $M_k$-algorithm computes a lower bound on the $k$-independence number if $\pi$ is the degree sequence of a graph for every positive integer $k$. Murphy [30] proved his result by showing inductively that the greedy algorithm always produces an independent set of size at least the Murphy's bound. Our proof is modeled on this approach.

The intuitive reasoning behind our statement is that the greedy algorithm *k-MIN* detects if the $k$ chosen vertices have common neighbors, while the $M_k$-algorithm counts the degree of each selected node. The number of nodes removed in each iteration of the greedy algorithm cannot exceed the number of eliminated terms determined by the $M_k$-algorithm. Besides, the degrees of the remaining graph, after a greedy iteration, cannot exceed the initial degrees.

Thus, the greedy heuristic is more effective, and the $M_k$-bound is at most the cardinality of the $k$-independent set produced by $k$-$MIN$.

**Theorem 5.3**
*Let $G$ be a graph with degree sequence $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ and $k$ a positive integer. Then*

$$M_k(G) \leq \alpha_k(G).$$

**Proof:**
First, we assume that $s = 0$. It suffices to prove that if $j \cdot k \leq M_k(G)$, then $k$-$MIN$ computes the sets $C_1, C_2, \ldots, C_j$ such that

$$|C_1 \cup C_2 \cup \ldots \cup C_j| \leq m_j + k - 1. \tag{5.1}$$

If this holds true, we conclude

$$\begin{aligned} |C_1 \cup C_2 \cup \ldots \cup C_j| < n, &\quad \text{if} \quad j \cdot k < M_k(G), \\ |C_1 \cup C_2 \cup \ldots \cup C_j| \leq n, &\quad \text{if} \quad j \cdot k = M_k(G) \end{aligned}$$

and thus, the $M_k$-bound is at most the size of the resulting $k$-independent set computed by the greedy procedure $k$-$MIN$:

$$M_k(G) \leq \mathcal{B}_k(G) \leq \alpha_k(G).$$

This will be done by induction on $j$. The statement is obviously true for $j = 0$. Assume that (5.1) holds for some $j$ with $j \cdot k < M_k(G)$. The assertion $j \cdot k < M_k(G)$ implies that $m_j + k - 1 < n$ and so

$$|C_1 \cup C_2 \cup \ldots \cup C_j| < n.$$

Thus, the while-condition is still true. Let $v_1, v_2, \ldots, v_k$ be $k$ vertices of smallest degree in $G$. Since the $k$-$MIN$ algorithm considers that these chosen vertices may share the same neighbors, the vertex set $C_{j+1}$ satisfies

$$|C_{j+1}| \leq d(v_1) + d(v_2) + \cdots + d(v_k) + k.$$

Since the degree sequence will be reduced at each iteration of $k$-$MIN$ and $d_1 \leq d_2 \leq \ldots \leq d_n$, we conclude

$$\begin{aligned} d(v_1) &\leq d_{m_j+1} \\ d(v_2) &\leq d_{m_j+2} \\ \vdots &\leq \vdots \\ d(v_k) &\leq d_{m_j+k}. \end{aligned}$$

It follows

$$|C_{j+1}| \leq \sum_{l=1}^{k} d_{m_j+l} + k.$$

Finally, we have the following chain of inequalities,

$$\begin{aligned} |C_1 \cup C_2 \cup \ldots \cup C_j \cup C_{j+1}| &\leq |C_1 \cup C_2 \cup \ldots \cup C_j| + |C_{j+1}| \\ &\leq m_j + k - 1 + |C_{j+1}| \\ &\leq m_j + \textstyle\sum_{l=1}^{k} d_{m_j+l} + k + k - 1 \\ &= m_{j+1} + k - 1, \end{aligned}$$

where the induction hypothesis is used for the second inequality and the construction of $m_{j+1}$ implies the last step.

If $s \neq 0$, then, in the last step, the remaining part of the partition has length $s \leq k-1$ and will be added to $M_k(G)$. Since the greedy algorithm $k$-$MIN$ takes into account that the chosen vertices $v_1, v_2, \ldots, v_k$ might have common neighbors, the remaining vertex set $r = |V|$ is at least $s$, the remaining part of the partition computed by the $M_k$-algorithm. Together with the above result, we obtain

$$M_k(G) = j \cdot k + s \leq j \cdot k + r \leq \mathcal{B}_k(G),$$

which completes the proof. ∎

We will now present some properties and investigate the quality of the new bound. The following result shows that the bound is sharp for complete graphs.

**Theorem 5.4**
*Let $K_n$ be a complete graph on $n \in \mathbb{N}$ vertices and $1 \leq k \leq n$. Then*

$$M_k(K_n) = \alpha_k(K_n).$$

**Proof:**
Suppose $\pi = ((n-1)^n)$ is the degree sequence of a complete graph $K_n$. Since $1 \leq k \leq n$, we mark $k$ terms in the first step of the $M_k$-algorithm. Then we move $k(n-1)+k$ positions to the right and leave the partition because of

$$k(n-1) + k = kn \geq n \quad \text{for} \quad k \geq 1.$$

It follows $M_k(K_n) = k$ and, by Remark 4.2, we obtain $\alpha_k(K_n) = k$, which was to be shown. ∎

We have seen that the Murphy bound of a regular graph can be calculated by an explicit formula. The following theorem generalizes this formula.

**Theorem 5.5**
*Let $\pi = (d^n)$ be the degree sequence of a d-regular graph G with $n \geq d+1$ and $m = \left\lceil \frac{n}{k(d+1)} \right\rceil$ for $1 \leq k \leq n$. Then*

$$M_k(G) = m \cdot k - r,$$

*where* $r = \max\{0, mk + (m-1)kd - n\}$.

**Proof:**
Since

$$m \geq \frac{n}{k(d+1)} \quad \text{and} \quad m - 1 < \frac{n}{k(d+1)},$$

by definition of $m$, the length of $\pi$ lies in the range of

$$mk + (m-1)kd - k + 1 \leq n \leq mk + mkd.$$

We distinguish two cases.

*Case 1:* $mk + (m-1)kd \leq n \leq mk + mkd$
Since the graph is regular, the step length does not change in every iteration of the $M_k$-algorithm. We mark $k$ terms of value $d$ and move $kd + k$ positions to the right. The length of $\pi$ has to be at least $mk + (m-1)kd$ and at most $m(k + kd)$ to perform $m$ steps until we move beyond the last term of the partition. This is exactly the range of $n$ in the considered case and

$$r = \max\{0, mk + (m-1)kd - n\} = 0,$$

which leads to $M_k(G) = mk - r$.

*Case 2:* $mk + (m-1)kd - k + 1 \leq n < mk + (m-1)kd$
The length of the partition $\pi$ is not sufficient to perform $m$ steps. So we have to reduce the $M_k$-bound by the difference $mk + (m-1)kd - n$, which is equal to $r$.
Since

$$\begin{aligned} r &= mk + (m-1)kd - n \\ &\leq mk + (m-1)kd - (mk + (m-1)kd - k + 1) \\ &= k - 1, \end{aligned}$$

the difference is at most $k - 1$, and we obtain

$$M_k(G) = mk - r.$$

■

**Corollary 5.6**
*If we set $k = 1$ in Theorem 5.5, then*

$$M_1(G) = \left\lceil \frac{n}{d+1} \right\rceil,$$

*which is precisely the formula to compute Murphy's bound for d-regular graphs of order n.*

**Proof:**
For $k = 1$, we obtain $m = \lceil \frac{n}{d+1} \rceil$ with

$$m(d+1) \geq n \quad \text{and} \quad m + (m-1)d \leq n.$$

This means that

$$m + (m-1)d - n \leq n - n = 0$$

and thus, $r = \max\{0, m + (m-1)d - n\} = 0$. We conclude

$$M_1(G) = \left\lceil \frac{n}{d+1} \right\rceil,$$

which is equal to $M(G)$ in Lemma 2.40. ■

**Theorem 5.7**
*Suppose $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ and $\sigma = (e_1 \leq e_2 \leq \ldots \leq e_n)$ are partitions with $\pi \trianglerighteq \sigma$ and $k$ is a positive integer. Then*

$$M_k(\pi) \leq M_k(\sigma).$$

**Proof:**
We use the notation of the $M_k$-algorithm (see Algorithm 4) and prove the statement by induction on the iterations $j \in \mathbb{N}_0$. If $m_j(\pi) \geq m_j(\sigma)$, the step-size in $\pi$ is at least the step-size in $\sigma$ and hence, $M_k(\pi) \leq M_k(\sigma)$. Since $m_0(\pi) = m_0(\sigma) = 0$, the statement holds for $j = 0$. Assume that the statement holds for some $j$ with $m_j(\sigma) + k - 1 \leq m_j(\pi) + k - 1 < n$, then we can perform at least one further step in the algorithm and obtain

$$\begin{aligned}
m_{j+1}(\sigma) &= m_j(\sigma) + \sum_{l=1}^{k} e_{m_j(\sigma)+l} + k \\
&\leq m_j(\pi) + \sum_{l=1}^{k} e_{m_j(\pi)+l} + k \\
&\leq m_j(\pi) + \sum_{l=1}^{k} d_{m_j(\pi)+l} + k \\
&= m_{j+1}(\pi).
\end{aligned}$$

The induction hypothesis is used for the first inequality and the majorization order $d_i \geq e_i$ for $1 \leq i \leq n$ is used for the second one. ■

## 5.2 Comparison with Known Bounds

We will start with some easy partitions and compare our results with known results on the $k$-independence number. It already shows that the considered bounds are mutually non-comparable.

**Example:**

$$
\begin{array}{lll}
\pi = (2^2,4^4,6^2): & M_2(\pi) = 4 & \lceil F_2(\pi) \rceil = 2 \\
\overline{\pi} = (2^6): & M_2(\overline{\pi}) = 2 & \lceil F_2(\overline{\pi}) \rceil = 3 \\
\\
\pi = (2^2,4^4,6^2): & M_2(\pi) = 4 & HS_2(\pi) = 2 \\
\overline{\pi} = (2^6): & M_2(\overline{\pi}) = 2 & HS_2(\overline{\pi}) = 3 \\
\\
\sigma = (2^2,4,6^5): & M_2(\sigma) = 4 & \lceil CH_2(\sigma) \rceil = 3 \\
\overline{\sigma} = (2^2,3^2,4^2): & M_2(\overline{\sigma}) = 2 & \lceil CH_2(\overline{\sigma}) \rceil = 3 \\
\\
\sigma = (2^2,4,6^5): & M_2(\sigma) = 4 & R_2(\sigma) = 3 \\
\overline{\sigma} = (2^2,3^2,4^2): & M_2(\overline{\sigma}) = 2 & R_2(\overline{\sigma}) = 3
\end{array}
$$

Now we take a closer look and compare our new bound with the bound of Favaron, Hopkins-Staton, Caro-Hansberg and the $k$-residue separately.

**Comparison of $M_k$ and $F_k$**

Let $k$ be a positive integer and

$$\pi_k = (k^k, (k+1)^{k^2+k})$$

a graphical partition. Since the length of $\pi_k$ is $2k+k^2$, we mark the first $k$ terms and move $k+k^2$ positions to the right and mark the last $k$ terms. This leads to $M_k(\pi_k) = 2k$. The Favaron bound can be estimated as follows:

$$F_k(\pi_k) = k\left(\frac{k}{1+k^2}\right) + (k^2+k)\left(\frac{k}{1+(k+1)k}\right) \leq \frac{k^2}{1+k^2} + \frac{k^2+k}{1+k} \leq 1 + \frac{k^2}{1+k} + \frac{k}{1+k} < k+2.$$

It follows

$$M_k(\pi_k) - F_k(\pi_k) > 2k - (k+2) = k-2 = \mathcal{O}(k).$$

Thus, the $M_k$-bound can become arbitrarily larger than Favaron's bound. Let $\sigma_n = (1^{kn})$ be the sequence of a 1-regular graph on $kn$ vertices for $2 \leq k \leq n$. By Theorem 5.5 the $M_k$-bound is at most

$$M_k(\pi_n) \leq \left\lceil \frac{kn}{2} \right\rceil.$$

On the other hand, $\frac{k}{1+k} \geq \frac{2}{3}$ for $k \geq 2$ and we obtain

$$F_k(\pi_n) = kn\left(\frac{k}{1+k}\right) \geq \frac{2}{3}kn.$$

$$F_k(\pi_n) - M_k(\pi_n) \geq \frac{2kn}{3} - \frac{kn}{2} - 1 = \frac{kn}{6} - 1,$$

which grows arbitrarily large as $n$ approaches infinity.

**Comparison of $M_k$ and $HS_k$**

Let $\sigma_n = (1^{kn}, kn)$ be the degree sequence of a star graph on $(kn+1)$ vertices and $n,k \in \mathbb{N}$. The $M_k$-bound yields

$$M_k(\sigma_n) \geq k \left\lceil \frac{kn}{2k} \right\rceil = k \left\lceil \frac{n}{2} \right\rceil .$$

Further, we obtain

$$HS_k(\sigma_n) = \frac{kn+1}{1 + \left\lfloor \frac{kn}{k} \right\rfloor} = \frac{kn+1}{1+n} \leq \frac{k(n+1)}{n+1} = k.$$

This leads to

$$M_k(\sigma_n) - HS_k(\sigma_n) > k(\frac{n}{2} - 1) = \mathcal{O}(n).$$

The comparison shows that the $M_k$-bound performs poorly on regular partitions. Suppose $\tau_k = (k^{(k^2)})$ is a graphical partition for $k \in \mathbb{N}$. To compute the $M_k$-bound, we mark the first $k$ terms and move $k^2 + k$ positions to the right and thus, we move beyond the sequence. So we obtain $M_k(\tau_k) = k$. The Hopkins-Staton bound yields

$$HS_k(\tau_k) = \frac{k^2}{2}.$$

$$HS_k(\tau_k) - M_k(\tau_k) = \frac{k^2}{2} - k = \mathcal{O}(k^2).$$

**Comparison of $M_k$ and $CH_k$**

We use the above partition $\tau_k = (k^{(k^2)})$. The Caro-Hansberg bound yields

$$CH_k(\tau_k) = \frac{k^3}{2k} = \frac{k^2}{2}.$$

Since $M_k(\tau_k) = k$, the bound of Caro and Hansberg can be arbitrarily larger. The following partition shows the reverse effect. Suppose

$$\rho_k = (k^k, (k^2)^{k+k^2})$$

is graphical for $k \geq 2$. Since the length of $\rho_k$ is $k^2 + 2k$, we mark the first $k$ terms and move $k + k^2$ positions to the right and mark the last $k$ terms and thus, $M_k(\rho_k) = 2k$. To compute the average degree, we carry out the following estimation:

$$k^2 + k^2(k + k^2) = k^4 + k^3 + k^2 > k^4 + k^3 - 2k^2 = (k^2 - k)(k^2 + 2k)$$

$$\Rightarrow \quad \overline{d} = \frac{k^2 + k^2(k + k^2)}{k^2 + 2k} > k^2 - k.$$

This leads to

$$CH_k(\rho_k) < \frac{k(k^2 + 2k)}{k + (k^2 - k)} = \frac{k^2(k+2)}{k^2} = k + 2.$$

$$M_k(\rho_k) - CH_k(\rho_k) > 2k - (k+2) = k - 2,$$

which grows arbitrarily large as $k$ tends to infinity.

**Comparison of $M_k$ and $R_k$**

The $k$-residue of a star graph with sequence $\sigma_n = (1^{kn}, kn)$ yields $R_k(\sigma_n) = kn$ for $n, k \in \mathbb{N}$. If we ignore the last term, we can use the formula from Theorem 5.5. For the last term we add 1 and obtain

$$M_k(\sigma_n) \le k \left\lceil \frac{kn}{2k} \right\rceil + 1 < \frac{kn}{2} + 2.$$

Therefore, we conclude

$$R_k(\sigma_n) - M_k(\sigma_n) > kn - \frac{kn}{2} - 2 = \frac{kn}{2} - 2 = \mathcal{O}(n).$$

Otherwise the $k$-residue of a complete graphs $K_n$ is

$$R_k(K_n) = \frac{k+1}{2}$$

for $1 \le k \le n$ by Proposition 4.19. The $M_k$-bound yields $M_k(K_n) = \alpha_k(K_n) = k$ by Theorem 5.4 and thus,

$$M_{\frac{n}{2}}(K_n) - R_{\frac{n}{2}}(K_n) \ge \frac{n}{4} - 1,$$

which grows arbitrarily large as $n$ approaches infinity.

As a consequence, and combined with the fact that the $k$-residue and the $M_k$-bound are always lower bounds on the $k$-independence number of a graph, we conclude:

**Corollary 5.8**
*There are graphical partitions $\pi$ for which the difference between*

$$\min\{\alpha_k(G) \mid G \text{ is a realization of } \pi\}$$

*and the k-residue $R_k(\pi)$ can become arbitrarily large.*

**Corollary 5.9**
*There are graphical partitions $\pi$ for which the difference between*

$$\min\{\alpha_k(G) \mid G \text{ is a realization of } \pi\}$$

*and the $M_k$-bound $M_k(\pi)$ can become arbitrarily large.*

The comparison clearly shows that there exist graphs in which the new $M_k$-bound and the considered known bound improve one another. However, for some graphs the $M_k$-algorithm computes a bound on the $k$-independence number which provides an improvement over all known bounds: $\pi = (4^2, 5^6, 6^4)$ with $k = 2$ yields $M_2(\pi) = 4$ and

$$R_2(\pi) = HS_2(\pi) = CH_2(\pi) = \lceil CT_2(\pi) \rceil = \lceil F_2(\pi) \rceil = 3.$$

Another partition is $\sigma = (3^3, 8^2, 10^4, 11, 12^4)$ with $k = 3$ yielding $M_3(\sigma) = 6$ and

$$R_3(\sigma) = 5, \quad \lceil CH_3(\sigma) \rceil = \lceil CT_3(\sigma) \rceil = 4, \quad HS_3(\sigma) = \lceil F_3(\sigma) \rceil = 3.$$

# 6 An Extremal Problem for Graphs with Prescribed $k$-Independence Number

An extremal problem in graph theory is to determine the size of the largest or smallest configuration with a given property. One of the fundamental results in extremal graph theory is the Theorem of Turán [34] from 1941. The result states that every graph $G$ with independence number $\alpha(G)$ has at least as many edges as some graph consisting of $\alpha(G)$ disjoint cliques. In the following, we will present an extension to it and answer the question: Suppose $G$ has a maximal $k$-independent set with prescribed size. How many edges can $G$ minimally have? Motivated by these results, we will present another new lower bound on the $k$-independence number for graphs which fulfill certain conditions.

## 6.1 Turán's Graph Theorem

Since some graph parameters are closely connected with each other, there exist different formulations of Turán's result. The original statement was the root of the so-called problem of forbidden subgraphs. Turán posed the following question: Suppose $G$ does not contain a clique of order $r+1$, that is $\omega(G) \leq r$ . How many edges can $G$ maximally have? The equivalent way to consider the question is: What is the minimum size of a graph $\overline{G}$ with independence number $\alpha(\overline{G}) \leq r$? Turán's theorem was rediscovered many times, and there exist many different proves. We will present the original proof and a variation by Erdős [12], who involves the degree sequence of $G$. Before proceeding, we will introduce some necessary tools.

**Definition 6.1**
*A graph $G = (V, E)$ is called* ***r-partite*** *if $V = V_1 \cup V_2 \cup \ldots \cup V_r$ is a partition and $V_1, V_2, \ldots, V_r$ are independent sets. If, additionally, every vertex in $V_i$ for $1 \leq i \leq r$ is connected to every vertex in $V_j$, $j \neq i$ by an edge, then $G$ is* ***complete r-partite****.*

Obviously, a $r$-partite graph does not contain a clique of order $r+1$. Thus, all complete $r$-partite graphs are edge maximal without containing a complete graph $K_{r+1}$. But which among these have the greatest number of edges? Suppose $G$ is a complete $r$-partite graph with partite sets differing by more than 1 in size. We move a vertex from the largest set $V_i$ to the smallest $V_j$. Through this, we gain $|V_i| - 1$ edges and lose $|V_j|$ edges. Since $|V_i| - |V_j| > 1$, the number of edges increases. Hence, we maximize the edges of $G$ if the partite sets are as close as possible. This leads to the following definition.

**Definition 6.2**
*The* ***Turán graph*** *$T_{n,r}$ is a complete $r$-partite graph formed by partitioning $n$ vertices into $r$ partite sets with sizes differing by at most 1.*

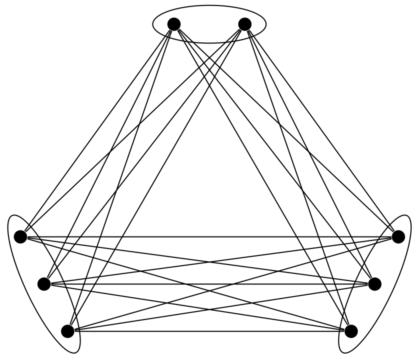


Figure 6.1: Turán graph $T_{8,3}$

Among all $r$-partite graphs on $n$ vertices, the Turán graph $T_{n,r}$ is the only graph with a maximum number of edges. If, in particular, $r$ divides the number of vertices $n$, then we may choose $|V_i| = \frac{n}{r}$ for all $1 \le i \le r$, and every vertex has exactly $(n - \frac{n}{r})$ neighbors. Since $|E| = \frac{1}{2}\sum_{i=1}^{n} d_i$, we obtain

$$|E| = \frac{n}{2}\left(n - \frac{n}{r}\right) = \frac{n^2}{2}\left(1 - \frac{1}{r}\right) = \frac{n^2}{2}\left(\frac{r-1}{r}\right).$$

Turán claims that this size is the greatest number of edges of any graph $G$ on $n$ vertices without a clique of order $r+1$.

**Theorem 6.3** (*Turán 1941, [34]*)
*Let $G = (V, E)$ be a graph on $n$ vertices and $\omega(G) \le r$ for $n, r \in \mathbb{N}$. Then*

$$|E| \le \left(\frac{r-1}{r}\right)\frac{n^2}{2}.$$

**Proof:**
We use induction on $n$. For $n \le r$, the condition $\omega(G) \le r$ has no effect on $G$, and the edge number is at most $\binom{n}{2}$. This yields

$$|E| \le \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \le \frac{n^2}{2} - \left(\frac{n}{2}\right)\left(\frac{n}{r}\right) = \frac{n^2}{2} - \frac{n^2}{2r} = \left(\frac{r-1}{r}\right)\frac{n^2}{2}.$$

Assume now $n \ge r+1$ and $G$ does not contain a clique of order $r+1$. $G$ certainly contains a complete graph $K_r$, otherwise we add edges. Let $A \subset V$ be a clique with $|A| = r$, and set $B = V \setminus A$ with $|B| = n - r$. Since $A$ is a clique of order $r$, A contains $|E_A| = \binom{r}{2}$ edges. We now estimate the edge numbers $|E_B|$ and $|E_{A,B}|$ between $A$ and $B$ (see Figure 6.2). The graph induced by $B$ does not contain a clique of order $r+1$ and, by the induction hypothesis, we have

$$|E_B| \le \left(\frac{r-1}{r}\right)\frac{(n-r)^2}{2}.$$

Since $\omega(G) \le r$, every vertex in $B$ is adjacent to at most $(r-1)$ vertices in $A$ and thus, $|E_{A,B}| \le (r-1)(n-r)$.

Altogether, we obtain

$$\begin{aligned}
|E| &\leq |E_A| + |E_B| + |E_{A,B}| = \binom{r}{2} + \left(\frac{r-1}{r}\right)\frac{(n-r)^2}{2} + (r-1)(n-r) \\
&= \left(\frac{r-1}{r}\right)\left(\frac{n^2-2rn+r^2}{2}\right) + \frac{r(r-1)}{2} + nr - n - r^2 + r \\
&= \left(\frac{r-1}{r}\right)\frac{n^2}{2} - n(r-1) + \frac{r(r-1)}{2} + \frac{r(r-1)}{2} + nr - n - r^2 + r \\
&= \left(\frac{r-1}{r}\right)\frac{n^2}{2} \underbrace{-nr + n + r^2 - r + nr - n - r^2 + r}_{=0} \\
&= \left(\frac{r-1}{r}\right)\frac{n^2}{2}.
\end{aligned}$$

■

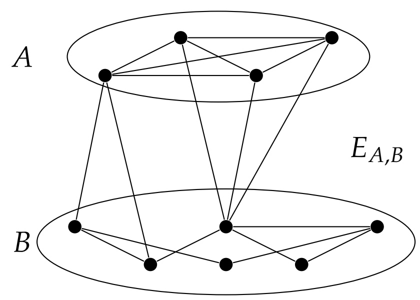



Figure 6.2: Sketch of Turán's proof

In 1970, Erdős proved a powerful result about all graphs containing no clique of order $r+1$, which states that the degree sequence of a $K_{r+1}$-free graph on $n$ vertices is majorized by the degree sequence of $T_{n,r}$. The proof makes use of the structure of the Turán graphs and implies Turán's theorem. Accordingly, we will present a variation of Erdős' result, which can also be found in Bauer et al. [4]. This variation considers the alternative formulation of Turán's theorem: What is the minimum size of a graph on $n$ vertices with independence number of at most $r$?

Due to the complementary formulation of Turán's result, we need some necessary tools.

**Lemma 6.4**
*A graph $G = (V, E)$ on $n$ vertices has a clique of order $r$ if and only if $\overline{G}$ has an independent set of order $r$, where $\overline{G}$ is the complement of $G$.*

**Proof:**
By definition, two distinct vertices of $G$ are adjacent if and only if they are not adjacent in $\overline{G}$. If $G$ contains a $K_r$, then these $r$ vertices build an empty graph in $\overline{G}$. Thus, $\overline{G}$ contains an independent set of size $r$. Similarly, if we start with an independent set in $\overline{G}$, there is a corresponding clique in $G$. ■

**Corollary 6.5**
*The complement of the Turán graph $T_{n,r}$ is a graph consisting of $r$ disjoint cliques and maximum independent set of size $r$.*

**Example:**

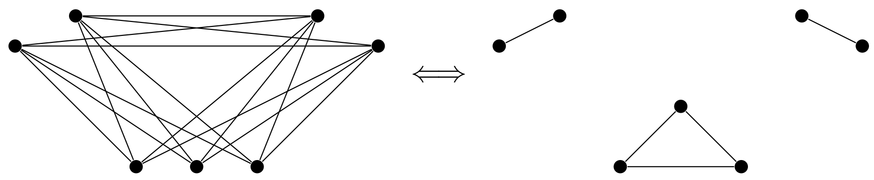


Figure 6.3: Turán graph $T_{7,3}$ and its complement consisting of 3 cliques

Among all graphs on $n$ vertices with $r$ disjoint cliques, the complement of $T_{n,r}$ has minimum size. In particular, if $r$ divides $n$, then we may choose $r$ cliques of order $\frac{n}{r}$, obtaining

$$|E| = r\binom{\frac{n}{r}}{2} = \frac{r}{2}\left(\frac{n}{r}\right)\left(\frac{n}{r}-1\right) = \frac{n^2}{2r} - \frac{n}{2}.$$

The result of Erdős reads as follows, regardless of the number of edges:

**Theorem 6.6** (*Erdős 1970, [12]*)
*Let $G$ be a graph with vertex set $V$ and $\alpha(G) \leq r$. Then $G$ majorizes a graph $H$ with vertex set $V$ consisting of $r$ disjoint cliques that is for every vertex $v \in V$, we have*

$$d_G(v) \geq d_H(v).$$

**Proof:**
We use induction on $r$. For $r = 1$ there is nothing to prove, since $G$ is a complete graph $K_n$ consisting of 1 clique. Thus, we set $H = G$. Assume now $r \geq 2$ and the assertion holds for smaller values of $r$.

Choose a vertex $v$ of minimum degree $\delta$ in $G$ and set

$$G' = G \setminus \{v \cup N_G(v)\}.$$

This yields $\alpha(G') \leq r - 1$. Otherwise $G'$ has an independent set $I(G')$ with $|I(G')| = r$. Then $\{I(G') \cup v\}$ is an independent set in $G$, which is a contradiction to $\alpha(G) \leq r$. We must have $\alpha(G') \leq r - 1$. By the induction hypothesis, the degree sequence of $G'$ majorizes the degree sequence of a graph $H'$ consisting of $r - 1$ disjoint cliques.

Suppose $K_{\{v \cup N_G(v)\}}$ is the complete graph on $\{v \cup N_G(v)\}$. Since every vertex in $K_{\{v \cup N_G(v)\}}$ has degree $\delta$, the graph $G$ majorizes the graph $G' \cup K_{\{v \cup N_G(v)\}}$. Finally, we set

$$H = H' \cup K_{\{v \cup N_G(v)\}}$$

on the vertex set $V$. Altogether, the graph $G$ majorizes $H$, which is a graph consisting of $r$ disjoint cliques. ∎

Since $d_G(v) \geq d_H(v)$ leads to

$$|E_G| = \frac{1}{2}\sum_{v\in V} d_G(v) \geq \frac{1}{2}\sum_{v\in V} d_H(v) = |E_H|,$$

Erdős' result implies Turán's theorem.

**Corollary 6.7**
*The complement of the Turán graph $T_{n,r}$ has the minimum size of any graphs $G$ on $n$ vertices with independence number at most $r$.*

**Corollary 6.8**
*The Turán graph $T_{n,r}$ has the maximum size of any graphs $G$ on $n$ vertices with clique number at most $r$.*

## 6.2 The Minimum Size of a Graph with given $k$-Independence Number

Following the idea of Turán, we will study the relation between the size of a graph and its bounded $k$-independence number. So we pose the following question: Suppose $G$ is a graph on $n$ vertices with $\alpha_k(G) \leq r$. How many edges can $G$ minimally have?

It makes sense here to consider Turán graphs and their complements and modify them in a manner reasonable for $k$-independence. Let $G' = (V, E)$ be a graph on $n$ vertices consisting of $r$ cliques $V_1, V_2, \ldots, V_r$ such that

$$V = V_1 \cup V_2 \cup \ldots \cup V_r \quad \text{and} \quad |V_1| + |V_2| \cup \ldots \cup |V_r| = n.$$

In contrast to the complement of a Turán graph, the $r$ cliques are not disjoint. Additionally, each vertex is adjacent to $k-1$ vertices outside its clique, such that for all $v \in V_i$, $1 \leq i \leq r$:

$$|N(v) \setminus V_i| = k - 1.$$

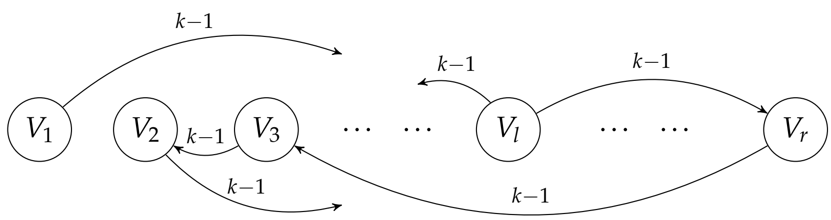



Figure 6.4: Graph consisting of $r$ cliques and additional $k-1$ neighbors for every vertex

Now we pick one vertex $v$ from every clique. Since each of them has $k-1$ neighbors outside the clique, these $r$ vertices build a $k$-independent set. Note that $k \leq r$ since every $k$-independent set has at least $k$ vertices. If $r$ divides $n$, we can compute the minimum size of the constructed graph:

$$\begin{aligned}
|E| &= r\binom{\frac{n}{r}}{2} + \frac{n}{2}(k-1) \\
&= \frac{r}{2}\left(\frac{n}{r}\right)\left(\frac{n}{r}-1\right) + \frac{n}{2}(k-1) \\
&= \frac{n^2}{2r} - \frac{n}{2} + \frac{n}{2}(k-1) \\
&= \frac{n^2}{2r} + \frac{n}{2}(k-2).
\end{aligned}$$

Based on the above considerations, we will present a lower bound on the edge size for any arbitrary graph with average degree at least $k \in \mathbb{N}$.

**Theorem 6.9**
*Let $G = (V,E)$ be a graph on $n$ vertices and $\alpha_k(G) \le r$ for $n, r \in \mathbb{N}$. Further, the average degree $\overline{d}_G \ge k$ for $k \ge 2$. Then*

$$|E| \ge \frac{n^2}{2r} + \frac{n}{2}(k-2).$$

**Proof:**
For $k = 1$, we have $\alpha(G) \le r$ and

$$|E| \ge \frac{n^2}{2r} - \frac{n}{2} = r\binom{\frac{n}{r}}{2}.$$

The graph $G$ has at least as many edges as some graph consisting of $r$ disjoint cliques. This is precisely Turán's theorem.

Suppose $k \ge 2$ and $\overline{d}_G = \overline{d} \ge k$. We distinguish between two cases:

*Case 1*: $n \le 2r$. Since $2|E| = n\overline{d}$, we obtain

$$\begin{aligned}
|E| &= \frac{n}{2}\overline{d} \\
&\ge \frac{n}{2}k \\
&= n + \frac{n}{2}(k-2) \\
&\overset{n\le 2r}{\ge} \frac{n}{2r}n + \frac{n}{2}(k-2) \\
&= \frac{n^2}{2r} + \frac{n}{2}(k-2).
\end{aligned}$$

*Case 2*: $n > 2r$. Since $\overline{d} \ge k$, the graph $G$ fulfills the assumption of Corollary 4.9:

$$\frac{k+1}{2(\overline{d}+1)}n \le \alpha_k(G).$$

The $k$-independence number of $G$ is bounded by $\alpha_k(G) \le r$. This leads to

$$\begin{aligned} \frac{k+1}{2(\overline{d}+1)}n &\leq r \\ \Leftrightarrow \quad \frac{n}{2}(k+1) &\leq r(\overline{d}+1) \\ \Leftrightarrow \quad \frac{n}{2r}(k+1)-1 &\leq \overline{d}. \end{aligned}$$

Now we can estimate the size of $G$:

$$\begin{aligned} |E| &= \frac{n}{2}\overline{d} \\ &\geq \frac{n}{2}\left(\frac{n}{2r}(k+1)-1\right) \\ &= \frac{n^2}{4r}(k+1)-\frac{n}{2} \\ &= \frac{n^2}{4r}k+\frac{n^2}{4r}-\frac{n}{2}+\left[\frac{n^2}{4r}-\frac{n^2}{4r}+\frac{n}{2}k-\frac{n}{2}k+\frac{n}{2}-\frac{n}{2}\right] \\ &= \frac{n^2}{2r}+\frac{n}{2}k-n+\frac{n^2}{4r}k-\frac{n^2}{4r}-\frac{n}{2}k+\frac{n}{2} \\ &= \frac{n^2}{2r}+\frac{n}{2}(k-2)+\frac{n^2}{4r}(k-1)-\frac{n}{2}(k-1) \\ &\overset{n>2r}{>} \frac{n^2}{2r}+\frac{n}{2}(k-2)+\frac{2rn}{4r}(k-1)-\frac{n}{2}(k-1) \\ &= \frac{n^2}{2r}+\frac{n}{2}(k-2)+\underbrace{\frac{n}{2}(k-1)-\frac{n}{2}(k-1)}_{=0} \\ &= \frac{n^2}{2r}+\frac{n}{2}(k-2), \end{aligned}$$

and the statement follows. ∎

**Remark 6.10**
The additional condition for the average degree, $\overline{d}_G \geq k$ for $k \geq 2$, is essential. Otherwise, the statement is false as the following example indicates:

Let $G=(V,E)$ be a star graph with $|V|=4$, $|E|=3$ and degree sequence $\pi=(1^3,3)$. If we choose $k=3$, this yields $\alpha_3(G)=3$. The average degree is $\overline{d}_G=\frac{3}{2}$, which does not satisfy the condition $d_G \geq 3$. Nevertheless, we apply Theorem 6.9 and obtain

$$|E| \geq \frac{n^2}{2r}+\frac{n}{2}(k-2)=\frac{16}{6}+\frac{4}{2}=\frac{14}{3}>4,$$

which is a contradiction to $|E|=3$.

Now we go one step further and pose the following question similar to Erdős: Suppose $G$ is a graph with $\alpha_k(G) \leq r$. Further, $G'$ is a graph on the same vertex set consisting of $r$ cliques such that each vertex has at least $k-1$ neighbors outside its clique. Does $G$ majorizes $G'$?

**Example:**
Let $G = (V,E)$ be a graph with $\pi(G) = (2,2,3,4,4,5)$ and $\alpha_2(G) \leq 3$. Then $G$ majorizes a graph $G'$ on $V$ consisting of $r = 3$ cliques and each $v \in V$ has additionally $k-1 = 1$ neighbor. The degree sequence of $G'$ is $\pi(G') = (2^6)$ and thus, $G \trianglerighteq G'$.

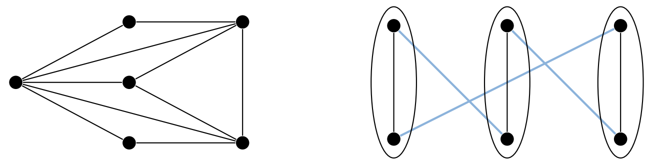


Figure 6.5: Graph $G$ (left) majorizes the extremal graph $G'$ (right)

**Remark 6.11**
In some cases the vertices in the extremal graph $G'$ cannot have exactly $(k-1)$ neighbors outside their clique as the following example indicates: Suppose $n$ and $k-1$ are both odd numbers, then $\frac{n}{2}(k-1)$ is not an integer. There must be vertices with more than $k-1$ neighbors.

Our conjecture is that any graph $G$ with $\alpha_k(G) \leq r$ majorizes a corresponding graph $G'$, where $G'$ consists of $r$ cliques, and each vertex has at least $k-1$ neighbors outside its clique:

**Conjecture 6.12**
*Let $G = (V,E)$ be a graph with $\overline{d}_G \geq k$ for $k \geq 2$ and $\alpha_k(G) \leq r$. Then $G$ majorizes a graph $G'$ on vertex set $V$ consisting of $r$ cliques with $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$, it holds*

$$|N(v) \setminus V_i| \geq k-1. \tag{6.1}$$

It is evident that the exact results for $k = 1$ is compatible with our conjecture. Now we will present a proof of our conjecture for regular graphs. However, we have not succeeded to prove the assumption in general.

**Theorem 6.13**
*Let $G = (V,E)$ be a $d$-regular graph with $d \geq k$ and $\alpha_k(G) \leq r$. Further, $r$ divides $|V| = n$. Then $G$ majorizes a graph $G'$ on the vertex set $V$ consisting of $r$ cliques with $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$, it holds*

$$|N(v) \setminus V_i| = k-1.$$

**Proof:**
Since $\alpha_k(G) \leq r$ implies $\alpha(G) \leq r$, the graph $G$ majorizes a graph $H'$, by Turán's theorem, consisting of $r$ disjoint cliques $V_1, V_2, \ldots, V_r$ with $|V_i| = \frac{n}{r}$ for all $1 \leq i \leq r$. Now we choose a vertex $v_1 \in V_1$ and connect $v_1$ to some vertices $v_2 \in V_2, \ldots, v_k \in V_k$. Since $k \leq r$ and $r$ divides $n$, this is realizable and $v_1$ has precisely $k-1$ neighbors outside $V_1$. We can repeat this process until we finally receive a graph $G'$ on the vertex set $V$ consisting of $r$ cliques $V_1, V_2, \ldots, V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$:

$$|N(v) \setminus V_i| = k - 1 \quad \text{and} \quad d_{G'}(v) = \frac{n}{r} + k - 2.$$

Since $\overline{d}_G = d \geq k$, the graph $G$ fulfills the assumption of Corollary 4.9:

$$\frac{k+1}{2(d+1)} n \leq \alpha_k(G) \leq r.$$

Rearranging the inequality leads to

$$\frac{2(d+1)}{k+1} \geq \frac{n}{r}.$$

It follows that

$$d_{G'}(v) = \frac{n}{r} + k - 2 \leq \frac{2(d+1)}{k+1} + k - 2.$$

Now we show that $d \geq \frac{2(d+1)}{k+1} + k - 2$:

$$\begin{array}{rlcl} & d & \geq & \frac{2(d+1)}{k+1} + k - 2 \\ \Leftrightarrow & d(k+1) & \geq & 2(d+1) + (k-2)(k+1) \\ \Leftrightarrow & dk + d & \geq & 2d + 2 + k^2 - k - 2 \\ \Leftrightarrow & dk - d & \geq & k^2 - k \\ \Leftrightarrow & d(k-1) & \geq & k(k-1) \\ \Leftrightarrow & d & \geq & k, \end{array}$$

which is obviously true. Altogether, we obtain

$$d_{G'}(v) = \frac{n}{r} + k - 2 \leq \frac{2(d+1)}{k+1} + k - 2 \leq d = d_G(v),$$

and the statement follows. ■

**Corollary 6.14**
*For every graph $G = (V, E)$ with $\alpha_k(G) \leq r$, $\delta(G) \geq \frac{n}{r} + k - 2$ and $r$ divides $n$, the Conjecture 6.12 is true.*

**Proof:**
Since $d_{G'}(v) = \frac{n}{r} + k - 2 \leq \delta(G)$ for every $v \in V$, it follows $d_{G'}(v) \leq d_G(v)$. ■

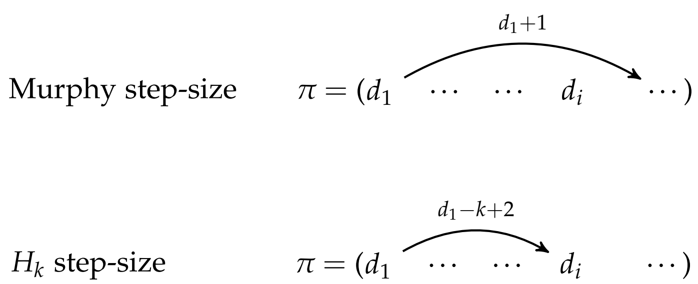
## 6.3 The $H_k$-Bound - Another New Lower Bound

We will present a new lower bound on the $k$-independence number for graphs which satisfy Conjecture 6.12. Again, the basis for the procedure is provided by Murphy's algorithm:

**Algorithm 6** $H_k$-Algorithm

**Input:** partition $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n),\ d_1 \geq k,\ k \in \mathbb{N}$
**Output:** $H_k(\pi) \geq 1$
  $j = 0$
  $h_0 = 0$
  **while** $h_j < n$ **do**
    $i = h_j$
    $h_{j+1} = h_j + d_{i+1} - k + 2$
    $j = j + 1$
  **end while**
  $h_j = n$
  $H_k(\pi) = j$

The idea behind the algorithm is to adapt the step length as a function of $k \in \mathbb{N}$, and, in contrast to Murphy's algorithm, the minimum degree must be at least $k$ to avoid the problem of negative step-size values. Then the procedure is similar: Mark the first term in $\pi$. If the marked vertex has degree $d_1 \geq k$, move $d_1 - k + 2$ positions to the right and mark the next degree. This process continues until we move beyond the last term of the sequence $\pi$. The sum of all marked degrees is $H_k(\pi)$. We state that the number of marked degrees builds a lower bound on the $k$-independence number of a graph with degree sequence $\pi$. As we will see, the additional assumption $d_i \geq k$ for all $1 \leq i \leq n$ is of particular importance.




**Definition 6.15**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be a partition and k a positive integer. The number $H_k(\pi)$ determined by the Algorithm 6 is called the **$\mathbf{H_k}$-bound** of the partition $\pi$. If $\pi$ is the degree sequence of a graph G, we write $H_k(G)$.*

**Example:**
Let $G$ be a graph with degree sequence $\pi = (3^4, 4^2, 6^2)$. We obtain

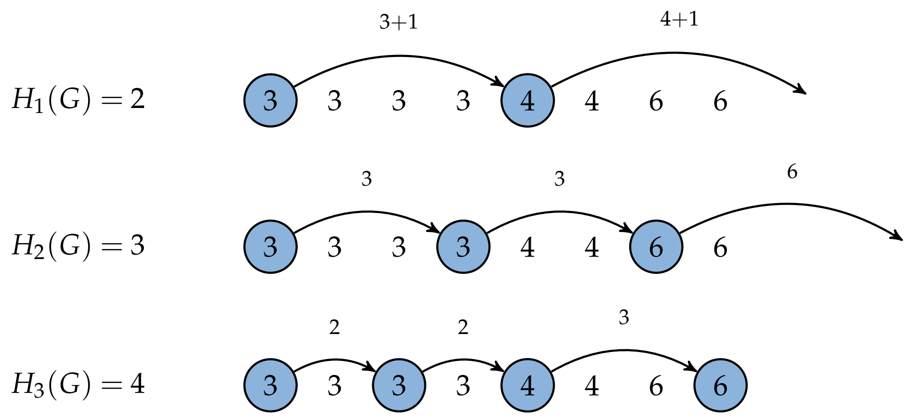



Figure 6.6: The principle of the $H_k$-algorithm

Before we proceed to the proof, that under certain assumptions the $H_k$-algorithm computes a lower bound on the $k$-independence number, we summarize important properties of the new algorithm.

**Lemma 6.16**
*For every graph $G$ with $\delta(G) \geq 1$*

$$H_1(G) = M(G),$$

*the $H_1$-bound is compatible with Murphy's bound.*

**Proof:**
Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be the degree sequence of $G$. If we set $k = 1$, then $d_1 \geq 1$ and the algorithm reads as follows:

**Algorithm 7** $H_1$-Algorithm

**Input:** partition $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$
**Output:** $H_1(\pi) \geq 1$
```
j = 0
h_0 = 0
while h_j < n do
    i = h_j
    h_{j+1} = h_j + d_{i+1} + 1
    j = j + 1
end while
H_1(π) = j
```

This is precisely Murphy's algorithm and the statement follows. ■

**Lemma 6.17**
*Suppose $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ and $\sigma = (e_1 \leq e_2 \leq \ldots \leq e_n)$ are partitions with $\pi \trianglerighteq \sigma$. Further, $e_1 \geq k$ for some positive integer $k$. Then*

$$H_k(\pi) \leq H_k(\sigma).$$

**Proof:**
Since $\pi$ majorizes $\sigma$ and $e_1 \geq k$, we have $d_1 \geq k$ and the $H_k$-algorithm is applicable. Similar to Theorem 5.7, we use induction on the iterations $j \in \mathbb{N}_0$ of the algorithm showing that $h_j(\sigma) \leq h_j(\pi)$.

Since $h_0(\pi) = h_0(\sigma) = 0$, the statement holds for $j = 0$. Assume that the statement holds for some $j$ with $h_j(\sigma) \leq h_j(\pi) < n$, then we can perform at least one further step in the algorithm and obtain

$$\begin{aligned} h_{j+1}(\sigma) &= h_j(\sigma) + e_{i+1} - k + 2 \\ &\leq h_j(\pi) + e_{i+1} - k + 2 \\ &\leq h_j(\pi) + d_{i+1} - k + 2 \\ &= h_{j+1}(\pi). \end{aligned}$$

Thus, the step-size in $\pi$ is at least the step-size in $\sigma$, and we conclude $H_k(\pi) \leq H_k(\sigma)$. ■

**Lemma 6.18**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be a partition with $d_1 \geq l$ for some positive integer $l$. If $k \leq l$, then*

$$H_k(\pi) \leq H_l(\pi).$$

**Proof:**
It suffices to show that $h_j^k \geq h_j^l$, where $h_j^k$ (resp. $h_j^l$) is the $j$-th step in the $H_k$-algorithm to compute $H_k(\pi)$ (resp. $H_l(\pi)$). This will be done by induction on $j$. The base case is trivial since $h_0^k = h_0^l = 0$. If the statement is true for $j < H_k(\pi)$, then $n > h_j^k \geq h_j^l$, and at least one step of the algorithm is executed. It follows

$$\begin{aligned} h_{j+1}^k &= h_j^k + d_{i+1} - k + 2 \\ &\geq h_j^l + d_{i+1} - k + 2 \\ &\geq h_j^l + d_{i+1} - l + 2 \\ &= h_{j+1}^l, \end{aligned}$$

which completes the proof. ■

The next result shows that the $H_k$-bound for regular graphs can be calculated by an explicit formula.

**Theorem 6.19**
*Let $G$ be a $d$-regular graph on $n \in \mathbb{N}$ vertices. If $n > d \geq k$, then*

$$H_k(G) = \left\lceil \frac{n}{d - k + 2} \right\rceil.$$

**Proof:**
Let $r = \lceil \frac{n}{d-k+2} \rceil$, then

$$r - 1 < \frac{n}{d-k+2} \quad \text{and} \quad r \geq \frac{n}{d-k+2}.$$

Since $G$ is $d$-regular, the step-size is $d - k + 2$ in every iteration of the $H_k$-algorithm. While $(r-1)(d-k+2) < n$, at least one further step is executed and since $r(d-k+2) \geq n$, we can perform exactly $r$ iterations until we move beyond the last term of the degree sequence. Thus $H_k(G) = \lceil \frac{n}{d-k+2} \rceil$. ■

**Remark 6.20**
Note that for $k = 1$, the formula in Theorem 6.19 yields

$$H_1(G) = \left\lceil \frac{n}{d+1} \right\rceil,$$

which is precisely Murphy's bound for $d$-regular graphs.

**Corollary 6.21**
*Let $G$ be $d$-regular on $n \in \mathbb{N}$ vertices. If $k = d$, then*

$$H_k(G) \leq \alpha_k(G).$$

**Proof:**
By Theorem 6.19 it immediately follows

$$H_k(G) = \left\lceil \frac{n}{2} \right\rceil,$$

which is equal to the Caro-Hansberg bound. Since this is a lower bound on the $k$-independence number, we obtain

$$H_k(G) = \left\lceil \frac{n}{2} \right\rceil = \lceil CH_k(G) \rceil \leq \alpha_k(G).$$

■

The above corollary indicates that the $H_k$-algorithm computes a lower bound on the $k$-independence number for certain graphs. We will show that, if a graph $G$ satisfies Conjecture 6.12, then $H_k(G) \leq \alpha_k(G)$. The proof is based on the following lemma.

**Lemma 6.22**
*Let $G' = (V, E)$ be the extremal graph of Conjecture 6.12, that is $G'$ consisting of $r \in \mathbb{N}$ cliques with $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$, it holds $|N(v) \setminus V_i| \geq k - 1$. Then*

$$H_k(G') \leq r.$$

**Proof:**
Suppose $G'$ consists of $r$ cliques such that each vertex has exactly $k - 1$ neighbors outside its

clique. Without loss of generality, we assume $|V_1| \leq |V_2| \leq \ldots \leq |V_r|$. The degree sequence of $G'$ reads as follows:

$$\pi := (\overbrace{|V_1|+k-2,\ldots,|V_1|+k-2}^{|V_1|} \leq \overbrace{|V_2|+k-2,\ldots,|V_2|+k-2}^{|V_2|} \leq \ldots$$
$$\leq \ldots \leq \overbrace{|V_r|+k-2,\ldots,|V_r|+k-2}^{|V_r|}).$$

Now we compute the $H_k$-bound for the graph $G$.

$$h_1 = h_0 + d_1 - k + 2 = |V_1| + k - 2 - k + 2 = |V_1|$$

In the first step we move $|V_1|$ positions to the right and thus, we leave the first clique:

$$\pi = (d_1 \overset{h_1=|V_1|}{\curvearrowright} \cdots \quad \cdots \; d_{|V_1|} \quad \cdots)$$

Analogously to that, we obtain

$$\begin{aligned}
h_2 &= h_1 + d_{|V_1|+1} - k + 2 = |V_1| + |V_2| + k - 2 - k + 2 = |V_1| + |V_2| \\
h_3 &= h_2 + d_{|V_1|+|V_2|+1} - k + 2 = |V_1| + |V_2| + |V_3| + k - 2 - k + 2 = |V_1| + |V_2| + |V_3| \\
\vdots &= \vdots \\
h_r &= |V_1| + |V_2| + \ldots + |V_r| = n.
\end{aligned}$$

Since $h_r = n$, the while-loop terminates, and the $H_k$-bound yields $H_k(G') = r$.

Suppose now the extremal graph $G'$ has vertices with more than $k-1$ neighbors outside their cliques. We denote the corresponding degree sequence with $\pi'$. Then $\pi'$ majorizes $\pi$ from above. This leads to $H_k(G') \leq r$ by Lemma 6.17, which completes the proof. ■

**Example:**
Let $G$ be a graph with degree sequence $\pi(G) = (2,2,2,3,3)$ and $\alpha_2(G) \leq 3$. The corresponding extremal graph $G'$ consists of 3 cliques $|V_1| = 1, |V_2| = 2, |V_3| = 2$ such that

$$|N(v) \setminus V_i| = \begin{cases} 2, & \text{for } v \in V_1, \\ 1, & \text{for } v \in V_2, V_3. \end{cases}$$

The graph $G'$ has degree sequence $\pi(G') = (2,2,2,2,2)$ with $G \trianglerighteq G'$ and $H_2(G') = 3$.

The following theorem shows that the $H_k$-algorithm computes a lower bound on the $k$-independence number for some graphs.

**Theorem 6.23**
*Suppose $G = (V, E)$ is a graph with $\delta(G) \geq k$ for $k \geq 2$. If $G$ satisfies Conjecture 6.12, then*

$$H_k(G) \leq \alpha_k(G).$$

**Proof:**
Since $\overline{d}(G) \geq \delta(G) \geq k$, the graph $G$ fulfills the assumptions of Conjecture 6.12. Let $r + 1 = H_k(G)$ and suppose on the contrary that the $H_k$-bound is not a lower bound, that is $\alpha(G) \leq r$.

If $G$ satisfies Conjecture 6.12, then $G$ majorizes a graph $G'$ consisting of $r$ cliques, and each vertex in $G'$ has at least $k - 1$ neighbors outside its clique. Using Lemma 6.22, we obtain $H_k(G') \leq r$ and due to $G \trianglerighteq G'$, we conclude

$$H_k(G) \leq H_k(G') \leq r,$$

a contradiction. ■

## 6.4 Comparison with Known Bounds and the $M_k$-Bound

We will show that all considered bounds in this work and the $H_k$-bound are mutually non-comparable except for one. The $H_k$-bound improves Favaron's bound. Moreover, we will give examples of graphs for which our new bound is an improvement on all known tractable lower bounds on the $k$-independence number.

**Theorem 6.24**
*Let $\pi = (d_1 \leq d_2 \leq \ldots \leq d_n)$ be the degree sequence of an arbitrary graph with $d_1 \geq k$ for a positive integer $k$. Then*

$$F_k(\pi) \leq H_k(\pi).$$

**Proof:**
Since $d_1 \geq k$ Favaron's bound reads as follows:

$$\begin{aligned}
F_k(\pi) = \sum_{i=1}^{n} \frac{k}{1+kd_i} &= \sum_{j=0}^{H_k(\pi)-1} \sum_{i=1+h_j}^{h_{j+1}} \frac{k}{1+kd_i} \\
&\leq \sum_{j=0}^{H_k(\pi)-1} \sum_{i=1+h_j}^{h_{j+1}} \frac{k}{1+kd_{1+h_j}} \\
&= \sum_{j=0}^{H_k(\pi)-1} \frac{k(h_{j+1}-h_j)}{1+kd_{h_j+1}} \\
&= \sum_{j=0}^{H_k(\pi)-1} \frac{k(d_{h_j+1}-k+2)}{1+kd_{h_j+1}} \\
&= \sum_{j=0}^{H_k(\pi)-1} \frac{-k^2+2k+kd_{h_j+1}}{1+kd_{h_j+1}} \\
&\leq \sum_{j=0}^{H_k(\pi)-1} 1 = H_k(\pi),
\end{aligned}$$

where $-k^2 + 2k \leq 1$ for all $k \in \mathbb{N}$ is used for the last inequality. ■

Now we take a closer look at our two new bounds. If not stated otherwise, we consider graphs with a minimum degree of at least $k \in \mathbb{N}$.

**Comparison of $H_k$ and $M_k$**

Let $k$ be a positive integer and $\pi_k = (k^{k^2+k})$. Since we mark the first $k$ terms and move $k^2 + k$ terms to the right, the $M_k$-bound yields $M_k(\pi_k) = k$. Using the explicit formula for regular graphs, the $H_k$-bound yields

$$H_k(\pi_k) = \left\lceil \frac{k^2+k}{2} \right\rceil.$$

Thus, the $H_k$-bound can be arbitrarily larger than the $M_k$-bound. Let

$$\sigma_k = ((2k)^k, (k^2)^{2k^2+k}),\ k \geq 2$$

be graphical. It follows immediately that $M_k(\sigma_k) = 2k$. The $H_k$-bound can be estimated as follows:

$$H_k(\sigma_k) = 1 + \left\lceil \frac{2k^2+k}{k^2-k+2} \right\rceil \leq 1 + \left\lceil \frac{2k^2+k}{k^2-k} \right\rceil \leq 1 + \left\lceil \frac{2k+1}{k-1} \right\rceil \leq 6,$$

for all $k \geq 2$. The $H_k$-bound is weak in this case.

**Comparison of $H_k$ and $HS_k$**

The $H_k$-bound of a $k$-regular graph on $n$ vertices is $\lceil \frac{n}{2} \rceil$ by Theorem 6.19. Thus, for the graphical partition $\tau_n = (k^n, 3k)$, $k < n$, we obtain

$$H_k(\tau_n) \geq \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad HS_k(\tau_n) = \frac{n+1}{1 + \left\lfloor \frac{3k}{k} \right\rfloor} = \frac{n+1}{4}.$$

Hence, the $H_k$-bound offers a significantly better lower bound on the $k$-independence number than Hopkins and Staton. On the other hand the partition $\rho_k = ((2k)^{3k^2})$ for $k \geq 2$ shows that the Hopkins-Staton bound improves the $H_k$-bound. Observe that

$$H_k(\rho_k) = \left\lceil \frac{3k^2}{k+2} \right\rceil \leq 3k \quad \text{and} \quad HS_k(\rho_k) = \frac{3k^2}{1 + \left\lfloor \frac{2k}{k} \right\rfloor} = \frac{3k^2}{3} = k^2.$$

**Comparison of $H_k$ and $CH_k$**

Consider the partition $\rho_k$ from above. The Caro-Hansberg bound yields

$$CH_k(\rho_k) = \frac{k \cdot 3k^2}{k+2k} = k^2.$$

Thus, the bound of Caro-Hansberg can be arbitrarily larger than the $H_k$-bound. Now we compute the bounds for the partition $\phi_k = (k^{100k}, (100k)^k)$. The average degree of $\phi_k$ is

$$\left\lceil \overline{d} \right\rceil = \left\lceil \frac{200k^2}{101k} \right\rceil = \left\lceil \frac{200}{101} \right\rceil k = 2k.$$

It follows

$$CH_k(\phi_k) = \frac{k \cdot 101k}{k + \left\lceil \overline{d} \right\rceil} = \frac{101k^2}{3k} = \frac{101}{3}k \quad \text{and} \quad H_k(\phi_k) \geq \frac{100}{2}k,$$

where the explicit formula for the first part ($k^{100k}$) is used for the estimation for $H_k$. Consequently, the $H_k$-bound is a much better lower bound in this case.

**Comparison of $H_k$ and $R_k$**

The $k$-residue and the $H_k$-bound behave in the same manner as the residue and the Murphy bound. There are graphical partitions in which the $k$-residue and the $H_k$-bound improve one another: $\pi = (4^6, 5)$ yields $R_4(\pi) = \frac{7}{2}$ and $H_4(\pi) = 4$ whereas $\sigma = (4^5, 8^4)$ leads to $R_4(\sigma) = 5$ and $M_4(\sigma) = 4$. The difference can be arbitrarily large as the following partition indicates:

$$\pi_k = (\underbrace{k^{k+2}}_{\text{part a}}, \underbrace{(2k+1)^k}_{\text{part b}})$$

In the $H_k$-algorithm we obtain step length 2 until we leave part a. Then we can perform at most one more iteration until we move beyond the last term of part b. We conclude

$$H_k(\pi_k) \leq \left\lceil \frac{k+2}{2} \right\rceil + 1.$$

To compute the $k$-residue, we use the Havel-Hakimi algorithm.

$$\mathcal{H}(\pi_k) = ((2k)^{k-1}, (k-1)^{k+2})$$

We obtain inductively

$$\mathcal{H}^i(\pi_k) = ((2k+1-i)^{k-i}, (k-i)^{k+2}) \quad \text{for } i = 0, 1, \ldots, k,$$

thus,

$$\mathcal{H}^{k-1}(\pi_k) = (k+2, 1^{k+2}) \quad \text{and} \quad \mathcal{H}^k(\pi_k) = (0^{k+2}).$$

The $k$-residue only depends on the resulting sequence of zeros.

$$R_k(\pi_k) = \frac{k(k+2)}{k} = k+2.$$

It follows

$$R_k(\pi_k) - H_k(\pi_k) \geq k + 2 - \frac{k+2}{2} - 1 = \frac{k}{2} = \mathcal{O}(k).$$

We close the comparison with the simple partition $\pi = (2, 2, 2, 3, 3)$ and $k = 2$. The $H_k$-bound yields $H_2(\pi) = 3$ and

$$R_2(\pi) = \frac{5}{2}, \quad HS_2(\pi) = \frac{5}{2}, \quad M_2(\pi) = CH_2(\pi) = CT_2(\pi) = F_2(\pi) = 2.$$

Thus, for some graphs, the $H_k$-bound means an improvement over all known bounds.

## 6.5 Numerical Evaluation

$\pi = (2,2,2,3,3) \quad n = 5$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ | $\alpha_{k,\min}(\pi)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.5 | 1.5 | 1.25 | 1.25 | 2 | 2 | 2 | 2 |
| 2 | 1.77 | 2.25 | 2.5 | 2 | 2.5 | 2 | 3 | 3 |
| 3 | 1.89 | 3 | 2.5 | 2.5 | 3 | 3 | - | 4 |

$\pi = (5,5,5,5,5,5) \quad n = 6$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ | $\alpha_{k,\min}(\pi)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1.1 | 1.5 | 2 | 1.71 | 1.5 | 2 | 2 | 2 |
| 3 | 1.13 | 2 | 3 | 2.25 | 2 | 3 | 2 | 3 |
| 4 | 1.14 | 2.5 | 3 | 2.67 | 2.5 | 4 | 2 | 4 |
| 5 | 1.15 | 3 | 3 | 4 | 3 | 5 | 3 | 5 |

$\pi = (4,4,5,5,5,5,6,6,8) \quad n = 9$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ | $\alpha_{k,\min}(\pi)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.46 | 1.46 | 1 | 1.29 | 2 | 2 | 2 | 2 |
| 2 | 1.6 | 2.2 | 1.8 | 2.25 | 3 | 2 | 2 | 4 |
| 3 | 1.65 | 2.93 | 3 | 3 | 3.67 | 3 | 3 | 5 |
| 4 | 1.67 | 3.66 | 3 | 3.6 | 4.25 | 4 | 4 | 6 |
| 5 | 1.69 | 4.4 | 4.5 | 4.1 | 4.8 | 5 | - | 7 |
| 6 | 1.7 | 5.1 | 4.5 | 4.5 | 5.33 | 6 | - | 8 |
| 7 | 1.71 | 5.59 | 4.5 | 4.85 | 5.71 | 7 | - | 8 |
| 8 | 1.72 | 6 | 4.5 | 5.14 | 6 | 8 | - | 8 |

$\pi = (4,4,4,4,4,8,8,8,8,8) \quad n = 10$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ | $\alpha_{k,\min}(\pi)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.56 | 1.56 | 1.11 | 1.43 | 5 | 2 | 2 | 5 |
| 2 | 1.7 | 2.33 | 2 | 2.5 | 5 | 2 | 2 | 5 |
| 3 | 1.75 | 3.11 | 3.33 | 3.33 | 5 | 3 | 3 | 5 |
| 4 | 1.78 | 3.89 | 3.33 | 4 | 5 | 4 | 4 | 5 |
| 5 | 1.8 | 4.67 | 5 | 4.55 | 5.2 | 5 | - | 6 |
| 6 | 1.81 | 5.3 | 5 | 5 | 5.5 | 6 | - | 7 |
| 7 | 1.82 | 5.8 | 5 | 5.38 | 5.86 | 7 | - | 8 |
| 8 | 1.73 | 6.25 | 5 | 5.71 | 6.25 | 8 | - | 8 |

$\pi = (5,5,5,6,6,6,6,7,7,7,7,8,8,8,8,8,8,8,8,9) \quad n = 20$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ |
|---|---|---|---|---|---|---|---|
| 1 | 2.56 | 2.56 | 2 | 2.5 | 3 | 3 | 3 |
| 2 | 2.74 | 3.84 | 4 | 4.44 | 4.5 | 4 | 4 |
| 3 | 2.81 | 5.12 | 5 | 6 | 5.67 | 5 | 4 |
| 4 | 2.84 | 6.4 | 6.67 | 7.27 | 6.75 | 5 | 5 |
| 5 | 2.86 | 7.68 | 10 | 8.33 | 8 | 5 | 6 |
| 6 | 2.88 | 8.96 | 10 | 9.23 | 9.17 | 6 | - |
| 7 | 2.89 | 10.17 | 10 | 10 | 10.29 | 7 | - |
| 8 | 2.89 | 11.26 | 10 | 10.67 | 11.38 | 8 | - |
| 9 | 2.9 | 12.22 | 10 | 11.25 | 12.22 | 9 | - |

$\pi = (6^3, 10^6, 13^7, 18^5, 19, 20^3) \quad n = 25$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1.93 | 1.93 | 1.19 | 1.67 | 3 | 3 | 3 |
| 2 | 2.01 | 2.9 | 2.27 | 3.13 | 4.5 | 4 | 3 |
| 3 | 2.05 | 3.9 | 3.57 | 4.41 | 6 | 6 | 3 |
| 4 | 2.07 | 4.83 | 4.17 | 5.56 | 7.5 | 6 | 4 |
| 5 | 2.08 | 5.8 | 5 | 6.58 | 8.8 | 6 | 4 |
| 6 | 2.08 | 6.76 | 6.25 | 7.5 | 10 | 6 | 5 |
| 7 | 2.09 | 7.72 | 8.33 | 8.33 | 10.86 | 7 | - |
| 8 | 2.09 | 8.63 | 8.33 | 9.09 | 11.5 | 8 | - |
| 9 | 2.1 | 9.51 | 8.33 | 9.78 | 12 | 9 | - |
| 10 | 2.1 | 10.36 | 8.33 | 10.42 | 12.4 | 10 | - |
| 11 | 2.1 | 11.19 | 12.5 | 11 | 12.82 | 11 | - |
| 12 | 2.1 | 11.96 | 12.5 | 11.54 | 13.25 | 12 | - |
| 13 | 2.1 | 12.69 | 12.5 | 12.04 | 13.69 | 13 | - |
| 14 | 2.1 | 13.38 | 12.5 | 12.5 | 14.14 | 14 | - |
| 15 | 2.1 | 14.01 | 12.5 | 12.93 | 14.6 | 15 | - |
| 18 | 2.11 | 15.64 | 12.5 | 14.1 | 15.78 | 18 | - |
| 20 | 2.11 | 16.55 | 12.5 | 14.71 | 16.55 | 20 | - |

$\pi = (80^{180}, 100^{20}) \quad n = 200$

| $k$ | $F_k(\pi)$ | $CT_k(\pi)$ | $HS_k(\pi)$ | $CH_k(\pi)$ | $R_k(\pi)$ | $M_k(\pi)$ | $H_k(\pi)$ |
|---|---|---|---|---|---|---|---|
| 10 | 2.45 | 13.31 | 18.18 | 21.74 | 14 | 10 | 3 |
| 20 | 2.45 | 25.41 | 33.33 | 39.22 | 26.5 | 20 | 4 |
| 30 | 2.45 | 37.51 | 50 | 53.57 | 39 | 30 | 4 |
| 40 | 2.45 | 49.62 | 66.67 | 65.57 | 54.45 | 40 | 5 |
| 50 | 2.45 | 61.72 | 66.67 | 75.76 | 63.86 | 50 | 7 |
| 60 | 2.45 | 73.82 | 100 | 84.51 | 76.3 | 60 | 10 |
| 70 | 2.45 | 85.92 | 100 | 92.11 | 88.76 | 70 | 16 |
| 80 | 2.45 | 98.02 | 100 | 98.77 | 100.13 | 80 | 91 |

# 7 Conclusions and Outlook

In this thesis we examined lower bounds on the independence number and on the generalized $k$-independence number of a graph in terms of degrees.

Therefore, we modified Murphy's algorithm. This lead to an improvement for graphs which satisfy certain properties and still guarantees a lower bound on the independence number. One question that must remain unanswered is whether it is possible to refine the algorithm for more classes of graphs.

Moreover, we constructed a new lower bound on the $k$-independence number based on Murphy's algorithm. To prove the assertion, we simultaneously studied the relation to a natural heuristic algorithm for constructing a $k$-independent set that has at least the size of our new bound. For some graphs, our new bound offers a genuine improvement over all known tractable bounds.

Motivated by Turán's famous theorem (see 6.3), we solved an extremal problem for graphs. Let $G$ be a graph on $n$ vertices and $k$-independence number at most $r$. Then the size of $G$ is at least

$$\frac{n^2}{2r} + \frac{n}{2}(k-2).$$

In particular, $k = 1$ is the result of Turán. With a lot of effort, we worked on an extension, that is that the above graph $G$ majorizes a corresponding graph $H$ consisting of $r$ cliques and additional conditions. But for a general proof, it seems more reasonable to look for a different approach. Hence, we conjecture (see 6.12):

**Conjecture:**
*Let $G = (V, E)$ be a graph with $\overline{d}_G \geq k$ for $k \geq 2$ and $\alpha_k(G) \leq r$. Then $G$ majorizes a graph $H$ on vertex set $V$ consisting of $r$ cliques with $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$, it holds*

$$|N(v) \setminus V_i| \geq k - 1.$$

It is these considerations which have led to another new lower bound on the $k$-independence number for graphs which satisfy the conjecture. We presented graphs for which our result is an improvement over all known bounds. Since we are convinced that our conjecture is true, this might be a lower bound for all graphs. A proof of this statement would imply our result. Another interesting question is if our new bound could be arbitrarily larger than the $k$-residue. An appropriate partition has not been found, yet.

There still remain unresolved issues in the field of $k$-independence in graphs, and all considered bounds still leave room for improvements.

# Appendix

## Matlab Codes

Listing 7.1: $M_k$-Algorithm in Matlab

```
%Input: grad=degree sequence in increasing order
%       k= positive integer

%Output: M_k-bound

function[]=M_k(grad,k)

j=0;
m=0;

while m+k-1 < length(grad)

      d=0;
      for i=1:k
      d=d+grad(m+i);
      end
      m=m+d+k;
      j=j+1;

end

s=length(grad)-m;
     if s<0
     s=0;
     end
j*k+s;

disp('The M_k Bound is')
j*k+s

end
```

Listing 7.2: $H_k$-Algorithm in Matlab

```
%Input: grad=degree sequence in increasing order
%        k= positive integer

%Output: H_k-bound

function[]=H_k(grad,k)

j=0;
h=0;

if min(grad) < k
    disp('The H_k algorithm is not applicable')

    else

    while h < length(grad)

         i=h;
         h=h+grad(i+1)-k+2;
         j=j+1;

    end
end
h=n;

disp('The H_k Bound is')
j

end
```

Listing 7.3: Refined Murphy algorithm in Matlab

```
%Input: grad=degree sequence in increasing order
%             under graphical pre-conditions

%Output: refined Murphy bound

function[]=Refined_Murphy(grad)

j=0;
a=0;
```

```
while a < length(grad)
  i=a;
  a=a+grad(i+1)+1;
     if graphical pre-conditions 1,2 or 3 true
        if 2*grad(i+1)-1<=grad(a+1)       % Refinement conditions
              a=a-grad(i+1)+1;            % Refinement-Step
              else
              a=a;
        end
     end

     j=j+1;
end

disp('The refined murphy number is')
j

end

% graphical pre-conditions:

1. Dual Partitions
if 2*min(grad)=max(grad) &&
   length(find(grad==min(grad)))=min(grad)+1 &&
   min(grad)<= length(find(grad==max(grad)))<=min(grad)+2

end

2. Dual Partitions relaxation
if 2*min(grad)-1 <= max(grad) &&
   length(find(grad==min(grad)))=min(grad)+1 &&
   min(grad)<= length(find(grad==max(grad)))<=min(grad)+2

end

3. Double Partitions grad=(d_1^k1,d_2^k2,...,d_n^kn)
if for i=1...n-1: 2*grad(i)-1 <= grad(a+1) &&
   for i=1...n-2:
   grad(i) <= length(find(grad==grad(i)))<=grad(i)+1 &&
   grad(n-1)<=length(find(grad==max(grad)))<=grad(n-1)+2 &&
   length(find(grad==grad(n-1)))<= 2/3*grad(n-1)+2

end
```

## Idea of a Proof of Conjecture 6.12

We present an idea how to prove our conjecture:

*Let $G = (V, E)$ be a graph with $\overline{d}_G \geq k$ for $k \geq 2$ and $\alpha_k(G) \leq r$. Then $G$ majorizes a graph $H$ on vertex set $V$ consisting of $r$ cliques $V_1, V_2, \ldots, V_r$ such that for all $v \in V_i$, $1 \leq i \leq r$:*

$$|N(v) \setminus V_i| \geq k - 1.$$

One possibility is to fix $k \in \mathbb{N}$ and to proceed by induction on $|V| = n$. The case $n = 1$ and $n = 2$ being trivial, since $k$ can take only the value 1, the statement follows immediately by Turán's theorem. The case $n = 3$ with $\pi(G) = (2,2,2)$ is the first non-trivial case with $\overline{d}_G = 2$. If $k = 1$, we apply Turáns theorem again. If $k = 2$, we have $\alpha_2(G) = 2$. Thus, we search for a graph $H$ consisting of 2 cliques such that every vertex has at least 1 neighbor outside its clique:

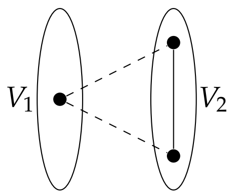



Figure 7.1: Extremal graph $H$ with $\pi(H) = (2,2,2)$

In this case we obtain $G = H$ and the statement is true. To perform the induction step we distinguish between two cases of which we are able to prove one part.

*Case:* $\overline{d}_G \geq \delta(G) > k$

Assume that the statement is valid for every graph on $n$ vertices, and suppose $G$ is a graph on $n + 1$ vertices with $\delta(G) = l > k$. Let $w \in V$ be the vertex with minimum degree $d_G(w) = l$. Now we choose a vertex $v \in N(w)$ and consider the graph

$$G_1 := G \setminus \{v\}.$$

Consequently, we obtain $d_{G_1}(w) = l - 1 \geq k$ and thus, the average degree of $G_1$ is at least $k$. Since every $k$-independent set in $G_1$ is a $k$-independent set in $G$, we conclude $\alpha_k(G_1) \leq r$. By induction, $G_1$ majorizes a graph $H_1$ consisting of $r$ cliques such that every vertex has at least $k - 1$ neighbors outside its clique. Suppose $V_1$ is a clique including the vertex $w$ in $H_1$. Since

$$l - 1 = d_{G_1}(w) \geq d_{H_1}(w),$$

the cardinality of $V_1$ is

$$|V_1| \leq l - (k - 1).$$

Now we construct the extremal graph $H$ by adding the missing vertex $v$ to the clique $V_1$. Thus, the vertex $v$ has at most $l - (k - 1)$ neighbors in $V_1$. The following process describes how the vertex $v$ receives successively $k - 1$ neighbors outside $V_1$: we choose two arbitrary vertices from different cliques which are adjacent and delete the edge. Then we connect these vertices with $v$. We repeat this process until $v$ has at least $k - 1$ neighbors (see Figure 7.2). The edge-switch does not change the degrees of the existing vertices.

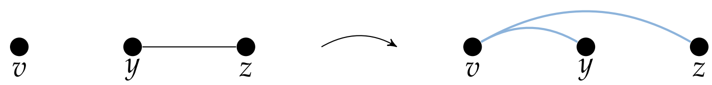



Figure 7.2: Edge-switch

This yields

$$d_H(v) = |V_1| + (k-1) \leq l - (k-1) + (k-1) = l = \delta(G) \leq d_G(v),$$

which shows, together with $G_1 \trianglerighteq H_1$, that $G$ majorizes $H$.

*Case:* $\overline{d}_G \geq k$ and $\delta(G) \leq k$

Except for the regular graphs, this case still remains open.

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# List of Symbols

# Index

**N**



**P**



**R**



**T**



**V**