Pullback Theory for Functions of Lattice-Index and Applications to Jacobi- and Modular Forms

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Introduction

In his *Fundamenta nova theoriae functionum ellipticarum* [12], C.G.J. Jacobi studied the classical theta function, i.e.

$$\vartheta(z,w) = \sum_{m\in\mathbb{Z}} e^{\pi i(m^2z+2mw)}, \quad z \in \mathcal{H}, w \in \mathbb{C},$$

as it is written nowadays. It obeys two substantially different transformation laws, namely

$$\vartheta\left(-\frac{1}{z},\frac{w}{z}\right) = e^{\frac{\pi i w^2}{z}} \cdot \sqrt{\frac{z}{i}} \cdot \vartheta(z,w), \quad \vartheta(z+2,w) = \vartheta(z,w),$$

of modular type as well as

$$\vartheta(z, w+az+b) = e^{-\pi i(a^2z+2aw)} \cdot \vartheta(z,w), \quad a,b \in \mathbb{Z},$$

of elliptic type. The latter can be interpreted as a certain invariance property of $\vartheta(z,w)$ with respect to the lattice $\underline{\mathbb{Z}} = (\mathbb{Z}, x^2)$, i.e. the abelian group $\mathbb{Z}$ with underlying quadratic form $x^2, x \in \mathbb{Z}$. Higher dimensional generalizations of such forms were studied for example by G. Shimura [24]. The upper half-plane $\mathcal{H}$ is replaced by an analog of higher degree, namely the Siegel upper half-space $\mathcal{H}_n$, and $\mathbb{C}$ by a matrix space $\mathbb{C}^{r\times n}$. The invariance property of elliptic type will then be considered with respect to a lattice in $\mathbb{Q}^{r\times n}$. In the case $n = r = 1$, a systematic treatment of functions $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ being holomorphic and satisfying both a transformation law of modular and elliptic type, so-called Jacobi forms, was initiated by M. Eichler and D. Zagier in their monograph [8]. V.A. Gritsenko [9] studied the case $n = 1$ and arbitrary r and lifting constructions to orthogonal modular forms. C. Ziegler [27] took up again the work of Shimura in order to develop a theory of Jacobi forms of higher degree in the spirit of [8]. In this thesis we will drop the modularity condition and focus on functions satisfying a transformation law of elliptic type. Therefore we introduce the general setting:

Let $\underline{L} = (L, Q)$ be a positive definite and even lattice, i.e. L is a free $\mathbb{Z}$-module of finite rank and $Q : L \longrightarrow \mathbb{N}_0$ denotes some definite quadratic form on L. Let $L^{(n)} := L^{1\times n}$ and $L^{(n)}_{\mathbb{C}} := L^{(n)} \otimes_{\mathbb{Z}} \mathbb{C}$ together with

$$Q^{(n)} : L^{(n)} \longrightarrow \operatorname{Sym}_n^{\sharp}(\mathbb{Z}), \quad Q^{(n)}(l) := \tfrac{1}{2}\big(B(l_i, l_j)\big)_{i,j},$$

for $l = (l_1, \ldots, l_n) \in L^{(n)}$, where B is the bilinear form associated to (L, Q). We will study complex-valued holomorphic functions $\phi : \mathcal{H}_n \times L^{(n)}_{\mathbb{C}} \to \mathbb{C}$ satisfying the elliptic transformation law

$$\phi(Z, W + lZ + l') = e^{-2\pi i \operatorname{tr}\left(Q^{(n)}(l)\cdot Z + B^{(n)}(l,W)\right)} \cdot \phi(Z,W), \quad l, l' \in L^{(n)},$$

for $(Z,W) \in \mathcal{H}_n \times L_{\mathbb{C}}^{(n)}$. These functions are called elliptic of degree n and index $\underline{L}$ and the set of such functions is denoted by $\mathcal{E}^{(n)}(\underline{L})$. The generic case is given by $L = (\mathbb{Z}^r, S[x])$, where S denotes some even and positive definite matrix. Then $Q^{(n)}(X) = S[X]$ for $X \in L^{(n)} = \mathbb{Z}^{r\times n}$ and we recover exactly the second transformation law as in [27, 1.1.] or [24, (3.11), p. 47]. Let $\underline{L_0} = (L_0, Q_0)$ be an another lattice and $\iota : L_0 \longrightarrow L$ a homomorphism satisfying $Q \circ \iota = Q_0$. In this case, ι is called an embedding. Then the pullback

$$\phi\left[\iota^{(n)}\right](Z,W_0) := \phi\left(Z,\iota^{(n)}(W_0)\right), \quad (Z,W_0) \in \mathcal{H}_n \times (L_0)_{\mathbb{C}}^{(n)},$$

where $\iota^{(n)}$ is defined componentwise, belongs to $\mathcal{E}^{(n)}(\underline{L_0})$. The aim of this thesis is to study the pullback operator $\left[\iota^{(n)}\right]$ concerning the following aspects:

i) injectivity, surjectivity, bijectivity,

ii) dependency of the pullback operator on the choice of the embedding,

iii) extent of determination of the embedding by the knowledge of certain values of the pullback operator.

We will take up these issues frequently within this thesis. In the following, we provide an overview of each chapter:

In chapter one, we summarize the theory of lattices with all its necessary terms, where we attach great importance to a basis independent description in order to keep the notation clean and handy. After that we deal with the theory of embeddings of lattices. The chapter ends with the construction of some irreducible root lattices of small rank, which will be utilized later.

Chapter two is devoted to elliptic functions of lattice-index. We introduce matrix-valued quadratic forms and define the real Heisenberg group $H_{\mathbb{R}}^{(n)}(\underline{L})$. This group acts on a certain space associated to $\underline{L}$, the Jacobi half-space $\mathcal{H}_n^J(L)$. The elliptic functions of index $\underline{L}$ turn out to be invariants with respect to the integral Heisenberg group $H^{(n)}(\underline{L})$, which is characterized by certain integrality conditions. As a prototype we consider the well-known Jacobi theta functions associated to $\underline{L}$, which provide a free basis for the $\mathcal{O}(\mathcal{H}_n)$-module of elliptic functions of index $\underline{L}$, cf. also [27]. In order to have a notion of boundedness at the cusps for such a class of functions, we define regularity and cuspidality conditions. We introduce the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{Z})$, which acts upon $\mathcal{E}^{(n)}(\underline{L})$. The transformation law of modular type then exactly correspond to a certain invariance property with respect to this group. Jacobi forms will be defined as invariants of both the modular and the elliptic action together with certain conditions of boundedness. We end the chapter by calculating certain determinant characters of Weil representations associated to lattices of small rank.

In chapter three we introduce the main object of our studies, namely the aforementioned pullback operator. It transforms elliptic functions of index $\underline{L}$ into elliptic functions of index $\underline{L_0}$. In matrix language, this operator already occured in [27, 3.5. Lemma], but for a different purpose. It commutes with the modular action and behaves well with respect to regularity. The question that arises naturally in this context is in what cases the pullback operator can be an isomorphism. In order to give an answer to this question, we will take the algebraic point of view, i.e. we consider the pullback operator as a homomorphism of free modules. It will turn out that its representation matrix, which we call automorphic transfer, is as a vector-valued modular form with respect to some tensor product representation. In the equidimensional case we can consider its determinant which turns out as a (scalar-valued) modular form. This leads to the astonishing result that the pullback operator is an isomorphism if and only if $n = 1$ and the determinant is a nonzero multiple of a certain power of Dedekind's η-function. We proceed further by an explicit determination of the automorphic transfer and the modular determinant in the case $n = 1$ for distinguished lattices of small rank. As a by-product of the theory developed in this chapter, we prove - by using a tensor product construction - the existence of an infinite family $(\chi_{3\cdot 2^n})_{n\in\mathbb{N}_0}$ of nontrivial Siegel cusp forms of degree n and weight $3\cdot 2^n$, satisfying $\chi_6 = \eta^{12}$ together with a remarkable recurrence relation

$$\chi_{2^n\cdot 3}\begin{pmatrix} Z_j & 0 \\ 0 & Z_{n-j} \end{pmatrix} = \chi_{2^j\cdot 3}(Z_j)^{2^{n-j}} \cdot \chi_{2^{n-j}\cdot 3}(Z_{n-j})^{2^j}$$

for $0 \leq j \leq n$. Especially, these forms do not vanish on the diagonal $\mathcal{H}^n \subseteq \mathcal{H}_n$. We end this chapter by formulating and proving some sufficient ad-hoc criteria for injectiveness of restrictions of the pullback operator to submodules as well as by formulating and proving certain separation theorems regarding embeddings. In both cases we will restrict only to $n = 1$.

The fourth chapter is a short one and deals with isomorphisms between spaces of Jacobi forms of degree 1 with respect to certain lattices of small rank. Here we derive benefit from the explicit determination of some automorphic transfer matrices in chapter three. These allow to construct explicit lifts from Jacobi forms of rank-1-index to Jacobi forms of higher rank index by using matrix-vector multiplication on the basis of the attached space of vector valued modular forms. Partly, such isomorphisms were also known before, cf. [15, 16].

From chapter five on, we draw attention to modular forms. Here we take up again a theory of G. Köhler [18], who considered embeddings of paramodular groups $\Gamma(T)$, also known as "Siegelsche Stufengruppen", into hermitian modular groups $\mathrm{U}_n(\mathfrak{o}_K)$, where $\mathfrak{o}_K$ denotes the integral closure of $\mathbb{Z}$ in some imaginary-quadratic number field K. He focused on the problem in what cases $\Gamma(T)$ can be conjugated into $\mathrm{U}_n(\mathfrak{o}_K)$ via some matrix $M \in \mathrm{U}_n(\mathbb{C})$. In this case, M is called a modular embedding. Köhler gave a necessary and sufficient criterion for the existence of such embeddings. We start by briefly recapturing the basic terms in the theory of orders, where we include also the noncommutative case. Then we define the modular group associated to an order $\mathcal{O}$ as well as the paramodular group of polarization T, where T is an elementary divisor matrix. After having all necessary terms at hand, we consider modular embeddings. We extend Köhlers work by defining a notion of equivalence in order to measure *substantially different* embeddings. Under reasonable prerequisits we can adapt the proof of his main result in order to fit also into the noncommutative setting. As a by-product, we obtain some sort of normal form for an embedding M, what we will call a embedding of principal type. They are connected to the representability of T by a diagonal matrix over $\mathcal{O}$, what we will call $\mathcal{O}$-models. Due to [17], $\Gamma(T)$ admits a maximal discrete extension and if $\mathcal{O}$ is a principal ideal domain, this group acts on the set of equivalence classes of $\mathcal{O}$-models. We end the chapter by defining a pullback theory for modular forms with respect to modular embeddings, which will turn modular forms into paramodular forms and is compatible with the equivalence relation.

The sixth and final chapter is conducted by the question to what extent the modular embedding M is determined - up to equivalence - by the family of functions $F\big|_k[M], F \in [\mathrm{U}_n(\mathcal{O}), k]$. Already in the case $n = 2$, the equivalence relation on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$ is to restrictive in the sense, that there are inequivalent embeddings which induce the same pullbacked functions. In order to handle this, we develop a notion of equivalence in the extended sense. We approach the problem by shortly introducing hermitian and quaternionic Jacobi forms as certain constituents of modular forms. We can solve the aforementioned question at least for certain orders and under certain divisibility assumptions on the polarization. Here we make use of the seperation theorems given in chapter three.

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0 Basic Notation

$\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0$ is the set of non-negative integers and $\mathbb{Z}$ is the ring of integers. $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers, respectively. Without exception, i will always denote the imaginary unit. Given $z = x + iy \in \mathbb{C}$, then x is the real part and y the imaginary part of z. By $\mathcal{H}$ we denote the upper half-plane in $\mathbb{C}$, i.e. $\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$. The unit circle in $\mathbb{C}$ is denoted by S^1, i.e. $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The principal branch of the square root on $\mathbb{C}$ is denoted by $\sqrt{\cdot}$. The symbol $\mathbb{H}$ denotes the skew field of real Hamilton quaternions with standard basis $1, i_1, i_2, i_3 = i_1 i_2$. The symbol $\mathbb{F}$ will be utilized as a placeholder for a (possibly skew-)field.

For a set $\mathcal{X}$ we denote its cardinality by $|\mathcal{X}|$, which will be finite throughout this thesis. For $n, m \in \mathbb{N}$ we denote by $\mathcal{X}^{n\times m}$ the set of n-by-m matrices with entries in $\mathcal{X}$. Given $X \in \mathcal{X}^{n\times m}$ we write $X = (x_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ or $X = (x_{ij})$ for short in order to refer to the entries of X. We use the convention $\mathcal{X}^n := \mathcal{X}^{n\times 1}$ for the set of column vectors over $\mathcal{X}$ as well as $\mathcal{X}^{(n)} := \mathcal{X}^{1\times n}$ for the set of row vectors over $\mathcal{X}$. Note that the latter notation will be a priori double assigned. Unless specified otherwise, the meaning should be clear from context. For $X = (x_{ij}) \in \mathcal{X}^{n\times m}$ we denote by $X^t := (x_{ji}) \in \mathcal{X}^{m\times n}$ its transposed matrix. The diagonal matrix with diagonal entries $x_1, \ldots, x_n \in \mathcal{X}$ is denoted by $\operatorname{diag}(x_1, \ldots, x_n)$. Given matrices $X_1, \ldots, X_m$ with $X_j \in \mathcal{X}^{n_j}$ we extend the definition by writing $\operatorname{diag}(X_1, \ldots, X_m)$ for the quadratic block-diagonal matrix of size $k \times k$, where $k = n_1 + \cdots + n_m$.

Let $n \in \mathbb{N}$. We denote by $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ the cyclic group of order n and by S_n the symmetric group on n points. Let G be a group. If H is a subgroup of G, we will write $H \leq G$ and $H \trianglelefteq G$ if H is a normal subgroup of G. For $\mathcal{X} \subseteq G$ we denote by $\langle \mathcal{X} \rangle$ the subgroup of G generated by the elements of $\mathcal{X}$. In the case $\mathcal{X} = \{g_1, \ldots, g_m\}$ we write $\langle g_1, \ldots, g_m \rangle := \langle \mathcal{X} \rangle$. If G is abelian, we will sometimes write $\langle X \rangle_{\mathbb{Z}}$ instead of $\langle X \rangle$. For a subset $\mathcal{X}$ of G we denote by $\mathcal{N}_G(\mathcal{X})$ the normalizer of $\mathcal{X}$ in G. The center of G is denoted by $\mathcal{C}(G) := \mathcal{N}_G(G)$. The commutator subgroup of G is written as G' and $G^{\mathrm{ab}} := G/G'$ is the commutator factor group of G, which is implicitly identified with the group of abelian characters $G \longrightarrow \mathbb{C}^\times$. If φ is a homomorphism of groups, we will denote its kernel by $\ker f$. For an abelian group A with $a, b, c \in A$ and a subgroup H of A we write $a \equiv b \bmod H$ if $a - b \in H$ and $a \equiv b \bmod c$, if $a \equiv b \bmod \langle c \rangle$. Furthermore, $\operatorname{Hom}(A, B)$ denotes the set of homomorphisms $A \longrightarrow B$.

Let V be a finite-dimensional $\mathbb{F}$-vector space. The dimension of V over $\mathbb{F}$ is denoted by $\dim_{\mathbb{F}}(V)$, which will be always finite throughout this thesis. $\mathrm{GL}(V)$ denotes the general linear group of V. The orthogonal and unitary group of V are denoted by $\mathrm{O}(V)$ and $\mathrm{U}(V)$, respectively.

Let $\mathcal{R}$ be a ring with 1, not necessary commutative. By $I_n \in R^{n\times n}$ we denote the identity matrix of size $n \times n$ and by J_n the block-matrix

$$J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathcal{R}^{2n\times 2n}, \quad J := J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $i, j \in \{1, \ldots, n\}$ let $I_{ij}^{(n)}$ denote the matrix whose (i, j)-entry is 1 and 0 otherwise. We suppress the superscript if the size is clear from the context. By $\mathrm{GL}_n(\mathcal{R})$ we denote the general linear group over $\mathcal{R}^{n\times n}$, i.e.

$$\mathrm{GL}_n(\mathcal{R}) = \{U \in \mathcal{R}^{n\times n} : UV = VU = I_n \text{ for some } V \in \mathcal{R}^{n\times n}\}.$$

We set $\mathcal{R}^\times := \mathrm{GL}_1(\mathcal{R})$, the so-called unit group of $\mathcal{R}$. We write U^{-1} for the inverse matrix of $U \in \mathrm{GL}_n(\mathcal{R})$. The definition is extended by $U^{-t} := (U^{-1})^t$. In the case if $\mathcal{R}$ is commutative, we denote by $\mathrm{SL}_n(\mathcal{R})$ the special linear group in $\mathcal{R}^{n\times n}$. Let $A \in \mathcal{R}^{n\times n}$. Then $\det(A)$ stands for the determinant of A and - unless specified otherwise - $\mathrm{tr}(A)$ for the usual matrix trace of A. The subset of symmetric matrices in $\mathcal{R}^{n\times n}$ is denoted by $\mathrm{Sym}_n(\mathcal{R})$. Let $B \in \mathcal{R}^{n\times m}$. If $\mathcal{R}$ admits - possibly trivial - involution $^-$, we define $\overline{B}$ by applying $^-$ pointwisely on each entry of B. In this case, $\mathrm{Her}_n(\mathcal{R})$ will denote the subset of hermitian matrices in $\mathcal{R}^{n\times n}$, i.e. $\mathrm{Her}_n(\mathcal{R}) = \{A \in \mathcal{R}^{n\times n} : \overline{A}^t = A\}$. Note that $\mathrm{Her}_n(\mathcal{R}) = \mathrm{Sym}_n(\mathcal{R})$ in the case of the trivial involution. Furthermore, we set $A[B] := \overline{B}^t AB \in \mathrm{Her}_m(\mathcal{R})$. If the involution satisfies $\{r \in \mathcal{R} : \overline{r} = r\} \subseteq \mathbb{R}$, then we call A positive definite, written $A > 0$, resp. positive semi-definite, written $A \geq 0$, if $A[x] > 0$ resp. $A[x] \geq 0$ for all $0 \neq x \in \mathcal{R}^n$. By $\mathrm{Pos}_n(\mathcal{R})$ we denote the set of positive definite matrices over $\mathcal{R}$. For $A, B \in \mathrm{Her}_n(\mathcal{R})$ we write $A < B$, if $B - A > 0$, and $A \leq B$, if $B - A \geq 0$.

If a set $\mathcal{X}$ admits a complex structure, we denote by $\mathcal{O}(\mathcal{X})$ the ring of holomorphic functions $\mathcal{X} \longrightarrow \mathbb{C}$ and by $\mathrm{Bih}\,\mathcal{X}$ the group of biholomorphic automorphism of $\mathcal{X}$.

For finite $\mathcal{X} \subseteq \mathbb{R}$ we denote by $\max \mathcal{X}$ the maximum of $\mathcal{X}$ with the convention $\max \emptyset := -\infty$. Finally, for $m, n \in \mathbb{N}$ we write $m|n$ if m divides n and $m||n$, if $m|n$ and $\gcd\left(\frac{n}{m}, n\right) = 1$, where gcd denotes the greatest common divisor. For $k \geq 0$, we set $\sigma_k(m) := \sum_{d|m} d^k$, where d runs through the positive divisors of m. By B_k we denote the k-th Bernoulli number.

1 Lattices and Embeddings

1.1 Lattices

We repeat the basic notions in the theory of lattices and quadratic forms, where we refer to [7], [22] or [3].

Definition 1.1.1. *A* lattice $\underline{L}$ *is a pair* (L, Q)*, where*

- L *is a free* $\mathbb{Z}$*-module of finite rank,*
- $Q : L \longrightarrow \mathbb{R}$ *is a* quadratic form*, i.e.*

$$Q(\alpha l) = \alpha^2 Q(l) \qquad \text{(Homogeneity)}$$
$$Q(l + l') + Q(l - l') = 2Q(l) + 2Q(l') \qquad \text{(Parallelogram law)}$$

holds for all $l, l' \in L$ *and* $\alpha \in \mathbb{Z}$*.*

The rank *of* $\underline{L}$ *is defined as the rank of the underlying* $\mathbb{Z}$*-module and denoted by* $r_{\underline{L}}$*.*

If $\underline{L} = (L, Q)$ is a lattice and Q is known from the context, we will sometimes refer to the lattice L instead of $\underline{L}$.

A direct verification yields the following

Lemma 1.1.2. *Let* L *be a free* $\mathbb{Z}$*-module of finite rank and* $Q : L \longrightarrow \mathbb{R}$*. Define* $B : L \times L \longrightarrow \mathbb{R}$ *by*

$$B(l, l') = Q(l + l') - Q(l) - Q(l') \qquad \text{(Polarization identity)}$$

for $l, l' \in L$*. Then the following statements are equivalent:*

a) Q *is a quadratic form on* L*, i.e.* (L, Q) *is a lattice,*

b) B *is a* $\mathbb{Z}$*-bilinear form on* L*.*

In this case, B *is called the* bilinear form *associated to* (L, Q)*.*

We introduce some constructions of lattices, namely orthogonal sums and scaling:

Definition 1.1.3. *a) Let $m \in \mathbb{N}$ and $\underline{L_1}, \ldots, \underline{L_m}$ be lattices with underlying quadratic forms $Q_1, \ldots, Q_m$. The* orthogonal sum *$\underline{L} := \bigoplus_{i=1}^m \underline{L_i}$ is the lattice $\underline{L} = (L, Q)$, where*

$$L := \bigoplus_{i=1}^{m} L_i, \quad Q := \bigoplus_{i=1}^{m} Q_i,$$

i.e. $Q(l_1, \ldots, l_m) := \sum_{i=1}^m Q_i(l_i)$ for $l_i \in L_i, i = 1, \ldots, m$. For a lattice $\underline{L} = (L, Q)$ we define $m\underline{L} := \bigoplus_{i=1}^m \underline{L}$.

b) Let $\underline{L} = (L, Q)$ be a lattice. For $t \in \mathbb{N}$ we define $\underline{L}(t) := (L, tQ)$. In thise case we say that $\underline{L}(t)$ arises from $\underline{L}$ by scaling *with t.*

We repeat some lattice-theoretic terms:

Definition 1.1.4. *Let $\underline{L} = (L, Q)$ be a lattice with associated bilinear form B.*

a) $\underline{L}$ is called non-degenerate*, if $B(l, \cdot) \not\equiv 0$ for all $0 \neq l \in L$,*

b) $\underline{L}$ is called integral*, if $B(l, l') \in \mathbb{Z}$ for all $l, l' \in L$,*

c) $\underline{L}$ is called even*, if $B(l, l) \in 2\mathbb{Z}$ for all $l \in L$,*

d) $\underline{L}$ is called positive definite*, if $B(l, l) > 0$ for all $0 \neq l \in L$.*

Unless specified otherwise, $\underline{L} = (L, Q)$ will always denote a positive definite and even lattice, i.e. $Q : L \longrightarrow \mathbb{N}_0$ and $Q(l) = 0$ if and only if $l = 0$.

Definition 1.1.5. *Let $\underline{L} = (L, Q)$ be a lattice and $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The $\mathbb{F}$-vector space*

$$L_{\mathbb{F}} := L \otimes_{\mathbb{Z}} \mathbb{F},$$

arising from L by extension of scalars of $\mathbb{F}$, is called the ambient space *of L over $\mathbb{F}$. By $\mathbb{F}$-linearity, the bilinear form B associated to $\underline{L}$ extends uniquely to $L_{\mathbb{F}}$ and is again denoted by B. By polarization, the quadratic form Q extends uniquely to a quadratic form on $L_{\mathbb{F}}$ and is again denoted by Q. The pair $\underline{L_{\mathbb{F}}} = (L_{\mathbb{F}}, Q)$ is called the* quadratic space *of $\underline{L}$ over $\mathbb{F}$.*

Note that $\dim_{\mathbb{F}} L_{\mathbb{F}} = r_{\underline{L}}$ for $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

The structure-preserving maps between lattices are described in the following

Definition 1.1.6. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices.*

a) An isometry *$\sigma : \underline{L_0} \longrightarrow \underline{L}$ is an isomorphism $\sigma : L_0 \longrightarrow L$ of abelian groups satisfying $Q \circ \sigma = Q_0$. In this case, $\underline{L}$ and $\underline{L_0}$ are called* isometric.

b) An isometry $\sigma : \underline{L} \longrightarrow \underline{L}$ is called orthogonal transformation *or* automorphism *of $\underline{L}$. The group of orthogonal transformations of $\underline{L}$ is denoted by $\mathrm{O}(\underline{L})$, called the* orthogonal group *of $\underline{L}$.*

Note that $\mathrm{O}(\underline{L})$ is a finite group.

Remark 1.1.7. *Let $\underline{L}$ be a lattice and $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$. Since L contains a $\mathbb{F}$-basis of $L_{\mathbb{F}}$, every orthogonal transformation of $\underline{L}$ extends in a unique way to an orthogonal transformation of the quadratic space $\underline{L_{\mathbb{F}}}$. Hence, the orthogonal group $\mathrm{O}(\underline{L})$ is characterized by*

$$\mathrm{O}(\underline{L}) = \{\sigma : L_{\mathbb{F}} \longrightarrow L_{\mathbb{F}} : \sigma \ \mathbb{F}\text{-linear}\,, Q \circ \sigma = Q \text{ and } \sigma(L) = L\}\,.$$

Definition 1.1.8. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) The pair $\underline{L}^* := (L^*, Q|_{L^*})$, where

$$L^* := \{v \in L_{\mathbb{Q}} : B(v, l) \in \mathbb{Z} \text{ for all } l \in L\}\,,$$

is called the dual lattice *of $\underline{L}$.*

b) The quotient group L^*/L is called the discriminant group *of $\underline{L}$.*

Similarly, if $\underline{L} = (L, Q)$ is a lattice and Q is given in the context, we will sometimes to refer to L^* as the dual lattice of L.

Remark 1.1.9. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) Every $\varphi \in \mathrm{Hom}(L, \mathbb{Z})$ can be uniquely extended to a $\mathbb{Q}$-linear functional $L_{\mathbb{Q}} \longrightarrow \mathbb{Q}$, which is again denoted by φ. Since B is non-degenerate, there is a unique $\mu \in L_{\mathbb{Q}}$ such that $\varphi(l) = B(\mu, l)$ holds for all $l \in L_{\mathbb{Q}}$. Hence, $\mu \in L^*$. Consequently, the map

$$L^* \longrightarrow \mathrm{Hom}(L, \mathbb{Z}), \quad \mu \mapsto B(\mu, \cdot)$$

is an isomorphism.

b) Let $(b_1, \ldots, b_r)$ denote some $\mathbb{Z}$-basis of L. Then $(b_1, \ldots, b_r)$ is also a $\mathbb{Q}$-basis of $L_{\mathbb{Q}}$. It is easily seen that the dual basis $(b_1^*, \ldots, b_r^*)$ of the vector space $L_{\mathbb{Q}}$ provides a free basis for L^*. Hence, $\underline{L}^*$ is again a lattice. Note that $\underline{L}^* = (L^*, Q|_{L^*})$ may not be integral in general.

c) In view of b), the level *of $\underline{L}$ is defined as*

$$N_{\underline{L}} := \min\{q \in \mathbb{N} : qQ(\mu) \in \mathbb{Z} \text{ for all } \mu \in L^*\}\,,$$

i.e. $N_{\underline{L}}$ is the smallest natural number q, such that $\underline{L}^*(q)$ is an even lattice.

Definition 1.1.10. *Let $\underline{L} = (L, Q)$ be a lattice and $(b_1, \ldots, b_r)$ a basis of L. The matrix*

$$S_{(b_1,\ldots,b_r)}(\underline{L}) := \big(B(b_i, b_j)\big)_{1 \le i,j \le r} \in \mathrm{Sym}_r(\mathbb{Z})$$

is called the Gram matrix *of $\underline{L}$ with respect to $(b_1, \ldots, b_r)$.*

The generic case is described in the following

Remark 1.1.11. *Let $S \in \operatorname{Sym}_r(\mathbb{Z})$ be positive definite and even, i.e $s_{ii} \in 2\mathbb{Z}$ for $i = 1, \dots, r$. Then the map*

$$Q_S(x) := \tfrac{1}{2} S[x], \quad x \in \mathbb{Z}^r,$$

is a quadratic form on $\mathbb{Z}^r$ and the pair $(\mathbb{Z}^r, Q_S)$ is a lattice. The dual lattice is given by $(S^{-1}\mathbb{Z}^r, Q_S)$. Conversely, let $\underline{L} = (L, Q)$ be a lattice and S denote its Gram matrix with respect to some basis $b_1, \dots, b_r$ of L. Let $\kappa : L \longrightarrow \mathbb{Z}^r$ be a coordinate system for L with respect to $b_1, \dots, b_r$. Then one has

$$B(l, l') = \kappa(l)^t S \kappa(l') \text{ for all } l, l' \in L,$$

i.e. κ is an isometry of the lattices (L, Q) and $(\mathbb{Z}^r, Q_S)$. Hence,

$$L^*/L \cong S^{-1}\mathbb{Z}^r/\mathbb{Z}^r \cong \mathbb{Z}^r/S\mathbb{Z}^r$$

and the cardinality $|L^*/L| = \det S$ is finite and independent from the choice of S.

This justifies the following

Definition 1.1.12. *Let $\underline{L}$ be a lattice. The finite number $\det \underline{L} := |L^*/L|$ is called the* determinant *or* discriminant *of $\underline{L}$.*

The identity

$$Q(\mu) - Q(\mu') = Q(\mu - \mu') - B(\mu' - \mu, \mu')$$

for $\mu, \mu' \in L_{\mathbb{R}}$ gives rise to the following

Definition 1.1.13. *Let $\underline{L} = (L, Q)$ be a lattice. Then the map*

$$\overline{Q} : L^*/L \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad \overline{Q}(\mu + L) := Q(\mu) + \mathbb{Z}, \quad \mu \in L^*,$$

is well-defined called the discriminant form *of $\underline{L}$.*

Remarks 1.1.14. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) The pair $\underline{L^*/L} = (L^*/L, \overline{Q})$ is called the finite quadratic module *associated to $\underline{L}$. It is easy to see, that the discriminant form takes its values in $N_{\underline{L}}^{-1}\mathbb{Z}/\mathbb{Z}$.*

b) Since $\mathrm{O}(\underline{L})$ acts naturally on L^*, it also acts on L^*/L via the assignment

$$(\sigma, \mu + L) \mapsto \sigma(\mu) + L, \quad \mu \in L^*, \sigma \in \mathrm{O}(L).$$

This action respects the discriminant form, i.e. $\overline{Q} \circ \sigma = \overline{Q}$.

c) The action given in b) induces a homomorphism $\mathrm{O}(\underline{L}) \longrightarrow \mathrm{O}\left(\underline{L^*/L}\right)$, where

$$\mathrm{O}\left(\underline{L^*/L}\right) := \{\sigma : L^*/L \to L^*/L : \sigma \text{ isomorphism}, \overline{Q} \circ \sigma = \overline{Q}\}$$

is the finite orthogonal group *of $\underline{L^*/L}$.*

Definition 1.1.15. *Let $\underline{L} = (L, Q)$ be a lattice. The kernel of the homomorphism given in 1.1.14 c) is called the* discriminant kernel *of $\underline{L}$ and is denoted by $O_d(\underline{L})$.*

Definition 1.1.16. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) An element $l \in L$ is called a root *of $\underline{L}$, if $Q(l) = 1$. The set of roots of $\underline{L}$ is denoted by $R(\underline{L})$.*

b) $\underline{L}$ is called a root lattice, *if L is generated by $R(\underline{L})$.*

Definition 1.1.17. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) For $l \in R(\underline{L})$ the reflection *$s_l \in O(\underline{L})$ along l is defined as*

$$s_l : L \longrightarrow L, \quad s_l(l') := l' - B(l, l')l, \quad l' \in L.$$

b) The Weyl group *of $\underline{L}$ is defined as*

$$W(\underline{L}) := \langle s_l : l \in R(\underline{L}) \rangle \leq O_d(\underline{L}),$$

with the convention $W(\underline{L}) := \{1\}$, if $R(\underline{L}) = \emptyset$.

Definition 1.1.18. *A lattice $\underline{L}$ is called* irreducible, *if it does not split into an orthogonal sum of two lattices.*

$\underline{L}$	$r_{\underline{L}}$	L^*/L	$\det \underline{L}$	$\lvert R(\underline{L})\rvert$
$\underline{A_r}$	r	$\mathbb{Z}_{r+1}$	$r+1$	$r(r+1)$
$\underline{D_r}$	r	$\begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2, & r \text{ even} \\ \mathbb{Z}_4, & r \text{ odd} \end{cases}$	4	$2r(r-1)$
$\underline{E_6}$	6	$\mathbb{Z}_3$	3	72
$\underline{E_7}$	7	$\mathbb{Z}_2$	2	126
$\underline{E_8}$	8	$\{0\}$	1	240

Table 1.1: Classification of the irreducible root lattices

Theorem 1.1.19. *(cf. [7, Thm. 1.2])*

a) Table 1.1 classifies the irreducible root lattices completely up to isometry.

b) Every root lattice is the orthogonal sum of irreducible root lattices given in table 1.1.

For concrete realizations confer [7, p. 14, p. 23ff]. Note that in the following we will use the constructions of $\underline{A_1}, \underline{A_2}, \underline{D_4}, \underline{E_6}, \underline{E_7}$ and $\underline{E_8}$ given in section 1.3.

Remark 1.1.20. *a) Table 1.2 classifies the irreducible root lattices, the structure of their discriminant groups and representatives for $\overline{Q}$.*

$\underline{L}$	$r_{\underline{L}}$	L^*/L	representatives	$\overline{Q}$	$N_{\underline{L}}$
$\underline{A_r}$	r	$\mathbb{Z}_{r+1}$	$\begin{cases} \mu_0 := 0 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_r \end{cases}$	$\begin{cases} 0 \\ \vdots \\ \frac{i(r+1-i)}{2(r+1)} \\ \vdots \\ \frac{r}{2(r+1)} \end{cases}$	$\begin{cases} r+1, & r \equiv 0 \bmod 2, \\ 2(r+1), & r \equiv 1 \bmod 2 \end{cases}$
$\underline{D_r}$ r even	r	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{cases} \mu_0 := 0 \\ \mu_1 \\ \mu_2 \\ \mu_3 := \mu_1 + \mu_2 \end{cases}$	$\begin{cases} 0 \\ \frac{r}{8} \\ \frac{1}{2} \\ \frac{r}{8} \end{cases}$	$\begin{cases} 1, & r \equiv 0 \bmod 8, \\ 2, & r \equiv 4 \bmod 8, \\ 4, & r \equiv 2 \bmod 8 \end{cases}$
$\underline{D_r}$ r odd	r	$\mathbb{Z}_4$	$\begin{cases} \mu_0 := 0 \\ \mu_1 \\ \mu_2 := 2\mu_1 \\ \mu_3 := 3\mu_1 \end{cases}$	$\begin{cases} 0 \\ \frac{r}{8} \\ \frac{1}{2} \\ \frac{r}{8} \end{cases}$	8
$\underline{E_6}$	6	$\mathbb{Z}_3$	$\begin{cases} \mu_0 := 0 \\ \mu_1 \\ \mu_2 := -\mu_1 \end{cases}$	$\begin{cases} 0 \\ \frac{2}{3} \\ \frac{2}{3} \end{cases}$	3
$\underline{E_7}$	7	$\mathbb{Z}_2$	$\begin{cases} \mu_0 := 0 \\ \mu_1 \end{cases}$	$\begin{cases} 0 \\ \frac{3}{4} \end{cases}$	4
$\underline{E_8}$	8	$\{0\}$	$\mu_0 := 0$	0	1

Table 1.2: Structure of L^*/L and values of $\overline{Q}$.

b) Table 1.3 classifies the irreducible root lattices, representatives for the quotient group $\mathrm{O}(\underline{L})/\mathrm{O}_d(\underline{L})$ *and the corresponding orbits on* L^*/L.

$\underline{L}$	$\mathrm{O}(\underline{L})\,/\,\mathrm{O}_d(\underline{L})$	representatives	orbits on L^*/L	$\lvert$orbits$\rvert$
$\underline{A_1}$	$\{0\}$	id	$\{0\}, \{\mu_1\}$	2
$\underline{A_r}$, $r > 1$	$\mathbb{Z}_2$	$\pm$id	$\{0\}, \{\mu_i, -\mu_i\}, 1 \leq i \leq \left\lfloor \frac{r+1}{2} \right\rfloor$	$1 + \left\lfloor \frac{r+1}{2} \right\rfloor$
$\underline{D_4}$	S_3	$\langle (1,2,3), (1,3) \rangle$	$\{0\}, \{\mu_1, \mu_2, \mu_3\}$	2
$\underline{D_r}$, $r \neq 4$	$\mathbb{Z}_2$	$\langle (1,3) \rangle$	$\{0\}, \{\mu_2\}, \{\mu_1, \mu_3\}$	3
$\underline{E_6}$	$\mathbb{Z}_2$	$\pm$id	$\{0\}, \{\mu_1, -\mu_1\}$	2
$\underline{E_7}$	$\{0\}$	id	$\{0\}, \{\mu_1\}$	2
$\underline{E_8}$	$\{0\}$	id	$\{0\}$	1

Table 1.3: Irreducible root lattices, representatives of the discriminant kernel and orbits

1.2 Embeddings

In this section, we define the fundamental object of our studies:

Definition 1.2.1. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices. An* embedding $\iota : \underline{L_0} \longrightarrow \underline{L}$ *of $\underline{L_0}$ into $\underline{L}$ is a $\mathbb{Z}$-homomorphism $\iota : L_0 \longrightarrow L$ such that $Q \circ \iota = Q_0$.*

Definition 1.2.2. *Let $\underline{L} = (L, Q)$ be a lattice. A lattice $\underline{L_0} = (L_0, Q_0)$ is called a* sublattice *of $\underline{L}$, if $L_0 \leq L$ and $Q_0 = Q\big|_{L_0}$.*

Conversely, since subgroups of free abelian groups are free itself, $(L_0, Q|_{L_0})$ is a sublattice of $\underline{L}$ for every subgroup $L_0 \leq L$.

We give a useful connection between embeddings and sublattices:

Proposition 1.2.3. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices. Then there is an one-to-one correspondence between the embeddings of $\underline{L_0}$ into $\underline{L}$ and the sublattices of $\underline{L}$, which are isometric to $\underline{L_0}$. More precisely, if $\iota : \underline{L_0} \longrightarrow \underline{L}$ is an embedding, then $\underline{\iota(L_0)} = (\iota(L_0), Q|_{\iota(L_0)})$ is a sublattice of $\underline{L}$ which is isometric to $\underline{L_0}$. Conversely, if $\underline{L}'$ is a sublattice of $\underline{L}$ and $\iota : \underline{L_0} \longrightarrow \underline{L}'$ is an isometry, then $\iota : \underline{L_0} \longrightarrow \underline{L}$ is an embedding.*

Some trivial observations are contained in the following

Remark 1.2.4. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices.*

a) Since Q_0 is definite, any embedding $\iota : \underline{L_0} \longrightarrow \underline{L}$ is necessarily injective.

b) The surjective embeddings of $\underline{L_0}$ into $\underline{L}$ are exactly the isometries $\underline{L_0} \longrightarrow \underline{L}$.

c) Every orthogonal transformation $\sigma \in \mathrm{O}(\underline{L})$ is an embedding of $\underline{L}$ into $\underline{L}$.

We introduce orthogonal complements:

Definition 1.2.5. *Let* $\underline{L} = (L, Q)$ *be a lattice.*

a) For a subset $K \subseteq L$*, the* $\mathbb{Q}$*-vector space*

$$K^{\perp} = \{l \in L_{\mathbb{Q}} : B(l, x) = 0 \text{ for all } x \in K\}$$

is called the orthogonal complement *of* K *in* $L_{\mathbb{Q}}$*.*

b) For a sublattice $\underline{L_0} = (L_0, Q_0)$ *we define* $L_0^{\perp,L} := L_0^{\perp} \cap L$*. The pair*

$$\underline{L_0}^{\perp,\underline{L}} := \left(L_0^{\perp,L}, Q|_{L_0^{\perp,L}}\right)$$

is called the orthogonal complement *of* $\underline{L_0}$ *in* $\underline{L}$*.*

Remark 1.2.6. *Let* $\underline{L} = (L, Q)$ *be a lattice and* $\underline{L_0}$ *a sublattice of* $\underline{L}$*. Then* $\underline{L_0}^{\perp,\underline{L}}$ *is a sublattice of* $\underline{L}$ *and one has*

$$r_{\underline{L}} = r_{\underline{L_0}} + r_{\underline{L_0}^{\perp,\underline{L}}}.$$

Remark 1.2.7. *Let* $\underline{L} = (L, Q)$ *and* $\underline{L_0} = (L_0, Q_0)$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. For* $\mu \in L^*$ *we consider the map*

$$L_0 \longrightarrow \mathbb{Z}, \quad l_0 \mapsto B(\iota(l_0), \mu), \quad l_0 \in L_0,$$

which belongs to $\mathrm{Hom}(L_0, \mathbb{Z})$*. By 1.1.9 there is a unique* $\mu_0 \in L_0^*$ *such that*

$$B(\iota(l_0), \mu) = B_0(l_0, \mu_0)$$

for all $l_0 \in L_0$*. The assignment* $\mu \to \mu_0$ *yields a* $\mathbb{Z}$*-homomorphism* $L^* \longrightarrow L_0^*$*.*

1.2.7 justifies the following

Definition 1.2.8. *Let* $\underline{L} = (L, Q)$ *and* $\underline{L_0} = (L_0, Q_0)$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. Then the map* $\iota^* : L^* \longrightarrow L_0^*$ *defined by the identity*

$$B(\iota(l_0), \mu) = B_0(l_0, \iota^*(\mu)) \text{ for all } l_0 \in L_0, \mu \in L^*,$$

is called the dual *of* ι*.*

Remark 1.2.9. *Let* $\underline{L} = (L, Q)$ *and* $\underline{L_0} = (L_0, Q_0)$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. For* $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ *the dual* $\iota^* : L^* \longrightarrow L_0^*$ *is uniquely extended by* $\mathbb{F}$*-linearity to a map* $L_{\mathbb{F}} \longrightarrow (L_0)_{\mathbb{F}}$*, which will again be denoted by* ι^**. It is uniquely determined by the identity*

$$B(\iota(l_0), \mu) = B_0(l_0, \iota^*(\mu)), \quad l_0 \in (L_0)_{\mathbb{F}}, \mu \in L_{\mathbb{F}}.$$

Obviously, $\iota^* \circ \iota = \mathrm{id}_L$*.*

For later use we need the following

Lemma 1.2.10. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then for $\mu \in L_{\mathbb{R}}$ the following assertions hold:*

a) $Q_0(\iota^*(\mu)) \leq Q(\mu)$,

b) $Q_0(\iota^*(\mu)) = Q(\mu)$ *if and only if* $\mu = \iota(\xi)$ *for some* $\xi \in L_0^*$.

Proof. a) There is nothing to prove in the case $Q_0(\iota^*(\mu)) = 0$. Otherwise, we apply the Cauchy-Schwarz inequality in order to obtain

$$\begin{aligned} B_0(\iota^*(\mu), \iota^*(\mu))^2 &= B(\iota\iota^*(\mu), \mu)^2 \\ &\leq B(\iota\iota^*(\mu), \iota\iota^*(\mu))B(\mu, \mu) \\ &= B_0(\iota^*(\mu), \iota^*(\mu))B(\mu, \mu). \end{aligned}$$

This yields

$$Q_0(\iota^*(\mu)) = \frac{1}{2}B_0(\iota^*(\mu), \iota^*(\mu)) \leq \frac{1}{2}B(\mu, \mu) = Q(\mu).$$

b) If $\iota(\xi) = \mu$, then one has $\iota^*(\mu) = \xi$ and

$$Q_0(\iota^*(\mu)) = Q_0(\xi) = Q(\mu).$$

holds. Assume that $Q_0(\iota^*(\mu)) = Q(\mu)$. Since there is nothing to prove in the case $\mu = 0$, we suppose $\mu \neq 0$. Hence, $\xi := \iota^*(\mu) \neq 0$. Due to the Cauchy-Schwarz inequality in a), $\iota(\xi)$ and μ must be linearly dependent over $\mathbb{Q}$. Thus, $\iota(\xi) = \alpha\mu$ for some $\alpha \in \mathbb{Q}$. We apply ι^* in order to obtain

$$\xi = \iota^*\iota(\xi) = \alpha\iota^*(\mu) = \alpha\xi,$$

i.e. $\alpha = 1$. □

Proposition 1.2.11. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then the kernel of $\iota^* : L^* \longrightarrow L_0^*$ equals $L^* \cap \iota(L_0)^\perp$.*

Proof. Since L_0 contains a $\mathbb{Q}$-basis of $(L_0)_{\mathbb{Q}}$, one has $\iota^*(\mu) = 0$ if and only if

$$B(\mu, \iota(l_0)) = B(\iota^*(\mu), l_0) = 0$$

for all $l_0 \in L_0$, i.e. if and only if $\mu \in L^* \cap \iota(L_0)^\perp$. □

The generic case is treated in the following

Example 1.2.12. *Let $S \in \mathrm{Sym}_r(\mathbb{Z}), S_0 \in \mathrm{Sym}_{r_0}(\mathbb{Z})$ be positive definite and even. For $A \in \mathbb{Z}^{r\times r_0}$ we define*

$$\iota_A : \mathbb{Z}^{r_0} \longrightarrow \mathbb{Z}^r, \quad \iota(\lambda) := A\lambda, \quad \lambda \in \mathbb{Z}^{r_0}.$$

Then the following statements are equivalent:

i) ι_A *is an embedding* $(\mathbb{Z}^{r_0}, Q_{S_0}) \longrightarrow (\mathbb{Z}^r, Q_S)$*,*

ii) $S[A] = S_0$.

In this case, the dual ι_A^* *is given by*

$$S^{-1}\mathbb{Z}^r \longrightarrow S_0^{-1}\mathbb{Z}^{r_0}, \quad \iota_A^*(\mu_S) = S_0^{-1}A^t S\mu_S, \quad \mu_S \in S^{-1}\mathbb{Z}^r$$

and one has

$$\ker(\iota_A^*) = G^{-1} \cdot \ker(A^t).$$

Definition 1.2.13. *Let* $\underline{L} = (L, Q)$ *be a lattice.*

a) *A sublattice* $\underline{L_0}$ *of* $\underline{L}$ *is called* primitive *in* $\underline{L}$*, if* L/L_0 *is a free* $\mathbb{Z}$*-module.*

b) *An element* $l \in L$ *is called primitive, if* $\underline{\langle l \rangle}$ *is primitive in* $\underline{L}$*.*

A characterization of primitive lattices is given in the next

Lemma 1.2.14. *Let* $\underline{L} = (L, Q)$ *be a lattice.*

a) *For a sublattice* $\underline{L_0}$ *of* $\underline{L}$ *the following statements are equivalent:*
   i) $\underline{L_0}$ *is primitive in* $\underline{L}$*,*
   ii) there is a sublattice $\underline{L_1}$ *of* $\underline{L}$*, such that* $L = L_0 \oplus L_1$ *as an inner direct sum.*

b) *For* $l \in L$ *the following statements are equivalent:*
   i) l *is primitive,*
   ii) $d^{-1}l \in L$ *for* $0 \neq d \in \mathbb{Z}$ *implies* $d = \pm 1$*,*
   iii) $\mathbb{Q}l \cap L = \mathbb{Z}l$*,*
   iv) $B(l, L^*) = \mathbb{Z}$*.*

Proof. a) Its obvious that ii) implies i). Hence assume that L/L_0 is free and let $l_1, \ldots, l_m \in L$ such that

$$L/L_0 = \bigoplus_{i=1}^{m} \langle l_i + L_0 \rangle.$$

Let $L_1 := \langle l_1, \ldots, l_m \rangle_{\mathbb{Z}}$. By construction, $L_1 \cap L_0 = \{0\}$ and $L = L_0 + L_1$.

b) The equivalence of i) and ii) follows from a) and ii), iii), iv) are simple reformulations. Note in iv) that $L^{**} = L$. □

Lemma 1.2.15. *Let* $\underline{L} = (L, Q)$ *be a lattice and* $\underline{L_0} \leq \underline{L}$*. Then the following assertions hold:*

a) *The sublattice* $\underline{L_0}^{\perp,\underline{L}}$ *is primitive.*

b) If $\underline{L_0}$ is primitive in $\underline{L}$, then one has $\left(\underline{L_0}^{\perp,\underline{L}}\right)^{\perp,\underline{L}} = \underline{L_0}$.

c) There is a unique lattice $\underline{L_1}$, such that $\underline{L} \geq \underline{L_1} \geq \underline{L_0}$ with $\underline{L_1}$ primitive in $\underline{L}$ and L_1/L_0 finite.

Proof. a) Let ι denote the inclusion $L_0 \hookrightarrow L$. We consider the restriction

$$\iota^*\big|_L : L \longrightarrow L_0^*,$$

whose kernel equals $L_0^{\perp,L}$. Hence, the quotient $L/L_0^{\perp,L}$ is isomorphic to a subgroup of the free abelian group L_0^* and thus is free itself.

b) Let $\underline{L_1} := \left(\underline{L_0}^{\perp,\underline{L}}\right)^{\perp,\underline{L}}$. The inclusion $L_0 \subseteq L_1$ holds by definition. L_1 splits L_0 as a direct summand, since $\underline{L_0}$ is also primitive in $\underline{L_1}$. From $r_{\underline{L_0}} = r_{\underline{L_1}}$ we conclude $L_0 = L_1$.

c) Since L_1/L_0 is finite we have $r_{\underline{L_1}} = r_{\underline{L_0}}$. Thus L_1 and L_0 span the same $\mathbb{Q}$-space, hence

$$\underline{L_1} = \left(\underline{L_1}^{\perp,\underline{L}}\right)^{\perp,\underline{L}} = \left(\underline{L_0}^{\perp,\underline{L}}\right)^{\perp,\underline{L}},$$

if we apply b) on $\underline{L_1}$. This proves both existence and uniqueness. □

In view of 1.2.15 c) we give the following

Remark 1.2.16. *If $L_1 \geq L_0$ and $|L_1/L_0| < \infty$, we call L_1 an* overlattice *of L_0. Note that the finiteness condition is equivalent to $r_{\underline{L_1}} = r_{\underline{L_0}}$.*

Definition 1.2.17. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. We call ι* primitive *or a* primitive embedding, *if $\underline{\iota(L_0)}$ is primitive in $\underline{L}$.*

Lemma 1.2.18. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then there exists a unique overlattice $\underline{L_0'} \geq \underline{L_0}$ in $(L_0)_\mathbb{Q}$ such that the extension $\iota : \underline{L_0'} \longrightarrow \underline{L}$ is a primitive embedding.*

Proof. By 1.2.15 there is a unique overlattice $\underline{L_1}$ of $\underline{\iota(L_0)}$ such that $\underline{L_1}$ is primitive in $\underline{L}$. Let $L_0' := \iota^{-1}(L_1)$. Then $\underline{L_0'}$ is an overlattice of $\underline{L_0}$ and $\underline{\iota(L_0')} = \underline{L_1}$ is primitive in $\underline{L}$, in other words, $\iota : \underline{L_0'} \longrightarrow \underline{L}$ is a primitive embedding. The uniqueness follows from the previous lemma. □

Lemma 1.2.19. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices. If $\iota : \underline{L_0} \longrightarrow \underline{L}$ is a primitive embedding, then its dual $\iota^* : L^* \longrightarrow L_0^*$ is surjective.*

Proof. Let $L' \leq L$ such that $L = \iota(L_0) \oplus L'$ as an inner direct sum. For $\mu_0 \in L_0^*$ we consider the $\mathbb{Z}$-linear functional

$$f_{\mu_0} : L_0 \longrightarrow \mathbb{Z}, \quad f_{\mu_0}(l_0) := B_0(\mu_0, l_0), \quad l_0 \in L_0.$$

We extend this to a $\mathbb{Z}$-linear functional $\widetilde{f_{\mu_0}} : L \longrightarrow \mathbb{Z}$ by defining

$$\widetilde{f_{\mu_0}}(l) := \begin{cases} f_{\mu_0}\left(\iota^{-1}(l)\right), & l \in \iota(L_0), \\ 0, & l \in L'. \end{cases}$$

Let $\mu \in L^*$ such that $\widetilde{f_{\mu_0}}(l) = B(\mu, l)$ for all $l \in L$. Especially we obtain

$$B(\mu, \iota(l_0)) = \widetilde{f_{\mu_0}}(\iota(l_0)) = f_{\mu_0}(l_0) = B_0(\mu_0, l_0)$$

for all $l_0 \in L_0$ and thus $\iota^*(\mu) = \mu_0$. □

Remark 1.2.20. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices. If $\iota : \underline{L_0} \longrightarrow \underline{L}$ is an embedding, then the map $\sigma \circ \iota : \underline{L_0} \longrightarrow \underline{L}$ for $\sigma \in \mathrm{O}(\underline{L})$ is also an embedding. Hence we obtain a natural action of $\mathrm{O}(\underline{L})$ on the set of embeddings $\{\iota : \underline{L_0} \longrightarrow \underline{L}\}$.*

This gives rise to the following

Definition 1.2.21. *Let $\underline{L} = (L, Q)$ and $\underline{L_0} = (L_0, Q_0)$ be lattices. Two embeddings $\iota, \kappa : \underline{L_0} \longrightarrow \underline{L}$ are called* equivalent, *if $\iota = \sigma \circ \kappa$ for some $\sigma \in \mathrm{O}(\underline{L})$. Furthermore, ι, κ are called* stably equivalent, *if $\iota = \sigma \circ \kappa$ for some $\sigma \in \mathrm{O}_d(\underline{L})$.*

We illustrate the theory by an

Example 1.2.22. *Let $\underline{L} = (L, Q)$ be a lattice. Then for every $t \in \mathbb{N}$ there is an one-to-one correspondence*

$$\{l \in L : Q(l) = t\} \longleftrightarrow \{\iota : \underline{\mathbb{Z}}(t) \longrightarrow \underline{L}\}.$$

More precisely, if $l \in L$ and $Q(l) = t$, then the map

$$\iota_l : \mathbb{Z} \longrightarrow L, \quad w \mapsto lw, \quad w \in \mathbb{Z}$$

is an embedding of $\underline{\mathbb{Z}}(t)$ into $\underline{L}$. Conversely, if $t \in \mathbb{N}$ and $\iota : \underline{\mathbb{Z}}(t) \longrightarrow \underline{L}$, then one has $\iota = \iota_l$ where $l = \iota(1)$. The dual ι_l^* is given by

$$\iota_l^* : L^* \longrightarrow \frac{1}{2t}\mathbb{Z}, \quad \iota_l^*(\mu) = \frac{1}{2t}B(\mu, l), \quad \mu \in L^*.$$

One has $\sigma \circ \iota_l = \iota_{\sigma(l)}$ for all $\sigma \in \mathrm{O}(\underline{L})$. Hence, two embeddings ι_l and $\iota_{l'}$ for $l, l' \in L$ are equivalent, if and only if $\sigma(l) = l'$ for some orthogonal transformation $\sigma \in \mathrm{O}(\underline{L})$.

Concerning irreducible root lattices, the special case $t = 1$ leads to the following

Corollary 1.2.23. *Let $\underline{L} = (L, Q)$ be an irreducible root lattice and $\iota, \kappa : \underline{\mathbb{Z}} \longrightarrow \underline{L}$ embeddings. Then ι and κ are stably equivalent.*

Proof. The Weyl group $W(\underline{L}) \leq \mathrm{O}_d(\underline{L})$ of $\underline{L}$ acts transitively on the roots of $\underline{L}$, cf. [7, Lemma 1.10]. The claim follows then from 1.2.22. □

1.3 Irreducible root lattices of low rank

In this section we will consider explicit realizations of certain irreducible root lattices of small rank in the quadratic space $\mathbb{R}^8$ equipped with the quadratic form

$$Q(x) := x^t x, \quad x \in \mathbb{R}^8.$$

The standard basis of $\mathbb{R}^8$ will be denoted by $e_1, \ldots, e_8$. For $i = 1, \ldots, 8$ let $\pi_i : \mathbb{R}^8 \to \mathbb{R}$ denote the projection on the i-th coordinate with respect to the standard basis.

The following construction will have the advantage of a simple determination of the pull-backs of the Jacobi theta functions as we will see later.

The irreducible root lattice $\underline{E_8}$

The irreducible root lattice $\underline{E_8}$ is realized as the underlying lattice of Coxeter's integral Cayley numbers, cf. [4, p. 101f]. To this end we define

$$\alpha_1 := \frac{1}{2}(e_2 + e_3 + e_4 + e_5), \quad \alpha_2 := \frac{1}{2}(e_1 + e_3 + e_5 + e_8),$$
$$\alpha_3 := \frac{1}{2}(e_1 + e_2 + e_5 + e_6), \quad \alpha_4 := \frac{1}{2}(e_1 + e_2 + e_3 + e_7)$$

and

$$E_8 := \langle e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle_{\mathbb{Z}}.$$

The quadratic form Q restricted to E_8 is integer-valued, hence $\underline{E_8} := (E_8, Q|_{E_8})$ is a positive definite and even lattice of rank 8. Hence, $\underline{E_8}$ is a root lattice and a direct calculation shows that $\det \underline{E_8} = 1$. The underlying $\mathbb{Z}$-module E_8 decomposes as a disjoint union

$$E_8 = \bigcup_{i=1}^{16} S_i$$

of sets $S_i, i = 1, \ldots, 16$, which are defined according to table 1.4 at the end of this section.

The irreducible root lattice $\underline{E_7}$

The irreducible root lattice $\underline{E_7}$ is constructed as the orthogonal complement of e_8 in $\underline{E_8}$, i.e.

$$\underline{E_7} = \langle e_8 \rangle^{\perp, \underline{E_8}}.$$

Hence, $\underline{E_7} = (E_7, Q|_{E_7})$ is a positive definite and even lattice of rank 7. As a module we have

$$E_7 = \langle e_1, e_2, e_3, e_4, \alpha_1, \alpha_3, \alpha_4 \rangle_{\mathbb{Z}}$$

i.e. $\underline{E_7}$ is a root lattice and a direct calculation shows that $\det \underline{E_7} = 2$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$E_7 = \bigcup_{i=1}^{8} S_i \cap \langle e_8 \rangle^{\perp}.$$

A complete set of representatives of the discriminant group E_7^*/E_7 is given by

$$\left\{ 0, \frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7}{2} \right\}.$$

The irreducible root lattice $\underline{E_6}$

The root lattice $\underline{E_6}$ is constructed as the orthogonal complement of $e_1 + e_3 + e_5$ in $\underline{E_7}$, i.e.

$$\underline{E_6} := \langle e_1 + e_3 + e_5 \rangle^{\perp, \underline{E_7}}.$$

Hence, $\underline{E_6} = (E_6, Q|_{E_6})$ is a positive definite and even lattice of rank 6. As a module we have

$$\begin{aligned} E_6 &= \langle e_2, e_4, e_6, \alpha_1 - e_3, \alpha_3 - e_1, \alpha_4 - e_1 \rangle_{\mathbb{Z}} \\ &= \left\langle e_2, e_4, e_6, \frac{e_2 + e_4 + e_6 + e_7}{2}, \alpha_3 - e_1, \alpha_4 - e_1 \right\rangle_{\mathbb{Z}}, \end{aligned}$$

i.e. $\underline{E_6}$ is a root lattice and a direct calculation shows that $\det \underline{E_6} = 3$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$E_6 = \bigcup_{i=1}^{8} S_i \cap \langle e_8, e_1 + e_3 + e_5 \rangle^{\perp}.$$

A complete set of representatives of the discriminant group E_6^*/E_6 is given by

$$\left\{ 0, \frac{e_1 + e_3 - 2e_5}{3}, -\frac{e_1 + e_3 - 2e_5}{3} \right\}.$$

The irreducible root lattice $\underline{D_4}$

The root lattice $\underline{D_4}$ is constructed as the orthogonal complement of $\{e_1 - e_3, e_3 - e_5\}$ in $\underline{E_6}$, i.e.

$$\underline{D_4} = \langle e_1 - e_3, e_3 - e_5 \rangle^{\perp, \underline{E_6}}.$$

Hence, $\underline{D_4} = (D_4, Q|_{D_4})$ is a positive definite and even lattice of rank 4. As a module we have

$$D_4 = \left\langle e_2, e_4, e_6, \frac{e_2 + e_4 + e_6 + e_7}{2} \right\rangle_{\mathbb{Z}},$$

i.e. $\underline{D_4}$ is a root lattice and a direct calculation shows that $\det \underline{D_4} = 4$. The underlying $\mathbb{Z}$-module decomposes as a disjoint union

$$D_4 = \bigcup_{i \in \{1,4\}} S_i \cap \langle e_1, e_3, e_5, e_8 \rangle^{\perp}.$$

A complete set of representatives of the discriminant group D_4^*/D_4 is given by

$$\left\{ 0, \frac{e_2 + e_4}{2}, \frac{e_2 + e_6}{2}, \frac{e_2 + e_7}{2} \right\}.$$

The irreducible root lattice $\underline{A_2}$

The root lattice $\underline{A_2}$ is constructed as the orthogonal complement of $\{e_4 - e_6, e_4 - e_7\}$ in $\underline{D_4}$, i.e.

$$\underline{A_2} = \langle e_4 - e_6, e_4 - e_7 \rangle^{\perp, \underline{D_4}}.$$

Hence, $\underline{A_2} := (A_2, Q|_{A_2})$ is a positive definite and even lattice of rank 2. As a module we have

$$A_2 = \left\langle e_2, \frac{e_2 + e_4 + e_6 + e_7}{2} \right\rangle_{\mathbb{Z}},$$

i.e. $\underline{A_2}$ is a root lattice and a direct calculation shows that $\det \underline{A_2} = 3$. A complete set of representatives of the discriminant group A_2^*/A_2 is given by

$$\left\{ 0, \frac{e_4 + e_6 + e_7}{3}, -\frac{e_4 + e_6 + e_7}{3} \right\}.$$

The irreducible root lattice $\underline{A_1}$

The root lattice $\underline{A_1}$ is constructed as the orthogonal complement of $e_4 + e_6 + e_7$ in $\underline{A_2}$, i.e.

$$\underline{A_1} = \langle e_4 + e_6 + e_7 \rangle^{\perp, \underline{A_2}}.$$

Hence, $\underline{A_1}$ is a positive definite and even lattice of rank 1. As a module,

$$A_1 = \langle e_2 \rangle_{\mathbb{Z}},$$

i.e. $\underline{A_1}$ is a root lattice and a direct calculation shows that $\det \underline{A_1} = 2$. Note that $\underline{A_1}$ is identified with $\underline{\mathbb{Z}} = (\mathbb{Z}, x^2)$, whenever it is convenient. A complete set of representatives of the discriminant group A_1^*/A_1 is given by $\left\{0, \frac{1}{2}e_2\right\}$.

S	$\pi_1(S)$	$\pi_2(S)$	$\pi_3(S)$	$\pi_4(S)$	$\pi_5(S)$	$\pi_6(S)$	$\pi_7(S)$	$\pi_8(S)$
S_1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S_2	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S_3	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$
S_4	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$
S_5	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S_6	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$
S_7	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S_8	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$
S_9	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{10}	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{11}	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{12}	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{13}	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{14}	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{15}	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
S_{16}	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$	$\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$

Table 1.4: Distribution of the integral and half-integral components of E_8 with respect to the standard basis $e_1, \ldots, e_8$ of $\mathbb{R}^8$

2 Elliptic Functions of Lattice-Index

2.1 Quadratic forms of higher degree and the Heisenberg group

Let $\mathrm{Sym}_n^\sharp(\mathbb{Z})$ denote the dual lattice of $\mathrm{Sym}_n(\mathbb{Z})$ with respect to the trace bilinear form, i.e.

$$\mathrm{Sym}_n^\sharp(\mathbb{Z}) := \left\{M \in \mathrm{Sym}_n(\mathbb{Q}) : \mathrm{tr}(SM) \in \mathbb{Z} \text{ for all } S \in \mathrm{Sym}_n(\mathbb{Z})\right\}.$$

Definition 2.1.1. *Let $\underline{L} = (L, Q)$ be a lattice. The map*

$$Q^{(n)} : L^{(n)} \longrightarrow \mathrm{Sym}_n^\sharp(\mathbb{Z}), \quad Q^{(n)}(l) := \tfrac{1}{2}\big(B(l_i, l_j)\big)_{1\le i,j\le n}$$

for $l = (l_1, \dots, l_n) \in L^{(n)}$ is called the quadratic form of degree *n associated to $\underline{L}$.*

Lemma 2.1.2. *Let $\underline{L} = (L, Q)$ be a lattice. Define $B^{(n)} : L^{(n)} \times L^{(n)} \longrightarrow \mathrm{Sym}_n^\sharp(\mathbb{Z})$ by*

$$B^{(n)}(l, l') = Q^{(n)}(l + l') - Q^{(n)}(l) - Q^{(n)}(l')$$

for $l, l' \in L^{(n)}$. Then one has

$$B^{(n)}(l, l') = \tfrac{1}{2}\big(B(l_i, l'_j) + B(l'_i, l_j)\big)_{1\le i,j\le n}.$$

We call $B^{(n)}$ the bilinear form of degree *n associated to (L, Q). The identity*

$$Q^{(n)}(l) = \tfrac{1}{2}B^{(n)}(l, l)$$

holds for all $l \in L^{(n)}$.

The generic case is treated in the following

Remark 2.1.3. *Let $S \in \mathrm{Sym}_r(\mathbb{Z})$ positive definite and even. Then $(\mathbb{Z}^r)^{(n)}$ can be naturally identified with $\mathbb{Z}^{r\times n}$. and we have*

$$Q_S^{(n)}(X) = \tfrac{1}{2}S[X], \quad B_S^{(n)}(X, Y) = \tfrac{1}{2}(X^tSY + Y^tSX), \quad X, Y \in \mathbb{Z}^{r\times n}.$$

Conversely, let $\underline{L} = (L, Q)$ be a lattice and S denote its Gram matrix with respect to some basis $(b_1, \ldots, b_r)$ of L. Let $\kappa : L \longrightarrow \mathbb{Z}^r$ denote the coordinate system with respect to $(b_1, \ldots, b_r)$. Let $\kappa^{(n)} : L^{(n)} \longrightarrow \mathbb{Z}^{r \times n}$ denote the map defined by

$$\kappa^{(n)}(l) := (\kappa(l_1), \ldots, \kappa(l_n)), \quad l = (l_1, \ldots, l_n) \in L^{(n)}.$$

Then the identity

$$Q^{(n)}(l) = \tfrac{1}{2}\kappa^{(n)}(l)^t S \kappa^{(n)}(l) = \tfrac{1}{2} S\left[\kappa^{(n)}(l)\right]$$

holds for all $l \in L^{(n)}$.

We will implicity assume that $B^{(n)}$ and $Q^{(n)}$ are extended by $\mathbb{F}$-linearity. All aforementioned identities are also valid in $L_{\mathbb{F}}^{(n)} \times L_{\mathbb{F}}^{(n)}$ resp. $L_{\mathbb{F}}^{(n)}$ for $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

By a direct verification, we obtain the following

Lemma 2.1.4. *Let $\underline{L} = (L, Q)$ be a lattice and $1 \leq j \leq n$. Then for all $l \in L_{\mathbb{R}}^{(n)}$ the following assertions hold:*

a) $Q^{(j)}(lX) = Q^{(n)}(l)[X]$ *for all* $X \in \mathbb{R}^{n \times j}$,

b) $\operatorname{tr}\left(B^{(n)}(l, l')S\right) = \operatorname{tr}\left(B^{(n)}(l, l'S)\right) = \operatorname{tr}\left(B^{(n)}(lS, l')\right)$ *for all* $S \in \operatorname{Sym}_n(\mathbb{R})$,

c) $Q^{(n)}(l) \geq 0$ *and* $Q^{(n)}(l) > 0$ *if and only if* $\dim_{\mathbb{R}}\langle l_1, \ldots, l_n\rangle = n$, *where* $l = (l_1, \ldots, l_n)$.

Remark 2.1.5. *Let $\underline{L} = (L, Q)$ be a lattice. Then the lattices $\left(L^{(n)}, \operatorname{tr} \circ Q^{(n)}\right)$ and $n\underline{L}$ are isometric.*

Definition 2.1.6. *Let $\underline{L} = (L, Q)$ be a lattice. The* real Heisenberg group *of degree n with respect to $\underline{L}$ is the group*

$$H_{\mathbb{R}}^{(n)}(\underline{L}) := L_{\mathbb{R}}^{(n)} \times L_{\mathbb{R}}^{(n)} \times S^1$$

with underlying group law

$$(\lambda, \mu, \zeta) \cdot (\lambda', \mu', \zeta') := \left(\lambda + \lambda', \mu + \mu', \zeta \cdot \zeta' \cdot e^{\pi i \operatorname{tr}\left(B^{(n)}(\lambda, \mu') - B^{(n)}(\lambda', \mu)\right)}\right)$$

for $\lambda, \lambda', \mu, \mu' \in L_{\mathbb{R}}^{(n)}$ and $\zeta, \zeta' \in S^1$. Furthermore, for $\lambda, \mu \in L_{\mathbb{R}}^{(n)}$ we define

$$[\lambda, \mu] := \left(\lambda, \mu, e^{\pi i \operatorname{tr}\left(B^{(n)}(\lambda, \mu)\right)}\right) \in H_{\mathbb{R}}^{(n)}(\underline{L}).$$

Remark 2.1.7. *Let $\underline{L} = (L, Q)$ be a lattice.*

a) S^1 *is identified in* $H_{\mathbb{R}}^{(n)}(\underline{L})$ *via*

$$S^1 \longrightarrow H_{\mathbb{R}}^{(n)}(\underline{L}), \quad \zeta \mapsto (0, 0, \zeta).$$

b) $L_{\mathbb{R}}^{(n)} \times L_{\mathbb{R}}^{(n)}$ *is identified in* $H_{\mathbb{R}}^{(n)}(\underline{L})$ *via* $(\lambda, \mu) \mapsto [\lambda, \mu]$. *Note that the map* $[\cdot, \cdot]$ *is no group homomorphism, in contrast to its restrictions*

$$L_{\mathbb{R}}^{(n)} \longrightarrow H_{\mathbb{R}}^{(n)}(\underline{L}), \quad \lambda \mapsto [\lambda, 0],$$
$$L_{\mathbb{R}}^{(n)} \longrightarrow H_{\mathbb{R}}^{(n)}(\underline{L}), \quad \mu \mapsto [0, \mu].$$

c) *For* $(\lambda, \mu), (\lambda', \mu') \in L^{*(n)} \times L^{(n)}$ *we have*

$$\begin{aligned}
[\lambda, \mu] \cdot [\lambda', \mu'] &= \left(\lambda, \mu, e^{\pi i \mathrm{tr}\left(B^{(n)}(\lambda,\mu)\right)}\right) \cdot \left(\lambda', \mu', e^{\pi i \mathrm{tr}\left(B^{(n)}(\lambda',\mu')\right)}\right) \\
&= \left(\lambda + \lambda', \mu + \mu', e^{\pi i \mathrm{tr}\left(B^{(n)}(\lambda,\mu) + B^{(n)}(\lambda',\mu')\right)} \cdot e^{\pi i \mathrm{tr}\left(B^{(n)}(\lambda,\mu') - B^{(n)}(\lambda',\mu)\right)}\right) \\
&= \left(\lambda + \lambda', \mu + \mu', e^{2\pi i \mathrm{tr}\left(B^{(n)}(\lambda+\lambda',\mu+\mu')\right)}\right) \\
&= [\lambda + \lambda', \mu + \mu'],
\end{aligned}$$

since $e^{2\pi i \mathrm{tr}\left(B^{(n)}(\lambda',\mu)\right)} = 1$. *Consequently, the restrictions of* $[\cdot, \cdot]$ *to* $L^{*(n)} \times L^{(n)}$ *and* $L^{(n)} \times L^{*(n)}$ *are monomorphisms of the groups.*

We introduce the integral Heisenberg group:

Definition 2.1.8. *Let* $\underline{L} = (L, Q)$ *be a lattice.*

a) *The* integral Heisenberg group *of degree n with respect to* $\underline{L}$ *is defined as*

$$H^{(n)}(\underline{L}) := \left[L^{(n)}, L^{(n)}\right].$$

b) *The* extended integral Heisenberg group *of degree n with respect to* $\underline{L}$ *is defined as*

$$H^{(n)}(\underline{L})^* := \left\{(\lambda, \mu, \zeta) \in H_{\mathbb{R}}^{(n)}(\underline{L}) : e^{2\pi i \mathrm{tr}\left(B^{(n)}(\lambda,l') - B^{(n)}(\mu,l)\right)} = 1 \text{ for all } l, l' \in L^{(n)}\right\}.$$

Remark 2.1.9. *Let* $\underline{L} = (L, Q)$ *be a lattice.*

a) *For each* $(\lambda, \mu, \zeta) \in H_{\mathbb{R}}^{(n)}(\underline{L})$ *the map*

$$\chi_{\lambda,\mu,\zeta} : H^{(n)}(\underline{L}) \longrightarrow \mathbb{C}^{\times}, \quad [l, l'] \mapsto e^{2\pi i \mathrm{tr}(B^{(n)}(\lambda,l') - B^{(n)}(\mu,l))}$$

is an abelian character of $H^{(n)}(\underline{L})$ *and the map*

$$H_{\mathbb{R}}^{(n)}(\underline{L}) \longrightarrow H^{(n)}(\underline{L})^{\mathrm{ab}}, \quad (\lambda, \mu, \zeta) \mapsto \chi_{\lambda,\mu,\zeta}$$

is a homomorphism of the groups with kernel $H^{(n)}(\underline{L})^*$.

b) The identity

$$(\lambda,\mu,\zeta)\cdot[l,l']\cdot(\lambda,\mu,\zeta)^{-1}=\chi_{\lambda,\mu,\zeta}\left([l,l']\right)\cdot[l,l']$$

for $[l,l'] \in H^{(n)}(\underline{L})$ *and* $(\lambda,\mu,\zeta)\in H_{\mathbb{R}}^{(n)}(\underline{L})$ *shows that* $H^{(n)}(\underline{L})^*$ *is precisely the centralizer of* $H^{(n)}(\underline{L})$ *in* $H_{\mathbb{R}}^{(n)}(\underline{L})$.

c) $H^{(n)}(\underline{L})\cdot S^1$ *is a normal subgroup of* $H^{(n)}(\underline{L})^*$ *and one has*

$$H^{(n)}(\underline{L})^*/\left(H^{(n)}(\underline{L})\cdot S^1\right)\cong (L^*/L)^{(2n)}.$$

d) We will see later, that $H^{(n)}(\underline{L})^*$ *is the precise invariance group of elliptic functions of index* $\underline{L}$ *and degree* n.

Definition 2.1.10. *Let* $\underline{L}=(L,Q)$ *be a lattice.*

a) The Siegel upper half-space *of degree* n *is defined as*

$$\mathcal{H}_n=\{Z=X+iY\in \operatorname{Sym}_n(\mathbb{C}):X=X^t,Y=Y^t>0\}.$$

We have $\mathcal{H}_1=\mathcal{H}$.

b) The Jacobi half-space *of degree* n *with respect to* $\underline{L}$ *is defined as*

$$\mathcal{H}_n^J(L):=\mathcal{H}_n\times L_{\mathbb{C}}^{(n)}=\left\{(Z,W):Z\in\mathcal{H}_n,W\in L_{\mathbb{C}}^{(n)}\right\}.$$

Some identifications are contained in the following

Remark 2.1.11. *The Siegel upper half-space* $\mathcal{H}_n$ *is implicitely identified with* $\mathcal{H}_n\times\{0\}$ *inside* $\mathcal{H}_n^J(L)$. *Furthermore, we will identify* $\mathcal{O}(\mathcal{H}_n)$ *as a subring of* $\mathcal{O}(H_n^J(L))$ *in a natural way.*

Definition 2.1.12. *Let* $\underline{L}=(L,Q)$ *be a lattice. Let* $\phi\in\mathcal{O}(H_n^J(L))$ *and* $(\lambda,\mu,\zeta)\in H_{\mathbb{R}}^{(n)}(\underline{L})$. *Then the function*

$$\phi\big|_{\underline{L}}(\lambda,\mu,\zeta):\mathcal{H}_n^J(L)\longrightarrow\mathbb{C}$$

is defined pointwisely by

$$\phi\big|_{\underline{L}}(\lambda,\mu,\zeta)(Z,W):=\zeta\cdot e^{2\pi i \operatorname{tr}\left(Q^{(n)}(\lambda)\cdot Z+B^{(n)}(\lambda,W)+\frac{1}{2}B^{(n)}(\lambda,\mu)\right)}\cdot\phi(Z,W+\lambda Z+\mu)$$

for $(Z,W)\in\mathcal{H}_n^J(L)$.

Remarks 2.1.13. *Let* $\underline{L}=(L,Q)$ *be a lattice.*

a) The assignment

$$((\lambda,\mu,\zeta),\phi)\mapsto\phi\big|_{\underline{L}}(\lambda,\mu,\zeta),\quad (\lambda,\mu,\zeta)\in H_{\mathbb{R}}^{(n)}(\underline{L}),\phi\in\mathcal{O}(H_n^J(L)),$$

defines an action of $H_{\mathbb{R}}^{(n)}(\underline{L})$ *on* $\mathcal{O}(H_n^J(L))$.

b) For $(\lambda,\mu)\in L^{*(n)}\times L^{(n)}$ *or* $(\lambda,\mu)\in L^{(n)}\times L^{*(n)}$ *one has*

$$\phi\big|_{\underline{L}}[\lambda,\mu](Z,W)=e^{2\pi i\operatorname{tr}\left(Q^{(n)}(\lambda)\cdot Z+B^{(n)}(\lambda,W)\right)}\cdot\phi(Z,W+\lambda Z+\mu)$$

for $(Z,W)\in\mathcal{H}_n^J(L)$.

2.2 Elliptic functions of lattice-index

Definition 2.2.1. *Let $\underline{L} = (L, Q)$ be a lattice. A function $\phi \in \mathcal{O}(\mathcal{H}_n^J(L))$ is called* elliptic of index $\underline{L}$ and degree n, *if*

$$\phi\big|_{\underline{L}}[l, l'] = \phi$$

for all $[l, l'] \in H^{(n)}(\underline{L})$. The set of elliptic functions of index $\underline{L}$ and degree n is denoted by $\mathcal{E}^{(n)}(\underline{L})$.

Regarding the structure of $\mathcal{E}^{(n)}(\underline{L})$, we have the following

Remark 2.2.2. *Let $\underline{L} = (L, Q)$ be a lattice. $\mathcal{E}^{(n)}(\underline{L})$ carries a canonical module structure over $\mathcal{O}(\mathcal{H}_n)$ via multiplication defined pointwisely by*

$$(h \cdot \phi)(Z, W) := h(Z) \cdot \phi(Z, W), \quad (Z, W) \in \mathcal{H}_n^J(L)$$

for $\phi \in \mathcal{E}^{(n)}(\underline{L})$ and $h \in \mathcal{O}(\mathcal{H}_n)$.

Again we treat the generic case:

Remark 2.2.3. *Let $\underline{L} = (L, Q)$ be a lattice and S denote its Gram matrix with respect to some basis $(b_1, \dots, b_r)$ of L. Let $\kappa : L \longrightarrow \mathbb{Z}^r$ denote the coordinate system with respect to $(b_1, \dots, b_r)$. Then the assignment*

$$\phi \mapsto \left(\widetilde{\phi} : \mathcal{H}_n \times \mathbb{C}^{r \times n} \to \mathbb{C}, \quad (Z, W) \mapsto \phi\left(Z, {\kappa^{(n)}}^{-1}(W)\right)\right)$$

is an $\mathcal{O}(\mathcal{H}_n)$-module isomorphism $\mathcal{E}^{(n)}(\underline{L}) \longrightarrow \mathcal{E}^{(n)}(\mathbb{Z}^r, Q_S)$.

Some functorial constructions are explained in the following

Definition 2.2.4. *Let $\underline{L}, \underline{L'}$ be lattices.*

a) For $\phi \in \mathcal{O}(\mathcal{H}_n^J(L)), \phi' \in \mathcal{O}(\mathcal{H}_{n'}^J(L'))$ we define its tensor product

$$\phi \otimes \phi' \in \mathcal{O}(\mathcal{H}_n^J(L) \times \mathcal{H}_{n'}^J(L'))$$

pointwisely by

$$\phi \otimes \phi'((Z, W), (Z', W')) := \phi(Z, W) \cdot \phi'(Z', W')$$

for $(Z, W) \in \mathcal{H}_n^J(L)$ and $(Z', W') \in \mathcal{H}_{n'}^J(L')$.

b) Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ and $\phi' \in \mathcal{E}^{(n)}(\underline{L'})$. Then the function

$$\phi \otimes \phi'\big|_{Z=Z'} : (Z, (W, W')) \mapsto \phi \otimes \phi'((Z, W), (Z, W'))$$

belongs to $\mathcal{E}^{(n)}(\underline{L} \oplus \underline{L'})$.

A first nontrivial example is given by the Jacobi theta functions of degree n associated to a lattice $\underline{L}$, cf. [27, § 3] or [24, (3.26) p. 49].

Definition 2.2.5. *Let $\underline{L} = (L, Q)$ be a lattice. Then the* Jacobi theta function $\vartheta^{(n)}_{\underline{L},\mu}$ *of degree n associated to $\underline{L}$ and $\mu \in L^{(n)}_{\mathbb{R}}$ is defined by*

$$\begin{aligned}\vartheta^{(n)}_{\underline{L},\mu}(Z,W) &:= \sum_{l\in\mu+L^{(n)}} 1\big|_{\underline{L}}[l,0](Z,W)\\ &= \sum_{l\in\mu+L^{(n)}} e^{2\pi i \mathrm{tr}(Q^{(n)}(l)Z+B^{(n)}(l,W))}, \quad (Z,W)\in\mathcal{H}^J_n(L),\end{aligned}$$

where we suppress the superscript if $n = 1$. The definition only depends on the coset $\mu + L^{(n)}$. The series converges absolutely and uniformly on each vertical strip

$$\left\{(Z,W)\in\mathcal{H}^J_n(L) : Y \geq \delta I_n, \mathrm{tr}(B(W,\overline{W})) \leq \delta^{-1}\right\}, \quad \delta > 0.$$

For $\mu \in L$ we will sometimes simply write $\vartheta^{(n)}_{\underline{L}}$ instead of $\vartheta^{(n)}_{\underline{L},\mu}$. Furthermore, by a slight abuse of notation, we define

$$\vartheta^{(n)}_{\underline{L},\mu}(Z) := \vartheta^{(n)}_{\underline{L},\mu}(Z,0), \quad Z\in\mathcal{H}_n.$$

Basic transformation properties of $\vartheta^{(n)}_{\underline{L},\mu}$ are given in the following

Proposition 2.2.6. *Let $\underline{L} = (L, Q)$ be a lattice and $\mu \in L^{(n)}_{\mathbb{R}}$. Then the following assertions hold:*

a) $\vartheta^{(n)}_{\underline{L},\mu}\big|_{\underline{L}}[l,l'] = e^{2\pi i \mathrm{tr}\left(B^{(n)}(\mu,l')\right)}\vartheta^{(n)}_{\underline{L},\mu}$ *for all $l, l' \in L^{(n)}$ and $\vartheta^{(n)}_{\underline{L},\mu} \in \mathcal{E}^{(n)}(\underline{L})$ if and only if $\mu \in L^{*(n)}$.*

b) $\vartheta^{(n)}_{\underline{L},\mu}(Z+S,W) = e^{2\pi i \mathrm{tr}(Q^{(n)}(\mu)S)} \cdot \vartheta^{(n)}_{\underline{L},\mu}$ *for $S \in \mathrm{Sym}_n(\mathbb{Z})$ such that $\mu S \in L^{*(n)}$,*

c) $\vartheta^{(n)}_{\underline{L},\mu}(Z[U], WU) = \vartheta^{(n)}_{\underline{L},\mu U^t}(Z,W)$ *for all $U \in \mathrm{GL}_n(\mathbb{Z})$.*

Proof. All properties follow from a straightforward calculation. Note in b) that the assertion $\mathrm{tr}(B^{(n)}(l,\mu)S) \in \mathbb{Z}$ holds for all $l \in L^{(n)}$ if and only if $\mu S \in L^{*(n)}$. In c) use that $L^{(n)}U = L^{(n)}$ for $U \in \mathrm{GL}_n(\mathbb{Z})$. □

Remark 2.2.7. *Let $\underline{L} = (L, Q)$ be a lattice and $\mu \in L^{*(n)}$. In view of the Fourier expansions, we have*

$$\dim_{\mathbb{C}}\left\langle \vartheta^{(n)}_{\underline{L},\mu}(Z,\cdot), \mu \in (L^*/L)^{(n)}\right\rangle = (\det \underline{L})^n$$

for every $Z \in \mathcal{H}_n$ as functions of W.

The algebraic structure of $\mathcal{E}^{(n)}(\underline{L})$ is revealed in the following theorem, cf. [24, Proposition 3.5] or [27, 3.1. Lemma]:

Theorem 2.2.8. *Let $\underline{L} = (L, Q)$ be a lattice. Then the space $\mathcal{E}^{(n)}(\underline{L})$ of elliptic functions of degree n and index $\underline{L}$ is a free $\mathcal{O}(\mathcal{H}_n)$-module of rank $(\det \underline{L})^n$ and decomposes as a direct sum*

$$\mathcal{E}^{(n)}(\underline{L}) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(\mathcal{H}_n) \cdot \vartheta^{(n)}_{\underline{L},\mu}.$$

Proof. Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ and $Z \in \mathcal{H}_n$. The function $\phi(Z, \cdot)$ is periodic in W with respect to $L^{(n)}$ and therefore has a Fourier expansion of the form

$$\phi(Z, W) = \sum_{\mu \in L^{*(n)}} c(\mu, Z) e^{2\pi i \mathrm{tr} B^{(n)}(\mu, W)}, \quad W \in L^{(n)}_{\mathbb{C}}$$

for certain functions $c(\mu, \cdot) : \mathcal{H}_n \longrightarrow \mathbb{C}$. For $l \in L^{(n)}$ we apply $\big|_{\underline{L}}[l, 0]$ on ϕ in order to obtain

$$\begin{aligned}
\phi\big|_{\underline{L}}[l, 0](Z, W) &= e^{2\pi i \mathrm{tr}(Q^{(n)}(l)Z + B^{(n)}(l,W))} \phi(Z, W + lZ) \\
&= e^{2\pi i \mathrm{tr}(Q^{(n)}(l)Z)} \cdot e^{2\pi i \mathrm{tr}(B^{(n)}(l,W))} \cdot \sum_{\mu \in L^{*(n)}} c(\mu, Z) e^{2\pi i \mathrm{tr}(B^{(n)}(\mu, W + lZ))} \\
&= \sum_{\mu \in L^{*(n)}} e^{2\pi i \mathrm{tr}((B^{(n)}(\mu,l) + Q^{(n)}(l))Z)} \cdot c(\mu, Z) \cdot e^{2\pi i \mathrm{tr}(B^{(n)}(\mu + l, W))} \\
&= \sum_{\mu \in L^{*(n)}} e^{2\pi i \mathrm{tr}((Q^{(n)}(\mu + l) - Q^{(n)}(\mu))Z)} \cdot c(\mu, Z) \cdot e^{2\pi i \mathrm{tr}(B^{(n)}(\mu + l, W))}.
\end{aligned}$$

A comparison of the Fourier coefficients yields

$$c(\mu + l, Z) = e^{2\pi i \mathrm{tr}\left((Q^{(n)}(\mu + l) - Q^{(n)}(\mu))Z\right)} \cdot c(\mu, Z), \quad Z \in \mathcal{H}_n, \mu \in L^{*(n)}, l \in L^{(n)}.$$

Hence, by a simple rearrangement we deduce

$$\begin{aligned}
\phi(Z, W) &= \sum_{\mu \in L^{*(n)}} c(\mu, Z) e^{2\pi i \mathrm{tr}(B^{(n)}(\mu, W))} \\
&= \sum_{\mu \in (L^*/L)^{(n)}} \sum_{l \in L^{(n)} + \mu} c(l, Z) e^{2\pi i \mathrm{tr}(B^{(n)}(l, W))} \\
&= \sum_{\mu \in (L^*/L)^{(n)}} \sum_{l \in L^{(n)} + \mu} e^{2\pi i \mathrm{tr}\left((Q^{(n)}(l) - Q^{(n)}(\mu))Z\right)} c(\mu, Z) e^{2\pi i \mathrm{tr}(B^{(n)}(l, W))} \\
&= \sum_{\mu \in (L^*/L)^{(n)}} c(\mu, Z) e^{-2\pi i \mathrm{tr}(Q^{(n)}(\mu)Z)} \left(\sum_{l \in L^{(n)} + \mu} e^{2\pi i \mathrm{tr}(Q^{(n)}(l)Z + B^{(n)}(l, W))} \right) \\
&= \sum_{\mu \in (L^*/L)^{(n)}} h_\mu(Z) \cdot \vartheta^{(n)}_{\underline{L},\mu}(Z, W),
\end{aligned}$$

where

$$h_\mu(Z) := c(\mu, Z) e^{-2\pi i \mathrm{tr}\left(Q^{(n)}(\mu)Z\right)}, \quad Z \in \mathcal{H}_n.$$

The uniqueness and holomorphicity of the functions h_μ follows from 2.2.7. For the latter fact confer also [24, Lemma 3.4]. □

2.2.8 gives rise to the following

Definition 2.2.9. *Let $\underline{L}$ be a lattice and $\phi \in \mathcal{E}^{(n)}(\underline{L})$. The decomposition*

$$\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_\mu \vartheta^{(n)}_{\underline{L},\mu}$$

according to 2.2.8 is called the theta decomposition *of ϕ.*

We revisit the centralizer $H^{(n)}(\underline{L})^*$ of $H^{(n)}(\underline{L})$ in $H^{(n)}_{\mathbb{R}}(\underline{L})$ and prove the following

Theorem 2.2.10. *Let $\underline{L} = (L, Q)$ be a lattice. Then the following assertions hold:*

a) $\mathcal{E}^{(n)}(\underline{L})$ is invariant under $H^{(n)}(\underline{L})^*$. More precisely, $H^{(n)}(\underline{L})^*$ is the maximal invariance group of $\mathcal{E}^{(n)}(\underline{L})$ in $H^{(n)}_{\mathbb{R}}(\underline{L})$, i.e. for $0 \neq \phi \in \mathcal{E}^{(n)}(\underline{L})$ and $(\lambda, \mu, \zeta) \in H^{(n)}_{\mathbb{R}}(\underline{L})$ one has

$$\phi\big|_{\underline{L}}(\lambda, \mu, \zeta) \in \mathcal{E}^{(n)}(\underline{L}) \text{ if and only if } (\lambda, \mu, \zeta) \in H^{(n)}(\underline{L})^*.$$

b) For $(\lambda, v, \zeta) \in H^{(n)}(\underline{L})^*$ and $\mu \in L^{*(n)}$ the transformation formula

$$\vartheta^{(n)}_{\underline{L},\mu}\Big|_{\underline{L}}(\lambda, v, \zeta) = \zeta \cdot e^{2\pi i \operatorname{tr}(B^{(n)}(\mu + \frac{1}{2}\lambda, v))} \cdot \vartheta^{(n)}_{\underline{L},\mu+\lambda}$$

holds.

Proof. a) For $(\lambda, v, \zeta) \in H^{(n)}_{\mathbb{R}}(\underline{L})$ and $[l, l'] \in H^{(n)}(\underline{L})$ we have

$$(\lambda, v, \zeta) \cdot [l, l'] = \chi_{\lambda,\mu,\zeta}\left([l, l']\right) \cdot [l, l'] \cdot (\lambda, v, \zeta).$$

Then both assertions follow immediately from

$$\phi\big|_{\underline{L}}(\lambda, v, \zeta)\big|_{\underline{L}}[l, l'] = \chi_{\lambda,\mu,\zeta}\left([l, l']\right) \cdot \phi\big|_{\underline{L}}(\lambda, v, \zeta).$$

b) For $\mu \in L^{*(n)}$ and $l \in L^{(n)}$ we obtain

$$[\mu + l, 0] \cdot (\lambda, v, \zeta) = \left(\mu + \lambda + l, v, \zeta \cdot e^{\pi i \operatorname{tr}(B^{(n)}(\mu + l, v))}\right) = \left(\mu + \lambda + l, v, \zeta \cdot e^{\pi i \operatorname{tr}(B^{(n)}(\mu, v))}\right)$$

from $v \in L^{*(n)}$. Hence,

$$\left(1\big|_{\underline{L}}[\mu + l, 0]\right)\Big|_{\underline{L}}(\lambda, v, \zeta) = \zeta \cdot e^{2\pi i \operatorname{tr}(B^{(n)}(\mu + \frac{1}{2}\lambda, v))} \cdot 1\big|_{\underline{L}}[\mu + \lambda + l, 0].$$

□

2.3 Siegel operators, regularity and cuspidality

In this section we define a generalization of the Siegel operator considered in [27, 1.9. Definition].

Definition 2.3.1. *Let $\underline{L} = (L, Q)$ be a lattice, $0 \leq j \leq n$ and $\phi \in \mathcal{O}(\mathcal{H}_n^J(L))$.*

a) We define $\mathcal{W}^{(j)}\phi \in \mathcal{O}\left(\mathcal{H}_j^J(L) \times \mathcal{H}_{n-j}^J(L)\right)$ pointwisely by

$$\mathcal{W}^{(j)}\phi((Z_j, W_j), (Z_{n-j}, W_{n-j})) := \phi\left(\begin{pmatrix} Z_j & 0 \\ 0 & Z_{n-j} \end{pmatrix}, (W_j, W_{n-j})\right)$$

for $(Z_j, W_j) \in \mathcal{H}_j^J(L), (Z_{n-j}, W_{n-j}) \in \mathcal{H}_{n-j}^J(L)$ with the convention

$$\mathcal{W}^{(0)} = \mathcal{W}^{(n)} = \mathrm{id}.$$

We call $\mathcal{W}^{(j)}$ the global Witt operator *of degree j.*

b) For $Y_{n-j} \in \mathrm{Pos}_{n-j}(\mathbb{R})$ we define $\mathcal{S}_{Y_{n-j}}^{(j)}\phi$ pointwisely by

$$\mathcal{S}_{Y_{n-j}}^{(j)}\phi(Z_j, W_j) := \lim_{t\to\infty} \mathcal{W}^{(j)}\phi((Z_j, W_j), (itY_{n-j}, 0))$$

for $(Z_j, W_j) \in \mathcal{H}_j^J(L)$, whenever this limit exists and is finite. We treat $\mathcal{S}_{Y_n}^{(0)}\phi$ as a constant and define $\mathcal{S}^{(n)} = \mathrm{id}$. Note that existence and value of the limit above in the case $j = n-1$ do not depend on the choice of $Y_1 = y_1 > 0$. In this case we will suppress the subscript Y_1 and simply write $\mathcal{S}^{(n-1)}$. We say that $\mathcal{S}_{Y_{n-j}}^{(j)}\phi$ exists, if $\mathcal{S}_{Y_{n-j}}^{(j)}\phi(Z_j, W_j)$ is defined for all $(Z_j, W_j) \in \mathcal{H}_j^J(L)$. The partial operator $\mathcal{S}_{Y_{n-j}}^{(j)}$ is called the Siegel operator *of degree j at Y_{n-j}.*

Definition 2.3.2. *Let $\underline{L} = (L, Q)$ be a lattice and $0 \leq j \leq n$. Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$.*

a) We say that ϕ satisfies the cusp condition *of degree $\geq j$ with respect to $H^{(n)}(\underline{L})$, if*

$$\mathcal{S}_{Y_{n-m}}^{(m)}\left(\phi|_{\underline{L}}[(0, \lambda_{n-m}), (0, \mu_{n-m})]\right)$$

exists for all $m = 0, \ldots, j$, $Y_{n-m} \in \mathrm{Pos}_{n-m}(\mathbb{R})$ and $\lambda_{n-m}, \mu_{n-m} \in L_{\mathbb{Q}}^{(n-m)}$. In this case,

$$\deg_{\mathrm{reg}}(\phi) := \max\{j \geq 0 : \phi \textit{ satisfies the cusp condition w.r.t. } H^{(n)}(\underline{L}) \textit{ of degree} \geq j\}$$

is called the degree of regularity *of ϕ. We call ϕ* regular, *if $\deg_{\mathrm{reg}}(\phi) \geq 0$. The space of regular elliptic functions of index $\underline{L}$ and degree n is denoted by $\mathcal{E}_{\mathrm{reg}}^{(n)}(\underline{L})$.*

b) We call $\phi \in \mathcal{E}^{(n)}(\underline{L})$ cuspidal *of degree* $\geq j$ *with respect to* $H^{(n)}(\underline{L})$*, if* $\deg_{\mathrm{reg}}(\phi) \geq j$ *and*

$$\mathcal{S}^{(m)}_{Y_{n-m}}\left(\phi\big|_{\underline{L}}[(0,\lambda_{n-m}),(0,\mu_{n-m})]\right) = 0$$

for all $m = 0,\dots,j$, $Y_{n-m} \in \mathrm{Pos}_{n-m}(\mathbb{R})$ *and* $\lambda_{n-m}, \mu_{n-m} \in L_{\mathbb{Q}}^{(n-m)}$*. In this case,*

$$\deg_{\mathrm{cusp}}(\phi) := \max\{j \geq 0 : \phi \text{ is cuspidal w.r.t. } H^{(n)}(\underline{L}) \text{ of degree} \geq j\}$$

is called the degree of cuspidality *of* ϕ*. We call* ϕ cuspidal*, if* $\deg_{\mathrm{cusp}}(\phi) \geq 0$*. The space of cuspidal elliptic functions of index* $\underline{L}$ *and degree n is denoted by* $\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L})$*.*

Remark 2.3.3. *Let* $\underline{L} = (L, Q)$ *be a lattice. Then both* $\mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L})$ *and* $\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L})$ *carry a natural module structure over the subring* $\mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}}$ *of* $\mathcal{O}(\mathcal{H}_n)$*, defined by*

$$\mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}} := \{f \in \mathcal{O}(\mathcal{H}_n) : \deg_{\mathrm{reg}}(f) \geq 0\}.$$

Furthermore, the subring $\mathcal{O}(\mathcal{H}_n)_{\mathrm{cusp}}$*, defined by*

$$\mathcal{O}(\mathcal{H}_n)_{\mathrm{cusp}} = \{f \in \mathcal{O}(\mathcal{H}_n) : \deg_{\mathrm{cusp}}(f) \geq 0\},$$

is an ideal in $\mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}}$ *and one has*

$$\mathcal{O}(\mathcal{H}_n)_{\mathrm{cusp}} \cdot \mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L}) \subseteq \mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L}).$$

Before we prove the main structure theorems, we need the following

Lemma 2.3.4. *Let* $\underline{L}$ *be a lattice and* $0 \leq j \leq n$*. Let* $\mu \in L^{*(n)}$ *and* $\lambda, v \in L_{\mathbb{R}}^{(n)}$*. We write* $\mu = (\mu_j, \mu_{n-j}), \lambda = (\lambda_j, \lambda_{n-j}), v = (v_j, v_{n-j})$ *for* $\mu_j \in L^{*(j)}, \mu_{n-j} \in L^{*(n-j)}$ *and* $\lambda_j, v_j \in L_{\mathbb{R}}^{(j)}, \lambda_{n-j}, v_{n-j} \in L_{\mathbb{R}}^{(n-j)}$*. Then the following assertions hold:*

a) $\mathcal{W}^{(j)}\left(\vartheta^{(n)}_{\underline{L},\mu}\Big|_{\underline{L}}[\lambda, v]\right) = \left(\vartheta^{(j)}_{\underline{L},\mu_j}\Big|_{\underline{L}}[\lambda_j, v_j]\right) \otimes \left(\vartheta^{(n-j)}_{\underline{L},\mu_{n-j}}\Big|_{\underline{L}}[\lambda_{n-j}, v_{n-j}]\right),$

b) $\mathcal{S}^{(j)}_{Y_{n-j}}\left(\vartheta^{(n)}_{\underline{L},\mu}\Big|_{\underline{L}}[\lambda, v]\right) = \begin{cases} \vartheta^{(j)}_{\underline{L},\mu_j}\Big|_{\underline{L}}[\lambda_j, v_j], & \text{if } \mu_{n-j} \equiv -\lambda_{n-j} \bmod L^{(n-j)}, \\ 0, & \text{else,} \end{cases}$

c) $\deg_{\mathrm{reg}}\left(\vartheta^{(n)}_{\underline{L},\mu}\right) = n.$

Proof. a) Follows from a straightforward calculation.

b) In view of a), it suffices to prove the assertion in the case $j = 0$. For $(Z, W) \in \mathcal{H}_n^J(L)$ we have

$$\vartheta^{(n)}_{\underline{L},\mu}\Big|_{\underline{L}}[\lambda, v](Z, W) = \sum_{l \in \mu + L^{(n)}} e^{2\pi i \mathrm{tr}(Q^{(n)}(l+\lambda)Z + B^{(n)}(l+\lambda, W))} \cdot e^{2\pi i \mathrm{tr}(B^{(n)}(l+\lambda, v))}.$$

Evaluating this expression at $(itY_n, 0)$ for $t > 0, Y_n > 0$ yields

$$\vartheta^{(n)}_{\underline{L},\mu}\Big|_{\underline{L}}[\lambda, v](itY_n, 0) = \sum_{l \in \mu + L^{(n)}} e^{-2\pi \mathrm{tr}\left(Q^{(n)}(l+\lambda)tY_n\right)} \cdot e^{2\pi i \mathrm{tr}(B^{(n)}(l+\lambda, v))}.$$

We have $\mathrm{tr}(Q^{(n)}(l+\lambda)Y_n) > 0$, whenever $Q^{(n)}(l+\lambda) \geq 0$ and $Q^{(n)}(l+\lambda) \neq 0$. As $t \to \infty$, all coefficients for $l \neq -\lambda$ vanish. This proves the second part of the statement. In the other case only the coefficient for $l = -\lambda$ remains and equals 1.

c) Follows immediately from b). □

Some useful identities are stated in the next

Lemma 2.3.5. *Let $\underline{L} = (L, Q)$ be a lattice. Let $\lambda, \mu \in L^{(n)}_{\mathbb{R}}$. Write $\lambda = (\lambda_j, \lambda_{n-j})$ and $\mu = (\mu_j, \mu_{n-j})$ for $\lambda_j, \mu_j \in L^{(j)}_{\mathbb{R}}, \lambda_{n-j}, \mu_{n-j} \in L^{(n-j)}_{\mathbb{R}}$. Then for $(Z_j, W_j) \in \mathcal{H}_j^J(L)$ the following assertions hold:*

a) $\mathcal{W}^{(j)}\left(\phi\big|_{\underline{L}}[\lambda, \mu]\right)((Z_j, W_j), \cdot) = \left(\mathcal{W}^{(j)}\left(\phi\big|_{\underline{L}}[(\lambda_j, 0), (\mu_j, 0)]\right)((Z_j, W_j), \cdot)\right)\Big|_{\underline{L}}[\lambda_{n-j}, \mu_{n-j}]$,

b) $\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\big|_{\underline{L}}[\lambda, \mu]\right)(Z_j, W_j) = \mathcal{S}^{(0)}_{Y_{n-j}}\left(\left(\mathcal{W}^{(j)}\left(\phi\big|_{\underline{L}}[(\lambda_j, 0), (\mu_j, 0)]\right)((Z_j, W_j), \cdot)\right)\Big|_{\underline{L}}[\lambda_{n-j}, \mu_{n-j}]\right)$.

Proof. Part a) is a verification and b) follows immediately from a). We omit the details. □

Proposition 2.3.6. *Let $\underline{L}$ be a lattice. Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ with theta decomposition*

$$\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_\mu \cdot \vartheta^{(n)}_{\underline{L},\mu}.$$

Then the following assertions are equivalent:

i) $\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\big|_{\underline{L}}[(0, \lambda_{n-j}), 0)]\right)$ *exists for all* $\lambda_{n-j} \in L^{*(n)}$,

ii) $\mathcal{S}^{(j)}_{Y_{n-j}} h_\mu$ *exists for all* $\mu \in (L^*/L)^{(n)}$.

In this case, $\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\big|_{\underline{L}}[\lambda, v]\right)$ *exists for all* $\lambda, v \in L^{(n)}_{\mathbb{R}}$. *As a special case, we have*

$$\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\big|_{\underline{L}}[(0, -\mu_{n-j}), 0]\right) = \sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{S}^{(j)}_{Y_{n-j}} h_{(\mu_j, \mu_{n-j})} \vartheta^{(j)}_{\underline{L},\mu_j}$$

for $\mu_{n-j} \in L^{*(n-j)}$. *The degrees of regularity and cuspidality are given by*

$$\deg_{\mathrm{reg}} \phi = \min\left\{\deg_{\mathrm{reg}} h_\mu : \mu \in (L^*/L)^{(n)}\right\},$$
$$\deg_{\mathrm{cusp}} \phi = \min\left\{\deg_{\mathrm{cusp}} h_\mu : \mu \in (L^*/L)^{(n)}\right\}.$$

Proof. We prove the nontrivial direction of the equivalence. At first, we treat the case $j = 0$. In this case, let $\lambda_1, \ldots, \lambda_{d^n}$ denote a system of representatives for $(L^*/L)^{(n)}$, where $d := \det \underline{L}$. We consider the holomorphic, matrix-valued function

$$Z \mapsto A(Z) := \left(\vartheta^{(n)}_{\underline{L},\lambda_j}\Big|_{\underline{L}}[-\lambda_i, 0](Z, 0) \right)_{1 \le i,j \le d^n}, \quad Z \in \mathcal{H}_n,$$

which satisfies the identity

$$\begin{pmatrix} \phi\big|_{\underline{L}}[-\lambda_1, 0](Z, 0) \\ \vdots \\ \phi\big|_{\underline{L}}[-\lambda_{d^n}, 0](Z, 0) \end{pmatrix} = A(Z) \begin{pmatrix} h_{\lambda_1} \\ \vdots \\ h_{\lambda_{d^n}} \end{pmatrix}.$$

We evaluate both sides at $Z = itY_n$ for $t > 0, Y_n > 0$. The limit of the left hand side as $t \to \infty$ exists by assumption on ϕ. Furthermore we have $\lim_{t\to\infty} A(itY_n) = I_{d^n}$ by 2.3.4. This shows that $A(itY_n)$ is invertible for all $t > \gamma$, where γ is sufficiently large. As a consequence, the limit $\lim\limits_{t\to\infty} h_\mu(itY_n)$ exists for all $\mu \in (L^*/L)^{(n)}$ and one has

$$\mathcal{S}^{(0)}_{Y_n} h_\mu = \lim_{t\to\infty} h_\mu(itY_n), \quad \mu \in (L^*/L)^{(n)}.$$

Hence, the formula stated above is valid in the case $j = 0$. For $j > 0$ we apply $\mathcal{W}^{(j)}$ on ϕ in order to obtain

$$\mathcal{W}^{(j)}\phi((Z_j, W_j), \cdot) = \sum_{\mu_{n-j} \in (L^*/L)^{(n-j)}} \left(\sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{W}^{(j)} h_{(\mu_j, \mu_{n-j})}(Z_j, \cdot) \vartheta^{(j)}_{\underline{L},\mu_j}(Z_j, W_j) \right) \vartheta^{(n-j)}_{\underline{L},\mu_{n-j}}.$$

By assumption as well as by application of 2.3.5, the function

$$\mathcal{S}^{(0)}_{Y_{n-j}} \left(\left(\mathcal{W}^{(j)}\phi((Z_j, W_j), \cdot) \right) \Big|_{\underline{L}} [\lambda_{n-j}, 0] \right)$$

exists for all $\lambda_{n-j} \in L^{*(n-j)}$. Hence we are reduced to the case $j = 0$. We apply the part proven previously on $\mathcal{W}^{(j)}\phi((Z_j, W_j), \cdot)$ in order to obtain that the function

$$\mathcal{S}^{(0)}_{Y_{n-j}} \left(\sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{W}^{(j)} h_{(\mu_j, \mu_{n-j})}(Z_j, \cdot) \vartheta^{(j)}_{\underline{L},\mu_j}(Z_j, W_j) \right)$$

exists for all $W_j \in \mathcal{H}^J_j(L)$ and all $\mu_{n-j} \in (L^*/L)^{(n-j)}$. The functions $\vartheta^{(j)}_{\underline{L},\mu_j}(Z_j, \cdot), \mu_j \in (L^*/L)^{(j)}$ are linearly independent for $Z_j \in \mathcal{H}_j$. Hence a simple argument shows, that

$$\mathcal{S}^{(j)}_{Y_{n-j}} h_\mu(Z_j) = \mathcal{S}^{(0)}_{Y_{n-j}} \left(\mathcal{W}^{(j)} h_\mu(Z_j, \cdot) \right), \quad Z_j \in \mathcal{H}_j$$

exists for all $\mu \in (L^*/L)^{(n)}$. The identity

$$\mathcal{S}^{(j)}_{Y_{n-j}} \left(\phi\big|_{\underline{L}}[(0, -\mu_{n-j}), 0] \right) = \sum_{\mu_j \in (L^*/L)^{(j)}} \mathcal{S}^{(j)}_{Y_{n-j}} h_{(\mu_j, \mu_{n-j})} \vartheta^{(j)}_{\underline{L},\mu_j}$$

follows from 2.3.4. □

As a direct consequence of 2.3.6, we obtain the following

Theorem 2.3.7. *Let $\underline{L} = (L, Q)$ be a lattice. Then the following assertions hold:*

a) $\mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L})$ *is a free* $\mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}}$*-module of rank* $(\det \underline{L})^n$ *and decomposes as a direct sum*

$$\mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L}) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}} \cdot \vartheta^{(n)}_{\underline{L},\mu}.$$

b) $\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L})$ *decomposes as a direct sum*

$$\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L}) = \bigoplus_{\mu \in (L^*/L)^{(n)}} \mathcal{O}(\mathcal{H}_n)_{\mathrm{cusp}} \cdot \vartheta^{(n)}_{\underline{L},\mu}.$$

We consider regularity and cuspidality conditions in a more familiar scenario, cf. also [27, 1.6. Lemma].

Proposition 2.3.8. *Let $\underline{L} = (L, Q)$ be a lattice and $\phi \in \mathcal{E}^{(n)}(\underline{L})$ with theta decomposition*

$$\phi = \sum_{\mu \in (L^*/L)^{(n)}} h_\mu \cdot \vartheta^{(n)}_{\underline{L},\mu}.$$

Suppose that ϕ has an absolutely and locally uniformly convergent Fourier expansion of the form

$$\phi(Z, W) = \sum_{S \in \frac{1}{q}\mathrm{Sym}^{\sharp}_n(\mathbb{Z})} \sum_{\mu \in L^{*(n)}} c(S, \mu) e^{2\pi i \mathrm{tr}(SZ + B^{(n)}(\mu, W))}, \quad (Z, W) \in \mathcal{H}^J_n(L).$$

for some $q \in \mathbb{N}$. Then the following statements hold:

a) For $\mu \in (L^*/L)^{(n)}$ one has

$$h_\mu(Z) = \sum_{S \in \frac{1}{q}\mathrm{Sym}^{\sharp}_n(\mathbb{Z})} c(S, \mu) e^{2\pi i \mathrm{tr}((S - Q^{(n)}(\mu))Z)}, \quad Z \in \mathcal{H}_n.$$

b) The following assertions are equivalent:

i) $\phi \in \mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L})$ *resp.* $\phi \in \mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L})$,

ii) for all $\mu \in L^{*(n)}$: $c(S, \mu) \neq 0$ implies $Q^{(n)}(\mu) \leq S$ resp. $Q^{(n)}(\mu) < S$.

In this case one has $\deg_{\mathrm{reg}}(\phi) = n$ and the formula

$$\mathcal{S}^{(j)}_{Y_{n-j}} \phi(Z_j, W_j) = \sum_{S_j \in \frac{1}{q}\mathrm{Sym}^{\sharp}_j(\mathbb{Z})} \sum_{\mu_j \in L^{*(j)}} c\left(\begin{pmatrix} S_j & 0 \\ 0 & 0 \end{pmatrix}, (\mu_j, 0)\right) e^{2\pi i \mathrm{tr}(S_j Z_j + B^{(j)}(\mu_j, W_j))}$$

holds for all $j = 0, \ldots, n$ and $Y_{n-j} > 0$. As a special case one has $\mathcal{S}^{(0)}_{Y_n} \phi = c(0, 0)$.

c) If $n \geq 2$ and ϕ satisfies the transformation property

$$\phi(Z[U], WU) = \phi(Z, W), \quad (Z, W) \in \mathcal{H}_n^J(L)$$

for all $U \in \mathrm{GL}_n(\mathbb{Z})[q]$, then the Koecher principle holds, i.e. $\phi \in \mathcal{E}_{\mathrm{reg}}^{(n)}(\underline{L})$.

Proof. a) In the notation given in the proof of 2.2.8, we have

$$c(\mu, Z) = \sum_{S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} c(S, \mu) e^{2\pi i \mathrm{tr}(SZ)}.$$

b) From 2.2.8 b) we conclude, that $\phi \in \mathcal{E}_{\mathrm{reg}}^{(n)}(\underline{L})$ if and only if $h_\mu \in \mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}}$ for all $\mu \in (L^*/L)^{(n)}$. The latter is equivalent to the existence of $\lim_{t \to \infty} h_\mu(itY_n)$ for all $\mu \in (L^*/L)^{(n)}$ and all $Y_n > 0$, hence to $\mathrm{tr}((S - Q^{(n)}(\mu))Y_n) \geq 0$ for all $Y_n > 0$, i.e. to $Q^{(n)}(\mu) \leq S$, whenever $c(S, \mu) \neq 0$. The claim for cuspidal functions follows from $Q^{(n)}(\mu) < S$ if and only if $\mathrm{tr}((S - Q^{(n)}(\mu))Y_n) > 0$ for all $Y_n > 0$. The supplement follows from the fact, that every principal minor of a positive semidefinite matrix is again positive semidefinite.

c) Let $n \geq 2$ and assume without loss of generality that $N_{\underline{L}} | q$. The transformation laws of ϕ imply

$$h_\mu(Z + S) = h_\mu(Z), \quad h_\mu(Z[U]) = h_\mu(Z), \quad \mu \in (L^*/L)^{(n)}, Z \in \mathcal{H}_n,$$

for all $S \in \mathrm{Sym}_n(q\mathbb{Z})$ and $U \in \mathrm{GL}_n(\mathbb{Z})[q]$. Then the claim follows from the classical Koecher principle applied on each function h_μ separately. □

2.4 Metaplectic group and the Weil representation

In this section, we introduce the metaplectic group of as double cover the symplectic group, cf. [1, § 2 Sec. 1, § 3 Sec.]. Furthermore, we define the modular action on elliptic functions of lattice-index and discuss the Weil representation of higher degree.

Definition 2.4.1. *The* real symplectic group *of degree n is defined as*

$$\mathrm{Sp}_n(\mathbb{R}) := \left\{ M \in \mathbb{R}^{2n \times 2n} : J_n[M] = J_n \right\}.$$

Definition 2.4.2. *The* integral symplectic group *of degree n is defined as*

$$\mathrm{Sp}_n(\mathbb{Z}) := \mathrm{Sp}_n(\mathbb{R}) \cap \mathbb{Z}^{2n \times 2n} = \left\{ M \in \mathbb{Z}^{2n \times 2n} : J_n[M] = J_n \right\}.$$

We collect basic facts:

Remark 2.4.3. *a)* $M \in \mathrm{Sp}_n(\mathbb{R})$ *will always be written in the form* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with matrix blocks* $A, B, C, D \in \mathbb{R}^{n\times n}$.

b) For $M_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \mathrm{Sp}_j(\mathbb{Z}), M_{n-j} = \begin{pmatrix} A_{n-j} & B_{n-j} \\ C_{n-j} & D_{n-j} \end{pmatrix} \in \mathrm{Sp}_{n-j}(\mathbb{Z})$ *we define its* cross product $M_j \times M_{n-j}$ *by*

$$M_j \times M_{n-j} := \begin{pmatrix} A_j & 0 & B_j & 0 \\ 0 & A_{n-j} & 0 & B_{n-j} \\ C_j & 0 & D_j & 0 \\ 0 & C_{n-j} & 0 & D_{n-j} \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}).$$

c) The matrices

$$J_n, \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Sym}_n(\mathbb{Z}), \quad \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \mathrm{GL}_n(\mathbb{Z})$$

belong to $\mathrm{Sp}_n(\mathbb{Z})$.

d) $\mathrm{Sp}_n(\mathbb{Z})$ *is generated by* J_n *and* $\begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Sym}_n(\mathbb{Z})$.

e) The determinant $\det(CZ + D)$ *is nonvanishing for* $Z \in \mathcal{H}_n$ *and* $M \in \mathrm{Sp}_n(\mathbb{R})$*. Furthermore,* $\mathrm{Sp}_n(\mathbb{R})$ *acts on* $\mathcal{H}_n$ *via* fractional-linear transformations

$$M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}, \quad M \in \mathrm{Sp}_n(\mathbb{R}), Z \in \mathcal{H}_n.$$

f) Since $\mathcal{H}_n$ *is a convex domain, the function* $Z \mapsto \det(CZ + D)$ *for* $M \in \mathrm{Sp}_n(\mathbb{R})$ *has two holomorphic holomorphic square roots on* $\mathcal{H}_n$*. We denote by* j_M *the distinguished branch, which is uniquely determined by the value*

$$j_M(iI_n) = \sqrt{\det(Ci + D)}.$$

Definition 2.4.4. *The* metaplectic group *of degree n is the group*

$$\mathrm{Mp}_{2n}(\mathbb{Z}) = \left\{(M, \varepsilon_M j_M) : M \in \mathrm{Sp}_n(\mathbb{Z}), \varepsilon_M \in \{\pm 1\}\right\}$$

with underlying multiplication

$$(M, \varepsilon_M j_M) \cdot (M', \varepsilon_{M'} j_{M'}) := \left(MM', \varepsilon_M \varepsilon_{M'} \varepsilon(M, M') j_{MM'}\right),$$

where

$$\varepsilon(M, M') := \frac{j_M(M'\langle Z\rangle) j_{M'}(Z)}{j_{MM'}(Z)} \in \{\pm 1\}$$

independent of $Z \in \mathcal{H}_n$*. For* $M \in \mathrm{Sp}_n(\mathbb{Z})$ *we write* $\widetilde{M} := (M, j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$.

We give some

Remarks 2.4.5. *a) By definition one has* $\widetilde{M}\widetilde{N} = \widetilde{MN}$ *if and only if* $\varepsilon(M,N) = 1$.

b) In $\mathrm{Mp}_{2n}(\mathbb{Z})$ *we distinguish the elements*

$$\widetilde{J_n} = \left(J_n, \sqrt{\det Z}\right), \quad \widetilde{\begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}}, S \in \mathrm{Sym}_n(\mathbb{Z}).$$

c) The kernel of the covering map

$$\mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_n(\mathbb{Z}),\ (M, \varepsilon_M j_M) \mapsto M$$

is cyclic of order two and the generated by the element $(I_{2n}, -j_{I_{2n}})$. *Hence,*

$$\mathrm{Mp}_{2n}(\mathbb{Z}) / \left\langle (I_{2n}, -j_{I_{2n}}) \right\rangle \cong \mathrm{Sp}_n(\mathbb{Z}).$$

By an explicit calculation we obtain the following

Lemma 2.4.6. *Let* $1 \leq j < n$. *Then the following assertions hold:*

a) $\varepsilon(J_n, J_n) = \begin{cases} i^n, & n \equiv 0 \bmod 2, \\ i^{n-1}, & n \equiv 1 \bmod 2, \end{cases}$

b) $\widetilde{J_n}^2 = \begin{cases} (-I_{2n}, i^n \cdot j_{-I_{2n}}), & n \equiv 0 \bmod 2, \\ (-I_{2n}, i^{n-1} \cdot j_{-I_{2n}}), & n \equiv 1 \bmod 2, \end{cases}$

c) $\widetilde{J_n}^4 = (I_{2n}, (-1)^n \cdot j_{I_{2n}})$,

d) $\varepsilon(M_j \times I_{2n-2j}, M_j' \times I_{2n-2j}) = \varepsilon(M_j, M_j')$ *for* $M_j, M_j' \in \mathrm{Sp}_j(\mathbb{Z})$,

e) $(\widetilde{J_j \times I_{2n-2j}})^4 = (I_{2n}, (-1)^j \cdot j_{I_{2n}})$.

Theorem 2.4.7. $\mathrm{Mp}_{2n}(\mathbb{Z})$ *is generated by the elements*

$$\widetilde{J_n}, \quad \widetilde{\begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}}, S \in \mathrm{Sym}_n(\mathbb{Z}).$$

Proof. Let $\widetilde{\Delta_n}$ denote the group generated by the elements stated above. From the surjectivity of the covering map and the corresponding result for $\mathrm{Sp}_n(\mathbb{Z})$ we deduce that $\mathrm{Mp}_{2n}(\mathbb{Z})$ is generated by $\widetilde{\Delta_n}$ and the central element $(I_{2n}, -j_{I_{2n}})$. In the case $n = 1$ we are done using the identity $\widetilde{J}^4 = (I_2, -j_{I_2})$. Let $n \geq 2$ and consider the element $(\widetilde{J \times I_{2n-2}})$. Then exactly one of two the cases

$$(I_{2n}, -j_{I_{2n}})^\delta (\widetilde{J \times I_{2(n-1)}}) \in \widetilde{\Delta_n}, \quad \delta = 0, 1,$$

occurs. But both already imply

$$(I_{2n}, -j_{I_{2n}}) = \left((I_{2n}, -j_{I_{2n}})^\delta (\widetilde{J \times I_{2(n-1)}}) \right)^4 \in \widetilde{\Delta_n}.$$

□

We introduce the modular action:

Definition 2.4.8. *Let $\underline{L} = (L, Q)$ be a lattice. For $\phi \in \mathcal{O}(\mathcal{H}_n^J(L)), (M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$ the function*

$$\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M) : \mathcal{H}_n^J(L) \longrightarrow \mathbb{C},$$

is defined pointwisely for $(Z, W) \in \mathcal{H}_n^J(L)$ by

$$\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M)(Z, W) := (\varepsilon_M j_M(Z))^{-2k} e^{-2\pi i \mathrm{tr}(Q^{(n)}(W)\cdot(CZ+D)^{-1}C)} \cdot \phi\big(M\langle Z\rangle, W(CZ+D)^{-1}\big).$$

We define $\phi\big|_{k,\underline{L}} M := \phi\big|_{k,\underline{L}} \widetilde{M}$ for $M \in \mathrm{Sp}_n(\mathbb{Z})$. The operator $\big|_{k,\underline{L}}$ is called the slash operator *of weight k and index $\underline{L}$.*

Remark 2.4.9. *a) One has $\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M) = \phi\big|_{k,\underline{L}} M$ for $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$ and $k \in \mathbb{Z}$.*

b) A straightforward calculation shows that for $k \in \frac{1}{2}\mathbb{Z}$ the assignment

$$((M, \varepsilon_M j_M), \phi) \mapsto \phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M), \quad (M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z}), \phi \in \mathcal{O}(\mathcal{H}_n(L))$$

defines an action of $\mathrm{Mp}_{2n}(\mathbb{Z})$ on $\mathcal{O}(\mathcal{H}_n^J(L))$. For $k \in \mathbb{Z}$ this action factors through an action of $\mathrm{Sp}_n(\mathbb{Z})$, while for half-integral k it factors through a projective action of $\mathrm{Sp}_n(\mathbb{Z})$, whose 2-cocycle equals $\varepsilon(M, M')^{2k}$ for $M, M' \in \mathrm{Sp}_n(\mathbb{Z})$.

Definition 2.4.10. *Let V be some finite-dimensional complex vector space. For a holomorphic function $\mathcal{H}_n \longrightarrow V$ and $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$ we define the function $f\big|_k(M, \varepsilon_M j_M)$ pointwisely by*

$$f\big|_k(M, \varepsilon_M j_M)(Z) := (\varepsilon_M j_M(Z))^{-2k} f(M\langle Z\rangle), \quad Z \in \mathcal{H}_n.$$

We define $\phi\big|_k M := \phi\big|_k \widetilde{M}$ for $M \in \mathrm{Sp}_n(\mathbb{Z})$. The operator $\big|_k$ is called the slash operator *of weight k.*

Proposition 2.4.11. *Let $\Gamma \leq \mathrm{Sp}_n(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$. For a map $\nu : \Gamma \longrightarrow \mathbb{C}^\times$ the following assertions are equivalent:*

i) $\nu(MM')j_{MM'}(Z)^{2k} = \nu(M)\nu(M')j_M(M'\langle Z\rangle)^{2k} j_{M'}(Z)^{2k}$ for all $M, M' \in \Gamma, Z \in \mathcal{H}_n$,

ii) $\nu(MM') = \varepsilon(M, M')^{2k}\nu(M)\nu(M')$ for all $M, M' \in \Gamma$,

iii) $\nu(MM')^{-1} \cdot 1\big|_k MM' = (\nu(M)\nu(M'))^{-1} \cdot (1\big|_k M)\big|_k M'$ for all $M, M' \in \Gamma$.

In this case, ν is called a multiplier system *of Γ of weight k and the assignment*

$$(M, \phi) \mapsto \nu(M)^{-1}\phi\big|_{k,\underline{L}} M, \quad M \in \Gamma, \phi \in \mathcal{O}(\mathcal{H}_n^J(L))$$

defines an action of Γ on $\mathcal{O}(\mathcal{H}_n^J(L))$ for every lattice $\underline{L}$. If $k \in \mathbb{Z}$, then $\nu \in \Gamma^{\mathrm{ab}}$.

A direct verification yields

Corollary 2.4.12. *Let* $\Gamma \leq \mathrm{Sp}_n(\mathbb{Z})$ *and* $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ *denote the pre-image of* Γ *under the covering map. Then there is a one-to-one correspondence between the abelian characters of* $\widetilde{\Gamma}$ *and the multiplier systems of* Γ*. More precisely, if* ν *is a multiplier system of* Γ *of weight* $k \in \frac{1}{2}\mathbb{Z}$*, then the map*

$$\widetilde{\nu} : \widetilde{\Gamma} \longrightarrow \mathbb{C}^\times, \quad \widetilde{\nu}(M, \varepsilon_M j_M) := \begin{cases} \nu(M), & k \in \mathbb{Z}, \\ \varepsilon_M \nu(M), & k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

is an abelian character of $\widetilde{\Gamma}$ *satisfying* $\widetilde{\nu}(\widetilde{M}) = \nu(M)$ *for* $M \in \Gamma$ *and* $\widetilde{\nu}(I_{2n}, -j_{I_{2n}}) = (-1)^{2k}$*. Conversely, if* $\widetilde{\nu} \in \widetilde{\Gamma}^{\mathrm{ab}}$*, then the map*

$$\nu : \Gamma \longrightarrow \mathbb{C}^\times, \quad \nu(M) := \widetilde{\nu}(\widetilde{M}), \quad M \in \Gamma,$$

is a multiplier system of Γ*. Furthermore,* $\nu \in \Gamma^{\mathrm{ab}}$ *if and only if* $\widetilde{\nu}(I_{2n}, -j_{I_{2n}}) = 1$*. Otherwise,* ν *is a multiplier system of weight* $\frac{1}{2}$ *and hence of weight* $\frac{k}{2}$ *for all odd* k*.*

Proposition 2.4.13. *Let* $n \geq 2$*. Then the following elements belong to* $\mathrm{Mp}_{2n}(\mathbb{Z})'$*:*

i) $\widetilde{M}^2$ *for* $M \in \mathrm{Sp}_n(\mathbb{Z})'$*,*

ii) $(I_{2n}, -j_{I_{2n}})$*,*

iii) $(-I_{2n}, j_{-I_{2n}})$*.*

Proof. i) Let $M \in \mathrm{Sp}_n(\mathbb{Z})'$. By surjectivity of the covering map exactly one of the two cases

$$(I_{2n}, -j_{I_{2n}})^\delta \widetilde{M} \in \mathrm{Mp}_{2n}(\mathbb{Z})', \quad \delta = 0, 1,$$

occurs. But both already imply $\widetilde{M}^2 \in \mathrm{Mp}_{2n}(\mathbb{Z})'$.

ii) In the case $n = 2$ consider $M_0 := \mathrm{diag}(-1, 1, -1, 1)$, which belongs to $\mathrm{Sp}_2(\mathbb{Z})'$. By i),

$$(I_4, -j_{I_4}) = \widetilde{\mathrm{diag}(-1, 1, -1, 1)}^2 = \widetilde{M_0}^2 \in \mathrm{Mp}_4(\mathbb{Z})'.$$

In the case $n \geq 3$ we obtain $M_0 \times I_{2n-4} \in \mathrm{Sp}_n(\mathbb{Z})'$ and hence

$$(I_{2n}, -j_{I_{2n}}) = \widetilde{M_0 \times I_{2n-4}}^2 \in \mathrm{Mp}_{2n}(\mathbb{Z})'.$$

iii) From $J_n \in \mathrm{Sp}_{2n}(\mathbb{Z})'$ we obtain $\widetilde{J_n}^2 = (-I_{2n}, \varepsilon(J_n, J_n) j_{-I_{2n}}) \in \mathrm{Mp}_{2n}(\mathbb{Z})'$ by i). In the case $\varepsilon(J_n, J_n) = 1$ we are done. In the case $\varepsilon(J_n, J_n) = -1$ we can multiply by $(I_{2n}, -j_{I_{2n}}) \in \mathrm{Mp}_{2n}(\mathbb{Z})'$. □

Remark 2.4.14. *Note that in view of 2.4.12, the fact* $(I_{2n}, -j_{I_{2n}}) \in \mathrm{Mp}_{2n}(\mathbb{Z})'$ *is the algebraic justification, that there exist no multiplier systems of half-integral weight of* $\mathrm{Mp}_{2n}(\mathbb{Z})$ *for* $n \geq 2$*.*

We discuss an example:

Example 2.4.15. *a) The* Dedekind eta function $\eta : \mathcal{H} \longrightarrow \mathbb{C}$*, defined by*

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right), \quad z \in \mathcal{H},$$

is a cusp form (cf. 2.5.11) of weight $\frac{1}{2}$ with respect to $\mathrm{Mp}_2(\mathbb{Z})$ and multiplier system ν_η, which is uniquely determined by the values

$$\nu_\eta(\widetilde{J}) = e^{-\frac{\pi i}{4}}, \quad \nu_\eta\left(\widetilde{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}\right) = e^{\frac{\pi i}{12}}.$$

Furthermore, ν_η is an abelian character of $\mathrm{Mp}_2(\mathbb{Z})$ of order 24 and satisfies $\nu_\eta(I_2, -j_{I_2}) = -1$.

b) It is well known that $\mathrm{Sp}_2(\mathbb{Z})$ has a unique nontrivial abelian character $\nu_{\mathrm{Maaß}}$. Since $(I_{2n}, -j_{I_{2n}})$ belongs to $\mathrm{Mp}_2(\mathbb{Z})'$, the character $\nu_{\mathrm{Maaß}}$ extends uniquely to $\mathrm{Mp}_2(\mathbb{Z})$ where it is again denoted by $\nu_{\mathrm{Maaß}}$. Furthermore,

$$\nu_{\mathrm{Maaß}}(\widetilde{J \times I_2}) = -1, \quad \nu_{\mathrm{Maaß}}\widetilde{\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}} = (-1)^{s_{11}+s_{12}+s_{22}}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z}).$$

This is enough to determine $\mathrm{Mp}_{2n}(\mathbb{Z})^{\mathrm{ab}}$ in the following way:

Lemma 2.4.16.

$$\mathrm{Mp}_{2n}(\mathbb{Z})^{\mathrm{ab}} = \begin{cases} \langle \nu_\eta \rangle, & n = 1, \\ \langle \nu_{\mathrm{Maaß}} \rangle, & n = 2, \\ \{1\}, & n \geq 3. \end{cases}$$

Proof. First we consider the case $n = 1$. Since $\mathrm{SL}_2(\mathbb{Z})^{\mathrm{ab}}$ is cyclic of order 12, we conclude that $\mathrm{Mp}_2(\mathbb{Z})^{\mathrm{ab}}$ has order at most 24. The characters $\widetilde{\nu}$ induced from $\mathrm{SL}_2(\mathbb{Z})$ are characterized by the condition $\widetilde{\nu}(I_2, -j_{I_2}) = 1$. Since $\nu_\eta(I_2, -j_{I_2}) = -1$, we obtain exactly 24 distinct characters. Let $n \geq 2$. Since $(I_{2n}, -j_{I_{2n}})$ belongs to $\mathrm{Mp}_{2n}(\mathbb{Z})'$, every character of $\mathrm{Mp}_{2n}(\mathbb{Z})$ is induced by a character of $\mathrm{Sp}_n(\mathbb{Z})$ and the claim follows. □

Definition 2.4.17. *Let G be a group and $\rho : G \longrightarrow \mathrm{GL}(V)$ a representation on some finite-dimensional complex vector space V. Then the map*

$$\det \rho : G \longrightarrow \mathbb{C}^\times, \quad g \mapsto \det(\rho(g))$$

is called the determinant character *of G with respect to ρ.*

Lemma 2.4.16 restricts the possible determinant characters arising from finite-dimensional representations of $\mathrm{Mp}_{2n}(\mathbb{Z})$:

Corollary 2.4.18. *Let V be a finite-dimensional vector space and $\rho : \mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{GL}(V)$ a representation. Then*

$$\det\rho = \begin{cases} \nu_\eta^d \text{ for some } d \in \{0,\dots,23\}, & n = 1, \\ \nu_{\text{Maaß}}^d \text{ for some } d \in \{0,1\}, & n = 2, \\ 1, & n \geq 3. \end{cases}$$

The compatibility of the actions of $H^{(n)}(\underline{L})$ and $\mathrm{Mp}_{2n}(\mathbb{Z})$ on $\mathcal{O}(\mathcal{H}_n^J(L))$ is explained in

Proposition 2.4.19. *Let $\underline{L}$ be a lattice. Let $[l,l'] \in H^{(n)}(\underline{L}), (M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$ and $\phi \in \mathcal{O}(\mathcal{H}_n^J(L))$. Then the following assertions hold:*

a) $\mathrm{Mp}_{2n}(\mathbb{Z})$ *acts from the right on* $H^{(n)}(\underline{L})$ *as automorphisms via*

$$[l,l'] \cdot (M, \varepsilon_M j_M) := [(l,l') \cdot M].$$

b) $\left(\phi\Big|_{k,\underline{L}}(M,\varepsilon_M j_M)\right)\Big|_{\underline{L}}\Big([l,l']\cdot(M,\varepsilon_M j_M)\Big) = \left(\phi\Big|_{\underline{L}}[l,l']\right)\Big|_{k,\underline{L}}(M,\varepsilon_M j_M).$

c) $\mathcal{E}^{(n)}(\underline{L})$ *is invariant under the slash operator* $\Big|_{k,\underline{L}}$.

This justifies the following

Definition 2.4.20. *The* metaplectic Jacobi group *of degree n associated to $\underline{L}$ is defined as the semidirect product*

$$\mathrm{Mp}_{2n}^J(\mathbb{Z})(\underline{L}) := \mathrm{Mp}_{2n}(\mathbb{Z}) \ltimes H^{(n)}(\underline{L}).$$

The definition is extended for arbitrary subgroups $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ by setting

$$\widetilde{\Gamma}^J(\underline{L}) := \widetilde{\Gamma} \ltimes H^{(n)}(\underline{L}).$$

Remark 2.4.21. *The groups $H^{(n)}(\underline{L})$ and $\mathrm{Mp}_{2n}(\mathbb{Z})$ are identified as subgroups of $\mathrm{Mp}_{2n}^J(\mathbb{Z})(\underline{L})$ via*

$$H^{(n)}(\underline{L}) \longrightarrow \mathrm{Mp}_{2n}^J(\mathbb{Z})(\underline{L}), \quad [l,l'] \mapsto \left(\widetilde{I}_n, [l,l']\right),$$
$$\mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{Mp}_{2n}^J(\mathbb{Z})(\underline{L}), \quad (M,\varepsilon_M j_M) \mapsto ((M,\varepsilon_M j_M), 0).$$

From the commutation relation in 2.4.19 we derive

Proposition 2.4.22. *Let $\underline{L}$ be a lattice and $k \in \frac{1}{2}\mathbb{Z}$. Then $\mathrm{Mp}^J_{2n}(\mathbb{Z})(\underline{L})$ acts on $\mathcal{O}(\mathcal{H}^J_n(L))$ by*

$$\phi\Big|_{k,\underline{L}}((M,\varepsilon_M j_M),[l,l']) := \left(\phi\Big|_{k,\underline{L}}(M,\varepsilon_M j_M)\right)\Big|_{\underline{L}}[l,l'],$$

where $(M,\varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z}), l,l' \in L^{(n)}, \phi \in \mathcal{O}(\mathcal{H}^J_n(L))$. Explicitly one has

$$\begin{aligned}\phi\big|_{k,\underline{L}}g(Z,W) =&(\varepsilon_M j_M(Z))^{-2k} \cdot e^{-2\pi i \mathrm{tr}\left(Q^{(n)}(W+lZ+l')(CZ+D)^{-1}C+Q^{(n)}(l)Z+B^{(n)}(l,W)\right)}\\ &\cdot \phi\big(M\langle Z\rangle,(W+lZ+l')(CZ+D)^{-1}\big), \quad (Z,W) \in \mathcal{H}^J_n(L),\end{aligned}$$

where $g = \big((M,\varepsilon_M j_M),[l,l']\big) \in \mathrm{Mp}^J_{2n}(\mathbb{Z})(\underline{L})$.

Definition 2.4.23. *Let $\underline{L}$ be a lattice. The complex vector space spanned by the Jacobi theta functions of degree n associated to $\underline{L}$ is defined as*

$$\Theta^{(n)}_{\underline{L}} := \left\langle \vartheta^{(n)}_{\underline{L},\mu} : \ \mu \in (L^*/L)^{(n)} \right\rangle_{\mathbb{C}}.$$

Note that $\dim_{\mathbb{C}} \Theta^{(n)}_{\underline{L}} = (\det \underline{L})^n$.

We apply the identification 2.2.3 on the Jacobi theta functions of degree n associated to $\underline{L}$ in order to obtain

$$\widetilde{\vartheta^{(n)}_{L,\mu}}(Z,W) = \Theta_{\kappa^{(n)}(\mu),SW}(Z,S), \quad (Z,W) \in \mathcal{H}_n \times \mathbb{C}^{r\times n},$$

where $\Theta_{P,Q}(Z,S)$ stands for the classical theta series in Z, S and characteristic (P,Q). By use of the general transformation formula for theta functions with characteristics, we obtain

Theorem 2.4.24. *Let $\underline{L} = (L,Q)$ be a lattice. Then the restriction of $|_{\frac{r_L}{2},\underline{L}}$ on $\Theta^{(n)}_{\underline{L}}$ induces an unitary representation*

$$\rho^{(n)}_{\underline{L}} : \mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{U}\left(\Theta^{(n)}_{\underline{L}}\right),$$

which is uniquely determined by the transformation laws

$$\rho^{(n)}_{\underline{L}}(\widetilde{J_n}) \cdot \vartheta^{(n)}_{\underline{L},\mu} := \vartheta^{(n)}_{\underline{L},\mu}\Big|_{\frac{r_L}{2},\underline{L}} J_n = \left(\frac{\sqrt{i}^{-r_L}}{\sqrt{\det \underline{L}}}\right)^n \sum_{\nu\in(L^*/L)^{(n)}} e^{-2\pi i \mathrm{tr}(B^{(n)}(\mu,\nu))} \cdot \vartheta^{(n)}_{\underline{L},\nu},$$

$$\rho^{(n)}_{\underline{L}}\widetilde{\begin{pmatrix} I_n & S\\ 0 & I_n\end{pmatrix}} \cdot \vartheta^{(n)}_{\underline{L},\mu} := \vartheta^{(n)}_{\underline{L},\mu}\Big|_{\frac{r}{2},\underline{L}}\begin{pmatrix} I_n & S\\ 0 & I_n\end{pmatrix} = e^{2\pi i \mathrm{tr}(Q^{(n)}(\mu)S)} \cdot \vartheta^{(n)}_{\underline{L},\mu}, \quad S \in \mathrm{Sym}_n(\mathbb{Z}),$$

for $\mu \in (L^*/L)^{(n)}$. The representation $\rho^{(n)}_{\underline{L}}$ is called the Weil representation *of* $\mathrm{Mp}_{2n}(\mathbb{Z})$ *associated to $\underline{L}$. In the case $n = 1$ we suppress the superscript and simply write $\rho_{\underline{L}}$.*

We give some

Remarks 2.4.25. *Let $\underline{L}$ be a lattice.*

a) By a slight abuse of notation we extend the definition by $\rho_{\underline{L}}^{(n)}(M) := \rho_{\underline{L}}^{(n)}(\widetilde{M})$ *for* $M \in \mathrm{Sp}_n(\mathbb{Z})$.

b) $\rho_{\underline{L}}^{(n)}(I_{2n}, -j_{I_{2n}}) = (-1)^{r_{\underline{L}}} \cdot \mathrm{id}_{\Theta_{\underline{L}}^{(n)}}$. *Hence,* $\rho_{\underline{L}}^{(n)}$ *factors through a representation of* $\mathrm{Sp}_n(\mathbb{Z})$ *if and only if* $r_{\underline{L}} \equiv 0 \bmod 2$. *Otherwise,* $\rho_{\underline{L}}^{(n)}$ *factors through a truly projective representation.*

c) $\rho_{\underline{L}}^{(n)}(-I_{2n}, j_{-I_{2n}}) \cdot \vartheta_{\underline{L},\mu}^{(n)} = \begin{cases} \vartheta_{\underline{L},-\mu}^{(n)}, & n \equiv 0 \bmod 2, \\ i^{-r_{\underline{L}}} \vartheta_{\underline{L},-\mu}^{(n)}, & n \equiv 1 \bmod 2. \end{cases}$

Definition 2.4.26. *Let $\underline{L}$ be a lattice.* $\phi \in \mathcal{E}^{(n)}(\underline{L})$ *is called* symmetric, *if*

$$\phi\left(Z, \sigma^{(n)}(W)\right) = \phi(Z, W)$$

holds for all $\sigma \in \mathrm{O}(\underline{L})$ *and all* $(Z, W) \in \mathcal{H}_n^J(L)$. *The submodule of symmetric elliptic functions of degree n and index $\underline{L}$ is denoted by* $\mathcal{E}^{(n)}(\underline{L})^{\mathrm{sym}}$.

Regarding the Jacobi theta functions we immediately obtain the following

Proposition 2.4.27. *Let $\underline{L}$ be a lattice and* $\sigma \in \mathrm{O}(\underline{L})$. *Then the following assertions hold:*

a) $\vartheta_{\underline{L},\mu}^{(n)}(Z, \sigma^{(n)}(W)) = \vartheta_{\underline{L},\sigma^{-1}(\mu)}^{(n)}(Z, W)$ *for all* $(Z, W) \in \mathcal{H}_n^J(L)$,

b) $\Theta_{\underline{L}}^{(n)}$ *is invariant under* $\mathrm{O}(\underline{L})$.

2.4.27 gives rise to the following

Definition 2.4.28. *Let $\underline{L}$ be a lattice and*

$$\Theta_{\underline{L}}^{(n),\mathrm{sym}} := \left\{ \phi \in \Theta_{\underline{L}}^{(n)} : \phi \text{ is symmetric} \right\}.$$

The induced subrepresentation

$$\rho_{\underline{L}}^{(n),\mathrm{sym}} : \mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{U}\left(\Theta_{\underline{L}}^{(n),\mathrm{sym}}\right)$$

is called the symmetric Weil representation *of degree n associated to $\underline{L}$.*

Explicit formulas are contained in the following

Proposition 2.4.29. *Let $\underline{L}$ be a lattice and $B_1, \dots, B_m$ denote the orbits of $\mathrm{O}(\underline{L})$ on $(L^*/L)^{(n)}$ given by diagonal action. Then a basis of $\Theta_{\underline{L}}^{(n),\mathrm{sym}}$ is given by*

$$\left\{ \sum_{\mu \in B_1} \vartheta_{\underline{L},\mu}^{(n)}, \dots, \sum_{\mu \in B_m} \vartheta_{\underline{L},\mu}^{(n)} \right\}.$$

For each $j = 1, \dots, m$ fix some $\mu_{B_j} \in B_j$. Then for $B \in \{B_1, \dots, B_m\}$ one has

$$\rho_{\underline{L}}^{(n),\mathrm{sym}}(J_n) \cdot \left(\sum_{\mu \in B} \vartheta_{\underline{L},\mu}^{(n)} \right) = \left(\frac{\sqrt{i}^{-r}}{\sqrt{\det(\underline{L})}} \right)^n \cdot \sum_{j=1}^{m} \left(\sum_{v \in B} e^{-2\pi i \mathrm{tr}(B^{(n)}(v, \mu_{B_j}))} \right) \left(\sum_{\mu \in B_j} \vartheta_{\underline{L},\mu}^{(n)} \right),$$

$$\rho_{\underline{L}}^{(n),\mathrm{sym}} \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix} \cdot \left(\sum_{\mu \in B} \vartheta_{\underline{L},\mu}^{(n)} \right) = e^{2\pi i \mathrm{tr}(Q^{(n)}(\mu_B))} \left(\sum_{\mu \in B} \vartheta_{\underline{L},\mu}^{(n)} \right), \quad S \in \mathrm{Sym}_n(\mathbb{Z}).$$

Lemma 2.4.30. *Let $\underline{L}$ be a lattice and $0 \le j \le n$. Then the global Witt operator $\mathcal{W}^{(j)}$ induces an isomorphism*

$$\mathcal{W}^{(j)} : \Theta_{\underline{L}}^{(n)} \longrightarrow \Theta_{\underline{L}}^{(j)} \otimes_{\mathbb{C}} \Theta_{\underline{L}}^{(n-j)}$$

of vector spaces.

Proof. The $(\det \underline{L})^n = (\det \underline{L})^j \cdot (\det \underline{L})^{n-j}$ maps

$$\mathcal{W}^{(j)} \vartheta_{\underline{L},\mu}^{(n)} = \vartheta_{\underline{L},\mu_j}^{(j)} \otimes \vartheta_{\underline{L},\mu_{n-j}}^{(n-j)}, \quad \mu = (\mu_j, \mu_{n-j}) \in (L^*/L)^{(n)},$$

are easily seen to be linearly independent. □

Proposition 2.4.31. *Let $\underline{L}$ be a lattice, $d := \det \underline{L}$ and $0 \le j \le n$. Then one has*

$$\mathcal{W}^j \circ \rho_{\underline{L}}^{(n)}(M_j \times M_{n-j}) \circ \mathcal{W}^{-j} = \rho_{\underline{L}}^{(j)}(M_j) \otimes \rho_{\underline{L}}^{(n-j)}(M_{n-j})$$

for all $M_j \in \mathrm{Sp}_j(\mathbb{Z}), M_{n-j} \in \mathrm{Sp}_{n-j}(\mathbb{Z})$. Furthermore, the determinant identity

$$\det\left(\rho_{\underline{L}}^{(n)}(M_j \times M_{n-j})\right) = \det\left(\rho_{\underline{L}}^{(j)}(M_j)\right)^{d^{n-j}} \cdot \det\left(\rho_{\underline{L}}^{(n-j)}(M_{n-j})\right)^{d^j}$$

holds.

Proof. Let $\mu = (\mu_j, \mu_{n-j})$ with $\mu_j \in (L^*/L)^{(j)}, \mu_{n-j} \in (L^*/L)^{(n-j)}$. Then one has

$$\mathcal{W}^{-j}\left(\vartheta_{\underline{L},\mu_j}^{(j)} \otimes \vartheta_{\underline{L},\mu_{n-j}}^{(n-j)}\right) = \vartheta_{\underline{L},\mu}^{(n)}.$$

Hence, the claim follows from

$$\mathcal{W}^{(j)}\left(\rho_{\underline{L}}^{(n)}(M_j \times M_{n-j})\vartheta_{\underline{L},\mu}^{(n)}\right) = \left(\rho_{\underline{L}}^{(j)}(M_j)\vartheta_{\underline{L},\mu_j}^{(j)}\right) \otimes \left(\rho_{\underline{L}}^{(n-j)}(M_{n-j})\vartheta_{\underline{L},\mu_{n-j}}^{(n-j)}\right).$$

The supplementary identity is then seen from the determinant formula for Kronecker products of matrices as well as $\dim \rho_{\underline{L}}^{(j)} = d^j$ and $\dim \rho_{\underline{L}}^{(n-j)} = d^{n-j}$. □

2.5 Jacobi forms and vector-valued modular forms

We define metaplectic Jacobi forms as invariants of the modular action on $\mathcal{E}^{(n)}(\underline{L})$:

Definition 2.5.1. *Let $\underline{L}$ be a lattice, $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ of finite index and $k \in \frac{1}{2}\mathbb{Z}$. Let $\widetilde{\nu}$ be an abelian character of $\widetilde{\Gamma}^J(\underline{L})$ of finite order, which acts trivially on $H^{(n)}(\underline{L})$. A holomorphic function*

$$\phi : \mathcal{H}_n^J(L) \longrightarrow \mathbb{C}$$

is called metaplectic Jacobi form *of degree n, weight k, index $\underline{L}$ and character $\widetilde{\nu}$, if*

i) $\phi\big|_{k,\underline{L}} g = \widetilde{\nu}(g)\phi$ *for all* $g \in \widetilde{\Gamma}^J(\underline{L})$,

ii) for all $g \in \mathrm{Mp}_{2n}^J(\mathbb{Z})$, the function $\phi\big|_{k,\underline{L}} g$ has a Fourier expansion of the form

$$\phi\big|_{k,\underline{L}} g(Z,W) = \sum_{0 \leq S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} \sum_{\substack{\mu \in L^{*(n)} \\ Q^{(n)}(\mu) \leq S}} c(S,\mu) e^{2\pi i \mathrm{tr}(SZ + B^{(n)}(\mu,W))}$$

for some $q = q(\widetilde{\Gamma}, \widetilde{\nu}, g) \in \mathbb{N}$.

We call ϕ a cusp form, *if $c(S,\mu) \neq 0$ in ii) already implies $Q^{(n)}(\mu) < S$. The space of metaplectic Jacobi forms of degree n, weight k, index $\underline{L}$ and character $\widetilde{\nu}$ is denoted by $J_{k,\underline{L}}^{(n)}(\widetilde{\Gamma}, \widetilde{\nu})$. The subspace of cusp forms is denoted by $J_{k,\underline{L}}^{(n)}(\widetilde{\Gamma}, \widetilde{\nu})^{\mathrm{cusp}}$. The space of symmetric Jacobi forms is denoted by $J_{k,\underline{L}}^{(n)}(\widetilde{\Gamma}, \widetilde{\nu})^{\mathrm{sym}}$.*

Consideration of the generators of $\widetilde{\Gamma}^J(\underline{L})$ yields

Proposition 2.5.2. *Let $\underline{L}$ be a lattice, $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and $\nu \in \widetilde{\Gamma}^{\mathrm{ab}}$ of finite order. Then one has $\phi \in J_{k,\underline{L}}^{(n)}(\widetilde{\Gamma}, \widetilde{\nu})$ if and only if the following assertions hold:*

i) $\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M) = \widetilde{\nu}(M, \varepsilon_M j_M)\phi$ *for all* $(M, \varepsilon_M j_M) \in \widetilde{\Gamma}$,

ii) $\phi\big|_{\underline{L}}[l, l'] = \phi$ *for all* $[l, l'] \in H^{(n)}(\underline{L})$,

iii) for all $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$, the function $\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M)$ has a Fourier expansion of the form

$$\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M)(Z,W) = \sum_{0 \leq S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} \sum_{\substack{\mu \in L^{*(n)} \\ Q^{(n)}(\mu) \leq S}} c(S,\mu) e^{2\pi i \mathrm{tr}(SZ + B^{(n)}(\mu,W))}$$

for some $q = q(\widetilde{\Gamma}, \widetilde{\nu}, (M, \varepsilon_M j_M)) \in \mathbb{N}$.

From 2.3.8 we obtain the following characterization:

Lemma 2.5.3. *Let $\underline{L}$ be a lattice, $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and $\widetilde{\nu} \in \widetilde{\Gamma}^{\mathrm{ab}}$ of finite order. Then the following assertions hold:*

i) $\phi \in J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma},\widetilde{\nu}) \iff \begin{cases} \phi|_{k,\underline{L}}(M,\varepsilon_M j_M) = \widetilde{\nu}(M,\varepsilon_M j_M)\phi & \text{for all } (M,\varepsilon_M j_M) \in \widetilde{\Gamma}, \\ \phi|_{k,\underline{L}}(M,\varepsilon_M j_M) \in \mathcal{E}^{(n)}(\underline{L})^{\mathrm{reg}} & \text{for all } (M,\varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z}). \end{cases}$

ii) $\phi \in J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma},\widetilde{\nu})^{\mathrm{cusp}} \iff \begin{cases} \phi|_{k,\underline{L}}(M,\varepsilon_M j_M) = \widetilde{\nu}(M,\varepsilon_M j_M)\phi & \text{for all } (M,\varepsilon_M j_M) \in \widetilde{\Gamma}, \\ \phi|_{k,\underline{L}}(M,\varepsilon_M j_M) \in \mathcal{E}^{(n)}(\underline{L})^{\mathrm{cusp}} & \text{for all } (M,\varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z}). \end{cases}$

Remark 2.5.4. *Loosely speaking, 2.5.3 states that a function $\phi \in \mathcal{E}^{(n)}(\underline{L})$ is a Jacobi form if and only if it satisfies a modular translation law with respect to $\widetilde{\Gamma}$ and all transformed functions $\phi|_{k,\underline{L}}(M,\varepsilon_M j_M)$ for $(M,\varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$ satisfy the cusp condition with respect to the Heisenberg group.*

Definition 2.5.5. *Let $\underline{L}$ be a lattice, $\Gamma \leq \mathrm{Sp}_n(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and ν a multiplier system of Γ of finite order. Then we define*

$$J^{(n)}_{k,\underline{L}}(\Gamma,\nu) := J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma},\widetilde{\nu}), \quad J^{(n)}_{k,\underline{L}}(\Gamma,\nu)^{\mathrm{cusp}} := J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma},\widetilde{\nu})^{\mathrm{cusp}}, \quad J^{(n)}_{k,\underline{L}}(\Gamma,\nu)^{\mathrm{cusp}} := J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma},\widetilde{\nu})^{\mathrm{sym}},$$

where $\widetilde{\Gamma}$ denotes the pre-image of Γ under the covering map and $\widetilde{\nu}$ is the abelian character of $\widetilde{\Gamma}$ according to 2.4.12. The elements of $J^{(n)}_{k,\underline{L}}(\Gamma,\nu)$ are called Jacobi forms *of degree n, weight k, index $\underline{L}$ and multiplier system ν. We will write*

$$J^{(n)}_{k,\underline{L}}(\Gamma) := J^{(n)}_{k,\underline{L}}(\Gamma,1), \quad J^{(n)}_{k,\underline{L}}(\nu) = J^{(n)}_{k,\underline{L}}(\mathrm{Sp}_n(\mathbb{Z}),\nu), \quad J^{(n)}_{k,\underline{L}} = J^{(n)}_{k,\underline{L}}(1).$$

Lemma 2.5.6. *Let $\underline{L}$ be a lattice, $\Gamma \leq \mathrm{Sp}_n(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and ν a multiplier system of Γ of finite order. Then the following assertions hold:*

a) $\dim_{\mathbb{C}} J^{(n)}_{k,\underline{L}}(\Gamma,\nu) < \infty$,

b) $\dim_{\mathbb{C}} J^{(n)}_{k,\underline{L}}(\Gamma,\nu) = 0$, *if* $k < 0$,

c) $\dim_{\mathbb{C}} J^{(n)}_{k,\underline{L}}(\nu) = 0$, *if $n \geq 2$ and k is half-integral.*

Proof. For a) and b) confer [27, Thm. 1.8]. The assertion c) follows from the fact that $(I_{2n},-1) \in \mathrm{Mp}_{2n}(\mathbb{Z})'$ if $n \geq 2$. □

For $q \in \mathbb{Z}$ we define congruence subgroups

$$\mathrm{Sp}_n(\mathbb{Z})_0[q] := \{M \in \mathrm{Sp}_n(\mathbb{Z}) : C \equiv 0 \bmod q\},$$
$$\mathrm{Sp}_n(\mathbb{Z})[q] := \{M \in \mathrm{Sp}_n(\mathbb{Z}) : M \equiv I_{2n} \bmod q\}.$$

From [1, II., § 4, Theorem 2.2] we adopt

Example 2.5.7. *Let $\underline{L} = (L, Q)$ be a lattice. Then one has*

$$\vartheta^{(n)}_{\underline{L},0} \in J^{(n)}_{\frac{r_{\underline{L}}}{2},\underline{L}} \left(\mathrm{Sp}_n(\mathbb{Z})_0[N_{\underline{L}}], \chi^{(n)}_Q\right),$$

where $\chi^{(n)}_Q$ is defined in [1, I., § 4, Thm. 4.10, Thm. 4.12]. Additionally,

$$\vartheta^{(n)}_{\underline{L},\mu} \in J^{(n)}_{\frac{r_{\underline{L}}}{2},\underline{L}} (\mathrm{Sp}_n(\mathbb{Z})[N_{\underline{L}}])$$

holds for all $\mu \in (L^*/L)^{(n)}$.

We construct Jacobi-Eisenstein series:

Remark 2.5.8. *Let $\underline{L} = (L, Q)$ be a lattice and $\mu \in (L^*/L)^{(n)}$ such that $Q^{(n)}(\mu) \in \mathrm{Sym}^{\sharp}_n(\mathbb{Z})$. Let $\mu^{\mathrm{GL}_n(\mathbb{Z})}$ denote the orbit of μ under the right action of $\mathrm{GL}_n(\mathbb{Z})$ on $(L^*/L)^{(n)}$. From*

$$Q^{(n)}(\mu U) = Q^{(n)}(\mu)[U] \in \mathrm{Sym}^{\sharp}_n(\mathbb{Z}), \quad U \in \mathrm{GL}_n(\mathbb{Z}),$$

and 2.2.6 we conclude that the averaged function

$$\frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{\lambda \in \mu^{\mathrm{GL}_n(\mathbb{Z})}} \vartheta^{(n)}_{\underline{L},\lambda}$$

is invariant under $|_{k,\underline{L}} M$ for $M = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})$ and $k \in \mathbb{Z}$ even.

This motivates the following

Example 2.5.9. *Let $\underline{L}$ be a lattice. Let $k \geq n + r_{\underline{L}} + 2$ even and $\mu \in (L^*/L)^{(n)}$ such that $Q^{(n)}(\mu) \in \mathrm{Sym}^{\sharp}_n(\mathbb{Z})$. Let $\mu^{\mathrm{GL}_n(\mathbb{Z})}$ denote the orbit of μ under the action of $\mathrm{GL}_n(\mathbb{Z})$ on $(L^*/L)^{(n)}$. The* Jacobi-Eisenstein series *of degree n, weight k and index $\underline{L}$ with respect to μ is defined by*

$$E^{(n)}_{k,\underline{L},\mu} := \frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{M:\mathrm{Sp}_n(\mathbb{Z})_0\backslash \mathrm{Sp}_n(\mathbb{Z})} \sum_{\lambda \in \mu^{\mathrm{GL}_n(\mathbb{Z})}} \vartheta^{(n)}_{\underline{L},\lambda}\Big|_{k,\underline{L}} M,$$

where $\mathrm{Sp}_n(\mathbb{Z})_0 := \left\{M = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z})\right\}$. The series converges absolutely and uniformly on each vertical strip

$$V_n(\delta) := \left\{(Z, W) \in \mathcal{H}^J_n(L) : Y \geq \delta I_n, \mathrm{tr}(X^2) \leq \delta^{-1}, \mathrm{tr}\left(B^{(n)}(W, \overline{W})\right) < \delta^{-1}\right\}, \quad \delta > 0.$$

By construction, $E^{(n)}_{k,\underline{L},\mu}$ is invariant under both $\mathrm{Sp}_n(\mathbb{Z})$ and $H^{(n)}(\underline{L})$, hence $E^{(n)}_{k,\underline{L},\mu} \in J^{(n)}_{k,\underline{L}}$.

The Jacobi-Eisenstein series $E^{(n)}_{k,\underline{L}} := E^{(n)}_{k,\underline{L},0}$ was studied in [27, p. 200f]. By using standard methods, cf. the proof of [27, 2.8 Theorem], we obtain:

Proposition 2.5.10. *Let $\underline{L}$ be a lattice. Let $k \geq n + r_{\underline{L}} + 2$ even and $\mu \in (L^*/L)^{(n)}$ such that $Q^{(n)}(\mu) \in \mathrm{Sym}_n^\sharp(\mathbb{Z})$. Let $0 \leq j \leq n$ and $\mu_j \in (L^*/L)^{(j)}, \mu_{n-j} \in (L^*/L)^{(n-j)}$ such that $\mu = (\mu_j, \mu_{n-j})$. Then the following assertions hold:*

a) $\mathcal{S}^{(j)}_{Y_{n-j}}\left(E^{(n)}_{k,\underline{L},\mu}\Big|_{\underline{L}}[(0,-\mu_{n-j}),0]\right) = E^{(j)}_{k,\underline{L},\mu_j}$ *for all* $Y_{n-j} > 0$ *with the convention* $E^{(0)}_{k,\underline{L},\mu_0} := 1$.

b) $E^{(n)}_{k,\underline{L},\mu}$ *does not vanish identically.*

c) The Jacobi-Eisenstein series $E^{(n)}_{k,\underline{L},\mu}$*, where* μ *runs through a system of representatives of the orbits of* $\mathrm{GL}_n(\mathbb{Z})$ *on*

$$\left\{\mu \in (L^*/L)^{(n)} : Q^{(n)}(\mu) \in \mathrm{Sym}_n^\sharp(\mathbb{Z})\right\},$$

are linearly independent.

Definition 2.5.11. *Let $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ of finite index. Let V be a complex vector space of finite dimension and $\rho : \widetilde{\Gamma} \longrightarrow \mathrm{GL}(V)$ a finite representation and $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $f : \mathcal{H}_n \longrightarrow V$ is called* vector-valued modular form *of degree n and weight k with respect to $\widetilde{\Gamma}$ and ρ, if the following assertions hold:*

i) $f\big|_k(M, \varepsilon_M j_M) = \rho(M, \varepsilon_M j_M) f$ *for all* $(M, \varepsilon_M j_M) \in \widetilde{\Gamma}$,

ii) for all $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$*, the function* $f\big|_k(M, \varepsilon_M j_M)$ *has a Fourier expansion of the form*

$$f\big|_k(M, \varepsilon_M j_M)(Z) = \sum_{j=1}^{\dim_{\mathbb{C}}(V)} \left(\sum_{0 \leq S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} c_j(S) \cdot e^{2\pi i \mathrm{tr}(SZ)} \right) v_j\,, \quad Z \in \mathcal{H}_n$$

for some basis $v_j, j = 1, \dots, \dim_{\mathbb{C}}(V)$ *of* V *and* $q = q(\widetilde{\Gamma}, \rho, (M, \varepsilon_M j_M)) \in \mathbb{N}$.

In thise case f a cusp form*, if $\mathcal{S}^{(n-1)}\left(f\big|_k(M, \varepsilon_M j_M)\right) \equiv 0$ holds for all $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$. If $V = \mathbb{C}$, we will we suppress the term 'vector-valued' and simply call f a* modular form*. The space of vector valued modular forms of degree n and weight k with respect to ρ is denoted by $[\widetilde{\Gamma}, k, \rho]$, the subspace of cusp forms by $[\widetilde{\Gamma}, k, \rho]^{\mathrm{cusp}}$.*

We end this section by briefly reviewing the well known correspondence between Jacobi forms and vector-valued modular forms, cf. [24, Proposition 3.5.].

Let $\underline{L}$ be a lattice and $d := \det \underline{L}$. Let $\mu_1, \dots, \mu_{d^n}$ denote a complete system of representatives for $(L^*/L)^{(n)}$. We define

$$\overrightarrow{\Theta^{(n)}_{\underline{L}}} := \left(\vartheta^{(n)}_{\underline{L},\mu_1}, \dots, \vartheta^{(n)}_{\underline{L},\mu_{d^n}}\right)^t : \mathcal{H}_n \longrightarrow \mathbb{C}^{d^n}.$$

Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ with theta decomposition

$$\phi = \sum_{j=1}^{d^n} h_j \cdot \vartheta^{(n)}_{\underline{L},\mu_j} = \left\langle h, \overrightarrow{\Theta^{(n)}_{\underline{L}}} \right\rangle,$$

where $h = (h_1, \dots, h_{d^n})^t : \mathcal{H}_n \longrightarrow \mathbb{C}^{d^n}$ is holomorphic and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{C}^{d^n}$. Let $\rho_{\underline{L}}^{(n)}$ denote the matrix representation of the Weil representation with respect to the basis of $\Theta_{\underline{L}}^{(n)}$ given by the Jacobi theta functions associated to the system $(\mu_1, \dots, \mu_{d^n})$. Hence, for $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$, we obtain

$$\begin{aligned}
\phi\big|_{k,\underline{L}}(M, \varepsilon_M j_M) &= \left\langle h\big|_{k-\frac{r_{\underline{L}}}{2}}(M, \varepsilon_M j_M), \overrightarrow{\Theta_{\underline{L}}^{(n)}}\Big|_{\frac{r_{\underline{L}}}{2}}(M, \varepsilon_M j_M) \right\rangle \\
&= \left\langle h\big|_{k-\frac{r_{\underline{L}}}{2}}(M, \varepsilon_M j_M), \rho_{\underline{L}}^{(n)}(M, \varepsilon_M j_M) \cdot \overrightarrow{\Theta_{\underline{L}}^{(n)}} \right\rangle \\
&= \left\langle \rho_{\underline{L}}^{(n)}(M, \varepsilon_M j_M)^t \cdot h\big|_{k-\frac{r_{\underline{L}}}{2}}(M, \varepsilon_M j_M), \overrightarrow{\Theta_{\underline{L}}^{(n)}} \right\rangle
\end{aligned}$$

and thus the connection between Jacobi forms and vector-valued modular forms:

Theorem 2.5.12. *Let $\underline{L}$ be a lattice, $\Gamma \leq \mathrm{Sp}_n(\mathbb{Z})$ of finite index, $k \in \frac{1}{2}\mathbb{Z}$ and ν a multiplier system of Γ of finite order. Let $\mu_1, \dots, \mu_{d^n}$ denote a complete system of representatives for $(L^*/L)^{(n)}$ and $\rho_{\underline{L}}^{(n)}$ denote the matrix representation of the Weil representation with respect to the basis of $\Theta_{\underline{L}}^{(n)}$ given by the Jacobi theta functions associated to the system $(\mu_1, \dots, \mu_{d^n})$. Then the assignment*

$$h \mapsto \left\langle h, \overrightarrow{\Theta_{\underline{L}}^{(n)}} \right\rangle$$

induces isomorphisms of vector spaces

i) $\left[\widetilde{\Gamma}, k - \frac{r_{\underline{L}}}{2}, \widetilde{\nu} \cdot \left(\rho_{\underline{L}}^{(n)}|_{\widetilde{\Gamma}}\right)^{-t}\right] \xrightarrow{\sim} J_{k,\underline{L}}^{(n)}(\Gamma, \nu)$,

ii) $\left[\widetilde{\Gamma}, k - \frac{r_{\underline{L}}}{2}, \widetilde{\nu} \cdot \left(\rho_{\underline{L}}^{(n)}|_{\widetilde{\Gamma}}\right)^{-t}\right]^{\mathrm{cusp}} \xrightarrow{\sim} J_{k,\underline{L}}^{(n)}(\Gamma, \nu)^{\mathrm{cusp}}$,

where $\widetilde{\Gamma}$ denotes the preimage of Γ under the covering map and $\widetilde{\nu}$ is the abelian character of $\widetilde{\Gamma}$ in accordance to 2.4.12.

As an example we return to the Jacobi-Eisenstein series $E_{k,\underline{L},\mu}^{(n)}$:

Example 2.5.13. *Let $\underline{L}$ be a lattice. Let $k \geq n + r_{\underline{L}} + 2$ even and $\mu \in (L^*/L)^{(n)}$ such that $Q^{(n)}(\mu) \in \mathrm{Sym}_n^\sharp(\mathbb{Z})$. Let $d := \det \underline{L}$ and $\mu_1, \dots, \mu_{d^n}$ denote a complete system of representatives for $(L^*/L)^{(n)}$ and $\rho_{\underline{L}}^{(n)}$ denote the matrix representation of the Weil representation with respect to the basis of $\Theta_{\underline{L}}^{(n)}$ given by the Jacobi theta functions associated to the system $(\mu_1, \dots, \mu_{d^n})$. By a slight abuse of notation we denote by e_λ the standard basis vector e_j of $\mathbb{C}^{d^n}$, if $\lambda = \mu_j$ for $1 \leq j \leq d^n$. By definition of $\rho_{\underline{L}}^{(n)}$ we have*

$$\vartheta_{\underline{L},\mu}^{(n)}\Big|_{\frac{r_{\underline{L}}}{2},\underline{L}} M = \left\langle \rho_{\underline{L}}^{(n)}(M) \overrightarrow{\Theta_{\underline{L}}^{(n)}}, e_\mu \right\rangle = \sum_{\upsilon \in (L^*/L)^{(n)}} \left\langle \rho_{\underline{L}}^{(n)}(M) e_\upsilon, e_\mu \right\rangle \cdot \vartheta_{\underline{L},\upsilon}^{(n)}.$$

Hence the theta decomposition of $E^{(n)}_{k,\underline{L},\mu}$ *is given by*

$$\begin{aligned}
E^{(n)}_{k,\underline{L},\mu} &= \frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{M:\mathrm{Sp}_n(\mathbb{Z})_0\backslash \mathrm{Sp}_n(\mathbb{Z})} \sum_{\lambda\in\mu^{\mathrm{GL}_n(\mathbb{Z})}} \vartheta^{(n)}_{\underline{L},\lambda}\Big|_{k,\underline{L}} M \\
&= \frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{M:\mathrm{Sp}_n(\mathbb{Z})_0\backslash \mathrm{Sp}_n(\mathbb{Z})} \sum_{\lambda\in\mu^{\mathrm{GL}_n(\mathbb{Z})}} \left\langle \rho^{(n)}_{\underline{L}}(M)\overrightarrow{\Theta^{(n)}_{\underline{L}}}, e_\lambda \right\rangle \cdot 1\Big|_{k-\frac{r_{\underline{L}}}{2},\underline{L}} M \\
&= \frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{M:\mathrm{Sp}_n(\mathbb{Z})_0\backslash \mathrm{Sp}_n(\mathbb{Z})} \sum_{\lambda\in\mu^{\mathrm{GL}_n(\mathbb{Z})}} \left(\sum_{v\in(L^*/L)^{(n)}} \left\langle \rho^{(n)}_{\underline{L}}(M)e_v, e_\lambda \right\rangle \cdot \vartheta^{(n)}_{\underline{L},v} \right) \cdot 1\Big|_{k-\frac{r_{\underline{L}}}{2},\underline{L}} M \\
&= \frac{1}{\left|\mu^{\mathrm{GL}_n(\mathbb{Z})}\right|} \sum_{v\in(L^*/L)^{(n)}} \left(\sum_{M:\mathrm{Sp}_n(\mathbb{Z})_0\backslash \mathrm{Sp}_n(\mathbb{Z})} \left(\sum_{\lambda\in\mu^{\mathrm{GL}_n(\mathbb{Z})}} \left\langle \rho^{(n)}_{\underline{L}}(M)e_v, e_\lambda \right\rangle \right) \cdot 1\Big|_{k-\frac{r_{\underline{L}}}{2},\underline{L}} M \right) \vartheta^{(n)}_{\underline{L},v}.
\end{aligned}$$

2.6 Determinant characters of Weil representations of degree 1

Utilizing a case-by-case study for the irreducible root lattices of small rank constructed in section 1.3, the tables 2.1 and 2.2 present the matrix representations of $\rho_{\underline{L}}(M)$ and $\rho_{\underline{L}}^{\mathrm{sym}}(M)$ for the canonical elements $M \in \left\{J, \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right\}$ with respect to suitable bases. In each case we determine the exact ν_η-power of the corresponding determinant character of $\mathrm{Mp}_2(\mathbb{Z})$.

$\underline{L}$	$\rho_{\underline{L}}(J)$	$\det\rho_{\underline{L}}(J)$	$\rho_{\underline{L}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\det\rho_{\underline{L}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\det\rho_{\underline{L}}$
$\underline{A_1}$	$\frac{\sqrt{i}^{-1}}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	i	$\begin{pmatrix}1&0\\0&i\end{pmatrix}$	i	ν_η^6
$\underline{A_2}$	$\frac{-i}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\rho&\overline{\rho}\\1&\overline{\rho}&\rho\end{pmatrix}$	1	$\begin{pmatrix}1&0&0\\0&\rho&0\\0&0&\rho\end{pmatrix}$	$\overline{\rho}$	ν_η^{16}
$\underline{D_4}$	$\frac{-1}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$	-1	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$	-1	ν_η^{12}
$\underline{E_6}$	$\frac{i}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\overline{\rho}&\rho\\1&\rho&\overline{\rho}\end{pmatrix}$	1	$\begin{pmatrix}1&0&0\\0&\overline{\rho}&0\\0&0&\overline{\rho}\end{pmatrix}$	ρ	ν_η^8
$\underline{E_7}$	$\frac{\sqrt{i}}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	$-i$	$\begin{pmatrix}1&0\\0&-i\end{pmatrix}$	$-i$	ν_η^{18}
$\underline{E_8}$	1	1	1	1	$\nu_\eta^{24}=1$

Table 2.1: Weil representations and their determinant characters

$\underline{L}$	$\rho_{\underline{L}}^{\mathrm{sym}}(J)$	$\det\rho_{\underline{L}}^{\mathrm{sym}}(J)$	$\rho_{\underline{L}}^{\mathrm{sym}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\det\rho_{\underline{L}}^{\mathrm{sym}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\det\rho_{\underline{L}}^{\mathrm{sym}}$
$\underline{A_1}$	$\frac{\sqrt{i}^{-1}}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	i	$\begin{pmatrix}1&0\\0&i\end{pmatrix}$	i	ν_η^6
$2\underline{A_1}$	$\frac{-i}{2}\begin{pmatrix}1&1&1\\2&0&-2\\1&-1&1\end{pmatrix}$	$-i$	$\begin{pmatrix}1&0&0\\0&i&0\\0&0&-1\end{pmatrix}$	$-i$	ν_η^{18}
$\underline{A_1}(2)$	$\frac{\sqrt{i}^{-1}}{2}\begin{pmatrix}1&1&1\\2&0&-2\\1&-1&1\end{pmatrix}$	$\sqrt{i}$	$\begin{pmatrix}1&0&0\\0&\sqrt{i}&0\\0&0&-1\end{pmatrix}$	$-\sqrt{i}$	ν_η^{15}
$\underline{A_2}$	$\frac{-i}{\sqrt{3}}\begin{pmatrix}1&1\\2&-1\end{pmatrix}$	1	$\begin{pmatrix}1&0\\0&\rho\end{pmatrix}$	ρ	ν_η^8
$2\underline{A_2}$	$\frac{-1}{3}\begin{pmatrix}1&1&1\\4&1&-2\\4&-2&1\end{pmatrix}$	1	$\begin{pmatrix}1&0&0\\0&\rho&0\\0&0&\overline{\rho}\end{pmatrix}$	1	1
$\underline{D_4}$	$-\frac{1}{2}\begin{pmatrix}1&1\\3&-1\end{pmatrix}$	-1	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	-1	ν_η^{12}
$\underline{E_6}$	$\frac{i}{\sqrt{3}}\begin{pmatrix}1&1\\2&-1\end{pmatrix}$	1	$\begin{pmatrix}1&0\\0&\overline{\rho}\end{pmatrix}$	$\overline{\rho}$	ν_η^{16}
$\underline{E_7}$	$\frac{\sqrt{i}}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$	$-i$	$\begin{pmatrix}1&0\\0&-i\end{pmatrix}$	$-i$	ν_η^{18}
$\underline{E_8}$	1	1	1	1	$\nu_\eta^{24}=1$

Table 2.2: Symmetric Weil representations and their determinant characters

3 Pullback Theory

3.1 The pullback operator

Definition 3.1.1. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then the map*

$$\iota^{(n)} : L_0^{(n)} \longrightarrow L^{(n)}, \quad \iota^{(n)}(l_1, \ldots, l_n) := (\iota(l_1), \ldots, \iota(l_n))$$

for $l_1, \ldots, l_n \in L$ is called the diagonal embedding *of degree n associated to ι. Its unique $\mathbb{F}$-linear extension $(L_0)_{\mathbb{F}}^{(n)} \longrightarrow L_{\mathbb{F}}^{(n)}$ will again be denoted by $\iota^{(n)}$. In the case $n = 1$ we will supress the superscript.*

Every embedding $\iota : \underline{L_0} \longrightarrow \underline{L}$ induces a series of further embeddings:

Remarks 3.1.2. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then the following assertions hold:*

a) The map

$$\iota^{(n)} : \mathcal{H}_n^J(L_0) \longrightarrow \mathcal{H}_n^J(L), \quad (Z, W_0) \mapsto \left(Z, \iota^{(n)}(W_0)\right)$$

is a holomorphic embedding of the corresponding Jacobi half-spaces.

b) The map

$$\iota^{(n)} : H_{\mathbb{R}}^{(n)}(\underline{L_0}) \longrightarrow H_{\mathbb{R}}^{(n)}(\underline{L}), \quad (\lambda_0, \mu_0, \zeta) \mapsto \left(\iota^{(n)}(\lambda_0), \iota^{(n)}(\mu_0), \zeta\right)$$

is a monomorphism of the Heisenberg groups and one has

$$\iota^{(n)}\big|_{H^{(n)}(\underline{L_0})} : H^{(n)}(\underline{L_0}) \longrightarrow H^{(n)}(\underline{L}).$$

We may now introduce the main operator of our studies:

Definition 3.1.3. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. For a holomorphic function $\phi \in \mathcal{O}(\mathcal{H}_n^J(L))$, we define*

$$\phi\left[\iota^{(n)}\right] : \mathcal{H}_n^J(L_0) \longrightarrow \mathbb{C}$$

pointwisely by

$$\phi\left[\iota^{(n)}\right](Z, W_0) := \phi\left(Z, \iota^{(n)}(W_0)\right), \quad (Z, W_0) \in \mathcal{H}_n^J(L_0).$$

The function $\phi\left[\iota^{(n)}\right]$ *is called the* pullback *of* ϕ *with respect to* $\iota^{(n)}$*. As an abbreviation we set*

$$\phi[l] := \phi\left[\iota_l^{(n)}\right], \quad l \in L,$$

where $\iota_l : \underline{\mathbb{Z}}(t) \longrightarrow L$*,* $t = Q(l)$*, as in* 1.2.22*.*

Regarding compatibility of the Heisenberg-action and the pullback operator, we obtain

Proposition 3.1.4. *Let* $\underline{L_0}$ *and* $\underline{L}$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. Let* $g \in H_{\mathbb{R}}^{(n)}(\underline{L_0})$ *and* $\phi \in \mathcal{O}(\mathcal{H}_n^J(L))$*. Then one has*

$$\phi\left[\iota^{(n)}\right]\Big|_{\underline{L_0}} g = \left(\phi\Big|_{\underline{L}}\, \iota^{(n)}(g)\right)\left[\iota^{(n)}\right],$$

i.e. the diagram

$$\begin{array}{ccc}
\mathcal{O}(\mathcal{H}_n^J(L)) & \xrightarrow{\left[\iota^{(n)}\right]} & \mathcal{O}(\mathcal{H}_n^J(L_0)) \\
\Big\downarrow \Big|_{\underline{L}}\, \iota^{(n)}(g) & & \Big|_{\underline{L_0}}\, g \Big\downarrow \\
\mathcal{O}(\mathcal{H}_n^J(L)) & \xrightarrow{\left[\iota^{(n)}\right]} & \mathcal{O}(\mathcal{H}_n^J(L_0))
\end{array}$$

is commutative for all $g \in H_{\mathbb{R}}^{(n)}(\underline{L_0})$*.*

As a by-product we obtain

Corollary 3.1.5. *Let* $\underline{L_0}, \underline{L}$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. Then the pullback operator*

$$\left[\iota^{(n)}\right] : \mathcal{E}^{(n)}(\underline{L}) \longrightarrow \mathcal{E}^{(n)}(\underline{L_0})$$

is a homomorphism of $\mathcal{O}(\mathcal{H}_n)$*-modules.*

The next aim is to prove that the pullback operator is compatible with respect to regularity and cuspidality. To this end we need the following technical lemma, which can the proved by a straightforward calculation.

Lemma 3.1.6. *Let* $\underline{L_0}$ *and* $\underline{L}$ *be lattices,* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding and* $0 \le j \le n$*. Let* $\mathcal{O}(\mathcal{H}_n^J(L))$*,* $Y_{n-j} \in \mathrm{Pos}_{n-j}(\mathbb{R})$*,* $(\lambda_0)_{n-j}, (\mu_0)_{n-j} \in (L_0)_{\mathbb{Q}}^{(n-j)}$ *and suppose that*

$$\mathcal{S}_{Y_{n-j}}^{(j)}\left(\phi\Big|_{\underline{L}}\left[\left(0, \iota^{(n-j)}((\lambda_0)_{n-j})\right), \left(0, \iota^{(n-j)}((\mu_0)_{n-j})\right)\right]\right)$$

exists. Then

$$\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\left[\iota^{(n)}\right]\Big|_{\underline{L_0}}\left[(0,(\lambda_0)_{n-j}),(0,(\mu_0)_{n-j})\right]\right)$$

exists and one has

$$\begin{aligned}&\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\left[\iota^{(n)}\right]\Big|_{\underline{L_0}}\left[(0,(\lambda_0)_{n-j}),(0,(\mu_0)_{n-j})\right]\right)\\ &=\mathcal{S}^{(j)}_{Y_{n-j}}\left(\phi\Big|_{\underline{L}}\left[\left(0,\iota^{(n-j)}((\lambda_0)_{n-j})\right),\left(0,\iota^{(n-j)}((\mu_0)_{n-j})\right)\right]\right)\left[\iota^{(j)}\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}\deg_{\mathrm{reg}}\left(\phi\left[\iota^{(n)}\right]\right)&\geq\deg_{\mathrm{reg}}(\phi),\\ \deg_{\mathrm{cusp}}\left(\phi\left[\iota^{(n)}\right]\right)&\geq\deg_{\mathrm{cusp}}(\phi).\end{aligned}$$

3.1.6 immediately implies

Theorem 3.1.7. *Let $\underline{L_0}$ and $\underline{L}$ be lattices, $\iota:\underline{L_0}\longrightarrow\underline{L}$ an embedding. Let $\phi\in\mathcal{E}^{(n)}(\underline{L})$. If ϕ is regular resp. cuspidal, then $\phi\left[\iota^{(n)}\right]$ is regular resp. cuspidal of degree at least $\deg_{\mathrm{reg}}(\phi)$ resp. $\deg_{\mathrm{cusp}}(\phi)$. The operators*

$$\begin{aligned}\left[\iota^{(n)}\right]&:\mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L})\longrightarrow\mathcal{E}^{(n)}_{\mathrm{reg}}(\underline{L_0}),\\ \left[\iota^{(n)}\right]&:\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L})\longrightarrow\mathcal{E}^{(n)}_{\mathrm{cusp}}(\underline{L_0}).\end{aligned}$$

are well defined and homomorphisms of $\mathcal{O}(\mathcal{H}_n)_{\mathrm{reg}}$-modules.

The commutation relation between $\left[\iota^{(n)}\right]$ and the slash action is explained in the following

Proposition 3.1.8. *Let $\underline{L_0},\underline{L}$ be lattices and $\iota:\underline{L_0}\longrightarrow\underline{L}$ an embedding. For $(M,\varepsilon_M j_M)\in\mathrm{Mp}_{2n}(\mathbb{Z}),\phi\in\mathcal{O}(\mathcal{H}^J_n(L))$ and $k\in\frac{1}{2}\mathbb{Z}$ one has*

$$\phi\left[\iota^{(n)}\right]\Big|_{k,\underline{L_0}}(M,\varepsilon_M j_M)=\left(\phi\Big|_{k,\underline{L}}(M,\varepsilon_M j_M)\right)\left[\iota^{(n)}\right],$$

i.e. the diagram

$$\begin{array}{ccc}\mathcal{O}(\mathcal{H}^J_n(L)) & \xrightarrow{\left[\iota^{(n)}\right]} & \mathcal{O}(\mathcal{H}^J_n(L_0))\\ \Big\downarrow{\scriptstyle |_{k,\underline{L}}(M,\varepsilon_M j_M)} & & \Big\downarrow{\scriptstyle |_{k,\underline{L_0}}(M,\varepsilon_M j_M)}\\ \mathcal{O}(\mathcal{H}^J_n(L)) & \xrightarrow{\left[\iota^{(n)}\right]} & \mathcal{O}(\mathcal{H}^J_n(L_0))\end{array}$$

is commutative for all $(M,\varepsilon_M j_M)\in\mathrm{Mp}_{2n}(\mathbb{Z})$.

From 3.1.8 we immediately obtain

Theorem 3.1.9. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Let $\widetilde{\Gamma} \leq \mathrm{Mp}_{2n}(\mathbb{Z})$ of finite index, $\widetilde{\nu} \in \widetilde{\Gamma}^{\mathrm{ab}}$ of finite order and $k \in \frac{1}{2}\mathbb{Z}$. Then the pullback operator $\left[\iota^{(n)}\right]$ induces a homomorphism*

$$\left[\iota^{(n)}\right] : J^{(n)}_{k,\underline{L}}(\widetilde{\Gamma}, \widetilde{\nu}) \longrightarrow J^{(n)}_{k,\underline{L_0}}(\widetilde{\Gamma}, \widetilde{\nu})$$

of vector spaces of metaplectic Jacobi forms. Furthermore, $\left[\iota^{(n)}\right]$ restricts to Jacobi cusp forms.

The pullback operator is functorial in the following sense:

Lemma 3.1.10. *Let $\underline{L}, \underline{L_0}$ and $\underline{L_1}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}, \kappa : \underline{L} \longrightarrow \underline{L_1}$ embeddings. Then for all $\phi \in \mathcal{O}(\mathcal{H}^J_n(L_1))$ one has*

$$\left(\phi\left[\kappa^{(n)}\right]\right)\left[\iota^{(n)}\right] = \phi\left[(\kappa \circ \iota)^{(n)}\right].$$

From 3.1.10 we obtain the following

Corollary 3.1.11. *Let $\underline{L}$ and $\underline{L_0}$ be lattices and $\iota, \kappa : \underline{L_0} \longrightarrow \underline{L}$ embeddings. Then the following assertions hold:*

a) If ι and κ are stably equivalent, then one has $\phi\left[\iota^{(n)}\right] = \phi\left[\kappa^{(n)}\right]$ for all $\phi \in \mathcal{E}^{(n)}(\underline{L})$.

b) If ι and κ are equivalent, then one has $\phi\left[\iota^{(n)}\right] = \phi\left[\kappa^{(n)}\right]$ for all $\phi \in \mathcal{E}^{(n)}(\underline{L})^{\mathrm{sym}}$.

We determine the effect of the pullback operator on Fourier expansions:

Proposition 3.1.12. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ such that ϕ has a Fourier expansion of the form*

$$\phi(Z, W) = \sum_{S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} \sum_{\mu \in L^{*(n)}} c(S, \mu) e^{2\pi i \mathrm{tr}(SZ + B^{(n)}(\mu, W))}, \quad (Z, W) \in \mathcal{H}^J_n(L)$$

for some $q \in \mathbb{N}$. Suppose that the finiteness condition

$$\left|\left\{\mu \in L^{*(n)} : \iota^{*(n)}(\mu) = \xi \text{ and } c(S, \mu) \neq 0\right\}\right| < \infty$$

is satisfied for all $S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})$ and $\xi \in L_0^{*(n)}$. Then the Fourier expansion of $\phi\left[\iota^{(n)}\right]$ is given by the formula

$$\phi\left[\iota^{(n)}\right](Z, W_0) = \sum_{S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})} \sum_{\mu_0 \in L_0^{*(n)}} \left(\sum_{\mu \in L^{*(n)},\ \iota^{*(n)}(\mu) = \xi} c(S, \mu)\right) e^{2\pi i \mathrm{tr}(SZ + B_0^{(n)}(\xi, W_0))}.$$

Proof. Let $(Z, W_0) \in \mathcal{H}_n^J(L_0)$. Then one has

$$\begin{aligned}\phi\left[\iota^{(n)}\right](Z, W_0) &= \phi\left(Z, \iota^{(n)}(W_0)\right) \\ &= \sum_{S\in\frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})}\sum_{\mu\in L^{*(n)}} c(S,\mu)e^{2\pi i \mathrm{tr}(SZ+B^{(n)}(\mu,\iota^{(n)}(W_0)))} \\ &= \sum_{S\in\frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})}\sum_{\mu\in L^{*(n)}} c(S,\mu)e^{2\pi i \mathrm{tr}(SZ+B^{(n)}(\iota^{*(n)}(\mu),W_0))}.\end{aligned}$$

Due to the finiteness condition we obtain the desired result after a rearrangement. □

We give some sufficient criteria:

Remark 3.1.13. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Let $\phi \in \mathcal{E}^{(n)}(\underline{L})$ such that ϕ has a Fourier expansion of the form*

$$\phi(Z, W) = \sum_{S\in\frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})}\sum_{\mu\in L^{*(n)}} c(S,\mu)e^{2\pi i \mathrm{tr}(SZ+B^{(n)}(\mu,W))}, \quad (Z, W) \in \mathcal{H}_n^J(L)$$

for some $q \in \mathbb{N}$. Then the finiteness condition in 3.1.12 holds for $\phi \in \mathcal{E}_{\mathrm{reg}}^{(n)}(\underline{L})$. Especially, the finiteness condition holds if the Fourier coefficients $c(S, \mu)$ satisfy the symmetry

$$c(S[U], \mu U) = c(S, \mu)$$

for all $S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z}), \mu \in L^{*(n)}$ and $U \in \mathrm{GL}_n(\mathbb{Z})[q]$.

Proof. The symmetry condition implies $\phi \in \mathcal{E}_{\mathrm{reg}}^{(n)}(\underline{L})$ by the Koecher principle, cf. [1, II, § 3, Sec. 2]. Fix $S \in \frac{1}{q}\mathrm{Sym}_n^\sharp(\mathbb{Z})$. Then $c(S, \mu) \neq 0$ for $\mu \in L^{*(n)}$ implies $Q^{(n)}(\mu) \leq S$. Since Q is positive definite, the set $\left\{\mu \in L^{*(n)} : Q^{(n)}(\mu) \leq S\right\}$ is finite. Hence, the finiteness condition is trivially satisfied. □

3.2 Automorphic transfer and modular determinant

In this section we will study the pullback operator from an algebraic point of view and investigate its modularity properties. As a motivational example we discuss

Example 3.2.1. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. According to 3.1.7 we define holomorphic functions*

$$h_{\mu\xi,\iota} : \mathcal{H}_n \longrightarrow \mathbb{C}, \quad \mu \in L^{*(n)}, \xi \in L_0^{*(n)},$$

via the theta decomposition

$$\vartheta_{\underline{L},\mu}^{(n)}\left[\iota^{(n)}\right] = \sum_{\xi\in(L_0^*/L_0)^{(n)}} h_{\mu\xi,\iota}^{(n)} \cdot \vartheta_{\underline{L_0},\xi}^{(n)}.$$

Then the following assertions hold:

a)

$$h^{(n)}_{\mu\xi,\iota}(Z) = \sum_{\substack{S\equiv Q^{(n)}(\mu)-Q_0^{(n)}(\xi) \\ \text{mod } \mathrm{Sym}_n^\sharp(\mathbb{Z})}} \left|\left\{l \in \mu + L^{(n)} : Q^{(n)}(l) = S + Q_0^{(n)}(\xi),\ \iota^{*(n)}(l) = \xi\right\}\right| e^{2\pi i \mathrm{tr}(SZ)}$$

for all $Z \in \mathcal{H}_n$.

b) Let $S^{(n)}_{\mu\xi}[\iota] = \left\{l \in \mu + L^{(n)} : \iota^{*(n)}(l) = \xi\right\}$. *Then one has*

$$S^{(n)}_{\mu\xi}[\iota] = \left(\iota^{(n)}(\xi) + \left(\iota(L_0)^\perp\right)^{(n)}\right) \cap \mu + L^{(n)}$$

and

$$h^{(n)}_{\mu\xi,\iota}(Z) = \sum_{l\in S^{(n)}_{\mu\xi}[\iota]} e^{2\pi i \mathrm{tr}\left(\left(Q^{(n)}(l)-Q_0^{(n)}(\xi)\right)Z\right)} = e^{-2\pi i \mathrm{tr}\left(Q_0^{(n)}(\xi)Z\right)} \cdot \vartheta^{(n)}_{S^{(n)}_{\mu\xi}[\iota]}(Z).$$

As a special case,

$$h^{(n)}_{\mu\xi,\iota}(Z) = \vartheta^{(n)}_{\underline{\iota(L_0)}^{\perp,\underline{L}}}(Z), \quad \mu \in L^{(n)}, \xi \in L_0^{(n)}, Z \in \mathcal{H}_n,$$

i.e. the theta series of degree n associated to the orthogonal complement of $\underline{\iota(L_0)}$ *in* $\underline{L}$.

c) Let $j = 1, \ldots, n$ *and* $\mu = (\mu_j, \mu_{n-j})$, *where* $\mu_j \in L^{*(j)}, \mu_{n-j} \in L^{*(n-j)}$. *Then one has*

$$\mathcal{S}^{(j)}_{Y_{n-j}} h^{(n)}_{\mu\xi,\iota} = \begin{cases} h^{(j)}_{\mu_j\xi_j,\iota}, & \iota^{(n-j)}(\xi_{n-j}) \equiv \mu_{n-j} \text{ mod } L^{(n-j)}, \\ 0, & \text{else.} \end{cases}$$

Proof. a) Let $l \in L^{*(n)}$ and $S \geq 0$. The coefficient of $e^{2\pi i \mathrm{tr}(SZ+B^{(n)}(l,W))}$ in $\vartheta^{(n)}_{\underline{L},\mu}$ is given by $\delta_{Q^{(n)}(l),S}$, where δ denotes the Kronecker delta. Due to 3.1.12 we have

$$c_{\vartheta^{(n)}_{\underline{L},\mu}[\iota^{(n)}]}(S,\xi) = \left|\left\{l \in \mu + L^{(n)} : Q^{(n)}(l) = S, \iota^{*(n)}(l) = \xi\right\}\right|$$

for all $\xi \in L_0^{*(n)}$. Hence, from 2.3.8, we obtain

$$\begin{aligned} h^{(n)}_{\mu\xi,\iota}(Z) &= \sum_{S\geq 0} c_{\vartheta^{(n)}_{\underline{L},\mu}[\iota^{(n)}]}\left(S + Q_0^{(n)}(\xi), \xi\right) e^{2\pi i \mathrm{tr}(SZ)} \\ &= \sum_{\substack{S\geq 0 \\ S\equiv Q^{(n)}(\mu)-Q_0^{(n)}(\xi) \\ \text{mod } \mathrm{Sym}_n^\sharp(\mathbb{Z})}} \left|\left\{l \in \mu + L^{(n)} : Q^{(n)}(l) = S + Q_0^{(n)}(\xi),\ \iota^{*(n)}(l) = \xi\right\}\right| e^{2\pi i \mathrm{tr}(SZ)}. \end{aligned}$$

Note that the congruence

$$Q^{(n)}(l) - Q_0^{(n)}(\xi) \equiv Q^{(n)}(\mu) - Q_0^{(n)}(\xi) \text{ mod } \mathrm{Sym}_n^\sharp(\mathbb{Z})$$

holds for all $l \in \mu + L^{(n)}$.

b) Follows immediately from a) by a rearrangement. The equality of the sets in question follows from the identity $\iota^{*(n)}(\iota^{(n)}(\xi)) = \xi$.

c) Follows from b) or can be seen from 2.3.6 together with 3.1.6. $\square$

Definition 3.2.2. *Let $\underline{L}$ be a lattice. For a subspace $V \leq \Theta_{\underline{L}}^{(n)}$ we denote by $\mathcal{O}(\mathcal{H}_n)V$ the module generated by V over $\mathcal{O}(\mathcal{H}_n)$.*

In order to define the pullback operator in a more general setting we need

Lemma 3.2.3. *Let $\underline{L}$ be a lattice and $V \leq \Theta_{\underline{L}}^{(n)}$.*

a) Let $\phi_1, \ldots, \phi_m$ be linearly independent in V. Then the functions $\phi_1(Z, \cdot), \ldots, \phi_m(Z, \cdot)$ are linearly independent for every $Z \in \mathcal{H}_n$.

b) $\mathcal{O}(\mathcal{H}_n)V$ is free of rank $\dim_{\mathbb{C}}(V)$ and every $\mathbb{C}$-basis of V is an $\mathcal{O}(\mathcal{H}_n)$-basis of $\mathcal{O}(\mathcal{H}_n)V$.

Proof. a) Let $d := \det \underline{L}$. By definition of $\Theta_{\underline{L}}^{(n)}$ we have

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} = A \begin{pmatrix} \vartheta_{\underline{L},\mu_1}^{(n)} \\ \vdots \\ \vartheta_{\underline{L},\mu_{d^n}}^{(n)} \end{pmatrix}$$

for some $A \in \mathbb{C}^{m \times d^n}$ if we fix a complete system of representatives $\mu_1, \ldots, \mu_{d^n}$ of $(L^*/L)^{(n)}$. By assumption on $\phi_1, \ldots, \phi_m$, the matrix A has rank m. The functions

$$\vartheta_{\underline{L},\mu}^{(n)}(Z, \cdot), \quad \mu \in (L^*/L)^n$$

are linearly independent for $Z \in \mathcal{H}_n$. Since the matrix identity is also valid for $Z \in \mathcal{H}_n$ as an identity of W, we conclude that

$$\phi_1(Z, \cdot), \ldots, \phi_m(Z, \cdot)$$

must be linearly independent.

b) This follows immediately from a). $\square$

We come to the main definition, which covers the cases we are interested in:

Definition 3.2.4. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Assume that the following setup holds:*

i) $V \leq \Theta_{\underline{L}}^{(n)}$ resp. $V_0 \leq \Theta_{\underline{L_0}}^{(n)}$ are $\rho_{\underline{L}}^{(n)}$- resp. $\rho_{\underline{L}}^{(n)}$-invariant subspaces,

ii) $\rho : \mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{GL}_d(\mathbb{C})$ and $\rho_0 : \mathrm{Mp}_{2n}(\mathbb{Z}) \longrightarrow \mathrm{GL}_{d_0}(\mathbb{C})$ are matrix representations of $\rho_{\underline{L}}^{(n)}\Big|_V$ resp. $\rho_{\underline{L_0}}^{(n)}\Big|_{V_0}$ with respect to certain bases of V resp. V_0, *iii)* $V\left[\iota^{(n)}\right] \subseteq \mathcal{O}(\mathcal{H}_n)V_0$.

In this setting we define

$$H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right] \in \mathcal{O}(\mathcal{H}_n)^{d\times d_0}$$

as the transposed $\mathcal{O}(\mathcal{H}_n)$*-representation matrix of the restricted map*

$$\left[\iota^{(n)}\right] : \mathcal{O}(\mathcal{H}_n)V \longrightarrow \mathcal{O}(\mathcal{H}_n)V_0$$

with respect to the bases in ii) according to 3.2.3. We call $H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$ *the* automorphic transfer *from* $\underline{L}$ *to* $\underline{L_0}$ *with respect to* $\iota^{(n)}, \rho, \rho_0$*. In the equidimensional case* $d = d_0$*, the determinant*

$$\det H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right] : \mathcal{H}_n \longrightarrow \mathbb{C}$$

is well-defined and called the modular determinant *with respect to* $\iota^{(n)}, \rho, \rho_0$*.*

Remark 3.2.5. *Let* $\underline{L_0}, \underline{L}$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. Assume that we are in the setting of 3.2.4.*

a) For bases $(\phi_1, \ldots, \phi_d)$ *of* V *resp.* $(\psi_1, \ldots, \psi_{d_0})$ *of* V_0 *in 3.2.4 ii) the automorphic transfer from* $\underline{L}$ *to* $\underline{L_0}$ *with respect to* $\iota^{(n)}, \rho, \rho_0$ *is uniquely characterized by the identity*

$$\begin{pmatrix}\phi_1\\ \vdots\\ \phi_d\end{pmatrix}\left[\iota^{(n)}\right] = H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]\begin{pmatrix}\psi_1\\ \vdots\\ \psi_{d_0}\end{pmatrix}.$$

b) With respect to the bases given by the Jacobi theta functions associated to $\underline{L}$ *resp.* $\underline{L_0}$ *the automorphic transfer* $H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho_{\underline{L}}^{(n)}, \rho_{\underline{L_0}}^{(n)}\right]$ *was determined explicitly in 3.2.1.*

c) There are matrices $A \in \mathbb{C}^{d\times(\det \underline{L})^n}$ *and* $B \in \mathbb{C}^{(\det \underline{L_0})^n \times d_0}$ *such that*

$$H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right] = A \cdot H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho_{\underline{L}}^{(n)}, \rho_{\underline{L_0}}^{(n)}\right] \cdot B.$$

d) Provided existence, the modular determinant $\det H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$ *obviously depends on the choice of the bases of* V *resp.* V_0 *and hence is uniquely determined only up to some nonzero complex scalar.*

Functoriality of the automorphic transfer is explained in the following

Proposition 3.2.6. *Let* $\underline{L_0}, \underline{L_1}, \underline{L}$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L_1}, \kappa : \underline{L_1} \longrightarrow \underline{L}$ *embeddings. Assume that we are in the setting of 3.2.4, such that*

$$H_{\underline{L_1}}^{\underline{L}}\left[\kappa^{(n)}, \rho, \rho_1\right], \quad H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right]$$

are defined. Then one has

$$H_{\underline{L_0}}^{\underline{L}}\left[(\kappa\circ\iota)^{(n)}, \rho, \rho_0\right] = H_{\underline{L_1}}^{\underline{L}}\left[\kappa^{(n)}, \rho, \rho_1\right] \cdot H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right]$$

with respect to suitable bases.

Proof. Follows directly from 3.1.10. □

Corollary 3.2.7. *Let $\underline{L_0}, \underline{L_1}$ and $\underline{L}$ be lattices and*

$$\iota, \iota' : \underline{L_0} \longrightarrow \underline{L_1}, \quad \kappa, \kappa' : \underline{L_1} \longrightarrow \underline{L}$$

embeddings. Assume that we are in the situation of 3.2.4. If

$$H_{\underline{L_0}}^{\underline{L}}\left[(\kappa \circ \iota)^{(n)}, \rho, \rho_0\right] = H_{\underline{L_0}}^{\underline{L}}\left[(\kappa' \circ \iota')^{(n)}, \rho, \rho_0\right], \quad H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right] = H_{\underline{L_0}}^{\underline{L_1}}\left[\iota'^{(n)}, \rho_1, \rho_0\right].$$

and $H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho, \rho_0\right]$ has full row-rank, then $H_{\underline{L_1}}^{\underline{L}}\left[\kappa^{(n)}, \rho, \rho_1\right] = H_{\underline{L_1}}^{\underline{L}}\left[\kappa'^{(n)}, \rho, \rho_1\right]$.

Proof. Due to 3.2.6 and by assumption on $\iota^{(n)}$ and $\iota'^{(n)}$ one has

$$H_{\underline{L_1}}^{\underline{L}}\left[\kappa^{(n)}, \rho, \rho_1\right] \cdot H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right] = H_{\underline{L_1}}^{\underline{L}}\left[\kappa'^{(n)}, \rho, \rho_1\right] \cdot H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right].$$

Due to the condition on the rank we can cancel $H_{\underline{L_0}}^{\underline{L_1}}\left[\iota^{(n)}, \rho_1, \rho_0\right]$ from the right and obtain the desired result. □

We repeat some representation theoretic constructions:

Remark 3.2.8. *Let G be a group, V a complex vector space and $\rho : G \longrightarrow \mathrm{GL}(V)$ a representation.*

a) The dual *or* contragredient representation *of G* with respect to *ρ is defined as*

$$\rho^* : G \longrightarrow \mathrm{GL}(V^*), \quad \rho^*(g) := \rho(g^{-1})^t, g \in G,$$

where V^* denotes the dual space of V and ^{t} means taking the transposed endomorphism. If $V = \mathbb{C}^m$ we can identify ^{t} with the usual matrix transpose.

b) Let $\rho' : G \longrightarrow W$ be another representation on some complex vector space W. Then there is a unique representation $\rho \otimes \rho' : G \longrightarrow \mathrm{GL}(V \otimes W)$, such that

$$(\rho \otimes \rho')(g)(v \otimes w) = \rho(g)v \otimes \rho'(g)w, \quad g \in G, v \in V, w \in W.$$

The representation $\rho \otimes \rho'$ is called the tensor product *of ρ and ρ'. If $V = \mathbb{C}^n, W = \mathbb{C}^m$, we can identify $\mathrm{GL}(V)$ resp. $\mathrm{GL}(W)$ with $\mathrm{GL}_n(\mathbb{C})$ resp. $\mathrm{GL}_m(\mathbb{C})$ and $V \otimes W$ with $\mathbb{C}^{n \times m}$ via the* dyadic product *$v \otimes w \mapsto vw^t$. In this case one has*

$$(\rho \otimes \rho')(g)X = \rho(g)X\rho'(g)^t, \quad X \in \mathbb{C}^{n \times m}, g \in G.$$

Now we can state the precise modularity properties of the automorphic transfer and the modular determinant provided its existence:

Theorem 3.2.9. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Assume that we are in the setting of 3.2.4. Then the following assertions hold:*

a) The transformation formula

$$H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right]\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2}}(M, \varepsilon_M j_M) = \rho(M, \varepsilon_M j_M) \cdot H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] \cdot \rho_0(M, \varepsilon_M j_M)^{-1}$$

holds for all $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$, i.e.

$$H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] \in \left[\mathrm{Mp}_{2n}(\mathbb{Z}), \frac{r_{\underline{L}} - r_{\underline{L_0}}}{2}, \rho \otimes \rho_0^*\right].$$

b) Suppose that $\dim \rho = \dim \rho_0$. In this case

$$\det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] \in \left[\mathrm{Mp}_{2n}(\mathbb{Z}), \frac{r_{\underline{L}} - r_{\underline{L_0}}}{2} \cdot \dim \rho, \det \rho \cdot (\det \rho_0)^{-1}\right].$$

Proof. a) The assertion is clear for $(I_{2n}, -j_{I_{2n}})$. We fix bases $(\phi_1, \ldots, \phi_d)$ of V and $(\psi_1, \ldots, \psi_{d_0})$ of V_0 that correspond to ρ resp. ρ_0. For reasons of readability we will simply write H instead of $H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right]$. The main idea is to apply the slash operator $\Big|_{\frac{r_L}{2}, \underline{L_0}} \widetilde{M}$ for every $M \in \mathrm{Sp}_n(\mathbb{Z})$ on the defining equation

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left[\iota^{(n)}\right] = H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}.$$

Considering the left-hand side we obtain

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left[\iota^{(n)}\right] \Big|_{\frac{r_L}{2}, \underline{L_0}} \widetilde{M} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \Big|_{\frac{r_L}{2}, \underline{L}} \widetilde{M} \left[\iota^{(n)}\right] = \rho(\widetilde{M}) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_d \end{pmatrix} \left[\iota^{(n)}\right] = \rho(\widetilde{M}) \cdot H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}.$$

On the right-hand side we obtain

$$H \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} \Big|_{\frac{r_L}{2}, \underline{L_0}} \widetilde{M} = \left(H\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2}} \widetilde{M}\right) \cdot \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix} \Big|_{\frac{r_{L_0}}{2}, \underline{L_0}} \widetilde{M} = \left(H\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2}} \widetilde{M}\right) \cdot \rho_0(M) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{d_0} \end{pmatrix}.$$

Hence, by linear independency, $H\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2}} \widetilde{M} = \rho(\widetilde{M}) \cdot H \cdot \rho_0(\widetilde{M})^{-1}$.

b) We use the notation as in a) and apply the determinant on the transformation formula. By the Leibniz formula, the identity

$$\det\left(H\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2}}(M, \varepsilon_M j_M)\right) = (\det H)\Big|_{\frac{r_{\underline{L}}-r_{\underline{L_0}}}{2} \cdot \dim \rho}(M, \varepsilon_M j_M)$$

holds for all $(M, \varepsilon_M j_M) \in \mathrm{Mp}_{2n}(\mathbb{Z})$. By the Koecher principle, holomorphicity at the cusps of $\mathrm{Mp}_{2n}(\mathbb{Z})$ is only to prove for $n = 1$. Here we note $V \leq \mathcal{E}^{(1)}_{\mathrm{reg}}(\underline{L})$ and $V_0 \leq \mathcal{E}^{(1)}_{\mathrm{reg}}(\underline{L_0})$. ☐

We need several lemmas:

Lemma 3.2.10. *Let* $\widetilde{v} \in \mathrm{Mp}_2(\mathbb{Z})^{\mathrm{ab}}$ *and* $f \in [\mathrm{Mp}_2(\mathbb{Z}), k, \widetilde{v}]$. *Then the following assertions are equivalent:*

i) f *is nonvanishing on* $\mathcal{H}$, *i.e.* $f \in \mathcal{O}(\mathcal{H})^\times$,

ii) $f \in \mathbb{C}^\times \eta^{2k}$.

Hence, we have

$$\mathcal{O}(\mathcal{H})^\times \cap [\mathrm{Mp}_2(\mathbb{Z}), k, \widetilde{v}] = \begin{cases} \mathbb{C}^\times \eta^{2k}, & \widetilde{v} = v_\eta^{2k}, \\ \emptyset, & \text{else.} \end{cases}$$

Proof. Clearly, ii) implies i). Let $\widetilde{v} = v_\eta^b$ for some $0 \leq b < 24$. We consider the function

$$g := \eta^{24-b} f \in \left[\mathrm{Mp}_2(\mathbb{Z}), k - \frac{b}{2} + 12\right],$$

which is still nonvanishing on $\mathcal{H}$, but vanishes at infinity of some order $m \geq 0$. Since $\eta^{24} \in [\mathrm{Mp}_2(\mathbb{Z}), 12]$ vanishes at infinity of order 1, we obtain that $g\eta^{-24m}$ is a modular form, nonvanishing on $\mathcal{H}$ including infinity. By the valence formula $g \cdot \eta^{-24m}$ has weight zero and hence is constant. It follows that $g \in \mathbb{C}^\times \eta^{24m}$. Consequently, f is a nonzero multiple of some η-power and due to the weight one has $f \in \mathbb{C}^\times \eta^{2k}$. □

In view of surjectivity we give a negative result:

Lemma 3.2.11. *Let* $\underline{L_0}, \underline{L}$ *be lattices and* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *an embedding. Assume that we are in the setting of 3.2.4. Suppose that* $n \geq 2, r_{\underline{L_0}} < r_{\underline{L}}$ *and* $\dim \rho = \dim \rho_0$. *Then the following assertions hold:*

a) The modular determinant $\det H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$ *has a zero in* $\mathcal{H}_n$ *and* $H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$ *is singular.*

b) The pullback $\left[\iota^{(n)}\right] : \mathcal{O}(\mathcal{H}_n)V \longrightarrow \mathcal{O}(\mathcal{H}_n)V_0$ *is not surjective.*

Proof. a) Let $F := \det H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$. By assumption on the ranks of $\underline{L}$ resp. $\underline{L_0}$, we conclude that F is a modular form of degree $n > 1$ and positive weight. Hence, by the Koecher principle, F has some zero $Z_0 \in \mathcal{H}_n$. Consequently, the matrix $H_{\underline{L_0}}^{\underline{L}}\left[\iota^{(n)}, \rho, \rho_0\right]$ is not invertible over $\mathcal{O}(\mathcal{H}_n)$.

b) Otherwise $\left[\iota^{(n)}\right]$ would extend to an epimorphism of equidimensional vector spaces by passing to the quotient field of the integral domain $\mathcal{O}(\mathcal{H}_n)$. Hence, $\left[\iota^{(n)}\right]$ would be a monomorphism of vector spaces, thus an isomorphism. This contradicts a). □

We prove a surprising necessary and sufficient criterion for isomorphy:

Theorem 3.2.12. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Assume that we are in the setting of 3.2.4. Suppose that $r_{\underline{L_0}} < r_{\underline{L}}$. Then the following assertions are equivalent:*

i) $\left[\iota^{(n)}\right] : \mathcal{O}(\mathcal{H}_n)V \longrightarrow \mathcal{O}(\mathcal{H}_n)V_0$ *is an isomorphism,*

ii) $\det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right]$ *is nonvanishing on $\mathcal{H}_n$,*

iii) $n = 1$ *and* $\det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] = c \cdot \eta^{\left(r_{\underline{L}} - r_{\underline{L_0}}\right)\cdot \dim \rho}$ *for some $c \in \mathbb{C}^\times$.*

In this case, the determinant characters are related by the formula

$$\det \rho = \nu_\eta^{\left(r_{\underline{L}} - r_{\underline{L_0}}\right)\cdot \dim \rho} \cdot \det \rho_0.$$

Proof. The equivalence of i) and ii) is immediate by linear algebra over $\mathcal{O}(\mathcal{H}_n)$. Furthermore, iii) obviously implies ii). In the remaining case we conclude $n = 1$ from 3.2.11 and $F = c \cdot \eta^{\left(r_{\underline{L}} - r_{\underline{L_0}}\right)\cdot \dim \rho}$ for some $c \neq 0$ from 3.2.10. Then one has $\det \rho \cdot (\det \rho_0)^{-1} = \nu_\eta^{\left(r_{\underline{L}} - r_{\underline{L_0}}\right)\cdot \dim \rho}$. □

We state necessary conditions for the existence of nontrivial modular determinants:

Proposition 3.2.13. *Let $\underline{L_0}, \underline{L}$ be lattices and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Assume that we are in the setting of 3.2.4. Suppose that $n \geq 2$ and $\dim \rho = \dim \rho_0$. Then the following assertions hold:*

a) If $\left(r_{\underline{L}} - r_{\underline{L_0}}\right) \cdot \dim \rho$ *is odd, then* $\det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] \equiv 0$.

b) If n is odd and $\left(r_{\underline{L}} - r_{\underline{L_0}}\right) \cdot \dim \rho \not\equiv 0 \bmod 4$, *then* $\det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right] \equiv 0$.

In all these cases, the pullback operator

$$\left[\iota^{(n)}\right] : \mathcal{O}(\mathcal{H}_n)V \longrightarrow \mathcal{O}(\mathcal{H}_n)V_0$$

is not injective.

Proof. a) Follows from the fact that for $n \geq 2$ every modular form with respect to $\mathrm{Mp}_{2n}(\mathbb{Z})$ of half-integral weight is trivial.

b) Let $F = \det H^{\underline{L}}_{\underline{L_0}}\left[\iota^{(n)}, \rho, \rho_0\right]$. Note that F has trivial character, since $n \geq 3$. Then the claim follows from

$$F = F\Big|_{\frac{r_{\underline{L}} - r_{\underline{L_0}}}{2}\cdot \dim \rho}(-I_{2n}) = i^{\left(r_{\underline{L}} - r_{\underline{L_0}}\right)\cdot \dim \rho} \cdot F.$$

□

We study the behaviour between the automorphic transfer and the Witt operator $\mathcal{W}^{(j)}$.

Given $\mathcal{X} = (x_1, \ldots, x_n)$ and $\mathcal{Y} = (y_1, \ldots, y_m)$ we define $\mathcal{X} \otimes \mathcal{Y}$ as

$$\mathcal{X} \otimes \mathcal{Y} := \left((x_1, y_1), \ldots, (x_1, y_m), \ldots, (x_2, y_1), \ldots, (x_n, y_m)\right).$$

Theorem 3.2.14. *Let $\underline{L_0}, \underline{L}$ be lattices, $d = \det \underline{L}$, $d_0 = \det \underline{L_0}$ and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. In the case $n = 1$ we distinguish complete systems of representatives*

$$\mathcal{R}_{\underline{L}}^{(1)} = (\mu_1, \ldots, \mu_d), \quad \mathcal{R}_{\underline{L_0}}^{(1)} = (\xi_1, \ldots, \xi_{d_0})$$

of L^*/L resp. L_0^*/L_0. In the case $n > 1$ we define systems of representatives of $(L^*/L)^{(n)}$ resp. $(L_0^*/L_0)^{(n)}$ recursively by

$$\mathcal{R}_{\underline{L}}^{(n)} := \mathcal{R}_{\underline{L}}^{(1)} \otimes \mathcal{R}_{\underline{L}}^{(n-1)}, \quad \mathcal{R}_{\underline{L_0}}^{(n)} := \mathcal{R}_{\underline{L_0}}^{(1)} \otimes \mathcal{R}_{\underline{L_0}}^{(n-1)}.$$

Let $\rho_{\underline{L}}^{(n)}$ resp. $\rho_{\underline{L_0}}^{(n)}$ denote the Weil representations with respect to the bases of $\Theta_{\underline{L}}^{(n)}$ resp. $\Theta_{\underline{L_0}}^{(n)}$ given by the Jacobi theta functions labeled by $\mathcal{R}_{\underline{L}}^{(n)}$ resp. $\mathcal{R}_{\underline{L_0}}^{(n)}$. Then one has

$$\mathcal{W}^{(j)} H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(n)}, \rho_{\underline{L}}^{(n)}, \rho_{\underline{L_0}}^{(n)}\right] = H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(j)}, \rho_{\underline{L}}^{(j)}, \rho_{\underline{L_0}}^{(j)}\right] \otimes H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(n-j)}, \rho_{\underline{L}}^{(n-j)}, \rho_{\underline{L_0}}^{(n-j)}\right],$$

where $\otimes$ denotes the Kronecker product of matrices. In the equidimensional case $d = d_0$ one has

$$\mathcal{W}^{(j)} \left(\det H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(n)}, \rho_{\underline{L}}^{(n)}, \rho_{\underline{L_0}}^{(n)}\right]\right) = \left(\det H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(j)}, \rho_{\underline{L}}^{(j)}, \rho_{\underline{L_0}}^{(j)}\right]\right)^{d^{(n-j)}} \otimes \left(\det H_{\underline{L_0}}^{\underline{L}} \left[\iota^{(n-j)}, \rho_{\underline{L}}^{(n-j)}, \rho_{\underline{L_0}}^{(n-j)}\right]\right)^{d^j}.$$

Proof. By an inductive argument we obtain natural identifications

$$\mathcal{R}_{\underline{L}}^{(n)} = \mathcal{R}_{\underline{L}}^{(j)} \otimes \mathcal{R}_{\underline{L}}^{(n-j)}, \quad \mathcal{R}_{\underline{L_0}}^{(n)} = \mathcal{R}_{\underline{L_0}}^{(j)} \otimes \mathcal{R}_{\underline{L_0}}^{(n-j)}$$

for $0 \leq j \leq n$. Then it is easy to see that

$$\mathcal{W}^{(j)} \left(\vartheta_{\underline{L},\mu}^{(n)} \left[\iota^{(n)}\right]\right) = \vartheta_{\underline{L},\mu_j}^{(j)} \left[\iota^{(j)}\right] \otimes \vartheta_{\underline{L},\mu_{n-j}}^{(n-j)} \left[\iota^{(n-j)}\right], \quad \mu = (\mu_j, \mu_{n-j}) \in \mathcal{R}_{\underline{L}}^{(n)}.$$

The claim follows by considering the corresponding representation matrices. The supplementary follows from the determinant formula for Kroneckers products of matrices. □

3.3 Examples of automorphic transfer and modular determinant of degree 1

In this section we shall determine $H^{\underline{L}}_{\underline{L_0}}[\iota, \rho, \rho_0]$ explicitly for some examples of embeddings of lattices of small rank and low-dimensional subrepresentations.

$\underline{L}$	representation	basis
$\underline{E_8}$	$\rho_{\underline{E_8}}$	$\begin{pmatrix} \vartheta_{\underline{E_8}} \end{pmatrix}$
$\underline{E_7}$	$\rho_{\underline{E_7}}$	$\begin{pmatrix} \vartheta_{\underline{E_7}} \\ \vartheta_{\underline{E_7}, \frac{1}{2}(e_1+e_2+e_3+e_4+e_5+e_6+e_7)} \end{pmatrix}$
$\underline{E_6}$	$\rho^{\mathrm{sym}}_{\underline{E_6}}$	$\begin{pmatrix} \vartheta_{\underline{E_6}} \\ \vartheta_{\underline{E_6}, \frac{1}{3}(e_1+e_3-2e_5)} + \vartheta_{\underline{E_6}, -\frac{1}{3}(e_1+e_3-2e_5)} \end{pmatrix}$
$\underline{D_4}$	$\rho^{\mathrm{sym}}_{\underline{D_4}}$	$\begin{pmatrix} \vartheta_{\underline{D_4}} \\ \vartheta_{\underline{D_4}, \frac{e_2+e_4}{2}} + \vartheta_{\underline{D_4}, \frac{e_2+e_6}{2}} + \vartheta_{\underline{D_4}, \frac{e_2+e_7}{2}} \end{pmatrix}$
$\underline{A_2}$	$\rho^{\mathrm{sym}}_{\underline{A_2}}$	$\begin{pmatrix} \vartheta_{\underline{A_2}} \\ \vartheta_{\underline{A_2}, \frac{e_4+e_6+e_7}{2}} + \vartheta_{\underline{A_2}, -\frac{e_4+e_6+e_7}{2}} \end{pmatrix}$
$2\underline{A_2}$	$\rho^{\mathrm{sym}}_{2\underline{A_2}}$	$\begin{pmatrix} \vartheta_{2\underline{A_2}} \\ \vartheta_{\underline{A_2}, \left(\frac{e_4+e_6+e_7}{2}, 0\right)} + \vartheta_{\underline{A_2}, \left(-\frac{e_4+e_6+e_7}{2}, 0\right)} + \vartheta_{\underline{A_2}, \left(0, \frac{e_4+e_6+e_7}{2}\right))} + \vartheta_{\underline{A_2}, \left(0, -\frac{e_4+e_6+e_7}{2}\right)} \\ \vartheta_{\underline{A_2}, \frac{e_4+e_6+e_7}{2} \cdot (1,1)} + \vartheta_{\underline{A_2}, \frac{e_4+e_6+e_7}{2} \cdot (1,-1)} + \vartheta_{\underline{A_2}, \frac{e_4+e_6+e_7}{2} \cdot (-1,1)} + \vartheta_{\underline{A_2}, \frac{e_4+e_6+e_7}{2} \cdot (-1,-1)} \end{pmatrix}$
$2\underline{A_1}$	$\rho^{\mathrm{sym}}_{2\underline{A_1}}$	$\begin{pmatrix} \vartheta_{2\underline{A_1}} \\ \vartheta_{2\underline{A_1}, \left(\frac{1}{2}, 0\right)} + \vartheta_{2\underline{A_1}, \left(0, \frac{1}{2}\right)} \\ \vartheta_{2\underline{A_1}, \left(\frac{1}{2}, \frac{1}{2}\right)} \end{pmatrix}$
$\underline{A_1}(2)$	$\rho^{\mathrm{sym}}_{\underline{A_1}(2)}$	$\begin{pmatrix} \vartheta_{\underline{A_1}(2)} \\ \vartheta_{\underline{A_1}(2), \frac{1}{4}} + \vartheta_{\underline{A_1}(2), -\frac{1}{4}} \\ \vartheta_{\underline{A_1}(2), \frac{1}{2}} \end{pmatrix}$
$\underline{A_1}$	$\rho_{\underline{A_1}}$	$\begin{pmatrix} \vartheta_{\underline{A_1}, 0} \\ \vartheta_{\underline{A_1}, \frac{1}{2}} \end{pmatrix}$

In order to fix some notation, we let for the rest of this section ϑ_0 and ϑ_1 denote the classical theta functions, i.e.

$$\vartheta_0(z,w) := \vartheta_{\underline{A_1},0} = \sum_{m\in\mathbb{Z}} e^{2\pi i(m^2z+2mw)}, \quad \vartheta_1(z,w) := \vartheta_{\underline{A_1},\frac{1}{2}} = \sum_{m\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i(m^2z+2mw)}$$

for $(z,w)\in\mathcal{H}\times\mathbb{C}$ with the conventions $\vartheta_0(z) := \vartheta_0(z,0)$ and $\vartheta_1(z) := \vartheta_1(z,0)$ for $z\in\mathcal{H}$.

Example 3.3.1. *Let $\iota : \underline{A_1} \longrightarrow \underline{E_7}$ be an embedding.*

a) *The automorphic transfer $H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right]$ is given by*

$$H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right](z) = \begin{pmatrix} \vartheta_0(z)^6+3\vartheta_0(z)^2\vartheta_1(z)^4 & 4\vartheta_0(z)^3\vartheta_1(z)^3 \\ 4\vartheta_0(z)^3\vartheta_1(z)^3 & \vartheta_1(z)^6+3\vartheta_0(z)^4\vartheta_1(z)^2 \end{pmatrix}$$

for $z\in\mathcal{H}$.

b) *The modular determinant* $\det H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right]$ *is given by*

$$\det H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right] = 3\vartheta_0(z)^2\vartheta_1(z)^2(\vartheta_0(z)^4-\vartheta_1(z)^4)^2 = 12\cdot\eta(z)^{12}$$

for $z\in\mathcal{H}$.

Proof. a) Since $\underline{E_7}$ is an irreducible root lattice it suffices to consider the embedding $\iota=\iota_{e_1}$. By a detailed analysis of table 1.4, we can determine the theta decomposition of $\vartheta_{\underline{E_7},0}[e_1]$, namely

$$\vartheta_{\underline{E_7},0}[e_1](z,w_0) = \left(\vartheta_0(z)^6+3\vartheta_0(z)^2\vartheta_1(z)^4\right)\vartheta_0(z,w_0) + 4\vartheta_0(z)^3\vartheta_1(z)^3\vartheta_1(z,w_0).$$

By using the identity

$$\vartheta_{\underline{E_7},1}[e_1](z,w_0) = -i\cdot\sqrt{2}i\vartheta_{\underline{E_7},0}[e_1]\Big|_{\frac{7}{2},\underline{A_1}} J(z,w_0) - \vartheta_{\underline{E_7},0}[e_1](z,w_0), \quad (z,w_0)\in\mathcal{H}\times\mathbb{C},$$

as well as the transformation formulas of ϑ_0 and ϑ_1, a straightforward calculation shows

$$\vartheta_{\underline{E_7},1}[e_1](z,w_0) = 4\vartheta_0(z)^3\vartheta_1(z)^3\vartheta_0(z,w_0) + \left(\vartheta_1(z)^6+3\vartheta_0(z)^4\vartheta_1(z)^2\right)\vartheta_1(z,w_0).$$

Note that the latter identity can also be seen from table 1.4.

b) Due to 3.2.9 one has

$$\det H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right] \in \left[\mathrm{Mp}_2(\mathbb{Z}),6,\nu_\eta^{12}\right] = \mathbb{C}\eta^{12}.$$

By expanding the determinant we see that its first nontrivial Fourier coefficient equals 12, hence

$$\det H_{\underline{A_1}}^{\underline{E_7}}\left[\iota,\rho_{\underline{E_7}},\rho_{\underline{A_1}}\right] = 12\cdot\eta^{12}.$$

□

Example 3.3.2. *Let $\iota : \underline{A_1} \longrightarrow \underline{E_6}$ be an embedding.*

a) The automorphic transfer $H^{E_6}_{\underline{A_1}}\left[\iota, \rho^{\mathrm{sym}}_{\underline{E_6}}, \rho_{\underline{A_1}}\right]$ is given by

$$H^{E_6}_{\underline{A_1}}\left[\iota, \rho^{\mathrm{sym}}_{\underline{E_6}}, \rho_{\underline{A_1}}\right](z) =$$
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \vartheta_{\underline{A_2}}(2z)\vartheta_0^3(z) + 3\psi(z)\vartheta_0(z)\vartheta_1^2(z) & \vartheta_{\underline{A_2}}(2z)\vartheta_1^3(z) + 3\psi(z)\vartheta_0^2(z)\vartheta_1(z) \\ \psi^*(z)\vartheta_0^3(z) + \frac{3}{2}\vartheta_0(z)\vartheta_1^2(z)\vartheta_1\left(\frac{z}{6}\right)\vartheta_1\left(\frac{z}{2}\right) & \psi^*(z)\vartheta_1^3(z) + \frac{3}{2}\vartheta_0^2(z)\vartheta_1(z)\vartheta_1\left(\frac{z}{6}\right)\vartheta_1\left(\frac{z}{2}\right) \end{pmatrix},$$

where

$$\psi(z) := \vartheta_1\left(\tfrac{z}{2}\right)\vartheta_1\left(\tfrac{3z}{2}\right) - \vartheta_0\left(\tfrac{z}{2}\right)\vartheta_0\left(\tfrac{3z}{2}\right) + \vartheta_{\underline{A_2}}(2z)$$

and

$$\psi^*(z) := \vartheta_0\left(\tfrac{z}{2}\right)\vartheta_0\left(\tfrac{z}{6}\right) - \tfrac{1}{2}\vartheta_1\left(\tfrac{z}{2}\right)\vartheta_1\left(\tfrac{z}{6}\right)$$

for $z \in \mathcal{H}$.

b) The modular determinant $\det H^{E_6}_{\underline{A_1}}\left[\iota, \rho^{\mathrm{sym}}_{\underline{E_6}}, \rho_{\underline{A_1}}\right]$ is given by

$$\det H^{E_6}_{\underline{A_1}}\left[\iota, \rho^{\mathrm{sym}}_{\underline{E_6}}, \rho_{\underline{A_1}}\right](z) = 12 \cdot \eta^{10}(z)$$

for $z \in \mathcal{H}$.

Proof. a) Using the notation from table 1.4 we have

$$E_6 = \bigcup_{i=1}^{8} S_i \cap \langle e_8, e_1 + e_3 + e_5 \rangle^{\perp}.$$

Since $\underline{E_6}$ is an irreducible root lattice, it suffices to consider the embedding given by $\iota = \iota_{e_2}$. In order to determine the theta decomposition of $\vartheta_{\underline{E_6},0}[e_2]$, we consider the pullbacks

$$\vartheta_{S_i \cap \langle e_8, e_1+e_3+e_5 \rangle^{\perp}}[e_2], \quad i = 1, \ldots, 8,$$

separately. For $l \in E_6$ let $l_1, \ldots, l_8$ denote the coordinates of l with respect to the standard basis $e_1, \ldots, e_8$ of $\mathbb{R}^8$. By definition we have $l \in E_6$ if and only if $l_8 = 0$ and $l_1 + l_3 + l_5 = 0$. In this case, $l_1^2 + l_3^2 + l_5^2 = 2(l_1^2 + l_1 l_3 + l_3^2)$. With respect to this coordinates we have

$$Q(l) = l_2^2 + l_4^2 + l_6^2 + l_7^2 + 2(l_1^2 + l_1 l_3 + l_3^2),\ B(l, e_2) = 2l_2, \quad l \in E_6.$$

Considering the sets S_1 and S_4 we immediately obtain

$$\sum_{l \in S_1 \cap \langle e_8, e_1+e_3+e_5 \rangle^{\perp}} e^{2\pi i (Q(l)z + B(l,e_2)w_0)} = \vartheta_{\underline{A_2}}(2z)\vartheta_0(z)^3\vartheta_0(z, w_0),$$
$$\sum_{l \in S_4 \cap \langle e_8, e_1+e_3+e_5 \rangle^{\perp}} e^{2\pi i (Q(l)z + B(l,e_2)w_0)} = \vartheta_{\underline{A_2}}(2z)\vartheta_1(z)^3\vartheta_1(z, w_0).$$

For $l \in S_2 \cap \langle e_8, e_1 + e_3 + e_5 \rangle^{\perp}$ we have $2l_1, 2l_3 \in 2\mathbb{Z} + 1$ and

$$l_1^2 + l_3^2 + l_5^2 = \tfrac{(2l_1)^2 + (2l_1)(2l_3) + (2l_3)^2}{2}.$$

This yields

$$\begin{aligned}\sum_{x,y\in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)z} &= \sum_{x\in\mathbb{Z},y\in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)z} - \sum_{x\in 2\mathbb{Z},y\in 2\mathbb{Z}+1} e^{2\pi i(x^2+xy+y^2)z}\\ &= \vartheta_1(z)\vartheta_1(3z) - \left(\sum_{\substack{x\in 2\mathbb{Z}\\ y\in\mathbb{Z}}} e^{2\pi i(x^2+xy+y^2)z} - \sum_{\substack{x\in 2\mathbb{Z}\\ y\in 2\mathbb{Z}}} e^{2\pi i(x^2+xy+y^2)z}\right)\\ &= \vartheta_1(z)\vartheta_1(3z) - \vartheta_0(z)\vartheta_0(3z) + \vartheta_{\underline{A_2}}(4z)\\ &= \psi(2z).\end{aligned}$$

Consequently,

$$\sum_{l\in S_2\cap\langle e_8,e_1+e_3+e_5\rangle^\perp} e^{2\pi i(Q(l)z+B(l,e_2)w_0)} = \psi(z)\vartheta_0(z)\vartheta_1(z)^2\vartheta_0(z,w_0)$$

and analogously

$$\vartheta_{S_2\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2] = \vartheta_{S_3\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2] = \vartheta_{S_8\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2].$$

A similar calculation yields

$$\sum_{l\in S_5\cap\langle e_8,e_1+e_3+e_5\rangle^\perp} e^{2\pi i(Q(l)z+B(l,e_2)w_0)} = \psi(z)\vartheta_0^2(z)\vartheta_1(z)\vartheta_1(z,w_0)$$

as well as

$$\vartheta_{S_5\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2] = \vartheta_{S_6\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2] = \vartheta_{S_7\cap\langle e_8,e_1+e_3+e_5\rangle^\perp}[e_2].$$

In summary, we obtain the decomposition

$$\begin{aligned}\vartheta_{\underline{E_6}}[e_2](z,w_0) = &\left[(\vartheta_{A_2}(2z)\vartheta_0(z)^3 + 3\psi(z)\vartheta_0(z)\vartheta_1(z)^2\right]\vartheta_0(z,w_0)\\ &+ \left[(\vartheta_{A_2}(2z)\vartheta_1(z)^3 + 3\psi(z)\vartheta_0^2(z)\vartheta_1(z)\right]\vartheta_1(z,w_0).\end{aligned}$$

In order to determine $(\vartheta_{\underline{E_6},\mu} + \vartheta_{\underline{E_6},-\mu})[e_2]$ for $\mu = \frac{1}{3}(e_1+e_3-2e_5)$ we make use of the identity

$$\vartheta_{\underline{E_6},\mu}[e_2] + \vartheta_{\underline{E_6},-\mu}[e_2] = -i\sqrt{3}\vartheta_{\underline{E_6}}[e_2]\Big|_{3,\underline{A_1}} J - \vartheta_{\underline{E_6}}[e_2].$$

Using the transformation properties of ϑ_0 and ϑ_1, an extensive calculation - which is omitted - finally yields the decomposition

$$\begin{aligned}\vartheta_{\underline{E_6}^*}[e_2](z,w_0) = &\left(\psi^*(z)\vartheta_0(z)^3 + \tfrac{3}{2}\vartheta_0(z)\vartheta_1(z)^2\vartheta_1\left(\tfrac{z}{6}\right)\vartheta_1\left(\tfrac{z}{2}\right)\right)\vartheta_0(z,w_0)\\ &+ \left(\psi^*(z)\vartheta_1(z)^3 + \tfrac{3}{2}\vartheta_0(z)^2\vartheta_1(z)\vartheta_1\left(\tfrac{z}{6}\right)\vartheta_1\left(\tfrac{z}{2}\right)\right)\vartheta_1(z,w_0).\end{aligned}$$

b) Due to 3.2.9 one has

$$\det H^{\underline{E_6}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{E_6}},\rho_{\underline{A_1}}\right] \in \left[\mathrm{Mp}_2(\mathbb{Z}),5,\nu_\eta^{10}\right] = \mathbb{C}\eta^{10}.$$

It remains to determine the constant. Expanding the determinant, we obtain

$$\det H^{\underline{E_6}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{E_6}},\rho_{\underline{A_1}}\right](z) = 3\vartheta_0(z)\vartheta_1(z)\left(\vartheta_0(z)^4-\vartheta_1(z)^4\right) \cdot \left[h(z)\left(\vartheta_0\left(\tfrac{z}{6}\right)\vartheta_0\left(\tfrac{z}{2}\right)+\vartheta_1\left(\tfrac{z}{6}\right)\vartheta_1\left(\tfrac{z}{2}\right)\right)+\tfrac{1}{2}g(z)\vartheta_1\left(\tfrac{z}{6}\right)\vartheta_1\left(\tfrac{z}{2}\right)\right],$$

where

$$h(z) := -\vartheta_0(6z)\vartheta_0(2z)-\vartheta_1(6z)\vartheta_1(2z)+\vartheta_0\left(\tfrac{z}{2}\right)\vartheta_0\left(\tfrac{3z}{2}\right)-\vartheta_1\left(\tfrac{z}{2}\right)\vartheta_1\left(\tfrac{3z}{2}\right)$$

and

$$g(z) := 4\vartheta_0(6z)\vartheta_0(2z)+4\vartheta_1(6z)\vartheta_1(2z)-3\vartheta_0\left(\tfrac{z}{2}\right)\vartheta_0\left(\tfrac{3z}{2}\right)+3\vartheta_1\left(\tfrac{z}{2}\right)\vartheta_1\left(\tfrac{3z}{2}\right).$$

A closer look at the Fourier expansions of the auxiliary functions h and g yields

$$h(z) = -4e^{\pi i z}+\dots \text{ and } g(z) = 1+\ \dots.$$

Hence the Fourier expansion of $\det H^{\underline{E_6}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{E_6}},\rho_{\underline{A_1}}\right]$ starts by

$$\det H^{\underline{E_6}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{E_6}},\rho_{\underline{A_1}}\right](z) = 3\cdot 2\cdot e^{\frac{\pi i z}{2}}\cdot\tfrac{1}{2}\cdot 2e^{\frac{\pi i z}{12}}\cdot 2e^{\frac{\pi i z}{4}}+\dots = 12\cdot e^{\frac{10\pi i z}{12}}+\dots,$$

i.e. the constant equals 12. □

Example 3.3.3. *Let* $\iota : \underline{A_1} \longrightarrow \underline{D_4}$ *be an embedding.*

a) The automorphic transfer $H^{\underline{D_4}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{D_4}},\rho_{\underline{A_1}}\right]$ *is given by*

$$H^{\underline{D_4}}_{\underline{A_1}}\left[\iota,\rho^{\mathrm{sym}}_{\underline{D_4}},\rho_{\underline{A_1}}\right](z) = \begin{pmatrix} \vartheta_0(z)^3 & \vartheta_1(z)^3 \\ 3\vartheta_0(z)\vartheta_1(z)^2 & 3\vartheta_0(z)^2\vartheta_1(z)\end{pmatrix}$$

for $z\in\mathcal{H}$.

b) The modular determinant $\det H^{\underline{D_4}}_{\underline{A_1}}[\iota,\rho^{\mathrm{sym}}_{\underline{D_4}},\rho_{\underline{A_1}}]$ *is given by*

$$\det H^{\underline{D_4}}_{\underline{A_1}}[\iota,\rho^{\mathrm{sym}}_{\underline{D_4}},\rho_{\underline{A_1}}](z) = 3\vartheta_0(z)\vartheta_1(z)(\vartheta_0(z)^4-\vartheta_1(z)^4) = 6\cdot\eta^6(z)$$

for $z\in\mathcal{H}$.

Proof. a) Since $\underline{D_4}$ is an irreducible root lattice, it suffices to consider the embedding given by $\iota = \iota_{e_2}$. By definition one has

$$D_4 = \left\langle e_2, e_4, e_6, \frac{e_2+e_4+e_6+e_7}{2}\right\rangle_{\mathbb{Z}}.$$

If l_2, l_4, l_6, l_7 denote the coordinates of $l \in D_4$ with respect to e_2, e_4, e_6, e_7, then one has

$$l \in \underline{D_4} \text{ if and only if } l_i \in \mathbb{Z}, i = 1, \ldots, 4 \quad \text{or} \quad l_i \in \mathbb{Z} + \frac{1}{2}, i = 1, \ldots, 4,$$

which immediately yields

$$\vartheta_{\underline{D_4}}[e_2](z, w_0) = \vartheta_0(z)^3 \vartheta_0(z, w_0) + \vartheta_1(z)^3 \vartheta_1(z, w_0).$$

From

$$\left(\vartheta_{\underline{D_4},\mu_1} + \vartheta_{\underline{D_4},\mu_2} + \vartheta_{\underline{D_4},\mu_3}\right)[e_2](z, w_0) = -2\vartheta_{\underline{D_4}}\Big|_{2,\underline{A_1}} J[e_2](z, w_0) - \vartheta_{\underline{D_4}}[e_2](z, w_0),$$

with $\mu_1 := \frac{e_2+e_4}{2}$, $\mu_2 := \frac{e_2+e_6}{2}$ and $\mu_3 := \frac{e_2+e_7}{2}$, we derive

$$\left(\vartheta_{\underline{D_4},\mu_1} + \vartheta_{\underline{D_4},\mu_2} + \vartheta_{\underline{D_4},\mu_3}\right)[e_2](z, w_0) = 3\vartheta_0(z)\vartheta_1(z)^2\vartheta_0(z, w_0) + 3\vartheta_0(z)^2\vartheta_1(z)\vartheta_1(z, w_0),$$

where we again used the transformation properties of ϑ_0 and ϑ_1.

b) Due to 3.2.9 one has

$$\det H_{\underline{A_1}}^{\underline{D_4}}\left[\iota, \rho_{\underline{D_4}}^{\text{sym}}, \rho_{\underline{A_1}}\right] \in \left[\text{Mp}_2(\mathbb{Z}), 3, \nu_\eta^6\right] = \mathbb{C}\eta^6.$$

Expanding the determinant, we obtain

$$\det H_{\underline{A_1}}^{\underline{D_4}}\left[\iota, \rho_{\underline{D_4}}^{\text{sym}}, \rho_{\underline{A_1}}\right](z) = 3\vartheta_0(z)\vartheta_1(z)(\vartheta_0(z)^4 - \vartheta_1(z)^4).$$

Hence the first Fourier coefficient equals 6. □

Example 3.3.4. *Let $\iota : \underline{A_1} \longrightarrow \underline{A_2}$ be an embedding.*

a) The automorphic transfer $H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\text{sym}}, \rho_{\underline{A_1}}\right]$ is given by

$$H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\text{sym}}, \rho_{\underline{A_1}}\right](z) = \begin{pmatrix} \vartheta_0(3z) & \vartheta_1(3z) \\ \vartheta_0(\frac{z}{3}) - \vartheta_0(3z) & \vartheta_1(\frac{z}{3}) - \vartheta_1(3z) \end{pmatrix}$$

for $z \in \mathcal{H}$.

b) The modular determinant $\det H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\text{sym}}, \rho_{\underline{A_1}}\right]$ is given by

$$\det H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\text{sym}}, \rho_{\underline{A_1}}\right](z) = \vartheta_0(3z)\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\vartheta_0\left(\frac{z}{3}\right) = 2 \cdot \eta^2(z)$$

for $z \in \mathcal{H}$.

Proof. a) Since $\underline{A_2}$ is an irreducible root lattice, it suffices to consider the embedding given by $\iota = \iota_{e_2}$. This time we shall use the explicit formulas given in 3.2.1. Thus we have

$$\vartheta_{\underline{A_2}}[e_2](z, w_0) = h_{0,0}(z)\vartheta_0(z, w_0) + h_{0,\frac{1}{2}}\vartheta_0(z, w_0),$$

where

$$\begin{aligned} h_{0,0}(z) &= \sum_{m=0}^{\infty} |\{l \in A_2 : Q(l) = m, B(l, e_2) = 0\}| \cdot e^{2\pi i m z} \\ &= \sum_{m=0}^{\infty} |\{(r,s) \in \mathbb{Z} : r^2 + sr + s^2 = m, 2r + s = 0\}| \cdot e^{2\pi i m z} \\ &= \sum_{m=0}^{\infty} |\{r \in \mathbb{Z} : 3r^2 = m\}| e^{2\pi i m z} \\ &= \vartheta_0(3z) \end{aligned}$$

as well as

$$\begin{aligned} h_{0,\frac{1}{2}}(z) &= \sum_{m \in -\frac{1}{4} + \mathbb{Z}} |\{l \in A_2 : Q(l) = m + \tfrac{1}{4}, B(l, e_2) = 1\}| \cdot e^{2\pi i m z} \\ &= \sum_{m \in -\frac{1}{4} + \mathbb{Z}} |\{(r,s) \in \mathbb{Z} : r^2 + sr + s^2 = n + \tfrac{1}{4}, 2r + s = 1\}| \cdot e^{2\pi i m z} \\ &= \sum_{m \in -\frac{1}{4} + \mathbb{Z}} |\{r \in \mathbb{Z} : 3(r + \tfrac{1}{2})^2 = m\}| \cdot e^{2\pi i m z} \\ &= \vartheta_1(3z). \end{aligned}$$

In order to determine $\left(\vartheta_{\underline{A_2},\mu} + \vartheta_{\underline{A_2},-\mu}\right)[e_2]$ for $\mu = \frac{e_4+e_6+e_7}{2}$ we use the identity

$$\left(\vartheta_{\underline{A_2},\mu} + \vartheta_{\underline{A_2},-\mu}\right)[e_2](z, w_0) = i\sqrt{3}\vartheta_{\underline{A_2}}[e_2]\Big|_{1,\underline{A_1}} J(z, w_0) - \vartheta_{\underline{A_2}}[e_2](z, w_0)$$

and a straightforward calculation yields

$$\left(\vartheta_{\underline{A_2},\mu} + \vartheta_{\underline{A_2},-\mu}\right)[e_2](z, w_0) = \left[\vartheta_0\left(\tfrac{z}{3}\right) - \vartheta_0(3z)\right]\vartheta_0(z, w_0) + \left[\vartheta_1\left(\tfrac{z}{3}\right) - \vartheta_1(3z)\right]\vartheta_1(z, w_0),$$

where we again used the transformation properties of ϑ_0 and ϑ_1.

b) Due to 3.2.9 one has

$$\det H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\mathrm{sym}}, \rho_{\underline{A_1}}\right] \in \left[\mathrm{Mp}_2(\mathbb{Z}), 1, \nu_\eta^2\right] = \mathbb{C}\eta^2.$$

Expanding the determinant, we obtain

$$\det H_{\underline{A_1}}^{\underline{A_2}}[\iota, \rho_{\underline{A_2}}^{\mathrm{sym}}, \rho_{\underline{A_1}}](z) = \vartheta_0(3z)\vartheta_1\left(\tfrac{z}{3}\right) - \vartheta_1(3z)\vartheta_0\left(\tfrac{z}{3}\right) = 2e^{\frac{\pi i z}{6}} + \dots.$$

Hence the constant equals 2. □

Regarding equivalence of embeddings of direct sums we give the following

Proposition 3.3.5. *Let $\underline{L}$ be an irreducible root lattice and $m \in \mathbb{N}$. Then the following assertions hold:*

a) The semidirect product $S_m \ltimes W(\underline{L})^m \leq \mathrm{O}(m\underline{L})$ acts m-fold transitively on $R(m\underline{L})$.

b) All embeddings $\iota : m\underline{A_1} \longrightarrow m\underline{L}$ are equivalent.

Example 3.3.6. *Let $\iota : 2\underline{A_1} \longrightarrow 2\underline{A_2}$ be an embedding.*

a) The automorphic transfer $H_{2\underline{A_1}}^{2\underline{A_2}}\left[\iota, \rho_{2\underline{A_2}}^{\mathrm{sym}}, \rho_{2\underline{A_1}}^{\mathrm{sym}}\right]$ is given by

$$H_{2\underline{A_1}}^{2\underline{A_2}}\left[\iota, \rho_{2\underline{A_2}}^{\mathrm{sym}}, \rho_{2\underline{A_1}}^{\mathrm{sym}}\right](z) =$$
$$\begin{pmatrix} \vartheta_0^2(3z) & \vartheta_0(3z)\vartheta_1(3z) & \vartheta_1^2(3z) \\ 2\vartheta_0(3z)(\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z)) & \vartheta_0(3z)\vartheta_1\left(\frac{z}{3}\right) + \vartheta_1(3z)\vartheta_0\left(\frac{z}{3}\right) - 2\vartheta_0(3z)\vartheta_1(3z) & 2\vartheta_1(3z)(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)) \\ (\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z))^2 & (\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z))(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)) & (\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z))^2 \end{pmatrix}$$

for $z \in \mathcal{H}$.

b) The modular determinant $\det H_{2\underline{A_1}}^{2\underline{A_2}}\left[\iota, \rho_{2\underline{A_2}}^{\mathrm{sym}}, \rho_{2\underline{A_1}}^{\mathrm{sym}}\right]$ is given by

$$\det H_{2\underline{A_1}}^{2\underline{A_2}}\left[\iota, \rho_{2\underline{A_2}}^{\mathrm{sym}}, \rho_{2\underline{A_1}}^{\mathrm{sym}}\right](z) = \left(\det H_{\underline{A_1}}^{\underline{A_2}}\left[\iota, \rho_{\underline{A_2}}^{\mathrm{sym}}, \rho_{\underline{A_1}}\right](z)\right)^3 = 8 \cdot \eta^6(z)$$

for $z \in \mathcal{H}$.

Proof. a) In view of 3.3.5 all embeddings $\iota : 2\underline{A_1} \longrightarrow 2\underline{A_2}$ are equivalent. Hence we write $[2\underline{A_1}]$ instead of $[\iota]$. With the decomposition $w_0 = (w_1, w_2) \in \mathbb{C}^{(2)}$ we have

$$\vartheta_{2\underline{A_2}}[2\underline{A_1}](z, w_0) = \vartheta_{\underline{A_2}}[\underline{A_1}](z, w_1) \cdot \vartheta_{\underline{A_2}}[\underline{A_1}](z, w_2).$$

From the identities

$$\begin{aligned} \vartheta_{2\underline{A_1},0}(z, w_0) &= \vartheta_0(z, w_1)\vartheta_0(z, w_2), \\ \vartheta_{2\underline{A_1},(\frac{1}{2},\frac{1}{2})}(z, w_0) &= \vartheta_1(z, w_1)\vartheta_1(z, w_2), \\ \left(\vartheta_{2\underline{A_1},(\frac{1}{2},0)} + \vartheta_{2\underline{A_1},(0,\frac{1}{2})}\right)(z, w_0) &= \vartheta_1(z, w_1)\vartheta_0(z, w_2) + \vartheta_0(z, w_1)\vartheta_1(z, w_2) \end{aligned}$$

we derive the theta decomposition

$$\begin{aligned} \vartheta_{2\underline{A_2}}[2\underline{A_1}](z, w_0) = {}& \vartheta_0^2(3z)\vartheta_{2\underline{A_1},0}(z, w_0) + \vartheta_0(3z)\vartheta_1(3z)\left(\vartheta_{2\underline{A_1},(\frac{1}{2},0)} + \vartheta_{2\underline{A_1},(0,\frac{1}{2})}\right)(z, w_0) \\ & + \vartheta_1^2(3z)\vartheta_{2\underline{A_1},(\frac{1}{2},\frac{1}{2})}(z, w_0). \end{aligned}$$

Let $\mu := \frac{e_4+e_6+e_7}{2}$. In a similar way, the identities

$$\begin{aligned} \vartheta_{2\underline{A_2},(\mu,0)}[2\underline{A_1}](z, w_0) &= \vartheta_{\underline{A_2},\mu}[\underline{A_1}](z, w_1) \cdot \vartheta_{\underline{A_2}}[\underline{A_1}](z, w_2), \\ \vartheta_{2\underline{A_2},(-\mu,0)}[2\underline{A_1}](z, w_0) &= \vartheta_{\underline{A_2},-\mu}[\underline{A_1}](z, w_1) \cdot \vartheta_{\underline{A_2}}[\underline{A_1}](z, w_2) \end{aligned}$$

yield

$$\left(\vartheta_{2\underline{A_2},(\mu,0)} + \vartheta_{2\underline{A_2},(-\mu,0)} + \vartheta_{2\underline{A_2},(0,\mu)} + \vartheta_{2\underline{A_2},(0,-\mu)}\right)[2\underline{A_1}](z,w_0)$$
$$= 2\vartheta_0(3z)\left(\vartheta_0\left(\tfrac{z}{3}big\right) - \vartheta_0(3z)\right)\vartheta_{2\underline{A_1},0}(z,w_0)$$
$$+ \left[\vartheta_0(3z)\vartheta_1\left(\tfrac{z}{3}\right) + \vartheta_1(3z)\vartheta_0\left(\tfrac{z}{3}\right) - 2\vartheta_0(3z)\vartheta_1(3z)\right]\left(\vartheta_{2\underline{A_1},(\frac{1}{2},0)} + \vartheta_{2\underline{A_1},(0,\frac{1}{2})}\right)(z,w_0)$$
$$+ 2\vartheta_1(3z)\left(\vartheta_1\left(\tfrac{z}{3}\right) - \vartheta_1(3z)\right)\vartheta_{2\underline{A_1},(\frac{1}{2},\frac{1}{2})}(z,w_0).$$

Completely analogous, by using the identities

$$\vartheta_{2\underline{A_2},(\mu,\mu)}[2\underline{A_1}](z,w_0) = \vartheta_{\underline{A_2},\mu}[\underline{A_1}](z,w_1)\cdot\vartheta_{\underline{A_2},\mu}[\underline{A_1}](z,w_2),$$
$$\vartheta_{2\underline{A_2},(\mu,-\mu)}[2\underline{A_1}](z,w_0) = \vartheta_{\underline{A_2},\mu}[\underline{A_1}](z,w_1)\cdot\vartheta_{\underline{A_2},-\mu}[\underline{A_1}](z,w_2),$$
$$\vartheta_{2\underline{A_2},(-\mu,\mu)}[2\underline{A_1}](z,w_0) = \vartheta_{\underline{A_2},-\mu}[\underline{A_1}](z,w_1)\cdot\vartheta_{\underline{A_2},\mu}[\underline{A_1}](z,w_2),$$
$$\vartheta_{2\underline{A_2},(-\mu,-\mu)}[2\underline{A_1}](z,w_0) = \vartheta_{\underline{A_2},-\mu}[\underline{A_1}](z,w_1)\cdot\vartheta_{\underline{A_2},-\mu}[\underline{A_1}](z,w_2),$$

we obtain the theta decomposition of

$$(\vartheta_{2\underline{A_2},(\mu,\mu)} + \vartheta_{2\underline{A_2},(\mu,-\mu)} + \vartheta_{2\underline{A_2},(-\mu,\mu)} + \vartheta_{2\underline{A_2},(-\mu,\mu)})[2\underline{A_1}]$$

and finish the proof.

b) Due to 3.2.9 one has

$$\det H^{2A_2}_{2\underline{A_1}}\left[\iota,\rho^{\text{sym}}_{2\underline{A_2}},\rho^{\text{sym}}_{2\underline{A_1}}\right] \in \left[\mathrm{Mp}_2(\mathbb{Z}),3,\nu_\eta^6\right] = \mathbb{C}\eta^6.$$

A straightforward calculation yields

$$\det H^{2A_2}_{2\underline{A_1}}\left[\iota,\rho^{\text{sym}}_{2\underline{A_2}},\rho^{\text{sym}}_{2\underline{A_1}}\right](z) = \left[\vartheta_0(3z)\vartheta_1\left(\tfrac{z}{3}\right) - \vartheta_1(3z)\vartheta_0\left(\tfrac{z}{3}\right)\right]^3 = 8\cdot\eta^6(z),$$

if we use the result of 3.3.4. □

Example 3.3.7. *Let $\iota : \underline{A_1}(2) \longrightarrow 2\underline{A_1}$ be an embedding.*

a) The automorphic transfer $H^{2A_1}_{\underline{A_1}(2)}\left[\iota,\rho^{\text{sym}}_{2\underline{A_1}},\rho^{\text{sym}}_{\underline{A_1}(2)}\right]$ is given by

$$H^{2A_1}_{\underline{A_1}(2)}\left[\iota,\rho^{\text{sym}}_{2\underline{A_1}},\rho^{\text{sym}}_{\underline{A_1}(2)}\right](z) = \begin{pmatrix} \vartheta_0(2z) & 0 & \vartheta_1(2z) \\ 2\vartheta_1(2z) & \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) & 2\vartheta_0(2z) \\ \vartheta_1(2z) & 0 & \vartheta_0(2z) \end{pmatrix}$$

for $z \in \mathcal{H}$.

b) The modular determinant $\det H^{2A_1}_{\underline{A_1}(2)}\left[\iota,\rho^{\text{sym}}_{2\underline{A_1}},\rho^{\text{sym}}_{\underline{A_1}(2)}\right]$ is given by

$$\det H^{2A_1}_{\underline{A_1}(2)}\left[\iota,\rho^{\text{sym}}_{2\underline{A_1}},\rho^{\text{sym}}_{\underline{A_1}(2)}\right](z) = \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z)\cdot\left(\vartheta_0^2(2z) - \vartheta_1^2(2z)\right) = \eta^3(z)$$

for $z \in \mathcal{H}$.

Proof. a) It is easy to check that every embedding $\iota : \underline{A_1}(2) \longrightarrow 2\underline{A_1}$ is equivalent to $\iota_{(1,1)}$. A system of representatives of $(2\underline{A_1})^*/2\underline{A_1}$ is given by

$$\xi : (0,0), \left(\tfrac{1}{2}, 0\right), \left(0, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

and a system of representatives for $\underline{A_1}(2)^*/\underline{A_1}(2)$ is given by

$$\mu : 0, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}.$$

In view of 3.2.1 it suffices to determine the theta series of the sets $S_{\mu\xi,(1,1)}$ for the given representatives. In all cases, an elementary calculation yields the desired result.

b) Due to 3.2.9 one has

$$\det H^{2\underline{A_1}}_{\underline{A_1}(2)}\left[\iota, \rho^{\mathrm{sym}}_{2\underline{A_1}}, \rho^{\mathrm{sym}}_{\underline{A_1}(2)}\right] \in \left[\mathrm{Mp}_2(\mathbb{Z}), \tfrac{3}{2}, \nu_\eta^3\right] = \mathbb{C}\eta^3.$$

By expanding the determinant, we obtain

$$\det H^{2\underline{A_1}}_{\underline{A_1}(2)}\left[\iota, \rho^{\mathrm{sym}}_{2\underline{A_1}}, \rho^{\mathrm{sym}}_{\underline{A_1}(2)}\right](z) = \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) \cdot \left(\vartheta_0^2(2z) - \vartheta_1^2(2z)\right).$$

Hence the constant equals 1. □

We summarize the above results in the following

Theorem 3.3.8. *Let $\left((\underline{L}, \rho), (\underline{L_0}, \rho_0)\right)$ be a pair, such that the following assumptions hold:*

i)

$$\underline{L}, \underline{L_0} \in \{\underline{A_1}, \underline{A_2}, \underline{D_4}, \underline{E_6}, \underline{E_7}, \underline{A_1}(2), 2\underline{A_1}, 2\underline{A_2}\}$$
$$\rho, \rho_0 \in \left\{\rho_{\underline{A_1}}, \rho^{\mathrm{sym}}_{\underline{A_2}}, \rho^{\mathrm{sym}}_{\underline{D_4}}, \rho^{\mathrm{sym}}_{\underline{E_6}}, \rho_{\underline{E_7}}, \rho^{\mathrm{sym}}_{\underline{A_1}(2)}, \rho^{\mathrm{sym}}_{2\underline{A_1}}, \rho^{\mathrm{sym}}_{2\underline{A_2}}\right\},$$

ii) there is an embedding $\iota : \underline{L_0} \longrightarrow \underline{L}$,

iii) $d := \dim \rho = \dim \rho_0 =: d_0$,

iv) $[\iota] : \mathcal{O}(\mathcal{H})V \longrightarrow \mathcal{O}(\mathcal{H})V_0$ is well-defined,

where V resp. V_0 denote the corresponding subspaces of $\Theta_{\underline{L}}$ resp. $\Theta_{\underline{L_0}}$ associated to ρ resp. ρ_0. Then the pullback operator in iv) is an isomorphism, which does not depend on the choice of the embedding in ii). With respect to suitable coordinates for V, the inverse map $[\iota]^{-1}$ is given by

$$\mathcal{O}(\mathcal{H})^d \longrightarrow \mathcal{O}(\mathcal{H})^d, \quad \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \mapsto H^{\underline{L}}_{\underline{L_0}}[\iota, \rho, \rho_0]^{-t} \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}.$$

Proof. For the subset $\{\underline{A_1}, \underline{A_2}, \underline{D_4}, \underline{E_6}, \underline{E_7}\}$ the independence of the choice of ι follows from 3.6.2. For the subset $\{\underline{A_1}(2), 2\underline{A_1}, 2\underline{A_2}\}$ the independence is a direct consequence of 3.3.5. The fact that the corresponding maps $[\iota]$ in iv) are actually isomorphisms follows from the concrete examples in this section as well as from 3.2.6 and 3.2.12. □

Remark 3.3.9. *Note that 3.2.6 allows to evaluate $H_{\underline{L_0}}^{\underline{L}}[\iota, \rho, \rho_0]$ in all possible cases of 3.3.8, which were not considered yet. For reasons of readability, we will not present these here, since the matrix entries are very longish.*

3.4 A distinguished infinite family of Siegel cusp forms

In this section we will utilize the embedding $\underline{A_1} \longrightarrow \underline{E_7}$ and 3.2.14 in order to construct an infinite series of nontrivial Siegel cusp forms of weight $3 \cdot 2^n, n \in \mathbb{N}$ satisfying a distinguished recurrence relation under the Witt operator.

Let $\Delta^* := \eta^{24}$ denote the normalized modular discriminant of weight 12. For formal reasons we define $(\Delta^*)^{\frac{1}{2}} := \eta^{12}$.

Theorem 3.4.1. *There exists an infinite family of nontrivial Siegel cusp forms*

$$\chi_{2^n \cdot 3} \in \begin{cases} \left[\mathrm{Mp}_2(\mathbb{Z}), 6, v_\eta^{12}\right]^{\mathrm{cusp}}, & n = 1, \\ \left[\mathrm{Mp}_{2n}(\mathbb{Z}), 2^n \cdot 3\right]^{\mathrm{cusp}}, & n > 1, \end{cases}$$

such that $\chi_6 = (\Delta^*)^{\frac{1}{2}}$ and the following assertions hold:

a) $\mathcal{W}^{(j)} \chi_{2^n \cdot 3} = \chi_{2^j \cdot 3}^{2^{n-j}} \otimes \chi_{2^{n-j} \cdot 3}^{2^j}$, *i.e.*

$$\chi_{2^n \cdot 3} \begin{pmatrix} Z_j & 0 \\ 0 & Z_{n-j} \end{pmatrix} = \chi_{2^j \cdot 3}(Z_j)^{2^{n-j}} \cdot \chi_{2^{n-j} \cdot 3}(Z_{n-j})^{2^j}$$

holds for all $Z_j \in \mathcal{H}_j, Z_{n-j} \in \mathcal{H}_{n-j}$ and $0 \le j \le n$,

b) $\chi_{2^n \cdot 3}(\mathrm{diag}(z_1, \dots, z_n)) = (\Delta^*(z_1) \cdots \Delta^*(z_n))^{2^{n-2}}$ *for all $z_1, \dots, z_n \in \mathcal{H}$,*

c) $\chi_{2^n \cdot 3}(zI_n) = \Delta^*(z)^{n \cdot 2^{n-2}}$ *for all $z \in \mathcal{H}$,*

d) $\chi_{2^n \cdot 3}$ *does not vanish on the diagonal $\mathcal{H}^n \subseteq \mathcal{H}_n$.*

Proof. For $n = 1$ we choose the systems of representatives of E_7^* / E_7 resp. A_1^* / A_1 used in 3.3.1. According to 3.2.14, we consider the systems of representatives $\mathcal{R}_{\underline{E_7}}^{(n)}$ of $(E_7^*/E_7)^{(n)}$ resp. $\mathcal{R}_{\underline{A_1}}^{(n)}$ of $(A_1^*/A_1)^{(n)}$. Let $\rho_{\underline{E_7}}^{(n)}$ resp. $\rho_{\underline{A_1}}^{(n)}$ denote the corresponding Weil representations with respect to the bases of $\Theta_{\underline{E_7}}^{(n)}$ resp. $\Theta_{\underline{A_1}}^{(n)}$, given by the Jacobi theta functions labeled by $\mathcal{R}_{\underline{E_7}}^{(n)}$ resp. $\mathcal{R}_{\underline{A_1}}^{(n)}$. For $n \in \mathbb{N}$ we define

$$\chi_{2^n \cdot 3} := \frac{1}{12^{n \cdot 2^{n-1}}} \cdot \det H_{\underline{A_1}}^{\underline{E_7}}\left[\iota^{(n)}, \rho_{\underline{E_7}}^{(n)}, \rho_{\underline{A_1}}^{(n)}\right].$$

Since $\mathrm{Mp}_{2n}(\mathbb{Z})$ is perfect for $n \geq 3$, we derive that $\chi_{2^n \cdot 3}$ has trivial character in this case. In the case $n = 2$ the unique nontrivial abelian character $\nu_{\mathrm{Maaß}}$ of $\mathrm{Mp}_4(\mathbb{Z})$ is determined by the value $\nu_{\mathrm{Maaß}}(J \times I_2) = -1$. From 2.4.31 we obtain

$$\det \rho^{(2)}_{\underline{E_7}}(J \times I_2) \cdot \left(\det \rho^{(2)}_{\underline{A_1}}(J \times I_2)\right)^{-1} = \det \rho_{\underline{E_7}}(J)^2 \cdot \det \rho_{\underline{A_1}}(J)^{-2}$$
$$= \nu_\eta^{36}(J) \cdot \nu_\eta^{-12}(J) = \nu_\eta^{24}(J) = 1.$$

Hence the character of $\chi_{2^n \cdot 3}$ is trivial for all $n \geq 2$ and the claim follows from 3.2.9. The identity $\chi_6 = (\Delta^*)^{\frac{1}{2}}$ is due to 3.3.1. Since η vanishes at infinity, we obtain $\mathcal{S}^{(n-1)}\chi_{3 \cdot 2^n} = 0$ from a) in the case $j = n - 1$.

a) Follows from 3.2.14. Note that the normalization factors $\gamma_n = 12^{-n \cdot 2^{n-1}}$ satisfy the identity $\gamma_j^{2^{n-j}} \cdot \gamma_{n-j}^{2^j} = \gamma_n$ for $j = 0, \ldots, n$.

b) Follows from a) by induction on n.

c) Follows from b) with $z_1 = \cdots = z_n = z$.

d) Follows from b) and the fact that η is nonvanishing on $\mathcal{H}$. □

Corollary 3.4.2. *Let $\iota : \underline{A_1} \longrightarrow \underline{E_7}$ be an embedding. Then the pullback operator*

$$\left[\iota^{(n)}\right] : \mathcal{E}^{(n)}(\underline{E_7}) \longrightarrow \mathcal{E}^{(n)}(\underline{A_1})$$

is a monomorphism for all $n \in \mathbb{N}$ and an isomorphism if and only if $n = 1$.

For the spaces of Jacobi forms we immediately obtain

Corollary 3.4.3. *Let $\iota : \underline{A_1} \longrightarrow \underline{E_7}$ be an embedding. Then the pullback operator*

$$\left[\iota^{(n)}\right] : J^{(n)}_{k,\underline{E_7}} \longrightarrow J^{(n)}_{k,\underline{A_1}}$$

is a monomorphism for all $n \in \mathbb{N}$ and an isomorphism in the case $n = 1$.

3.5 Ad-hoc criteria for injectivity in degree 1

Let $\underline{L}$ be an unimodular lattice. Then $\mathcal{E}^{(n)}(\underline{L})$ is a module of rank one, i.e.

$$\mathcal{E}^{(n)}(\underline{L}) = \mathcal{O}(\mathcal{H}_n) \cdot \vartheta^{(n)}_{\underline{L},0}.$$

Hence it is easy to see that for all embeddings $\iota : \underline{L_0} \longrightarrow \underline{L}$, where $\underline{L_0}$ is a lattice, the corresponding pullback operator

$$\left[\iota^{(n)}\right] : \mathcal{E}^{(n)}(\underline{L}) \longrightarrow \mathcal{E}^{(n)}(\underline{L_0})$$

is clearly injective. In this section we will formulate and prove certain ad-hoc criteria for injectiveness of pullback operators induced by embeddings into non-unimodular lattices, where we restrict to the case of degree $n = 1$ only.

We start by a simple

Lemma 3.5.1. *Let $\underline{L}$ and $\underline{L_0}$ be lattices, $\underline{L}$ non-unimodular, and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Suppose that there is some $\xi \in L_0^*$ such that $\iota(\xi) \in L^* \setminus L$. Then the pullback operator*

$$[\iota] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{L_0})$$

is a monomorphism.

Proof. If suffices to show that the functions $\vartheta_{\underline{L}}[\iota]$ and $\vartheta_{\underline{L}^*}[\iota]$ are linearly independent over $\mathcal{O}(\mathcal{H})$. To this end, let $h, g \in \mathcal{O}(\mathcal{H})$ such that

$$h \cdot \vartheta_{\underline{L}}[\iota] + g \cdot \vartheta_{\underline{L}^*}[\iota] = 0.$$

We apply $\Big|_{\underline{L_0}} [\xi, 0]$ on this equation in order to obtain

$$h \cdot \vartheta_{\underline{L},\iota(\xi)}[\iota] + g \cdot \vartheta_{\underline{L}^*}[\iota] = 0.$$

by 2.2.10 and 3.1.4. Note that the map $\mu \mapsto \mu + \iota(\xi)$ is a permutation of L^*. Hence,

$$h \cdot \Big(\vartheta_{\underline{L}}[\iota] - \vartheta_{\underline{L},\iota(\xi)}[\iota] \Big) = 0.$$

But $\vartheta_{\underline{L}}[\iota] - \vartheta_{\underline{L},\iota(\xi)}[\iota] = 1 + \dots$, since $\iota(\xi) \notin L$. Hence $h = g = 0$. □

The following lemma is rather technical:

Lemma 3.5.2. *Let $\underline{L}$ and $\underline{L_0}$ be lattices, $\underline{L}$ non-unimodular, and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Suppose that there is a vector μ^* of minimal quadratic form in $L^* \setminus L$ such that $\mu^* \notin \iota(L_0)^{\perp,\underline{L}}$. Then the pullback operator*

$$[\iota] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{L_0})$$

is a monomorphism.

Proof. Since L is non-unimodular, the value

$$m^* := \min_{\mu \in L^* \setminus L} Q(\mu)$$

is well-defined in $\mathbb{Q}^\times$. We show that the functions $\vartheta_{\underline{L}}[\iota]$ and $\left(\vartheta_{\underline{L}^*} - \vartheta_{\underline{L}}\right)[\iota]$ are linearly independent over $\mathcal{O}(\mathcal{H})$. By 3.1.12 we have Fourier expansions

$$\vartheta_{\underline{L}}[\iota](z, w_0) = 1 + \sum_{0 \neq m \in \mathbb{Q}_+} \left(\sum_{\xi \in L_0^*} |\{\mu \in L : Q(\mu) = m, \iota^*(\mu) = \xi\}| \cdot e^{2\pi i B(\xi, w_0)} \right) e^{2\pi i m z},$$

$$\vartheta_{\underline{L}^* - \underline{L}}[\iota](z, w_0) = \left(\sum_{\xi \in L_0^*} |\{\mu \in L^* - L : Q(\mu) = m^*, \iota^*(\mu) = \xi\}| \cdot e^{2\pi i B(\xi, w_0)} \right) e^{2\pi i m^* z} + \dots .$$

Hence, we obtain a local representation

$$\frac{(\vartheta_{\underline{L}^*} - \vartheta_{\underline{L}})\,[\iota]}{\vartheta_{\underline{L}}[\iota]} = \left(\sum_{\xi \in L_0^*} |\{\mu \in L^* \setminus L : Q(\mu) = m^*, \iota^*(\mu) = \xi\}| \cdot e^{2\pi i B(\xi, w_0)} \right) \cdot e^{2\pi i m^* z} + \dots .$$

By assumption, $Q(\mu^*) = m^*$ and $\iota^*(\mu^*) \neq 0$. Hence the function $\frac{\vartheta_{\underline{L}^*-\underline{L}}[\iota]}{\vartheta_{\underline{L}}[\iota]}$ depends nontrivially on w_0 and thus can not be the quotient of two functions $f, g \in \mathcal{O}(\mathcal{H})$. □

We apply this on irreducible root lattices:

Corollary 3.5.3. *Let $\underline{L}$ and $\underline{L_0}$ be lattices, $\underline{L}$ a non-unimodular irreducible root lattice, and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Then the pullback operator*

$$[\iota] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{L_0})$$

is a monomorphism.

Proof. Since $\underline{L}$ is non-unimodular, the set of elements of minimal quadratic form in $L^* \setminus L$ is nonempty. As a consequence, the $\mathbb{Q}$-vector space spanned by this set is nontrivial, $W(\underline{L})$-invariant and consequently must equal $L_{\mathbb{Q}}$, since $W(\underline{L})$ acts irreducibly. Hence, there must be some element of minimal quadratic form μ^* in $L^* \setminus L$, which is non-perpendicular to $\iota(L_0)$. The claim follows then from 3.5.2. □

We summarize our results in the following

Theorem 3.5.4. *Let $\underline{L}$ and $\underline{L_0}$ be lattices, $\underline{L}$ non-unimodular, and $\iota : \underline{L_0} \longrightarrow \underline{L}$ an embedding. Suppose that one of the three assertions holds:*

i) $\iota(\xi) \in L^* \setminus L$ for some $\xi \in L_0^*$,

ii) $\mu^* \notin \iota(L_0)^{\perp, \underline{L}}$ for some $\mu^* \in L^* \setminus L$ of minimal quadratic form,

iii) $\underline{L}$ is an irreducible root lattice,

Then the pullback operator

$$[\iota] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{L_0})$$

is injective.

With a bit more effort, we can formulate and prove a further criterion for injectivity of the pullback operator:

Theorem 3.5.5. *Let $\underline{L}$ be a lattice and $0 \neq \mu_1 \in L^*/L$ be a fixed-point of $\mathrm{O}(\underline{L})$. Assume that $\mathrm{O}(\underline{L})$ acts transitively on the remaining non-zero cosets. If $\iota(\xi) \equiv \mu_1 \bmod L$ for some $\xi \in L_0^*$, then the pullback operator*

$$[\iota] : \mathcal{E}(\underline{L})^{\mathrm{sym}} \to \mathcal{E}(\underline{L_0}),$$

is injective.

Proof. Under the assumptions on $\underline{L}$, a basis of $\mathcal{E}(\underline{L})^{\mathrm{sym}}$ is given by

$$\left\{ \vartheta_{\underline{L}}, \vartheta_{\underline{L},\mu_1}, \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu} \right\}.$$

Let $\phi \in \mathcal{E}(\underline{L})^{\mathrm{sym}}$ with theta decomposition

$$\phi = h_0 \cdot \vartheta_{\underline{L}} + h_1 \cdot \vartheta_{\underline{L},\mu_1} + h \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu}, \quad h_0, h_1, h \in \mathcal{O}(\mathcal{H})$$

such that

$$0 = \phi[\iota] = h_0 \cdot \vartheta_{\underline{L}}[\iota] + h_1 \cdot \vartheta_{\underline{L},\mu_1}[\iota] + h \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu}[\iota].$$

From $-\mathrm{id}_L \in \mathrm{O}(\underline{L})$, we conclude $\mathrm{ord}_{L^*/L}(\mu_1) = 2$. We apply $\big|_{\underline{L_0}}[\xi, 0]$ on the previous equation in order to obtain

$$h_1 \cdot \vartheta_{\underline{L}}[\iota] + h_0 \cdot \vartheta_{\underline{L},\mu_1}[\iota] + h \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu}[\iota] = 0,$$

since the map $\mu \mapsto \mu + \mu_1$ is a permutation of the subset $\{\mu : \mu \neq 0, \mu_1\}$. Substraction of both equations yields

$$(h_0 - h_1)(\vartheta_{\underline{L}} - \vartheta_{\underline{L},\mu_1})[\iota] = 0.$$

Since $(\vartheta_{\underline{L}} - \vartheta_{\underline{L},\mu_1})[\iota] = 1 + \ldots$ does not vanish identically, we obtain $h_0 = h_1$. Thus

$$h_0 \cdot \left(\vartheta_{\underline{L}} + \vartheta_{\underline{L},\mu_1}\right)[\iota] + h \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu}[\iota] = 0.$$

Since μ_1 has order 2 in L^*/L, we have $B(\mu_1, \mu) \in \{0, \frac{1}{2}\} \bmod \mathbb{Z}$ for all $\mu \in L^*/L$. By assumption, the set $\{\mu : \mu \neq 0, \mu_1\}$ is an $\mathrm{O}(\underline{L})$-orbit and the coset μ_1 is a fixed point of $\mathrm{O}(\underline{L})$. Hence the value $\delta := e^{2\pi i B(\mu_1,\mu)} \in \{\pm 1\}$ is independent of the choice of $\mu \notin \{0, \mu_1\}$. We apply $\big|_{\underline{L_0}}[0, \xi]$ on the previous equation in order to obtain

$$h_0 \cdot \left(\vartheta_{\underline{L}} + e^{2\pi i B(\mu_1,\mu_1)} \vartheta_{\underline{L},\mu_1}\right)[\iota] + \delta \cdot h \sum_{\mu\neq 0,\mu_1} \vartheta_{\underline{L},\mu}[\iota] = 0.$$

In the case $\delta = -1$ we add the two equations and conclude that

$$h_0 \cdot \left(2\vartheta_{\underline{L}} + \left(1 + e^{2\pi i B(\mu_1,\mu_1)}\right) \vartheta_{\underline{L},\mu_1}\right)[\iota] = 0.$$

The same argument as in the beginning shows that the second factor does not vanish identically. Thus $h_0 = h_1 = h = 0$, hence $\phi = 0$. We consider the remaining case $\delta = 1$. Since $\mu_1 \notin L = L^{**}$, we necessarily have $e^{2\pi i B(\mu_1,\mu_1)} = -1$. This time we substract both equations from each other and obtain

$$2 \cdot h_0 \cdot \vartheta_{\underline{L},\mu_1}[\iota] = 0.$$

Hence, $h_0 = h_1 = 0$ and $h = 0$, i.e. $\phi = 0$. □

3.6 Separation theorems in degree 1

As a consequence of 3.1.11 and 1.2.23, we obtain the following simple

Lemma 3.6.1. *Let* $\underline{L}$ *be an irreducible root lattice. Then for all roots* $l, l' \in R(\underline{L})$ *and all* $\phi \in \mathcal{E}(\underline{L})$ *one has* $\phi[l] = \phi[l']$.

For embeddings of irreducible root lattices, we can derive a result of independency of the pullback operator - restricted to submodules of small rank - of the choice of the embedding:

Proposition 3.6.2. *Let* $\underline{L}$ *and* $\underline{L_0}$ *be irreducible root lattices and* $\iota, \kappa : \underline{L_0} \longrightarrow \underline{L}$ *embeddings. Assume that the pullback operators*

$$[\iota], [\kappa] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L^*}} \rangle_{\mathbb{C}} \longrightarrow \mathcal{O}(\mathcal{H}) \cdot \left\langle \vartheta_{\underline{L_0}}, \vartheta_{\underline{L_0^*}} \right\rangle_{\mathbb{C}}$$

are well-defined. Then one has $[\iota] = [\kappa]$.

Proof. Fix some arbitrary root $l_0 \in R(\underline{L_0})$. Let $l := \iota(l_0) \in R(\underline{L})$ and $l' := \kappa(l_0) \in R(\underline{L})$. The corresponding embeddings $\iota_l : \underline{\mathbb{Z}} \longrightarrow \underline{L}$ and $\iota_{l_0} : \underline{\mathbb{Z}} \longrightarrow \underline{L_0}$ then satisfy

$$\iota_r = \iota \circ \iota_{r_0} \text{ and } \iota_{r'} = \kappa \circ \iota_{r_0}.$$

Since $\underline{L}$ is an irreducible root lattice, the pullbacks

$$[\iota_r], [\iota_{r'}] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L^*}} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{\mathbb{Z}})$$

coincide by 3.6.1. Furthermore, the map

$$[\iota_{r_0}] : \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L_0}}, \vartheta_{\underline{L_0^*}} \rangle_{\mathbb{C}} \longrightarrow \mathcal{E}(\underline{\mathbb{Z}})$$

is injective due to 3.5.4 and the result follows from 3.2.7. □

Theorem 3.6.3. *Let* $\underline{L}$ *and* $\underline{L_0}$ *be lattices and* $\iota_1, \ldots, \iota_m : \underline{L_0} \longrightarrow \underline{L}$ *embeddings. Suppose that there is some* $\xi \in L_0^*$ *such that*

$$\iota_i(\xi) \in L^* \setminus L \text{ for all } i = 1, \ldots, m, \quad \overline{Q_0}(\xi) \neq 0,$$

hold. Then for all $(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^{(m)}$ *the following assertions are equivalent:*

i) $\sum_{i=1}^{m} \alpha_i \cdot \vartheta_{\underline{L}}[\iota_i] = 0,$

ii) $\sum_{i=1}^{m} \alpha_i \cdot \phi[\iota_i] = 0$ *for some* $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L^*}} \rangle_{\mathbb{C}}$,

iii) $\sum_{i=1}^{m} \alpha_i \cdot \phi[\iota_i] = 0$ *for all* $\phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L^*}} \rangle_{\mathbb{C}}$.

In this case, the dimension formula

$$\dim_{\mathbb{C}} \langle \phi[\iota_i], i = 1, \dots, m \rangle_{\mathbb{C}} = \dim_{\mathbb{C}} \langle \vartheta_{\underline{L}}[\iota_i], i = 1, \dots, m \rangle_{\mathbb{C}}$$

holds for all $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}}$.

Proof. From the identity

$$\vartheta_{\underline{L}}\big|_{\frac{r_{\underline{L}}}{2},\underline{L}} J = \frac{\sqrt{i}^{-r_{\underline{L}}}}{\sqrt{\det \underline{L}}} \cdot \vartheta_{\underline{L}^*}$$

we obtain the equivalence of i) and iii) as well as that i) implies ii). It suffices to show the nontrivial direction, i.e. that ii) implies i). To this end we write

$$\phi = h_0 \cdot \vartheta_{\underline{L}} + h_1 \cdot \vartheta_{\underline{L}^*} \text{ for } h, g \in \mathcal{O}(\mathcal{H}).$$

By assumption on ϕ at least one of the functions h and g does not vanish identically. We consider the auxilliary functions

$$D := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_{\underline{L}}[\iota_i], \quad D_\xi := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_{\underline{L},\iota_i(\xi)}[\iota_i], \quad D^* := \sum_{i=1}^{m} \alpha_i \cdot \vartheta_{\underline{L}^*}[\iota_i].$$

In this notation, $\sum_{i=1}^{m} \alpha_i \cdot \phi[\iota_i] = 0$ implies

$$h \cdot D + g \cdot D^* = 0.$$

We apply the operator $\big|_{\underline{L_0}}[\xi, 0]$ on this equation. By 3.1.4 and 2.2.10 we obtain

$$h \cdot D_\xi + g \cdot D^* = 0,$$

since the maps $\mu \mapsto \mu + \iota_i(\xi)$ for $i = 1, \dots, m$ are permutations of L^*/L. From this we deduce

$$h \cdot (D - D_\xi) = 0.$$

In the case $h \neq 0$ we obtain $D = D_\xi$ and the transformation $z \mapsto z + 1$ yields

$$D(z) = D(z+1) = D_\xi(z+1) = e^{2\pi i Q_0(\xi)} \cdot D(z),$$

since $Q(\iota_i(\xi)) = Q_0(\xi)$ for $i = 1, \dots, m$. From $e^{2\pi i Q_0(\xi)} \neq 1$ we obtain $D \equiv 0$. In the case $h = 0$ the function g does not vanish identically. This implies $D^* = 0$, hence $D = 0$. □

As a corollary we specify the case $L_0 = \underline{\mathbb{Z}}(t)$ for $t \in \mathbb{N}$:

Corollary 3.6.4. *Let* $\underline{L}$ *be a lattice and* N *denote the exponent of the discriminant group* L^*/L. *Let* $l_1, \dots, l_m \in NL^* \setminus NL$ *such that*

$$Q(l_1) = \dots = Q(l_m) = t, \quad N^2 \nmid t.$$

Then for all $(\alpha_1, \dots, \alpha_m) \in \mathbb{C}^{1 \times m}$ *the following statements are equivalent:*

a) $\sum_{i=1}^{m} \alpha_i \cdot \vartheta_{\underline{L}}[l_i] = 0,$

b) $\sum_{i=1}^{m} \alpha_i \cdot \phi[l_i] = 0$ *for some* $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}},$

c) $\sum_{i=1}^{m} \alpha_i \cdot \phi[l_i] = 0$ *for all* $\phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}}.$

In this case, the dimension formula

$$\dim_{\mathbb{C}} \left\langle \phi[l_i], i = 1, \ldots, m \right\rangle_{\mathbb{C}} = \dim_{\mathbb{C}} \left\langle \vartheta_{\underline{L}}[l_i], i = 1, \ldots, m \right\rangle_{\mathbb{C}}$$

holds for all $0 \neq \phi \in \mathcal{O}(\mathcal{H}) \cdot \langle \vartheta_{\underline{L}}, \vartheta_{\underline{L}^*} \rangle_{\mathbb{C}}.$

Proof. The embeddings $\iota_{l_i} : \underline{\mathbb{Z}}(t) \longrightarrow \underline{L}$ for $i = 1, \ldots, m$ and $\xi := N^{-1}$ satisfy the assumptions of 3.6.3, since

$$\iota_i(\xi) = \frac{1}{N} l_i \in L^* \setminus L \text{ for } i = 1, \ldots, m, \quad Q_0(\xi) = \frac{t}{N^2} \notin \mathbb{Z}.$$

□

4 Isomorphisms of Spaces of Jacobi Forms

We apply the results of section 3.3 and theorem 3.3.8 on Jacobi forms of degree 1.

Therefore we fix the notation $J_{k,m} := J^{(1)}_{k,\underline{\mathbb{Z}}(m)}$ for $k \in \mathbb{Z}$ and $m \geq 0$ in accordance to [8, p. 10].

4.1 The A_1, A_2, E_6, E_7-tower

Theorem 4.1.1. *Let $\iota : \underline{A_1} \longrightarrow \underline{E_7}$ be an embedding.*

a) For all $k \geq 0$ even, the pullback

$$[\iota] : J_{k,\underline{E_7}} \longrightarrow J_{k,1}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,1} \longrightarrow J_{k,\underline{E_7}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \mapsto \frac{1}{12 \cdot \eta^{12}(z)} \begin{pmatrix} \vartheta_1(z)^6 + 3\vartheta_0(z)^4\vartheta_1(z)^2 & -4\vartheta_0(z)^3\vartheta_1(z)^3 \\ -4\vartheta_0(z)^3\vartheta_1(z)^3 & \vartheta_0(z)^6 + 3\vartheta_0(z)^2\vartheta_1(z)^4 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}.$$

b) For all $k \in \mathbb{N}_0$ the following dimension formula holds:

$$\dim_{\mathbb{C}} J_{k,\underline{E_7}} = \dim_{\mathbb{C}} \left[\mathrm{Mp}_2(\mathbb{Z}), k - \tfrac{7}{2}, \overline{\rho_{\underline{E_7}}}\right] = \begin{cases} \left\lfloor \frac{k+2}{6} \right\rfloor, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Proof. a) Let $(f_0, f_1)^t$ denote the image of $(h_0, h_1)^t$ under $[\iota]^{-1}$. By definition, $(f_0, f_1)^t$ transforms correctly with respect to the dual representation of $\rho_{\underline{E_7}}$. Hence it suffices to show, that both f_0 and f_1 are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$\eta^{12}(z) = e^{\pi i z} + \dots,$$
$$\vartheta_1^2(z) = 4e^{\pi i z} + \dots,$$
$$h_1(z) = c \cdot e^{3\pi i z/2} + \dots$$

for some $c \in \mathbb{C}$.

b) Follows from a) and the well-known dimension formula for $J_{k,1}$. □

For $k \geq 4$ let $G_k^* \in [\mathrm{SL}_2(\mathbb{Z}), k]$ denote the classical normalized Eisenstein series of weight k, i.e.

$$G_k^*(z) = 1 - \frac{2k}{B_k} \sum_{m=1}^{\infty} \sigma_{k-1}(m) e^{2\pi i m z}, \quad z \in \mathcal{H}.$$

Furthermore, let

$$E_{k,m} := E^{(1)}_{k,\underline{\mathbb{Z}}(m),0}$$

for $k \geq 4$ even and $m \in \mathbb{N}_0$ denote the Jacobi-Eisenstein series of weight k and index m as in [8, § 2]. From the structure theorem for the graded vector space $\bigoplus_{k=0}^{\infty} J_{2k,1}$ in [8, Thm. 3.5] we obtain the following

Corollary 4.1.2. *The graded vector space*

$$J_{2*,\underline{E_7}} := \bigoplus_{k=0}^{\infty} J_{2k,\underline{E_7}}$$

is a free $\mathbb{C}[G_4^*, G_6^*]$*-module with basis* $\left([\iota]^{-1}E_{4,1}, [\iota]^{-1}E_{6,1}\right)$ *and one has* $[\iota]^{-1}E_{4,1} = \vartheta_{\underline{E_8}}[\underline{E_7}]$.

The corresponding result for $\underline{A_2}$ and $\underline{E_6}$, where in the latter case we use the same notation as in 3.3.2, is stated in the following

Theorem 4.1.3. *a) Let* $\iota : \underline{A_1} \longrightarrow \underline{E_6}$ *be an embedding. Then for all* $k \geq 0$ *even, the pullback*

$$[\iota] : J_{k,\underline{E_6}} \longrightarrow J_{k,1}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,1} \longrightarrow J_{k,\underline{E_6}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \mapsto \frac{1}{12 \cdot \eta^{10}(z)} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \psi^*(z)\vartheta_1^3(z) + \frac{3}{2}\vartheta_0^2(z)\vartheta_1(z)\vartheta_1\left(\frac{z}{6}\right)\vartheta_1\left(\frac{z}{2}\right) & -\psi^*(z)\vartheta_0^3(z) + \frac{3}{2}\vartheta_0(z)\vartheta_1^2(z)\vartheta_1\left(\frac{z}{6}\right)\vartheta_1\left(\frac{z}{2}\right) \\ -\vartheta_{\underline{A_2}}(2z)\vartheta_1^3(z) + 3\psi(z)\vartheta_0^2(z)\vartheta_1(z) & \vartheta_{\underline{A_2}}(2z)\vartheta_0^3(z) + 3\psi(z)\vartheta_0(z)\vartheta_1^2(z) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}.$$

b) Let $\iota : \underline{A_1} \longrightarrow \underline{A_2}$ *be an embedding. Then for all* $k \geq 0$ *even, the pullback*

$$[\iota] : J_{k,\underline{A_2}} \longrightarrow J_{k,1}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,1} \longrightarrow J_{k,\underline{A_2}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \mapsto \frac{1}{2 \cdot \eta^2(z)} \begin{pmatrix} \vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z) & -\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z) \\ -\vartheta_1(3z) & \vartheta_0(3z) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}.$$

Regarding the graded vector space of Jacobi forms, we obtain the following

Corollary 4.1.4. *a) The graded vector space*

$$J_{2*,\underline{E_6}} := \bigoplus_{k=0}^{\infty} J_{2k,\underline{E_6}}$$

is a free $\mathbb{C}[G_4^*, G_6^*]$*-module with basis* $\left([\iota]^{-1}E_{4,1}, [\iota]^{-1}E_{6,1}\right)$ *and one has* $[\iota]^{-1}E_{4,1} = \vartheta_{\underline{E_8}}[\underline{E_6}]$.

b) The graded vector space

$$J_{2*,\underline{A_2}} := \bigoplus_{k=0}^{\infty} J_{2k,\underline{A_2}}$$

is a free $\mathbb{C}[G_4^*, G_6^*]$*-module with basis* $\left([\iota]^{-1}E_{4,1}, [\iota]^{-1}E_{6,1}\right)$ *and one has* $[\iota]^{-1}E_{4,1} = \vartheta_{\underline{E_8}}[\underline{A_2}]$.

From the dimension formula for $J_{k,1}$ in [8, p. 105] we obtain the following

Corollary 4.1.5. *For all* $k \geq 0$ *even, the following dimension formulas hold:*

$$\dim_{\mathbb{C}} J_{k,\underline{E_7}} = \dim_{\mathbb{C}} J_{k,\underline{E_6}} = \dim_{\mathbb{C}} J_{k,\underline{A_2}} = \dim_{\mathbb{C}} J_{k,1} = \left\lfloor \frac{k+2}{6} \right\rfloor .$$

Remark 4.1.6. *For all* $k \geq 0$ *even and all defined embeddings* $\iota : \underline{L_0} \longrightarrow \underline{L}$ *for lattices* $\underline{L_0}, \underline{L} \in \{\underline{A_1}, \underline{A_2}, \underline{E_6}, \underline{E_7}\}$*, the pullbacks*

$$[\iota] : J_{k,\underline{L}} \longrightarrow J_{k,\underline{L_0}}$$

of Jacobi forms are isomorphisms of the vector spaces and independent of the choice of ι.

4.2 The A_1, A_2, D_4-tower

Theorem 4.2.1. *Let* $\iota : \underline{A_1} \longrightarrow \underline{D_4}$ *be an embedding.*

a) For all $k \geq 0$ *even, the pullback*

$$[\iota] : J^{\mathrm{sym}}_{k,\underline{D_4}} \longrightarrow J_{k,1}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,1} \longrightarrow J^{\mathrm{sym}}_{k,\underline{D_4}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \mapsto \frac{1}{6 \cdot \eta^6(z)} \begin{pmatrix} 3\vartheta_0(z)^2\vartheta_1(z) & -3\vartheta_0(z)\vartheta_1(z)^2 \\ -\vartheta_1(z)^3 & \vartheta_0(z)^3 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} .$$

b) For all $k \in \mathbb{N}_0$ the following dimension formula holds:

$$\dim_{\mathbb{C}} J^{\mathrm{sym}}_{k,\underline{D_4}} = \dim_{\mathbb{C}} \left[\mathrm{Mp}_2(\mathbb{Z}), k-2, \left(\rho^{\mathrm{sym}}_{\underline{D_4}}\right)^t\right] = \begin{cases} \left\lfloor \frac{k+2}{6} \right\rfloor, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Proof. a) Let $(f_0, f_1)^t$ denote the image of $(h_0, h_1)^t$ under $[\iota]^{-1}$. By definition, $(f_0, f_1)^t$ transforms correctly with respect to the dual representation of $\rho^{\mathrm{sym}}_{\underline{D_4}}$. Hence it suffices to show, that both f_0 and f_1 are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$\eta^6(z) = e^{\pi i z/2} + \ldots,$$
$$\vartheta_1(z) = 2e^{\pi i z/2} + \ldots,$$
$$h_1(z) = c \cdot e^{3\pi i z/2} + \ldots$$

for some $c \in \mathbb{C}$.

b) Follows from a) and the well known dimension formula for $J_{k,1}$, cf. [8, p. 105]. □

Corollary 4.2.2. *For all $k \geq 0$ even and all embeddings $\iota : \underline{A_2} \longrightarrow \underline{D_4}$, the pullback*

$$[\iota] : J^{\mathrm{sym}}_{k,\underline{D_4}} \longrightarrow J_{k,\underline{A_2}}$$

of Jacobi forms is an isomorphism of the vector spaces and independent of the choice of ι.

4.3 The $A_1(2), 2A_1, 2A_2$-tower

Theorem 4.3.1. *Let $\iota : 2\underline{A_1} \longrightarrow 2\underline{A_2}$ be an embedding. Then for all $k \geq 0$ even, the pullback*

$$[\iota] : J^{\mathrm{sym}}_{k,2\underline{A_2}} \longrightarrow J^{\mathrm{sym}}_{k,2\underline{A_1}}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J^{\mathrm{sym}}_{k,2\underline{A_1}} \longrightarrow J^{\mathrm{sym}}_{k,2\underline{A_2}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \mapsto \frac{1}{4 \cdot \eta^4(z)}$$
$$\begin{pmatrix} \left(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\right)^2 & -2\left(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\right) \cdot \left(\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z)\right) & \left(\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z)\right)^2 \\ -\vartheta_1(3z) \cdot \left(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\right) & \vartheta_0(3z)\vartheta_1\left(\frac{z}{3}\right) + \vartheta_1(3z)\vartheta_0\left(\frac{z}{3}\right) - 2\vartheta_0(3z)\vartheta_1(3z) & -\vartheta_0(3z)\left(\vartheta_0\left(\frac{z}{3}\right) - \vartheta_0(3z)\right) \\ \vartheta_1^2(3z) & -2\vartheta_0(3z)\vartheta_1(3z) & \vartheta_0^2(3z) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}.$$

Proof. Let $(f_0, f_1, f_2)^t$ denote the image of $(h_0, h_1, h_2)^t$ under $[\iota]^{-1}$. By definition, $(f_0, f_1, f_2)^t$ transforms correctly with respect to the dual representation of $\rho^{\mathrm{sym}}_{2\underline{A_2}}$. Hence it suffices to show, that f_0, f_1 and f_2 are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$\begin{aligned}
\eta^4(z) &= e^{\pi i \frac{z}{3}} + \dots, \\
h_1(z) &= c_1 e^{\pi i \frac{3z}{2}} + \dots, \\
h_2(z) &= c_2 e^{\pi i z} + \dots, \\
\left(\vartheta_1\left(\frac{z}{3}\right) - \vartheta_1(3z)\right)^2 &= 4e^{\pi i \frac{z}{3}} + \dots, \\
\vartheta_1(3z) &= 2e^{\pi i \frac{3z}{2}} + \dots
\end{aligned}$$

for some $c_1, c_2 \in \mathbb{C}$. □

Theorem 4.3.2. *Let $\iota : \underline{A_1}(2) \longrightarrow 2\underline{A_1}$ be an embedding. Then for all $k \geq 0$ even, the pullback*

$$[\iota] : J^{\mathrm{sym}}_{k,2\underline{A_1}} \longrightarrow J_{k,2}$$

of Jacobi forms is an isomorphism of the vector spaces. On the corresponding space of vector valued modular forms, the inverse map

$$[\iota]^{-1} : J_{k,2} \longrightarrow J^{\mathrm{sym}}_{k,2\underline{A_1}}$$

is given explicitly by

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \mapsto \frac{1}{\eta^3(z)} \begin{pmatrix} \vartheta_0(2z) \cdot \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) & 0 & -\vartheta_1(2z) \cdot \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) \\ 0 & \vartheta_0^2(2z) - \vartheta_1^2(2z) & 0 \\ -\vartheta_1(2z) \cdot \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) & -2(\vartheta_0^2(2z) - \vartheta_1^2(2z)) & \vartheta_0(2z) \cdot \vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) \end{pmatrix} \cdot \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}.$$

Proof. Let $(f_0, f_1, f_2)^t$ denote the image of $(h_0, h_1, h_2)^t$ under $[\iota]^{-1}$. By definition, $(f_0, f_1, f_2)^t$ transforms correctly with respect to the dual representation of $\rho^{\mathrm{sym}}_{2\underline{A_1}}$. Hence it suffices to show, that f_0, f_1 and f_2 are holomorphic at the cusp. But this follows immediately from the Fourier expansions

$$\begin{aligned}
\eta^3(z) &= e^{\pi i \frac{z}{4}} + \dots, \\
h_1(z) &= c_1 e^{\pi i \frac{7z}{4}} + \dots, \\
h_2(z) &= c_2 e^{\pi i z} + \dots, \\
\vartheta_{\underline{A_1}(2),\frac{1}{4}}(2z) &= e^{\pi i \frac{z}{4}} + \dots
\end{aligned}$$

for some $c_1, c_2 \in \mathbb{C}$. □

From the dimension formula for $J_{k,2}$ in [8, p. 105] we obtain the following

Corollary 4.3.3. *For all $k \geq 0$ even, the following dimension formulas hold:*

$$\dim_{\mathbb{C}} J^{\mathrm{sym}}_{k,2\underline{A_2}} = \dim_{\mathbb{C}} J^{\mathrm{sym}}_{k,2\underline{A_1}} = \dim_{\mathbb{C}} J_{k,2} = \left\lfloor \frac{k}{4} \right\rfloor.$$

5 Modular Embeddings of Paramodular Groups

5.1 (Non-)commutative orders

Unless specified otherwise let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Denote by $^- : \mathbb{F} \longrightarrow \mathbb{F}$ the *standard involution* on $\mathbb{F}$. This involution is uniquely determined by the condition $\overline{x} = x$ if and only if $x \in \mathbb{R}$. For $x \in \mathbb{F}$ we call $\overline{x}$ the *conjugate* of x, $N(x) := x\overline{x}$ the *(reduced) norm* of x and $\operatorname{tr}(x) := x + \overline{x}$ the *(reduced) trace* of x. The *norm form*

$$N : \mathbb{F} \longrightarrow \mathbb{R}_+$$

turns $\mathbb{F}$ into a positive definite quadratic space satisfying the *composition law*

$$N(xy) = N(x) \cdot N(y) \quad \text{for all } x, y \in \mathbb{F}.$$

The associated bilinear form obtained by polarization is called the *trace bilinear form* and is given by $\operatorname{tr}(x\overline{y}), x, y \in \mathbb{F}$, i.e.

$$\operatorname{tr}(x\overline{y}) = N(x + y) - N(x) - N(y) \quad \text{for all } x, y \in \mathbb{F}.$$

It is easy to see that every $x \in \mathbb{F}$ satisfies the quadratic equation

$$x^2 - \operatorname{tr}(x)x + N(x) = 0.$$

We compare norm and trace with norm and trace with respect to field extensions:

Remark 5.1.1. *Let $x \in \mathbb{F} \setminus \mathbb{R}$. The algebra $\mathbb{R}[x]$, generated by x over $\mathbb{R}$, is a commutative subfield of $\mathbb{F}$ and one has $\mathbb{R}[x] = \mathbb{R} \oplus \mathbb{R}x$. The matrix of the left-multiplication $y \mapsto xy, y \in \mathbb{R}[x]$ with respect to the basis $(1, x)$ then equals*

$$\begin{pmatrix} 0 & 1 \\ -N(x) & \operatorname{tr}(x) \end{pmatrix}.$$

Hence, usual norm and trace of x with respect to the field extension $\mathbb{R}[x]/\mathbb{R}$ are given by $N(x)$ and $\operatorname{tr}(x)$.

We will introduce the notion of an order in $\mathbb{F}$:

Definition 5.1.2. *A subset $\mathcal{O} \subseteq \mathbb{F}$ is called an* order *in $\mathbb{F}$, if the following assertions hold:*

i) *$\mathcal{O}$ is a discrete subring with $1 \in \mathcal{O}$,*

ii) *$\mathcal{O}$ generates $\mathbb{F}$ as a $\mathbb{R}$-vector space, i.e. $\mathcal{O}\mathbb{R} = \mathbb{F}$.*

An order is called maximal, *if it is a maximal element in the set of orders in $\mathbb{F}$.*

For basic references in the theory of orders in non-necessary commutative rings confer the books of Reiner [23] or Vignéras [26].

It is well-known, that every discrete additive subgroup of a finite-dimensional real vector space is a free abelian group over $\mathbb{Z}$. Hence we obtain another characterization given in the following

Lemma 5.1.3. *For a subring $\mathcal{O} \subseteq \mathbb{F}$ with $1 \in \mathcal{O}$ the following assertions are equivalent:*

a) *$\mathcal{O}$ is an order in $\mathbb{F}$,*

b) *$\mathcal{O}$ is a free abelian group and generated by a $\mathbb{R}$-basis of $\mathbb{F}$.*

Regarding integrality of norm and trace we have to following

Proposition 5.1.4. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. Then the following assertions hold:*

a) *$\mathcal{O} \cap \mathbb{R} = \mathbb{Z}$,*

b) *$N(a), \mathrm{tr}(a) \in \mathbb{Z}$ for all $a \in \mathcal{O}$,*

c) *$\overline{a} \in \mathcal{O}$ for all $a \in \mathcal{O}$.*

Proof. a) Since $\mathcal{O}$ is discrete in $\mathbb{F}$, the intersection $\mathcal{O} \cap \mathbb{R}$ is a discrete additive subgroup of $\mathbb{R}$, hence cyclic, i.e. equals $\mathbb{Z}v$ for some $v \in \mathbb{R}$. From $1 \in \mathcal{O}$ we can assume that $v = \frac{1}{s}$ for some $s \in \mathbb{N}$. From $\lim_{n\to\infty} v^n = 0$ for $s > 1$ we obtain $s = 1$ from the discreteness of $\mathcal{O}$.

b) The claim is immediate for $a \in \mathbb{Z}$. For $a \in \mathcal{O} \setminus \mathbb{R}$ consider the subfield $\mathbb{R}[a] = \mathbb{R} \oplus \mathbb{R}a$, generated by a over $\mathbb{R}$. Let $\mathcal{O}_a := \mathcal{O} \cap \mathbb{R}[a]$. Then $\mathcal{O}_a$ is a discrete subring of $\mathbb{R}[a]$ with $1 \in \mathcal{O}_a$. Since $\mathcal{O}_a$ contains the $\mathbb{R}$-basis $(1, a)$ of $\mathbb{R}[a]$, we conclude that $\mathcal{O}_a$ is an order in $\mathbb{R}[a]$, i.e. $\mathcal{O}_a = \mathbb{Z}x + \mathbb{Z}y$ for some $\mathbb{R}$-basis (x, y) of $\mathbb{R}[a]$. Hence the usual norm and trace of $a \in \mathcal{O}$ with respect to the field extension $\mathbb{R}[a]/\mathbb{R}$ lie in $\mathbb{Z}$. Since norm and trace are independent from choice of the basis of $\mathbb{R}[a]$, we can apply 5.1.1 and obtain $N(a), \mathrm{tr}(a) \in \mathbb{Z}$.

c) Follows from $\overline{a} = \mathrm{tr}(a) - a$ and b). □

Corollary 5.1.5. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. Then $(\mathcal{O}, N|_{\mathcal{O}})$ is a positive definite, even lattice. In the case $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, the map $a \mapsto \overline{a}$ is a nontrivial automorphism of $(\mathcal{O}, N|_{\mathcal{O}})$ and $\mathbb{Z}$ is the largest sublattice of $\mathcal{O}$, on which the standard involution of $\mathbb{F}$ acts trivially.*

We repeat some ring-theoretic terms:

Definition 5.1.6. *Let $\mathcal{O}$ be an order in $\mathbb{F}$.*

a) *An element $\varepsilon \in \mathcal{O}$ is called a* unit, *if there is some $\delta \in \mathcal{O}$ such that $\varepsilon\delta = 1$. In this case, δ is uniquely determined and one has $\delta\varepsilon = 1$. The set of units $\mathcal{O}$ is a group and one has*

$$\mathcal{O}^\times = \{a \in \mathcal{O} : N(a) = 1\}.$$

b) *Let $a, b \in \mathcal{O}, a \neq 0$. Then a is called a* left *resp.* right divisor *of b, if $a^{-1}b \in \mathcal{O}$ resp. $ba^{-1} \in \mathcal{O}$. In this case we write $a|_l b$ resp. $a|_r b$.*

c) *Let I be an additive subgroup of $\mathcal{O}$. We call I a* left *resp.* right ideal, *if $\mathcal{O}I \subset I$ resp. $I\mathcal{O} \subset I$. We call I a* two-sided ideal, *if I is both a left and right ideal. We call I a* left *resp.* right principal ideal, *if there is $a \in \mathcal{O}$ such that $I = \mathcal{O}a$ resp. $I = a\mathcal{O}$. For $a, b \in \mathcal{O}$ one has $b\mathcal{O} \subset a\mathcal{O}$ resp. $\mathcal{O}b \subset a\mathcal{O}$ if and only if $a|_l b$ resp. $a|_r b$.*

d) *We call $\mathcal{O}$ a* principal ideal domain, *if every left and every right ideal of $\mathcal{O}$ is principal.*

e) *We call $\mathcal{O}$* norm-euclidean, *if for every $a \in \mathbb{F}$ there exists $g \in \mathcal{O}$ such that $N(a - g) < 1$.*

Remark 5.1.7. *Let $\mathcal{O}$ be a norm-euclidean order in $\mathbb{F}$. Then $\mathcal{O}$ admits a* left *resp.* right euclidean algorithm. *Hence, every nontrivial left resp. right ideal I is principal and is generated by an element of nonzero minimal norm in I. Every norm-euclidean order is a principal ideal domain.*

We need some identifications:

Example 5.1.8. *We consider the skewfield of quaternions $\mathbb{H}$, i.e.*

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i_1 + \mathbb{R}i_2 + \mathbb{R}i_3$$

with multiplication linearly extended via the defining relations

$$i_1^2 = i_2^2 = -1, i_3 = i_1 i_2.$$

Every $a \in \mathbb{H}$ has a unique representation $a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3$ with $a_0, \dots, a_3 \in \mathbb{R}$. The complex field $\mathbb{C}$ will be identified in $\mathbb{H}$ via $\mathbb{C} := \mathbb{R} + \mathbb{R}i_1$. It is the maximal commutative subfield of $\mathbb{H}$ and hence a splitting field *for $\mathbb{H}$. Via this identification, $\mathbb{H}$ carries a natural structure of a* left *$\mathbb{C}$-vector space with basis $1, i_2$. Explicitly one has*

$$a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 = (a_0 + a_1 i_1) + (a_2 + a_3 i_1) i_2 \in \mathbb{C} \oplus \mathbb{C}i_2.$$

Right multiplication by $a = \alpha + \beta i_2, \alpha, \beta \in \mathbb{C}$ induces a monomorphism of $\mathbb{R}$-algebras

$$^\vee : \mathbb{H} \longrightarrow \mathbb{C}^{2\times 2}, \quad \alpha + \beta i_2 \mapsto \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \alpha, \beta \in \mathbb{C},$$

where the definition is implicitly extended to matrices componentwisely. The map

$$\overline{} : \mathbb{H} \longrightarrow \mathbb{H}, \quad a \mapsto a_0 - a_1 i_1 - a_2 i_2 - a_3 i_3$$

is the unique standard involution on $\mathbb{H}$*. For* $a \in \mathbb{H}$*, we call* $\overline{a}$ *its* conjugate*. It is straightforward to check that reduced trace resp. norm are given by*

$$\mathrm{tr}(a) = 2a_0, \quad N(a) = a\overline{a} = \overline{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

for $a \in \mathbb{H}$*. The map*

$$\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}, \quad (a, b) \mapsto \mathrm{tr}(\overline{a}b)$$

is a symmetric, positive definite $\mathbb{R}$*-bilinear form on* $\mathbb{H}$*. The decomposition* $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}i_2$ *is orthogonal with respect to this bilinear form.*

We describe some lattice-theoretic objects in the new setting:

Definition 5.1.9. *Let* $\mathcal{O}$ *be an order in* $\mathbb{F}$*.*

a) The dual lattice

$$\mathcal{O}^\sharp := \{x \in \mathbb{F} : \mathrm{tr}(a\overline{x}) \in \mathbb{Z} \text{ for all } a \in \mathcal{O}\}$$

with respect to the trace bilinear form is called the inverse different *of* $\mathcal{O}$*.*

b) The finite abelian group $\mathcal{O}^\sharp/\mathcal{O}$ *is called the* discriminant group *of* $\mathcal{O}$*. The number*

$$d(\mathcal{O}) := |\mathcal{O}^\sharp/\mathcal{O}| < \infty$$

is called the discriminant *of* $\mathcal{O}$*.*

c) The two-sided ideal

$$\mathcal{D}(\mathcal{O}) := \left\{x \in \mathcal{O} : x\mathcal{O}^\sharp \subseteq \mathcal{O}\right\}$$

is called the different *of* $\mathcal{O}$*.*

Examples 5.1.10. *a) Let* K *be an* imaginary-quadratic number field *of discriminant* $-D_K$*, i.e.*

$$K = \mathbb{Q}(\sqrt{-D_K}) \subseteq \mathbb{C}, \quad D_K > 0,$$

such that

- $D_K \equiv 3 \bmod 4$ *and* D_K *squarefree or*
- $D_K \equiv 0 \bmod 4, D_K/4 \equiv 1, 2 \bmod 4$ *and* $D_K/4$ *squarefree.*

Let $\mathfrak{o}_K$ *denote the* integral closure *of* $\mathbb{Z}$ *in* K*, i.e.*

$$\mathfrak{o}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\frac{\sqrt{-D_K}}{2}, & D_K \equiv 0 \bmod 4, \\ \mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{-D_K}}{2}, & D_K \equiv 3 \bmod 4. \end{cases}$$

The inverse different $\mathfrak{o}_K^\sharp$ *is given by*

$$\mathfrak{o}_K^\sharp = \frac{i_1}{\sqrt{D_K}}\mathfrak{o}_K = \begin{cases} \frac{i_1}{\sqrt{D_K}}\mathbb{Z} + \frac{1}{2}\mathbb{Z}, & D_K \equiv 0 \bmod 4, \\ \frac{i_1}{\sqrt{D_K}}\mathbb{Z} + \frac{1}{2}\left(1 + \frac{i_1}{\sqrt{D_K}}\right)\mathbb{Z}, & D_K \equiv 3 \bmod 4, \end{cases}$$

and $d(\mathfrak{o}_K) = D_K$*. The different* $\mathcal{D}(\mathfrak{o}_K)$ *is the principal ideal generated by* $i_1\sqrt{D_K}$*. The unit groups* $\mathfrak{o}_K^\times$ *are given by*

$$\mathfrak{o}_K^\times = \begin{cases} \{\pm 1, \pm\rho, \pm\overline{\rho}\}, & D_K = 3, \rho := \frac{1}{2}(-1 + i_1\sqrt{3}), \\ \{\pm 1, \pm i_1\}, & D_K = 4, \\ \{\pm 1\}, & D_K > 4. \end{cases}$$

b) *Let* K *as in a) and define* $\mathcal{O}_K := \mathfrak{o}_K + \mathfrak{o}_K i_2$*. From* $i_2 w = \overline{w} i_2$ *for all* $w \in \mathfrak{o}_K$ *we conclude that* $\mathcal{O}_K$ *is an order in* $\mathbb{H}$*. The inverse different is given by*

$$\mathcal{O}_K^\sharp = \mathfrak{o}_K^\sharp + \mathfrak{o}_K^\sharp i_2 = \frac{i_1}{\sqrt{D_K}}\mathcal{O}_K$$

and $d(\mathcal{O}_K) = D_K^2$*. The different* $\mathcal{D}(\mathcal{O}_K)$ *is the two-sided principal ideal generated by* $i_1\sqrt{D_K}$*. The unit groups* $\mathcal{O}_K^\times$ *are given by*

$$\mathcal{O}_K^\times = \begin{cases} \{\pm 1, \pm\rho, \pm\overline{\rho}, \pm i_2, \pm\rho i_2, \pm\overline{\rho} i_2\}, & D_K = 3, \\ \{\pm 1, \pm i_1, \pm i_2, \pm i_3\}, & D_K = 4, \\ \{\pm 1, \pm i_2\}, & D_K > 4. \end{cases}$$

c) *The ring of* Hurwitz quaternions

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}\omega, \quad \omega := \frac{1 + i_1 + i_2 + i_3}{2},$$

is a maximal order in $\mathbb{H}$*. The inverse different of* $\mathcal{O}$ *is given by* $\mathcal{O}^\sharp = (1+i_1)^{-1}\mathcal{O}$ *and* $d(\mathcal{O}) = 4$*. The different* $\mathcal{D}(\mathcal{O})$ *is the two-sided principal ideal generated by* $1 + i_1$ *and is denoted by* $\wp$*. It can be shown that*

$$\wp = \{a \in \mathcal{O} : N(a) \equiv 0 \bmod 2\}.$$

We call $\wp$ *the ideal of* even quaternions*. The elements* $0, 1, \omega, \overline{\omega}$ *form a complete set of representatives for* $\wp$ *in* $\mathcal{O}$*. The unit group of* $\mathcal{O}$ *is given by*

$$\mathcal{O}^\times = \left\{\pm 1, \pm i_1, \pm i_2, \pm i_3, \frac{\pm 1 \pm i_1 \pm i_2 \pm i_3}{2}\right\} = \langle \omega, i_1 \rangle.$$

It is a noncommutative group of order 24*. The identities*

$$i_1^2 = -1, \quad (i_1 + i_2)^2 = -2, \quad (-\omega)^3 = 1$$

show that the orders $\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}, \mathfrak{o}_{\mathbb{Q}(\sqrt{-2})}$ *and* $\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$ *can be naturally embedded into* $\mathcal{O}$*.*

d) *The ring of* Lipschitz quaternions

$$\Lambda := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}i_2 + \mathbb{Z}i_3$$

is an order in $\mathbb{H}$*. Indeed,* $\Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$*. The inverse different of* Λ *is given by* $\Lambda^\sharp = \frac{1}{2}\Lambda$ *and* $d(\Lambda) = 16$*. The different* $\mathcal{D}(\Lambda)$ *is the two-sided principal ideal generated by* 2*. It is easy to see that* $\wp \subseteq \Lambda \subseteq \mathcal{O}$ *and that the index of each inclusion equals two. Especially,* Λ *is a non-maximal order. The elements* $0, 1$ *form a complete system of representatives for* $\wp$ *in* Λ*. Furthermore one has*

$$\Lambda = \{a \in \mathcal{O} : a \equiv 0, 1 \bmod \wp\}$$

as well as

$$\mathcal{O} = \Lambda \cup \omega\Lambda \cup \overline{\omega}\Lambda = \Lambda \cup \Lambda\omega \cup \Lambda\overline{\omega}.$$

e) $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ *is a maximal order in* $\mathbb{H}$*.*

f) $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ *is not maximal. Indeed,* $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ *is strictly contained in the order*

$$\mathcal{O}_2 := \mathbb{Z} + \mathbb{Z}i_1 + \mathbb{Z}\frac{1 + i_1 + \sqrt{2}i_2}{2} + \mathbb{Z}\frac{1 + i_1 + \sqrt{2}i_3}{2},$$

which is seen to be isomorphic to Hurwitz order $\mathcal{O}$ *in c) via the map*

$$i_1 \mapsto i_1, \quad \frac{1 + i_1 + \sqrt{2}i_2}{2} \mapsto \frac{1 + i_1 + i_2 + i_3}{2}, \quad \frac{1 + i_1 + \sqrt{2}i_3}{2} \mapsto \frac{1 + i_1 + i_2 - i_3}{2}.$$

Hence, $\mathcal{O}_2$ *is a maximal order.*

In order to deduce a *suitable description* of the orthogonal group of the lattice $(\mathcal{O}, N|_{\mathcal{O}})$, we introduce the notion of invariant elements:

Definition 5.1.11. *Let* $\mathcal{O}$ *be an order in* $\mathbb{H}$*. An element* $0 \neq a \in \mathcal{O}$ *is called* invariant*, if it generates a two-sided ideal in* $\mathcal{O}$*, i.e.* $a\mathcal{O} = \mathcal{O}a$*. In other words, a is invariant, if and only if the map* $u \mapsto a^{-1}ua$ *is a* $\mathbb{Z}$*-automorphism of* $\mathcal{O}$*. The set of invariant elements of* $\mathcal{O}$ *carries the structure of a multiplicative semigroup, which will be denoted by* $\mathcal{I}(\mathcal{O})$*. For formal reasons we extend the definition for orders* $\mathcal{O}$ *in* $\mathbb{C}$ *by setting* $\mathcal{I}(\mathcal{O}) := \{1\}$ *in this case.*

An explicit determination of $\mathcal{I}(\mathcal{O})$ for certain orders is given in the following

Lemma 5.1.12. *a) Let* $\mathcal{O}$ *denote the Hurwitz order. Then one has*

$$\mathcal{I}(\mathcal{O}) = \mathbb{N} \cdot \langle 1 + i_1, \varepsilon \in \mathcal{O}^\times \rangle.$$

b) Let $K = \mathbb{Q}(\sqrt{-D_K})$. Then one has

$$\mathcal{I}(\mathcal{O}_K) = \mathbb{N} \cdot \begin{cases} \langle i_1\sqrt{3}, \varepsilon \in \mathcal{O}_K^\times \rangle, & D_K = 3, \\ \langle 2\omega, 1+i_1, \varepsilon \in \mathcal{O}_K^\times \rangle, & D_K = 4, \\ \langle i_1\sqrt{D_K}, \varepsilon \in \mathcal{O}_K^\times \rangle & D_K > 4. \end{cases}$$

Proof. a) Confer [11, p. 29] or [14, I. Lemma 1.5].

b) For $0 \neq a \in \mathcal{I}(\mathcal{O}_K)$ consider the automorphism

$$\varphi_a : \mathcal{O}_K \to \mathcal{O}_K, \quad w \mapsto a^{-1}wa.$$

Since $\mathcal{O}_K$ contains a $\mathbb{R}$-basis of $\mathbb{H}$ and $\mathcal{O} \cap \mathbb{R} = \mathbb{Z}$, we have $\varphi_a = \mathrm{id}$ for $a \in \mathcal{I}(\mathcal{O})$ if and only if $a \in \mathbb{Z} \setminus \{0\}$. Of course, $\varphi_a(\pm 1) = \pm 1$. In the case $D_K = 3$ one has

$$\begin{aligned} \mathrm{ord}(1) = 1, \mathrm{ord}(-1) = 2, \mathrm{ord}(\rho) = \mathrm{ord}(\overline{\rho}) = 3, \\ \mathrm{ord}(\pm i_2) = \mathrm{ord}(\pm\rho i_2) = \mathrm{ord}(\pm\overline{\rho} i_2) = 4, \\ \mathrm{ord}(-\rho) = \mathrm{ord}(-\overline{\rho}) = 6. \end{aligned}$$

Hence, $\varphi_a(\rho) \in \{\rho, \overline{\rho}\}$ and $\varphi_a(i_2) \in \{\pm i_2, \pm\rho i_2, \pm\overline{\rho} i_2\}$. The value $\varphi_a(\rho i_2)$ is uniquely determined by $\varphi_a(i_2)$ and $\varphi_a(\rho)$. Consequently, the number of different maps φ_a is less or equal than 12. A direct verfication shows that any map of the form above is induced by conjugation by ε or $i_1\sqrt{3}\varepsilon$ for $\varepsilon \in \mathcal{O}_K^\times$. In the case $D_K = 4$, it is easily seen that every automorphism of determinant 1 of the lattice (Λ, N_Λ) is given by a signed permutation of sign 1. This group has order 192. A direct verification shows that it is generated by the maps

$$\varphi_a, \quad a \in \{2\omega, 1+i_1, \pm i_1, \pm i_2, \pm i_3\}.$$

In the case $D_K > 4$ we obtain $\varphi_a(i_2) = \pm i_2$, since φ restricts to units. Furthermore,

$$\varphi_a(i_1\sqrt{D_K}/2) \in \left\{\pm i_1\sqrt{D_K}/2, \pm i_3\sqrt{D_K}/2\right\}$$

is due to norm reasons and the fact that $-D_K$ is a discriminant. Again a direct verification shows that φ_a is a product of maps of the form

$$w \mapsto a^{-1}wa, \quad a \in \{i_2, i_1\sqrt{D_K}\}.$$

□

The following result is well-known. For sake of completeness we include a proof:

Proposition 5.1.13. *Let $K = \mathbb{Q}(\sqrt{-3})$. Then the following assertions hold:*

a) $\{u \in \mathfrak{o}_K : N(u) \equiv 0 \bmod 3\} = i\sqrt{3}\mathfrak{o}_K$,

b) $\{u \in \mathcal{O}_K : N(u) \equiv 0 \bmod 3\} = i\sqrt{3}\mathcal{O}_K$.

Proof. a) The trivial inclusion follows from $N(i_1\sqrt{3}) = 3$ and the multiplicativity of the norm form. For the nontrivial inclusion let $u = a + b\rho \in \mathfrak{o}_K$ such that $N(u) \equiv 0 \bmod 3$. By definition,

$$N(u) = \left(a + \frac{b}{2}\right)^2 + \frac{3}{4}b^2 = a^2 + ab + b^2 \equiv a^2 - 2ab + b^2 = (a-b)^2 \bmod 3.$$

Hence, $a \equiv b \bmod 3$. We conclude $u = b + 3l + b\rho = 3l + b(1+\rho)$ for some $l \in \mathbb{Z}$. Hence it suffices to show $1 + \rho \in i\sqrt{3}\mathfrak{o}_K$. This is seen from the identity $1 + \rho = i_1\sqrt{3}(-\rho + 1)$.

b) Let $u = u_1 + u_2 i_2, u_1, u_2 \in \mathfrak{o}_K$ such that

$$N(u) = N(u_1) + N(u_2) \equiv 0 \bmod 3.$$

In the proof of a) it was shown that both $N(u_1)$ and $N(u_2)$ are quadratic residues mod 3 and hence congruent to $0, 1 \bmod 3$. Thus, $N(u) \equiv 0 \bmod 3$ already implies $N(u_1), N(u_2) \equiv 0 \bmod 3$. From a) we deduce $u_1, u_2 \in i_1\sqrt{3}\mathfrak{o}_K$ and thus $u \in i_1\sqrt{3}\mathcal{O}_K$. □

We cite a classical theorem of Cayley, cf. [6, 7 §3, p. 215], that characterizes the orthogonal groups of the quadratic spaces $(\mathbb{F}, N)$ in the following manner:

Theorem 5.1.14 (Cayley). *Let* $\mathbb{F}_1 := \{w \in \mathbb{F} : N(w) = 1\}$. *Then one has*

$$\mathrm{O}(\mathbb{F}) = \begin{cases} \langle w \mapsto -w \rangle, & \mathbb{F} = \mathbb{R}, \\ \langle w \to \varepsilon w : \varepsilon \in \mathbb{C}_1 \rangle \rtimes \langle w \to \overline{w} \rangle, & \mathbb{F} = \mathbb{C}, \\ \left(\{w \mapsto \delta w : \delta \in \mathbb{H}_1\} \cdot \{w \mapsto \bar{\varepsilon} w \varepsilon : \varepsilon \in \mathbb{H}_1\} \right) \rtimes \langle w \mapsto \overline{w} \rangle, & \mathbb{F} = \mathbb{H}. \end{cases}$$

We describe the normalizer of an quaternionic order $\mathcal{O}$ in terms of its invariant elements:

Lemma 5.1.15. *Let $\mathcal{O}$ be an order in $\mathbb{H}$. Then the following assertions hold:*

a) $\mathcal{O}\mathbb{Q}$ is a central simple $\mathbb{Q}$-algebra.

b) $\mathcal{N}_{\mathbb{F}_1}(\mathcal{O}) = \left\{ \frac{u}{\sqrt{N(u)}} : u \in \mathcal{I}(\mathcal{O}) \right\}$.

Proof. a) For $0 \neq u \in \mathcal{O}$ one has $u^{-1} = N(u)^{-1}\overline{u} \in \mathcal{O}\mathbb{Q}$. Hence $\mathcal{O}\mathbb{Q}$ is a skewfield over $\mathbb{Q}$ and trivially simple. In order to prove that $\mathcal{O}\mathbb{Q}$ is central with center $\mathbb{Q}$, note that we have

$$\mathcal{C}(\mathcal{O}\mathbb{Q}) = \mathcal{C}(\mathbb{H}) \cap \mathcal{O}\mathbb{Q} = \mathbb{R} \cap \mathcal{O}\mathbb{Q},$$

since $\mathcal{O}\mathbb{Q}$ contains a $\mathbb{R}$-basis of $\mathbb{H}$. Thus it suffices to show that $\mathbb{R} \cap \mathcal{O}\mathbb{Q} = \mathbb{Q}$. But for $x \in \mathbb{R} \cap \mathcal{O}\mathbb{Q}$ we find $r \in \mathbb{Z}$ such that $rx \in \mathcal{O} \cap \mathbb{R} = \mathbb{Z}$, i.e. $x \in \mathbb{Q}$.

b) Let $x \in N_{\mathbb{F}_1}(\mathcal{O})$, i.e. $x \in \mathbb{F}$, $N(x) = 1$ and $x^{-1}\mathcal{O}x = \mathcal{O}$. Hence, conjugation by x extends to a $\mathbb{Q}$-automorphism of the central simple algebra $\mathcal{O}\mathbb{Q}$. Due to the Skolem-Noether theorem, cf. [23, Chap. 1, § 7d, Thm. (7.21)], this automorphism is inner, i.e. there is $0 \neq u \in \mathcal{O}\mathbb{Q}$, such that $x^{-1}ax = u^{-1}au$ for all $a \in \mathcal{O}\mathbb{Q}$. Since this equation also holds for the multiples $m \cdot u$ for $0 \neq m \in \mathbb{Z}$, we can assume that $u \in \mathcal{O}$, i.e. $u \in \mathcal{I}(\mathcal{O})$. Since $\mathcal{O}$ contains a $\mathbb{R}$-basis of $\mathbb{H}$, the element xu^{-1} is central in $\mathbb{H}$ and consequently there is $r \in \mathbb{R}$ such that $x = ru$. From $N(x) = 1$ we obtain $r = \pm\frac{1}{\sqrt{N(u)}}$. □

From 5.1.14 and 5.1.15 we derive an explicit description of the orthogonal group of the lattice $(\mathcal{O}, N|_{\mathcal{O}})$ in terms of the multiplicative structure of the order and its invariant elements:

Theorem 5.1.16. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. Then the structure of $\mathrm{O}(\mathcal{O})$ is given by*

$$\mathrm{O}(\mathcal{O}) = \begin{cases} \langle w \mapsto -w \rangle, & \mathbb{F} = \mathbb{R}, \\ \{w \mapsto \delta w : \delta \in \mathcal{O}^\times\} \rtimes \langle w \mapsto \overline{w} \rangle, & \mathbb{F} = \mathbb{C}, \\ \big(\{w \mapsto \delta w : \delta \in \mathcal{O}^\times\} \cdot \{w \mapsto u^{-1}wu : u \in \mathcal{I}(\mathcal{O})\}\big) \rtimes \langle w \mapsto \overline{w} \rangle, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Proof. The claim is obvious in the case $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and for the left-multiplications if $\mathbb{F} = \mathbb{H}$, since $1 \in \mathcal{O}$. In the remaining case let $\varepsilon, \delta \in \mathbb{H}_1$ such that $\delta\mathcal{O}\varepsilon = \mathcal{O}$. This implies $\delta\varepsilon \in \mathcal{O}^\times$, i.e. $\delta = \delta'\overline{\varepsilon}$ for some $\delta' \in \mathcal{O}^\times$. As a consequence,

$$\mathcal{O} = \varepsilon\mathcal{O}\delta = \varepsilon\mathcal{O}\delta'\overline{\varepsilon} = \varepsilon\mathcal{O}\overline{\varepsilon},$$

since $\delta' \in \mathcal{O}^\times$. Thus $\varepsilon \in \mathcal{N}_{\mathbb{H}_1}(\mathcal{O})$ and by 5.1.15, $\varepsilon = \frac{u}{\sqrt{N(u)}}$ for some $u \in \mathcal{I}(\mathcal{O})$. □

We give some explicit

Examples 5.1.17. *Let $\mathcal{O}$ denote the Hurwitz order and $\pi := 1 + i_1$.*

a) $\mathrm{O}(\mathcal{O}) = \big(\{w \mapsto \delta w, \delta \in \mathcal{O}^\times\} \cdot \langle w \mapsto u^{-1}wu : u \in \mathcal{O}^\times \cup \{\pi\}\rangle\big) \rtimes \langle w \mapsto \overline{w} \rangle,$

b) $\mathrm{O}(\Lambda) = \big(\{w \mapsto \delta w, \delta \in \Lambda^\times\} \cdot \langle w \mapsto u^{-1}wu : u \in \Lambda^\times \cup \{\pi, \omega\}\rangle\big) \rtimes \langle w \mapsto \overline{w} \rangle,$

c) $\mathrm{O}(\mathcal{O}_K) = \big(\{w \mapsto \delta w, \delta \in \mathcal{O}_K^\times\} \cdot \langle w \mapsto u^{-1}wu : u \in \mathcal{O}_K^\times \cup \{i_1\sqrt{D_K}\}\rangle\big) \rtimes \langle w \mapsto \overline{w} \rangle$ *for $D_K = 3$ or $D_K > 4$.*

Regarding the discriminant kernel of the lattice $(\mathcal{O}, N|_{\mathcal{O}})$ we obtain the following

Proposition 5.1.18. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. Then the map $(w \mapsto -\overline{w})$ belongs to $\mathrm{O}_d(\mathcal{O})$.*

Proof. Follows from $\mu - (-\overline{\mu}) = \mu + \overline{\mu} = \mathrm{tr}(\mu \cdot 1) \in \mathbb{Z}$ for all $\mu \in \mathcal{O}^\sharp$. □

5.2 Unitary symplectic groups

Definition 5.2.1. *The* unitary symplectic group *of degree n over $\mathbb{F}$ is defined as*

$$\mathrm{U}_n(\mathbb{F}) := \left\{ M \in \mathbb{F}^{2n \times 2n} : J_n[M] = J_n \right\}.$$

Note that $\mathrm{U}_n(\mathbb{R}) = \mathrm{Sp}_n(\mathbb{R})$. In order to avoid ambiguities, we will use the term *unitary* only in the case $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$.

As in [14, II. § 2 Lemma (1.1)] we obtain the following characterization of $\mathrm{U}_n(\mathbb{F})$:

Theorem 5.2.2. *The unitary symplectic group $\mathrm{U}_n(\mathbb{F})$ is a subgroup of $\mathrm{GL}_{2n}(\mathbb{F})$. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with blocks $A, B, C, D \in \mathbb{F}^{n \times n}$ the following assertions are equivalent:*

i) $M \in \mathrm{U}_n(\mathbb{F})$,

ii) $\overline{M}^t \in \mathrm{U}_n(\mathbb{F})$,

iii) $\overline{A}^t C - \overline{C}^t A = \overline{B}^t D - \overline{D}^t B = 0, \overline{A}^t D - \overline{C}^t B = I_n$,

iv) $A\overline{B}^t - B\overline{A}^t = C\overline{D}^t - D\overline{C}^t = 0, A\overline{D}^t - B\overline{C}^t = I_n$.

In this case, one has

$$M^{-1} = \begin{pmatrix} \overline{D}^t & -\overline{B}^t \\ -\overline{C}^t & \overline{A}^t \end{pmatrix}.$$

As a generalization of the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$ we introduce the modular group with respect to an order $\mathcal{O}$ in $\mathbb{F}$:

Definition 5.2.3. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. Then the* modular group *of degree n with respect to $\mathcal{O}$ is defined as*

$$\mathrm{U}_n(\mathcal{O}) := \mathrm{U}_n(\mathbb{F}) \cap \mathcal{O}^{2n \times 2n} = \left\{ M \in \mathcal{O}^{2n \times 2n} : J_n[M] = J_n \right\}.$$

We distinguish certain elements in the modular group:

Remarks 5.2.4. *Let $\mathcal{O}$ be an order in $\mathbb{F}$.*

a) The matrices

$$J_n, \quad \begin{pmatrix} \overline{U}^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \mathrm{GL}_n(\mathcal{O}), \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Her}_n(\mathcal{O})$$

belong to $\mathrm{U}_n(\mathcal{O})$.

b) $\mathrm{U}_1(\mathcal{O}) = \mathcal{O}^\times \cdot \mathrm{SL}_2(\mathbb{Z})$.

This gives rise to the following

Definition 5.2.5. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. The modular group $\mathrm{U}_n(\mathcal{O})$ is called* standardly generated*, if*

$$\mathrm{U}_n(\mathcal{O}) = \left\langle J_n, \begin{pmatrix} \overline{U}^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \mathrm{GL}_n(\mathcal{O}), \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Her}_n(\mathcal{O}) \right\rangle.$$

A class of examples is contained in the following

Theorem 5.2.6. *Let $\mathcal{O}$ be an order in $\mathbb{F}$. If $\mathcal{O}$ is a principal ideal domain, then $\mathrm{U}_n(\mathcal{O})$ is standardly generated.*

Proof. Let $x \in \mathcal{O}^n$. As a special case of the elementary divisor theorem for principal ideal domains, where we refer to [25] in the noncommutative case, yields the existence of a matrix $U \in \mathrm{GL}_n(\mathcal{O})$ such that $Ux = (\gamma, 0, \ldots, 0)^t$, where γ is a greatest common right divisor of the entries of x. Then rest of the proof is along the same lines as in [14, II. §2, Prop. 2.2, Thm. 2.3]. □

Under more rigid assumptions on the order one can determine a smaller set of generators of $\mathrm{U}_n(\mathcal{O})$. By utilizing the euclidean algorithm, one can show that $\mathrm{GL}_n(\mathcal{O})$ is generated by elementary matrices. Hence, we obtain

Corollary 5.2.7. *Let $\mathcal{O}$ be a norm-euclidean order in $\mathbb{F}$. Then the following assertions hold:*

a) $\mathrm{U}_n(\mathcal{O})$ *is generated by*

$$J_n, \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Her}_n(\mathcal{O}), \quad \begin{pmatrix} \overline{U}^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U = \mathrm{diag}(\varepsilon, 1, \ldots, 1), \varepsilon \in \mathcal{O}^\times.$$

b) $\mathrm{U}_n(\mathcal{O})$ *is generated by*

$$J_2 \times I_{2n-2}, \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Her}_n(\mathcal{O}), \quad \begin{pmatrix} \overline{U}^t & 0 \\ 0 & U^{-1} \end{pmatrix},$$

where $U = \mathrm{diag}(\varepsilon, 1, \ldots, 1), \varepsilon \in \mathcal{O}^\times$ *or U is a permutation matrix.*

5.3 Paramodular groups

Definition 5.3.1. *A matrix $T \in \mathbb{Z}^{n \times n}$ is called* elementary divisor matrix*, if*

$$T = \mathrm{diag}(t_1, \ldots, t_n), \quad t_i | t_{i+1} \text{ for } i = 1, \ldots, n-1.$$

Definition 5.3.2. *The* integral paramodular group *of* polarization T *is defined as*

$$\widehat{\Gamma}(T) = \left\{ M \in \mathbb{Z}^{2n \times 2n} : M^t \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \right\}.$$

Of course one has $\widehat{\Gamma}(I_n) = \mathrm{Sp}_n(\mathbb{Z})$.

The content of [2, § 2] is summarized in the following

Theorem 5.3.3. *For* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{Z}^{2n \times 2n}$ *with blocks* $A, B, C, D \in \mathbb{Z}^{n \times n}$ *the following assertions are equivalent:*

i) $M \in \widehat{\Gamma}(T)$*,*

ii) $A^tTC, D^tTB \in \mathrm{Sym}_n(\mathbb{Z}), A^tTD - C^tTB = T$*,*

iii) $I_T M I_T^{-1} \in \mathrm{Sp}_n(\mathbb{Q})$*, where* $I_T := \mathrm{diag}(I_n, T)$*,*

iv) $\widetilde{I_T} M \widetilde{I_T}^{-1} \in \mathrm{Sp}_n(\mathbb{Q})$*, where* $\widetilde{I_T} := \mathrm{diag}(T, I_n) = J_n^{-1} I_T J_n$*,*

v) $\left(\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right)^{-1} M^t \left(\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right) \in \widehat{\Gamma}(T)$*.*

In this case, one has

$$M^{-1} = \begin{pmatrix} T^{-1}D^tT & -T^{-1}B^tT \\ -T^{-1}C^tT & T^{-1}A^tT \end{pmatrix}.$$

In view of 5.3.3 iii) we have the following

Definition 5.3.4. *The group*

$$\Gamma(T) := I_T \widehat{\Gamma}(T) I_T^{-1} = \left\{ M \in \mathrm{Sp}_n(\mathbb{Q}) : I_T^{-1} M I_T \in \mathbb{Z}^{2n \times 2n} \right\}$$

is called the paramodular group *of polarization* T*. For* $t \in \mathbb{N}$ *we will simply write*

$$\Gamma(t) := \Gamma \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Remark 5.3.5. *a) In view of 5.3.3 iv) one has* $\widetilde{I_T} \widehat{\Gamma}(T) \widetilde{I_T}^{-1} = \Gamma(T)^t$*.*

b) $\Gamma(T)$ *and* $\Gamma(T)^t$ *are conjugate as subgroups of* $\mathrm{Sp}_n(\mathbb{Q})$*. Explicitly one has*

$$\Gamma(T)^t = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma(T) \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1}.$$

c) From b) we obtain, that the assignment

$$M \mapsto \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}^{-1} M^t \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$$

is a nontrivial involutive automorphism of $\Gamma(T)$.

Generators of $\widehat{\Gamma}(T)$ were determined in [13, Thm. 1.12]. A translation yields

Theorem 5.3.6. *The group* $\Gamma(T)$ *is generated by*

$$J_T := \begin{pmatrix} 0 & -T^{-1} \\ T & 0 \end{pmatrix}, \quad \begin{pmatrix} I_n & G_{ij} \\ 0 & I_n \end{pmatrix}, G_{ij} := \begin{cases} \frac{1}{t_i} \cdot I_{ii}, & i = j, \\ \frac{1}{t_i} \cdot (I_{ij} + I_{ji}), & i < j. \end{cases}$$

The paramodular group admits nontrivial discrete extensions in $\mathrm{Sp}_n(\mathbb{R})$. From [17, Satz 4′] we cite the following

Theorem 5.3.7. *The group*

$$\Gamma(T)^{\max} := \mathcal{N}_{\mathrm{Sp}_n(\mathbb{R})}(\Gamma(T))$$

is the maximal discrete extension *of* $\Gamma(T)$ *in* $\mathrm{Sp}_n(\mathbb{R})$.

5.4 Modular embeddings of paramodular groups

In the following let $\mathcal{O}$ always denote an order in $\mathbb{F}$ and $T = \mathrm{diag}(t_1, \ldots, t_n)$ an elementary divisor matrix.

In the spirit of [18] we give the following

Definition 5.4.1. *Let* $M \in \mathrm{U}_n(\mathbb{F})$. *We say that* M *is a* modular embedding *of* $\Gamma(T)$ *into* $\mathrm{U}_n(\mathcal{O})$, *if* $M^{-1}\Gamma(T)M \subseteq \mathrm{U}_n(\mathcal{O})$. *In this case, we define*

$$\Phi_M : \Gamma(T) \longrightarrow \mathrm{U}_n(\mathcal{O}), \quad \Phi_M(H) := M^{-1}HM, \quad H \in \Gamma(T).$$

Furthermore, we set

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) := \left\{ M \in \mathrm{U}_n(\mathbb{F}) : M^{-1}\Gamma(T)M \subseteq \mathrm{U}_n(\mathcal{O}) \right\}.$$

Some facts are contained in the following

Remarks 5.4.2. *a) The map*

$$\Gamma(T) \longrightarrow \Gamma\left(t_1^{-1}T\right), \quad H \mapsto \begin{pmatrix} \sqrt{t_1}I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}}I_n \end{pmatrix}^{-1} H \begin{pmatrix} \sqrt{t_1}I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}}I_n \end{pmatrix}$$

is an isomorphism of paramodular groups. Hence the map

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \longrightarrow \mathrm{Mod}\left(\Gamma\left(t_1^{-1}T\right), \mathrm{U}_n(\mathcal{O})\right), \quad M \mapsto \begin{pmatrix} \sqrt{t_1}I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}}I_n \end{pmatrix}^{-1} \cdot M$$

is a bijection.

b) Modular embeddings of $\Gamma(T)^t$ *are defined in the same manner of 5.4.1. Indeed, the map*

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \longrightarrow \mathrm{Mod}\left(\Gamma(T)^t, \mathrm{U}_n(\mathcal{O})\right), \quad M \mapsto \left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1} \end{smallmatrix}\right) M$$

is a bijection.

c) The group $\mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$ *acts on* $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ *via the assignment*

$$M \mapsto \varepsilon M R, \quad \varepsilon \in \mathbb{F}_1, R \in \mathrm{U}_n(\mathcal{O}).$$

Part c) of 5.4.2 gives rise to

Definition 5.4.3. *Let* $M, M' \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$*. We call* M, M' equivalent, *if there is* $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$ *such that* $M' = \varepsilon M R$*. In this case, we write* $M \sim M'$*. The equivalence class of* M *is denoted by* $[M]_\sim$ *and the set of equivalence classes by* $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})/\sim$*.*

Definition 5.4.4. *Let* $a_1, \ldots, a_n \in \mathcal{O}$*. The matrix* $A := \mathrm{diag}(a_1, a_2, \ldots, a_n)$ *is called an* $\mathcal{O}$-model *of* T*, if*

$$A\overline{A} = T \text{ and } a_i\big|_r a_{i+1} \text{ for } i = 1, \ldots, n-1.$$

We define a notion of equivalence for $\mathcal{O}$-models:

Definition 5.4.5. *Let* A, B *be* $\mathcal{O}$*-models of* T*. We call* A, B equivalent, *if there are* $\delta_1, \ldots, \delta_n, \varepsilon \in \mathcal{O}^\times$ *and*

$$B = \mathrm{diag}(\delta_1, \ldots, \delta_n) \cdot A \cdot \overline{\varepsilon},$$

i.e. $b_i = \delta_i a_i \overline{\varepsilon}$ *for* $i = 1, \ldots, n$*.*

In view of the constructing $\mathcal{O}$-models of T, we state the following easy

Lemma 5.4.6. *Let* $a_1, \ldots, a_n \in \mathcal{O}$ *and* $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$*. Then the following assertions are equivalent:*

i) A is an $\mathcal{O}$-model for T,

ii) there are $\gamma_1,\ldots,\gamma_n \in \mathcal{O}$ such that $\gamma_1 = 1$ and

$$a_k a_1^{-1} = \gamma_k \cdots \gamma_2, \quad N(\gamma_k) = t_k t_{k-1}^{-1} \text{ for } k = 2,\ldots,n.$$

In this case, $\gamma_1,\ldots,\gamma_n$ are uniquely determined by A and one has

$$\gamma_k = a_k a_{k-1}^{-1} \text{ for } k = 2,\ldots,n.$$

In [18] the set $\mathrm{Mod}(\Gamma(T)^t, \mathrm{U}_n(\mathfrak{o}_K))$ was studied and some prototype was introduced, which is also suitable in the noncommutative setting:

Proposition 5.4.7. *Let $a_1,\ldots,a_n \in \mathcal{O}$ and $A := \mathrm{diag}(a_1,\ldots,a_n)$. Define*

$$M_A := \begin{pmatrix} A^{-1} & 0 \\ 0 & \overline{A} \end{pmatrix} \in \mathrm{U}_n(\mathbb{F}).$$

Then the following assertions are equivalent:

i) A is an $\mathcal{O}$-model of T,

ii) $M_A \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

Proof. In view of 5.3.6 it suffices to show, that the condition

$$M_A^{-1} J_T M_A \in \mathrm{U}_n(\mathcal{O}) \text{ and } M_A^{-1}\begin{pmatrix} I_n & G_{ij} \\ 0 & I_n \end{pmatrix} M_A \in \mathrm{U}_n(\mathcal{O}), \quad 1 \le i \le j \le n,$$

is equivalent to A being an $\mathcal{O}$-model for T. The assertion

$$M_A^{-1} J_T M_A = \begin{pmatrix} 0 & -AT^{-1}\overline{A} \\ \overline{A}^{-1} T A^{-1} & 0 \end{pmatrix} \in \mathrm{U}_n(\mathcal{O})$$

is equivalent to $t_i | N(a_i)$ and $N(a_i) | t_i$, i.e. $N(a_i) = t_i$ for $i = 1,\ldots,n$. The assertion

$$M_A^{-1}\begin{pmatrix} I_n & G_{ij} \\ 0 & I_n \end{pmatrix} M_A \in \mathrm{U}_n(\mathcal{O}), \quad 1 \le i \le j \le n,$$

is equivalent to

$$\frac{1}{t_i} A I_{ji} \overline{A} = \frac{1}{t_i} a_j \overline{a}_i I_{ji} = a_j a_i^{-1} I_{ji} \in \mathcal{O}^{n\times n}, \quad 1 \le i \le j \le n,$$

i.e. to $a_i |_r a_j$ for $1 \le i \le j \le n$. □

Proposition 5.4.7 leads to the following

Definition 5.4.8. *Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. We say that M is of* principal type, *if $M = M_A$ for some $\mathcal{O}$-model A of T. The set of modular embeddings of principal type is denoted by* $\mathrm{PMod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

Remark 5.4.9. *Let T, T' be elementary divisor matrices with $\mathcal{O}$-models A resp. A'. Then AA' is an $\mathcal{O}$-model of TT' if and only if*

$$a_{i+1}a'_{i+1}{a'_i}^{-1}a_i^{-1} \in \mathcal{O}, \quad i = 1, \dots, n-1.$$

Note that this condition is trivially fulfilled for $\mathbb{F}\{\mathbb{R}, \mathbb{C}\}$, while in the case $\mathbb{F} = \mathbb{H}$ it is more restrictive.

In the proof of [18, Satz 2], Köhler gave an implicit classification of $\mathrm{Mod}(\Gamma(T)^t, \mathrm{U}_n(\mathfrak{o}_K))$ in the case, where K has class number one. Therefore it is reasonable to assume that $\mathcal{O}$ is a principal ideal domain. To this end we will analyse his proof carefully and ensure that the arguments used there also hold in the noncommutative case. As a by-product it turns out, provided $t_1 = 1$, that the embeddings of principal type contain a complete system of representatives of the equivalence classes of $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

We introduce coprime matrix pairs:

Definition 5.4.10. *Let $C, D \in \mathcal{O}^{n\times n}$ and $(C, D) \in \mathcal{O}^{n\times 2n}$ the corresponding* matrix pair.

a) (C, D) is called coprime, *if for all $V \in \mathbb{F}^{n\times n}$ one has $V(C, D) \in \mathcal{O}^{n\times 2n}$ if and only if $V \in \mathcal{O}^{n\times n}$.*

b) (C, D) is called hermitian, *if $C\overline{D}^t \in \mathrm{Her}_n(\mathcal{O})$.*

Remark 5.4.11. *It is easy to see, that the group $\mathrm{GL}_n(\mathcal{O}) \times \mathrm{GL}_{2n}(\mathcal{O})$ acts naturally on the set of coprime pairs (C, D) via $U(C, D)V^{-1}$ for $U \in \mathrm{GL}_n(\mathcal{O}), V \in \mathrm{GL}_{2n}(\mathcal{O})$.*

Proposition 5.4.12. *Let $\mathcal{O}$ be a principal ideal domain and $C, D \in \mathcal{O}^{n\times n}$. Then the following assertions hold:*

a) There is some coprime matrix pair $(C_1, D_1) \in \mathcal{O}^{n\times 2n}$ and some $G \in \mathcal{O}^{n\times n}$ such that $(C, D) = G(C_1, D_1)$.

b) (C, D) is the second block row of some $M \in \mathrm{U}_n(\mathcal{O})$ if and only if (C, D) is coprime and hermitian.

Proof. a) Due to [25] the elementary divisor theorem also holds for noncommutative principal ideal domains, i.e. we find $U \in \mathrm{GL}_n(\mathcal{O})$ and $V \in \mathrm{GL}_{2n}(\mathcal{O})$ such that

$$(C, D) = U(F, 0)V = UF(I_n, 0)V,$$

where $F = \mathrm{diag}(f_{11}, \dots, f_{nn}) \in \mathcal{O}^{n\times n}$ and f_{ii} is a two-sided divisor of f_{jj} for all $i \le j$. The claim follows then with $G := UF$ and $(C_1, D_1) := (I_n, 0)V$.

b) Assume that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{U}_n(\mathcal{O})$ holds for some $A, B \in \mathcal{O}^{n\times n}$. From 5.2.2 iv) we conclude $D\overline{A}^t - C\overline{B}^t = I_n$ and $C\overline{D}^t \in \mathrm{Her}_n(\mathcal{O})$. Hence, the pair (C, D) is coprime and hermitian. Conversely, using the elementary divisor theorem, we find $U \in \mathrm{GL}_n(\mathcal{O})$ and $V \in \mathrm{GL}_{2n}(\mathcal{O})$ such that

$$U(C, D)V = (C_1, 0), \quad C_1 = \mathrm{diag}(c_{11}, \ldots, c_{nn}), c_{ii} \neq 0.$$

Since $(C_1, 0)$ is necessarily coprime, we have $c_{ii} \in \mathcal{O}^\times$ for $i = 1, \ldots, n$. Hence we can assume $C_1 = I_n$. With the definition

$$\begin{pmatrix} X \\ Y \end{pmatrix} := V \begin{pmatrix} U \\ 0 \end{pmatrix}$$

we obtain

$$CX + DY = (C, D)\begin{pmatrix} X \\ Y \end{pmatrix} = (C, D)V\begin{pmatrix} U \\ 0 \end{pmatrix} = I_n.$$

Finally, we set

$$A := \overline{Y}^t + \overline{X}^t YC, \quad B := -\overline{X}^t + \overline{X}^t YD$$

and an explicit calculation shows that the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfies 5.2.2 iv). □

Now we are able to prove the main classification theorem for modular embeddings. The proof is adapted from [18]. The assertions in the theorem are formulated accordingly in order to fit into the new setting.

Theorem 5.4.13. *Let $\mathcal{O}$ be a principal ideal domain and $M \in \mathrm{U}_n(\mathbb{F})$. Then the following assertions are equivalent:*

i) $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$,

ii) there exist $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$ and some $\mathcal{O}$-model A for $t_1^{-1}T$ such that

$$\begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M = \varepsilon M_A R.$$

Proof. Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. After replacing M by $\begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M$ we can assume that $t_1 = 1$. Furthermore, after replacing M by $\left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1} \end{smallmatrix}\right) M$ we can assume that $M \in \mathrm{Mod}(\Gamma(T)^t, \mathrm{U}_n(\mathcal{O}))$, i.e.

$$M^{-1}\Gamma(T)^t M \subseteq \mathrm{U}_n(\mathcal{O}). \tag{1}$$

By [17, Hilfssatz 1′], the $\mathbb{Z}$-lattice $\widehat{\Gamma}(T)_{\mathbb{Z}}$ over $\mathbb{Z}$, generated by $\widehat{\Gamma}(T)$, is a free abelian group with basis $f_{kl} I_{kl}, 1 \leq k, l \leq 2n$, where

$$f_{k+n,l+n} := f_{k+n,l} := f_{k,l+n} := f_{k,l} := \begin{cases} 1, & \text{for } 1 \leq l \leq k \leq n, \\ t_l t_k^{-1}, & \text{for } 1 \leq k < l \leq n. \end{cases}$$

Hence we have

$$M^{-1}\widetilde{I_T}\widehat{\Gamma}(T)_{\mathbb{Z}}\widetilde{I_T}^{-1}M \subseteq \mathcal{O}^{2n\times 2n}. \tag{2}$$

Considering the generators $f_{kl}I_{kl}$ in (2), we obtain that the numbers

$$\begin{matrix} f_{kl}t_kt_l^{-1}\overline{d}_{k\mu}a_{l\nu} & f_{kl}t_kt_l^{-1}\overline{d}_{k\mu}b_{l\nu} & f_{kl}t_kt_l^{-1}\overline{c}_{k\mu}a_{l\nu} & f_{kl}t_kt_l^{-1}\overline{c}_{k\mu}b_{l\nu} \\ f_{kl}t_k\overline{d}_{k\mu}c_{l\nu} & f_{kl}t_k\overline{d}_{k\mu}d_{l\nu} & f_{kl}t_k\overline{c}_{k\mu}c_{l\nu} & f_{kl}t_k\overline{c}_{k\mu}d_{l\nu} \\ f_{kl}t_l^{-1}\overline{b}_{k\mu}a_{l\nu} & f_{kl}t_l^{-1}\overline{b}_{k\mu}b_{l\nu} & f_{kl}t_l^{-1}\overline{a}_{k\mu}a_{l\nu} & f_{kl}t_l^{-1}\overline{a}_{k\mu}b_{l\nu} \\ f_{kl}\overline{b}_{k\mu}c_{l\nu} & f_{kl}\overline{b}_{k\mu}d_{l\nu} & f_{kl}\overline{a}_{k\mu}c_{l\nu} & f_{kl}\overline{a}_{k\mu}d_{l\nu} \end{matrix} \tag{3}$$

belong to $\mathcal{O}$ for all $k,l,\mu,\nu \in \{1,\dots,n\}$. From $M \in \mathrm{U}_n(\mathbb{F})$ we obtain that there is at least one nonzero entry of M, say $\overline{\rho}$. Hence (3) implies, that the product of ρ with every entry of M belongs to $\mathcal{O}\mathbb{Q}$, i.e. we can write

$$M = \rho^{-1}M', \quad \rho \in \mathbb{F}^\times, N(\rho) \in \mathbb{Q}, M' \in \mathbb{Q}\cdot\mathcal{O}^{2n\times 2n}.$$

After multiplying ρ and M' by some suitable rational number, we can assume

$$M = \rho^{-1}M' = \rho^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad r^{-1} := N(\rho) \in \mathbb{N}$$

with blocks $A = (a_{kl}), B = (b_{kl}), C = (c_{kl}), D = (d_{kl}) \in \mathcal{O}^{n\times n}$. The matrix $\sqrt{r}\cdot\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\mathrm{U}_n(\mathbb{F})$. The identity

$$\overline{\rho^{-1}x}\rho^{-1}y = \overline{x}\overline{\rho^{-1}}\rho^{-1}y = r\overline{x}y,$$

valid for arbitrary $x, y \in \mathbb{F}$, shows that the numbers

$$\begin{matrix} rf_{kl}t_kt_l^{-1}\overline{d}_{k\mu}a_{l\nu} & rf_{kl}t_kt_l^{-1}\overline{d}_{k\mu}b_{l\nu} & rf_{kl}t_kt_l^{-1}\overline{c}_{k\mu}a_{l\nu} & rf_{kl}t_kt_l^{-1}\overline{c}_{k\mu}b_{l\nu} \\ rf_{kl}t_k\overline{d}_{k\mu}c_{l\nu} & rf_{kl}t_k\overline{d}_{k\mu}d_{l\nu} & rf_{kl}t_k\overline{c}_{k\mu}c_{l\nu} & rf_{kl}t_k\overline{c}_{k\mu}d_{l\nu} \\ rf_{kl}t_l^{-1}\overline{b}_{k\mu}a_{l\nu} & rf_{kl}t_l^{-1}\overline{b}_{k\mu}b_{l\nu} & rf_{kl}t_l^{-1}\overline{a}_{k\mu}a_{l\nu} & rf_{kl}t_l^{-1}\overline{a}_{k\mu}b_{l\nu} \\ rf_{kl}\overline{b}_{k\mu}c_{l\nu} & rf_{kl}\overline{b}_{k\mu}d_{l\nu} & rf_{kl}\overline{a}_{k\mu}c_{l\nu} & rf_{kl}\overline{a}_{k\mu}d_{l\nu} \end{matrix} \tag{4}$$

belong to $\mathcal{O}$ for all $k,l,\mu,\nu \in \{1,\dots,n\}$. Due to 5.4.12 a), we can write

$$(A,B) = G(A_1,B_1), \quad G \in \mathcal{O}^{n\times n}, (A_1,B_1) \in \mathcal{O}^{n\times 2n} \text{ coprime}.$$

Part b) of 5.4.12 yields the existence of $R_1 = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \in \mathrm{U}_n(\mathcal{O})$. The matrix MR_1^{-1} has the form $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$. After substitution of M by MR_1^{-1}, we can assume that M is of the form

$$M = \rho^{-1}\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad N(\rho) = \frac{1}{r} \in \mathbb{N}. \tag{5}$$

Inductively, the elementary divisor theorem yields the existence of $U \in \mathrm{GL}_n(\mathcal{O})$ such that AU is of lower triangular shape. Then

$$R_2 := \begin{pmatrix} U^{-1} & 0 \\ 0 & \overline{U}^t \end{pmatrix} \in \mathrm{U}_n(\mathcal{O}).$$

By construction, the A-block of MR_2^{-1} is of lower triangular shape and the D-block of MR_2^{-1} is of upper triangular shape, since $r\overline{A}^t D = I_n$. Now $r\overline{A}^t D = I_n$ implies

$$\overline{a_{kk}}d_{kk} = r^{-1}, \quad \text{i.e. } a_{kk}^{-1} = r\overline{d_{kk}} \quad \text{for } 1 \leq k \leq n. \tag{6}$$

From (6) and (4), we obtain

$$a_{kk}^{-1}a_{k\mu} = r\overline{d_{kk}}a_{k\mu} = rf_{kk}\overline{d}_{kk}a_{k\mu} \in \mathcal{O} \quad \text{for } 1 \leq \mu \leq k \leq n,$$

that is $a_{kk}\big|_l a_{k\mu}$ for $1 \leq \mu \leq k \leq n$. As a consequence,

$$U := \operatorname{diag}(a_{11}, \dots, a_{nn})^{-1}A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O}).$$

Hence

$$R_3 = \begin{pmatrix} U^{-1} & 0 \\ 0 & \overline{U}^t \end{pmatrix} \in \mathrm{U}_n(\mathcal{O})$$

and we observe that the corresponding A- and D-block of MR_3 are diagonal matrices. Thus we can assume A and D in (5) have diagonal form, say

$$A = \operatorname{diag}(a_{11}, \dots, a_{nn}), \quad D = \operatorname{diag}(d_{11}, \dots, d_{nn}),$$

which satisfy (6). From (4) we obtain

$$d_{\mu\mu}^{-1}c_{\mu\nu} = r\overline{a}_{\mu\mu}c_{\mu\nu} = rf_{\mu\mu}\overline{a}_{\mu\mu}c_{\mu\nu} \in \mathcal{O}.$$

This implies $H := -D^{-1}C \in \mathcal{O}^{n\times n}$, and $C\overline{D}^t \in \mathrm{Her}_n(\mathcal{O})$ implies $H \in \mathrm{Her}_n(\mathcal{O})$. Therefore

$$R_4 := \begin{pmatrix} I_n & 0 \\ H & I_n \end{pmatrix} \in \mathrm{U}_n(\mathcal{O}).$$

By construction, MR_4 is a diagonal matrix. Now (4) implies

$$rt_k^{-1}\overline{a}_{kk}a_{kk} \in \mathcal{O}, \quad (rt_k^{-1}\overline{a}_{kk}a_{kk})^{-1} = rt_k^{-1}\overline{d}_{kk}d_{kk} \in \mathcal{O}, \quad 1 \leq k \leq n.$$

Therefore, $N(a_{kk}) = r^{-1}t_k$. Especially, $N(a_{11}) = r^{-1}$. Again, (4) implies

$$a_{k-1,k-1}^{-1}a_{kk} = r\overline{d}_{k-1,k-1}a_{kk} = r\underbrace{f_{k-1,k}t_{k-1}t_k^{-1}}_{=1} d_{k-1,k-1}a_{kk} \in \mathcal{O},$$

thus $a_{k-1,k-1}|_l a_{k,k}$ for $k = 2, \dots, n$. Finally, we set $\varepsilon := \rho^{-1}a_{11}$. Then

$$N(\varepsilon) = N(\rho)^{-1}N(a_{11}) = 1, \text{ i.e. } \varepsilon \in \mathbb{F}_1.$$

Using the notation $a_k := a_{11}^{-1}a_{kk}$ for $k = 1, \dots, n$, we can write

$$M = \varepsilon \begin{pmatrix} \overline{A} & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A = \operatorname{diag}(1, a_2, \dots, a_n), N(a_k) = t_k, N(\varepsilon) = 1.$$

Recapitulating the proof, we have constructed

$$\varepsilon \in \mathbb{F}_1, A = \operatorname{diag}(1, a_2, \dots, a_n) \in \mathcal{O}^{n \times n} \text{ and } R \in \mathrm{U}_n(\mathcal{O})$$

such that

$$\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M = \varepsilon \begin{pmatrix} \overline{A} & 0 \\ 0 & A^{-1} \end{pmatrix} R,$$

i.e.

$$\begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M = \varepsilon M_A R$$

and A is an $\mathcal{O}$-model of $t_1^{-1}T$. □

A reformulation of 5.4.13 and the fact that the number of $\mathcal{O}$-models of T is finite yields:

Corollary 5.4.14. *Let $\mathcal{O}$ be a principal ideal domain and $M \in \operatorname{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. Then the following assertions hold:*

a) $\begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M$ *is equivalent to some modular embedding of principal type.*

b) The set of equivalence classes $\operatorname{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/_\sim$ *is finite.*

The dependency of T on some modular embedding M of $\Gamma(T)$ is revealed in the following

Corollary 5.4.15. *Let $\mathcal{O}$ be a principal ideal domain. Then the following assertions hold:*

a) $\operatorname{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \neq \emptyset$ *if and only if $t_1^{-1}T$ has an $\mathcal{O}$-model.*

b) Let T' be another elementary divisor matrix and $t_1 = t_1'$. Then

$$\operatorname{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \cap \operatorname{Mod}(\Gamma(T'), \mathrm{U}_n(\mathcal{O})) \neq \emptyset$$

implies $T = T'$.

Proof. a) Follows from 5.4.13,5.4.7 and 5.4.2.

b) Let $M \in \mathrm{U}_n(\mathbb{F})$ be a modular embedding of both $\Gamma(T)$ and $\Gamma(T')$. By 5.4.13,

$$\begin{pmatrix} \sqrt{t_1} I_n & 0 \\ 0 & \frac{1}{\sqrt{t_1}} I_n \end{pmatrix}^{-1} M \sim M_A$$

for some $\mathcal{O}$-model A of $t_1^{-1}T$. From $t_1 = t_1'$ and 5.4.7 we obtain, that A is also an $\mathcal{O}$-model of $t_1^{-1}T'$. As a consequence,

$$t_1^{-1}T = \overline{A}A = t_1^{-1}T',$$

i.e. $T = T'$. □

The equivalence relation between embeddings of principal type is characterized in the following

Theorem 5.4.16. *Let A and B be $\mathcal{O}$-models of $t_1^{-1}T$. Then the following assertions are equivalent:*

i) M_A and M_B are equivalent as modular embeddings,

ii) A and B are equivalent as $\mathcal{O}$-models.

Proof. Let $\varepsilon \in \mathbb{F}_1$. Then we have $R := M_B^{-1}\varepsilon M_A \in \mathrm{U}_n(\mathcal{O})$ if and only if

$$B\varepsilon A^{-1} = \mathrm{diag}(\delta_1, \dots, \delta_n) \in \mathrm{GL}_n(\mathcal{O})$$

for certain units $\delta_1, \dots, \delta_n \in \mathcal{O}^\times$. From $a_1, b_1 \in \mathcal{O}^\times$ we derive $\varepsilon \in \mathcal{O}^\times$. □

As a consequence of the classification theorem 5.4.13, 5.4.14 and 5.4.16 we can express the number of nonequivalent modular embeddings:

Corollary 5.4.17. *Let $\mathcal{O}$ be a principal domain. Then the number of nonequivalent modular embeddings of $\Gamma(T)$ into $\mathrm{U}_n(\mathcal{O})$ equals the number of orbits of the group ${\mathcal{O}^\times}^{(n+1)}$ on the set*

$$\{a \in \mathcal{O}^{(n)} : N(a_i) = t_i t_1^{-1}, i = 1, \dots, n\}$$

via the action

$$(a_1, \dots, a_n) \mapsto (\delta_1 a_1, \dots, \delta_n a_n)\overline{\varepsilon}, \quad (\delta_1, \delta_2, \dots, \delta_n, \varepsilon) \in {\mathcal{O}^\times}^{(n+1)}.$$

In view of [18], we give the following

Definition 5.4.18. *Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. We call M* maximal*, if*

$$\Gamma(T) = M\mathrm{U}_n(\mathcal{O})M^{-1} \cap \mathrm{Sp}_n(\mathbb{R}).$$

Remark 5.4.19. *Maximality of $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ only depends on its equivalence class.*

A characterization of maximality is explained in the following

Proposition 5.4.20. *Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. Then the following assertions are equivalent:*

i) M is a maximal modular embedding,

ii) for all $H \in \mathrm{Sp}_n(\mathbb{R})$ one has $H \in \Gamma(T)$ if and only if $M^{-1}HM \in \mathrm{U}_n(\mathcal{O})$,

iii) for all $\Gamma \leq \mathrm{Sp}_n(\mathbb{R})$ such that $M^{-1}\Gamma M \subseteq \mathrm{U}_n(\mathcal{O})$, one has $\Gamma \leq \Gamma(T)$.

In this case, $\Gamma(T)$ is the maximal subgroup of $\mathrm{Sp}_n(\mathbb{R})$, which embeds into $\mathrm{U}_n(\mathcal{O})$ via Φ_M.

From [18, pp. 75-76] we adapt a correspondence between maximal modular embeddings of principal type and primitive lattice embeddings $\mathbb{Z}(t_n t_1^{-1}) \longrightarrow (\mathcal{O}, N|_{\mathcal{O}})$.

Proposition 5.4.21. *Let $A = \mathrm{diag}(a_1, \ldots, a_n)$ be an $\mathcal{O}$-model for T. Then the following assertions are equivalent:*

i) M_A is a maximal embedding,

ii) $a_k a_1^{-1}$ is primitive for $k = 1, \ldots, n$,

iii) $a_n a_1^{-1}$ is primitive.

Corollary 5.4.22. *Let $\mathcal{O}$ be a principal ideal domain and $t_n t_1^{-1}$ squarefree. Then every embedding $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ is maximal.*

In order to study the effect of the maximal discrete extension $\Gamma(T)^{\max}$ on the set of modular embeddings of $\Gamma(T)$ we obtain the following

Theorem 5.4.23. *The following assertions hold:*

a) $\Gamma(T)^{\max}$ acts on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ by multiplication from the left.

b) The action defined in a) respects the equivalence relation on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ and hence induces an action of $\Gamma(T)^{\max}$ on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/_\sim$.

c) $\Gamma(T)$ lies in the kernel of the action defined in b) and this induces an action of $\Gamma(T)^{\max}/\Gamma(T)$ on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/_\sim$.

Proof. Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

a) Let $H \in \Gamma(T)^{\max} = \mathcal{N}_{\mathrm{Sp}_n(\mathbb{R})}(\Gamma(T))$. Then one has

$$(HM)^{-1}\Gamma(T)HM = M^{-1}H^{-1}\Gamma(T)HM = M^{-1}\Gamma(T)M \subseteq \mathrm{U}_n(\mathcal{O}),$$

i.e. $HM \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

b) $M' = \varepsilon MR$ for some $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$ implies $HM' = \varepsilon(HM)R$, since ε and H commute.

c) For $H \in \Gamma(T)$ one has $R := M^{-1}HM \in \mathrm{U}_n(\mathcal{O})$ by assumption on M. Hence, trivially $HM = MR$ and thus $HM \sim M$. □

Corollary 5.4.24. *Let $t_1 = 1$. If $\mathcal{O}$ is a principal ideal domain, then the action of $\Gamma(T)^{\max}/\Gamma(T)$ on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/_\sim$ induces an action of $\Gamma(T)^{\max}/\Gamma(T)$ on the set of equivalence classes of $\mathcal{O}$-models of T.*

Proof. Let $H \in \Gamma(T)^{\max}/\Gamma(T)$ and A be an $\mathcal{O}$-model for T. By 5.4.13, we find some $\mathcal{O}$-model B of T such that $HM_A \sim M_B$. Since H respects the equivalence relation, this induces an action on the equivalence classes of $\mathcal{O}$-models of T. □

We will determine this action in section 6.1 explicitly.

5.5 Pullback theory for modular forms

We develop a pullback theory for modular forms with respect to modular embeddings of paramodular groups.

First we briefly define the basic concepts and terms in the theory of hermitian and quaternionic modular forms. For a general reference confer [14].

Definition 5.5.1. *The* half-space *of degree n over $\mathbb{F}$ is the set*

$$\mathcal{H}_n(\mathbb{F}) = \left\{X + iY \in \mathbb{F}^{n\times n} \otimes_{\mathbb{R}} \mathbb{C} : X = \overline{X}^t, Y = \overline{Y}^t > 0\right\}.$$

The set $\mathcal{H}_n(\mathbb{C})$ is called the hermitian half-space *and $\mathcal{H}_n(\mathbb{H})$ is called the* quaternionic half-space.

Clearly, $\mathcal{H}_n(\mathbb{R}) = \mathcal{H}_n$ and $\mathcal{H}_1(\mathbb{F}) = \mathcal{H}$.

Definition 5.5.2. *Let $k \in \mathbb{Z}$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{U}_n(\mathbb{F})$, where we assume $k \equiv 0 \bmod 2$ if $\mathbb{F} = \mathbb{H}$.*

a) For $Z \in \mathcal{H}_n(\mathbb{F})$ we define

$$(\det M\{Z\})^k := \begin{cases} \det(CZ+D)^k, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ (\det(CZ+D)^\vee)^{k/2}, & \mathbb{F} = \mathbb{H}, \end{cases}$$

where $^\vee$ denotes the embedding $\mathbb{H}^{n\times n} \longrightarrow \mathbb{C}^{2n\times 2n}$ as in [14, I., § 2, p.14]. The map

$$(M, Z) \mapsto (\det M\{Z\})^k, \quad M \in \mathrm{U}_n(\mathbb{F}), Z \in \mathcal{H}_n(\mathbb{F})$$

defines a factor of automorphy for $\mathrm{U}_n(\mathbb{F})$, cf. [14, II. Theorem 1.7 and p.77f].

b) For $Z \in \mathcal{H}_n(\mathbb{F})$ we define $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$.

c) For a holomorphic function $F : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{C}$ we define $F\big|_k M$ pointwisely by

$$F\big|_k M(Z) := (\det M\{Z\})^{-k} \cdot F(M\langle Z\rangle), \quad Z \in \mathcal{H}_n(\mathbb{F}).$$

Again, we call $\big|_k$ the slash operator of weight k.

We introduce modular forms:

Definition 5.5.3. *Let $k \in \mathbb{Z}$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Let $\Gamma \leq \mathrm{U}_n(\mathcal{O})$ of finite index and ν an abelian character of Γ of finite order. A function $F : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{C}$ is called a* modular form *of weight k and degree n with respect to Γ and ν, if the following conditions are satisfied:*

i) F is holomorphic,

ii) $F\big|_k M = \nu(M)F$ for all $M \in \Gamma$, *iii) in the case $n = 1$, the functions $F|_k M$, $M \in \mathrm{U}_1(\mathcal{O})$, are bounded on each set*

$$\{z \in \mathcal{H} : y \geq \delta\}, \quad \delta > 0.$$

In this case, F is called a Siegel modular form *if $\mathbb{F} = \mathbb{R}$, a* hermitian modular form *if $\mathbb{F} = \mathbb{C}$ and a* quaternionic modular form *if $\mathbb{F} = \mathbb{H}$. The space of modular forms of weight k and degree n with respect to Γ and ν is denoted by $[\Gamma, k, \nu]$.*

For $S, Z \in \mathrm{Her}_n(\mathbb{F}) \otimes_{\mathbb{R}} \mathbb{C}$ we define $\mathrm{tr}(SZ) := \frac{1}{2}\mathrm{trace}(SZ + ZS) \in \mathbb{C}$, where in this case, trace denotes the usual matrix trace. Let $\mathrm{Her}_n^\sharp(\mathcal{O})$ denote the dual lattice of $\mathrm{Her}_n(\mathcal{O})$ with respect to the trace form, i.e.

$$\mathrm{Her}_n^\sharp(\mathcal{O}) = \{M \in \mathrm{Her}_n(\mathbb{F}) : \mathrm{tr}(SM) \in \mathbb{Z} \text{ for all } S \in \mathrm{Her}_n(\mathcal{O})\}.$$

From [14, III., §1, Thm. (1.2)] we cite the following

Theorem 5.5.4. *Let $k \in \mathbb{Z}$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Then every modular form $F \in [\mathrm{U}_n(\mathcal{O}), k]$ has a Fourier expansion of the form*

$$F(Z) = \sum_{0 \leq S \in \mathrm{Her}_n^\sharp(\mathcal{O})} \alpha_F(S) e^{2\pi i \mathrm{tr}(SZ)}, \quad Z \in \mathcal{H}_n(\mathbb{F}).$$

The Fourier series converges absolutely and uniformly in every domain

$$\{Z \in \mathcal{H}_n(\mathbb{F}) : Y \geq \delta I_n\}, \quad \delta > 0.$$

We introduce paramodular forms:

Definition 5.5.5. *Let $k \in \mathbb{Z}, \Gamma \leq \Gamma(T)^{\max}$ of finite index and ν an abelian character with respect to Γ of finite order. A function $F : \mathcal{H}_n \longrightarrow \mathbb{C}$ is called a* paramodular form *of weight k and polarization T with respect to Γ and ν, if the following conditions are satisfied:*

i) F is holomorphic,

ii) $F|_k M = \nu(M) f$ for all $M \in \Gamma$,

iii) in the case $n = 1$, the functions $F|_k M$, $M \in \Gamma(T)^{\max}$, are bounded on each set

$$\{z \in \mathcal{H} : y \geq \delta\}, \quad \delta > 0.$$

The space of paramodular forms of weight k with respect to Γ and ν is denoted by $[\Gamma, k, \nu]$.

Modular embeddings transform modular forms into paramodular forms:

Proposition 5.5.6. *a) Let $F \in [\mathrm{U}_n(\mathcal{O}), k, \nu]$. Then the map*

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \longrightarrow [\Gamma(T), k, \nu \circ \Phi_M], \quad M \mapsto \left(F\big|_k M^{-1}\right)\Big|_{\mathcal{H}_n}$$

is well-defined.

b) Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. Then the map

$$[\mathrm{U}_n(\mathcal{O}), k, \nu] \longrightarrow [\Gamma(T), k, \nu \circ \Phi_M], \quad F \mapsto \left(F\big|_k M^{-1}\right)\Big|_{\mathcal{H}_n}$$

is a homomorphism of the vector spaces.

Proof. This follows from the identity

$$(F\big|_k M^{-1})\big|_k H = \left(F\big|_k \Phi_M(H)\right)\Big|_k M^{-1} = (\nu \circ \Phi_M)(H) \cdot F\big|_k M^{-1}$$

for $H \in \Gamma(T)$ and $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. □

In view of the equivalence relation on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$, we motivate our definition of a pullback theory for modular forms with respect to modular embeddings in the following

Remark 5.5.7. *Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ and $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$. Then for every modular form $F \in [\mathrm{U}_n(\mathcal{O}), k]$ one has*

$$\left(F\big|_k (\varepsilon M R)^{-1}\right)\Big|_{\mathcal{H}_n} = \begin{cases} \varepsilon^{nk} \cdot \left(F\big|_k M^{-1}\right)\Big|_{\mathcal{H}_n}, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ \left(F\big|_k M^{-1}\right)\Big|_{\mathcal{H}_n}, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Hence, except the case $\mathbb{F} = \mathbb{H}$, the map

$$M \mapsto \left(F\big|_k M^{-1}\right)\Big|_{\mathcal{H}_n}$$

is not compatible with the equivalence relation on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$.

Definition 5.5.8. *Let $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$ and $k \equiv 0 \bmod 2$. For a holomorphic function $F : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{C}$, the* pullback *$F\big|_k[M] : \mathcal{H}_n \longrightarrow \mathbb{C}$ of F with respect to M and weight k is defined pointwisely by*

$$F\big|_k[M](Z) := \begin{cases} N(\det M^{-1}\{Z\})^{-k/2} \cdot F(M^{-1}\langle Z\rangle), & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ F\big|_k M^{-1}(Z), & \mathbb{F} = \mathbb{H}, \end{cases}$$

for $Z \in \mathcal{H}_n$. Here, the norm form N is extended to $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity.

Definition 5.5.9. *We define* $w_{\mathcal{O}} := |\mathcal{C}(\mathbb{F}) \cap \mathcal{O}^\times|$, *i.e.*

$$w_{\mathcal{O}} = \begin{cases} |\mathcal{O}^\times|, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ 2, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Lemma 5.5.10. *Let* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $k \equiv 0 \bmod 2$. *Assume that* $\mathrm{U}_n(\mathcal{O})$ *is standardly generated. Then the two factors of automorphy*

$$(M, Z) \mapsto N(\det M\{Z\})^{k/2}, \quad (M, Z) \mapsto (\det M\{Z\})^k$$

coincide on $\mathrm{U}_n(\mathcal{O}) \times \mathcal{H}_n$.

Proof. Due to the multiplicativity of the norm form N, the map

$$(M, Z) \mapsto N(\det M\{Z\})^{k/2}$$

satisfies the cocycle condition. Hence it suffices to prove that for fixed $Z \in \mathcal{H}_n$ both maps coincide for matrices of the type

$$J_n, \quad \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}, S \in \mathrm{Her}_n(\mathcal{O}), \quad \begin{pmatrix} \overline{U}^t & 0 \\ 0 & U^{-1} \end{pmatrix}, U \in \mathrm{GL}_n(\mathcal{O}).$$

For the translations this is obvious and for the inversion J_n it follows from the identity

$$N(\det Z)^{k/2} = \left((\det Z)^2\right)^{k/2} = (\det Z)^k.$$

For the rotations, we obtain

$$N\left(\det U^{-1}\right)^{k/2} = N(\det U)^{-k/2} = 1 = (\det U)^{-k} = \left(\det U^{-1}\right)^k,$$

since $\det U \in \mathcal{O}^\times = \{u \in \mathcal{O} : N(u) = 1\}$ and k is divisible by $|\mathcal{O}^\times|$. □

Now we are in charge to prove the following

Theorem 5.5.11. *Let* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *where in the case* $\mathbb{F} = \mathbb{C}$ *the group* $\mathrm{U}_n(\mathcal{O})$ *is assumed to be standardly generated. Let* $k \equiv 0 \bmod w_{\mathcal{O}}$. *Then the following assertions hold:*

a) Let $F \in [\mathrm{U}_n(\mathcal{O}), k, \nu]$. *Then the map*

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O})) \longrightarrow [\Gamma(T), k, \nu \circ \Phi_M], \quad M \mapsto F\big|_k[M],$$

is well-defined.

b) Let $F \in [\mathrm{U}_n(\mathcal{O}), k]$. *Then* $M \sim N$ *implies* $F\big|_k[M] = F\big|_k[N]$. *In other words, the map*

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/\sim \;\longrightarrow [\Gamma(T), k], \quad [M]_\sim \mapsto F\big|_k[M],$$

is well-defined.

Proof. There is nothing to prove in the case $\mathbb{F} = \mathbb{H}$.

a) Let $H \in \Gamma(T), R := M^{-1}HM \in \mathrm{U}_n(\mathcal{O})$ and $Z \in \mathcal{H}_n$. From 5.5.10 we obtain

$$\begin{aligned}
F\big|_k[M]\big|_k H(Z) &= \det H\{Z\}^{-k} \cdot F\big|_k[M](H\langle Z\rangle) \\
&= \det H\{Z\}^{-k} \cdot N(\det M^{-1}\{H\langle Z\rangle\})^{-k/2} \cdot F(M^{-1}H\langle Z\rangle) \\
&= N(\det H\{Z\})^{-k/2} \cdot N(\det M^{-1}\{H\langle Z\rangle\})^{-k/2} \cdot F(M^{-1}H\langle Z\rangle) \\
&= N(\det M^{-1}H\{Z\})^{-k/2} \cdot F(M^{-1}H\langle Z\rangle) \\
&= N(\det RM^{-1}\{Z\})^{-k/2} \cdot F(RM^{-1}\langle Z\rangle) \\
&= N(\det R\{M^{-1}\langle Z\rangle\})^{-k/2} \cdot N(\det M^{-1}\{Z\})^{-k/2} \cdot F(RM^{-1}\langle Z\rangle) \\
&= \det R\{M^{-1}\langle Z\rangle\}^{-k} \cdot N(\det M^{-1}\{Z\})^{-k/2} \cdot F(R\langle M^{-1}\langle Z\rangle\rangle) \\
&= (F\big|_k R)\big|_k[M](Z) \\
&= (\nu \circ \Phi_M)(H) \cdot F\big|_k[M].
\end{aligned}$$

b) Let $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_n(\mathcal{O})$ and $Z \in \mathcal{H}_n$. From 5.5.10 and $\varepsilon^{-1}\langle Z\rangle = Z$ we obtain

$$\begin{aligned}
F\big|_k[\varepsilon MR](Z) &= N(\det R^{-1}M^{-1}\varepsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\varepsilon^{-1}\langle Z\rangle) \\
&= N(\det R^{-1}\{M^{-1}\varepsilon^{-1}\langle Z\rangle\} \cdot \det M^{-1}\varepsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\varepsilon^{-1}\langle Z\rangle) \\
&= N(\det R^{-1}\{M^{-1}\langle Z\rangle\} \cdot \det M^{-1}\varepsilon^{-1}\{Z\})^{-k/2} \cdot F(R^{-1}M^{-1}\langle Z\rangle) \\
&= N(\det M^{-1}\varepsilon^{-1}\{Z\})^{-k/2} \cdot \Big(N(\det R^{-1}\{M^{-1}\langle Z\rangle\})^{-k/2} \cdot F(R^{-1}M^{-1}\langle Z\rangle)\Big) \\
&= N(\det M^{-1}\{\varepsilon^{-1}\langle Z\rangle\})^{-k/2} \cdot N(\det \varepsilon^{-1}I_n\{Z\})^{-k/2} \cdot F\big|_k R(M^{-1}\langle Z\rangle) \\
&= N(\varepsilon^{-n})^{-k/2} \cdot N(\det M^{-1}\{Z\})^{-k/2} \cdot F\big|_k R(M^{-1}\langle Z\rangle) \\
&= N(\varepsilon)^{nk/2} \cdot (F\big|_k R)\big|_k[M] \\
&= F\big|_k[M].
\end{aligned}$$
□

In view of the construction of paramodular forms with respect to the maximal discrete extension of $\Gamma(T)$ we obtain the following corollary by a simple averaging argument:

Corollary 5.5.12. *Let $\mathcal{O}$ be a principal domain and $k \equiv 0 \bmod w_{\mathcal{O}}$. Then one has*

$$\sum_{[M]_\sim \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/\sim} F\big|_k[M] \in [\Gamma(T)^{\max}, k].$$

for all $F \in [\mathrm{U}_n(\mathcal{O}), k]$.

Proof. Follows directly from 5.5.11, 5.4.23, 5.4.14. □

Example 5.5.13. *Let A be an $\mathcal{O}$-model for T. Then one has $M_A^{-1}\langle Z\rangle = AZ\overline{A} = Z[\overline{A}]$ for $Z \in \mathcal{H}_n$. Evaluation of the corresponding factor of automorphy yields in both cases*

$$(\det T)^{k/2} = \begin{cases} N(\det M_A^{-1}\{Z\})^{-k/2}, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ (\det M_A^{-1}\{Z\})^{-k}, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Thus for a holomorphic function $F : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{C}$ *and* $k \in 2\mathbb{Z}$ *we obtain*

$$F\big|_k[M_A](Z) = (\det T)^{k/2} \cdot F(AZ\overline{A}), \quad Z \in \mathcal{H}_n.$$

Definition 5.5.14. *Let* A *be an* $\mathcal{O}$*-model for* T*. For a holomorphic function* $F : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{C}$ *we define the* pullback $F[A] : \mathcal{H}_n \longrightarrow \mathbb{C}$ *with respect to* A *pointwisely by*

$$F[A](Z) := F(AZ\overline{A}) = F(Z[\overline{A}]), \quad Z \in \mathcal{H}_n.$$

6 Separation Theorems for Modular Embeddings of Degree 2

In this chapter we will restrict to the degree two case only and assume that the elementary divisor matrix T is given by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \quad t \in \mathbb{N}.$$

Up to equivalence every $\mathcal{O}$-model A of T has the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \overline{a} \end{pmatrix}, \quad a \in \mathcal{O}, N(a) = t.$$

By a slight abuse of notation, we define

$$M_a := M_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \overline{a} \end{smallmatrix}\right)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \overline{a}^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \in \mathrm{PMod}(\Gamma(t), \mathrm{U}_2(\mathcal{O})).$$

Definition 6.0.15. *Let $a \in \mathcal{O}$. For a holomorphic function $F : \mathcal{H}_2(\mathbb{F}) \longrightarrow \mathbb{C}$ we define the function $F[a] : \mathcal{H}_2 \longrightarrow \mathbb{C}$ pointwisely by*

$$F[a](Z) := F\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & \overline{a} \end{smallmatrix}\right)\right](Z) = F\begin{pmatrix} z & aw \\ \overline{a}w & N(a)z' \end{pmatrix}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.$$

We call $F[a]$ the pullback *of F with respect to a.*

6.1 Modular embeddings and maximal discrete extensions

In 5.4.23 and 5.4.24 we have seen that $\Gamma(T)^{\max}/\Gamma(T)$ acts naturally on the set of equivalence classes $\mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))/_\sim$, which induces an action of $\Gamma(T)^{\max}/\Gamma(T)$ on the set of equivalence classes of $\mathcal{O}$-models for T, if we assume $\mathcal{O}$ to be a principal ideal domain. In this section, we will determine an explicit description of this action in the case $n = 2$. Suprisingly, the result is deeply connected to factorization theory in the order $\mathcal{O}$.

To proceed, we briefly develop factorization theory in non-necessary commutative orders.

Definition 6.1.1. *We say that $\mathcal{O}$ has the* exact factorization lifting property, *if for every $a \in \mathcal{O}$ and every exact divisor $d||N(a)$, there are $\alpha, \beta \in \mathcal{O}$ such that $N(\alpha) = d$ and $a = \alpha\beta$.*

The existence of such orders is shown in the following

Proposition 6.1.2. *Let $\mathcal{O}$ be a principal ideal domain. Then $\mathcal{O}$ has the exact factorization lifting property.*

Proof. For $a \in \mathcal{O}$ consider the right ideal $a\mathcal{O} + d\mathcal{O}$. Since $\mathcal{O}$ is a principal ideal domain, there is $\alpha \in \mathcal{O}$ such that $\alpha\mathcal{O} = a\mathcal{O} + d\mathcal{O}$. Let $\beta, \gamma \in \mathcal{O}$ such that $a = \alpha\beta$ and $d = \alpha\gamma$. Then

$$N(\alpha)|d^2, \quad N(\alpha)|N(a) = d \cdot \frac{t}{d}, \quad \gcd\left(d, \frac{t}{d}\right) = 1$$

imply $N(\alpha)|d$. Conversely,

$$\alpha = ax + dy \qquad \text{for some } x, y \in \mathcal{O},$$
$$N(\alpha) = N(a)N(x) + d \cdot \mathrm{tr}(\overline{ax}y) + d^2 N(y)$$

imply $d|N(\alpha)$ and $N(\alpha) > 0$ yields $N(\alpha) = d$. Then $N(\beta) = \frac{t}{d}$ follows necessarily. $\square$

Example 6.1.3. $\Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ *has the exact factorization property.*

Proof. Let $\Lambda := \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. Let $\mathcal{O}$ denote the Hurwitz order and $\wp$ the ideal of even quaternions. Let $a \in \Lambda$. Since $\mathcal{O}$ is a principal ideal domain, it suffices to show, that every factorization $a = \alpha\beta$ with $\alpha, \beta \in \mathcal{O}$ can be migrated to a factorization in Λ. In the case $a \in \wp$ we can assume $\alpha \in \wp$, since $\wp$ is a prime ideal. Furthermore, there is $\varepsilon \in \{1, \omega, \overline{\omega}\}$ such that $\varepsilon\beta \in \Lambda$. The claim follows then from $\alpha\overline{\varepsilon} \in \wp \subseteq \Lambda$. In the remaining case $a \notin \wp$, we have $a \equiv 1 \bmod \wp$, since $a \in \Lambda$. This implies $\alpha \equiv \overline{\beta} \bmod \wp$ and hence that there is some $\varepsilon \in \{1, \omega, \overline{\omega}\}$ such that $\alpha\varepsilon, \overline{\varepsilon}\beta \in \Lambda$. $\square$

In view of uniqueness, we give the following

Definition 6.1.4. *Let $\alpha, \beta \in \mathcal{O}$. The product $\alpha\beta$ is called* unique up to unit-migration, *if*

$$\alpha\beta = \alpha'\beta', \quad N(\alpha) = N(\alpha'), \quad N(\beta) = N(\beta'),$$

for some $\alpha', \beta' \in \mathcal{O}$ implies $\alpha' = \alpha\varepsilon$ and $\beta' = \overline{\varepsilon}\beta$ for some unit $\varepsilon \in \mathcal{O}^\times$.

Lemma 6.1.5. *Let $\alpha, \beta \in \mathcal{O}$ such that $\gcd(N(\alpha), N(\beta)) = 1$. Then $\alpha\beta$ is unique up to unit-migration.*

Proof. First we prove that for $\alpha, \beta \in \mathcal{O}$ and $m \in \mathbb{N}$, the conditions

$$m|\alpha\beta, \quad \gcd(m, N(\alpha)) = 1,$$

already imply $m|\beta$. But this follows from

$$\begin{aligned} &rN(\alpha) + sm = r\overline{\alpha}\alpha + sm = 1 \quad \text{for some } r, s \in \mathbb{Z}, \\ &r\overline{\alpha}\alpha\beta + sm\beta = \beta. \end{aligned}$$

For the rest of the proof assume that

$$\alpha\beta = \alpha_1\beta_1, \quad N(\alpha_1) = N(\alpha), \quad N(\beta_1) = N(\beta),$$

for some $\alpha_1, \beta_1 \in \mathcal{O}$. Then the assertions

$$\alpha\beta = \alpha_1\beta_1, \quad \alpha N(\beta) = \alpha\beta\overline{\beta} = \alpha_1\beta_1\overline{\beta}$$

imply $N(\beta)|\alpha_1\beta_1\overline{\beta}$. From $\gcd(N(\beta), N(\alpha_1)) = 1$ we obtain $N(\beta)|\beta_1\overline{\beta}$ and hence the existence of $\varepsilon \in \mathcal{O}$ such that

$$\varepsilon N(\beta) = \varepsilon\beta\overline{\beta} = \beta_1\overline{\beta}, \varepsilon\beta = \beta_1.$$

From $N(\beta) = N(\beta_1)$ we obtain $N(\varepsilon) = 1$, i.e. $\varepsilon \in \mathcal{O}^\times$ and $\alpha_1 = \alpha\overline{\varepsilon}$. □

Lemma 6.1.6. *Let $\alpha, \alpha_1, \beta \in \mathcal{O}$ and $N(\alpha) = N(\alpha_1)$. Then one has $\alpha_1\big|_r\alpha\beta$ if and only if $\alpha\big|_r\alpha_1\overline{\beta}$.*

Proof. Let $\gamma \in \mathcal{O}$. Then one has $\alpha\beta = \gamma\alpha_1$ if and only if $\alpha = \gamma\alpha_1\beta^{-1}$ if and only if $\overline{\gamma}\alpha = N(\gamma)\alpha_1\beta^{-1} = \alpha_1\overline{\beta}$, since $N(\gamma) = N(\beta) = \overline{\beta}\beta$. □

From [10] we cite

Theorem 6.1.7. *Let $t \in \mathbb{N}$ with prime factor decomposition $t = \prod_{p|t} p^{\nu_p(t)}$. For every exact divisor $d||t$ let $t_d := \frac{t}{d}$ denote its complementary divisor. Choose $x, y \in \mathbb{Z}$ such that $xd - yt_d = 1$ and define matrices $V_d \in \mathrm{Sp}_2(\mathbb{R})$ by*

$$V_d := \begin{pmatrix} U_d & 0 \\ 0 & U_d^{-t} \end{pmatrix}, \quad U_d := \frac{1}{\sqrt{d}} \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then $\Gamma(t)^* := \langle \Gamma(t), V_d, d||t \rangle \leq \mathrm{Sp}_2(\mathbb{R})$ is a discrete extension of $\Gamma(t)$ of index $2^{\nu(t)}$, where $\nu(t)$ denotes the number of distinct prime divisors of t. If t is squarefree, one has

$$\Gamma(t)^{\max} = \langle \Gamma(t), V_d, d||t \rangle .$$

Now we are in charge to describe to action of $V_d, d||t$ explicitly:

Theorem 6.1.8. *Let $\alpha, \beta \in \mathcal{O}, t := N(\alpha\beta)$ and $d := N(\alpha)$. Suppose that $d||t$ and that there are $\alpha', \beta' \in \mathcal{O}$ such that*

$$\alpha\beta = \beta'\alpha', \quad N(\alpha) = N(\alpha'), \quad N(\beta) = N(\beta').$$

Then one has

$$V_d \cdot [M_{\alpha\beta}]_\sim = [M_{\overline{\alpha}\beta'}]_\sim.$$

Furthermore, the following special cases hold:

i) $V_t \cdot [M_{\alpha\beta}]_\sim = [M_{\overline{\beta\overline{\alpha}}}]_\sim$,

ii) $V_d \cdot [M_{\alpha\beta}]_\sim = [M_{\overline{\alpha}\beta}]_\sim$ *, if* $\beta^{-1}\alpha\beta \in \mathcal{O}$,

iii) $V_d \cdot [M_{\alpha\beta}]_\sim = [M_{\beta\overline{\alpha}}]_\sim$ *, if* $\alpha\beta\alpha^{-1} \in \mathcal{O}$,

iv) if $\beta^{-1}\alpha\beta \in \mathcal{O}$ *and* $\beta^{-1}\mathcal{O}^\times\beta = \mathcal{O}^\times$*, then one has* $V_d \cdot [M_{\alpha\beta}]_\sim = [M_{\alpha\beta}]_\sim$*, if and only if* α *and* $\overline{\alpha}$ *are associated.*

Proof. Let

$$U := \begin{pmatrix} x\overline{\alpha} & -\beta \\ -y\overline{\beta'} & (\overline{\beta'}\alpha)\overline{\beta}^{-1} \end{pmatrix}, \quad V := \begin{pmatrix} \alpha & \beta' \\ y\overline{\beta} & x(\overline{\beta\overline{\alpha}})\overline{\beta'}^{-1} \end{pmatrix}.$$

From $\beta'\big|_l \alpha\beta$ we obtain $\overline{\beta'}\big|_r \overline{\beta}\overline{\alpha}$ and 6.1.6 implies $\overline{\beta}\big|_r \overline{\beta'}\alpha$. Hence, $U, V \in \mathcal{O}^{2\times 2}$. The assumptions $xd - y\frac{t}{d} = 1, N(\alpha) = N(\alpha')$ and $N(\beta) = N(\beta')$ imply

$$UV = VU = I_2$$

by an explicit calculation. Hence, $U, V \in \mathrm{GL}_2(\mathcal{O})$ and $U^{-1} = V$. We define $\varepsilon := \overline{\alpha}/\sqrt{d}$. Then $N(\varepsilon) = 1$ and from the four identities

- $\sqrt{d}x\varepsilon = \sqrt{d}x\overline{\alpha}/\sqrt{d} = x\overline{\alpha}$,
- $-\frac{t}{\sqrt{d}}\varepsilon\overline{\alpha\beta}^{-1} = -\frac{t}{\sqrt{d}}\frac{\overline{\alpha}}{\sqrt{d}}\overline{\alpha}^{-1}\overline{\beta}^{-1} = -\frac{t}{d}\overline{\beta}^{-1} = -\beta$, since $N(\beta) = \frac{t}{d}$,
- $-\frac{y}{\sqrt{d}}\overline{\overline{\alpha}\beta'}\varepsilon = -\frac{y}{\sqrt{d}}(\overline{\beta'}\alpha)\frac{\overline{\alpha}}{\sqrt{d}} = -y\overline{\beta'}\frac{N(\alpha)}{d} = -y\overline{\beta'}$, since $N(\alpha) = d$,
- $\sqrt{d}(\overline{\overline{\alpha}\beta'})\varepsilon(\overline{\alpha\beta})^{-1} = \sqrt{d}(\overline{\beta'}\alpha)\frac{\overline{\alpha}}{\sqrt{d}}(\overline{\alpha}^{-1}\overline{\beta}^{-1}) = (\overline{\beta'}\alpha)\overline{\beta}^{-1}$

we derive that

$$\begin{pmatrix} 1 & 0 \\ 0 & \overline{\overline{\alpha}\beta'} \end{pmatrix} \varepsilon U_d \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha\beta}^{-1} \end{pmatrix} = \begin{pmatrix} x\overline{\alpha} & -\beta \\ -y\overline{\beta'} & (\overline{\beta'}\alpha)\overline{\beta}^{-1} \end{pmatrix} = U \in \mathrm{GL}_2(\mathcal{O}),$$

i.e.

$$\varepsilon U_d \begin{pmatrix} 1 & 0 \\ 0 & \overline{\alpha\beta}^{-1} \end{pmatrix} U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \overline{\overline{\alpha}\beta'}^{-1} \end{pmatrix}.$$

From this we obtain

$$\varepsilon V_d M_{\alpha\beta} \begin{pmatrix} U^{-1} & 0 \\ 0 & \overline{U}^t \end{pmatrix} = M_{\overline{\alpha}\beta'}, \quad \begin{pmatrix} U^{-1} & 0 \\ 0 & \overline{U}^t \end{pmatrix} \in \mathrm{U}_2(\mathcal{O}), \varepsilon \in \mathbb{F}_1.$$

Hence, $V_d M_{\alpha\beta}$ is equivalent to $M_{\overline{\alpha}\beta'}$, i.e. $V_d \cdot [M_{\alpha\beta}]_\sim = [M_{\overline{\alpha}\beta'}]_\sim$. For i) we choose $\beta' = \beta = 1$. For the proof of ii) write $\alpha\beta = \beta(\beta^{-1}\alpha\beta)$, i.e. we can choose $\beta' = \beta$. For the proof of iii) write $\alpha\beta = (\alpha\beta\alpha^{-1})\alpha$, i.e. we can choose $\beta' = \alpha\beta\alpha^{-1}$. Note that $\overline{\alpha}\alpha\beta\alpha^{-1} = \beta\overline{\alpha}$. Finally, to prove iv) note that in view of ii) one has $[M_{\alpha\beta}]_\sim = [M_{\overline{\alpha}\beta}]_\sim$ if and only if $\varepsilon\alpha\beta\delta = \overline{\alpha}\beta$, i.e. $\varepsilon\alpha\beta\delta\beta^{-1} = \overline{\alpha}$ for certain units $\delta, \varepsilon \in \mathcal{O}^\times$. The claim follows then from $\beta\mathcal{O}^\times\beta^{-1} = \mathcal{O}^\times$. □

If $\mathcal{O}$ is a principal ideal domain, then action of $\Gamma(t)^{\max}/\Gamma(t)$ can be described as follows:

Corollary 6.1.9. *Let $\mathcal{O}$ be a principal ideal domain. Let $t \in \mathbb{N}$ squarefree and $M \in \mathrm{Mod}(\Gamma(t), \mathrm{U}_2(\mathcal{O}))$. According to 5.4.13 let $a \in \mathcal{O}$, $N(a) = t$ such that $M \sim M_a$. Let $d||t$ and $\alpha, \alpha', \beta, \beta' \in \mathcal{O}$ such that $\alpha\beta = a = \beta'\alpha'$ and $N(\alpha) = d = N(\alpha')$. Then one has*

$$V_d \cdot [M]_\sim = [M_{\overline{\alpha}\beta'}]_\sim$$

and this does only depend on d and not on the choice of $a, \alpha, \alpha', \beta, \beta'$.

From 6.1.8 we obtain the behaviour of pullbacks under the matrices V_d:

Theorem 6.1.10. *Let $\alpha, \beta \in \mathcal{O}, t := N(\alpha\beta)$ and $d := N(\alpha)$. Suppose that $d||t$ and that there are $\alpha', \beta' \in \mathcal{O}$ such that*

$$\alpha\beta = \beta'\alpha', \quad N(\alpha) = N(\alpha'), \quad N(\beta) = N(\beta').$$

Let $k \equiv 0 \bmod w_{\mathcal{O}}$. Then for every $F \in [\mathrm{U}_2(\mathcal{O}), k]$ one has

$$F[\alpha\beta]\big|_k V_d = F[\overline{\alpha}\beta'].$$

Furthermore, the following special cases hold:

i) $F[\alpha\beta]\big|_k V_t = F[\overline{\beta}\overline{\alpha}]$,

ii) $F[\alpha\beta]\big|_k V_d = F[\overline{\alpha}\beta]$, *if* $\beta^{-1}\alpha\beta \in \mathcal{O}$,

iii) $F[\alpha\beta]\big|_k V_d = F[\beta\overline{\alpha}]$, *if* $\alpha\beta\alpha^{-1} \in \mathcal{O}$,

iv) if $\beta^{-1}\alpha\beta \in \mathcal{O}, \beta^{-1}\mathcal{O}^\times\beta = \mathcal{O}^\times$ *and* $\alpha, \overline{\alpha}$ *are associated, then one has* $F[\alpha\beta]\big|_k V_d = F[\alpha\beta]$.

6.2 Equivalence in the extended sense

In this section we introduce an extension of the projective modular group $\mathrm{PU}_2(\mathcal{O})$ by certain biholomorphic automorphisms of the half-space $\mathcal{H}_2(\mathbb{F})$, that are induced by elements of the orthogonal group of the positive definite lattice $(\mathcal{O}, N|_{\mathcal{O}})$.

Remark 6.2.1. *It is well known, that there is a monomorphism of groups*

$$\mathrm{PU}_2(\mathbb{F}) := \mathrm{U}_2(\mathbb{F})/\mathcal{C}(\mathbb{F}_1)\cdot I_4 \longrightarrow \mathrm{Bih}\,\mathcal{H}_2(\mathbb{F})$$

induced by the fractional-linear action of $\mathrm{U}_2(\mathbb{F})$ *on* $\mathcal{H}_2(\mathbb{F})$. *For sake of simplicity, we will identify* $M \in \mathrm{U}_2(\mathbb{F})$ *with its image in* $\mathrm{PU}_2(\mathbb{F})$, *whenever it is convenient. Due to 5.1.14, every* $\sigma \in \mathrm{O}(\mathbb{F})$ *is either of the form*

$$w \mapsto \sigma(w) = \varepsilon w \delta \text{ or } w \mapsto \sigma(w) = \varepsilon \overline{w} \delta, \quad \varepsilon, \delta \in \mathbb{F}_1.$$

For $\varepsilon, \delta \in \mathbb{F}_1$ *set* $U_{\varepsilon,\delta} := \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \delta \end{pmatrix}$. *Then for* $Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F})$ *one has*

$$\begin{pmatrix} \overline{U_{\varepsilon,\delta}}^t & 0 \\ 0 & U_{\varepsilon,\delta}^{-1} \end{pmatrix} \langle Z \rangle = Z[U_{\varepsilon,\delta}] = \begin{pmatrix} z & \sigma(w) \\ \overline{\sigma(w)} & z' \end{pmatrix}$$

in the first case, as well as

$$\left(\begin{pmatrix} \overline{U_{\varepsilon,\delta}}^t & 0 \\ 0 & U_{\varepsilon,\delta}^{-1} \end{pmatrix} \circ I_{\mathrm{tr}} \right) \langle Z \rangle = Z^t[U_{\varepsilon,\delta}] = \begin{pmatrix} z & \sigma(w) \\ \overline{\sigma(w)} & z' \end{pmatrix}$$

in the second case. Here, $I_{\mathrm{tr}} : Z \mapsto Z^t$ *denotes the* exceptional automorphism *of* $\mathcal{H}_2(\mathbb{F})$ *in the case* $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. *Hence we obtain a monomorphism of groups*

$$\mathrm{O}(\mathbb{F}) \longrightarrow \mathrm{Bih}\,\mathcal{H}_2(\mathbb{F}).$$

Thereby we will identify $\mathrm{O}(\mathbb{F})$ *as a subgroup of* $\mathrm{Bih}\,\mathcal{H}_2(\mathbb{F})$. *Indeed,*

$$\mathrm{Bih}\,\mathcal{H}_2(\mathbb{F}) \cong \begin{cases} \mathrm{PSp}_2(\mathbb{R}), & \mathbb{F} = \mathbb{R}, \\ \mathrm{PU}_2(\mathbb{F}) \rtimes \langle I_{\mathrm{tr}} \rangle, & \mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}. \end{cases}$$

For more details confer [14, II., § 1].

It is easy to see, that the group $\mathrm{U}_2(\mathcal{O})$ is normalized by the matrices $\frac{u}{\sqrt{N(u)}} I_4 \in \mathrm{U}_2(\mathbb{F})$ for $u \in \mathcal{I}(\mathcal{O})$. This gives rise to the following

Definition 6.2.2. *The* special modular group $\mathrm{U}_2^*(\mathcal{O})$ *is defined as*

$$\mathrm{U}_2^*(\mathcal{O}) := \left\langle \mathrm{U}_2(\mathcal{O}), \frac{u}{\sqrt{N(u)}} I_4 : u \in \mathcal{I}(\mathcal{O}) \right\rangle \leq \mathrm{U}_2(\mathbb{F}).$$

We give some trivial

Remarks 6.2.3. *a) By construction,* $\mathrm{U}_2(\mathcal{O})$ *is a normal subgroup of* $\mathrm{U}_2^*(\mathcal{O})$. *Furthermore,*

$$\mathrm{U}_2^*(\mathcal{O}) = \mathrm{U}_2(\mathcal{O}) \cdot \left\{ \frac{u}{\sqrt{N(u)}} I_4 : u \in \mathcal{I}(\mathcal{O}) \right\}.$$

b) If $\mathbb{F} = \mathbb{C}$*, then one has* $\mathrm{U}_2^*(\mathcal{O}) = \mathrm{U}_2(\mathcal{O})$.

Definition 6.2.4. *a) The* projective modular group $\mathrm{PU}_2(\mathcal{O})$ *is defined as*

$$\mathrm{PU}_2(\mathcal{O}) := \mathrm{U}_2(\mathcal{O})/(\mathcal{C}(\mathbb{F}_1) \cap \mathcal{O}) \cdot I_4 = \begin{cases} \mathrm{U}_2(\mathcal{O})/\mathcal{O}^\times I_4, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ \mathrm{U}_2(\mathcal{O})/\{\pm I_4\}, & \mathbb{F} = \mathbb{H}. \end{cases}$$

b) The special projective modular group $\mathrm{PU}_2^*(\mathcal{O})$ *is defined as*

$$\mathrm{PU}_2^*(\mathcal{O}) := \mathrm{U}_2^*(\mathcal{O})/(\mathcal{C}(\mathbb{F}_1) \cap \mathcal{O}) \cdot I_4 = \begin{cases} \mathrm{U}_2^*(\mathcal{O})/\mathcal{O}^\times I_4, & \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, \\ \mathrm{U}_2^*(\mathcal{O})/\{\pm I_4\}, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Definition 6.2.5. *Let* $\mathcal{O}$ *be an order. Then the* extended modular group $\Gamma(\mathcal{O})$ *is defined as*

$$\Gamma(\mathcal{O}) := \left\langle \mathrm{PU}_2^*(\mathcal{O}), I_{\mathrm{tr}} \right\rangle = \langle I_{\mathrm{tr}}, Z \mapsto M\langle Z \rangle : M \in \mathrm{U}_2^*(\mathcal{O}) \rangle.$$

Remark 6.2.6. *From the explicit description of* $\mathrm{O}(\mathcal{O})$ *in* 5.1.16 *and the identification of* $\mathrm{O}(\mathcal{O})$ *in* $\mathrm{Bih}\,\mathcal{H}_2(\mathbb{F})$ *according to* 6.2.1 *we obtain*

$$\Gamma(\mathcal{O}) = \left\langle \mathrm{PU}_2(\mathcal{O}), \mathrm{O}(\mathcal{O}) \right\rangle.$$

Proposition 6.2.7. *Let* $\mathcal{O}$ *be an order. If* $\mathbb{F} = \mathbb{H}$ *assume additionally that* $\mathrm{U}_2(\mathcal{O})$ *is generated by the matrices*

$$J_2, \quad \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}, S \in \mathrm{Her}_2(\mathcal{O}), \quad \mathrm{diag}(\varepsilon, 1, \varepsilon, 1), \varepsilon \in \mathcal{O}^\times.$$

Then both $\mathrm{PU}_2(\mathcal{O})$ *and* $\mathrm{PU}_2^*(\mathcal{O})$ *are normal subgroups of* $\Gamma(\mathcal{O})$*. Furthermore, one has*

$$\Gamma(\mathcal{O}) = \begin{cases} \mathrm{PU}_2(\mathcal{O}) \rtimes \langle I_{\mathrm{tr}} \rangle, & \mathbb{F} = \mathbb{C}, \\ \mathrm{PU}_2^*(\mathcal{O}) \rtimes \langle I_{\mathrm{tr}} \rangle, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Proof. It suffices to show that $\mathrm{PU}_2(\mathcal{O})$ is normalized by I_{tr}. In the case $\mathbb{F} = \mathbb{C}$ this follows from the identity $M\langle Z^t \rangle^t = \overline{M}\langle Z \rangle$ for all $M \in \mathrm{U}_2(\mathbb{C})$ and $Z \in \mathcal{H}_2(\mathbb{C})$. In the case $\mathbb{F} = \mathbb{H}$ it suffices to demonstrate the properties on the generators quoted above. Trivially, I_{tr} fixes the maps $Z \mapsto u^{-1}Zu$ for $u \in \mathcal{I}(\mathcal{O})$. □

Examples 6.2.8. *a) Let* $\mathfrak{o}$ *be an order in* $\mathbb{C}$*. Then* $\Gamma(\mathfrak{o})$ *is an extension of index* 2 *of* $\mathrm{PU}_2(\mathfrak{o})$.

b) Let $\mathcal{O}$ *denote the Hurwitz order. Since* $\mathcal{O}$ *is norm-euclidean,* $\Gamma(\mathcal{O})$ *is a normal extension of index* 4 *of* $\mathrm{PU}_2(\mathcal{O})$ *and index* 2 *of* $\mathrm{PU}_2^*(\mathcal{O})$*. Explicitly one has*

$$\Gamma(\mathcal{O}) = \left(\mathrm{PU}_2(\mathcal{O}) \cdot \left\langle \frac{1 + i_1}{\sqrt{2}} \cdot I_4 \right\rangle \right) \rtimes \langle I_{\mathrm{tr}} \rangle.$$

c) Let K be an imaginary-quadratic number field of discriminant $-D_K$ *for* $D_K \in \{3,4\}$*. Then* $\Gamma(\mathcal{O}_K)$ *is a normal extension of* $\mathrm{PU}_2(\mathcal{O}_K)$ *of index*

$$[\Gamma(\mathcal{O}_K) : \mathrm{PU}_2(\mathcal{O}_K)] = \begin{cases} 4, & D_K = 3, \\ 12, & D_K = 4. \end{cases}$$

From the explicit description in 5.1.12 we obtain

$$\Gamma(\mathcal{O}_K) = \begin{cases} \left(\mathrm{PU}_2(\mathcal{O}_K) \cdot \langle i_1 I_4 \rangle\right) \rtimes \langle I_{\mathrm{tr}} \rangle, & D_K = 3, \\ \left(\mathrm{PU}_2(\mathcal{O}_K) \cdot \left\langle \omega I_4, \frac{1+i_1}{\sqrt{2}} I_4 \right\rangle\right) \rtimes \langle I_{\mathrm{tr}} \rangle, & D_K = 4. \end{cases}$$

Note that it can be shown by using a weak version of the euclidean algorithm that the order $\Lambda = \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ *satisfies the assumptions of 6.2.7*

In order to motivate definition 6.2.10, we give the following

Remark 6.2.9. *Let* $\mathcal{O}$ *be an order and assume that* $\mathrm{PU}_2(\mathcal{O})$ *is normalized by the automorphism* I_{tr}*. Let* $M \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$*. Then the element* $M^* := I_{\mathrm{tr}} M I_{\mathrm{tr}}$ *is determined in* $\mathrm{U}_2(\mathbb{F})$ *up to some multiplicative constant* $\varepsilon \in \mathcal{C}(\mathbb{F}_1)$*. Hence, the matrices*

$$M^{*-1} H M^*, \quad H \in \Gamma(T) \leq \mathrm{Sp}_2(\mathbb{R})$$

are uniquely determined in $\mathrm{U}_2(\mathbb{F})$*. Furthermore, we have*

$$M^{*-1} H M^* = I_{\mathrm{tr}} M^{-1} I_{\mathrm{tr}} H I_{\mathrm{tr}} M I_{\mathrm{tr}} = I_{\mathrm{tr}} M^{-1} H M I_{\mathrm{tr}} \in \mathrm{PU}_2(\mathcal{O}),$$

since $I_{\mathrm{tr}} H I_{\mathrm{tr}} = H$ *holds in* $\mathrm{PU}_2(\mathbb{F})$ *and* $M^{-1} H M \in \mathrm{U}_2(\mathcal{O})$*. Thus,* $M^{*-1} H M^*$ *is uniquely determined in* $\mathrm{U}_2(\mathcal{O})$*, i.e. every representative of* M^* *is an element of* $\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$*. In other words,* M^* *can be considered as an element of* $\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))/\sim$*.*

6.2.9 gives rise to the following

Definition 6.2.10. *Let* $\mathcal{O}$ *be an order and assume that* $\mathrm{PU}_2(\mathcal{O})$ *is normalized by* I_{tr}*. Let* $M, N \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$*. We call M and N equivalent in the* extended sense*, written* $M \sim_* N$*, if there is* $(\varepsilon, R) \in \mathbb{F}_1 \times \mathrm{U}_2^*(\mathcal{O})$ *such that* $N = \varepsilon M R$ *or* $N = \varepsilon M^* R$*. Note that in the second case, the condition is independent from the choice of the representative of* M^* *modulo* $\mathcal{C}(\mathbb{F}_1)$*. The equivalence class of M is denoted by* $[M]_{\sim_*}$ *and the set of equivalence classes by* $\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))/_{\sim_*}$*.*

We characterize equivalence in the extended sense between embeddings of principal type:

Lemma 6.2.11. *Let* $a \in \mathcal{O}$ *and* $u \in \mathcal{I}(\mathcal{O})$*. Then the following assertions hold:*

a) $M_{\overline{a}} \sim_* M_a$*,*

b) $M_{u^{-1}au} \sim_* M_a$*.*

Proof. a) Let $t = N(a)$. For arbitrary $Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F})$ we have

$$I_{\mathrm{tr}} M_a I_{\mathrm{tr}} \langle Z \rangle = I_{\mathrm{tr}} M_a \left\langle \begin{pmatrix} z & \overline{w} \\ w & z' \end{pmatrix} \right\rangle = I_{\mathrm{tr}} \left\langle \begin{pmatrix} z & \overline{w} a^{-1} \\ \overline{a}^{-1} w & z'/t \end{pmatrix} \right\rangle = \begin{pmatrix} z & \overline{a}^{-1} w \\ \overline{w} a^{-1} & z'/t \end{pmatrix}.$$

Hence,

$$\frac{\overline{a}}{\sqrt{t}} M_a^* \langle Z \rangle = \frac{\overline{a}}{\sqrt{t}} I_{\mathrm{tr}} M_a I_{\mathrm{tr}} \langle Z \rangle = \begin{pmatrix} z & w \overline{a}^{-1} \\ a^{-1} \overline{w} & z'/t \end{pmatrix} = M_{\overline{a}} \langle Z \rangle.$$

b) Follows from $M_{u^{-1}au} = \frac{\overline{u}}{\sqrt{N(u)}} M_a \left(\frac{u}{\sqrt{N(u)}} I_4 \right)$. □

Now we can characterize equivalence in the extended sense of principal embeddings:

Theorem 6.2.12. *Let $\mathcal{O}$ be an order and assume that $\mathrm{PU}_2(\mathcal{O})$ is normalized by I_{tr}. Let $a, b \in \mathcal{O}$ such that $N(a) = N(b)$. Then the following assertions are equivalent:*

i) $M_a \sim_* M_b$*,*

ii) $\sigma(a) = b$ *for some* $\sigma \in \mathrm{O}(\mathcal{O})$*.*

Proof. The assertion $\sigma(a) = b$ for some $\sigma \in \mathrm{O}(\mathcal{O})$ is equivalent to the existence of $\delta \in \mathcal{O}^\times$ and $u \in \mathcal{I}(\mathcal{O})$ such that $a = u^{-1} \delta b u$ or $a = u^{-1} \delta \overline{b} u$. The first case is equivalent to $M_{uau^{-1}} \sim M_b$ in the usual sense, the second equivalent to $M_{uau^{-1}} \sim M_{\overline{b}}$ in the usual sense. Then the claim follows from 6.2.11. □

Concerning embeddability of the Fricke-involution V_t, we give the following

Remark 6.2.13. *Let $u \in \mathcal{O}$ such that $N(u) = t$ and $u^2 \in N(u) \cdot \mathcal{O}$. Then one has*

$$M_u^{-1} V_t M_u = \frac{u}{\sqrt{t}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{U}_2^*(\mathcal{O}), \quad \delta := u^{-1} \overline{u} \in \mathcal{O}^\times.$$

Consequently,

$$M_u^{-1} \langle \Gamma(t), V_t \rangle M_u \subseteq \mathrm{U}_2^*(\mathcal{O}).$$

Examples 6.2.14. *a) Let $\mathcal{O}$ denote the Hurwitz order. Then one has*

$$M_{1+i_1}^{-1} \cdot \Gamma(2)^{\max} \cdot M_{1+i_1} \subseteq \mathrm{U}_2^*(\mathcal{O}).$$

b) Let $D_K > 0$ be a prime discriminant. Then one has

$$M_{i_1 \sqrt{D_K}}^{-1} \cdot \Gamma(D_K)^{\max} \cdot M_{i_1 \sqrt{D_K}} \subseteq \mathrm{U}_2^*(\mathcal{O}_{\mathbb{Q}(\sqrt{-D_K})}).$$

6.3 Hermitian and quaternionic Jacobi forms

We introduce Fourier-Jacobi expansions:

Definition 6.3.1. *Let $F \in [\mathrm{U}_2(\mathcal{O}), k]$ with Fourier expansion*

$$F(Z) = \sum_{0 \leq S = \left(\begin{smallmatrix} n & r \\ \overline{r} & m \end{smallmatrix}\right) \in \mathrm{Her}_2^\sharp(\mathcal{O})} \alpha_F(S) e^{2\pi i \mathrm{tr}(SZ)}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F}).$$

The rearrangement

$$F(Z) = \sum_{m=0}^{\infty} \phi_{m,F}(z,w) e^{2\pi i m z'}, \qquad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F})$$

is called the Fourier-Jacobi expansion *of F. For $m \in \mathbb{N}_0$ the function*

$$\phi_{m,F}(z,w) := \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^\sharp \\ N(r) \leq nm}} \alpha_F \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix} e^{2\pi i (nz + \mathrm{tr}(\overline{r} w))}, \quad z \in \mathcal{H}, w \in \mathbb{F}_\mathbb{C} := \mathbb{F} \otimes_\mathbb{R} \mathbb{C},$$

*is called the m-*th Jacobi form *associated to F.*

Definition 6.3.2. *The* parabolic subgroup $\mathrm{U}_{2,1}(\mathcal{O})$ *is defined by*

$$\mathrm{U}_{2,1}(\mathcal{O}) := \left\{ M \in \mathrm{U}_2(\mathcal{O}) : M = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}.$$

For $M_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu, v \in \mathcal{O}$ with $v - \overline{\lambda}\mu \in \mathbb{Z}$ we distinguish the special elements

$$M_1 \times I_2 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [\lambda, \mu, v] := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \overline{\lambda} & 1 & \overline{\mu} & v \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of $\mathrm{U}_{2,1}(\mathcal{O})$.

The next lemma is folklore:

Lemma 6.3.3. *Every $M \in \mathrm{U}_{2,1}(\mathcal{O})$ has a unique representation of the form*

$$M = \mathrm{diag}(\varepsilon, \delta, \varepsilon, \delta) \cdot (M_1 \times I_2) \cdot [\lambda, \mu, v],$$

with $\delta, \varepsilon \in \mathcal{O}^\times, M_1 \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu, v \in \mathcal{O}$ such that $v - \overline{\lambda}\mu \in \mathbb{Z}$.

We introduce hermitian and quaternionic Jacobi forms. Basically they obey the same transformation laws as Jacobi forms considered in Chapter 4, but have some additional behaviour under the maps $w \mapsto \delta w \varepsilon$ for units $\varepsilon, \delta \in \mathcal{O}^\times$.

Definition 6.3.4. *Let $k, m \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. A holomorphic function $\phi : \mathcal{H}_1 \times \mathbb{F}_\mathbb{C} \longrightarrow \mathbb{C}$ is called* hermitian Jacobi form *in the case $\mathbb{F} = \mathbb{C}$ resp.* quaternionic Jacobi form *in the case $\mathbb{F} = \mathbb{H}$ of weight k and index m with respect to $\mathcal{O}$, if the associated function*

$$\phi_m^* : \mathcal{H}_2(\mathbb{F}) \longrightarrow \mathbb{C}, \quad \phi_m^*(Z) := \phi(z,w)e^{2\pi i m z'}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F}),$$

satisfies the following conditions:

i) $\phi_m^*\big|_k M = \phi_m^*$ *for all* $M \in \mathrm{U}_{2,1}(\mathcal{O})$,

ii) ϕ *has an absolutely and locally uniformly convergent Fourier expansion of the form*

$$\phi(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^\sharp \\ N(r) \le nm}} c(n,r) e^{2\pi i (nz + \mathrm{tr}(\overline{r}w))}, \quad (z,w) \in \mathcal{H} \times \mathbb{F}_\mathbb{C}.$$

The space of hermitian resp. quaternionic Jacobi forms of weight k and index m with respect to $\mathcal{O}$ is denoted by $J_{k,m}^{\mathbb{C}}(\mathcal{O})$ resp. $J_{k,m}^{\mathbb{H}}(\mathcal{O})$.

Considering the action of generators quoted in 6.3.3 on the associated function ϕ_m^*, we obtain the following characterization of Jacobi forms:

Lemma 6.3.5. *Let $k, m \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. A holomorphic function $\phi : \mathcal{H}_1 \times \mathbb{F}_\mathbb{C} \longrightarrow \mathbb{C}$ is a hermitian Jacobi form resp. quaternionic Jacobi form of weight k and index m with respect to $\mathcal{O}$ if and only if the following assertions hold:*

i) $\phi(z,w) = \phi\big|_{k,t} M(z,w) := (cz+d)^{-k} \cdot e^{2\pi i \frac{-cmN(w)}{cz+d}} \phi\left(Mz, \frac{w}{cz+d}\right)$ *for all* $M \in \mathrm{SL}_2(\mathbb{Z})$,

ii) $\phi(z,w) = \phi\big|_m [\lambda,\mu](z,w) := e^{2\pi i m \left(N(\lambda)z + \mathrm{tr}(\overline{\lambda}w)\right)} \phi\left(z, w + \lambda z + \mu\right)$ *for all* $\lambda, \mu \in \mathcal{O}$,

iii)

$$\phi(z, \varepsilon w) = \varepsilon^k \phi(z,w) \quad \textit{for all } \varepsilon \in \mathcal{O}^\times, \quad \mathbb{F} = \mathbb{C},$$
$$\phi(z, \varepsilon w \delta) = \phi(z,w) \quad \textit{for all } \varepsilon, \delta \in \mathcal{O}^\times, \quad \mathbb{F} = \mathbb{H},$$

iv) ϕ *has absolutely and locally uniformly convergent Fourier expansion of the form*

$$\phi(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^\sharp \\ N(r) \le nm}} c(n,r) e^{2\pi i (nz + \mathrm{tr}(\overline{r}w))}.$$

We sort hermitian and quaternionic Jacobi forms into the context of Jacobi forms of lattice-index:

Remark 6.3.6. *Let $k, m \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Then one has*

$$J^{\mathbb{F}}_{k,m}(\mathcal{O}) \subseteq J_{k,\underline{\mathcal{O}}(m)}, \quad \underline{\mathcal{O}} := (\mathcal{O}, N|_{\mathcal{O}}).$$

Every modular form induces a family of Jacobi forms:

Example 6.3.7. *Let $k, m \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Let $F \in [\mathrm{U}_2(\mathcal{O}), k]$. Then $\phi_{m,F} \in J^{\mathbb{F}}_{k,m}(\mathcal{O})$, where $\phi_{m,F}$ denotes the m-th Jacobi form associated to F.*

We introduce the operator V_m defined in [8, I. § 4] in the general setting:

Definition 6.3.8. *Let $m \in \mathbb{N}$ and $\mathcal{T}(m) := \{M \in \mathrm{SL}_2(\mathbb{Z}) : \det M = m\}$. For a holomorphic function $\phi : \mathcal{H} \times \mathbb{F}_{\mathbb{C}} \longrightarrow \mathbb{C}$ we define*

$$\phi|_{k,t} V_m(z,w) := m^{\frac{k}{2}-1} \sum_{M:\mathcal{T}(m)\backslash \mathrm{SL}_2(\mathbb{Z})} \phi|_{k,t} \left[\frac{1}{\sqrt{m}} M\right](z, \sqrt{m}w), \quad (z,w) \in \mathcal{H} \times \mathbb{F}_{\mathbb{C}}.$$

From [8, I. § 4] we cite

Theorem 6.3.9. *Let $m, t \in \mathbb{N}$. Then V_m induces a linear operator*

$$\big|_{k,t} V_m : J^{\mathbb{F}}_{k,t}(\mathcal{O}) \longrightarrow J^{\mathbb{F}}_{k,mt}(\mathcal{O}).$$

For $\phi \in J^{\mathbb{F}}_{k,t}(\mathcal{O})$ with Fourier expansion

$$\phi(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^{\sharp} \\ N(r) \leq nt}} c(n,r) e^{2\pi i (nz + \mathrm{tr}(\bar{r}w))}$$

one has

$$\phi|_{k,t} V_m(z,w) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}^{\sharp} \\ N(r) \leq nmt}} \left(\sum_{d|(n,r,m)} d^{k-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right)\right) e^{2\pi i (nz + \mathrm{tr}(\bar{r}w))},$$

where $d|(n,r,m)$ means $d^{-1}(n,r,m) \in \mathbb{N}_0 \times \mathcal{O}^{\sharp} \times \mathbb{N}_0$.

We consider Jacobi theta functions associated to the lattice $(\mathcal{O}, N|_{\mathcal{O}})$:

Definition 6.3.10. *Let $\mathcal{O}$ be an order. For $u \in \mathcal{O}^{\sharp}/\mathcal{O}$ define*

$$\vartheta_{\mathcal{O},u}(z,w) := \vartheta^{(1)}_{(\mathcal{O},N|_{\mathcal{O}}),u}(z,w) = \sum_{g \in u + \mathcal{O}} e^{2\pi i (N(g) z + \mathrm{tr}(\bar{g}w))}, \quad (z,w) \in \mathcal{H} \times \mathbb{F}_{\mathbb{C}}.$$

Some immediate transformation laws are given in the following

Lemma 6.3.11. *Let $u \in \mathcal{O}^\sharp/\mathcal{O}$ and $\varepsilon, \delta \in \mathcal{O}^\times$. Then the following assertions hold:*

a) $\vartheta_{\mathcal{O},u}(z, \varepsilon w \delta) = \vartheta_{\mathcal{O},\overline{\varepsilon} u \overline{\delta}}(z, w)$,

b) $\vartheta_{\mathcal{O},u}(z, \overline{w}) = \vartheta_{\mathcal{O},\overline{u}}(z, w)$,

c) $\vartheta_{\mathcal{O},u}(z, -\overline{w}) = \vartheta_{\mathcal{O},u}(z, w)$.

Proof. Note that by 5.1.18, the map $w \mapsto -\overline{w}$ belongs to the discriminant kernel $\mathrm{O}_d(\mathcal{O})$, i.e. $u + \mathcal{O} = -\overline{u} + \mathcal{O}$. □

A reformulation of 2.3.7 yields that every $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$ has a unique theta decomposition of the form

$$\phi(z, w) = \sum_{u:\mathcal{O}^\sharp/\mathcal{O}} f_u(z) \vartheta_{\mathcal{O},u}(z, w), \quad (z, w) \in \mathcal{H} \times \mathbb{F}_{\mathbb{C}}.$$

From 6.3.11 we obtain the following

Lemma 6.3.12. *Let $k \in \mathbb{N}_0$ and $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$. Then one has*

a) $\phi(z, -\overline{w}) = \phi(z, w)$,

b) $\phi(z, \overline{w}) = (-1)^k \phi(z, w)$.

Especially, ϕ is invariant under the map $w \mapsto \overline{w}$, if k is even.

We use the explicit description of $\mathrm{O}(\mathfrak{o})$ for orders $\mathfrak{o}$ in $\mathbb{C}$:

Corollary 6.3.13. *Let $\mathfrak{o}$ be an order in $\mathbb{C}$. Let $k \in \mathbb{N}_0$ such that $k \equiv 0 \bmod |\mathfrak{o}^\times|$. Then one has*

$$J^{\mathbb{C}}_{k,1}(\mathfrak{o}) = J^{\mathrm{sym}}_{k,\underline{\mathfrak{o}}}.$$

In general, a characterization as in 6.3.13 does not hold for quaternionic orders, since the orthogonal group $\mathrm{O}(\mathcal{O})$ contains conjugations by invariant elements $u \in \mathcal{I}(\mathcal{O})$. But in the cases we are dealing with, the invariance of quaternionic Jacobi forms under the maps $w \mapsto u^{-1} w u$ is already implied by the invariance under $w \mapsto \varepsilon w \delta$ for $\varepsilon, \delta \in \mathcal{O}^\times$.

We need the following lemma, which is easy to prove:

Lemma 6.3.14. *Let $k \in \mathbb{N}_0$. Let $G \leq \mathrm{O}(\mathcal{O})$ and $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$ invariant under G. If G acts transitively on the sets*

$$A_d := \{u \in \mathcal{O}^\sharp/\mathcal{O} : N(u) \equiv d \bmod \mathbb{Z}\}, \quad d \in \mathbb{Q},$$

then one has $\phi \in J^{\mathbb{F},\mathrm{sym}}_{k,1}(\mathcal{O})$.

Proof. By definition, the sets A_d for $d \in \mathbb{Q}$ are $\mathrm{O}(\mathcal{O})$-invariant. From the assumption on G, also $\mathrm{O}(\mathcal{O})$ acts transitively on A_d for $d \in \mathbb{Q}$. Since these sets are pairwise disjoint, the set $\{A_d : A_d \neq \emptyset\}$ is exactly the orbit space of $\mathrm{O}(\mathcal{O})$ on the discriminant group $\mathcal{O}^\sharp/\mathcal{O}$. By assumption, ϕ is invariant under G. Hence the function $u \mapsto f_u$ is constant on the sets A_d and thus on all $\mathrm{O}(\mathcal{O})$-orbits, i.e. $\phi \in J_{k,1}^{\mathbb{F},\mathrm{sym}}(\mathcal{O})$. □

We apply the lemma in the proof of

Theorem 6.3.15. *Let $k \in \mathbb{N}_0$ and $k \equiv 0 \bmod w_{\mathcal{O}}$. Let $\mathcal{O}$ be the Hurwitz order or $\mathcal{O} = \mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-D})$ for $D = 3, 4$. Then for all $F \in [\mathrm{U}_2(\mathcal{O}), k]$ one has $\phi_{1,F} \in J_{k,1}^{\mathbb{H},\mathrm{sym}}(\mathcal{O})$.*

Proof. The invariance under $w \mapsto \overline{w}$ follows from 6.3.12, since k is necessarily even. First let $\mathcal{O}$ be the Hurwitz order. The group generated by $w \mapsto \overline{\omega} w \omega$ acts transitively on the set

$$\left\{ u \in \mathcal{O}^\sharp/\mathcal{O} : N(u) \equiv \tfrac{1}{2} \bmod \mathbb{Z} \right\},$$

which is directly verified by considering the representatives $\frac{1+i_1}{2}$, $\frac{1+i_2}{2}$ and $\frac{1+i_3}{2}$. Since $N(u)$ mod $\mathbb{Z}$ takes its values in $\{0, \frac{1}{2}\}$, the claim follows from 6.3.14. For $D_K = 3$ a system of representatives of $\mathcal{O}_K^\sharp/\mathcal{O}_K$ is given by

$$0, \pm\mu, \pm\mu i_2, \pm\mu \pm \mu i_2, \quad \mu := \frac{i_1}{\sqrt{3}}.$$

A direct verification shows that the group generated by $w \mapsto w i_2$ acts transitively on both sets

$$\left\{ u \in \mathcal{O}_K^\sharp/\mathcal{O}_K : N(u) \equiv \tfrac{1}{3} \bmod \mathbb{Z} \right\}, \quad \left\{ u \in \mathcal{O}_K^\sharp/\mathcal{O}_K : N(u) \equiv \tfrac{2}{3} \bmod \mathbb{Z} \right\}.$$

In this case the claim again follows from 6.3.14. For $D_K = 4$ a system of representatives of $\mathcal{O}_K^\sharp/\mathcal{O}_K$ is given by the sixteen elements

$$0, \frac{1}{2}, \frac{i_1}{2}, \frac{i_2}{2}, \frac{i_3}{2}, \frac{1+i_1}{2}, \frac{1+i_2}{2}, \frac{1+i_3}{2}, \frac{i_1+i_2}{2}, \frac{i_1+i_3}{2}, \frac{i_2+i_3}{2}, \frac{1+i_1+i_2}{2}, \frac{1+i_1+i_3}{2}, \frac{1+i_2+i_3}{2}, \frac{i_1+i_2+i_3}{2}, \omega.$$

A direct calculation shows that the group generated by the transformations $w \mapsto w i_1$ and $w \mapsto w i_2$ acts transitively on

$$\left\{ u \in \mathcal{O}_K^\sharp/\mathcal{O}_K : N(u) \equiv \tfrac{k}{4} \bmod \mathbb{Z} \right\}, \quad k = 1, 2, 3.$$

Hence, these sets are $\mathrm{O}(\mathcal{O}_K)$ orbits. The class $\omega + \mathcal{O}_K$ is a fixed point of $\mathrm{O}(\mathcal{O}_K)$ and thus a single orbit. Together with the zero class $\mathcal{O}_K$, we have determined all $\mathrm{O}(\mathcal{O}_K)$-orbits and the claim follows. Note that 6.3.14 is not applicable in this case. □

The compatibility of the pullback operator with respect to modular embeddings and the pullback operator with respect to Jacobi forms is explained in the following

Proposition 6.3.16. *Let $k \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Let $a, b \in \mathcal{O}$ and $F \in [\mathrm{U}_2(\mathcal{O}), k]$. Then the following assertions hold:*

a) $F[a](Z) = \sum_{m=0}^{\infty} \phi_{m,F}[a](z,w) e^{2\pi i N(a) m z'}$,

b) $F[a] = F[b]$ *implies* $\phi_{m,F}[a] = \phi_{m,F}[b]$ *for all* $m \in \mathbb{N}_0$.

Proof. Let $F \in [\mathrm{U}_2(\mathcal{O}), k]$ with Fourier-Jacobi expansion

$$F(Z) = \sum_{m=0}^{\infty} \phi_{m,F}(z,w) e^{2\pi i m z'}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F}).$$

Hence,

$$F[a](Z) = F\begin{pmatrix} z & aw \\ \overline{a}\overline{w} & N(a)z' \end{pmatrix} = \sum_{m=0}^{\infty} \phi_{m,F}(z, aw) e^{2\pi i N(a) m z'}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2.$$

Part b) follows from the uniqueness of the Fourier-Jacobi expansion. □

6.4 Maaß spaces and lifting constructions

Let $k \in \mathbb{Z}$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Then $F \in [\mathrm{U}_2(\mathcal{O}), k]$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le S \in \mathrm{Her}_2^{\sharp}(\mathcal{O})} \alpha_F(S) e^{2\pi i \operatorname{tr}(SZ)}, \quad Z \in \mathcal{H}_2(\mathbb{F}).$$

We introduce an arithmetically motivated subspace of $[\mathrm{U}_2(\mathcal{O}), k]$, cf. [19, 20, 21]:

Definition 6.4.1. *Let $k \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. The Maaß space $\mathcal{M}_k(\mathcal{O})$ with respect to $\mathcal{O}$ consists of all $F \in [\mathrm{U}_2(\mathcal{O}), k]$ such that*

$$\alpha_F(S) = \sum_{d \in \mathbb{N}, d | \varepsilon(S)} d^{k-1} \alpha_F \begin{pmatrix} mn/d^2 & r/d \\ \overline{r}/d & 1 \end{pmatrix}$$

holds for all $0 \neq S = \left(\begin{smallmatrix} n & r \\ \overline{r} & m \end{smallmatrix}\right), S \geq 0$, *where*

$$\varepsilon(S) := \max\left\{ m \in \mathbb{N} : m^{-1} S \in \mathrm{Her}_2^{\sharp}(\mathcal{O}) \right\}.$$

Note that $\mathcal{M}_k(\mathbb{Z})$ is the space which was considered initially by H. Maaß.

Basic facts are collected in the following

Remarks 6.4.2. *Let $k \in \mathbb{N}_0$, where $k \equiv 0 \bmod 2$ in the case $\mathbb{F} = \mathbb{H}$. Let $F \in \mathcal{M}_k(\mathcal{O})$ with Fourier-Jacobi expansion*

$$F(Z) = \sum_{m=0}^{\infty} \phi_{m,F}(z,w) e^{2\pi i m z'}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F}).$$

a) If k is odd or $k = 2$, then $\phi_{0,F} \equiv 0$. Furthermore,

$$\phi_{0,F}(z,w) = \alpha_F \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} G_k^*(z), \quad (z,w) \in \mathcal{H} \times \mathbb{F}_{\mathbb{C}},$$

for all even $k \geq 4$ and

$$\phi_{m,F}(z,w) = \phi_{1,F}\big|_{k,1} V_m(z,w), \quad (z,w) \in \mathcal{H} \times \mathbb{F}_{\mathbb{C}},$$

for all $m > 0$. For odd k or $k = 2$, the space $\mathcal{M}_k(\mathcal{O})$ consists of cups forms only and one has $\mathcal{M}_0(\mathcal{O}) = \mathbb{C}$.

b) The map $\mathcal{M}_k(\mathcal{O}) \longrightarrow J_{k,1}^{\mathbb{F}}(\mathcal{O})$, $F \mapsto \phi_{1,F}$ is injective.

c) $F \in \mathcal{M}_k(\mathcal{O})$ satisfies

$$F\big|I_{\mathrm{tr}} = (-1)^k F(Z), \quad Z \in \mathcal{H}_2(\mathbb{F}).$$

Hence, if k is even, $\mathcal{M}_k(\mathcal{O})$ consists of symmetric modular forms only. If k is odd and $\mathbb{F} = \mathbb{C}$, the space $\mathcal{M}_k(\mathcal{O})$ consists of skew-symmetric forms only. If k is odd and $\mathbb{F} = \mathbb{R}$, then $\mathcal{M}_k(\mathcal{O})$ is trivial.

d) $\mathrm{O}(\mathcal{O})$ acts on $\mathcal{M}_k(\mathcal{O})$ via

$$F^{\sigma}(Z) := F\begin{pmatrix} z & \sigma^{-1}(w) \\ \overline{\sigma^{-1}(w)} & z' \end{pmatrix}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix}, \sigma \in \mathrm{O}(\mathcal{O}).$$

Furthermore, $\phi_{m,F^{\sigma}} = \phi_{m,F}[\sigma]$ for all $m \in \mathbb{N}_0$.

Remark 6.4.3. *Let $k \in \mathbb{N}_0$ such that $k \equiv 0 \bmod w_{\mathcal{O}}$. For a holomorphic function $F : \mathcal{H}_2(\mathbb{F}) \longrightarrow \mathbb{C}$ and $M \in \mathrm{U}_2(\mathcal{O})$ the function $F\big|_k M$ only depends on $M \bmod \mathcal{C}(\mathbb{F}_1)$. Hence, the slash operator*

$$(M, F) \mapsto F\big|_k M$$

is well-defined and defines an action of the extended modular group $\Gamma(\mathcal{O})$ on the set of holomorphic functions $\{F : \mathcal{H}_2(\mathbb{F}) \longrightarrow \mathbb{C}\}$, where the definition is extended by

$$f\big|_k I_{\mathrm{tr}}(Z) := f(Z^t), \quad Z \in \mathcal{H}_2(\mathbb{F}).$$

Definition 6.4.4. *Let $k \in \mathbb{N}_0$ such that $k \equiv 0 \bmod w_{\mathcal{O}}$. A function $F : \mathcal{H}_2(\mathbb{F}) \longrightarrow \mathbb{C}$ is called a modular form of weight k with respect to $\Gamma(\mathcal{O})$, if the following assertions hold:*

i) F is holomorphic,

ii) $F\big|_k M = F$ *for all* $M \in \Gamma(\mathcal{O})$.

The space of modular forms of weight k with respect to $\Gamma(\mathcal{O})$ *is denoted by* $[\Gamma(\mathcal{O}), k]$.

As a direct consequence of 5.5.11 we obtain:

Proposition 6.4.5. *Let* $\mathcal{O}$ *be an order and assume that* $\mathrm{PU}_2(\mathcal{O})$ *is normalized by* I_{tr}. *If* $\mathbb{F} = \mathbb{C}$, *suppose additionally that* $\mathrm{U}_2(\mathcal{O})$ *is standardly generated. Let* $k \equiv 0 \bmod w_{\mathcal{O}}, F \in [\Gamma(\mathcal{O}), k]$ *and* $M, N \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$. *Then* $M \sim_* N$ *implies* $F\big|_k[M] = F\big|_k[N]$. *In other words, the map*

$$\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))/\sim_* \longrightarrow [\Gamma(T), k], \quad [M]_{\sim_*} \mapsto F\big|_k[M]$$

is well-defined.

Remark 6.4.6. *Note that the assumptions of 6.4.5 hold for example if* $\mathbb{F} = \mathbb{C}$ *and* $\mathcal{O}$ *is a principal ideal domain or* $\mathcal{O} = \mathfrak{o}_K$ *for some imaginary-quadratic number field K (cf. [5, Lemma (1.1)]) or if* $\mathbb{F} = \mathbb{H}$ *and* $\mathcal{O}$ *is norm-euclidean.*

Remark 6.4.7. *Modular forms with respect to subgroups of* $\Gamma(\mathcal{O})$ *and characters are defined in the same manner. Note that* $[\mathrm{PU}_2(\mathcal{O}), k] = [\mathrm{U}_2(\mathcal{O}), k]$ *holds for all* $k \equiv 0 \bmod w_{\mathcal{O}}$. *Hence the theory of modular forms with respect to the extended modular group contains the one for* $\mathrm{U}_2(\mathcal{O})$ *in this case.*

Remark 6.4.8. *Let* $\mathfrak{o}$ *be an order in* $\mathbb{C}$ *and* $k \equiv 0 \bmod |\mathfrak{o}^\times|$. *Then* $[\Gamma(\mathfrak{o}), k] = [\mathrm{U}_2(\mathfrak{o}), k]^{\mathrm{sym}}$.

Corollary 6.4.9. *Let* $k \in \mathbb{N}_0$ *such that* $k \equiv 0 \bmod w_{\mathcal{O}}$. *Let* $\mathcal{O}$ *denote the Hurwitz order or let* $\mathcal{O} = \mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$ *for* $D = 3, 4$ *or* $\mathcal{O} = \mathfrak{o}_K$ *for* $K = \mathbb{Q}(\sqrt{-D_K})$. *Then* $\mathcal{M}_k(\mathcal{O})$ *is contained in* $[\Gamma(\mathcal{O}), k]$.

Proof. Let $F \in \mathcal{M}_k(\mathcal{O})$ and $\sigma \in \mathrm{O}(\mathcal{O})$. From 6.3.15 we obtain $\phi_{1,F^\sigma} = \phi_{1,F}[\sigma] = \phi_{1,F}$. Since the map $F \mapsto \phi_{1,F}$ is injective, we conclude $F^\sigma = F$, i.e. $F \in [\Gamma(\mathcal{O}), k]$. □

Under certain assumptions on $\mathrm{U}_2(\mathcal{O})$, the projection onto the first Jacobi form has a right inverse, namely the Maaß lift:

Theorem 6.4.10. *Suppose that* $\mathrm{U}_2(\mathcal{O})$ *is generated by the parabolic subgroup* $\mathrm{U}_{2,1}(\mathcal{O})$ *and* $\mathrm{diag}(P, P)$, *where* $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. *Let* $k \geq 4$ *such that* $k \equiv 0 \bmod w_{\mathcal{O}}$. *For* $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$ *with constant Fourier coefficient* $c_\phi(0,0)$ *define*

$$\text{M-Lift}(\phi)(Z) := -\frac{B_k}{2k} c_\phi(0,0) G_k^*(z) + \sum_{m=1}^{\infty} \left(\phi\big|_{k,1} V_m\right)(z, w) e^{2\pi i m z'}, \quad Z = \begin{pmatrix} z & w \\ \overline{w} & z' \end{pmatrix} \in \mathcal{H}_2(\mathbb{F}).$$

Then the following assertions hold:

a) $\text{M-Lift}(\phi) \in [\mathrm{U}_2(\mathcal{O}), k]$ *for all* $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$.

b) $\text{M-Lift}: J_{k,1}^{\mathbb{F}}(\mathcal{O}) \longrightarrow [\text{U}_2(\mathcal{O}), k]$ *is injective.*

c) For $\phi \in J_{k,1}^{\mathbb{F}}(\mathcal{O})$ *the Fourier coefficients of* $\text{M-Lift}(\phi)$ *are given by the formula*

$$\alpha_{\text{M-Lift}(\phi)}(S) = \begin{cases} -\frac{B_k}{2k} c_\phi(0,0), & S = 0, \\ c_\phi(0,0)\sigma_{k-1}(n), & S = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right), n > 0, \\ \sum\limits_{d \mid \varepsilon(S)} d^{k-1} c_\phi\left(\frac{mn}{d^2}, \frac{r}{d}\right), & S = \left(\begin{smallmatrix} n & r \\ \overline{r} & m \end{smallmatrix}\right), m > 0. \end{cases}$$

d) $\text{M-Lift}: J_{k,1}^{\mathbb{F}}(\mathcal{O}) \longrightarrow \mathcal{M}_k(\mathcal{O})$ *is an isomorphism.*

The operator M-Lift *is called the* Maaß lift *with respect to* $\mathcal{O}$.

Proof. We give a sketch of the proof, since the methods are similar to the literature. Injectivity follows from $V_1 = \text{id}$. The invariance of $\text{M-Lift}(\phi)$ under the parabolic subgroup $\text{U}_{2,1}(\mathcal{O})$ follows from construction. The explicit formula for the Fourier coefficients is a direct consequence of 6.3.8. Finally, part c) together with the invariance of ϕ under the transformation $w \mapsto \overline{w}$ implies, that $\text{M-Lift}(\phi)$ is invariant under $Z \mapsto Z[P]$. By assumption on $\text{U}_2(\mathcal{O})$ this proves a). For part d) let $F \in \mathcal{M}_k(\mathcal{O})$ with first Jacobi form $\phi_{1,F}$. Then F and $\text{M-Lift}(\phi_{1,F})$ have the same first Jacobi form. Hence, $F = \text{M-Lift}(\phi_{1,F})$ by injectivity. This proves surjectivity. □

Let $t \in \mathbb{N}$. Since $\Gamma(t)$ contains the transformation

$$\begin{pmatrix} z & w \\ w & z' \end{pmatrix} \mapsto \begin{pmatrix} z & w \\ w & z' + \frac{1}{t} \end{pmatrix},$$

we obtain that any $F \in [\Gamma(t), k]$ has a Fourier expansion of the form

$$F(Z) = \sum_{S \in \text{Sym}_2^\sharp(\mathbb{Z}), S = \left(\begin{smallmatrix} n & r \\ r & tm \end{smallmatrix}\right) \geq 0} \alpha_F(S) e^{2\pi i \text{tr}(SZ)}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.$$

The Maaß space with respect to the paramodular group is then defined in a similar way:

Definition 6.4.11. *Let* $k \in \mathbb{N}_0$ *and* $t \in \mathbb{N}$. *The* Maaß space $\mathcal{M}_{k,t}$ *of polarization* t *consists of all* $F \in [\Gamma(t), k]$ *such that*

$$\alpha_F(S) = \sum_{d \mid \varepsilon(S)} d^{k-1} \alpha_F \begin{pmatrix} mn/d^2 & r/d \\ r/d & t \end{pmatrix}$$

holds for all $0 \neq S = \left(\begin{smallmatrix} n & r \\ r & tm \end{smallmatrix}\right) \geq 0$.

Some basis facts are collected in the following

Remark 6.4.12. *Let* $k \in \mathbb{N}_0$ *and* $t \in \mathbb{N}$. *Let* $F \in \mathcal{M}_{k,t}$ *with Fourier-Jacobi expansion*

$$F(Z) = \sum_{m=0}^{\infty} \phi_{tm,F}(z,w) e^{2\pi i t m z'}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.$$

a) *If* k *is odd or* $k = 2$*, then* $\phi_{0,F} \equiv 0$*. Furthermore,*

$$\phi_{0,F}(z,w) = \alpha_F \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} G_k^*(z)$$

for all even $k \geq 4$ *and*

$$\phi_{mt,F}(z,w) = \phi_{t,F}\big|_{k,t} V_m(z,w)$$

for all $m > 0$*. For odd* k *or* $k = 2$*, the space* $\mathcal{M}_{k,t}$ *consists of cups forms only and one has* $\mathcal{M}_{0,t} = \mathbb{C}$*.*

b) *The map* $\mathcal{M}_{k,t} \longrightarrow J_{k,t}, F \mapsto \phi_{t,F}$ *is injective.*

c) $F\big|_k V_t = (-1)^k F$*.*

Under certain assumptions on the weight, the projection onto the t-th Fourier-Jacobi coefficient has a right inverse, the so-called Gritsenko lift. From [10, Hauptsatz 2.1] we cite:

Theorem 6.4.13. *Let* $t \in \mathbb{N}$ *and* $k \geq 4$*. For* $\phi \in J_{k,t}$ *with constant Fourier coefficient* $c_\phi(0,0)$ *define*

$$\text{G-Lift}_t(\phi)(Z) := -\frac{B_k}{2k} c_\phi(0,0) G_k^*(z) + \sum_{m=1}^{\infty} \left(\phi\big|_{k,t} V_m\right)(z,w) e^{2\pi i t m z'}, \quad Z = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathcal{H}_2.$$

Then the following assertions hold:

a) $\text{G-Lift}_t(\phi) \in [\Gamma(t), k]$ *for all* $\phi \in J_{k,t}$ *and* $\text{G-Lift}_t(\phi)\big|_k V_t = (-1)^k \text{G-Lift}_t(\phi)$*.*

b) $\text{G-Lift}_t : J_{k,t} \longrightarrow [\Gamma(t), k]$ *is injective.*

c) *For* $\phi \in J_{k,t}$ *the Fourier coefficients of* $\text{G-Lift}_t(\phi)$ *are given by the formula*

$$\alpha_{\text{G-Lift}_t(\phi)}(S) = \begin{cases} -\frac{B_k}{2k} c_\phi(0,0), & S = 0, \\ c_\phi(0,0)\sigma_{k-1}(n), & S = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right), n > 0, \\ \sum\limits_{d|(n,r,m)} d^{k-1} c_\phi\left(\frac{mn}{d^2}, \frac{r}{d}\right), & S = \left(\begin{smallmatrix} n & r \\ r & tm \end{smallmatrix}\right), m > 0. \end{cases}$$

d) $\text{G-Lift}_t : J_{k,t} \longrightarrow \mathcal{M}_{k,t}$ *is an isomorphism.*

The operator G-Lift_t *is called* Gritsenko lift *of index* t*.*

We end this section by proving a commutation relation between the two lifting constructions and the pullback operator:

Theorem 6.4.14. *Let $\mathcal{O}$ be an order. Suppose that $\mathrm{U}_2(\mathcal{O})$ is generated by the parabolic subgroup $\mathrm{U}_{2,1}(\mathcal{O})$ and $\mathrm{diag}(P,P)$, where $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $a \in \mathcal{O}, k \geq 4$ and $k \equiv 0 \bmod w_{\mathcal{O}}$. Then one has*

$$\text{G-Lift}_{N(a)}(\phi[a]) = \text{M-Lift}(\phi)[a]$$

for all $\phi \in J^{\mathbb{F}}_{k,1}(\mathcal{O})$ and the pullback operator $[a]$ maps $\mathcal{M}_k(\mathcal{O})$ into $\mathcal{M}_{k,t}$. In other words, the diagram

$$\begin{array}{ccc} J^{\mathbb{F}}_{k,1}(\mathcal{O}) & \xrightarrow{\text{M-Lift}} & \mathcal{M}_k(\mathcal{O}) \\ {\scriptstyle [a]}\downarrow & & \downarrow{\scriptstyle [a]} \\ J_{k,N(a)} & \xrightarrow{\text{G-Lift}_{N(a)}} & \mathcal{M}_{k,N(a)} \end{array}$$

is commutative.

Proof. Let $Z \in \mathcal{H}_2$. Then one has

$$\begin{aligned} \text{M-Lift}(\phi)[a](Z) &= -\frac{B_k}{2k} c_\phi(0,0) G^*_k(z) + \sum_{m=1}^{\infty} \left(\phi\big|_{k,1} V_m\right)(z, aw) e^{2\pi i m N(a) z'} \\ &= -\frac{B_k}{2k} c_{\phi[a]}(0,0) G^*_k(z) + \sum_{m=1}^{\infty} \left(\phi[a]\big|_{k,N(a)} V_m\right)(z, w) e^{2\pi i m N(a) z'} \\ &= \text{G-Lift}_{N(a)}(\phi[a])(Z), \end{aligned}$$

since the operators $[a]$ and V_m commute. Note that $c_{\phi[a]}(0,0) = c_\phi(0,0)$. Since $\text{G-Lift}_{N(a)}$ and M-Lift are isomorphisms, we conclude that the diagram commutes. □

From 4.2.1, 4.3.1 and 4.3.2 we derive the following isomorphisms of Maaß spaces, which can partly be found already in [15, Sec. 4, Theorem 2 a)] and [16, Sec. 9, Corollary]. Note that the methods developed in Chapter 4 allow to construct the inverse maps on the basis of the theta decomposition of the first Jacobi form.

Corollary 6.4.15. *Let $k \in \mathbb{N}_0$ and $\mathcal{O}$ denote the Hurwitz order.*

a) $\mathcal{M}_k(\mathcal{O}) \cong \mathcal{M}_{k,1}$, *if* $k \equiv 0 \bmod 2$ *and* $\mathcal{M}_k(\mathcal{O}) \cong \mathcal{M}_k(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$, *if* $k \equiv 0 \bmod 6$.

b) $\mathcal{M}_k(\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}) \cong \mathcal{M}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}) \cong \mathcal{M}_{k,2}$, *if* $k \equiv 0 \bmod 4$.

6.5 Separation theorems

Let $F \in [\mathrm{U}_n(\mathcal{O}), k]$ and $M, N \in \mathrm{Mod}(\Gamma(T), \mathrm{U}_n(\mathcal{O}))$. In 5.5.11 we saw that $F\big|_k[M] = F\big|_k[N]$ if $M \sim N$. In this section we consider the converse problem, i.e. the question if a modular embedding M is determined up to equivalence by the family of pullbacked functions $F\big|_k[M]$, where $F \in [\mathrm{U}_n(\mathcal{O}), k]$. Already in the case $n = 2$, the usual equivalence relation on $\mathrm{Mod}(\Gamma(T), \mathrm{U}_2(\mathcal{O}))$ is to restrictive as one has $F[a] = F[\overline{a}]$ for $a \in \mathcal{O}$ and symmetric forms F, but in general $M_a \not\sim M_{\overline{a}}$. By retaking the up concept of equivalence in the extended sense, we can solve the converse problem at least for certain orders and under divisibility conditions on the polarization t.

6.5.1 $\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$

Regarding pullbacks of the Jacobi theta functions we need the following

Lemma 6.5.1. *Let $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$. Then the following statements are equivalent:*

i) $\vartheta_{\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})},0}[a] = \vartheta_{\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})},0}[b]$,

ii) a and b lie in the same $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$*-orbit,*

iii) there is $\varepsilon \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-1})}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon \overline{b}$.

Proof. First note that in all cases one has $N(a) = N(b)$. The equivalence of ii) and iii) is clear from the explicit description of $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$. Furthermore, ii) implies i). Hence we are left to prove that i) implies ii). By comparing the Fourier coefficients of the index $(1, s), s \in \mathbb{Z}$, we obtain

$$|\{x \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-1})} : \mathrm{tr}(\overline{a}x) = s\}| = |\{x \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-1})} : \mathrm{tr}(\overline{b}x) = s\}|$$

i.e.

$$|\{x \in \{\pm 1, \pm i_1\} : \mathrm{tr}(\overline{a}x) = s\}| = |\{x \in \{\pm 1, \pm i_1\} : \mathrm{tr}(\overline{b}x) = s\}|$$

for all $s \in \mathbb{Z}$. Write $a = a_0 + a_1 i_1, b = b_0 + b_1 i_1$ and assume without loss of generality that $a_1, a_2, b_1, b_2 \geq 0$ after some simple orthogonal transformation. Considering $s = 0$, we obtain

$$2 \cdot |\{j \in \{0, 1\} : a_j = 0\}| = 2 \cdot |\{j \in \{0, 1\} : b_j = 0\}|$$

and

$$|\{j \in \{0, 1\} : a_j = s\}| = |\{j \in \{0, 1\} : b_j = s\}|$$

for $s > 0$. Hence, a and b have the same set of components with respect to $1, i_1$ including multiplicity. Thus, there must be a permutation σ, which can be realized in $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$, such that $\sigma(a) = b$. □

We characterize certain elements in $\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$:

Lemma 6.5.2. *a)* $\{a \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})} : N(a) \equiv 0 \bmod 2\} = (1 + i_1)\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$.

b) $\{a \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})} : a \equiv (1 + i_1) \bmod 2\} = \{a \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})} : \nu_2(N(a)) \equiv 1 \bmod 2\}$.

Theorem 6.5.3. *Let $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$ such that $N(a) = N(b)$ and $\nu_2(N(a)) \equiv 1 \bmod 2$. Let $k \in \mathbb{N}$ and $k \equiv 0 \bmod 4$. For $0 \neq \phi \in J^{\mathbb{C}}_{k,1}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$ the following assertions are equivalent:*

i) $\phi[a] = \phi[b]$,

ii) a and b lie in the same $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$*-orbit,*

iii) there exists $\varepsilon \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-1})}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon \overline{b}$, *iv)* ι_a *and* ι_b *are equivalent.*

Proof. The equivalence of ii), iii) and iv) is obvious from the explicit description of $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$ and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The theta decomposition of ϕ is of the form

$$\phi = f_0\vartheta_{\mathfrak{o},0} + f_{\frac{1}{2}}\left(\vartheta_{\mathfrak{o},\frac{1}{2}} + \vartheta_{\mathfrak{o},\frac{1}{2}i_1}\right) + f_{\frac{1+i_1}{2}}\vartheta_{\mathfrak{o},\frac{1+i_1}{2}}.$$

Let

$$D_0 := \vartheta_{\mathfrak{o},0}[a] - \vartheta_{\mathfrak{o},0}[b], \quad D_1 := \vartheta_{\mathfrak{o},0}[a] - \vartheta_{\mathfrak{o},0}[b], \quad D_2 := \vartheta_{\mathfrak{o},\frac{1+i_1}{2}}[a] - \vartheta_{\mathfrak{o},\frac{1+i_1}{2}}[b].$$

Then $\phi[a] = \phi[b]$ implies

$$f_0D_0 + f_{\frac{1}{2}}D_1 + f_{\frac{1+i_1}{2}}D_2 = 0.$$

We apply the elliptic transformation $|_{N(a)}\left[\frac{1}{2},0\right]$ on both sides of this equation. By assumption on $N(a)$ and 6.5.2 we have $\frac{a}{2} \equiv \frac{1+i_1}{2} \bmod \mathfrak{o}$. Hence, by 2.2.10 and 3.1.4,

$$f_0D_2 + f_{\frac{1}{2}}D_1 + f_{\frac{1+i_1}{2}}D_0 = 0.$$

We substract both equations from each other in order to obtain

$$\left(f_0 - f_{\frac{1+i_1}{2}}\right) \cdot (D_0 - D_2) = 0.$$

From $\phi \neq 0$ we obtain $f_0 \neq 0$, hence $f_0 - f_{\frac{1+i_1}{2}} \neq 0$, since $f_{\frac{1+i_1}{2}}(z+1) = -f_{\frac{1+i_1}{2}}(z)$. Consequently, we have $D_0 = D_2$. But the identities $D_0(z+1) = D_0(z)$ and $D_2(z+1) = -D_2(z)$ already imply $D_0 = 0$. By 6.5.1 this proves ii). □

The corresponding result for modular forms is given in the following

Theorem 6.5.4. *Let* $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}$ *such that* $N(a) = N(b)$ *and* $\nu_2(N(a)) \equiv 1 \bmod 2$. *Let* $k \in \mathbb{N}$ *and* $k \equiv 0 \bmod 4$. *Then the following assertions are equivalent:*

i) $F[a] = F[b]$ *for some* $F \in \left[\mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}), k\right]$ *and* $\phi_{1,F} \neq 0$,

ii) $F[a] = F[b]$ *for all* $F \in [\Gamma(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}), k]$,

iii) a *and* b *lie in the same* $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$*-orbit,*

iv) there exists $\varepsilon \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-1})}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon\bar{b}$,

v) M_a *and* M_b *are equivalent in the extended sense.*

Proof. The equivalence of iii), iv) and v) follows from the explicit description of the group $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$ and 6.2.12. Each of them implies ii). Assume that ii) holds. Since $k \equiv 0 \bmod 4$, there exists a hermitian Jacobi form $0 \neq \phi \in J^{\mathbb{C}}_{k,1}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})$. Hence, i) holds with $F = \text{M-Lift}(\phi)$. Note that $\phi_{1,F} = \phi \neq 0$. Furthermore, i) implies iii) by 6.3.16 and 6.5.3. □

Theorem 5.4.13 directly implies

Theorem 6.5.5. *Let $t \in \mathbb{N}$ such that $\nu_2(t) \equiv 1 \bmod 2$. Let $M, N \in \mathrm{Mod}\Big(\Gamma(t), \mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})})\Big)$. Let $k \in \mathbb{N}$ and $k \equiv 0 \bmod 4$. Then the following assertions are equivalent:*

i) $F\big|_k[M] = F\big|_k[N]$ *for some* $F \in [\mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}), k]$ *and* $\phi_{1,F} \neq 0$,

ii) $F\big|_k[M] = F\big|_k[N]$ *for all* $F \in [\Gamma(\mathfrak{o}_{\mathbb{Q}(\sqrt{-1})}), k]$,

iii) M and N are equivalent in the extended sense.

6.5.2 $\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$

Let $\rho := -\frac{1}{2} + i_1\frac{\sqrt{3}}{2}$.

Lemma 6.5.6. *a)* $\mathbb{Z}[i_1\sqrt{3}] = \{x + y\rho \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} : x \in \mathbb{Z}, y \in 2\mathbb{Z}\}$,

b) $\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[i\sqrt{3}] \dot\cup \left(\rho + \mathbb{Z}[i\sqrt{3}]\right)$,

c) for all $w \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$ *there is some* $\varepsilon \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-3})}$ *such that* $w\varepsilon \in \mathbb{Z}[i\sqrt{3}]$.

Proof. a) Follows from $2\rho \in \mathbb{Z}[i_1\sqrt{3}]$ and $x + yi_1\sqrt{3} = (a+1) + 2b\rho$.

b) Follows directly from a).

c) Let $w = x + y\rho$. In the case $y \equiv 0 \bmod 2$ the claim follows from ii). In the case

$$x \equiv 0 \bmod 2, y \equiv 1 \bmod 2$$

one has

$$w\overline{\rho} = x\overline{\rho} + y = -x - x\rho + y = (y - x) - x\rho \in \mathbb{Z}[i_1\sqrt{3}].$$

Let both x, y be odd. Then

$$w\rho = x\rho + y\rho^2 = x\rho + y\overline{\rho} = x\rho + y(-1-\rho) = -y + (x-y)\rho \in \mathbb{Z}[i_1\sqrt{3}]. \qquad \square$$

We characterize the elements of even norm in $\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$:

Lemma 6.5.7.

$$\begin{aligned} 2\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} &= \{w \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} : N(w) \equiv 0 \bmod 2\} \\ &= \{w \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} : N(w) \equiv 0 \bmod 4\}. \end{aligned}$$

Proof. The value $N(x + y\rho) = x^2 - xy + y^2$ for $x, y \in \mathbb{Z}$ is even, if and only if all summands are even, i.e. if and only if $x + y\rho \in 2\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$. Note that in this case one necessarily has $N(x + y\rho) \equiv 0 \bmod 4$. $\square$

Lemma 6.5.8. *Let* $a \in \mathbb{Z}[i_1\sqrt{3}]$ *such that* $N(a) \equiv 1 \bmod 2$*. Then* $\mathrm{tr}(\overline{a}b) \equiv 1 \bmod 2$ *for all* $b \in \rho + \mathbb{Z}[i_1\sqrt{3}]$.

Proof. Let $a = x + y\rho \in \mathbb{Z}[i\sqrt{3}]$. Then $y \in 2\mathbb{Z}$ and from $N(a) \equiv 1 \bmod 2$ we obtain $x \equiv 1 \bmod 2$. As a consequence, $\mathrm{tr}(\overline{a}\rho) = -x + 2y \equiv 1 \bmod 2$. The claim follows then from $\mathrm{tr}(\overline{a}b) \equiv 0 \bmod 2$ for all $b \in \mathbb{Z}[i\sqrt{3}]$. $\square$

Regarding pullbacks of the Jacobi theta functions we need the following

Lemma 6.5.9. *Let* $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$*. Then the following assertions are equivalent:*

i) $\vartheta_{\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})},0}[a] = \vartheta_{\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})},0}[b]$,

ii) *a and b lie in the same* $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$*-orbit,*

iii) *there is* $\varepsilon \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\times}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon \overline{b}$.

Proof. First note that in all cases one has $N(a) = N(b)$. The equivalence of ii) and iii) is clear from the explicit description of $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$. Each of them implies i). Hence we are left to prove that i) implies ii). By 6.5.7 and 6.5.6 we may substitute a, b by $2^{-r}\varepsilon_a a$ resp. $2^{-r}\varepsilon_b b$ for suitable $\varepsilon_a, \varepsilon_b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\times}$ and $r \in \mathbb{N}_0$. Hence we can assume $a, b \in \mathbb{Z}[i\sqrt{3}]$ and $N(a) = N(b) \equiv 1 \bmod 2$. By comparing the Fourier coefficients of index $(1, s), s \in \mathbb{Z}$, we obtain

$$\left|\left\{w \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\times} : \mathrm{tr}(\bar{a}w) = s\right\}\right| = \left|\left\{w \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\times} : \mathrm{tr}(\bar{b}w) = s\right\}\right|.$$

By 6.5.8 we obtain

$$\left|\left\{w \in \mathbb{Z}[i\sqrt{3}]^{\times} : \mathrm{tr}(\bar{a}w) = s\right\}\right| = \left|\left\{w \in \mathbb{Z}[i\sqrt{3}] : \mathrm{tr}(\bar{b}w) = s\right\}\right|,$$

thus

$$|\{w \in \{\pm 1\} : \mathrm{tr}(\bar{a}w) = s\}| = |\{w \in \{\pm 1\} : \mathrm{tr}(\bar{b}w) = s\}|$$

for all even s. Write $a = x + yi_1\sqrt{3}$ and $b = u + vi_1\sqrt{3}$. After some orthogonal transformations applied on a and b we can assume that $x, y, u, v \geq 0$. Then we have $\delta_{s,x} = \delta_{s,u}$ for all $s \geq 0$, which implies $x = u$. From $N(a) = N(b)$ we conclude $y^2 = v^2$, thus $y = v$. □

Theorem 6.5.10. *Let* $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$ *such that* $N(a) = N(b)$ *and* $\nu_3(N(a)) \equiv 1 \bmod 2$. *Let* $k \in \mathbb{N}$ *and* $k \equiv 0 \bmod 6$. *For* $0 \neq \phi \in J_{k,1}^{\mathbb{C}}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$ *the following assertions are equivalent:*

i) $\phi[a] = \phi[b]$,

ii) *a and b lie in the same* $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$*-orbit,*

iii) *there exists* $\varepsilon \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\times}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon \overline{b}$,

iv) ι_a *and* ι_b *are equivalent.*

Proof. The equivalence of ii), iii) and iv) is obvious from the explicit description of $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$ and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The exponent of the discriminant group $\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\sharp}/\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$ is 3 and one has

$$3\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\sharp} \setminus 3\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} = \left\{a \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})} : N(a) \equiv 0 \bmod 3, N(a) \not\equiv 0 \bmod 9\right\}$$

by 5.1.13. By assumption,

$$\nu_3(N(a)) = \nu_3(N(b)) = 2r + 1$$

for some $r \geq 0$. After replacing a resp. b by $3^{-r}a$ resp. $3^{-r}b$, we can assume that

$$a, b \in 3\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}^{\sharp} \setminus 3\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}.$$

Then the claim follows from 3.6.4 together with 6.5.9. □

The corresponding result for modular forms is given in the following

Theorem 6.5.11. *Let* $a, b \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}$ *such that* $N(a) = N(b)$ *and* $\nu_3(N(a)) \equiv 1 \bmod 2$. *Let* $k \in \mathbb{N}$ *and* $k \equiv 0 \bmod 6$. *Then the following assertions are equivalent:*

i) $F[a] = F[b]$ *for some* $F \in [\mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}), k]$ *and* $\phi_{1,F} \neq 0$,

ii) $F[a] = F[b]$ *for all* $F \in [\Gamma(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}), k]$,

iii) a *and* b *lie in the same* $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$*-orbit,*

iv) there exists $\varepsilon \in \mathfrak{o}^{\times}_{\mathbb{Q}(\sqrt{-3})}$ *such that* $a = \varepsilon b$ *or* $a = \varepsilon \overline{b}$,

v) M_a *and* M_b *are equivalent in the extended sense.*

Proof. The equivalence of iii), iv) and v) follows from the explicit description of the group $\mathrm{O}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$ and 6.2.12. Each of them implies ii). Assume that ii) holds. Since $k \equiv 0 \bmod 6$, there exists a hermitian Jacobi form $0 \neq \phi \in J^{\mathbb{C}}_{k,1}(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})$. Hence, i) holds with $F = \text{M-Lift}(\phi)$. Note that $\phi_{1,F} = \phi \neq 0$. Furthermore, i) implies iii) by 6.3.16 and 6.5.10. □

Theorem 5.4.13 directly implies

Theorem 6.5.12. *Let* $t \in \mathbb{N}$ *such that* $\nu_3(t) \equiv 1 \bmod 2$. *Let* $M, N \in \mathrm{Mod}\Big(\Gamma(t), \mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})})\Big)$. *Let* $k \in \mathbb{N}$ *and* $k \equiv 0 \bmod 6$. *Then the following assertions are equivalent:*

i) $F\big|_k[M] = F\big|_k[N]$ *for some* $F \in [\mathrm{U}_2(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}), k]$ *and* $\phi_{1,F} \neq 0$,

ii) $F\big|_k[M] = F\big|_k[N]$ *for all* $F \in [\Gamma(\mathfrak{o}_{\mathbb{Q}(\sqrt{-3})}), k]$,

iii) M *and* N *are equivalent in the extended sense.*

6.5.3 Hurwitz order

Let $\mathcal{O}$ denote the Hurwitz quaternions and $\pi := (1 + i_1)$ a slight abuse of notation.

Lemma 6.5.13. *Let $a \in \mathcal{O}$ and $r \in \mathbb{N}_0$. Then for all $(z, w) \in \mathcal{H} \times \mathbb{H}_\mathbb{C}$ one has*

$$\vartheta_\mathcal{O}[\pi^r a](z, w) = \begin{cases} \vartheta_\mathcal{O}[a](z, 2^{r/2} \cdot w), & r \equiv 0 \bmod 2, \\ -2 \cdot \vartheta_\mathcal{O}[a]\big|_{2,N(a)} J\big(2z, 2^{\frac{r-1}{2}+1} \cdot w\big), & r \equiv 1 \bmod 2. \end{cases}$$

Proof. Assume that $r \equiv 0 \bmod 2$. Then one has $\pi^r = 2^{r/2}\varepsilon$ for some $\varepsilon \in \mathcal{O}^\times$. Hence,

$$\vartheta_\mathcal{O}[\pi^r a](z, w) = \vartheta_\mathcal{O}(z, \pi^r a w) = \vartheta_\mathcal{O}(z, 2^{r/2}\varepsilon a w) = \vartheta_\mathcal{O}[\varepsilon a](z, 2^{r/2} w) = \vartheta_\mathcal{O}[a](z, 2^{r/2} w).$$

In the remaining case, $r - 1$ is even and $\pi^{r-1} = 2^{\frac{r-1}{2}}\varepsilon$ for some $\varepsilon \in \mathcal{O}^\times$, that means $\pi^r a = 2^{\frac{r-1}{2}}\varepsilon\pi a$. Hence,

$$\vartheta_\mathcal{O}[\pi^r a](z, w) = \vartheta_\mathcal{O}\left[2^{\frac{r-1}{2}}\varepsilon\pi a\right](z, w) = \vartheta_\mathcal{O}[\pi a](z, 2^{\frac{r-1}{2}} w).$$

Therefore it suffices to prove the claim in case $r = 1$. In this case, we calculate

$$\begin{aligned} \vartheta_\mathcal{O}[\pi a](z, w) &= \sum_{g \in \mathcal{O}} e^{2\pi i (N(g) z + \operatorname{tr}(\overline{g}\pi a) w)} \\ &= \sum_{g \in \mathcal{O}} e^{2\pi i (N(g) z + \operatorname{tr}(\overline{\overline{\pi} g} a) w)} \\ &= \sum_{g \in \mathcal{O}} e^{2\pi i \left(N(\frac{1}{2}\overline{\pi} g) \cdot 2z + \operatorname{tr}(\overline{\frac{1}{2}\overline{\pi} g} a) \cdot 2w \right)}, \end{aligned}$$

since $N(\frac{1}{2}\overline{\pi}) = \frac{1}{2}$. From $\frac{1}{2}\overline{\pi}\mathcal{O} = \mathcal{O}^\sharp$ we deduce, that the last sum equals

$$\sum_{g \in \mathcal{O}^\sharp} e^{2\pi i (N(g) \cdot 2z + \operatorname{tr}(\overline{g} a) \cdot 2w)} = -2 \cdot \vartheta_\mathcal{O}[a]\big|_{2,N(a)} J(2z, 2w).$$

□

Corollary 6.5.14. *Let $a, b \in \mathcal{O}$ and $r \in \mathbb{N}_0$. Then the following assertions are equivalent:*

i) $\vartheta_\mathcal{O}[a] = \vartheta_\mathcal{O}[b]$,

ii) $\vartheta_\mathcal{O}[\pi^r a] = \vartheta_\mathcal{O}[\pi^r b]$.

Regarding pullbacks of the Jacobi theta functions we need the following

Lemma 6.5.15. *Let $a, b \in \mathcal{O}$. The following assertions are equivalent:*

i) $\vartheta_\mathcal{O}[a] = \vartheta_\mathcal{O}[b]$,

ii) a and b lie in the same $\mathrm{O}(\mathcal{O})$*-orbit,* *iii) there are* $\delta, \varepsilon \in \mathcal{O}^\times$ *and* $r \in \{0,1\}$ *such that* $a = \pi^{-r}\overline{\varepsilon}\delta b \varepsilon \pi^r$ *or* $a = \pi^{-r}\overline{\varepsilon}\delta \overline{b} \varepsilon \pi^r$*,*

iv) ι_a *and* ι_b *are equivalent.*

Proof. First note that in all cases one has $N(a) = N(b)$. The equivalence of ii), iii) and iv) is obvious from the explicit description of $\mathrm{O}(\mathcal{O})$ and 1.2.22. Each of them implies i). Hence we are left to prove that i) implies ii). First assume that $N(a) = N(b)$ is odd. After multiplication with some suitable unit, we can assume that $a, b \in \Lambda$, say

$$a = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3, \quad b = b_0 + b_1 i_1 + b_2 i_2 + b_3 i_3.$$

Furthermore, $\mathrm{tr}(\overline{\omega} a) \equiv 1 \bmod 2\mathbb{Z}$. Consequently, $\mathrm{tr}((\overline{g + \omega})a) \equiv 1 \bmod 2\mathbb{Z}$ for all $g \in \Lambda$. By comparing the Fourier coefficients of index $(1, s), s \in \mathbb{Z}$, we obtain

$$|\{w \in \mathcal{O}^\times : \mathrm{tr}(\bar{a}w) = s\}| = |\{w \in \mathcal{O}^\times : \mathrm{tr}(\bar{b}w) = s\}|.$$

Consequently, we deduce

$$\left|\{x \in \Lambda^\times : \mathrm{tr}(\bar{a}w) = 2s\}\right| = \left|\{g \in \Lambda^\times : \mathrm{tr}(\bar{a}w) = 2s\}\right|$$

for all $s \in \mathbb{Z}$. Since $\mathrm{O}(\Lambda)$ contains all signed permutations, we can assume $a_i, b_i \geq 0$ for all $i = 0, \ldots, 3$. In this case, the previous identity simplifies to

$$|\{i \in \{0,1,2,3\} : a_i = s\}| = |\{i \in \{0,1,2,3\} : b_i = s\}|$$

for all $s \geq 0$. Hence the sets of coefficients of a and b with respect to $1, i_1, i_2, i_3$ coincide including multiplicity. Since $\mathrm{O}(\Lambda)$ contains all permutations we have $\sigma(a) = b$ for some $\sigma \in \mathrm{O}(\Lambda)$, which can also be considered as an element of $\mathrm{O}(\mathcal{O})$. In the remaining case, we assume that both $N(a)$ and $N(b)$ are even. Let $r = \nu_2(N(a)) = \nu_2(N(b))$. From 6.5.14 and the first part, there is $\sigma \in \mathrm{O}(\mathcal{O})$ such that $\sigma(\pi^{-r}a) = \pi^{-r}b$ or equivalently

$$\pi^r \sigma(\pi^{-r}a) = b, \text{ i.e. } (l_\pi^r \sigma l_\pi^{-r})(a) = b,$$

if l_π denotes the left-multiplication by π. Since

$$\pi^r \mathcal{O} = \{g \in \mathcal{O} : \nu_2(N(g)) \geq r\}$$

and $\pi \in \mathcal{I}(\mathcal{O})$ we have

$$l_\pi^{-r} \mathrm{O}(\mathcal{O}) l_\pi^r = \mathrm{O}(\mathcal{O})$$

due to the finiteness of $\mathrm{O}(\mathcal{O})$. Thus, $l_\pi^r \sigma l_\pi^{-r} \in \mathrm{O}(\mathcal{O})$. □

Theorem 6.5.16. *Let* $a, b \in \mathcal{O}$ *such that* $N(a) = N(b)$ *and* $\nu_2(N(a)) \equiv 1 \bmod 2$*. Let* $k \geq 4$ *and* $k \equiv 0 \bmod 2$*. For* $0 \neq \phi \in J_{k,1}^{\mathbb{H}}(\mathcal{O})$ *the following assertions are equivalent:*

i) $\phi[a] = \phi[b]$*,*

ii) a *and* b *lie in the same* $\mathrm{O}(\mathcal{O})$*-orbit,* *iii) there are* $\delta, \varepsilon \in \mathcal{O}^\times$ *and* $r \in \mathbb{N}_0$ *such that* $a = \pi^{-r}\overline{\varepsilon}\delta b \varepsilon \pi^r$ *or* $a = \pi^{-r}\overline{\varepsilon}\delta\overline{b}\varepsilon\pi^r$*,*

iv) ι_a *and* ι_b *are equivalent.*

Proof. The equivalence of ii), iii) and iv) is obvious from the explicit description of $\mathrm{O}(\mathcal{O})$ and 1.2.22. Each of them implies i). Hence it remains to show, that i) implies one of the remaining assertions. The exponent of the discriminant group $\mathcal{O}^\sharp/\mathcal{O}$ is 2 and one has

$$2\mathcal{O}^\sharp \setminus 2\mathcal{O} = \{g \in \mathcal{O} : N(g) \equiv 0 \bmod 2, N(g) \not\equiv 0 \bmod 4\}.$$

By assumption,

$$\nu_2(N(a)) = \nu_2(N(b)) = 2r + 1$$

for some $r \geq 0$. After replacing a resp. b by $2^{-r}a$ resp. $2^{-r}b$, we can assume, that

$$a, b \in 2\mathcal{O}^\sharp \setminus 2\mathcal{O}.$$

Then the claim follows from 3.6.4 together with 6.5.15. □

The corresponding result for modular forms is given in the following

Theorem 6.5.17. *Let* $a, b \in \mathcal{O}$ *such that* $N(a) = N(b)$ *and* $\nu_2(N(a)) \equiv 1 \bmod 2$*. Let* $k \geq 4$ *and* $k \equiv 0 \bmod 2$*. Then the following assertions are equivalent:*

i) $F[a] = F[b]$ *for some* $F \in [\mathrm{U}_2(\mathcal{O}, k]$ *and* $\phi_{1,F} \neq 0$*,*

ii) $F[a] = F[b]$ *for all* $F \in [\Gamma(\mathcal{O}), k]$*,*

iii) a and b lie in the same $\mathrm{O}(\mathcal{O})$*-orbit,*

iv) there are $\delta, \varepsilon \in \mathcal{O}^\times$ *and* $r \in \{0, 1\}$ *such that* $a = \pi^{-r}\overline{\varepsilon}\delta b \varepsilon \pi^r$ *or* $a = \pi^{-r}\overline{\varepsilon}\delta\overline{b}\varepsilon\pi^r$*,*

v) M_a *and* M_b *are equivalent in the extended sense.*

Proof. The equivalence of iii), iv) and v) follows from the explicit description of the group $\mathrm{O}(\mathcal{O})$ and 6.2.12. Each of them implies ii). Assume that ii) holds. Since $k \equiv 0 \bmod 2$ and $k \geq 4$, there exists a quaternionic Jacobi form $0 \neq \phi \in J^{\mathbb{H}}_{k,1}(\mathcal{O})$. Hence, i) holds with $F = \text{M-Lift}(\phi)$. Note that $\phi_{1,F} = \phi \neq 0$. Furthermore, i) implies iii) by 6.3.16 and 6.5.16. □

Theorem 5.4.13 directly implies

Theorem 6.5.18. *Let* $t \in \mathbb{N}$ *such that* $\nu_2(t) \equiv 1 \bmod 2$*. Let* $M, N \in \mathrm{Mod}\,(\Gamma(t), \mathrm{U}_2(\mathcal{O}))$*. Let* $k \geq 4$ *and* $k \equiv 0 \bmod 2$*. Then the following assertions are equivalent:*

i) $F\big|_k[M] = F\big|_k[N]$ *for some* $F \in [\mathrm{U}_2(\mathcal{O}), k]$ *and* $\phi_{1,F} \neq 0$*,*

ii) $F\big|_k[M] = F\big|_k[N]$ *for all* $F \in [\Gamma(\mathcal{O}), k]$*,*

iii) M and N are equivalent in the extended sense.

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Symbols

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