On upper bounds for waiting times for
doubly nonlinear parabolic equations

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der
Rheinisch-Westfälischen Technischen Hochschule Aachen zur Erlangung des
akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

vorgelegt von

Diplom-Mathematiker

Kianhwa Colin Djie

aus Aachen

Berichter: Universitätsprofessor Dr. rer. nat. Michael Wiegner
Universitätsprofessor Dr. rer. nat. Josef Bemelmans

Tag der mündlichen Prüfung: 14. Februar 2008

Diese Dissertation ist auf den Internetseiten der Hochschulbibliothek online verfügbar.
Contents

Introduction .............................................. 5
Notation .................................................... 6

1 The model type of the doubly nonlinear diffusion equation 9
1.1 Existence of a radially symmetric solution .................. 10
1.2 The doubly nonlinear diffusion equation with weak absorption resp. reaction term ........................................... 13
1.3 HARDY’s inequality ..................................... 13
1.4 The case $\alpha > q - 1$ .................................. 18

2 Non-weak absorption and reaction terms and variants 27
2.1 The critical case $\alpha = q - 1$ ............................ 27
2.2 The supercritical case $\alpha < q - 1$ ......................... 30
2.3 Variant: Convection and advection terms ................... 37

3 On a system with coupled reaction terms 43

Bibliography ............................................. 49
Introduction

One of the first examples for modelling time-dependent natural processes by partial differential equations are solutions \( u : \mathbb{R}^N \times [0, \infty) \to \mathbb{R} \) of the classical heat equation

\[
  u_t - \Delta u = 0 \, , \, u(\cdot,0) = u_0
\]

for the Laplacian \( \Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \). The study of their qualitative behaviour gives an indication for the quality of the model. One criterion which apparently fails in this linear model is the finite speed of propagation: If \( u_0 \geq 0 \), \( u_0 \not\equiv 0 \) is a nonnegative initial value, then one easily obtains \( u(x,t) > 0 \) for all \( x \in \mathbb{R}^N \) and \( t > 0 \). Therefore solutions of the classical heat equation have no finite speed of propagation. These missing more realistic phenomena can be achieved by the use of nonlinearities. As a typical model example serves the porous medium equation

\[
  u_t - \Delta u^m = 0 \, , \, m > 1 \, ,
\]

which is meanwhile sufficiently examined.

In case of finite speed of propagation one can hope for the existence of waiting time phenomena: Let \( \Omega \) be the support of the initial value

\[
  \Omega := \text{supp}(u_0) := \{x \in \mathbb{R}^N \mid u_0(x) \neq 0\} .
\]

Then the waiting time \( t^*_\Omega \) denotes the time in which the solution remains within the initial support \( \text{supp}(u_0) \) until it starts to spread out of it:

\[
  t^*_\Omega := \sup \{ t \geq 0 \mid u(x,\tau) = 0 \ \forall x \in \mathbb{R}^N \setminus \Omega \ \forall \tau \in [0, t] \} .
\]

Solutions have the waiting time phenomenon, if one can guarantee that the solution remains in the initial support for a certain explicitly given positive time (depending on the initial value near \( \partial \Omega \)). This can be proven for the doubly nonlinear differential equation

\[
  u_t - \Delta_p u^m = 0
\]

with the \( p \)-Laplacian operator \( \Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) \), which generalizes the porous medium equation \( (p = 2) \). This equation can be rewritten in the form \( (u^{q-1})_t - \Delta_p v = 0 \) by means of the transformation \( u^m = v \) and \( q - 1 = \frac{1}{m} \). One is interested in quantitative upper bounds for the waiting time depending on the growth of the initial value \( u_0 \) near the boundary of its support \( \Omega \). Whereas everything is already known for the porous medium
equation with respect to waiting times (lower and upper bounds), nothing is known about quantitative upper bounds for the waiting time beyond the porous medium equation.

For this equation we will obtain a result in the following form: If $x_0 \in \partial \Omega$ is a boundary point of the initial support $\Omega$ and the initial value $u_0$ grows near $x_0$ as $u_0(x) \geq A \|x - x_0\|^\gamma$ in a ball $B \subset \Omega$ with $x_0 \in \partial B$, then there exists a nonnegative solution of $(u^{q-1})_t - \Delta_p u = 0$ (with $1 < q < p$ and $p \geq 2$) whose waiting time can be estimated in the following way:

- $0 < \gamma < \frac{p}{p-q}$ implies that the waiting time is zero.
- $\gamma = \frac{p}{p-q}$ implies $t_\Omega^* \leq C A^{-(p-q)}$.

Because there exist lower bounds of the type $t_\Omega^* \geq C B^{-q(p-q)}$, if $u_0(x) \leq B \|x - x_0\|^\gamma$ for $\gamma = \frac{p}{p-q}$, the exponent $\frac{p}{p-q}$ turns out to be critical.

The lower bound approach does not work anymore, if one adds a reaction term like in

$$(u^{q-1})_t - \Delta_p u - u^\alpha = 0.$$

Nevertheless, we will achieve upper bounds for the waiting time for the more general equations of the type

$$(u^{q-1})_t - \Delta_p u \pm \lambda u^\alpha = 0.$$

We will see that the new reaction/absorption term has significant influence, if $\alpha \leq q - 1$. Especially the critical exponent $\frac{p}{p-q}$ changes to $\frac{p}{p-1-\alpha}$.

These (and other similar) results will be obtained with the aid of energy methods. This technique also works for the coupled system with reaction terms

$$\begin{cases} u_t - \Delta u^m - v^\alpha & = 0, \\ v_t - \Delta v^n - u^\beta & = 0, \end{cases}$$

as one can see in the third chapter.

Parts of the first chapter have already been published in the journal *Interfaces and Free Boundaries*, [10].

Now, it is my pleasure to express my gratitude to Prof. Dr. Michael Wiegner for his supervision of my work. I am also grateful for Dr. Michael Winkler whose discussions were often helpful to me. Moreover I like to thank Ellen Behnke, Tatjana Gerzen, Dr. Hans Jürgen Heep and especially Christian Stinner who created the pleasant atmosphere in which I really enjoyed working.

### Notation

We let $\mathbb{R}$ denote the field of the real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ the natural numbers without zero. Adding 0 leads to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The vector space $\mathbb{R}^N := \mathbb{R} \times \cdots \times \mathbb{R}$ as the cartesian product with $N$ factors ($N \in \mathbb{N}$) is usually equipped with the standard scalar
product \( \langle x, y \rangle := \sum_{i=1}^{N} x_i y_i \). Therefore we use the Euclidean norm \( \| x \| := \sqrt{\sum_{i=1}^{N} x_i^2} \) if not otherwise stated. All balls are usually denoted in this norm, i.e.

\[
B(x, R) := \{ y \in \mathbb{R}^N \mid \| x - y \| < R \}
\]

the open ball with center \( x \in \mathbb{R}^N \) and radius \( R > 0 \) resp.

\[
\overline{B}(x, R) := \{ y \in \mathbb{R}^N \mid \| x - y \| \leq R \}
\]

the closed ball with center \( x \in \mathbb{R}^N \) and radius \( R > 0 \). For a given set \( A \subset \mathbb{R}^N \) let \( A^\circ \) denote the interior, \( \partial A \) the boundary and \( \overline{A} \) the closure of \( A \) in this topology. We write \( B \subset \subset A \) if the closure \( B \) is a compact subset of \( A \). The distance \( \text{dist}(x_0, A) \) between a point \( x_0 \in \mathbb{R}^N \) and a set \( A \subset \mathbb{R}^N \) is defined by \( \text{dist}(x_0, A) := \inf\{\| x_0 - y \| \mid y \in A \} \).

Let \( u : D \to \mathbb{R} \) for \( D \subset \mathbb{R}^N \) be an arbitrary real-valued function. Then we call \( \text{supp}(u) := \{ x \in D \mid u(x) \neq 0 \} \) the support of \( u \). If \( v : D' \subset \mathbb{R}^N \to \mathbb{R} \) is another real-valued function, then we set \( \{ u < v \} := \{ x \in D \cap D' \mid u(x) < v(x) \} \). Similarly one defines \( \{ u \leq v \} \) and so on.

For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N \) we set \( |\alpha| := \sum_{i=1}^{N} \alpha_i \). We can write partial derivatives with the aid of multi-indices: \( D^\alpha u := \frac{\partial^{||\alpha||}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} u \). The gradient is abbreviated as \( \nabla u := (\frac{\partial}{\partial x_1} u, \ldots, \frac{\partial}{\partial x_N} u) \). We denote the Laplacian of a vector field as \( u : D \subset \mathbb{R}^N \to \mathbb{R}^N \) as

\[
\Delta u := \text{div}(\nabla u) = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} u_i
\]

and the \( p \)-Laplacian as

\[
\Delta_p u := \text{div}(\| \nabla u \|^{p-2} \nabla u)
\]

for \( p \geq 2 \). If \( u \) depends on a “spatial” variable \( x \in D \subset \mathbb{R}^N \) and on a “time” variable \( t \in I \subset \mathbb{R} \), then the expressions \( \nabla u, \Delta u, \Delta_p u \) apply only with respect to the “spatial” variable \( x \).

For \( 1 \leq p \leq \infty \) and a Lebesgue-measurable set \( \Omega \subset \mathbb{R}^N \) we denote in the standard way with \( L^p(\Omega) \) the Lebesgue-space of real-valued measurable functions with finite norm \( \| u \|_{L^p(\Omega)} := (\int_{\Omega} |u|^p dx)^{1/p} \), if \( p < \infty \), and \( \| u \|_{L^\infty(\Omega)} := \text{esssup}_{x \in \Omega} |u(x)| \). We denote the Sobolev spaces

\[
W^{m,p}(\Omega) := \{ u : \Omega \to \mathbb{R} \mid D^\alpha u \in L^p(\Omega) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \leq m \}
\]

which shall be equipped with the norm

\[
\| u \|_{W^{m,p}(\Omega)} := \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^p(\Omega)}
\]

The Lebesgue spaces and Sobolev spaces are of course Banach spaces together with the norms given above.
Let $A \subset \mathbb{R}^N$ be open. Then we denote

$$C^k(A) := \{ u : A \to \mathbb{R} \mid D^\alpha u \text{ exists and is continuous for every multi-index } \alpha \text{ with } |\alpha| \leq k \}$$

and $C^\infty(A) := \bigcap_{k=1}^{\infty} C^k(A)$. $C^\infty_0(A)$ stands for the set of all smooth functions whose support is a compact subset of $A$.

If we are concerned with vector-valued functions $u : \Omega \subset \mathbb{R}^N \to X$ wherein $X$ denotes a Banach space, we define the function spaces in a similar way and arrive at $L^p(\Omega; X)$, $W^{m,p}(\Omega; X)$, $C^k(A; X)$ and so on. One has only to replace the Lebesgue integral by a Lebesgue-Bochner integral.
Chapter 1

The model type of the doubly nonlinear diffusion equation

We are concerned with the doubly nonlinear diffusion equation in $\mathbb{R}^N$

$$
\begin{aligned}
\begin{cases}
(|u|^{q-2}u)_t - \Delta_p(u) &= 0 &\text{in } \mathbb{R}^N \times [0, \infty), \\
u(x, 0) &= u_0(x) &\text{for all } x \in \mathbb{R}^N,
\end{cases}
\end{aligned}
$$

(1.1)

with $\Delta_p(u) = \text{div}(\| \nabla u \|^{p-2} \nabla u)$, parameters

$$p \geq 2, \ 1 < q < p,$$

and an initial value $u_0 : \mathbb{R}^N \to \mathbb{R}$ which fulfils

$$|u_0|^{q-1} \in L^1(\mathbb{R}^N).$$

The special case $p = 2$ leads to the porous medium equation and the case $q = 2$ is the parabolic $p$-LAPLACIAN equation. By means of a simple transformation this equation is equivalent to the type $v_t - \Delta_p(|v|^{m-1}v) = 0$ for $m = \frac{1}{q-1}$.

**Definition 1.0.1 (Weak solution of (1.1))**

A measurable function $u : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ is called weak solution of (1.1), if for all bounded open sets $\Omega \subset \mathbb{R}^N$ and $T > 0$ holds

$$\| \nabla u \|^{p-1} \in L^1(\Omega \times (0, T))$$

and

$$|u|^{q-2}u \in C^0([0, \infty); L^1(\Omega))$$

as well as

$$
\int_\Omega |u|^{q-2}u \varphi(x, t)dx - \int_0^t \int_\Omega (|u|^{q-2}u)\varphi_t + \int_0^t \int_\Omega \| \nabla u \|^{p-2} \langle \nabla u, \nabla \varphi \rangle = \int_\Omega |u_0|^{q-2}u_0 \varphi(x, 0)dx
$$

for all $\varphi \in W^{1,\infty}([0, \infty); L^\infty(\Omega)) \cap L^\infty([0, \infty); W^{1,\infty}_0(\Omega))$ and $t > 0$. 
The existence of a weak solution (for \( |u_0|^{r-1} \in L^1(\mathbb{R}^N) \)) is proven in [13], see below. However, the uniqueness of a weak solution is hitherto unknown.

We will supplement the proof of the existence in [19, 13] by the following arguments, because one needs additional properties of the solution, namely the conservation of radial symmetry as well as the sign of the radial derivative of the initial value and the existence of comparable solutions for comparable initial values.

### 1.1 Existence of a radially symmetric solution

**Lemma 1.1.1**

For all \( \gamma \in O(N) := \{ A \in \mathbb{R}^{N \times N} \mid A^T = A^{-1} \}, \nu \geq 0, \) and the notation \( f_\gamma(x) := f(\gamma^{-1}x) \) for smooth \( f \) holds

\[
\text{div}(\|\nabla u_\gamma\|^2 + \nu \frac{p-2}{2} \nabla u_\gamma) = (\text{div}(\|\nabla u\|^2 + \nu \frac{p-2}{2} \nabla u))_\gamma.
\]

**Proof:** It is

\[
\text{div}(\|\nabla u_\gamma\|^2 + \nu \frac{p-2}{2} \nabla u_\gamma) = \langle \nabla (\|\nabla u_\gamma\|^2 + \nu \frac{p-2}{2} \nabla u_\gamma), \nabla u_\gamma \rangle + (\|\nabla u_\gamma\|^2 + \nu \frac{p-2}{2} \Delta u_\gamma)
\]

\[
= \langle \nabla ((\nabla u_\gamma)^{-1})^2 + \nu \frac{p-2}{2}, (\nabla u_\gamma)^{-1} \rangle + (\|\nabla u_\gamma\|^2 + \nu \frac{p-2}{2} (\sum_{j=1}^{\infty} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} (u_\gamma))
\]

\[
= \langle \nabla (\|\nabla u\|^2 + \nu \frac{p-2}{2}), (\nabla u)^{-1} \rangle + (\|\nabla u\|^2 + \nu \frac{p-2}{2} (\sum_{j=1}^{\infty} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} (u))
\]

\[
= ((\nabla (\|\nabla u\|^2 + \nu \frac{p-2}{2}), (\nabla u)^{-1} \rangle) (\|\nabla u\|^2 + \nu \frac{p-2}{2} (\Delta u))
\]

\[
= (\text{div}(\|\nabla u\|^2 + \nu \frac{p-2}{2} \nabla u))_\gamma
\]

because the orthogonal matrix \( \gamma^{-1} \) induces an isometry as a linear map. \qed

To the definition of \( H^{2+\beta,1+\frac{\beta}{4}}(\Omega \times [0,T]) \) one might have a look at page 7, [14].

**Theorem 1.1.2 (Existence of a classical solution of the regularized problem)**

Let \( \varepsilon, \nu \in (0,1], \Omega := B(0,n) \) and \( u_0 \in C_0^\infty(\Omega) \) with \( u_0 \geq 0 \). Then there exists a classical solution \( u = u_{\varepsilon,\nu} \in \bigcap_{T>0} \bigcap_{\beta>0} H^{2+\beta,1+\frac{\beta}{4}}(\Omega \times [0,T]) \) of the regularized initial-boundary-value problem

\[
\begin{cases}
(u^{q-2}u)_t - \text{div}(\|\nabla u\|^2 + \nu \frac{p-2}{2} \nabla u) = 0, & (x,t) \in \Omega \times (0,\infty), \\
u u(x,t) = \varepsilon, & (x,t) \in \partial\Omega \times [0,\infty), \\
u u(x,0) = u_0(x)+\varepsilon, & x \in \Omega,
\end{cases}
\]

with \( \varepsilon \leq u \leq ||u_0||_{L^\infty(\Omega)} + \varepsilon \).

If \( v_0 \in C_0^\infty(\Omega) \) with \( u_0 \leq v_0 \), then \( u \leq v \) for the solution \( v \) of (1.2) with initial value \( v_0 \) which is constructed in this proof.

If \( u_0 \) is additionally radially symmetric with nonpositive radial derivative \( (u_0)_r \leq 0 \), then the solution \( u \) constructed in this proof is radially symmetric with \( u_0 \leq 0 \).
The differential equation (1.4) has the form

\[ \varphi_\varepsilon(z) = \begin{cases} (\frac{z}{\varepsilon})^{2-q}, & \text{if } z \leq \frac{\varepsilon}{2}, \\ z^{2-q}, & \text{if } z \in [\varepsilon, M+\varepsilon], \\ (M+2\varepsilon)^{2-q}, & \text{if } z \geq M+2\varepsilon. \end{cases} \]

Then we examine the problem

\[ \begin{align*}
& u_t - \frac{1}{q-1} \varphi_\varepsilon(u) \text{div}(\| \nabla u \|^2 + \nu)^{\frac{p-2}{2}} \nabla u = 0, & (x,t) \in \Omega \times (0,\infty), \\
& u(x,t) = \varepsilon, & (x,t) \in \partial \Omega \times [0,\infty), \\
& u(x,0) = u_0(x) + \varepsilon, & x \in \Omega.
\end{align*} \tag{1.3} \]

The differential equation in (1.3) has the form

\[ u_t - \sum_{i,j=1}^N A_{ij}(x,t,u,\nabla u)u_{x_i x_j} = 0 \]

with \( A = (A_{ij})_{i,j=1,...,n} := B + C \) as well as

\[ B(x,t,z,\eta) := \frac{p-2}{q-1} \varphi_\varepsilon(z)(\| \eta \|^2 + \nu)^{\frac{p-2}{2}} \cdot (\eta \cdot \eta^T) \]

and

\[ C(x,t,z,\eta) := \frac{1}{q-1} \varphi_\varepsilon(z)(\| \eta \|^2 + \nu)^{\frac{p-2}{2}} I_N \]

for \( x \in \Omega, t \in [0,\infty), z \in \mathbb{R}, \) and \( \eta \in \mathbb{R}^N. \)

Due to Theorem VI.4.1, [14], the equation (1.3) possesses a unique classical solution \( u_{\varepsilon,\nu} \in \bigcap_{T>0} \bigcap_{\beta>0} H^{2+\beta,1+\frac{\beta}{2}}(\Omega \times [0,T]) \) (and analogously \( v_{\varepsilon,\nu} \) for the initial value \( v_0 \)).

According to the usual comparison theorem, e. g. Theorem 3.12, [16], one obtains \( \varepsilon \leq u_{\varepsilon,\nu} \leq v_{\varepsilon,\nu} \leq M + \varepsilon. \) Therefore both \( u_{\varepsilon,\nu} \) and \( v_{\varepsilon,\nu} \) are classical solutions of (1.2).

Let \( u_0 \) be radially symmetric with nonpositive radial derivative. Because of the radial symmetry of \( u_0 + \varepsilon \) and due to Lemma 1.1.1 \( (u_{\varepsilon,\nu})_\gamma \) is also a solution of (1.3) for all \( \gamma \in O(N). \)

One obtains from the uniqueness \( u_{\varepsilon,\nu} = (u_{\varepsilon,\nu})_\gamma. \) Therefore \( u_{\varepsilon,\nu} \) is radially symmetric, i. e. \( u_{\varepsilon,\nu}(x,t) = \tilde{u}(\| x \|, t) \) for a suitable \( \tilde{u} = \tilde{u}(r, t) : [0,n] \times [0,\infty) \rightarrow [\varepsilon, M+\varepsilon]. \)

Because \( u \) is smooth (also in the origin), it holds \( \frac{\partial u}{\partial r}(0,0) = 0 \) for all \( k \in \mathbb{N} \) and \( t \in [0,\infty). \)

Now, \( \tilde{u} \) is a classical solution of the following differential equation for every \( T > 0): \)

\[ \begin{align*}
& \left( \tilde{u}^{q-1} \right)_t r^{N-1} - \left[ \left( \frac{\tilde{u}(r,t)}{\tilde{u}_r(t)} \right)^{\frac{2}{2-q}} r^{N-1} \tilde{u}_r \right]_r = 0, & \text{if } (r,t) \in (0,n) \times (0,T), \\
& \tilde{u}(r,t) = u_{\varepsilon,\nu}(r,t), & \text{if } r \in \{0,n\}, t \in [0,T], \\
& \tilde{u}(r,0) = \tilde{u}_0(r) + \varepsilon, & r \in (0,n).
\end{align*} \tag{1.4} \]

The differential equation (1.4) has the form

\[ \tilde{u}_t - a_{11}(r,t,\tilde{u}_r) \tilde{u}_{rr} + a(r,t,\tilde{u}_r) = 0 \]
Chapter 1  The model type of the doubly nonlinear diffusion equation

\[ a_{11}(r, t, \eta) := \frac{1}{q-1} \varphi_\varepsilon(\bar{u}(r, t))(\eta^2 + \nu)^{\frac{p-2}{2}}\left\{ (p-1)\eta^2 + \nu \right\} > 0 \]

and

\[ a(r, t, \eta) := -\frac{N-1}{q-1} \frac{\bar{u}_r(r, t)}{r} \cdot \varphi_\varepsilon(\bar{u}(r, t))(\eta^2 + \nu)^{\frac{p-2}{2}} \]

for \( r \in (0, n) \), \( t \in (0, T] \), and \( z, \eta \in \mathbb{R} \). Moreover is \( \bar{u}_r(n, t) \leq 0 \), because \( \bar{u}(n, t) = \varepsilon \leq \bar{u} \).

Now, one differentiates (1.4) with respect to \( \bar{r} \) and obtains thus as differential equation for \( \bar{u}_r \)

\[
\begin{cases}
\left( \bar{u}_r(t) - a_{11}(r, t, \bar{u}_r)(\bar{u}_r)_{rr} + b(r, t, \bar{u}_r, (\bar{u}_r)_r) \right) = 0, & \text{if } (r, t) \in (0, n) \times (0, T], \\
\bar{u}_r(r, t) \leq 0, & \text{if } r \in \{0, n\}, t \in [0, T], \\
\bar{u}_r(r, 0) = (\bar{u}_0)_r(r) \leq 0, & r \in (0, n),
\end{cases}
\]

with

\[ b(r, t, \bar{u}_r, (\bar{u}_r)_r) = (a(r, t, \bar{u}_r))_r - (a_{11}(r, t, \bar{u}_r))(\bar{u}_r)_r, \]

hence

\[ b(r, t, \bar{u}_r, (\bar{u}_r)_r) = \frac{N-1}{q-1} (z^2 + \nu)^{\frac{p-2}{2}}\left\{ (\eta^2 - \frac{z^2}{\nu}) \varphi_\varepsilon(\bar{u}(r, t))(z^2 + \nu) + \frac{\bar{u}_r(r, t)}{r} \right\} z^2 + \nu \]

\[ \frac{1}{q-1} (z^2 + \nu)^{\frac{p-2}{2}} \eta \left[ \varphi'_\varepsilon(\bar{u}(r, t)) z (z^2 + \nu) (p-1)z^2 + \nu \right] + \varphi_\varepsilon(\bar{u}(r, t)) z \eta((p-1)(p-2)z^2 + 3(p-2)\nu) \]

Due to the comparison principle, e.g. Theorem 3.12, [16], one concludes for every \( \gamma \in (0, n) \)

\[ \bar{u}_r(r, t) \leq \max\{\max\{0, \bar{u}_r(\gamma, \tau)\} | \tau \in [0, T]\} \text{ for all } (r, t) \in [\gamma, n] \times [0, T], \]

therefore it is \( \bar{u}_r \leq 0 \) on \([0, n] \times [0, T]\) with \( \gamma \searrow 0 \) and because of the uniform continuity of \( \bar{u}_r \).

Taking the limit \( \varepsilon \searrow 0 \) and \( \nu \searrow 0 \) like in [19, 13] one obtains this way the following existence theorem:

**Theorem 1.1.3 (Existence of a weak solution in \( \mathbb{R}^N \), [19, 13])**

For every \( u_0 \geq 0 \) with \( u_0^{g-1} \in L^1(\mathbb{R}^N) \) there is a weak solution \( u \geq 0 \) of (1.1) with initial value \( u_0 \). If \( u_0 \leq v_0 \) with \( v_0^{g-1} \in L^1(\mathbb{R}^N) \), then one has the existence of weak solutions \( u, v \geq 0 \) with initial values \( u_0, v_0 \) and the additional property \( u \leq v \) pointwise almost everywhere.

If additionally \( u_0 \) is radially symmetric with \( (u_0)_r \leq 0 \), then the solution \( u \) constructed here has a radially symmetric representative with \( u_r \leq 0 \).
1.2 The doubly nonlinear diffusion equation with weak absorption resp. reaction term

Let $\Omega := \text{supp}(u_0) \subset \mathbb{R}^N$. We are interested in an upper bound of the waiting time

$$t_*^\Omega:= \sup\{t \geq 0 \mid u(x, \tau) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega \text{ and } \tau \in [0, t]\}$$

for weak solutions of the doubly nonlinear diffusion equation (eventually also called non-Newtonian polytropical filtration equation) with absorption resp. reaction term – according to the sign of $\lambda$

$$\begin{cases} (|u|^{q-2}u)_t - \Delta_p(u) + \lambda |u|^\alpha u = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) & \text{for all } x \in \mathbb{R}^N, \end{cases}$$ (1.6)

with $\Delta_p(u) = \text{div}(\|\nabla u\|^{p-2}\nabla u)$ and parameters

$$p \geq 2, \quad 1 < q < p, \quad \alpha > 0.$$ We call the absorption resp. reaction term weak, if $\alpha > q - 1$, critical, if $\alpha = q - 1$, and supercritical, if $\alpha < q - 1$. We will experience for the weak case, which is treated in this chapter, that it has no influence on the upper bound of the waiting time derived here. The other cases will be investigated in the following chapter.

To be more precise: For $x_0 \in \partial \Omega$ we want to estimate the local waiting time

$$t^*_{\Omega, x_0}:= \sup\{t \geq 0 \mid \text{For all } \tau \in [0, t] \exists \varepsilon > 0 \text{ with } u(x, \tau) = 0 \text{ for all } x \in B(x_0, \varepsilon) \setminus \Omega\}$$

depending on the growth of $u_0$ in the neighbourhood of $x_0$. Estimations of the waiting time from below depending on the growth of the initial value $u_0$ are known for many different types of equations, cf. [8, 11]. Upper bounds however are only known for the porous medium equation, cf. [7].

The transformation

$$v(x, t) := |\lambda|^\frac{1}{2} \cdot u(|\lambda|^{-\frac{q-1}{q-2}} x, |\lambda|^{-\frac{q-1}{p-1}} t), \quad \lambda \neq 0,$$

leads to the equation $(|v|^{q-2}v)_t - \Delta_p(v) + (\text{sgn } \lambda)|v|^\alpha v = 0$, so that the influence of $\lambda$ can be rescaled this way. We will eventually take use of this scaling, if we do not want to break down the influence of $\lambda$ into single steps.

1.3 Hardy’s inequality

Hardy’s inequality is of important interest, because it serves as substitute for the pointwise estimates in [7] which do not work any more for our differential equation. This inequality is virtually the key in order to generalize the arguments for the Laplacian operator to the $p$-Laplacian operator.

The proof of Theorem 4.1. (i), (ii), [6] (which we formulate here for nondecreasing functions) leads in one dimension to a more easy manageable criterion written down as follows:
Theorem 1.3.1 (Hardy’s inequality for nondecreasing functions, [6])

Let \( u, v : (a, b) \to (0, \infty) \) be measurable, \( 0 < r < s \), and \( -\infty \leq a < b \leq \infty \) arbitrary. If 
\[
\int_t^b v(x)dx < \infty \text{ for all } t \in (a, b) \text{ and } 
\]
\[
B := \left( \int_a^b \left( \int_t^b v(x)dx \right)^{-\frac{r}{s}} \left( \int_t^b u(x)dx \right)^{-\frac{r}{s}} u(t)dt \right)^{\frac{1}{r}} < \infty
\]
with \( \frac{1}{r} = \frac{1}{s} - \frac{1}{s} > 0 \), then for all measurable and nondecreasing \( f : (a, b) \to [0, \infty) \) holds the inequality 
\[
\left( \int_a^b f^s u \right)^{\frac{1}{s}} \leq C \left( \int_a^b f^r v \right)^{\frac{1}{r}}
\]
for a suitable \( C = C(r, s) \cdot B < \infty \).

PROOF: Due to the monotone convergence theorem one only needs to prove this theorem only for the case \( -\infty < a < b < \infty \), because \( C = C(r, s) \) is independent of \( a \) and \( b \). Equally one may assume \( \int_a^b v(t)dt < \infty \).

Without loss of generality let \( f \neq 0 \). We define \( f(a) := \lim_{x \downarrow a} f(x) \). Then there exists a sequence of functions \((f_m)_{m \in \mathbb{N}}\) continuous on \([a, b]\) with \( 0 \leq f_m \leq f \) and \( f_m \not\equiv f \) pointwise almost everywhere on \((a, b)\) as well as \( f_m \) increasing on \( \text{supp} f \). For example one chooses \( f_m := \phi_m \cdot g_m \) for piecewise affine functions which fulfil \( \phi_m(a) := 1 - \frac{1}{m} \), \( \phi_m(b) := 1 \), and \( g_m(x^{(m)}_k) := f(x^{(m)}_{k-1}) \) for \( x^{(m)}_k := a \) as well as \( x^{(m)}_k := a + \frac{k}{2^m}(b - a), \ k \in \{0, 2^m\} \). Then \( f_m \) converges to \( f \) pointwise in every continuity point of \( f \), therefore almost everywhere, because \( f \) is monotone.

Therefore one may also assume thanks to the monotone convergence theorem that \( f \) is continuous on \([a, b]\) and increasing on \( \text{supp} f \).

Define for \( t \geq 0 \)
\[
U(t) := \int_{f > t} u(x)dx, \ V(t) := \int_{f > t} v(x)dx.
\]
Because \( B < \infty \) and \( u, v > 0 \), it follows \( U(t) \in [0, \infty) \) (and also \( V(t) \in [0, \infty) \) by assumption) for all \( t \geq 0 \). Especially is \( \lim_{t \to \infty} U(t) = \lim_{t \to \infty} V(t) = 0 \) by the dominated convergence theorem. Apparently \( U \) and \( V \) are nonincreasing. Furthermore \( U \) and \( V \) are continuous, because for every \( t \in [0, \infty) \) and \( \varepsilon > 0 \) thanks to the strict monotony of \( f \) on \( \text{supp} f \) there exists a \( \delta > 0 \) with \( \mathcal{L}^1(\{|f - t| \leq \delta\} \cap \{f > 0\}) < \varepsilon \).

Set \( \tau_0 := f(a) \). Then it is \( U(\tau_0), V(\tau_0) \in (0, \infty) \). One defines the increasing sequence 
\[
\tau_{k+1} := \min \left\{ t \in (\tau_k, f(b)) \mid \min \left\{ \frac{U(\tau_k)}{U(t)}, \frac{V(\tau_k)}{V(t)} \right\} = 2 \right\}, \ k \in \mathbb{N}_0.
\]

It follows \( 0 < U(\tau_k) \leq \frac{1}{2^k} U(0) \) and \( 0 < V(\tau_k) \leq \frac{1}{2^k} V(0) \) for all \( k \in \mathbb{N}_0 \) by taking advantage of \( u, v > 0 \) for the positivity. Now, \( U \) is nonincreasing with \( U(t) > 0 \) on \([0, f(b)]\) and \( U(t) = 0 \) for \( t \geq f(b) \). From \( U(\tau_k) \leq \frac{1}{2^k} U(0) \) one concludes therefore \( \lim_{k \to \infty} \tau_k = f(b) \).
1.3 Hardy’s inequality

Set

\[ Z_1 := \left\{ k \in \mathbb{N}_0 \mid V(\tau_{k+1}) = \frac{1}{2} V(\tau_k) \right\}, \quad Z_2 := \left\{ k \in \mathbb{N}_0 \setminus Z_1 \mid U(\tau_{k+1}) = \frac{1}{2} U(\tau_k) \right\}. \]

Then it holds \( Z_1 \cap Z_2 = \emptyset \) and \( Z_1 \cup Z_2 = \mathbb{N}_0 \). Now, one calculates

\[
I := \int_a^b f r u = \sum_{k \in \mathbb{N}} \tau_k \int_{\tau_{k-1}}^{\tau_k} u \quad \text{because} \quad \int_{f \in \{f(a), f(b)\}} f r u = 0, \\
\leq \sum_{k \in \mathbb{N}} \tau_k \int_{\tau_{k-1}}^{\tau_k} u \leq \sum_{k \in \mathbb{N}} \tau_k \left( \frac{(\tau_k - \tau_{k-1}) u}{(\sum_{n \leq k} (\tau_n - \tau_{n-1}) u)^{\frac{\tau_k - \tau_{k-1}}{\tau_n - \tau_{n-1}}}} \right) \sum_{n \leq k} (\tau_n - \tau_{n-1}) u \sum_{n \leq k} (\tau_n - \tau_{n-1}) u V(\tau_n)^{-\frac{d}{2}} \right) \frac{\tau_k - \tau_{k-1}}{\tau_n - \tau_{n-1}}}
\leq \sum_{k \in \mathbb{N}} \tau_k \left( \frac{(\tau_k - \tau_{k-1}) u}{(\sum_{n \leq k} (\tau_n - \tau_{n-1}) u)^{\frac{\tau_k - \tau_{k-1}}{\tau_n - \tau_{n-1}}}} \right) \sum_{n \leq k} (\tau_n - \tau_{n-1}) u \sum_{n \leq k} (\tau_n - \tau_{n-1}) u V(\tau_n)^{-\frac{d}{2}} \right) \frac{\tau_k - \tau_{k-1}}{\tau_n - \tau_{n-1}}
=: I_1^2 \cdot I_2^3
\]
due to the (discrete) Hölder inequality. First we estimate \( I_1 \). In order to do this one notes

\[
\sum_{n \leq k} \left( \int_{\tau_{n-1}}^{\tau_n} u \right)^\frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \geq \left( \int_{\tau_{k-1}}^{\tau_k} u \right)^\frac{d}{2} V(\tau_k)^{-\frac{d}{2}}.
\]

That implies immediately

\[
I_1 \leq \sum_{k \in \mathbb{N}} \tau_k^d V(\tau_k).
\]

By construction of the \( \tau_k \) it follows

\[
V(\tau_k) \geq 2V(\tau_{k+1}), \quad U(\tau_k) \geq 2U(\tau_{k+1}) \quad \text{for all} \quad k \in \mathbb{N}_0
\]

and therefore

\[
2V(\tau_{k+1}) \leq V(\tau_k) = \int_{f > \tau_k} v = \int_{f < f \leq \tau_{k+1}} v + \int_{f > \tau_{k+1}} v = \int_{\tau_k < f \leq \tau_{k+1}} v + V(\tau_{k+1}),
\]

hence \( V(\tau_{k+1}) \leq \int_{\tau_k < f \leq \tau_{k+1}} v \) and so

\[
V(\tau_k) = \int_{\tau_k < f \leq \tau_{k+1}} v + V(\tau_{k+1}) \leq 2 \int_{\tau_k < f \leq \tau_{k+1}} v.
\]

By substitution \( V, v \) by \( U, u \) in this argumentation one obtains analogously

\[
U(\tau_k) \leq 2 \int_{\tau_k < f \leq \tau_{k+1}} u \quad \text{with equality, if} \quad 2U(\tau_{k+1}) = U(\tau_k).
\]

This implies the desired estimation for \( I_1 \), namely

\[
I_1 \leq 2 \sum_{k \in \mathbb{N}} \tau_k^d \int_{\tau_k < f \leq \tau_{k+1}} v \leq 2 \sum_{k \in \mathbb{N}} \int_{\tau_k < f \leq \tau_{k+1}} f^s v \leq 2 \int_a^b f^s v.
\]
Now one has to estimate $I_2$ appropriately. It holds

$$I_2 = \sum_{k \in \mathbb{N}} \left( \int_{\tau_{k-1} < f \leq \tau_k} u \right) \cdot \sum_{n \leq k} \left( \int_{\tau_{k-1} < f \leq \tau_n} u \right)^\frac{d}{2} V(\tau_n)^{-\frac{d}{2}}$$

$$= \sum_{n \in \mathbb{N}} \left( \int_{\tau_{n-1} < f \leq \tau_n} u \right) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \cdot \sum_{k \geq n} \left( \int_{\tau_{k-1} < f \leq \tau_n} u \right)$$

$$= \sum_{n \in \mathbb{N}} \left( \int_{\tau_{n-1} < f \leq \tau_n} u \right) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} U(\tau_{n-1})$$

$$=: I_{2,1} + I_{2,2}$$

wherein the summation in $I_{2,1}$ spreads over all $n \in \mathbb{N}$ with $n - 1 \in \mathbb{Z}$. Therefore it is

$$I_{2,1} \leq 2 \sum_{n \in Z_1} \left( \int_{\tau_{n-1} < f \leq \tau_n} u \right)^{1+\frac{d}{2}} V(\tau_n)^{-\frac{d}{2}}$$

$$= 2 \sum_{n \in Z_1} \left( (U(\tau_n) - U(\tau_{n-1})) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \right)$$

$$= 2^{1+\frac{d}{2}} \sum_{n \in Z_1} \left( (U(\tau_n) - U(\tau_{n-1})) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \right) \frac{d}{2} V(\tau_{n-1})^{-\frac{d}{2}} \right].$$

Correspondingly one estimates $I_{2,2}$:

$$I_{2,2} = \sum_{n \in Z_2} \left( U(\tau_n) \frac{d}{2} U(\tau_{n-1}) V(\tau_n)^{-\frac{d}{2}} \right)$$

$$= 2 \sum_{n \in Z_2} \left( U(\tau_n) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \right)$$

$$\leq 2^{1+\frac{d}{2}} \sum_{n \in Z_2} \left( (U(\tau_n) - U(\tau_{n+1})) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \right) \frac{d}{2} V(\tau_{n+1})^{-\frac{d}{2}} \right].$$

Setting

$$B := \left( \sum_{n \in \mathbb{N}_0} \left[ (U(\tau_n) - U(\tau_{n+1})) \frac{d}{2} V(\tau_n)^{-\frac{d}{2}} \right] \right)^\frac{1}{\frac{d}{2}}$$

it follows

$$I_2 = I_{2,1} + I_{2,2} \leq C(r, s) \cdot B^\frac{1}{2}.$$

This implies

$$\left( \int_a^b f^s u \right)^\frac{1}{s} \leq C(r, s) B \cdot \left( \int_a^b f^s v \right)^\frac{1}{s}.$$

One has therefore only to show $B \leq C(r, s) \cdot B$. Set $x_0 := \max \{ x < b \mid f(x) = f(a) \}$ and $x_k := f^{-1}(\tau_k)$. Then $(x_k)_{k \in \mathbb{N}_0}$ is an increasing sequence with $\tau_k < f(x) < \tau_{k+1}$ for all $x \in (x_k, x_{k+1})$. With the aid of the absolutely continuous functions

$$\mathcal{U}(t) := \int_t^b u(x) dx$$

and $\mathcal{V}(t) := \int_t^b v(x) dx$

one obtains

$$U(\tau_n) = \int_{f(\tau_n)}^x u = \mathcal{U}(x_n)$$

as well as $V(\tau_n) = \int_{f(\tau_n)}^x v = \mathcal{V}(x_n)$,
1.3 Hardy’s inequality

hence

\[ B^d = \sum_{n \in \mathbb{N}_0} \left[ (\mathcal{U}(x_n) - \mathcal{U}(x_{n+1}))^{\frac{d}{r}} \mathcal{V}(x_n)^{-\frac{d}{s}} \right]. \]

Set finally \( \mathcal{U}_{n+1}(x) := (\int_{x_n}^{x_{n+1}} u(t)\,dt)^{\frac{d}{r}} \), then it holds \( -\mathcal{U}_{n+1}'(x) = \frac{d}{r} (\int_{x_n}^{x_{n+1}} u(t)\,dt)^{\frac{d}{r}} u \leq \frac{d}{r} \mathcal{U}(x)^{\frac{d}{r}} u(x) \) almost everywhere, hence

\[ B^d \leq \sum_{n \in \mathbb{N}_0} \left[ \frac{d}{r} \int_{x_n}^{x_{n+1}} (\mathcal{U}(t)^{\frac{d}{r}} u(t))\,dt \cdot \mathcal{V}(x_n)^{-\frac{d}{s}} \right]. \]

Since \( \mathcal{V} \) is nonincreasing, one concludes

\[ B^d \leq \sum_{n \in \mathbb{N}_0} \left[ \frac{d}{r} \int_{x_n}^{x_{n+1}} (\mathcal{U}(t)^{\frac{d}{r}} u(t))\,dt \right], \]

hence \( B \leq C(r, s) \cdot B \), as asserted. \( \blacksquare \)

Now, the main work for HARDY’s inequality in the case \( s = 1 \) is done. We will present here a simple generalization of this case to higher derivatives, even though we will not need this version at present:

**Theorem 1.3.2 (HARDY’s inequality for the case \( 0 < r < s, \ s \geq 1 \), \cite{12, 18}**

Let \( k \in \mathbb{N}, f \in AC^{k-1}([a, b]) \), i.e. \( f \) is absolutely continuous and possesses \( k-1 \) derivatives that are all absolutely continuous with \( f^{(l)}(a) = 0 \) for all \( l \in \{0, \ldots, k-1\} \) and let \( u, v : (a, b) \to \mathbb{R}^{>0} \) be measurable. Furthermore let \( s \geq 1 \) and \( 0 < r < s \) be arbitrary.

If in case \( s = 1 \) also hold \( \int_a^b u(t)\,dt < \infty \) for all \( x \in (a, b) \) as well as the inequality

\[ A := \left( \int_a^b [\inf_{a \leq t \leq x} v(t)]^{-\frac{1}{s-r}} \left( \int_a^b u(t)(t-a)^{(k-1)r}\,dt \right)^{\frac{1}{1-s}} u(x)(x-a)^{(k-1)r} \,dx \right)^{\frac{1}{1-s}} < \infty \]

resp. in case \( s > 1 \) the inequality

\[ A := \left( \int_a^b \left( \int_a^b u(t)(t-a)^{(k-1)r}\,dt \right)^{\frac{s}{s-r}} \left( \int_a^x v(t)^{-\frac{1}{s-1}}\,dt \right)^{\frac{s-1}{s-r}} v(x)^{-\frac{1}{s-1}} \,dx \right)^{\frac{s-1}{s-r}} < \infty , \]

then HARDY’s inequality

\[ \left( \int_a^b |f(x)|^r u(x)\,dx \right)^{\frac{1}{r}} \leq C \left( \int_a^b |f^{(k)}(x)|^s v(x)\,dx \right)^{\frac{1}{s}} \]

is valid with a suitable \( C = C(k, r, s) \cdot A < \infty \).
Proof: We will prove only the case $s = 1$, because $s > 1$ is a classical result.
The function $\tilde{\nu}(x) := \text{ess inf}_{a < x < b} v(t)$ is apparently finite and nonincreasing. Since $A < \infty$
and $u, v > 0$, it follows $\tilde{\nu}(x) > 0$ for all $x \in (a, b)$.
Then there exists a sequence of measurable functions $g_m : (a, b) \to (0, \infty)$ with

$$0 < \tilde{\nu}(x) \leq v_m(x) := \int_x^b g_m(t)dt < \infty \text{ for all } x \in (a, b),$$

such that $v_m \searrow \tilde{\nu}$ converges pointwisely almost everywhere on $(a, b)$. (This sequence can
be constructed analogously to the $f_m$ in the beginning of the proof of Theorem 1.3.1.)
Now, define the nondecreasing and nonnegative function $F_m(x) := \int_a^x \frac{|f(k)(t)|}{v_m(t)} \tilde{\nu}(t)dt$. Then by Theorem 1.3.1 one obtains

$$\left(\int_a^b F_m(x)^r [(x-a)^{(k-1)r}u(x)] dx \right)^{\frac{1}{r}} \leq C(r, s) A \cdot \int_a^b F_m(x)g_m(x)dx.$$ 

Therefore it follows

$$\left(\int_a^b \left(\frac{(x-t)^{(k-1)}f(k)(t)}{(k-1)!} \tilde{\nu}(t)\right) \tilde{\nu}(t)dt \right)^r u(x)dx \right)^{\frac{1}{r}} \leq \left(\int_a^b F_m(x)^r (x-a)^{(k-1)r}u(x)dx \right)^{\frac{1}{r}} \leq C(r, s) A \cdot \int_a^b \left(\frac{|f(k)(t)|}{v_m(t)} \tilde{\nu}(t)\right) \tilde{\nu}(t)dt \leq C(r, s) A \cdot \frac{1}{(k-1)!} \int_a^b \frac{|f(k)(t)|}{v_m(t)} \tilde{\nu}(t)dt.$$ 

Now, $\int_a^b |f(x)|^r u(x)dx = \int_a^b \left|\int_a^x \frac{(x-t)^{(k-1)}f(k)(t)}{(k-1)!} dt\right|^r u(x)dx \leq \int_a^b \left(\int_a^x \frac{(x-t)^{(k-1)}f(k)(t)}{(k-1)!} dt\right)^r u(x)dx$
by TAYLOR’s Theorem with LAGRANGE remainder. Apply the monotone convergence theorem.

1.4 The case $\alpha > q - 1$

In the following we will extend the method of $[7]$, which was developed for the porous medium equation, in order to estimate $t^\ast_{\Omega, x_0}$. We will experience that the absorption term is negligible for $\alpha > q - 1$.

Lemma 1.4.1

Let $A, B, C \geq 0$, $t^\ast \in (0, \infty]$, $\gamma > -1$, $\varepsilon > 1$, $\delta < 1 + (\varepsilon - 1)(\vartheta(\gamma + 1) - \eta + 1)$, $\vartheta > 1$,
$\eta < 1 + (\gamma + 1)(\vartheta - 1)$, and

$$F \in C^0([0, t^\ast)) \cap C^1((0, t^\ast))$$

with

$$F(0) = 0, \quad F(\tau) \geq 0 \text{ for all } \tau \in (0, t^\ast).$$
and
\[ F'(\tau) \geq A\tau^\gamma + B\tau^{-\delta} F(\tau)^\epsilon + C\tau^{-\eta}F(\tau)^\theta. \]

Then it holds
\[ A^{\nu - 1} \beta(t^*)^{1 - \delta + (\gamma + 1)\epsilon - 1} + A^{\nu - 1} C(t^*)^{1 - \eta + (\gamma + 1)\theta - 1} \leq K \]

with a suitable \( K = K(\gamma, \delta, \epsilon, \eta, \theta) > 0. \)

**Proof:** Because of an indirect argument it is sufficient to prove the assertion for \( t^* < \infty. \) Since \( F \) is nonnegative, one obtains \( F'(\tau) \geq A\tau^\gamma. \) It follows from the continuity in the origin
\[ F(\tau) = \lim_{\varepsilon \searrow 0} F(\tau) - F(\varepsilon) = \lim_{\varepsilon \searrow 0} \int_0^\tau F'(t)dt \geq \int_0^\tau A\tau^\gamma dt = \frac{A}{\gamma + 1} \tau^{\gamma + 1}. \]

Particularly it is \( F(\tau) > 0 \) for \( \tau \in (0, t^*). \) Now, choose a
\[ \nu = \nu(\gamma, \delta, \epsilon, \eta, \theta) \in \left(0, \min\left\{\varepsilon - 1, \theta - 1, \varepsilon - 1 - \frac{\delta - 1}{\gamma + 1}, \theta - 1 - \frac{\eta - 1}{\gamma + 1}\right\}\right). \]

Then it is
\[ F'(\tau) \geq F(\tau)^{1 + \nu} [B\tau^{-\delta} F(\tau)^{\epsilon - 1 - \nu} + C\tau^{-\eta} F(\tau)^{\theta - 1 - \nu}] \]
\[ \geq K F(\tau)^{1 + \nu} [A^{\nu - 1} \beta(t^*)^{(\gamma + 1)(\epsilon - 1 - \nu) - \delta} + A^{\nu - 1} C(t^*)^{(\gamma + 1)(\theta - 1 - \nu) - \eta}]. \]

Because of \( F(\tau) > 0 \) one obtains
\[ -\frac{1}{\nu}(F^{-\nu})(\tau) = F(\tau)^{-1 - \nu} F'(\tau) \geq K [A^{\nu - 1} \beta(t^*)^{(\gamma + 1)(\epsilon - 1 - \nu) - \delta} + A^{\nu - 1} C(t^*)^{(\gamma + 1)(\theta - 1 - \nu) - \eta}], \]

for all \( \tau \in (0, t^*). \) Now, choose \( \alpha \in (0, 1), \beta \in (\alpha, 1) \) and integrate over \([\alpha t^*, \beta t^*]. \) Then one concludes
\[ \frac{1}{\nu} F(\alpha t^*)^{-\nu} \geq \frac{1}{\nu} \int_{\alpha t^*}^{\beta t^*} \frac{1}{\nu}(F^{-\nu})(\tau)d\tau \]
\[ \geq K [A^{\nu - 1} \beta B(t^*)^{\gamma(\gamma + 1)(\epsilon - 1 - \nu) - \delta} + A^{\nu - 1} C(t^*)^{(\gamma + 1)(\theta - 1 - \nu) - \eta}]. \]

hence
\[ \frac{1}{\nu} F(\alpha t^*)^{-\nu} \geq K [A^{\nu - 1} \beta B(t^*)^{\gamma(\gamma + 1)(\epsilon - 1 - \nu) - \delta} + A^{\nu - 1} C(t^*)^{(\gamma + 1)(\theta - 1 - \nu) - \eta}]. \]

because \( \beta \in (\alpha, 1) \) is arbitrary. With
\[ F(\alpha t^*) \geq \frac{A}{\gamma + 1} (\alpha t^*)^{\gamma + 1}, \]

it follows
\[ \frac{1}{\nu} \left[ \frac{A}{\gamma + 1} (\alpha t^*)^{\gamma + 1} \right]^{-\nu} \geq K [A^{\nu - 1} \beta B(t^*)^{(\gamma + 1)(\epsilon - 1 - \nu) - \delta} + A^{\nu - 1} C(t^*)^{(\gamma + 1)(\theta - 1 - \nu) - \eta}]. \]
therefore it is
\[ A^{1-\epsilon}B(t^*)^{1-\delta+(\gamma+1)(\epsilon-1)} + A^{q-1}C(t^*)^{1-\gamma+(\gamma+1)(\delta-1)} \leq K. \]

**Theorem 1.4.2 (radially symmetric version)**

Let \( R > 0 \) and \( u : \mathbb{R}^N \times [0, \infty) \to [0, \infty) \) be a nonnegative radially symmetric weak solution of (1.6) for \( \lambda \leq 0, \alpha > q - 1 \) with \( \Omega = B(0, R) \). Let \( u(x, t) =: \tilde{u}(||x||, t) \) and \( \tilde{u}_r \leq 0 \).

Then there exists for every \( \delta \in (1, p - 1) \) for \( p > 2 \) resp. \( \delta = 1 \) for \( p = 2 \) a constant \( C = C(p, q, \alpha, \delta) > 0 \), independent of the spatial dimension, such that for every \( \epsilon \in (0, R) \) the estimation
\[
\tau^* \leq C(p, q, \alpha, \delta) \left[ C(u_0; \delta, \epsilon) \frac{R^\alpha}{(\frac{R}{\epsilon})^{q-1}} \right]^{\frac{(N-1)(p-1)}{q-1}} (R - \tau)^{\frac{(N-1)(p-q)}{q-1}} - \frac{\epsilon^{1-\delta}(\gamma+1)}{q-1} - \frac{\epsilon^{1-p}(\gamma+1)}{q-1}
\]
holds wherein
\[
C(u_0; \delta, \epsilon) := \int_0^R \tilde{u}_0(r) q^{-1}(\epsilon^{\delta} - (R - r) \delta)r^{N-1}dr > 0.
\]

**Proof:** Choose \( \tau \in (0, \tau^*) \) arbitrary. Then we test the differential equation with the function \( \varphi(x, t) := g(t) \tilde{h}(x) \) with \( g(t) := (\tau - t)^\gamma \) and \( \tilde{h}(x) := \tilde{h}(||x||) \) as well as \( \tilde{h}(r) := (\epsilon^\delta - |R - r|^\delta)^+ \) wherein \( \delta \in (1, p - 1) \) for \( p > 2 \) resp. \( \delta = 1 \) for \( p = 2 \). Set \( \Omega_\epsilon := B(0, R) \setminus B(0, R - \epsilon) \). One obtains
\[
\int_{\Omega_\epsilon} u_0^{q-1}(\varphi(x, 0)dx) + \int_{\Omega_\epsilon} \int_0^{\tau} u^{q-1}\varphi dt dx
= \left[ \int_{\Omega_\epsilon} hu^{q-1}dx \right]^{\frac{\gamma}{\tau}} + \int_0^{\tau} g_t \int_{\Omega_\epsilon} u^{q-1}hdx dt
= N\omega_N \int_0^{R - \epsilon} \int_{\Omega_\epsilon} u^{q-1}h^{\gamma} dx dt
+ N\omega_N \int_0^{\tau} g_t \int_{R - \epsilon}^{R} \tilde{u}^{q-1}h^{\gamma} dx dt.
\]

Besides we have \( \frac{\partial}{\partial x_j} u(x, t) = \frac{\partial}{\partial x_j} \tilde{u}(||x||, t) = \tilde{u}_r(||x||, t) \frac{x_j}{||x||} \), hence \( \nabla u = \tilde{u}_r(||x||, t) \frac{x}{||x||} \). Moreover it is \( \nabla \tilde{h}(x) = \tilde{h}'(||x||) \frac{x}{||x||} \). This implies
\[
- \int_0^{\tau} f_{\Omega_\epsilon} \| \nabla u \|^{p-2} (\nabla u, \nabla \varphi) dx dt
= - \int_0^{\tau} g \int_{\Omega_\epsilon} \| \nabla u \|^{p-2} (\nabla u, \nabla h) dx dt
= - \int_0^{\tau} g \int_{\Omega_\epsilon} |\tilde{u}_r|^{p-2} \tilde{u}_r \tilde{h}'(||x||) dx dt
= \int_0^{\tau} g \int_{\Omega_\epsilon} |\tilde{u}_r|^{p-1} \tilde{h}'(||x||) dx dt
= N\omega_N \int_0^{\tau} g \int_{R - \epsilon}^{R} |\tilde{u}_r|^{p-1} \tilde{h}'(r^{N-1}) dr dt.
\]

Altogether we have the equation
\[
\int_0^{\tau} g_t \int_{R - \epsilon}^{R} \tilde{u}^{q-1}h^{\gamma} r^{N-1} dr dt + \int_0^{\tau} g \int_{R - \epsilon}^{R} |\tilde{u}_r|^{p-1} \tilde{h}'(r^{N-1}) dr dt + |\lambda| \int_0^{\tau} g \int_{R - \epsilon}^{R} \tilde{u}^{\alpha} \tilde{h}^{\gamma} r^{N-1} dr dt + C(u_0; \delta, \epsilon) \frac{\tau^{\gamma}}{\tau} = 0.
\]

We want to apply HARDY’s inequality with exponents \( s := p - 1 \) and \( r := q - 1 \) as well as \( u(t) := \tilde{h}(R - t) \cdot (R - t)^{N-1} \) and \( v(t) := |\tilde{h}'(R - t)| \cdot (R - t)^{N-1} \) for \( t \in (0, \epsilon) \) to the inner integral in the second summand.
1.4 The case $\alpha > q - 1$

In the case $p = 2$ we use (with $\delta = 1$) the estimation

\[
A = (\int_0^\varepsilon (\text{ess inf}_{0 < t < \varepsilon} (R - t)^{N-1})^{-\frac{q-1}{2-q}} \left[ \int_t^\varepsilon (\varepsilon - t)(R - t)^{N-1} dt \right]^{\frac{q-1}{q}} (\varepsilon - x)(R - x)^{N-1} dx)^{\frac{2-q}{1-q}} 
\leq (\int_0^\varepsilon (R - x) \left( \int_{\varepsilon}^\varepsilon (R - t)^{N-1} dt \right)^{\frac{q-1}{2-q}} (\varepsilon - x)(R - x)^{N-1} dx)^{\frac{2-q}{1-q}} 
\leq (R - \varepsilon)^{-N(q-1)} R^{N-1+\frac{(q-1)(q-2)}{2-q}} \varepsilon^{1+\frac{(q-1)(q-2)}{2-q}} 
= (R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}}.
\]

In the case $p > 2$ one takes the following inequality ($C = C(p, q, \delta)$)

\[
A \leq C(\int_0^\varepsilon (\int_x^\varepsilon R^{N-1} dt)^{\frac{p-1}{p-q}} (\int_0^x \frac{1}{p-1} \left( (R - t)^{-N(q-1)} \right) dt)^{\frac{p-1}{p-q}} x^{-\frac{q-1}{p-1}} R^{-\frac{N-1}{p-1}} dx)^{\frac{p-1}{p-q}} 
\leq C \cdot R^{-\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} (R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}} 
= C(R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}}.
\]

We obtain in all cases thanks to HARDY's inequality

\[
\left( \int_{R-\varepsilon}^R |u|^{-1} h^N_{-1} \right)^{\frac{1}{N-1}} \leq C \left( \frac{R}{R - \varepsilon} \right)^{\frac{N-1}{q-1} - \delta} (R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}} \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}}
\]

hence

\[
\int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \geq C(p, q, \delta) \left( \frac{R}{R - \varepsilon} \right)^{\frac{N-1}{q-1} - \delta} (R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}} \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}}
\]

and therefore

\[
\int_0^\sigma \int_{R-\varepsilon}^R g |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \geq C(p, q, \delta) \left( \frac{R}{R - \varepsilon} \right)^{\frac{N-1}{q-1} - \delta} (R - \varepsilon)^{\frac{N-1}{q-1} + \frac{p-1}{p-1} - \delta} \varepsilon^{\frac{1}{q-1} + \frac{p-1}{p-1}} \left( \int_{R-\varepsilon}^R g |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}}
\]

One notes \[ \int |f|^\frac{1}{r} \leq (\int |fg|)^{\frac{1}{r}} (\int |g|^{\frac{r}{r-1}})^{\frac{r-1}{r}} , \] hence \[ \int |fg| \geq \left( \int |f|^{\frac{1}{r}} \right)^r \left( \int |g|^{-\frac{1}{r-1}} \right)^{-\frac{1}{r-1}} \] for \( r > 1 \).

Now, we apply this inverse HÖLDER inequality to the second and third summand:

\[
\int_0^{\tau} (\tau - t)^{\gamma} \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}} 
= \int_0^{\tau} ((\tau - t)^r \gamma) \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}} 
\geq \left[ \int_0^{\tau} (\tau - t)^r \gamma \right] \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}} 
= (\gamma + 1)^{\frac{1}{p-1}} \left[ \int_0^{\tau} (\tau - t)^r \right] \left( \int_{R-\varepsilon}^R |u|^p |h^N_{-1}|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}} 
\]
and
\[
\int_0^\tau \int_{R-\varepsilon}^R |\tilde{u}|^\alpha (\tau - t)^\gamma \hat{h}r^{N-1} = \int_0^\tau \int_{R-\varepsilon}^R |\tilde{u}|q^{-1}(\tau - t)^\gamma \hat{h}r^{N-1}1 - \frac{\alpha}{q-1} \\
\geq \left[ \int_0^\tau (\tau - t)^\gamma \int_{R-\varepsilon}^R |\tilde{u}|q^{-1}\hat{h}r^{N-1} \right] \frac{\alpha}{q-1} \left[ \int_0^\tau (\tau - t)^\gamma \hat{h}r^{N-1}1 - \frac{\alpha}{q-1} \right].
\]

Setting
\[
F(\tau) := \int_0^\tau (\tau - t)^\gamma \int_{R-\varepsilon}^R |\tilde{u}|q^{-1}\hat{h}r^{N-1} \geq 0
\]
it follows
\[
F'(\tau) = \int_0^\tau \gamma(\tau - t)^{\gamma - 1} \int_{R-\varepsilon}^R |\tilde{u}|^{q-1}\hat{h}r^{N-1}
\]
and therefore the differential inequality
\[
-F'(\tau) + C(p, q, \delta) \left( \frac{R}{R-\varepsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} (R - \varepsilon)^{\frac{(N-1)(p-q)}{q-1}} \varepsilon^{\delta+1-p} - \left( \frac{p-1}{q-1} \right) F(\tau) + C(u_0; \delta, \varepsilon)\tau^\gamma \leq 0,
\]
hence
\[
F'(\tau) \geq C(u_0; \delta, \varepsilon)\tau^\gamma + C(p, q, \delta) \left( \frac{R}{R-\varepsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} (R - \varepsilon)^{\frac{(N-1)(p-q)}{q-1}} \varepsilon^{\delta+1-p} - \left( \frac{p-1}{q-1} \right) F(\tau) + C(p, q, \delta)\tau^\gamma
\]
Now, one applies Lemma 1.4.1 in order to achieve the desired estimation for the waiting time.

**Corollary 1.4.3 (radially symmetric version, reaction term, \(\alpha > q - 1\))**

Let \(A, R, \gamma > 0\) and \(u : \mathbb{R}^N \times [0, \infty) \rightarrow [0, \infty)\) be a nonnegative radially symmetric weak solution of (1.6) with \(\lambda \leq 0, \alpha > q - 1\) and initial value \(u_0(x) = A(R - \|x\|)\) as well as \(\bar{u}_t \leq 0\).

If \(\gamma = \frac{p}{p-q}\), then it follows \(t_\Omega^* \leq CA^{-(p-q)}\) for a constant \(C = C(p, q, \alpha) > 0\), independent of \(\lambda\) and the spatial dimension \(N\). If however \(\gamma < \frac{p}{p-q}\), then it follows \(t_\Omega^* = 0\).

**Proof:** It is for \(\varepsilon \in (0, R)\)
\[
C(u_0; \delta, \varepsilon) = \int_{R-\varepsilon}^R \tilde{u}_0(r)^{q-1}(\varepsilon^\delta - (R - r)^\delta)r^{N-1}\,dr
\]
\[
= A^{q-1} \int_0^\tau \varepsilon^{(q-1)}(\varepsilon^\delta - t^\delta)(R - t)^{N-1}\,dt
\]
\[
\geq CA^{q-1}(R - \varepsilon)^{N-1}\varepsilon^{\delta+\gamma(q-1)+1}. 
\]
Due to Theorem 1.4.2 it follows
\[
t_\Omega^* \leq C[A^{p-q} \left( \frac{R}{R-\varepsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} \varepsilon^{\delta+\gamma(q-1)+1+p-\left( \frac{p-1}{q-1} \right) + \delta+1-p} - \left( \frac{p-1}{q-1} \right) \varepsilon^{\alpha-(q-1)}\right]^{-1}.
\]
1.4 The case \( \alpha > q - 1 \)

Noticing that the exponent of the first \( \varepsilon \) vanishes, if and only if \( \gamma = \frac{p}{p-q} \), and that the exponent of the second \( \varepsilon \) is always strictly positive the assertion follows by taking to the limit \( \varepsilon \searrow 0 \).

**Theorem 1.4.4 (Upper bound of the waiting time)**

Let \( u : \mathbb{R}^N \times [0, \infty) \to [0, \infty) \) be the weak solution of (1.1) for the initial value \( u_0 \geq 0 \) constructed in [13]. Let further be \( x_0 \in \partial \Omega \) and let there exist a ball \( B = B(y_0, r_0) \subseteq \Omega \) with \( x_0 \in \partial B \) and \( A, \gamma > 0 \), such that the initial value \( u_0 \) fulfills the estimation \( u_0(x) \geq A\|x - x_0\|^\gamma \) for all \( x \in B \).

If \( \gamma = \frac{p}{p-q} \), then it follows \( t_{\Omega, x_0}^* \leq CA^{\frac{p}{1-(p-q)}} \) for a suitable \( C = C(p, q) > 0 \), independent of the spatial dimension \( N \). If this estimation of the initial value is fulfilled for a \( \gamma < \frac{p}{p-q} \), then it even follows \( t_{\Omega, x_0}^* = 0 \).

**Proof:** Set \( v_0(x) := A(r_0 - \|x\|)^\gamma \). Then it is \( v_0(x - y_0) = A(\|x_0 - y_0\| - \|x - y_0\|)^\gamma \leq A\|x - x_0\|^\gamma = u_0(x) \) for all \( x \in B \). Apparently it is \( v_0 \geq 0 \) radially symmetric with nonpositive radial derivative. Now, apply Theorem 1.1.3 and let \( v \) denote the corresponding radially symmetric solution of the doubly nonlinear equation with initial value \( v_0 \). Then \( w(x) := v(x - y_0) \) is also a solution of the doubly nonlinear equation with \( w_0 \leq v_0 \). Therefore the waiting time of \( v_0 \) is lower or equal to the waiting time of \( w_0 \) which we can estimate from above.

The lower bound of the waiting time for (1.1) which is obtained in Section 3.1, [11] coincides with the upper bound given here, whereas the constant here does not depend on the spatial dimension. This proves the optimality of this estimation.

One obtains \( v_t = \text{div} (\|\nabla v^m\|^{p-2} \nabla v^m) \) with \( m := \frac{1}{q-1} \) by means of \( v := |u|^{q-2}u \). In the case \( p = 2 \) it is \( v_t = \Delta v^m \) the porous medium equation and we receive the result of [7].

Now, we examine the other sign \( \lambda > 0 \). We will experience that the sign of \( \lambda \) has no influence for \( q - 1 < \alpha < p - 1 \) at all. The technique is in principle the same. The difference is that one obtains this way a negative term in the differential inequality which can be hidden behind both positive summands due to Young’s inequality for \( \varepsilon \searrow 0 \), \( q - 1 \leq \alpha < p - 1 \), and suitably small \( \nu \). We will demonstrate the proof for the case \( p > 2 \) in which we will postulate additionally \( \tilde{u}_r \leq 0 \) for technical reasons. In the case \( p = 2 \) one can immediately integrate by parts twice and does therefore not need Hardy’s inequality at all.

By testing the differential equation with \( (\varepsilon^\delta - |R - \|x\||^\delta)^+(T - t)^\gamma \) for \( \delta \in (1, p - 1) \) it follows

\[
\begin{align*}
\gamma \int_0^T \int_{R-\varepsilon} |\tilde{u}|^{p-1}(R-r)^{\delta-1}(T-t)^{\gamma}r^{N-1} + \int_{R-\varepsilon}^{R} \tilde{u}^{q-1}(\varepsilon^\delta - (R-r)^{\delta})r^{N-1}T^{\gamma} \\
= \gamma \int_0^T \int_{R-\varepsilon} \tilde{u}^{q-1}(\varepsilon^\delta - (R-r)^{\delta})(T-t)^{\gamma}r^{N-1} \\
+ \lambda \int_0^T \int_{R-\varepsilon} \tilde{u}^{q-1}(\varepsilon^\delta - (R-r)^{\delta})(T-t)^{\gamma}r^{N-1}.
\end{align*}
\]
One fixes $\beta \in (\alpha, p - 1)$ and obtains by Hardy’s inequality, Hölder’s inequality and the inverse Hölder inequality by setting

$$F(T) := \int_0^T \int_{R-\varepsilon}^R \tilde{u}(\varepsilon R - (R - r)\delta)(T - t)T^{-\gamma}r^{N-1}$$

the differential inequality

$$[F'(T)]^{\frac{p-1}{\alpha}} \varepsilon^{\frac{p+1}{\alpha}(\beta-q+1)}T^{\frac{p}{\alpha}(\beta-q+1)} R^{(N-1)\frac{\beta-\alpha-n}{\beta}}$$

$$\geq \frac{C}{2} A^{q-1}(R - \varepsilon)^{N-1} \varepsilon T^{\frac{p-1}{\alpha}(\beta-q+1)} R^{(N-1)\frac{\beta-\alpha-n}{\beta}}$$

$$\leq \frac{C}{2} A^{q-1}(R - \varepsilon)^{N-1} T^{\frac{p-1}{\alpha}(\beta-q+1)} R^{(N-1)\frac{\beta-\alpha-n}{\beta}} F(T)^{\frac{p-1}{\alpha}}$$

$$\cdot (R - \varepsilon)^{\frac{p-1}{\alpha}(N-1)\varepsilon^{\frac{p-1}{\alpha}(\beta-q+1)} R^{(N-1)\frac{\beta-\alpha-n}{\beta}}}$$

such that one obtains

$$[F'(T)]^{\frac{p-1}{\alpha}} \varepsilon^{\frac{p+1}{\alpha}(\beta-q+1)} T^{\frac{p}{\alpha}(\beta-q+1)} R^{(N-1)\frac{\beta-\alpha-n}{\beta}}$$

$$\geq \frac{C}{2} A^{q-1} \varepsilon^{1+\delta+(q-1)\nu}(R - \varepsilon)^{N-1} T^{\gamma}$$

$$\cdot (R - \varepsilon)^{\frac{p-1}{\alpha}(N-1)(1+\delta+(q-1)\nu)} R^{\frac{p-1}{\alpha}(1+\delta+(q-1)\nu)}$$

Furthermore it holds

$$A^{-(q-1)(\frac{p-1}{\alpha}-1)} T^{\frac{p-1}{\alpha}[(\gamma+1)\frac{\beta-\alpha-n}{\beta} - \frac{p-1}{\alpha}(1+\delta+(q-1)\nu)]}$$

$$\cdot (R - \varepsilon)^{\frac{p-1}{\alpha}(N-1)(1-\delta+(q-1)\nu)} R^{\frac{p-1}{\alpha}(1+\delta+(q-1)\nu)}$$

which is equivalent to

$$T \leq C A^{q-1} \varepsilon^{(q-1)\nu} \cdot \left(\frac{R - \varepsilon}{R}\right)^{(N-1)}.$$

The exponent at $\varepsilon$ is negative, if and only if $\nu < \frac{p\alpha}{(q-1)(p-1-\alpha)}$. Now, one notices

$$\frac{p}{p - q} < \frac{p\alpha}{(q-1)(p-1-\alpha)},$$

if and only if $\alpha > q - 1$.
Theorem 1.4.5 (absorption term, proof of the) waiting time. Therefore the equation behaves for $u$ with $\nu = \frac{p}{p - q}$, $q - 1 < \alpha < p - 1$, then it certainly holds $\nu < \frac{p}{p - q}$, $q - 1 \leq \alpha < p - 1$ or $\nu = \frac{p}{p - q}$, $q - 1 < \alpha < p - 1$, for $\epsilon \rightarrow 0$. On the one hand one can calculate the exact dependency on $\epsilon$ said to be independent of $\nu$. In this final formulation. It is possible to see this in two ways: On the one hand one can calculate the exact dependency on $\lambda$ in every single step (as in the last Theorem for $\lambda < 0$) and experiences that this dependency vanishes in the limit $\epsilon \rightarrow 0$. On the other hand one can take this result first for the single case $\lambda = 1$ and gets

$F''(T) \geq CA^\beta\frac{\beta(1 + \delta + (q - 1)\nu)}{q - 1} - \frac{\beta(1 + \delta + (q - 1)\nu)}{q - 1} R - \frac{\beta(1 + \delta + (q - 1)\nu)}{q - 1}$. 

Now, one can calculate that the conditions for Lemma 2.1.1 are fulfilled. Hence one obtains from this lemma

$\nu \leq KA^\nu$ which implies for $\epsilon \rightarrow 0$ the first two points of the following result:

**Theorem 1.4.5 (absorption term, $\alpha \geq q - 1$)**

Let $u$ be a nonnegative radially symmetric weak solution of

$$ (u^{q-1})_t - \text{div}(\|\nabla u\|^{p-2}\nabla u) + \lambda u^\alpha = 0 , \quad q - 1 \leq \alpha < p - 1 , \quad u(\cdot , 0) = u_0 , \quad \lambda > 0 $$

with $u(x, t) =: \tilde{u}(\|x\|, t)$ wherein $\partial B(0, R) \subset \text{supp}(u_0)$ with $u_0(x) \geq A\|x - x_0\|^\nu$ for $x_0 \in \partial B(0, R), x \in B(0, R) \setminus B(0, R - \epsilon_0)$, and suitable $A, \nu > 0, \epsilon_0 \in (0, R)$. For technical reasons we postulate $u_r \leq 0$ in the case $p > 2$. Then it holds for the waiting time $T^*$ on $\partial B(0, R)$:

- In the case $\nu < \frac{p}{p - q}$ it is $T^* = 0$.
- In the case $\alpha > q - 1$, $\nu = \frac{p}{p - q}$ it is $T^* \leq CA^{\nu(p - q)}$ for a constant $C = C(p, q, \alpha) > 0$.
- In the case $\alpha = q - 1$, $\nu = \frac{p}{p - q}$ there exists an $A_0 = A_0(p, q) > 0$ and $C = C(p, q) > 0$, such that $T^* \leq CA^{\nu(p - q)}$, if $A \geq A_0\lambda^{\frac{1}{p - q}}$.

Therefore the equation behaves for $\alpha > q - 1$ as the doubly nonlinear differential equation for $\lambda = 0$, i.e. the absorption term has a negligible effect with respect to the (upper bound of the) waiting time.

**Proof:** The first two points are just proven above. One notices that $C = C(p, q, \alpha)$ is said to be independent of $\lambda$ in this final formulation. It is possible to see this in two ways: On the one hand one can calculate the exact dependency on $\lambda$ in every single step (as in the last Theorem for $\lambda < 0$) and experiences that this dependency vanishes in the limit $\epsilon \rightarrow 0$. On the other hand one can take this result first for the single case $\lambda = 1$ and gets
Chapter 1  The model type of the doubly nonlinear diffusion equation

after that general \( \lambda \) by means of rescaling (see Section 1.2). It is easy to calculate that the rescaled waiting time has nevertheless no dependency on \( \lambda \).

For the last assertion, namely the case \( \alpha = q - 1 \) and \( \nu = \frac{p}{p-q} \), one shall calculate the same steps as in the first two points. One has to notice that from the assumption \( T^* > 0 \) and \( T \leq CA^{q-1}\left(\frac{R-\varepsilon}{R}\right)^{(N-1)} \) automatically follows \( T \leq KA^{-(p-q)}\left(\frac{R}{R-\varepsilon}\right)^{(N-1)(p-q)} \). Now, take first the limit \( \varepsilon \downarrow 0 \). This would already imply a contradiction, if one could choose a \( 0 < T < T^* \) which fulfils \( KA^{-(p-q)} < T \leq CA^{q-1} \). Now, it is \( KA^{-(p-q)} < CA^{q-1} \) for \( A \geq A_0 := \left(\frac{K}{C}\right)^{\frac{1}{p-1}} \). Therefore one can avoid this contradiction only, if \( T^* \leq KA^{-(p-q)} \). This finishes the third point.

The sign of \( \bar{u}_r \) is not required in the case \( p = 2 \) any more, because one can integrate by parts twice due to the second Green’s Formula, so one can resign Hardy’s inequality. The other steps can be done analogously. Details can be found at the end of the proof of Theorem 2.2.1.
Chapter 2

Non-weak absorption and reaction terms and variants

2.1 The critical case $\alpha = q - 1$

We have in principle done the case $\alpha = q - 1, \lambda > 0$ by Theorem 1.4.5. It remains now to examine the case $\alpha = q - 1$ and $\lambda \leq 0$. This works in substance similarly to $\alpha > q - 1$. But now we receive the estimation of the waiting time from a differential inequality whose second additive term does not grow superlinearly any more. Therefore one has to estimate more precisely in the following lemma. As result we obtain that the waiting time for critical growth converges to zero with $\lambda \rightarrow -\infty$, i.e. the waiting time depends now significantly on $\lambda$.

Lemma 2.1.1
Let $A, B > 0$, $C \geq 0$, $t^* \in (0, \infty]$, $\gamma \in \mathbb{N}_0$, $\varepsilon > 1$, $\delta < 1 + (\varepsilon - 1)(\gamma + 1)$, and a function

$$F \in C^0([0, t^*)) \cap C^1((0, t^*))$$

with

$$F(0) = 0, \quad F(\tau) \geq 0 \text{ for all } \tau \in (0, t^*)$$

and

$$F'(\tau) \geq A\tau^\gamma + B\tau^{-\delta}F(\tau)^{\varepsilon} + CF(\tau) .$$

Then it holds

$$t^* \leq \frac{K}{C} \ln \left(1 + KC(A^{\varepsilon-1}B)^{-\frac{1}{1+\gamma+\frac{1}{\varepsilon-1}}} \right)$$

with $K = K(\gamma, \delta, \varepsilon) > 0$. In case $C = 0$ this has to be interpreted as

$$t^* \leq K(A^{\varepsilon-1}B)^{-\frac{1}{1+\gamma+\frac{1}{\varepsilon-1}}} .$$
Chapter 2  Non-weak absorption and reaction terms and variants

Proof: Because of an indirect argument it is sufficient to prove the lemma under the assumption \( t^* < \infty \). We claim

\[
F(\tau) \geq A\tau^{\gamma+1} \sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{k}(\gamma + i)} (C\tau)^{k-1} \text{ for all } n \in \mathbb{N}_0 \text{ and } \tau \in [0, t^*) .
\]

This can be shown by a simple induction. Because \( F \) is nonnegative, the basis is done. If one has already proven

\[
F(\tau) \geq A\tau^{\gamma+1} \sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{k}(\gamma + i)} (C\tau)^{k-1} ,
\]

it follows

\[
F'(\tau) \geq A\tau^{\gamma} + CF(\tau) \geq A\tau^{\gamma+1} \sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{k}(\gamma + i)} (C\tau)^{k-1} .
\]

Thus one obtains

\[
F(\tau) = F(\tau) - F(0) = \lim_{\varepsilon \searrow 0} [F(\tau) - F(\varepsilon)] = \lim_{\varepsilon \searrow 0} \int_{\tau}^{\tau} F'(t)dt \\
\geq \int_{\tau}^{\tau} A\tau^{\gamma} + AC\tau^{\gamma+1} \sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{k}(\gamma + i)} (C\tau)^{k-1}dt = A\tau^{\gamma+1} \sum_{k=1}^{n} \frac{1}{\prod_{i=1}^{k}(\gamma + i)} (C\tau)^{k-1} .
\]

In particular we note at this place

\[
F(\tau) \geq \frac{A}{\gamma + 1} \tau^{\gamma+1}
\]
as well as

\[
F(\tau) > 0 \text{ for all } \tau \in (0, t^*) .
\]

Now, we simplify this sum by means of

\[
F(\tau) \geq A\tau^{\gamma+1} \sum_{k=0}^{\infty} \frac{1}{\prod_{i=1}^{k+1}(\gamma + i)} (C\tau)^{k} = \gamma!A\tau^{\gamma+1} \frac{e^{C\tau} - \sum_{k=0}^{\gamma} (C\tau)^{k}}{(C\tau)^{\gamma+1}}
\]

and obtain thus by estimating again

\[
F(\tau) \geq K\tau^{\gamma+1} \frac{e^{C\tau}}{(C\tau)^{\gamma+1} + 1} \geq K\tau^{\gamma+1} e^{\frac{\nu}{\gamma+1}} .
\]

Choose a

\[
\nu = \nu(\gamma, \delta, \varepsilon) \in \left(0, \min \left\{ \varepsilon - 1, \varepsilon - 1 - \frac{\delta - 1}{\gamma + 1} \right\} \right) .
\]

Then it is

\[
F'(\tau) \geq F(\tau)^{1+\nu} [B\tau^{-\delta} F(\tau)^{\varepsilon-1-\nu}] \geq KF(\tau)^{1+\nu} [A^{\varepsilon-1-\nu} B\tau^{(\gamma+1)(\varepsilon-1-\nu)-\delta}] .
\]
2.1 The critical case $\alpha = q - 1$

Because of $F(\tau) > 0$ it follows

$$-rac{1}{\nu}(F^{-\nu})'(\tau) = F(\tau)^{-1-\nu}F'(\tau) \geq KA^{\epsilon-1-\nu}B\tau(\gamma+1)(\epsilon-1-\nu)$$

for all $\tau \in (0,t^*)$. Choose an $\alpha \in (0,1)$, $\beta \in (\alpha,1)$ and integrate over $[\alpha t^*, \beta t^*]$. Then one concludes

$$\frac{1}{\nu}F(\alpha t^*)^{-\nu} \geq \frac{1}{\nu}[F(\alpha t^*)^{-\nu} - F(\beta t^*)^{-\nu}] = \int_{\alpha t^*}^{\beta t^*} \frac{1}{\nu}(F^{-\nu})'(\tau)d\tau$$

$$\geq KA^{\epsilon-1-\nu}B\int_{\alpha t^*}^{\beta t^*} \tau(\gamma+1)(\epsilon-1-\nu-\delta)d\tau,$$

hence

$$\frac{1}{\nu}F(\alpha t^*)^{-\nu} \geq KA^{\epsilon-1-\nu}B\int_{\alpha t^*}^{\beta t^*} \tau(\gamma+1)(\epsilon-1-\nu-\delta)d\tau$$

$$= KA^{\epsilon-1-\nu}B(t^*)^{1-\delta+(\gamma+1)(\epsilon-1-\nu)}(1-\alpha^{1-\delta+(\gamma+1)(\epsilon-1-\nu)}),$$

because $\beta \in (\alpha,1)$ is arbitrary. With $F(\alpha t^*) \geq KA(\alpha t^*)^{\gamma+1}e^{\frac{1}{2}\nu \alpha t^*}$ one obtains

$$\frac{1}{\nu}\left[KA(\alpha t^*)^{\gamma+1}e^{\frac{1}{2}\nu \alpha t^*}\right]^{-\nu} \geq K[A^{\epsilon-1-\nu}B(t^*)^{1-\delta+(\gamma+1)(\epsilon-1-\nu)}(1-\alpha^{1-\delta+(\gamma+1)(\epsilon-1-\nu)})],$$

thus it is

$$(t^*)^{1-\delta+(\gamma+1)(\epsilon-1)}e^{\frac{1}{2}\nu \alpha t^*} \leq \frac{K}{A^{\epsilon-1}B}.$$
with \( C = C(p, q) > 0 \), independent of \( \lambda \) and the spatial dimension \( N \). In case \( \lambda = 0 \) this has to be interpreted as
\[
t^*_\Omega \leq CA^{-(p-q)}.
\]
If it even holds \( \gamma < \frac{p}{p-q} \), then one obtains \( t^*_\Omega = 0 \).

**Proof:** Because the arguments are very similar to those from the first chapter, we will be more brief at this place: From the differential inequality
\[
F'(\tau) \geq C(u_0; \delta, \epsilon) \tau^{\tilde{\gamma}} + C(p, q, \delta) \left( \frac{R}{R-\epsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} (R-\epsilon)^{-\frac{(N-1)(p-q)}{q-1}} \epsilon^{\delta+1-p-\frac{(p-1)(\delta+1)}{q-1}} \tau^{\frac{(q+1)(p-q)}{q-1}} F(\tau)^{\frac{q-1}{q-1}}
\]
it follows
\[
t^* \leq \frac{C}{|\lambda|} \ln \left( 1 + \frac{|\lambda|}{A^{\epsilon-1}B} \right)
\]
with
\[
A^{\epsilon-1}B = C(u_0; \delta, \epsilon)^{\frac{p-q}{q-1}} \left( \frac{R}{R-\epsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} (R-\epsilon)^{-\frac{(N-1)(p-q)}{q-1}} \epsilon^{\delta+1-p-\frac{(p-1)(\delta+1)}{q-1}} F(\tau)^{\frac{q-1}{q-1}}
\]
It holds for \( \epsilon \in (0, R) \)
\[
C(u_0; \delta, \epsilon) = \int_{R-\epsilon}^{R} \tilde{u}_0(r)^{q-1}(\epsilon^\delta - (R-r)^\delta) r^{N-1} dr = A^{\epsilon-1} \int_{0}^{\epsilon} t^{q/(q-1)}(\epsilon^\delta - t^\delta)(R-t)^{N-1} dt \geq CA^{q-1}(R-\epsilon)^{N-1}\epsilon^\delta \gamma(q-1)+1.
\]
Thus one obtains
\[
A^{\epsilon-1}B \geq CA^{p-q} \left( \frac{R}{R-\epsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} \epsilon^{\frac{p-q}{q-1}(\delta+\gamma(q-1)+1)+\delta+1-p-\frac{(p-1)(\delta+1)}{q-1}}
\]
By noticing that the exponent of \( \epsilon \) vanishes, if and only if \( \gamma = \frac{p}{p-q} \), the assertion follows by taking to the limit \( \epsilon \downarrow 0 \).

### 2.2 The supercritical case \( \alpha < q - 1 \)

We will experience now that the absorption term \( \lambda u^\alpha \) for \( \lambda > 0 \), \( \alpha < q - 1 \), is important enough in order to take influence on the critical growth exponent. Admittedly the method used here does not lead any more to a differential inequality, but to a functional inequality (without derivatives). Moreover one has to start the argument indirectly now.

**Theorem 2.2.1 (absorption term, \( \alpha < q - 1 \))**

Let
\[
\lambda > 0, \ p \geq 2, \ 1 < q < p, \ \alpha < q - 1,
\]
2.2 The supercritical case \( \alpha < q - 1 \)

and \( u \) be an arbitrary nonnegative radially symmetric solution of the differential equality

\[
(u^{q-1})_t - \text{div}(\|\nabla u\|^{p-2}\nabla u) + \lambda u^\alpha = 0 , \ u(\cdot, 0) = u_0
\]

with \( u(x, t) =: \tilde{u}(\|x\|, t) \) wherein \( \partial B(0, R) \subset \partial \text{supp}(u_0) \) with \( u_0(x) \geq A\|x - x_0\|^{\nu} \) for \( x_0 \in \partial B(0, R) \), \( x \in B(0, R) \setminus B(0, R - \varepsilon_0) \), and suitable \( A, \nu > 0 \). We postulate \( \tilde{u}_r \leq 0 \) in the case \( p > 2 \) for technical reasons. Then it holds for the waiting time:

- In the case \( \nu \in (0, \frac{p}{p-1-\alpha}) \) the waiting time is zero.
- In the case \( \nu = \frac{p}{p-1-\alpha} \) there is an \( A_0 = A_0(p, q, \alpha) > 0 \), such that the waiting time is zero for all \( A \geq A_0\lambda^{\frac{1}{p-1-\alpha}} \).

**Remark 2.2.2**

Because

\[
u(t, x) := A \cdot x^\frac{p-1-\alpha}{p} \quad \text{with} \quad A := \left[ \lambda \left( \frac{p-1-\alpha}{p} \right)^p \frac{p}{(p-1)(1+\alpha)} \right]^{\frac{1}{p-1-\alpha}} > 0
\]

is a stationary solution of the above differential equation for \( N = 1 \), the condition for \( \nu \) is reasonable.

It is proven in [2] by means of other methods that the waiting time is zero in the special case \( N = 1, p = 2, \nu < \frac{2}{1-\alpha} \).

**Proof:** We will first show the case \( p > 2 \); this proof would also work for \( p = 2 \) only under the additional assumption \( \tilde{u}_r \leq 0 \). Therefore we will give another proof for \( p = 2 \) below.

Supposed it was \( T^* > 0 \). Let \( 0 < T \leq T^* \) and \( \varepsilon \in (0, \varepsilon_0) \). Starting point remains the test with the test function

\[
\varphi(t, x) = (\varepsilon^\delta - |R - \|x\|^{\delta})_+(T - t)^\gamma .
\]

Then one obtains the identity

\[
\delta \int_0^T \int_{R-\varepsilon}^{R} \tilde{u}_{r}^{p-1}(R - r)^{\delta-1}(T - t)^\gamma r^{N-1} + [\int_{R-\varepsilon}^{R} \tilde{u}_0^{q-1}(\varepsilon^\delta - (R - r)^{\delta})_+ r^{N-1}] T^\gamma
\]

\[
= \gamma \int_0^T \int_{R-\varepsilon}^{R} \tilde{u}_{r}^{q-1}(T - t)^\gamma r^{N-1} + \lambda \int_0^T \int_{R-\varepsilon}^{R} \tilde{u}_0^{q-1}(\varepsilon^\delta - (R - r)^{\delta})_+ r^{N-1} \gamma r^{N-1} .
\]

Now, fix a \( \beta \in (q - 1, p - 1) \)

and set

\[
F(T) := \int_0^T \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon^\delta - (R - r)^{\delta})(T - t)^\gamma r^{N-1} \geq 0 .
\]
Our goal is to derive a functional inequality for $F(T)$ that has to imply a contradiction. First one has the trivial estimation
\[ \left[ \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} \right]^\frac{1}{\beta} \leq H \left[ \int_{R-\varepsilon}^{R} |\tilde{u}|^{p-1}(R - r)^{\delta - 1} r^{N-1} \right]^\frac{1}{p-1}. \]

The constant $H$ can be obtained for $p > 2$ from the following calculation
\[ \left[ \int_{\varepsilon}^{\beta} \int_0^T (\varepsilon - t)^{(R - t)^{N-1}} dt \right]^{\frac{p-1}{\beta}} \cdot \left[ \int_0^{\beta} \int_0^T (t^{(R - t)^{N-1}})^{\frac{1}{\beta}} dt \right]^{\frac{p-1}{\beta}} \leq C \varepsilon^{\frac{1}{\beta} \left[ (N-1)(\frac{p}{\beta} - 1) - \right]} \left[ F(T) \right]^{\frac{p-1}{\beta}}. \]

It follows
\[ \int_{R-\varepsilon}^{R} |\tilde{u}|^{p-1}(R - r)^{\delta - 1} r^{N-1} \geq CR^{-\frac{(N-1)(p-1-\delta)}{\beta}} \varepsilon^{-\frac{p-1}{\beta} \left[ (1 + \frac{p}{\beta} - 1) - \delta \right]} \cdot \left[ \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} \right]^{\frac{p-1}{\beta}}. \]

Now, we combine this together with the inverse HÖLDER inequality:
\[ \int_{0}^{T} \left( \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} dr \right)^{\frac{p-1}{\beta}} (T - t)^{1/(p-1)} \geq CT^{-\left( (\gamma + 1) \gamma \right)} \left[ F(T) \right]^{\frac{p-1}{\beta}}. \]

Thus we can express the left hand side of the identity by means of $F(T)$ after the testing. Now, we want to estimate the right hand side from above in order to obtain only terms with $F(T)$ This will be done with the aid of the “common” HÖLDER inequality with exponents $\frac{\beta}{q-1}$ and $\frac{\beta}{j-q+1}$
\[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/j} \]
\[ \leq \left[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/j} \right]^{\frac{1}{j}} \cdot \left[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/j} \right]^{\frac{1}{j}} \]
\[ \leq CF(T) \frac{q}{\beta} \varepsilon^{\left( j \beta \frac{q}{\beta} - 1 \right)} \frac{\beta}{j-q+1} T^{\gamma - 1} \frac{\beta}{j-q+1} \frac{\beta}{j-q+1} R^{\frac{(N-1)(\beta - 1)}{\beta}}. \]

as well as with exponents $\frac{\beta}{\alpha}$ and $\frac{\beta}{\beta - \alpha}$
\[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/\alpha} \]
\[ \leq \left[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/\alpha} \right]^{\frac{1}{\alpha}} \cdot \left[ \int_{0}^{T} \int_{R-\varepsilon}^{R} \tilde{u}^{\beta}(\varepsilon - (R - r)^{\delta}) r^{N-1} (T - t)^{1/\alpha} \right]^{\frac{1}{\alpha}} \]
\[ \leq CF(T) \frac{\beta}{\alpha} \varepsilon^{\left( \beta \frac{\beta}{\alpha} - 1 \right)} T^{\gamma - 1} \frac{\beta}{\alpha} \frac{\beta}{j-q+1} R^{\frac{(N-1)(\beta - 1)}{\beta}}. \]
2.2 The supercritical case $\alpha < q - 1$

Altogether we are standing at this functional inequality

$T^{-(\gamma + 1)b^{-1}} R^{(N-1)(\beta - \gamma - \delta)} \varepsilon^{\frac{p-1}{\beta}} \left[ F(T) \right]^{\frac{p-1}{\beta}} + 2AF^{-1} \varepsilon^{1+\delta+(q-1)\nu} T^\gamma (R - \varepsilon)^{N-1} \leq CF(T)^{\frac{q-1}{\beta}} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} + 2AF^{-1} \varepsilon^{1+\delta+(q-1)\nu} T^\gamma (R - \varepsilon)^{N-1}$

Now, we use Young's inequality with exponents $\frac{q-1}{\alpha}$ and $\frac{q-1}{q-1-\alpha}$ and obtain thus

$CF(T)^{\frac{q-1}{\beta}} T^{(\gamma + 1)b^{-1}} \varepsilon^{\frac{\gamma \alpha}{\alpha}} T^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} R^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} (R - \varepsilon)^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \leq A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^\gamma (R - \varepsilon)^{N-1}$

Therefore one gets the inequality

$T^{-(\gamma + 1)b^{-1}} \varepsilon^{\frac{p-1}{\beta}} \left[ F(T) \right]^{\frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} + A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^\gamma (R - \varepsilon)^{N-1} \leq CF(T)^{\frac{q-1}{\beta}} T^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} R^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \left[ F(T) \right]^{\frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \leq A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^\gamma (R - \varepsilon)^{N-1}$

Now, it is

$A^{q-1} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} R^{\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \left[ F(T) \right]^{\frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \leq \frac{A^{q-1} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}}}{A^{q-1} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}}}$

equivalent to

$A^{q-1} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \leq T$, as one sees after some longer calculations. Supposing this to be true one would obtain from this

$T^{-(\gamma + 1)b^{-1}} \varepsilon^{\frac{p-1}{\beta}} \left[ F(T) \right]^{\frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}} \leq A^{q-1} \varepsilon^{(1+\delta)(\beta - \gamma - \delta) + \frac{p-1}{\beta}} T^{-1-\gamma \frac{q-1}{\beta} + \frac{p-1}{\beta}}$
and by neglecting the $F(T)$ term at the left hand side one gets after a tedious calculation of all exponents
\[ T \geq CA^{q-1}e^{(q-1)\nu} \left( \frac{R - \varepsilon}{R} \right)^{\frac{(N-1)(q-1)-\alpha p}{q-1}}. \]
Altogether one would arrive at a contradiction, if we could choose a sufficiently small $\varepsilon > 0$ and $T > 0$ which suffices the inequalities
\[ A^{q-1-\alpha e^{(q-1)\nu}} \left( \frac{R - \varepsilon}{R} \right)^{\frac{(N-1)(q-1)-\alpha p}{q-1}} \leq T \]
and
\[ T < CA^{q-1}e^{(q-1)\nu} \left( \frac{R - \varepsilon}{R} \right)^{(q-1)(N-1)}. \]
This is possible, if and only if
\[ A^{q-1}e^{(q-1)\nu} \left( \frac{R - \varepsilon}{R} \right)^{\frac{(N-1)(q-1)-\alpha p}{q-1}} < CA^{q-1}e^{(q-1)\nu} \left( \frac{R - \varepsilon}{R} \right)^{(q-1)(N-1)} \]
which is equivalent to
\[ \varepsilon^{-\alpha \nu + \frac{\alpha p}{p-1-\alpha}} < CA^{p} \left( \frac{R - \varepsilon}{R} \right)^{(q-1)(N-1)-\frac{(N-1)(q-1)-\alpha p}{q-1}}. \]
Now, we notice
\[-\alpha \nu + \frac{\alpha p}{p-1-\alpha} > 0 \iff \nu < \frac{p}{p-1-\alpha} \]
resp.
\[-\alpha \nu + \frac{\alpha p}{p-1-\alpha} = 0 \iff \nu = \frac{p}{p-1-\alpha}. \]
This implies the desired contradiction to $T^* > 0$.

The case $p = 2$ could be done analogously by the special case of HARDY’s inequality. But there is an alternative way which does not require a sign of $\vec{u}_r$. This way will be sketched here briefly: By means of the second GREEN’s Formula and $[\Delta \varphi](x, t) = \left[ \frac{N-1}{\|x\|} \vec{\varphi}_r + \vec{\varphi}_{rr}\right](\|x\|, t)$ one obtains by testing with $\varphi$ defined by
\[ \varphi(x, t) := (\|x\| - R + \varepsilon)^{\delta}\gamma(T - t)^{\gamma} \]
for $\delta > 2$ the identity
\[ \delta(\delta - 1) \int_0^T \int_{R-\varepsilon}^R \tilde{u}(r - R + \varepsilon)^{\delta-2}r^{N-1}(T - t)^{\gamma} + \delta(N - 1) \int_0^T \int_{R-\varepsilon}^R \tilde{u}(r - R + \varepsilon)^{\delta-1}r^{N-2}(T - t)^{\gamma} + \int_{R-\varepsilon}^R \tilde{u}_0^{\delta-1}(r - R + \varepsilon)^{\delta}r^{N-1}T^{\gamma} \]
\[ = \gamma \int_0^T \int_{R-\varepsilon}^R \tilde{u}^{\gamma-1}(T - t)^{\gamma-1}(r - R + \varepsilon)^{\delta}r^{N-1} + \lambda \int_0^T \int_{R-\varepsilon}^R \tilde{u}^{\alpha}(r - R + \varepsilon)^{\delta}(T - t)^{\gamma}r^{N-1}. \]
2.2 The supercritical case $\alpha < q - 1$

By setting

$$F(T) := \int_0^T \int_{R-\varepsilon}^R \tilde{u}(r - R + \varepsilon)^{\delta-2} r^{N-1} (T - t)^{\gamma} \geq 0$$

and using Hölder’s inequality with exponents $\frac{1}{q-1}$ and $\frac{1}{2-q}$

$$\int_0^T \int_{R-\varepsilon}^R \tilde{u}^{q-1}(r - R + \varepsilon)^{\delta} r^{N-1} (T - t)^{\gamma-1} \leq CF(T)^{q-1} \cdot \varepsilon^{\delta(2-q)+q\gamma(2-q)+1-q} R^{(N-1)(2-q)}$$

as well as with exponents $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$

$$\int_0^T \int_{R-\varepsilon}^R \tilde{u}(r - R + \varepsilon)^{\delta} r^{N-1} (T - t)^{\gamma} \leq CF(T)^{q-1} \cdot \varepsilon^{\delta(1-\alpha)+1+\alpha T(1-\alpha)} R^{(N-1)(1-\alpha)}$$

and Young’s inequality with exponents $\frac{2-1}{\alpha}$ and $\frac{q-1}{q-1-\alpha}$

$$\begin{align*}
F(T)^{q-1} T^{(\gamma)(1-\alpha)} & \leq \left[ C_2 A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)}(R - \varepsilon)^{N-1} 
+ CA^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)(1-\alpha)} \frac{2}{\alpha} + \left(\frac{1}{\alpha} - 1\right) \cdot \frac{1}{\alpha} T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma \right] .
\end{align*}$$

one obtains

$$\begin{align*}
F(T) + A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)}(R - \varepsilon)^{N-1} & \leq CF(T)^{q-1} T^{(\gamma)(2-q)+1-q} \varepsilon^{(2-q)+q} R^{(N-1)(2-q)} 
+ A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)(1-\alpha)} \frac{2}{\alpha} + \left(\frac{1}{\alpha} - 1\right) \cdot \frac{1}{\alpha} T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma \right] .
\end{align*}$$

Under the assumption

$$A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} \left(\frac{R - \varepsilon}{R}\right)^{(N-1)(1-\alpha)} \leq T$$

it follows

$$\begin{align*}
F(T) + A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)}(R - \varepsilon)^{N-1} & \leq CF(T)^{q-1} A^{q-1} \varepsilon^{1+\delta+(q-1)\nu} T^{(\gamma)(1-\alpha)} \frac{2}{\alpha} + \left(\frac{1}{\alpha} - 1\right) \cdot \frac{1}{\alpha} T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma T^{(\gamma)(1-\alpha)} \frac{2-1}{\alpha} \gamma \right] .
\end{align*}$$

Exactly as in the above case $p \neq 2$ one concludes from this the inequality

$$CA^{q-1} \varepsilon^{-2} \frac{1}{\alpha} \cdot \frac{1}{\alpha} + \nu(q-1) \left(\frac{R - \varepsilon}{R}\right)^{(N-1)(1-\alpha)} \leq T .$$
In particular one would arrive at a contradiction, if
\[ A^{\nu-1} e^{(q-1-\alpha)\nu} \left( \frac{R - \varepsilon}{R} \right)^{(N-1)(q-1-\alpha)} < C A^{\nu-1} e^{-\frac{2}{1+\alpha}(q-1-\alpha)} \left( \frac{R - \varepsilon}{R} \right)^{(N-1)(1-\alpha)} \]
which is equivalent to \( e^{\alpha \left( \frac{2}{1+\alpha} - \nu \right)} < C A^{\alpha} \left( \frac{R - \varepsilon}{R} \right)^{(N-1)(1-\alpha) - \frac{(N-1)(q-1-\alpha)}{q-1}} \).

In the case \( \lambda < 0, \alpha < q - 1 \) we would require again the differential inequality technique. We only want to sketch the needed estimation of the blow-up time, because the essential steps are similar to those of the last lemmas:

**Lemma 2.2.3**

Let \( A, B, C > 0, t^* \in (0, \infty), 0 < \vartheta < 1, \gamma, \eta > -1, \varepsilon > 1, m \in (\gamma + 1, \frac{1+\gamma}{1+\gamma} + 1) \) with the convention \( \langle a, b \rangle := [\min\{a, b\}, \max\{a, b\}] \), \( 1 + \eta - (1+\gamma)(1-\vartheta) \neq 0 \), and \( (1-\delta) + (\varepsilon-1)m > 0 \) as well as

\[ F \in C^0([0, t^*)) \cap C^1((0, t^*)) \]

with
\[ F(0) = 0, \; F(\tau) \geq 0 \text{ for all } \tau \in (0, t^*), \]

and
\[ F'(\tau) \geq A\tau^\gamma + B\tau^{-\delta} F(\tau)^\varepsilon + C\tau^\eta F(\tau)^\vartheta. \]

Then it holds
\[ t^* \leq KA^{-\frac{(\varepsilon-1)(1+\eta-m(1-\vartheta))}{(1+\eta)-(1+\gamma)(1-\vartheta)+(1-\delta)+(\varepsilon-1)m}} B \]
for a suitable \( K = K(m, \gamma, \delta, \varepsilon, \eta, \vartheta) > 0. \)

**Proof:** In the first step one scales the inequality by means of
\[ G(t) := [A^{-(1+\eta)} C^{1+\gamma}]^{1\over 1+\eta-(1+\gamma)(1-\vartheta)} \cdot F([A^{1-\varepsilon} C^{-1}]^{1\over 1+\eta-(1+\gamma)(1-\vartheta)} t) \]
into the differential inequality
\[ G'(t) \geq t^\gamma + B\tau^{-\delta} G(t)^\varepsilon + t^\eta G(t)^\vartheta \]
with
\[ B := [A^{-(1+\eta)(1-\varepsilon)+(1-\vartheta)(1-\delta)} G^{(1+\gamma)(1-\varepsilon)-(1-\delta)}]^{1\over 1+\eta-(1+\gamma)(1-\vartheta)} B. \]

In particular it holds
\[ t^*(F) \leq [A^{1-\varepsilon} C^{-1}]^{1\over 1+\eta-(1+\gamma)(1-\vartheta)} \cdot t^*(G). \]

In the second step we derive an a priori growth for \( G \). On the one hand it is \( G'(t) \geq t^\gamma \) and \( G(0) = 0 \), thus \( G(t) \geq K t^{\gamma+1} \). In particular it holds \( G(t) > 0 \) for \( t \in (0, t^*(G)) \) and we obtain from \( G'(t) G(t)^\vartheta \geq t^\vartheta \) and \( G(0) = 0 \) finally \( G(t)^{1-\vartheta} \geq K t^{1+\eta} \). Therefore one concludes in every case \( G(t) \geq K t^m \).
2.3 Variant: Convection and advection terms

In the third step we take advantage of the superlinearity in order to prove a blow-up. This can be done analogously to the last lemmas. Briefly: From

\[ G'(t) \geq G(t)^{1+\nu}[\tilde{B}t^{-\delta}G(t)^{\varepsilon-1-\nu}] \geq KG(t)^{1+\nu}[\tilde{B}t^{-\delta+(\varepsilon-1-\nu)m}] \]

for suitably small \( \nu > 0 \) one obtains

\[ \frac{1}{\nu}[K\alpha^{-\nu m}(t^*)^{-\nu m}] \geq K\tilde{B}(t^*)^{1-\delta+(\varepsilon-1-\nu)m}(1-\alpha)^{1-\delta+(\varepsilon-1-\nu)m} \]

for \( 0 < \alpha < 1 \) which implies

\[ t^*(\tau) \leq K\tilde{B}^{-\frac{1}{1-\delta+(\varepsilon-1)m}}. \]

Rescaling implies the assertion. \( \Box \)

Applying this result to the differential inequality

\[ F'(\tau) \geq C(u_0; \delta, \varepsilon)\tau^\gamma \]

\[ + C(p, q, \delta)
\]

\[ \left( \frac{R}{R - \varepsilon} \right)^{\frac{(N-1)(p-1)}{q-1}} (R - \varepsilon)^{\frac{(N-1)(p-q)}{q-1}} \left( R - \varepsilon \right)^{\frac{(\gamma+1)(p-q)}{q-1}} - (\gamma+1)(p-q) \right)^{\frac{1}{q-1}} \]

\[ + C(p, q, \delta) |\lambda|^{-\frac{p-q}{p-1-\alpha}} \epsilon^{\frac{p-1-\alpha}{p-1}} \]

for the parameter \( m := \gamma + 1 + \frac{q-1}{q-1-\alpha} \) wherein \( R > 0 \) is chosen so large that \( C(u_0; \delta, \varepsilon) > 0 \) one obtains

\[ t^* \leq C \left( \frac{R}{R - \varepsilon} \right)^{-\frac{(N-1)(p-q)}{(q-1)(p-1-\alpha)}} (R - \varepsilon)^{-\frac{(p-q)(N-1)(q-1-\alpha)}{(q-1)(p-1-\alpha)}} |\lambda|^{-\frac{p-q}{p-1-\alpha}} \epsilon^{\frac{p-1-\alpha}{p-1-\alpha}}. \]

Taking \( \epsilon \searrow 0 \) we receive

\[ t^* = 0 \text{, if } u_0 \neq 0. \]

Therefore we have proven in the case of a supercritical reaction term, i.e. \( \alpha < q - 1 \):

**Theorem 2.2.4 (reaction term, \( \alpha < q - 1 \))**

Let \( u \) be a nonnegative radially symmetric weak solution of

\[ (u^{q-1})_t - \text{div}(\|\nabla u\|^{p-2}\nabla u) + \lambda u^\alpha = 0 \text{, } u(\cdot, 0) = u_0 \]

for \( \lambda < 0 \) and \( \alpha < q - 1 \) with initial value \( u_0 \neq 0 \) and \( \tilde{u}_t \leq 0 \). Then the waiting time is everywhere zero.

2.3 Variant: Convection and advection terms

The techniques presented here also work for nonnegative solutions of the doubly degenerate diffusion-convection equation (resp. diffusion-advection equation)

\[
\begin{cases}
(u^{q-1})_t - (|u_x|^{p-2}u_x)_x + \lambda (u^\alpha)_x = 0, & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x) & \text{for all } x \in \mathbb{R},
\end{cases}
\tag{2.1}
\]
wherein \( u_0(x) = 0 \) for \( x \in (-\infty, 0) \) and \( u_0(x) \geq Ax^\nu \) for \( x \in [0, \varepsilon_0] \) and suitable \( A, \nu > 0 \). We postulate \( u_x \geq 0 \) on \([0, \varepsilon_0] \times [0, T^*] \) in the case \( p > 2 \) for technical reasons. Now, \( \alpha = q^{p-1} \) turns out to be the new critical exponent. More precisely one obtains the following facts:

**Theorem 2.3.1**

- In the case \( \alpha \geq q^{p-1}/p, \lambda < 0, \nu < p/(p-q), \) resp. \( q^{p-1}/p \leq \alpha < p-1, \lambda > 0, \nu < p/(p-q), \) it holds \( T^* = 0 \).

- In the case \( \alpha > q^{p-1}/p, \lambda < 0, \nu = p/(p-q), \) resp. \( q^{p-1}/p < \alpha < p-1, \lambda > 0, \nu = p/(p-q), \) it holds \( T^* \leq CA^{-(p-q)} \) for a suitable \( C = C(p,q,\alpha) > 0 \).

Therefore the equation behaves for \( \alpha > q^{p-1}/p \) as the doubly nonlinear differential equation for \( \lambda = 0 \).

- In the case \( 0 < \alpha < q^{p-1}/p, \lambda > 0, \nu < p/(p-1-\alpha), \) it holds \( T^* = 0 \).

- In the case \( 0 < \alpha < q^{p-1}/p, \lambda > 0, \nu = p/(p-1-\alpha), \) there is an \( A_0 = A_0(p,q,\alpha) > 0 \), such that \( T^* = 0 \), if \( A \geq A_0 \lambda \frac{p-1}{p-\alpha} \).

- In the case \( \alpha = q^{p-1}/p, \lambda > 0, \nu = p/(p-q), \) there is an \( A_0 = A_0(p,q) > 0 \), such that \( T^* \leq CA^{-(p-q)} \lambda \) for a \( C = C(p,q) > 0 \), if \( A \geq A_0 \lambda \frac{p-1}{p-\alpha} \).

- In the case \( q-1 < \alpha < q^{p-1}/p, \lambda < 0, \nu < 1/(\alpha-q+1), \) it holds \( T^* = 0 \).

- In the case \( q-1 < \alpha < q^{p-1}/p, \lambda < 0, \nu = 1/(\alpha-q+1), \) it holds \( T^* \leq CA^{-(\alpha-q+1)} \lambda \) for a \( C = C(p,q,\alpha) > 0 \).

- In the case \( \alpha = q^{p-1}/p, \lambda < 0, \nu = p/(p-q), \) it holds \( T^* \leq \frac{C A^{-(p-q)}}{|\lambda|} \lambda \) for a suitable \( C = C(p,q) > 0 \).

The cases which imply \( T^* = 0 \) under the assumption \( p = 2 \) were already proven in [1], though with other methods. However, the other cases are new.

We want to remark at this place that in case \( \lambda > 0, 0 < \alpha < p-1 \) there is the following stationary solution

\[ u(t, x) = A x^{\frac{p-1}{p-1-\alpha}} \text{ with } A := \left[ \lambda \left( \frac{p-1-\alpha}{p-1} \right)^{p-1} \right]^{\frac{1}{p-1-\alpha}} > 0 \]

wherein \( \frac{p-1}{p-1-\alpha} \leq \frac{p}{p-q} \), if and only if \( \alpha \leq q^{p-1}/p \). In particular this explains why the critical growth exponent must change at \( \alpha = q^{p-1}/p \).
2.3 Variant: Convection and advection terms

**Remark 2.3.2**

The analogous results hold by passage from $x$ to $-x$, if 0 is the right boundary point of $\Omega$. One has only to change the sign of $\lambda$.

**Proof:** We want to notice first that one can scale out the influence of the parameter $\lambda$ by means of the transformation

$$v(x, t) := u(|\lambda|^{-\frac{1}{p-1}}x, |\lambda|^{-\frac{p}{p-1}}t),$$

because $v$ is then a solution of $(v^{q-1})_{t} - (|v_{x}|^{p-2}v_{x})_{x} + (\text{sgn } \lambda)(v^{\alpha})_{x} = 0$. With this in mind we may work with $|\lambda| = 1$ and obtain the assertion for general $\lambda$ by rescaling.

Testing with $(\varepsilon - x)^{\delta}(T - t)^{\gamma}$ leads to

$$\delta \int_{0}^{T} \int_{0}^{\varepsilon} u^{q-1}(\varepsilon - x)^{\delta-1}(T - t)^{\gamma} + [\int_{0}^{\varepsilon} u^{q-1}(\varepsilon - x)^{\delta}]T^{\gamma} = \gamma \int_{0}^{T} \int_{0}^{\varepsilon} u^{q-1}(\varepsilon - x)^{\delta}(T - t)^{\gamma-1} + (\text{sgn } \lambda)\delta \int_{0}^{T} \int_{0}^{\varepsilon} u^{\alpha}(\varepsilon - x)^{\delta-1}(T - t)^{\gamma}.$$ 

Because all the techniques are discussed in the former sections, we want to present here only some sketches: The case $\lambda = -1$ can be proven in principle without distinction of cases. By using Hardy’s inequality and the inverse Hölder inequality one obtains for

$$F(T) := \int_{0}^{T} \int_{0}^{\varepsilon} u^{q-1}(\varepsilon - x)^{\delta-1}(T - t)^{\gamma}$$

the differential inequality

$$F'(T) \geq CA^{q-1}\varepsilon^{1+\delta+(q-1)\nu}T^{\gamma} + CF(T)\varepsilon^{\delta(1-\frac{\alpha}{q-1})-\frac{\alpha}{q-1}T^{\frac{\gamma}{q-1}(\alpha-q+1)}} + CF(T)\varepsilon^{-\frac{\alpha}{q-1}T^{\frac{\gamma(q+1)}{q-1}(p-q)}}.$$ 

Applying Lemma 1.4.1 implies

$$T^{\alpha}[A^{p-q}\varepsilon^{(p-q)\nu}-p + A^{p-q+1}\varepsilon^{(p-q+1)\nu}] \leq K$$

which proves the assertion in the given subcases for $\varepsilon \searrow 0$.

Now, let $\lambda = 1$ and $q^{\frac{p-1}{p}} \leq \alpha < p - 1$. Fix a $\beta \in (\alpha, p - 1)$. With the aid of Hardy’s inequality, Hölder’s inequality and the inverse Hölder inequality it follows for

$$F(T) := \int_{0}^{T} \int_{0}^{\varepsilon} u^{\beta}(\varepsilon - x)^{\delta}(T - t)^{\gamma}$$

the differential inequality

$$[F'(T)]^{\frac{\alpha-1}{\beta}} \leq A^{\frac{p-1}{\beta}(\beta - q + 1)}T^{\frac{\gamma}{p}(\beta - q + 1)}$$

$$CT^{\frac{\gamma}{p}(p-1-\beta)}\varepsilon^{-\frac{\alpha-1}{\beta}[1+(1-\frac{\alpha-1}{p})\frac{p-1}{\beta}+(1+\delta)]\frac{\gamma}{p(1-\beta)}F(T)^{\frac{p-1}{\beta}} + CT^{\frac{\gamma}{p}(\beta - q + 1)}\varepsilon^{-\frac{\alpha-1}{\beta}[1+(1-\frac{\alpha-1}{p})\frac{p-1}{\beta}+(1+\delta)]\frac{\gamma}{p(1-\beta)}F(T)^{\frac{p-1}{\beta}}.$$
By using Young’s inequality with exponents $\frac{p-1}{\alpha}$ and $\frac{p-1}{p-1-\alpha}$

$$CT^{(\gamma+1)}\frac{\beta}{\alpha} \leq \frac{\hat{C}}{2} A_{\tau_1}^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

$$+ CF(T) \left( \frac{\beta}{\alpha} \cdot A_{(q-1)(\frac{\beta}{\alpha} - 1)}^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}} \right)$$

one obtains

$$[F'(T)]^{\frac{\beta}{\alpha} T^{-\frac{\beta}{\alpha} \tau_1} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

$$\geq CA^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

$$+ CF(T) \left( \frac{\beta}{\alpha} \cdot A_{(q-1)(\frac{\beta}{\alpha} - 1)}^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}} \right)$$

Furthermore it is

$$A_{-(q-1)(\frac{\beta}{\alpha} - 1)}^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

$$\leq \frac{1}{2} T^{-\frac{\beta}{\alpha} \tau_1} T^{(\gamma+1)} \frac{\beta}{\alpha}$$

equivalent to

$$T \leq CA^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

The exponent at $\varepsilon$ is negative, if and only if $\nu < \frac{\alpha p - p + 1}{(q-1)(p-1-\alpha)}$. But this is fulfilled in the first both given cases in Theorem 2.3.1, because

$$\frac{p}{p - q} < \frac{\alpha p - p + 1}{(q-1)(p-1-\alpha)}$$

and of course $\frac{p}{p - q} = \frac{\alpha p - p + 1}{(q-1)(p-1-\alpha)}$ for $\alpha = \frac{q(p-1)}{p}$. Thus one obtains for fixed $T > 0$ and sufficiently small $\varepsilon > 0$

$$F'(T) \geq CA^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}}$$

$$+ CF(T) \left( \frac{\beta}{\alpha} \cdot A_{(q-1)(\frac{\beta}{\alpha} - 1)}^{\frac{\beta}{\alpha} T^{(\gamma+1)} \frac{\beta}{\alpha}} \right)$$

One can calculate that the conditions for Lemma 2.1.1 are fulfilled. Therefore one gets from this lemma

$$T^* \leq KA^{\frac{p-1}{p}} \varepsilon^{(\nu(p-1))}$$

which implies the assertion for taking $\varepsilon \downarrow 0$.

The case $0 < \alpha \leq q \frac{p+1}{p}$, $\lambda = 1$, must be split into the three subcases $\alpha < q - 1$, $\alpha = q - 1$ resp. $\alpha > q - 1$. Choose

$$\beta \in (\max \{ \alpha, q - 1 \}, p - 1)$$

and set for $T \in (0, T^*)$ under the (contradiction) assumption $T^* > 0$

$$F(T) := \int_0^T \int_0^T u^\beta (\varepsilon - x)^2 (T - t)^\gamma.$$
2.3 Variant: Convection and advection terms

By using Hölder’s inequality, Hardy’s inequality and the inverse Hölder inequality all cases lead to the same functional inequality

\[ 2A^q \varepsilon^{1+\delta+(q-1)\nu} T^\gamma + T\varepsilon^{-(\gamma+1)(p-1-\beta)} \varepsilon^{-(\frac{\alpha-1}{\alpha})[(1+\delta)\frac{p-1}{p-1-\gamma}+(1+\delta)\frac{\beta-1}{\beta-\gamma}] - \frac{\delta-1}{\gamma-1}} F(T)^{\frac{\alpha-1}{\alpha}} \leq CF(T)^{\frac{\alpha-1}{\alpha}} \varepsilon^{(\delta+1)\frac{\beta-1}{\beta} - \frac{\alpha-1+1}{\alpha}(1+\delta+(q-1)\nu)} T^{\frac{\alpha}{\alpha}} \left( \frac{\gamma+1}{\gamma} \right)^{-\frac{\alpha-1}{\alpha}} F(T)^{\frac{\alpha-1}{\alpha}}. \]

The case \( \alpha = q - 1 \) follows from \( T \geq \varepsilon \) in the usual manner \( T \geq C A^q \varepsilon^{1-(q-1)(\nu(q-\nu)+p-1)} \) which implies under the given assumptions a contradiction for \( \varepsilon \searrow 0 \). In the cases \( \alpha \neq q - 1 \) one can hide the smaller \( F(T) \)-power with the aid of Young’s inequality behind the greater \( F(T) \)-power with remainder term \( A^q \varepsilon^{1+\delta+(q-1)\nu} T^\gamma \), i.e. in the case \( q - 1 < \alpha \leq q \frac{p-1}{p} \)

\[ CF(T)^{\frac{\alpha-1}{\alpha}} \varepsilon^{((\delta+1)\beta-1)\frac{\beta-1}{\beta} - \frac{\alpha-1+1}{\alpha}(1+\delta+(q-1)\nu)) T^{\frac{\alpha}{\alpha}} \left( \frac{\gamma+1}{\gamma} \right)^{-\frac{\alpha-1}{\alpha}} F(T)^{\frac{\alpha-1}{\alpha}} \]

resp.

\[ CF(T)^{\frac{\alpha-1}{\alpha}} \varepsilon^{((\delta+1)\beta-1)\frac{\beta-1}{\beta} - \frac{\alpha-1+1}{\alpha}(1+\delta+(q-1)\nu)) T^{\frac{\alpha}{\alpha}} \left( \frac{\gamma+1}{\gamma} \right)^{-\frac{\alpha-1}{\alpha}} F(T)^{\frac{\alpha-1}{\alpha}} \]

in the case \( 0 < \alpha < q - 1 \). After that one obtains similarly as in the case \( \alpha = q - 1 \) the desired contradiction by taking \( \varepsilon \searrow 0 \).

The case \( p = 2 \) works as in Theorem 2.2.1: By integrating by parts twice one can resign Hardy’s inequality and thus requires no local sign of \( u_x \) near 0 any more. 

\[ \blacksquare \]
Chapter 3

On a system with coupled reaction terms

In the following we want to examine nonnegative weak solutions $u, v$ of the system

$$\begin{align*}
\begin{cases}
    u_t - \Delta u^m - v^\alpha &= 0, \\
v_t - \Delta v^n - u^\beta &= 0,
\end{cases}
\end{align*}$$

with initial values

$$u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0,$$

and parameters

$$m, n > 1, \quad \alpha, \beta \geq 1.$$ 

Our goal is to derive an upper bound for the (common) waiting time $t^*_\Omega$ of $u$ and $v$. In absence of comparison principles one has to work this time without using any radial symmetries. We first fix

$$\Omega := \text{supp}(u_0) \cup \text{supp}(v_0), \quad y_0 \in \mathbb{R}^N \setminus \Omega, \quad \varepsilon \in (\text{dist}(y_0, \Omega), \sqrt{\frac{N + 2}{N + 1}} \text{dist}(y_0, \Omega)), \quad \tau \in (0, t^*_\Omega).$$

Due to Lemma 1, [7], it holds for

$$h(x) := (\varepsilon^2 - \|x - y_0\|^2)^2$$

the inequality

$$(\Delta h)(x) \geq \varepsilon^{-4k+2}[h(x)]^k$$

for all $k > 1$ and $\frac{N + 1}{N + 2} \varepsilon^2 \leq \|x - y_0\|^2 < \varepsilon^2$.

We will test the system with the function $\varphi(x, t) := h(x)(\tau - t)^2$ and obtain as weak formulation by setting $\Omega_x := \Omega \cap B_\varepsilon(y_0)$

$$0 = \left( \int_{\Omega_x} u_0 h(x) \right) \tau^2 - 2 \int_{\Omega_x} \int_0^\tau u(\tau - t) h(x) + \int_0^\tau (\tau - t)^2 \int_{\Omega_x} u^m(\Delta h) + \int_0^\tau \int_{\Omega_x} v^\alpha(\tau - t)^2 h$$
as well as
\[
0 = \left( \int_{\Omega_e} v_\theta h(x) \right)^2 - 2 \int_{\Omega_e} \int_0^\tau v(\tau - t) h(x) + \int_0^\tau (\tau - t)^2 \int_{\Omega_e} v^n(\Delta h) + \int_0^\tau \int_{\Omega_e} u^\beta (\tau - t)^2 h.
\]

Define the functionals
\[
F(\tau) := \int_0^\tau \int_{\Omega_e} (\tau - t)^2 h(x) u(x,t) \quad \text{and} \quad G(\tau) := \int_0^\tau \int_{\Omega_e} (\tau - t)^2 h(x) v(x,t),
\]
then one obtains for their derivatives
\[
F'(\tau) := 2 \int_0^\tau \int_{\Omega_e} (\tau - t) h(x) u(x,t) \quad \text{and} \quad G'(\tau) := 2 \int_0^\tau \int_{\Omega_e} (\tau - t) h(x) v(x,t).
\]
Notice now \( \Omega_\varepsilon \subset \{ x \in \mathbb{R}^N \mid \frac{N+1}{N-1} \varepsilon^2 \leq \| x - y_0 \|^2 < \varepsilon^2 \} \), so we can apply the differential inequality for \( h \) from Lemma 1, [7], to \( \Omega_\varepsilon \) and obtain therefore
\[
F'(\tau) \geq \left[ \int_0^\tau \int_{\Omega_e} (\tau - t)^2 h(x)^n u^n \right]\varepsilon^{-4m+2} + \left[ \int_{\Omega_e} u_0 h(x) \right]\tau^2 + \int_0^\tau \int_{\Omega_e} v^\alpha (\tau - t)^2 h
\]
and
\[
G'(\tau) \geq \left[ \int_0^\tau \int_{\Omega_e} (\tau - t)^2 h(x)^n v^n \right]\varepsilon^{-4n+2} + \left[ \int_{\Omega_e} v_0 h(x) \right]\tau^2 + \int_0^\tau \int_{\Omega_e} u^\beta (\tau - t)^2 h.
\]
The inverse Hölder inequality implies the integral inequalities
\[
\int_0^\tau \int_{\Omega_e} (\tau - t)^2 h^m u^m \geq \left[ \int_0^\tau h u(\tau - t)^2 \right]^m \left[ \int_0^\tau (\tau - t)^2 \right]^{-(m-1)} = F(\tau)^m \left( \frac{3}{\tau^3 \mathcal{L}^n(\Omega_\varepsilon)} \right)^{m-1} ,
\]
\[
\int_0^\tau \int_{\Omega_e} (\tau - t)^2 h^n v^n \geq \left[ \int_0^\tau h v(\tau - t)^2 \right]^n \left[ \int_0^\tau (\tau - t)^2 \right]^{-(n-1)} = G(\tau)^n \left( \frac{3}{\tau^3 \mathcal{L}^n(\Omega_\varepsilon)} \right)^{n-1} ,
\]
\[
\int_0^\tau \int_{\Omega_e} (\tau - t)^2 v^\alpha h \geq \left[ \int_0^\tau h v(\tau - t)^2 \right]^{\alpha} \left[ \int_0^\tau (\tau - t)^2 h \right]^{-(\alpha-1)} = G(\tau)^{\alpha} \left( \frac{3}{\tau^3} \right)^{\alpha-1} \left[ \int_{\Omega_e} h \right]^{-(\alpha-1)} ,
\]
and analogously
\[
\int_0^\tau \int_{\Omega_e} (\tau - t)^2 u^\beta h \geq F(\tau)^{\beta} \left( \frac{3}{\tau^3} \right)^{\beta-1} \left[ \int_{\Omega_e} h \right]^{-(\beta-1)} .
\]
Finally by using
\[
\int_{\Omega_e} h(x) = \int_{\Omega_e} \left( \varepsilon^2 - \| x - y_0 \|^2 \right)^2 \leq C\varepsilon^{4+N}
\]
one arrives at the coupled differential inequalities
\[
F'(\tau) \geq C[F(\tau)^m \tau^{3(m-1)} \varepsilon^{-4m+2-(m-1)N} + \left( \int_{\Omega_e} u_0 h \right)^2 + G(\tau)^{\alpha} \tau^{-3(\alpha-1)} \varepsilon^{-(4+N)(\alpha-1)}]
\]
and
\[ G'(\tau) \geq C[G(\tau)^{\mathcal{n}} - \mathcal{3}(n-1) \varepsilon^{-4n+2-(-n)N} + (\int_{\Omega_{\epsilon}} v_0 h)\tau^2 + F(\tau)^{\mathcal{3}} - \mathcal{3}(\beta-1)\varepsilon^{-(4+N)(\beta-1)}]. \]

On the one hand one could neglect the last summand and one would obtain by Lemma 1.4.1 from
\[ F'(\tau) \geq C[F(\tau)^{\mathcal{m}} - \mathcal{3}(m-1) \varepsilon^{-4m+2-(-m)N} + (\int_{\Omega_{\epsilon}} u_0 h)\tau^2] \]
and
\[ G'(\tau) \geq C[G(\tau)^{\mathcal{n}} - \mathcal{3}(n-1) \varepsilon^{-4n+2-(-n)N} + (\int_{\Omega_{\epsilon}} v_0 h)\tau^2] \]
the estimation
\[ t^* \leq C \min\{(\int_{\Omega_{\epsilon}} u_0 h)^{-(m-1)} \varepsilon^{4m-2+(m-1)N}, (\int_{\Omega_{\epsilon}} v_0 h)^{-(n-1)} \varepsilon^{4n-2+(n-1)N}\}. \]

On the other hand one has to consider the coupling of the differential equations. In order to do so one might neglect the first and third summand in the differential inequalities and this implies
\[ F(\tau) \geq C\tau^3(\int_{\Omega_{\epsilon}} u_0 h), \quad G(\tau) \geq C\tau^3(\int_{\Omega_{\epsilon}} v_0 h). \]
Therefore is follows
\[ F'(\tau) \geq C[F(\tau)^{\mathcal{m}} - \mathcal{3}(m-1) \varepsilon^{-4m+2-(-m)N} + (\int_{\Omega_{\epsilon}} v_0 h)\alpha\tau^2\varepsilon^{-(4+N)(\alpha-1)}] \]
and
\[ G'(\tau) \geq C[G(\tau)^{\mathcal{n}} - \mathcal{3}(n-1) \varepsilon^{-4n+2-(-n)N} + (\int_{\Omega_{\epsilon}} u_0 h)\beta\tau^3\varepsilon^{-(4+N)(\beta-1)}]. \]
Lemma 1.4.1 implies
\[ t^* \leq C \min\{(\int_{\Omega_{\epsilon}} v_0 h)^{-\frac{\alpha(m-1)}{m}} \varepsilon^{-\frac{1}{m}(-4+N)(\alpha-1)(m-1)-4m+2-(-m)N}], \]
\[ (\int_{\Omega_{\epsilon}} u_0 h)^{-\frac{\beta(n-1)}{n}} \varepsilon^{-\frac{1}{n}(-4+N)(\beta-1)(n-1)-4n+2-(-n)N}]. \]
Finally we need a suitable initial growth of \( u_0, v_0 \) in the neighbourhood of a point \( x_0 \in \partial \Omega \) and topological assumptions for \( \Omega \). Let for this reason be given two cones \( \mathcal{C}, \mathcal{C}' \) with vertex \( x_0 \) and opening angle \( \theta > 0, \theta' > \arcsin \sqrt{\frac{N+1}{N+2}}, \) such that \( \Omega \cap \mathcal{C}' \cap B_R(x_0) = \emptyset \) for a suitable \( R > 0 \) as well as
\[ u_0(x) \geq A \|x - x_0\|^\nu + o(\|x - x_0\|^\nu), \quad v_0(x) \geq B \|x - x_0\|^\mu + o(\|x - x_0\|^\mu) \]
wherein the \( o \)-notation is postulated only for taking to the limit \( x \to x_0, x \in \mathcal{C} \).
Now, choose a sequence \((y_k)_{k \in \mathbb{N}} \in \mathbb{R}^N \setminus \Omega\) with \(y_k \to x_0\), \(\delta_k := \|y_k - x_0\|\), \(\varepsilon_k := \sqrt{\frac{N+1}{N+2}} \sin(\theta') \delta_k\), dist\((y_k, \Omega) \geq \sin(\theta') \delta_k\), and
\[
u_0(x) \geq (A - \frac{1}{k})\|x - x_0\|^\nu, \quad v_0(x) \geq (B - \frac{1}{k})\|x - x_0\|^\mu
\]
for all \(x \in \mathcal{C} \cap B_{\frac{1}{k}}(\sqrt{\frac{N+1}{N+2}} \sin(\theta') - \frac{1}{k})((x_0)\). We want to apply the estimation for the waiting time given above for \(y_k\) and \(\varepsilon_k\) instead of \(y_0\), \(\varepsilon\) and take to the limit \(k \to \infty\). Therefore one needs the following chain of inequalities:
\[
\int_{\mathcal{C} \cap B_{\frac{1}{k}}(y_k)} u_0(x)(\varepsilon_k^2 - \|x - y_k\|^2)^2 \geq \int_{\mathcal{C} \cap B_{\frac{1}{k}}(\sqrt{\frac{N+1}{N+2}} \sin(\theta') - \frac{1}{k})((x_0)) u_0(x)(\varepsilon_k^2 - \|x - y_k\|^2)^2 \\
\geq C\delta_k^4 \int_{\mathcal{C} \cap B_{\frac{1}{k}}(\sqrt{\frac{N+1}{N+2}} \sin(\theta') - \frac{1}{k})((x_0)) \|x - x_0\|^\nu \\
= C(A - \frac{1}{k})\delta_k^4 \int_0^{1} \mu + N - 1 dr \\
= C(A - \frac{1}{k})\delta_k^4 \xi^{\nu + N - 1}_k.
\]
Analogously it certainly holds \(\int_{\mathcal{C} \cap B_{\frac{1}{k}}(y_k)} v_0(x)(\varepsilon_k^2 - \|x - y_k\|^2)^2 \geq C(B - \frac{1}{k})\delta_k^{4 + \mu + N}\). This leads to the following estimation for the common waiting time
\[
t_0^* \leq C \liminf_{k \to \infty} \min\{(A - \frac{1}{k})^{1-m}\xi_k^{2-\nu(m-1)}, (B - \frac{1}{k})^{1-n}\xi_k^{2-\mu(n-1)}\}
\]
as well as
\[
t_0^* \leq C \liminf_{k \to \infty} \min\{(B - \frac{1}{k})^{-\frac{\alpha(m-1)}{m}}\xi_k^{-\frac{1}{m}[\alpha m(m-1) - 2]}, (A - \frac{1}{k})^{-\frac{\beta(n-1)}{n}}\xi_k^{-\frac{1}{n} [\beta n(n-1) - 2]}\}.
\]
In order to take to the limit \(k \to \infty\) one should notice at last, when the exponent at \(\varepsilon_k\) is positive resp. when it vanishes.
Thus we have proven the following Theorem:

**Theorem 3.1**

Let \(\Omega := \text{supp}(u_0) \cup \text{supp}(v_0)\), \(x_0 \in \partial \Omega\), and \(u, v\) be nonnegative weak solutions of the system
\[
\begin{cases}
  u_t - \Delta u^m - v^\alpha = 0, \\
v_t - \Delta v^n - u^\beta = 0,
\end{cases}
\]
for \(m, n > 1\), \(\alpha, \beta \geq 1\), and let there exist \(A, B, \mu, \nu, R > 0\), and cones \(\mathcal{C}, \mathcal{C}'\) with vertex \(x_0\) and opening angles \(\theta > 0\), \(\theta' > \arcsin(\sqrt{\frac{N+1}{N+2}})\), such that \(\Omega \cap \mathcal{C}' \cap B_R(x_0) = \emptyset\) as well as
\[
u_0(x) \geq A\|x - x_0\|^\nu + o(\|x - x_0\|^\nu), \quad v_0(x) \geq B\|x - x_0\|^\mu + o(\|x - x_0\|^\mu)
\]
wherein the \(o\)-notation is postulated only for taking to the limit \(x \to x_0\), \(x \in \mathcal{C}\). In the case of dimension one let \(\mathcal{C}, \mathcal{C}'\) be intervals and the assumption for the opening angles shall be ignored. Then it holds:
(a) \( t^*_\Omega = 0 \), if \( \nu < \max\{\frac{2}{m-1}, \frac{2}{\beta(n-1)}\} \) or \( \mu < \max\{\frac{2}{n-1}, \frac{2}{\alpha(m-1)}\} \).

(b) \( t^*_\Omega \leq CA^{-(m-1)} \), if \( \nu = \frac{2}{m-1} \), and \( t^*_\Omega \leq CB^{-(n-1)} \), if \( \mu = \frac{2}{n-1} \).

(c) \( t^*_\Omega \leq CA^{-\frac{\beta(n-1)}{n}} \), if \( \nu = \frac{2}{\beta(n-1)} \), and \( t^*_\Omega \leq CB^{-\frac{\alpha(m-1)}{m}} \), if \( \mu = \frac{2}{\alpha(m-1)} \).

\( C = C(\alpha, \beta, m, n, N, \theta, \theta') > 0 \) denotes always a constant depending only on the parameters.
Bibliography


Lebenslauf


