Pullback Theory for Functions of Lattice-Index and Applications to Jacobi- and Modular Forms

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Introduction

In his *Fundamenta nova theoriae functionum ellipticarum* [12], C.G.J. Jacobi studied the classical theta function, i.e.
\[
\theta(z, w) = \sum_{m \in \mathbb{Z}} e^{\pi i (m^2 z + 2mw)}, \quad z \in \mathcal{H}, w \in \mathbb{C},
\]
as it is written nowadays. It obeys two substantially different transformation laws, namely
\[
\theta \left( -\frac{1}{z} w \right) = e^{\pi i \frac{w^2}{z}} \sqrt{\frac{z}{i}} \cdot \theta(z, w), \quad \theta(z + 2, w) = \theta(z, w),
\]
of modular type as well as
\[
\theta(z, w + az + b) = e^{-\pi i (a^2 z + 2aw)} \cdot \theta(z, w), \quad a, b \in \mathbb{Z},
\]
of elliptic type. The latter can be interpreted as a certain invariance property of \(\theta(z, w)\) with respect to the lattice \(\mathbb{Z} = (\mathbb{Z}, x^2)\), i.e. the abelian group \(\mathbb{Z}\) with underlying quadratic form \(x^2, x \in \mathbb{Z}\). Higher dimensional generalizations of such forms were studied for example by G. Shimura [24]. The upper half-plane \(\mathcal{H}\) is replaced by an analog of higher degree, namely the Siegel upper half-space \(\mathcal{H}_n\), and \(\mathbb{C}\) by a matrix space \(\mathbb{C}^{r \times n}\). The invariance property of elliptic type will then be considered with respect to a lattice in \(\mathbb{Q}^{r \times n}\). In the case \(n = r = 1\), a systematic treatment of functions \(\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}\) being holomorphic and satisfying both a transformation law of modular and elliptic type, so-called Jacobi forms, was initiated by M. Eichler and D. Zagier in their monograph [8]. V.A. Gritsenko [9] studied the case \(n = 1\) and arbitrary \(r\) and lifting constructions to orthogonal modular forms. C. Ziegler [27] took up again the work of Shimura in order to develop a theory of Jacobi forms of higher degree in the spirit of [8]. In this thesis we will drop the modularity condition and focus on functions satisfying a transformation law of elliptic type. Therefore we introduce the general setting:

Let \(L = (L, Q)\) be a positive definite and even lattice, i.e. \(L\) is a free \(\mathbb{Z}\)-module of finite rank and \(Q : L \to \mathbb{N}_0\) denotes some definite quadratic form on \(L\). Let \(L^{(n)} := L^{1 \times n}\) and \(L^{(n)}_{\mathbb{C}} := L^{(n)} \otimes_{\mathbb{Z}} \mathbb{C}\) together with
\[
Q^{(n)} : L^{(n)} \to \text{Sym}^n_\mathbb{Z}(\mathbb{Z}), \quad Q^{(n)}(l) := \frac{1}{2} \left( B(l_i, l_j) \right)_{i,j},
\]
for \(l = (l_1, \ldots, l_n) \in L^{(n)}\), where \(B\) is the bilinear form associated to \((L, Q)\). We will study complex-valued holomorphic functions \(\phi : \mathcal{H}_n \times L^{(n)}_{\mathbb{C}} \to \mathbb{C}\) satisfying the elliptic transformation law
\[
\phi(Z, W + lZ + l') = e^{-2\pi i \text{tr} \left( Q^{(n)}(l) \cdot Z + B^{(n)}(l, W) \right)} \cdot \phi(Z, W), \quad l, l' \in L^{(n)},
\]
for \((Z, W) \in \mathcal{H}_n \times L_{c}^{(n)}\). These functions are called elliptic of degree \(n\) and index \(L\) and the set of such functions is denoted by \(\mathcal{E}^{(n)}(L)\). The generic case is given by \(L = (\mathbb{Z}^r, S[x])\), where \(S\) denotes some even and positive definite matrix. Then \(Q^{(n)}(X) = S[X]\) for \(X \in L^{(n)} = \mathbb{Z}^{r \times n}\) and we recover exactly the second transformation law as in [27, 1.1.] or [24, (3.11), p. 47]. Let \(L_0 = (L_0, Q_0)\) be another lattice and \(i : L_0 \rightarrow L\) a homomorphism satisfying \(Q \circ i = Q_0\). In this case, \(i\) is called an embedding. Then the pullback

\[
\phi \left[ i^{(n)} \right] (Z, W_0) := \phi \left( Z, i^{(n)}(W_0) \right), \quad (Z, W_0) \in \mathcal{H}_n \times (L_0)^{(n)}_c,
\]

where \(i^{(n)}\) is defined componentwise, belongs to \(\mathcal{E}^{(n)}(L_0)\). The aim of this thesis is to study the pullback operator \(\left[ i^{(n)} \right]\) concerning the following aspects:

i) injectivity, surjectivity, bijectivity,

ii) dependency of the pullback operator on the choice of the embedding,

iii) extent of determination of the embedding by the knowledge of certain values of the pullback operator.

We will take up these issues frequently within this thesis. In the following, we provide an overview of each chapter:

In chapter one, we summarize the theory of lattices with all its necessary terms, where we attach great importance to a basis independent description in order to keep the notation clean and handy. After that we deal with the theory of embeddings of lattices. The chapter ends with the construction of some irreducible root lattices of small rank, which will be utilized later.

Chapter two is devoted to elliptic functions of lattice-index. We introduce matrix-valued quadratic forms and define the real Heisenberg group \(H_{\mathbb{R}}^{(n)}(L)\). This group acts on a certain space associated to \(L\), the Jacobi half-space \(\mathcal{H}_n(L)\). The elliptic functions of index \(L\) turn out to be invariants with respect to the integral Heisenberg group \(H^{(n)}(L)\), which is characterized by certain integrality conditions. As a prototype we consider the well-known Jacobi theta functions associated to \(L\), which provide a free basis for the \(O(\mathcal{H}_n)\)-module of elliptic functions of index \(L\), cf. also [27]. In order to have a notion of boundedness at the cusps for such a class of functions, we define regularity and cuspidality conditions. We introduce the metaplectic group \(Mp_{2n}(Z)\), which acts upon \(\mathcal{E}^{(n)}(L)\). The transformation law of modular type then exactly correspond to a certain invariance property with respect to this group. Jacobi forms will be defined as invariants of both the modular and the elliptic action together with certain conditions of boundedness. We end the chapter by calculating certain determinant characters of Weil representations associated to lattices of small rank.

In chapter three we introduce the main object of our studies, namely the aforementioned pullback operator. It transforms elliptic functions of index \(L\) into elliptic functions of index \(L_0\). In matrix language, this operator already occurred in [27, 3.5. Lemma], but for a different purpose. It commutes with the modular action and behaves well with respect to regularity. The
question that arises naturally in this context is in what cases the pullback operator can be an isomorphism. In order to give an answer to this question, we will take the algebraic point of view, i.e. we consider the pullback operator as a homomorphism of free modules. It will turn out that its representation matrix, which we call automorphic transfer, is as a vector-valued modular form with respect to some tensor product representation. In the equidimensional case we can consider its determinant which turns out as a (scalar-valued) modular form. This leads to the astonishing result that the pullback operator is an isomorphism if and only if \( n = 1 \) and the determinant is a nonzero multiple of a certain power of Dedekind’s \( \eta \)-function.

We proceed further by an explicit determination of the automorphic transfer and the modular determinant in the case \( n = 1 \) for distinguished lattices of small rank. As a by-product of the theory developed in this chapter, we prove - by using a tensor product construction - the existence of an infinite family \((\chi_{3 \cdot 2^n})_{n \in \mathbb{N}_0}\) of nontrivial Siegel cusp forms of degree \( n \) and weight \( 3 \cdot 2^n \), satisfying \( \chi_6 = \eta^{12} \) together with a remarkable recurrence relation

\[
\chi_{2^n \cdot 3} \left( \begin{array}{cc} Z_j & 0 \\ 0 & z_{n-j} \end{array} \right) = \chi_{2^n \cdot 3}(Z_j)^{2^{n-j}} \cdot \chi_{2^{n-j} \cdot 3}(Z_{n-j})^{2^j}
\]

for \( 0 \leq j \leq n \). Especially, these forms do not vanish on the diagonal \( \mathcal{H}^n \subseteq \mathcal{H}_n \). We end this chapter by formulating and proving some sufficient ad-hoc criteria for injectiveness of restrictions of the pullback operator to submodules as well as by formulating and proving certain separation theorems regarding embeddings. In both cases we will restrict only to \( n = 1 \).

The fourth chapter is a short one and deals with isomorphisms between spaces of Jacobi forms of degree 1 with respect to certain lattices of small rank. Here we derive benefit from the explicit determination of some automorphic transfer matrices in chapter three. These allow to construct explicit lifts from Jacobi forms of rank-1-index to Jacobi forms of higher rank index by using matrix-vector multiplication on the basis of the attached space of vector valued modular forms. Partly, such isomorphisms were also known before, cf. [15, 16].

From chapter five on, we draw attention to modular forms. Here we take up again a theory of G. Köhler [18], who considered embeddings of paramodular groups \( \Gamma(T) \), also known as "Siegelsche Stufengruppen", into hermitian modular groups \( \text{U}_n(\sigma_K) \), where \( \sigma_K \) denotes the integral closure of \( \mathbb{Z} \) in some imaginary-quadratic number field \( K \). He focused on the problem in what cases \( \Gamma(T) \) can be conjugated into \( \text{U}_n(\sigma_K) \) via some matrix \( M \in \text{U}_n(\mathbb{C}) \). In this case, \( M \) is called a modular embedding. Köhler gave a necessary and sufficient criterion for the existence of such embeddings. We start by briefly recapturing the basic terms in the theory of orders, where we include also the noncommutative case. Then we define the modular group associated to an order \( \mathcal{O} \) as well as the paramodular group of polarization \( T \), where \( T \) is an elementary divisor matrix. After having all necessary terms at hand, we consider modular embeddings. We extend Köhlers work by defining a notion of equivalence in order to measure substantially different embeddings. Under reasonable prerequisites we can adapt the proof of his main result in order to fit also into the noncommutative setting. As a by-product, we obtain some sort of normal form for an embedding \( M \), what we will call a embedding of principal type. They are connected to the representability of \( T \) by a diagonal matrix over \( \mathcal{O} \),
what we will call $\mathcal{O}$-models. Due to [17], $\Gamma(T)$ admits a maximal discrete extension and if $\mathcal{O}$ is a principal ideal domain, this group acts on the set of equivalence classes of $\mathcal{O}$-models.

We end the chapter by defining a pullback theory for modular forms with respect to modular embeddings, which will turn modular forms into paramodular forms and is compatible with the equivalence relation.

The sixth and final chapter is conducted by the question to what extent the modular embedding $M$ is determined - up to equivalence - by the family of functions $F\mid_{\mathcal{M}}$, $F \in [U_n(\mathcal{O}), k]$.

Already in the case $n = 2$, the equivalence relation on $\text{Mod}(\Gamma(T), U_2(\mathcal{O}))$ is too restrictive in the sense, that there are inequivalent embeddings which induce the same pullbacked functions. In order to handle this, we develop a notion of equivalence in the extended sense. We approach the problem by shortly introducing hermitian and quaternionic Jacobi forms as certain constituents of modular forms. We can solve the aforementioned question at least for certain orders and under certain divisibility assumptions on the polarization. Here we make use of the separation theorems given in chapter three.

This thesis was developed and written at Lehrstuhl A für Mathematik, RWTH Aachen University. The work on this topic was initiated and supervised by Prof. Dr. Aloys Krieg. He suggested revisiting the theory of modular embeddings again. I am indebted to him for his valuable suggestions and encouragement.

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0 Basic Notation

\( \mathbb{N} \) is the set of positive integers, \( \mathbb{N}_0 \) is the set of non-negative integers and \( \mathbb{Z} \) is the ring of integers. \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) denote the fields of rational, real and complex numbers, respectively. Without exception, \( i \) will always denote the imaginary unit. Given \( z = x + iy \in \mathbb{C} \), then \( x \) is the real part and \( y \) the imaginary part of \( z \). By \( \mathcal{H} \) we denote the upper half-plane in \( \mathbb{C} \), i.e. \( \mathcal{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} \). The unit circle in \( \mathbb{C} \) is denoted by \( S^1 \), i.e. \( S^1 = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \). The principal branch of the square root on \( \mathbb{C} \) is denoted by \( \sqrt{\cdot} \). The symbol \( \mathbb{H} \) denotes the skew field of real Hamilton quaternions with standard basis \( 1, i, j, k \). For a subset \( X \) we denote by \( \langle X \rangle \) the cyclic group of order \( n \) generated by the elements of \( X \). Unless specified otherwise, the meaning should be clear from context.

For a set \( \mathcal{X} \) we denote its cardinality by \( |\mathcal{X}| \), which will be finite throughout this thesis. For \( n, m \in \mathbb{N} \) we denote by \( \mathcal{X}^{n \times m} \) the set of \( n \)-by-\( m \) matrices with entries in \( \mathcal{X} \). Given \( X \in \mathcal{X}^{n \times m} \) we write \( X = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \) or \( X = (x_{ij}) \) for short in order to refer to the entries of \( X \). We use the convention \( \mathcal{X}^n := \mathcal{X}^{n \times 1} \) for the set of column vectors over \( \mathcal{X} \) as well as \( \mathcal{X}^\langle n \rangle := \mathcal{X}^\langle 1 \times n \rangle \) for the set of row vectors over \( \mathcal{X} \). Note that the latter notation will be a priori double assigned.

Let \( n \in \mathbb{N} \). We denote by \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \) the cyclic group of order \( n \) and by \( S_n \) the symmetric group on \( n \) points. Let \( G \) be a group. If \( H \) is a subgroup of \( G \), we will write \( H \triangleleft G \) and \( H \leq G \) if \( H \) is a normal subgroup of \( G \). For \( \mathcal{X} \subseteq G \) we denote by \( \langle \mathcal{X} \rangle \) the subgroup of \( G \) generated by the elements of \( \mathcal{X} \). In the case \( \mathcal{X} = \{ g_1, \ldots, g_m \} \) we write \( \langle g_1, \ldots, g_m \rangle := \langle \mathcal{X} \rangle \). If \( G \) is abelian, we will sometimes write \( \langle X \rangle_Z \) instead of \( \langle X \rangle \). For a subset \( \mathcal{X} \) of \( G \) we denote by \( \mathcal{N}_G(\mathcal{X}) \) the normalizer of \( \mathcal{X} \) in \( G \). The center of \( G \) is denoted by \( \mathcal{Z}(G) := \mathcal{N}_G(\mathcal{C} \langle G \rangle) \). The commutator subgroup of \( G \) is written as \( G' \) and \( G^{\text{ab}} := G/G' \) is the commutator factor group of \( G \), which is implicitly identified with the group of abelian characters \( G \rightarrow \mathbb{C}^\times \). If \( \varphi \) is a homomorphism of groups, we will denote its kernel by \( \ker \varphi \). For an abelian group \( A \) with \( a, b, c \in A \) and a subgroup \( H \) of \( A \) we write \( a \equiv b \text{ mod } H \) if \( a - b \in H \) and \( a \equiv b \text{ mod } c \), if \( a \equiv b \text{ mod } \langle c \rangle \). Furthermore, \( \text{Hom}(A, B) \) denotes the set of homomorphisms \( A \rightarrow B \).

Let \( V \) be a finite-dimensional \( \mathbb{F} \)-vector space. The dimension of \( V \) over \( \mathbb{F} \) is denoted by \( \text{dim}_\mathbb{F}(V) \), which will be always finite throughout this thesis. \( \text{GL}(V) \) denotes the general linear group of \( V \). The orthogonal and unitary group of \( V \) are denoted by \( \text{O}(V) \) and \( \text{U}(V) \), respectively.
Let \( R \) be a ring with 1, not necessary commutative. By \( I_n \in R^{n \times n} \) we denote the identity matrix of size \( n \times n \) and by \( J_n \) the block-matrix

\[
J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in R^{2n \times 2n}, \quad J := J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

For \( i, j \in \{1, \ldots, n\} \) let \( I_{ij}^{(n)} \) denote the matrix whose \((i,j)\)-entry is 1 and 0 otherwise. We suppress the superscript if the size is clear from the context. By \( GL_n(R) \) we denote the general linear group over \( R^{n \times n} \), i.e.

\[
GL_n(R) = \{ U \in R^{n \times n} : UV = VU = I_n \text{ for some } V \in R^{n \times n} \}.
\]

We set \( R^\times := GL_1(R) \), the so-called unit group of \( R \). We write \( U^{-1} \) for the inverse matrix of \( U \in GL_n(R) \). The definition is extended by \( U^{-1} := (U^{-1})^t \). In the case if \( R \) is commutative, we denote by \( SL_n(R) \) the special linear group in \( R^{n \times n} \). Let \( A \in R^{n \times n} \). Then \( \det(A) \) stands for the determinant of \( A \) and - unless specified otherwise - \( \text{tr}(A) \) for the usual matrix trace of \( A \). The subset of symmetric matrices in \( R^{n \times n} \) is denoted by \( \text{Sym}_n(R) \). Let \( B \in R^{n \times m} \).

If \( R \) admits - possibly trivial - involution \(-\), we define \( \overline{B} \) by applying \(-\) pointwisely on each entry of \( B \). In this case, \( \text{Her}_n(R) \) will denote the subset of hermitian matrices in \( R^{n \times n} \), i.e.

\[
\text{Her}_n(R) = \{ A \in R^{n \times n} : \overline{A}^t = A \}.
\]

Note that \( \text{Her}_n(R) = \text{Sym}_n(R) \) in the case of the trivial involution. Furthermore, we set \( A[B] := \overline{B} AB \in \text{Her}_m(R) \). If the involution satisfies \( \{ r \in R : \overline{r} = r \} \subseteq R \), then we call \( A \) positive definite, written \( A > 0 \), resp. positive semi-definite, written \( A \geq 0 \), if \( A[x] > 0 \) resp. \( A[x] \geq 0 \) for all \( 0 \neq x \in R^n \). By \( \text{Pos}_n(R) \) we denote the set of positive definite matrices over \( R \). For \( A, B \in \text{Her}_n(R) \) we write \( A < B \), if \( B - A > 0 \), and \( A \leq B \), if \( B - A \geq 0 \).

If a set \( \mathcal{X} \) admits a complex structure, we denote by \( \mathcal{O}(\mathcal{X}) \) the ring of holomorphic functions \( \mathcal{X} \to \mathbb{C} \) and by \( \text{Bih} \mathcal{X} \) the group of biholomorphic automorphism of \( \mathcal{X} \).

For finite \( \mathcal{X} \subseteq \mathbb{R} \) we denote by \( \max \mathcal{X} \) the maximum of \( \mathcal{X} \) with the convention \( \max \emptyset := -\infty \).

Finally, for \( m, n \in \mathbb{N} \) we write \( m|n \text{ if } m \text{ divides } n \) and \( m||n \), if \( m|n \) and \( \gcd \left( \frac{n}{m}, n \right) = 1 \), where \( \gcd \) denotes the greatest common divisor. For \( k \geq 0 \), we set \( \sigma_k(m) := \sum_{d|m} d^k \), where \( d \) runs through the positive divisors of \( m \). By \( B_k \) we denote the \( k \)-th Bernoulli number.
1 Lattices and Embeddings

1.1 Lattices

We repeat the basic notions in the theory of lattices and quadratic forms, where we refer to [7], [22] or [3].

**Definition 1.1.1.** A lattice $L$ is a pair $(L, Q)$, where

- $L$ is a free $\mathbb{Z}$-module of finite rank,
- $Q : L \rightarrow \mathbb{R}$ is a quadratic form, i.e.

  \[
  Q(\alpha l) = \alpha^2 Q(l) \quad \text{(Homogeneity)}
  \]
  \[
  Q(l + l') + Q(l - l') = 2Q(l) + 2Q(l') \quad \text{(Parallelogram law)}
  \]

holds for all $l, l' \in L$ and $\alpha \in \mathbb{Z}$.

The rank of $L$ is defined as the rank of the underlying $\mathbb{Z}$-module and denoted by $r_L$.

If $L = (L, Q)$ is a lattice and $Q$ is known from the context, we will sometimes refer to the lattice $L$ instead of $L$.

A direct verification yields the following

**Lemma 1.1.2.** Let $L$ be a free $\mathbb{Z}$-module of finite rank and $Q : L \rightarrow \mathbb{R}$. Define $B : L \times L \rightarrow \mathbb{R}$ by

\[
B(l, l') = Q(l + l') - Q(l) - Q(l') \quad \text{(Polarization identity)}
\]

for $l, l' \in L$. Then the following statements are equivalent:

a) $Q$ is a quadratic form on $L$, i.e. $(L, Q)$ is a lattice,

b) $B$ is a $\mathbb{Z}$-bilinear form on $L$.

In this case, $B$ is called the bilinear form associated to $(L, Q)$.

We introduce some constructions of lattices, namely orthogonal sums and scaling:
**Definition 1.1.3.** a) Let $m \in \mathbb{N}$ and $L_1, \ldots, L_m$ be lattices with underlying quadratic forms $Q_1, \ldots, Q_m$. The orthogonal sum $L := \bigoplus_{i=1}^{m} L_i$ is the lattice $L = (L, Q)$, where

\[ L := \bigoplus_{i=1}^{m} L_i, \quad Q := \bigoplus_{i=1}^{m} Q_i, \]

i.e. $Q(l_1, \ldots, l_m) := \sum_{i=1}^{m} Q_i(l_i)$ for $l_i \in L_i, i = 1, \ldots, m$. For a lattice $L = (L, Q)$ we define $mL := \bigoplus_{i=1}^{m} L$.

b) Let $L = (L, Q)$ be a lattice. For $t \in \mathbb{N}$ we define $L(t) := (L, tQ)$. In this case we say that $L(t)$ arises from $L$ by scaling with $t$.

We repeat some lattice-theoretic terms:

**Definition 1.1.4.** Let $L = (L, Q)$ be a lattice with associated bilinear form $B$.

a) $L$ is called non-degenerate, if $B(l, \cdot) \neq 0$ for all $0 \neq l \in L$.

b) $L$ is called integral, if $B(l, l') \in \mathbb{Z}$ for all $l, l' \in L$.

c) $L$ is called even, if $B(l, l) \in 2\mathbb{Z}$ for all $l \in L$.

d) $L$ is called positive definite, if $B(l, l) > 0$ for all $0 \neq l \in L$.

Unless specified otherwise, $L = (L, Q)$ will always denote a positive definite and even lattice, i.e. $Q : L \to \mathbb{N}_0$ and $Q(l) = 0$ if and only if $l = 0$.

**Definition 1.1.5.** Let $L = (L, Q)$ be a lattice and $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The $F$-vector space

\[ L_F := L \otimes_{\mathbb{Z}} F, \]

arising from $L$ by extension of scalars of $F$, is called the ambient space of $L$ over $F$. By $F$-linearity, the bilinear form $B$ associated to $L$ extends uniquely to $L_F$ and is again denoted by $B$. By polarization, the quadratic form $Q$ extends uniquely to a quadratic form on $L_F$ and is again denoted by $Q$. The pair $L_F = (L_F, Q)$ is called the quadratic space of $L$ over $F$.

Note that $\dim_F L_F = r_L$ for $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

The structure-preserving maps between lattices are described in the following

**Definition 1.1.6.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices.

a) An isometry $\sigma : L_0 \to L$ is an isomorphism $\sigma : L_0 \to L$ of abelian groups satisfying $Q \circ \sigma = Q_0$. In this case, $L$ and $L_0$ are called isometric.

b) An isometry $\sigma : L \to L$ is called orthogonal transformation or automorphism of $L$. The group of orthogonal transformations of $L$ is denoted by $O(L)$, called the orthogonal group of $L$.
1.1 Lattices

Note that $O(L)$ is a finite group.

**Remark 1.1.7.** Let $L$ be a lattice and $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$. Since $L$ contains a $\mathbb{F}$-basis of $L_{\mathbb{F}}$, every orthogonal transformation of $L$ extends in a unique way to an orthogonal transformation of the quadratic space $L_{\mathbb{F}}$. Hence, the orthogonal group $O(L)$ is characterized by

$$O(L) = \{\sigma : L_{\mathbb{F}} \to L_{\mathbb{F}} : \sigma \text{ $\mathbb{F}$-linear, } Q \circ \sigma = Q \text{ and } \sigma(L) = L\}.$$

**Definition 1.1.8.** Let $L = (L, Q)$ be a lattice.

a) The pair $L^* := (L^*, Q|_{L^*})$, where

$$L^* := \{v \in L_Q : B(v, l) \in \mathbb{Z} \text{ for all } l \in L\},$$

is called the dual lattice of $L$.

b) The quotient group $L^*/L$ is called the discriminant group of $L$.

Similarly, if $L = (L, Q)$ is a lattice and $Q$ is given in the context, we will sometimes refer to $L^*$ as the dual lattice of $L$.

**Remark 1.1.9.** Let $L = (L, Q)$ be a lattice.

a) Every $\varphi \in \text{Hom}(L, \mathbb{Z})$ can be uniquely extended to a $\mathbb{Q}$-linear functional $L_Q \to \mathbb{Q}$, which is again denoted by $\varphi$. Since $B$ is non-degenerate, there is a unique $\mu \in L_Q$ such that $\varphi(l) = B(\mu, l)$ holds for all $l \in L_Q$. Hence, $\mu \in L^*$. Consequently, the map

$$L^* \to \text{Hom}(L, \mathbb{Z}), \quad \mu \mapsto B(\mu, \cdot)$$

is an isomorphism.

b) Let $(b_1, \ldots, b_r)$ denote some $\mathbb{Z}$-basis of $L$. Then $(b_1, \ldots, b_r)$ is also a $\mathbb{Q}$-basis of $L_Q$. It is easily seen that the dual basis $(b_1^*, \ldots, b_r^*)$ of the vector space $L_Q$ provides a free basis for $L^*$. Hence, $L^*$ is again a lattice. Note that $L^*_+ = (L^*, Q|_{L^*})$ may not be integral in general.

c) In view of b), the level of $L$ is defined as

$$N_L := \min \{q \in \mathbb{N} : qQ(\mu) \in \mathbb{Z} \text{ for all } \mu \in L^*\},$$

i.e. $N_L$ is the smallest natural number $q$, such that $L^*_+(q)$ is an even lattice.

**Definition 1.1.10.** Let $L = (L, Q)$ be a lattice and $(b_1, \ldots, b_r)$ a basis of $L$. The matrix

$$S_{(b_1, \ldots, b_r)}(L) := (B(b_i, b_j))_{1 \leq i, j \leq r} \in \text{Sym}_r(\mathbb{Z})$$

is called the Gram matrix of $L$ with respect to $(b_1, \ldots, b_r)$.

The generic case is described in the following
Remark 1.1.11. Let $S \in \text{Sym}_r(\mathbb{Z})$ be positive definite and even, i.e. $s_{ii} \in 2\mathbb{Z}$ for $i = 1, \ldots, r$. Then the map 

$$Q_S(x) := \frac{1}{2} S[x], \quad x \in \mathbb{Z}^r,$$

is a quadratic form on $\mathbb{Z}^r$ and the pair $(\mathbb{Z}^r, Q_S)$ is a lattice. The dual lattice is given by $(S^{-1}\mathbb{Z}^r, Q_S)$. Conversely, let $L = (L, Q)$ be a lattice and $S$ denote its Gram matrix with respect to some basis $b_1, \ldots, b_r$ of $L$. Let $\kappa : L \rightarrow \mathbb{Z}^r$ be a coordinate system for $L$ with respect to $b_1, \ldots, b_r$. Then one has 

$$B(l, l') = \kappa(l)^t S \kappa(l')$$

for all $l, l' \in L$, i.e. $\kappa$ is an isometry of the lattices $(L, Q)$ and $(\mathbb{Z}^r, Q_S)$. Hence, 

$$L^* / L \cong S^{-1}\mathbb{Z}^r / \mathbb{Z}^r \cong \mathbb{Z}^r / S\mathbb{Z}^r$$

and the cardinality $|L^* / L| = \det S$ is finite and independent from the choice of $S$.

This justifies the following

Definition 1.1.12. Let $L$ be a lattice. The finite number $\det L := |L^* / L|$ is called the determinant or discriminant of $L$.

The identity 

$$Q(\mu) - Q(\mu') = Q(\mu - \mu') - B(\mu' - \mu, \mu')$$

for $\mu, \mu' \in L_{\mathbb{R}}$ gives rise to the following

Definition 1.1.13. Let $L = (L, Q)$ be a lattice. Then the map 

$$\overline{Q} : L^* / L \rightarrow Q / \mathbb{Z}, \quad \overline{Q}(\mu + L) := Q(\mu) + \mathbb{Z}, \quad \mu \in L^*,$$

is well-defined called the discriminant form of $L$.

Remarks 1.1.14. Let $L = (L, Q)$ be a lattice.

a) The pair $L^* / L = (L^* / L, \overline{Q})$ is called the finite quadratic module associated to $L$. It is easy to see, that the discriminant form takes its values in $N_{L^{-1}}^{-1} \mathbb{Z} / \mathbb{Z}$.

b) Since $O(L)$ acts naturally on $L^*$, it also acts on $L^* / L$ via the assignment 

$$(\sigma, \mu + L) \mapsto \sigma(\mu) + L, \quad \mu \in L^*, \sigma \in O(L).$$

This action respects the discriminant form, i.e. $\overline{Q} \circ \sigma = \overline{Q}$.

c) The action given in b) induces a homomorphism $O(L) \rightarrow O\left( L^* / L \right)$, where 

$$O\left( L^* / L \right) := \{ \sigma : L^* / L \rightarrow L^* / L : \sigma \text{ isomorphism, } \overline{Q} \circ \sigma = \overline{Q} \}$$

is the finite orthogonal group of $L^* / L$.
Definition 1.1.15. Let $L = (L, Q)$ be a lattice. The kernel of the homomorphism given in 1.1.14 c) is called the discriminant kernel of $L$ and is denoted by $O_d(L)$.

Definition 1.1.16. Let $L = (L, Q)$ be a lattice.

1. An element $l \in L$ is called a root of $L$ if $Q(l) = 1$. The set of roots of $L$ is denoted by $R(L)$.
2. $L$ is called a root lattice, if $L$ is generated by $R(L)$.

Definition 1.1.17. Let $L = (L, Q)$ be a lattice.

1. For $l \in R(L)$ the reflection $s_l \in O(L)$ along $l$ is defined as
   $$s_l : L \rightarrow L, \quad s_l(l') := l' - B(l, l')l, \quad l' \in L.$$

2. The Weyl group of $L$ is defined as
   $$W(L) := \langle s_l : l \in R(L) \rangle \leq O_d(L),$$
   with the convention $W(L) := \{1\}$, if $R(L) = \emptyset$.

Definition 1.1.18. A lattice $L$ is called irreducible, if it does not split into an orthogonal sum of two lattices.

| $L$ | $r_L$ | $L^*/L$ | $\det L$ | $|R(L)|$ |
|-----|-------|---------|----------|--------|
| $A_r$ | $r$ | $\mathbb{Z}_{r+1}$ | $r + 1$ | $r(r + 1)$ |
| $D_r$ | $r$ | $\begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2, & r \text{ even} \\ \mathbb{Z}_4, & r \text{ odd} \end{cases}$ | 4 | $2r(r - 1)$ |
| $E_6$ | 6 | $\mathbb{Z}_3$ | 3 | 72 |
| $E_7$ | 7 | $\mathbb{Z}_2$ | 2 | 126 |
| $E_8$ | 8 | $\{0\}$ | 1 | 240 |

Table 1.1: Classification of the irreducible root lattices

Theorem 1.1.19. (cf. [7, Thm. 1.2])

1. Table 1.1 classifies the irreducible root lattices completely up to isometry.
2. Every root lattice is the orthogonal sum of irreducible root lattices given in table 1.1.

For concrete realizations confer [7, p. 14, p. 23ff]. Note that in the following we will use the constructions of $A_1, A_2, D_4, E_6, E_7$ and $E_8$ given in section 1.3.
Remark 1.1.20. a) Table 1.2 classifies the irreducible root lattices, the structure of their discriminant groups and representatives for $\mathbb{Q}$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$r_L$</th>
<th>$L^*/L$</th>
<th>representatives</th>
<th>$Q$</th>
<th>$N_L$</th>
</tr>
</thead>
</table>
| $A_r$ | $r$ | $\mathbb{Z}_{r+1}$ | $\begin{cases} \mu_0 := 0 \\
 \vdots \\
 \mu_i \\
 \vdots \\
 \mu_r \end{cases}$ | $\begin{cases} 0 \\
 \vdots \\
 i(r+1-i) \\
 \vdots \\
 r(r+1) \end{cases}$ | $\begin{cases} r+1, & r \equiv 0 \text{ mod } 2, \\
 2(r+1), & r \equiv 1 \text{ mod } 2 \end{cases}$ |
| $D_r$ \, $r$ \, even | $r$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\begin{cases} \mu_0 := 0 \\
 \mu_1 \\
 \mu_2 \\
 \mu_3 := \mu_1 + \mu_2 \end{cases}$ | $\begin{cases} 0 \\
 \frac{r}{8} \\
 \frac{1}{2} \\
 \frac{1}{8} \end{cases}$ | $\begin{cases} 1, & r \equiv 0 \text{ mod } 8, \\
 2, & r \equiv 4 \text{ mod } 8, \\
 4, & r \equiv 2 \text{ mod } 8 \end{cases}$ |
| $D_r$ \, $r$ \, odd | $r$ | $\mathbb{Z}_4$ | $\begin{cases} \mu_0 := 0 \\
 \mu_1 \\
 \mu_2 := 2\mu_1 \\
 \mu_3 := 3\mu_1 \end{cases}$ | $\begin{cases} 0 \\
 \frac{r}{8} \\
 \frac{1}{2} \\
 \frac{1}{8} \end{cases}$ | $8$ |
| $E_6$ | 6 | $\mathbb{Z}_3$ | $\begin{cases} \mu_0 := 0 \\
 \mu_1 \\
 \mu_2 := -\mu_1 \end{cases}$ | $\begin{cases} 0 \\
 \frac{2}{3} \\
 \frac{2}{3} \end{cases}$ | $3$ |
| $E_7$ | 7 | $\mathbb{Z}_2$ | $\begin{cases} \mu_0 := 0 \\
 \mu_1 \end{cases}$ | $\begin{cases} 0 \\
 \frac{3}{4} \end{cases}$ | $4$ |
| $E_8$ | 8 | $\{0\}$ | $\mu_0 := 0$ | $0$ | $1$ |

Table 1.2: Structure of $L^*/L$ and values of $\overline{Q}$.

b) Table 1.3 classifies the irreducible root lattices, representatives for the quotient group $\mathcal{O}(L)/\mathcal{O}_d(L)$ and the corresponding orbits on $L^*/L$. 
1.2 Embeddings

In this section, we define the fundamental object of our studies:

**Definition 1.2.1.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices. An embedding $\iota : L_0 \rightarrow L$ of $L_0$ into $L$ is a $\mathbb{Z}$-homomorphism $\iota : L_0 \rightarrow L$ such that $Q \circ \iota = Q_0$.

**Definition 1.2.2.** Let $L = (L, Q)$ be a lattice. A lattice $L_0 = (L_0, Q_0)$ is called a sublattice of $L$, if $L_0 \leq L$ and $Q_0 = Q|_{L_0}$.

Conversely, since subgroups of free abelian groups are free itself, $(L_0, Q|_{L_0})$ is a sublattice of $L$ for every subgroup $L_0 \leq L$.

We give a useful connection between embeddings and sublattices:

**Proposition 1.2.3.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices. Then there is an one-to-one correspondence between the embeddings of $L_0$ into $L$ and the sublattices of $L$, which are isometric to $L_0$. More precisely, if $\iota : L_0 \rightarrow L$ is an embedding, then $\iota(L_0) = (\iota(L_0), Q|_{\iota(L_0)})$ is a sublattice of $L$ which is isometric to $L_0$. Conversely, if $L'$ is a sublattice of $L$ and $\iota : L_0 \rightarrow L'$ is an isometry, then $\iota : L_0 \rightarrow L$ is an embedding.

Some trivial observations are contained in the following

**Remark 1.2.4.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices.

a) Since $Q_0$ is definite, any embedding $\iota : L_0 \rightarrow L$ is necessarily injective.

b) The surjective embeddings of $L_0$ into $L$ are exactly the isometries $L_0 \rightarrow L$.

c) Every orthogonal transformation $\sigma \in O(L)$ is an embedding of $L_0$ into $L$.

Table 1.3: Irreducible root lattices, representatives of the discriminant kernel and orbits

| $L$   | $O(L) / O_d(L)$ | representatives | orbits on $L^* / L$ | $|\text{orbits}|$ |
|-------|-----------------|-----------------|---------------------|-----------------|
| $A_1$ | {0}             | id              | {0}, {µ_1}          | 2               |
| $A_r$, $r > 1$ | $\mathbb{Z}_2$ | ±id             | {0}, {µ_i - µ_1}, 1 ≤ i ≤ $\frac{r+1}{2}$ | $1 + \frac{r+1}{2}$ |
| $D_4$ | $S_3$           | $\langle (1,2,3), (1,3) \rangle$ | {0}, {µ_1, µ_2, µ_3} | 2               |
| $D_r$, $r \neq 4$ | $\mathbb{Z}_2$ | $\langle (1,3) \rangle$ | {0}, {µ_2}, {µ_1, µ_3} | 3               |
| $E_6$ | $\mathbb{Z}_2$ | ±id             | {0}, {µ_1, -µ_1}   | 2               |
| $E_7$ | {0}             | id              | {0}, {µ_1}          | 2               |
| $E_8$ | {0}             | id              | {0}                | 1               |
We introduce orthogonal complements:

**Definition 1.2.5.** Let $L = (L, Q)$ be a lattice.

a) For a subset $K \subseteq L$, the $Q$-vector space

$$K^\perp = \{ l \in L_Q : B(l, x) = 0 \text{ for all } x \in K \}$$

is called the orthogonal complement of $K$ in $L_Q$.

b) For a sublattice $L_0 = (L_0, Q_0)$ we define $L_0^\perp L := L_0^\perp \cap L$. The pair

$$L_0^\perp L := (L_0^\perp L, Q|_{L_0^\perp L})$$

is called the orthogonal complement of $L_0$ in $L$.

**Remark 1.2.6.** Let $L = (L, Q)$ be a lattice and $L_0$ a sublattice of $L$. Then $L_0^\perp L$ is a sublattice of $L$ and one has

$$r_L = r_{L_0} + r_{L_0^\perp L}.$$

**Remark 1.2.7.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices and $\iota : L_0 \longrightarrow L$ an embedding. For $\mu \in L^*$ we consider the map

$$L_0 \longrightarrow \mathbb{Z}, \quad l_0 \mapsto B(\iota(l_0), \mu), \quad l_0 \in L_0,$$

which belongs to $\text{Hom}(L_0, \mathbb{Z})$. By 1.1.9 there is a unique $\mu_0 \in L_0^*$ such that

$$B(\iota(l_0), \mu) = B_0(l_0, \mu_0)$$

for all $l_0 \in L_0$. The assignment $\mu \rightarrow \mu_0$ yields a $\mathbb{Z}$-homomorphism $L^* \longrightarrow L_0^*$.

1.2.7 justifies the following

**Definition 1.2.8.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices and $\iota : L_0 \longrightarrow L$ an embedding. Then the map $\iota^* : L^* \longrightarrow L_0^*$ defined by the identity

$$B(\iota(l_0), \mu) = B_0(l_0, \iota^*(\mu)) \text{ for all } l_0 \in L_0, \mu \in L^*,$$

is called the dual of $\iota$.

**Remark 1.2.9.** Let $L = (L, Q)$ and $L_0 = (L_0, Q_0)$ be lattices and $\iota : L_0 \longrightarrow L$ an embedding. For $F \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ the dual $\iota^* : L^* \longrightarrow L_0^*$ is uniquely extended by $F$-linearity to a map $L_F \longrightarrow (L_0)_F$, which will again be denoted by $\iota^*$. It is uniquely determined by the identity

$$B(\iota(l_0), \mu) = B_0(l_0, \iota^*(\mu)), \quad l_0 \in (L_0)_F, \mu \in L_F.$$

Obviously, $\iota^* \circ \iota = \text{id}_L$. 

For later use we need the following

**Lemma 1.2.10.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices and \( \iota : L_0 \rightarrow L \) an embedding. Then for \( \mu \in L_\mathbb{R} \) the following assertions hold:

a) \( Q_0(\iota^*(\mu)) \leq Q(\mu) \),

b) \( Q_0(\iota^*(\mu)) = Q(\mu) \) if and only if \( \mu = \iota(\xi) \) for some \( \xi \in L_0^\perp \).

**Proof.**
a) There is nothing to prove in the case \( Q_0(\iota^*(\mu)) = 0 \). Otherwise, we apply the Cauchy-Schwarz inequality in order to obtain

\[
B_0(\iota^*(\mu), \iota^*(\mu))^2 = B(u^*(\mu), \mu)^2 
\leq B(u^*(\mu), u^*(\mu))B(\mu, \mu) 
= B_0(\iota^*(\mu), \iota^*(\mu))B(\mu, \mu).
\]

This yields

\[
Q_0(\iota^*(\mu)) = \frac{1}{2} B_0(\iota^*(\mu), \iota^*(\mu)) \leq \frac{1}{2} B(\mu, \mu) = Q(\mu).
\]

b) If \( \iota(\xi) = \mu \), then one has \( \iota^*(\mu) = \xi \) and

\[
Q_0(\iota^*(\mu)) = Q_0(\xi) = Q(\mu).
\]

holds. Assume that \( Q_0(\iota^*(\mu)) = Q(\mu) \). Since there is nothing to prove in the case \( \mu = 0 \), we suppose \( \mu \neq 0 \). Hence, \( \xi := \iota^*(\mu) \neq 0 \). Due to the Cauchy-Schwarz inequality in a), \( \iota(\xi) \) and \( \mu \) must be linearly dependent over \( Q \). Thus, \( \iota(\xi) = \alpha \mu \) for some \( \alpha \in Q \). We apply \( \iota^* \) in order to obtain

\[
\xi = \iota^* \iota(\xi) = \alpha \iota^*(\mu) = \alpha \xi,
\]

i.e. \( \alpha = 1 \). \( \square \)

**Proposition 1.2.11.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices and \( \iota : L_0 \rightarrow L \) an embedding. Then the kernel of \( \iota^* : \mathcal{L}^* \rightarrow L_0^\perp \) equals \( \mathcal{L}^* \cap \iota(L_0)^\perp \).

**Proof.** Since \( L_0 \) contains a \( Q \)-basis of \( (L_0)_Q \), one has \( \iota^*(\mu) = 0 \) if and only if

\[
B(\mu, \iota(l_0)) = B(\iota^*(\mu), l_0) = 0
\]

for all \( l_0 \in L_0 \), i.e. if and only if \( \mu \in \mathcal{L}^* \cap \iota(L_0)^\perp \). \( \square \)

The generic case is treated in the following

**Example 1.2.12.** Let \( S \in \text{Sym}_r(\mathbb{Z}), S_0 \in \text{Sym}_{r_0}(\mathbb{Z}) \) be positive definite and even. For \( A \in \mathbb{Z}^{r \times r_0} \) we define

\[
\iota_A : \mathbb{Z}^{r_0} \rightarrow \mathbb{Z}^r, \quad \iota(\lambda) := A\lambda, \quad \lambda \in \mathbb{Z}^{r_0}.
\]

Then the following statements are equivalent:
i) $\iota_A$ is an embedding $(\mathbb{Z}^{r_0}, Q_{S_0}) \to (\mathbb{Z}^r, Q_S)$.

ii) $S[A] = S_0$.

In this case, the dual $\iota_A^*$ is given by

$$S^{-1}Z^r \to S_0^{-1}Z^{r_0}, \quad \iota_A^* (\mu_S) = S_0^{-1} A^t\mu_S, \quad \mu_S \in S^{-1}Z^r$$

and one has

$$\ker (\iota_A^*) = G^{-1} \cdot \ker (A^t).$$

**Definition 1.2.13.** Let $L = (L, Q)$ be a lattice.

a) A sublattice $L_0$ of $L$ is called primitive in $L$, if $L / L_0$ is a free $\mathbb{Z}$-module.

b) An element $l \in L$ is called primitive, if $\langle l \rangle$ is primitive in $L$.

A characterization of primitive lattices is given in the next

**Lemma 1.2.14.** Let $L = (L, Q)$ be a lattice.

a) For a sublattice $L_0$ of $L$ the following statements are equivalent:

i) $L_0$ is primitive in $L$,

ii) there is a sublattice $L_1$ of $L$ such that $L = L_0 \oplus L_1$ as an inner direct sum.

b) For $l \in L$ the following statements are equivalent:

i) $l$ is primitive,

ii) $d^{-1}l \in L$ for $0 \neq d \in \mathbb{Z}$ implies $d = \pm 1$,

iii) $Ql \cap L = Zl$,

iv) $B(l, L^*) = Z$.

**Proof.** a) Its obvious that ii) implies i). Hence assume that $L / L_0$ is free and let $l_1, \ldots, l_m \in L$ such that

$$L / L_0 = \bigoplus_{i=1}^m \langle l_i + L_0 \rangle.$$

Let $L_1 := \langle l_1, \ldots, l_m \rangle Z$. By construction, $L_1 \cap L_0 = \{0\}$ and $L = L_0 + L_1$.

b) The equivalence of i) and ii) follows from a) and ii), iii), iv) are simple reformulations. Note in iv) that $L^{**} = L$.  

**Lemma 1.2.15.** Let $L = (L, Q)$ be a lattice and $L_0 \leq L$. Then the following assertions hold:

a) The sublattice $L_0^⊥L$ is primitive.
b) If \( L_0 \) is primitive in \( L \), then one has \( \left( L_0 \perp L \right)^\perp = L_0 \).

c) There is a unique lattice \( L_1 \), such that \( L_0 \leq L_1 \geq L_0 \) with \( L_1 \) primitive in \( L \) and \( L_1 / L_0 \) finite.

**Proof.**

a) Let \( \iota \) denote the inclusion \( L_0 \hookrightarrow L \). We consider the restriction

\[
\iota^*|_L : L \rightarrow L_0^*,
\]

whose kernel equals \( L_0 \perp L \). Hence, the quotient \( L / L_0 \perp L \) is isomorphic to a subgroup of the free abelian group \( L_0^* \) and thus is free itself.

b) Let \( L_1 := \left( L_0 \perp L \right)^\perp \). The inclusion \( L_0 \subseteq L_1 \) holds by definition. \( L_1 \) splits \( L_0 \) as a direct summand, since \( L_0 \) is also primitive in \( L_1 \). From \( r_{L_0} = r_{L_1} \) we conclude \( L_0 = L_1 \).

c) Since \( L_1 / L_0 \) is finite we have \( r_{L_1} = r_{L_0} \). Thus \( L_1 \) and \( L_0 \) span the same \( \mathbb{Q} \)-space, hence

\[
L_1 = \left( L_1 \perp L \right)^\perp = \left( L_0 \perp L \right)^\perp,
\]

if we apply b) on \( L_1 \). This proves both existence and uniqueness. \( \square \)

In view of 1.2.15 c) we give the following

**Remark 1.2.16.** If \( L_1 \geq L_0 \) and \( |L_1 / L_0| < \infty \), we call \( L_1 \) an overlattice of \( L_0 \). Note that the finiteness condition is equivalent to \( r_{L_1} = r_{L_0} \).

**Definition 1.2.17.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices and \( \iota : L_0 \hookrightarrow L \) an embedding. We call \( \iota \) primitive or a primitive embedding, if \( \iota(L_0) \) is primitive in \( L \).

**Lemma 1.2.18.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices and \( \iota : L_0 \hookrightarrow L \) an embedding. Then there exists a unique overlattice \( L_0' \geq L_0 \) in \( (L_0)Q \) such that the extension \( \tilde{\iota} : L_0' \rightarrow L \) is a primitive embedding.

**Proof.** By 1.2.15 there is a unique overlattice \( L_1 \) of \( \iota(L_0) \) such that \( L_1 \) is primitive in \( L \). Let \( L_0' := \iota^{-1}(L_1) \). Then \( L_0' \) is an overlattice of \( L_0 \) and \( \iota(L_0') = L_1 \) is primitive in \( L \), in other words, \( \tilde{\iota} : L_0' \rightarrow L \) is a primitive embedding. The uniqueness follows from the previous lemma. \( \square \)

**Lemma 1.2.19.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices. If \( \iota : L_0 \hookrightarrow L \) is a primitive embedding, then its dual \( \iota^* : L^* \rightarrow L_0^* \) is surjective.

**Proof.** Let \( L' \leq L \) such that \( L = \iota(L_0) \oplus L' \) as an inner direct sum. For \( \mu_0 \in L_0^* \) we consider the \( \mathbb{Z} \)-linear functional

\[
f_{\mu_0} : L_0 \rightarrow \mathbb{Z}, \quad f_{\mu_0}(l_0) := B_0(\mu_0, l_0), \quad l_0 \in L_0.
\]
We extend this to a \( \mathbb{Z} \)-linear functional \( \tilde{f}_{\mu} : L \to \mathbb{Z} \) by defining
\[
\tilde{f}_{\mu}(l) := \begin{cases} 
  f_{\mu}\left(\iota^{-1}(l)\right), & l \in \iota(L_0), \\
  0, & l \in L'. 
\end{cases}
\]
Let \( \mu \in L^* \) such that \( \tilde{f}_{\mu}(l) = B(\mu, l) \) for all \( l \in L \). Especially we obtain
\[
B(\mu, \iota(l_0)) = \tilde{f}_{\mu}(\iota(l_0)) = f_{\mu}(l_0) = B_0(\mu_0, l_0)
\]
for all \( l_0 \in L_0 \) and thus \( \iota^*(\mu) = \mu_0 \).

**Remark 1.2.20.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices. If \( \iota : L_0 \to L \) is an embedding, then the map \( \sigma \circ \iota : L_0 \to L \) for \( \sigma \in O(L) \) is also an embedding. Hence we obtain a natural action of \( O(L) \) on the set of embeddings \( \{ \iota : L_0 \to L \} \).

This gives rise to the following

**Definition 1.2.21.** Let \( L = (L, Q) \) and \( L_0 = (L_0, Q_0) \) be lattices. Two embeddings \( \iota, \kappa : L_0 \to L \) are called equivalent, if \( \iota = \sigma \circ \kappa \) for some \( \sigma \in O(L) \). Furthermore, \( \iota, \kappa \) are called stably equivalent, if \( \iota = \sigma \circ \kappa \) for some \( \sigma \in O_d(L) \).

We illustrate the theory by an

**Example 1.2.22.** Let \( L = (L, Q) \) be a lattice. Then for every \( t \in \mathbb{N} \) there is an one-to-one correspondence
\[
\{ l \in L : Q(l) = t \} \longleftrightarrow \{ \iota : \mathbb{Z}(t) \to L \}.
\]
More precisely, if \( l \in L \) and \( Q(l) = t \), then the map
\[
\iota_l : \mathbb{Z} \to L, \quad w \mapsto lw, \quad w \in \mathbb{Z}
\]
is an embedding of \( \mathbb{Z}(t) \) into \( L \). Conversely, if \( t \in \mathbb{N} \) and \( \iota : \mathbb{Z}(t) \to L \), then one has \( \iota = \iota_l \) where \( l = \iota(1) \). The dual \( \iota_l^* \) is given by
\[
\iota_l^* : L^* \to \frac{1}{2t} \mathbb{Z}, \quad \iota_l^*(\mu) = \frac{1}{2t} B(\mu, l), \quad \mu \in L^*.
\]
One has \( \sigma \circ \iota_l = \iota_{\sigma(l)} \) for all \( \sigma \in O(L) \). Hence, two embeddings \( \iota_l \) and \( \iota_{l'} \) for \( l, l' \in L \) are equivalent, if and only if \( \sigma(l) = l' \) for some orthogonal transformation \( \sigma \in O(L) \).

Concerning irreducible root lattices, the special case \( t = 1 \) leads to the following

**Corollary 1.2.23.** Let \( L = (L, Q) \) be an irreducible root lattice and \( \iota, \kappa : \mathbb{Z} \to L \) embeddings. Then \( \iota \) and \( \kappa \) are stably equivalent.

**Proof.** The Weyl group \( W(L) \leq O_d(L) \) of \( L \) acts transitively on the roots of \( L \), cf. [7, Lemma 1.10]. The claim follows then from 1.2.22. \( \square \)
1.3 Irreducible root lattices of low rank

In this section we will consider explicit realizations of certain irreducible root lattices of small rank in the quadratic space $\mathbb{R}^8$ equipped with the quadratic form

$$Q(x) := x^t x, \quad x \in \mathbb{R}^8.$$

The standard basis of $\mathbb{R}^8$ will be denoted by $e_1, \ldots, e_8$. For $i = 1, \ldots, 8$ let $\pi_i : \mathbb{R}^8 \to \mathbb{R}$ denote the projection on the $i$-th coordinate with respect to the standard basis.

The following construction will have the advantage of a simple determination of the pull-backs of the Jacobi theta functions as we will see later.

The irreducible root lattice $E_8$

The irreducible root lattice $E_8$ is realized as the underlying lattice of Coxeter’s integral Cayley numbers, cf. [4, p. 101f]. To this end we define

$$\alpha_1 := \frac{1}{2}(e_2 + e_3 + e_4 + e_5), \quad \alpha_2 := \frac{1}{2}(e_1 + e_3 + e_5 + e_8),$$

$$\alpha_3 := \frac{1}{2}(e_1 + e_2 + e_5 + e_6), \quad \alpha_4 := \frac{1}{2}(e_1 + e_2 + e_3 + e_7)$$

and

$$E_8 := \langle e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \mathbb{Z}.$$

The quadratic form $Q$ restricted to $E_8$ is integer-valued, hence $E_8 := (E_8, Q|_{E_8})$ is a positive definite and even lattice of rank 8. Hence, $E_8$ is a root lattice and a direct calculation shows that $\det E_8 = 1$. The underlying $\mathbb{Z}$-module $E_8$ decomposes as a disjoint union

$$E_8 = \bigcup_{i=1}^{16} S_i$$

of sets $S_i$, $i = 1, \ldots, 16$, which are defined according to table 1.4 at the end of this section.

The irreducible root lattice $E_7$

The irreducible root lattice $E_7$ is constructed as the orthogonal complement of $e_8$ in $E_8$, i.e.

$$E_7 = \langle e_8 \rangle^\perp \mathbb{Z} \mathbb{Z}.$$

Hence, $E_7 := (E_7, Q|_{E_7})$ is a positive definite and even lattice of rank 7. As a module we have

$$E_7 = \langle e_1, e_2, e_3, e_4, \alpha_1, \alpha_3, \alpha_4 \rangle \mathbb{Z}.$$
i.e. $E_7$ is a root lattice and a direct calculation shows that $\det E_7 = 2$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$E_7 = \bigcup_{i=1}^{8} S_i \cap \langle e_8 \rangle^\perp.$$ 

A complete set of representatives of the discriminant group $E_7^*/E_7$ is given by

$$\left\{ 0, \frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7}{2} \right\}.$$ 

**The irreducible root lattice $E_6$**

The root lattice $E_6$ is constructed as the orthogonal complement of $e_1 + e_3 + e_5$ in $E_7$, i.e.

$$E_6 := \langle e_1 + e_3 + e_5 \rangle^\perp_{E_7}.$$ 

Hence, $E_6 = (E_6, Q|_{E_6})$ is a positive definite and even lattice of rank 6. As a module we have

$$E_6 = \langle e_2, e_4, e_6, \alpha_1 - e_3, \alpha_3 - e_1, \alpha_4 - e_1 \rangle_{\mathbb{Z}}$$

$$= \left\langle e_2, e_4, e_6, \frac{e_2 + e_4 + e_6 + e_7}{2}, \alpha_3 - e_1, \alpha_4 - e_1 \right\rangle_{\mathbb{Z}},$$

i.e. $E_6$ is a root lattice and a direct calculation shows that $\det E_6 = 3$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$E_6 = \bigcup_{i=1}^{8} S_i \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp.$$ 

A complete set of representatives of the discriminant group $E_6^*/E_6$ is given by

$$\left\{ 0, \frac{e_1 + e_3 - 2e_5}{3}, -\frac{e_1 + e_3 - 2e_5}{3} \right\}.$$ 

**The irreducible root lattice $D_4$**

The root lattice $D_4$ is constructed as the orthogonal complement of $\{e_1 - e_3, e_3 - e_5\}$ in $E_6$, i.e.

$$D_4 = \langle e_1 - e_3, e_3 - e_5 \rangle^\perp_{E_6}.$$
Hence, $D_4 = (D_4, Q|_{D_4})$ is a positive definite and even lattice of rank 4. As a module we have

$$D_4 = \left\langle e_2, e_4, e_6, \frac{e_2 + e_4 + e_6 + e_7}{2} \right\rangle_{\mathbb{Z}},$$

i.e. $D_4$ is a root lattice and a direct calculation shows that $\det D_4 = 4$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$D_4 = \bigcup_{i \in \{1, 4\}} S_i \cap (e_1, e_3, e_5, e_8)^\perp.$$

A complete set of representatives of the discriminant group $D_4^*/D_4$ is given by

$$\left\{0, \frac{e_2 + e_4}{2}, \frac{e_2 + e_6}{2}, \frac{e_2 + e_7}{2}\right\}.$$

### The irreducible root lattice $A_2$

The root lattice $A_2$ is constructed as the orthogonal complement of $\{e_4 - e_6, e_4 - e_7\}$ in $D_4$, i.e.

$$A_2 = (e_4 - e_6, e_4 - e_7)^\perp_{D_4}.$$ 

Hence, $A_2 := (A_2, Q|_{A_2})$ is a positive definite and even lattice of rank 2. As a module we have

$$A_2 = \left\langle e_2, \frac{e_2 + e_4 + e_6 + e_7}{2} \right\rangle_{\mathbb{Z}},$$

i.e. $A_2$ is a root lattice and a direct calculation shows that $\det A_2 = 3$. A complete set of representatives of the discriminant group $A_2^*/A_2$ is given by

$$\left\{0, \frac{e_4 + e_6 + e_7}{3}, -\frac{e_4 + e_6 + e_7}{3}\right\}.$$

### The irreducible root lattice $A_1$

The root lattice $A_1$ is constructed as the orthogonal complement of $e_4 + e_6 + e_7$ in $A_2$, i.e.

$$A_1 = (e_4 + e_6 + e_7)^\perp_{A_2}.$$ 

Hence, $A_1$ is a positive definite and even lattice of rank 1. As a module,

$$A_1 = \langle e_2 \rangle_{\mathbb{Z}}.$$
i.e. $A_1$ is a root lattice and a direct calculation shows that $\det A_1 = 2$. Note that $A_1$ is identified with $\mathbb{Z} = (\mathbb{Z}, \Omega^2)$, whenever it is convenient. A complete set of representatives of the discriminant group $A_1^\ast / A_1$ is given by $\{0, \frac{1}{2} \varepsilon_2\}$.

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<th>$S$</th>
<th>$\pi_1(S)$</th>
<th>$\pi_2(S)$</th>
<th>$\pi_3(S)$</th>
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Table 1.4: Distribution of the integral and half-integral components of $E_8$ with respect to the standard basis $e_1, \ldots, e_8$ of $\mathbb{R}^8$.
Elliptic Functions of Lattice-Index

2.1 Quadratic forms of higher degree and the Heisenberg group

Let $\text{Sym}_n^\#(\mathbb{Z})$ denote the dual lattice of $\text{Sym}_n(\mathbb{Z})$ with respect to the trace bilinear form, i.e.

$$\text{Sym}_n^\#(\mathbb{Z}) := \{ M \in \text{Sym}_n(\mathbb{Q}) : \text{tr}(SM) \in \mathbb{Z} \text{ for all } S \in \text{Sym}_n(\mathbb{Z}) \}.$$ 

Definition 2.1.1. Let $L = (L, Q)$ be a lattice. The map

$$Q^{(n)} : L^{(n)} \to \text{Sym}_n^\#(\mathbb{Z}), \quad Q^{(n)}(l) := \frac{1}{2} \left( B(l_i, l_j) \right)_{1 \leq i, j \leq n}$$

for $l = (l_1, \ldots, l_n) \in L^{(n)}$ is called the quadratic form of degree $n$ associated to $L$.

Lemma 2.1.2. Let $L = (L, Q)$ be a lattice. Define $B^{(n)} : L^{(n)} \times L^{(n)} \to \text{Sym}_n^\#(\mathbb{Z})$ by

$$B^{(n)}(l, l') = Q^{(n)}(l + l') - Q^{(n)}(l) - Q^{(n)}(l')$$

for $l, l' \in L^{(n)}$. Then one has

$$B^{(n)}(l, l') = \frac{1}{2} \left( B(l_i, l'_i) + B(l'_i, l_i) \right)_{1 \leq i, j \leq n}.$$ 

We call $B^{(n)}$ the bilinear form of degree $n$ associated to $(L, Q)$. The identity

$$Q^{(n)}(l) = \frac{1}{2} B^{(n)}(l, l)$$

holds for all $l \in L^{(n)}$.

The generic case is treated in the following

Remark 2.1.3. Let $S \in \text{Sym}_r(\mathbb{Z})$ positive definite and even. Then $(\mathbb{Z}^r)^{(n)}$ can be naturally identified with $\mathbb{Z}^{r \times n}$. and we have

$$Q^{(n)}_S(X) = \frac{1}{2} S[X], \quad B^{(n)}_S(X, Y) = \frac{1}{2} (X^tSY + Y^tSX), \quad X, Y \in \mathbb{Z}^{r \times n}.$$
Conversely, let \( L = (L, Q) \) be a lattice and \( S \) denote its Gram matrix with respect to some basis \((b_1, \ldots, b_r)\) of \( L \). Let \( \kappa : L \rightarrow \mathbb{Z}^r \) denote the coordinate system with respect to \((b_1, \ldots, b_r)\). Let \( \kappa^{(n)} : L^{(n)} \rightarrow \mathbb{Z}^{r \times n} \) denote the map defined by

\[
\kappa^{(n)}(l) := (\kappa(l_1), \ldots, \kappa(l_n)), \quad l = (l_1, \ldots, l_n) \in L^{(n)}.
\]

Then the identity

\[
Q^{(n)}(l) = \frac{1}{2} \kappa^{(n)}(l)^t S \kappa^{(n)}(l) = \frac{1}{2} S \kappa^{(n)}(l)
\]

holds for all \( l \in L^{(n)} \).

We will implicitly assume that \( B^{(n)} \) and \( Q^{(n)} \) are extended by \( \mathbb{F} \)-linearity. All aforementioned identities are also valid in \( L_F^{(n)} \times L_F^{(n)} \) resp. \( L_F^{(n)} \) for \( F \in \{ \mathbb{Q}, \mathbb{R}, \mathbb{C} \} \).

By a direct verification, we obtain the following

**Lemma 2.1.4.** Let \( L = (L, Q) \) be a lattice and \( 1 \leq j \leq n \). Then for all \( l \in L^{(n)}_\mathbb{R} \) the following assertions hold:

a) \( Q^{(j)}(lX) = Q^{(n)}(l)[X] \) for all \( X \in \mathbb{R}^{n \times j} \),

b) \( \text{tr}(B^{(n)}(l, l')) = \text{tr}(B^{(n)}(lS, l')) \) for all \( S \in \text{Sym}_n(\mathbb{R}) \),

c) \( Q^{(n)}(l) \geq 0 \) and \( Q^{(n)}(l) > 0 \) if and only if \( \dim_\mathbb{R}(l_1, \ldots, l_n) = n \), where \( l = (l_1, \ldots, l_n) \).

**Remark 2.1.5.** Let \( L = (L, Q) \) be a lattice. Then the lattices \( (L^{(n)}, \text{tr} \circ Q^{(n)}) \) and \( nL \) are isometric.

**Definition 2.1.6.** Let \( L = (L, Q) \) be a lattice. The real Heisenberg group of degree \( n \) with respect to \( L \) is the group

\[
H^{(n)}_\mathbb{R}(L) := L^{(n)}_\mathbb{R} \times L^{(n)}_\mathbb{R} \times S^1
\]

with underlying group law

\[
(\lambda, \mu, \zeta) \cdot (\lambda', \mu', \zeta') := \left(\lambda + \lambda', \mu + \mu', \zeta \cdot \zeta' \cdot e^{\pi i \text{tr}(B^{(n)}(\lambda, \mu') - B^{(n)}(\lambda', \mu))}\right)
\]

for \( \lambda, \lambda', \mu, \mu' \in L^{(n)}_\mathbb{R} \) and \( \zeta, \zeta' \in S^1 \). Furthermore, for \( \lambda, \mu \in L^{(n)}_\mathbb{R} \) we define

\[
[\lambda, \mu] := \left(\lambda, \mu, e^{\pi i \text{tr}(B^{(n)}(\lambda, \mu))}\right) \in H^{(n)}_\mathbb{R}(L).
\]

**Remark 2.1.7.** Let \( L = (L, Q) \) be a lattice.

a) \( S^1 \) is identified in \( H^{(n)}_\mathbb{R}(L) \) via

\[
S^1 \hookrightarrow H^{(n)}_\mathbb{R}(L), \quad \zeta \mapsto (0, 0, \zeta).
\]
Let $L^{(n)}_\mathbb{R} \times L^{(n)}_\mathbb{R}$ be identified in $H^{(n)}_\mathbb{R}(L)$ via $(\lambda, \mu) \mapsto [\lambda, \mu]$. Note that the map $[\cdot, \cdot]$ is no group homomorphism, in contrast to its restrictions

$$
L^{(n)}_\mathbb{R} \rightarrow H^{(n)}_\mathbb{R}(L), \quad \lambda \mapsto [\lambda, 0],
$$
$$
L^{(n)}_\mathbb{R} \rightarrow H^{(n)}_\mathbb{R}(L), \quad \mu \mapsto [0, \mu].
$$

For $(\lambda, \mu), (\lambda', \mu') \in L^{*(n)} \times L^{(n)}$ we have

$$
[\lambda, \mu] \cdot [\lambda', \mu'] = (\lambda, \mu, e^{\pi i \text{tr}(B^{(n)}(\lambda, \mu))}) \cdot (\lambda', \mu', e^{\pi i \text{tr}(B^{(n)}(\lambda', \mu'))})
$$
$$
= (\lambda + \lambda', \mu + \mu', e^{\pi i \text{tr}(B^{(n)}(\lambda, \mu) + B^{(n)}(\lambda', \mu'))} \cdot e^{\pi i \text{tr}(B^{(n)}(\lambda, \mu) - B^{(n)}(\lambda', \mu'))})
$$
$$
= (\lambda + \lambda', \mu + \mu', e^{2\pi i \text{tr}(B^{(n)}(\lambda + \lambda', \mu + \mu'))})
$$
$$
= [\lambda + \lambda', \mu + \mu'],
$$

since $e^{2\pi i \text{tr}(B^{(n)}(\lambda', \mu'))} = 1$. Consequently, the restrictions of $[\cdot, \cdot]$ to $L^{*(n)} \times L^{(n)}$ and $L^{(n)} \times L^{*(n)}$ are monomorphisms of the groups.

We introduce the integral Heisenberg group:

**Definition 2.1.8.** Let $L = (L, Q)$ be a lattice.

a) The integral Heisenberg group of degree $n$ with respect to $L$ is defined as

$$
H^{(n)}(L) := \left[ L^{(n)}, L^{(n)} \right].
$$

b) The extended integral Heisenberg group of degree $n$ with respect to $L$ is defined as

$$
H^{(n)}(L)^* := \left\{ (\lambda, \mu, \xi) \in H^{(n)}_\mathbb{R}(L) : e^{2\pi i \text{tr}(B^{(n)}(\lambda, \mu', \mu) - B^{(n)}(\mu, \mu'))} = 1 \text{ for all } l, l' \in L^{(n)} \right\}.
$$

**Remark 2.1.9.** Let $L = (L, Q)$ be a lattice.

a) For each $(\lambda, \mu, \xi) \in H^{(n)}_\mathbb{R}(L)$ the map

$$
\chi_{\lambda, \mu, \xi} : H^{(n)}(L) \rightarrow \mathbb{C}^\times, \quad [l, l'] \mapsto e^{2\pi i \text{tr}(B^{(n)}(\lambda, l') - B^{(n)}(\mu, l'))}
$$

is an abelian character of $H^{(n)}(L)$ and the map

$$
H^{(n)}_\mathbb{R}(L) \rightarrow H^{(n)}(L)^{\text{ab}}, \quad (\lambda, \mu, \xi) \mapsto \chi_{\lambda, \mu, \xi}
$$

is a homomorphism of the groups with kernel $H^{(n)}(L)^*$.
b) The identity
\[(\lambda, \mu, \zeta) \cdot [l, l'] \cdot (\lambda, \mu, \zeta)^{-1} = \chi_{\lambda, \mu, \zeta} ([l, l']) \cdot [l, l']\]
for \([l, l'] \in H^{(n)}(\mathbb{L})\) and \((\lambda, \mu, \zeta) \in H^{(n)}_{\mathbb{R}}(\mathbb{L})\) shows that \(H^{(n)}(\mathbb{L})^*\) is precisely the centralizer of \(H^{(n)}(\mathbb{L})\) in \(H^{(n)}_{\mathbb{R}}(\mathbb{L})\).

c) \(H^{(n)}(\mathbb{L}) \cdot S^1\) is a normal subgroup of \(H^{(n)}(\mathbb{L})^*\) and one has
\[H^{(n)}(\mathbb{L})^*/(H^{(n)}(\mathbb{L}) \cdot S^1) \cong (L^* / L)^{2n} \]
for \((L, Q)\) be a lattice.

d) We will see later, that \(H^{(n)}(\mathbb{L})^*\) is the precise invariance group of elliptic functions of index \(L\) and degree \(n\).

**Definition 2.1.10.** Let \(\mathbb{L} = (L, Q)\) be a lattice.

a) The Siegel upper half-space of degree \(n\) is defined as
\[\mathcal{H}_n = \{ Z = X + iY \in \text{Sym}_n(\mathbb{C}) : X = X^t, Y = Y^t > 0 \}.\]
We have \(\mathcal{H}_1 = \mathcal{H}\).

b) The Jacobi half-space of degree \(n\) with respect to \(\mathbb{L}\) is defined as
\[\mathcal{H}^J_n(\mathbb{L}) := \mathcal{H}_n \times L^{(n)}_{\mathbb{C}} = \{ (Z, W) : Z \in \mathcal{H}_n, W \in L^{(n)}_{\mathbb{C}} \} \]
for \((Z, W)\) is precisely the centralizer of \(L\).

**Remark 2.1.11.** The Siegel upper half-space \(\mathcal{H}_n\) is implicitly identified with \(\mathcal{H}_n \times \{0\}\) inside \(\mathcal{H}^J_n(\mathbb{L})\). Furthermore, we will identify \(\mathcal{O}(\mathcal{H}_n)\) as a subring of \(\mathcal{O}(\mathcal{H}^J_n(\mathbb{L}))\) in a natural way.

**Definition 2.1.12.** Let \(\mathbb{L} = (L, Q)\) be a lattice. Let \(\phi \in \mathcal{O}(\mathcal{H}^J_n(\mathbb{L}))\) and \((\lambda, \mu, \zeta) \in H^{(n)}_{\mathbb{R}}(\mathbb{L})\). Then the function
\[\phi|_{\mathbb{L}}(\lambda, \mu, \zeta) : \mathcal{H}^J_n(\mathbb{L}) \to \mathbb{C}\]
is defined pointwisely by
\[\phi|_{\mathbb{L}}(\lambda, \mu, \zeta)(Z, W) := \zeta \cdot e^{2\pi i \text{tr}(Q^{(n)}(\lambda) Z + B^{(n)}(\lambda, W) + \frac{i}{2} B^{(n)}(\lambda, \mu))} \cdot \phi(Z, W + \lambda Z + \mu)\]
for \((Z, W) \in \mathcal{H}^J_n(\mathbb{L})\).

**Remarks 2.1.13.** Let \(\mathbb{L} = (L, Q)\) be a lattice.

a) The assignment
\[((\lambda, \mu, \zeta), \phi) \mapsto \phi|_{\mathbb{L}}(\lambda, \mu, \zeta), \quad (\lambda, \mu, \zeta) \in H^{(n)}_{\mathbb{R}}(\mathbb{L}), \phi \in \mathcal{O}(\mathcal{H}^J_n(\mathbb{L})),\]
defines an action of \(H^{(n)}_{\mathbb{R}}(\mathbb{L})\) on \(\mathcal{O}(\mathcal{H}^J_n(\mathbb{L}))\).

b) For \((\lambda, \mu) \in L^{(n)} \times L^{(n)}\) or \((\lambda, \mu) \in L^{(n)} \times L^{* (n)}\) one has
\[\phi|_{\mathbb{L}}(\lambda, \mu)(Z, W) = e^{2\pi i \text{tr}(Q^{(n)}(\lambda) Z + B^{(n)}(\lambda, W))} \cdot \phi(Z, W + \lambda Z + \mu)\]
for \((Z, W) \in \mathcal{H}^J_n(\mathbb{L})\).
2.2 Elliptic functions of lattice-index

Definition 2.2.1. Let $L = (L, Q)$ be a lattice. A function $\phi \in \mathcal{O}(\mathcal{H}^1_n(L))$ is called elliptic of index $L$ and degree $n$, if

$$\phi|_{[l, l']} = \phi$$

for all $[l, l'] \in H(n)(L)$. The set of elliptic functions of index $L$ and degree $n$ is denoted by $\mathcal{E}^{(n)}(L)$.

Regarding the structure of $\mathcal{E}^{(n)}(L)$, we have the following

Remark 2.2.2. Let $L = (L, Q)$ be a lattice. $\mathcal{E}^{(n)}(L)$ carries a canonical module structure over $\mathcal{O}(\mathcal{H}_n)$ via multiplication defined pointwisely by

$$(h \cdot \phi)(Z, W) := h(Z) \cdot \phi(Z, W), \quad (Z, W) \in \mathcal{H}^1_n(L)$$

for $\phi \in \mathcal{E}^{(n)}(L)$ and $h \in \mathcal{O}(\mathcal{H}_n)$.

Again we treat the generic case:

Remark 2.2.3. Let $L = (L, Q)$ be a lattice and $S$ denote its Gram matrix with respect to some basis $(b_1, \ldots, b_r)$ of $L$. Let $\kappa : L \rightarrow \mathbb{Z}^r$ denote the coordinate system with respect to $(b_1, \ldots, b_r)$. Then the assignment

$$\phi \mapsto \left(\tilde{\phi} : \mathcal{H}_n \times \mathbb{C}^r \rightarrow \mathbb{C}, \quad (Z, W) \mapsto \phi(Z, \kappa^{-1}(W))\right)$$

is an $\mathcal{O}(\mathcal{H}_n)$-module isomorphism $\mathcal{E}^{(n)}(L) \rightarrow \mathcal{E}^{(n)}(\mathbb{Z}^r, QS)$.

Some functorial constructions are explained in the following

Definition 2.2.4. Let $L, L'$ be lattices.

a) For $\phi \in \mathcal{O}(\mathcal{H}^1_n(L)), \phi' \in \mathcal{O}(\mathcal{H}^1_{n'}(L'))$ we define its tensor product

$$\phi \otimes \phi' \in \mathcal{O}(\mathcal{H}^1_n(L) \times \mathcal{H}^1_{n'}(L'))$$

pointwisely by

$$\phi \otimes \phi'((Z, W), (Z', W')) := \phi(Z, W) \cdot \phi'(Z', W')$$

for $(Z, W) \in \mathcal{H}^1_n(L)$ and $(Z', W') \in \mathcal{H}^1_{n'}(L')$.

b) Let $\phi \in \mathcal{E}^{(n)}(L)$ and $\phi' \in \mathcal{E}^{(n)}(L')$. Then the function

$$\phi \otimes \phi'|_{Z=Z'} : (Z, (W, W')) \mapsto \phi \otimes \phi'((Z, W), (Z, W'))$$

belongs to $\mathcal{E}^{(n)}(L \oplus L')$. 
A first nontrivial example is given by the Jacobi theta functions of degree \( n \) associated to a lattice \( L \), cf. [27, § 3] or [24, (3.26) p. 49].

**Definition 2.2.5.** Let \( L = (L, Q) \) be a lattice. Then the Jacobi theta function \( \vartheta_{L,\mu}^{(n)} \) of degree \( n \) associated to \( L \) and \( \mu \in L_{\mathbb{R}}^{(n)} \) is defined by

\[
\vartheta_{L,\mu}^{(n)}(Z, W) := \sum_{l \in \mu + L^{(n)}} 1_{L} |l, 0|(Z, W)
\]

\[
= \sum_{l \in \mu + L^{(n)}} e^{2\pi i \operatorname{tr}(Q^{(n)}(l)Z + B^{(n)}(l, W))}, \quad (Z, W) \in \mathcal{H}_{n}(L),
\]

where we suppress the superscript if \( n = 1 \). The definition only depends on the coset \( \mu + L^{(n)} \). The series converges absolutely and uniformly on each vertical strip

\[
\left\{ (Z, W) \in \mathcal{H}_{n}(L) : Y \geq \delta I_{n}, \operatorname{tr}(B(W, \overline{W})) \leq \delta^{-1} \right\}, \quad \delta > 0.
\]

For \( \mu \in L \) we will sometimes simply write \( \vartheta_{L}^{(n)} \) instead of \( \vartheta_{L,\mu}^{(n)} \). Furthermore, by a slight abuse of notation, we define

\[
\vartheta_{L}^{(n)}(Z) := \vartheta_{L,\mu}^{(n)}(Z, 0), \quad Z \in \mathcal{H}_{n}.
\]

Basic transformation properties of \( \vartheta_{L,\mu}^{(n)} \) are given in the following

**Proposition 2.2.6.** Let \( L = (L, Q) \) be a lattice and \( \mu \in L_{\mathbb{R}}^{(n)} \). Then the following assertions hold:

a) \( \vartheta_{L,\mu}^{(n)} |_{L} [l, l'] = e^{2\pi i \operatorname{tr}(B^{(n)}(\mu, l'))} \vartheta_{L,\mu}^{(n)} \) for all \( l, l' \in L^{(n)} \) and \( \vartheta_{L,\mu}^{(n)} \in \mathcal{E}^{(n)}(L) \) if and only if \( \mu \in L^{*} \).

b) \( \vartheta_{L,\mu}^{(n)}(Z + S, W) = e^{2\pi i \operatorname{tr}(Q^{(n)}(\mu)S)} \cdot \vartheta_{L,\mu}^{(n)} \) for \( S \in \operatorname{Sym}_{n}(\mathbb{Z}) \) such that \( \mu S \in L^{*} \).

c) \( \vartheta_{L,\mu}^{(n)}(Z[U], WU) = \vartheta_{L,\mu U}^{(n)}(Z, W) \) for all \( U \in \operatorname{GL}_{n}(\mathbb{Z}) \).

**Proof.** All properties follow from a straightforward calculation. Note in b) that the assertion \( \operatorname{tr}(B^{(n)}(l, \mu)S) \in \mathbb{Z} \) holds for all \( l \in L^{(n)} \) if and only if \( \mu S \in L^{*} \). In c) use that \( L^{(n)}U = L^{(n)} \) for \( U \in \operatorname{GL}_{n}(\mathbb{Z}) \). \( \square \)

**Remark 2.2.7.** Let \( L = (L, Q) \) be a lattice and \( \mu \in L^{*} \). In view of the Fourier expansions, we have

\[
\dim \left\langle \vartheta_{L,\mu}^{(n)}(Z, \cdot), \mu \in (L^{*}/L)^{(n)} \right\rangle = (\det L)^{n}
\]

for every \( Z \in \mathcal{H}_{n} \) as functions of \( W \).

The algebraic structure of \( \mathcal{E}^{(n)}(L) \) is revealed in the following theorem, cf. [24, Proposition 3.5] or [27, 3.1. Lemma]:
2.2 Elliptic functions of lattice-index

**Theorem 2.2.8.** Let \( \mathcal{L} = (L, Q) \) be a lattice. Then the space \( \mathcal{E}^{(n)}(\mathcal{L}) \) of elliptic functions of degree \( n \) and index \( \mathcal{L} \) is a free \( \mathcal{O}(\mathcal{H}_n) \)-module of rank \( (\det \mathcal{L})^n \) and decomposes as a direct sum

\[
\mathcal{E}^{(n)}(\mathcal{L}) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(\mathcal{H}_n) \cdot \varphi_{\mathcal{L}/\mu}^{(n)}.
\]

**Proof.** Let \( \phi \in \mathcal{E}^{(n)}(\mathcal{L}) \) and \( Z \in \mathcal{H}_n \). The function \( \phi(Z, \cdot) \) is periodic in \( W \) with respect to \( L^{(n)} \) and therefore has a Fourier expansion of the form

\[
\phi(Z, W) = \sum_{\mu \in L^{(n)}} c(\mu, Z)e^{2\pi i \text{tr}B^{(n)}(\mu, W)}, \quad W \in L_C^{(n)}
\]

for certain functions \( c(\mu, \cdot) : \mathcal{H}_n \to \mathbb{C} \). For \( l \in L^{(n)} \), we apply \( \big|_{\mathcal{L}^{[l, 0]}} \) on \( \phi \) in order to obtain

\[
\phi \big|_{\mathcal{L}^{[l, 0]}}(Z, W) = e^{2\pi i \text{tr}(Q^{(n)}(l)Z + B^{(n)}(l, W))} \phi(Z, W + lZ)
\]

\[
= e^{2\pi i \text{tr}(Q^{(n)}(l)Z)} \cdot e^{2\pi i \text{tr}(B^{(n)}(l, W))} \sum_{\mu \in L^{(n)}} c(\mu, Z)e^{2\pi i \text{tr}(B^{(n)}(\mu, W + lZ))}
\]

\[
= \sum_{\mu \in L^{(n)}} e^{2\pi i \text{tr}((B^{(n)}(\mu, l + Q^{(n)}(l)))Z)} \cdot c(\mu, Z) \cdot e^{2\pi i \text{tr}(B^{(n)}(\mu + l, W))}
\]

\[
= \sum_{\mu \in L^{(n)}} e^{2\pi i \text{tr}((Q^{(n)}(\mu + l) - Q^{(n)}(\mu))Z)} \cdot c(\mu, Z) \cdot e^{2\pi i \text{tr}(B^{(n)}(\mu + l, W))}.
\]

A comparison of the Fourier coefficients yields

\[
c(\mu + l, Z) = e^{2\pi i \text{tr}((Q^{(n)}(\mu + l) - Q^{(n)}(\mu))Z)} \cdot c(\mu, Z), \quad Z \in \mathcal{H}_n, \mu \in L^{(n)}^*, l \in L^{(n)}.
\]

Hence, by a simple rearrangement we deduce

\[
\phi(Z, W) = \sum_{\mu \in L^{(n)}^*} c(\mu, Z)e^{2\pi i \text{tr}(B^{(n)}(\mu, W))}
\]

\[
= \sum_{\mu \in (L^*/L)^{(n)}} \sum_{l \in (L^{(n)})^* + \mu} c(l, Z)e^{2\pi i \text{tr}(B^{(n)}(l, W))}
\]

\[
= \sum_{\mu \in (L^*/L)^{(n)}} \sum_{l \in (L^{(n)})^* + \mu} e^{2\pi i \text{tr}((Q^{(n)}(l) - Q^{(n)}(\mu))Z)} c(\mu, Z)e^{2\pi i \text{tr}(B^{(n)}(l, W))}
\]

\[
= \sum_{\mu \in (L^*/L)^{(n)}} c(\mu, Z)e^{-2\pi i \text{tr}(Q^{(n)}(\mu)Z)} \left( \sum_{l \in (L^{(n)})^* + \mu} e^{2\pi i \text{tr}(Q^{(n)}(l)Z + B^{(n)}(l, W))} \right)
\]

\[
= \sum_{\mu \in (L^*/L)^{(n)}} h_{\mu}(Z) \cdot \varphi_{\mathcal{L}/\mu}^{(n)}(Z, W),
\]

where

\[
h_{\mu}(Z) := c(\mu, Z)e^{-2\pi i \text{tr}(Q^{(n)}(\mu)Z)}, \quad Z \in \mathcal{H}_n.
\]

The uniqueness and holomorphicity of the functions \( h_{\mu} \) follows from 2.2.7. For the latter fact confer also [24, Lemma 3.4]. \( \square \)
2.2.8 gives rise to the following

**Definition 2.2.9.** Let \( L \) be a lattice and \( \phi \in \mathcal{E}^{(n)}(L) \). The decomposition

\[
\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_{\mu} \vartheta_{L,\mu}^{(n)}
\]

according to 2.2.8 is called the theta decomposition of \( \phi \).

We revisit the centralizer \( H^{(n)}(L) \ast \) of \( H^{(n)}(L) \) in \( H^{(n)}_{\mathbb{R}}(L) \) and prove the following

**Theorem 2.2.10.** Let \( L = (L, Q) \) be a lattice. Then the following assertions hold:

a) \( \mathcal{E}^{(n)}(L) \) is invariant under \( H^{(n)}(L) \ast \). More precisely, \( H^{(n)}(L)^{\ast} \) is the maximal invariance group of \( \mathcal{E}^{(n)}(L) \) in \( H^{(n)}_{\mathbb{R}}(L) \), i.e. for \( 0 \neq \phi \in \mathcal{E}^{(n)}(L) \) and \( (\lambda, \mu, \zeta) \in H^{(n)}_{\mathbb{R}}(L) \) one has

\[
\phi \mid_{L}(\lambda, \mu, \zeta) \in \mathcal{E}^{(n)}(L) \text{ if and only if } (\lambda, \mu) \in H^{(n)}(L)^{\ast}.
\]

b) For \( (\lambda, \nu, \zeta) \in H^{(n)}(L)^{\ast} \) and \( \mu \in L^{(n)} \ast \) the transformation formula

\[
\vartheta_{L,\mu}^{(n)} \mid_{L}(\lambda, \nu, \zeta) = \zeta \cdot e^{2 \pi i \text{tr}(B^{(n)}(\mu + \frac{1}{2} \lambda, \nu))} \cdot \vartheta_{L,\mu+\lambda}^{(n)}
\]

holds.

**Proof.** a) For \( (\lambda, \nu, \zeta) \in H^{(n)}_{\mathbb{R}}(L) \) and \( [l, l'] \in H^{(n)}(L) \) we have

\[
(\lambda, \nu, \zeta) \cdot [l, l'] = \chi_{\lambda, \nu, \zeta}([l, l']) \cdot [l, l'] \cdot (\lambda, \nu, \zeta).
\]

Then both assertions follow immediately from

\[
\phi \mid_{L}(\lambda, \nu, \zeta) \mid_{L}[l, l'] = \chi_{\lambda, \nu, \zeta}(l', l) \cdot \phi \mid_{L}(\lambda, \nu, \zeta).
\]

b) For \( \mu \in L^{(n)} \ast \) and \( l \in L^{(n)} \) we obtain

\[
[\mu + l, 0] \cdot (\lambda, \nu, \zeta) = \left( \mu + \lambda + l, \nu, \zeta \cdot e^{\pi i \text{tr}(B^{(n)}(\mu + l, \nu))} \right) = \left( \mu + \lambda + l, \nu, \zeta \cdot e^{\pi i \text{tr}(B^{(n)}(\mu, \nu))} \right)
\]

from \( v \in L^{(n)} \ast \). Hence,

\[
\left( 1 \mid_{L} [\mu + l, 0] \right) \mid_{L}(\lambda, \nu, \zeta) = \zeta \cdot e^{2 \pi i \text{tr}(B^{(n)}(\mu + \frac{1}{2} \lambda, \nu))} \cdot 1 \mid_{L}[\mu + \lambda + l, 0].
\]
2.3 Siegel operators, regularity and cuspidality

In this section we define a generalization of the Siegel operator considered in [27, 1.9. Definition].

**Definition 2.3.1.** Let \( L = (L, Q) \) be a lattice, \( 0 \leq j \leq n \) and \( \phi \in \mathcal{O}(\mathcal{H}^L_n(L)) \).

a) We define \( \mathcal{W}^{(j)} \in \mathcal{O}(\mathcal{H}^L_j(L) \times \mathcal{H}^L_{n-j}(L)) \) pointwisely by

\[
\mathcal{W}^{(j)} \phi((Z_j, W_j), (Z_{n-j}, W_{n-j})) := \phi \left( \begin{pmatrix} Z_j & 0 \\ 0 & Z_{n-j} \end{pmatrix}, (W_j, W_{n-j}) \right)
\]

for \((Z_j, W_j) \in \mathcal{H}^L_j(L), (Z_{n-j}, W_{n-j}) \in \mathcal{H}^L_{n-j}(L)\) with the convention

\[
\mathcal{W}^{(0)} = \mathcal{W}^{(n)} = \text{id.}
\]

We call \( \mathcal{W}^{(j)} \) the global Witt operator of degree \( j \).

b) For \( Y_{n-j} \in \text{Pos}_{n-j}(\mathbb{R}) \) we define \( S^{(j)}_{Y_{n-j}} \phi \) pointwisely by

\[
S^{(j)}_{Y_{n-j}} \phi(Z_j, W_j) := \lim_{t \to \infty} \mathcal{W}^{(j)} \phi((Z_j, W_j), (i t Y_{n-j}, 0))
\]

for \((Z_j, W_j) \in \mathcal{H}^L_j(L)\), whenever this limit exists and is finite. We treat \( S^{(0)}_{Y_n} \phi \) as a constant and define \( S^{(n)} = \text{id.} \). Note that existence and value of the limit above in the case \( j = n - 1 \) do not depend on the choice of \( Y_1 = y_1 > 0 \). In this case we will suppress the subscript \( Y_1 \) and simply write \( S^{(n-1)} \). We say that \( S^{(j)}_{Y_{n-j}} \phi \) exists, if \( S^{(j)}_{Y_{n-j}} \phi(Z_j, W_j) \) is defined for all \((Z_j, W_j) \in \mathcal{H}^L_j(L)\).

The partial operator \( S^{(j)}_{Y_{n-j}} \) is called the Siegel operator of degree \( j \) at \( Y_{n-j} \).

**Definition 2.3.2.** Let \( L = (L, Q) \) be a lattice and \( 0 \leq j \leq n \). Let \( \phi \in \mathcal{E}^{(n)}(L) \).

a) We say that \( \phi \) satisfies the cusp condition of degree \( \geq j \) with respect to \( H^{(n)}(L) \), if

\[
S^{(m)}_{Y_{n-m}} \left. \phi \right|_L((0, \lambda_{n-m}), (0, \mu_{n-m}))
\]

exists for all \( m = 0, \ldots, j \), \( Y_{n-m} \in \text{Pos}_{n-m}(\mathbb{R}) \) and \( \lambda_{n-m}, \mu_{n-m} \in \mathbb{L}^{n-m}_Q \). In this case,

\[
\deg_{\text{reg}}(\phi) := \max\{ j \geq 0 : \phi \text{ satisfies the cusp condition w.r.t. } H^{(n)}(L) \text{ of degree } \geq j \}
\]

is called the degree of regularity of \( \phi \). We call \( \phi \) regular, if \( \deg_{\text{reg}}(\phi) \geq 0 \). The space of regular elliptic functions of index \( L \) and degree \( n \) is denoted by \( \mathcal{E}^{(n)}_{\text{reg}}(L) \).
b) We call \( \phi \in \mathcal{E}^{(n)}(L) \) cuspidal of degree \( \geq j \) with respect to \( H^{(n)}(L) \), if \( \deg_{\text{reg}}(\phi) \geq j \) and

\[
S_{Y_{n-m}}^{(m)}\left( \phi \big|_{L}[(0, \lambda_{n-m}), (0, \mu_{n-m})] \right) = 0
\]

for all \( m = 0, \ldots, j \), \( Y_{n-m} \in \text{Pos}_{n-m}(\mathbb{R}) \) and \( \lambda_{n-m}, \mu_{n-m} \in L_Q^{(n-m)} \). In this case,

\[
\deg_{\text{cusp}}(\phi) := \max\{ j \geq 0 : \phi \text{ is cuspidal w.r.t. } H^{(n)}(L) \text{ of degree } \geq j \}
\]

is called the degree of cuspidality of \( \phi \). We call \( \phi \) cuspidal, if \( \deg_{\text{cusp}}(\phi) \geq 0 \). The space of cuspidal elliptic functions of index \( L \) and degree \( n \) is denoted by \( \mathcal{E}^{(n)}_{\text{cusp}}(L) \).

**Remark 2.3.3.** Let \( L = (L, Q) \) be a lattice. Then both \( \mathcal{E}^{(n)}_{\text{reg}}(L) \) and \( \mathcal{E}^{(n)}_{\text{cusp}}(L) \) carry a natural module structure over the subring \( \mathcal{O}(H_n)_{\text{reg}} \) of \( \mathcal{O}(H_n) \), defined by

\[
\mathcal{O}(H_n)_{\text{reg}} := \{ f \in \mathcal{O}(H_n) : \deg_{\text{reg}}(f) \geq 0 \}.
\]

Furthermore, the subring \( \mathcal{O}(H_n)_{\text{cusp}} \), defined by

\[
\mathcal{O}(H_n)_{\text{cusp}} = \{ f \in \mathcal{O}(H_n) : \deg_{\text{cusp}}(f) \geq 0 \},
\]

is an ideal in \( \mathcal{O}(H_n)_{\text{reg}} \) and one has

\[
\mathcal{O}(H_n)_{\text{cusp}} \cdot \mathcal{E}^{(n)}_{\text{reg}}(L) \subseteq \mathcal{E}^{(n)}_{\text{cusp}}(L).
\]

Before we prove the main structure theorems, we need the following

**Lemma 2.3.4.** Let \( L \) be a lattice and \( 0 \leq j \leq n \). Let \( \mu \in L^{*(n)} \) and \( \lambda, v \in L^{(n)}_R \). We write \( \mu = (\mu_j, \mu_{n-j}), \lambda = (\lambda_j, \lambda_{n-j}), v = (v_j, v_{n-j}) \) for \( \mu_j \in L^{*(j)}, \mu_{n-j} \in L^{*(n-j)} \) and \( \lambda_j, v_j \in L^{(j)}_R, \lambda_{n-j}, v_{n-j} \in L^{(n-j)}_R \). Then the following assertions hold:

a) \( \mathcal{W}^{(j)}(\theta^{(n)}_{L, \mu}|_{L} [\lambda, v]) = (\theta^{(j)}_{L, \mu_j}|_{L} [\lambda_j, v_j]) \otimes (\theta^{(n-j)}_{L, \mu_{n-j}}|_{L} [\lambda_{n-j}, v_{n-j}]) \),

b) \( \mathcal{S}^{(j)}_{L, \mu} \theta^{(n)}_{L, \mu}|_{L} [\lambda, v] = \begin{cases} \theta^{(j)}_{L, \mu_j}|_{L} [\lambda_j, v_j], & \text{if } \mu_{n-j} \equiv -\lambda_{n-j} \mod L^{(n-j)}, \\ 0, & \text{else}, \end{cases} \)

c) \( \deg_{\text{reg}}(\theta^{(n)}_{L, \mu}) = n \).

**Proof.** a) Follows from a straightforward calculation.
b) In view of a), it suffices to prove the assertion in the case \( j = 0 \). For \((Z, W) \in \mathcal{H}_n^{(1)}(L)\) we have
\[
\vartheta_{L^{(n)}}^{(n)}(\lambda, \nu)(Z, W) = \sum_{l \in \mu + L^{(n)}} e^{2\pi i \text{tr}(Q^{(n)}(l+\lambda)Z + B^{(n)}(l+\lambda, W))} \cdot e^{2\pi i \text{tr}(B^{(n)}(l+\lambda, \nu))}.
\]
Evaluating this expression at \((itY_n, 0)\) for \( t > 0, Y_n > 0 \) yields
\[
\vartheta_{L^{(n)}}^{(n)}(\lambda, \nu)(itY_n, 0) = \sum_{l \in \mu + L^{(n)}} e^{-2\pi i \text{tr}(Q^{(n)}(l+\lambda)Y_n)} \cdot e^{2\pi i \text{tr}(B^{(n)}(l+\lambda, \nu))}.
\]
We have \( \text{tr}(Q^{(n)}(l+\lambda)Y_n) > 0 \), whenever \( Q^{(n)}(l+\lambda) \geq 0 \) and \( Q^{(n)}(l+\lambda) \neq 0 \). As \( t \to \infty \), all coefficients for \( l \neq -\lambda \) vanish. This proves the second part of the statement. In the other case only the coefficient for \( l = -\lambda \) remains and equals 1.

c) Follows immediately from b). \qed

Some useful statements are identified in the next

**Lemma 2.3.5.** Let \( L = (L, Q) \) be a lattice. Let \( \lambda, \mu \in L^{(n)}_{\mathbb{R}} \). Write \( \lambda = (\lambda_j, \lambda_{n-j}) \) and \( \mu = (\mu_j, \mu_{n-j}) \) for \( \lambda_j, \mu_j \in L^{(n)}_{\mathbb{R}}, \lambda_{n-j}, \mu_{n-j} \in L^{(n-j)}_{\mathbb{R}} \). Then for \((Z_j, W_j) \in \mathcal{H}_j^{(1)}(L)\) the following assertions hold:

a) \( \mathcal{W}^{(j)} \left( \phi \big|_{L} [\lambda, \mu] \right) \left( (Z_j, W_j), \cdot \right) = \left( \mathcal{W}^{(j)} \left( \phi \big|_{L} [(\lambda_j, 0), (\mu_j, 0)] \right) \left( (Z_j, W_j), \cdot \right) \right) \big|_{L} [\lambda_{n-j}, \mu_{n-j}] \)

b) \( S_{Y_{n-j}}^{(j)} \left( \phi \big|_{L} [\lambda, \mu] \right) (Z_j, W_j) = S_{Y_{n-j}}^{(0)} \left( \mathcal{W}^{(j)} \left( \phi \big|_{L} [(\lambda_j, 0), (\mu_j, 0)] \right) \left( (Z_j, W_j), \cdot \right) \right) \big|_{L} [\lambda_{n-j}, \mu_{n-j}] \).

**Proof.** Part a) is a verification and b) follows immediately from a). We omit the details. \qed

**Proposition 2.3.6.** Let \( L \) be a lattice. Let \( \phi \in \mathcal{E}^{(n)}(L) \) with theta decomposition
\[
\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_{\mu} \cdot \vartheta_{L^{(n)}}^{(n)}.
\]
Then the following assertions are equivalent:

i) \( S_{Y_{n-j}}^{(j)} \left( \phi \big|_{L} [(0, \lambda_{n-j}), 0] \right) \) exists for all \( \lambda_{n-j} \in L^{*(n)} \),

ii) \( S_{Y_{n-j}}^{(j)} h_{\mu} \) exists for all \( \mu \in (L^*/L)^{(n)} \).

In this case, \( S_{Y_{n-j}}^{(j)} \left( \phi \big|_{L} [\lambda, \nu] \right) \) exists for all \( \lambda, \nu \in L^{(n)}_{\mathbb{R}} \). As a special case, we have
\[
S_{Y_{n-j}}^{(j)} \left( \phi \big|_{L} [(0, -\mu_{n-j}), 0] \right) = \sum_{\mu_j \in (L^*/L)^{(j)}} S_{Y_{n-j}}^{(j)} h_{(\mu_j, \mu_{n-j})} \cdot \vartheta_{L^{(j)}}^{(j)}.
\]
for \( \mu_{n-j} \in L^{*(n-j)} \). The degrees of regularity and cuspidality are given by
\[
\deg_{\text{reg}} \phi = \min \left\{ \deg_{\text{reg}} h_{\mu} : \mu \in (L^*/L)^{(n)} \right\},
\]
\[
\deg_{\text{cusp}} \phi = \min \left\{ \deg_{\text{cusp}} h_{\mu} : \mu \in (L^*/L)^{(n)} \right\}.
\]
Proof. We prove the nontrivial direction of the equivalence. At first, we treat the case \( j = 0 \). In this case, let \( \lambda_1, \ldots, \lambda_{d^n} \) denote a system of representatives for \((L^*/L)^{(n)}\), where \( d := \det L \).

We consider the holomorphic, matrix-valued function

\[
Z \mapsto A(Z) := \left( \theta^{(n)}_{L,\lambda_j}|_L \left[ -\lambda_j, 0 \right](Z, 0) \right)_{1 \leq i, j \leq d^n}, \quad Z \in \mathcal{H}_n,
\]

which satisfies the identity

\[
\begin{bmatrix}
\phi|_L \left[ -\lambda_1, 0 \right](Z, 0) \\
\vdots \\
\phi|_L \left[ -\lambda_{d^n}, 0 \right](Z, 0)
\end{bmatrix} = A(Z) \begin{bmatrix} h_{\lambda_1} \\
\vdots \\
h_{\lambda_{d^n}} \end{bmatrix}.
\]

We evaluate both sides at \( Z = it\gamma \) for \( t > 0, \gamma > 0 \). The limit of the left hand side as \( t \to \infty \) exists by assumption on \( \phi \). Furthermore we have \( \lim_{t \to \infty} A(it\gamma) = \lambda_{d^n} \) by 2.3.4. This shows that \( \lambda_{d^n} \) is invertible for all \( t > \gamma \), where \( \gamma \) is sufficiently large. As a consequence, the limit \( \lim_{t \to \infty} h_{\mu}(it\gamma) \) exists for all \( \mu \in (L^*/L)^{(n)} \) and one has

\[
S^{(n)}_{Y^n} h_{\mu} = \lim_{t \to \infty} h_{\mu}(it\gamma), \quad \mu \in (L^*/L)^{(n)}.
\]

Hence, the formula stated above is valid in the case \( j = 0 \). For \( j > 0 \) we apply \( \mathcal{W}^{(j)} \) on \( \phi \) in order to obtain

\[
\mathcal{W}^{(j)} \phi((Z_j, W_j), \cdot) = \sum_{\mu_{n-j} \in (L^*/L)^{(n-j)}} \left( \sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{W}^{(j)} h_{(\mu_j, \mu_{n-j})}(Z_j, \cdot) \theta^{(j)}_{L,\mu_j}(Z_j, W_j) \right) \theta^{(n-j)}_{L,\mu_{n-j}}.
\]

By assumption as well as by application of 2.3.5, the function

\[
S^{(n-j)}_{Y_{n-j}} \left( \left( \mathcal{W}^{(j)} \phi((Z_j, W_j), \cdot) \right)|_L \left[ \lambda_{n-j}, 0 \right] \right)
\]

exists for all \( \lambda_{n-j} \in L^*/L \). Hence we are reduced to the case \( j = 0 \). We apply the part proven previously on \( \mathcal{W}^{(j)} \phi((Z_j, W_j), \cdot) \) in order to obtain that the function

\[
S^{(0)}_{Y_{n-j}} \left( \sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{W}^{(j)} h_{(\mu_j, \mu_{n-j})}(Z_j, \cdot) \theta^{(j)}_{L,\mu_j}(Z_j, W_j) \right)
\]

exists for all \( W_j \in \mathcal{H}_j(L) \) and all \( \mu_{n-j} \in (L^*/L)^{(n-j)} \). The functions \( \theta^{(j)}_{L,\mu_j}(Z_j, \cdot), \mu_j \in (L^*/L)^{(j)} \) are linearly independent for \( Z_j \in \mathcal{H}_j \). Hence a simple argument shows, that

\[
S^{(j)}_{Y_{n-j}} h_{\mu}(Z_j) = S^{(0)}_{Y_{n-j}} \left( \mathcal{W}^{(j)} h_{\mu}(Z_j, \cdot) \right), \quad Z_j \in \mathcal{H}_j
\]

exists for all \( \mu \in (L^*/L)^{(n)} \). The identity

\[
S^{(j)}_{Y_{n-j}} \left( \phi|_L \left[ 0, -\mu_{n-j} \right], 0 \right) = \sum_{\mu_j \in (L^*/L)^{(j)}} S^{(j)}_{Y_{n-j}} h_{(\mu_j, \mu_{n-j})} \theta^{(j)}_{L,\mu_j}
\]

follows from 2.3.4.
As a direct consequence of 2.3.6, we obtain the following

**Theorem 2.3.7.** Let \( L = (L,Q) \) be a lattice. Then the following assertions hold:

a) \( \mathcal{E}^{(n)}_{\text{reg}}(L) \) is a free \( \mathcal{O}(H_n)_{\text{reg}} \)-module of rank \( (\det L)^n \) and decomposes as a direct sum

\[
\mathcal{E}^{(n)}_{\text{reg}}(L) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(H_n)_{\text{reg}} \cdot \vartheta^{(n)}_{L^*,\mu}.
\]

b) \( \mathcal{E}^{(n)}_{\text{cusp}}(L) \) decomposes as a direct sum

\[
\mathcal{E}^{(n)}_{\text{cusp}}(L) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(H_n)_{\text{cusp}} \cdot \vartheta^{(n)}_{L^*,\mu}.
\]

We consider regularity and cuspidality conditions in a more familiar scenario, cf. also [27, 1.6. Lemma].

**Proposition 2.3.8.** Let \( L = (L,Q) \) be a lattice and \( \phi \in \mathcal{E}^{(n)}(L) \) with theta decomposition

\[
\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_{\mu} \cdot \vartheta^{(n)}_{L^*,\mu}.
\]

Suppose that \( \phi \) has an absolutely and locally uniformly convergent Fourier expansion of the form

\[
\phi(Z,W) = \sum_{S \in \text{Sym}_n^{(n)}} \sum_{\mu \in L^*(n)} c(S,\mu)e^{2\pi ir \text{tr}(SZ+B^{(n)}(\mu, W))}, \quad (Z,W) \in H_n(L).
\]

for some \( q \in \mathbb{N} \). Then the following statements hold:

a) For \( \mu \in (L^*/L)^{(n)} \) one has

\[
h_{\mu}(Z) = \sum_{S \in \text{Sym}_n^{(n)}(Z)} c(S,\mu)e^{2\pi ir \text{tr}((S-Q^{(n)}(\mu))Z)}, \quad Z \in H_n.
\]

b) The following assertions are equivalent:

i) \( \phi \in \mathcal{E}^{(n)}_{\text{reg}}(L) \) resp. \( \phi \in \mathcal{E}^{(n)}_{\text{cusp}}(L) \).

ii) for all \( \mu \in L^*(n) \): \( c(S,\mu) \neq 0 \) implies \( Q^{(n)}(\mu) \leq S \) resp. \( Q^{(n)}(\mu) < S \).

In this case one has \( \deg_{\text{reg}}(\phi) = n \) and the formula

\[
S_{Y_{n-j}}^{(j)}(Z_j,W_j) = \sum_{S_j \in \text{Sym}_{n-j}^{(n)}} \sum_{\mu_j \in L^*(i)} c \left( \begin{array}{c} S_j \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ \mu_j \end{array} \right) e^{2\pi ir \text{tr}(S_jZ_j+B^{(j)}(\mu_j,W_j))}
\]

holds for all \( j = 0, \ldots, n \) and \( Y_{n-j} > 0 \). As a special case one has \( S_{Y_{n}}^{(0)}(\phi) = c(0,0) \).
c) If \( n \geq 2 \) and \( \phi \) satisfies the transformation property
\[
\phi(Z[U], WU) = \phi(Z, W), \quad (Z, W) \in \mathcal{H}_n^L \n
\]
for all \( U \in \text{GL}_n(Z)[q] \), then the Koecher principle holds, i.e. \( \phi \in \mathcal{E}_\text{reg}^{(n)}(L) \).

Proof. a) In the notation given in the proof of 2.2.8, we have
\[
c(\mu, Z) = \sum_{S \in \frac{1}{2}\text{Sym}_n(qZ)} c(S, \mu) e^{2\pi i \text{tr}(SZ)}.
\]

b) From 2.2.8 b) we conclude, that \( \phi \in \mathcal{E}_\text{reg}^{(n)}(L) \) if and only if \( h_\mu \in \mathcal{O}(\mathcal{H}_n)_{\text{reg}} \) for all \( \mu \in (L^*/L)^{(n)} \). The latter is equivalent to the existence of \( \lim_{t \to \infty} h_\mu(itY_n) \) for all \( Y_n > 0 \), hence to \( \text{tr}((S - Q^{(n)}(\mu))Y_n) \geq 0 \) for all \( Y_n > 0 \), i.e. to \( Q^{(n)}(\mu) \leq S \), whenever \( c(S, \mu) \neq 0 \). The claim for cuspidal functions follows from the classical Koecher principle applied on each function \( h_\mu \) separately.

c) Let \( n \geq 2 \) and assume without loss of generality that \( N_L | q \). The transformation laws of \( \phi \) imply
\[
h_\mu(Z + S) = h_\mu(Z), \quad h_\mu(Z[U]) = h_\mu(Z), \quad \mu \in (L^*/L)^{(n)}, Z \in \mathcal{H}_n,
\]
for all \( S \in \text{Sym}_n(q\mathbb{Z}) \) and \( U \in \text{GL}_n(Z)[q] \). Then the claim follows from the classical Koecher principle applied on each function \( h_\mu \) separately.

\[\square\]

2.4 Metaplectic group and the Weil representation

In this section, we introduce the metaplectic group of as double cover the symplectic group, cf. [1, § 2 Sec. 1, § 3 Sec.]. Furthermore, we define the modular action on elliptic functions of lattice-index and discuss the Weil representation of higher degree.

Definition 2.4.1. The real symplectic group of degree \( n \) is defined as
\[
\text{Sp}_n(\mathbb{R}) := \left\{ M \in \mathbb{R}^{2n \times 2n} : J_n[M] = J_n \right\}.
\]

Definition 2.4.2. The integral symplectic group of degree \( n \) is defined as
\[
\text{Sp}_n(\mathbb{Z}) := \text{Sp}_n(\mathbb{R}) \cap \mathbb{Z}^{2n \times 2n} = \left\{ M \in \mathbb{Z}^{2n \times 2n} : J_n[M] = J_n \right\}.
\]
We collect basic facts:

**Remark 2.4.3.**  

a) $M \in \text{Sp}_n(\mathbb{R})$ will always be written in the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrix blocks $A, B, C, D \in \mathbb{R}^{n \times n}$.

b) For $M_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \text{Sp}_j(\mathbb{Z})$, $M_{n-j} = \begin{pmatrix} A_{n-j} & B_{n-j} \\ C_{n-j} & D_{n-j} \end{pmatrix} \in \text{Sp}_{n-j}(\mathbb{Z})$ we define its cross product $M_j \times M_{n-j}$ by

$$M_j \times M_{n-j} := \begin{pmatrix} A_j & 0 & B_j & 0 \\ 0 & A_{n-j} & 0 & B_{n-j} \\ C_j & 0 & D_j & 0 \\ 0 & C_{n-j} & 0 & D_{n-j} \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}).$$

c) The matrices $J_n$, \((I_n S) I_n\), $S \in \text{Sym}_n(\mathbb{Z})$, \(U^t U^{-1}\), $U \in \text{GL}_n(\mathbb{Z})$ belong to $\text{Sp}_n(\mathbb{Z})$.

d) $\text{Sp}_n(\mathbb{Z})$ is generated by $J_n$ and \((I_n S) I_n\), $S \in \text{Sym}_n(\mathbb{Z})$.

e) The determinant $\det(CZ + D)$ is nonvanishing for $Z \in \mathcal{H}_n$ and $M \in \text{Sp}_n(\mathbb{R})$. Furthermore, $\text{Sp}_n(\mathbb{R})$ acts on $\mathcal{H}_n$ via fractional-linear transformations

$$M(Z) := (AZ + B)(CZ + D)^{-1}, \quad M \in \text{Sp}_n(\mathbb{R}), Z \in \mathcal{H}_n.$$

f) Since $\mathcal{H}_n$ is a convex domain, the function $Z \mapsto \det(CZ + D)$ for $M \in \text{Sp}_n(\mathbb{R})$ has two holomorphic holomorphic square roots on $\mathcal{H}_n$. We denote by $j_M$ the distinguished branch, which is uniquely determined by the value

$$j_M(iI_n) = \sqrt{\det(Ci + D)}.$$

**Definition 2.4.4.** The metaplectic group of degree $n$ is the group

$$\text{Mp}_{2n}(\mathbb{Z}) = \{(M, \varepsilon_M j_M) : M \in \text{Sp}_n(\mathbb{Z}), \varepsilon_M \in \{\pm 1\}\}$$

with underlying multiplication

$$(M, \varepsilon_M j_M) \cdot (M', \varepsilon_M' j_M') := (MM', \varepsilon_M \varepsilon_M' \varepsilon(M, M') j_{MM'}),$$

where

$$\varepsilon(M, M') := \frac{j_M(M'(Z)) j_M(Z)}{j_M(Z)} \in \{\pm 1\}$$

independent of $Z \in \mathcal{H}_n$. For $M \in \text{Sp}_n(\mathbb{Z})$ we write $\widetilde{M} := (M, j_M) \in \text{Mp}_{2n}(\mathbb{Z})$. 
We give some

**Remarks 2.4.5.**  
a) By definition one has \( \overset{\sim}{M} \overset{\sim}{N} = \overset{\sim}{MN} \) if and only if \( \varepsilon(M, N) = 1 \).

b) In \( \text{Mp}_{2n}(\mathbb{Z}) \) we distinguish the elements
\[
\widetilde{J}_n = \left( I_n, \sqrt{\det Z} \right), \quad \left( \begin{array}{c} I_n \\ S \\ 0 \\ I_n \end{array} \right), \quad S \in \text{Sym}_n(\mathbb{Z}).
\]

c) The kernel of the covering map
\[
\text{Mp}_{2n}(\mathbb{Z}) \longrightarrow \text{Sp}_n(\mathbb{Z}), \quad (M, \varepsilon_{MjM}) \mapsto M
\]
is cyclic of order two and the generated by the element \((I_{2n}, -j_{2n})\). Hence,
\[
\text{Mp}_{2n}(\mathbb{Z})/\langle(I_{2n}, -j_{2n})\rangle \cong \text{Sp}_n(\mathbb{Z}).
\]

By an explicit calculation we obtain the following

**Lemma 2.4.6.** Let \( 1 \leq j < n \). Then the following assertions hold:

a) \( \varepsilon(I_n, J_n) = \begin{cases} i_n, & n \equiv 0 \text{ mod } 2, \\ i_n^{-1}, & n \equiv 1 \text{ mod } 2, \end{cases} \)

b) \( \widetilde{J}_n^2 = \begin{cases} (I_{2n}, i_n \cdot j_{-l_{2n}}), & n \equiv 0 \text{ mod } 2, \\ (I_{2n}, i_n^{-1} \cdot j_{-l_{2n}}), & n \equiv 1 \text{ mod } 2, \end{cases} \)

c) \( \widetilde{J}_n^4 = (I_{2n}, (-1)^n \cdot j_{l_{2n}}) \)

d) \( \varepsilon(M_j \times I_{2n-2j}, M'_j \times I_{2n-2j}) = \varepsilon(M_j, M'_j) \) for \( M_j, M'_j \in \text{Sp}_j(\mathbb{Z}) \),

e) \( (J_j \times I_{2n-2j})^4 = (I_{2n}, (-1)^j \cdot j_{l_{2n}}) \).

**Theorem 2.4.7.** \( \text{Mp}_{2n}(\mathbb{Z}) \) is generated by the elements
\[
\widetilde{J}_n, \quad \left( \begin{array}{c} I_n \\ S \\ 0 \\ I_n \end{array} \right), \quad S \in \text{Sym}_n(\mathbb{Z}).
\]

**Proof.** Let \( \widetilde{\Delta}_n \) denote the group generated by the elements stated above. From the surjectivity of the covering map and the corresponding result for \( \text{Sp}_n(\mathbb{Z}) \) we deduce that \( \text{Mp}_{2n}(\mathbb{Z}) \) is generated by \( \widetilde{\Delta}_n \) and the central element \((I_{2n}, -j_{2n})\). In the case \( n = 1 \) we are done using the identity \( \widetilde{J}^4 = (I_2, -j_2) \). Let \( n \geq 2 \) and consider the element \((J \times I_{2n-2})\). Then exactly one of the cases
\[
(I_{2n}, -j_{2n})^\delta (J \times I_{2(n-1)}) \in \widetilde{\Delta}_n, \quad \delta = 0, 1,
\]
occurs. But both already imply
\[
(I_{2n}, -j_{2n}) = (I_{2n}, -j_{2n})^\delta (J \times I_{2(n-1)})^4 \in \widetilde{\Delta}_n.
\]
We introduce the modular action:

**Definition 2.4.8.** Let \( L = (L, Q) \) be a lattice. For \( \phi \in \mathcal{O}(\mathcal{H}_n(L)), (M, \epsilon_Mj_M) \in \text{Mp}_{2n}(\mathbb{Z}) \) and \( k \in \frac{1}{2}\mathbb{Z} \) the function
\[
\phi|_{k,L}(M, \epsilon_Mj_M) : \mathcal{H}_n(L) \rightarrow \mathbb{C},
\]
is defined pointwisely for \( (Z, W) \in \mathcal{H}_n(L) \) by
\[
\phi|_{k,L}(M, \epsilon_Mj_M)(Z, W) := (\epsilon_Mj_M(Z))^{-2k} e^{-2\pi i \text{tr}(Q^*(W) \cdot (CZ + D)^{-1}C)} : \phi(M(Z), W(CZ + D)^{-1}).
\]

We define \( \phi|_{k,L}M := \phi|_{k,L}\tilde{M} \) for \( M \in \text{Sp}_n(\mathbb{Z}) \). The operator \( |_{k,L} \) is called the slash operator of weight \( k \) and index \( L \).

**Remark 2.4.9.**

a) One has \( \phi|_{k,L}(M, \epsilon_Mj_M) = \phi|_{k,L}M \) for \( (M, \epsilon_Mj_M) \in \text{Mp}_{2n}(\mathbb{Z}) \) and \( k \in \mathbb{Z} \).

b) A straightforward calculation shows that for \( k \in \frac{1}{2}\mathbb{Z} \) the assignment

\[
((M, \epsilon_Mj_M), \phi) \mapsto \phi|_{k,L}(M, \epsilon_Mj_M), \quad (M, \epsilon_Mj_M) \in \text{Mp}_{2n}(\mathbb{Z}), \phi \in \mathcal{O}(\mathcal{H}_n(L))
\]
defines an action of \( \text{Mp}_{2n}(\mathbb{Z}) \) on \( \mathcal{O}(\mathcal{H}_n(L)) \). For \( k \in \mathbb{Z} \) this action factors through an action of \( \text{Sp}_n(\mathbb{Z}) \), while for half-integral \( k \) it factors through a projective action of \( \text{Sp}_n(\mathbb{Z}) \), whose 2-cocycle equals \( \epsilon(M, M')^{2k} \) for \( M, M' \in \text{Sp}_n(\mathbb{Z}) \).

**Definition 2.4.10.** Let \( V \) be some finite-dimensional complex vector space. For a holomorphic function \( \mathcal{H}_n \rightarrow V \) and \( (M, \epsilon_Mj_M) \in \text{Mp}_{2n}(\mathbb{Z}) \) we define the function \( f|_k(M, \epsilon_Mj_M) \) pointwisely by
\[
f|_k(M, \epsilon_Mj_M)(Z) := (\epsilon_Mj_M(Z))^{-2k} f(M(Z)), \quad Z \in \mathcal{H}_n.
\]

We define \( \phi|_kM := \phi|_k\tilde{M} \) for \( M \in \text{Sp}_n(\mathbb{Z}) \). The operator \( |_k \) is called the slash operator of weight \( k \).

**Proposition 2.4.11.** Let \( \Gamma \leq \text{Sp}_n(\mathbb{Z}) \) and \( k \in \frac{1}{2}\mathbb{Z} \). For a map \( \nu : \Gamma \rightarrow \mathbb{C}^\times \) the following assertions are equivalent:

i) \( \nu(MM')j_{MM'}(Z)^{2k} = \nu(M)\nu(M')j_{MM'}(Z)^{2k}j_{MM'}(Z)^{2k} \) for all \( M, M' \in \Gamma, Z \in \mathcal{H}_n \),

ii) \( \nu(MM') = \epsilon(M, M')^{2k} \nu(M') \) for all \( M, M' \in \Gamma \),

iii) \( \nu(MM')^{-1} \cdot 1|_kMM' = (\nu(M)\nu(M'))^{-1} \cdot (1|_kM) |_kM' \) for all \( M, M' \in \Gamma \).

In this case, \( \nu \) is called a multiplier system of \( \Gamma \) of weight \( k \) and the assignment
\[
(M, \phi) \mapsto \nu(M)^{-1}\phi|_{k,L}M, \quad M \in \Gamma, \phi \in \mathcal{O}(\mathcal{H}_n(L))
\]
defines an action of \( \Gamma \) on \( \mathcal{O}(\mathcal{H}_n(L)) \) for every lattice \( L \). If \( k \in \mathbb{Z} \), then \( \nu \in \Gamma^{ab} \).

A direct verification yields
Corollary 2.4.12. Let $\Gamma \leq \text{Sp}_n(\mathbb{Z})$ and $\widetilde{\Gamma} \leq \text{Mp}_{2n}(\mathbb{Z})$ denote the pre-image of $\Gamma$ under the covering map. Then there is a one-to-one correspondence between the abelian characters of $\widetilde{\Gamma}$ and the multiplier systems of $\Gamma$. More precisely, if $\nu$ is a multiplier system of $\Gamma$ of weight $k \in \frac{1}{2}\mathbb{Z}$, then the map

$$\nu : \Gamma \rightarrow \mathbb{C}^\times, \quad \nu(M, \varepsilon_{MjM}) := \begin{cases} \nu(M), & k \in \mathbb{Z}, \\ \varepsilon_{MjM}(M), & k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

is an abelian character of $\widetilde{\Gamma}$ satisfying $\nu(\widetilde{M}) = \nu(M)$ for $M \in \Gamma$ and $\nu(I_{2n}, -j_{I_{2n}}) = (-1)^k$. Conversely, if $\nu \in \widetilde{\Gamma}^{ab}$, then the map

$$\nu : \Gamma \rightarrow \mathbb{C}^\times, \quad \nu(M) := \nu(\widetilde{M}), \quad M \in \Gamma,$

is a multiplier system of $\Gamma$. Furthermore, $\nu \in \Gamma^{ab}$ if and only if $\nu(I_{2n}, -j_{I_{2n}}) = 1$. Otherwise, $\nu$ is a multiplier system of weight $\frac{1}{2}$ and hence of weight $\frac{k}{2}$ for all odd $k$.

Proposition 2.4.13. Let $n \geq 2$. Then the following elements belong to $\text{Mp}_{2n}(\mathbb{Z})'$:

i) $\widetilde{M}^2$ for $M \in \text{Sp}_n(\mathbb{Z})'$,

ii) $(I_{2n}, -j_{I_{2n}})$,

iii) $(-I_{2n}, j_{-I_{2n}})$.

Proof. i) Let $M \in \text{Sp}_n(\mathbb{Z})'$. By surjectivity of the covering map exactly one of the two cases

$$(I_{2n}, -j_{I_{2n}})^{\delta} \widetilde{M} \in \text{Mp}_{2n}(\mathbb{Z})', \quad \delta = 0, 1,$$

occurs. But both already imply $\widetilde{M}^2 \in \text{Mp}_{2n}(\mathbb{Z})'$.

ii) In the case $n = 2$ consider $M_0 := \text{diag}(-1, 1, -1, 1)$, which belongs to $\text{Sp}_2(\mathbb{Z})'$. By i),

$$(I_4, -j_4) = \text{diag}(\overbrace{-1, 1}^{\infty}, -1, 1)^2 = \widetilde{M_0}^2 \in \text{Mp}_4(\mathbb{Z})'.$$

In the case $n \geq 3$ we obtain $M_0 \times I_{2n-4} \in \text{Sp}_n(\mathbb{Z})'$ and hence

$$(I_{2n}, -j_{I_{2n}}) = M_0 \overbrace{\times I_{2n-4}}^{2} \in \text{Mp}_{2n}(\mathbb{Z})'.$$

iii) From $J_n \in \text{Sp}_{2n}(\mathbb{Z})'$ we obtain $\overbrace{J_n}^2 = (-I_{2n}, \varepsilon(J_n, J_n)j_{-I_{2n}}) \in \text{Mp}_{2n}(\mathbb{Z})'$ by i). In the case $\varepsilon(J_n, J_n) = 1$ we are done. In the case $\varepsilon(J_n, J_n) = -1$ we can multiply by $(I_{2n}, -j_{I_{2n}}) \in \text{Mp}_{2n}(\mathbb{Z})'$.

\[\square\]

Remark 2.4.14. Note that in view of 2.4.12, the fact $(I_{2n}, -j_{I_{2n}}) \in \text{Mp}_{2n}(\mathbb{Z})'$ is the algebraic justification, that there exist no multiplier systems of half-integral weight of $\text{Mp}_{2n}(\mathbb{Z})$ for $n \geq 2$.

We discuss an example:
Example 2.4.15. a) The Dedekind eta function $\eta : \mathcal{H} \rightarrow \mathbb{C}$, defined by

$$\eta(z) = e^{\frac{2\pi i}{12} \sum_{n=1}^{\infty} \left(1 - e^{2\pi i nz}\right)}, \quad z \in \mathcal{H},$$

is a cusp form (cf. 2.5.11) of weight $\frac{1}{2}$ with respect to $\text{Mp}_2(\mathbb{Z})$ and multiplier system $\nu_\eta$, which is uniquely determined by the values

$$\nu_\eta(\tilde{J}) = e^{-\frac{\pi i}{3}}, \quad \nu_\eta\left(\frac{1}{1} 1\right) = e^{\frac{\pi i}{12}}.$$

Furthermore, $\nu_\eta$ is an abelian character of $\text{Mp}_2(\mathbb{Z})$ of order 24 and satisfies $\nu_\eta(I_2, -jI_2) = -1$.

b) It is well known that $\text{Sp}_2(\mathbb{Z})$ has a unique nontrivial abelian character $\nu_{\text{Maass}}$. Since $(I_{2n}, -jI_{2n})$ belongs to $\text{Mp}_2(\mathbb{Z})'$, the character $\nu_{\text{Maass}}$ extends uniquely to $\text{Mp}_2(\mathbb{Z})$ where it is again denoted by $\nu_{\text{Maass}}$. Furthermore,

$$\nu_{\text{Maass}}(\tilde{J} \times I_2) = -1, \quad \nu_{\text{Maass}}\left(I_2 \begin{pmatrix} S & \cdot \\ \cdot & S \end{pmatrix} I_2\right) = (-1)^{s_{11} + s_{12} + s_{22}}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \in \text{Sym}_2(\mathbb{Z}).$$

This is enough to determine $\text{Mp}_{2n}(\mathbb{Z})^{ab}$ in the following way:

Lemma 2.4.16.

$$\text{Mp}_{2n}(\mathbb{Z})^{ab} = \begin{cases} \langle \nu_\eta \rangle, & n = 1, \\ \langle \nu_{\text{Maass}} \rangle, & n = 2, \\ \{1\}, & n \geq 3. \end{cases}$$

Proof. First we consider the case $n = 1$. Since $\text{SL}_2(\mathbb{Z})^{ab}$ is cyclic of order 12, we conclude that $\text{Mp}_2(\mathbb{Z})^{ab}$ has order at most 24. The characters $\tilde{\nu}$ induced from $\text{SL}_2(\mathbb{Z})$ are characterized by the condition $\tilde{\nu}(I_2, -jI_2) = 1$. Since $\nu_\eta(I_2, -jI_2) = -1$, we obtain exactly 24 distinct characters. Let $n \geq 2$. Since $(I_{2n}, -jI_{2n})$ belongs to $\text{Mp}_{2n}(\mathbb{Z})'$, every character of $\text{Mp}_{2n}(\mathbb{Z})$ is induced by a character of $\text{Sp}_n(\mathbb{Z})$ and the claim follows.

Definition 2.4.17. Let $G$ be a group and $\rho : G \rightarrow \text{GL}(V)$ a representation on some finite-dimensional complex vector space $V$. Then the map

$$\text{det} \rho : G \rightarrow \mathbb{C}^\times, \quad g \mapsto \text{det}(\rho(g))$$

is called the determinant character of $G$ with respect to $\rho$.

Lemma 2.4.16 restricts the possible determinant characters arising from finite-dimensional representations of $\text{Mp}_{2n}(\mathbb{Z})$: 
Corollary 2.4.18. Let $V$ be a finite-dimensional vector space and $\rho : \text{Mp}_{2n}(\mathbb{Z}) \rightarrow \text{GL}(V)$ a representation. Then

$$
\det \rho = \begin{cases} 
\nu_d^n & \text{for some } d \in \{0, \ldots, 23\}, \quad n = 1, \\
\nu_{\text{Maaß}}^d & \text{for some } d \in \{0, 1\}, \quad n = 2, \\
1, & \text{for } n \geq 3.
\end{cases}
$$

The compatibility of the actions of $H^{(n)}(L)$ and $\text{Mp}_{2n}(\mathbb{Z})$ on $\mathcal{O}(\mathcal{H}_n^I(L))$ is explained in

Proposition 2.4.19. Let $\underline{L}$ be a lattice. Let $[l, l'] \in H^{(n)}(\underline{L}), (M, \varepsilon_{MJM}) \in \text{Mp}_{2n}(\mathbb{Z})$ and $\phi \in \mathcal{O}(\mathcal{H}_n^I(L))$. Then the following assertions hold:

a) $\text{Mp}_{2n}(\mathbb{Z})$ acts from the right on $H^{(n)}(\underline{L})$ as automorphisms via

$$
[l, l'] \cdot (M, \varepsilon_{MJM}) := [(l, l') \cdot M].
$$

b) \( \left( \phi \right)_{k_{\underline{L}}} (M, \varepsilon_{MJM}) \bigg|_{k_{\underline{L}}} \left( [l, l'] \cdot (M, \varepsilon_{MJM}) \right) = \left( \phi \right)_{k_{\underline{L}}} [l, l'] \bigg|_{k_{\underline{L}}} (M, \varepsilon_{MJM}).
$$

c) $\mathcal{E}^{(n)}(\underline{L})$ is invariant under the slash operator $\bigg|_{k_{\underline{L}}}$.\n
This justifies the following

Definition 2.4.20. The metaplectic Jacobi group of degree $n$ associated to $\underline{L}$ is defined as the semidirect product

$$
\text{Mp}_{2n}^I(\mathbb{Z})(\underline{L}) := \text{Mp}_{2n}(\mathbb{Z}) \rtimes H^{(n)}(\underline{L}).
$$

The definition is extended for arbitrary subgroups $\tilde{\Gamma} \leq \text{Mp}_{2n}(\mathbb{Z})$ by setting

$$
\tilde{\Gamma}^I(\underline{L}) := \tilde{\Gamma} \rtimes H^{(n)}(\underline{L}).
$$

Remark 2.4.21. The groups $H^{(n)}(\underline{L})$ and $\text{Mp}_{2n}(\mathbb{Z})$ are identified as subgroups of $\text{Mp}_{2n}^I(\mathbb{Z})(\underline{L})$ via

$$
H^{(n)}(\underline{L}) \rightarrow \text{Mp}_{2n}^I(\mathbb{Z})(\underline{L}), \quad [l, l'] \mapsto \left( \tilde{I}_n, [l, l'] \right),
$$

$$
\text{Mp}_{2n}(\mathbb{Z}) \rightarrow \text{Mp}_{2n}^I(\mathbb{Z})(\underline{L}), \quad (M, \varepsilon_{MJM}) \mapsto ((M, \varepsilon_{MJM}), 0).
$$

From the commutation relation in 2.4.19 we derive
Proposition 2.4.22. Let $L$ be a lattice and $k \in \frac{1}{2} \mathbb{Z}$. Then $\text{Mp}_{2n}(\mathbb{Z})(L)$ acts on $\mathcal{O}(\mathcal{H}^j_n(L))$ by

$$\phi_{k,L}((M, \varepsilon_{Mj}M), [l, l']) := \left( \phi_{k,L}(M, \varepsilon_{Mj}M) \right)_{[l, l'], L},$$

where $(M, \varepsilon_{Mj}M) \in \text{Mp}_{2n}(\mathbb{Z}), l, l' \in L^{(n)}, \phi \in \mathcal{O}(\mathcal{H}^j_n(L))$. Explicitly one has

$$\phi_{k,L}(Z, W) = (\varepsilon_{Mj}M(Z))^{-2k} \cdot e^{-2\pi i r_\varepsilon(W + Iz + l') (CZ + D)^{-1} C + Q(n)(i)Z + B(n)(i, W)} \cdot \phi(M(Z), (W + Iz + l')(CZ + D)^{-1}), \quad (Z, W) \in \mathcal{H}^j_n(L),$$

where $g = ((M, \varepsilon_{Mj}M), [l, l']) \in \text{Mp}_{2n}(\mathbb{Z})(L)$.

Definition 2.4.23. Let $L$ be a lattice. The complex vector space spanned by the Jacobi theta functions of degree $n$ associated to $L$ is defined as

$$\Theta^{(n)}_L := \left\langle \theta_{L, \mu}^{(n)} : \mu \in (L^* / L)^{(n)} \right\rangle_C.$$

Note that $\text{dim}_C \Theta^{(n)}_L = (\det L)^n$.

We apply the identification 2.2.3 on the Jacobi theta functions of degree $n$ associated to $L$ in order to obtain

$$\tilde{\theta}_{L, \mu}^{(n)}(Z, W) = \Theta_{x^{(n)}(\mu), SW}(Z, S), \quad (Z, W) \in \mathcal{H}_n \times \mathbb{C}^{r \times n},$$

where $\Theta_{P, Q}(Z, S)$ stands for the classical theta series in $Z, S$ and characteristic $(P, Q)$. By use of the general transformation formula for theta functions with characteristics, we obtain

Theorem 2.4.24. Let $L = (L, Q)$ be a lattice. Then the restriction of $\bigg|_{\frac{1}{2}L} \left( \frac{r_\varepsilon}{\sqrt{\det L}} \right)^n \sum_{v \in (L^* / L)^{(n)}} e^{-2\pi i r_\varepsilon (B^{(n)}(\mu, \nu))} \theta_{L, \mu}^{(n)} \theta_{L, \nu}^{(n)} \bigg|_{\frac{1}{2}L}$

which is uniquely determined by the transformation laws

$$\rho^{(n)}_L(\tilde{I}_n) \cdot \theta_{L, \mu}^{(n)} : \theta_{L, \mu}^{(n)} \bigg|_{\frac{1}{2}L} = \left( \frac{\sqrt{1 - r_\varepsilon}}{\sqrt{\det L}} \right)^n \sum_{v \in (L^* / L)^{(n)}} e^{-2\pi i r_\varepsilon (B^{(n)}(\mu, \nu))} \theta_{L, \nu}^{(n)}$$

$$\rho^{(n)}_L \left( \begin{array}{cc} I_n & S \\ 0 & I_n \end{array} \right) \cdot \theta_{L, \mu}^{(n)} : \theta_{L, \mu}^{(n)} \bigg|_{\frac{1}{2}L} = e^{2\pi i r_\varepsilon(Q^{(n)}(\mu)S)} \theta_{L, \mu}^{(n)} \cdot S \in \text{Sym}_n(Z),$$

for $\mu \in (L^* / L)^{(n)}$. The representation $\rho^{(n)}_L$ is called the Weil representation of $\text{Mp}_{2n}(\mathbb{Z})$ associated to $L$. In the case $n = 1$ we suppress the superscript and simply write $\rho_L$.

We give some
Remarks 2.4.25. Let \( L \) be a lattice.

a) By a slight abuse of notation we extend the definition by \( \rho^{(n)}_L(M) := \rho^{(n)}_L(\tilde{M}) \) for \( M \in \text{Sp}_n(\mathbb{Z}) \).

b) \( \rho^{(n)}_L(I_{2n}, -j_{2n}) = (-1)^{r_L} \cdot \text{id}_{\Theta^{(n)}_L}. \) Hence, \( \rho^{(n)}_L \) factors through a representation of \( \text{Sp}_n(\mathbb{Z}) \) if and only if \( r_L \equiv 0 \mod 2 \). Otherwise, \( \rho^{(n)}_L \) factors through a truly projective representation.

c) \( \rho^{(n)}_L(-I_{2n}, j_{2n}) \cdot \Theta^{(n)}_L = \begin{cases} \Theta^{(n)}_L & n \equiv 0 \mod 2, \\ i^{-r_L} \Theta^{(n)}_L & n \equiv 1 \mod 2. \end{cases} \)

Definition 2.4.26. Let \( L \) be a lattice. \( \phi \in \mathcal{E}^{(n)}(L) \) is called symmetric, if

\[
\phi \left( Z, \sigma^{(n)}(W) \right) = \phi(Z, W)
\]

holds for all \( \sigma \in O(L) \) and all \( (Z, W) \in \mathcal{H}^I_n(L) \). The submodule of symmetric elliptic functions of degree \( n \) and index \( L \) is denoted by \( \mathcal{E}^{(n)}(L)^{\text{sym}} \).

Regarding the Jacobi theta functions we immediately obtain the following

Proposition 2.4.27. Let \( L \) be a lattice and \( \sigma \in O(L) \). Then the following assertions hold:

a) \( \theta^{(n)}_{L, \mu}(Z, \sigma^{(n)}(W)) = \theta^{(n)}_{L, \sigma^{-1}(\mu)}(Z, W) \) for all \( (Z, W) \in \mathcal{H}^I_n(L) \),

b) \( \Theta^{(n)}_L \) is invariant under \( O(L) \).

2.4.27 gives rise to the following

Definition 2.4.28. Let \( L \) be a lattice and

\[
\Theta^{(n), \text{sym}}_L := \left\{ \phi \in \Theta^{(n)}_L : \phi \text{ is symmetric} \right\}.
\]

The induced subrepresentation

\[
\rho^{(n), \text{sym}}_L : \text{Mp}_2n(\mathbb{Z}) \rightarrow U(\Theta^{(n), \text{sym}}_L)
\]

is called the symmetric Weil representation of degree \( n \) associated to \( L \).

Explicit formulas are contained in the following
Proposition 2.4.29. Let $L$ be a lattice and $B_1, \ldots, B_m$ denote the orbits of $O(L)$ on $(L^*/L)^{(n)}$ given by diagonal action. Then a basis of $\Theta_L^{(n), \text{sym}}$ is given by
\[
\left\{ \sum_{\mu \in B_1} \theta_L^{(n)}(\mu_1), \ldots, \sum_{\mu \in B_m} \theta_L^{(n)}(\mu_m) \right\}.
\]
For each $j = 1, \ldots, m$ fix some $\mu_{B_j} \in B_j$. Then for $B \in \{B_1, \ldots, B_m\}$ one has
\[
\rho^{(n), \text{sym}}_L(J_n) \cdot \left( \sum_{\mu \in B} \theta_L^{(n)}(\mu) \right) = \left( \frac{\sqrt{r}}{\sqrt{\det(L)}} \right)^n \cdot \prod_{j=1}^m \left( \sum_{v \in B_j} e^{-2\pi i r \text{tr}(B^{(n)}(v, \mu_{B_j}))} \right) \left( \sum_{\mu \in B_j} \theta_L^{(n)}(\mu) \right),
\]
\[
\rho^{(n), \text{sym}}_L(I_n) \cdot \left( \sum_{\mu \in B} \theta_L^{(n)}(\mu) \right) = e^{2\pi i r \text{tr}(Q^{(n)}(\mu_B))} \left( \sum_{\mu \in B} \theta_L^{(n)}(\mu) \right), \quad S \in \text{Sym}_n(\mathbb{Z}).
\]

Lemma 2.4.30. Let $L$ be a lattice and $0 \leq j \leq n$. Then the global Witt operator $\mathcal{W}^{(j)}$ induces an isomorphism
\[
\mathcal{W}^{(j)} : \Theta_L^{(n)} \longrightarrow \Theta_L^{(j)} \otimes \Theta_L^{(n-j)}
\]
of vector spaces.

Proof. The $(\det L)^n = (\det L)^j \cdot (\det L)^{n-j}$ maps
\[
\mathcal{W}^{(j)} \theta^{(n)}_L = \theta^{(j)}_L \otimes \theta^{(n-j)}_L, \quad \mu = (\mu_j, \mu_{n-j}) \in (L^*/L)^{(n)},
\]
are easily seen to be linearly independent. \hfill \Box

Proposition 2.4.31. Let $L$ be a lattice, $d := \det L$ and $0 \leq j \leq n$. Then one has
\[
\mathcal{W}^j \circ \rho^{(n)}_L(M_j \times M_{n-j}) \circ \mathcal{W}^{-j} = \rho^{(j)}_L(M_j) \otimes \rho^{(n-j)}_L(M_{n-j})
\]
for all $M_j \in \text{Sp}_j(\mathbb{Z})$, $M_{n-j} \in \text{Sp}_{n-j}(\mathbb{Z})$. Furthermore, the determinant identity
\[
\det \left( \rho^{(n)}_L(M_j \times M_{n-j}) \right) = \det \left( \rho^{(j)}_L(M_j) \right)^{d^{n-j}} \cdot \det \left( \rho^{(n-j)}_L(M_{n-j}) \right)^{d^j}
\]
holds.

Proof. Let $\mu = (\mu_j, \mu_{n-j})$ with $\mu_j \in (L^*/L)^{(j)}$, $\mu_{n-j} \in (L^*/L)^{(n-j)}$. Then one has
\[
\mathcal{W}^{-j} \left( \theta^{(j)}_L \otimes \theta^{(n-j)}_L \right) = \theta^{(n)}_L.
\]
Hence, the claim follows from
\[
\mathcal{W}^{(j)} \left( \rho^{(n)}_L(M_j \times M_{n-j}) \theta^{(n)}_L \right) = \left( \rho^{(j)}_L(M_j) \theta^{(j)}_L \right) \otimes \left( \rho^{(n-j)}_L(M_{n-j}) \theta^{(n-j)}_L \right).
\]
The supplementary identity is then seen from the determinant formula for Kronecker products of matrices as well as $\dim \rho^{(j)}_L = d^j$ and $\dim \rho^{(n-j)}_L = d^{n-j}$. \hfill \Box
2.5 Jacobi forms and vector-valued modular forms

We define metaplectic Jacobi forms as invariants of the modular action on $E(n)(L)$:

**Definition 2.5.1.** Let $L$ be a lattice, $\tilde{\Gamma} \leq \Mp_{2n}(\mathbb{Z})$ of finite index and $k \in \frac{1}{2}\mathbb{Z}$. Let $\tilde{v}$ be an abelian character of $\tilde{\Gamma}(L)$ of finite order, which acts trivially on $H^{(n)}(L)$. A holomorphic function

$$\phi : \mathcal{H}^{(n)}(L) \rightarrow \mathbb{C}$$

is called metaplectic Jacobi form of degree $n$, weight $k$, index $L$ and character $\tilde{v}$, if

i) $\phi|_{k,L}g = \tilde{v}(g)\phi$ for all $g \in \tilde{\Gamma}(L)$,

ii) for all $g \in \Mp_{2n}(\mathbb{Z})$, the function $\phi|_{k,L}g$ has a Fourier expansion of the form

$$\phi|_{k,L}g(Z,W) = \sum_{0 \leq S \in \frac{1}{2}\Sym^n_s(\mathbb{Z})} \sum_{\mu \in L^+(n)\in \Q(n)(\mu) \leq S} c(S,\mu)e^{2\pi itr(SZ+B^{(n)}(\mu,W))}$$

for some $q = q(\tilde{\Gamma},\tilde{v},g) \in \mathbb{N}$.

We call $\phi$ a cusp form, if $c(S,\mu) \neq 0$ in ii) already implies $Q^{(n)}(\mu) < S$. The space of metaplectic Jacobi forms of degree $n$, weight $k$, index $L$ and character $\tilde{v}$ is denoted by $J^{(n)}_{k,L}(\tilde{\Gamma},\tilde{v})$. The subspace of cusp forms is denoted by $J^{(n)}_{k,L}(\tilde{\Gamma},\tilde{v})_{\text{cusp}}$. The space of symmetric Jacobi forms is denoted by $J^{(n)}_{k,L}(\tilde{\Gamma},\tilde{v})_{\text{sym}}$.

Consideration of the generators of $\tilde{\Gamma}(L)$ yields

**Proposition 2.5.2.** Let $L$ be a lattice, $\tilde{\Gamma} \leq \Mp_{2n}(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and $\nu \in \tilde{\Gamma}^{ab}$ of finite order. Then one has $\phi \in J^{(n)}_{k,L}(\tilde{\Gamma},\tilde{v})$ if and only if the following assertions hold:

i) $\phi|_{k,L}(M,\varepsilon_{MjM}) = \tilde{v}(M,\varepsilon_{MjM})\phi$ for all $(M,\varepsilon_{MjM}) \in \tilde{\Gamma}$,

ii) $\phi|_{L}[l,l'] = \phi$ for all $[l,l'] \in H^{(n)}(L)$,

iii) for all $(M,\varepsilon_{MjM}) \in \Mp_{2n}(\mathbb{Z})$, the function $\phi|_{k,L}(M,\varepsilon_{MjM})$ has a Fourier expansion of the form

$$\phi|_{k,L}(M,\varepsilon_{MjM})(Z,W) = \sum_{0 \leq S \in \frac{1}{2}\Sym^n_s(\mathbb{Z})} \sum_{\mu \in L^+(n)\in \Q(n)(\mu) \leq S} c(S,\mu)e^{2\pi itr(SZ+B^{(n)}(\mu,W))}$$

for some $q = q(\tilde{\Gamma},\tilde{v},(M,\varepsilon_{MjM})) \in \mathbb{N}$.

From 2.3.8 we obtain the following characterization:

**Lemma 2.5.3.** Let $L$ be a lattice, $\tilde{\Gamma} \leq \Mp_{2n}(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and $\tilde{v} \in \tilde{\Gamma}^{ab}$ of finite order. Then the following assertions hold:


2.5 Jacobi forms and vector-valued modular forms

\[ 2.5.6 \]

\[ \begin{align*}
   i) \phi \in j_{k,L}^{(n)}(\tilde{\Gamma}, \tilde{v}) & \iff \left\{ \begin{array}{l}
   \phi|_{k,L}(M, \epsilon_{M\mid M}) = \tilde{v}(M, \epsilon_{M\mid M})\phi \\
   \phi|_{k,L}(M, \epsilon_{M\mid M}) \in E^{(n)}(L)^{\text{reg}} 
   \end{array} \right. \\
   \quad \text{for all } (M, \epsilon_{M\mid M}) \in \tilde{\Gamma}, \\
   \quad \text{for all } (M, \epsilon_{M\mid M}) \in \text{Mp}_{2n}(\mathbb{Z}).
   
   ii) \phi \in j_{k,L}^{(n)}(\tilde{\Gamma}, \tilde{v})^{\text{cusp}} & \iff \left\{ \begin{array}{l}
   \phi|_{k,L}(M, \epsilon_{M\mid M}) = \tilde{v}(M, \epsilon_{M\mid M})\phi \\
   \phi|_{k,L}(M, \epsilon_{M\mid M}) \in E^{(n)}(L)^{\text{cusp}}
   \end{array} \right. \\
   \quad \text{for all } (M, \epsilon_{M\mid M}) \in \tilde{\Gamma}, \\
   \quad \text{for all } (M, \epsilon_{M\mid M}) \in \text{Mp}_{2n}(\mathbb{Z}).
\end{align*} \]

Remark 2.5.4. Loosely speaking, 2.5.3 states that a function \( \phi \in E^{(n)}(L) \) is a Jacobi form if and only if it satisfies a modular translation law with respect to \( \tilde{\Gamma} \) and all transformed functions \( \phi|_{k,L}(M, \epsilon_{M\mid M}) \) for \( (M, \epsilon_{M\mid M}) \in \text{Mp}_{2n}(\mathbb{Z}) \) satisfy the cusp condition with respect to the Heisenberg group.

Definition 2.5.5. Let \( L \) be a lattice, \( \Gamma \leq \text{Sp}_n(\mathbb{Z}) \) of finite index, \( k \in \frac{1}{2} \mathbb{Z} \) and \( \nu \) a multiplier system of \( \Gamma \) of finite order. Then we define

\[ j_{k,L}^{(n)}(\Gamma, \nu) := j_{k,L}^{(n)}(\tilde{\Gamma}, \tilde{v}), \quad j_{k,L}^{(n)}(\Gamma, \nu)^{\text{cusp}} := j_{k,L}^{(n)}(\tilde{\Gamma}, \tilde{v})^{\text{cusp}}, \quad j_{k,L}^{(n)}(\Gamma, \nu)^{\text{sym}} := j_{k,L}^{(n)}(\tilde{\Gamma}, \tilde{v})^{\text{sym}}, \]

where \( \tilde{\Gamma} \) denotes the pre-image of \( \Gamma \) under the covering map and \( \tilde{v} \) is the abelian character of \( \tilde{\Gamma} \) according to 2.4.12. The elements of \( j_{k,L}^{(n)}(\Gamma, \nu) \) are called Jacobi forms of degree \( n \), weight \( k \), index \( L \) and multiplier system \( \nu \). We will write

\[ j_{k,L}^{(n)}(\tilde{\Gamma}) := j_{k,L}^{(n)}(\tilde{\Gamma}, 1), \quad j_{k,L}^{(n)}(\nu) := j_{k,L}^{(n)}(\text{Sp}_n(\mathbb{Z}), \nu), \quad j_{k,L}^{(n)} := j_{k,L}^{(n)}(1). \]

Lemma 2.5.6. Let \( L \) be a lattice, \( \Gamma \leq \text{Sp}_n(\mathbb{Z}) \) of finite index, \( k \in \frac{1}{2} \mathbb{Z} \) and \( \nu \) a multiplier system of \( \Gamma \) of finite order. Then the following assertions hold:

a) \( \dim_{\mathbb{C}} j_{k,L}^{(n)}(\Gamma, \nu) < \infty, \)

b) \( \dim_{\mathbb{C}} j_{k,L}^{(n)}(\Gamma, \nu) = 0, \) if \( k < 0, \)

c) \( \dim_{\mathbb{C}} j_{k,L}^{(n)}(\nu) = 0, \) if \( n \geq 2 \) and \( k \) is half-integral.

Proof. For a) and b) confer [27, Thm. 1.8]. The assertion c) follows from the fact that \( (I_{2n}, -1) \in \text{Mp}_{2n}(\mathbb{Z})' \) if \( n \geq 2. \)

For \( q \in \mathbb{Z} \) we define congruence subgroups

\[ \text{Sp}_n(\mathbb{Z})_0[q] := \{ M \in \text{Sp}_n(\mathbb{Z}) : C \equiv 0 \mod q \}, \]

\[ \text{Sp}_n(\mathbb{Z})[q] := \{ M \in \text{Sp}_n(\mathbb{Z}) : M \equiv I_{2n} \mod q \}. \]

From [1, II., § 4, Theorem 2.2] we adopt
Example 2.5.7. Let $L = (L, Q)$ be a lattice. Then one has
\[\vartheta_{L_0}^{(n)} \in \int_{\frac{1}{2} + L} (\text{Sp}_n(Z)_0 | N_L, \chi_Q^{(n)}) ,\]
where $\chi_Q^{(n)}$ is defined in [1, I., § 4, Thm. 4.10, Thm. 4.12]. Additionally,
\[\vartheta_{L, \mu}^{(n)} \in \int_{\frac{1}{2} + L} (\text{Sp}_n(Z) | N_L)\]
holds for all $\mu \in (L^* / L)^{(n)}$.

We construct Jacobi-Eisenstein series:

Remark 2.5.8. Let $L = (L, Q)$ be a lattice and $\mu \in (L^* / L)^{(n)}$ such that $Q^{(n)}(\mu) \in \text{Sym}_n^\sharp(Z)$. Let $\mu^{\text{GL}_n(Z)}$ denote the orbit of $\mu$ under the right action of $\text{GL}_n(Z)$ on $(L^* / L)^{(n)}$. From
\[Q^{(n)}(\mu U) = Q^{(n)}(\mu)[U] \in \text{Sym}_n^\sharp(Z), \ U \in \text{GL}_n(Z),\]
and 2.2.6 we conclude that the averaged function
\[
\frac{1}{|\mu^{\text{GL}_n(Z)}|} \sum_{\lambda \in \mu^{\text{GL}_n(Z)}} \vartheta_{L, \lambda}^{(n)}
\]
is invariant under $|kL| M$ for $M = \left( \begin{array}{cc} A & B \\ \delta & 0 \end{array} \right) \in \text{Sp}_n(Z)$ and $k \in Z$ even.

This motivates the following

Example 2.5.9. Let $L$ be a lattice. Let $k \geq n + r_L + 2$ even and $\mu \in (L^* / L)^{(n)}$ such that $Q^{(n)}(\mu) \in \text{Sym}_n^\sharp(Z)$. Let $\mu^{\text{GL}_n(Z)}$ denote the orbit of $\mu$ under the action of $\text{GL}_n(Z)$ on $(L^* / L)^{(n)}$. The Jacobi-Eisenstein series of degree $n$, weight $k$ and index $L$ with respect to $\mu$ is defined by
\[E_{k, L, \mu}^{(n)} := \frac{1}{|\mu^{\text{GL}_n(Z)}|} \sum_{M : \text{Sp}_n(Z)_0} \sum_{\lambda \in \mu^{\text{GL}_n(Z)}} \vartheta_{L, \lambda}^{(n)} |M|,\]
where $\text{Sp}_n(Z)_0 := \{ M = \left( \begin{array}{cc} A & B \\ \delta & 0 \end{array} \right) \in \text{Sp}_n(Z) \}$. The series converges absolutely and uniformly on each vertical strip
\[V_n(\delta) := \{ (Z, W) \in H_n^*(L) : Y \geq \delta I_n, \text{tr}(X^2) \leq \delta^{-1}, \text{tr}(B^{(n)}(W, W)) < \delta^{-1} \}, \ \delta > 0.\]
By construction, $E_{k, L, \mu}^{(n)}$ is invariant under both $\text{Sp}_n(Z)$ and $H^{(n)}(L)$, hence $E_{k, L, \mu}^{(n)} \in I_{k, L}^{(n)}$.

The Jacobi-Eisenstein series $E_{k, L}^{(n)} := E_{k, L, 0}^{(n)}$ was studied in [27, p. 200f]. By using standard methods, cf. the proof of [27, 2.8 Theorem], we obtain:
Proposition 2.5.10. Let \( L \) be a lattice. Let \( k \geq n + r_L + 2 \) even and \( \mu \in (L^*/L)^{(n)} \) such that \( Q^{(n)}(\mu) \in \text{Sym}^n_\mu(\mathbb{Z}) \). Let \( 0 \leq j \leq n \) and \( \mu_j \in (L^*/L)^{(j)} \), \( \mu_{n-j} \in (L^*/L)^{(n-j)} \) such that \( \mu = (\mu_j, \mu_{n-j}) \). Then the following assertions hold:

(a) \( S_{Y_{n-j}}^{(j)} \left( E_{k,L;\mu}^{(n)} \big| (0, -\mu_{n-j}), 0 \right) = E_{k,L;\mu}^{(j)} \) for all \( Y_{n-j} > 0 \) with the convention \( E_{k,L;\mu}^{(0)} := 1 \).

(b) \( E_{k,L;\mu}^{(n)} \) does not vanish identically.

c) The Jacobi-Eisenstein series \( E_{k,L;\mu}^{(n)} \) where \( \mu \) runs through a system of representatives of the orbits of \( GL_n(\mathbb{Z}) \) on

\[
\left\{ \mu \in (L^*/L)^{(n)} : Q^{(n)}(\mu) \in \text{Sym}^n_\mu(\mathbb{Z}) \right\},
\]

are linearly independent.

Definition 2.5.11. Let \( \tilde{\Gamma} \leq \text{Mp}_{2n}(\mathbb{Z}) \) of finite index. Let \( V \) be a complex vector space of finite dimension and \( \rho : \tilde{\Gamma} \rightarrow GL(V) \) a finite representation and \( k \in \frac{1}{2} \mathbb{Z} \). A holomorphic function \( f : \mathcal{H}_n \rightarrow V \) is called vector-valued modular form of degree \( n \) and weight \( k \) with respect to \( \tilde{\Gamma} \) and \( \rho \), if the following assertions hold:

i) \( f|_k(M, \varepsilon_MJ_M) = \rho(M, \varepsilon_MJ_M)f \) for all \( (M, \varepsilon_MJ_M) \in \tilde{\Gamma} \),

ii) for all \( (M, \varepsilon_MJ_M) \in \text{Mp}_{2n}(\mathbb{Z}) \), the function \( f|_k(M, \varepsilon_MJ_M) \) has a Fourier expansion of the form

\[
f|_k(M, \varepsilon_MJ_M)(Z) = \sum_{j=1}^{\text{dim}_c(V)} \sum_{0 \leq S \in \frac{1}{2}\text{Sym}^*_\mu(\mathbb{Z})} c_j(S) \cdot e^{2\pi i r(S)Z} v_j, \quad Z \in \mathcal{H}_n
\]

for some basis \( v_j, j = 1, \ldots, \text{dim}_c(V) \) of \( V \) and \( q = q(\tilde{\Gamma}, \rho, (M, \varepsilon_MJ_M)) \in \mathbb{N} \).

In this case \( f \) a cusp form, if \( S^{(n-1)} \left( f|_k(M, \varepsilon_MJ_M) \right) \equiv 0 \) holds for all \( (M, \varepsilon_MJ_M) \in \text{Mp}_{2n}(\mathbb{Z}) \). If \( V = \mathbb{C} \), we will we suppress the term ‘vector-valued’ and simply call \( f \) a modular form. The space of vector valued modular forms of degree \( n \) and weight \( k \) with respect to \( \rho \) is denoted by \( [\tilde{\Gamma}, k, \rho] \), the subspace of cusp forms by \( [\tilde{\Gamma}, k, \rho]^{\text{cusp}} \).

We end this section by briefly reviewing the well known correspondence between Jacobi forms and vector-valued modular forms, cf. [24, Proposition 3.5.].

Let \( L \) be a lattice and \( d := \det L \). Let \( \mu_1, \ldots, \mu_d^{(n)} \) denote a complete system of representatives for \( (L^*/L)^{(n)} \). We define

\[
\Theta^{(n)}_L := \left( \theta^{(n)}_{L,\mu_1}, \ldots, \theta^{(n)}_{L,\mu_d} \right)^t : \mathcal{H}_n \rightarrow \mathbb{C}^d.
\]

Let \( \phi \in \mathcal{E}^{(n)}(L) \) with theta decomposition

\[
\phi = \sum_{j=1}^d h_j \cdot \theta^{(n)}_{L,\mu_j} = \left< h, \Theta^{(n)}_L \right>,
\]
where \( h = (h_1, \ldots, h_d)^t : \mathcal{H}_n \to \mathbb{C}^d \) is holomorphic and \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product on \( \mathbb{C}^d \). Let \( \varphi_L^{(n)} \) denote the matrix representation of the Weil representation with respect to the basis of \( \Theta_L^{(n)} \) given by the Jacobi theta functions associated to the system \((\mu_1, \ldots, \mu_d)\). Hence, for \((M, \varepsilon_{M^j_M}) \in \text{Mp}_{2n}(\mathbb{Z})\), we obtain

\[
\phi|_{kL}(M, \varepsilon_{M^j_M}) = \left\langle h|_{k-r_L^n} (M, \varepsilon_{M^j_M}), \Theta_L^{(n)} \right\rangle_{\text{cusp}} \rho_L^{(n)} (M, \varepsilon_{M^j_M}) \cdot \Theta_L^{(n)}
\]

and thus the connection between Jacobi forms and vector-valued modular forms:

**Theorem 2.5.12.** Let \( L \) be a lattice, \( \Gamma \leq \text{Sp}_n(\mathbb{Z}) \) of finite index, \( k \in \frac{1}{2} \mathbb{Z} \) and \( \nu \) a multiplier system of \( \Gamma \) of finite order. Let \( \mu_1, \ldots, \mu_d \) denote a complete system of representatives for \((L^*/L)^{(n)}\) and \( \rho_L^{(n)} \) denote the matrix representation of the Weil representation with respect to the basis of \( \Theta_L^{(n)} \) given by the Jacobi theta functions associated to the system \((\mu_1, \ldots, \mu_d)\). Then the assignment

\[
h \mapsto \left\langle h, \Theta_L^{(n)} \right\rangle
\]

induces isomorphisms of vector spaces

\[
i) \quad \left[ \tilde{\Gamma}, k - r_L^n, \tilde{\nu} \cdot \left( \rho_L^{(n)} \right)^{-t} \right] \sim j_{kL}^{(n)}(\Gamma, \nu),
\]

\[
j) \quad \left[ \tilde{\Gamma}, k - r_L^n, \tilde{\nu} \cdot \left( \rho_L^{(n)} \right)^{-t} \right]_{\text{cusp}} \sim j_{kL}^{(n)}(\Gamma, \nu)_{\text{cusp}},
\]

where \( \tilde{\Gamma} \) denotes the preimage of \( \Gamma \) under the covering map and \( \tilde{\nu} \) is the abelian character of \( \tilde{\Gamma} \) in accordance to 2.4.12.

As an example we return to the Jacobi-Eisenstein series \( E_{kL}^{(n)} \):

**Example 2.5.13.** Let \( L \) be a lattice. Let \( k \geq n + r_L + 2 \) even and \( \mu \in (L^*/L)^{(n)} \) such that \( Q^{(n)}(\mu) \in \text{Sym}_n^2(\mathbb{Z}) \). Let \( d := \det L \) and \( \mu_1, \ldots, \mu_d \) denote a complete system of representatives for \((L^*/L)^{(n)}\) and \( \rho_L^{(n)} \) denote the matrix representation of the Weil representation with respect to the basis of \( \Theta_L^{(n)} \) given by the Jacobi theta functions associated to the system \((\mu_1, \ldots, \mu_d)\). By a slight abuse of notation we denote by \( e_\lambda \) the standard basis vector \( e_j \) of \( \mathbb{C}^d \), if \( \lambda = \mu_j \) for \( 1 \leq j \leq d \). By definition of \( \rho_L^{(n)} \) we have

\[
\theta_L^{(n)}|_{r_L^n} M = \left\langle \rho_L^{(n)}(M) \Theta_L^{(n)}, e_\mu \right\rangle = \sum_{\nu \in (L^*/L)^{(n)}} \left\langle \rho_L^{(n)}(M)e_\nu, e_\mu \right\rangle \cdot \theta_L^{(n)}.
\]
Hence the theta decomposition of $E_{k,L}^{(n)}$ is given by

$$E_{k,L}^{(n)}(z) = \frac{1}{|\mu_{\text{GL}_n(\mathbb{Z})}|} \sum_{M: \text{Sp}_n(\mathbb{Z})_0 \setminus \text{Sp}_n(\mathbb{Z})} \sum_{\lambda \in \mu_{\text{GL}_n(\mathbb{Z})}} \frac{\theta^{(n)}_{\mathcal{L}^\lambda} |_k M}{1 \mid_{k-r_{\mathcal{L}}}^z} \cdot 1 \mid_{k-r_{\mathcal{L}}}^z$$

$$= \frac{1}{|\mu_{\text{GL}_n(\mathbb{Z})}|} \sum_{M: \text{Sp}_n(\mathbb{Z})_0 \setminus \text{Sp}_n(\mathbb{Z})} \sum_{\lambda \in \mu_{\text{GL}_n(\mathbb{Z})}} \left( \sum_{\nu \in (L^*/L)^{(n)}} \langle \rho_{\mathcal{L}}^{(n)}(M)e_\nu, e_\lambda \rangle \cdot \vartheta^{(n)}_{\mathcal{L}^\nu} \right) \cdot 1 \mid_{k-r_{\mathcal{L}}}^z$$

$$= \frac{1}{|\mu_{\text{GL}_n(\mathbb{Z})}|} \sum_{\nu \in (L^*/L)^{(n)}} \left( \sum_{M: \text{Sp}_n(\mathbb{Z})_0 \setminus \text{Sp}_n(\mathbb{Z})} \left( \sum_{\lambda \in \mu_{\text{GL}_n(\mathbb{Z})}} \langle \rho_{\mathcal{L}}^{(n)}(M)e_\nu, e_\lambda \rangle \right) \cdot 1 \mid_{k-r_{\mathcal{L}}}^z \right) \vartheta^{(n)}_{\mathcal{L}^\nu}.$$
2.6 Determinant characters of Weil representations of degree 1

Utilizing a case-by-case study for the irreducible root lattices of small rank constructed in section 1.3, the tables 2.1 and 2.2 present the matrix representations of $\rho_L(M)$ and $\rho_L^{\text{sym}}(M)$ for the canonical elements $M \in \{J, \left(\frac{1}{0} \frac{1}{1}\right)\}$ with respect to suitable bases. In each case we determine the exact $\nu_\eta$-power of the corresponding determinant character of $\text{Mp}_2(\mathbb{Z})$.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
L & \rho_L(J) & \det \rho_L(J) & \rho_L(\frac{1}{0} \frac{1}{1}) & \det \rho_L(\frac{1}{0} \frac{1}{1}) & \det \rho_L \\
\hline
A_1 & \frac{\sqrt{-1}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & i & \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} & i & \nu_\eta^6 \\
A_2 & -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho & \bar{\rho} \\ 1 & \bar{\rho} & \rho \end{pmatrix} & 1 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix} & \bar{\rho} & \nu_\eta^{16} \\
D_4 & -\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} & -1 & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & -1 & \nu_\eta^{12} \\
E_6 & \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{\rho} & \rho \\ 1 & \rho & \bar{\rho} \end{pmatrix} & 1 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\rho} & 0 \\ 0 & 0 & \rho \end{pmatrix} & \rho & \nu_\eta^8 \\
E_7 & \frac{\sqrt{i}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -i & \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} & -i & \nu_\eta^{18} \\
E_8 & 1 & 1 & 1 & 1 & \nu_\eta^{24} = 1 \\
\hline
\end{array}
\]

Table 2.1: Weil representations and their determinant characters
### 2.6 Determinant characters of Weil representations of degree 1

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\rho^\text{sym}_L(j)$</th>
<th>$\det \rho^\text{sym}_L(j)$</th>
<th>$\rho^\text{sym}_L(\begin{smallmatrix}1 &amp; 1 \ 0 &amp; 1 \end{smallmatrix})$</th>
<th>$\det \rho^\text{sym}_L(\begin{smallmatrix}1 &amp; 1 \ 0 &amp; i \end{smallmatrix})$</th>
<th>$\det \rho^\text{sym}_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$\frac{\sqrt{i}^{-1}}{\sqrt{2}} \begin{pmatrix} 1 &amp; 1 \ 1 &amp; -1 \end{pmatrix}$</td>
<td>$i$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; i \end{pmatrix}$</td>
<td>$i$</td>
<td>$\nu^6_\eta$</td>
</tr>
<tr>
<td>$2A_1$</td>
<td>$\frac{-i}{2} \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 2 &amp; 0 &amp; -2 \ 1 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$-i$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; i &amp; 0 \ 0 &amp; 0 &amp; -1 \end{pmatrix}$</td>
<td>$-i$</td>
<td>$\nu^{18}_\eta$</td>
</tr>
<tr>
<td>$A_1(2)$</td>
<td>$\frac{\sqrt{i}^{-1}}{2} \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 2 &amp; 0 &amp; -2 \ 1 &amp; -1 &amp; 1 \end{pmatrix}$</td>
<td>$\sqrt{i}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \sqrt{i} &amp; 0 \ 0 &amp; 0 &amp; -1 \end{pmatrix}$</td>
<td>$-\sqrt{i}$</td>
<td>$\nu^{15}_\eta$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\frac{-i}{\sqrt{3}} \begin{pmatrix} 1 &amp; 1 \ 2 &amp; -1 \end{pmatrix}$</td>
<td>$1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; \rho \end{pmatrix}$</td>
<td>$\rho$</td>
<td>$\nu^8_\eta$</td>
</tr>
<tr>
<td>$2A_2$</td>
<td>$\frac{-1}{3} \begin{pmatrix} 1 &amp; 1 &amp; 1 \ 4 &amp; 1 &amp; -2 \ 4 &amp; -2 &amp; 1 \end{pmatrix}$</td>
<td>$1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; \rho &amp; 0 \ 0 &amp; 0 &amp; \bar{\rho} \end{pmatrix}$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$\frac{-1}{2} \begin{pmatrix} 1 &amp; 1 \ 3 &amp; -1 \end{pmatrix}$</td>
<td>$-1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{pmatrix}$</td>
<td>$-1$</td>
<td>$\nu^{12}_\eta$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\frac{i}{\sqrt{3}} \begin{pmatrix} 1 &amp; 1 \ 2 &amp; -1 \end{pmatrix}$</td>
<td>$1$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; \bar{\rho} \end{pmatrix}$</td>
<td>$\bar{\rho}$</td>
<td>$\nu^{16}_\eta$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\frac{\sqrt{i}}{\sqrt{2}} \begin{pmatrix} 1 &amp; 1 \ 1 &amp; -1 \end{pmatrix}$</td>
<td>$-i$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; -i \end{pmatrix}$</td>
<td>$-i$</td>
<td>$\nu^{18}_\eta$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\nu^{24}_\eta = 1$</td>
</tr>
</tbody>
</table>

Table 2.2: Symmetric Weil representations and their determinant characters
3 Pullback Theory

3.1 The pullback operator

Definition 3.1.1. Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Then the map

$$\iota^{(n)} : L_0^{(n)} \rightarrow L^{(n)}, \quad \iota^{(n)}(l_1, \ldots, l_n) := (\iota(l_1), \ldots, \iota(l_n))$$

for $l_1, \ldots, l_n \in L$ is called the diagonal embedding of degree $n$ associated to $\iota$. Its unique $\mathbb{F}$-linear extension $(L_0)_{\mathbb{F}}^{(n)} \rightarrow L_{\mathbb{F}}^{(n)}$ will again be denoted by $\iota^{(n)}$. In the case $n = 1$ we will suppress the superscript.

Every embedding $\iota : L_0 \rightarrow L$ induces a series of further embeddings:

Remarks 3.1.2. Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Then the following assertions hold:

a) The map

$$\iota^{(n)} : H^I_n(L_0) \rightarrow H^I_n(L), \quad (Z, W_0) \mapsto (Z, \iota^{(n)}(W_0))$$

is a holomorphic embedding of the corresponding Jacobi half-spaces.

b) The map

$$\iota^{(n)} : H^R_n(L_0) \rightarrow H^R_n(L), \quad (\lambda_0, \mu_0, \zeta) \mapsto (\iota^{(n)}(\lambda_0), \iota^{(n)}(\mu_0), \zeta)$$

is a monomorphism of the Heisenberg groups and one has

$$\iota^{(n)}|_{H^R_n(L_0)} : H^R_n(L_0) \rightarrow H^R_n(L).$$

We may now introduce the main operator of our studies:

Definition 3.1.3. Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. For a holomorphic function $\phi \in \mathcal{O}(H^I_n(L))$, we define

$$\phi \left[ \iota^{(n)} \right] : H^I_n(L_0) \rightarrow \mathbb{C}$$

pointwisely by

$$\phi \left[ \iota^{(n)} \right] (Z, W_0) := \phi \left( Z, \iota^{(n)}(W_0) \right), \quad (Z, W_0) \in H^I_n(L_0).$$
The function $\phi \left[ t^{(n)} \right]$ is called the pullback of $\phi$ with respect to $t^{(n)}$. As an abbreviation we set

$$\phi[l] := \phi \left[ t^{(n)}_l \right], \quad l \in L,$$

where $t_l : \mathbb{Z}(t) \rightarrow L$, $t = Q(l)$, as in 1.2.22.

Regarding compatibility of the Heisenberg-action and the pullback operator, we obtain

**Proposition 3.1.4.** Let $L_0$ and $L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Let $g \in H_R^{(n)}(L_0)$ and $\phi \in \mathcal{O}(\mathcal{H}_n(L))$. Then one has

$$\phi \left[ t^{(n)} \right] \mid L_0 g = \left( \phi \mid L^{(n)}(g) \right) \left[ t^{(n)} \right],$$

i.e. the diagram

$$\begin{array}{ccc}
\mathcal{O}(\mathcal{H}_n(L)) & \xrightarrow{[t^{(n)}]} & \mathcal{O}(\mathcal{H}_n(L_0)) \\
|_{L^{(n)}(g)} & & |_{L_0 g} \\
\mathcal{O}(\mathcal{H}_n(L)) & \xrightarrow{[t^{(n)}]} & \mathcal{O}(\mathcal{H}_n(L_0))
\end{array}$$

is commutative for all $g \in H_R^{(n)}(L_0)$.

As a by-product we obtain

**Corollary 3.1.5.** Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Then the pullback operator

$$\left[ t^{(n)} \right] : \mathcal{E}^{(n)}(L) \rightarrow \mathcal{E}^{(n)}(L_0)$$

is a homomorphism of $\mathcal{O}(\mathcal{H}_n)$-modules.

The next aim is to prove that the pullback operator is compatible with respect to regularity and cuspidality. To this end we need the following technical lemma, which can the proved by a straightforward calculation.

**Lemma 3.1.6.** Let $L_0, L$ be lattices, $\iota : L_0 \rightarrow L$ an embedding and $0 \leq j \leq n$. Let $\mathcal{O}(\mathcal{H}_n(L))$, $Y_{n-j} \in \text{Pos}_{n-j}(R), (\lambda_0)_{n-j}, (\mu_0)_{n-j} \in (L_0)^{n-j}_Q$ and suppose that

$$S_{Y_{n-j}}^{(j)} \left( \phi \mid_L \left[ (0, t^{(n-j)}((\lambda_0)_{n-j})), (0, t^{(n-j)}((\mu_0)_{n-j})) \right] \right)$$
exists. Then

\[ S_{Y_{n-j}}^{(j)} \left( \phi \left[ \iota^{(n)} \right] \bigg|_{L_0} \left[ (0, (\lambda_0)_{n-j}), (0, (\mu_0)_{n-j}) \right] \right) \]

exists and one has

\[ S_{Y_{n-j}}^{(j)} \left( \phi \left[ \iota^{(n)} \right] \bigg|_{L_0} \left[ (0, (\lambda_0)_{n-j}), (0, (\mu_0)_{n-j}) \right] \right) = S_{Y_{n-j}}^{(j)} \left( \phi \left|_{L} \left[ (0, \iota^{(n-j)}((\lambda_0)_{n-j})), (0, \iota^{(n-j)}((\mu_0)_{n-j})) \right] \right| \iota^{(j)} \right). \]

Furthermore,

\begin{align*}
\deg_{\text{reg}} \left( \phi \left[ \iota^{(n)} \right] \right) & \geq \deg_{\text{reg}} (\phi), \\
\deg_{\text{cusp}} \left( \phi \left[ \iota^{(n)} \right] \right) & \geq \deg_{\text{cusp}} (\phi).
\end{align*}

3.1.6 immediately implies

**Theorem 3.1.7.** Let \( L_0 \) and \( L \) be lattices, \( \iota : L_0 \rightarrow L \) an embedding. Let \( \phi \in E^{(n)}(L) \). If \( \phi \) is regular resp. cuspidal, then \( \phi \left[ \iota^{(n)} \right] \) is regular resp. cuspidal of degree at least \( \deg_{\text{reg}} (\phi) \) resp. \( \deg_{\text{cusp}} (\phi) \).

The operators

\[ \left[ \iota^{(n)} \right] : E^{(n)}(L) \rightarrow E^{(n)}(L_0), \]

\[ \left[ \iota^{(n)} \right] : E^{(n)}(L) \rightarrow E^{(n)}(L_0). \]

are well defined and homomorphisms of \( \mathcal{O}(\mathcal{H}_n)_{\text{reg}} \)-modules.

The commutation relation between \( \left[ \iota^{(n)} \right] \) and the slash action is explained in the following

**Proposition 3.1.8.** Let \( L_0, L \) be lattices and \( \iota : L_0 \rightarrow L \) an embedding. For \((M, \epsilon_{M\iota M}) \in \text{Mp}_{2n}(\mathbb{Z}), \phi \in \mathcal{O}(\mathcal{H}_n(L)) \) and \( k \in \frac{1}{2}\mathbb{Z} \) one has

\[ \phi \left[ \iota^{(n)} \right] \bigg|_{k,L_0} (M, \epsilon_{M\iota M}) = \left( \phi \bigg|_{k,L} (M, \epsilon_{M\iota M}) \right) \left[ \iota^{(n)} \right], \]

i.e. the diagram

\[
\begin{align*}
\mathcal{O}(\mathcal{H}_n(L)) & \xrightarrow{[\iota^{(n)}]} \mathcal{O}(\mathcal{H}_n(L_0)) \\
& \xrightarrow{|_{k,L} (M, \epsilon_{M\iota M})} \mathcal{O}(\mathcal{H}_n(L)) \\
& \xrightarrow{|_{k,L_0} (M, \epsilon_{M\iota M})} \mathcal{O}(\mathcal{H}_n(L_0))
\end{align*}
\]

is commutative for all \((M, \epsilon_{M\iota M}) \in \text{Mp}_{2n}(\mathbb{Z}).\)
From 3.1.8 we immediately obtain

**Theorem 3.1.9.** Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Let $\bar{\Gamma} \leq \text{Mp}_{2n}(\mathbb{Z})$ of finite index, $\bar{\nu} \in \bar{\Gamma}^{ab}$ of finite order and $k \in \mathbb{Q}$. Then the pullback operator $[\iota^{(n)}]$ induces a homomorphism $[\iota^{(n)}] : J_k(L, \bar{\Gamma}, \bar{\nu}) \rightarrow J_k(L_0, \bar{\Gamma}, \bar{\nu})$ of vector spaces of metaplectic Jacobi forms. Furthermore, $[\iota^{(n)}]$ restricts to Jacobi cusp forms.

The pullback operator is functorial in the following sense:

**Lemma 3.1.10.** Let $L, L_0$ and $L_1$ be lattices and $\iota : L_0 \rightarrow L, \kappa : L_1 \rightarrow L_1$ embeddings. Then for all $\phi \in \mathcal{O}(\mathcal{H}_n(L_1))$ one has $$(\phi [\kappa^{(n)}]) [\iota^{(n)}] = \phi [(\kappa \circ \iota)^{(n)}].$$

From 3.1.10 we obtain the following

**Corollary 3.1.11.** Let $L$ and $L_0$ be lattices and $\iota, \kappa : L_0 \rightarrow L$ embeddings. Then the following assertions hold:

a) If $\iota$ and $\kappa$ are stably equivalent, then one has $\phi [\iota^{(n)}] = \phi [\kappa^{(n)}]$ for all $\phi \in \mathcal{E}^{(n)}(L)$.

b) If $\iota$ and $\kappa$ are equivalent, then one has $\phi [\iota^{(n)}] = \phi [\kappa^{(n)}]$ for all $\phi \in \mathcal{E}^{(n)}(L)^{\text{sym}}$.

We determine the effect of the pullback operator on Fourier expansions:

**Proposition 3.1.12.** Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Let $\phi \in \mathcal{E}^{(n)}(L)$ such that $\phi$ has a Fourier expansion of the form

$$\phi(Z, W) = \sum_{S \in \frac{1}{q} \text{Sym}^n \mathbb{Z}(Z)} \sum_{\mu \in L^*^{(n)}} c(S, \mu) e^{2\pi i tr(SZ + B^{(n)}(\mu, W))}, \quad (Z, W) \in \mathcal{H}_n^I(L)$$

for some $q \in \mathbb{N}$. Suppose that the finiteness condition

$$\left| \left\{ \mu \in L^*^{(n)} : \iota^*(\mu) = \xi \text{ and } c(S, \mu) \neq 0 \right\} \right| < \infty$$

is satisfied for all $S \in \frac{1}{q} \text{Sym}^n \mathbb{Z}$ and $\xi \in L_0^{*^{(n)}}$. Then the Fourier expansion of $\phi [\iota^{(n)}]$ is given by the formula

$$\phi [\iota^{(n)}](Z, W_0) = \sum_{S \in \frac{1}{q} \text{Sym}^n \mathbb{Z}} \sum_{\mu_0 \in L_0^*^{(n)}} \left( \sum_{\mu \in L^*^{(n)}, \iota^*(\mu) = \xi} c(S, \mu) \right) e^{2\pi i tr(SZ + B_0^{(n)}(\xi, W_0))}.$$
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Proof. Let \((Z, W_0) \in \mathcal{H}_n^1(L_0)\). Then one has
\[
\phi \left[ l^{(n)} \right] (Z, W_0) = \phi \left( Z, l^{(n)}(W_0) \right) = \sum_{S \in \frac{1}{q} \text{Sym}_n^\mathbb{Z}(Z)} \sum_{\mu \in L^*^{(n)}} c(S, \mu) e^{2\pi i \text{tr}(SZ + B^{(n)}(\mu, l^{(n)}(W_0)))} = \sum_{S \in \frac{1}{q} \text{Sym}_n^\mathbb{Z}(Z)} \sum_{\mu \in L^*^{(n)}} c(S, \mu) e^{2\pi i \text{tr}(SZ + B^{(n)}(\mu^{*^{(n)}}(\mu), W_0))}.
\]

Due to the finiteness condition we obtain the desired result after a rearrangement. \(\square\)

We give some sufficient criteria:

**Remark 3.1.13.** Let \(L_0, L\) be lattices and \(i : L_0 \to L\) an embedding. Let \(\phi \in \mathcal{E}^{(n)}(L)\) such that \(\phi\) has a Fourier expansion of the form
\[
\phi(Z, W) = \sum_{S \in \frac{1}{q} \text{Sym}_n^\mathbb{Z}(Z)} \sum_{\mu \in L^*^{(n)}} c(S, \mu) e^{2\pi i \text{tr}(SZ + B^{(n)}(\mu, W))}, \quad (Z, W) \in \mathcal{H}_n^1(L)
\]
for some \(q \in \mathbb{N}\). Then the finiteness condition in 3.1.12 holds for \(\phi \in \mathcal{E}_{\text{reg}}^{(n)}(L)\). Especially, the finiteness condition holds if the Fourier coefficients \(c(S, \mu)\) satisfy the symmetry
\[
c(S[U], \mu U) = c(S, \mu)
\]
for all \(S \in \frac{1}{q} \text{Sym}_n^\mathbb{Z}(Z), \mu \in L^*^{(n)}\) and \(U \in \text{GL}_n(Z)[q]\).

**Proof.** The symmetry condition implies \(\phi \in \mathcal{E}_{\text{reg}}^{(n)}(L)\) by the Koecher principle, cf. [1, II, § 3, Sec. 2]. Fix \(S \in \frac{1}{q} \text{Sym}_n^\mathbb{Z}(Z)\). Then \(c(S, \mu) \neq 0\) for \(\mu \in L^*^{(n)}\) implies \(Q^{(n)}(\mu) \leq S\). Since \(Q\) is positive definite, the set \(\left\{\mu \in L^*^{(n)} : Q^{(n)}(\mu) \leq S\right\}\) is finite. Hence, the finiteness condition is trivially satisfied. \(\square\)

3.2 Automorphic transfer and modular determinant

In this section we will study the pullback operator from an algebraic point of view and investigate its modularity properties. As a motivational example we discuss

**Example 3.2.1.** Let \(L_0, L\) be lattices and \(i : L_0 \to L\) an embedding. According to 3.1.7 we define holomorphic functions
\[
h_{\mu, \xi, \iota} : \mathcal{H}_n \to \mathbb{C}, \quad \mu \in L^*^{(n)}, \xi \in L_0^*^{(n)},
\]
via the theta decomposition
\[
\theta_{L_0}^{(n)} \left[ l^{(n)} \right] = \sum_{\xi \in (L_0/L_0)(n)} h_{\mu, \xi, \iota}^{(n)} \cdot \theta_{L_0}^{(n)}\xi.
\]

Then the following assertions hold:
Proof.  a) Let $h_{\mu,\xi}^{(n)}(Z) = \sum_{S \equiv Q_0^{(n)}(\mu) - Q_0^{(n)}(\xi) \mod \text{Sym}_n^*(Z)} \left| \left\{ l \in \mu + L^{(n)} : Q^{(n)}(l) = S + Q_0^{(n)}(\xi), \ i^*(n)(l) = \xi \right\} \right| e^{2\pi i \text{tr}(SZ)}$

for all $Z \in H_n$.

b) Let $S_{\mu,\xi}^{(n)}[l] = \left\{ l \in \mu + L^{(n)} : i^*(n)(l) = \xi \right\}$. Then one has

$$S_{\mu,\xi}^{(n)}[l] = \left( i^{(n)}(\xi) + \left( i(L_0) \right)^{(n)} \right) \cap \mu + L^{(n)}$$

and

$$h_{\mu,\xi}^{(n)}(Z) = \sum_{l \in S_{\mu,\xi}^{(n)}[l]} e^{2\pi i \text{tr}((Q^{(n)}(l) - Q_0^{(n)}(\xi))Z)} = e^{-2\pi i \text{tr}(Q_0^{(n)}(\xi)Z)} \cdot \theta_{S_{\mu,\xi}^{(n)}[l]}(Z).$$

As a special case,

$$h_{\mu,\xi}^{(n)}(Z) = \delta_{\mu,\xi}^{(n)}(Z), \quad \mu \in L^{(n)}, \xi \in L_0^{(n)}, Z \in H_n,$n

i.e. the theta series of degree $n$ associated to the orthogonal complement of $(L_0)$ in $L$.

c) Let $j = 1, \ldots, n$ and $\mu = (\mu_j, \mu_{n-j})$, where $\mu_j \in L^{(j)}, \mu_{n-j} \in L^{(n-j)}$. Then one has

$$S_{\mu_{n-j}}^{(j)} h_{\mu,\xi}^{(n)} = \left\{ \begin{array}{ll} h_{\mu_{j},\xi}^{(n)}, & i^{(n-j)}(\xi_{n-j}) \equiv \mu_{n-j} \mod L^{(n-j)}, \\ 0, & \text{else.} \end{array} \right.$$
b) Follows immediately from a) by a rearrangement. The equality of the sets in question follows from the identity \( \iota_n^*(\iota_n(\xi)) = \xi \).

c) Follows from b) or can be seen from 2.3.6 together with 3.1.6.

**Definition 3.2.2.** Let \( L \) be a lattice. For a subspace \( V \leq \Theta_L^{(n)} \) we denote by \( O_H(n) V \) the module generated by \( V \) over \( O_H(n) \).

In order to define the pullback operator in a more general setting we need

**Lemma 3.2.3.** Let \( L \) be a lattice and \( V \leq \Theta_L^{(n)} \).

a) Let \( \phi_1, \ldots, \phi_m \) be linearly independent in \( V \). Then the functions \( \phi_1(Z, \cdot), \ldots, \phi_m(Z, \cdot) \) are linearly independent for every \( Z \in H_n \).

b) \( O(H_n)V \) is free of rank \( \dim_C(V) \) and every \( C \)-basis of \( V \) is an \( O(H_n) \)-basis of \( O(H_n)V \).

**Proof.**

a) Let \( d := \det L \). By definition of \( \Theta_L^{(n)} \) we have

\[
\begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_m
\end{pmatrix} = A
\begin{pmatrix}
\theta^{(n)}_{L,\mu_1} \\
\vdots \\
\theta^{(n)}_{L,\mu_d}
\end{pmatrix}
\]

for some \( A \in \mathbb{C}^{m \times d^n} \) if we fix a complete system of representatives \( \mu_1, \ldots, \mu_d \) of \( (L^*/L)^{(n)} \). By assumption on \( \phi_1, \ldots, \phi_m \), the matrix \( A \) has rank \( m \). The functions

\[
\theta^{(n)}_{L,\mu}(Z, \cdot), \quad \mu \in (L^*/L)^n
\]

are linearly independent for \( Z \in H_n \). Since the matrix identity is also valid for \( Z \in H_n \) as an identity of \( W \), we conclude that

\[
\phi_1(Z, \cdot), \ldots, \phi_m(Z, \cdot)
\]

must be linearly independent.

b) This follows immediately from a).

We come to the main definition, which covers the cases we are interested in:

**Definition 3.2.4.** Let \( L_0 \rightarrow L \) be lattices and \( \iota : L_0 \rightarrow L \) an embedding. Assume that the following setup holds:

i) \( V \leq \Theta_L^{(n)} \) resp. \( V_0 \leq \Theta_{L_0}^{(n)} \) are \( \rho_L^{(n)} \)- resp. \( \rho_{L_0}^{(n)} \)-invariant subspaces,

ii) \( \rho : Mp_{2n}(Z) \rightarrow GL_d(C) \) and \( \rho_0 : Mp_{2n}(Z) \rightarrow GL_{d_0}(C) \) are matrix representations of \( \rho_L^{(n)} \) resp. \( \rho_{L_0}^{(n)} \) with respect to certain bases of \( V \) resp. \( V_0 \),
iii) $V[i^{(n)}] \subseteq \mathcal{O}(\mathcal{H}_n)V_0$.

In this setting we define

$$H_{L_0}^L[i^{(n)}, \rho, \rho_0] \in \mathcal{O}(\mathcal{H}_n)^{d \times d_0}$$

as the transposed $\mathcal{O}(\mathcal{H}_n)$-representation matrix of the restricted map

$$[i^{(n)}] : \mathcal{O}(\mathcal{H}_n)V \rightarrow \mathcal{O}(\mathcal{H}_n)V_0$$

with respect to the bases in ii) according to 3.2.3. We call $H_{L_0}^L[i^{(n)}, \rho, \rho_0]$ the automorphic transfer from $L$ to $L_0$ with respect to $i^{(n)}$, $\rho$, $\rho_0$. In the equidimensional case $d = d_0$, the determinant

$$\det H_{L_0}^L[i^{(n)}, \rho, \rho_0] : \mathcal{H}_n \rightarrow \mathbb{C}$$

is well-defined and called the modular determinant with respect to $i^{(n)}$, $\rho$, $\rho_0$.

**Remark 3.2.5.** Let $L_0$, $L$ be lattices and $i : L_0 \rightarrow L$ an embedding. Assume that we are in the setting of 3.2.4.

a) For bases $(\phi_1, \ldots, \phi_d)$ of $V$ resp. $(\psi_1, \ldots, \psi_{d_0})$ of $V_0$ in 3.2.4 ii) the automorphic transfer from $L$ to $L_0$ with respect to $i^{(n)}$, $\rho$, $\rho_0$ is uniquely characterized by the identity

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} [i^{(n)}] = H_{L_0}^L \begin{pmatrix} i^{(n)} \\ \rho_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}.$$  

b) With respect to the bases given by the Jacobi theta functions associated to $L$ resp. $L_0$ the automorphic transfer $H_{L_0}^L[i^{(n)}, \rho_L^{(n)}, \rho_L^{(n)}]$ was determined explicitly in 3.2.1.

c) There are matrices $A \in \mathbb{C}^{d \times (\det L)^n}$ and $B \in \mathbb{C}^{(\det L_0)^n \times d_0}$ such that

$$H_{L_0}^L[i^{(n)}, \rho, \rho_0] = A \cdot H_{L_0}^L[i^{(n)}, \rho_L^{(n)}, \rho_L^{(n)}] \cdot B.$$  

d) Provided existence, the modular determinant $\det H_{L_0}^L[i^{(n)}, \rho, \rho_0]$ obviously depends on the choice of the bases of $V$ resp. $V_0$ and hence is uniquely determined only up to some nonzero complex scalar.

Functoriality of the automorphic transfer is explained in the following

**Proposition 3.2.6.** Let $L_0$, $L_1$, $L$ be lattices and $i : L_0 \rightarrow L_1$, $\kappa : L_1 \rightarrow L$ embeddings. Assume that we are in the setting of 3.2.4, such that

$$H_{L_1}^{L_1}[\kappa^{(n)}, \rho, \rho_1], \quad H_{L_0}^{L_1}[i^{(n)}, \rho_1, \rho_0]$$

are defined. Then one has

$$H_{L_0}^{L_0}[(\kappa \circ i)^{(n)}, \rho, \rho_0] = H_{L_1}^{L_1}[\kappa^{(n)}, \rho, \rho_1] \cdot H_{L_0}^{L_1}[i^{(n)}, \rho_1, \rho_0]$$

with respect to suitable bases.
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Proof. Follows directly from 3.1.10.

Corollary 3.2.7. Let $L_0, L_1$ and $L$ be lattices and

$$
\iota, \iota' : L_0 \longrightarrow L_1, \quad \kappa, \kappa' : L_1 \longrightarrow L
$$

embeddings. Assume that we are in the situation of 3.2.4. If

$$
H_{L_0}^{L} [(\kappa \circ \iota)^{(n)}(\rho_0, \rho_0)] = H_{L_0}^{L} [(\kappa' \circ \iota')^{(n)}(\rho_0, \rho_0)], \quad H_{L_1}^{L} [(\iota(n), \rho_0, \rho_0)] = H_{L_1}^{L} [(\iota(n), \rho_1, \rho_0)].
$$

and $H_{L_1}^{L} [(\iota(n), \rho_0, \rho_0)]$ has full row-rank, then $H_{L_1}^{L} [(\kappa(n), \rho_0, \rho_1)] = H_{L_1}^{L} [(\kappa'(n), \rho, \rho_1)]$.

Proof. Due to 3.2.6 and by assumption on $\iota(n)$ and $\iota'(n)$ one has

$$
H_{L_1}^{L} [(\kappa(n), \rho_0, \rho_1)] \cdot H_{L_0}^{L} [(\iota(n), \rho_1, \rho_0)] = H_{L_1}^{L} [(\kappa'(n), \rho, \rho_1)] \cdot H_{L_2}^{L} [(\iota(n), \rho_1, \rho_0)].
$$

Due to the condition on the rank we can cancel $H_{L_2}^{L} [(\iota(n), \rho_1, \rho_0)]$ from the right and obtain the desired result.

We repeat some representation theoretic constructions:

Remark 3.2.8. Let $G$ be a group, $V$ a complex vector space and $\rho : G \longrightarrow \text{GL}(V)$ a representation.\n
a) The dual or contragredient representation of $G$ with respect to $\rho$ is defined as

$$
\rho^* : G \longrightarrow \text{GL}(V^*), \quad \rho^*(g) := \rho(g^{-1})^t, g \in G,
$$

where $V^*$ denotes the dual space of $V$ and $^t$ means taking the transposed endomorphism. If $V = \mathbb{C}^m$ we can identify $^t$ with the usual matrix transpose.

b) Let $\rho' : G \longrightarrow W$ be another representation on some complex vector space $W$. Then there is a unique representation $\rho \otimes \rho' : G \longrightarrow \text{GL}(V \otimes W)$, such that

$$(\rho \otimes \rho')(g)(v \otimes w) = \rho(g)v \otimes \rho'(g)w, \quad g \in G, v \in V, w \in W.$$

The representation $\rho \otimes \rho'$ is called the tensor product of $\rho$ and $\rho'$. If $V = \mathbb{C}^n, W = \mathbb{C}^m$, we can identify $\text{GL}(V)$ resp. $\text{GL}(W)$ with $\text{GL}_n(\mathbb{C})$ resp. $\text{GL}_m(\mathbb{C})$ and $V \otimes W$ with $\mathbb{C}^{n \times m}$ via the dyadic product $v \otimes w \mapsto vw^t$. In this case one has

$$
(\rho \otimes \rho')(g)X = \rho(g)X\rho'(g)^t, \quad X \in \mathbb{C}^{n \times m}, g \in G.
$$

Now we can state the precise modularity properties of the automorphic transfer and the modular determinant provided its existence:
Theorem 3.2.9. Let $L_0, L$ be lattices and $\iota : L_0 \to L$ an embedding. Assume that we are in the setting of 3.2.4. Then the following assertions hold:

a) The transformation formula

$$H^L_{L_0} \left[ \iota(n), \rho, \rho_0 \right] \bigg|_{\frac{r_L - r_{L_0}}{2}} (M, \varepsilon_{M\mathcal{J}M}) = \rho(M, \varepsilon_{M\mathcal{J}M}) \cdot H^L_{L_0} \left[ \iota(n), \rho, \rho_0 \right] \cdot \rho_0(M, \varepsilon_{M\mathcal{J}M})^{-1}$$

holds for all $(M, \varepsilon_{M\mathcal{J}M}) \in \text{Mp}_{2n}(\mathbb{Z})$, i.e.

$$H^L_{L_0} \left[ \iota(n), \rho, \rho_0 \right] \in \left[ \text{Mp}_{2n}(\mathbb{Z}), \frac{r_L - r_{L_0}}{2}, \rho \otimes \rho_0 \right].$$

b) Suppose that $\dim \rho = \dim \rho_0$. In this case

$$\det H^L_{L_0} \left[ \iota(n), \rho, \rho_0 \right] \in \left[ \text{Mp}_{2n}(\mathbb{Z}), \frac{r_L - r_{L_0}}{2} \cdot \dim \rho, \det \rho \cdot (\det \rho_0)^{-1} \right].$$

Proof. a) The assertion is clear for $(\iota_{2n}, -\iota_{2n})$. We fix bases $(\varphi_1, \ldots, \varphi_d)$ of $V$ and $(\psi_1, \ldots, \psi_{d_0})$ of $V_0$ that correspond to $\rho$ resp. $\rho_0$. For reasons of readability we will simply write $H$ instead of $H^L_{L_0} \left[ \iota(n), \rho, \rho_0 \right]$. The main idea is to apply the slash operator $\left| \frac{r_L}{L_0} \right. \tilde{M}$ for every $M \in \text{Sp}_n(\mathbb{Z})$ on the defining equation

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} = \left( \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \right) \left| \frac{r_L}{L_0} \right. \tilde{M}.$$ 

Considering the left-hand side we obtain

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left| \frac{r_L}{L_0} \right. \tilde{M} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left| \frac{r_L}{L_0} \right. \tilde{M} \left[ \iota(n) \right] = \rho(\tilde{M}) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left| \frac{r_L}{L_0} \right. \tilde{M} = \rho(\tilde{M}) \cdot H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}.$$ 

On the right-hand side we obtain

$$H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} \left| \frac{r_L}{L_0} \right. \tilde{M} = H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} \left| \frac{r_{L_0}}{L_0} \right. \tilde{M} = H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} \rho_0(M) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}. $$

Hence, by linear independency, $H \left| \frac{r_L - r_{L_0}}{2} \right. \tilde{M} = \rho(\tilde{M}) \cdot H \cdot \rho_0(\tilde{M})^{-1}$.

b) We use the notation as in a) and apply the determinant on the transformation formula. By the Leibniz formula, the identity

$$\det \left( H \left| \frac{r_L - r_{L_0}}{2} \right. (M, \varepsilon_{M\mathcal{J}M}) \right) = (\det H) \left| \frac{r_L - r_{L_0}}{2} \right. \dim \rho (M, \varepsilon_{M\mathcal{J}M})$$

holds for all $(M, \varepsilon_{M\mathcal{J}M}) \in \text{Mp}_{2n}(\mathbb{Z})$. By the Koecher principle, holomorphicity at the cusps of $\text{Mp}_{2n}(\mathbb{Z})$ is only to prove for $n = 1$. Here we note $V \leq \mathcal{E}_{\text{reg}}(1)$ and $V_0 \leq \mathcal{E}_{\text{reg}}(1)$. \qed
We need several lemmas:

**Lemma 3.2.10.** Let $\tilde{\nu} \in Mp_2(\mathbb{Z})^{ab}$ and $f \in [Mp_2(\mathbb{Z}), k, \tilde{\nu}]$. Then the following assertions are equivalent:

i) $f$ is nonvanishing on $\mathcal{H}$, i.e. $f \in \mathcal{O}(\mathcal{H})^\times$,

ii) $f \in C^\times \eta^{2k}$.

Hence, we have

$$\mathcal{O}(\mathcal{H})^\times \cap [Mp_2(\mathbb{Z}), k, \tilde{\nu}] = \begin{cases} C^\times \eta^{2k}, & \tilde{\nu} = \nu_2^b, \\ \emptyset, & \text{else.} \end{cases}$$

**Proof.** Clearly, ii) implies i). Let $\tilde{\nu} = \nu^{b_2} \eta$ for some $0 \leq b < 24$. We consider the function

$$g := \eta^{24-b} f \in \left[ Mp_2(\mathbb{Z}), k - \frac{b}{2} + 12 \right],$$

which is still nonvanishing on $\mathcal{H}$, but vanishes at infinity of some order $m \geq 0$. Since $\eta^{24} \in [Mp_2(\mathbb{Z}), 12]$ vanishes at infinity of order 1, we obtain that $g \eta^{-24m}$ is a modular form, nonvanishing on $\mathcal{H}$ including infinity. By the valence formula $g \cdot \eta^{-24m}$ has weight zero and hence is constant. It follows that $g \in C^\times \eta^{24m}$. Consequently, $f$ is a nonzero multiple of some $\eta$-power and due to the weight one has $f \in C^\times \eta^{2k}$.

In view of surjectivity we give a negative result:

**Lemma 3.2.11.** Let $L_0, L$ be lattices and $\iota : L_0 \rightarrow L$ an embedding. Assume that we are in the setting of 3.2.4. Suppose that $n \geq 2, r_{L_0} < r_L$ and $\dim \rho = \dim \rho_0$. Then the following assertions hold:

a) The modular determinant $\det H_{L_0}^L \left[ \iota(n), \rho, \rho_0 \right]$ has a zero in $\mathcal{H}_n$ and $H_{L_0}^L \left[ \iota(n), \rho, \rho_0 \right]$ is singular.

b) The pullback $\left[ \iota(n) \right] : \mathcal{O}(\mathcal{H}_n)V \rightarrow \mathcal{O}(\mathcal{H}_n)V_0$ is not surjective.

**Proof.** a) Let $F := \det H_{L_0}^L \left[ \iota(n), \rho, \rho_0 \right]$. By assumption on the ranks of $L$ resp. $L_0$, we conclude that $F$ is a modular form of degree $n > 1$ and positive weight. Hence, by the Koecher principle, $F$ has some zero $Z_0 \in \mathcal{H}_n$. Consequently, the matrix $H_{L_0}^L \left[ \iota(n), \rho, \rho_0 \right]$ is not invertible over $\mathcal{O}(\mathcal{H}_n)$.

b) Otherwise $\left[ \iota(n) \right]$ would extend to an epimorphism of equidimensional vector spaces by passing to the quotient field of the integral domain $\mathcal{O}(\mathcal{H}_n)$. Hence, $\left[ \iota(n) \right]$ would be a monomorphism of vector spaces, thus an isomorphism. This contradicts a).
Theorem 3.2.12. Let $L_0, L$ be lattices and $i : L_0 \longrightarrow L$ an embedding. Assume that we are in the setting of 3.2.4. Suppose that $r_{L_0} < r_L$. Then the following assertions are equivalent:

i) $[i^{(n)}] : \mathcal{O}(\mathcal{H}_n) V \longrightarrow \mathcal{O}(\mathcal{H}_n) V_0$ is an isomorphism,

ii) $\det H_{L_0}^L \left[ i^{(n)}, \rho, \rho_0 \right]$ is nonvanishing on $\mathcal{H}_n$.

iii) $n = 1$ and $\det H_{L_0}^L \left[ i^{(n)}, \rho, \rho_0 \right] = c \cdot \eta \left( r_L - r_{L_0} \right)^{\dim \rho}$ for some $c \in \mathbb{C}^\times$.

In this case, the determinant characters are related by the formula

$$\det \rho = \nu \left( r_L - r_{L_0} \right)^{\dim \rho} \cdot \det \rho_0.$$

Proof. The equivalence of i) and ii) is immediate by linear algebra over $\mathcal{O}(\mathcal{H}_n)$. Furthermore, iii) obviously implies ii). In the remaining case we conclude $n = 1$ from 3.2.11 and $F = c \cdot \eta \left( r_L - r_{L_0} \right)^{\dim \rho}$ for some $c \neq 0$ from 3.2.10. Then one has $\det \rho \cdot (\det \rho_0)^{-1} = \nu \left( r_L - r_{L_0} \right)^{\dim \rho}$.

We state necessary conditions for the existence of nontrivial modular determinants:

Proposition 3.2.13. Let $L_0, L$ be lattices and $i : L_0 \longrightarrow L$ an embedding. Assume that we are in the setting of 3.2.4. Suppose that $n \geq 2$ and $\dim \rho = \dim \rho_0$. Then the following assertions hold:

a) If $(r_L - r_{L_0}) \cdot \dim \rho$ is odd, then $\det H_{L_0}^L \left[ i^{(n)}, \rho, \rho_0 \right] \equiv 0$.

b) If $n$ is odd and $(r_L - r_{L_0}) \cdot \dim \rho \not\equiv 0 \mod 4$, then $\det H_{L_0}^L \left[ i^{(n)}, \rho, \rho_0 \right] \equiv 0$.

In all these cases, the pullback operator $[i^{(n)}] : \mathcal{O}(\mathcal{H}_n) V \longrightarrow \mathcal{O}(\mathcal{H}_n) V_0$ is not injective.

Proof. a) Follows from the fact that for $n \geq 2$ every modular form with respect to $\text{Mp}_{2n}(\mathbb{Z})$ of half-integral weight is trivial.

b) Let $F = \det H_{L_0}^L \left[ i^{(n)}, \rho, \rho_0 \right]$. Note that $F$ has trivial character, since $n \geq 3$. Then the claim follows from

$$F = F \big|_{r_L - r_{L_0} \cdot \dim \rho = \frac{1}{2} \cdot (r_L - r_{L_0}) \cdot \dim \rho} \left( -I_{2n} \right) = i^{(r_L - r_{L_0}) \cdot \dim \rho} \cdot F.$$

We study the behaviour between the automorphic transfer and the Witt operator $\mathcal{W}^{(j)}$.

Given $\mathcal{X} = (x_1, \ldots, x_n)$ and $\mathcal{Y} = (y_1, \ldots, y_m)$ we define $\mathcal{X} \otimes \mathcal{Y}$ as

$$\mathcal{X} \otimes \mathcal{Y} := \{(x_1, y_1), \ldots, (x_1, y_m), \ldots, (x_2, y_1), \ldots, (x_n, y_m)\}.$$
Theorem 3.2.14. Let $L_0, L$ be lattices, $d = \det L$, $d_0 = \det L_0$ and $\iota : L_0 \rightarrow L$ an embedding. In the case $n = 1$ we distinguish complete systems of representatives

$$\mathcal{R}_L^{(1)} = (\mu_1, \ldots, \mu_d), \quad \mathcal{R}_{L_0}^{(1)} = (\xi_1, \ldots, \xi_{d_0})$$

of $L^*/L$ resp. $L_0^*/L_0$. In the case $n > 1$ we define systems of representatives of $(L^*/L)^{(n)}$ resp. $(L_0^*/L_0)^{(n)}$ recursively by

$$\mathcal{R}_L^{(n)} := \mathcal{R}_L^{(1)} \otimes \mathcal{R}_L^{(n-1)}, \quad \mathcal{R}_{L_0}^{(n)} := \mathcal{R}_{L_0}^{(1)} \otimes \mathcal{R}_{L_0}^{(n-1)}.$$

Let $\rho_L^{(n)}$ resp. $\rho_{L_0}^{(n)}$ denote the Weil representations with respect to the bases of $\Theta_L^{(n)}$ resp. $\Theta_{L_0}^{(n)}$ given by the Jacobi theta functions labeled by $R_L^{(n)}$ resp. $R_{L_0}^{(n)}$. Then one has

$$W^{(j)} \left( \Theta_L^{(n)} \left[ i^{(n)}, \rho_L^{(n)}, \rho_{L_0}^{(n)} \right] \right) = \Theta_L^{(j)} \left[ i^{(j)}, \rho_L^{(j)}, \rho_{L_0}^{(j)} \right] \otimes \left( \det H_L^{(n)} \left[ i^{(n)}, \rho_L^{(n)}, \rho_{L_0}^{(n)} \right] \right)^d.$$

where $\otimes$ denotes the Kronecker product of matrices. In the equidimensional case $d = d_0$ one has

$$W^{(j)} \left( \det H_L^{(n)} \left[ i^{(n)}, \rho_L^{(n)}, \rho_{L_0}^{(n)} \right] \right) = \left( \det H_L^{(n)} \left[ i^{(j)}, \rho_L^{(j)}, \rho_{L_0}^{(j)} \right] \right)^{d(n-j)} \otimes \left( \det H_L^{(n)} \left[ i^{(n-j)}, \rho_L^{(n-j)}, \rho_{L_0}^{(n-j)} \right] \right)^d.$$

Proof. By an inductive argument we obtain natural identifications

$$\mathcal{R}_L^{(n)} = \mathcal{R}_L^{(j)} \otimes \mathcal{R}_L^{(n-j)}, \quad \mathcal{R}_{L_0}^{(n)} = \mathcal{R}_{L_0}^{(j)} \otimes \mathcal{R}_{L_0}^{(n-j)}$$

for $0 \leq j \leq n$. Then it is easy to see that

$$W^{(j)} \left( \vartheta_L^{(n)} \left[ i^{(n)} \right] \right) = \vartheta_L^{(j)} \left[ i^{(j)} \right] \otimes \vartheta_L^{(n-j)} \left[ i^{(n-j)} \right], \quad \mu = (\mu_j, \mu_{n-j}) \in \mathcal{R}_L^{(n)}.$$

The claim follows by considering the corresponding representation matrices. The supplementary follows from the determinant formula for Kronecker products of matrices. \qed
### 3.3 Examples of automorphic transfer and modular determinant of degree 1

In this section we shall determine $H^L_{\mathcal{L}_0}[\iota, \rho, \rho_0]$ explicitly for some examples of embeddings of lattices of small rank and low-dimensional subrepresentations.

<table>
<thead>
<tr>
<th>$L$</th>
<th>representation</th>
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<tbody>
<tr>
<td>$E_8$</td>
<td>$\rho_{E_8}$</td>
<td>$\left( \vartheta_{E_8} \right)$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\rho_{E_7}$</td>
<td>$\left( \vartheta_{E_7}^{\frac{1}{2} \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\rho_{E_6}^{\text{sym}}$</td>
<td>$\left( \vartheta_{E_6}^{\frac{1}{2} \left( \varepsilon_1 + \varepsilon_3 - 2 \varepsilon_5 \right)} + \vartheta_{E_6}^{\frac{1}{2} \left( \varepsilon_1 + \varepsilon_3 - 2 \varepsilon_5 \right)} \right)$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$\rho_{D_4}^{\text{sym}}$</td>
<td>$\left( \vartheta_{D_4}^{\frac{1}{2} \left( \varepsilon_2 + \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} + \vartheta_{D_4}^{\frac{1}{2} \left( \varepsilon_2 + \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\rho_{A_2}^{\text{sym}}$</td>
<td>$\left( \vartheta_{A_2}^{\frac{1}{2} \left( \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} + \vartheta_{A_2}^{\frac{1}{2} \left( \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$2A_2$</td>
<td>$\rho_{2A_2}^{\text{sym}}$</td>
<td>$\left( \vartheta_{2A_2}^{\frac{1}{2} \left( \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} + \vartheta_{2A_2}^{\frac{1}{2} \left( \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$2A_1$</td>
<td>$\rho_{2A_1}^{\text{sym}}$</td>
<td>$\left( \vartheta_{2A_1}^{\frac{1}{2} \left( \frac{1}{2} \varepsilon_4 + \frac{1}{2} \varepsilon_6 + \frac{1}{2} \varepsilon_7 \right)} + \vartheta_{2A_1}^{\frac{1}{2} \left( \frac{1}{2} \varepsilon_4 + \frac{1}{2} \varepsilon_6 + \frac{1}{2} \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$A_1(2)$</td>
<td>$\rho_{A_1(2)}^{\text{sym}}$</td>
<td>$\left( \vartheta_{A_1(2)}^{\frac{1}{2} \left( \frac{1}{2} \varepsilon_4 + \frac{1}{2} \varepsilon_6 + \frac{1}{2} \varepsilon_7 \right)} + \vartheta_{A_1(2)}^{\frac{1}{2} \left( \frac{1}{2} \varepsilon_4 + \frac{1}{2} \varepsilon_6 + \frac{1}{2} \varepsilon_7 \right)} \right)$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$\rho_{A_1}$</td>
<td>$\left( \vartheta_{A_1}^{\frac{1}{2} \left( \varepsilon_4 + \varepsilon_6 + \varepsilon_7 \right)} \right)$</td>
</tr>
</tbody>
</table>
In order to fix some notation, we let for the rest of this section \( \vartheta_0 \) and \( \vartheta_1 \) denote the classical theta functions, i.e.

\[
\vartheta_0(z, w) := \vartheta_{\mathbb{A}_1, 0} = \sum_{m \in \mathbb{Z}} e^{2 \pi i (m^2 z + 2 mw)}, \quad \vartheta_1(z, w) := \vartheta_{\mathbb{A}_1, 1} = \sum_{m \in \mathbb{Z} + \frac{1}{2}} e^{2 \pi i (m^2 z + 2 mw)}
\]

for \((z, w) \in \mathcal{H} \times \mathbb{C}\) with the conventions \( \vartheta_0(z) := \vartheta_0(z, 0) \) and \( \vartheta_1(z) := \vartheta_1(z, 0) \) for \( z \in \mathcal{H} \).

**Example 3.3.1.** Let \( \iota : \mathbb{A}_1 \rightarrow E_7 \) be an embedding.

a) The automorphic transfer \( H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] \) is given by

\[
H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] (z) = \begin{pmatrix}
\vartheta_0(z)^6 + 3 \vartheta_0(z)^2 \vartheta_1(z)^4 & 4 \vartheta_0(z)^3 \vartheta_1(z)^3 \\
4 \vartheta_0(z)^3 \vartheta_1(z)^3 & \vartheta_1(z)^6 + 3 \vartheta_0(z)^4 \vartheta_1(z)^2
\end{pmatrix}
\]

for \( z \in \mathcal{H} \).

b) The modular determinant \( \det H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] \) is given by

\[
\det H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] = 3 \vartheta_0(z)^2 \vartheta_1(z)^2 (\vartheta_0(z)^4 - \vartheta_1(z)^4)^2 = 12 \cdot \eta(z)^{12}
\]

for \( z \in \mathcal{H} \).

**Proof.** a) Since \( E_7 \) is an irreducible root lattice it suffices to consider the embedding \( \iota = \iota_{\mathbb{A}_1} \).

By a detailed analysis of table 1.4, we can determine the theta decomposition of \( \vartheta_{E_7, 0}[e_1] \), namely

\[
\vartheta_{E_7, 0}[e_1](z, w) = \begin{pmatrix}
\vartheta_0(z)^6 + 3 \vartheta_0(z)^2 \vartheta_1(z)^4 & \vartheta_0(z, w_0) + 4 \vartheta_0(z)^3 \vartheta_1(z)^3 \vartheta_1(z, w_0)
\end{pmatrix}
\]

By using the identity

\[
\vartheta_{E_7, 1}[e_1](z, w_0) = -i \cdot \sqrt{2} \vartheta_{E_7, 0}[e_1] \cdot J(z, w_0) - \vartheta_{E_7, 0}[e_1](z, w_0), \quad (z, w_0) \in \mathcal{H} \times \mathbb{C},
\]

as well as the transformation formulas of \( \vartheta_0 \) and \( \vartheta_1 \), a straightforward calculation shows

\[
\vartheta_{E_7, 1}[e_1](z, w_0) = 4 \vartheta_0(z)^3 \vartheta_1(z)^3 \vartheta_0(z, w_0) + \begin{pmatrix}
\vartheta_1(z)^6 + 3 \vartheta_0(z)^4 \vartheta_1(z)^2
\end{pmatrix} \vartheta_1(z, w_0).
\]

Note that the latter identity can also be seen from table 1.4.

b) Due to 3.2.9 one has

\[
\det H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] \in \begin{bmatrix} \text{Mp}_2(\mathbb{Z}), 6, \nu_\eta^{12} \end{bmatrix} = C \eta^{12}.
\]

By expanding the determinant we see that its first nontrivial Fourier coefficient equals 12, hence

\[
\det H_{E_7}^{[\mathbb{A}_1]} \left[ \iota, \rho_{E_7}, \rho_{\mathbb{A}_1} \right] = 12 \cdot \eta^{12}.
\]
Example 3.3.2. Let \( \iota : A_1 \longrightarrow E_6 \) be an embedding.

a) The automorphic transfer \( H^{E_6}_{A_1}[\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] \) is given by

\[
H^{E_6}_{A_1}[\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}](z) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \vartheta_{A_2}(2z) \vartheta_0^3(z) + 3\vartheta(z) \vartheta_0(z) \vartheta_1^2(z) \\ \vartheta_{A_2}(2z) \vartheta_1^3(z) + 3\vartheta(z) \vartheta_0^2(z) \vartheta_1(z) \\ \psi(z) \vartheta_0^3(z) + \frac{3}{2} \vartheta_0(z) \vartheta_1^2(z) \vartheta_1(\frac{z}{2}) \vartheta(\frac{z}{2}) \\ \psi(z) \vartheta_1^3(z) + \frac{3}{2} \vartheta_0^2(z) \vartheta_1(z) \vartheta_1(\frac{z}{2}) \vartheta(\frac{z}{2}) \end{pmatrix},
\]

where

\[
\psi(z) := \vartheta_1(z) \vartheta_1\left(\frac{3z}{2}\right) - \vartheta_0(z) \vartheta_0\left(\frac{3z}{2}\right) + \vartheta_{A_2}(2z)
\]

and

\[
\psi^*(z) := \vartheta_0\left(\frac{z}{2}\right) \vartheta_0\left(\frac{z}{2}\right) \vartheta_1\left(\frac{z}{2}\right) - \frac{1}{2} \vartheta_1\left(\frac{z}{2}\right) \vartheta_1\left(\frac{z}{2}\right)
\]

for \( z \in \mathcal{H} \).

b) The modular determinant \( \det H^{E_6}_{A_1}[\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] \) is given by

\[
\det H^{E_6}_{A_1}[\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}](z) = 12 \cdot \eta^10(z)
\]

for \( z \in \mathcal{H} \).

**Proof.** a) Using the notation from table 1.4 we have

\[
E_6 = \bigcup_{i=1}^{8} S_i \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp.
\]

Since \( E_6 \) is an irreducible root lattice, it suffices to consider the embedding given by \( \iota = \iota_{e_2} \).

In order to determine the theta decomposition of \( \vartheta_{E_6,0}[e_2] \), we consider the pullbacks

\[
\vartheta_{S_i \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp}[e_2], \quad i = 1, \ldots, 8,
\]

separately. For \( l \in E_6 \) let \( l_1, \ldots, l_8 \) denote the coordinates of \( l \) with respect to the standard basis \( e_1, \ldots, e_8 \) of \( \mathbb{R}^8 \). By definition we have \( l \in E_6 \) if and only if \( l_8 = 0 \) and \( l_1 + l_3 + l_5 = 0 \).

In this case, \( l_1^2 + l_3^2 + l_5^2 = 2(l_1^2 + l_3^2 + l_5^2) \). With respect to this coordinates we have

\[
Q(l) = l_1^2 + l_4^2 + l_6^2 + l_7^2 + 2(l_1^2 + l_3^2 + l_5^2), \quad B(l, e_2) = 2l_2, \quad l \in E_6.
\]

Considering the sets \( S_1 \) and \( S_4 \) we immediately obtain

\[
\sum_{l \in S_1 \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp} e^{2\pi i Q(l)z + B(l, e_2)w_0} = \vartheta_{A_2}(2z) \vartheta_0(z)^3 \vartheta_0(z, w_0),
\]

\[
\sum_{l \in S_4 \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp} e^{2\pi i Q(l)z + B(l, e_2)w_0} = \vartheta_{A_2}(2z) \vartheta_1(z)^3 \vartheta_1(z, w_0).
\]

For \( l \in S_2 \cap \langle e_8, e_1 + e_3 + e_5 \rangle^\perp \) we have \( 2l_1, 2l_3 \in 2\mathbb{Z} + 1 \) and

\[
l_1^2 + l_3^2 + l_5^2 = \frac{(2l_1)^2 + (2l_3)^2 + (2l_5)^2}{2}.
\]
3.3 Examples of automorphic transfer and modular determinant of degree 1

This yields

\[
\sum_{x,y \in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)}z = \sum_{x \in \mathbb{Z}, y \in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)}z - \sum_{x \in 2\mathbb{Z}, y \in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)}z
\]

\[
= \vartheta_1(z) \vartheta_1(3z) - \left( \sum_{x \in 2\mathbb{Z}} e^{2\pi i(x^2+xy+y^2)}z - \sum_{y \in 2\mathbb{Z}} e^{2\pi i(x^2+xy+y^2)}z \right)
\]

\[
= \vartheta_1(z) \vartheta_1(3z) - \vartheta_0(z) \vartheta_0(3z) + \vartheta_{A_2}(4z)
\]

\[
= \psi(2z).
\]

Consequently,

\[
\sum_{l \in S_2 \cap (e_6, e_1 + 3 + e_5)^\perp} e^{2\pi i(Q(l)z + B(l, e_2)w_0)} = \psi(z) \vartheta_0(z) \vartheta_1(z)^2 \vartheta_0(z, w_0)
\]

and analogously

\[
\vartheta_{S_2 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2] = \vartheta_{S_5 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2] = \vartheta_{S_6 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2].
\]

A similar calculation yields

\[
\sum_{l \in S_5 \cap (e_6, e_1 + 3 + e_5)^\perp} e^{2\pi i(Q(l)z + B(l, e_2)w_0)} = \psi(z) \vartheta_0^2(z) \vartheta_1(z) \vartheta_1(z, w_0)
\]

as well as

\[
\vartheta_{S_5 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2] = \vartheta_{S_6 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2] = \vartheta_{S_7 \cap (e_6, e_1 + 3 + e_5)^\perp} [e_2].
\]

In summary, we obtain the decomposition

\[
\vartheta_{E_6}(e_2)(z, w_0) = \left( (\vartheta_{A_2}(2z) \vartheta_0(z)^3 + 3\psi(z) \vartheta_0(z) \vartheta_1(z)^2 \right) \vartheta_0(z, w_0)
\]

\[
+ \left( (\vartheta_{A_2}(2z) \vartheta_1(z)^3 + 3\psi(z) \vartheta_0^2(z) \vartheta_1(z) \right) \vartheta_1(z, w_0).
\]

In order to determine \((\vartheta_{E_6, \mu} + \vartheta_{E_6, -\mu})[e_2]\) for \(\mu = \frac{1}{3}(e_1 + e_3 - 2e_5)\) we make use of the identity

\[
\vartheta_{E_6, \mu}[e_2] + \vartheta_{E_6, -\mu}[e_2] = -i\sqrt{3} \vartheta_{E_6}[e_2] \bigg|_{3, A_1} f - \vartheta_{E_6}[e_2].
\]

Using the transformation properties of \(\vartheta_0\) and \(\vartheta_1\), an extensive calculation - which is omitted - finally yields the decomposition

\[
\vartheta_{E_6}(e_2)(z, w_0) = \left( \psi^*(z) \vartheta_0(z)^3 + \frac{3}{2} \vartheta_0(z) \vartheta_1(z)^2 \vartheta_1\left(\frac{z}{2}\right) \vartheta_1\left(\frac{z}{2}\right) \right) \vartheta_0(z, w_0)
\]

\[
+ \left( \psi^*(z) \vartheta_1(z)^3 + \frac{3}{2} \vartheta_0(z)^2 \vartheta_1(z) \vartheta_1\left(\frac{z}{2}\right) \vartheta_1\left(\frac{z}{2}\right) \right) \vartheta_1(z, w_0).
\]
b) Due to 3.2.9 one has
\[
\det H_{E_6/A_1}^{E_6} [\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] \in \left[ \text{Mp}_2(Z), 5, \nu_{10} \right] = C\nu^{10}.
\]
It remains to determine the constant. Expanding the determinant, we obtain
\[
\det H_{E_6/A_1}^{E_6} [\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] (z) = 3 \theta_0(z) \theta_1(z) \left( \theta_0(z)^4 - \theta_1(z)^4 \right)
\cdot \left[ h(z) \left( \theta_0 \left( \frac{z}{6} \right) \theta_0 \left( \frac{z}{2} \right) + \theta_1 \left( \frac{z}{6} \right) \theta_1 \left( \frac{z}{2} \right) \right) + \frac{1}{2} g(z) \theta_1 \left( \frac{z}{6} \right) \theta_1 \left( \frac{z}{2} \right) \right],
\]
where
\[
h(z) := -\theta_0(6z) \theta_0(2z) - \theta_1(6z) \theta_1(2z) + \theta_0 \left( \frac{3z}{2} \right) - \theta_1 \left( \frac{3z}{2} \right)
\]
and
\[
g(z) := 4 \theta_0(6z) \theta_0(2z) + 4 \theta_1(6z) \theta_1(2z) - 3 \theta_0 \left( \frac{3z}{2} \right) + 3 \theta_1 \left( \frac{3z}{2} \right).
\]
A closer look at the Fourier expansions of the auxiliary functions \( h \) and \( g \) yields
\[
h(z) = -4e^{\pi iz} + \ldots \text{ and } g(z) = 1 + \ldots.
\]
Hence the Fourier expansion of \( \det H_{E_6/A_1}^{E_6} [\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] \) starts by
\[
\det H_{E_6/A_1}^{E_6} [\iota, \rho_{E_6}^{\text{sym}}, \rho_{A_1}] (z) = 3 \cdot 2 \cdot e^{\frac{\pi iz}{2}} \cdot \frac{1}{2} \cdot 2e^{\frac{\pi iz}{4}} \cdot 2e^{\frac{\pi iz}{8}} + \ldots = 12 \cdot e^{\frac{2\pi iz}{4}} + \ldots,
\]
i.e. the constant equals 12. \( \square \)

**Example 3.3.3.** Let \( \iota : A_1 \rightarrow D_4 \) be an embedding.

a) The automorphic transfer \( H_{A_1}^{D_4} [\iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1}] \) is given by
\[
H_{A_1}^{D_4} [\iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1}] (z) = \begin{pmatrix} \theta_0(z)^3 \\ 3 \theta_0(z) \theta_1(z)^2 \\ 3 \theta_0(z)^2 \theta_1(z) \end{pmatrix}
\]
for \( z \in \mathcal{H} \).

b) The modular determinant \( \det H_{A_1}^{D_4} [\iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1}] \) is given by
\[
\det H_{A_1}^{D_4} [\iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1}] (z) = 3 \theta_0(z) \theta_1(z) \left( \theta_0(z)^4 - \theta_1(z)^4 \right) = 6 \cdot \eta^6(z)
\]
for \( z \in \mathcal{H} \).

**Proof.** a) Since \( D_4 \) is an irreducible root lattice, it suffices to consider the embedding given by \( \iota = \iota_{e_2} \). By definition one has
\[
D_4 = \left\langle e_2, e_4, e_6, \frac{e_2 + e_4 + e_6 + e_7}{2} \right\rangle_Z.
\]
If \( l_2, l_4, l_6, l_7 \) denote the coordinates of \( l \in D_4 \) with respect to \( e_2, e_4, e_6, e_7 \), then one has

\[
l \in D_4 \text{ if and only if } l_i \in \mathbb{Z}, i = 1, \ldots, 4 \quad \text{or} \quad l_i \in \mathbb{Z} + \frac{1}{2}, i = 1, \ldots, 4,
\]

which immediately yields

\[
\vartheta_{D_4}^{(2)}(z, w_0) = \vartheta_0(z)^3 \vartheta_0(z, w_0) + \vartheta_1(z)^3 \vartheta_1(z, w_0).
\]

From

\[
\left( \vartheta_{D_4}^{(2)} + \vartheta_{D_4}^{(2)} + \vartheta_{D_4}^{(2)} \right)(z, w_0) = -2\vartheta_{D_4}(z, w_0) - \vartheta_{D_4}^{(2)}(z, w_0),
\]

with \( \mu_1 := \frac{e_2 + e_4}{2}, \mu_2 := \frac{e_2 + e_6}{2} \) and \( \mu_3 := \frac{e_2 + e_7}{2} \), we derive

\[
\left( \vartheta_{D_4}^{(2)} + \vartheta_{D_4}^{(2)} + \vartheta_{D_4}^{(2)} \right)(z, w_0) = 3\vartheta_0(z)\vartheta_1(z)^2 \vartheta_0(z, w_0) + 3\vartheta_0(z)^2 \vartheta_1(z) \vartheta_1(z, w_0),
\]

where we again used the transformation properties of \( \vartheta_0 \) and \( \vartheta_1 \).

b) Due to 3.2.9 one has

\[
\det H_{A_1}^{D_4} \left[ \iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1} \right] \in \left[ \text{Mp}_2(\mathbb{Z}), 3, \eta^6 \right] = C\eta^6.
\]

Expanding the determinant, we obtain

\[
\det H_{A_1}^{D_4} \left[ \iota, \rho_{D_4}^{\text{sym}}, \rho_{A_1} \right](z) = 3\vartheta_0(z)\vartheta_1(z)(\vartheta_0(z)^4 - \vartheta_1(z)^4).
\]

Hence the first Fourier coefficient equals 6. \( \square \)

**Example 3.3.4.** Let \( i : A_1 \longrightarrow A_2 \) be an embedding.

a) The automorphic transfer \( H_{A_1}^{A_2} \left[ \iota, \rho_{A_2}^{\text{sym}}, \rho_{A_1} \right] \) is given by

\[
H_{A_1}^{A_2} \left[ \iota, \rho_{A_2}^{\text{sym}}, \rho_{A_1} \right](z) = \begin{pmatrix}
\vartheta_0(3z) \\
\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z) \\
\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)
\end{pmatrix}
\]

for \( z \in \mathcal{H} \).

b) The modular determinant \( \det H_{A_1}^{A_2} \left[ \iota, \rho_{A_2}^{\text{sym}}, \rho_{A_1} \right] \) is given by

\[
\det H_{A_1}^{A_2} \left[ \iota, \rho_{A_2}^{\text{sym}}, \rho_{A_1} \right](z) = \vartheta_0(3z)\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\vartheta_0\left(\frac{z}{3}\right) = 2 \cdot \eta^2(z)
\]

for \( z \in \mathcal{H} \).
Proof. a) Since \( A_2 \) is an irreducible root lattice, it suffices to consider the embedding given by \( t = i_{e_2} \). This time we shall use the explicit formulas given in 3.2.1. Thus we have
\[
\vartheta_{A_2}[e_2](z, w_0) = h_{0,0}(z) \vartheta_0(z, w_0) + h_{0, \frac{1}{2}} \vartheta_0(z, w_0),
\]
where
\[
h_{0,0}(z) = \sum_{m=0}^{\infty} |\{l \in A_2 : Q(l) = m, B(l, e_2) = 0\}| \cdot e^{2\pi i mz} = \sum_{m=0}^{\infty} |\{(r,s) \in \mathbb{Z} : r^2 + sr + s^2 = 2m, 2r + s = 0\}| \cdot e^{2\pi i mz} = \sum_{m=0}^{\infty} |\{r \in \mathbb{Z} : 3r^2 = m\}| e^{2\pi i mz} = \vartheta_0(3z)
\]
as well as
\[
h_{0, \frac{1}{2}}(z) = \sum_{m \in \frac{1}{4} + \mathbb{Z}} |\{l \in A_2 : Q(l) = m + \frac{1}{4}, B(l, e_2) = 1\}| \cdot e^{2\pi i mz} \]
\[
= \sum_{m \in \frac{1}{4} + \mathbb{Z}} |\{(r,s) \in \mathbb{Z} : r^2 + sr + s^2 = n + \frac{1}{4}, 2r + s = 1\}| \cdot e^{2\pi i mz} = \sum_{m \in \frac{1}{4} + \mathbb{Z}} |\{r \in \mathbb{Z} : 3(r + \frac{1}{2})^2 = m\}| e^{2\pi i mz} = \vartheta_1(3z).
\]
In order to determine \( (\vartheta_{A_2, \mu} + \vartheta_{A_2, -\mu})[e_2] \) for \( \mu = \frac{e_4 + e_5 + e_7}{2} \), we use the identity
\[
(\vartheta_{A_2, \mu} + \vartheta_{A_2, -\mu})[e_2](z, w_0) = i\sqrt{3}\vartheta_{A_2}[e_2]|_{A_1} J(z, w_0) - \vartheta_{A_2}[e_2](z, w_0)
\]
and a straightforward calculation yields
\[
(\vartheta_{A_2, \mu} + \vartheta_{A_2, -\mu})[e_2](z, w_0) = [\vartheta_0(\frac{z}{3}) - \vartheta_0(3z)] \vartheta_0(z, w_0) + [\vartheta_1(\frac{z}{3}) - \vartheta_1(3z)] \vartheta_1(z, w_0),
\]
where we again used the transformation properties of \( \vartheta_0 \) and \( \vartheta_1 \).

b) Due to 3.2.9 one has
\[
\det H_{A_2}^{A_1} [\nu, \rho_{A_2}^\text{sym}, \rho_{A_1}] \in \left\{ \text{Mp}_2(\mathbb{Z}), 1, \nu_0^2 \right\} = C\eta^2.
\]
Expanding the determinant, we obtain
\[
\det H_{A_2}^{A_1} [\nu, \rho_{A_2}^\text{sym}, \rho_{A_1}](z) = \vartheta_0(3z)\vartheta_1(\frac{z}{3}) - \vartheta_1(3z)\vartheta_0(\frac{z}{3}) = 2e^{\pi i \frac{1}{12}} + \ldots.
\]
Hence the constant equals 2. □
3.3 Examples of automorphic transfer and modular determinant of degree 1

Regarding equivalence of embeddings of direct sums we give the following

**Proposition 3.3.5.** Let \( \mathbb{L} \) be an irreducible root lattice and \( m \in \mathbb{N} \). Then the following assertions hold:

a) The semidirect product \( S_m \ltimes W(\mathbb{L})^m \leq O(m\mathbb{L}) \) acts \( m \)-fold transitively on \( R(m\mathbb{L}) \).

b) All embeddings \( \iota : mA_1 \rightarrow mL \) are equivalent.

**Example 3.3.6.** Let \( \iota : 2A_1 \rightarrow 2A_2 \) be an embedding.

a) The automorphic transfer \( H^{2A_2}_{2A_1} [\iota, \rho^{\text{sym}}_{2A_2}, \rho^{\text{sym}}_{2A_1}] \) is given by

\[
H^{2A_2}_{2A_1} [\iota, \rho^{\text{sym}}_{2A_2}, \rho^{\text{sym}}_{2A_1}] (z) = \begin{pmatrix}
\frac{\theta_0(3z)}{(3z)(\frac{3}{2} - \theta_0(3z))} & \frac{\theta_0(3z)\theta_1(3z)}{(3z)(\frac{3}{2} - \theta_0(3z))^2} & \frac{\theta_1(3z)}{(\theta_1(\frac{3}{2}) - \theta_0(3z))^2} \\
2\theta_0(3z)(\frac{3}{2} - \theta_0(3z)) & \theta_0(3z)\theta_1(3z) - 2\theta_0(3z)\theta_1(3z) & 2\theta_1(3z)(\theta_1(\frac{3}{2}) - \theta_0(3z)) \\
(\theta_0(\frac{3}{2}) - \theta_0(3z))^2 & (\theta_0(\frac{3}{2}) - \theta_0(3z))(\theta_1(\frac{3}{2}) - \theta_1(3z)) & (\theta_1(\frac{3}{2}) - \theta_0(3z))^2
\end{pmatrix}
\]

for \( z \in \mathbb{H} \).

b) The modular determinant \( \det H^{2A_2}_{2A_1} [\iota, \rho^{\text{sym}}_{2A_2}, \rho^{\text{sym}}_{2A_1}] \) is given by

\[
\det H^{2A_2}_{2A_1} [\iota, \rho^{\text{sym}}_{2A_2}, \rho^{\text{sym}}_{2A_1}] (z) = \left( \det H^{A_2}_{A_1} [\iota, \rho^{\text{sym}}_{A_2}, \rho^{\text{sym}}_{A_1}] (z) \right)^3 = 8 \cdot \eta(z)
\]

for \( z \in \mathbb{H} \).

**Proof.** a) In view of 3.3.5 all embeddings \( \iota : 2A_1 \rightarrow 2A_2 \) are equivalent. Hence we write \( [2A_1] \) instead of \([\iota]\). With the decomposition \( \mathbf{w}_0 = (w_1, w_2) \in \mathbb{C}^2 \) we have

\[
\theta_{2A_1}[2A_1](z, w_0) = \theta_{A_2}[A_1](z, w_1) \cdot \theta_{A_2}[A_1](z, w_2).
\]

From the identities

\[
\theta_{2A_1,0}(z, w_0) = \theta_0(z, w_1)\theta_0(z, w_2),\\
\theta_{2A_1,\frac{1}{2}}(z, w_0) = \theta_1(z, w_1)\theta_1(z, w_2),\\
\left( \theta_{2A_1,\frac{1}{2}}(z, w_0) + \theta_{2A_1,0}(z, w_0) \right)(z, w_0) = \theta_1(z, w_1)\theta_0(z, w_2) + \theta_0(z, w_1)\theta_1(z, w_2)
\]

we derive the theta decomposition

\[
\theta_{2A_2}[2A_1](z, w_0) = \theta_0^2(3z)\theta_{2A_1,0}(z, w_0) + \theta_0(3z)\theta_1(3z)\left( \theta_{2A_1,\frac{1}{2}}(z, w_0) + \theta_{2A_1,0}(z, w_0) \right)(z, w_0)
\]

\[
+ \theta_0^2(3z)\theta_{2A_1,\frac{1}{2}}(z, w_0).
\]

Let \( \mu := \frac{\epsilon_1 + \epsilon_2}{2} \). In a similar way, the identities

\[
\theta_{2A_2,\mu}(z, w_0) = \theta_{2A_2,\mu}[A_1](z, w_1) \cdot \theta_{2A_2}[A_1](z, w_2),\\
\theta_{2A_2,(-\mu, 0)}(z, w_0) = \theta_{2A_2,-\mu}[A_1](z, w_1) \cdot \theta_{2A_2}[A_1](z, w_2)
\]
yield
\[
\left( \vartheta_{2A_2}(\mu,0) + \vartheta_{2A_2}(\mu,0) + \vartheta_{2A_2}(0,\mu) + \vartheta_{2A_2}(0,-\mu) \right) [2A_1](z, w_0)
= 2\vartheta_0(3z) \left( \vartheta_0 \left( \frac{\zeta}{3} \right) - \vartheta_0(3z) \right) \vartheta_{2A_1,0}(z, w_0)
+ \left[ \vartheta_0(3z) \vartheta \left( \frac{z}{3} \right) + \vartheta_1(3z) \vartheta \left( \frac{z}{3} \right) - 2\vartheta_0(3z) \vartheta_1(3z) \right] \left( \vartheta_{2A_1,1,0} + \vartheta_{2A_1,0,1} \right)(z, w_0)
+ 2\vartheta_1(3z) \left( \vartheta \left( \frac{z}{3} \right) - \vartheta_1(3z) \right) \vartheta_{2A_1,1,1}(z, w_0).
\]
Completely analogous, by using the identities
\[
\vartheta_{2A_2,\mu}[2A_1](z, w_0) = \vartheta_{2A_2,\mu}[A_1](z, w_1) \cdot \vartheta_{2A_2,\mu}[A_1](z, w_2),
\vartheta_{2A_2,\mu}[2A_1](z, w_0) = \vartheta_{2A_2,\mu}[A_1](z, w_1) \cdot \vartheta_{2A_2,\mu}[A_1](z, w_2),
\vartheta_{2A_2,-\mu}[2A_1](z, w_0) = \vartheta_{2A_2,-\mu}[A_1](z, w_1) \cdot \vartheta_{2A_2,-\mu}[A_1](z, w_2),
\vartheta_{2A_2,-\mu}[2A_1](z, w_0) = \vartheta_{2A_2,-\mu}[A_1](z, w_1) \cdot \vartheta_{2A_2,-\mu}[A_1](z, w_2),
\]
we obtain the theta decomposition of
\[
(\vartheta_{2A_2,\mu} + \vartheta_{2A_2,-\mu} + \vartheta_{2A_2,\mu} + \vartheta_{2A_2,-\mu})[2A_1]
\]
and finish the proof.

b) Due to 3.2.9 one has
\[
\det H^2_{2A_2} \left[ \iota, \rho_{2A_2}^\text{sym}, \rho_{2A_1}^\text{sym} \right] \in \left[ \text{Mp}_2(\mathbb{Z}), 3, \eta^6 \right] = C\eta^6.
\]
A straightforward calculation yields
\[
\det H^2_{2A_2} \left[ \iota, \rho_{2A_2}^\text{sym}, \rho_{2A_1}^\text{sym} \right] (z) = \left[ \vartheta_0(3z) \vartheta_1 \left( \frac{z}{3} \right) - \vartheta_1(3z) \vartheta_0 \left( \frac{z}{3} \right) \right]^3 = 8 \cdot \eta^6(z),
\]
if we use the result of 3.3.4. \qed

Example 3.3.7. Let $\iota : A_1(2) \to 2A_1$ be an embedding.

a) The automorphic transfer $H^2_{A_1} \left[ \iota, \rho_{A_1}^\text{sym}, \rho_{A_1}^\text{sym} \right]$ is given by
\[
H^2_{A_1} \left[ \iota, \rho_{A_1}^\text{sym}, \rho_{A_1}^\text{sym} \right] (z) = \begin{pmatrix}
\vartheta_0(2z) & 0 & \vartheta_1(2z) \\
2\vartheta_1(2z) & \vartheta_{A_1}(2z) & 2\vartheta_0(2z) \\
\vartheta_1(2z) & 0 & \vartheta_0(2z)
\end{pmatrix}
\]
for $z \in \mathcal{H}$.

b) The modular determinant $\det H^2_{A_1} \left[ \iota, \rho_{A_1}^\text{sym}, \rho_{A_1}^\text{sym} \right]$ is given by
\[
\det H^2_{A_1} \left[ \iota, \rho_{A_1}^\text{sym}, \rho_{A_1}^\text{sym} \right] (z) = \vartheta_{A_1}(2z) \cdot \left( \vartheta_0^2(2z) - \vartheta_1^2(2z) \right) = \eta^3(z)
\]
for $z \in \mathcal{H}$. 

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3.3 Examples of automorphic transfer and modular determinant of degree 1

Proof. a) It is easy to check that every embedding \( \iota : A_1(2) \longrightarrow 2A_1 \) is equivalent to \( \iota_{(1,1)} \). A system of representatives of \( (2A_1)^*/2A_1 \) is given by

\[ \xi : (0,0), \left( \frac{1}{2},0 \right), \left( 0,\frac{1}{2} \right), \left( \frac{1}{2},\frac{1}{2} \right) \]

and a system of representatives for \( A_1(2)^*/A_1(2) \) is given by

\[ \mu : 0, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}. \]

In view of 3.2.1 it suffices to determine the theta series of the sets \( S_{\mu\xi,(1,1)} \) for the given representatives. In all cases, an elementary calculation yields the desired result.

b) Due to 3.2.9 one has

\[ \det H^2_{A_1(2)}[\iota, \rho_{A_1(2)}^\text{sym}, \rho_{A_1(2)}] \in \left[ \mathfrak{M}_2(\mathbb{Z}), \frac{3}{2}, v_3^2 \right] = C\eta^3. \]

By expanding the determinant, we obtain

\[ \det H^2_{A_1(2)}[\iota, \rho_{A_1(2)}^\text{sym}, \rho_{A_1(2)}](z) = \vartheta_{A_1(2)}(2z) \cdot \left( \vartheta_{A_1(2)}^2(2z) - \vartheta_{A_1(2)}^2(2z) \right). \]

Hence the constant equals 1.

We summarize the above results in the following

**Theorem 3.3.8.** Let \( ((L, \rho), (L_0, \rho_0)) \) be a pair, such that the following assumptions hold:

i) \( L, L_0 \in \{ A_1, A_2, D_4, E_6, E_7, A_1(2), 2A_1, 2A_2 \} \)

\( \rho, \rho_0 \in \{ \rho_{A_1}, \rho_{A_2}^\text{sym}, \rho_{D_4}^\text{sym}, \rho_{E_6}^\text{sym}, \rho_{E_7}^\text{sym}, \rho_{A_1(2)}^\text{sym}, \rho_{2A_1}^\text{sym}, \rho_{2A_2}^\text{sym} \} \),

ii) there is an embedding \( \iota : L_0 \longrightarrow L \),

iii) \( d : = \dim \rho = \dim \rho_0 =: d_0 \),

iv) \([\iota] : \mathcal{O}(H)V \longrightarrow \mathcal{O}(H)V_0 \) is well-defined,

where \( V \) resp. \( V_0 \) denote the corresponding subspaces of \( \Theta_L \) resp. \( \Theta_{L_0} \) associated to \( \rho \) resp. \( \rho_0 \). Then the pullback operator in iv) is an isomorphism, which does not depend on the choice of the embedding in ii). With respect to suitable coordinates for \( V \), the inverse map \([\iota]^{-1} \) is given by

\[ \mathcal{O}(H)^d \longrightarrow \mathcal{O}(H)^d, \quad \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \mapsto H^2_{L_0}(\iota, \rho, \rho_0)^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}. \]
Proof. For the subset \( \{ A_1, A_2, D_4, E_6, E_7 \} \) the independence of the choice of \( i \) follows from 3.6.2. For the subset \( \{ A_1(2), 2A_1, 2A_2 \} \) the independence is a direct consequence of 3.3.5. The fact that the corresponding maps \([i]\) in iv) are actually isomorphisms follows from the concrete examples in this section as well as from 3.2.6 and 3.2.12.

\[ \square \]

Remark 3.3.9. Note that 3.2.6 allows to evaluate \( H_{E_6}^{\mathbb{Z}}[i, \rho, \rho_0] \) in all possible cases of 3.3.8, which were not considered yet. For reasons of readability, we will not present these here, since the matrix entries are very longish.

### 3.4 A distinguished infinite family of Siegel cusp forms

In this section we will utilize the embedding \( A_1 \rightarrow E_7 \) and 3.2.14 in order to construct an infinite series of nontrivial Siegel cusp forms of weight \( 3 \cdot 2^n \), \( n \in \mathbb{N} \) satisfying a distinguished recurrence relation under the Witt operator.

Let \( \Delta^* := \eta^{24} \) denote the normalized modular discriminant of weight 12. For formal reasons we define \( (\Delta^*)^{\frac{1}{2}} := \eta^{12} \).

**Theorem 3.4.1.** There exists an infinite family of nontrivial Siegel cusp forms

\[
\chi_{2^n, 3} \in \begin{cases} 
[\text{Mp}_2(\mathbb{Z}), 6, \eta^{12}]_{\text{cusp}}, & n = 1, \\
[\text{Mp}_{2n}(\mathbb{Z}), 2^n \cdot 3]_{\text{cusp}}, & n > 1,
\end{cases}
\]

such that \( \chi_6 = (\Delta^*)^{\frac{1}{2}} \) and the following assertions hold:

a) \( W^{(i)} \chi_{2^n, 3} = \chi_{2^{n-j}, 3} \otimes \chi_{2^{n-i}, 3}^{j} \), i.e.

\[
\chi_{2^n, 3}(Z_j) = \chi_{2^{n-j}, 3}(Z_j)^{2^n-j} \cdot \chi_{2^{n-i}, 3}^{j}(Z_{n-j})
\]

holds for all \( Z_j \in \mathcal{H}_j, Z_{n-j} \in \mathcal{H}_{n-j} \) and \( 0 \leq j \leq n \),

b) \( \chi_{2^n, 3}(\text{diag}(z_1, \ldots, z_n)) = (\Delta^*(z_1) \cdots \Delta^*(z_n))^{2^{n-2}} \) for all \( z_1, \ldots, z_n \in \mathcal{H} \),

c) \( \chi_{2^n, 3}(zI_n) = \Delta^*(z)^{2^{n-2}} \) for all \( z \in \mathcal{H} \),

d) \( \chi_{2^n, 3} \) does not vanish on the diagonal \( \mathcal{H}^n \subseteq \mathcal{H}_n \).

**Proof.** For \( n = 1 \) we choose the systems of representatives of \( E_7^* / E_7 \) resp. \( A_1^* / A_1 \) used in 3.3.1. According to 3.2.14, we consider the systems of representatives \( \mathcal{R}^{(n)}_{E_7} \) of \( E_7^* / E_7 \) resp. \( \mathcal{R}^{(n)}_{A_1} \) of \( (A_1^* / A_1)^{(n)} \). Let \( \rho^{(n)}_{E_7} \) resp. \( \rho^{(n)}_{A_1} \) denote the corresponding Weil representations with respect to the bases of \( \Theta^{(n)}_{E_7} \) resp. \( \Theta^{(n)}_{A_1} \), given by the Jacobi theta functions labeled by \( \mathcal{R}^{(n)}_{E_7} \) resp. \( \mathcal{R}^{(n)}_{A_1} \). For \( n \in \mathbb{N} \) we define

\[
\chi_{2^n, 3} := \frac{1}{12^{2^{n-1}}} \cdot \det H_{E_7}^{\mathbb{Z}}[\rho^{(n)}_{E_7}, \rho^{(n)}_{A_1}] .
\]
3.5 Ad-hoc criteria for injectivity in degree 1

Since $\text{Mp}_{2n}(\mathbb{Z})$ is perfect for $n \geq 3$, we derive that $\chi_{2^n.3}$ has trivial character in this case. In the case $n = 2$ the unique nontrivial abelian character $\nu_{\text{Maaß}}$ of $\text{Mp}_4(\mathbb{Z})$ is determined by the value $\nu_{\text{Maaß}}(J \times I_2) = -1$. From 2.4.31 we obtain

$$\det \rho_{E_7}^{(2)}(J \times I_2) \cdot \left( \det \rho_{A_1}^{(2)}(J \times I_2) \right)^{-1} = \det \rho_{E_7}(J)^2 \cdot \det \rho_{A_1}(J)^{-2} = \nu_\eta^{36}(J) \cdot \nu_\eta^{-12}(J) = \nu_\eta^{24}(J) = 1.$$ 

Hence the character of $\chi_{2^n.3}$ is trivial for all $n \geq 2$ and the claim follows from 3.2.9. The identity $\chi_6 = (\Delta^*)^{\frac{1}{2}}$ is due to 3.3.1. Since $\eta$ vanishes at infinity, we obtain $S^{(n-1)} \chi_{3.2^n} = 0$ from a) in the case $j = n - 1$.

a) Follows from 3.2.14. Note that the normalization factors $\gamma_n = 12^{-n} \cdot 2^{n-1}$ satisfy the identity $\gamma_j^{2^{n-j}} \cdot \gamma_{n-j} = \gamma_n$ for $j = 0, \ldots, n$.

b) Follows from a) by induction on $n$.

c) Follows from b) with $z_1 = \cdots = z_n = z$.

d) Follows from b) and the fact that $\eta$ is nonvanishing on $\mathcal{H}$.

**Corollary 3.4.2.** Let $\iota : A_1 \to E_7$ be an embedding. Then the pullback operator

$$\left[ \iota^{(n)} \right] : \mathcal{E}^{(n)}(E_7) \to \mathcal{E}^{(n)}(A_1)$$

is a monomorphism for all $n \in \mathbb{N}$ and an isomorphism if and only if $n = 1$.

For the spaces of Jacobi forms we immediately obtain

**Corollary 3.4.3.** Let $\iota : A_1 \to E_7$ be an embedding. Then the pullback operator

$$\left[ \iota^{(n)} \right] : J_{k,E_7}^{(n)} \to J_{k,A_1}^{(n)}$$

is a monomorphism for all $n \in \mathbb{N}$ and an isomorphism in the case $n = 1$.

### 3.5 Ad-hoc criteria for injectivity in degree 1

Let $L$ be an unimodular lattice. Then $\mathcal{E}^{(n)}(L)$ is a module of rank one, i.e.

$$\mathcal{E}^{(n)}(L) = \mathcal{O}(\mathcal{H}_n) \cdot \theta^{(n)}_{L_0}.$$ 

Hence it is easy to see that for all embeddings $\iota : L_0 \to L$, where $L_0$ is a lattice, the corresponding pullback operator

$$\left[ \iota^{(n)} \right] : \mathcal{E}^{(n)}(L) \to \mathcal{E}^{(n)}(L_0)$$

is clearly injective. In this section we will formulate and prove certain ad-hoc criteria for injectivity of pullback operators induced by embeddings into non-unimodular lattices, where we restrict to the case of degree \( n = 1 \) only.

We start by a simple

**Lemma 3.5.1.** Let \( L \) and \( L_0 \) be lattices, \( L \) non-unimodular, and \( \iota : L_0 \to L \) an embedding. Suppose that there is some \( \xi \in L_0^* \) such that \( \iota(\xi) \in L^* \setminus L \). Then the pullback operator

\[
[i] : \mathcal{O}(\mathcal{H}) \cdot \langle \theta_L, \theta_L^* \rangle \to \mathcal{E}(L_0)
\]

is a monomorphism.

**Proof.** If suffices to show that the functions \( \theta_L[i] \) and \( \theta_L^*[i] \) are linearly independent over \( \mathcal{O}(\mathcal{H}) \). To this end, let \( h, g \in \mathcal{O}(\mathcal{H}) \) such that

\[
h \cdot \theta_L[i] + g \cdot \theta_L^*[i] = 0.
\]

We apply \( \bigg|_{L_0} [\xi, 0] \) on this equation in order to obtain

\[
h \cdot \theta_{L,\iota(\xi)}[i] + g \cdot \theta_{L^*,\iota}^*[i] = 0.
\]

by 2.2.10 and 3.1.4. Note that the map \( \mu \mapsto \mu + \iota(\xi) \) is a permutation of \( L^* \). Hence,

\[
h \cdot \left( \theta_L[i] - \theta_{L,\iota(\xi)}[i] \right) = 0.
\]

But \( \theta_L[i] - \theta_{L,\iota(\xi)}[i] = 1 + \ldots \), since \( \iota(\xi) \notin L \). Hence \( h = g = 0 \). \( \square \)

The following lemma is rather technical:

**Lemma 3.5.2.** Let \( L \) and \( L_0 \) be lattices, \( L \) non-unimodular, and \( \iota : L_0 \to L \) an embedding. Suppose that there is a vector \( \mu^* \) of minimal quadratic form in \( L^* \setminus L \) such that \( \mu^* \notin \iota(L_0)^{\perp} \). Then the pullback operator

\[
[i] : \mathcal{O}(\mathcal{H}) \cdot \langle \theta_L, \theta_L^* \rangle \to \mathcal{E}(L_0)
\]

is a monomorphism.

**Proof.** Since \( L \) is non-unimodular, the value

\[
m^* := \min_{\mu \in L^* \setminus L} Q(\mu)
\]

is well-defined in \( Q^\times \). We show that the functions \( \theta_L[i] \) and \( (\theta_L^* - \theta_L) [i] \) are linearly independent over \( \mathcal{O}(\mathcal{H}) \). By 3.1.12 we have Fourier expansions

\[
\theta_L[i](z, w_0) = 1 + \sum_{0 \neq \mu \in Q^+_+} \left( \sum_{\xi \in L_0^*} |\{ \mu \in L : Q(\mu) = m^*, \iota^*(\mu) = \xi \}| \cdot e^{2\pi i B(\xi, w_0)} \right) e^{2\pi i m z},
\]

\[
\theta_L^* - \theta_L[i](z, w_0) = \left( \sum_{\xi \in L_0^*} |\{ \mu \in L^* - L : Q(\mu) = m^*, \iota^*(\mu) = \xi \}| \cdot e^{2\pi i B(\xi, w_0)} \right) e^{2\pi i m^* z} + \ldots .
\]
Hence, we obtain a local representation
\[
\left( \frac{\theta_{L^*} - \theta_L}{\theta_L[i]} \right) = \left( \sum_{\zeta \in L_0^*} |\{ \mu \in L^* \setminus L : Q(\mu) = m^*, \iota^*(\mu) = \zeta \}| \cdot e^{2\pi i B(\zeta, w_0)} \right) \cdot e^{2\pi i m^* z} + \ldots .
\]

By assumption, \(Q(\mu^*) = m^*\) and \(\iota^*(\mu^*) \neq 0\). Hence the function \(\frac{\theta_{L^*} - \theta_L}{\theta_L[i]}\) depends nontrivially on \(w_0\) and thus can not be the quotient of two functions \(f, g \in \mathcal{O}(\mathcal{H})\).

We apply this on irreducible root lattices:

**Corollary 3.5.3.** Let \(L\) and \(L_0\) be lattices, \(L\) a non-unimodular irreducible root lattice, and \(\iota : L_0 \to L\) an embedding. Then the pullback operator
\[
[i] : \mathcal{O}(\mathcal{H}) \cdot \langle \theta_L^*, \theta_{L^*}^* \rangle_{\mathbb{C}} \to \mathcal{E}(L_0)
\]
is a monomorphism.

**Proof.** Since \(L\) is non-unimodular, the set of elements of minimal quadratic form in \(L^* \setminus L\) is nonempty. As a consequence, the \(\mathbb{Q}\)-vector space spanned by this set is nontrivial, \(W(L)\)-invariant and consequently must equal \(L_0\), since \(W(L)\) acts irreducibly. Hence, there must be some element of minimal quadratic form \(\mu^* \in L^* \setminus L\), which is non-perpendicular to \(\iota(L_0)\). The claim follows then from 3.5.2.

We summarize our results in the following

**Theorem 3.5.4.** Let \(L\) and \(L_0\) be lattices, \(L\) non-unimodular, and \(\iota : L_0 \to L\) an embedding. Suppose that one of the three assertions holds:

i) \(\iota(\xi) \in L^* \setminus L\) for some \(\xi \in L_0^*\),

ii) \(\mu^* \not\in \iota(L_0) \perp L\) for some \(\mu^* \in L^* \setminus L\) of minimal quadratic form,

iii) \(L\) is an irreducible root lattice,

Then the pullback operator
\[
[i] : \mathcal{O}(\mathcal{H}) \cdot \langle \theta_L^*, \theta_{L^*}^* \rangle_{\mathbb{C}} \to \mathcal{E}(L_0)
\]
is injective.

With a bit more effort, we can formulate and prove a further criterion for injectivity of the pullback operator:

**Theorem 3.5.5.** Let \(L\) be a lattice and \(0 \neq \mu_1 \in L^* / L\) be a fixed-point of \(\mathcal{O}(L)\). Assume that \(\mathcal{O}(L)\) acts transitively on the remaining non-zero cosets. If \(\iota(\xi) \equiv \mu_1 \mod L\) for some \(\xi \in L_0^*\), then the pullback operator
\[
[i] : \mathcal{E}(L)^{\text{sym}} \to \mathcal{E}(L_0),
\]
is injective.
Proof. Under the assumptions on $L$, a basis of $\mathcal{E}(L)^{\text{sym}}$ is given by

$$
\left\{ \vartheta_L, \vartheta_{L \mu_1}, \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu} \right\}.
$$

Let $\phi \in \mathcal{E}(L)^{\text{sym}}$ with theta decomposition

$$
\phi = h_0 \cdot \vartheta_L + h_1 \cdot \vartheta_{L \mu_1} + h \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu}, \quad h_0, h_1, h \in \mathcal{O}(H)
$$

such that

$$
0 = \phi[i] = h_0 \cdot \vartheta_L[i] + h_1 \cdot \vartheta_{L \mu_1}[i] + h \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu}[i].
$$

From $- \text{id}_L \in \mathcal{O}(L)$, we conclude $\text{ord}_{L^*/L}(\mu_1) = 2$. We apply $|_{L_0}[\xi, 0]$ on the previous equation in order to obtain

$$
h_1 \cdot \vartheta_L[i] + h_0 \cdot \vartheta_{L \mu_1}[i] + h \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu}[i] = 0,
$$

since the map $\mu \mapsto \mu + \mu_1$ is a permutation of the subset $\{\mu : \mu \neq 0, \mu_1\}$. Subtraction of both equations yields

$$
(h_0 - h_1)(\vartheta_L - \vartheta_{L \mu_1})[i] = 0.
$$

Since $(\vartheta_L - \vartheta_{L \mu_1})[i] = 1 + \ldots$ does not vanish identically, we obtain $h_0 = h_1$. Thus

$$
h_0 \cdot (\vartheta_L + \vartheta_{L \mu_1})[i] + h \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu}[i] = 0.
$$

Since $\mu_1$ has order 2 in $L^*/L$, we have $B(\mu_1, \mu) \equiv \{0, \frac{1}{2}\} \mod \mathbb{Z}$ for all $\mu \in L^*/L$. By assumption, the set $\{\mu : \mu \neq 0, \mu_1\}$ is an $\mathcal{O}(L)$-orbit and the coset $\mu_1$ is a fixed point of $\mathcal{O}(L)$. Hence the value $\delta := e^{2\pi i B(\mu_1, \mu)} \in \{\pm 1\}$ is independent of the choice of $\mu \notin \{0, \mu_1\}$. We apply $|_{L_0}[0, \xi]$ on the previous equation in order to obtain

$$
h_0 \cdot \left( \vartheta_L + e^{2\pi i B(\mu_1, \mu)} \vartheta_{L \mu_1} \right)[i] + \delta \cdot h \sum_{\mu \neq 0, \mu_1} \vartheta_{L \mu}[i] = 0.
$$

In the case $\delta = -1$ we add the two equations and conclude that

$$
h_0 \cdot \left( 2 \vartheta_L + \left( 1 + e^{2\pi i B(\mu_1, \mu_1)} \right) \vartheta_{L \mu_1} \right)[i] = 0.
$$

The same argument as in the beginning shows that the second factor does not vanish identically. Thus $h_0 = h_1 = h = 0$, hence $\phi = 0$. We consider the remaining case $\delta = 1$. Since $\mu_1 \notin L = L^{**}$, we necessarily have $e^{2\pi i B(\mu_1, \mu_1)} = -1$. This time we subtract both equations from each other and obtain

$$
2 \cdot h_0 \cdot \vartheta_{L \mu_1}[i] = 0.
$$

Hence, $h_0 = h_1 = 0$ and $h = 0$, i.e. $\phi = 0$. \qed
3.6 Separation theorems in degree 1

As a consequence of 3.1.11 and 1.2.23, we obtain the following simple

**Lemma 3.6.1.** Let \( L \) be an irreducible root lattice. Then for all roots \( l, l' \in R(L) \) and all \( \phi \in \mathcal{E}(L) \) one has \( \phi[l] = \phi[l'] \).

For embeddings of irreducible root lattices, we can derive a result of independency of the pullback operator - restricted to submodules of small rank - of the choice of the embedding:

**Proposition 3.6.2.** Let \( L \) and \( L_0 \) be irreducible root lattices and \( \iota, \kappa : L_0 \to L \) embeddings. Assume that the pullback operators

\[
[i], [\kappa] : \mathcal{O}(H) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C \to \mathcal{O}(H) \cdot \langle \vartheta_{L_0}, \vartheta_{L_0}^* \rangle_C
\]

are well-defined. Then one has \([i] = [\kappa] \).

**Proof.** Fix some arbitrary root \( l_0 \in R(L_0) \). Let \( l := \iota(l_0) \in R(L) \) and \( l' := \kappa(l_0) \in R(L) \). The corresponding embeddings \( \iota_l : \mathbb{Z} \to L \) and \( \iota_{l_0} : \mathbb{Z} \to L_0 \) then satisfy

\[
i_r = \iota \circ \iota_{l_0} \text{ and } i_{r'} = \kappa \circ \iota_{l_0}.
\]

Since \( L \) is an irreducible root lattice, the pullbacks

\[
[i_r], [i_{r'}] : \mathcal{O}(H) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C \to \mathcal{E}(\mathbb{Z})
\]

coincide by 3.6.1. Furthermore, the map

\[
[i_{r_0}] : \mathcal{O}(H) \cdot \langle \vartheta_{L_0}, \vartheta_{L_0}^* \rangle_C \to \mathcal{E}(\mathbb{Z})
\]

is injective due to 3.5.4 and the result follows from 3.2.7.

**Theorem 3.6.3.** Let \( L \) and \( L_0 \) be lattices and \( \iota_1, \ldots, \iota_m : L_0 \to L \) embeddings. Suppose that there is some \( \xi \in L_0^* \) such that

\[
i_i(\xi) \in L^* \setminus L \text{ for all } i = 1, \ldots, m, \quad \overline{Q}_0(\xi) \neq 0,
\]

hold. Then for all \((\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m\) the following assertions are equivalent:

i) \[ \sum_{i=1}^m \alpha_i \cdot \vartheta_L[i_i] = 0, \]

ii) \[ \sum_{i=1}^m \alpha_i \cdot \phi[i_i] = 0 \text{ for some } 0 \neq \phi \in \mathcal{O}(H) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C, \]

iii) \[ \sum_{i=1}^m \alpha_i \cdot \phi[i_i] = 0 \text{ for all } \phi \in \mathcal{O}(H) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C. \]
In this case, the dimension formula
\[ \dim_C \langle \phi[t_i], i = 1, \ldots, m \rangle_C = \dim_C \langle \vartheta_L[t_i], i = 1, \ldots, m \rangle_C \]
holds for all \( 0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_L, \vartheta_{L^*} \rangle_C \).

Proof. From the identity
\[ \vartheta_L |_{T_L} \frac{1}{\vartheta_{L^*}} J = \frac{\sqrt{\det L}}{\sqrt{\det L}} \cdot \vartheta_{L^*} \]
we obtain the equivalence of i) and iii) as well as that i) implies ii). It suffices to show the nontrivial direction, i.e. that ii) implies i). To this end we write
\[ \phi = h_0 \cdot \vartheta_L + h_1 \cdot \vartheta_{L^*} \] for \( h, g \in \mathcal{O}(\mathcal{H}) \).

By assumption on \( \phi \) at least one of the functions \( h \) and \( g \) does not vanish identically. We consider the auxiliary functions
\[ D := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_L[t_i], \quad D_{\xi} := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_{L(i)}(\xi)[t_i], \quad D^* := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_{L^*}[t_i]. \]

In this notation, \( \sum_{i=1}^{m} \alpha_i \cdot \phi[t_i] = 0 \) implies
\[ h \cdot D + g \cdot D^* = 0. \]

We apply the operator \( |_{L_0}^{[\xi, 0]} \) on this equation. By 3.1.4 and 2.2.10 we obtain
\[ h \cdot D_{\xi} + g \cdot D^* = 0, \]
since the maps \( \mu \mapsto \mu + i_t(\xi) \) for \( i = 1, \ldots, m \) are permutations of \( L^*/L \). From this we deduce
\[ h \cdot (D - D_{\xi}) = 0. \]

In the case \( h \neq 0 \) we obtain \( D = D_{\xi} \) and the transformation \( z \mapsto z + 1 \) yields
\[ D(z) = D(z + 1) = D_{\xi}(z + 1) = e^{2\pi i Q_0(\xi)} \cdot D(z), \]
since \( Q(i_t(\xi)) = Q_0(\xi) \) for \( i = 1, \ldots, m \). From \( e^{2\pi i Q_0(\xi)} \neq 1 \) we obtain \( D = 0 \). In the case \( h = 0 \) the function \( g \) does not vanish identically. This implies \( D^* = 0 \), hence \( D = 0 \).

As a corollary we specify the case \( L_0 = \mathbb{Z}(t) \) for \( t \in \mathbb{N} \):

**Corollary 3.6.4.** Let \( L \) be a lattice and \( N \) denote the exponent of the discriminant group \( L^*/L \). Let \( l_1, \ldots, l_m \in NL^*/NL \) such that
\[ Q(l_1) = \cdots = Q(l_m) = t, \quad N^2 \nmid t. \]

Then for all \( (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^{1 \times m} \) the following statements are equivalent:
3.6 Separation theorems in degree 1

a) $\sum_{i=1}^{m} \alpha_i \cdot \vartheta_L[l_i] = 0,$

b) $\sum_{i=1}^{m} \alpha_i \cdot \phi[l_i] = 0$ for some $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C$, 

c) $\sum_{i=1}^{m} \alpha_i \cdot \phi[l_i] = 0$ for all $\phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C$.

In this case, the dimension formula

$$\dim_C \langle \phi[l_i], i = 1, \ldots, m \rangle_C = \dim_C \langle \vartheta_L[l_i], i = 1, \ldots, m \rangle_C$$

holds for all $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_L, \vartheta_L^* \rangle_C$.

Proof. The embeddings $\iota_i : \mathbb{Z}(t) \rightarrow L$ for $i = 1, \ldots, m$ and $\xi := N^{-1}$ satisfy the assumptions of 3.6.3, since

$$\iota_i(\xi) = \frac{1}{N} l_i \in L^* \setminus L \text{ for } i = 1, \ldots, m, \quad Q_0(\xi) = \frac{t}{N^2} \notin \mathbb{Z}. \qed$$
4 Isomorphisms of Spaces of Jacobi Forms

We apply the results of section 3.3 and theorem 3.3.8 on Jacobi forms of degree 1. Therefore we fix the notation $J_{k,m} := J_{k,Z(m)}^{(1)}$ for $k \in \mathbb{Z}$ and $m \geq 0$ in accordance to [8, p. 10].

4.1 The $A_1, A_2, E_6, E_7$-tower

**Theorem 4.1.1.** Let $\iota : A_1 \to E_7$ be an embedding.

a) For all $k \geq 0$ even, the pullback $[\iota] : J_{k,E_7} \to J_{k,1}$ of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map $[\iota]^{-1} : J_{k,1} \to J_{k,E_7}$ is given explicitly by

$$
\begin{pmatrix}
    h_0 \\
    h_1
\end{pmatrix}
\mapsto
\frac{1}{12 \cdot \eta^{12}(z)} \begin{pmatrix}
    \vartheta_1(z)^6 + 3\vartheta_0(z)^4\vartheta_1(z)^2 & -4\vartheta_0(z)^3\vartheta_1(z)^3 \\
    -4\vartheta_0(z)^3\vartheta_1(z)^3 & \vartheta_0(z)^6 + 3\vartheta_0(z)^2\vartheta_1(z)^4
\end{pmatrix}
\begin{pmatrix}
    h_0 \\
    h_1
\end{pmatrix}.
$$

b) For all $k \in \mathbb{N}_0$ the following dimension formula holds:

$$
\dim_{\mathbb{C}} J_{k,E_7} = \dim_{\mathbb{C}} \left[ \text{Mp}_2(\mathbb{Z}), k - \frac{7}{2}, \rho_{E_7} \right] = \begin{cases} 
\lfloor \frac{k+2}{6} \rfloor, & k \text{ even}, \\
0, & k \text{ odd}.
\end{cases}
$$

**Proof.** a) Let $(f_0, f_1)^t$ denote the image of $(h_0, h_1)^t$ under $[\iota]^{-1}$. By definition, $(f_0, f_1)^t$ transforms correctly with respect to the dual representation of $\rho_{E_7}$. Hence it suffices to show, that both $f_0$ and $f_1$ are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$
\eta^{12}(z) = e^{\pi iz} + \ldots,
\vartheta_1^2(z) = 4e^{\pi iz} + \ldots,
\vartheta_1(z)^4 = c \cdot e^{3\pi iz/2} + \ldots
$$

for some $c \in \mathbb{C}$. 
b) Follows from a) and the well-known dimension formula for $J_{k,1}$. \hfill \Box

For $k \geq 4$ let $G_k^* \in \text{SL}_2(\mathbb{Z}, k)$ denote the classical normalized Eisenstein series of weight $k$, i.e.

$$G_k^*(z) = 1 - \frac{2k}{B_k} \sum_{m=1}^\infty \sigma_{k-1}(m) e^{2\pi i m z}, \quad z \in \mathcal{H}.$$ 

Furthermore, let

$$E_{k,m} := E^{(1)}_{k,\mathbb{Z}(m),0}$$

for $k \geq 4$ even and $m \in \mathbb{N}_0$ denote the Jacobi-Eisenstein series of weight $k$ and index $m$ as in [8, § 2]. From the structure theorem for the graded vector space $\bigoplus_{k=0}^\infty J_{2k,1}$ in [8, Thm. 3.5] we obtain the following

**Corollary 4.1.2.** The graded vector space

$$J_{2*,E_2} := \bigoplus_{k=0}^\infty J_{2k,E_2}$$

is a free $\mathbb{C}[G_4^*, G_6^*]$-module with basis $([i]^{-1} E_{4,1}, [i]^{-1} E_{6,1})$ and one has $[i]^{-1} E_{4,1} = \theta_{E_{2}}[E_{2}]$.

The corresponding result for $A_2$ and $E_6$, where in the latter case we use the same notation as in 3.3.2, is stated in the following

**Theorem 4.1.3.** a) Let $i : A_1 \longrightarrow E_6$ be an embedding. Then for all $k \geq 0$ even, the pullback $[i] : J_{k,E_6} \longrightarrow J_{k,1}$ of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map $[i]^{-1} : J_{k,1} \longrightarrow J_{k,E_6}$ is given explicitly by

$$
\begin{pmatrix}
h_0 \\
h_1
\end{pmatrix}
\mapsto
\frac{1}{12 \cdot \eta^{10}(z)} \cdot
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi^*(z) \theta_1^2(z) + \frac{3}{2} \theta_0^2(z) \theta_1(z) \theta_1(\frac{z}{2}) \theta_1(\frac{z}{5}) & -\psi^*(z) \theta_0^2(z) + \frac{3}{2} \theta_0(z) \theta_1^2(z) \theta_1(\frac{z}{2}) \theta_1(\frac{z}{5}) \\
-\theta_{A_2}(2z) \theta_0^2(z) + 3 \psi(z) \theta_0^2(z) \theta_1(z) & \theta_{A_2}(2z) \theta_0^2(z) + 3 \psi(z) \theta_0(z) \theta_1^2(z)
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1
\end{pmatrix}.
$$

b) Let $i : A_1 \longrightarrow A_2$ be an embedding. Then for all $k \geq 0$ even, the pullback $[i] : J_{k,A_2} \longrightarrow J_{k,1}$ of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map $[i]^{-1} : J_{k,1} \longrightarrow J_{k,A_2}$ is given explicitly by

$$
\begin{pmatrix}
h_0 \\
h_1
\end{pmatrix}
\mapsto
\frac{1}{2 \cdot \eta^{2}(z)} \cdot
\begin{pmatrix}
\theta_1(\frac{z}{2}) - \theta_1(3z) & -\theta_0(\frac{z}{2}) - \theta_0(3z) \\
-\theta_1(3z) & \theta_0(3z)
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1
\end{pmatrix}.
$$
Regarding the graded vector space of Jacobi forms, we obtain the following

**Corollary 4.1.4.** a) The graded vector space

\[ J_{2*,E_6} := \bigoplus_{k=0}^{\infty} J_{2k,E_6} \]

is a free \( \mathbb{C}[G^*_4,G^*_6] \)-module with basis \( ([i]^{-1}E_{4,1},[i]^{-1}E_{6,1}) \) and one has \( [i]^{-1}E_{4,1} = \vartheta_{E_8}[E_6] \).

b) The graded vector space

\[ J_{2*,A_2} := \bigoplus_{k=0}^{\infty} J_{2k,A_2} \]

is a free \( \mathbb{C}[G^*_4,G^*_6] \)-module with basis \( ([i]^{-1}E_{4,1},[i]^{-1}E_{6,1}) \) and one has \( [i]^{-1}E_{4,1} = \vartheta_{E_8}[A_2] \).

From the dimension formula for \( J_{k,1} \) in [8, p. 105] we obtain the following

**Corollary 4.1.5.** For all \( k \geq 0 \) even, the following dimension formulas hold:

\[
\dim_{\mathbb{C}} J_{k,E_7} = \dim_{\mathbb{C}} J_{k,E_6} = \dim_{\mathbb{C}} J_{k,A_2} = \dim_{\mathbb{C}} J_{k,1} = \left\lfloor \frac{k+2}{6} \right\rfloor.
\]

**Remark 4.1.6.** For all \( k \geq 0 \) even and all defined embeddings \( \iota : L_0 \rightarrow L \) for lattices \( L_0,L \in \{A_1, A_2, E_6, E_7\} \), the pullbacks

\[ \iota : J_{k,L} \rightarrow J_{k,L_0} \]

of Jacobi forms are isomorphisms of the vector spaces and independent of the choice of \( \iota \).

### 4.2 The \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{D}_4 \)-tower

**Theorem 4.2.1.** Let \( \iota : \mathbf{A}_1 \rightarrow \mathbf{D}_4 \) be an embedding.

a) For all \( k \geq 0 \) even, the pullback

\[ [\iota] : J^\text{sym}_{k,D_4} \rightarrow J_{k,1} \]

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

\[ [\iota]^{-1} : J_{k,1} \rightarrow J^\text{sym}_{k,D_4} \]

is given explicitly by

\[
\begin{pmatrix}
    h_0 \\
    h_1
\end{pmatrix} \mapsto \frac{1}{6 \cdot \eta^6(z)} \begin{pmatrix}
    3 \vartheta_0(z)^2 \vartheta_1(z) & -3 \vartheta_0(z) \vartheta_1(z)^2 \\
    -\vartheta_1(z)^3 & \vartheta_0(z)^3
\end{pmatrix}
\begin{pmatrix}
    h_0 \\
    h_1
\end{pmatrix}.
\]
b) For all $k \in \mathbb{N}_0$ the following dimension formula holds:

$$\text{dim}_C \text{sym} f_{k,D_4} = \text{dim}_C \left[ \text{Mp}_2(\mathbb{Z}), k-2, \left( \rho_{\text{sym}}^{D_4} \right)^t \right] = \begin{cases} \frac{k+2}{6}, & \text{if } k \text{ even}, \\ 0, & \text{if } k \text{ odd}. \end{cases}$$

**Proof.**

a) Let $(f_0, f_1)^t$ denote the image of $(h_0, h_1)^t$ under $[i]^{-1}$. By definition, $(f_0, f_1)^t$ transforms correctly with respect to the dual representation of $\rho_{\text{sym}}^{D_4}$. Hence it suffices to show, that both $f_0$ and $f_1$ are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$\eta^6(z) = e^{\pi i z/2} + \ldots,$$

$$\vartheta_1(z) = 2 e^{\pi i z/2} + \ldots,$$

$$h_1(z) = c \cdot e^{3\pi i z/2} + \ldots$$

for some $c \in \mathbb{C}$.

b) Follows from a) and the well known dimension formula for $J_{k,1}$, cf. [8, p. 105].

**Corollary 4.2.2.** For all $k \geq 0$ even and all embeddings $\iota : A_2 \rightarrow D_4$, the pullback

$$[\iota] : J_{k,D_4}^{\text{sym}} \rightarrow J_{k,A_2}^{\text{sym}}$$

of Jacobi forms is an isomorphism of the vector spaces and independent of the choice of $\iota$.

4.3 The $A_1(2), 2A_1, 2A_2$-tower

**Theorem 4.3.1.** Let $\iota : 2A_1 \rightarrow 2A_2$ be an embedding. Then for all $k \geq 0$ even, the pullback

$$[\iota] : J_{k,2A_2}^{\text{sym}} \rightarrow J_{k,2A_1}^{\text{sym}}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,2A_1}^{\text{sym}} \rightarrow J_{k,2A_2}^{\text{sym}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \mapsto \frac{1}{4 \cdot \eta^4(z)} \begin{pmatrix} \frac{\vartheta_1(\frac{z}{2}) - \vartheta_1(3z)}{\vartheta_1(3z)} \\ -2 \left( \frac{\vartheta_1(\frac{z}{2}) - \vartheta_1(3z)}{\vartheta_1(3z)} \right)^2 - \frac{\vartheta_0(\frac{z}{2}) - \vartheta_0(3z)}{\vartheta_0(3z)} \vartheta_1(3z) + \vartheta_1(3z) \vartheta_0(\frac{z}{2}) - 2 \vartheta_0(3z) \vartheta_1(3z) + \vartheta_0(3z) \left( \frac{\vartheta_0(\frac{z}{2}) - \vartheta_0(3z)}{\vartheta_0(3z)} \right)^2 \\ -2 \vartheta_0(3z) \vartheta_1(3z) - \vartheta_0(3z) \vartheta_1(3z) \\ \vartheta_0(3z) \vartheta_1(3z) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}.$$
Proof. Let $(f_0, f_1, f_2)^t$ denote the image of $(h_0, h_1, h_2)^t$ under $[i]^{-1}$. By definition, $(f_0, f_1, f_2)^t$ transforms correctly with respect to the dual representation of $\rho_{2A_2}$. Hence it suffices to show, that $f_0, f_1$ and $f_2$ are holomorphic at the cusp. But this follows immediately from the Fourier expansions
\[
\eta^4(z) = e^{\pi i z} + \ldots,
\]
\[
h_1(z) = c_1 e^{\pi i 3z} + \ldots,
\]
\[
h_2(z) = c_2 e^{\pi i z} + \ldots,
\]
\[
\left(\frac{\theta_1\left(\frac{z}{3}\right)}{3} - \theta_1(3z)\right)^2 = 4e^{\pi i \frac{z}{3}} + \ldots,
\]
\[
\theta_1(3z) = 2e^{\pi i \frac{3z}{2}} + \ldots
\]
for some $c_1, c_2 \in \mathbb{C}$.

\[\square\]

**Theorem 4.3.2.** Let $i : A_1(2) \to 2A_1$ be an embedding. Then for all $k \geq 0$ even, the pullback
\[\left[i\right] : J_{k,2A_1}^{\text{sym}} \to J_{k,2}
\]
of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map
\[\left[i\right]^{-1} : J_{k,2} \to J_{k,2A_1}^{\text{sym}}
\]
is given explicitly by
\[
\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \mapsto \frac{1}{\eta^3(z)} \begin{pmatrix} \theta_0(2z) \cdot \theta_{A_1(2),\frac{1}{2}}(2z) & 0 & -\theta_1(2z) \cdot \theta_{A_1(2),\frac{1}{2}}(2z) \\ -\theta_1(2z) \cdot \theta_{A_1(2),\frac{1}{2}}(2z) & \theta_0(2z) - \theta_1(2z) - \theta_0(2z) - \theta_1(2z) \\ 0 & 0 & \theta_0(2z) \cdot \theta_{A_1(2),\frac{1}{2}}(2z) \end{pmatrix} \cdot \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}.
\]

Proof. Let $(f_0, f_1, f_2)^t$ denote the image of $(h_0, h_1, h_2)^t$ under $[i]^{-1}$. By definition, $(f_0, f_1, f_2)^t$ transforms correctly with respect to the dual representation of $\rho_{2A_1}^{\text{sym}}$. Hence it suffices to show, that $f_0, f_1$ and $f_2$ are holomorphic at the cusp. But this follows immediately from the Fourier expansions
\[
\eta^3(z) = e^{\pi i \frac{z}{3}} + \ldots,
\]
\[
h_1(z) = c_1 e^{\pi i 2z} + \ldots,
\]
\[
h_2(z) = c_2 e^{\pi i z} + \ldots,
\]
\[
\theta_{A_1(2),\frac{1}{2}}(2z) = e^{\pi i \frac{z}{3}} + \ldots
\]
for some $c_1, c_2 \in \mathbb{C}$.

\[\square\]

From the dimension formula for $J_{k,2}$ in [8, p. 105] we obtain the following

**Corollary 4.3.3.** For all $k \geq 0$ even, the following dimension formulas hold:
\[
\dim \mathbb{C} J_{k,2A_2}^{\text{sym}} = \dim \mathbb{C} J_{k,2A_1}^{\text{sym}} = \dim \mathbb{C} J_{k,2} = \left\lfloor \frac{k}{4} \right\rfloor.
\]
5 Modular Embeddings of Paramodular Groups

5.1 (Non-)commutative orders

Unless specified otherwise let $F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. Denote by $- : F \to F$ the standard involution on $F$. This involution is uniquely determined by the condition $\bar{x} = x$ if and only if $x \in \mathbb{R}$. For $x \in F$ we call $\bar{x}$ the conjugate of $x$, $N(x) := x\bar{x}$ the (reduced) norm of $x$ and $\text{tr}(x) := x + \bar{x}$ the (reduced) trace of $x$. The norm form

$$N : F \to \mathbb{R}_+$$

turns $F$ into a positive definite quadratic space satisfying the composition law

$$N(xy) = N(x) \cdot N(y) \quad \text{for all } x, y \in F.$$

The associated bilinear form obtained by polarization is called the trace bilinear form and is given by $\text{tr}(xy), x, y \in F$, i.e.

$$\text{tr}(xy) = N(x + y) - N(x) - N(y) \quad \text{for all } x, y \in F.$$

It is easy to see that every $x \in F$ satisfies the quadratic equation

$$x^2 - \text{tr}(x)x + N(x) = 0.$$

We compare norm and trace with norm and trace with respect to field extensions:

Remark 5.1.1. Let $x \in F \setminus \mathbb{R}$. The algebra $\mathbb{R}[x]$, generated by $x$ over $\mathbb{R}$, is a commutative subfield of $F$ and one has $\mathbb{R}[x] = \mathbb{R} \oplus \mathbb{R}x$. The matrix of the left-multiplication $y \mapsto xy, y \in \mathbb{R}[x]$ with respect to the basis $(1, x)$ then equals

$$\begin{pmatrix} 0 & 1 \\ -N(x) & \text{tr}(x) \end{pmatrix}.$$

Hence, usual norm and trace of $x$ with respect to the field extension $\mathbb{R}[x]/\mathbb{R}$ are given by $N(x)$ and $\text{tr}(x)$.

We will introduce the notion of an order in $F$:
Definition 5.1.2. A subset $\mathcal{O} \subseteq F$ is called an order in $F$, if the following assertions hold:

i) $\mathcal{O}$ is a discrete subring with $1 \in \mathcal{O}$,

ii) $\mathcal{O}$ generates $F$ as a $R$-vector space, i.e. $\mathcal{O}R = F$.

An order is called maximal, if it is a maximal element in the set of orders in $F$.

For basic references in the theory of orders in non-necessary commutative rings confer the books of Reiner [23] or Vignéras [26].

It is well-known, that every discrete additive subgroup of a finite-dimensional real vector space is a free abelian group over $\mathbb{Z}$. Hence we obtain another characterization given in the following

Lemma 5.1.3. For a subring $\mathcal{O} \subseteq F$ with $1 \in \mathcal{O}$ the following assertions are equivalent:

a) $\mathcal{O}$ is an order in $F$,

b) $\mathcal{O}$ is a free abelian group and generated by a $R$-basis of $F$.

Regarding integrality of norm and trace we have to following

Proposition 5.1.4. Let $\mathcal{O}$ be an order in $F$. Then the following assertions hold:

a) $\mathcal{O} \cap R = \mathbb{Z}$,

b) $N(a), \text{tr}(a) \in \mathbb{Z}$ for all $a \in \mathcal{O}$,

c) $\overline{a} \in \mathcal{O}$ for all $a \in \mathcal{O}$.

Proof. a) Since $\mathcal{O}$ is discrete in $F$, the intersection $\mathcal{O} \cap R$ is a discrete additive subgroup of $R$, hence cyclic, i.e. equals $\mathbb{Z}v$ for some $v \in R$. From $1 \in \mathcal{O}$ we can assume that $v = \frac{1}{s}$ for some $s \in \mathbb{N}$. From $\lim_{n \to \infty} v^n = 0$ for $s > 1$ we obtain $s = 1$ from the discreteness of $\mathcal{O}$.

b) The claim is immediate for $a \in \mathbb{Z}$. For $a \in \mathcal{O} \setminus R$ consider the subfield $R[a] = R \oplus Ra$, generated by $a$ over $R$. Let $\mathcal{O}_a := \mathcal{O} \cap R[a]$. Then $\mathcal{O}_a$ is a discrete subring of $R[a]$ with $1 \in \mathcal{O}_a$. Since $\mathcal{O}_a$ contains the $R$-basis $(1, a)$ of $R[a]$, we conclude that $\mathcal{O}_a$ is an order in $R[a]$, i.e. $\mathcal{O}_a = \mathbb{Z}x + \mathbb{Z}y$ for some $R$-basis $(x, y)$ of $R[a]$. Hence the usual norm and trace of $a \in \mathcal{O}$ with respect to the field extension $R[a]/R$ lie in $\mathbb{Z}$. Since norm and trace are independent from choice of the basis of $R[a]$, we can apply 5.1.1 and obtain $N(a), \text{tr}(a) \in \mathbb{Z}$.

c) Follows from $\overline{a} = \text{tr}(a) - a$ and b). \qed

Corollary 5.1.5. Let $\mathcal{O}$ be an order in $F$. Then $(\mathcal{O}, N|_{\mathcal{O}})$ is a positive definite, even lattice. In the case $F \in \{\mathbb{C}, \mathbb{H}\}$, the map $a \mapsto \overline{a}$ is a nontrivial automorphism of $(\mathcal{O}, N|_{\mathcal{O}})$ and $\mathbb{Z}$ is the largest sublattice of $\mathcal{O}$, on which the standard involution of $F$ acts trivially.
We repeat some ring-theoretic terms:

**Definition 5.1.6.** Let \( \mathcal{O} \) be an order in \( \mathbb{F} \).

a) An element \( \varepsilon \in \mathcal{O} \) is called a unit, if there is some \( \delta \in \mathcal{O} \) such that \( \varepsilon \delta = 1 \). In this case, \( \delta \) is uniquely determined and one has \( \delta \varepsilon = 1 \). The set of units \( \mathcal{O} \) is a group and one has
\[
\mathcal{O}^\times = \{ a \in \mathcal{O} : N(a) = 1 \}.
\]

b) Let \( a, b \in \mathcal{O}, a \neq 0 \). Then \( a \) is called a left resp. right divisor of \( b \), if \( a^{-1}b \in \mathcal{O} \) resp. \( ba^{-1} \in \mathcal{O} \). In this case we write \( a \mid_l b \) resp. \( a \mid_r b \).

c) Let \( I \) be an additive subgroup of \( \mathcal{O} \). We call \( I \) a left resp. right ideal, if \( \mathcal{O}I \subset I \) resp. \( I \mathcal{O} \subset I \). We call \( I \) a two-sided ideal, if \( I \) is both a left and right ideal. We call \( I \) a left resp. right principal ideal, if there is \( a \in \mathcal{O} \) such that \( I = \mathcal{O}a \) resp. \( I = a \mathcal{O} \). For \( a, b \in \mathcal{O} \) one has \( b \mathcal{O} \subset a \mathcal{O} \) resp. \( \mathcal{O}b \subset a \mathcal{O} \) if and only if \( a \mid_l b \) resp. \( a \mid_r b \).

d) We call \( \mathcal{O} \) a principal ideal domain, if every left and every right ideal of \( \mathcal{O} \) is principal.

e) We call \( \mathcal{O} \) norm-euclidean, if for every \( a \in \mathbb{F} \) there exists \( g \in \mathcal{O} \) such that \( N(a - g) < 1 \).

**Remark 5.1.7.** Let \( \mathcal{O} \) be a norm-euclidean order in \( \mathbb{F} \). Then \( \mathcal{O} \) admits a left resp. right euclidean algorithm. Hence, every nontrivial left resp. right ideal \( I \) is principal and is generated by an element of nonzero minimal norm in \( I \). Every norm-euclidean order is a principal ideal domain.

We need some identifications:

**Example 5.1.8.** We consider the skewfield of quaternions \( \mathbb{H} \), i.e.
\[
\mathbb{H} = \mathbb{R} + \mathbb{R}i_1 + \mathbb{R}i_2 + \mathbb{R}i_3
\]
with multiplication linearly extended via the defining relations
\[
i_1^2 = i_2^2 = -1, \quad i_3 = i_1i_2.
\]
Every \( a \in \mathbb{H} \) has a unique representation \( a = a_0 + a_1i_1 + a_2i_2 + a_3i_3 \) with \( a_0, \ldots, a_3 \in \mathbb{R} \). The complex field \( \mathbb{C} \) will be identified in \( \mathbb{H} \) via \( \mathbb{C} := \mathbb{R} + \mathbb{R}i_1 \). It is the maximal commutative subfield of \( \mathbb{H} \) and hence a splitting field for \( \mathbb{H} \). Via this identification, \( \mathbb{H} \) carries a natural structure of a left \( \mathbb{C} \)-vector space with basis \( 1, i_2 \). Explicitly one has
\[
a_0 + a_1i_1 + a_2i_2 + a_3i_3 = (a_0 + a_1i_1) + (a_2 + a_3i_1)i_2 \in \mathbb{C} \oplus \mathbb{C}i_2.
\]
Right multiplication by \( a = \alpha + \beta i_2, \alpha, \beta \in \mathbb{C} \) induces a monomorphism of \( \mathbb{R} \)-algebras
\[
\gamma : \mathbb{H} \longrightarrow \mathbb{C}^2 \times 2, \quad \alpha + \beta i_2 \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \alpha, \beta \in \mathbb{C}.
\]
where the definition is implicitly extended to matrices componentwisely. The map
\[ - : \mathbb{H} \rightarrow \mathbb{H}, \quad a \mapsto a_0 - a_1 i - a_2 i_2 - a_3 i_3 \]
is the unique standard involution on \( \mathbb{H} \). For \( a \in \mathbb{H} \), we call \( \overline{a} \) its conjugate. It is straightforward to check that reduced trace resp. norm are given by
\[ \text{tr}(a) = 2a_0, \quad N(a) = a \overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \]
for \( a \in \mathbb{H} \). The map
\[ \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}, \quad (a, b) \mapsto \text{tr}(\overline{a}b) \]
is a symmetric, positive definite \( \mathbb{R} \)-bilinear form on \( \mathbb{H} \). The decomposition \( \mathbb{H} = \mathbb{C} \oplus \mathbb{C}i_2 \) is orthogonal with respect to this bilinear form.

We describe some lattice-theoretic objects in the new setting:

**Definition 5.1.9.** Let \( \mathcal{O} \) be an order in \( \mathbb{F} \).

a) The dual lattice
\[ \mathcal{O}^\sharp := \{ x \in \mathbb{F} : \text{tr}(ax) \in \mathbb{Z} \text{ for all } a \in \mathcal{O} \} \]
with respect to the trace bilinear form is called the inverse different of \( \mathcal{O} \).

b) The finite abelian group \( \mathcal{O}^\sharp / \mathcal{O} \) is called the discriminant group of \( \mathcal{O} \). The number
\[ d(\mathcal{O}) := |\mathcal{O}^\sharp / \mathcal{O}| < \infty \]
is called the discriminant of \( \mathcal{O} \).

c) The two-sided ideal
\[ \mathcal{D}(\mathcal{O}) := \{ x \in \mathcal{O} : x\mathcal{O}^\sharp \subseteq \mathcal{O} \} \]
is called the different of \( \mathcal{O} \).

**Examples 5.1.10.** a) Let \( K \) be an imaginary-quadratic number field of discriminant \( -D_K \), i.e.
\[ K = \mathbb{Q}(\sqrt{-D_K}) \subseteq \mathbb{C}, \quad D_K > 0, \]
such that
- \( D_K \equiv 3 \text{ mod } 4 \) and \( D_K \) squarefree or
- \( D_K \equiv 0 \text{ mod } 4, D_K / 4 \equiv 1, 2 \text{ mod } 4 \) and \( D_K / 4 \) squarefree.
Let \( \mathfrak{o}_K \) denote the integral closure of \( \mathbb{Z} \) in \( K \), i.e.
\[
\mathfrak{o}_K = \begin{cases} 
\mathbb{Z} + \mathbb{Z}\sqrt{-D_K}, & D_K \equiv 0 \text{ mod } 4, \\
\mathbb{Z} + \mathbb{Z}\frac{1 + \sqrt{-D_K}}{2}, & D_K \equiv 3 \text{ mod } 4.
\end{cases}
\]

The inverse different \( \mathfrak{o}_K^\ast \) is given by
\[
\mathfrak{o}_K^\ast = \frac{i_1}{\sqrt{D_K}} \mathfrak{o}_K = \begin{cases} 
\frac{i_1}{\sqrt{D_K}}\mathbb{Z} + \frac{1}{2}\mathbb{Z}, & D_K \equiv 0 \text{ mod } 4, \\
\frac{i_1}{\sqrt{D_K}}\mathbb{Z} + \frac{1}{2}\left(1 + \frac{i_1}{\sqrt{D_K}}\right)\mathbb{Z}, & D_K \equiv 3 \text{ mod } 4,
\end{cases}
\]
and \( d(\mathfrak{o}_K) = D_K \). The different \( D(\mathfrak{o}_K) \) is the principal ideal generated by \( i_1\sqrt{D_K} \). The unit groups \( \mathfrak{o}_K^\times \) are given by
\[
\mathfrak{o}_K^\times = \begin{cases} 
\{\pm 1, \pm \rho, \pm \bar{\rho}\}, & D_K = 3, \\
\{\pm 1, \pm i_1\}, & D_K = 4, \\
\{\pm 1\}, & D_K > 4.
\end{cases}
\]

b) Let \( K \) as in a) and define \( \mathfrak{O}_K := \mathfrak{o}_K + \mathfrak{o}_K i_2 \). From \( i_2w = \overline{w}i_2 \) for all \( w \in \mathfrak{o}_K \) we conclude that \( \mathfrak{O}_K \) is an order in \( \mathbb{H} \). The inverse different is given by
\[
\mathfrak{O}_K^\ast = \mathfrak{o}_K^\ast + \mathfrak{o}_K^\ast i_2 = \frac{i_1}{\sqrt{D_K}} \mathfrak{O}_K
\]
and \( d(\mathfrak{O}_K) = D_K^2 \). The different \( D(\mathfrak{O}_K) \) is the two-sided principal ideal generated by \( i_1\sqrt{D_K} \). The unit groups \( \mathfrak{O}_K^\times \) are given by
\[
\mathfrak{O}_K^\times = \begin{cases} 
\{\pm 1, \pm \rho, \pm \bar{\rho}, \pm i_2, \pm \rho i_2, \pm \bar{\rho}i_2\}, & D_K = 3, \\
\{\pm 1, \pm i_1, \pm i_2, \pm i_3\}, & D_K = 4, \\
\{\pm 1, \pm i_2\}, & D_K > 4.
\end{cases}
\]

c) The ring of Hurwitz quaternions
\[
\mathfrak{O} = \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}\omega, \quad \omega := \frac{1 + i_1 + i_2 + i_3}{2},
\]
is a maximal order in \( \mathbb{H} \). The inverse different of \( \mathfrak{O} \) is given by \( \mathfrak{O}^\ast = (1 + i_1)^{-1} \mathfrak{O} \) and \( d(\mathfrak{O}) = 4 \). The different \( D(\mathfrak{O}) \) is the two-sided principal ideal generated by \( 1 + i_1 \) and is denoted by \( \varphi \). It can be shown that
\[
\varphi = \{a \in \mathfrak{O} : N(a) \equiv 0 \text{ mod } 2\}.
\]
We call \( \varphi \) the ideal of even quaternions. The elements \( 0, 1, \omega, \overline{\omega} \) form a complete set of representatives for \( \varphi \) in \( \mathfrak{O} \). The unit group of \( \mathfrak{O} \) is given by
\[
\mathfrak{O}^\times = \left\{\pm 1, \pm i_1, \pm i_2, \pm i_3, \frac{\pm 1 + i_1 \pm i_2 \pm i_3}{2}\right\} = \langle \omega, i_1 \rangle.
\]
It is a noncommutative group of order 24. The identities

\[ i_1^2 = -1, \quad (i_1 + i_2)^2 = -2, \quad (-\omega)^3 = 1 \]

show that the orders \( \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}, \mathcal{O}_{\mathbb{Q}(\sqrt{-2})} \) and \( \mathcal{O}_{\mathbb{Q}(\sqrt{-3})} \) can be naturally embedded into \( \mathcal{O} \).

d) The ring of Lipschitz quaternions

\[ \Lambda := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}i_3 \]

is an order in \( \mathbb{H} \). Indeed, \( \Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})} \). The inverse different of \( \Lambda \) is given by \( \Lambda^2 = \frac{1}{2}\Lambda \) and \( d(\Lambda) = 16 \). The different \( D(\Lambda) \) is the two-sided principal ideal generated by 2. It is easy to see that \( \varphi \subseteq \Lambda \subseteq \mathcal{O} \) and that the index of each inclusion equals two. Especially, \( \Lambda \) is a non-maximal order. The elements 0, 1 form a complete system of representatives for \( \varphi \) in \( \Lambda \). Furthermore one has

\[ \Lambda = \{ a \in \mathcal{O} : a \equiv 0, 1 \mod \varphi \} \]

as well as

\[ \mathcal{O} = \Lambda \cup \omega \Lambda \cup \overline{\omega} \Lambda = \Lambda \cup \Lambda \omega \cup \Lambda \overline{\omega}. \]

e) \( \mathcal{O}_{\mathbb{Q}(\sqrt{-3})} \) is a maximal order in \( \mathbb{H} \).

f) \( \mathcal{O}_{\mathbb{Q}(\sqrt{-2})} \) is not maximal. Indeed, \( \mathcal{O}_{\mathbb{Q}(\sqrt{-2})} \) is strictly contained in the order

\[ \mathcal{O}_2 := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z} \frac{1 + i_1 + \sqrt{2}i_2}{2} + \mathbb{Z} \frac{1 + i_1 + \sqrt{2}i_3}{2}, \]

which is seen to be isomorphic to Hurwitz order \( \mathcal{O} \) in c) via the map

\[ i_1 \mapsto i_1, \quad \frac{1 + i_1 + \sqrt{2}i_2}{2} \mapsto \frac{1 + i_1 + i_2 + i_3}{2}, \quad \frac{1 + i_1 + \sqrt{2}i_3}{2} \mapsto \frac{1 + i_1 + i_2 - i_3}{2}. \]

Hence, \( \mathcal{O}_2 \) is a maximal order.

In order to deduce a suitable description of the orthogonal group of the lattice \((\mathcal{O}, N|_{\mathcal{O}})\), we introduce the notion of invariant elements:

**Definition 5.1.11.** Let \( \mathcal{O} \) be an order in \( \mathbb{H} \). An element \( 0 \neq a \in \mathcal{O} \) is called invariant, if it generates a two-sided ideal in \( \mathcal{O} \), i.e. \( a\mathcal{O} = \mathcal{O}a \). In other words, \( a \) is invariant, if and only if the map \( u \mapsto a^{-1}ua \) is a \( \mathbb{Z} \)-automorphism of \( \mathcal{O} \). The set of invariant elements of \( \mathcal{O} \) carries the structure of a multiplicative semigroup, which will be denoted by \( \mathcal{I}(\mathcal{O}) \). For formal reasons we extend the definition for orders \( \mathcal{O} \) in \( \mathbb{C} \) by setting \( \mathcal{I}(\mathcal{O}) := \{1\} \) in this case.

An explicit determination of \( \mathcal{I}(\mathcal{O}) \) for certain orders is given in the following

**Lemma 5.1.12. a)** Let \( \mathcal{O} \) denote the Hurwitz order. Then one has

\[ \mathcal{I}(\mathcal{O}) = \mathbb{N} \cdot \langle 1 + i_1, \epsilon \in \mathcal{O}^\times \rangle. \]
b) Let $K = \mathbb{Q}(\sqrt{-D_K})$. Then one has

$$I(O_K) = N \cdot \begin{cases} 
\langle i_1 \sqrt{3}, \varepsilon \in O_K^\times \rangle, & D_K = 3, \\
\langle 2\omega, 1 + i_1, \varepsilon \in O_K^\times \rangle, & D_K = 4, \\
\langle i_1 \sqrt{D_K}, \varepsilon \in O_K^\times \rangle & D_K > 4.
\end{cases}$$

**Proof.** a) Confer [11, p. 29] or [14, I. Lemma 1.5].

b) For $0 \neq a \in I(O_K)$ consider the automorphism

$$\varphi_a : O_K \to O_K, \quad w \mapsto a^{-1}wa.$$  

Since $O_K$ contains a $\mathbb{R}$-basis of $H$ and $O \cap \mathbb{R} = \mathbb{Z}$, we have $\varphi_a = \text{id}$ for $a \in I(O)$ if and only if $a \in \mathbb{Z} \setminus \{0\}$. Of course, $\varphi_a(\pm 1) = \pm 1$. In the case $D_K = 3$ one has

$$\text{ord}(1) = 1, \text{ord}(-1) = 2, \text{ord}(\rho) = \text{ord}(\overline{p}) = 3,$$

$$\text{ord}(\pm i_2) = \text{ord}(\pm \rho i_2) = \text{ord}(\pm \overline{p}i_2) = 4,$$

$$\text{ord}(-\rho) = \text{ord}(-\overline{p}) = 6.$$  

Hence, $\varphi_a(\rho) \in \{\rho, \overline{p}\}$ and $\varphi_a(i_2) \in \{\pm i_2, \pm \rho i_2, \pm \overline{p}i_2\}$. The value $\varphi_a(\rho i_2)$ is uniquely determined by $\varphi_a(i_2)$ and $\varphi_a(\rho)$. Consequently, the number of different maps $\varphi_a$ is less or equal than 12. A direct verification shows that any map of the form above is induced by conjugation by $\varepsilon$ or $i_1 \sqrt{3} \varepsilon$ for $\varepsilon \in O_K^\times$. In the case $D_K = 4$, it is easily seen that every automorphism of determinant 1 of the lattice $(\Lambda, N_\Lambda)$ is given by a signed permutation of sign 1. This group has order 192. A direct verification shows that it is generated by the maps

$$\varphi_a, \quad a \in \{2\omega, 1 + i_1, \pm i_1, \pm i_2, \pm i_3\}.$$  

In the case $D_K > 4$ we obtain $\varphi_a(i_2) = \pm i_2$, since $\varphi$ restricts to units. Furthermore,

$$\varphi_a(i_1 \sqrt{D_K}/2) \in \left\{ \pm i_1 \sqrt{D_K}/2, \pm i_3 \sqrt{D_K}/2 \right\}$$

is due to norm reasons and the fact that $-D_K$ is a discriminant. Again a direct verification shows that $\varphi_a$ is a product of maps of the form

$$w \mapsto a^{-1}wa, \quad a \in \{i_2, i_1 \sqrt{D_K}\}. \quad \Box$$

The following result is well-known. For sake of completeness we include a proof:

**Proposition 5.1.13.** Let $K = \mathbb{Q}(\sqrt{-3})$. Then the following assertions hold:

a) $\{u \in o_K : N(u) \equiv 0 \mod 3\} = i\sqrt{3}o_K$,

b) $\{u \in O_K : N(u) \equiv 0 \mod 3\} = i\sqrt{3}O_K$. 


Proof. a) The trivial inclusion follows from $N(i_1 \sqrt{3}) = 3$ and the multiplicativity of the norm form. For the nontrivial inclusion let $u = a + b \rho \in \mathfrak{o}_K$ such that $N(u) \equiv 0 \mod 3$. By definition,

$$N(u) = \left( a + \frac{b}{2} \right)^2 + \frac{3}{4} b^2 = a^2 + ab + b^2 \equiv a^2 - 2ab + b^2 = (a - b)^2 \mod 3.$$ 

Hence, $a \equiv b \mod 3$. We conclude $u = b + 3l + b \rho \in \mathfrak{o}_K$ for some $l \in \mathbb{Z}$. Hence it suffices to show $1 + \rho \in i_1 \sqrt{3} \mathfrak{o}_K$. This is seen from the identity $1 + \rho = i_1 \sqrt{3} (1 + \rho)$.

b) Let $u = u_1 + u_2 i_2, u_1, u_2 \in \mathfrak{o}_K$ such that $N(u) = N(u_1) + N(u_2) \equiv 0 \mod 3$.

In the proof of a) it was shown that both $N(u_1)$ and $N(u_2)$ are quadratic residues mod 3 and hence congruent to 0, 1 mod 3. Thus, $N(u) \equiv 0 \mod 3$ already implies $N(u_1), N(u_2) \equiv 0 \mod 3$. From a) we deduce $u_1, u_2 \in i_1 \sqrt{3} \mathfrak{o}_K$ and thus $u \in i_1 \sqrt{3} \mathfrak{o}_K$.

We cite a classical theorem of Cayley, cf. [6, 7 §3, p. 215], that characterizes the orthogonal groups of the quadratic spaces $(\mathbb{F}, N)$ in the following manner:

**Theorem 5.1.14 (Cayley).** Let $\mathbb{F}_1 := \{ w \in \mathbb{F} : N(w) = 1 \}$. Then one has

$$O(\mathbb{F}) = \begin{cases} 
\langle w \mapsto -w \rangle, & \mathbb{F} = \mathbb{R}, \\
\langle w \mapsto \varepsilon w : \varepsilon \in C_1 \rangle \times \langle w \mapsto \overline{w} \rangle, & \mathbb{F} = \mathbb{C}, \\
\left( \{ w \mapsto \delta w : \delta \in \mathbb{H}_1 \} \cdot \{ w \mapsto \varepsilon w : \varepsilon \in \mathbb{H}_1 \} \right) \times \langle w \mapsto \overline{w} \rangle, & \mathbb{F} = \mathbb{H}. 
\end{cases}$$

We describe the normalizer of an quaternionic order $\mathcal{O}$ in terms of its invariant elements:

**Lemma 5.1.15.** Let $\mathcal{O}$ be an order in $\mathbb{H}$. Then the following assertions hold:

a) $\mathcal{O}Q$ is a central simple $\mathbb{Q}$-algebra.

b) $N_{\mathbb{F}_1}(\mathcal{O}) = \left\{ \frac{u}{\sqrt{N(u)}} : u \in \mathcal{I}(\mathcal{O}) \right\}$.

Proof. a) For $0 \neq u \in \mathcal{O}$ one has $u^{-1} = N(u)^{-1} \overline{u} \in \mathcal{O}Q$. Hence $\mathcal{O}Q$ is a skewfield over $\mathbb{Q}$ and trivially simple. In order to prove that $\mathcal{O}Q$ is central with center $\mathbb{Q}$, note that we have

$$\mathcal{C}(\mathcal{O}Q) = \mathcal{C}(\mathbb{H}) \cap \mathcal{O}Q = \mathbb{R} \cap \mathcal{O}Q,$$

since $\mathcal{O}Q$ contains a $\mathbb{R}$-basis of $\mathbb{H}$. Thus it suffices to show that $\mathbb{R} \cap \mathcal{O}Q = \mathbb{Q}$. But for $x \in \mathbb{R} \cap \mathcal{O}Q$ we find $r \in \mathbb{Z}$ such that $rx \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z}$, i.e. $x \in \mathbb{Q}$. 
b) Let \( x \in N_{\mathbb{F}_1}(\mathcal{O}) \), i.e. \( x \in \mathbb{F} \), \( N(x) = 1 \) and \( x^{-1} \mathcal{O} x = \mathcal{O} \). Hence, conjugation by \( x \) extends to a \( \mathbb{Q} \)-automorphism of the central simple algebra \( \mathcal{O} \mathbb{Q} \). Due to the Skolem-Noether theorem, cf. [23, Chap. 1, § 7d, Thm. (7.21)], this automorphism is inner, i.e. there is \( 0 \neq u \in \mathcal{O} \mathbb{Q} \), such that \( x^{-1} ax = u^{-1} au \) for all \( a \in \mathcal{O} \mathbb{Q} \). Since this equation also holds for the multiples \( m \cdot u \) for \( m \neq 0 \in \mathbb{Z} \), we can assume that \( u \in \mathcal{O} \), i.e. \( u \in \mathcal{I}(\mathcal{O}) \). Since \( \mathcal{O} \) contains a \( \mathbb{R} \)-basis of \( \mathcal{H} \), the element \( xu^{-1} \) is central in \( \mathcal{H} \) and consequently there is \( r \in \mathbb{R} \) such that \( x = ru \). From \( N(x) = 1 \) we obtain \( r = \pm \frac{1}{\sqrt{N(u)}} \). \( \Box \)

From 5.1.14 and 5.1.15 we derive an explicit description of the orthogonal group of the lattice \( (\mathcal{O}, N|_{\mathcal{O}}) \) in terms of the multiplicative structure of the order and its invariant elements:

**Theorem 5.1.16.** Let \( \mathcal{O} \) be an order in \( \mathbb{F} \). Then the structure of \( \mathcal{O}(\mathcal{O}) \) is given by

\[
\mathcal{O}(\mathcal{O}) = \begin{cases} 
\langle w \mapsto -w \rangle, & \mathbb{F} = \mathbb{R}, \\
\{ w \mapsto \delta w : \delta \in \mathcal{O}^\times \} \times \langle w \mapsto w \rangle, & \mathbb{F} = \mathbb{C}, \\
\{ \{ w \mapsto \delta w : \delta \in \mathcal{O}^\times \} \cdot \{ w \mapsto u^{-1}wu : u \in \mathcal{I}(\mathcal{O}) \} \} \times \langle w \mapsto w \rangle, & \mathbb{F} = \mathbb{H}.
\end{cases}
\]

**Proof.** The claim is obvious in the case \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \) and for the left-multiplications if \( \mathbb{F} = \mathbb{H} \), since \( 1 \in \mathcal{O} \). In the remaining case let \( \varepsilon, \delta \in \mathbb{H}_1 \) such that \( \delta \mathcal{O} \varepsilon = \mathcal{O} \). This implies \( \delta \varepsilon \in \mathcal{O}^\times \), i.e. \( \delta = \delta' \varepsilon \) for some \( \delta' \in \mathcal{O}^\times \). As a consequence,

\[
\mathcal{O} = \varepsilon \delta \mathcal{O} = \varepsilon \mathcal{O} \delta' = \varepsilon \mathcal{O} \varepsilon,
\]

since \( \delta' \in \mathcal{O}^\times \). Thus \( \varepsilon \in \mathcal{N}_{\mathbb{H}_1}(\mathcal{O}) \) and by 5.1.15, \( \varepsilon = \frac{u}{\sqrt{N(u)}} \) for some \( u \in \mathcal{I}(\mathcal{O}) \). \( \Box \)

We give some explicit

**Examples 5.1.17.** Let \( \mathcal{O} \) denote the Hurwitz order and \( \pi := 1 + i_1 \).

a) \( \mathcal{O}(\mathcal{O}) = \{ w \mapsto \delta w, \delta \in \mathcal{O}^\times \} \cdot \langle w \mapsto u^{-1}wu : u \in \mathcal{O}^\times \cup \{ \pi \} \rangle \times \langle w \mapsto w \rangle \),

b) \( \mathcal{O}(\Lambda) = \{ w \mapsto \delta w, \delta \in \Lambda^\times \} \cdot \langle w \mapsto u^{-1}wu : u \in \Lambda^\times \cup \{ \pi, \omega \} \rangle \times \langle w \mapsto w \rangle \),

c) \( \mathcal{O}(\mathcal{O}_K) = \{ w \mapsto \delta w, \delta \in \mathcal{O}_K^\times \} \cdot \langle w \mapsto u^{-1}wu : u \in \mathcal{O}_K^\times \cup \{ i_1 \sqrt{D_K} \} \rangle \times \langle w \mapsto w \rangle \)

for \( D_K = 3 \) or \( D_K > 4 \).

Regarding the discriminant kernel of the lattice \( (\mathcal{O}, N|_{\mathcal{O}}) \) we obtain the following

**Proposition 5.1.18.** Let \( \mathcal{O} \) be an order in \( \mathbb{F} \). Then the map \( (w \mapsto -\overline{w}) \) belongs to \( \mathcal{O}_d(\mathcal{O}) \).

**Proof.** Follows from \( \mu - (-\overline{\mu}) = \mu + \overline{\mu} = \text{tr}(\mu \cdot 1) \in \mathbb{Z} \) for all \( \mu \in \mathcal{O}_d \). \( \Box \)
5.2 Unitary symplectic groups

Definition 5.2.1. The unitary symplectic group of degree \( n \) over \( \mathbb{F} \) is defined as
\[
U_n(\mathbb{F}) := \left\{ M \in \mathbb{F}^{2n \times 2n} : J_n[M] = J_n \right\}.
\]
Note that \( U_n(\mathbb{R}) = \text{Sp}_n(\mathbb{R}) \). In order to avoid ambiguities, we will use the term *unitary* only in the case \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{F} = \mathbb{H} \).

As in [14, II. § 2 Lemma (1.1)] we obtain the following characterization of \( U_n(\mathbb{F}) \):

Theorem 5.2.2. The unitary symplectic group \( U_n(\mathbb{F}) \) is a subgroup of \( \text{GL}_{2n}(\mathbb{F}) \). For \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with blocks \( A, B, C, D \in \mathbb{F}^{n \times n} \) the following assertions are equivalent:

i) \( M \in U_n(\mathbb{F}) \),

ii) \( M^t \in U_n(\mathbb{F}) \),

iii) \( \overline{A}^t C - \overline{C}^t A = \overline{B}^t D - \overline{D}^t B = 0, \overline{A}^t D - \overline{D}^t C = I_n \),

iv) \( A\overline{B}^t - B\overline{A}^t = C\overline{D}^t - D\overline{C}^t = 0, A\overline{D}^t - B\overline{C}^t = I_n \).

In this case, one has
\[
M^{-1} = \begin{pmatrix} \overline{D}^t & -\overline{B}^t \\ -\overline{C}^t & \overline{A}^t \end{pmatrix}.
\]

As a generalization of the Siegel modular group \( \text{Sp}_n(\mathbb{Z}) \) we introduce the modular group with respect to an order \( \mathcal{O} \) in \( \mathbb{F} \):

Definition 5.2.3. Let \( \mathcal{O} \) be an order in \( \mathbb{F} \). Then the modular group of degree \( n \) with respect to \( \mathcal{O} \) is defined as
\[
U_n(\mathcal{O}) := U_n(\mathbb{F}) \cap \mathcal{O}^{2n \times 2n} = \left\{ M \in \mathcal{O}^{2n \times 2n} : J_n[M] = J_n \right\}.
\]

We distinguish certain elements in the modular group:

Remarks 5.2.4. Let \( \mathcal{O} \) be an order in \( \mathbb{F} \).

a) The matrices
\[
J_n, \quad \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \text{GL}_n(\mathcal{O}), \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \text{Her}_n(\mathcal{O})
\]
b) \( U_1(\mathcal{O}) = \mathcal{O}^\times \cdot \text{SL}_2(\mathbb{Z}) \).
This gives rise to the following

**Definition 5.2.5.** Let $\mathcal{O}$ be an order in $\mathbb{F}$. The modular group $U_n(\mathcal{O})$ is called standardly generated, if

$$U_n(\mathcal{O}) = \left\langle J_n, \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \text{GL}_n(\mathcal{O}), \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \text{Her}_n(\mathcal{O}) \right\rangle.$$

A class of examples is contained in the following

**Theorem 5.2.6.** Let $\mathcal{O}$ be an order in $\mathbb{F}$. If $\mathcal{O}$ is a principal ideal domain, then $U_n(\mathcal{O})$ is standardly generated.

**Proof.** Let $x \in \mathcal{O}^n$. As a special case of the elementary divisor theorem for principal ideal domains, where we refer to [25] in the noncommutative case, yields the existence of a matrix $U \in \text{GL}_n(\mathcal{O})$ such that $Ux = (\gamma, 0, \ldots, 0)^t$, where $\gamma$ is a greatest common right divisor of the entries of $x$. Then rest of the proof is along the same lines as in [14, II. §2, Prop. 2.2, Thm. 2.3].

Under more rigid assumptions on the order one can determine a smaller set of generators of $U_n(\mathcal{O})$. By utilizing the euclidean algorithm, one can show that $\text{GL}_n(\mathcal{O})$ is generated by elementary matrices. Hence, we obtain

**Corollary 5.2.7.** Let $\mathcal{O}$ be a norm-euclidean order in $\mathbb{F}$. Then the following assertions hold:

a) $U_n(\mathcal{O})$ is generated by

$$J_n, \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \text{Her}_n(\mathcal{O}), \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U = \text{diag}(\varepsilon, 1, \ldots, 1), \varepsilon \in \mathcal{O}^\times.$$

b) $U_n(\mathcal{O})$ is generated by

$$J_{2 \times 2n-2}, \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \text{Her}_n(\mathcal{O}), \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix},$$

where $U = \text{diag}(\varepsilon, 1, \ldots, 1), \varepsilon \in \mathcal{O}^\times$ or $U$ is a permutation matrix.

### 5.3 Paramodular groups

**Definition 5.3.1.** A matrix $T \in \mathbb{Z}^{n \times n}$ is called elementary divisor matrix, if

$$T = \text{diag}(t_1, \ldots, t_n), \quad t_i | t_{i+1} \text{ for } i = 1, \ldots, n - 1.$$
Definition 5.3.2. The integral paramodular group of polarization $T$ is defined as

$$\hat{\Gamma}(T) = \left\{ M \in \mathbb{Z}^{2n \times 2n} : M^t \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\}.$$ 

Of course one has $\hat{\Gamma}(I_n) = \text{Sp}_n(\mathbb{Z})$.

The content of [2, § 2] is summarized in the following

Theorem 5.3.3. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}^{2n \times 2n}$ with blocks $A, B, C, D \in \mathbb{Z}^{n \times n}$ the following assertions are equivalent:

i) $M \in \hat{\Gamma}(T)$,

ii) $A^tTC, D^tTB \in \text{Sym}_n(\mathbb{Z}), A^tTD - C^tTB = T$,

iii) $I_TMI_T^{-1} \in \text{Sp}_n(\mathbb{Q})$, where $I_T := \text{diag}(I_n, T)$,

iv) $\tilde{I}_TMI_T^{-1} \in \text{Sp}_n(\mathbb{Q})$, where $\tilde{I}_T := \text{diag}(T, I_n) = J_n^{-1}I_TJ_n$,

v) $\begin{pmatrix} I_n & 0 \\ 0 & 1/T \end{pmatrix}^{-1}M^t \begin{pmatrix} I_n & 0 \\ 0 & 1/T \end{pmatrix} \in \hat{\Gamma}(T)$.

In this case, one has

$$M^{-1} = \begin{pmatrix} T^{-1}D^tT & -T^{-1}B^tT \\ -T^{-1}C^tT & T^{-1}A^tT \end{pmatrix}.$$

In view of 5.3.3 iii) we have the following

Definition 5.3.4. The group

$$\Gamma(T) := I_T \hat{\Gamma}(T)I_T^{-1} = \left\{ M \in \text{Sp}_n(\mathbb{Q}) : I_T^{-1}MI_T \in \mathbb{Z}^{2n \times 2n} \right\}$$

is called the paramodular group of polarization $T$. For $t \in \mathbb{N}$ we will simply write

$$\Gamma(t) := \Gamma \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Remark 5.3.5. a) In view of 5.3.3 iv) one has $\tilde{I}_T \hat{\Gamma}(T) \tilde{I}_T^{-1} = \Gamma(T)^t$.

b) $\Gamma(T)$ and $\Gamma(T)^t$ are conjugate as subgroups of $\text{Sp}_n(\mathbb{Q})$. Explicitly one has

$$\Gamma(T)^t = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma(T) \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1}.$$
c) From b) we obtain, that the assignment
\[ M \mapsto \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1} M^t \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \]
is a nontrivial involutive automorphism of \( \Gamma(T) \).

Generators of \( \hat{\Gamma}(T) \) were determined in [13, Thm. 1.12]. A translation yields

Theorem 5.3.6. The group \( \Gamma(T) \) is generated by
\[ I_T := \begin{pmatrix} 0 & -T^{-1} \\ T & 0 \end{pmatrix}, \quad \left( I_n \begin{pmatrix} G_{ij} \\ I_n \end{pmatrix} \right), \quad G_{ij} := \begin{cases} \frac{1}{t_i} \cdot I_{ii}, & i = j, \\ \frac{1}{t_i} \cdot (I_{ij} + I_{ji}), & i < j. \end{cases} \]

The paramodular group admits nontrivial discrete extensions in \( \text{Sp}_n(\mathbb{R}) \). From [17, Satz 4'] we cite the following

Theorem 5.3.7. The group \( \Gamma(T)^{\text{max}} := N_{\text{Sp}_n(\mathbb{R})}(\Gamma(T)) \) is the maximal discrete extension of \( \Gamma(T) \) in \( \text{Sp}_n(\mathbb{R}) \).

### 5.4 Modular embeddings of paramodular groups

In the following let \( \mathcal{O} \) always denote an order in \( \mathbb{F} \) and \( T = \text{diag}(t_1, \ldots, t_n) \) an elementary divisor matrix.

In the spirit of [18] we give the following

Definition 5.4.1. Let \( M \in \text{U}_n(\mathbb{F}) \). We say that \( M \) is a modular embedding of \( \Gamma(T) \) into \( \text{U}_n(\mathcal{O}) \), if \( M^{-1} \Gamma(T) M \subseteq \text{U}_n(\mathcal{O}) \). In this case, we define
\[ \Phi_M : \Gamma(T) \rightarrow \text{U}_n(\mathcal{O}), \quad \Phi_M(H) := M^{-1}HM, \quad H \in \Gamma(T). \]

Furthermore, we set
\[ \text{Mod}(\Gamma(T), \text{U}_n(\mathcal{O})) := \left\{ M \in \text{U}_n(\mathbb{F}) : M^{-1} \Gamma(T) M \subseteq \text{U}_n(\mathcal{O}) \right\}. \]

Some facts are contained in the following
Remarks 5.4.2.  a) The map
\[ \Gamma(T) \longrightarrow \Gamma(t^{-1} T), \quad H \mapsto \left( \begin{array}{cc} \sqrt{t} I_n & 0 \\ 0 & \frac{1}{\sqrt{t}} I_n \end{array} \right)^{-1} H \left( \begin{array}{cc} \sqrt{t} I_n & 0 \\ 0 & \frac{1}{\sqrt{t}} I_n \end{array} \right) \]
is an isomorphism of paramodular groups. Hence the map
\[ \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \longrightarrow \text{Mod}(\Gamma(t^{-1} T), U_n(\mathcal{O})), \quad M \mapsto \left( \begin{array}{cc} \sqrt{t} I_n & 0 \\ 0 & \frac{1}{\sqrt{t}} I_n \end{array} \right)^{-1} \cdot M \]
is a bijection.

b) Modular embeddings of \( \Gamma(T) \) are defined in the same manner of 5.4.1. Indeed, the map
\[ \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \longrightarrow \text{Mod}(\Gamma(T)^l, U_n(\mathcal{O})), \quad M \mapsto \left( \begin{array}{cc} T & 0 \\ 0 & T^{-1} \end{array} \right) M \]
is a bijection.

c) The group \( \mathbb{F}_1 \times U_n(\mathcal{O}) \) acts on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \) via the assignment
\[ M \mapsto \varepsilon MR, \quad \varepsilon \in \mathbb{F}_1, R \in U_n(\mathcal{O}). \]

Part c) of 5.4.2 gives rise to

Definition 5.4.3. Let \( M, M' \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \). We call \( M, M' \) equivalent, if there is \( (\varepsilon, R) \in \mathbb{F}_1 \times U_n(\mathcal{O}) \) such that \( M' = \varepsilon MR \). In this case, we write \( M \sim M' \). The equivalence class of \( M \) is denoted by \( [M] \) and the set of equivalence classes by \( \text{Mod}(\Gamma(T), U_n(\mathcal{O}))/\sim \).

Definition 5.4.4. Let \( a_1, \ldots, a_n \in \mathcal{O} \). The matrix \( A := \text{diag}(a_1, a_2, \ldots, a_n) \) is called an \( \mathcal{O} \)-model of \( T \), if
\[ A\overline{\mathbb{A}} = T \text{ and } a_i | a_{i+1} \text{ for } i = 1, \ldots, n-1. \]

We define a notion of equivalence for \( \mathcal{O} \)-models:

Definition 5.4.5. Let \( A, B \) be \( \mathcal{O} \)-models of \( T \). We call \( A, B \) equivalent, if there are \( \delta_1, \ldots, \delta_n, \varepsilon \in \mathcal{O}^\times \) and
\[ B = \text{diag}(\delta_1, \ldots, \delta_n) \cdot A \cdot \varepsilon, \]
i.e. \( b_i = \delta_i a_i \varepsilon \) for \( i = 1, \ldots, n \).

In view of the constructing \( \mathcal{O} \)-models of \( T \), we state the following easy

Lemma 5.4.6. Let \( a_1, \ldots, a_n \in \mathcal{O} \) and \( A = \text{diag}(a_1, a_2, \ldots, a_n) \). Then the following assertions are equivalent:
i) $A$ is an $\mathcal{O}$-model for $T$,

ii) there are $\gamma_1, \ldots, \gamma_n \in \mathcal{O}$ such that $\gamma_1 = 1$ and

$$a_k a_i^{-1} = \gamma_k \cdots \gamma_2, \quad N(\gamma_k) = t_k t_{k-1}^{-1} \text{ for } k = 2, \ldots, n.$$ 

In this case, $\gamma_1, \ldots, \gamma_n$ are uniquely determined by $A$ and one has

$$\gamma_k = a_k a_{k-1}^{-1} \text{ for } k = 2, \ldots, n.$$ 

In [18] the set Mod$(\Gamma(T)^t, U_n(\mathcal{O}_K))$ was studied and some prototype was introduced, which is also suitable in the noncommutative setting:

**Proposition 5.4.7.** Let $a_1, \ldots, a_n \in \mathcal{O}$ and $A := \text{diag}(a_1, \ldots, a_n)$. Define

$$M_A := \begin{pmatrix} A^{-1} & 0 \\ 0 & \overline{A} \end{pmatrix} \in U_n(\mathbb{F}).$$

Then the following assertions are equivalent:

i) $A$ is an $\mathcal{O}$-model of $T$,

ii) $M_A \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})).$

**Proof.** In view of 5.3.6 it suffices to show, that the condition

$$M_A^{-1} J_T M_A \in U_n(\mathcal{O}) \text{ and } M_A^{-1} \begin{pmatrix} I_n & G_{ij} \\ 0 & I_n \end{pmatrix} M_A \in U_n(\mathcal{O}), \quad 1 \leq i \leq j \leq n,$$

is equivalent to $A$ being an $\mathcal{O}$-model for $T$. The assertion

$$M_A^{-1} J_T M_A = \begin{pmatrix} 0 & -AT^{-1} \overline{A} \\ \overline{A}^{-1} TA^{-1} & 0 \end{pmatrix} \in U_n(\mathcal{O})$$

is equivalent to $t_i | N(a_i)$ and $N(a_i)| t_i$, i.e. $N(a_i) = t_i$ for $i = 1, \ldots, n$. The assertion

$$M_A^{-1} \begin{pmatrix} I_n & G_{ij} \\ 0 & I_n \end{pmatrix} M_A \in U_n(\mathcal{O}), \quad 1 \leq i \leq j \leq n,$$

is equivalent to

$$\frac{1}{t_i} A I_{ji} \overline{A} = \frac{1}{t_i} a_i a_i^{-1} I_{ji} = a_i a_i^{-1} I_{ji} \in \mathcal{O}^{n \times n}, \quad 1 \leq i \leq j \leq n,$$

i.e. to $a_i | t_i$ for $1 \leq i \leq j \leq n$. □

Proposition 5.4.7 leads to the following
Definition 5.4.8. Let $M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O}))$. We say that $M$ is of principal type, if $M = M_A$ for some $\mathcal{O}$-model $A$ of $T$. The set of modular embeddings of principal type is denoted by $\text{PMod}(\Gamma(T), U_n(\mathcal{O}))$.

Remark 5.4.9. Let $T, T'$ be elementary divisor matrices with $\mathcal{O}$-models $A$ resp. $A'$. Then $AA'$ is an $\mathcal{O}$-model of $TT'$ if and only if
\[
a_{i+1}a'_{i+1}a'^{-1}_{i}a^{-1}_i \in \mathcal{O}, \quad i = 1, \ldots, n - 1.
\]
Note that this condition is trivially fulfilled for $F\{\mathbb{R}, \mathbb{C}\}$, while in the case $F = \mathbb{H}$ it is more restrictive.

In the proof of [18, Satz 2], Köhler gave an implicit classification of $\text{Mod}(\Gamma(T)^t, U_n(\mathfrak{o}_K))$ in the case, where $K$ has class number one. Therefore it is reasonable to assume that $\mathcal{O}$ is a principal ideal domain. To this end we will analyse his proof carefully and ensure that the arguments used there also hold in the noncommutative case. As a by-product it turns out, provided $t_1 = 1$, that the embeddings of principal type contain a complete system of representatives of the equivalence classes of $\text{Mod}(\Gamma(T), U_n(\mathcal{O}))$.

We introduce coprime matrix pairs:

Definition 5.4.10. Let $C, D \in \mathcal{O}^{n \times n}$ and $(C, D) \in \mathcal{O}^{n \times 2n}$ the corresponding matrix pair.

a) $(C, D)$ is called coprime, if for all $V \in \mathbb{F}^{n \times n}$ one has $V(C, D) \in \mathcal{O}^{n \times 2n}$ if and only if $V \in \mathcal{O}^{n \times n}$.

b) $(C, D)$ is called hermitian, if $CD^t \in \text{Her}_n(\mathcal{O})$.

Remark 5.4.11. It is easy to see, that the group $\text{GL}_n(\mathcal{O}) \times \text{GL}_{2n}(\mathcal{O})$ acts naturally on the set of coprime pairs $(C, D)$ via $U(C, D)V^{-1}$ for $U \in \text{GL}_n(\mathcal{O}), V \in \text{GL}_{2n}(\mathcal{O})$.

Proposition 5.4.12. Let $\mathcal{O}$ be a principal ideal domain and $C, D \in \mathcal{O}^{n \times n}$. Then the following assertions hold:

a) There is some coprime matrix pair $(C_1, D_1) \in \mathcal{O}^{n \times 2n}$ and some $G \in \mathcal{O}^{n \times n}$ such that $(C, D) = G(C_1, D_1)$.

b) $(C, D)$ is the second block row of some $M \in U_n(\mathcal{O})$ if and only if $(C, D)$ is coprime and hermitian.

Proof. a) Due to [25] the elementary divisor theorem also holds for noncommutative principal ideal domains, i.e. we find $U \in \text{GL}_n(\mathcal{O})$ and $V \in \text{GL}_{2n}(\mathcal{O})$ such that
\[
(C, D) = U(F, 0)V = UF(I_n, 0)V,
\]
where $F = \text{diag}(f_{11}, \ldots, f_{nn}) \in \mathcal{O}^{n \times n}$ and $f_{ii}$ is a two-sided divisor of $f_{jj}$ for all $i \leq j$. The claim follows then with $G := UF$ and $(C_1, D_1) := (I_n, 0)V$. 
b) Assume that \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_n(\mathcal{O}) \) holds for some \( A, B \in \mathcal{O}^{n \times n} \). From 5.2.2 iv) we conclude \( D\overline{A}^t - C\overline{B}^t = I_n \) and \( C\overline{D}^t \in \text{Her}_n(\mathcal{O}) \). Hence, the pair \((C, D)\) is coprime and hermitian. Conversely, using the elementary divisor theorem, we find \( U \in \text{GL}_n(\mathcal{O}) \) and \( V \in \text{GL}_{2n}(\mathcal{O}) \) such that
\[
U(C, D)V = (C_1, 0), \quad C_1 = \text{diag}(c_{11}, \ldots, c_{nn}), c_{ii} \neq 0.
\]
Since \((C_1, 0)\) is necessarily coprime, we have \( c_{ii} \in \mathcal{O}^\times \) for \( i = 1, \ldots, n \). Hence we can assume \( C_1 = I_n \). With the definition
\[
\begin{pmatrix} X \\ Y \end{pmatrix} := V \begin{pmatrix} U \\ 0 \end{pmatrix}
\]
we obtain
\[
CX + DY = (C, D) \begin{pmatrix} X \\ Y \end{pmatrix} = (C, D)V \begin{pmatrix} U \\ 0 \end{pmatrix} = I_n.
\]
Finally, we set
\[
A := \overline{X}^t + \overline{X}^tYC, \quad B := -\overline{X}^t + \overline{X}^tYD
\]
and an explicit calculation shows that the matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) satisfies 5.2.2 iv).

Now we are able to prove the main classification theorem for modular embeddings. The proof is adapted from [18]. The assertions in the theorem are formulated accordingly in order to fit into the new setting.

**Theorem 5.4.13.** Let \( \mathcal{O} \) be a principal ideal domain and \( M \in U_n(\mathcal{F}) \). Then the following assertions are equivalent:

i) \( M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \),

ii) there exist \((\epsilon, R) \in \mathbb{F}_1 \times U_n(\mathcal{O})\) and some \( \mathcal{O} \)-model \( A \) for \( t_1^{-1}T \) such that
\[
\begin{pmatrix} \sqrt{T_1}I_n & 0 \\ 0 & \frac{1}{\sqrt{T_1}}I_n \end{pmatrix}^{-1} M = \epsilon M A R.
\]

**Proof.** Let \( M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \). After replacing \( M \) by \( \begin{pmatrix} \sqrt{T_1}I_n & 0 \\ 0 & \frac{1}{\sqrt{T_1}}I_n \end{pmatrix}^{-1} M \) we can assume that \( t_1 = 1 \). Furthermore, after replacing \( M \) by \( \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} M \) we can assume that \( M \in \text{Mod}(\Gamma(T)^4, U_n(\mathcal{O})) \), i.e.
\[
M^{-1} \Gamma(T)^4 M \subseteq U_n(\mathcal{O}). \quad (1)
\]
By [17, Hilfssatz 1'], the \( \mathbb{Z} \)-lattice \( \widehat{\Gamma}(T)\mathbb{Z} \) over \( \mathbb{Z} \), generated by \( \widehat{\Gamma}(T) \), is a free abelian group with basis \( f_{kl}I_{kl}, 1 \leq k, l \leq 2n \), where
\[
f_{k+n,l+n} := f_{k+n,l} := f_{k,l+n} := f_{k,l} := \begin{cases} 1, & \text{for } 1 \leq l \leq k \leq n, \\ t_1 t_k^{-1}, & \text{for } 1 \leq k < l \leq n. \end{cases}
\]
Hence we have
\[ M^{-1} \tilde{T} \tilde{\Theta} (T) Z \tilde{T}^{-1} M \subseteq \mathcal{O}^{2n \times 2n}. \] (2)

Considering the generators \( f_{kl} l_{kl} \) in (2), we obtain that the numbers
\[ \begin{align*}
& f_{kl} t_l t_l^{-1} d_{kl} a_{kl} l_{kl} \
& f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} c_{kl} \
& f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} d_{kl} \
& f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} f_{kl} t_l t_l^{-1} d_{kl} a_{kl} \end{align*} \]
(3)

belong to \( \mathcal{O} \) for all \( k, l, \mu, \nu \in \{1, \ldots, n\} \). From \( M \in U_n(\mathbb{F}) \) we obtain that there is at least one nonzero entry of \( M \), say \( \overline{\rho} \). Hence (3) implies, that the product of \( \rho \) with every entry of \( M \) belongs to \( \mathcal{O} \mathbb{Q} \), i.e. we can write
\[ M = \rho^{-1} M', \quad \rho \in \mathbb{F}^*, N(\rho) \in \mathbb{Q}, M' \in \mathbb{Q} \cdot \mathcal{O}^{2n \times 2n}. \]

After multiplying \( \rho \) and \( M' \) by some suitable rational number, we can assume
\[ M = \rho^{-1} M' = \rho^{-1} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right), \quad r^{-1} := N(\rho) \in \mathbb{N} \]
with blocks \( A = (a_{kl}), B = (b_{kl}), C = (c_{kl}), D = (d_{kl}) \in \mathcal{O}^{n \times n} \). The matrix \( \sqrt{\mathbb{r}} \cdot \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) belongs to \( U_n(\mathbb{F}) \). The identity
\[ \overline{\rho^{-1} x \rho^{-1} y} = \overline{x} \rho^{-1} \rho^{-1} y = rxy, \]
valid for arbitrary \( x, y \in \mathbb{F} \), shows that the numbers
\[ \begin{align*}
& r f_{kl} t_l t_l^{-1} d_{kl} a_{kl} l_{kl} \
& r f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} c_{kl} \
& r f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} d_{kl} \
& r f_{kl} t_l t_l^{-1} d_{kl} a_{kl} c_{kl} f_{kl} t_l t_l^{-1} d_{kl} a_{kl} \end{align*} \]
(4)

belong to \( \mathcal{O} \) for all \( k, l, \mu, \nu \in \{1, \ldots, n\} \). Due to 5.4.12 a), we can write
\[ (A, B) = G(A_1, B_1), \quad G \in \mathcal{O}^{n \times n}, (A_1, B_1) \in \mathcal{O}^{n \times 2n} \text{ coprime}. \]

Part b) of 5.4.12 yields the existence of \( R_1 = \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right) \in U_n(\mathcal{O}). \) The matrix \( MR_1^{-1} \) has the form
\[ \left( \begin{array}{cc} a & 0 \\ c & d \end{array} \right). \] After substitution of \( M \) by \( MR_1^{-1}, \) we can assume that \( M \) is of the form
\[ M = \rho^{-1} \left( \begin{array}{cc} A & 0 \\ C & D \end{array} \right), \quad N(\rho) = \frac{1}{r} \in \mathbb{N}. \] (5)

Inductively, the elementary divisor theorem yields the existence of \( U \in \text{GL}_n(\mathcal{O}) \) such that \( AU \) is of lower triangular shape. Then
\[ R_2 := \left( \begin{array}{cc} U^{-1} & 0 \\ 0 & \overline{U} \end{array} \right) \in U_n(\mathcal{O}). \]
By construction, the $A$-block of $MR_2^{-1}$ is of lower triangular shape and the $D$-block of $MR_2^{-1}$ is of upper triangular shape, since $rA^tD = I_n$. Now $rA^tD = I_n$ implies

$$a_{kk}^{-1}d_{kk} = r^{-1}, \quad \text{i.e.} \quad a_{kk}^{-1} = rd_{kk} \quad \text{for } 1 \leq k \leq n. \quad (6)$$

From (6) and (4), we obtain

$$a_{kk}^{-1}a_{k\mu} = rd_{kk}a_{k\mu} = rf_{kk}d_{kk}a_{k\mu} \in \mathcal{O} \quad \text{for } 1 \leq \mu \leq k \leq n,$$

that is $a_{kk}|a_{k\mu}$ for $1 \leq \mu \leq k \leq n$. As a consequence,

$$U := \text{diag}(a_{11}, \ldots, a_{nn})^{-1}A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \in \text{GL}_n(\mathcal{O}).$$

Hence

$$R_3 = \begin{pmatrix} U^{-1} & 0 \\ 0 & U^t \end{pmatrix} \in U_n(\mathcal{O})$$

and we observe that the corresponding $A$- and $D$-block of $MR_3$ are diagonal matrices. Thus we can assume $A$ and $D$ in (5) have diagonal form, say

$$A = \text{diag}(a_{11}, \ldots, a_{nn}), \quad D = \text{diag}(d_{11}, \ldots, d_{nn}),$$

which satisfy (6). From (4) we obtain

$$d_{\mu\mu}^{-1}c_{\mu\nu} = r\bar{a}_{\mu\mu}c_{\mu\nu} = rf_{\mu\mu}\bar{a}_{\mu\mu}c_{\mu\nu} \in \mathcal{O}.$$ 

This implies $H := -D^{-1}C \in \mathcal{O}^{n \times n}$, and $CD^t \in \text{Her}_n(\mathcal{O})$ implies $H \in \text{Her}_n(\mathcal{O})$. Therefore

$$R_4 := \begin{pmatrix} I_n & 0 \\ H & I_n \end{pmatrix} \in U_n(\mathcal{O}).$$

By construction, $MR_4$ is a diagonal matrix. Now (4) implies

$$rt_k^{-1}\bar{a}_{kk}a_{kk} \in \mathcal{O}, \quad (rt_k^{-1}\bar{a}_{kk}a_{kk})^{-1} = rt_k^{-1}\bar{d}_{kk}d_{kk} \in \mathcal{O}, \quad 1 \leq k \leq n.$$ 

Therefore, $N(a_{kk}) = r^{-1}t_k$. Especially, $N(a_{11}) = r^{-1}$. Again, (4) implies

$$a_{k-1,k-1}^{-1}a_{kk} = r\bar{d}_{k-1,k-1}a_{kk} = rf_{k-1,k-1}t_{k-1}^{-1}d_{k-1,k-1}a_{kk} \in \mathcal{O},$$

thus $a_{k-1,k-1}|a_{kk}$ for $k = 2, \ldots, n$. Finally, we set $\varepsilon := \rho^{-1}a_{11}$. Then

$$N(\varepsilon) = N(\rho)^{-1}N(a_{11}) = 1, \text{ i.e. } \varepsilon \in \mathbb{F}_1.$$
Using the notation \( a_k := a_{11}^{-1}a_{kk} \) for \( k = 1, \ldots, n \), we can write
\[
M = \epsilon \begin{pmatrix} \tilde{A} & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A = \text{diag}(1, a_2, \ldots, a_n), N(a_k) = t_k, N(\epsilon) = 1.
\]

Recapitulating the proof, we have constructed \( \epsilon \in \mathbb{F}_1, A = \text{diag}(1, a_2, \ldots, a_n) \in \mathcal{O}^{n \times n} \) and \( R \in \text{U}_n(\mathcal{O}) \) such that
\[
\left( \begin{array}{cc} T & 0 \\ 0 & T^{-1} \end{array} \right) \left( \sqrt{t_1} I_n \begin{array}{cc} 0 \\ \frac{1}{\sqrt{t_1}} I_n \end{array} \right)^{-1} M = \epsilon \begin{pmatrix} \tilde{A} & 0 \\ 0 & A^{-1} \end{pmatrix} R,
\]
i.e.
\[
\left( \sqrt{t_1} I_n \begin{array}{cc} 0 \\ \frac{1}{\sqrt{t_1}} I_n \end{array} \right)^{-1} M = \epsilon M_A R
\]
and \( A \) is an \( \mathcal{O} \)-model of \( t_1^{-1}T \).

A reformulation of 5.4.13 and the fact that the number of \( \mathcal{O} \)-models of \( T \) is finite yields:

**Corollary 5.4.14.** Let \( \mathcal{O} \) be a principal ideal domain and \( M \in \text{Mod}(\Gamma(T), \text{U}_n(\mathcal{O})) \). Then the following assertions hold:

a) \( \left( \sqrt{t_1} I_n \begin{array}{cc} 0 \\ \frac{1}{\sqrt{t_1}} I_n \end{array} \right)^{-1} M \) is equivalent to some modular embedding of principal type.

b) The set of equivalence classes \( \text{Mod}(\Gamma(T), \text{U}_n(\mathcal{O})) \) is finite.

The dependency of \( T \) on some modular embedding \( M \) of \( \Gamma(T) \) is revealed in the following

**Corollary 5.4.15.** Let \( \mathcal{O} \) be a principal ideal domain. Then the following assertions hold:

a) \( \text{Mod}(\Gamma(T), \text{U}_n(\mathcal{O})) \neq \emptyset \) if and only if \( t_1^{-1}T \) has an \( \mathcal{O} \)-model.

b) Let \( T' \) be another elementary divisor matrix and \( t_1 = t_1' \). Then
\[
\text{Mod}(\Gamma(T), \text{U}_n(\mathcal{O})) \cap \text{Mod}(\Gamma(T'), \text{U}_n(\mathcal{O})) \neq \emptyset
\]
implies \( T = T' \).

**Proof.** a) Follows from 5.4.13, 5.4.7 and 5.4.2.

b) Let \( M \in \text{U}_n(\mathbb{F}) \) be a modular embedding of both \( \Gamma(T) \) and \( \Gamma(T') \). By 5.4.13,
\[
\left( \sqrt{t_1} I_n \begin{array}{cc} 0 \\ \frac{1}{\sqrt{t_1}} I_n \end{array} \right)^{-1} M \sim M_A
\]
for some \( \mathcal{O} \)-model \( A \) of \( t_1^{-1}T \). From \( t_1 = t_1' \) and 5.4.7 we obtain, that \( A \) is also an \( \mathcal{O} \)-model of \( t_1^{-1}T' \). As a consequence,
\[
t_1^{-1}T = \tilde{A}A = t_1^{-1}T',
\]
i.e. \( T = T' \).
The equivalence relation between embeddings of principal type is characterized in the following

**Theorem 5.4.16.** Let $A$ and $B$ be $\mathcal{O}$-models of $t_1^{-1}T$. Then the following assertions are equivalent:

i) $M_A$ and $M_B$ are equivalent as modular embeddings,

ii) $A$ and $B$ are equivalent as $\mathcal{O}$-models.

**Proof.** Let $\varepsilon \in \mathbb{F}_1$. Then we have

$$R := M_B^{-1}\varepsilon M_A \in U_n(\mathcal{O})$$

if and only if

$$B\varepsilon A^{-1} = \text{diag}(\delta_1, \ldots, \delta_n) \in \text{GL}_n(\mathcal{O})$$

for certain units $\delta_1, \ldots, \delta_n \in \mathcal{O}^\times$. From $a_1, b_1 \in \mathcal{O}^\times$ we derive $\varepsilon \in \mathcal{O}^\times$. \hfill $\Box$

As a consequence of the classification theorem 5.4.13, 5.4.14 and 5.4.16 we can express the number of nonequivalent modular embeddings:

**Corollary 5.4.17.** Let $\mathcal{O}$ be a principal domain. Then the number of nonequivalent modular embeddings of $\Gamma(T)$ into $U_n(\mathcal{O})$ equals the number of orbits of the group $\mathcal{O}^\times(n+1)$ on the set

$$\{ a \in \mathcal{O}^{(n)} : N(a_i) = t_1t_1^{-1}, i = 1, \ldots, n \}$$

via the action

$$(a_1, \ldots, a_n) \mapsto (\delta_1a_1, \ldots, \delta_na_n)\overline{\varepsilon}, \quad (\delta_1, \delta_2, \ldots, \delta_n, \varepsilon) \in \mathcal{O}^\times(n+1).$$

In view of [18], we give the following

**Definition 5.4.18.** Let $M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O}))$. We call $M$ maximal, if

$$\Gamma(T) = MU_n(\mathcal{O})M^{-1} \cap \text{Sp}_n(\mathbb{R}).$$

**Remark 5.4.19.** Maximality of $M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O}))$ only depends on its equivalence class.

A characterization of maximality is explained in the following

**Proposition 5.4.20.** Let $M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O}))$. Then the following assertions are equivalent:

i) $M$ is a maximal modular embedding,

ii) for all $H \in \text{Sp}_n(\mathbb{R})$ one has $H \in \Gamma(T)$ if and only if $M^{-1}HM \in U_n(\mathcal{O})$,

iii) for all $\Gamma \leq \text{Sp}_n(\mathbb{R})$ such that $M^{-1}\Gamma M \subseteq U_n(\mathcal{O})$, one has $\Gamma \leq \Gamma(T)$.

In this case, $\Gamma(T)$ is the maximal subgroup of $\text{Sp}_n(\mathbb{R})$, which embeds into $U_n(\mathcal{O})$ via $\Phi_M$. 

5.4 Modular embeddings of paramodular groups

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From [18, pp. 75-76] we adapt a correspondence between maximal modular embeddings of principal type and primitive lattice embeddings \( \mathbb{Z}(t_n t_1^{-1}) \rightarrow (\mathcal{O}, N|_{\mathcal{O}}) \).

**Proposition 5.4.21.** Let \( A = \text{diag}(a_1, \ldots, a_n) \) be an \( \mathcal{O} \)-model for \( T \). Then the following assertions are equivalent:

i) \( M_A \) is a maximal embedding,

ii) \( a_k a_1^{-1} \) is primitive for \( k = 1, \ldots, n \),

iii) \( a_n a_1^{-1} \) is primitive.

**Corollary 5.4.22.** Let \( \mathcal{O} \) be a principal ideal domain and \( t_n t_1^{-1} \) squarefree. Then every embedding \( M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \) is maximal.

In order to study the effect of the maximal discrete extension \( \Gamma(T)^{\text{max}} \) on the set of modular embeddings of \( \Gamma(T) \) we obtain the following

**Theorem 5.4.23.** The following assertions hold:

a) \( \Gamma(T)^{\text{max}} \) acts on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \) by multiplication from the left.

b) The action defined in a) respects the equivalence relation on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \) and hence induces an action of \( \Gamma(T)^{\text{max}} \) on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O}))/\sim \).

c) \( \Gamma(T) \) lies in the kernel of the action defined in b) and this induces an action of \( \Gamma(T)^{\text{max}}/\Gamma(T) \) on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O}))/\sim \).

**Proof.** Let \( M \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \).

a) Let \( H \in \Gamma(T)^{\text{max}} = N_{\text{Sp}_n(\mathbb{R})}(\Gamma(T)) \). Then one has

\[
(HM)^{-1}\Gamma(T)HM = M^{-1}H^{-1}\Gamma(T)HM = M^{-1}\Gamma(T)M \subseteq U_n(\mathcal{O}),
\]

i.e. \( HM \in \text{Mod}(\Gamma(T), U_n(\mathcal{O})) \).

b) \( M' = \epsilon MR \) for some \( (\epsilon, R) \in F_1 \times U_n(\mathcal{O}) \) implies \( HM' = \epsilon(HM)R \), since \( \epsilon \) and \( H \) commute.

c) For \( H \in \Gamma(T) \) one has \( R := M^{-1}HM \in U_n(\mathcal{O}) \) by assumption on \( M \). Hence, trivially \( HM = MR \) and thus \( HM \sim M \).

**Corollary 5.4.24.** Let \( t_1 = 1 \). If \( \mathcal{O} \) is a principal ideal domain, then the action of \( \Gamma(T)^{\text{max}}/\Gamma(T) \) on \( \text{Mod}(\Gamma(T), U_n(\mathcal{O}))/\sim \) induces an action of \( \Gamma(T)^{\text{max}}/\Gamma(T) \) on the set of equivalence classes of \( \mathcal{O} \)-models of \( T \).

**Proof.** Let \( H \in \Gamma(T)^{\text{max}}/\Gamma(T) \) and \( A \) be an \( \mathcal{O} \)-model for \( T \). By 5.4.13, we find some \( \mathcal{O} \)-model \( B \) of \( T \) such that \( HM_A \sim M_B \). Since \( H \) respects the equivalence relation, this induces an action on the equivalence classes of \( \mathcal{O} \)-models of \( T \).

We will determine this action in section 6.1 explicitly.
5.5 Pullback theory for modular forms

We develop a pullback theory for modular forms with respect to modular embeddings of paramodular groups.

First we briefly define the basic concepts and terms in the theory of hermitian and quaternionic modular forms. For a general reference confer [14].

**Definition 5.5.1.** The half-space of degree $n$ over $F$ is the set
\[ \mathcal{H}_n(F) = \left\{ X + iY \in F^{n\times n} \otimes_R C : X = X^t, Y = Y^t > 0 \right\}. \]

The set $\mathcal{H}_n(C)$ is called the hermitian half-space and $\mathcal{H}_n(H)$ is called the quaternionic half-space.

Clearly, $\mathcal{H}_n(R) = \mathcal{H}_n$ and $\mathcal{H}_1(F) = \mathcal{H}$.

**Definition 5.5.2.** Let $k \in \mathbb{Z}$ and $M = (\begin{array}{cc} A & B \\ C & D \end{array}) \in U_n(F)$, where we assume $k \equiv 0 \pmod{2}$ if $F = H$.

a) For $Z \in \mathcal{H}_n(F)$ we define
\[ (\det M(Z))^k := \begin{cases} \det(CZ + D)^k, & F = \{R, C\}, \\ (\det(CZ + D)^\vee)^{k/2}, & F = H, \end{cases} \]

where $\vee$ denotes the embedding $H^{n\times n} \rightarrow C^{2n\times 2n}$ as in [14, I., § 2, p.14]. The map
\[ (M, Z) \mapsto (\det M(Z))^k, \quad M \in U_n(F), Z \in \mathcal{H}_n(F) \]

defines a factor of automorphy for $U_n(F)$, cf. [14, II. Theorem 1.7 and p.77f].

b) For $Z \in \mathcal{H}_n(F)$ we define $M(Z) := (AZ + B)(CZ + D)^{-1}$.

c) For a holomorphic function $F : \mathcal{H}_n(F) \rightarrow \mathbb{C}$ we define $F|_k M$ pointwisely by
\[ F|_k M(Z) := (\det M(Z))^{-k} \cdot F(M(Z)), \quad Z \in \mathcal{H}_n(F). \]

Again, we call $|_k$ the slash operator of weight $k$.

We introduce modular forms:

**Definition 5.5.3.** Let $k \in \mathbb{Z}$, where $k \equiv 0 \pmod{2}$ in the case $F = H$. Let $\Gamma \leq U_n(O)$ of finite index and $\nu$ an abelian character of $\Gamma$ of finite order. A function $F : \mathcal{H}_n(F) \rightarrow \mathbb{C}$ is called a modular form of weight $k$ and degree $n$ with respect to $\Gamma$ and $\nu$, if the following conditions are satisfied:

i) $F$ is holomorphic,

ii) $F|_k M = \nu(M)F$ for all $M \in \Gamma$,.
iii) in the case $n = 1$, the functions $F|_k M, M \in U_1(\mathcal{O})$, are bounded on each set
\[ \{z \in \mathcal{H} : y \geq \delta\}, \quad \delta > 0. \]

In this case, $F$ is called a Siegel modular form if $\mathbb{F} = \mathbb{R}$, a hermitian modular form if $\mathbb{F} = \mathbb{C}$ and a quaternionic modular form if $\mathbb{F} = \mathbb{H}$. The space of modular forms of weight $k$ and degree $n$ with respect to $\Gamma$ and $\nu$ is denoted by $[\Gamma, k, \nu]$.

For $S, Z \in \text{Her}_n(\mathbb{F}) \otimes_{\mathbb{R}} \mathbb{C}$ we define $\text{tr}(SZ) := \frac{1}{2} \text{trace}(SZ + ZS) \in \mathbb{C}$, where in this case, trace denotes the usual matrix trace. Let $\text{Her}_n(\mathcal{O})^\#$ denote the dual lattice of $\text{Her}_n(\mathcal{O})$ with respect to the trace form, i.e.
\[ \text{Her}_n(\mathcal{O})^\# = \{ M \in \text{Her}_n(\mathbb{F}) : \text{tr}(SM) \in \mathbb{Z} \text{ for all } S \in \text{Her}_n(\mathcal{O}) \}. \]

From [14, III., §1, Thm. (1.2)] we cite the following

**Theorem 5.5.4.** Let $k \in \mathbb{Z}$, where $k \equiv 0 \mod 2$ in the case $\mathbb{F} = \mathbb{H}$. Then every modular form $F \in [U_n(\mathcal{O}), k]$ has a Fourier expansion of the form
\[ F(Z) = \sum_{0 \leq S \in \text{Her}_n(\mathcal{O})^\#} \alpha_F(S) e^{2\pi i \text{tr}(SZ)}, \quad Z \in \mathcal{H}_n(\mathbb{F}). \]

The Fourier series converges absolutely and uniformly in every domain
\[ \{Z \in \mathcal{H}_n(\mathbb{F}) : Y \geq \delta I_n\}, \quad \delta > 0. \]

We introduce paramodular forms:

**Definition 5.5.5.** Let $k \in \mathbb{Z}, \Gamma \leq \Gamma(T)^{\text{max}}$ of finite index and $\nu$ an abelian character with respect to $\Gamma$ of finite order. A function $F : \mathcal{H}_n \longrightarrow \mathbb{C}$ is called a paramodular form of weight $k$ and polarization $T$ with respect to $\Gamma$ and $\nu$, if the following conditions are satisfied:

i) $F$ is holomorphic,

ii) $F|_k M = \nu(M)f$ for all $M \in \Gamma$,

iii) in the case $n = 1$, the functions $F|_k M, M \in \Gamma(T)^{\text{max}}$, are bounded on each set
\[ \{z \in \mathcal{H} : y \geq \delta\}, \quad \delta > 0. \]

The space of paramodular forms of weight $k$ with respect to $\Gamma$ and $\nu$ is denoted by $[\Gamma, k, \nu]$.

Modular embeddings transform modular forms into paramodular forms:
Proposition 5.5.6. a) Let $F \in [U_n(O), k, \nu]$. Then the map

$$\text{Mod}(\Gamma(T), U_n(O)) \rightarrow [\Gamma(T), k, \nu \circ \Phi_M], \quad M \mapsto \left| F|_k M^{-1} \right|_{H_n}$$

is well-defined.

b) Let $M \in \text{Mod}(\Gamma(T), U_n(O))$. Then the map

$$[U_n(O), k, \nu] \rightarrow [\Gamma(T), k, \nu \circ \Phi_M], \quad F \mapsto \left| F|_k M^{-1} \right|_{H_n}$$

is a homomorphism of the vector spaces.

Proof. This follows from the identity

$$\left| F|_k M^{-1} \right|_{k} H = \left| F|_k \Phi_M(H) \right|_{k} M^{-1} = (\nu \circ \Phi_M)(H) \cdot F|_k M^{-1}$$

for $H \in \Gamma(T)$ and $M \in \text{Mod}(\Gamma(T), U_n(O))$. \qed

In view of the equivalence relation on $\text{Mod}(\Gamma(T), U_n(O))$, we motivate our definition of a pullback theory for modular forms with respect to modular embeddings in the following

Remark 5.5.7. Let $M \in \text{Mod}(\Gamma(T), U_n(O))$ and $(\varepsilon, R) \in F_1 \times U_n(O)$. Then for every modular form $F \in [U_n(O), k]$ one has

$$\left| F|_k (\varepsilon M R)^{-1} \right|_{H_n} = \begin{cases} \varepsilon^{nk} \cdot \left| F|_k M^{-1} \right|_{H_n}, & F \in \{\mathbb{R}, \mathbb{C}\}, \\ \left| F|_k M^{-1} \right|_{H_n}, & F = \mathbb{H}. \end{cases}$$

Hence, except the case $F = \mathbb{H}$, the map

$$M \mapsto \left| F|_k M^{-1} \right|_{H_n}$$

is not compatible with the equivalence relation on $\text{Mod}(\Gamma(T), U_n(O))$.

Definition 5.5.8. Let $M \in \text{Mod}(\Gamma(T), U_n(O))$ and $k \equiv 0 \mod 2$. For a holomorphic function $F : H_n(\mathbb{F}) \rightarrow \mathbb{C}$, the pullback $F|_k[M] : H_n \rightarrow \mathbb{C}$ of $F$ with respect to $M$ and weight $k$ is defined pointwisely by

$$F|_k[M](Z) := \begin{cases} N(\det M^{-1}\{Z\})^{-k/2} \cdot F(M^{-1}\{Z\}), & F \in \{\mathbb{R}, \mathbb{C}\}, \\ F|_k M^{-1}(Z), & F = \mathbb{H}, \end{cases}$$

for $Z \in H_n$. Here, the norm form $N$ is extended to $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity.
Definition 5.5.9. We define \( w_{\mathcal{O}} := |\mathcal{O}(\mathbb{F}) \cap \mathcal{O}^\times| \), i.e.
\[
\begin{cases}
|\mathcal{O}^\times|, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\
2, & \mathbb{F} = \mathbb{H}.
\end{cases}
\]

Lemma 5.5.10. Let \( \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\} \) and \( k \equiv 0 \mod 2 \). Assume that \( U_n(\mathcal{O}) \) is standardly generated. Then the two factors of automorphy
\[
(M, Z) \mapsto N(\det M\{Z\})^{k/2}, \quad (M, Z) \mapsto (\det M\{Z\})^k
\]
coincide on \( U_n(\mathcal{O}) \times \mathcal{H}_n \).

Proof. Due to the multiplicativity of the norm form \( N \), the map
\[
(M, Z) \mapsto N(\det M\{Z\})^{k/2}
\]
satisfies the cocycle condition. Hence it suffices to prove that for fixed \( Z \in \mathcal{H}_n \) both maps coincide for matrices of the type
\[
J_n, \quad \left( \begin{array}{cc} I_n & S \\ 0 & I_n \end{array} \right), S \in \text{Her}_n(\mathcal{O}), \quad \left( \begin{array}{cc} U^t & 0 \\ 0 & U^{-1} \end{array} \right), U \in \text{GL}_n(\mathcal{O}).
\]
For the translations this is obvious and for the inversion \( J_n \) it follows from the identity
\[
N(\det Z)^{k/2} = ((\det Z)^2)^{k/2} = (\det Z)^k.
\]
For the rotations, we obtain
\[
N \left( \det U^{-1} \right)^{k/2} = N(\det U)^{-k/2} = 1 = (\det U)^{-k} = \left( \det U^{-1} \right)^k,
\]
since \( \det U \in \mathcal{O}^\times = \{ u \in \mathcal{O} : N(u) = 1 \} \) and \( k \) is divisible by \( |\mathcal{O}^\times| \).

Now we are in charge to prove the following

Theorem 5.5.11. Let \( \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \), where in the case \( \mathbb{F} = \mathbb{C} \) the group \( U_n(\mathcal{O}) \) is assumed to be standardly generated. Let \( k \equiv 0 \mod w_{\mathcal{O}} \). Then the following assertions hold:

a) Let \( F \in [U_n(\mathcal{O}), k,v] \). Then the map
\[
\text{Mod}(\Gamma(T), U_n(\mathcal{O})) \longrightarrow [\Gamma(T), k,v \circ \Phi_M], \quad M \mapsto F|_k[M],
\]
is well-defined.

b) Let \( F \in [U_n(\mathcal{O}), k] \). Then \( M \sim N \) implies \( F|_k[M] = F|_k[N] \). In other words, the map
\[
\text{Mod}(\Gamma(T), U_n(\mathcal{O}))/_\sim \longrightarrow [\Gamma(T), k], \quad [M]_\sim \mapsto F|_k[M],
\]
is well-defined.
Proof. There is nothing to prove in the case $F = \mathbb{H}$.

a) Let $H \in \Gamma(T), R := M^{-1}HM \in U_n(O)$ and $Z \in \mathcal{H}_n$. From 5.5.10 we obtain

$$F|_k[M]|_k H(Z) = det H\{Z\}^{-k} \cdot F|_k[M](H\{Z\})$$

$$= det H\{Z\}^{-k} \cdot N(det M^{-1}\{H\{Z\}\})^{-k/2} \cdot F(M^{-1}H\{Z\})$$

$$= N(det M^{-1}\{H\{Z\}\})^{-k/2} \cdot N(det M^{-1}\{H\{Z\}\})^{-k/2} \cdot F(M^{-1}H\{Z\})$$

$$= N(det M^{-1}H\{Z\})^{-k/2} \cdot F(M^{-1}H\{Z\})$$

$$= N(det R\{M^{-1}\{Z\}\})^{-k/2} \cdot N(det M^{-1}\{Z\})^{-k/2} \cdot F(RM^{-1}\{Z\})$$

$$= det R\{M^{-1}\{Z\}\}^{-k} \cdot N(det M^{-1}\{Z\})^{-k/2} \cdot F(RM^{-1}\{Z\})$$

$$= (F|_k[R]|_k[M](Z)$$

$$= (\nu \circ \Phi_M)(H) \cdot F|_k[M].$$

b) Let $(\epsilon, R) \in F_1 \times U_n(O)$ and $Z \in \mathcal{H}_n$. From 5.5.10 and $\epsilon^{-1}\{Z\} = Z$ we obtain

$$F|_k[\epsilon MR](Z) = N(det R^{-1}M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\epsilon^{-1}\{Z\})$$

$$= N(det R^{-1}\{M^{-1}\epsilon^{-1}\{Z\}\} \cdot det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\epsilon^{-1}\{Z\})$$

$$= N(det R^{-1}\{M^{-1}\{Z\}\} \cdot det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\{Z\})$$

$$= N(det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot (N(det R^{-1}\{M^{-1}\{Z\}\})^{-k/2} \cdot F(R^{-1}M^{-1}\{Z\}))$$

$$= N(det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot (N(det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\{Z\}))$$

$$= N(\epsilon^{-1}\{Z\})^{-k/2} \cdot N(det M^{-1}\epsilon^{-1}\{Z\})^{-k/2} \cdot F|_k R(M^{-1}\{Z\})$$

$$= N(\epsilon^{-1}\{Z\})^{-k/2} \cdot F|_k [M]$$

$$= F|_k[M].$$

In view of the construction of paramodular forms with respect to the maximal discrete extension of $\Gamma(T)$ we obtain the following corollary by a simple averaging argument:

**Corollary 5.5.12.** Let $O$ be a principal domain and $k \equiv 0 \mod w_O$. Then one has

$$\sum_{\sim \in Mod(\Gamma(T), U_n(O))/\sim} F|_k[M] \in [\Gamma(T)_{\text{max}}, k].$$

for all $F \in [U_n(O), k]$.

*Proof.* Follows directly from 5.5.11, 5.4.23, 5.4.14. 

**Example 5.5.13.** Let $A$ be an $O$-model for $T$. Then one has $M_A^{-1}\{Z\} = AZ \overline{A} = Z[\overline{A}]$ for $Z \in \mathcal{H}_n$. Evaluation of the corresponding factor of automorphy yields in both cases

$$(\det T)^{k/2} = \begin{cases} N(det M_A^{-1}\{Z\})^{-k/2}, & \text{if } F \in \{\mathbb{R}, \mathbb{C}\}, \\ (det M_A^{-1}\{Z\})^{-k}, & \text{if } F = \mathbb{H}. \end{cases}$$
Thus for a holomorphic function $F : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathbb{C}$ and $k \in 2\mathbb{Z}$ we obtain

$$F|_k[M_A](Z) = (\det T)^{k/2} \cdot F(AZ\overline{A}), \quad Z \in \mathcal{H}_n.$$ 

**Definition 5.5.14.** Let $A$ be an $\mathcal{O}$-model for $T$. For a holomorphic function $F : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathbb{C}$ we define the pullback $F[A] : \mathcal{H}_n \rightarrow \mathbb{C}$ with respect to $A$ pointwisely by

$$F[A](Z) := F(AZ\overline{A}) = F(Z\overline{A}), \quad Z \in \mathcal{H}_n.$$
6 Separation Theorems for Modular Embeddings of Degree 2

In this chapter we will restrict to the degree two case only and assume that the elementary divisor matrix $T$ is given by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad t \in \mathbb{N}. $$

Up to equivalence every $\mathcal{O}$-model $A$ of $T$ has the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \in \mathcal{O}, N(a) = t. $$

By a slight abuse of notation, we define

$$M_a := M(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \overline{a}^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \in \text{PMod}(\Gamma(t), U_2(\mathcal{O})).$$

**Definition 6.0.15.** Let $a \in \mathcal{O}$. For a holomorphic function $F : \mathcal{H}_2(\mathbb{F}) \rightarrow \mathbb{C}$ we define the function $F[a] : \mathcal{H}_2 \rightarrow \mathbb{C}$ pointwisely by

$$F[a](Z) := F \left[ \left( \begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix} \right) \right] (Z) = F\left( \frac{Z}{\overline{a}w}, \frac{aw}{N(a)z'} \right), \quad Z = \left( \begin{smallmatrix} z \\ w \\ w \end{smallmatrix} \right) \in \mathcal{H}_2. $$

We call $F[a]$ the pullback of $F$ with respect to $a$.

6.1 Modular embeddings and maximal discrete extensions

In 5.4.23 and 5.4.24 we have seen that $\Gamma(T)^{\text{max}}/\Gamma(T)$ acts naturally on the set of equivalence classes $\text{Mod}(\Gamma(T), U_n(\mathcal{O}))/\sim$, which induces an action of $\Gamma(T)^{\text{max}}/\Gamma(T)$ on the set of equivalence classes of $\mathcal{O}$-models for $T$, if we assume $\mathcal{O}$ to be a principal ideal domain. In this section, we will determine an explicit description of this action in the case $n = 2$. Surprisingly, the result is deeply connected to factorization theory in the order $\mathcal{O}$.

To proceed, we briefly develop factorization theory in non-necessary commutative orders.
Definition 6.1.1. We say that \( \mathcal{O} \) has the exact factorization lifting property, if for every \( a \in \mathcal{O} \) and every exact divisor \( d \mid |N(a)\), there are \( \alpha, \beta \in \mathcal{O} \) such that \( N(\alpha) = d \) and \( a = \alpha \beta \).

The existence of such orders is shown in the following

Proposition 6.1.2. Let \( \mathcal{O} \) be a principal ideal domain. Then \( \mathcal{O} \) has the exact factorization lifting property.

Proof. For \( a \in \mathcal{O} \) consider the right ideal \( a\mathcal{O} + d\mathcal{O} \). Since \( \mathcal{O} \) is a principal ideal domain, there is \( \alpha \in \mathcal{O} \) such that \( \alpha \mathcal{O} = a\mathcal{O} + d\mathcal{O} \). Let \( \beta, \gamma \in \mathcal{O} \) such that \( a = \alpha \beta \) and \( d = \alpha \gamma \). Then

\[
N(\alpha) | d^2, \quad N(\alpha) | N(a) = d \cdot \frac{t}{d}, \quad \gcd\left( d, \frac{t}{d}\right) = 1
\]

imply \( N(\alpha) | d \). Conversely,

\[
\alpha = ax + dy \quad \text{for some } x, y \in \mathcal{O},
\]

\[
N(\alpha) = N(a)N(x) + d \cdot \text{tr}(\overline{ax}y) + d^2N(y)
\]

imply \( d | N(\alpha) \) and \( N(\alpha) > 0 \) yields \( N(\alpha) = d \). Then \( N(\beta) = \frac{t}{d} \) follows necessarily.

Example 6.1.3. \( \Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})} \) has the exact factorization property.

Proof. Let \( \Lambda := \mathcal{O}_{\mathbb{Q}(\sqrt{-1})} \). Let \( \mathcal{O} \) denote the Hurwitz order and \( \wp \) the ideal of even quaternions. Let \( a \in \Lambda \). Since \( \mathcal{O} \) is a principal ideal domain, it suffices to show, that every factorization \( a = \alpha \beta \) with \( \alpha, \beta \in \mathcal{O} \) can be migrated to a factorization in \( \Lambda \). In the case \( a \in \wp \) we can assume \( \alpha \in \wp \), since \( \wp \) is a prime ideal. Furthermore, there is \( \varepsilon \in \{1, \omega, \overline{\omega}\} \) such that \( \varepsilon \beta \in \Lambda \). The claim follows then from \( \alpha \overline{\varepsilon} \in \wp \subseteq \Lambda \). In the remaining case \( a \notin \wp \), we have \( a \equiv 1 \mod \wp \), since \( a \in \Lambda \). This implies \( \alpha \equiv \overline{\beta} \mod \wp \) and hence that there is some \( \varepsilon \in \{1, \omega, \overline{\omega}\} \) such that \( \alpha \varepsilon, \overline{\varepsilon} \beta \in \Lambda \).

In view of uniqueness, we give the following

Definition 6.1.4. Let \( \alpha, \beta \in \mathcal{O} \). The product \( \alpha \beta \) is called unique up to unit-migration, if

\[
\alpha \beta = \alpha' \beta', \quad N(\alpha) = N(\alpha'), \quad N(\beta) = N(\beta'),
\]

for some \( \alpha', \beta' \in \mathcal{O} \) implies \( \alpha' = \alpha \varepsilon \) and \( \beta' = \overline{\varepsilon} \beta \) for some unit \( \varepsilon \in \mathcal{O}^\times \).

Lemma 6.1.5. Let \( \alpha, \beta \in \mathcal{O} \) such that \( \gcd(N(\alpha), N(\beta)) = 1 \). Then \( \alpha \beta \) is unique up to unit-migration.
Proof. First we prove that for \( \alpha, \beta \in \mathcal{O} \) and \( m \in \mathbb{N} \), the conditions
\[
m | \alpha \beta, \quad \gcd(m, N(\alpha)) = 1,
\]
already imply \( m | \beta \). But this follows from
\[
\begin{align*}
rN(\alpha) + sm &= r\alpha + sm = 1 \quad \text{for some } r, s \in \mathbb{Z}, \\
r\alpha \beta + sm\beta &= \beta.
\end{align*}
\]
For the rest of the proof assume that
\[
\alpha \beta = \alpha_1 \beta_1, \quad N(\alpha_1) = N(\alpha), \quad N(\beta_1) = N(\beta),
\]
for some \( \alpha_1, \beta_1 \in \mathcal{O} \). Then the assertions
\[
\alpha \beta = \alpha_1 \beta_1, \quad \alpha N(\beta) = \alpha \beta \beta = \alpha_1 \beta_1 \beta
\]
imply \( N(\beta) | \alpha_1 \beta_1 \beta \). From \( \gcd(N(\beta), N(\alpha_1)) = 1 \) we obtain \( N(\beta) | \beta_1 \beta \) and hence the existence of \( \varepsilon \in \mathcal{O} \) such that
\[
\varepsilon N(\beta) = \varepsilon \beta \beta = \beta_1 \beta, \quad \varepsilon \beta = \beta_1.
\]
From \( N(\beta) = N(\beta_1) \) we obtain \( N(\varepsilon) = 1 \), i.e. \( \varepsilon \in \mathcal{O}^\times \) and \( \alpha_1 = \alpha \varepsilon \).

Lemma 6.1.6. Let \( \alpha, \alpha_1, \beta \in \mathcal{O} \) and \( N(\alpha) = N(\alpha_1) \). Then one has \( \alpha_1 | \alpha \beta \) if and only if \( \alpha | \alpha_1 \beta \).

Proof. Let \( \gamma \in \mathcal{O} \). Then one has \( \alpha \beta = \gamma \alpha_1 \) if and only if \( \alpha = \gamma \alpha_1 \beta^{-1} \) if and only if \( \gamma \alpha = N(\gamma) \alpha_1 \beta^{-1} = \alpha_1 \beta \), since \( N(\gamma) = N(\beta) = \beta \beta \).

From [10] we cite

Theorem 6.1.7. Let \( t \in \mathbb{N} \) with prime factor decomposition \( t = \prod_{p | t} p^{v_p(t)} \). For every exact divisor \( d | t \) let \( t_d := \frac{t}{d} \) denote its complementary divisor. Choose \( x, y \in \mathbb{Z} \) such that \( xd - yt_d = 1 \) and define matrices \( V_d \in \text{Sp}_2(\mathbb{R}) \) by
\[
V_d := \begin{pmatrix} U_d & 0 \\ 0 & U_d^{-t} \end{pmatrix}, \quad U_d := \frac{1}{\sqrt{d}} \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]
Then \( \Gamma(t)^* := \langle \Gamma(t), V_d, d | t \rangle \leq \text{Sp}_2(\mathbb{R}) \) is a discrete extension of \( \Gamma(t) \) of index \( 2^{v(t)} \), where \( v(t) \) denotes the number of distinct prime divisors of \( t \). If \( t \) is squarefree, one has
\[
\Gamma(t)^\max = \langle \Gamma(t), V_d, d | t \rangle.
\]

Now we are in charge to describe to action of \( V_d, d | t \) explicitly:
**Theorem 6.1.8.** Let \( \alpha, \beta \in \mathcal{O}, t := N(\alpha \beta) \) and \( d := N(\alpha) \). Suppose that \( d || t \) and that there are \( \alpha', \beta' \in \mathcal{O} \) such that

\[
\alpha \beta = \beta' \alpha', \quad N(\alpha) = N(\alpha'), \quad N(\beta) = N(\beta').
\]

Then one has

\[
V_d \cdot [M_{\alpha\beta}]_{\sim} = [M_{\pi\beta'}]_{\sim}.
\]

Furthermore, the following special cases hold:

1. \( V_d \cdot [M_{\alpha\beta}]_{\sim} = [M_{\overline{\beta}}]_{\sim} \),
2. \( V_d \cdot [M_{\alpha\beta}]_{\sim} = [M_{\pi\beta}]_{\sim} \), if \( \beta^{-1} \alpha \beta \in \mathcal{O} \),
3. \( V_d \cdot [M_{\alpha\beta}]_{\sim} = [M_{\overline{\beta}}]_{\sim} \), if \( \alpha \beta \alpha^{-1} \in \mathcal{O} \),
4. if \( \beta^{-1} \alpha \beta \in \mathcal{O} \) and \( \beta^{-1} \mathcal{O} \times \beta = \mathcal{O} \times \), then one has \( V_d \cdot [M_{\alpha\beta}]_{\sim} = [M_{\alpha\beta}]_{\sim} \), if and only if \( \alpha \) and \( \overline{\alpha} \) are associated.

**Proof.** Let

\[
U := \begin{pmatrix} x \overline{\alpha} & -\beta \\ \overline{\alpha} & \beta \overline{\alpha}^{-1} \end{pmatrix}, \quad V := \begin{pmatrix} \alpha & \beta' \\ \frac{y}{\sqrt{d}} & \frac{y}{\sqrt{d}} \overline{\alpha} \beta^{-1} \overline{\beta} \end{pmatrix}.
\]

From \( \beta' | \alpha \beta \) we obtain \( \overline{\beta'} | \overline{\alpha} \overline{\beta} \) and 6.1.6 implies \( \overline{\beta'} | \overline{\beta} \alpha \). Hence, \( U, V \in \mathcal{O}^{2 \times 2} \). The assumptions \( xd - y^2 = 1, N(\alpha) = N(\alpha') \) and \( N(\beta) = N(\beta') \) imply

\[
UV = VU = I_2
\]

by an explicit calculation. Hence, \( U, V \in \text{GL}_2(\mathcal{O}) \) and \( U^{-1} = V \). We define \( \varepsilon := \pi / \sqrt{d} \). Then \( N(\varepsilon) = 1 \) and from the four identities

- \( \sqrt{d} \alpha = \sqrt{d} \alpha / \sqrt{d} = x \overline{\alpha} \)
- \( -\frac{t}{\sqrt{d}} \varepsilon \overline{\alpha} \overline{\beta}^{-1} = -\frac{t}{\sqrt{d}} \frac{x}{\overline{\alpha}} \overline{\alpha}^{-1} \overline{\beta}^{-1} = -\frac{t}{\sqrt{d}} \overline{\beta}^{-1} = -\beta \), since \( N(\beta) = \frac{t}{\sqrt{d}} \)
- \( -\frac{y}{\sqrt{d}} \overline{\alpha} \overline{\beta} \varepsilon = -\frac{y}{\sqrt{d}} (\overline{\beta} \alpha) \frac{x}{\overline{d}} = -y \overline{\beta} \frac{N(\alpha)}{d} = -y \overline{\beta}, \) since \( N(\alpha) = d \)
- \( \sqrt{d} (\overline{\alpha} \overline{\beta}) \varepsilon (\overline{\alpha} \overline{\beta})^{-1} = \sqrt{d} (\overline{\beta} \alpha) \frac{x}{\sqrt{d}} (\overline{\alpha}^{-1} \overline{\beta}^{-1}) = (\overline{\beta} \alpha) \overline{\alpha} \overline{\beta}^{-1} \)

we derive that

\[
\begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha} \overline{\beta} \end{pmatrix} \varepsilon U_d \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha} \overline{\beta} \end{pmatrix} = \begin{pmatrix} x \overline{\alpha} & -\beta \\ \overline{\alpha} & \beta \overline{\alpha}^{-1} \end{pmatrix} = U \in \text{GL}_2(\mathcal{O}),
\]

i.e.

\[
\varepsilon U_d \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha} \overline{\beta} \end{pmatrix} U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha} \overline{\beta}^{-1} \end{pmatrix}.
\]

From this we obtain

\[
\varepsilon V_d M_{\alpha\beta} \begin{pmatrix} U^{-1} & 0 \\ 0 & U' \end{pmatrix} = M_{\pi\beta'}, \quad \begin{pmatrix} U^{-1} & 0 \\ 0 & U' \end{pmatrix} \in U_2(\mathcal{O}), \varepsilon \in \mathbb{F}_1.
\]
Hence, $V_dM_{a\beta}$ is equivalent to $M_{\pi \beta'}$, i.e. $V_d \cdot [M_{a\beta}]_\sim = [M_{\pi \beta'}]_\sim$. For i) we choose $\beta' = \beta = 1$. For the proof of ii) write $a\beta = \beta(\beta^{-1}a\beta)$, i.e. we can choose $\beta' = \beta$. For the proof of iii) write $a\beta = (\alpha \beta \alpha^{-1})a$, i.e. we can choose $\beta' = a\beta \alpha^{-1}$. Note that $\pi a\beta \alpha^{-1} = \beta \pi$. Finally, to prove iv) note that in view of ii) one has $[M_{a\beta}]_\sim = [M_{\pi \beta}]_\sim$ if and only if $\varepsilon a\beta \delta = \pi \beta$, i.e. $\varepsilon a\beta \delta \beta^{-1} = \pi$ for certain units $\delta, \varepsilon \in \mathcal{O}^\times$. The claim follows then from $\beta \mathcal{O}^\times \beta^{-1} = \mathcal{O}^\times$. □

If $\mathcal{O}$ is a principal ideal domain, then action of $\Gamma(t)^\max / \Gamma(t)$ can be described as follows:

**Corollary 6.1.9.** Let $\mathcal{O}$ be a principal ideal domain. Let $t \in \mathbb{N}$ squarefree and $M \in \text{Mod}(\Gamma(t), U_2(\mathcal{O}))$. According to 5.4.13 let $a \in \mathcal{O}, N(a) = t$ such that $M \sim M_a$. Let $d || t$ and $\alpha, \alpha', \beta, \beta' \in \mathcal{O}$ such that $a\beta = a = \beta' \alpha'$ and $N(\alpha) = d = N(\alpha')$. Then one has $V_d \cdot [M]_\sim = [M_{\pi \beta'}]_\sim$
and this does only depend on $d$ and not on the choice of $a, \alpha, \alpha', \beta, \beta'$.

From 6.1.8 we obtain the behaviour of pullbacks under the matrices $V_d$:

**Theorem 6.1.10.** Let $\alpha, \beta \in \mathcal{O}, t := N(\alpha \beta)$ and $d := N(\alpha)$. Suppose that $d || t$ and that there are $\alpha', \beta' \in \mathcal{O}$ such that $a\beta = \beta' \alpha'$, $N(\alpha) = N(\alpha')$, $N(\beta) = N(\beta')$.

Let $k \equiv 0 \mod w_\mathcal{O}$. Then for every $F \in [U_2(\mathcal{O}), k]$ one has $F[a\beta]_k V_d = F[\bar{\pi} \beta']$.

Furthermore, the following special cases hold:

i) $F[a\beta]_k V_t = F[\bar{\beta} \pi],$

ii) $F[a\beta]_k V_d = F[\bar{\pi} \beta]$, if $\beta^{-1} a\beta \in \mathcal{O},$

iii) $F[a\beta]_k V_d = F[\bar{\beta} \pi]$, if $a\beta \alpha^{-1} \in \mathcal{O},$

iv) if $\beta^{-1} a\beta \in \mathcal{O}, \beta^{-1} \mathcal{O}^\times \beta = \mathcal{O}^\times$ and $a, \pi$ are associated, then one has $F[a\beta]_k V_d = F[a\beta]$.

### 6.2 Equivalence in the extended sense

In this section we introduce an extension of the projective modular group $PU_2(\mathcal{O})$ by certain biholomorphic automorphisms of the half-space $\mathcal{H}_2(\mathbb{F})$, that are induced by elements of the orthogonal group of the positive definite lattice $(\mathcal{O}, N|_{\mathcal{O}})$. 
Remark 6.2.1. It is well known, that there is a monomorphism of groups

$$PU_2(F) := U_2(F)/C(F_1) \cdot I_4 \twoheadrightarrow \text{Bih} \, \mathcal{H}_2(F)$$

induced by the fractional-linear action of $U_2(F)$ on $\mathcal{H}_2(F)$. For sake of simplicity, we will identify $M \in U_2(F)$ with its image in $PU_2(F)$, whenever it is convenient. Due to 5.1.14, every $\sigma \in O(F)$ is either of the form

$$w \mapsto \sigma(w) = \epsilon w \delta, \quad \epsilon, \delta \in F_1.$$

For $\epsilon, \delta \in F_1$ set $U_{\epsilon, \delta} = \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix}$. Then for $Z = \begin{pmatrix} z & w \\ \bar{w} & z' \end{pmatrix} \in \mathcal{H}_2(F)$ one has

$$\left( \begin{pmatrix} U_{\epsilon, \delta} & 0 \\ 0 & U_{\epsilon, \delta}^{-1} \end{pmatrix} \right) \langle Z \rangle = Z[U_{\epsilon, \delta}] = \begin{pmatrix} z & \sigma(w) \\ \sigma(w) & z' \end{pmatrix}$$

in the first case, as well as

$$\left( \begin{pmatrix} U_{\epsilon, \delta} & 0 \\ 0 & U_{\epsilon, \delta}^{-1} \end{pmatrix} \circ I_{tr} \right) \langle Z \rangle = Z'[U_{\epsilon, \delta}] = \begin{pmatrix} z & \sigma(w) \\ \sigma(w) & z' \end{pmatrix}$$

in the second case. Here, $I_{tr} : Z \mapsto Z'$ denotes the exceptional automorphism of $\mathcal{H}_2(F)$ in the case $F \in \{C, H\}$. Hence we obtain a monomorphism of groups

$$O(F) \twoheadrightarrow \text{Bih} \, \mathcal{H}_2(F).$$

Thereby we will identify $O(F)$ as a subgroup of $\text{Bih} \, \mathcal{H}_2(F)$. Indeed,

$$\text{Bih} \, \mathcal{H}_2(F) \cong \begin{cases} \text{PSp}_2(R), & F = R, \\ PU_2(F) \rtimes \langle I_{tr} \rangle, & F \in \{C, H\}. \end{cases}$$

For more details confer [14, II., § 1].

It is easy to see, that the group $U_2(O)$ is normalized by the matrices $\frac{u}{\sqrt{N(u)}} I_4 \in U_2(F)$ for $u \in \mathcal{I}(O)$. This gives rise to the following

Definition 6.2.2. The special modular group $U_2^*(O)$ is defined as

$$U_2^*(O) := \left< U_2(O), \frac{u}{\sqrt{N(u)}} I_4 : u \in \mathcal{I}(O) \right> \leq U_2(F).$$

We give some trivial

Remarks 6.2.3. a) By construction, $U_2(O)$ is a normal subgroup of $U_2^*(O)$. Furthermore,

$$U_2^*(O) = U_2(O) \cdot \left\{ \frac{u}{\sqrt{N(u)}} I_4 : u \in \mathcal{I}(O) \right\}.$$
6.2 Equivalence in the extended sense

b) If \( F = \mathbb{C} \), then one has \( U_2^+(\mathcal{O}) = U_2(\mathcal{O}) \).

Definition 6.2.4. a) The projective modular group \( PU_2(\mathcal{O}) \) is defined as

\[
PU_2(\mathcal{O}) := U_2(\mathcal{O}) / (C(F_1) \cap \mathcal{O}) \cdot I_4 = \begin{cases}
U_2(\mathcal{O}) / O^\times I_4, & F \in \{ \mathbb{R}, \mathbb{C} \}, \\
U_2(\mathcal{O}) / \{ \pm I_4 \}, & F = \mathbb{H}.
\end{cases}
\]

b) The special projective modular group \( PU_2^+(\mathcal{O}) \) is defined as

\[
PU_2^+(\mathcal{O}) := U_2^+(\mathcal{O}) / (C(F_1) \cap \mathcal{O}) \cdot I_4 = \begin{cases}
U_2^+(\mathcal{O}) / O^\times I_4, & F \in \{ \mathbb{R}, \mathbb{C} \}, \\
U_2^+(\mathcal{O}) / \{ \pm I_4 \}, & F = \mathbb{H}.
\end{cases}
\]

Definition 6.2.5. Let \( \mathcal{O} \) be an order. Then the extended modular group \( \Gamma(\mathcal{O}) \) is defined as

\[
\Gamma(\mathcal{O}) := \left< PU_2^+(\mathcal{O}), I_{tr} \right> = \left< I_{tr}, Z \mapsto M(Z) : M \in U_2^+(\mathcal{O}) \right>.
\]

Remark 6.2.6. From the explicit description of \( O(\mathcal{O}) \) in 5.1.16 and the identification of \( O(\mathcal{O}) \) in Bih \( H_2(\mathbb{F}) \) according to 6.2.1 we obtain

\[
\Gamma(\mathcal{O}) = \left< PU_2(\mathcal{O}), O(\mathcal{O}) \right>.
\]

Proposition 6.2.7. Let \( \mathcal{O} \) be an order. If \( F = \mathbb{H} \) assume additionally that \( U_2(\mathcal{O}) \) is generated by the matrices

\[
I_2, \quad \left( \begin{array}{cc}
I_2 & S \\
0 & I_2
\end{array} \right), \quad S \in \text{Her}_2(\mathcal{O}), \quad \text{diag}(\epsilon, 1, \epsilon, 1), \epsilon \in O^\times.
\]

Then both \( PU_2(\mathcal{O}) \) and \( PU_2^+(\mathcal{O}) \) are normal subgroups of \( \Gamma(\mathcal{O}) \). Furthermore, one has

\[
\Gamma(\mathcal{O}) = \begin{cases}
PU_2(\mathcal{O}) \rtimes \langle I_{tr} \rangle, & F = \mathbb{C}, \\
PU_2^+(\mathcal{O}) \rtimes \langle I_{tr} \rangle, & F = \mathbb{H}.
\end{cases}
\]

Proof. It suffices to show that \( PU_2(\mathcal{O}) \) is normalized by \( I_{tr} \). In the case \( F = \mathbb{C} \) this follows from the identity \( M(Z^t)^t = \overline{M}(Z) \) for all \( M \in U_2(\mathbb{C}) \) and \( Z \in H_2(\mathbb{C}) \). In the case \( F = \mathbb{H} \) it suffices to demonstrate the properties on the generators quoted above. Trivially, \( I_{tr} \) fixes the maps \( Z \mapsto u^{-1} Zu \) for \( u \in \mathcal{I}(\mathcal{O}) \).

 Examples 6.2.8. a) Let \( \mathfrak{o} \) be an order in \( \mathbb{C} \). Then \( \Gamma(\mathfrak{o}) \) is an extension of index 2 of \( PU_2(\mathfrak{o}) \).

b) Let \( \mathcal{O} \) denote the Hurwitz order. Since \( \mathcal{O} \) is norm-euclidean, \( \Gamma(\mathcal{O}) \) is a normal extension of index 4 of \( PU_2(\mathcal{O}) \) and index 2 of \( PU_2^+(\mathcal{O}) \). Explicitly one has

\[
\Gamma(\mathcal{O}) = \left< PU_2(\mathcal{O}) \cdot \left< \frac{1 + i_1}{\sqrt{2}} \cdot I_4 \right> \right> \rtimes \langle I_{tr} \rangle.
\]
c) Let $K$ be an imaginary-quadratic number field of discriminant $-D_K$ for $D_K \in \{3, 4\}$. Then $\Gamma(\mathcal{O}_K)$ is a normal extension of $\text{PU}_2(\mathcal{O}_K)$ of index

$$[\Gamma(\mathcal{O}_K) : \text{PU}_2(\mathcal{O}_K)] = \begin{cases} 4, & D_K = 3, \\ 12, & D_K = 4. \end{cases}$$

From the explicit description in 5.1.12 we obtain

$$\Gamma(\mathcal{O}_K) = \begin{cases} (\text{PU}_2(\mathcal{O}_K) \cdot \langle i_1 I_4 \rangle) \rtimes \langle I_{\text{tr}} \rangle, & D_K = 3, \\ (\text{PU}_2(\mathcal{O}_K) \cdot \langle \omega I_4, \frac{1+i}{\sqrt{2}} I_4 \rangle) \rtimes \langle I_{\text{tr}} \rangle, & D_K = 4. \end{cases}$$

Note that it can be shown by using a weak version of the euclidean algorithm that the order $\Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ satisfies the assumptions of 6.2.7.

In order to motivate definition 6.2.10, we give the following

**Remark 6.2.9.** Let $\mathcal{O}$ be an order and assume that $\text{PU}_2(\mathcal{O})$ is normalized by the automorphism $I_{\text{tr}}$. Let $M \in \text{Mod}(\Gamma(T), U_2(\mathcal{O}))$. Then the element $M^* := I_{\text{tr}} M I_{\text{tr}}$ is determined in $U_2(\mathbb{F})$ up to some multiplicative constant $\varepsilon \in \mathcal{C}(\mathbb{F}_1)$. Hence, the matrices

$$M^*^{-1} H M^*, \quad H \in \Gamma(T) \leq \text{Sp}_2(\mathbb{R})$$

are uniquely determined in $U_2(\mathbb{F})$. Furthermore, we have

$$M^*^{-1} H M^* = I_{\text{tr}} M^{-1} I_{\text{tr}} H I_{\text{tr}} M I_{\text{tr}} = I_{\text{tr}} M^{-1} H M I_{\text{tr}} \in \text{PU}_2(\mathcal{O}),$$

since $I_{\text{tr}} H I_{\text{tr}} = H$ holds in $\text{PU}_2(\mathbb{F})$ and $M^{-1} H M \in U_2(\mathcal{O})$. Thus, $M^*^{-1} H M^*$ is uniquely determined in $U_2(\mathcal{O})$, i.e. every representative of $M^*$ is an element of $\text{Mod}(\Gamma(T), U_2(\mathcal{O}))$. In other words, $M^*$ can be considered as an element of $\text{Mod}(\Gamma(T), U_2(\mathcal{O}))/\sim$.

6.2.9 gives rise to the following

**Definition 6.2.10.** Let $\mathcal{O}$ be an order and assume that $\text{PU}_2(\mathcal{O})$ is normalized by $I_{\text{tr}}$. Let $M, N \in \text{Mod}(\Gamma(T), U_2(\mathcal{O}))$. We call $M$ and $N$ equivalent in the extended sense, written $M \sim_* N$, if there is $(\varepsilon, R) \in \mathbb{F}_1 \times U_2^*(\mathcal{O})$ such that $N = \varepsilon M R$ or $N = \varepsilon M^* R$. Note that in the second case, the condition is independent from the choice of the representative of $M^*$ modulo $\mathcal{C}(\mathbb{F}_1)$. The equivalence class of $M$ is denoted by $[M]_{\sim_*}$ and the set of equivalence classes by $\text{Mod}(\Gamma(T), U_2(\mathcal{O}))/\sim_*$.

We characterize equivalence in the extended sense between embeddings of principal type:

**Lemma 6.2.11.** Let $a \in \mathcal{O}$ and $u \in \mathcal{I}(\mathcal{O})$. Then the following assertions hold:

a) $M_{a} \sim_* M_a$,

b) $M_{a^{-1} au} \sim_* M_a$.
**Proof.** a) Let \( t = N(a) \). For arbitrary \( Z = \left( \begin{array}{cc} z & w \\ w & z' \end{array} \right) \in \mathcal{H}_2(\mathbb{F}) \) we have

\[
I_{\text{tr}}M_a I_{\text{tr}} \langle Z \rangle = I_{\text{tr}} \left( \begin{array}{cc} z & wa^{-1} \\ w & z'/t \end{array} \right) = \left( \begin{array}{cc} z & a^{-1}w \\ w & z'/t \end{array} \right).
\]

Hence,

\[
\sqrt{t} M_a^* I_{\text{tr}} \langle Z \rangle = \sqrt{t} M_a \left( \begin{array}{cc} z & wa^{-1} \\ w & z'/t \end{array} \right) = M_a \langle Z \rangle.
\]

b) Follows from \( M_{u^{-1}au} = \frac{u}{\sqrt{N(u)}} M_a \left( \frac{u}{\sqrt{N(u)}} I_4 \right) \).

Now we can characterize equivalence in the extended sense of principal embeddings:

**Theorem 6.2.12.** Let \( O \) be an order and assume that \( \text{PU}_2(O) \) is normalized by \( I_{\text{tr}} \). Let \( a, b \in O \) such that \( N(a) = N(b) \). Then the following assertions are equivalent:

i) \( M_a \sim_* M_b \),

ii) \( \sigma(a) = b \) for some \( \sigma \in O(O) \).

**Proof.** The assertion \( \sigma(a) = b \) for some \( \sigma \in O(O) \) is equivalent to the existence of \( \delta \in O^\times \) and \( u \in I(O) \) such that \( a = u^{-1}\delta bu \) or \( a = u^{-1}\delta bu \). The first case is equivalent to \( M_{uau^{-1}} \sim M_b \) in the usual sense, the second equivalent to \( M_{uau^{-1}} \sim M_b \) in the usual sense. Then the claim follows from 6.2.11.

Concerning embeddability of the Fricke-involution \( V_t \), we give the following

**Remark 6.2.13.** Let \( u \in O \) such that \( N(u) = t \) and \( u^2 \in N(u) \cdot O \). Then one has

\[
M^{-1}_u V_t M_u = \frac{u}{\sqrt{t}} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta \end{array} \right) \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right) \in U^+_2(O), \quad \delta := u^{-1}u \in O^\times.
\]

Consequently,

\[
M^{-1}_u \langle \Gamma(t), V_t \rangle M_u \subseteq U^+_2(O).
\]

**Examples 6.2.14.** a) Let \( O \) denote the Hurwitz order. Then one has

\[
M^{-1}_{i+1} \cdot \Gamma(2)^{\max} \cdot M_{1+i} \subseteq U^+_2(O).
\]

b) Let \( D_K > 0 \) be a prime discriminant. Then one has

\[
M^{-1}_{i\sqrt{D_K}} \cdot \Gamma(D_K)^{\max} \cdot M_{i\sqrt{D_K}} \subseteq U^+_2(O_{Q(\sqrt{-D_K})}).
\]
6.3 Hermitian and quaternionic Jacobi forms

We introduce Fourier-Jacobi expansions:

**Definition 6.3.1.** Let $F \in [U_2(O), k]$ with Fourier expansion

$$F(Z) = \sum_{0 \leq S = (\frac{n}{r}, \frac{m}{r}) \in \text{Her}_2^\sharp(O)} \alpha_F(S) e^{2\pi i \text{tr}(SZ)}, \quad Z = \left(\begin{array}{c} z \\ \bar{w} \end{array}\right) \in \mathcal{H}_2(F).$$

The rearrangement

$$F(Z) = \sum_{m=0}^\infty \phi_{m,F}(z, w) e^{2\pi i m z'}, \quad Z = \left(\begin{array}{c} z \\ \bar{w} \end{array}\right) \in \mathcal{H}_2(F)$$

is called the Fourier-Jacobi expansion of $F$. For $m \in \mathbb{N}_0$ the function

$$\phi_{m,F}(z, w) := \sum_{n=0}^\infty \sum_{r \in O, N(r) \leq nm} \alpha_F \left(\begin{array}{c} n \\ \overline{r} \end{array}\right) e^{2\pi i (nz + \text{tr}(rw))}, \quad z \in \mathcal{H}, w \in F_C := F \otimes \mathbb{R},$$

is called the $m$-th Jacobi form associated to $F$.

**Definition 6.3.2.** The parabolic subgroup $U_{2,1}(O)$ is defined by

$$U_{2,1}(O) := \left\{ M \in U_2(O) : M = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

For $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $\lambda, \mu, v \in O$ with $v - \overline{\lambda} \mu \in \mathbb{Z}$ we distinguish the special elements

$$M_1 \times I_2 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [\lambda, \mu, v] := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \overline{\lambda} & 1 & \overline{\mu} & v \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of $U_{2,1}(O)$.

The next lemma is folklore:

**Lemma 6.3.3.** Every $M \in U_{2,1}(O)$ has a unique representation of the form

$$M = \text{diag}(\varepsilon, \delta, \varepsilon, \delta) \cdot (M_1 \times I_2) \cdot [\lambda, \mu, v],$$

with $\delta, \varepsilon \in O^\times, M_1 \in \text{SL}_2(\mathbb{Z})$ and $\lambda, \mu, v \in O$ such that $v - \overline{\lambda} \mu \in \mathbb{Z}$. 
We introduce hermitian and quaternionic Jacobi forms. Basically they obey the same transformation laws as Jacobi forms considered in Chapter 4, but have some additional behaviour under the maps \( w \mapsto \delta w \) for units \( \epsilon, \delta \in O^\times \).

**Definition 6.3.4.** Let \( k, m \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( F = H \). A holomorphic function \( \phi : \mathcal{H}_1 \times F_C \rightarrow \mathbb{C} \) is called hermitian Jacobi form in the case \( F = \mathbb{C} \) resp. quaternionic Jacobi form in the case \( F = H \) of weight \( k \) and index \( m \) with respect to \( O \), if the associated function

\[
\phi_m^* : \mathcal{H}_2(F) \rightarrow \mathbb{C}, \quad \phi_m^*(Z) := \phi(z,w)e^{2\pi i mz'}, \quad Z = \left( \begin{array}{c} z \\ w \\ z' \end{array} \right) \in \mathcal{H}_2(F),
\]

satisfies the following conditions:

i) \( \phi_m^* |_{k} M = \phi_m^* \) for all \( M \in U_{2,1}(O) \),

ii) \( \phi \) has an absolutely and locally uniformly convergent Fourier expansion of the form

\[
\phi(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in O \times \mathbb{N} \mid N(r) \leq nm \times O \times \mathbb{N} \mid N(r) \leq nm}} c(n,r)e^{2\pi i (nz + \text{tr}(rw))}, \quad (z,w) \in \mathcal{H} \times F_C.
\]

The space of hermitian resp. quaternionic Jacobi forms of weight \( k \) and index \( m \) with respect to \( O \) is denoted by \( J_{k,m}^C(O) \) resp. \( J_{k,m}^H(O) \).

Considering the action of generators quoted in 6.3.3 on the associated function \( \phi_m^* \), we obtain the following characterization of Jacobi forms:

**Lemma 6.3.5.** Let \( k, m \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( F = H \). A holomorphic function \( \phi : \mathcal{H}_1 \times F_C \rightarrow \mathbb{C} \) is a hermitian Jacobi form resp. quaternionic Jacobi form of weight \( k \) and index \( m \) with respect to \( O \) if and only if the following assertions hold:

i) \( \phi(z,w) = \phi(z,w) \) for all \( \epsilon \in O^\times \), \( F = \mathbb{C} \), \( \phi(z,\epsilon w) = \phi(z,w) \) for all \( \epsilon, \delta \in O^\times \), \( F = H \),

ii) \( \phi(z,w) = e^{2\pi i \mu (\text{tr}(w))} \phi(z,w + \lambda z + \mu) \) for all \( \lambda, \mu \in O \),

iii) \( \phi(z,\epsilon w) = \epsilon^k \phi(z,w) \) for all \( \epsilon \in O^\times \), \( F = \mathbb{C} \), \( \phi(z,\epsilon w \delta) = \phi(z,w) \) for all \( \epsilon, \delta \in O^\times \), \( F = H \),

iv) \( \phi \) has absolutely and locally uniformly convergent Fourier expansion of the form

\[
\phi(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in O \times \mathbb{N} \mid N(r) \leq nm \times O \times \mathbb{N} \mid N(r) \leq nm}} c(n,r)e^{2\pi i (nz + \text{tr}(rw))}.
\]
We sort hermitian and quaternionic Jacobi forms into the context of Jacobi forms of lattice-index:

**Remark 6.3.6.** Let \( k, m \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( F = \mathbb{H} \). Then one has

\[
J_{k,m}^F(O) \subseteq J_{k,O(m)}^\mathcal{Q}, \quad \mathcal{Q} := (O, N|_O).
\]

Every modular form induces a family of Jacobi forms:

**Example 6.3.7.** Let \( k, m \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( F = \mathbb{H} \). Let \( F \in [U_2(O), k] \). Then \( \phi_{m,F} \in J_{k,m}^F(O) \), where \( \phi_{m,F} \) denotes the \( m \)-th Jacobi form associated to \( F \).

We introduce the operator \( V_m \) defined in [8, I. § 4] in the general setting:

**Definition 6.3.8.** Let \( m \in \mathbb{N} \) and \( T(m) := \{ M \in \text{SL}_2(\mathbb{Z}) : \det M = m \} \). For a holomorphic function \( \phi : \mathcal{H} \times F_\mathbb{C} \to \mathbb{C} \) we define

\[
\phi|_{k,t} V_m(z, w) := m^{k-1} \sum_{M \in T(m) \setminus \text{SL}_2(\mathbb{Z})} \phi|_{k,t} \left( \frac{1}{\sqrt{m}} M \right) (z, \sqrt{m}w), \quad (z, w) \in \mathcal{H} \times F_\mathbb{C}.
\]

From [8, I. § 4] we cite

**Theorem 6.3.9.** Let \( m, t \in \mathbb{N} \). Then \( V_m \) induces a linear operator

\[
|_{k,t} V_m : J_{k,t}^F(O) \to J_{k,mt}^F(O).
\]

For \( \phi \in J_{k,t}^F(O) \) with Fourier expansion

\[
\phi(z, w) = \sum_{n=0}^{\infty} \sum_{r \in O^2 : N(r) \leq nt} c(n, r) e^{2\pi i (nz + tr(w))}
\]

one has

\[
\phi|_{k,t} V_m(z, w) = \sum_{n=0}^{\infty} \sum_{r \in O^2 : N(r) \leq nmt} \left( \sum_{d \mid (n, r, m)} d^{k-1} c \left( \frac{nm}{d^2}, \frac{r}{d} \right) \right) e^{2\pi i (nz + tr(w))},
\]

where \( d \mid (n, r, m) \) means \( d^{-1} (n, r, m) \in \mathbb{N}_0 \times O^2 \times \mathbb{N}_0 \).

We consider Jacobi theta functions associated to the lattice \((O, N|_O)\):

**Definition 6.3.10.** Let \( O \) be an order. For \( u \in O^2 / O \) define

\[
\vartheta_{O,u}(z, w) := \vartheta_{(O,N|_O),u}^{(1)}(z, w) = \sum_{g \in u + O} e^{2\pi i (Ngz + tr(gw))}, \quad (z, w) \in \mathcal{H} \times F_\mathbb{C}.
\]
Some immediate transformation laws are given in the following

**Lemma 6.3.11.** Let \( u \in \mathcal{O}^\#:\mathcal{O} \) and \( \varepsilon, \delta \in \mathcal{O}^\times \). Then the following assertions hold:

a) \( \vartheta_{\mathcal{O}, u}(z, \varepsilon w \delta) = \vartheta_{\mathcal{O}, \varepsilon u \delta}(z, w) \),

b) \( \vartheta_{\mathcal{O}, u}(z, \overline{w}) = \vartheta_{\mathcal{O}, \overline{u}}(z, w) \),

c) \( \vartheta_{\mathcal{O}, u}(z, -\overline{w}) = \vartheta_{\mathcal{O}, u}(z, w) \).

**Proof.** Note that by 5.1.18, the map \( w \mapsto -\overline{w} \) belongs to the discriminant kernel \( \mathcal{O}_d(\mathcal{O}) \), i.e. \( u + \mathcal{O} = -\overline{u} + \mathcal{O} \). 

A reformulation of 2.3.7 yields that every \( \phi \in \mathcal{J}_{\mathcal{F}}^{k,1}(\mathcal{O}) \) has a unique theta decomposition of the form

\[
\phi(z, w) = \sum_{u: \mathcal{O}^\#:\mathcal{O}} f_u(z) \vartheta_{\mathcal{O}, u}(z, w), \quad (z, w) \in \mathcal{H} \times \mathbb{I}_\mathbb{C}.
\]

From 6.3.11 we obtain the following

**Lemma 6.3.12.** Let \( k \in \mathbb{N}_0 \) and \( \phi \in \mathcal{J}_{\mathcal{F}}^{k,1}(\mathcal{O}) \). Then one has

a) \( \phi(z, -\overline{w}) = \phi(z, \overline{w}) \),

b) \( \phi(z, \overline{w}) = (-1)^k \phi(z, w) \).

Especially, \( \phi \) is invariant under the map \( w \mapsto \overline{w} \), if \( k \) is even.

We use the explicit description of \( \mathcal{O}(\mathfrak{o}) \) for orders \( \mathfrak{o} \) in \( \mathbb{C} \):

**Corollary 6.3.13.** Let \( \mathfrak{o} \) be an order in \( \mathbb{C} \). Let \( k \in \mathbb{N}_0 \) such that \( k \equiv 0 \mod |\mathfrak{o}^\times| \). Then one has

\[
\mathcal{J}_{\mathcal{F}}^{\mathfrak{o}}(k,1) = \mathcal{J}_{\mathcal{F}}^{\mathfrak{o},\text{sym}}(k,1).
\]

In general, a characterization as in 6.3.13 does not hold for quaternionic orders, since the orthogonal group \( \mathcal{O}(\mathcal{O}) \) contains conjugations by invariant elements \( u \in \mathcal{I}(\mathcal{O}) \). But in the cases we are dealing with, the invariance of quaternionic Jacobi forms under the maps \( w \mapsto u^{-1}wu \) is already implied by the invariance under \( w \mapsto \varepsilon w \delta \) for \( \varepsilon, \delta \in \mathcal{O}^\times \).

We need the following lemma, which is easy to prove:

**Lemma 6.3.14.** Let \( k \in \mathbb{N}_0 \). Let \( G \leq \mathcal{O}(\mathcal{O}) \) and \( \phi \in \mathcal{J}_{\mathcal{F}}^{k,1}(\mathcal{O}) \) invariant under \( G \). If \( G \) acts transitively on the sets

\[
A_d := \{ u \in \mathcal{O}^\#:\mathcal{O} : N(u) \equiv d \mod \mathbb{Z} \}, \quad d \in \mathbb{Q},
\]

then one has \( \phi \in \mathcal{J}_{\mathcal{F}}^{k,1,\text{sym}}(\mathcal{O}) \).
Thus on all $O$-assumption, $\phi$ also $O$-back operator with respect to Jacobi forms is explained in the following.

Hence, these sets are $O$-invariant under $G$. Hence the function $u \mapsto f_u$ is constant on the sets $A_d$ and thus on all $O(O)$-orbits, i.e. $\phi \in J_{k,1}^{F,sym}(O)$.

We apply the lemma in the proof of

**Theorem 6.3.15.** Let $k \in \mathbb{N}_0$ and $k \equiv 0 \mod w_O$. Let $O$ be the Hurwitz order and $O = O_K$ for $K = Q(\sqrt{-D})$ for $D = 3, 4$. Then for all $F \in [U_2(O), k]$ one has $\phi_{1,F} \in \hat{J}_{k,1}^{symp}(O)$.

**Proof.** The invariance under $w \mapsto \pi w$ follows from 6.3.12, since $k$ is necessarily even. First let $O$ be the Hurwitz order. The group generated by $w \mapsto \omega w w$ acts transitively on the set

$$\left\{ u \in O^2/O : N(u) \equiv \frac{1}{2} \mod Z \right\},$$

which is directly verified by considering the representatives $\frac{1+i}{2}, \frac{1+2i}{2}$ and $\frac{1+3i}{2}$. Since $N(u) \mod Z$ takes its values in $\{0, \frac{1}{2}\}$, the claim follows from 6.3.14. For $D_K = 3$ a system of representatives of $O_K^2/O_K$ is given by

$$0, \pm \mu, \pm \mu i_2, \pm \mu \pm \mu i_2, \quad \mu := \frac{i_1}{\sqrt{3}}.$$

A direct verification shows that the group generated by $w \mapsto wi_2$ acts transitively on both sets

$$\left\{ u \in O_K^2/O_K : N(u) \equiv \frac{1}{2} \mod Z \right\}, \quad \left\{ u \in O_K^2/O_K : N(u) \equiv \frac{2}{3} \mod Z \right\}.$$

In this case the claim again follows from 6.3.14. For $D_K = 4$ a system of representatives of $O_K^2/O_K$ is given by the sixteen elements

$$0, \frac{1}{2}, \frac{i_1}{2}, \frac{i_2}{2}, \frac{i_3}{2}, \frac{1+i_1}{2}, \frac{1+i_2}{2}, \frac{1+i_3}{2}, \frac{i_1+i_2}{2}, \frac{i_1+i_3}{2}, \frac{i_2+i_3}{2}, \frac{1+i_1+i_2}{2}, \frac{1+i_1+i_3}{2}, \frac{1+i_2+i_3}{2}, \frac{i_1+i_2+i_3}{2}, \omega.$$

A direct calculation shows that the group generated by the transformations $w \mapsto wi_1$ and $w \mapsto wi_2$ acts transitively on

$$\left\{ u \in O_K^2/O_K : N(u) \equiv \frac{k}{4} \mod Z \right\}, \quad k = 1, 2, 3.$$

Hence, these sets are $O(O_K)$ orbits. The class $\omega + O_K$ is a fixed point of $O(O_K)$ and thus a single orbit. Together with the zero class $O_K$, we have determined all $O(O_K)$-orbits and the claim follows. Note that 6.3.14 is not applicable in this case.

The compatibility of the pullback operator with respect to modular embeddings and the pullback operator with respect to Jacobi forms is explained in the following.
Proposition 6.3.16. Let \( k \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( \mathcal{F} = \mathcal{H} \). Let \( a, b \in \mathcal{O} \) and \( F \in [\mathcal{U}_2(\mathcal{O}), k] \). Then the following assertions hold:

a) \( F[a](Z) = \sum_{m=0}^{\infty} \phi_{m,F}[a](z,w)e^{2\pi i N(a) mz'} \),

b) \( F[a] = F[b] \) implies \( \phi_{m,F}[a] = \phi_{m,F}[b] \) for all \( m \in \mathbb{N}_0 \).

Proof. Let \( F \in [\mathcal{U}_2(\mathcal{O}), k] \) with Fourier-Jacobi expansion
\[
F(Z) = \sum_{m=0}^{\infty} \phi_{m,F}(z,w)e^{2\pi i mz'}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2(\mathcal{F}).
\]

Hence,
\[
F[a](Z) = F\left( \frac{z}{aw} N(a) z' \right) = \sum_{m=0}^{\infty} \phi_{m,F}(z,aw)e^{2\pi i N(a) mz'}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.
\]

Part b) follows from the uniqueness of the Fourier-Jacobi expansion. \( \square \)

6.4 Maaß spaces and lifting constructions

Let \( k \in \mathbb{Z} \), where \( k \equiv 0 \mod 2 \) in the case \( \mathcal{F} = \mathcal{H} \). Then \( F \in [\mathcal{U}_2(\mathcal{O}), k] \) has a Fourier expansion of the form
\[
F(Z) = \sum_{0 \leq S \in \text{Her}_2(\mathcal{O})} a_F(S)e^{2\pi i \text{tr}(SZ)}, \quad Z \in \mathcal{H}_2(\mathcal{F}).
\]

We introduce an arithmetically motivated subspace of \([\mathcal{U}_2(\mathcal{O}), k]\), cf. [19, 20, 21]:

Definition 6.4.1. Let \( k \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( \mathcal{F} = \mathcal{H} \). The Maaß space \( \mathcal{M}_k(\mathcal{O}) \) with respect to \( \mathcal{O} \) consists of all \( F \in [\mathcal{U}_2(\mathcal{O}), k] \) such that
\[
a_F(S) = \sum_{d \in \mathbb{N}, d \mid \varepsilon(S)} d^{k-1} a_F\left( \frac{mn/d^2}{r/d} \right) \left( \frac{r/d}{1} \right)
\]
holds for all \( 0 \neq S = \left( \frac{r}{m} \right), S \geq 0 \), where
\[
\varepsilon(S) := \max \left\{ m \in \mathbb{N} : m^{-1} S \in \text{Her}_2(\mathcal{O}) \right\}.
\]

Note that \( \mathcal{M}_k(\mathcal{Z}) \) is the space which was considered initially by H. Maaß.

Basic facts are collected in the following
Remarks 6.4.2. Let \( k \in \mathbb{N}_0 \), where \( k \equiv 0 \mod 2 \) in the case \( \mathbb{F} = \mathbb{H} \). Let \( F \in \mathcal{M}_k(\mathcal{O}) \) with Fourier-Jacobi expansion
\[
F(Z) = \sum_{m=0}^{\infty} \phi_{m,F}(z,w)e^{2\pi imz'}, \quad Z = \left( \begin{array}{c} z \\ w \\ z' \end{array} \right) \in \mathcal{H}_2(\mathbb{F}).
\]
a) If \( k \) is odd or \( k = 2 \), then \( \phi_{0,F} \equiv 0 \). Furthermore,
\[
\phi_{0,F}(z,w) = \alpha_F \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) G^e_k(z), \quad (z,w) \in \mathcal{H} \times \mathbb{F}_C,
\]
for all even \( k \geq 4 \) and
\[
\phi_{m,F}(z,w) = \phi_{1,F} \mid_{k,1} V_m(z,w), \quad (z,w) \in \mathcal{H} \times \mathbb{F}_C,
\]
for all \( m > 0 \). For odd \( k \) or \( k = 2 \), the space \( \mathcal{M}_k(\mathcal{O}) \) consists of cups forms only and one has \( \mathcal{M}_0(\mathcal{O}) = \mathbb{C} \).

b) The map \( \mathcal{M}_k(\mathcal{O}) \rightarrow J_{k,1}^F(\mathcal{O}), \ F \mapsto \phi_{1,F} \) is injective.

c) \( F \in \mathcal{M}_k(\mathcal{O}) \) satisfies
\[
F \mid_{I_{tr}} = (-1)^k F(Z), \quad Z \in \mathcal{H}_2(\mathbb{F}).
\]
Hence, if \( k \) is even, \( \mathcal{M}_k(\mathcal{O}) \) consists of symmetric modular forms only. If \( k \) is odd and \( \mathbb{F} = \mathbb{C} \), the space \( \mathcal{M}_k(\mathcal{O}) \) consists of skew-symmetric forms only. If \( k \) is odd and \( \mathbb{F} = \mathbb{R} \), then \( \mathcal{M}_k(\mathcal{O}) \) is trivial.

d) \( \mathcal{O}(\mathcal{O}) \) acts on \( \mathcal{M}_k(\mathcal{O}) \) via
\[
F^\sigma(Z) := F \left( \sigma^{-1}(w), \sigma^{-1}(w) \right) \begin{pmatrix} z \\ w \\ z' \end{pmatrix}, \quad Z = \begin{pmatrix} z \\ w \\ z' \end{pmatrix}, \sigma \in \mathcal{O}(\mathcal{O}).
\]
Furthermore, \( \phi_{m,F^\sigma} = \phi_{m,F}[\sigma] \) for all \( m \in \mathbb{N}_0 \).

Remark 6.4.3. Let \( k \in \mathbb{N}_0 \) such that \( k \equiv 0 \mod \omega_\mathcal{O} \). For a holomorphic function \( F : \mathcal{H}_2(\mathbb{F}) \rightarrow \mathbb{C} \) and \( M \in \mathcal{U}_2(\mathcal{O}) \) the function \( F \mid_{k,M} \) only depends on \( M \mod \mathbb{C}(\mathcal{F}_1) \). Hence, the slash operator
\[
(M,F) \mapsto F \mid_{k,M}
\]
is well-defined and defines an action of the extended modular group \( \Gamma(\mathcal{O}) \) on the set of holomorphic functions \( \{ F : \mathcal{H}_2(\mathbb{F}) \rightarrow \mathbb{C} \} \), where the definition is extended by
\[
f \mid_{k,I_{tr}}(Z) := f(Z^t), \quad Z \in \mathcal{H}_2(\mathbb{F}).
\]

Definition 6.4.4. Let \( k \in \mathbb{N}_0 \) such that \( k \equiv 0 \mod \omega_\mathcal{O} \). A function \( F : \mathcal{H}_2(\mathbb{F}) \rightarrow \mathbb{C} \) is called a modular form of weight \( k \) with respect to \( \Gamma(\mathcal{O}) \), if the following assertions hold:
6.4 Maaß spaces and lifting constructions

i) $F$ is holomorphic,

ii) $F|_k M = F$ for all $M \in \Gamma(O)$.

The space of modular forms of weight $k$ with respect to $\Gamma(O)$ is denoted by $[\Gamma(O), k]$.

As a direct consequence of 5.5.11 we obtain:

**Proposition 6.4.5.** Let $O$ be an order and assume that $PU_2(O)$ is normalized by $I_{tr}$. If $F = \mathbb{C}$, suppose additionally that $U_2(O)$ is standardly generated. Let $k \equiv 0 \mod w_O, F \in [\Gamma(O), k]$ and $M, N \in \text{Mod}(\Gamma(T), U_2(O))$. Then $M \sim N$ implies $F|_k[M] = F|_k[N]$. In other words, the map

$$\text{Mod}(\Gamma(T), U_2(O))/\sim \rightarrow [\Gamma(T), k], \quad [M] \sim \mapsto F|_k[M]$$

is well-defined.

**Remark 6.4.6.** Note that the assumptions of 6.4.5 hold for example if $F = \mathbb{C}$ and $O$ is a principal ideal domain or $O = \mathfrak{o}_K$ for some imaginary-quadratic number field $K$ (cf. [5, Lemma (1.1)]) or if $F = \mathbb{H}$ and $O$ is norm-euclidean.

**Remark 6.4.7.** Modular forms with respect to subgroups of $\Gamma(O)$ and characters are defined in the same manner. Note that $[PU_2(O), k] = [U_2(O), k]$ holds for all $k \equiv 0 \mod w_O$. Hence the theory of modular forms with respect to the extended modular group contains the one for $U_2(O)$ in this case.

**Remark 6.4.8.** Let $\mathfrak{o}$ be an order in $\mathbb{C}$ and $k \equiv 0 \mod |\mathfrak{o}^\times|$. Then $[\Gamma(\mathfrak{o}), k] = [U_2(\mathfrak{o}), k]^{\text{sym}}$.

**Corollary 6.4.9.** Let $k \in \mathbb{N}_0$ such that $k \equiv 0 \mod w_O$. Let $O$ denote the Hurwitz order or let $O = O_{\mathbb{Q}(\sqrt{-D})}$ for $D = 3, 4$ or $O = \mathfrak{o}_K$ for $K = \mathbb{Q}(\sqrt{-D_K})$. Then $\mathcal{M}_k(O)$ is contained in $[\Gamma(O), k]$.

**Proof.** Let $F \in \mathcal{M}_k(O)$ and $\sigma \in O(O)$. From 6.3.15 we obtain $\varphi_{1,F} = \varphi_{1,F}[\sigma] = \varphi_{1,F}$. Since the map $F \mapsto \varphi_{1,F}$ is injective, we conclude $F^\sigma = F$, i.e. $F \in [\Gamma(O), k]$.

Under certain assumptions on $U_2(O)$, the projection onto the first Jacobi form has a right inverse, namely the Maaß lift:

**Theorem 6.4.10.** Suppose that $U_2(O)$ is generated by the parabolic subgroup $U_{2,1}(O)$ and $\text{diag}(P, P)$, where $P = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. Let $k \geq 4$ such that $k \equiv 0 \mod w_O$. For $\varphi \in J_{k,1}^F(O)$ with constant Fourier coefficient $c_0(0, 0)$ define

$$\text{M-Lift}(\varphi)(Z) := -\frac{B_k}{2k} c_\varphi(0, 0) G_k^*(z) + \sum_{m=1}^{\infty} \left( \varphi|_{k,1} V_m \right)(z, w) e^{2\pi i m z'}, \quad Z = \left( \begin{array}{c} z \\ w \end{array} \right) \in \mathcal{H}_2(\mathbb{F}).$$

Then the following assertions hold:

a) $\text{M-Lift}(\varphi) \in [U_2(O), k]$ for all $\varphi \in J_{k,1}^F(O)$.
b) **M-Lift**: \( J_{k,1}^\mathbb{F}(O) \rightarrow [U_2(O), k] \) is injective.

c) For \( \phi \in J_{k,1}^\mathbb{F}(O) \) the Fourier coefficients of \( \text{M-Lift}(\phi) \) are given by the formula

\[
\alpha_{\text{M-Lift}(\phi)}(S) = \begin{cases} 
- \frac{B_k}{2\pi} c_\phi(0,0), & S = 0, \\
- \sum_{d|e(S)} d^{k-1} c_\phi \left( \frac{mn}{d^2}, \frac{r}{d} \right), & S = (n^r_m, m > 0).
\end{cases}
\]

d) **M-Lift**: \( J_{k,1}^\mathbb{F}(O) \rightarrow \mathcal{M}_k(O) \) is an isomorphism.

The operator **M-Lift** is called the Maaß lift with respect to \( O \).

**Proof.** We give a sketch of the proof, since the methods are similar to the literature. Injectivity follows from \( V_1 = \text{id} \). The invariance of \( \text{M-Lift}(\phi) \) under the parabolic subgroup \( U_{2,1}(O) \) follows from construction. The explicit formula for the Fourier coefficients is a direct consequence of 6.3.8. Finally, part c) together with the invariance of \( \phi \) under the transformation \( w \mapsto w \) implies, that \( \text{M-Lift}(\phi) \) is invariant under \( Z \mapsto Z[P] \). By assumption on \( U_2(O) \) this proves a). For part d) let \( F \in \mathcal{M}_k(O) \) with first Jacobi form \( \phi_{1,F} \). Then \( F \) and \( \text{M-Lift}(\phi_{1,F}) \) have the same first Jacobi form. Hence, \( F = \text{M-Lift}(\phi_{1,F}) \) by injectivity. This proves surjectivity. □

Let \( t \in \mathbb{N} \). Since \( \Gamma(t) \) contains the transformation

\[
\begin{pmatrix} z & w \\
w & z' \end{pmatrix} \mapsto \begin{pmatrix} z & w \\
\frac{w}{z'} & \frac{z}{z'} + \frac{1}{t} \end{pmatrix},
\]

we obtain that any \( F \in [\Gamma(t), k] \) has a Fourier expansion of the form

\[
F(Z) = \sum_{S \in \text{Sym}_2^g(Z), S = (n^r_m, m > 0)} \alpha_F(S) e^{2\pi i tr(SZ)}, \quad Z = \begin{pmatrix} z & w \\
w & z' \end{pmatrix} \in \mathcal{H}_2.
\]

The Maaß space with respect to the paramodular group is then defined in a similar way:

**Definition 6.4.11.** Let \( k \in \mathbb{N}_0 \) and \( t \in \mathbb{N} \). The Maaß space \( \mathcal{M}_{k,t} \) of polarization \( t \) consists of all \( F \in [\Gamma(t), k] \) such that

\[
\alpha_F(S) = \sum_{d|e(S)} d^{k-1} \alpha_F \left( \frac{mn}{d^2}, \frac{r}{d} \right)
\]

holds for all \( 0 \neq S = (n^r_m, m > 0) \).

Some basis facts are collected in the following

**Remark 6.4.12.** Let \( k \in \mathbb{N}_0 \) and \( t \in \mathbb{N} \). Let \( F \in \mathcal{M}_{k,t} \) with Fourier-Jacobi expansion

\[
F(Z) = \sum_{m=0}^\infty \phi_{lm,F}(z,w) e^{2\pi i mw}, \quad Z = \begin{pmatrix} z & w \\
w & z' \end{pmatrix} \in \mathcal{H}_2.
\]
a) If $k$ is odd or $k = 2$, then $\phi_{0,F} \equiv 0$. Furthermore,

$$\phi_{0,F}(z, w) = \alpha_F \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} G_k^t(z)$$

for all even $k \geq 4$ and

$$\phi_{mt,F}(z, w) = \phi_{t,F}|_{k,t} V_m(z, w)$$

for all $m > 0$. For odd $k$ or $k = 2$, the space $M_{k,t}$ consists of cusps forms only and one has $M_{0,t} = \mathbb{C}$.

b) The map $M_{k,t} \rightarrow J_{k,t}, F \mapsto \phi_{t,F}$ is injective.

c) $F|_k V_t = (-1)^k F$.

Under certain assumptions on the weight, the projection onto the $t$-th Fourier-Jacobi coefficient has a right inverse, the so-called Gritsenko lift. From [10, Hauptsatz 2.1] we cite:

**Theorem 6.4.13.** Let $t \in \mathbb{N}$ and $k \geq 4$. For $\phi \in J_{k,t}$ with constant Fourier coefficient $c_\phi(0,0)$ define

$$\text{G-Lift}_t(\phi)(Z) := -\frac{B_k}{2k} c_\phi(0,0) G_k^t(z) + \sum_{m=1}^{\infty} \left( \phi|_{k,t} V_m \right)(z, w) e^{2\pi i m z'}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.$$

Then the following assertions hold:

a) $\text{G-Lift}_t(\phi) \in \Gamma(t), k$ for all $\phi \in J_{k,t}$ and $\text{G-Lift}_t(\phi)|_k V_t = (-1)^k \text{G-Lift}_t(\phi)$.

b) $\text{G-Lift}_t : J_{k,t} \rightarrow \Gamma(t), k$ is injective.

c) For $\phi \in J_{k,t}$ the Fourier coefficients of $\text{G-Lift}_t(\phi)$ are given by the formula

$$\alpha_{\text{G-Lift}_t(\phi)}(S) = \begin{cases} -\frac{B_k}{2k} c_\phi(0,0), & S = 0, \\ c_\phi(0,0) \sigma_{k-1}(n), & S = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}, n > 0, \\ \sum_{d|\frac{m}{n},r|m} d^{k-1} c_\phi \left( \frac{mn}{d^2}, \frac{r}{d} \right), & S = \begin{pmatrix} n & r \\ r & tm \end{pmatrix}, m > 0. \end{cases}$$

d) $\text{G-Lift}_t : J_{k,t} \rightarrow M_{k,t}$ is an isomorphism.

The operator $\text{G-Lift}_t$ is called Gritsenko lift of index $t$.

We end this section by proving a commutation relation between the two lifting constructions and the pullback operator:
**Theorem 6.4.14.** Let $\mathcal{O}$ be an order. Suppose that $U_2(\mathcal{O})$ is generated by the parabolic subgroup $U_{2,1}(\mathcal{O})$ and $\text{diag}(P,P)$, where $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $a \in \mathcal{O}, k \geq 4$ and $k \equiv 0 \mod w_\mathcal{O}$. Then one has

$$G\text{-Lift}_{N(a)}(\phi[a]) = M\text{-Lift}(\phi)[a]$$

for all $\phi \in J_{k,1}^F(\mathcal{O})$ and the pullback operator $[a]$ maps $\mathcal{M}_k(\mathcal{O})$ into $\mathcal{M}_{k,t}$. In other words, the diagram

$$
\begin{array}{ccc}
J_{k,1}^F(\mathcal{O}) & \xrightarrow{\text{M-Lift}} & \mathcal{M}_k(\mathcal{O}) \\
[a] & \downarrow & [a] \\
J_{k,N(a)} & \xrightarrow{\text{G-Lift}_{N(a)}} & \mathcal{M}_{k,N(a)}
\end{array}
$$

is commutative.

**Proof.** Let $Z \in \mathcal{H}_2$. Then one has

$$
\text{M-Lift}(\phi)[a](Z) = -\frac{B_{k}}{2^k} c_{\phi}(0,0) G_\kappa(z) + \sum_{m=1}^{\infty} \left( \phi|_{k,1} V_m \right)(z, aw) e^{2\pi i m N(a) z'}
$$

$$
= -\frac{B_{k}}{2^k} c_{\phi[a]}(0,0) G_\kappa(z) + \sum_{m=1}^{\infty} \left( \phi[a]|_{k,N(a)} V_m \right)(z, w) e^{2\pi i m N(a) z'}
$$

$$
= G\text{-Lift}_{N(a)}(\phi[a])(Z),
$$

since the operators $[a]$ and $V_m$ commute. Note that $c_{\phi[a]}(0,0) = c_{\phi}(0,0)$. Since $G\text{-Lift}_{N(a)}$ and M-Lift are isomorphisms, we conclude that the diagram commutes. 

From 4.2.1, 4.3.1 and 4.3.2 we derive the following isomorphisms of Maaß spaces, which can partly be found already in [15, Sec. 4, Theorem 2 a)] and [16, Sec. 9, Corollary]. Note that the methods developed in Chapter 4 allow to construct the inverse maps on the basis of the theta decomposition of the first Jacobi form.

**Corollary 6.4.15.** Let $k \in \mathbb{N}_0$ and $\mathcal{O}$ denote the Hurwitz order.

a) $\mathcal{M}_k(\mathcal{O}) \cong \mathcal{M}_{k,1}$, if $k \equiv 0 \mod 2$ and $\mathcal{M}_k(\mathcal{O}) \cong \mathcal{M}_k(\mathcal{O}(\sqrt{-3}))$, if $k \equiv 0 \mod 6$.

b) $\mathcal{M}_k(\mathcal{O}(\sqrt{-3})) \cong \mathcal{M}(\mathcal{O}(\sqrt{-1})) \cong \mathcal{M}_{k,2}$, if $k \equiv 0 \mod 4$.

### 6.5 Separation theorems

Let $F \in [U_n(\mathcal{O}),k]$ and $M, N \in \text{Mod}(\Gamma(T),U_n(\mathcal{O}))$. In 5.5.11 we saw that $F|_k[M] = F|_k[N]$ if $M \sim N$. In this section we consider the converse problem, i.e. the question if a modular embedding $M$ is determined up to equivalence by the family of pullbacked functions $F|_k[M]$, where $F \in [U_n(\mathcal{O}),k]$. Already in the case $n = 2$, the usual equivalence relation on $\text{Mod}(\Gamma(T),U_2(\mathcal{O}))$ is too restrictive as one has $F[a] = F[\bar{a}]$ for $a \in \mathcal{O}$ and symmetric forms $F$, but in general $M_a \not\sim M_{\bar{a}}$. By retaking the up concept of equivalence in the extended sense, we can solve the converse problem at least for certain orders and under divisibility conditions on the polarization $t$. 

6.5 Separation theorems

6.5.1 \( \mathfrak{o}_{Q(\sqrt{-1})} \)

Regarding pullbacks of the Jacobi theta functions we need the following

**Lemma 6.5.1.** Let \( a, b \in \mathfrak{o}_{Q(\sqrt{-1})} \). Then the following statements are equivalent:

i) \( \vartheta_{\mathfrak{o}_{Q(\sqrt{-1})},0}[a] = \vartheta_{\mathfrak{o}_{Q(\sqrt{-1})},0}[b] \),

ii) \( a \) and \( b \) lie in the same \( \mathcal{O}(\mathfrak{o}_{Q(\sqrt{-1})}) \)-orbit,

iii) there is \( \epsilon \in \mathfrak{o}_{Q(\sqrt{-1})}^\times \) such that \( a = \epsilon b \) or \( a = \epsilon \overline{b} \).

**Proof.** First note that in all cases one has \( N(a) = N(b) \). The equivalence of ii) and iii) is clear from the explicit description of \( \mathcal{O}(\mathfrak{o}_{Q(\sqrt{-1})}) \). Furthermore, ii) implies i). Hence we are left to prove that i) implies ii). By comparing the Fourier coefficients of the index \( (1, s), s \in \mathbb{Z} \), we obtain

\[
\left| \{ x \in \mathfrak{o}_{Q(\sqrt{-1})}^\times : \text{tr}(\overline{a}x) = s \} \right| = \left| \{ x \in \mathfrak{o}_{Q(\sqrt{-1})}^\times : \text{tr}(\overline{b}x) = s \} \right|
\]

i.e.

\[
\left| \{ x \in \{ \pm 1, \pm i_1 \} : \text{tr}(\overline{a}x) = s \} \right| = \left| \{ x \in \{ \pm 1, \pm i_1 \} : \text{tr}(\overline{b}x) = s \} \right|
\]

for all \( s \in \mathbb{Z} \). Write \( a = a_0 + a_1 i_1, b = b_0 + b_1 i_1 \) and assume without loss of generality that \( a_1, a_2, b_1, b_2 \geq 0 \) after some simple orthogonal transformation. Considering \( s = 0 \), we obtain

\[
2 \cdot \left| \{ j \in \{0,1\} : a_j = 0 \} \right| = 2 \cdot \left| \{ j \in \{0,1\} : b_j = 0 \} \right|
\]

and

\[
\left| \{ j \in \{0,1\} : a_j = s \} \right| = \left| \{ j \in \{0,1\} : b_j = s \} \right|
\]

for \( s > 0 \). Hence, \( a \) and \( b \) have the same set of components with respect to \( 1, i_1 \) including multiplicity. Thus, there must be a permutation \( \sigma \), which can be realized in \( \mathcal{O}(\mathfrak{o}_{Q(\sqrt{-1})}) \), such that \( \sigma(a) = b \). \( \square \)

We characterize certain elements in \( \mathfrak{o}_{Q(\sqrt{-1})} \):

**Lemma 6.5.2.** a) \( \{ a \in \mathfrak{o}_{Q(\sqrt{-1})} : N(a) \equiv 0 \mod 2 \} = (1 + i_1)\mathfrak{o}_{Q(\sqrt{-1})} \).

b) \( \{ a \in \mathfrak{o}_{Q(\sqrt{-1})} : a \equiv (1 + i_1) \mod 2 \} = \{ a \in \mathfrak{o}_{Q(\sqrt{-1})} : \nu_2(N(a)) \equiv 1 \mod 2 \} \).

**Theorem 6.5.3.** Let \( a, b \in \mathfrak{o}_{Q(\sqrt{-1})} \) such that \( N(a) = N(b) \) and \( \nu_2(N(a)) \equiv 1 \mod 2 \). Let \( k \in \mathbb{N} \) and \( k \equiv 0 \mod 4 \). For \( \phi \neq \phi \in \mathcal{J}_{k,1}^C(\mathfrak{o}_{Q(\sqrt{-1})}) \) the following assertions are equivalent:

i) \( \phi[a] = \phi[b] \),

ii) \( a \) and \( b \) lie in the same \( \mathcal{O}(\mathfrak{o}_{Q(\sqrt{-1})}) \)-orbit,

iii) there exists \( \epsilon \in \mathfrak{o}_{Q(\sqrt{-1})}^\times \) such that \( a = \epsilon b \) or \( a = \epsilon \overline{b} \),
iv) \( \iota_a \) and \( \iota_b \) are equivalent.

**Proof.** The equivalence of ii), iii) and iv) is obvious from the explicit description of \( \mathcal{O}(\mathfrak{a}_Q(\sqrt{-1})) \) and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The theta decomposition of \( \phi \) is of the form

\[
\phi = f_0 \vartheta_{\mathfrak{a},0} + f_1 \left( \vartheta_{\mathfrak{a},\frac{1}{2}} + \vartheta_{\mathfrak{a},\frac{1}{2}i_1} \right) + f_{1+i_1} \vartheta_{\mathfrak{a},\frac{1}{2}i_1}.
\]

Let

\[
D_0 := \vartheta_{\mathfrak{a},0}[a] - \vartheta_{\mathfrak{a},0}[b], \quad D_1 := \vartheta_{\mathfrak{a},0}[a] - \vartheta_{\mathfrak{a},0}[b], \quad D_2 := \vartheta_{\mathfrak{a},\frac{1}{2}i_1}[a] - \vartheta_{\mathfrak{a},\frac{1}{2}i_1}[b].
\]

Then \( \phi[a] = \phi[b] \) implies

\[
f_0 D_0 + f_1 D_1 + f_{1+i_1} D_2 = 0.
\]

We apply the elliptic transformation \( |_{N(a)} \left[ \frac{1}{2}, 0 \right] \) on both sides of this equation. By assumption on \( N(a) \) and 6.5.2 we have \( \frac{a}{2} \equiv \frac{1+i_1}{2} \mod \mathfrak{a} \). Hence, by 2.2.10 and 3.1.4,

\[
f_0 D_2 + f_1 D_1 + f_{1+i_1} D_0 = 0.
\]

We substract both equations from each other in order to obtain

\[
\left( f_0 - f_{1+i_1} \right) \cdot (D_0 - D_2) = 0.
\]

From \( \phi \neq 0 \) we obtain \( f_0 \neq 0 \), hence \( f_0 - f_{1+i_1} \neq 0 \), since \( f_{1+i_1}(z+1) = -f_{1+i_1}(z) \). Consequently, we have \( D_0 = D_2 \). But the identities \( D_0(z+1) = D_0(z) \) and \( D_2(z+1) = -D_2(z) \) already imply \( D_0 = 0 \). By 6.5.1 this proves ii). \( \square \)

The corresponding result for modular forms is given in the following

**Theorem 6.5.4.** Let \( a, b \in \mathfrak{a}_Q(\sqrt{-1}) \) such that \( N(a) = N(b) \) and \( \nu_2(N(a)) \equiv 1 \mod 2 \). Let \( k \in \mathbb{N} \) and \( k \equiv 0 \mod 4 \). Then the following assertions are equivalent:

i) \( F[a] = F[b] \) for some \( F \in \left[ U_2(\mathfrak{a}_Q(\sqrt{-1})), k \right] \) and \( \phi_{1,F} \neq 0 \),

ii) \( F[a] = F[b] \) for all \( F \in \left[ \Gamma(\mathfrak{a}_Q(\sqrt{-1})), k \right] \),

iii) \( a \) and \( b \) lie in the same \( \mathcal{O}(\mathfrak{a}_Q(\sqrt{-1})) \)-orbit,

iv) there exists \( \varepsilon \in \mathfrak{a}_Q^\times(\sqrt{-1}) \) such that \( a = \varepsilon b \) or \( a = \varepsilon \bar{b} \),

v) \( M_a \) and \( M_b \) are equivalent in the extended sense.
Proof. The equivalence of iii), iv) and v) follows from the explicit description of the group $O(Q(\sqrt{-1}))$ and 6.2.12. Each of them implies ii). Assume that ii) holds. Since $k \equiv 0 \mod 4$, there exists a hermitian Jacobi form $0 \neq \phi \in J_{k,1}^C(\mathfrak{o}_{Q(\sqrt{-1})})$. Hence, i) holds with $F = \text{M-Lift}^r(\phi)$. Note that $\phi_{1,F} = \phi \neq 0$. Furthermore, i) implies iii) by 6.3.16 and 6.5.3. □

Theorem 6.5.13 directly implies

**Theorem 6.5.5.** Let $t \in \mathbb{N}$ such that $\nu_2(t) \equiv 1 \mod 2$. Let $M, N \in \text{Mod}\left(\Gamma(t), U_2(\mathfrak{o}_{Q(\sqrt{-1})})\right)$. Let $k \in \mathbb{N}$ and $k \equiv 0 \mod 4$. Then the following assertions are equivalent:

i) $F\mid_{k}[M] = F\mid_{k}[N]$ for some $F \in [U_2(\mathfrak{o}_{Q(\sqrt{-1})}), k]$ and $\phi_{1,F} \neq 0$,

ii) $F\mid_{k}[M] = F\mid_{k}[N]$ for all $F \in \Gamma(\mathfrak{o}_{Q(\sqrt{-1})}, k]$,

iii) $M$ and $N$ are equivalent in the extended sense.
6.5.2 \( o_{\mathbb{Q}(\sqrt{-3})} \)

Let \( \rho := -\frac{1}{2} + i_1 \frac{\sqrt{3}}{2} \).

**Lemma 6.5.6.**

a) \( \mathbb{Z}[i_1 \sqrt{3}] = \{ x + y \rho \in o_{\mathbb{Q}(\sqrt{-3})} : x \in \mathbb{Z}, y \in 2\mathbb{Z} \} \),

b) \( o_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[i \sqrt{3}] \cup (\rho + \mathbb{Z}[i \sqrt{3}]) \),

c) for all \( w \in o_{\mathbb{Q}(\sqrt{-3})} \) there is some \( \epsilon \in o_{\mathbb{Q}(\sqrt{-3})} \) such that \( we \in \mathbb{Z}[i \sqrt{3}] \).

**Proof.**

a) Follows from \( 2\rho \in \mathbb{Z}[i_1 \sqrt{3}] \) and \( x + yi_1 \sqrt{3} = (a + 1) + 2bp \).

b) Follows directly from a).

c) Let \( w = x + y \rho \). In the case \( y \equiv 0 \mod 2 \) the claim follows from ii). In the case \( x \equiv 0 \mod 2, y \equiv 1 \mod 2 \) one has

\[
w\rho = x\rho + y\rho^2 = x\rho + y(1 - \rho) = -y + (x - y)\rho \in \mathbb{Z}[i_1 \sqrt{3}].
\]

Let both \( x, y \) be odd. Then

\[
w\rho = x\rho + y\rho^2 = x\rho + y\rho = (y - x) - x\rho \in \mathbb{Z}[i_1 \sqrt{3}].
\]

We characterize the elements of even norm in \( o_{\mathbb{Q}(\sqrt{-3})} \):

**Lemma 6.5.7.**

\[
2o_{\mathbb{Q}(\sqrt{-3})} = \{ w \in o_{\mathbb{Q}(\sqrt{-3})} : N(w) \equiv 0 \mod 2 \} = \{ w \in o_{\mathbb{Q}(\sqrt{-3})} : N(w) \equiv 0 \mod 4 \}.
\]

**Proof.** The value \( N(x + y \rho) = x^2 - xy + y^2 \) for \( x, y \in \mathbb{Z} \) is even, if and only if all summands are even, i.e. if and only if \( x + y \rho \in 2o_{\mathbb{Q}(\sqrt{-3})} \). Note that in this case one necessarily has \( N(x + y \rho) \equiv 0 \mod 4 \). \( \square \)

**Lemma 6.5.8.** Let \( a \in \mathbb{Z}[i_1 \sqrt{3}] \) such that \( N(a) \equiv 1 \mod 2 \). Then \( \text{tr}(\bar{a}b) \equiv 1 \mod 2 \) for all \( b \in \rho + \mathbb{Z}[i_1 \sqrt{3}] \).

**Proof.** Let \( a = x + y \rho \in \mathbb{Z}[i \sqrt{3}] \). Then \( y \in 2\mathbb{Z} \) and from \( N(a) \equiv 1 \mod 2 \) we obtain \( x \equiv 1 \mod 2 \). As a consequence, \( \text{tr}(\bar{a}\rho) = -x + 2y \equiv 1 \mod 2 \). The claim follows then from \( \text{tr}(\bar{a}b) \equiv 0 \mod 2 \) for all \( b \in \mathbb{Z}[i \sqrt{3}] \). \( \square \)

Regarding pullbacks of the Jacobi theta functions we need the following

**Lemma 6.5.9.** Let \( a, b \in o_{\mathbb{Q}(\sqrt{-3})} \). Then the following assertions are equivalent:
6.5 Separation theorems

\(\theta_{\mathcal{O}(\sqrt{-3})} \equiv \theta_{\mathcal{O}(\sqrt{-3})} \) for all \(s \in \mathbb{Z}\), we obtain

\[\left| \left\{ w \in \mathcal{O}(\sqrt{-3}) : \text{tr}(\bar{a}w) = s \right\} \right| = \left| \left\{ w \in \mathcal{O}(\sqrt{-3}) : \text{tr}(\bar{b}w) = s \right\} \right|\]

for all even \(s\). Write \(a = x + y \sqrt{3}\) and \(b = u + v \sqrt{3}\). After some orthogonal transformations applied on \(a\) and \(b\) we can assume that \(x, y, u, v \geq 0\). Then we have \(\delta_{s,x} = \delta_{s,u}\) for all \(s \geq 0\), which implies \(x = u\). From \(N(a) = N(b)\) we conclude \(y^2 = v^2\), thus \(y = v\).

**Theorem 6.5.10.** Let \(a, b \in \mathcal{O}(\sqrt{-3})\) such that \(N(a) = N(b)\) and \(\nu_3(N(a)) \equiv 1 \mod 2\). Let \(k \in \mathbb{N}\) and \(k \equiv 0 \mod 6\). For \(0 \neq \phi \in \mathcal{C}_k(\mathcal{O}(\sqrt{-3}))\) the following assertions are equivalent:

i) \(\phi[a] = \phi[b]\),

ii) \(a\) and \(b\) lie in the same \(\mathcal{O}(\mathcal{O}(\sqrt{-3}))\)-orbit,

iii) there exists \(\varepsilon \in \mathcal{O}(\sqrt{-3})\) such that \(a = \varepsilon b\) or \(a = \varepsilon b\),

iv) \(\iota_a\) and \(\iota_b\) are equivalent.

**Proof.** The equivalence of ii), iii) and iv) is obvious from the explicit description of \(\mathcal{O}(\mathcal{O}(\sqrt{-3}))\) and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The exponent of the discriminant group \(\mathcal{O}_3(\sqrt{-3})/\mathcal{O}(\sqrt{-3})\) is 3 and one has

\[3\mathcal{O}_3(\sqrt{-3})/3\mathcal{O}(\sqrt{-3}) = \left\{ a \in \mathcal{O}(\sqrt{-3}) : N(a) \equiv 0 \mod 3, N(a) \neq 0 \mod 9 \right\}\]

by 5.1.13. By assumption,

\[\nu_3(N(a)) = \nu_3(N(b)) = 2r + 1\]

for some \(r \geq 0\). After replacing \(a\) resp. \(b\) by \(3^{-r}a\) resp. \(3^{-r}b\), we can assume that

\[a, b \in 3\mathcal{O}_3(\sqrt{-3})/3\mathcal{O}(\sqrt{-3})\].

Then the claim follows from 3.6.4 together with 6.5.9.
The corresponding result for modular forms is given in the following

**Theorem 6.5.11.** Let \(a, b \in \mathcal{O}_Q(\sqrt{-3})\) such that \(N(a) = N(b)\) and \(v_3(N(a)) \equiv 1 \mod 2\). Let \(k \in \mathbb{N}\) and \(k \equiv 0 \mod 6\). Then the following assertions are equivalent:

i) \(F[a] = F[b]\) for some \(F \in \Gamma(\mathcal{O}_Q(\sqrt{-3}), k)\) and \(\phi_{1,F} \neq 0\),

ii) \(F[a] = F[b]\) for all \(F \in [\Gamma(\mathcal{O}_Q(\sqrt{-3}), k)]\),

iii) \(a\) and \(b\) lie in the same \(\mathcal{O}(\mathcal{O}_Q(\sqrt{-3}))\)-orbit,

iv) there exists \(\varepsilon \in \mathcal{O}_Q^\times(\sqrt{-3})\) such that \(a = \varepsilon b\) or \(a = \varepsilon \bar{b}\),

v) \(M_a\) and \(M_b\) are equivalent in the extended sense.

**Proof.** The equivalence of iii), iv) and v) follows from the explicit description of the group \(\mathcal{O}(\mathcal{O}_Q(\sqrt{-3}))\) and 6.2.12. Each of them implies ii). Assume that ii) holds. Since \(k \equiv 0 \mod 6\), there exists a hermitian Jacobi form \(0 \neq \phi \in J_{k,1}(\mathcal{O}_Q(\sqrt{-3}))\). Hence, i) holds with \(F = M\cdot\text{Lift}(\phi)\). Note that \(\phi_{1,F} = \phi \neq 0\). Furthermore, i) implies iii) by 6.3.16 and 6.5.10. \(\square\)

**Theorem 5.4.13** directly implies

**Theorem 6.5.12.** Let \(t \in \mathbb{N}\) such that \(v_3(t) \equiv 1 \mod 2\). Let \(M, N \in \text{Mod}(\Gamma(t), \Gamma(\mathcal{O}_Q(\sqrt{-3})))\).

Let \(k \in \mathbb{N}\) and \(k \equiv 0 \mod 6\). Then the following assertions are equivalent:

i) \(F|_k[M] = F|_k[N]\) for some \(F \in [U_2(\mathcal{O}_Q(\sqrt{-3})), k]\) and \(\phi_{1,F} \neq 0\),

ii) \(F|_k[M] = F|_k[N]\) for all \(F \in [\Gamma(\mathcal{O}_Q(\sqrt{-3})), k]\),

iii) \(M\) and \(N\) are equivalent in the extended sense.
6.5.3 Hurwitz order

Let $\mathcal{O}$ denote the Hurwitz quaternions and $\pi := (1+i_1)$ a slight abuse of notation.

**Lemma 6.5.13.** Let $a \in \mathcal{O}$ and $r \in \mathbb{N}_0$. Then for all $(z, w) \in \mathcal{H} \times \mathbb{H}_C$ one has

$$\vartheta_{\mathcal{O}}[\pi^r a](z, w) = \begin{cases} \vartheta_{\mathcal{O}}[a](z, 2^{r/2} \cdot w), & r \equiv 0 \mod 2, \\ -2 \cdot \vartheta_{\mathcal{O}}[a]|_{2N(a)}(2z, 2^{r/2} + 1 \cdot w), & r \equiv 1 \mod 2. \end{cases}$$

**Proof.** Assume that $r \equiv 0 \mod 2$. Then one has $\pi^r = 2^{r/2} \varepsilon$ for some $\varepsilon \in \mathcal{O}^\times$. Hence,

$$\vartheta_{\mathcal{O}}[\pi^r a](z, w) = \vartheta_{\mathcal{O}}(z, \pi^r aw) = \vartheta_{\mathcal{O}}(z, 2^{r/2} \varepsilon aw) = \vartheta_{\mathcal{O}}[a](z, 2^{r/2} w) = \vartheta_{\mathcal{O}}[a](z, 2^{r/2} w).$$

In the remaining case, $r-1$ is even and $\pi^{r-1} = 2^{r-1} \varepsilon$ for some $\varepsilon \in \mathcal{O}^\times$, that means $\pi^r a = 2^{r-1} \varepsilon \pi a$. Hence,

$$\vartheta_{\mathcal{O}}[\pi^r a](z, w) = \vartheta_{\mathcal{O}}[\pi a](z, 2^{r-1} \varepsilon w).$$

Therefore it suffices to prove the claim in case $r = 1$. In this case, we calculate

$$\vartheta_{\mathcal{O}}[\pi a](z, w) = \sum_{g \in \mathcal{O}} e^{2\pi i (N(g)z + \text{tr}(g)a)w}$$

$$= \sum_{g \in \mathcal{O}} e^{2\pi i (N(g)z + \text{tr}(g)a)w}$$

$$= \sum_{g \in \mathcal{O}} e^{2\pi i \left(\frac{1}{2} N(g) 2z + \text{tr}(g) 2aw\right)},$$

since $N(\frac{1}{2} \pi) = \frac{1}{2}$. From $\frac{1}{2} \pi \mathcal{O} = \mathcal{O}^2$ we deduce, that the last sum equals

$$\sum_{g \in \mathcal{O}^2} e^{2\pi i (N(g) 2z + \text{tr}(g) 2aw)} = -2 \cdot \vartheta_{\mathcal{O}}[a]|_{2N(a)}(2z, 2w).$$

**Corollary 6.5.14.** Let $a, b \in \mathcal{O}$ and $r \in \mathbb{N}_0$. Then the following assertions are equivalent:

i) $\vartheta_{\mathcal{O}}[a] = \vartheta_{\mathcal{O}}[b]$,

ii) $\vartheta_{\mathcal{O}}[\pi^r a] = \vartheta_{\mathcal{O}}[\pi^r b]$.

Regarding pullbacks of the Jacobi theta functions we need the following

**Lemma 6.5.15.** Let $a, b \in \mathcal{O}$. The following assertions are equivalent:

i) $\vartheta_{\mathcal{O}}[a] = \vartheta_{\mathcal{O}}[b]$,

ii) $a$ and $b$ lie in the same $\mathcal{O}(\mathcal{O})$-orbit,
iii) there are \( \delta, \varepsilon \in \mathcal{O}^\times \) and \( r \in \{0,1\} \) such that \( a = \pi^{-r} \varepsilon \delta \varepsilon b \pi^r \) or \( a = \pi^{-r} \varepsilon \delta \varepsilon b \pi^r \),

iv) \( \iota_a \) and \( \iota_b \) are equivalent.

Proof. First note that in all cases one has \( N(a) = N(b) \). The equivalence of ii), iii) and iv) is obvious from the explicit description of \( \mathcal{O}(\mathcal{O}) \) and 1.2.22. Each of them implies i). Hence we are left to prove that i) implies ii). First assume that \( N(a) = N(b) \) is odd. After multiplication with some suitable unit, we can assume that \( a, b \in \Lambda \), say

\[
a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3, \quad b = b_0 + b_1 i_1 + b_2 i_2 + b_3 i_3.
\]

Furthermore, \( \text{tr}(\overline{a}a) \equiv 1 \mod 2\mathbb{Z} \). Consequently, \( \text{tr}((g + \overline{a})a) \equiv 1 \mod 2\mathbb{Z} \) for all \( g \in \Lambda \). By comparing the Fourier coefficients of index \((1,s)\), \( s \in \mathbb{Z} \), we obtain

\[
|\{ w \in \mathcal{O}^\times : \text{tr}(\overline{a}w) = s \}| = |\{ w \in \mathcal{O}^\times : \text{tr}(\overline{b}w) = s \}|.
\]

Consequently, we deduce

\[
|\{ x \in \Lambda^\times : \text{tr}(\overline{a}w) = 2s \}| = |\{ g \in \Lambda^\times : \text{tr}(\overline{a}w) = 2s \}|
\]

for all \( s \in \mathbb{Z} \). Since \( \mathcal{O}(\Lambda) \) contains all signed permutations, we can assume \( a_i, b_i \geq 0 \) for all \( i = 0, \ldots, 3 \). In this case, the previous identity simplifies to

\[
|\{ i \in \{0,1,2,3\} : a_i = s \}| = |\{ i \in \{0,1,2,3\} : b_i = s \}|
\]

for all \( s \geq 0 \). Hence the sets of coefficients of \( a \) and \( b \) with respect to \( 1, i_1, i_2, i_3 \) coincide including multiplicity. Since \( \mathcal{O}(\Lambda) \) contains all permutations we have \( \sigma(a) = b \) for some \( \sigma \in \mathcal{O}(\Lambda) \), which can also be considered as an element of \( \mathcal{O}(\mathcal{O}) \). In the remaining case, we assume that both \( N(a) \) and \( N(b) \) are even. Let \( r = \nu_2(N(a)) = \nu_2(N(b)) \). From 6.5.14 and the first part, there is \( \sigma \in \mathcal{O}(\mathcal{O}) \) such that \( \sigma(\pi^{-r}a) = \pi^{-r}b \) or equivalently

\[
\pi^r \sigma(\pi^{-r}a) = b, \quad \text{i.e. } (l_{\pi}^{-r} \sigma l_{\pi}^{-r})(a) = b,
\]

if \( l_{\pi} \) denotes the left-multiplication by \( \pi \). Since

\[
\pi^r \mathcal{O} = \{ g \in \mathcal{O} : \nu_2(N(g)) \geq r \}
\]

and \( \pi \in \mathcal{I}(\mathcal{O}) \) we have

\[
l_{\pi}^{-r} \mathcal{O}(\mathcal{O}) l_{\pi}^{-r} = \mathcal{O}(\mathcal{O})
\]

due to the finiteness of \( \mathcal{O}(\mathcal{O}) \). Thus, \( l_{\pi}^{-r} \sigma l_{\pi}^{-r} \in \mathcal{O}(\mathcal{O}) \).

\[ \square \]

**Theorem 6.5.16.** Let \( a, b \in \mathcal{O} \) such that \( N(a) = N(b) \) and \( \nu_2(N(a)) \equiv 1 \mod 2 \). Let \( k \geq 4 \) and \( k \equiv 0 \mod 2 \). For \( 0 \neq \phi \in H^{1}_{\text{et},1}(\mathcal{O}) \) the following assertions are equivalent:

i) \( \phi[a] = \phi[b] \),

ii) \( a \) and \( b \) lie in the same \( \mathcal{O}(\mathcal{O}) \)-orbit,
iii) there are $\delta, \epsilon \in \mathcal{O}^\times$ and $r \in \mathbb{N}_0$ such that $a = \pi^{-r}\epsilon\delta b\epsilon\pi^r$ or $a = \pi^{-r}\epsilon\delta\bar{b}\epsilon\pi^r$.

iv) $\iota_a$ and $\iota_b$ are equivalent.

**Proof.** The equivalence of ii), iii) and iv) is obvious from the explicit description of $\mathcal{O}(\mathcal{O})$ and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The exponent of the discriminant group $\mathcal{O}^\sharp/\mathcal{O}$ is 2 and one has

$$2\mathcal{O}^\sharp \setminus 2\mathcal{O} = \{g \in \mathcal{O} : N(g) \equiv 0 \mod 2, N(g) \not\equiv 0 \mod 4\}.$$  

By assumption, 

$$\nu_2(N(a)) = \nu_2(N(b)) = 2r + 1$$

for some $r \geq 0$. After replacing $a$ resp. $b$ by $2^r a$ resp. $2^{-r}b$, we can assume, that 

$$a, b \in 2\mathcal{O}^\sharp \setminus 2\mathcal{O}.$$ 

Then the claim follows from 3.6.4 together with 6.5.15. □

The corresponding result for modular forms is given in the following

**Theorem 6.5.17.** Let $a, b \in \mathcal{O}$ such that $N(a) = N(b)$ and $\nu_2(N(a)) \equiv 1 \mod 2$. Let $k \geq 4$ and $k \equiv 0 \mod 2$. Then the following assertions are equivalent:

i) $F[a] = F[b]$ for some $F \in \mathcal{U}_2(\mathcal{O}, k)$ and $\phi_{1,F} \neq 0$,

ii) $F[a] = F[b]$ for all $F \in \mathcal{U}_2(\mathcal{O}, k)$,

iii) $a$ and $b$ lie in the same $\mathcal{O}(\mathcal{O})$-orbit,

iv) there are $\delta, \epsilon \in \mathcal{O}^\times$ and $r \in \{0, 1\}$ such that $a = \pi^{-r}\epsilon\delta b\epsilon\pi^r$ or $a = \pi^{-r}\epsilon\delta\bar{b}\epsilon\pi^r$,

v) $M_a$ and $M_b$ are equivalent in the extended sense.

**Proof.** The equivalence of iii), iv) and v) follows from the explicit description of the group $\mathcal{O}(\mathcal{O})$ and 6.2.12. Each of them implies ii). Assume that ii) holds. Since $k \equiv 0 \mod 2$ and $k \geq 4$, there exists a quaternionic Jacobi form $0 \neq \phi \in J_{\mathcal{H},1}^H(\mathcal{O})$. Hence, i) holds with $F = M\text{-Lift}(\phi)$. Note that $\phi_{1,F} = \phi \neq 0$. Furthermore, i) implies iii) by 6.3.16 and 6.5.16. □

Theorem 5.4.13 directly implies

**Theorem 6.5.18.** Let $t \in \mathbb{N}$ such that $\nu_2(t) \equiv 1 \mod 2$. Let $M, N \in \mathcal{M}(\mathcal{O})$. Let $k \geq 4$ and $k \equiv 0 \mod 2$. Then the following assertions are equivalent:

i) $F|_k[M] = F|_k[N]$ for some $F \in \mathcal{U}_2(\mathcal{O}, k)$ and $\phi_{1,F} \neq 0$,

ii) $F|_k[M] = F|_k[N]$ for all $F \in \mathcal{U}_2(\mathcal{O}, k)$,

iii) $M$ and $N$ are equivalent in the extended sense.
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