$W$-pair production near threshold in unstable-particle effective theory

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

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Abstract

In this thesis we present a dedicated study of the four-fermion production process $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u \bar{d} X$ near the $W$-pair production threshold, in view of its importance for a precise determination of the $W$-boson mass at the ILC. The calculation is performed in the framework of unstable-particle effective theory, which allows for a gauge-invariant inclusion of instability effects, and for a systematic approximation of the full cross section with an expansion in the coupling constants, the ratio $\Gamma_W/M_W$, and the non-relativistic velocity $v$ of the $W$ boson. The effective-theory result, computed to next-to-leading order in the expansion parameters $\Gamma_W/M_W \sim \alpha_{ew} \sim v^2$, is compared to the full numerical next-to-leading order calculation of the four-fermion production cross section, and agreement to better than 0.5\% is found in the region of validity of the effective theory. Furthermore, we estimate the contributions of missing higher-order corrections to the four-fermion process, and how they translate into an error on the $W$-boson mass determination. We find that the dominant theoretical uncertainty on $M_W$ is currently due to an incomplete treatment of initial-state radiation, while the remaining combined uncertainty of the two NLO calculations translates into $\delta M_W \approx 5$ MeV. The latter error is removed by an explicit computation of the dominant missing terms, which originate from the expansion in $v$ of next-to-next-to-leading order Standard Model diagrams. The effect of resummation of logarithmically-enhanced terms is also investigated, but found to be negligible.
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Chapter 1
Introduction

In this chapter we will discuss the phenomenological relevance of the process of $W$-pair production near threshold for a precise determination of the $W$-boson mass $M_W$. We will also review the theoretical issues related to the calculation of this process, and give a short overview of the methods and results available at present. Finally we will present the outline of this thesis.

1.1 Measurement of $M_W$

The mass of the $W$ gauge boson has a central role in tests of the Standard Model (SM) and searches for virtual-particle effects through electroweak precision measurements, and is of pivotal importance for understanding the physics underlying the electroweak symmetry breaking. In the Standard Model $M_W$ is related to the top quark mass $m_t$ and the Higgs boson mass $M_H$ through loop corrections [1], as schematically shown in Figure 1.1, and the direct measurements of the first two masses give informations on the third one. This complements bounds on $M_H$ from direct searches, and allows to discriminate between different models. This is shown in Figure 1.2 [2], where the regions of the $(M_W, m_t)$ plane compatible with direct searches of the Higgs boson for the Standard Model and the Minimal Supersymmetric Standard Model (MSSM) are compared to the measured values of $M_W$ and $m_t$. With the imminent startup of the Large Hadron Collider (LHC) at Cern, we will soon be able to directly access the mass range where the Standard Model (and MSSM) Higgs boson appears more likely to be. If the Higgs boson is found at the LHC, its mass will be measured at the proposed International Linear Collider (ILC) with a precision comparable to the top quark mass. Hence the combined measurements of $M_W$, $m_t$ and $M_H$ will enable us to test the SM at the loop level, and to indirectly investigate the nature of possible new physics. If no Higgs boson or other signals of new physics are observed at the LHC or ILC, electroweak precision measurements will probably be the only way to obtain informations on the mechanism behind electroweak symmetry breaking. In both cases, a precise knowledge of the $W$ mass plays a key role.

The current value of the $W$ mass, $\hat{M}_W = (80.403 \pm 0.029)$ GeV [3], is determined from
a combination of measurements at LEPII [4–8] and at the Tevatron [9–11]. At LEPII $W$ bosons were pair-produced via $e^-e^+$ annihilation, with subsequent decays to four-fermion final states. $M_W$ has been extracted from fits of the four-fermion cross section, measured at ten different values of the centre-of-mass energy (from the $WW$ production threshold at $\sqrt{s} = 161.3$ GeV up to $\sqrt{s} = 206.6$ GeV), and for different leptonic, semi-leptonic and hadronic final states. At the Tevatron $W$ bosons are mainly single-produced in $q\bar{q}$ annihilation, and the $W$ mass is extracted from transverse-mass distributions of the decay products. Further measurements of single-$W$ production at the LHC are expected to reduce the error by a factor of two.

It has been estimated that at ILC an error of 6 MeV could be achieved with a total integrated luminosity of $L = 100\text{fb}^{-1}$ (corresponding to one year of running), by measuring the four-fermion cross section at different centre-of-mass energies in the vicinity of the $W$-pair production threshold [12]. Unlike LEPII, where only one energy below 170 GeV was considered, at ILC the experimental points would predominantly be in the narrow region $\sqrt{s} \sim 160 - 165$ GeV. Such a threshold scan would exploit the high sensitivity to the $W$ mass of the $WW$ line-shape in the threshold region to convert accurate measurements of the cross section into a precise prediction for $M_W$. The estimate of [12] is based on the expected statistics and the performance of a future linear collider [13], and it assumes that the theoretical cross section is known to an accuracy of $\sim 1\%$. In reality, achieving this accuracy is a difficult theoretical task, requiring the calculation of loop and radiative corrections. Also, since the $W$ bosons decay rapidly, this calculation should be done for a final state of sufficiently long-lived particles, rather than for on-shell $W$-pair production. A systematic treatment of finite-width effects is therefore needed.

In the following section we will give a brief review of the theoretical status of $W$-pair production, with particular emphasis on the threshold region that will be explored at ILC.

---

1 This value refers to the definition of the $W$ mass from a Breit-Wigner parameterisation with a running width as it is adopted in the experimental analyses. It is related to the pole mass $M_W$ defined later in this thesis by $\Delta M_W = M_W - M_W = \Gamma_W^2/(2M_W) + O(\alpha^3_{ew})$. 

2
Figure 1.2: $M_W$ as a function of $m_t$ in the SM (red region) and the MSSM (green region) for different values of $M_H$. The MSSM result is obtained by scanning over the SUSY parameter space [2]. The ellipses represent the present central values and errors of $m_t$ and $M_W$ (blue ellipse), and the one expected at LHC (black ellipse) and ILC (red ellipse).

1.2 Theoretical status of $W$-pair production

The tree-level cross section for the on-shell $W$-pair production process

$$e^- e^+ \rightarrow W^- W^+$$

has been known since thirty years [14, 15], and next-to-leading order (NLO) corrections to the production process [16–21], and to the decay of an on-shell $W$ [22–24] have been computed. However, as pointed out in the previous section, the small experimental error ($\sim \%$) on the cross sections foreseen at ILC does not allow us to treat the $W$ boson as a stable particle. Therefore, instead of (1.1), one should consider a physical process with sufficiently long-lived final states. For $W$-pair production this process is represented by four-fermion production in $e^- e^+$ collision. For definiteness here and in the rest of this thesis we will focus on the semileptonic process

$$e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u \bar{d} + X.$$  (1.2)

The tree-level Feynman diagrams contributing to (1.2) are shown in Figure 1.3. They
Figure 1.3: Tree-level Feynman diagrams for the four-fermion production process $e^-e^+ \to \mu^-\bar{\nu}_\mu u\bar{d}$.

Figure 1.3 consists of diagrams effectively containing a pair of $W$ bosons (so called double-resonant or “CC03” diagrams, (1), (2) and (3) in Figure 1.3), and of topologies with only one internal $W$ line (single-resonant or background diagrams). For some flavour-specific final states different from the one considered here also diagrams with no internal $W$ lines contribute to the Born cross section. The matrix element of (1.2) computed in standard fixed-order perturbation theory contains singularities coming from phase-space regions where the intermediate $W$ lines approach the mass shell. Thus, some regularisation of the propagator must be supplied in order to compute the cross section of the four-fermion process (1.2). Note that this problem is not specific to the process considered here, but is common to all processes involving intermediate unstable particles. The well-known remedy consists in the Dyson resummation of self-energy insertions along the unstable-particle line, that
translates in the substitution (for the simplest case of a scalar particle)

\[
\frac{1}{p^2 - M^2} - \frac{1}{p^2 - M^2} \sum_{n=0}^{\infty} \left( \frac{\Pi(p^2)}{p^2 - M^2} \right)^n = \frac{1}{p^2 - M^2 - \Pi(p^2)}.
\]

(1.3)

The self-energy \( \Pi(p^2) \) has an imaginary part of order \( M \Gamma \), where \( M \) is the mass of the unstable-particle and \( \Gamma \sim g^2 M \) its on-shell decay width, that regularises the singularities of the matrix element. We note that for narrow resonances, \( \Gamma \ll M \), the resummed propagator can be approximated by

\[
\left| \frac{1}{p^2 - M^2 + iM \Gamma} \right|^2 \sim \frac{2\pi}{\Gamma} \delta(p^2 - M^2).
\]

(1.4)

This corresponds to the so-called narrow-width approximation, in which the physical cross section reduces to the product of the squared matrix element for on-shell production of the unstable particle and the branching ratio for the specific decay products under consideration. The left and right-hand side of (1.4) differ by terms of order \( \Gamma/M \). For the \( W \) boson, where \( \Gamma_W/M_W \sim 2.5\% \), the error of the approximation is thus much larger than the target accuracy for ILC.

The substitution (1.3) sums a subset of terms of order \( (g^2 M^2/(p^2 - M^2))^n \sim 1 \) (near the resonance) to all orders in the expansion parameter \( g^2 \). This procedure naturally raises the question of how to identify all the terms, and only those, contributing to the scattering amplitude at a given order in \( g^2 \) and \( \Gamma/M \). The failure in addressing such a question may lead to a lack of gauge invariance and unitarity of the resummed amplitude [25, 26], since these properties are guaranteed only order-by-order in standard perturbation theory and for the full amplitude. Clearly this issue cannot be ignored in a calculation that aims to a total theoretical accuracy at the per-mille level. Besides the problems related to violation of gauge invariance and unitarity due to resummation of finite-width effects, an additional complication in obtaining accurate theoretical predictions for (1.2) comes from the necessity of a complete calculation of electroweak and QCD radiative corrections to the full \( 2 \to 4 \) process.

Many of the current approaches to unstable particles take the restoration of gauge invariance as a starting point. The fermion-loop scheme [27–29] is based on the observation that the dominant contribution to the width of the \( W \) and \( Z \) gauge bosons comes from fermion loops. Therefore, the gauge invariance of the resummed amplitude can be recovered by including fermion-loop corrections to propagators and vertices, since all fixed-order terms proportional to \( g^2 N_f \), where \( N_f \) is the number of fermion flavours, are then included. The disadvantage of this scheme is given by its limitation to gauge bosons, and by the necessity to compute one-loop vertices even for a leading-order approximation. An other example is given by the scheme presented in [30, 31], where a gauge-invariant non-local effective action is matched onto the two-point functions of the underlying theory. The gauge Ward identities are satisfied by construction in this scheme. The two schemes, both implemented in the calculation of four-fermion production, solve not only the gauge-invariance problem but also capture some sophisticated features such as the running of
the couplings. However, it is unclear how to extend them to a systematic approximation of the scattering amplitude in powers of $g^2$ and $\Gamma/M$.

The pole approximation [32,33] can be considered as the first step towards a systematic approximation scheme to the scattering amplitude based on the separation of scales. Roughly speaking, it consists in a kinematical expansion of the full matrix element around the physical complex pole of the unstable particle propagator. Consider the simplified case of a single intermediate resonance. The full matrix element can be generically written as

$$\mathcal{M}(k^2) = \frac{A(k^2)}{k^2 - \bar{s}} + B(k^2),$$

(1.5)

where the term $A(k^2)/(k^2 - \bar{s})$ accounts for the contributions of diagrams with an internal unstable-particle line, while $B$ encodes the remaining background contributions. For the case of pair-production of unstable particle equation (1.5) would contain terms with two, one and, possibly, no unstable-particle propagators. The dominant contribution to the scattering amplitude is expected to originate from resonant kinematical configurations with $k^2 - \bar{s} \sim M\Gamma$. Thus $\mathcal{M}(k^2)$ is systematically expanded around the pole $\bar{s} \equiv M^2 - iM\Gamma$:

$$\mathcal{M}(k^2) = \frac{A(\bar{s})}{k^2 - \bar{s}} + \frac{\partial A}{\partial k^2}(\bar{s}) + B(\bar{s}) + O(k^2 - \bar{s}).$$

(1.6)

The first term clearly accounts for the leading resonant contributions. The second can be interpreted as the contribution of diagrams containing an internal unstable particle, but with invariant mass far from the mass shell. The third term encodes the correction from true background diagrams. Equation (1.6) results in two simplifications. While the expansion of the functions $A$ and $B$ around $\bar{s}$ makes the calculation simpler, the relation $(k^2 - \bar{s}) \sim M\Gamma$ enables to identify at once those terms in the expansion that contribute at a given order in $\alpha$ and $\Gamma/M$. The gauge-invariance of the full matrix element and of the location of the physical pole $\bar{s}$ ensures that the expansion is also gauge invariant order-by-order in $k^2 - \bar{s}$.

The pair-production formulation of the pole approximation, the so-called double-pole approximation, has been applied to the computation of the four-fermion cross section [34–36]. The calculation correctly includes all the contributions to the cross section up to next-to-leading order in an expansion in $\delta$, where $\delta$ can be either the ratio $\Gamma_W/M_W \sim 2.5\%$ or the electromagnetic coupling $\alpha$. These contributions include the leading term in the pole-expansion of the tree-level double-resonant diagrams in Figure 1.3, and a subleading correction, suppressed by $\Gamma_W/M_W$ (corresponding to the second term in (1.6)). The single-resonant diagrams contain at most one resonant $W$ propagator, and are suppressed by $\Gamma_W/M_W$ with respect to the double-resonant diagrams. Thus only their leading-order expansion around the complex pole $\bar{s}$ (the third term in (1.6)) has to be included in the calculation. Radiative corrections to the tree diagrams in Figure 1.3 introduce an extra power of $\alpha$. Therefore, only $O(\alpha)$ corrections to the leading double-resonant contributions have to be computed, since other terms are suppressed, compared to the leading Born result, by at least $\alpha\Gamma_W/M_W < 0.1\%$. The radiative corrections to the double-resonant diagrams can be divided into two classes, given by factorisable and non-factorisable contributions. Loosely speaking, factorisable corrections are represented by those corrections...
that affect only one of the three hard subprocesses in (1.2), namely the production of the two $W$ bosons, $e^-e^+ \to W^-W^+$, their propagation and their decays into the final states. This is exemplified by the diagram shown in Figure 1.4, where factorisable corrections are indicated by grey blobs located at the production and decay vertices, and on the two $W$ lines. Non-factorisable corrections correspond to radiative corrections to the four-fermion process that connect different hard subprocesses, like, for example, production and decay (correction (a) in Figure 1.4) or two different decays (correction (b) in Figure 1.4). At next-to-leading order in $\alpha$ and leading order in $\Gamma_W/M_W$ non-factorisable corrections are represented by photonic corrections computed in the soft approximation, with the photon energy scaling as $E_\gamma \sim \Gamma_W$. This is justified by the observation that exchange of modes with $E \gg \Gamma_W$ between different hard subprocesses set the $W$ propagators off-shell, leading to a further suppression by powers of $\Gamma_W/M_W$. Note that the soft approximation results in a substantial simplification of the calculation.

Radiative corrections to the four-fermion production process in the double-pole approximation were implemented in the Monte-Carlo generators RacoonWW [37, 38] and YFSWW3 [39,40]. The error of the calculation in the double-pole approximation amounts to few per-milles in the continuum, i.e. for centre-of-mass energies much larger than the $W$-pair production threshold. This accuracy was enough for the analysis of the four-fermion cross section at LEPII. However the method is not reliable near threshold, where terms of the form $M_W/(\sqrt{s}-2M_W)$ cause a poor convergence of the pole expansion. Thus, until few years ago, there existed only LO calculations in the threshold region, as well as studies of the effect of Coulomb photon exchanges [41,42].
Figure 1.5: Examples of one-loop six-point diagrams contributing to the four-fermion process (1.2).

Recently a full NLO calculation of the four-fermion production cross section has been performed in the complex-mass scheme [43,44]. In the complex-mass scheme (CMS) [45,46] the resummation of the finite-width effects is obtained by splitting the bare parameters of the Lagrangian into complex renormalised quantities and counterterms. For example, for the $W$ mass one has

$$M_W^{(0)} = \mu_W^2 + \delta_W^2,$$

where $\mu_W^2$ contains an imaginary part of order $\alpha$. $\mu_W^2$ is included in the $W$ propagator, and regularises the singularities, while $\delta_W^2$ is included in the calculation as a perturbation. Since (1.7) corresponds to a reorganisation of the gauge-invariant bare Lagrangian, in this approach Ward identities are exactly preserved order-by-order in perturbation theory if the complex masses are used consistently everywhere in the calculation. This in particular implies that the couplings of the Lagrangian have to be complex, since $\cos \theta_W = \mu_W/\mu_Z$, and loop contributions to the renormalisation of the electromagnetic coupling $e$ contain complex masses. While this could in principle lead to a violation of unitarity, in a $O(\alpha)$ calculation the unitarity-violating terms are of higher-order, and are never parametrically enhanced [46].

The NLO computation of the four-fermion process (1.2) in the complex mass scheme does not imply any kinematic expansion, and is thus valid for arbitrary centre-of-mass energies, including the threshold region. However, this a very difficult calculation, requiring complex algorithms to reduce a large number ($O(10^3)$) of non-trivial spinor chains and new numerical techniques to evaluate one-loop six-point tensor integrals with complex masses (see Figure 1.5). In particular, in such a calculation, the evaluation of higher-order corrections beyond NLO is practically not feasible. Given the high theoretical accuracy required at ILC, a simpler approach is desirable in which higher-order contributions can be systematically included, and the effect of these corrections on the determination of $M_W$ easily estimated.

Such an approach is based on an effective-theory description of the four-fermion production process near threshold [47], that allows a systematic (gauge-invariant) computation of the full cross section in powers of the coupling $\alpha$ and the ratios $\Gamma_W/M_W$ and $(\sqrt{s} - 2M_W)/M_W$. Being tailored to the threshold region, this method involves a number
of kinematical approximations that make the calculation much simpler than the full computa-
tion in the complex-mass scheme, leading to fully analytic results, and allowing the in-
clusion of corrections beyond next-to-leading order in $\alpha$. In the rest of this thesis we will
discuss this formalism, and apply it to the explicit computation of the cross section of
the four-fermion production process, equation (1.2).

1.3 Outline

The organisation of the thesis, which is an extension of results presented in recent publi-
cations [47, 48], is as follows.

In Chapter 2 we review the effective-theory formalism and methods on which com-
putations of later chapters are based. To this end, we consider a toy model containing a
single scalar resonance, and later discuss how to extend the formalism to the description of
$W$-pair production near threshold. We also list all the relevant corrections necessary for a
NLO ($\propto \Gamma_W/M_W \sim (\sqrt{s} - 2M_W)/M_W$) computation of the four-fermion production
cross section.

In Chapter 3 we test the formalism by comparing an effective-theory calculation of the
total Born cross section for the process (1.2) with a numerical result from the Monte Carlo
generator WHIZARD. We also discuss to which extent kinematical cuts can be included
in the effective-theory calculation.

In Chapter 4 we compute all the radiative corrections to the four-fermion cross section
needed for a NLO calculation in $\propto \Gamma_W/M_W \sim (\sqrt{s} - 2M_W)/M_W$. These are combined
in Chapter 5, where we present our numerical results for the NLO cross section and
compare the effective-theory approximation with the full NLO calculation in the complex-
mass scheme. In the same Chapter we identify the sources of the dominant remaining
theoretical uncertainties, and estimate their effects on the experimental measurement of
the $W$ mass.

Some of the higher-order corrections responsible for the uncertainties discussed in
Chapter 5, given by a subset of NNLO Standard Model corrections parametrically en-
hanced by inverse powers of $(\sqrt{s} - 2M_W)/M_W$, are explicitly computed in Chapter 6.
Their effect on the four-fermion cross section and the determination of the $W$ mass is dis-
cussed. In Chapter 7 we investigate the effect of another set of higher-order corrections,
related to the resummation of logarithms of $\Gamma_W/M_W$.

Finally, in Chapter 8 we summarise and draw our conclusions. Details about specific
parts of the calculation are presented in Appendices A, B, C, D, E and F.
Chapter 2

Effective Field Theory description of unstable-particle production

This chapter introduces the effective-field theory (EFT) approach to unstable-particle production employed in the rest of the thesis for the calculation of the cross section of the process $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu \bar{u}d$ near the $W$-pair production threshold. In Section 2.1 we review the essential features of the approach, using the simplified example of a single scalar resonance. The generalisation of the method to pair-production of electroweak gauge bosons is presented in Section 2.2, where we also compute the leading-order EFT contribution to the four-fermion total cross section. Finally in Section 2.3 we list all the terms relevant for the calculation of the NLO cross section, which will be presented in Chapters 3 and 4.

2.1 Unstable-particle effective theory

Processes involving an unstable particle close to resonance are characterised by two different, well-separated scales, $\Gamma \ll M$, where $\Gamma$ and $M$ are respectively the width and mass of the unstable particle. In [49] it was suggested that the hierarchy $\Gamma \ll M$ could be exploited to construct an effective field theory from which hard modes with virtuality of order $M^2$ or larger are removed. The effect of these modes is included in the coefficients of the effective Lagrangian and corresponds to factorisable corrections. Non-factorisable corrections arise from loop diagrams containing the dynamical degrees of freedom of the effective theory. The idea was pursued and developed in [50, 51], and applied to a toy model containing a single scalar resonance. In this section we review the model and introduce the basic features of the effective-theory approach. For a more detailed description of the model and of unstable-particle effective theory we refer the reader to [51].

The toy model describes a massive unstable scalar $\phi$ and two fermion fields $\psi, \chi$. The scalar and one of the fermion fields (which in the following we will call the “electron” for sake of simplicity) are charged under an Abelian gauge symmetry, while the second fermion field (the “neutrino”) is neutral. The model allows for the scalar to be produced by and to decay into an electron-neutrino pair through a Yukawa interaction. The Lagrangian of
the model reads

\[
\mathcal{L} = (D_\mu \phi)\dagger D^\mu \phi - \hat{M}^2 \phi \dagger \phi + \bar{\psi} i D \psi + \bar{\chi} i D \chi - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} - \frac{1}{2 \xi} (\partial_\mu A^\mu)^2 \\
+ y \phi \bar{\psi} \chi + y^* \phi \dagger \bar{\chi} \psi - \frac{\lambda}{4} (\phi \dagger \phi)^2 + \mathcal{L}_{\text{ct}},
\]

(2.1)

where \(\hat{M}\) and \(\mathcal{L}_{\text{ct}}\) denote the renormalised mass and the counterterm Lagrangian, and the covariant derivative \(D^\mu\) is defined by \(D_\mu = \partial_\mu - ig A_\mu\). Here the notation \(\hat{M}\) is used to distinguish the renormalised mass (in a not-yet specified renormalisation scheme based on dimensional regularisation) from the pole mass \(M\) defined below. The gauge coupling \(g\) and the Yukawa coupling \(y\) are supposed to be of the same order, and for counting purposes the two fine structure constants \(\alpha_g \equiv g^2/(4\pi)\) and \(\alpha_y \equiv |y|^2/(4\pi)\) are collectively referred to as \(\alpha\). The quartic coupling \(\lambda\), which must be included for the renormalisability of the theory, is for simplicity assumed to be of order \(\alpha^2\), since the leading counterterm is of this order. The tree-level decay width of the heavy scalar can be computed from the Lagrangian (2.1), and is given by \(\Gamma = \alpha y / 2 \hat{M} \sim \alpha M \ll M\).

We want to compute the total cross section for electron-neutrino scattering,

\[
e^- (p) + \bar{\nu} (q) \rightarrow X,
\]

(2.2)

near the production threshold of the scalar \(\phi\). The vicinity to the threshold is parametrised by the quantity \(s - M^2\), which is assumed to scale as \(s - M^2 \sim M \Gamma \sim \alpha M^2 \ll M^2\). The total cross section for (2.2) is extracted from the imaginary part of the forward-scattering amplitude

\[
\delta \equiv \frac{s - \hat{M}^2}{M^2} \sim \frac{\Gamma}{\hat{M}} \sim \alpha.
\]

(2.4)

As pointed out in Chapter 1, this cannot be done in standard weak-coupling perturbation theory, since at every order in \(\alpha\) there are kinematic enhancements due to resonant scalar propagators, which are proportional to \(\hat{M}^2/(s - M^2) \sim 1/\alpha\). These enhancements are the origin of the well-known need for resummation, or, in other words, of a reorganised loop expansion and kinematic expansion of the amplitude,

\[
\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \ldots,
\]

(2.5)

where the leading contribution \(\mathcal{A}^{(0)}\) includes all the terms that are dominant near threshold \(((\alpha/\delta)^n \sim 1)\), while the subleading contributions, \(\mathcal{A}^{(1)} + \ldots\), account for terms that are suppressed by extra powers of \(\alpha\) or \(\delta\). Note that in a theory that formulates the combined
expansion in $\alpha$ and $\delta$ correctly, issues like resummation of self-energy insertions and gauge invariance are taken care of automatically, since all the terms relevant at a given order in $\alpha \sim \delta$, and only those, are included in the calculation.

The process (2.3) is primarily (but not exclusively) mediated by production of an intermediate resonant scalar, and its subsequent decay into the final state $e^- \bar{\nu}$. The time scale characterising the production of the heavy scalar is of order $\sim 1/M$. After being produced the particle propagates for a much longer time of order $1/\Gamma \gg 1/M$ and then decays, again within a short time of order $1/M$. One therefore expects some kind of factorisation between production, propagation and decay of the resonance. This factorisation must persist quantum-mechanically, since only long-wavelength fluctuations with $k \sim \Gamma$ can resolve simultaneously the details of production and decay, which are separated by the long time interval $1/\Gamma$. This situation is represented schematically by the left-hand graph of Figure 2.1, where only long-distance modes, denoted by (S), connect different hard subprocesses separated by the time interval $1/\Gamma$ (the precise nature of these modes will be clarified in the following). The right-hand side of Figure 2.1 represents non-resonant contributions mediated by momentum configurations with virtuality of order $M^2$. These interactions cannot be resolved by long-distance modes with time scale of order $1/\Gamma$, and are effectively described by contact four-fermion vertices.

The physical picture given above reflects into the construction of the effective theory, where “hard” effects related to quantum fluctuations with momenta $k \sim M$ are integrated out. These modes do not explicitly enter the effective Lagrangian $\mathcal{L}_{\text{EFT}}$, which describes only long-distance degrees of freedom with virtuality of order $M\Gamma$ or smaller. The contribution of hard modes is instead incorporated in the coefficients of $\mathcal{L}_{\text{EFT}}$, which are determined by matching the effective theory onto the full one. This is explained in Subsection 2.1.3. At the level of single Feynman diagrams, the separation of modes corresponds to performing an expansion by regions [52] (see also Appendix A). Each Feynman diagram is split into regions according to the scaling of the loop momentum, and the integrand is expanded in the parameters and momenta that are small in a given region. The hard loop-momentum region ($k \sim M$) corresponds to contributions included in the coefficient.
functions of the effective operators. Since the typical virtuality is of order $M^2$ or larger, the hard loops can be computed in standard perturbation theory, without resummation of self-energy insertions, because they do not contain terms enhanced by inverse powers of $\delta$. The remaining regions ($k \ll M$) correspond to momentum configurations near the mass-shell, and are reproduced by the diagrams in the effective theory. This is schematically described in Figure 2.2. More details about the relation between the effective-theory formulation and the expansion by regions are given in Chapter 3.

The effective Lagrangian is, roughly speaking, split into three pieces. The first, $L_{HSET}$, describes the unstable scalar field and its interaction with the gauge field. The second part, $L_{SCET}$, accounts for gauge interactions of energetic fermions. Finally, the third part, $L_{int}$, describes the external fermions and how they interact to produce the final state. These three contributions will be considered in turn in the following subsections.

### 2.1.1 Effective Lagrangian for soft and collinear interactions

The construction of $L_{HSET}$ follows closely the construction of the effective Lagrangian of heavy-quark effective theory (HQET) [53–55]. Near the resonance the momentum of the scalar field is parametrised as $P = \hat{M}v + k$, where the velocity vector $v$ satisfies $v^2 = 1$ and all the components of the residual momentum $k$ scale as $\hat{M}\delta \sim \Gamma$. This scaling is determined by the condition $P^2 - \hat{M}^2 \sim M\Gamma$ which defines the resonant region. In the following such field will be called a “soft field” (in [49] the term “resonant” was used). The virtuality of the field $\phi$ remains of order $M\Gamma$ if the scalar interacts with soft gauge bosons with momentum $M\delta$. Also, soft interactions do not modify the velocity vector $v$, since $k \ll \hat{M}v$. Thus, in analogy with HQET, the rapid spatial variation $e^{-i\hat{M}v \cdot x}$ is removed from $\phi$ and the new field

$$\phi_v(x) \equiv e^{i\hat{M}v \cdot x} \mathcal{P}_+ \phi(x),$$

which describes soft fluctuations of the heavy scalar around the on-shell configuration $\hat{M}v$, is introduced. $\mathcal{P}_+$ projects onto the positive-frequency part of the fields $\phi(x)$, and ensures that $\phi_v(x)$ is a pure destruction field. Thus the field $\phi_v(x)$ describes the scalar particle $\phi$, but not the corresponding antiparticle. Note also that, even though the initial field

Figure 2.2: Correspondence between the effective-theory calculation and the terms contributing to an expansion by regions of the full amplitude. The hard loop-momentum region (first diagram from the left) reproduces the effect of the short-distance matching coefficients of the effective Lagrangian (second diagram), while small-momentum regions (third diagram) correspond to loop diagrams in the effective theory (fourth diagram).
\( \phi(x) \) might have been real, the field \( \phi_v(x) \) is in general complex. As already pointed out, \( \phi_v(x) \) carries only the residual long-distance component of the momentum \( P \), and thus \( \partial \phi_v(x) \sim [M\delta] \phi_v(x) \). The bilinear terms for the field \( \phi_v(x) \) are constructed in such a way to reproduce the full two-point function close to resonance order-by-order in \( \alpha \) and \( \delta \).

Denoting the gauge-invariant complex pole of the propagator as \( \bar{s} \equiv M^2 - i M \Gamma \), where \( M \) and \( \Gamma \) are the pole mass and width, and the residue at the pole by \( R \phi \), the full propagator can be written as

\[
i R_{\phi} \frac{P^2 - \bar{s}}{2Mv \cdot k + k^2 - (\bar{s} - M^2)},
\]

where the right-hand side of (2.7) is obtained by replacing \( P \) with its threshold parametrisation, \( P = \hat{M}v + k \). By introducing the notation

\[
\Delta = \bar{s} - \hat{M}^2,
\]

and \( a^{\mu}_\perp = a^{\mu} - v \cdot a^{\mu} \) for any vector \( a^{\mu} \), equation (2.7) can be recast in the form

\[
i R_{\phi} \frac{P^2 - \bar{s}}{(v \cdot k)^2 + 2\hat{M}v \cdot k + k^2 - \hat{M} \Delta} = i R_{\phi} \frac{P^2 - \bar{s}}{((v \cdot k) - s_1)((v \cdot k) - s_2)},
\]

where \( s_{1,2} \) are the two solutions of the quadratic equation \((v \cdot k)^2 + 2\hat{M}v \cdot k + k^2 - \hat{M} \Delta = 0\).

An explicit calculation gives

\[
s_1 = -\hat{M} + \sqrt{\hat{M}^2 + \hat{M} \Delta - k^2_\perp} = -2\hat{M} - \Delta \frac{\Delta^2 + 4k^2_\perp}{8M} + O(\delta^3),
\]

\[
s_2 = -\hat{M} + \sqrt{\hat{M}^2 + \hat{M} \Delta - k^2_\perp} = \Delta \frac{\Delta^2 + 4k^2_\perp}{8M} + O(\delta^3),
\]

where \( \Delta \sim k_\perp \sim \delta \) has been used to expand the two solutions up to order \( \delta^2 \). The first solution \( s_1 \) is of order \( \hat{M} \) and cannot describe the propagation of the soft field \( \phi_v \). The second solution has the correct scaling, \( s_2 \sim M \delta \), and determines the form of the bilinear terms for the unstable scalar \( \phi_v \) in \( \mathcal{L}_{\text{HSET}} \).

\[
\mathcal{L}_{\phi \phi} = 2\hat{M} \phi_v^{\dagger} \left( iv \cdot \partial - \frac{\Delta}{2} \right) \phi_v + 2\hat{M} \phi_v^{\dagger} \left( -\frac{\partial^2}{2\hat{M}} + \frac{\Delta^2}{8M} \right) \phi_v + \ldots
\]

The two terms in the first brackets are both of order \( M \delta \), and are included in the propagator of the resonant field, which in momentum space reads

\[
i \frac{1}{v \cdot k - \frac{\Delta}{2}}.
\]

Thus in the effective-theory formulation one is naturally led to a fixed-width prescription for the regularisation of the unstable-particle propagator. The remaining terms in equation (2.12) are suppressed by a power of \( \delta \) compared to the leading-order propagator, and are included in the calculation as perturbations. Note that in (2.12) we have limited
ourselves to the operators necessary for a NLO (in $\alpha \sim \delta$) computation of the forward-scattering amplitude, but the effective Lagrangian can be easily extended to include higher-dimensional operators by keeping more terms in the expansion in $\delta$ of the solution (2.11).

The quantity $\Delta$ is determined entirely by hard contributions [51], and is thus classified as a matching coefficient. $\Delta$ can be expressed in terms of the hard part of the heavy-scalar self-energy $\Pi_h(s)$. If one expands $\Pi_h(s)$ in the number of loops $k$ and powers of $\delta$,

$$\Pi_h(s) = \hat{M}^2 \sum_{k,l} \delta^l \Pi^{(k,l)},$$

the first two terms in the expansion in $\alpha \sim \delta$ of the matching coefficient $\Delta$ are [51]

$$\Delta \equiv \Delta^{(1)} + \Delta^{(2)} + ... = \hat{M} \Pi^{(1,0)} + \hat{M} \left( \Pi^{(2,0)} + \Pi^{(1,1)} \Pi^{(1,0)} \right) + ....$$

(2.15)

The leading term $\Delta^{(1)}$ coincides with the one-loop self-energy $\Pi_h^{(1)}(s)$ evaluated at $s = \hat{M}^2$, while $\Delta^{(2)}$ accounts for contributions that are suppressed by one power of $\alpha$ or $\delta = (s - \hat{M}^2)/\hat{M}^2$. Though in equation (2.15) only the hard part of the self-energy $\Pi_h(s)$ enters the expressions of $\Delta^{(1)}$ and $\Delta^{(2)}$, one can freely replace it with the full self-energy $\Pi(s)$, because the latter does not receive contributions from soft loops, since they give scaleless integrals that vanish in dimensional regularisation. Explicit results for $\Delta^{(1)}$ and $\Delta^{(2)}$ in the $\overline{\text{MS}}$ and pole scheme were given in [51]. Note that in the pole scheme, where $\hat{M} \equiv M$, the matching coefficient $\Delta$ is a purely-imaginary gauge-independent quantity, $\Delta = -i\Gamma$.

Interactions of the scalar $\phi_v$ with soft photons are included in (2.12) by replacing ordinary derivatives with the soft covariant derivative $D_s = \partial - igA_s$. The prescription follows from the observation that the full Lagrangian is gauge invariant, and so is the separation into hard and soft contributions. Thus the effective Lagrangian must have a residual soft $U(1)$ gauge symmetry. If in equation (2.12) we substitute $\Delta$ with its two-loop expression $\Delta = \Delta^{(1)} + \Delta^{(2)}$, and add kinetic terms for soft photons and fermions, the soft effective Lagrangian up to NLO in $\delta$ reads

$$\mathcal{L}_{\text{HSET}} = 2\hat{M}\phi_v^\dagger \left( iv \cdot D_s - \frac{\Delta^{(1)}}{2} \right) \phi_v + 2\hat{M}\phi_v^\dagger \left( \frac{(iD_{s\perp})^2}{2\hat{M}} + \frac{[\Delta^{(1)}]^2}{8\hat{M}} - \frac{\Delta^{(2)}}{2} \right) \phi_v$$

$$- \frac{1}{4} F_{s,\mu\nu} F_{s}^{\mu\nu} + \bar{\psi}_s i D_s \psi_s + \bar{\chi}_s i \partial \chi_s.$$

(2.16)

Each field in equation (2.16) can be assigned a specific scaling power in $\delta$ [56]. Since in momentum space the propagator of $\phi_v(x)$ scales as $\delta^{-1}$, and for a soft field $\int d^4k$ counts as $\delta^4$, the field $\phi_v$ scales as $\delta^{3/2}$. $D_s$ and $\Delta$ scale both as $\delta$, and soft photons and fermion fields scale respectively as $\delta$ and $\delta^{3/2}$ [56]. Thus the first bilinear term in (2.16) scales as $\delta^4$, while the second one is suppressed by one power of $\delta$ or $\alpha$, hence counting as a NLO corrections. The kinetic terms for soft photon and fermions both scale as $\delta^4$.

An important thing to note is that the bilinear terms in (2.16) do not reproduce the NLO expansion of full propagator (2.7), but instead that of the quantity $i\varpi^{-1} R_{\phi}/(P^2 - \hat{s})$, where $\varpi$ is the Poincaré invariant mass of the soft photon, $R_{\phi}$ the bare axial current renormalisation constant, and $P$ is the momentum of the soft photon.
where
\[ \varpi^{-1} = \left( 1 + \frac{M\Delta - k_\perp^2}{M^2} \right)^{1/2} = 1 + O(\alpha, \delta). \]  

(2.17)

The normalisation factor \( \varpi^{-1} \) originates from the expansion of the solution \( s_1 \), equation (2.10), that, being a hard effect related to the scale \( \hat{M} \), is not reproduced by the operators in the effective Lagrangian (2.16). Instead this hard contribution is taken into account in the calculation of the matching coefficients of production and decay operators. More precisely, whenever one computes an amputated Green function in the effective theory one multiplies every external \( \phi_v \)-line by an additional wave-function renormalisation factor \( \varpi^{-1/2} \). This issue is discussed in Subsection 2.1.3 and later in Chapter 3.

We now turn to the construction of the part of the effective Lagrangian describing energetic \( (E \sim M) \) fermions, \( \mathcal{L}_{\text{SCET}} \). Such “collinear” modes have been discussed extensively in the context of soft-collinear effective theory (SCET) [57, 58]. Since the total cross section is extracted from the forward-scattering amplitude, one does not need to explicitly include final states in the effective Lagrangian. Thus the only relevant direction is set by the incoming particles. It is here assumed that the electron moves with large momentum in the direction \( \vec{n} \) representing the beam axis. This defines the two light-like vectors \( n^\pm = (1, \pm \vec{n}) \), with \( n^2_+ = n^2_- = 0 \) and \( n^- \cdot n^+ = 2 \). A collinear momentum in the direction \( \vec{n} \) is decomposed according to
\[ p^\mu = (n^+ \cdot p) \frac{n^\mu}{2} + p^\mu_\perp + (n^- \cdot p) \frac{n^\mu}{2}, \]

(2.18)

with the three components scaling as \( n^+ \cdot p \sim M, n^- \cdot p \sim M\delta \) and \( p_\perp \sim M\delta^{1/2} \). The scaling of the small component is determined by the interaction of collinear modes with soft modes, which implies that a general collinear fluctuation has an offshellness of order \( M\delta \). The scaling of the transverse component is then fixed by the poles of collinear propagators [56].

The collinear Lagrangian has been worked out to order \( \delta \) in [56] \(^1\). Again, only the terms relevant for the NLO computation of the line shape (2.3) are given here. For the fields collinear with the direction \( n^- \) (that in the following will be identified by the label \( c1 \) these terms read
\[ \mathcal{L}_{\text{SCET}} = \bar{\psi}_{c1} \left( i n^- \cdot D + iP_{\perp,c1} \frac{1}{i n^+ \cdot D_{c1} + i \epsilon} i D_{\perp,c1} \right) \frac{\gamma_+}{2} \psi_{c1} - \frac{1}{4} F_{c1,\mu\nu} F_{c1}^{\mu\nu} + \ldots, \]  

(2.19)

where the collinear electron field \( \psi_{c1} \) satisfies \( \gamma^- \psi_{c1} = 0 \) [56–58]. The covariant derivative \( D_{c1} = \partial - igA_{c1} \) contains the interaction with collinear photons \( A_{c1} \), while the soft photon field appears in \( n^- \cdot D = n^- \cdot (\partial - igA_{c1} + igA_s) \). The inverse covariant derivative \( (in^+ \cdot D_{c1} + i \epsilon)^{-1} \) can be written in terms of Wilson lines
\[ (in^+ \cdot D_{c1} + i \epsilon)^{-1} = W_{c1}(in^+ \cdot \partial + i \epsilon)^{-1} W_{c1}^\dagger, \]  

(2.20)

\(^1\)Note that the ultrasoft modes defined in [56] correspond to what are here called soft modes.
where $W_{c1}$ is

$$W_{c1}(x) = \exp \left( ig \int_{-\infty}^{0} ds n_+ \cdot A_{c1}(x + s n_+) \right). \quad (2.21)$$

Therefore the Lagrangian (2.19) contains effective vertices with any number of $A_{c1}$ fields, which are all leading order in $\delta$, but suppressed by a gauge-coupling factor $g$. The soft-interaction term $g(n_- \cdot A_s) \bar{\psi}_{c1} \psi_{c1}$ generates the leading-order vertex $i g n_\mu$, that corresponds to the “eikonal” approximation for the coupling of soft photons to energetic fermions.

The complete SCET Lagrangian is obtained by adding to (2.19) a second term describing a set of collinear fields along the direction of the incoming neutrino (these will be denoted by the subscript $c2$). Working in the centre-of-mass frame, in which electron and neutrino collide head-on, this term is obtained from equation (2.19) with the obvious replacements $\psi_{c1} \to \chi_{c2}$, $n_- \leftrightarrow n_+$ and $D \to \partial$. Hence the relevant effective Lagrangian for collinear fields reads

$$L_{\text{SCET}} = \bar{\psi}_{c1} \left( \frac{i n_- \cdot D + i D_{\perp c1}}{i n_+ \cdot D_{\perp c1} + i e D_{\perp c1}} \right) \frac{\gamma_+}{2} \psi_{c1} - \frac{1}{4} F_{c1,\mu \nu} F_{c1}^{\mu \nu}$$

$$+ \bar{\chi}_{c2} \left( \frac{i n_+ \cdot \partial + i D_{\perp c2}}{i n_+ \cdot \partial_{\perp c2} + i e D_{\perp c2}} \right) \frac{\gamma_-}{2} \chi_{c2} \quad (2.22)$$

In complete generality, one should also introduce a $c2$-collinear photon and electron, $A_{c2}$ and $\psi_{c2}$, and a $c1$-collinear neutrino, $\chi_{c1}$, but these fields enter terms which are highly suppressed in $\delta$, and do not contribute to the calculation of the next-to-leading order forward-scattering amplitude.

### 2.1.2 Effective vertices

The last ingredient to be added to the effective Lagrangian are interaction terms that allow the production and decay of the resonant unstable particle. Since $L_{\text{HSET}}$ does not contain collinear fields, and $L_{\text{SCET}}$ does not contain the heavy scalar $\phi_v$, these fields can only interact indirectly through soft fields. In particular, there is no vertex in the Lagrangian that allows the production of the soft scalar in the collision of two collinear fermions with opposite directions. These vertices cannot be included in the effective Lagrangian as interaction terms without introducing additional modes [51]. This can be intuitively understood by considering that the collision of two generic collinear fields with opposite directions, $\psi_{c1}$ and $\chi_{c2}$, produces a configuration with off-shellness of order $M^2$, which cannot be described by effective fields with virtuality of order $M \Gamma$ or smaller. Instead the momenta of the incoming collinear modes should be prearranged in such a way that the invariant mass of the configuration produced in the collision is exactly $M^2$, within a small amount of order $M^2 \delta$.

To avoid the definition of new collinear fields, effective operators $J^{(n)}(x)$ and $T^{(k)}(x)$ are introduced. These operators are not included in the effective Lagrangian as interaction terms, but appear only in the calculation of the forward-scattering amplitude (2.3), which
is given by a sum of matrix elements of currents,

$$iA = \sum_{m,n} \langle \bar{\nu} e | \int d^4 x T \{ iJ^{(m)\dagger} (0) iJ^{(n)} (x) \} | \bar{\nu} e \rangle + \sum_{k} \langle \bar{\nu} e | iT^{(k)} (0) | \bar{\nu} e \rangle .$$

(2.23)

The $T$ in the first term denotes time-ordering of operators, and the sums extend over the sets of effective vertices of the two types. The matrix elements in equation (2.23) are evaluated using the Lagrangian $L_{\text{EFT}} = L_{\text{HSET}} + L_{\text{SCET}}$, as given in equations (2.16) and (2.22).

The first type of vertex, $J^{(n)} (J^{(m)\dagger})$, describes the production (decay) of a resonant scalar $\phi_v$. The leading-order vertex is simply the original Yukawa coupling re-expressed in terms of the effective-Lagrangian fields,

$$J^{(0)} (x) = e^{-iMv \cdot x} Cy[\phi_v \bar{\psi} c_1 \chi c_2](x) ,$$

(2.24)

where $C = 1 + O(\alpha)$. At higher orders in $\delta$ and $\alpha$ a larger set of operators is generated by integrating-out hard fluctuations, and the coefficient function $C$ will in general receive loop corrections. However these operators have in common the feature of containing the field $\phi_v$ exactly once.

The operators $T^{(k)}$ account for non-resonant contributions, in which the intermediate configurations have off-shellness of order $M^2$. They correspond, for example, to the physical situation in which the electron radiates a hard or collinear photon before interacting with the neutrino. Note that in the present toy model this still requires an intermediate scalar line, because the neutrino has only Yukawa interactions, but in a generic model $T^{(k)}$ includes also contributions from background processes that do not contain the intermediate unstable particle. The simplest operator of this type is

$$T^{(1)} (x) = F \frac{yy^*}{M^2} (\bar{\psi} c_1 \chi c_2)(\bar{\chi} c_2 \psi c_1)(x) ,$$

(2.25)

with $F = 1/4 + O(\alpha)$, as it will be shown below.

In terms of Feynman diagrams, the two classes of contributions in equation (2.23) are represented by the two topologies shown in Figure 2.1. The graph with a resonant scalar line on the left-hand side of the figure corresponds to the contraction of two operators $J^{(n)}$. Collinear interactions (C) take place at both vertices, but only soft modes (S) can connect different hard subprocesses, because exchange of a collinear mode between initial and final states would put the scalar off resonance. The right-hand side diagram represents non-resonant configurations. In this case both soft and collinear modes can connect the external legs to each other and to the vertex. In both topologies the matching coefficients at the vertices are determined by hard modes (H) that have been integrated out.

Note that the coefficient function of the operator (2.25) has a non-vanishing contribution even at the tree level. This seems to contradict the physical picture given in Figure 2.1, that suggests that radiation of an energetic photon is necessary to set the scalar off-shell. In fact the leading-order coefficient $F^{(0)} = 1/4$ accounts for a kinematical configuration with invariant mass equal to $M^2$, in which the resonant propagator $i/(v \cdot k - \Delta/2)$
is however cancelled by higher-order terms in the expansion of the full propagator (2.7). This expansion reads

\[ \frac{i}{P^2 - s} = \frac{1}{2M} \frac{i}{v \cdot k - \frac{\Delta}{2}} \left( 1 - \frac{\Delta}{2M} \right) - \frac{i}{4M^2} \]

\[ + \frac{1}{2M} \frac{i}{v \cdot k - \frac{\Delta}{2}} \left( \frac{ik^2}{2M^2} + \frac{i\Delta^2}{8M} \right) \frac{i}{v \cdot k - \frac{\Delta}{2}} + ..., \]  

(2.26)

where the different terms have been rearranged in such a way that no powers of \( v \cdot k \) appear at the numerator. The first term in equation (2.26) reproduces the leading-order effective propagator, multiplied by the normalisation factor \( \hat{\varpi} \), while the last term accounts for higher-order contribution to the soft Lagrangian (2.16). The term \(-i/4M^2\) does not contain a resonant propagator and is not reproduced by any operator in the Lagrangian (2.16). It is thus associated to the four-fermion operator in equation (2.25).

It is interesting to consider the relative weight of resonant and non-resonant contributions. The leading resonant diagram contains two vertices \( y\phi_v \bar{\psi}_{c1} \chi_{c2} \) and a resonant propagator that counts as \( \delta^{-1} \). Thus the leading resonant contribution scales as \( \alpha/\delta \sim 1 \). The leading four-fermion operator \( \frac{yy^*}{4M^2} (\bar{\psi}_{c1} \chi_{c2})(\chi_{c2} \bar{\psi}_{c1}) \) does not contain a resonant propagator, and scales as \( \alpha \). Thus, as expected, the forward-scattering amplitude receives contributions from non-resonant configurations only starting at NLO in \( \alpha \sim \delta \).

In the rest of the thesis we will adopt the procedure just illustrated for the inclusion of interactions in the effective-theory description, i.e. we will parametrise the forward-scattering amplitude as (2.23), and compute the matrix elements with (2.16) and (2.22). However, for completeness, we briefly remind how vertices analogous to (2.24) and (2.25) can be included in the effective Lagrangian as explicit interaction terms by defining new collinear fields. To this aim, one introduces “external-collinear” modes in the direction \( n_- \), and assigns them a momentum \( \hat{M}n_- + k \), where \( k \sim M\delta \). These modes have the same virtuality of a generic \( c1 \)-collinear mode, but their momentum is not described by (2.18), since its large component has been fixed to the value \( \hat{M}/2 \) required to produce an unstable scalar near the mass-shell. The large component of the momentum is thus extracted from \( \psi_{c1} \), and the new collinear field defined as [50, 51]

\[ \psi_{n_-}(x) \equiv e^{i\hat{M}/2(n_- \cdot x)} \mathcal{P}_+ \psi_{c1}(x), \]

(2.27)

and similarly for \( \chi_{n_+} \). The field \( \psi_{n_-} \) describes soft fluctuations around the external collinear configuration, whereas \( \psi_{c1} \) is kept to describe generic collinear fluctuations around \( \hat{M}/2n_- \).

With the introduction of the new fields the collinear Lagrangian (2.22) assumes a more complicated expression, but the only terms needed for the NLO computation of the line shape turn out to be

\[ \mathcal{L}_{\pm} = \bar{\psi}_{n_-} i n_- \cdot D_s \frac{\gamma_\pm}{2} \psi_{n_-} + \bar{\chi}_{n_+} i n_+ \cdot \partial \frac{\gamma_\pm}{2} \chi_{n_+}, \]

(2.28)

while the now allowed interactions read [50]

\[ \mathcal{L}_{\text{int}} = C y \phi_v \bar{\psi}_{n_-} \chi_{n_+} + Cy^* \phi_v \bar{\psi}_{n_+} \chi_{n_-} + F \frac{yy^*}{M^2} (\bar{\psi}_{n_-} \chi_{n_+})(\chi_{n_+} \bar{\psi}_{n_-}). \]

(2.29)
As before $C = 1 + O(\alpha)$ and $F = 1/4 + O(\alpha)$. The first two vertices describe the production and decay of one resonant heavy scalar. Note that there are no similar vertices with field combination $\phi_v \bar{\psi}_c \chi_{n+}^{1}$, $\phi_v \bar{\psi}_c \chi_{c2}$, or $\phi_v \bar{\psi}_c \chi_{c2}$, because the collision of an external-collinear field with a collinear field $\psi_{c1}$ or $\chi_{c2}$, or the collision of $c1$ and $c2$-collinear fields, produces a configuration which is off-shell of an amount of order $M^2$. The third operator in equation (2.29) describes non-resonant contributions. In this case the field combinations $(\bar{\psi}_c \chi_{n+}^{1})(\bar{\chi}_{n+}^{1}\psi_{c1})$, $(\bar{\psi}_c \chi_{c2})(\bar{\chi}_{c2}\psi_{n-}^{1})$ and $(\bar{\psi}_c \chi_{c2})(\bar{\chi}_{c2}\psi_{c1}^{1})$ are allowed, but their matching coefficients are suppressed by at least one power of $\alpha$ with respect to $F(0)$, and can be neglected in the NLO calculation.

### 2.1.3 Matching of $L_{\text{EFT}}$

In the previous section we have discussed the bilinear terms of the effective Lagrangian and the allowed interactions, and presented the terms necessary for the computation of the LO and NLO term in the expansion of the forward-scattering amplitude, equation (2.5). We stress here once more that the inclusion of higher-order operators in the effective Lagrangian is straightforward, and does not present new conceptual problems. The matching coefficients $\Delta$, $C$ and $F$ in equations (2.16), (2.24) and (2.25) have to be computed at the order in $\alpha$ required to match the total target accuracy for the forward-scattering amplitude (2.2). For the NLO computation the matching coefficient $\Delta = \Delta^{(1)} + \Delta^{(2)} + \ldots$ is needed at the two-loop level, the production-vertex coefficient $C = 1 + \frac{\alpha}{2\pi} C^{(1)} + \ldots$ at the one-loop level, while only the tree-level coefficient of the four-fermion operator in (2.29), $F = F^{(0)} + \ldots$, is required, since non-resonant diagrams contribute first at next-to-leading order in $\delta$.

The matching coefficients $\Delta$ and $C$ are determined by the requirement that the full calculation and the EFT calculation agree order by order in an expansion in $\alpha \sim \delta$. This condition translates into the matching equation

$$\mathcal{F}_{\text{full}} \sqrt{R_{\text{LSZ}}^{\text{full}}} = C_F \mathcal{F}_{\text{EFT}} \sqrt{R_{\text{LSZ}}^{\text{EFT}}} \varpi^{-n_\phi/2}, \quad (2.30)$$

where $\mathcal{F}_{\text{full}}$ and $\mathcal{F}_{\text{EFT}}$ are the same amputated $n$-point function, containing the vertex whose matching coefficient $C_F$ has to be determined, computed respectively in the full theory and with the effective Lagrangian $L_{\text{EFT}}$. The quantities $R_{\text{LSZ}}^{\text{full}}$ and $R_{\text{LSZ}}^{\text{EFT}}$ represent the product of the LSZ residue factors for the external legs, and $\varpi^{-n_\phi/2}$, where $n_\phi$ is the number of external scalar legs, accounts for the different normalisation of the original scalar field $\phi(x)$ and the non-relativistic field $\phi_v(x)$, as explained below equation (2.17). The $n$-point functions $\mathcal{F}_{\text{full}}$ and $\mathcal{F}_{\text{EFT}}$ are computed in standard fixed-order perturbation theory and for on-shell external momenta, with the consequence that the matching coefficient $C_F$ is gauge invariant by construction. Note that the on-shell condition for the heavy scalar implies that its momentum is evaluated at the gauge-invariant complex pole, $P^2 = M^2 - iM\Gamma$, rather than at $P^2 = \hat{M}^2$. For the coefficient $\Delta$, $\mathcal{F}_{\text{full}}$ and $\mathcal{F}_{\text{EFT}}$ are given by the two-point function of the scalar field computed in the full theory and in the effective theory. As already anticipated below equation (2.15), the effective-theory result vanishes in dimensional regularisation, and $\Delta$
can be directly extracted from the full self-energy $\Pi(s)$, as shown in equation (2.15). The
detailed calculation of $\Pi(s)$ at one-loop and two-loop accuracy was presented in [51].

$C^{(1)}$ is obtained from the matching of the on-shell three-point function of a scalar field,
an electron and a neutrino at order $\alpha$ and at leading order in $\delta$. This is illustrated in Figure
2.3, where for simplicity we neglect the corrections from the LSZ residue factors and $\varpi$,
that are proportional to the leading-order vertex. The full one-loop result (first diagram
in Figure 2.3) must be reproduced by the sum of the contribution from the matching
coefficient $C^{(1)}$ (second diagram) and collinear and soft loop corrections (third and fourth
diagram) computed with the effective-theory Feynman rules. If dimensional regularisation
is used, all collinear and soft loops vanish for on-shell external momenta, with the result
that the matching equation (2.30) simplifies to (at $O(\alpha)$ and leading order in $\delta$)

$$F_{\text{full},(1)} + \frac{1}{2} F^{(0)} R_{\text{LSZ}}^{\text{full},(1)} = \left( C^{(1)} - \frac{\varpi^{(1)}}{2} \right) F^{(0)},$$

(2.31)

where all the quantities in (2.30) have been explicitly replaced with their expansions in $\alpha$,
$F^{\text{full}} = F^{(0)} + \frac{\alpha}{2\pi} F^{\text{full},(1)} + \ldots$, $R_{\text{LSZ}}^{\text{full}} = 1 + \frac{\alpha}{2\pi} R_{\text{LSZ}}^{\text{full},(1)} + \ldots$, $\varpi = 1 + \frac{\alpha}{2\pi} \varpi^{(1)} + \ldots$. Alternatively,
$C^{(1)}$ can be directly computed by performing an expansion by regions and extracting the
hard contributions to the full one-loop vertex diagram, as illustrated in Figure 2.2. At
NLO accuracy, this is actually equivalent to compute the full one-loop graph for $s = M^2$.

The results of the calculation of $C^{(1)}$ can be found in [51]. Details about the matching
procedure and renormalisation convention for the specific case of $W$-pair production will
be given in Subsection 4.1.1.
To summarise, Feynman diagrams contributing to the forward-scattering amplitude (2.3) near the heavy-scalar production threshold receive contributions from four different momentum regions:

- **hard (h):** \( p \sim M \),
- **soft (s):** \( p \sim M\delta \),
- **collinear (c1):** \( p_\perp \sim M\delta^{1/2}, \ n_+ \cdot p \sim M, \ n_- \cdot p \sim M\delta \),
- **collinear (c2):** \( p_\perp \sim M\delta^{1/2}, \ n_+ \cdot p \sim M\delta, \ n_- \cdot p \sim M \). (2.32)

After integrating out the hard modes, the effective Lagrangian contains only long-distance degrees of freedom. These are represented by a soft heavy scalar \( \phi_v, c_1 \)-collinear and soft photons, \( A_{c_1} \) and \( A_s \) respectively, \( c_1 \)-collinear and soft electrons, \( \psi_{c_1} \) and \( \psi_s \), and \( c_2 \)-collinear and soft neutrinos, \( \chi_{c_2} \) and \( \chi_s \). The contribution of hard modes is encoded in the coefficients of the effective Lagrangian. These are determined by fixed-order computations of the hard part of the relevant on-shell matrix elements in the full theory. As we will show in the next section, the extension of the model to pair-production of unstable particles requires only minor modifications.

### 2.2 Unstable-particle effective theory for pair production near threshold

The generalisation of the formalism presented in the previous section from a scalar to a pair of vector-boson resonances is straightforward. Apart from the obvious modifications related to the different spins of the unstable particles involved, the main difference is that the pair-production threshold kinematics implies a change in power counting, and enforces the introduction of a new dynamical mode. This is analogous to the difference between heavy-quark effective theory and non-relativistic QCD [59, 60]. As before, the momentum of the vector resonance is parametrised as \( P = \hat{M}_W v + r \), where \( \hat{M}_W \) is the resormalised \( W \) mass. In the centre-of-mass frame the requirement that both vector bosons are close to the mass-shell, \( P^2 - \hat{M}_W^2 \sim M_W \Gamma_W \), where \( \Gamma_W \) is the \( W \) width, implies that the velocity vector \( v^\mu \) is equal to \( (1, 0) \), while the small residual momentum \( r \) scales as a potential momentum, \( r_0 \sim \Gamma_W, \ r = \sqrt{\hat{M}_W \Gamma_W} \) [61], whereas the residual momentum of the single resonance discussed in the previous section scaled as a soft momentum, \( k_0 \sim |\vec{k}| \sim \Gamma \).

In \( W \)-pair production the hard fluctuations are given by modes whose momentum components are all of order \( \hat{M}_W \). As in the single-resonance case, they do not represent dynamical degrees of freedom of the effective theory, and their effect is reproduced by short-distance matching coefficients. The effective Lagrangian describes the propagation and interactions of two non-relativistic, spin-1 fields \( \Omega^1 \pm_\pm \) representing the nearly on-shell potential \( W^\pm \) modes, two sets of collinear fields for the incoming electron and positron respectively, and potential, soft and collinear photon fields. The corresponding momentum
The small parameter $\delta$ is either $(s - 4M^2_W)/(4M^2_W)$, which is related to the square of the non-relativistic velocity $v^2$ of the resonant $W$ bosons (not to be confused with the large vector component $v^\mu$ defined above), or $\Gamma_W/M_W \sim \alpha_{em}$, since the characteristic virtuality is never parametrically smaller than $M_W \Gamma_W$ for an unstable $W$. The interactions of collinear modes are described by the SCET Lagrangian [56–58], equation (2.22), where $\chi^c_2(x)$ has to be replaced by a field describing a highly-energetic positron moving in the direction $-\vec{n}$, and the covariant derivatives modified consistently. As far as the next-to-leading order calculation is concerned, the soft-collinear Lagrangian (2.22) allows one to perform the standard eikonal approximation $\pm i e n^\mu_\mp$ for the interaction of soft photons with the electron (positron) in the soft one-loop correction.

The Lagrangian for the resonance fields is given by the non-relativistic Lagrangian $L_{\text{NRQED}}$, generalised to account for the instability of the $W$ bosons [61, 62]. The terms relevant at NLO in $\delta$ are

$$L_{\text{NRQED}} = \sum_{a=\mp} \left[ \Omega^\dagger_i a \left( iD^0 + \frac{\vec{D}^2}{2M_W} - \frac{\Delta}{2} \right) \Omega_i a + \Omega^\dagger_i a \frac{(\vec{D}^2 - M_W^2 \Delta)^2}{8M^4_W} \Omega_i a \right].$$

Here $\Omega^c_+$ and $\Omega^c_-$ ($i = 1, 2, 3$) are non-relativistic, spin-1 destruction fields for particles with electric charge $\pm 1$, respectively. Note that, in contrast to equation (2.12), a factor $(2M_W)^{1/2}$ has been absorbed in the definition of the fields $\Omega_{\pm}$, whose mass dimension is $3/2$. The interactions with soft and potential photons is contained in the covariant derivative $D_\mu \Omega^\dagger_\mp \equiv (\partial_\mu \mp ieA_\mu)\Omega^\dagger_\mp$. The effective theory does not contain fields for the other heavy particles in the Standard Model, the $Z$ and Higgs bosons, and the top quark. Their propagators are always off-shell by amounts of order $M^2_W$ and therefore their effect is encoded in the short-distance matching coefficients. In a general $R_\xi$-gauge this also applies to the pseudo-Goldstone (unphysical Higgs) fields, except in 't Hooft-Feynman gauge $\xi = 1$, where the scalar $W$ and unphysical charged pseudo-Goldstone modes have masses $M_W$ and can also be resonant. However, the two degrees of freedom cancel each other, leading to the same Lagrangian (2.34) describing the three polarisation states of a massive spin-1 particle. The effective Lagrangian has only a U(1) electromagnetic gauge symmetry as should be expected at scales far below $M_W$. However, since the short-distance coefficients of the Lagrangian and all other operators are determined by fixed-order matching of on-shell matrix elements to the full Standard Model, as explained in Subsection 2.1.3, they are independent of the gauge parameter in $R_\xi$-gauge by construction.

The matching coefficient $\Delta$ in (2.34) is obtained from the on-shell two-point function of a transverse $W$ boson. As pointed out in Subsection 2.1.3, “on-shell” here refers to the complex pole determined from

$$\bar{s} - \hat{M}_W^2 - \Pi^W_T (\hat{s}) = 0,$$

(2.35)
with \( \tilde{M}_W \) any renormalised mass parameter, and \( \Pi_W^T(q^2) \) the renormalised, transverse self-energy. The solution to this equation,

\[
\bar{s} = M_W^2 - iM_W\Gamma_W,
\]

(2.36)
defines the pole mass and the pole width of the \( W \) boson. The matching coefficient is then given by

\[
\Delta \equiv \frac{\bar{s} - \tilde{M}_W^2}{M_W}.
\]

(2.37)

In the remainder of the thesis, we will adopt the renormalisation convention where \( \tilde{M}_W \) is the pole mass \( M_W \), and consequently the matching coefficient \( \Delta \) is purely imaginary, \( \Delta = -i\Gamma_W \). Since \( D^0 \sim M_W\delta \), \( \vec{D}^2 \sim M_W^2\delta \), and \( \Delta \sim M_W\delta \), one sees that the first bilinear term in (2.34) consists of leading-order operators, while the second is suppressed by one factor of \( \delta \), and can be treated as a perturbation. Accordingly, the propagator of the \( \Omega_\pm \) fields is

\[
\frac{i\delta^{ij}}{k^0 - \frac{k^2}{2M_W} - \Delta}.
\]

(2.38)

Note that it would be sufficient to resum only the one-loop expression for \( \Delta \) in the propagator, and to include higher-order corrections perturbatively, as in equation (2.16). However in the following we will often keep \( \Delta \) unexpanded.

Loop diagrams calculated using the Lagrangian (2.34) receive contributions from soft and potential photons. Since the potential photons do not correspond to on-shell particles \( (k^2 \sim -|\vec{k}|^2 \sim M_W^2\delta) \), they can be integrated out, resulting in a non-local Coulomb potential. This is analogous to the matching of non-relativistic QED onto potential non-relativistic QED [63]. Up to NLO the required PNRQED Lagrangian is

\[
L_{PNRQED} = \sum_{a=\pm} \left[ \Omega_a^{ij} \left( iD^0_a + \frac{\vec{D}^2}{2M_W} - \frac{\Delta}{2} \right) \Omega_a^{ji} + \Omega_a^{ij} \left( \frac{\vec{D}^2}{2} - \frac{M_W\Delta}{8M_W^3} \right) \Omega_a^{ji} \right] + \int d^3\vec{r} \left[ \Omega_+^{ij} \Omega_-^{ji} \right] (x + \vec{r}) \left( -\frac{\alpha}{\vec{r}} \right) \left[ \Omega_+^{ij} \Omega_-^{ji} \right] (x).
\]

(2.39)

After removing the explicit potential-photon field from the Lagrangian, only the soft photon \( A_0^J(t,0) \) appears in the covariant derivative \( D^0_a \). From the scaling of a potential momentum, equation (2.33), it follows that the resonant \( W \) field has support in a region \( \sim \delta^{-1} \) in the time direction and in a region \( \sim \delta^{-1/2} \) in each space direction. Hence the measure \( d^4x \) in the action scales as \( \delta^{-5/2} \). Together with \( \partial_0 \sim \delta \) we find from the leading-order kinetic term that \( \Omega_+^{ij} \sim \delta^{3/4} \). Consequently the non-local Coulomb potential scales as \( \delta^{-5/2} \delta^{-3/2} [\delta^{3/2} (\alpha \delta^{1/2}) [\delta^{3/2}] = \alpha / \sqrt{\delta} \). Since we count \( \alpha \sim \delta \), the Coulomb potential is suppressed by \( \sqrt{\delta} \), or \( \alpha^{1/2} \), and need not be resummed, in contrast to the case of top-quark

\footnote{What we call “soft” here, is usually termed “ultrasoft” in the literature on non-relativistic QCD. There are further modes, called “soft” there, with momentum \( k \sim M_W\sqrt{\delta} \) [52]. In the present context these modes cause, for instance, a small modification of the QED Coulomb potential due to the one-loop photon self-energy, but these effects are beyond NLO. See however Chapter 6.}
pair-production near threshold. However, with this counting, the Coulomb enhancement introduces an expansion in half-integer powers of the electromagnetic coupling, the one-loop Coulomb correction being a "N$^{1/2}$LO" term.

### 2.2.1 Effective vertices and the leading-order cross section

In analogy with (2.23), here we adopt the following representation of the forward-scattering amplitude [51]:

$$ iA = \sum_{k,l} \int d^4x \langle e^-e^+|T[iO_p^{(k)}(x)iO_p^{(l)}(x)]|e^-e^+\rangle + \sum_k \langle e^-e^+|iO^{(k)}_4(0)|e^-e^+\rangle. \quad (2.40) $$

The operators $O_p^{(l)}(x)$ ($O_p^{(k)}(x)$) in the first term on the right-hand side produce (destroy) a pair of non-relativistic, resonant $W$ bosons. The second term accounts for the remaining non-resonant contributions, analogously to the four-fermion operators $T^{(k)}$. The matrix elements have to be computed with the effective Lagrangian discussed above and the operators should include short-distance coefficients determined by the hard fluctuations. Note that in (2.40) there is no term for production of one resonant and one off-shell $W$. These configurations are effectively short-distance and included in the non-resonant production-decay operators $O^{(k)}_4(0)$.

The lowest-dimension production operator, $O_p^{(0)}(0)$, must clearly have field content $(\bar{e}_2e_1)(\Omega_1^0\Omega_2^0)$, where the subscripts on the electron fields denote the two different direction labels of the collinear fields. The short-distance coefficient follows from matching the renormalised on-shell matrix elements for $e^-e^+ \rightarrow W^-W^+$, expanded in the small relative $W$ momentum $r$, to the desired order in ordinary weak-coupling perturbation theory. As seen in Subsection 2.1.3, the on-shell condition for the $W$ lines implies that their momenta satisfy $k_1^2 = k_2^2 = \bar{s} = M_W^2 + M_W \Delta$, but in a perturbative matching calculation this condition must be fulfilled only to the appropriate order in $\alpha$ and $\delta$ (see Section 4.1 for details). On the effective-theory side of the matching equation one also has to add a factor $\sqrt{2M_W} \varpi^{-1/2}$ for each external $\Omega$ line, as indicated in equation (2.30). Explicitly one finds [51] \(^3\)

$$ \varpi^{-1} \equiv \left(1 + \frac{M_W \Delta + \vec{\omega}^2}{M_W^2}\right)^{1/2}, \quad (2.41) $$

which at tree-level, and at leading order in $\delta$, reads $\varpi^{-1} = 1$.

For the tree-level matching of the production operator, we are led to consider the three on-shell $W$-pair production diagrams shown in Figure 2.4. To leading order in the non-relativistic expansion we can set the momenta of the external fermions to $M_W(1, \pm \vec{n})$, with $\vec{n}$ the unit vector defining the direction of the incoming electron, and the $W$ momenta to $k_1 = k_2 = (M_W, \vec{0})$. In this approximation the $s$-channel diagrams vanish, and only the $t$-channel contribution to the helicity configuration $e_L^+e_R^-$ survives. The corresponding

\(^3\)This is the well-known $(E/M)^{1/2}$ factor, which accounts for the normalisation of non-relativistic fields, generalised to unstable particles and general mass renormalisation conventions.
operator, including its tree-level coefficient function, reads
\[ O_p^{(0)}(0) = \frac{\pi \alpha_{\text{ew}}}{M_W^2} \left( \bar{e}_{c_2} L \gamma^j n^j e_{c_1} L \right) \left( \Omega_+^{ij} \Omega_+^{ij} \right), \tag{2.42} \]
where we have introduced the notation \( a^{[ij]} b^{[ji]} = a^i b^j + a^j b^i \). For completeness we note that the emission of collinear photons from the W or collinear fields of some other direction, which leads to off-shell propagators, can be incorporated by adding Wilson lines to the collinear fields, as anticipated in Subsection 2.1.1. This would modify \( O_p^{(0)} \) to \( \frac{\pi \alpha_{\text{ew}}}{M_W^2} \left( \bar{e}_{c_2} L W_\gamma n^j W^\dagger_{c_1} e_{c_1} L \right) \left( \Omega_+^{ij} \Omega_+^{ij} \right) \). However, these Wilson lines will not be needed for the NLO calculation, since the collinear loop integrals vanish in dimensional regularisation (see Section 4.4).

Having determined the tree-level matching coefficient of the production operator \( O_p^{(0)} \), we are able to compute the leading-order contribution to the forward-scattering amplitude (2.40) from resonant configurations, which is given by the expression
\[ iA_{LR}^{(0)} = \int d^4x \langle e^- e^+ | T[iO_p^{(0)\dagger}(0)iO_p^{(0)}(x)] | e^- e^+ \rangle. \tag{2.43} \]
This corresponds to the one-loop diagram shown in Figure 2.5, computed with the vertex (2.42) and the propagator (2.38). Since the momentum of the resonant Ws circulating in the loop is potential, in the following we will often use the terms “resonant” contributions and “potential” contributions interchangeably. We can use a power-counting argument to estimate the magnitude of the leading-order amplitude prior to its calculation. Each of the vertices of the diagram in Figure 2.5 is proportional to \( \alpha \), while a resonant W-propagator counts as \( \delta^{-1} \). Since the loop-momentum is potential, the integration measure scales as \( d^4 r \sim \delta^{5/2} \). Hence \( iA_{LR}^{(0)} \sim \alpha^2 \delta^{1/2} \).

This expectation is confirmed by the explicit calculation of the one-loop diagram:
\[ iA_{LR}^{(0)} = \frac{\pi^2 \alpha_{\text{ew}}^2}{M_W^4} \left( p_2 - |n^{[i} \gamma^{j]} p_1] \right) \left( p_1 - |n^{[i} \gamma^{j]} p_2] \right) \quad \times \quad \int \frac{d^4 r}{(2\pi)^4} \left( \frac{E - r^0 - \frac{r^2}{2M_W^2} + i\Gamma_W/2}{E - r^0 - \frac{r^2}{2M_W^2} + i\Gamma_W/2} \right)^4, \tag{2.44} \]
where \( p_1, p_2 \) indicate the momenta of the electron and positron respectively. Here we have defined \( E = \sqrt{s} - 2M_W \), and adopted the standard helicity notation \( |p\pm\rangle = \frac{1\pm\gamma^5}{2} u(p) \). We
Figure 2.5: Leading-order effective-theory diagram for the forward-scattering amplitude.

have also used $\Delta = -i\Gamma_W$, valid in the pole scheme. The fermion energies are set to $M_W$ in the external spinors, leading to $(p_2 - |n\rho\gamma\beta|p_1 - ) = 16(1 - \epsilon)M_W^2$. The calculation is performed by first evaluating the $r^0$ integral using Cauchy’s theorem, and eliminating the trivial angular integration

$$iA_{LR}^{(0)} = -{16(1 - \epsilon)\pi^2\alpha_{ew}^2 \over M_W^2} {i2^{2\epsilon -2}\pi^{3/2} \over \Gamma(3/2 - \epsilon)} \int_0^\infty d|\vec{r}| \left| \vec{r} \right|^{-2\epsilon} \left( E - {\vec{r}^2 \over 2M_W} + i\Gamma_W \right).$$

The remaining $|\vec{r}|$ integral contains a linear divergence that is, however, rendered finite by dimensional regularisation:

$$iA_{LR}^{(0)} = -{16(1 - \epsilon)\pi^2\alpha_{ew}^2 \over M_W^2} {i2^{2\epsilon -2}\pi^{3/2} \over \Gamma(3/2 - \epsilon)} \left[ -M_W \int_0^\infty d|\vec{r}| \left| \vec{r} \right|^{-2\epsilon} + M_W(E + i\Gamma_W) \int_0^\infty d|\vec{r}| {\left| \vec{r} \right|^{-2\epsilon} \over E - {\vec{r}^2 \over 2M_W} + i\Gamma_W} \right]$$

$$= -4i\pi\alpha_{ew}^2 {\sqrt{E - i\Gamma_W} \over M_W}.$$ (2.46)

The first term in square brackets is a scaleless integral, and vanishes in dimensional regularisation, while the second can be evaluated in the $\epsilon \to 0$ limit, and gives the final expression for $iA_{LR}^{(0)}$. The numerical comparison of (2.46) to the full tree-level result and the convergence of the effective theory approximation will be discussed in Chapter 3.

Taking the imaginary part of (2.46) does not yield the cross section of the flavour-specific four-fermion production process (1.2). At leading order the correct result is obtained by simply multiplying the imaginary part with the leading-order branching ratio product $Br^{(0)}(W^+ \to u\bar{d}) = 1/27$. This procedure can be justified as follows. The imaginary part of the non-relativistic propagator obtained by cutting an $\Omega$ line is given by

$$\text{Im} {1 \over E - {\vec{r}^2 \over 2M_W} + i\Gamma_W} = -{\Gamma_W^{(0)} \over 2} \left( E - {\vec{r}^2 \over 2M_W} \right)^2 + {\Gamma_W^{(0)}}^2 \over 4.$$ (2.47)

The propagator of the $\Omega_\mp$ line implicitly includes a string of self-energy insertions. Taking the imaginary part amounts to performing all possible cuts of the self-energy insertions while the unstable particle is not cut [64, 65]. To obtain the total cross section for a
flavour-specific four-fermion final state, only the cuts through these specific fermion lines have to be taken into account. At the leading order this amounts to replacing $\Gamma^{(0)}_W$ in the numerator of (2.47) by the corresponding partial width, here $\Gamma^{(0)}_{\mu\bar{\nu}}$ and $\Gamma^{(0)}_{\nu\bar{\nu}}$, respectively, while the total width is retained in the denominator. The leading-order cross section is therefore

$$\sigma^{(0)}_{LR} = \frac{1}{27s} \text{Im} A^{(0)}_{LR} = \frac{4\pi\alpha^2}{27s_\text{w}^4} \text{Im} \left[ -\sqrt{-E + i\Gamma^{(0)}_W/M_W} \right].$$  (2.48)

The unpolarised cross section is given by $\sigma^{(0)}_{LR}/4$, since the other three helicity combinations vanish at this order in $\delta$.

We now consider the leading contribution from non-resonant production-decay operators $O_{4e}^{(k)}$ to (2.40), that arises from four-electron operators of the form

$$O_{4e}^{(k)} = \frac{C_{4e}^{(k)}}{M_W^2} (\bar{e}_c_1 \Gamma_1 e_{c_2})(\bar{e}_{c_2} \Gamma_2 e_{c_1}),$$  (2.49)

where $\Gamma_1$, $\Gamma_2$ are generic Dirac matrices. Note that if $C_{4e}^{(k)} \sim \alpha^n$, the contribution to the forward-scattering amplitude scales as $\alpha^n$. This should be compared to the scaling of the leading-order forward-scattering amplitude, $A^{(0)}_{LR} \sim \alpha^2 \delta^{1/2}$. The calculation of the short-distance coefficients $C_{4e}^{(k)}$ is performed in standard fixed-order perturbation theory in the full electroweak theory. The $W$ propagator is the unresummed propagator, since the self-energy insertions are treated perturbatively. In the language of the method of regions, the result of the matching calculation coincides with the expansion of the full forward-scattering amplitude under the assumption that the loop momentum $k$ is hard, $k^2 \sim M_W^2$. The leading contribution to the forward-scattering amplitude arises from the one-loop diagrams shown in Figure 2.6. Since the real part of the short-distance matching coefficient does not enter the definition of the four-fermion process cross section, we directly calculate the imaginary part of $C_{4e}^{(k)}$ by evaluating cut diagrams. From the explicit calculation one finds that the matching coefficient vanish at $O(\alpha^2)$ (see Subsection 3.2.1). This was predictable, since at leading order in the expansion in $\delta$ the cut one-loop diagrams in Figure 2.6 correspond to the production cross section of two on-shell $W$ bosons directly at threshold, which vanishes. In fact, from an explicit representation of these one-loop diagrams it can be seen that the imaginary parts from the hard region vanish in
dimensional regularisation to all orders in the $\delta$ expansion. Thus the leading imaginary parts of $C^{(k)}_{4e}$ arise from two-loop diagrams of order $\alpha^3$. Just as the Coulomb correction the leading non-resonant (hard) contribution provides another $N^{1/2}\text{LO}$ correction relative to (2.46).

2.3 Classification of corrections up to NLO

In this section we list all the contributions to the four-fermion cross section at $N^{1/2}\text{LO}$ and NLO in the EFT counting scheme, $\alpha \sim \delta$. These corrections can be divided into two classes. To the first class belong the short-distance coefficient $\Delta$ in the Lagrangian (2.39), and the matching coefficients of the production operators $O^{(k)}_p$ and the four-electron operators $O^{(k)}_{4e}$. The second class contains corrections that arise from the calculation of loop contributions to the matrix elements in (2.40) within the effective theory.

2.3.1 Hard matching coefficients

The first class of $N^{1/2}\text{LO}$ and NLO corrections is represented by the hard-matching coefficients of the effective Lagrangian and of the operators entering the definition of the forward-scattering amplitude, equation (2.40). They are given by:

**Short-distance coefficient of the effective Lagrangian.** The effective Lagrangian (2.39) contains already all the operators relevant to the NLO computation of the four-fermion cross section. The only non-trivial matching coefficient is $\Delta$, which is related to the location of the $W$ complex pole, $\bar{s} = M_W^2 - i M_W \Gamma_W$, and can be computed from the expansion of the self-energy, as illustrated in equation 2.15. In the pole scheme ($\hat{M}_W = M_W$), the matching coefficient $\Delta$ coincides with the $W$ decay width $\Gamma_W$, defined as the imaginary part of the pole location, as follows from the definition (2.37). At leading order $\Delta^{(1)} = -i \Gamma_W^{(0)}$, where the leading-order decay width is $^4$

$$\Gamma_W^{(0)} = \frac{3}{4} \alpha_{ew} M_W.$$  \hspace{1cm} (2.50)

The $W$ self-energy receives electroweak as well QCD corrections. In the following we will count the strong coupling $\alpha_s$ as $\alpha_{ew}^{1/2}$. Thus the mixed QCD-electroweak two-loop self-energy provides a $N^{1/2}\text{LO}$ correction to $\Delta$, while at NLO we need the self-energy at orders $\alpha_{ew}^2$ and $\alpha_{ew} \alpha_s^2$. The QCD effects are included by multiplying the leading-order hadronic partial decay widths by the universal QCD correction for massless quarks [66],

$$\delta_{\text{QCD}} = 1 + \frac{\alpha_s}{\pi} + 1.409 \frac{\alpha_s^2}{\pi^2},$$  \hspace{1cm} (2.51)

with $\alpha_s = \alpha_s(M_W)$ in the \MS scheme. We denote the electroweak correction to the pole-scheme decay width by $\Gamma_W^{(1, \text{ew})}$, whose explicit expression will be provided in Section 4.1.2.

\footnote{Here the masses of the light fermions are neglected, and the CKM matrix has been set to the unit matrix.}
We therefore have
\[ \Delta_{(3/2)} = -i \Gamma_W^{(1/2)} = -i \frac{2\alpha_s}{3\pi} \Gamma_W^{(0)} , \quad \Delta^{(2)} = -i \Gamma_W^{(1)} = -i \left[ \Gamma_W^{(1, ew)} + 1.409 \frac{2\alpha_s^2}{3\pi^2} \Gamma_W^{(0)} \right]. \]

These results refer to the total width, which appears in the propagator and the forward-scattering amplitude. The extraction of the flavour-specific process \( e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u \bar{d} X \) beyond the leading-order prescription given below equation (2.46) will be discussed in Subsection 4.1.2.

**Matching coefficients of the production operators.** There are two kinds of corrections related to production operators: higher-dimensional operators suppressed by powers of \( \delta \), and one-loop corrections to the matching coefficients of operators of lowest dimension such as (2.42).

The higher-dimension production operators have the generic form
\[
O_p^{(k)} = \frac{C_p^{(k)}}{M_W^{1+\delta(k)}} \left( \bar{e}_{c_2,L/R} \Gamma e_{c_1,L/R} \right) \left( \Omega_{-i} \Gamma_{+i} \right),
\]
where \( \Gamma \) stands for some combination of Dirac matrices and \( F \) and \( G \) are polynomial functions of the covariant derivative \( D \) acting on the fields. As explained in Subsection 2.1.3, the short-distance coefficients of these operators are extracted from the expansion of appropriate on-shell amplitudes around the threshold, where the expansion parameter is given by \( |\vec{r}| \sim \delta^{1/2} \). For the inclusive cross section there is no interference of the \( |\vec{r}| \)-suppressed operators with the leading one. Hence the corrections from higher-dimension operators begin at NLO, as explicitly proven in Appendix B.1 (however full results for the tree-level matching of the \( N^{1/2} \) LO production operators can be found in [61]). The NLO contribution to the inclusive cross section will be computed in Section 3.1.

The \( O(\alpha) \) correction to the matching coefficient of the production vertex (2.42), and of the corresponding operator with right-handed electrons, is determined from the renormalised scattering amplitudes for \( e^- e^+_L \rightarrow W^+W^- \) and \( e^- e^+_R \rightarrow W^+W^- \), computed at NLO in ordinary weak coupling perturbation theory, and at leading order in \( \delta \). This corresponds to evaluating the scattering amplitude directly at threshold, i.e. for the momentum configuration \( (p_1 + p_2)^2 = 4M_W^2 \). The NLO production operators read
\[
O_p^{(1)} = \frac{\pi\sin\theta}{M_W^2} \left[ C^{(1)}_{p,LR} \left( \bar{e}_{c_2,L} \gamma^{[\mu} j_{\nu]} e_{c_1,L} \right) + C^{(1)}_{p,RL} \left( \bar{e}_{c_2,R} \gamma^{[\mu} j_{\nu]} e_{c_1,R} \right) \right] \left( \Omega_{-i} \Gamma_{+i} \right). \]

The calculation of the coefficients \( C^{(1)}_{p,LR}, C^{(1)}_{p,RL} \) is discussed in Subsection 4.1.1. Note, however, that the one-loop correction \( C^{(1)}_{p,RL} \) does in fact not contribute to the NLO cross section, since there is no leading-order contribution from the \( e^- e^+_L \) helicity initial state, and no interference between LR and RL configurations.

**Matching coefficients of four-electron operators.** As discussed above, contributions from the non-resonant production-decay operators to the imaginary part of the
forward scattering amplitude arise first at $N^{1/2}$LO from two-loop cut diagrams (see Figure 3.3). The half-integer scaling arises from the absence of the threshold suppression $\sqrt{E/M_W} \sim \delta^{1/2}$ \footnote{This is analogous to the suppression of the production cross section of a pair of stable $W$ by the relative non-relativistic velocity $v$ of the pair.} present in the LO cross section, equation (2.48). The calculation of the cut 2-loop diagrams amounts to the calculation of the squared and phase-space integrated matrix element of the on-shell processes $e^- e^+ \rightarrow W^- u \bar{d}$ and $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu W^+$ in ordinary perturbation theory, with no resummation of self-energy insertions in internal $W$ propagators. This includes contributions of what are usually called double-resonant (or CC03) diagrams, but where one of the $W$ propagators is in fact off-shell, as well as genuine single-resonant processes. In the terminology of the method of regions, these corrections are given by the hard-hard part of the two-loop forward-scattering amplitude. Since all diagrams contributing to the tree-level scattering processes $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu W^+$ and $e^- e^+ \rightarrow W^- u \bar{d}$ are included in the calculation, the matching coefficients are clearly gauge invariant. The calculation is presented in Subsection 3.2.2 and details are given in Appendix B.2.

To NLO in the power counting $\alpha_s^2 \sim \alpha_{ew}$ also the NLO QCD correction to the process $e^- e^+ \rightarrow W^- u \bar{d} (+g)$ would have to be computed. The corrections to the “double-resonant” (CC03) diagrams can be taken into account approximately by multiplying them with the one-loop QCD correction to the hadronic decay width, given by the second term in equation (2.51), but the corrections to the single-resonant diagrams require a non-trivial calculation of three-loop diagrams. However, as we will find in Subsection 3.2.2, the contribution of the single-resonant diagrams to $e^- e^+ \rightarrow W^- u \bar{d}$ turns out to be numerically already small, so that the QCD corrections are negligible.

2.3.2 Loop calculations in the effective theory

The second class of $N^{1/2}$LO and NLO corrections in $\alpha$ and $\delta$ to the leading forward-scattering amplitude (2.46) arises from loop corrections in the effective theory. These are represented by:

One-loop diagrams with insertions of subleading operators. These contributions arise from evaluating the first term in (2.40) at one loop, but with one insertion of the subleading bilinear terms in the Lagrangian (2.39) (see Figure 2.7a), which correspond to kinetic energy and width corrections, or with production-operator products $\mathcal{O}^{(0)}_p \mathcal{O}^{(1)}_p$ and $\mathcal{O}^{(1/2)}_p \mathcal{O}^{(1/2)}_p$, where $\mathcal{O}^{(1)}_p$ is either a higher-dimension operator (2.53) or the one-loop correction (2.54), as in Figure 2.7b. These corrections coincide with the expansion by regions of the SM diagrams in Figure 2.7e and 2.7f, where the fermion loops are taken to be hard, the momentum in the $WW$ loop is potential and the additional loop in 2.7f is hard. In the calculation discussed further in Chapter 3 we actually follow this approach, and directly expand by regions the full forward-scattering amplitude. This yields the products of operators discussed above rather than individual production vertices.
Figure 2.7: NLO corrections from effective-theory loops. Upper line: correction from higher-order bilinear (a) and production-operator insertion (b), Coulomb correction (c) and contribution from soft-photon exchange (d). Lower line: corresponding Standard-Model diagrams.

**Coulomb corrections.** As explained below (2.39), a single insertion of the Coulomb potential interaction in the Lagrangian (2.39) contributes a $N^{1/2}\text{LO}$ correction to (2.46), due to the threshold enhancement $v^{-1} \sim \delta^{-1/2}$ of Coulomb-photon exchange. To NLO in $\alpha \sim \delta$ one has to calculate the single (Figure 2.7c) and double insertion into the leading-order amplitude for the production-operator product $O_p^{(0)}O_p^{(0)}$ and a single insertion for $O_p^{(0)}O_p^{(1/2)}$. The latter vanishes for the total cross section. Since there is no coupling of the potential photons to the collinear electrons and positrons, there are no Coulomb corrections to the four-fermion operators. The single Coulomb exchange coincides with the expansion of the SM diagram in Figure 2.7g, where the photon loop-momentum is potential. The calculation of the single and double Coulomb correction is presented in Section 4.2.

**NLO corrections from soft and collinear photons.** To NLO one also has to calculate two-loop diagrams in the effective theory arising from the coupling of the collinear modes and the potential $W$ bosons to the soft and collinear photons contained in the NRQED Lagrangian (2.34) and the SCET Lagrangian. Their imaginary parts correspond to one-loop virtual corrections and bremsstrahlung corrections to the leading-order cross section. One sample (soft) diagram is shown in Figure 2.7d. In the terminology of the method of regions these are contributions from four-loop cut diagrams with two hard fermion loops, one potential and one soft loop (Figure 2.7h). They represent “non-factorisable corrections”, and their calculation is discussed in Section 4.3.
Chapter 3

The four-fermion Born cross section

In this chapter we calculate all the $N^{1/2}\text{LO}$ and NLO corrections in the effective theory (EFT) listed in Section 2.3, except for those related to loop corrections, which will be added in Chapter 4. We also investigate the convergence of the successive EFT approximations to what is usually referred to as the Born four-fermion production cross section. Since the implementation of the $W$ width in the Born cross section is not unique, we define the “exact” Born cross section by the ten tree-level diagrams for $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu \bar{u}d$ shown in Figure 1.3, where the $W$ propagators are regularised by a fixed-width prescription, $i/(k^2 - M_W^2) \rightarrow i/(k^2 - M_W^2 + iM_W\Gamma_W)$. The EFT calculation is done by expanding directly the forward-scattering amplitude in $\delta$, according to the method of regions (see Appendix A), rather than computing the matrix elements in equation (2.40). The relevant loop-momentum regions are either all hard, or hard and potential. The contribution of the latter regions coincides with the matrix element of higher-dimension production operators $O^{(1/2)}_p$ and $O^{(1)}_p$, and with corrections from subleading terms in the PNRQED Lagrangian, while the all-hard contributions correspond to the matching coefficient and matrix element of the four-electron operators, equation (2.49).

3.1 Contributions from the potential region

We first reconsider the one-loop diagrams (before cutting) shown in Figure 2.6, where the loop momentum is now assumed to be potential, $k^2 - M_W^2 \sim M_W\Gamma_W$. The contribution of these diagrams to the forward-scattering amplitude may be written as

$$iA = \sum_{c_a,c_b} \int \frac{d^4r}{(2\pi)^2} [iM_{eeWW,c_a}^{\mu\nu}(E,r)P_{\mu\nu'}(k_1)P_{\nu\nu'}(k_2)][iM_{eeWW,c_b}^{\mu\nu}(E,r)],$$

where $E = \sqrt{s} - 2M_W$ is the total non-relativistic kinetic energy of the $WW$ system, $k_1 = M_Wv + r$ and $k_2 = P - M_Wv - r$, with $v^\mu = (1, \vec{0})$, are the four-momenta of the $W$s and $P = p_1 + p_2$ the sum of the initial-state momenta. $M_{eeWW,c_a}^{\mu\nu}$ is the tree-level matrix
element for the off-shell subprocess $e^-(p_1)e^+(p_2) \rightarrow W^-(k_1)W^+(k_2)$ in the production channel $c_a$ (in this case $t$ or $s$-channel), and

$$P^{\mu\nu}(k) = \frac{i \left(-g^{\mu\nu} + \frac{k_\mu k'_\nu}{k^2}\right)}{k^2 - M_W^2 - \Pi_T^W(k^2)}$$  \hspace{1cm} (3.2)$$

is the full renormalised transverse $W$ propagator \(^1\). Writing the amplitude in the full theory with a resummed propagator is contrary to the spirit of an effective field theory calculation, where the matching coefficients are obtained by fixed-order calculations. However, this allows us to compare the EFT expansion with the standard calculation of the fixed-width Born cross section. Furthermore, any gauge-invariance violating term possibly included in (3.2) is discarded upon expansion of the resummed amplitude, since the expansion by regions automatically select a subset of corrections with homogeneous scaling in $\delta$ (and only those). By introducing the quantity

$$\Phi(E, r) = -\sum_{c_a, c_b} [\mathcal{M}_{\mu
u}^{eeWW, c_a, c_b}, \mathcal{M}_{\mu
u'}^{eeWW, c_a, c_b}](E, r) \left(-g_{\mu\nu} + \frac{k_1\mu k_1\nu'}{k_1^2}\right) \left(-g_{\mu'\nu'} + \frac{k_2\mu k_2\nu'}{k_2^2}\right)$$  \hspace{1cm} (3.3)$$

we recast equation (3.1) in the simple and compact form

$$iA = \int \frac{d^d r}{(2\pi)^d} \Phi(E, r) P(k_1) P(k_2),$$  \hspace{1cm} (3.4)$$

where $P(k)$ is the scalar $W$ propagator

$$P(k) = \frac{i}{k^2 - M_W^2 - \Pi_T^W(k^2)}.$$  \hspace{1cm} (3.5)$$

### 3.1.1 Threshold expansion of the resummed propagator

To see the correspondence between the direct expansion of the amplitude (3.4) and the EFT calculation, we insert in (3.5) the parametrisation $k^\mu = M_W v^\mu + r^\mu$ of the $W$ momentum given above, and expand $P(k)$ in $\delta$, remembering that the residual potential momentum $r^\mu$ scales as $r_0 \sim M_W \delta$, $\vec{r} \sim M_W \delta^{1/2}$. This includes an expansion of the self-energy around $M_W^2$ and in the number of loops,

$$\Pi_T^W(k^2) = M_W^2 \sum_{m,n} \delta^n \Pi^{(m,n)};$$  \hspace{1cm} (3.6)$$

with $\delta = (k^2 - M_W^2)/M_W^2$, and $m$ denoting the loop order. The result of the expansion of (3.5) is

$$P(r) = \frac{i(1 + \Pi^{(1,1)})}{2M_W \left(r_0 - \frac{r^2}{2M_W} - \frac{\Delta^{(1)}}{2}\right)} - \frac{i(r^2_0 - M_W \Delta^{(1)})}{4M_W^2 \left(r_0 - \frac{r^2}{2M_W} - \frac{\Delta^{(1)}}{2}\right)^2} + O\left(\frac{\delta}{M_W^2}\right),$$  \hspace{1cm} (3.7)$$

\(^1\)The longitudinal part of the propagator is cancelled by the transverse projector from the decay into massless fermions.

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where, to make the notation simpler, the QCD correction $\Delta^{(3/2)}$ from (2.52) has been included into the matching coefficient $\Delta^{[1]} = \Delta^{(1)} + \Delta^{(3/2)}$ instead of being expanded, and $\Delta^{(2)} = M_W (\Pi^{(2,0)} + \Pi^{(1,1)} \Pi^{(1,0)})$ (see (2.15)). $r_0$ can be eliminated from the numerator in (3.7) by completing the square,

$$r_0^2 = \left( r_0 - \frac{r^2}{2M_W} - \frac{\Delta^{[1]}}{2} \right)^2 + 2 \left( \frac{r^2}{2M_W} + \frac{\Delta^{[1]}}{2} \right) \left( r_0 - \frac{r^2}{2M_W} - \frac{\Delta^{[1]}}{2} \right)$$

$$+ \left( \frac{r^2}{2M_W} + \frac{\Delta^{[1]}}{2} \right)^2. \quad (3.8)$$

Thus (3.7) can be written as

$$P(r) = \frac{i}{2M_W} \left( r_0 - \frac{r^2}{2M_W} - \frac{\Delta^{[1]}}{2} \right) \left( 1 + \Pi^{(1,1)} - \frac{M_W \Delta^{[1]} + r^2}{2M_W^2} \right)$$

$$- \frac{i}{4M_W^2} \left( \frac{r^2}{2M_W} + \frac{\Delta^{[1]}}{2} \right)^2 - \frac{M_W \Delta^{(2)}}{2M_W^2} + O\left( \frac{\delta M_W^2}{M_W^2} \right). \quad (3.9)$$

Each individual term can be given a clear interpretation in the EFT formalism. The first term in the second line corresponds to an insertion into a $W$ line of the NLO kinetic-energy correction and second-order width correction to the non-relativistic Lagrangian (2.39). The second term, $-i/(4M_W^2)$, is analogous to the corresponding term in single-resonance production discussed in Subsection 2.1.2, where it was included in the tree-level matching coefficient of a four-fermion effective vertex. Here this term leads to potential loop integrals with only one or no non-relativistic $W$ propagator, which vanish in dimensional regularisation. Thus we can freely drop it. In the first line of (3.9) we recognise the non-relativistic $W$ propagator (2.38) multiplied by a correction to the residue, corresponding to the expansion around $k^2 = M_W^2$ of the field normalisation factor $\varpi$ defined in (2.41), and of the hard part of the residue of the full propagator (3.5), $R_{hW} = 1 + \Pi^{(1,1)} + ... [50]$. In an EFT calculation these residue corrections enter the matching relation (2.30) of the one-loop and higher-dimension production and decay vertices, as explained in Subsection 2.1.3. In order to compare the effective-theory prediction with the “exact” Born cross section, where these terms are included, we keep these residue corrections here rather than including them in the matching calculation of Section 4.1.

The real part of $\Pi^{(1,1)}$ depends on the $W$ field-renormalisation convention chosen in the full theory. Here and in the following we adopt the on-shell scheme for field renormalisation, $\text{Re} \, \Pi^{(1,1)} = 0$, and the pole scheme for mass renormalisation, which implies $\Delta = -i \Gamma_W$ (see definition 2.37). At one-loop the imaginary part of $\Pi^W_T (k^2)$ is determined by the $W$ decay into massless fermions, and reads $\text{Im} \, \Pi^W_T (k^2) = -k^2 \Gamma_W (0) / M_W \theta (k^2)$.

2In dimensional regularisation soft loop contributions to the full renormalised self-energy $\Pi^W_T (k^2)$ vanish, and the residue of the full propagator $R_W$ coincides with its hard part, $R_W = R_{hW}$. 37
Therefore, from \( \text{Re} \Pi^{(1,1)} = 0 \) it follows that \( \Pi^{(1,1)} = i\partial [\text{Im} \Pi_W^{(2)}(k^2)]/\partial k^2 = -i\Gamma_W^{(0)}/M_W \). Furthermore, in the pole mass renormalisation scheme, \( \Delta^{(1)} = M_W \Pi^{(1,0)} \equiv -i\Gamma_W^{(0)} \) and \( \Delta^{(2)} = M_W (\Pi^{(2,0)} + \Pi^{1(1)} \Pi^{(0,1)}) \equiv -i\Gamma_W^{(1)} \), which implies \( \text{Re} \Pi^{(2,0)} = -\text{Re}[\Pi^{(1,1)} \Pi^{(0,1)}] = (\Gamma_W^{(0)}/M_W)^2, \text{Im} \Pi^{(2,0)} = -\Gamma_W^{(1)}/M_W \). The QCD correction \( \Delta^{(3/2)} = -i\Gamma_W^{(1/2)} \) can be included into \(-i\Gamma_W^{(0)} \) as before.

### 3.1.2 Relation between the effective-theory approximation and the fixed-width prescription

To understand how the fixed-width implementation of the Born cross section and the EFT result are related, we write the resummed propagator (3.5) (in the renormalisation scheme just defined) in the following form

\[
P(k) = i \left( \frac{k^2 - M_W^2 - \Gamma_W^{(0)} - iM_W}{k^2 - M_W^2 - \Gamma_W^{(0)}} \right)^2 + M_W^2 \left( \frac{k^2 \Gamma_W^{(0)} - \Gamma_W^{(1)}}{k^2 - M_W^2} \right)^2 + O \left( \frac{\delta}{M_W^2} \right). \tag{3.10}
\]

The fixed-width prescription in the Born calculation corresponds to replacing \( k^2 \Gamma_W^{(0)}/M_W^2 \) by \( \Gamma_W^{(0)} \) in the denominator, but not in the numerator, where the term \( k^2 \) arises from explicitly integrating over the two-particle phase space of the \( W \) decay products. In addition one drops the \( \Gamma_W^{(0)} \) terms, which come from \( \text{Re} \Pi^{(2,0)} \), and \( \Gamma_W^{(1)} \), since they both originate from the expansion of the two-loop renormalised self-energy \( \Pi_W^{(2)}(k^2) \), and are not included in a Born calculation. We thus have

\[
P(k)_{\text{fixed-width}} = i \left( \frac{k^2 - M_W^2 - iM_W}{k^2 - M_W^2} \right)^2 + M_W^2 \left( \frac{k^2 \Gamma_W^{(0)} - \Gamma_W^{(1)}}{k^2 - M_W^2} \right)^2 + O \left( \frac{\delta}{M_W^2} \right). \tag{3.11}
\]

Repeating the derivation of the expansion (3.9) with the modified expression we obtain

\[
P(k)_{\text{fixed-width}} = \left[ \text{Eq. (3.9) with } \Delta^{(2)} = -i\Gamma_W^{(1)} \to 0 \right]
+ \frac{\Gamma_W^{(0)}/M_W^2}{4M_W^2 \left( r_0 - \frac{r_0^2}{2M_W} + \frac{\Gamma_W^{(0)}}{4} \right)} + O \left( \frac{\delta}{M_W^2} \right). \tag{3.12}
\]

The additional term is purely real and does not contribute to the cut propagator \( \text{Im} P(k) \) relevant to the cross-section calculation. We therefore arrive at the interesting conclusion that, if \( M_W \) is the pole mass, the fixed-width prescription coincides with the EFT approximation in the potential region up to the next-to-leading order (if we exclude a trivial term related to the one-loop correction \( \Gamma_W^{(1)} \) to the pole scheme decay width).
3.1.3 Expansion of the production squared matrix elements

To calculate the NLO corrections to the forward-scattering amplitude in the potential region, we replace the two propagators in (3.4) with the expansion (3.9), dropping all the terms that are beyond NLO. In this way we already account for all the NLO terms in the effective Lagrangian (2.39) not associated with loop corrections, and for some contributions from higher-dimension production operators with tree-level short-distance coefficients, encoded in the residue factor $\varpi R = 1 + \Pi^{(1,1)}(1) - (M_W^2 \Delta^{(1)} + \bar{r}^2)/(2M_W^2) + \ldots$. Other NLO corrections come from the expansion of the squared matrix element $\Phi(E, r)$, and correspond to contributions of higher-dimensional production operators to the first matrix element in (2.40). The square of the production amplitude of two off-shell $W$ bosons depends on four kinematic invariants, that may be chosen to be $r^2$, $p_1 \cdot r$, $k_1^2 - M_W^2$, and $k_2^2 - M_W^2$. This choice is convenient, since all four invariants are small with respect to $M_W^2$ in the potential region. In the expansion of $\Phi(E, r)$ to NLO, we may further approximate $r^2$ by $-\bar{r}^2$, since $r_0 \sim r^2/M_W \ll |\bar{r}|$. The detailed calculation of the expansion of $\Phi(E, r)$ is given in Appendix B.1. The helicity combinations $e^-_L e^+_L$ and $e^-_R e^+_R$ vanish for massless incoming fermions. For the $e^-_L e^+_R$ and $e^-_R e^+_L$ helicity initial states we find

$$
\Phi_{LR}(E, r) = -64\pi^2\alpha^2_{\text{ew}} \left[ 1 + \frac{11}{6} + 2\xi^2(s) + \frac{38}{9}\xi(s) \right] \frac{\bar{r}^2}{M_W^2} + O(\delta^2),
$$

$$
\Phi_{RL}(E, r) = -128\pi^2\alpha^2_{\text{ew}} \chi^2(s) \frac{\bar{r}^2}{M_W^2} + O(\delta^2). 
\quad (3.13)
$$

To obtain 3.13 we have exploited the fact that $P(k)$ does not depend on the direction of $\vec{r}$ to average $\Phi$ over the solid angle $\Omega$ (see Appendix B.1),

$$
\int d\Omega P(k_1)P(k_2)\Phi(E, r) = P(k_1)P(k_2) \int d\Omega \Phi(E, r). \quad (3.14)
$$

The functions

$$
\xi(s) = -\frac{3M_W^2(s - 2M_Z^2s_w^2)}{s(s - M_Z^2)}, \quad \chi(s) = \frac{6M_W^2M_Z^2s_w^2}{s(s - M_Z^2)} 
\quad (3.15)
$$

originate from linear combinations of the $s$-channel photon and $Z$-boson propagators, with coefficients given by the couplings of the electron to the electroweak neutral gauge bosons. The NLO terms proportional to $\bar{r}^2$ can be identified with products of tree-level production operators $O^{(0)}_p O^{(1)}_p$ and $O^{(1)}_p O^{(1)}_p$. In the effective-theory language $\xi(s)$ and $\chi(s)$ enter the short-distance matching coefficients of the operators $O^{(1/2)}_p$ and $O^{(1)}$, and they would be naturally evaluated at $s = 4M_W^2$. Here we keep the exact $s$-dependence, since this can be done at no calculational cost. No $N^{1/2}$LO contribution to the total cross section arises from the operator product $O^{(0)}_p O^{(1)}_p$, since this correction vanishes upon the angular integration (3.14).

The coefficient functions of production operators in the EFT are determined by on-shell matching, which implies an expansion of the amplitude around the complex pole position $\bar{s} = M_W^2 + M_W \Delta$ rather than $M_W^2$ [32,33]. The difference cannot be neglected.
in a NLO calculation. In principle the expansions (3.13) could have yielded terms such as \( k_1^2 - \bar{\Delta} \), which should be written as \( k_1^2 - \bar{s} + M_W \Delta \). The difference \( k_1^2 - \bar{s} \) cancels a resonant propagator, possibly giving rise to a production-decay operator matching coefficient, while the remaining \( M_W \Delta \) term must be combined with other contributions to the loop correction to the leading-order production vertex. This complication can be ignored here, since the expansion of \( \Phi(E, r) \) is independent of \( k_1^2 - \bar{s} \) up to NLO, as explicitly proven in Appendix B.1.

### 3.1.4 NLO potential contributions to the Born cross section

We now insert the expansions (3.9), (3.13) given in the previous subsections into (3.4) and perform the loop integration. The integral corresponding to a generic term in the expansions (3.9) or (3.13) has the expression (after performing an appropriate shift of the loop momentum)

\[
I(\epsilon; n_1, n_2) = \int \frac{d^{3-2\epsilon} r}{(2\pi)^{3-2\epsilon}} \left( r_0 - \frac{r^2}{2M_W} + i\frac{\Gamma(0)}{2} \right)^{1+n_2} \left( E - r_0 - \frac{r^2}{2M_W} + i\frac{\Gamma(0)}{2} \right)^{1/2-n_1-n_2},
\]

with \( n_1 = 0, 1, 2 \) and \( n_2 = 0, 1 \). In general, \( I(\epsilon; n_1, n_2) \) has odd-power divergences, that are however rendered finite by dimensional regularisation, as already seen for the LO amplitude. Taking the limit \( \epsilon \to 0 \) after the integration, we obtain

\[
\lim_{\epsilon \to 0} I(\epsilon; n_1, n_2) = \frac{(-1)^{n_2} \Gamma(3/2 + n_1) \Gamma(n_2 - n_1 - 1/2)}{2\Gamma(1 + n_2)} \frac{1}{2} M_W^{2+2n_1-n_2} \left( \frac{-E + i\Gamma(0)}{M_W} \right)^{1/2+n_1-n_2}.
\]

(3.17)

The LO cross section has already been given in (2.48). The NLO potential contributions are easily computed using the result (3.17):

\[
\sigma^{(1)}_{LR,\text{Born}} = \frac{4\pi\alpha^2}{27s_{\text{lab}}^4 s} \left\{ \left( \frac{11}{6} + 2\xi^2(s) + \frac{38}{9} \xi(s) \right) \mathrm{Im} \left[ \left( \frac{-E + i\Gamma_W(0)}{M_W} \right)^{3/2} \right] \right\}
\]

\[
+ \mathrm{Im} \left[ \left( \frac{3E}{8M_W} + \frac{17 i\Gamma_W(0)}{8M_W} \right) \sqrt{-E + i\Gamma_W(0)} \right] + \left. \left( \frac{\Gamma_W(0)^2}{8M_W} - \frac{i\Gamma_W(1)}{2M_W} \right) \sqrt{-M_W} \right] \frac{E}{i\Gamma_W(0)} \mathrm{Im} \left[ \left( \frac{M_W}{E} \right)^{3/2} \right],
\]

\[
\sigma^{(1)}_{RL,\text{Born}} = \frac{8\pi\alpha^2}{27s_{\text{lab}}^4 s} \lambda^2(s) \mathrm{Im} \left[ \left( \frac{-E + i\Gamma_W(0)}{M_W} \right)^{3/2} \right].
\]

(3.18)

Since \( E/M_W \sim \Gamma_W(0)/M_W \sim \delta \) and \( \Gamma_W^{(1)}/M_W \sim \delta^2 \) every term is suppressed by \( \delta \) relative to the leading order as it should be. The unpolarised cross section is one fourth of the sum of
the LR and RL contributions. The factor $1/27$ comes from the tree-level branching ratio for the final state $\mu^- \bar{\nu}_\mu \mu u \bar{d}$ in the conversion from the forward-scattering amplitude to the partial cross section. As discussed above, when we use this expression to compare with the standard Born cross section in the fixed-width scheme, we set $\Gamma^{(1)}_{W}$ to zero. When we use the expression (3.18) in the complete NLO calculation including radiative corrections, we have to keep in mind that multiplying all terms by the product $1/27$ of leading-order branching ratios as in (3.18) is actually not correct. The required modification is discussed in Section 4.1.2.

The relative effect of (3.18) on the total unpolarised cross section is shown in Figure 3.1, where we plot separately the corrections arising from the expansion of the squared matrix element, equation (3.13), and of the propagators, equation (3.9). At $\sqrt{s} = 170$ GeV the two corrections amount to $\sim 3\%$ each, while below threshold the total correction is dominated by the contribution from the expansion of the propagator, that grows up to $20\%$ of the leading-order term at $\sqrt{s} = 158$ GeV.

Beside the $\delta$-suppressed terms from the potential region of the one-loop diagrams shown in Figure 2.6, another NLO contribution could arise from the leading terms in the expansion of two-loop diagrams with one hard and one potential loop, which may be also associated with the Born cross section. An example is displayed in Figure 3.2. Cut (1) does not correspond to a four-fermion final state and must be dropped. Cut (3) corresponds to the interference of the tree-level production operator with a hard one-loop correction to the production operator. Since the $s$-channel diagrams do not contribute to the $O(\alpha\delta^0)$ production operator, as shown later in Subsection 4.1.1, this cut is beyond NLO. Cut (2) is a contribution to what is usually termed the “Born cross section”, and corresponds

![Figure 3.1: Relative corrections to the leading-order cross section (2.48) from NLO potential contributions: propagator correction (dotted red), matrix-element correction (dashed blue) and total NLO potential correction (solid green).](image-url)
to the interference of single and double-resonant diagrams in the kinematic region where both fermion pairs have invariant mass of order $M^2_W$. The contribution from this cut is contained in the imaginary parts of the hard one-loop correction to the production operators. However the threshold suppression of the s-channel diagrams discussed before equation (2.42) applies also here, and this contribution is also not relevant at NLO, and will not be considered further here.

3.2 Contributions from the hard region

We now consider the hard contributions to the Born cross section, which in the effective-theory description determine the matching coefficients of the four-electron production-decay operators (2.49). As for the potential contributions, we extract these corrections from the threshold expansion of full Standard-Model diagrams, where the loop momenta are now taken to be all hard, $k^2 \sim M^2_W$. These diagrams are to be calculated in standard perturbation theory with no width added to the $W$ propagator, since the $W$’s are far off-shell. In the hard region it is actually simpler to calculate the four-fermion cross section directly as the sum over the relevant cuts of the forward-scattering amplitude, which are given by the one-loop cut diagrams in Figure 2.6 and two-loop cut diagrams in Figure 3.3. Note that this calculation implies cutting $W$ lines as well as diagrams with self-energy insertions into the $W$ propagator. This can be interpreted as an expansion of the resummed propagator in powers of $\Gamma_W/M_W^3$, with the complication that the terms of the expansion are distributions rather than functions \cite{67, 68}:

\[
\frac{M_W \Gamma_W}{(k^2 - M_W^2)^2 + M_W^2 \Gamma_W} = \pi \delta(k^2 - M_W^2) + \text{PV} \frac{M_W \Gamma_W}{(k^2 - M_W^2)^2} + O\left(\frac{\delta^2}{M_W^2}\right). \quad (3.19)
\]

“PV” denotes the principal value. The left-hand side arises from cutting fermion-loops, but not the $W$ lines itself, into a string of self-energy insertions \cite{65}. But the leading term in the expansion of this expression, equivalent to the narrow-width approximation, looks as if a $W$ line with no self-energy insertions is cut, as displayed in Figure 3.3.

\footnote{This is analogous to the expansion (3.9), with the difference that now $k^2 - M_W^2 \sim M_W^2$, rather than $k^2 - M_W^2 \sim M_W^3$.}
3.2.1 Vanishing of the leading-order hard correction

As already discussed in Subsection 2.2.1, the one-loop diagrams in Figure 2.6 do not provide imaginary parts of the forward-scattering amplitude. We now show this explicitly. In complete analogy with equation (3.1), we can write the contribution of the four diagrams (before cutting) to the forward-scattering amplitude as

\[
iA = \sum_{c_a,c_b} \int \frac{d^dk_1}{(2\pi)^d} iM^{\mu\nu}_{eeWW,c_a}(k_1,k_2) \frac{i(-g_{\mu\nu'} + \frac{k_{1\mu}k_{1\nu'}}{M^2_W})}{k_1^2 - M^2_W + i\epsilon} \times \frac{i(-g_{\nu\nu'} + \frac{k_{2\nu}k_{2\nu'}}{M^2_W})}{k_2^2 - M^2_W + i\epsilon} [iM'^{\mu\nu'}_{eeWW,c_b}](k_1,k_2),
\]

where \(k_1\) and \(k_2 = P - k_1\) are the momenta of the two \(W\) lines, and, as before, \(c_a, c_b\) denote different production channels. Note that in (3.20) the resummed transverse propagator (3.2) is replaced by the fixed-order \(W\) propagator, chosen for simplicity in the unitary gauge to avoid diagrams with unphysical degrees of freedom. Using Cutkosky rules [64,65] to extract the imaginary part of (3.20), we obtain

\[
2\text{Im}A = \int \frac{d^dk_1}{(2\pi)^d} \Phi^{\text{hard}}(k_1,k_2) \delta(k_1^2 - M^2_W) \delta(k_1^0) \delta(k_2^2 - M^2_W) \delta(k_2^0),
\]

where the function \(\Phi^{\text{hard}}(k_1,k_2)\) is defined, analogously to (3.3), as

\[
\Phi^{\text{hard}}(k_1,k_2) = \sum_{c_a,c_b} [M^{\mu\nu}_{eeWW,c_a}M'^{\mu\nu'}_{eeWW,c_b}](k_1,k_2) \left(-g_{\mu\nu'} + \frac{k_{1\mu}k_{1\nu'}}{M^2_W}\right) \left(-g_{\nu\nu'} + \frac{k_{2\nu}k_{2\nu'}}{M^2_W}\right),
\]

and the two terms \(2\pi\delta(k_1^2 - M^2_W)\delta(k_1^0)\delta(k_2^2 - M^2_W)\delta(k_2^0)\) originate from cutting the two \(W\) lines. (3.21) has to be expanded in \(E = \sqrt{s} - 2M_W\), which is the only quantity that is small near threshold for hard diagrams, since \(k_1^2 - M^2_W \sim k_2^2 - M^2_W \sim M^2_W\). The expansion in \(E\) produces derivatives of distributions, such as \(\frac{d^k}{d\kappa^k} \delta(k^2 - M_W^2)\), but these are converted to ordinary derivatives of the function \(\Phi^{\text{hard}}\) using the definition

\[
\int \left[\frac{d^k}{d\kappa^k} \delta(x)\right] f(x) = (-1)^k \int \delta(x) \frac{d^k f(x)}{d\kappa^k}.
\]

Hence the threshold expansion of (3.21) reads

\[
2\text{Im}A = \sum_n E^n \int \frac{d^dk_1}{(2\pi)^d} \Phi^{\text{hard},(n)}(k_1,\tilde{k}_2) \delta(k_1^2 - M^2_W) \delta(\tilde{k}_2^2 - M^2_W) \theta(\tilde{k}_2^0),
\]

where \(\tilde{k}_2 = 2M_W v - k_1\), and \(\Phi^{\text{hard},(n)}\) includes all the terms scaling as \(E^n\), including the ones originating from derivatives of \(\delta(k_2^2 - M^2_W)\). The product of \(\delta\)-functions and step functions \(\theta\) in (3.24) can be recast into the simple form

\[
\delta(k_1^2 - M^2_W) \delta(k_1^0) \delta(\tilde{k}_2^2 - M^2_W) \theta(\tilde{k}_2^0) = \frac{1}{8M_W|\tilde{k}_1|} \delta(k_1^0 - M_W) \delta(|\tilde{k}_1|),
\]

43
with the result that expression (3.24) reduces to

$$2\text{Im} \mathcal{A} = \frac{1}{8M_W(2\pi)^2} \sum_n \lim_{|k_1| \to 0} E^n |k_1|^{1-2\epsilon} \tilde{\Phi}^{\text{hard},(n)}(|k_1|).$$  \hspace{1cm} (3.26)$$

Here \( \tilde{\Phi}^{\text{hard},(n)}(|k_1|) \) represents the original function \( \Phi^{\text{hard},(n)}(k_1^0 = M_W, |k_1|, \Omega) \) integrated over the total solid angle \( \Omega \). In dimensional regularisation \( \lim_{\epsilon \to 0} x^\lambda(\epsilon) = 0 \), as long as \( \lambda \) depends on \( \epsilon \) in a way that allows one to analytically continue \( \lambda \) to positive values [69]. This is in particular true when \( \lambda = a - 2\epsilon \), where \( a \) is any real number. For \( |k_1| \to 0 \) one can easily show that \( \tilde{\Phi}^{\text{hard},(n)}(|k_1|) \propto |k_1|^m \), for some integer \( m \). We therefore conclude that (3.26) vanishes, and the cut diagrams in Figure 2.6 do not contribute to \( \text{Im} \mathcal{A} \) at any order in \( E \).

### 3.2.2 \( N^{1/2} \text{LO} \) contribution from the hard region

The leading hard contributions originate from the two-loop diagrams in Figure 3.3. The result must be of order \( \alpha^3 \), which results in a \( N^{1/2} \text{LO} \) correction relative to the leading-order cross section, that scales as \( \alpha^2 \delta^{1/2} \). Higher-order contributions from the hard region come from higher-order terms in the expansion in \( E = \sqrt{s} - 2M_W \) near threshold, and from diagrams with more hard loops, all of which count as \( N^{3/2} \text{LO} \) or smaller. The calculation of the cuts in Figure 3.3 is straightforward but lengthy, and is presented in Appendix B.2. Here we only remark that, as in the calculation of the potential contributions in Section 3.1, extra divergences arise as a consequence of factorising hard and potential regions in the threshold expansion. As in the potential region, the integrals are performed in dimensional regularisation, and analytically continued to finite values, since the divergences are odd power divergences. After taking the limit \( \epsilon \to 0 \), the result for the non-vanishing helicity combinations LR and RL can be written as

\[
\sigma^{(1/2)}_{LR,\text{Born}} = \frac{4\alpha^3}{27\sqrt{s}} \left[ K_{h1} + K_{h2} \xi(s) + K_{h3} \xi^2(s) + \sum_{i=4}^{b7} \sum_{f} C^{f}_{i,LR}(s) K^{f}_{i} \right],
\]

\[
\sigma^{(1/2)}_{RL,\text{Born}} = \frac{4\alpha^3}{27\sqrt{s}} \left[ K_{h3} \chi^2(s) + \sum_{i=4}^{b7} \sum_{f} C^{f}_{i,RL}(s) K^{f}_{i} \right],
\]

(3.27)

where the functions \( \xi(s) \) and \( \chi(s) \) are the same introduced in Section 3.1. Here the first sum extends over the diagrams as labelled in Figure 3.3, the second over the fermions \( f \in u, d, \mu, \nu_\mu \) in the internal fermion loops. The explicit values of the coefficients arising from the diagrams h1-h3 are

\[
K_{h1} = -2.35493, \quad K_{h2} = 3.86286, \quad K_{h3} = 1.88122.
\]

(3.28)

The three coefficients contain the contribution of the diagrams h1-h3 shown in Figure 3.3 and of the symmetric diagrams with self-energy insertions on the lower \( W \) line. \( K_{h2} \) contains also the contribution of the complex conjugate of \( h2 \). The explicit expressions
of the coefficients \(K_f^i\) and \(C_{i,h}^f\), with \(h = LR, RL\), for the diagrams \(h4-h7\) are given in Appendix B.2. Similarly to (3.15) the \(s\)-dependence of the \(C_{i,h}^f\) arises trivially from photon and \(Z\) propagators, and we could simply set \(s = 4M_W^2\) at \(N^{1/2}\)LO. Since all other terms in (3.27) are energy-independent, we conclude that the leading hard contribution results in a (almost) constant \(N^{1/2}\)LO shift of the cross section.

The contribution (3.27) can be interpreted as arising from a final state where one fermion pair originates from a nearly on-shell \(W\) decay, while the other is produced non-resonantly, either from a highly virtual \(W\), or as in the truly single-resonant diagrams \(h4-h7\). An explicit numerical evaluation of (3.27) reveals that the contribution from \(h4-h7\) is rather small, below 0.5% of the full tree cross section in the energy range \(\sqrt{s} = 155\) GeV and 180 GeV. The smallness of the true single-resonant contributions is in part due to large cancellations between the diagrams \(h4\) and \(h5\). Note that the cuts shown in Figure 3.3 correspond to all the possible interferences of tree-level diagrams for the process \(e^-e^+ \rightarrow W^-ud\) or \(e^-e^+ \rightarrow W^+ \mu^- \bar{\nu}_\mu\). Since no resummation of the \(W\) self-energy is performed in the hard region, we conclude that the result (3.27) is gauge invariant.

The comparison with the Born cross section performed below shows that the region of validity of the EFT expansion is significantly enlarged if the energy-dependent \(N^{3/2}\)LO terms are included. These can only arise from the next-to-leading order terms of the expansion in the hard region (the expansion in the potential region produces only integer-power corrections in \(\delta\)). The energy-dependent terms are related to the next order in the threshold expansion of the cut diagrams in Figure 3.3. The computation for the

Figure 3.3: Two-loop cut diagrams. Symmetric diagrams are not shown.
numerically dominant diagrams h1-h3 gives (see Appendix B.2)

\[ \sigma^{(3/2),a}_{LR,\text{Born}} = \frac{4\alpha^3E}{27s^6WsM_W} [K_{h1}^a + K_{h2}^a \xi(s) + K_{h3}^a \xi^2(s)], \]

\[ \sigma^{(3/2),a}_{RL,\text{Born}} = \frac{4\alpha^3E}{27s^6WsM_W} K_{h3}^a \deltachi^2(s), \]

(3.29)

where

\[ K_{h1}^a = -5.87912, \quad K_{h2}^a = -19.15095, \quad K_{h3}^a = -6.18662. \]

(3.30)

Note that the quantity \( E/M_W \), which is formally of order \( \Gamma_W/M_W \sim 0.025 \), can be numerically quite larger than this value. For example, at \( \sqrt{s} = 170 \text{ GeV}, E/M_W \sim 0.11 \).

Other \( N^{3/2}\)LO corrections related to the Born cross section arise from cut three-loop diagrams of the type h1-h3, but with two self-energy insertions (two on the same W line or one for each line), and of type h4-h7 with one insertion. This \( N^{3/2}\)LO term is (almost) energy-independent and can be parameterised by

\[ \sigma^{(3/2),b}_{h,\text{Born}} = \frac{4\alpha^4}{27s^8Ws} \sum_{i=h1}^{h3} C^b_{i,h}(s) K^b_i. \]

(3.31)

The coefficients \( C^b_{i,h}(s) \) are equal to the factors multiplying \( K^a_{hi} \) in (3.29) and we omitted the small contributions from h4-h7. The calculation of the numerical coefficients \( K^b_i \) is non-trivial, since it contains products of distributions arising from the expansion of the W propagators as given in (3.19). A rough estimate of these corrections is \( \sigma^{(3/2),b}_{h,\text{Born}} \sim \sigma^{(1/2)}_{h,\text{Born}} \Gamma_W^0/M_W \sim 0.025 \sigma^{(1/2)}_{h,\text{Born}} \), resulting in an energy-independent contribution to the cross section of order 2 fb. The comparison below suggests that it is actually significantly smaller.
Figure 3.4: Successive EFT approximations: LO (long-dashed/blue), $N^{1/2}\text{LO}$ (dash-dotted/red) and NLO (short-dashed/green). The solid/black curve is the full Born result computed with WHIZARD/CompHep. The $N^{3/2}\text{LO}$ EFT approximation is indistinguishable from the full Born result on the scale of this plot.

3.3 Comparison of the EFT prediction to the four-fermion Born cross section

In this section we compare successive EFT approximations to the four-fermion Born cross section in the fixed-width scheme. Here we only give results for the unpolarised cross section given by $(\sigma_{LR} + \sigma_{RL})/4$. The relevant EFT contribution to the Born cross section were given in (2.48), (3.18), (3.27), and (3.29). Here and in the following Chapters we adopt the $G_\mu$-scheme definition of the fine-structure constant, $\alpha \equiv \sqrt{2} G_\mu M_W^3 s^2_w / \pi$, and the on-shell Weinberg angle $c_w \equiv \cos \theta_w = M_W / M_Z$. We therefore choose our independent input parameters to be [3]

$$\hat{M}_W = 80.403 \text{ GeV}, \quad M_Z = 91.188 \text{ GeV}, \quad G_\mu = 1.16637 \cdot 10^{-5} \text{ GeV}^{-2}, \quad$$

(3.32)

where $\hat{M}_W$ is the experimentally measured *on-shell* $W$ mass, which is related to the pole mass $M_W$ through the relation (valid to $O(\Gamma_W^2)$)

$$\hat{M}_W = M_W + \frac{\Gamma_W^2}{2M_W}, \quad$$

(3.33)

with the tree-level $W$ decay width given by

$$\Gamma_W = \frac{3}{4} \frac{G_\mu}{s^2_w} M_W = \frac{3G_\mu M_W^3}{2\sqrt{2}\pi}. \quad$$

(3.34)
Inserting (3.34) into (3.33), and solving the equation for $M_W$, we get the following pole parameters:

$$M_W = 80.377 \text{ GeV}, \quad \Gamma_W = 2.04483 \text{ GeV}. \quad (3.35)$$

The value of the $W$ width used here is the leading-order result (2.50), excluding one-loop QCD and electroweak corrections. This is appropriate for a tree-level calculation and ensures that the branching ratios add up to one. Correspondingly we set $\Delta^{(2)} = 0$ in the effective theory expression, equation 3.18.

In Figure 3.4 we plot the numerical result obtained with WHIZARD [70, 71] for the tree-level cross-section, implemented with a fixed-width prescription for the propagator, and the successive effective-theory approximations. We have checked that the numerical results from O’Mega [73], CompHep [74, 75] and MadGraph [76, 77] matrix elements agree within the numerical error of the Monte-Carlo integration. The large constant shift of about 100 fb corresponding to the $N^{1/2}\text{LO}$ correction (3.27) from the hard region is clearly visible, but the NLO approximation is already close to the full Born calculation. Table 3.1 presents more detailed results, now including also the $N^{3/2}\text{LO}$ approximation (with the missing energy-independent $N^{3/2}\text{LO}$ terms (3.31) set to zero). As expected, the convergence of the expansion is very good close to the threshold at $\sqrt{s} \approx 161 \text{ GeV}$, where the difference between the EFT prediction and the full numerical result is $\sim 0.1\%$ already at NLO. The accuracy of the approximation degrades as one moves away from threshold, particularly below threshold, where the double-resonant potential configurations are kinematically suppressed. Clearly, for a target theoretical error on the cross section of 0.5%, the NLO approximation is sufficiently accurate only in a rather narrow region around the threshold. However, we observe that the inclusion of the $N^{3/2}\text{LO}$ correction from the first subleading term in the expansion of the cross section in the hard region leads to a clear improvement both above ($\sim 0.1\%$ at 170 GeV) and below threshold ($\sim 10\%$ at 155 GeV). In this case, the energy range where the target accuracy is met covers the region of interest for the $W$ mass determination.

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<th>EFT(NLO)</th>
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</tr>
</tbody>
</table>

Table 3.1: Comparison of the numerical computation of the full Born result with WHIZARD with successive effective-theory approximations.
3.4 Implementation of kinematical cuts in the EFT formalism

In Section 3.3 we have presented results for the totally inclusive (on the momenta and directions of the final states) Born cross section. This clearly does not correspond to an experimentally measurable quantity, since cuts on the momenta and angles of the observed particles have to be applied in order to disentangle signal and background processes, and additional constraints are imposed by the geometry of the detector. While in this thesis we do not aim at a general treatment of such kinematical cuts, here we will show how at least some of them can be implemented in the effective-theory formalism.

For orientation, we consider the selection cuts used for the measurement of the four-fermion production cross section at $\sqrt{s} = 161$ GeV at LEP [5–8], and for definiteness, we adopt the set of cuts used by the L3 collaboration in [7] to select the $q\bar{q}\mu\nu(\gamma)$ final state.

The cuts can be summarised as follows:

(i) The muon momentum has to satisfy $|p_{\mu}| > 20$ GeV;

(ii) the jet-jet invariant mass $M_{jj}$ and the invariant mass $M_{\mu\nu}$ of the muon-neutrino system have to satisfy $40$ GeV $< M_{jj} < 120$ GeV and $M_{\mu\nu} > 55$ GeV, respectively;

(iii) the angle between the muon and both hadronic jets must satisfy $\theta_{\mu j} > 15$ degrees to suppress backgrounds from $q\bar{q}(\gamma)$ production where the muon arises as a decay product of hadrons;

(iv) the polar angle of the missing-momentum vector has to satisfy $|\cos \theta_{\nu}| < 0.95$ to suppress $q\bar{q}(\gamma)$ events where the missing energy arises from a photon lost in the beam pipe.

The effect of the cuts on the full SM Born cross section have been computed using WHIZARD [71], and the numerical effects are shown in Table 3.2. We note that individual cuts are not very restrictive, but their combined effect reduces the cross section by about 9 percent, and therefore cannot be neglected. The cut (i) on the muon momentum is in the range of few per-milles and will not be considered further. The remaining cuts are represented by invariant-mass cuts on the products of the $W$ decays, and cuts on the relative angles of the final-state particles and of a single particle with the beam axis. These will be discussed in turn in Subsections 3.4.1 and 3.4.2 respectively.

3.4.1 Invariant-mass cuts

To illustrate the correct implementation of invariant-mass cuts in the effective-theory formalism, we start noting that an invariant-mass cut on the final states of the $W$ decays, $-\Lambda_{f_1}^2 < M_{h_1}^2_f - M_W^2 < \Lambda_{f_2}^2$, translates into a cut on the momenta of the $W$ bosons circulating in the loops of the forward-scattering diagrams whose imaginary part contributes to the four-fermion cross section. This is only partially true for the truly single-resonant hard cuts $h_4$-$h_7$ in Figure 3.3, where one of the two final-fermion pairs does not come from the decay of a $W$ boson. However, as noted in Section 3.2, these corrections are numerically
very suppressed, and we expect the effect of the cuts on these contributions to be negligible. We thus implement invariant-mass cuts in the effective theory by inserting a product of step functions \( \theta(\Lambda^2 - k^2_{1,2} + M_W^2) \theta(\Lambda^2 + k^2_{1,2} - M_W^2) \) in the full cut loop diagrams, and expanding the integral according to the scaling of the loop momentum (potential or hard), as in Sections 3.1 and 3.2, including the step functions encoding the cuts. The last step clearly depends on the scaling assigned to the ratio \( \Lambda/M_W \) with respect to the threshold-expansion parameter \( \delta \). For the loose cuts (ii) under considerations, where \( \Lambda_{1,\mu\bar{\nu}} \sim 0.7M_W \), \( \Lambda_{1,jj} \sim 0.9M_W \) and \( \Lambda_{2,jj} \sim 1.1M_W \), it is appropriate to count \( \Lambda/M_W \sim 1 \). However, it is also interesting to consider the case of tighter cuts of the order \( \Lambda/M_W \sim \sqrt{\Gamma_W/M_W} \sim \sqrt{\delta} \).

The modification of the effective-theory calculation in the two different cases will be discussed in the following.

\[ \text{Loose cuts: } \Lambda \sim M_W \quad \text{Consider first the contribution of the potential region, where the } W\text{-loop momentum is decomposed as } k = M_W v + r \text{ with } v = (1, \vec{0}) \text{ and a residual potential momentum } r = (r^0, \vec{r}) \sim (\delta, \sqrt{\delta}). \text{ In terms of the momentum } r, \text{ the step functions implementing the cuts read} \]

\[ \theta(\Lambda^2 \pm (r^2_0 + 2M_Wr_0 - \bar{r}^2)) . \]

Since by assumption \( \Lambda \gg r_0, \bar{r}^2 \sim \delta \) the momentum can be dropped in the step function, so that at leading order in \( \delta \) we have

\[ \theta(\Lambda^2 \pm (r^2_0 + 2M_Wr_0 - \bar{r}^2)) \sim \theta(\Lambda^2) = 1 , \]

and the cut can be neglected in the loop integrals in the potential region. On the other hand, in the calculation of the matching coefficients of the four-electron operators the step functions are operative, since the \( W\)-loop momenta are hard and taken to be off-shell of an amount of order \( k^2 - M_W^2 \sim M_W^2 \sim \Lambda^2 \). Therefore, a loose cut with \( \Lambda \sim M_W \) is taken into account entirely by modifying the matching coefficient of the four-electron operator,

\[ \text{Table 3.2: Effects of the phase-space cuts used in [7] on the full SM Born cross section at } \sqrt{s} = 161 \text{ GeV. The numbers have been computed with WHIZARD.} \]

<table>
<thead>
<tr>
<th>Cut</th>
<th>( \sigma_{\text{Born}}(e^-e^+ \rightarrow \mu^-\nu\mu\bar{d})(\text{fb}) )</th>
<th>( \sigma_{\text{cut}}/\sigma_{\text{tot}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\vec{p}_\mu</td>
<td>&gt; 20 \text{ GeV} )</td>
</tr>
<tr>
<td>( M_{\mu\nu} &gt; 55 \text{ GeV, } 40 \text{ GeV} &lt; M_{jj} &lt; 120 \text{ GeV} )</td>
<td>157.07(6)</td>
<td>97.79(7) %</td>
</tr>
<tr>
<td>( \theta_{\nu j} &gt; 15 \text{ degrees} )</td>
<td>155.60(6)</td>
<td>96.87(7) %</td>
</tr>
<tr>
<td>(</td>
<td>\cos \theta_{\nu j}</td>
<td>&lt; 0.95 )</td>
</tr>
<tr>
<td>all</td>
<td>145.95(6)</td>
<td>90.87(7) %</td>
</tr>
</tbody>
</table>

\[ \text{4 In this respect it is interesting to mention that in the effective-theory treatment of top-pair production near threshold [72] the assumed hierarchy } \Gamma \ll \Lambda \ll M \text{ is closer to the second case.} \]
Figure 3.5: Comparison of the Born cross section in the full SM computed with WHIZARD (red dots) with the effective-theory result for loose-cut implementation (dashed blue curve) and tight-cut implementation (solid black curve).

equation (3.28)\(^5\), while loop integrals in the effective theory are to be performed without constraints on the \(W\)-momenta.

**Tight cuts:** \(\Lambda \sim M_W \sqrt{\delta}\) When the cuts are tight the situation is quite different. Since by assumption \(\Lambda^2 \sim M_W \Gamma_W\), the constraints imposed by the step function cannot be neglected in the evaluation of potential loops, where at lowest order we have

\[
\theta(\Lambda^2 \pm (r_0^2 + 2M_W r_0 - \vec{r}^2)) \sim \theta(\Lambda^2 \pm (2M_W r_0 - \vec{r}^2)) \sim 0.
\]

(3.38)

In the calculation of the matching coefficients of the four-electron operators, the \(W\) momenta are taken to be hard and satisfy \(|k^2 - M_W^2| \gg \Lambda^2\). Since \(\theta(\Lambda^2 - |k^2 - M_W^2|) \sim \theta(|k^2 - M_W^2|) = 0\), tight cuts lead to vanishing matching coefficients. Therefore the invariant-mass cuts have to be taken into account in the loop calculations in the effective theory, while four-electron operators do not contribute to the cross section at all. This was to be expected, since in the case of tight cuts the \(W\) momenta are always constrained inside the resonant region.

To verify that our prescriptions are the correct ones to include invariant-mass cuts in the effective theory, in Figure 3.5 we compare the full tree-level cross section at \(\sqrt{s} = 161\,\text{GeV}\) for the process (1.2), computed with WHIZARD for symmetric invariant-mass

\[5\]The effect of the cuts reduces to the modification of the upper limit in the integrations in equation B.10, \(\Lambda \left/ \sqrt{2M_W} \right. \sqrt{1 + \Lambda^2 / 8M_W^2}\), while the lower limit is not modified by the cuts. See Appendix B.2 for details.
cuts $|M_{ud}^2 - M_W^2| < \Lambda^2$ and $|M_{\mu\bar{\nu}} - M_W^2| < \Lambda^2$, to the effective theory result obtained using either the counting $\Lambda \sim M_W$ or $\Lambda \sim M_W \sqrt{\delta}$. The result reveals that the effective-theory predictions are in good agreement with the SM cut cross section in the regions where the respective counting rule is appropriate. In the intermediate region $M_W \sqrt{\delta} \ll \Lambda \ll M_W$, where both predictions start diverging from the true result, the cross section should be expanded independently in the two parameters $\Lambda/M_W$ and $\delta$. Implementing the cuts of [7] quoted at point (ii), by modifying the matching coefficient of the four-electron operator according to the loose-cut prescription, we obtain for the NLO EFT approximation to the cut Born cross section $\sigma_{\text{EFT}}(161\text{GeV}) = 157.24\text{ fb}$, which is in good agreement with the WHIZARD result of $\sigma(161\text{GeV}) = 157.07 \pm 0.06\text{ fb}$.

### 3.4.2 Angular cuts

The cuts (iii) and (iv) are sensitive to the angular distributions of the decay products of the $W$ bosons and it is more problematic to include them in the approach followed in [47] and in this thesis, where final states do not explicitly appear in the SCET Lagrangian, equation (2.22). To study the effects of angular cuts we can use a variant of the effective theory that includes the decay products [61], that, at leading order in $\alpha$ and $\delta$, interact with the resonant fields through the vertices

$$L^{(0)}_{\text{decay}} = -\frac{g}{2\sqrt{M_W}} (\Omega^i_+ \bar{\mu}_L \gamma^i \nu_L + \Omega^i_+ \bar{u}_L \gamma^i d_L) .$$

(3.39)

We start considering cuts on the angle between one of final-state particles and the beam axis, like in (iv). In the leading-order effective-theory description the $W$ bosons are produced in a $s$-wave by the operator (2.42). At this order the angular distribution of the $W$s is predicted to be isotropic. However, due to spin correlations, the decay-product distributions are not isotropic, and an explicit calculation gives (we consider for definiteness the neutrino angular distribution)

$$\frac{1}{\sigma^{(0)}_{\text{EFT}}} \frac{d\sigma^{(0)}_{\text{EFT}}}{d\cos \theta_\nu} = \frac{3}{16} (1 - \cos \theta_\nu)(3 + \cos \theta_\nu) .$$

(3.40)

Computing the effect of the cut (iv) using (3.40) gives a result $\kappa = \frac{\sigma^{(0)}_{\text{EFT, cut}}}{\sigma^{(0)}_{\text{EFT}}} = 96.16\%$ which is already in good agreement with the number given in table 3.2. As another example, consider the neutrino forward-backward asymmetry $|\sigma(\cos \theta_\nu > 0) - \sigma(\cos \theta_\nu < 0)|/\sigma_{\text{tot}} = 3/8$ computed from (3.40) as compared to the result $\sim 30\%$ obtained using WHIZARD.

A closer look at the angular distributions generated by WHIZARD reveals some deviations from the theoretical leading-order results, for example a forward-backward asymmetry of the $W$-bosons $|\sigma(\cos \theta_W > 0) - \sigma(\cos \theta_W < 0)|/\sigma_{\text{tot}} \sim 30\%$. These deviations are mainly to be attributed to non-resonant momentum configurations that are incorporated in the four-fermion operators that are formally of order $N^{1/2}$LO. At threshold the effect of the four-fermion operators is about 40 percent of the leading order EFT contributions, which is consistent with the order of magnitude of the observed asymmetries. Unfortunately the implementation of angular distributions analogous to (3.40) for non-resonant
momentum configurations would imply the inclusion in the effective Lagrangian of effective six-fermion operators containing collinear fields describing the two incoming electrons and the four final-state fermions, which is beyond the scope of this thesis.
Chapter 4

Radiative corrections

In this chapter we compute the remaining EFT NLO contributions, representing genuine loop corrections to the four-fermion production cross section. These have been listed in Section 2.3, and correspond to an electroweak correction to the matching coefficient of the leading production operator and to $W$ decay (Section 4.1), a correction from Coulomb-photon exchange (Section 4.2), and soft and collinear photon effects (Sections 4.3 and 4.4 respectively). For a non-vanishing electron mass $m_e$, contributions from additional collinear modes must be added to the NLO cross section, as explained in Section 4.5.

4.1 Hard corrections to production and decay

As explained in Section 2.3, a complete next-to-leading order EFT description in $\alpha \sim \delta$ of the four-fermion production process requires the calculation of hard electroweak contributions to the one-loop matching coefficients $C_{p,LR}^{(1)}$ and $C_{p,RL}^{(1)}$ of the production operators (2.54), and of the two-loop electroweak $W$ self-energy entering the definition of $\Delta^{(2)}$, see (2.52). These are extracted from conventional perturbative calculations performed in a strict expansion in $\alpha_{ew}$, as illustrated in Section 2.1.3.

Before presenting the calculation of these two contributions, we briefly review the renormalisation conventions for the parameters and fields of the electroweak Standard Model. The one-loop counterterm for a scattering amplitude which, at the tree level, is proportional to $g_{ew}^n = (4\pi \alpha_{ew})^{n/2} = (4\pi \alpha/s_w^2)^{n/2}$ is given by

$$[\text{tree} \left( -n \frac{\delta s_w}{s_w} + n \delta Z_e + \frac{1}{2} \sum_{\text{ext}} \delta Z_{\text{ext}} \right),$$

where the sum extends over all external lines. As in Section 3.3 we choose the $W$ and $Z$ boson mass, and the Fermi constant $G_\mu$ as the three independent parameters of the electroweak SM, while $c_w \equiv M_W/M_Z$ and $\alpha \equiv \alpha_{ew} s_w^2 \equiv \sqrt{2} G_\mu M_W^2 s_w^2 / \pi$ are derived quantities. The choice of the $G_\mu$-scheme has the advantage that in the one-loop calculation the light-fermion masses can be set to zero [78,79]. In this scheme, the counterterm for
\( s_w \) is related to the \( W^- \) and \( Z^- \)-boson self-energies through

\[
- \frac{\delta s_w}{s_w} + \delta Z_e = \frac{1}{s_w c_w} \frac{\Pi^{AZ}_{T}(0)}{M_Z^2} + \frac{\Pi^{W}_{T}(0) - \Re \Pi^{W}_{T}(M_W^2)}{2M_W^2} - \frac{\delta r}{2},
\]  

(4.2)

where \( \Pi^{AZ}_{T} \) denotes the photon-\( Z^- \) mixing, \( \Pi^{W}_{T} \) is the transverse self-energy of the \( W^- \) boson \(^1\) and

\[
\delta r = \frac{\alpha}{4\pi s_w^2} \left( 6 + \frac{7 - 4s_w^2}{2s_w^2} \ln c_w^2 \right),
\]  

(4.3)

appears in the explicit expression for the electroweak correction to muon decay, \( \Delta r \) (see e.g. [79]). Consistently with the choice made in Section 3.1 for the renormalised propagator, here we adopt the conventional on-shell renormalisation scheme for the wave-function counterterm \( \delta Z_{\text{ext}} \) of the external fields [79]. In particular, for the \( W^- \)-boson and fermion wave-function renormalisation one has

\[
\delta Z_{W} = \Re \frac{\partial \Pi^{W}_{T}(p^2)}{\partial p^2}\bigg|_{p^2 = M_W^2}, \quad \delta Z_{f} = \Re \Pi^{f}(0),
\]  

(4.4)

where \( \Pi^{f} \) denotes the self-energy of the fermion. Note that, since we never consider a physical process with external \( W^- \) bosons, the final result is independent of the particular renormalisation convention chosen for \( \delta Z_{W} \). However, the matching coefficient of the production operator calculated below does depend on this convention, and this dependence is cancelled by the dependence of (3.9) on the on-shell derivative of the renormalised one-loop self-energy \( \Pi^{(1,1)} \), whose value depends in turn on \( \delta Z_{W} \).

### 4.1.1 Production vertices

The general matching relation needed to calculate the short-distance coefficients of production operators has been given in Subsection 2.1.3, and for the case under consideration is analogous to equation (2.31). \( C_{\rho,L,R}^{(1)} \) and \( C_{\rho,R,L}^{(1)} \) are extracted from the full SM scattering amplitude for \( \epsilon^{-}_{L/R} e^{+}_{L/R} \rightarrow W^- W^+ \), computed at \( O(\alpha) \) and at leading order in the non-relativistic approximation, and using dimensional regularisation in \( d = 4 - 2\epsilon \) dimensions. Since no gauge-violating resummation of self-energy insertions is performed, and all the non-vanishing diagram of order \( \alpha\delta^0 \) are taken into account, the two matching coefficients are gauge invariant by construction, provided that the scattering amplitude is calculated with the external \( W^- \) boson momenta at the complex pole position, \( k^2 = M_W^2 - iM_W \Gamma_W \). Note however that this condition must be satisfied only at the appropriate accuracy in \( \delta \), and at the lowest order it is correct to set \( k^2 = M_W^2 \). On the effective-theory side, the matching prescription (2.30) includes in principle an additional factor \( \sqrt{2M_W \varpi^{-1/2} } \) (see equation 2.41) for each external \( \Omega \) field [51]. However, here we depart from the “correct” matching procedure and omit the factor \( \varpi^{-1/2} \), since it was already included in Section 3.1 for the comparison of the EFT approximation with the true Born result (see discussion after (3.9)).

---

\(^1\)In the conventions used here and in [47] the sum of the amputated 1PI graphs is given by \((-i\Pi)\), which has the opposite sign compared to [79].
The matching coefficients of the production operators have been computed in [47]. The diagrams for the \( e^- (p_1) e^+ (p_2) \rightarrow W^- (k_1) W^+ (k_2) \) scattering process were generated with FeynArts [80] and the algebra performed with FeynCalc [81]. At one loop, there are 65 two-point diagrams, 84 three-point diagrams and 31 four-point diagrams, some of which are shown in Figure 4.1. The complete set of relevant SM diagrams is given in Appendix C. At lowest order in the non-relativistic expansion, and consistently with the leading-order on-shell condition for the external \( W \) lines, \( k_1^2 = k_2^2 = M_W^2 \), the \( W \) momenta can be set to \( k_1 = k_2 = M_W v \), with \( v = (1, 0) \) as usual, whereas the incoming lepton momenta are parameterised as \( p_1 = (M_W, \vec{p}) \) and \( p_2 = (M_W, -\vec{p}) \) with \( |\vec{p}| = M_W \). This results in two simplifications. First, many diagrams vanish consistent with the fact that the tree-level \( s \)-channel diagrams do not contribute at leading order in the non-relativistic expansion (see Appendix B.1). Second, the number of scales present in the loop integrals is reduced to one, \( \Lambda = M_W \), since \( \sqrt{s} \) is set to \( 2M_W \). Due to the simplified kinematics, all box integrals can be reduced to triangle diagrams and the one-loop correction to the amplitude for the process \( e^- L e^+ R \rightarrow W^- W^+ \) takes the simple form

\[
A_{WW} = \frac{\pi \alpha_{ew}^2}{M_W^2} C^{(1)}_{p,LR} (p_1 - p_2)\mu \, \langle p_2 - |\not\epsilon_3^\mu + |\not\epsilon_4^\mu |p_1 - \rangle, \tag{4.5}
\]

where \( \epsilon_3^\mu, 4 \) denote the polarisation vectors of the \( W \) bosons. Note that the structure of (4.5) is consistent with the effective-theory side of the matching equation 2.30, represented by (2.54). For the \( RL \) helicity combination one obtains a similar expression, with the fermion helicities reversed. The scalar coefficients \( C^{(1)}_{p,h} \) can be obtained by projections of the full amplitude. Thus, one is left with the calculation of a scalar quantity and standard techniques for the reduction of tensor and scalar integrals can be applied.

After adding the counterterm (4.1) with \( n = 2 \) to (4.5), the production-operator matching coefficient for the \( LR \) helicity state takes the form

\[
C^{(1)}_{p,LR} = \frac{\alpha}{2\pi} \left[ \left( \frac{2e^2}{3} - \frac{1}{2e^2} \right) \left( -\frac{4M_W^2}{\mu^2} \right) \right]^{-\epsilon} + C^{(1,\text{fin})}_{p,LR}, \tag{4.6}
\]

where the finite part \( C^{(1,\text{fin})}_{p,LR} \) together with the expression for \( C^{(1)}_{p,RL} \) is given explicitly in Appendix C. For the final expression of the matching coefficient, the surviving (infrared) poles have to be subtracted. However, here we keep them in order to explicitly demonstrate their cancellation against poles from the soft contribution and poles related to initial-state

Figure 4.1: Sample diagrams contributing to the matching of the production operator \( O_p \) at one loop.
collinear singularities. Numerically one finds,

\[ c_{\text{fin}}^{(1)} = -10.076 + 0.205i \]  

(4.7)

for \( M_W = 80.377 \text{ GeV} \), \( M_Z = 91.188 \text{ GeV} \), top-quark mass \( m_t = 174.2 \text{ GeV} \) and Higgs mass \( M_H = 115 \text{ GeV} \). In the expression of \( C_{p,RL}^{(1)} \) all poles cancel. This is to be expected, since the corresponding Born term vanishes, as indicated in (2.42).

Note that both matching coefficients \( C_{p,LR}^{(1)} \) and \( C_{p,RL}^{(1)} \) have a non-vanishing imaginary part, that contributes to the imaginary part of the forward-scattering amplitude \( A \) and, therefore, to the total cross section. If we denote with \( A_{\Delta C}^{(1)} \) the contribution to \( A \) of the NLO matching coefficient \( C_p^{(1)} \), we have that

\[ \text{Im} A_{\Delta C}^{(1)} = \text{Im} \left( 2C_p^{(1)} A^{(0)} \right) = 2 \text{Re} C_p^{(1)} \text{Im} A^{(0)} + 2 \text{Im} C_p^{(1)} \text{Re} A^{(0)}. \]  

(4.8)

The second term in (4.8) is induced by cuts that do not correspond to the final state we are interested in, such as the \( Z\gamma \) intermediate state in the fourth diagram of Figure 4.1. In fact, at leading order in the non-relativistic expansion, none of the diagrams that contribute to the hard matching coefficients contains either a quark or a muon. Thus, to obtain the flavour-specific cross section we are concerned with, we have to drop the second term in (4.8) and in what follows it is always understood that we take the real part of the matching coefficients \( C_{p,LR}^{(1)} \) and \( C_{p,RL}^{(1)} \). The situation becomes more complicated beyond NLO in \( \alpha \sim \delta \). If we consider again cut (2) of the \( \delta \)-suppressed s-channel diagram in Figure 3.2, we note that some of the cuts contributing to \( \text{Im} C_p^{(1)} \) do correspond to the flavour-specific cross section we are interested in, and must be included in the final result for the cross-section.

The contribution to the cross section resulting from the NLO correction to the production operators is thus obtained by multiplying the imaginary part of the leading-order polarised forward-scattering amplitude \( A_h^{(0)} \) by twice the real part of the corresponding matching coefficient. For the \( e^-_L e^+_R \) initial state we have

\[ \Delta \sigma_{\text{hard}}^{(1)} = \frac{1}{27s} 2 \text{Re} C_{p,LR}^{(1)} \text{Im} A_{LR}^{(0)}. \]  

(4.9)

where, as usual, we have included the leading-order branching ratio product, \( 1/27 \), to select the semileptonic flavour-specific final state. Since there is no tree-level contribution to the \( RL \) helicity combination, and no interference arises between \( e_R e_L^c \) and \( e^+_L e_R^- \), the coefficient \( C_{p,RL}^{(1)} \) contributes first at NNLO in the effective-theory counting. The explicit expression of (4.9) follows from equations (2.46) and (4.6). However, to show the cancellation of the poles of (4.9) against the corrections computed in the following sections, it is useful to keep in (4.9) the unintegrated expression of \( A_{LR}^{(0)} \), equation (2.44). Introducing the abbreviations

\[ \eta_+ = E - r^0 - \frac{r^2}{2M_W} + i \frac{\Gamma_W^{(0)}}{2}, \quad \eta_- = r^0 - \frac{r^2}{2M_W} + i \frac{\Gamma_W^{(0)}}{2} \]  

(4.10)
for the non-relativistic propagators in the leading-order diagram, Figure 2.5, and \( \tilde{\mu}^2 = \mu^2 e^{\gamma_E}/(4\pi) \), we can rewrite (4.9) as

\[
\Delta\sigma^{(1)}_{\text{hard}} = \frac{16\pi^2\alpha_{\text{ew}}^2}{27M_W^2} \Im \left\{ (-i) \tilde{\mu}^{2e} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \right\} \\
\times 2 \Re \frac{\alpha}{2\pi} \left[ \left( -\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} \right) \left( -\frac{4M_W^2}{\mu^2} \right)^{-\epsilon} + c^{(1,\text{fin})}_{\mu,LR} \right].
\] (4.11)

### 4.1.2 Decay corrections and selection of flavour-specific final states at NLO

Before presenting the next-to-leading order matching of the coefficient \( \Delta \), we briefly discuss the implementation of the flavour-specific final state \( \mu^- \bar{\nu}_\mu u \bar{d} \) beyond the leading-order prescription given in Chapter 3.

In the comparison of the effective-theory prediction with the Born SM result presented in Chapter 3, we set the two-loop matching coefficient \( \Delta^{(2)} \) in equation (3.18) to zero, and extracted the flavour-specific cross section by multiplying the imaginary part of the forward-scattering amplitude by the leading-order branching-ratio product \( \text{Br}^{(0)}(W^- \rightarrow \mu^- \bar{\nu}_\mu) \text{Br}^{(0)}(W^+ \rightarrow u \bar{d}) = 1/27 \). This is the correct treatment at the Born level, since all terms appearing in the expansions (3.9) and (3.13) (except for the contribution proportional to \( \Delta^{(2)} \)) are flavour-independent corrections, except for trivial colour factors. When we consider the full NLO prediction, including the radiative corrections computed in this chapter, the term proportional to \( \Delta^{(2)} \) in the expansion of the propagator (3.9) must be also taken into account, but the cross section (3.18) must be modified in such a way that only cuts contributing to the flavour-specific final state \( \mu^- \bar{\nu}_\mu u \bar{d} \) are included.

Consider the contribution arising from cutting a \( W \) line with an insertion of the two-loop matching coefficient \( \Delta^{(2)} \), given by the imaginary part of the term in (3.9),

\[
\Im \left[ \frac{-i}{\eta_-} \frac{i\Delta^{(2)}}{2} \frac{i}{\eta_-} \right] = -\Im \left[ \frac{1}{\eta_-} \left( \frac{\Delta^{(2)}}{2} \right)^* \frac{1}{\eta_-} - \frac{1}{\eta_-} \frac{\Delta^{(2)}}{2} \Im \left[ \frac{1}{\eta_-} \right] - \frac{1}{\eta_-} \left[ \Im \Delta^{(2)} \right] \frac{1}{\eta_-} \right],
\] (4.12)

where \( \eta_- \) is the inverse propagator of the non-relativistic \( W \) boson defined above, \( \eta_- = r_0 - \vec{r}^2/(2M_W) + i\Gamma^{(0)}_W/2 \). The first two terms correspond to cutting the \( W \) line to the left and right of the \( \Delta^{(2)} \) insertion. The flavour-specific final states are extracted from these cuts as discussed below (2.47). This amounts to multiplying the NLO correction (3.18) by the leading-order branching ratios, so these two terms are treated correctly by the factor 1/27. The last term corresponds to a cut two-loop self-energy insertion, where only the cuts leading to the desired final state must be taken into account. Therefore here \( -\Im \Delta^{(2)} = \Gamma^{(1)}_W \) has to be replaced by \( \Gamma^{(1)}_{\mu^- \bar{\nu}_\mu} = \Gamma^{(1,\text{ew})}_{\mu^- \bar{\nu}_\mu} \) and \( \Gamma^{(1)}_{u \bar{d}} = \Gamma^{(1,\text{ew})}_{u \bar{d}} + 1.409 \frac{\alpha^2}{\pi} \Gamma^{(0)}_{u \bar{d}} \), respectively, to obtain the NLO cross section for the four-fermion final state.

To understand how to implement these replacements, let us consider again the full cut...
propagator
\[
\mathrm{Im} \left[ \frac{1}{k^2 - M_W^2 - \Pi_T^W(k^2)} \right] = -\frac{\mathrm{Im}\Pi_T^{W*}(k^2)}{(k^2 - M_W^2 - \mathrm{Re}\Pi_T^W(k^2))^2 + (\mathrm{Im}\Pi_T^W(k^2))^2}. \tag{4.13}
\]

In a calculation of the four-fermion cross section from the squared matrix element the term \(\mathrm{Im}\Pi_T^{W*}(k^2)\) in the numerator would arise from integrating over the two-body phase-space of the \(W\) decay products. Therefore, the contributions to a specific final state can be extracted from (4.13) by replacing the imaginary part of the full transverse self-energy, \(\mathrm{Im}\Pi_T^W(k^2)\), with its flavour-specific counterpart, \(\mathrm{Im}\Pi_{T,X}^W(k^2)\), where \(X\) denotes a particular \(W\)-decay channel. The correct modification of equation (3.18) is thus obtained by inserting a factor \(\mathrm{Im}\Pi_{T,X}^{W*}(k^2)/\mathrm{Im}\Pi_T^{W*}(k^2)\) for each decaying \(W\) in the total amplitude (3.4), and expanding it at the required accuracy. To NLO in \(\delta\) the explicit expression of \(\mathrm{Im}\Pi_{T,X}^{W*}(k^2)\) is
\[
\mathrm{Im}\Pi_{T,X}^{W*}(k^2) = M_W \Gamma_X^{(0)} \left( 1 + \frac{r_0}{M_W} - \frac{r^2}{2M_W^2} \right) + M_W \Gamma_X^{(1)} + O(\delta^3), \tag{4.14}
\]

where \(\Gamma_X\) is the partial \(W\) decay width. The result for \(\mathrm{Im}\Pi_T^{W*}(k^2)\) is analogous. The cross section for the final state \(\mu^-\nu_\mu u\bar{d}\) is therefore obtained by multiplying (3.18) by
\[
\frac{\mathrm{Im}\Pi_{T,\mu^-\nu_\mu}(k^2)\mathrm{Im}\Pi_{T,\nu_\mu u\bar{d}}(k^2)}{(\mathrm{Im}\Pi_T^{W*}(k^2))^2} = \frac{\Gamma^{(0)}_{\mu^-\nu_\mu} \Gamma^{(0)}_{u\bar{d}}}{(M_W)^2} \left[ 1 + \left( \frac{\Gamma^{(1)}_{\mu^-\nu_\mu} \Gamma^{(0)}_{u\bar{d}}}{\Gamma^{(0)}_{\mu^-\nu_\mu} \Gamma^{(0)}_{u\bar{d}}} - 2 \frac{\Gamma^{(1)}_W}{\Gamma^{(0)}_W} \right) \right] + O(\delta^2)
\]
\[
= \frac{1}{27} \left[ 1 + \left( \frac{\Gamma^{(1)}_{\mu^-\nu_\mu}}{\Gamma^{(0)}_{\mu^-\nu_\mu}} + \frac{\Gamma^{(1)}_{u\bar{d}}}{\Gamma^{(0)}_{u\bar{d}}} - 2 \frac{\Gamma^{(1)}_W}{\Gamma^{(0)}_W} \right) \right] + O(\delta^2), \tag{4.15}
\]
or simply adding the next-to-leading order correction
\[
\Delta\sigma^{(1)}_{\text{decay}} = \left( \frac{\Gamma^{(1)}_{\mu^-\nu_\mu}}{\Gamma^{(0)}_{\mu^-\nu_\mu}} + \frac{\Gamma^{(1)}_{u\bar{d}}}{\Gamma^{(0)}_{u\bar{d}} - 2 \frac{\Gamma^{(1)}_W}{\Gamma^{(0)}_W} \right) \sigma^{(0)}_{LR} \tag{4.16}
\]
to (3.18). Note that to NLO in \(\delta\) the prefactor (4.15) is a constant, but at higher order it exhibits a non-trivial dependence on \(r_0\) and \(r^2\). An alternative derivation of the result (4.16) can be found in [47].

We now discuss the electroweak correction to the matching coefficient \(\Delta\), and to the flavour-specific on-shell widths \(\Gamma_{\mu^-\nu_\mu}\) and \(\Gamma_{u\bar{d}}\), as presented in [47]. In the pole mass and on-shell field renormalisation scheme \(\Delta^{(2,\text{ew})} = -i\Gamma^{(1,\text{ew})} = iM_W \mathrm{Im}\Pi^{(2,0)}_W\). The cuts of the 2-loop electroweak \(W\) self-energy consist of two parts, corresponding to the virtual and real hard corrections to the pole \(W\) decay width.

The virtual one-loop correction to the pole-scheme decay width into a single lepton (\(l\)) or quark (\(h\)) doublet can be written as
\[
\Gamma_{W,l/h}^{(1,\text{virt})} = 2\Gamma_{W,l/h}^{(0)} \Re C_{d,l/h}^{(1)}, \tag{4.17}
\]
where the tree-level widths in $d$ dimensions are $\Gamma_{W,l}^{(0)} = \Gamma_{\mu-\bar{\nu}_\mu}^{(0)} = \alpha_{ew} M_W / 12 + O(\epsilon)$ and $\Gamma_{W,h}^{(0)} = \Gamma_{ud}^{(0)} = 3 \Gamma_{W,l}^{(0)}$. The calculation of $C_{d,h}^{(1)}$ involves the evaluation of the diagrams depicted in Figure 4.2 with obvious modifications for the leptonic decay. After adding the counterterm (4.1) with $n = 1$ one obtains

$$C_{d,l/h}^{(1)} = \frac{\alpha^2}{2\pi} \left[ \left( -\frac{1}{2\epsilon^2} - \frac{5}{4\epsilon} \right) \left( \frac{M_W^2}{\mu^2} \right)^{-\epsilon} + Q_f \bar{Q}_f \left( -\frac{1}{e^2} - \frac{3}{2e} \right) \left( -\frac{M_W^2}{\mu^2} \right)^{-\epsilon} + c_{d,l/h}^{(1,\text{fin})} \right],$$

(4.18)

where for the leptonic (hadronic) decay we have to set the electric charges to $Q_f = -1$, $\bar{Q}_f = 0$ ($Q_f = 2/3$, $\bar{Q}_f = -1/3$). The finite parts $c_{d,l/h}^{(1,\text{fin})}$ of the matching coefficients are given explicitly in Appendix C. Numerically,

$$c_{d,l}^{(1,\text{fin})} = -2.709 - 0.552i, \quad c_{d,h}^{(1,\text{fin})} = -2.034 - 0.597i,$$

(4.19)

for $M_W = 80.377$ GeV, $M_Z = 91.188$ GeV, $m_t = 174.2$ GeV, and $M_H = 115$ GeV.

To this we have to add the correction due to hard real radiation of a single photon. Since the corresponding soft corrections vanish, the hard real corrections are equivalent to the full real corrections evaluated in the standard electroweak theory and their calculation is straightforward. The squared matrix element (divided by $2 M_W$) corresponding to bremsstrahlung is integrated over the $d$-dimensional phase-space [82]. The expression thus obtained contains infrared (double) poles which cancel the poles in (4.17) and we are left with finite expressions for the flavour-specific leptonic and hadronic matching coefficients. Including the two-loop QCD correction $^2$ to the hadronic decay, they read

$$\Delta^2_l = -i \Gamma_{W,l}^{(1,\text{ew})},$$

$$\Delta^2_h = -i \left[ \Gamma_{W,h}^{(1,\text{ew})} + 1.409 \frac{\alpha^2}{\pi^2} \Gamma_{W,h}^{(0)} \right],$$

$$\Gamma_{W,l/h}^{(1,\text{ew})} = \Gamma_{W,l/h}^{(0)} \frac{\alpha}{2\pi} \left[ 2 \Re c_{d,l/h}^{(1,\text{fin})} + \frac{101}{12} + \frac{19}{2} Q_f \bar{Q}_f - \frac{7\pi^2}{12} - \frac{\pi^2}{6} Q_f \bar{Q}_f \right].$$

(4.20)

Strictly speaking, for the computation of these matching coefficients, as for (4.6), one has to expand around the complex pole $\bar{s}$ and not around $M_W^2$. However, the difference in the width is of order $\alpha^3$ and thus beyond NLO [83].

\footnote{The strong coupling constant counts as $\alpha_s \sim \sqrt{\alpha_{ew}} \sim \sqrt{\delta}$. Thus at NLO in the EFT expansion we have to include $O(\alpha_s^2) \sim \delta$ QCD corrections.}
4.2 Coulomb corrections

We now consider the exchange of potential photons shown in Figure 4.4, where the loop momenta scales as $k_0 \sim M_W \delta$, $\mathbf{k} \sim M_W \sqrt{\delta}$. These corrections correspond to insertions of the non-local four-boson interaction in the PNRQED effective Lagrangian, equation (2.39), and can be resummed to all orders in terms of the zero-distance Coulomb Green function, i.e. the Green function $G(0)$ of the Schrödinger operator $-\nabla^2/M_W - \alpha/r$ evaluated at $\mathbf{r}_1 = \mathbf{r}_2 = 0$, as shown in Figure 4.3.

Using the representation of the Green function given in [84], the pure Coulomb contribution to the forward-scattering amplitude is

$$A_{LR}^C = \frac{16\pi^2}{M_W^2} \frac{\alpha}{e^2} G_C(0; 0, 0; E + i\Gamma_W(0)), \quad (4.21)$$

where the explicit expression of the MS-subtracted Green function reads [85]

$$G_C(0; 0, 0; E + i\Gamma_W(0)) = -\frac{M_W^2}{4\pi} \left\{ \sqrt{-\frac{E + i\Gamma_W(0)}{M_W}} + \alpha \left[ \frac{1}{2} \ln \left( -\frac{4M_W(E + i\Gamma_W(0))}{\mu^2} \right) \right] \right. $$

$$- \frac{1}{2} + \gamma_E + \psi \left( 1 - \frac{\alpha}{2\sqrt{(E + i\Gamma_W(0))/M_W}} \right) \right\} \right\}, \quad (4.22)$$

Here $\gamma_E$ is the Euler-Mascheroni constant, $\mu$ the 't Hooft unit of mass, and $\psi$ the Euler psi-function. Taking the imaginary part, and multiplying it by the leading-order branching-ratio product $1/27$, we obtain the pure Coulomb contribution to the four-fermion cross section

$$\sigma_{LR}^C = -\frac{4\pi\alpha^2}{27s_{\theta,8}} \text{Im} \left\{ \sqrt{-\frac{E + i\Gamma_W(0)}{M_W}} + \alpha \left[ \frac{1}{2} \ln \left( -\frac{E + i\Gamma_W(0)}{M_W} \right) \right] \right. $$

$$+ \psi \left( 1 - \frac{\alpha}{2\sqrt{-(E + i\Gamma_W(0))/M_W}} \right) \right\} \right\}, \quad (4.23)$$

where the dependence of the unphysical scale $\mu$ has dropped out. The first term in (4.23) coincides with the leading-order polarised cross section, equation (2.48), while the logarithmic term accounts for single Coulomb exchange, Figure 2.7c, and $\psi$ contains the
contribution from multiple Coulomb exchange. Since the single-Coulomb exchange contribution counts as $\alpha/\sqrt{\delta} \sim \sqrt{\alpha}$ with respect to the leading-order cross section, as explained in Section 2.2 and confirmed by (4.23), Coulomb corrections can be actually included in the four-fermion calculation perturbatively. Expanding equation (4.23) to NLO in $\delta \sim \alpha$ we obtain the contribution from the single ($\sim \sqrt{\alpha}$) and double-exchange ($\sim \alpha$) diagrams in Figure 4.4:

$$\Delta\sigma_{\text{Coulomb}}^{(1)} = \frac{4\pi\alpha^2}{27s_w^4s} \text{Im} \left[ -\frac{\alpha}{2} \ln \left( \frac{-E + i\Gamma^{(0)}_W}{M_W} \right) + \frac{\alpha^2\pi^2}{12} \sqrt{-\frac{M_W}{E + i\Gamma^{(0)}_W}} \right].$$  

(4.24)

This contributes only to the LR helicity cross section, since the production operator at the vertices in Figure 4.4 is the leading order one (2.42). Directly at threshold ($E = 0$) the one-photon exchange $N^{1/2}$LO term (the logarithm in (4.24)) is of order 5% relative to the leading order. The two-photon exchange is only a few-per-mille correction, confirming the expectation that Coulomb exchanges do not have to be summed to all orders. The one and two Coulomb-exchange terms have already been discussed in [41, 42].

4.3 Soft-photon corrections

We now turn to the corrections originating from soft-photon exchange. These are $O(\alpha)$ contributions to the forward-scattering amplitude, and correspond to two-loop diagrams in the effective theory containing a photon with momentum components $q_0 \sim |\vec{q}| \sim M_W \delta$. The relevant Feynman rule for the coupling of the soft photon to the $\Omega_{\pm}$ fields is given in the PNRQED Lagrangian (2.39), and reads

$$\mp ie v^\mu \delta^{ij},$$  

(4.25)

where $\mu, i, j$ are the polarisation indices of $\gamma$ and $\Omega_{\pm}$, and $v^\mu = (1, \vec{0})$. Note that (4.25) coincides with the threshold expansion of the full SM trilinear $WW\gamma$ vertex. The coupling to the collinear electrons and positrons is given by the SCET Lagrangian. At lowest order this simply leads to the eikonal approximation $\pm ien^\mu$, where $n^\mu$ is the direction of the four-momentum of the electron or positron, or equivalently, for a generic collinear vector $p_i$,

$$\frac{ie\gamma^\mu(p_i \pm \vec{q})}{(p_i \pm q)^2 + i\epsilon} \rightarrow \frac{2iep_i^\mu}{\pm 2p_i \cdot q + i\epsilon}.$$  

(4.26)
Figure 4.5: Soft-photon diagrams in the effective theory: Initial-initial state interference (ii), initial-intermediate state interference (im) and intermediate-intermediate state interference (mm). Symmetric diagrams are not shown.

Also, since the residual momentum of the fields $\Omega_\mp$ is potential, $r_0 \sim M_W \delta$, $|r|^2 \sim M_W \sqrt{\delta}$, the non-relativistic propagators inside the soft loops depend on $q$ only through its time-like component,

$$i\delta^{ij} \frac{r_0 \pm q_0 - |\vec{r}|^2/2M_W + i \Gamma_W/2}{r_0 \pm q_0 - |\vec{r}|^2/2M_W + i \Gamma_W/2},$$

because $\vec{r} \gg \vec{q}$. The topologies contributing to the two-loop forward-scattering amplitude are shown in Figure 4.5. The $W$-boson vertices are leading-order production vertices, hence at NLO the soft correction applies only to the left-right $e^-e^+$ helicity forward-scattering amplitude. Note that (mm2) is not a double-counting of the Coulomb-exchange diagram in Figure 4.4, since the two diagrams refer to different loop momentum regions.

Using the above Feynman rules, the correction to the forward-scattering amplitude from diagram (ii1) can be easily shown to be

$$\Delta A^{(1)}_{\text{soft,ii1}} = \frac{16\pi^2\alpha^2}{M_W^2} 4\pi \alpha (p_1 \cdot p_2) (1 - \epsilon) \tilde{\mu}^{4\epsilon} \int \frac{d^d r}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\eta_+} \frac{1}{(-q \cdot p_1 + i \epsilon)} \frac{1}{(-q \cdot p_2 + i \epsilon)} \frac{1}{(\eta_- - q_0)^2},$$

(4.28)

where we have used the usual abbreviations $\eta_- = r_0 - |q|^2/2M_W + i \Gamma_W/2$, $\eta_+ = E - r_0 - |q|^2/2M_W + i \Gamma_W/2$, and $\tilde{\mu} = \mu e^{\gamma_E/2}/\sqrt{4\pi}$. One possible way to compute (4.28) is to use the identity

$$\frac{1}{a^r b^s} = \frac{\Gamma(r+s)}{\Gamma(r) \Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{(a + 2b\lambda)^{r+s}}$$

(4.29)
to combine quadratic and linear propagators in (4.28). After the standard loop-momentum
redefinition, the momentum integration and the Feynman-parameter integrals can be
straightforwardly performed. Alternatively, one can first perform the $q_0$-integration by
means of the Cauchy theorem, and then the $d - 1$ remaining integrations. We will show
how to proceed in the second case.

The integrand (4.28) has five poles in the complex $q_0$ plane, given by

\begin{align*}
q_0 &= \pm |\vec{q}| \mp i\epsilon \\
q_0 &= \pm \vec{q} \cdot \vec{n} + i\epsilon \\
q_0 &= \eta_- ,
\end{align*}

(4.30)

where we have set the external momenta to $p_{1,2} = M_W(1, \pm \vec{n})$. Of these poles, four lay
in the upper half-plane and one in the lower. We therefore choose to close the contour in
the lower half-plane, and extract the residue of the integrand at $q_0 = |\vec{q}| - i\epsilon$, which reads

\begin{equation}
\Delta A^{(1)}_{\text{soft,ii1}} = \frac{16\pi^2 \alpha_s^2}{M_W^2} (-4\pi i\alpha) (1 - \epsilon) \mu^{4\epsilon} \int \frac{d^d \tau}{(2\pi)^d} \int \frac{d^{d-1} q}{(2\pi)^{d-1}} \times \frac{1}{\eta_+} \frac{1}{|\vec{q}|} \frac{1}{(|\vec{q} - \vec{q}' \cdot \vec{n}) (|\vec{q} + \vec{q}' \cdot \vec{n}) (\eta_+ - |\vec{q}|)} ,
\end{equation}

where the redundant $i\epsilon$ prescriptions have been dropped. It is useful to rewrite the
$d - 1$-dimensional loop-momentum volume element in polar coordinates,

\begin{equation}
\int d^{d-1} q = \int_0^{2\pi} d\phi_{1-2\epsilon} \int_{-1}^1 dy (1 - y^2)^{\epsilon} \int_0^{\infty} d|\vec{q}| |\vec{q}|^{2-2\epsilon},
\end{equation}

where $y$ is the cosine of the azimuthal angle $\theta_{\vec{q}}, y = \vec{n} \cdot \vec{q} / |\vec{q}|$, and $\int_0^{2\pi} d\phi_{1-2\epsilon}$ represents
the integration over the remaining $1 - 2\epsilon$ polar angles. With this choice of coordinates
equation (4.31) assumes the simple factorised form

\begin{equation}
\Delta A^{(1)}_{\text{soft,ii1}} = \frac{16\pi^2 \alpha_s^2}{M_W^2} (-4\pi i\alpha) (1 - \epsilon) \mu^{4\epsilon} \int \frac{d^d \tau}{(2\pi)^d} \frac{1}{\eta_+} \times \int_0^{2\pi} d\phi_{1-2\epsilon} \int_{-1}^1 dy (1 - y)^{-1-\epsilon}(1 + y)^{-1-\epsilon} \int_0^{\infty} d|\vec{q}| |\vec{q}|^{-1-2\epsilon} \eta_- - |\vec{q}| .
\end{equation}

The individual integrals can be easily computed,

\begin{align*}
\int_0^{2\pi} d\phi_{1-2\epsilon} &= \frac{2\pi^{1-\epsilon}}{\Gamma(1 - \epsilon)} \\
\int_{-1}^1 dy (1 - y)^{-1-\epsilon}(1 + y)^{-1-\epsilon} &= \frac{\sqrt{\pi}\Gamma(-\epsilon)}{\Gamma(1 - \epsilon)} \\
\int_0^{\infty} d|\vec{q}| |\vec{q}|^{-1-2\epsilon} \eta_- - |\vec{q}| &= \frac{\pi}{\sin(2\pi\epsilon)} (-\eta_-)^{-1-2\epsilon} ,
\end{align*}

(4.34)
and lead to the result

\[
\Delta A^{(1)}_{\text{soft,ii1}} = \frac{16\pi^2\alpha_{\text{em}}^2}{M_W^2} \frac{i\alpha e^{\gamma_E} \sqrt{\pi \Gamma(-\epsilon)}}{\sin(2\pi\epsilon) \Gamma(1-\epsilon) \Gamma(\frac{1}{2}-\epsilon)} \tilde{\mu}^{2\epsilon} \int \frac{d^d r}{(2\pi)^d} \frac{(1-\epsilon)(-\eta_-)}{\eta_-\eta_+} \left(-\frac{\eta_-}{\mu}\right)^{-2\epsilon}.
\]

We can further simplify equation (4.35) using the identity

\[
\frac{\sqrt{\pi}}{\sin(2\pi\epsilon) \Gamma(1-\epsilon) \Gamma(\frac{1}{2}-\epsilon)} = 2 \frac{-2\epsilon}{\pi} \Gamma(2\epsilon),
\]

which gives the final expression

\[
\Delta A^{(1)}_{\text{soft,ii1}} = \frac{16\pi^2\alpha_{\text{em}}^2}{M_W^2} \frac{i\alpha}{\pi} e^{\gamma_E} \Gamma(2\epsilon) \Gamma(-\epsilon) \tilde{\mu}^{2\epsilon} \int \frac{d^d r}{(2\pi)^d} \frac{(1-\epsilon)(-2\eta_-)}{\eta_-\eta_+} \left(-\frac{2\eta_-}{\mu}\right)^{-2\epsilon}.
\]

The computation of the other contributions is analogous to the calculation of diagram (ii1). Diagram (ii2) corresponds to a scaleless integral, while (ii3) is proportional to \( p_i^2 = 0 \). We thus conclude that

\[
\Delta A^{(1)}_{\text{soft,ii2}} = \Delta A^{(1)}_{\text{soft,ii3}} = 0.
\]

The contribution of diagram (im) to the forward-scattering amplitude is

\[
\Delta A^{(1)}_{\text{soft,im}} = \frac{16\pi^2\alpha_{\text{em}}^2}{M_W^2} \frac{i\alpha Q_i Q_\Omega}{2\pi} e^{\gamma_E} \Gamma(1+2\epsilon) \Gamma(-\epsilon) \tilde{\mu}^{2\epsilon} \int \frac{d^d r}{(2\pi)^d} \frac{(1-\epsilon)(-2\eta_\Omega)}{\eta_-\eta_+} \left(-\frac{2\eta_\Omega}{\mu}\right)^{-2\epsilon},
\]

where \( Q_i \) and \( Q_\Omega \) denote the electric charge of the fermion line and non-relativistic line the photon is attached to, and \( \eta_\Omega = \eta_\pm \) for \( \Omega = \Omega_\pm \) respectively. Such correction clearly cancels when we sum over all possible attachments of a soft photon to one external and one \( \Omega \) line.

The correction corresponding to the sum of (mm1) and of the symmetric diagram with a soft-photon loop on the upper \( \Omega \) line reads

\[
\Delta A^{(1)}_{\text{soft,mm1}} = \frac{16\pi^2\alpha_{\text{em}}^2}{M_W^2} \frac{i\alpha e^{\gamma_E} \Gamma(1+2\epsilon) \Gamma(-\epsilon)}{2\pi} \tilde{\mu}^{2\epsilon} \int \frac{d^d r}{(2\pi)^d} \frac{(1-\epsilon)}{\eta_-\eta_+} \left(-\frac{2\eta_-}{\mu}\right)^{-2\epsilon} + \left(-\frac{2\eta_+}{\mu}\right)^{-2\epsilon}.
\]

\[
= \frac{16\pi^2\alpha_{\text{em}}^2}{M_W^2} \frac{i\alpha e^{\gamma_E} \Gamma(1+2\epsilon) \Gamma(-\epsilon)}{\pi} \tilde{\mu}^{2\epsilon} \int \frac{d^d r}{(2\pi)^d} \frac{(1-\epsilon)(-2\eta_-)}{\eta_-\eta_+} \left(-\frac{2\eta_-}{\mu}\right)^{-2\epsilon}.\]
while the contribution of (mm2) to the forward-scattering amplitude is

\[
\Delta A^{(1)}_{\text{soft,mm2}} = \frac{16\pi^2\alpha_{\text{ew}}^2}{M_W^2} \left( \frac{-i\alpha}{2\pi} \right) \frac{e^{\gamma_E} \Gamma(1 + 2\epsilon)\Gamma(-\epsilon)\tilde\mu^{2\epsilon}}{1 - 2\epsilon} \int \frac{d^dr}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- + \eta_+} \times \left[ \frac{1}{\eta_+} \left( -\frac{2\eta_-}{\mu} \right)^{-2\epsilon} + \frac{1}{\eta_-} \left( -\frac{2\eta_+}{\mu} \right)^{-2\epsilon} \right]
\]

\[
= \frac{16\pi^2\alpha_{\text{ew}}^2}{M_W^2} \left( \frac{-i\alpha}{\pi} \right) \frac{e^{\gamma_E} \Gamma(1 + 2\epsilon)\Gamma(-\epsilon)\tilde\mu^{2\epsilon}}{1 - 2\epsilon} \int \frac{d^dr}{(2\pi)^d} \frac{1 - \epsilon}{(\eta_- + \eta_+)(\eta_- + \eta_+)} \left( -\frac{2\eta_-}{\mu} \right)^{-2\epsilon},
\]

(4.39)

with \( \eta_- + \eta_+ = E - r^2/M_W + i\Gamma_{W}^{(0)} \). In both equations we have used the symmetry of the integral under the exchange \( \eta_- \leftrightarrow \eta_+ \) to write the sum in brackets as a single term. The sum of (4.38) and (4.39),

\[
\frac{16\pi^2\alpha_{\text{ew}}^2}{M_W^2} \left( \frac{i\alpha}{\pi} \right) \frac{e^{\gamma_E} \Gamma(1 + 2\epsilon)\Gamma(-\epsilon)\tilde\mu^{2\epsilon}}{1 - 2\epsilon} \int \frac{d^dr}{(2\pi)^d} \frac{1 - \epsilon}{(\eta_- + \eta_+)(\eta_- + \eta_+)} \left( -\frac{2\eta_-}{\mu} \right)^{-2\epsilon},
\]

(4.40)

has neither poles not cuts in the \( r_0 \) upper half-plane, and vanishes once the integration over \( r \) is performed. We therefore arrive to the conclusion that all the corrections originating from the coupling of a soft photon to a non-relativistic line, equations (4.37), (4.38) and (4.39), vanish. In the effective theory this cancellation can be seen from the outset, since it follows from the particular form of the leading coupling of a soft photon to non-relativistic \( W \) bosons in the effective Lagrangian (2.39), which involves only \( A_0^W(t,0) \). Since the residual gauge invariance of the effective Lagrangian allows one to set the time-like component of the photon field to zero, at leading order the \( \gamma \Omega_+ \Omega_- \) coupling can be removed from the Lagrangian.

Therefore the soft-photon correction in the effective theory is given solely by the initial-initial state interference diagram (ii1), and the corresponding crossed diagram where the electron and positron lines are exchanged. The sum of the two diagrams is equal to twice the expression (4.36),

\[
\Delta A^{(1)}_{\text{soft}} = \frac{16\pi^2\alpha_{\text{ew}}^2}{M_W^2} \frac{2i\alpha^2}{\pi} e^{\gamma_E} \Gamma(2\epsilon)\Gamma(-\epsilon)\tilde\mu^{2\epsilon} \int \frac{d^dr}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \left( -\frac{2\eta_-}{\mu} \right)^{-2\epsilon}\]

\[
= \frac{16\pi^2\alpha_{\text{ew}}^2}{M_W^2} \frac{\alpha}{\pi} (-i) \tilde\mu^{2\epsilon} \int \frac{d^dr}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \times \left[ \frac{1}{\epsilon^2} - \frac{2}{\epsilon} \ln \left( -\frac{2\eta_-}{\mu} \right) + 2\ln^2 \left( -\frac{2\eta_-}{\mu} \right) + \frac{5\pi^2}{12} \right].
\]

(4.41)

The double \( \epsilon \)-pole in (4.41) cancels against the pole contained in contribution of the hard matching coefficient to the cross section, equation (4.11), while the single pole can be
Figure 4.6: Collinear-photon diagrams in the effective theory. Two symmetric diagrams are not shown.

factorised into the initial-state electron (positron) structure function, as shown in Section 4.5. Subtracting the pole part of the integrand (4.41) before performing the integration, one obtains

$$\Delta A_{\text{soft}}^{(1,\text{fin})} = A_{L,R}^{(0)} \frac{2 \alpha}{\pi} \left[ \ln^2 \left( -\frac{8(E + i \Gamma_W^{(0)})}{\mu} \right) - 4 \ln \left( -\frac{8(E + i \Gamma_W^{(0)})}{\mu} \right) + 8 + \frac{13}{24} \pi^2 \right]. \quad (4.42)$$

As before, the $r^0$ integration has been performed by closing the $r^0$ integration contour in the upper half-plane and picking up the pole at $r^0 = E - t^2/(2M_W) + i \Gamma_W^{(0)}/2$. Because of the absence of soft corrections related to the final state [86, 87] (see also Appendix D), at NLO the soft corrections to the flavour-specific process (1.2) can be obtained by multiplying the soft two-loop contributions to the forward-scattering amplitude by the leading-order branching ratios, thus leading to

$$\Delta \sigma_{\text{soft}}^{(1)} = \frac{1}{27s} \text{Im} \Delta A_{\text{soft}}^{(1)}. \quad (4.43)$$

As a check of the above results, we have also calculated the soft corrections directly for the four-fermion process (1.2), and found agreement with the simpler calculation of the forward-scattering amplitude. This is explicitly presented in Appendix D.

### 4.4 Collinear-photon corrections

Finally we come to collinear-photon corrections, corresponding to photon energies of order $M_W$, and photon virtuality of order $M_W \Gamma_W$. The four-momentum of the photon is proportional to the initial-state electron or positron momentum. The collinear photon couplings arise from the SCET Lagrangian, while their couplings to the $W$ bosons is encoded in the collinear Wilson lines included in the production operator (2.42). The diagrams corresponding to NLO contributions are shown in Figure 4.6. As discussed in [51] all these diagrams are scaleless for on-shell, massless initial-state particles. However, we shall have to say more about collinear effects in Section 4.5, where we discuss the resummation of large initial-state radiation logarithms.
4.5 Initial-state radiation

The total next-to-leading order contribution to the four-fermion cross section originating from radiative corrections is given by the sum of the results (4.11), (4.43), (4.24) and (4.16) presented in the previous sections,

\[ \hat{\sigma}_{LR}^{(1)} = \Delta \sigma_{\text{hard}}^{(1)} + \Delta \sigma_{\text{soft}}^{(1)} + \Delta \sigma_{\text{Coulomb}}^{(1)} + \Delta \sigma_{\text{decay}}^{(1)}. \]  

(4.44)

Note that this refers to the $e^-e^+$ helicity initial state, while there are no radiative corrections to the other helicity combinations at NLO in $\alpha \sim \delta$. Therefore, the radiative correction to the unpolarised cross section is one fourth of the LR contribution.

Because of the approximation $m_e = 0$, the total NLO cross section is not infrared-safe, as can be seen by inserting in (4.44) the explicit expression \( s \) of the four contributions. The Coulomb and decay corrections are free of infrared singularities, but for the sum of the soft (4.41) and hard (4.11) terms we obtain the following expression:

\[
\Delta \sigma_{\text{hard}}^{(1)} + \Delta \sigma_{\text{soft}}^{(1)} = \frac{16\pi^2\alpha_{\text{ew}}^2}{27M_W^2s} \frac{\alpha}{\pi} \text{Im} \left\{ (-i) \tilde{\mu}^2 \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \right. \\
\times \left[ -\frac{1}{\epsilon} \left( 2 \ln \left( \frac{-\eta_+}{M_W} \right) + \frac{3}{2} \right) + 2 \ln \left( \frac{-2\eta_+}{\mu} \right) - 2 \ln \left( \frac{2M_W}{\mu} \right) \right. \\
+ 3 \ln \left( \frac{2M_W}{\mu} \right) + \text{Re} \left[ c_{\mu,LR}^{(1,\text{fin})} \right] + \frac{11\pi^2}{12} \right\}.
\]  

(4.45)

The remaining $\epsilon$-poles are associated with emission of photons collinear to the incoming electron or positron, and can be factorised into the electron distribution function $\Gamma_{ee}^{\text{MS}}$. The cross section $\hat{\sigma}_{LR}^{(1)}$ can thus be viewed as a “partonic” cross section, and should be convoluted with the electron and positron structure functions, containing the infrared effects associated with the electron-mass scale $m_e$.

The physical infrared-finite cross section $\sigma$ reads [88,89]

\[
\sigma_h(s) = \int_0^1 dx_1 \int_0^1 dx_2 \Gamma_{ee}^{\text{MS}}(x_1) \Gamma_{ee}^{\text{MS}}(x_2) \hat{\sigma}_{h}^{\text{MS}}(x_1x_2s),
\]  

(4.46)

where $\hat{\sigma}_{h}^{\text{MS}}(s) = \hat{\sigma}_{h,\text{Born}}(s) + \hat{\sigma}_{h,\text{radiative}}^{(1)}(s)$ is our result for the NLO helicity-specific cross section after adding the Born cross section from Section 3 and the radiative correction from (4.44), with the infrared $\epsilon$-poles minimally subtracted. The partonic cross section $\hat{\sigma}_{h}^{\text{MS}}$ depends on the scales $Q = \{ M_W, E, \Gamma_W \}$ and the factorisation scale $\mu$, whereas $\Gamma_{ee}^{\text{MS}}(x_1)$, the electron distribution function in the $\overline{\text{MS}}$ scheme, depends on $\mu$ and on the very-long distance scale $m_e$. The physical cross section is independent of $\mu$ and includes the electron-mass dependence up to effects suppressed by powers of $m_e/Q$. By evolving the electron distribution function from the low scale $m_e$ to the scale $Q$ with the corresponding renormalisation-group equations, one can sum large collinear logarithms $\alpha^n \ln^{n_2} \left( Q^2/m_e^2 \right)$, with $n_1 = 1, ..., \infty$ and $n_2 = 1, ..., n_1$, from initial-state radiation of photons to all orders in $\alpha$. 

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perturbation theory. A NLO calculation of the partonic cross section should go along with a next-to-leading logarithmic approximation, where all terms with $n_2 = n_1$ and $n_2 = n_1 - 1$ are resummed in $\Gamma_{ee}^{\text{MS}}(x_1)$. Note that here we do not attempt to sum logarithms of $M_W / \Gamma_W$, which are less important, although the effective theory formalism is ideally suited for this summation as well. This will be discussed later in Chapter 7.

Unfortunately the structure functions $\Gamma_{ee}^{\text{LL}}(x)$ available in the literature do not correspond to the $\overline{\text{MS}}$ scheme and sum only leading logarithms $\alpha^n \ln^n \left( Q^2 / m^2 \right)$ [90,91]. Setting $\beta_\text{exp} = \beta_s = \beta_H = \beta_e = 2 \alpha / \pi (2 \ln(\mu / m_e) - 1)$ in the notation of [91], they read

$$\Gamma_{ee}^{\text{LL}}(x; m_e, \mu) = \frac{\exp \left( -\frac{1}{2} \beta_e \gamma_E + \frac{3}{2} \beta_e \right) \beta_e (1 - x)^{\beta_e / 2 - 1} - \frac{\beta_e}{4} (1 + x)}{\Gamma \left( 1 + \frac{1}{2} \beta_e \right)}$$

$$- \frac{\beta^2_e}{32} \left\{ \frac{1 + 3x^2}{1 - x} \ln(x) + 4(1 + x) \ln(1 - x) + 5 + x \right\}$$

$$- \frac{\beta^3_e}{384} \{ (1 + x) \left[ 6 \text{Li}_2(x) + 12 \ln^2(1 - x) - 3\pi^2 \right]$$

$$+ \frac{1}{1 - x} \left[ \frac{3}{2} (1 + x^2 + 3x^2) \ln(x) + 6(x + 5)(1 - x) \ln(1 - x) \right.$$

$$+ 12(1 + x^2) \ln(x) \ln(1 - x) - \frac{1}{2} (1 + 7x^2) \ln^2(x)$$

$$+ \frac{1}{4} (39 - 24x - 15x^2) \} \}. \tag{4.47}$$

The scale $\mu$ is not fixed in the context of leading-logarithmic resummation, and should be set to one of the typical scales of the short-distance partonic process. In the following we adopt the conventional choice $\mu = \sqrt{s}$.

To convert our result $\hat{\sigma}_h^{\text{MS}}(s)$ to this scheme and sum the leading-logarithmic initial-state radiation effects we proceed as follows: first, using the expansion $\Gamma_{ee}^{\text{MS}}(x) = \delta(1 - x) + \Gamma_{ee}^{\text{LL}}(x) + O(\alpha^2)$, we compute the scheme-independent NLO physical cross section without summation of collinear logarithms,

$$\sigma_h^{\text{NLO}}(s) = \sigma_h^{\text{Born}}(s) + \hat{\sigma}_h^{(1)\text{MS}}(s) + 2 \int_0^1 dx \Gamma_{ee}^{\text{MS}}(x) \sigma_h^{\text{Born}}(xs). \tag{4.48}$$

Then, by comparing this to the corresponding equation in the conventional scheme,

$$\sigma_h^{\text{NLO}}(s) = \sigma_h^{\text{Born}}(s) + \hat{\sigma}_h^{(1)\text{conv}}(s) + 2 \int_0^1 dx \Gamma_{ee}^{\text{LL}}(x) \sigma_h^{\text{Born}}(xs), \tag{4.49}$$

we determine $\hat{\sigma}_h^{(1)\text{conv}}(s)$, and hence $\hat{\sigma}_h^{\text{conv}}(s) = \sigma_h^{\text{Born}}(s) + \hat{\sigma}_h^{(1)\text{conv}}(s)$. Finally, we calculate the initial-state radiation resummed cross section

$$\sigma_h(s) = \int_0^1 dx_1 \int_0^1 dx_2 \Gamma_{ee}^{\text{LL}}(x_1) \Gamma_{ee}^{\text{LL}}(x_2) \hat{\sigma}_h^{\text{conv}}(x_1 x_2 s) \tag{4.50}$$
in the conventional scheme for the electron (positron) distribution functions. The first step is presented in Subsection 4.5.1, while the last two steps are discussed in Subsection 4.5.2. Note that since the Born cross section for the RL helicity combination is already a NLO effect, the scheme conversion must be performed only for $h = LR$. For $h = RL$ we simply have $\hat{\sigma}_{RL}^{conv}(s) = \hat{\sigma}_{RL}^{MS}(s) = \sigma_{RL,\text{Born}}(s)$.

### 4.5.1 The physical next-to-leading order cross section

Rather than computing the last term on the right-hand side of (4.48), we compute directly the radiative correction to the physical cross section, $\sigma_{LR}^{(1)}(s)$, by converting $\hat{\sigma}_{h,\text{MS}}^{(1)}(s)$, where the collinear divergences are regularised dimensionally, into the expression obtained when the electron mass itself is used as the regulator.

In the presence of the new scale $m_e \ll \Gamma_W, E, M_W$ there are two new momentum regions that give non-zero contributions to the radiative corrections. They correspond to hard-collinear photon momentum ($q^0 \sim M_W, q^2 \sim m_e^2$) and soft-collinear photons ($q^0 \sim \Gamma_W, q^2 \sim m_e^2 \Gamma_W^2/M_W^2$) $^3$. The corresponding loop integrals are scaleless when $m_e = 0$; for $m_e \neq 0$, they supply the difference

$$\sigma_{LR}^{(1)}(s) - \hat{\sigma}_{LR}^{(1)} = \Delta \sigma_{\text{s-coll}}^{(1)} + \Delta \sigma_{\text{h-coll}}^{(1)}.$$  \hspace{1cm} (4.51)

In other words $\sigma_{LR}^{(1)}(s)$ is the sum of the four contributions in (4.44) plus those from the two new momentum regions. Note that the new collinear modes are different from the ones discussed in Section 4.4, whose virtuality is of order of the much larger scale $M_W \Gamma_W$, whereas here $q^2 \lesssim m_e^2$.

Only a small subset of all the radiative-correction diagrams has hard-collinear or soft-collinear contributions, namely those containing a photon line connecting to an external electron or positron. For hard-collinear modes the contribution to the forward-scattering amplitude assumes the simple factorised form

$$\Delta A^{(1)} = 2C_{h-c}^{(1)} A^{(0)}.$$  \hspace{1cm} (4.52)

The result is consistent with the observation that hard-collinear interaction can take place only at the vertices of the forward-scattering diagram in Figure 2.5, since a hard-collinear mode connecting different vertices would set the intermediate $W$s off resonance. The coefficient $C_{h-c}^{(1)}$ is computed in Appendix E, and its explicit expression reads

$$C_{h-c}^{(1)} = \frac{\alpha}{4\pi} e^{\gamma_E} \Gamma(\epsilon) \frac{2 - \epsilon}{\epsilon(1 - 2\epsilon)} \left(\frac{m_e^2}{\mu^2}\right)^{-\epsilon}$$

$$= \frac{\alpha}{2\pi} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[-2\ln\left(\frac{m_e}{\mu}\right) + \frac{3}{2}\right] + 2\ln^2\left(\frac{m_e}{\mu}\right) - 3\ln\left(\frac{m_e}{\mu}\right) + \frac{\pi^2}{12} + 3\right].$$  \hspace{1cm} (4.53)

$^3$The existence of two collinear momentum regions is related to the fact that the $W$ pair-production threshold region probes the electron distribution function near $x = 1$, where hard-collinear real radiation is inhibited.
Therefore the NLO correction to the four-fermion cross section originating from hard-collinear modes is

$$
\Delta \sigma^{(1)}_{\text{h-coll}} = \frac{16\pi^2 \alpha_s \alpha^2}{27 M_W^2 s} \pi \text{Im} \left\{ (-i) \hat{\mu}^{2e} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_\parallel \eta_\perp} \right.
\times \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[ -2 \ln \left( \frac{m_e}{\mu} \right) + \frac{3}{2} \right] + 2 \ln^2 \left( \frac{m_e}{\mu} \right) - 3 \ln \left( \frac{m_e}{\mu} \right) + \frac{\pi^2}{12} + 3 \right]\}.
$$

(4.54)

The soft-collinear diagrams contributing next-to-leading order corrections to the forward-scattering amplitude are shown in Figure 4.7. These are computed by threshold-expanding the corresponding two-loop Standard-Model diagrams, where the photon-loop momentum is assumed to be soft-collinear, while the $W$-loop momentum is potential. We parameterise a generic momentum $p$ as

$$
p = p_+ n_- + p_+ + \frac{p_-}{2} n_+ ,
$$

(4.55)

where the two lightlike vectors $n_\pm$ are as usual defined by $n_\pm = (1, \pm \vec{n})$, with $\vec{n}$ the direction of the incoming electron, and $p_\perp$ represents a space-like vector perpendicular to $\vec{n}$. The coefficients $p_\pm$ are given by $p_\pm = (p \cdot n_\pm)/M_W$. In this parameterisation the momenta of the incoming electron and positron $p_{1,2} = (M_W, \pm \sqrt{M_W^2 - m_e^2} \vec{n})$ (where we have set $\sqrt{s} = 2M_W$) are

$$
p_{1,2} = \left( M_W - \frac{m_e^2}{4M_W} \right) n_\parallel + \frac{m_e^2}{4M_W} n_\perp + O \left( \frac{m_e^4}{M_W^4} \right).
$$

(4.56)

For a soft-collinear momentum $(q_0 \sim \Gamma_W, q^2 \sim m_e^2 \Gamma_W^2 / M_W^2)$ along the direction $n_-$ the three components of (4.55) scale as

$$
q_+ \sim \Gamma_W, \quad q_\perp \sim \Gamma_W \frac{m_e}{M_W}, \quad q_- \sim \Gamma_W \frac{m_e^2}{M_W^2}.
$$

(4.57)

The scaling for soft-collinear momenta in the direction $n_+$ is obtained by exchanging $q_+$ and $q_-$. According to (4.57), and to the scaling of the potential $W$ momentum, $r_0 \sim \frac{1}{\epsilon^2}$. 

---

**Figure 4.7:** Two-loop diagrams containing one soft-collinear photon along the direction $n_-$. Symmetric diagrams are not shown.
\[ q^2 + i \epsilon \to q_+ q_- + q_\perp^2 + i \epsilon , \]
\[ (p_1 - q)^2 - m_\gamma^2 + i \epsilon \to -2M_W \left(q_- + \frac{m_\gamma^2}{4M_W^2} q_+ \right) + i \epsilon , \]
\[ (p_2 \pm q)^2 - m_\gamma^2 + i \epsilon \to \pm 2M_W q_+ + i \epsilon , \]
\[ (k - q)^2 - M_W^2 + \Pi^W_W ((k - q)^2) \to 2M_W \left( -\frac{q_+}{2} + \eta_- \right) , \] \hspace{1cm} (4.58)
while, in terms of \( q_+, q_- \) and \( q_\perp \), the \( d \)-dimensional volume element reads
\[ \int dq^d \to -\frac{1}{2} \int dq_+ dq_- d^{d-2} q_\perp , \] \hspace{1cm} (4.59)
with \( q_\perp^2 = -|q_\perp|^2 \).

In the approximation (4.58), the contribution of diagram (a) in Figure (4.7) to the next-to-leading order forward-scattering amplitude is
\[ \Delta A^{(1)}_{s-coll,(a)} = \frac{16 \pi^2 \alpha^2 s_\gamma^2}{M_W^2} \frac{16 \pi \alpha M_W^2 \tilde{\mu}^{4\epsilon}}{4 \sqrt{\pi}} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \int \frac{dq_+ dq_- d^{d-2} q_\perp}{(2\pi)^d} \frac{1}{(q_\perp^2 + q_- q_+ + i \epsilon) (2M_W q_+ + i \epsilon) \left( -2M_W q_- - \frac{m_\gamma^2}{2M_W} q_+ + i \epsilon \right)} . \] \hspace{1cm} (4.60)

Introducing the two dimensionless quantities \( x = \frac{q_+}{2M_W} , y = \frac{q_-}{2M_W} \), and defining \( l^2 = -q_\perp^2 \), equation (4.60) can be cast in the form
\[ \Delta A^{(1)}_{s-coll,(a)} = \frac{16 \pi^2 \alpha^2 s_\gamma^2}{M_W^2} \frac{16 \pi \alpha M_W^2 \tilde{\mu}^{4\epsilon}}{4 \sqrt{\pi}} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \int \frac{d\phi}{-\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt d^{1-2\epsilon} \]
\[ \times \frac{1}{\left( -l^2 + 4M_W^2 xy + i \epsilon \right) (x + i \epsilon) \left( -y - \frac{m_\gamma^2}{4M_W} x + i \epsilon \right)} \]
\[ = \frac{16 \pi^2 \alpha^2 s_\gamma^2}{M_W^2} \frac{16 \pi \alpha M_W^2 \tilde{\mu}^{4\epsilon}}{2 \pi^2 \Gamma(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta_- \eta_+} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{0}^{\infty} dt d^{1-2\epsilon} \]
\[ \times \frac{1}{\left( -l^2 + 4M_W^2 xy + i \epsilon \right) (x + i \epsilon) \left( -y - \frac{m_\gamma^2}{4M_W} x + i \epsilon \right)} , \] \hspace{1cm} (4.61)
where we have explicitly solved the \((1 - 2\epsilon)\)-dimensional integration over the polar angles, \( \int d\phi = 2\pi^{1-\epsilon} / \Gamma(1 - \epsilon) \). The \( y \) integration is performed by closing the contour in the lower complex plane, and using Cauchy theorem to evaluate the integral. If \( x < 0 \) both poles
are in the upper complex half-plane, and the integral vanishes, while for \( x > 0 \) we pick the residue of the pole at \( y = \frac{l^2}{4M_W^2} - i\epsilon \), and obtain

\[
\Delta A^{(1)}_{s-coll,(a)} = \frac{16\pi^2 \alpha^2}{M_W^2} \frac{\mu^2}{\mu^2} \frac{\alpha e^{\epsilon \gamma_E}}{\alpha(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta} \int_0^\infty dx \int_0^\infty x l \, dx^{1-2\epsilon} = \frac{1}{(-l^2 - x^2 \mu^2 - i\epsilon)(x + i\epsilon)}.
\]

The \( l \)-integration can be easily performed, and gives the result

\[
\Delta A^{(1)}_{s-coll,(a)} = \frac{16i\pi^2 \alpha^2}{M_W^2} \frac{\mu^2}{\mu^2} \frac{\alpha e^{\epsilon \gamma_E}}{\alpha(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta} \int_0^\infty dq_\perp dq_{\perp'} \frac{d\epsilon^{\perp-2}q_{\perp}}{(2\pi)^d}
\]

\[
\times \frac{1}{(q_\perp^2 + q_{-q_+} + i\epsilon)(-2M_W q_+ + i\epsilon)(-q_+^2 + \eta - \eta_+)} (-2M_W q_- - \frac{m_e^2}{2M_W} q_+ + i\epsilon).
\]

After the \( y \) and \( l \) integration, that are performed exactly as in (4.60), equation (4.64) reduces to

\[
\Delta A^{(1)}_{s-coll,(d)} = -\frac{16i\pi^2 \alpha^2}{M_W^2} \frac{\mu^2}{\mu^2} \frac{\alpha e^{\epsilon \gamma_E}}{\alpha(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta} \int_0^\infty dx \frac{x^{1-2\epsilon}}{-xM_W + \eta - \eta_+}.
\]

The remaining \( x \) integration is easily solved, and gives

\[
\Delta A^{(1)}_{s-coll,(d)} = \frac{16i\pi^2 \alpha^2}{M_W^2} \frac{\mu^2}{\mu^2} \frac{\alpha e^{\epsilon \gamma_E}}{\alpha(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta} \int_0^\infty \left( \frac{m_e \eta}{\mu M_W} \right)^{1-2\epsilon}
\]

\[
= \frac{16\pi^2 \alpha^2}{M_W^2} \frac{\mu^2}{\mu^2} \frac{\alpha e^{\epsilon \gamma_E}}{\alpha(1 - \epsilon)} \int \frac{d^d r}{(2\pi)^d} \frac{1 - \epsilon}{\eta} \frac{\epsilon}{4\pi} \left[ \frac{1}{\eta} + \frac{2}{\ln} \left( \frac{m_e \eta}{\mu M_W} \right) - 2 \ln^2 \left( \frac{m_e \eta}{\mu M_W} \right) - \frac{3\pi^2}{4} \right].
\]

Note that the total contribution of the soft-collinear region to \( \Delta A^{(1)} \) is four times the expression (4.66), since diagram (d) contains an analogous correction from soft-collinear
photons along the direction \( n_+ \), and an extra factor 2 comes from considering the cross diagram with electron and positron legs exchanged. Therefore, the soft-collinear correction to the four-fermion cross section is

\[
\Delta \sigma_{s-coll}^{(1)} = \frac{16\pi^2\alpha^2_{em}}{27M_W^2s} \frac{\alpha}{\pi} \Im \left\{ (-i) \hat{\mu}^2 e^i \int \frac{d^d r}{(2\pi)^d} \frac{1}{\eta - \eta_+} \right\} \times \left[ -\frac{1}{e^2} + \frac{2}{\epsilon} \ln \left( -\frac{m_+ \eta_+}{\mu M_W} \right) - 2 \ln^2 \left( -\frac{m_+ \eta_+}{\mu M_W} \right) - \frac{3\pi^2}{4} \right].
\]  

(4.67)

The structure of the logarithms in (4.54) and (4.67) makes it clear that the two contributions arise each from a single scale, \( \mu \sim m_+ \) and \( \mu \sim m_+^2/M_W \), respectively.

Adding (4.44), (4.67), (4.54), and making use of (4.45), results in the factorisation-scheme independent radiative correction to the physical cross section,

\[
\sigma_{LR}^{(1)}(s) = \frac{16\pi^2\alpha^2_{em}}{27M_W^2s} \frac{\alpha}{\pi} \Im \left\{ (-i) \hat{\mu}^2 e^i \int \frac{d^d r}{(2\pi)^d} \frac{1}{\eta - \eta_+} \right\} \left[ 4 \ln \left( -\frac{\eta_+}{M_W} \right) \ln \left( \frac{2M_W}{m_+} \right) \right.
\]

\[
+ 3 \ln \left( \frac{2M_W}{m_+} \right) + \Re \left[ c_{p,LR}^{(1,fn)} \right] + \frac{\pi^2}{4} + 3 \right\} + \Delta \sigma_{\text{Coulomb}} + \Delta \sigma_{\text{decay}}
\]

\[
= \frac{4\alpha^3}{27s^{\frac{1}{2}} s} \Im \left\{ (-1) \sqrt{\frac{E + i\Gamma^{(0)}_{LR}}{M_W}} \left[ 4 \ln \left( -\frac{4(E + i\Gamma^{(0)}_{LR})}{M_W} \right) \ln \left( \frac{2M_W}{m_+} \right) \right. \right. 
\]

\[
- 5 \ln \left( \frac{2M_W}{m_+} \right) + \Re \left[ c_{p,LR}^{(1,fn)} \right] + \frac{\pi^2}{4} + 3 \right\} + \Delta \sigma_{\text{Coulomb}} + \Delta \sigma_{\text{decay}}.
\]  

(4.68)

After performing the \( r \)-integral we may set \( d \) to four and obtain a finite result. As expected the \( \epsilon \)-poles have cancelled, but the infrared-sensitivity of the cross section is reflected in the large logarithms \( \ln(2M_W/m_+) \).

### 4.5.2 Resummation of initial-state radiation

The conventional “partonic” cross section \( \hat{\sigma}_{LR,\text{conv}}^{(1)}(s) \) is obtained by comparing the right-hand sides of (4.48) and (4.49), and using equation (4.51):

\[
\hat{\sigma}_{LR,\text{conv}}^{(1)}(s) = \sigma_{LR}^{(1)}(s) - 2 \int_0^1 dx \Gamma^{LL}_{ee}^{(1)}(x) \sigma_{LR,\text{Born}}(xs). 
\]  

(4.69)

\( \Gamma_{ee}^{LL}(1)(x) \) is the \( O(\alpha) \) term in the expansion of the electron structure function 4.47. To calculate the subtraction term in (4.69) it is sufficient to approximate \( \sqrt{s} = 2M_W \) in the expression for \( \hat{\beta}_e \), to set \( \sigma_{LR,\text{Born}}(xs) \) to the leading-order Born term (2.48) with the replacement of \( E \) by \( E - M_W(1 - x) \) \(^4\), and to use \( \Gamma_{ee}^{LL}(1)(x) \) in the limit \( x \to 1 \),

\[
\Gamma_{ee}^{LL}(1)(x) \xrightarrow{x \to 1} \frac{\beta_e}{4} \left( \frac{2}{1 - x} + \frac{3}{2} \delta(1 - x) \right).
\]  

(4.70)

\(^4\)This follows from the requirement that \( xs - 4M_W^2 \) is of order \( M^2_\gamma \), that implies \( 1 - x \sim \delta \). Then \( E' \equiv \sqrt{xs} - 2M_W = \sqrt{s} + (1 - x) \delta \sim \sqrt{s} - 2M_W - \sqrt{s}(1 - x)/2 \sim E - M_W(1 - x) \).
To show the cancellation of large logarithms, we reintroduce the integral over $r$, and exchange the $r$- and $x$-integration,

$$-2\int_0^1 dx\, \Gamma_{ee}^{(1)}(x) \sigma^{(0)}_{LR}(xs) =$$

$$-\frac{16\pi^2\alpha_s^2}{27\hat{M}_W^4 s}\text{Im}\left\{ (-i) \hat{\mu}^2 e^\varepsilon \int \frac{d^d r}{(2\pi)^d} \frac{1}{\eta_- \eta_+} \frac{\beta_e}{2} \left[ 2 \ln \left( -\frac{\eta_-}{\hat{M}_W} \right) + \frac{3}{2} \right] \right\}, \tag{4.71}$$

where to obtain the final expression we have shifted the integration variable $r_0$ to $E - r_0$.

Equation (4.71), with the explicit expression of $\beta_e = \frac{2\alpha_s}{\pi} (2\ln(2\hat{M}_W/m_e) - 1)$, shows that $\hat{\sigma}^{conv}_{LR}(s)$ is free from the large electron-mass logarithms. Summing (4.68) and (4.71), and performing the $r$-integration, gives the final result for the next-to-leading order radiative correction to the conventional “partonic” cross section

$$\hat{\sigma}^{(1)}_{LR,conv}(s) = \frac{4\alpha^3}{27s^{4/2}} \text{Im}\left\{ (-1) \sqrt{-\frac{E + i\Gamma_0}{\hat{M}_W}} \left( 2 \ln \left( -\frac{4(E + i\Gamma_0)}{\hat{M}_W} \right) + \text{Re} \left[ c_{\rho,LR}^{(1,fin)} \right] \right) + \frac{\pi^2}{4} + \frac{1}{2} \right\} + \Delta\sigma^{(1)}_{\text{Coulomb}} + \Delta\sigma^{(1)}_{\text{decay}}. \tag{4.72}$$

The summation of collinear logarithms from initial-state radiation is completed by performing the convolution (4.50) using the Born cross section and the radiative correction (4.72) together with the electron structure functions from [90, 91], equation (4.47). This constitutes our final result, which we shall discuss in detail in the following section.
Chapter 5

NLO four-fermion production cross section

In this chapter we present our NLO predictions for the total cross section of the process $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u d X$, based on the results computed in Chapters 3 and 4. We also assess the theoretical error on the $W$-mass measurement due to the remaining uncertainties on the four-fermion cross-section.

5.1 Input parameters

In addition to the input parameters (3.32) used for the comparison of the EFT prediction to the tree-level cross section we use $\alpha_s = \alpha_s^{\overline{\text{MS}}}(80.4 \text{ GeV}) = 0.1199$ and the masses $m_t = 174.2$ GeV, $M_H = 115$ GeV, $m_e = 0.51099892$ MeV. \hfill \text{(5.1)}

The choice of the strong coupling constant is dictated by the universal QCD correction (2.51), which requires that $\alpha_s$ is evaluated at the scale $M_W$. We use the fine structure constant $\alpha$ in the $G_\mu$ scheme everywhere, including the electron distribution function (4.47). With these input parameters we obtain from (4.20) the numerical value of the $W$ width to NLO,

$$\Gamma_W = 3 \left( \Gamma^{(0)}_{W,l} + \Gamma^{(1,ew)}_{W,l} \right) + 2 \left( \Gamma^{(0)}_{W,h} + \Gamma^{(1,ew)}_{W,h} \right) \delta_{\text{QCD}} = 2.09201 \text{ GeV}. \hfill \text{(5.2)}$$

Here we have chosen to multiply not only the leading order, but also the electroweak correction to the hadronic decay by the factor $\delta_{\text{QCD}}$ defined in (2.51). Furthermore, in the numerical evaluation presented below we will resum the full NLO width (5.2) in the effective-theory propagator (2.38), rather than performing a strict expansion of the propagator in the perturbative corrections to the matching coefficient $\Delta$. This requires a modification of the contributions to the NLO cross section, equations (3.18), (3.27) and (4.72), as we will explain now.

Keeping $\Delta = -i \Gamma_W$ unexpanded in the propagator amounts to setting $\Gamma^{(1)}_W$ to zero in the NLO tree cross section (3.18) and to replacing $\Gamma^{(0)}_W$ by $\Gamma_W$ wherever it appears,
including equation (4.72). Some care has to be taken in order to obtain the correct cross section for the flavour-specific four-fermion final state from the calculation of the forward-scattering amplitude. Let us ignore for a moment the correction to the $W$ decay $\Delta \sigma_{\text{decay}}^{(1)}$, and focus on the remaining contributions. A cut through a single effective-theory propagator yields the factor

$$\frac{M_W \Gamma_W}{(r_0 - \frac{\mathbf{r}^2}{2M_W})^2 + \frac{\Gamma_W^2}{4}}.$$  \hfill (5.3)

As pointed out in Subsection 4.1.2, in the direct calculation of the four-fermion production cross section the term $\Gamma_W$ in the numerator would arise from the integration over a two-body phase space of the $W$-decay tree-level squared matrix element, which corresponds to the leading-order partial decay width, $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}$ or $\Gamma^{(0)}_{u\bar{d}}$. Hence, to correctly reproduce the flavour-specific cross section we multiply all contributions to the forward-scattering amplitude containing two cut effective-theory propagators by a factor $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}/\Gamma_W^2$ instead of the factor $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu} \Gamma^{(0)}_{u\bar{d}}/\Gamma_W^2 = 1/27$ used in the tree level analysis. This applies to the potential contributions of Section 3.1, the Coulomb and soft radiative corrections, and the contribution from the one-loop correction to the production operator. In the calculation of the matching coefficient of the four-electron production-decay operators performed in Section 3.2, one of the two decays is already correctly included at the lowest order by cutting through fermion loops containing only the correct final states. The other $W$ decay is effectively treated in the narrow-width approximation

$$\frac{M_W \Gamma_W}{(k^2 - M_W^2)^2 + M_W^2 \Gamma_W^2} \rightarrow \frac{\Gamma_W}{\Gamma_W} \delta(k^2 - M_W^2).$$ \hfill (5.4)

Therefore, to obtain the correct flavour-specific final state we have in this case to include only a single prefactor $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}/\Gamma_W$ or $\Gamma^{(0)}_{u\bar{d}}/\Gamma_W$, depending on the charge of the cut $W$ line. As shown in Table 5.1, with these new prescriptions the $N^{3/2}$LO effective-theory approximation and the full Born cross section (in the fixed-width definition but now using (5.2)) are in very good agreement, as in the earlier comparison, where only $\Gamma_W^{(0)}$ was resummed in the propagator.

As mentioned above all the electroweak radiative corrections to the flavour-specific four-fermion cross section are correctly reproduced by multiplying the inclusive forward-scattering amplitude by $\Gamma^{(0)}_{W^-\mu^-\bar{\nu}_\tau} \Gamma^{(0)}_{W^+\bar{u}\bar{d}}/\Gamma_W^2$, except for the correction to $W$ decay itself. These contributions are included by adding the decay correction

$$\Delta \sigma_{\text{decay}}^{(1)} = \left( \frac{\Gamma^{(1, \text{ew})}_{\mu^-\bar{\nu}_\mu}}{\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}} + \frac{\Gamma^{(1, \text{ew})}_{u\bar{d}}}{\Gamma^{(0)}_{u\bar{d}}} \right) \sigma^{(0)}$$ \hfill (5.5)

instead of (4.16). This can be understood by noting that multiplying the Born cross section by the prefactor $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu} \Gamma^{(0)}_{u\bar{d}}/\Gamma_W^2 \sim 1/27(1 - 2\Gamma^{(1)}_W/\Gamma_W^{(0)})$ is equivalent to incorporating the last term of (4.16), $-2\Gamma^{(1)}_W/\Gamma_W^{(0)} \sigma^{(0)}$, into the Born cross section. Because of the large NLO corrections to the tree cross section and the large effect of ISR, it is sensible to
apply the QCD decay correction to the full NLO electroweak cross section. This amounts to multiplying $\Gamma^{(0)}_{\mu\nu\bar{u}\bar{d}}$, $\Gamma^{(1,ew)}_{\mu\nu\bar{u}\bar{d}}$ by the radiative correction factor $\delta_{QCD}$ as given in (2.51), wherever they appear, which is consistent with the definition of the NLO W width (5.2). If in addition we also account for the QCD decay correction to the non-resonant contributions from Section 3.2, this is equivalent to multiplying the entire NLO electroweak cross section by $\delta_{QCD}$ and using the QCD corrected width (5.2). Note that this is not the correct treatment for the truly single-resonant diagrams entering the matching of the four-electron operators, but these contributions are numerically so suppressed that the error of this approximation is beyond our target accuracy.

### Table 5.1: Comparison of the numerical computation of the full Born result with WHIZARD with successive effective-theory approximations as in Table 3.1, but now with the full NLO decay width $\Gamma^W$, as given in(5.2), resummed in the propagator.

<table>
<thead>
<tr>
<th>$\sqrt{s}$ [GeV]</th>
<th>EFT Tree (NLO)</th>
<th>EFT Tree (N^{3/2}LO)</th>
<th>exact Born</th>
</tr>
</thead>
<tbody>
<tr>
<td>155</td>
<td>42.25</td>
<td>30.54</td>
<td>33.58(1)</td>
</tr>
<tr>
<td>158</td>
<td>65.99</td>
<td>60.83</td>
<td>61.67(2)</td>
</tr>
<tr>
<td>161</td>
<td>154.02</td>
<td>154.44</td>
<td>154.19(6)</td>
</tr>
<tr>
<td>164</td>
<td>298.6</td>
<td>303.7</td>
<td>303.0(1)</td>
</tr>
<tr>
<td>167</td>
<td>400.3</td>
<td>409.3</td>
<td>408.8(2)</td>
</tr>
<tr>
<td>170</td>
<td>469.4</td>
<td>481.7</td>
<td>481.7(2)</td>
</tr>
</tbody>
</table>

5.2 NLO four-fermion production cross section in the effective theory

We start our numerical analysis showing results for the $O(\alpha)$ “partonic” cross section, given by equation (4.72). In Figure 5.1 we plot the N^{1/2}LO and NLO electroweak radiative corrections to the “partonic” cross section, represented by first and second Coulomb corrections (equation (4.24)), the sum of hard and soft contributions (terms in curly brackets in (4.72)), and decay corrections (equation (5.5)). The three terms are given as relative corrections to the NLO tree-level effective-theory approximation (second column of Table 5.1). At the nominal threshold, $\sqrt{s} = 161$ GeV, the bulk of the electroweak correction is represented by the first Coulomb correction, which contributes a positive shift of $\sim 8\%$ (the second Coulomb correction is only $\sim 0.3\%$ of the NLO tree-level cross section), while the sum of soft and hard contributions and the decay corrections are both negative and amount respectively to $\sim -3\%$ and $\sim -1\%$ of the NLO tree approximation.

We now turn to the physical cross section, equation (4.50). The convolution of the “partonic” cross section with the electron structure functions contains integrations over partonic centre-of-mass energies far below threshold, where the effective field theory ap-
proximation eventually breaks down. Thus, the EFT calculation should be matched to a full cross section calculation below some centre-of-mass energy, for example $\sqrt{s} = 155$ GeV. Note that at such energies a Born treatment for the full calculation is sufficient, because the cross section is small below threshold, and the experimental error large due to the low statistics. Since the $N_3^3/2$ LO EFT approximation provides a very good approximation to the Born cross section, except significantly below threshold, we found it more convenient to replace the EFT approximation to the Born cross section, convoluted according to (4.50), by the full ISR-improved Born cross section (i.e. the full Born result convoluted with the electron structure functions) as generated by the WHIZARD program [70, 71], rather than performing the matching of the EFT result to the full Born cross section. To the numerical result generated by WHIZARD we add the NLO radiative correction (4.72), replacing the leading-order cross section $\sigma^{(0)}$ by the full Born cross section $\sigma_{\text{Born}}$ in the decay correction (5.5). This is also convoluted with the electron distribution functions. To avoid contributions from regions where the NLO EFT result is not valid, for the radiative corrections we simply cut off the integration region $\sqrt{x_1 x_2 s} < 155$ GeV. The dependence on the cut-off is negligible. Lowering it from to 155 GeV to 150 GeV (140 GeV), changes the cross section at $\sqrt{s} = 161$ GeV from 117.81 fb to 117.87 fb (117.91 fb), while the dependence on the cut-off for higher centre-of-mass energy is even smaller.

Our results for the NLO four-fermion cross section are given in Table 5.2, where we compare the exact Born cross section (second column, identical to the last column in Table 5.1), the ISR-improved Born cross section (third column) and the NLO result (fourth column). It is well known that the bulk of the full radiative correction is represented by initial-state radiation effects, which account for a large negative correction of about $-30\%$ at threshold. The size of the genuine radiative correction is assessed by comparing the “NLO” column to the “Born(ISR)” column, and it is seen to be $\sim +6\%$. The large

Figure 5.1: Electroweak radiative corrections to the partonic cross section: Coulomb-photon corrections (solid black), soft and hard corrections (dashed blue), and decay corrections (dotted red). The results are normalised to the NLO Born approximation.
\[ \sigma(e^-e^+ \rightarrow \mu^-\bar{\nu}_\mu ud X)(\text{fb}) \]

<table>
<thead>
<tr>
<th>( \sqrt{s} ) [GeV]</th>
<th>Born</th>
<th>Born(ISR)</th>
<th>NLO</th>
<th>NLO(ISR-tree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>158</td>
<td>61.67(2)</td>
<td>45.64(2)</td>
<td>49.19(2)</td>
<td>50.02(2)</td>
</tr>
<tr>
<td></td>
<td>[-26.0%]</td>
<td>[-20.2%]</td>
<td>[-18.9%]</td>
<td></td>
</tr>
<tr>
<td>161</td>
<td>154.19(6)</td>
<td>108.60(4)</td>
<td>117.81(5)</td>
<td>120.00(5)</td>
</tr>
<tr>
<td></td>
<td>[-29.6%]</td>
<td>[-23.6%]</td>
<td>[-22.2%]</td>
<td></td>
</tr>
<tr>
<td>164</td>
<td>303.0(1)</td>
<td>219.7(1)</td>
<td>234.9(1)</td>
<td>236.8(1)</td>
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<tr>
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<td>[-27.5%]</td>
<td>[-22.5%]</td>
<td>[-21.8%]</td>
<td></td>
</tr>
<tr>
<td>167</td>
<td>408.8(2)</td>
<td>310.2(1)</td>
<td>328.2(1)</td>
<td>329.1(1)</td>
</tr>
<tr>
<td></td>
<td>[-24.1%]</td>
<td>[-19.7%]</td>
<td>[-19.5%]</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>481.7(2)</td>
<td>378.4(2)</td>
<td>398.0(2)</td>
<td>398.3(2)</td>
</tr>
<tr>
<td></td>
<td>[-21.4%]</td>
<td>[-17.4%]</td>
<td>[-17.3%]</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2: Two NLO implementations of the effective-theory calculation, which differ by the treatment of initial-state radiation compared to the “exact” Born cross section without (second column) and with (third column) ISR improvement. The relative correction in brackets is given with respect to the Born cross section in the second column.

correction of about 8% due to Coulomb photon exchange is reduced by the convolution with ISR and partly cancelled by the negative corrections from hard and soft contributions and decay correction, though the positive QCD contribution (~ 2%) somehow compensates this effect. Note that, given that we aim at a theoretical accuracy at the sub-percent level, the genuine radiative correction is an important effect. We will comment further on this below.

In Section 4.5 we pointed out that a NLO calculation of the partonic cross section should be accompanied by a next-to-leading logarithmic resummation of the electron structure functions. However, the conventional implementation of ISR used here sums only leading logarithms \( \alpha^n \ln^n(2M_W/m_e) \). Thus, rather than convoluting the full NLO partonic cross section with the structure functions as done above and indicated in (4.50), one could choose to convolute only the Born cross section with the structure functions, and add the radiative corrections without ISR improvement, as done in previous NLO calculations [38,43]. The two implementations are formally equivalent, because the difference is given by next-to-leading logarithmic terms, \( \alpha^{n+1} \ln^n(2M_W/m_e) \), that are not controlled by the leading logarithmic approximation. The difference in the two implementations of the ISR convolution is obtained by comparing our NLO prediction with the results in the fifth column of Table 5.2, where we show the NLO cross section computed from the expression

\[
\sigma_{\text{ISR-tree}}(s) = \int_0^1 dx_1 \int_0^1 dx_2 \Gamma^{LL}_{ee}(x_1)\Gamma^{LL}_{ee}(x_2) \sigma_{\text{Born}}(x_1 x_2 s) + \hat{\sigma}_{\text{conv}}(s),
\]

\[ (5.6) \]
where the NLO correction to the “partonic” cross section, $\hat{\sigma}^{(1)}_{\text{conv}}(s)$, is given in (4.72) (with $1/27$ replaced by $\Gamma^{(0)}_{\mu-\bar{\nu}_\mu}/\Gamma^{(0)}_W$). The two implementations, along with the ISR-improved Born cross section, are plotted in Figure 5.2 for the centre-of-mass energy range $155 \text{ GeV} < \sqrt{s} < 170 \text{ GeV}$. The comparison shows that the difference between the two cross sections reaches almost two percent at threshold and is therefore much larger than the target accuracy in the per-mille range. The difference between the two implementations becomes smaller at higher energies and is negligible at $\sqrt{s} = 170 \text{ GeV}$. The impact of this difference on the accuracy of the $W$-mass measurement will be investigated further in Section 5.4.

### 5.3 Comparison to the full four-fermion calculation

We now compare the NLO prediction of the four-fermion production process (1.2) obtained with the effective theory method to the full NLO calculation performed in [43] in the complex-mass scheme. For this comparison we have to adjust our input parameters to those of [43],

$$M_W = 80.425 \text{ GeV}, \quad \Gamma_W = 2.0927 \text{ GeV}, \quad m_t = 178 \text{ GeV}, \quad \alpha_s = 0.1187, \quad (5.7)$$

and use $\alpha(0) = 1/137.03599911$ in the relative radiative corrections as in [43] (this amounts to multiply the contributions computed in Chapter 4 by a factor $\alpha(0)/\alpha_G$). We first compare the strict electroweak NLO calculation, i.e. the cross section without the QCD correction $\delta_{\text{QCD}}$ and without initial-state radiation beyond the first-order term. In the effective-theory calculation the corresponding radiative correction is given by (4.68), omit-
\[ \sigma(e^-e^+ \rightarrow \mu^-\bar{\nu}_\mu u\bar{d}X)(fb) \]

<table>
<thead>
<tr>
<th>(\sqrt{s}) [GeV]</th>
<th>Born</th>
<th>NLO(EFT)</th>
<th>ee4f [43]</th>
<th>DPA [43]</th>
</tr>
</thead>
<tbody>
<tr>
<td>161</td>
<td>150.05(6)</td>
<td>104.97(6)</td>
<td>105.71(7)</td>
<td>103.15(7)</td>
</tr>
<tr>
<td>170</td>
<td>481.2(2)</td>
<td>373.74(2)</td>
<td>377.1(2)</td>
<td>376.9(2)</td>
</tr>
</tbody>
</table>

Table 5.3: Comparison of the strict electroweak NLO results (without QCD corrections and ISR resummation).

\[ \sigma(e^-e^+ \rightarrow \mu^-\bar{\nu}_\mu u\bar{d}X)(fb) \]

<table>
<thead>
<tr>
<th>(\sqrt{s}) [GeV]</th>
<th>Born(ISR)</th>
<th>NLO(EFT)</th>
<th>ee4f [43]</th>
<th>DPA [43]</th>
</tr>
</thead>
<tbody>
<tr>
<td>161</td>
<td>107.06(4)</td>
<td>117.38(4)</td>
<td>118.12(8)</td>
<td>115.48(7)</td>
</tr>
<tr>
<td>170</td>
<td>381.0(2)</td>
<td>399.9(2)</td>
<td>401.8(2)</td>
<td>402.1(2)</td>
</tr>
</tbody>
</table>

Table 5.4: Comparison of NLO results with QCD corrections and ISR resummation included.

The discrepancies between the EFT result and the full four-fermion calculation are only 0.7% at \(\sqrt{s} = 161\) GeV and grows to about 1% at \(\sqrt{s} = 170\) GeV.

Next, in Table 5.4, we compare the full result including the QCD correction and the resummation of ISR corrections with [43]. Here we implement the QCD correction as in [43], by multiplying the entire electroweak NLO result by the overall factor \((1 + \alpha_s/\pi)\). Furthermore, we include ISR corrections only to the Born cross section as in (5.6), in agreement with the treatment of [43]. Again the second-order Coulomb correction is set to zero, because [43] does not include any two-loop effects. As before, the Table shows the two NLO calculations, the Born cross section (now ISR improved) and the double-pole approximation. The discrepancy between the EFT calculation and the full four-fermion calculation is around 0.6% at threshold. Note that the EFT approximation is significantly better than the double-pole approximation directly at threshold, while at higher energies the quality of the DPA improves relative to the EFT approximation, since no threshold expansion is performed in the DPA.

### 5.4 Theoretical error of the \(M_W\) determination

As discussed in the introduction of this thesis, one of the options for determining the \(W\) mass at ILC is the measurement of the four-fermion production cross section at a few selected centre-of-mass energies near the \(W\)-pair production threshold. In this
section we want to estimate the error on the $W$ mass from various sources of theoretical uncertainty. To this end we assume that measurements $O_i$ will be taken at $\sqrt{s} = 160, 161, 162, 163, 164$ GeV, and at $\sqrt{s} = 170$ GeV, and that the measured values coincide with our NLO calculation (labelled “NLO(EFT)” in Table 5.2), corresponding to the “true” value of the $W$ pole mass of $M_W = 80.377$ GeV. By $E_i(\delta M_W)$ we will denote the cross section at the six centre-of-mass energy points chosen above, for any other theoretical calculation of the four-fermion production process. $E_i(\delta M_W)$ are functions of the input value chosen for the $W$ mass, which is parametrised as $M'_W = 80.377$ GeV $+ \delta M_W$. We thus regard the value $\delta M_W$ at which the minimum of the $\chi^2$ function,

$$\chi^2(\delta M_W) = \sum_{i=1}^{6} \frac{(O_i - E_i(\delta M_W))^2}{\sigma_i^2},$$

(5.8)

is located as an estimate of the error on the determination of $M_W$ due to the difference between the theoretical prediction $E_i$ and the “measured” cross section $O_i$. For simplicity we assume that each point carries the same weight, so $\sigma_i \equiv \sigma$ is an arbitrary constant of mass dimension $-2$. We explicitly checked that a more realistic assignment $\sigma_i \sim \sqrt{O_i}$ does not lead to significantly different results.

If, for instance, we set $E_i(\delta M_W)$ to the ISR-improved Born cross section without genuine electroweak corrections and QCD corrections (labelled “Born(ISR)” in Table 5.2), we obtain $\delta M_W = -201$ MeV. This tells us that comparing measurements to a theoretical calculation that does not include the genuine radiative corrections would result in a value of $M_W$ that is about 200 MeV too low. The NLO calculation is therefore crucial for an accurate $M_W$ determination. We will now try to estimate whether its accuracy is enough.

**Treatment of initial-state radiation.** As pointed out in the previous section, the two different implementations of ISR, given in the last and next-to-last columns of Table 5.2, are formally equivalent at the leading-logarithmic level. However, the difference in the predicted cross section at $\sqrt{s} = 161$ GeV, where the sensitivity to $M_W$ is largest, is about 2%. Here we take this difference as a measure of the uncertainty related to the missing next-to-leading logarithmic corrections to the structure function. To estimate how this uncertainty affects the error on $M_W$, we apply the procedure discussed above and find

$$[\delta M_W]_{\text{ISR}} \approx 31 \text{ MeV}.$$  

(5.9)

This large error could be avoided by measuring the cross section predominantly around 170 GeV rather than around 162 GeV, but the sensitivity to $M_W$ is significantly smaller at higher energies, as shown by Figure 5.3 below. Thus, this error should be eliminated by a consistent treatment of the electron structure functions at the next-to-leading logarithmic level, in which all NLL corrections are taken into account by convoluting the NLO cross section with the NLL structure functions. A related effect is connected to the choice of scheme and scale of the electromagnetic coupling. The difference in the cross section between using $\alpha(0)$ and $\alpha$ in the $G_{\mu}$-scheme in the radiative correction, including, in particular, initial-state radiation, is about 1%, which translates into another error of about
15 MeV on the $W$ mass. The scale ambiguity of the coupling used in initial-state radiation can be resolved only in the context of a next-to-leading logarithmic resummation, which takes the evolution of $\alpha$ between $m_e$ and $\Gamma_W$ into account. Here, the choice of $\alpha$ in the $G\mu$ scheme was motivated by the observation that its value is numerically close to the electromagnetic running coupling at the typical scales of the short-distance cross section, $\mu \gtrsim \Gamma_W \approx 2$ GeV.

**Missing corrections to the “partonic” cross section.** Further uncertainties on the four-fermion cross section come from the leading missing higher-order terms in the expansion in $\alpha$ and $\delta$. These are $N^{3/2}\text{LO}$ corrections to the forward-scattering amplitude from four-loop potential diagrams (third Coulomb correction), three-loop diagrams with two potential loops and one soft loop (interference of single-Coulomb and soft radiative corrections), two-loop potential diagrams with $O(\alpha)$ matching coefficients or $O(\delta)$ higher-dimensional production operators, and the $O(\alpha)$ correction to the matching coefficients of the four-electron production-decay operators. Of these contributions, the latter is expected to be the largest one, since the non-resonant $N^{1/2}\text{LO}$ contributions are large at the Born level ($\sim 40\%$ at threshold, see Table 3.1). Presumably, this contribution is also the origin of the 0.6% difference between the EFT result “NLO(EFT)” and the full four-fermion calculation “ee4f” at $\sqrt{s} = 161$ GeV in Table 5.4. A rough estimate of this correction to
the helicity-averaged cross section is
\[ \Delta \hat{\sigma} = \frac{\alpha^4}{27 s_{\text{in}}^8 s} K, \]  
(5.10)
where \( K \) is an \( s \)-independent constant of order 1. In fact, if we attributed the difference between our calculation ("NLO(EFT)") and that of \([43]\) ("ee4f") at \( \sqrt{s} = 161 \text{ GeV} \) exclusively to this contribution, we would obtain \( K = 0.96 \). We thus choose \( K = 1 \), add (5.10) to the "NLO(EFT)" calculation, and minimise the \( \chi^2 \) function. From this we obtain an error
\[ [\delta M_W]_{\text{non-res}} \approx 8 \text{ MeV}. \]  
(5.11)
This error could be removed by using the full NLO four-fermion calculation, where the correction (5.10) is included. We expect the second largest uncalculated correction to the partonic cross section to come from diagrams with single-Coulomb exchange and a soft photon or a hard correction to the production vertex. A naive estimate of the sum of the two terms is
\[ \Delta \hat{\sigma} = \frac{\hat{\sigma}^{(1)}_{LR} - \Delta \sigma^{(1)}_{\text{Coulomb}} - \Delta \sigma^{(1)}_{\text{decay}}}{\sigma^{(0)}_{LR}} \delta \sigma^{(1)}_{\text{Coulomb}}, \]  
(5.12)
where the quantities involved have been defined in Chapter 4. Estimating the corresponding uncertainty on the \( W \) mass as before, we find
\[ [\delta M_W]_{\text{Coulomb \times (hard+soft)}} \approx -5 \text{ MeV}. \]  
(5.13)
Since the correction 5.12 corresponds to a genuine NNLO (in the conventional counting scheme) contribution, the corresponding error on \( M_W \) is not eliminated by the full NLO four-fermion calculation. Thus, to reach a total error of \( \sim 6 \text{ MeV} \) it is necessary to include at least some \( N_3^{3/2} \) LO corrections in the EFT approach. This will be discussed in Chapter 6.

The discussion above is summarised in Figure 5.3, where we plot \( \kappa = \sigma(s,M_W + \delta M_W)/\sigma(s,M_W) \) for different values of \( \delta M_W \) as function of the centre-of-mass energy, \( \sigma \) being our NLO result, "NLO(EFT)". The relative change in the cross section is shown as dashed lines for \( \delta M_W = \pm 15, \pm 30, \pm 45 \text{ MeV} \). The shape of these curves shows that the sensitivity of the cross section to the \( W \) mass is largest around the nominal threshold \( \sqrt{s} \approx 161 \text{ GeV} \), as expected, and rapidly decreases for larger \( \sqrt{s} \). The loss in sensitivity is partially compensated by a larger cross section, implying smaller relative statistical errors of the anticipated experimental data.

The dark-shaded area in Figure 5.3 corresponds to the uncertainty on the cross section from (5.12), while the light-shaded area adds (linearly) the uncertainty from (5.10). The theoretical error decreases with \( \sqrt{s} \), since \( \Delta \sigma \) in (5.10) is roughly energy-independent, while \( \sigma \) increases. The largest current uncertainty is, however, due to ambiguities in the implementation of ISR. The solid red curve gives the ratio of the two different implementations of ISR, NLO(EFT) vs. NLO(ISR-tree), both evaluated at \( M_W = 80.377 \text{ GeV} \).
As mentioned above this uncertainty can be removed with further work on a next-to-leading-logarithmic ISR resummation, that will be required for many other processes at a high-energy $e^-e^+$ collider.
In the previous chapter we have shown that, aside from a large uncertainty due to an incomplete treatment of next-to-leading logarithmic effects, the biggest remaining theoretical error on the NLO effective-theory result is related to missing higher-order contributions that are suppressed by $\delta^{3/2}$ ($N^{3/2}\text{LO}$) compared to the leading Born cross section. Some of these corrections (interference of Coulomb-photon exchange and higher-dimensional production operators, $O(\alpha^3)$ four-fermion operators) have to be ascribed to higher-order terms in the threshold expansion of two-loop forward-scattering diagrams, and can be accounted for by the full four-fermion calculation [43, 44]. The remaining ones are represented by three-loop (interference of Coulomb-photon exchange with hard and soft effects) and four-loop (triple Coulomb exchange) forward-scattering diagrams. In a complete four-fermion calculation these would correspond to genuine NNLO (and NNNLO) Standard-Model diagrams, and are thus not included in [43, 44]. In Section 5.4 we estimated that these corrections are responsible for an uncertainty of 5 MeV on the $W$-mass determination. We therefore conclude that these missing contributions must be included in any prediction of the four-fermion cross section near the $W$-pair threshold that aims to a total error on $M_W$ of $\sim 6$ MeV, as required for ILC phenomenology.

In this chapter we present the calculation of all the $N^{3/2}\text{LO}$ corrections to the process $e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u\bar{d} + X$ not already included in the full four-fermion calculation in the complex-mass scheme [43, 44]. These results were first presented in [48], to which we refer the reader for details.

### 6.1 Relevant momentum regions and classification of $N^{3/2}\text{LO}$ corrections

The total cross section of the four-fermion production process is extracted, as before, from the appropriate unitarity cuts of the $e^- e^+$ forward-scattering amplitude. The $N^{3/2}\text{LO}$ corrections are computed by expanding by regions full NNLO Standard-Model diagrams and retaining only the dominant terms. The momentum regions (in the centre-of-mass
frame) contributing to the expansion include the four already introduced in Section 2.2,

\[
\begin{align*}
\text{hard} : & \quad k_0 \sim |\mathbf{k}| \sim M_W, & \text{potential} : & \quad k_0 \sim M_W \delta, & |\mathbf{k}| \sim M_W \sqrt{\delta}, \\
\text{soft} : & \quad k_0 \sim |\mathbf{k}| \sim M_W \delta, & \text{collinear} : & \quad k_0 \sim M_W, & k^2 \sim M^2_W \delta,
\end{align*}
\]

where \( k \) is an arbitrary loop-integration momentum. Starting from NNLO diagrams another mode, that will be called ‘semi-soft’ here, has to be included in addition to the ones given in (6.1) \(^1\):

\[
\text{semi-soft} : \quad k_0 \sim |\mathbf{k}| \sim M_W \sqrt{\delta}.
\]

As it will be shown below, semi-soft modes contribute to the renormalisation of the Coulomb force between the two slowly-moving \( W \) bosons. For a non-vanishing electron mass \( m_e \), also the hard-collinear and soft-collinear regions discussed in Section 4.5.1 contribute to the \( N^{3/2} \)LO cross section.

Before presenting the actual calculation, we briefly remind how the non-standard scaling \( \delta^{3/2} \) of the dominant NNLO SM diagrams arises. As seen in Section 2.2.1, the leading-order term from the threshold expansion of the SM diagram in Figure 2.7e, reproduced by the leading-order EFT cross section, equation (2.48), counts as \( \alpha^2_{\text{ew}} \sqrt{\delta} \). Adding a potential photon loop (Figure 2.7g) to the one-loop forward-scattering diagram, introduces an additional suppression \( \alpha/\sqrt{\delta} \sim \sqrt{\delta} \), as explained below equation 2.39 and explicitly found in the calculation in Section 4.2. Hard corrections (Section 4.1) and soft corrections (Section 4.3), corresponding to the SM cut diagrams in Figure 2.7f and 2.7h respectively, are both suppressed by \( \alpha \sim \delta \) compared to the leading term. As we will see below, the same is true for semi-soft loops. We thus expect that the dominant NNLO Standard-Model corrections to the relevant cuts of the forward-scattering amplitude arise from three-loop diagrams in which two of the loop momenta are potential, and the remaining one is hard, semi-soft or soft, and that their scaling, compared to the leading-order forward-scattering amplitude, is \( \delta^{1/2} \times \delta = \delta^{3/2} \), thus identifying them as effective \( N^{3/2} \)LO corrections. Note that those corrections include more terms that the one naively estimated in Section 5.4, where semi-soft momenta have not been considered. We repeat that contributions of the same order in the EFT power counting that are already included in the SM LO or NLO diagrams are not considered here. Those \( N^{3/2} \)LO radiative corrections that correspond to SM NNLO diagrams can be readily organised in the following classes:

**Mixed hard/Coulomb corrections:** This class is given by diagrams with a single-Coulomb photon and one insertion of a hard NLO correction to the

- **production stage:** Here the LO operator (2.42) is replaced by the NLO expression (Figure 6.1a). A representative diagram of the full-SM counterpart is shown in Figure 6.1d; this correction is computed in Subsection 6.2.2.
- **decay stage:** A sample diagram in the standard loop expansion is shown in Figure 6.1e. The implementation of this correction in the EFT, discussed in Subsection 6.2.6, is denoted by the black dot labelled \( \delta_{\text{decay}} \) in Figure6.1b.

---

\(^1\)This is analogous to the ‘soft’ mode in the NRQCD literature whose ‘ultrasoft’ mode corresponds to the ‘soft’ mode in our conventions.
Figure 6.1: Sample $N^{3/2}$LO $\text{EFT}^\text{EFT}$ diagrams in the EFT (first line) with mixed hard/Coulomb corrections and corresponding NNLO diagrams in the full SM (second line). External fermionic lines are electrons and positrons; internal fermionic lines appearing in the diagrams in the second line represent the $\mu^- \bar{\nu}_\mu$ and $u \bar{d}$ doublets. Note the insertion of a NLO production operator in diagram a.

- **propagation stage:** A sample diagram in the standard loop expansion is shown in Figure 6.1f. The implementation of this correction in the EFT, discussed in Subsection 6.2.7, is denoted by the black dot labelled $\delta_{\text{residue}}$ in Figure 6.1b.

**Interference of Coulomb and radiative corrections:** There are two contributions in this class:

- **Single-Coulomb exchange and soft photons.** An EFT diagram in this class is shown in Figure 6.2a. A representative diagram of the full-SM counterpart is shown in Figure 6.2d. This correction is computed in Subsection 6.2.1.

- **Single-Coulomb exchange and collinear photons.** These corrections (see Figure 6.2b for a representative diagram in the EFT, and Figure 6.2e for a counterpart in the full SM) vanish if the electron mass is set to zero. If a finite electron mass is used as infrared (IR) regulator there are further contributions from soft-collinear and hard-collinear modes that are computed in Subsections 6.2.1 and 6.2.2, respectively.

**NLO corrections to the Coulomb potential:** The relevant diagram is given by a semi-soft fermion bubble insertion into the Coulomb photon (see Figure 6.2c for the EFT
Figure 6.2: Sample N\(^{3/2}\)LO\(^{\text{EFT}}\) diagrams with single-Coulomb exchange and radiative corrections in the EFT (first line) and corresponding NNLO diagrams in the full SM (second line). The same conventions of Figure 6.1 are adopted.

diagram and Figure 6.2f for the counterpart in the full SM). This correction is computed in Subsection 6.2.5 and Appendix F.

This concludes our survey of the dominant NNLO corrections. In the next section we will compute the different contributions identified above. At the same order in the EFT power counting as the above contributions we also encounter triple-Coulomb exchange that is a NNNLO correction in the standard loop expansion. This effect can be straightforwardly computed by expanding the all-order Coulomb contribution (4.23) computed in Section (4.2) to order \(\alpha^5\):  
\[
\Delta \sigma_{C3}^{\text{LR}} = \frac{\pi a_{ew}^2}{27 s} \alpha^3 \zeta(3) \text{Im} \left[ -\frac{M_W}{E + i\Gamma_W} \right].
\]  
(6.3)

This correction turns out to be negligible. The contribution to the helicity-averaged cross section \(\Delta \sigma_{C3} = \Delta \sigma_{C3}^{\text{LR}}/4\) directly at threshold is \(\Delta \sigma_{C3}(\sqrt{s} = 161\text{GeV}) = 0.01\text{fb}\), while the effect is even smaller away from threshold.
6.2 Evaluation of $N^{3/2}$LO corrections

In this section we compute all the relevant contributions identified in the previous section. Interference of single-Coulomb exchange with corrections from soft and soft-collinear photons are computed in Subsection 6.2.1. Subsection 6.2.2 contains the results for interference of single-Coulomb exchange with hard corrections to the production stage and with hard-collinear photon exchange. Both computations are carried out in a form that allows in principle to include all-order Coulomb exchange as well. This subset of $N^{3/2}$LO corrections is combined to a finite partial result in Subsection 6.2.3 and large logarithms of the electron mass are absorbed in electron structure functions in Subsection 4.5. Radiative corrections to the single-Coulomb exchange potential itself are discussed in Subsection 6.2.5. Interference effects of single-Coulomb exchange with corrections to the decay and propagation stages are subject of Subsections 6.2.6 and 6.2.7.

6.2.1 Soft and soft-collinear corrections

We consider here the contribution of single-Coulomb exchange diagrams with an additional soft-photon loop, as in Figure 6.2a. As a consequence of the soft-photon approximation, and of the analytic properties of the Coulomb corrected amplitude (4.21), it is actually possible to compute the correction arising from inserting a single soft photon into the all-order corrected forward-scattering amplitude, Figure 4.3. We thus consider diagrams with the topology shown in Figure 6.3, where the gray circle stands for all-order Coulomb exchange, as encoded in the Green function (4.22), rather than Figure 6.2a. Analogously to the NLO calculation presented in Chapter 4 and Appendix (D), diagrams with a coupling of soft photons to Ω lines cancel, as can be shown using a gauge-invariance argument (see also Section 7.3). Thus, the only $O(\alpha)$ soft contributions to the Coulomb-corrected forward-scattering amplitude arise from initial-initial state interference diagrams, as was the case for the calculation presented in Section 4.3.

Employing the soft-photon approximation discussed in Section 4.3, the sum of the two diagrams shown in Figure 6.3 can be written, in close analogy with (4.21), as

$$iA_{LR}^{C\times S} = \frac{16\pi^2\alpha^2}{M_W^2} (1 - \epsilon) 2I_S,$$

where $I_S$ denotes the convolution of the zero-distance Green function and a single soft-photon exchange correction. Including the correct prefactors it reads

$$I_S \equiv -32\pi\alpha M_W^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{G_C^{(0)}(0,0;\xi_W - k^0)}{(k^2 + i0)(-2p_1 \cdot k + i0)(-2p_2 \cdot k + i0)},$$

where, as usual, $d \equiv 4 - 2\epsilon$, $\mu^2 \equiv \mu^2 e^{7\pi i}/(4\pi)$, and $\xi_W = E + i\Gamma_W$. $G_C^{(0)}$ is given, to all orders in $\alpha$ but in $d = 4$, in equation (4.22). The soft integral (6.5) can be easily evaluated with the same techniques used in Section 4.3. We perform the $k^0$ integration closing the integration contour in the lower complex $k^0$ half-plane, using Cauchy theorem and picking up the residue of the pole at $k^0 = |\vec{k}| - i0$. Note that all singularities of $G_C^{(0)}$ are located in
Figure 6.3: Soft-photon corrections to the all-order Coulomb-corrected forward-scattering amplitude.

the upper $k^0$ half plane. Setting the external momenta to $p_1 \equiv M_W n_-$ and $p_2 \equiv M_W n_+$, the result reads

$$I_S = -i \left( \frac{\alpha}{\pi} \right) \frac{\sqrt{\pi}}{\epsilon \Gamma(1/2 - \epsilon)} (e^{\gamma E} \mu^2)^{\epsilon} \int_0^\infty dk G_C^{(0)}(0, 0; E_W - k) \frac{k^{1+2\epsilon}}{k^{1+2\epsilon}}. \quad (6.6)$$

The single pole in the $\epsilon$ plane in the prefactor of (6.6) is associated with the emission of photons collinear to the incoming electron or positron, whose mass must be neglected in the soft region. The finite electron mass requires the introduction of the two further regions, soft- and hard-collinear, discussed in Section 4.5, which convert the collinear $1/\epsilon$ pole into a large logarithm containing the electron mass.

The soft-collinear correction to the forward-scattering amplitude is

$$i A_{L R}^{C \times SC} = \frac{16\pi^2 \alpha_{em}^2}{M_W^2} (1 - \epsilon) \frac{4}{I_{SC}}, \quad (6.7)$$

where the integral $I_{SC}$, denoting the convolution of the zero-distance Green function with soft-collinear emission, is the same expression as (6.5), but threshold expanded according to (4.56) and (4.57). The result for the soft-collinear configuration reads

$$I_{SC} = \frac{i}{2} \left( \frac{\alpha}{\pi} \right) \Gamma(\epsilon) \left( \frac{M_W}{m_e} \right)^{2\epsilon} (e^{\gamma E} \mu^2)^{\epsilon} \int_0^\infty dk \frac{G_C^{(0)}(0, 0; E_W - k)}{k^{1+2\epsilon}}. \quad (6.8)$$

Summing up the soft and soft-collinear corrections (6.4) and (6.7) and using the explicit results for the integrals $I_S$ and $I_{SC}$ (6.6) and (6.8), we obtain

$$A_{L R}^{C \times [S+SC]} \equiv A_{L R}^{C \times S} + A_{L R}^{C \times SC} = \frac{32\pi \alpha_{em}^2 \alpha}{M_W^2} (1 - \epsilon) \left[ -\frac{\sqrt{\pi}}{\epsilon \Gamma(1/2 - \epsilon)} + \Gamma(\epsilon) \left( \frac{M_W}{m_e} \right)^{2\epsilon} \right]

\times (e^{\gamma E} \mu^2)^{\epsilon} \int_0^\infty dk \frac{G_C^{(0)}(0, 0; E_W - k)}{k^{1+2\epsilon}}. \quad (6.9)$$

As expected, the $\epsilon$ pole cancels in the prefactor of (6.9). The infrared sensitivity of the result is reflected in the large logarithms $\ln (M_W/m_e)$. 

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6.2.2 Hard and hard-collinear corrections

We turn now to the radiative corrections to the total cross section of (1.2) obtained replacing the LO production operator (2.42) with the NLO expression, equation (2.54). As for the NLO calculation, the one-loop coefficient $C_{p,RL}^{(1)}$ is irrelevant, since the $e^{-R}e^{+L}$ helicity configuration does not give any LO contribution, and no interference between the $RL$ and $LR$ configurations arises. Thus, analogously to equation (4.9), the hard production-vertex contributions to the Coulomb-corrected forward-scattering amplitude can be readily obtained multiplying the expression in (4.21) by twice the one-loop coefficient (4.6),

$$\text{Im} \mathcal{A}^{C\times H}_{LR} = \frac{16\pi^2\alpha_{ew}^2}{M_W^2}(1 - \epsilon)2\text{Re}C_{p,LR}^{(1)}\text{Im} G^{(0)}_{C}(0,0;\mathcal{E}_W).$$  

To (6.10) we have to add hard-collinear photon corrections, associated with momentum scalings $k^0 \sim M_W$ and $k^2 \sim m_e^2$, and illustrated in Figure 6.4. As explained in Appendix E, hard-collinear photon corrections to the forward-scattering amplitude have a factorised form, and their contribution can be obtained multiplying (4.21) by twice the hard-collinear factor derived in Appendix E,

$$\mathcal{A}^{C\times HC}_{LR} = \frac{16\pi^2\alpha_{ew}^2}{M_W^2}(1 - \epsilon)2C_{h-c}^{(1)}G^{(0)}_{C}(0,0;\mathcal{E}_W).$$  

(6.11)

Summing up the production-vertex and hard-collinear photon corrections (6.10) and (6.11) one obtains

$$\text{Im} \mathcal{A}^{C\times [H+HC]}_{LR} \equiv \text{Im} \left( \mathcal{A}^{C\times H}_{LR} + \mathcal{A}^{C\times HC}_{LR} \right) = \frac{16\pi^2\alpha_{ew}^2\alpha}{M_W^2}(1 - \epsilon)\text{Im} G^{(0)}_{C}(0,0;\mathcal{E}_W)$$

$$\times \left\{ 2\ln \left( \frac{2M_W}{m_e} \right) \left[ \frac{1}{\epsilon} + \ln \left( \frac{2M_W}{m_e} \right) - \ln \left( \frac{4M_W^2}{\mu^2} \right) + \frac{3}{2} \right] + 3 + \frac{7\pi^2}{12} + \text{Re} C_{p,LR}^{(1,\text{fin})} \right\}.$$  

(6.12)

6.2.3 The total cross section

We now combine the two results (6.9) and (6.12), and show how the sum is free of $\epsilon$-poles. In order to show how the cancellation of poles takes place, we introduce a modified ‘+’
distribution, defined by
\[
\int_0^\infty \frac{dk}{k} f(k) \equiv \int_0^a \frac{dk}{k} f(k) - f(0) + \int_a^\infty \frac{dk}{k} f(k),
\]  
(6.13)
where \(a\) is an arbitrary positive real parameter. By recasting the integral containing the zero-distance Coulomb Green function on the right-hand side of (6.9) into
\[
\mu^2 \int_0^\infty \frac{dk}{k^{1+2\epsilon}} \left[ \ln \left( \frac{a}{\mu} \right) \right] f(k) + \int_a^\infty \frac{dk}{k} f(k),
\]  
(6.14)
where \(f(k) \equiv G_C^{(0)}(0,0;E_W - k)\), we can isolate the \(\epsilon\)-pole implicitly contained in the convolution, and prove that the sum of (6.9) and (6.12) is free of \(\epsilon\) poles. Note that the final result does not depend on the particular choice of \(a\). The full correction to the total cross section (1.2) reads
\[
\Delta \sigma_{C \times [S+H]}^{[S+H]} \equiv \frac{1}{27 s} \text{Im} \left( A_{LR}^{[S+SC]} + A_{LR}^{[H+HC]} \right)
\]  
(6.15)
Since all poles in \(\epsilon\) have cancelled, we are justified to set \(G_C^{(0)}(0,0;E_W - k)\) to its four-dimensional expression given in (4.22). By inserting the zero-Coulomb exchange term of (4.22) into (6.15), and using the result
\[
\text{Im} \int_0^\infty \frac{1}{|k|_{a+}} \sqrt{-\frac{E_W - k}{M_W}} = \text{Im} \left\{ \sqrt{-\frac{E_W}{M_W}} \left[ \ln \left( -\frac{4E_W}{a} \right) - 2 \right] \right\},
\]  
(6.16)
one can easily rederive the NLO correction to the total cross section computed in Chapter 4, equation (4.68) (except for the terms \(\Delta \sigma_\text{Coulomb}^{(1)}\) and \(\Delta \sigma_\text{decay}^{(1)}\)).

The new \(N^{3/2}\)LO correction is obtained by inserting the one-Coulomb exchange term of (4.22) into (6.15), and using the result
\[
\text{Im} \int_0^\infty \frac{1}{|k|_{a+}} \sqrt{-\frac{E_W - k}{M_W}} = \text{Im} \left\{ \sqrt{-\frac{E_W}{M_W}} \left[ \ln \left( -\frac{4E_W}{a} \right) - 2 \right] \right\},
\]  
(6.16)
We thus have
\[
\Delta \sigma_{C \times [S+H]}^{[S+H]} = \frac{4 \alpha_e^2 \alpha^2}{27 s} \text{Im} \left\{ -\frac{1}{2} \ln \left( -\frac{E_W}{M_W} \right) - 2 \ln \left( -\frac{E_W}{M_W} \right) \ln \left( \frac{2M_W}{m_e} \right) \right\},
\]  
(6.17)
We thus have
\[
\Delta \sigma_{C \times [S+H]}^{[S+H]} = \frac{4 \alpha_e^2 \alpha^2}{27 s} \text{Im} \left\{ -\frac{1}{2} \ln \left( -\frac{E_W}{M_W} \right) - 2 \ln \left( -\frac{E_W}{M_W} \right) \ln \left( \frac{2M_W}{m_e} \right) \right\},
\]  
(6.18)
As expected, the dependence on the \(a\) regulator has cancelled. The final result is finite in the \(\epsilon\) plane, but the infrared sensitivity of the cross section is reflected in the large logarithms \(\ln(2M_W/m_e)\).

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6.2.4 ISR resummation

As explained in Section 4.5, large collinear logarithms \(\ln(2M_W/m_e)\) are resummed to all orders (at leading logarithmic accuracy) by convoluting the “partonic” cross section \(\hat{\sigma}_h\) with electron distribution functions \(\Gamma_{ee}^{LL}\), as indicated in equation (4.50). Following the procedure outlined there, we construct \(\hat{\sigma}^{C\times[S+H]}_{LR}\) starting from \(\sigma^{C\times[S+H]}_{LR}\) of (6.15) and performing the subtraction,

\[
\hat{\sigma}^{C\times[S+H]}_{LR}(s) = \sigma^{C\times[S+H]}_{LR}(s) - 2 \int_0^1 dx \Gamma_{ee}^{LL,(1)}(x)\sigma^{C}_{LR}(xs). \tag{6.19}
\]

\(\Gamma_{ee}^{LL,(1)}\) was defined in equation (4.70), while \(\sigma^{C}_{LR}\) is the all-order Coulomb-corrected cross section, equation (4.23). Note that the zero-distance Coulomb Green function appearing in (4.21) has now to be evaluated with the replacement \(E \rightarrow E - M_W(1 - x)\), as for the leading-order cross section in (4.71). Using (4.70), we can write the subtraction term appearing in (6.19) as

\[
-2 \int_0^1 dx \Gamma_{ee}^{LL,(1)}(x)\sigma^{C}_{LR}(xs) = -\frac{16\pi\alpha_e^2\alpha}{27sM_W^2} \left[ 2 \ln \left( \frac{2M_W}{m_e} \right) - 1 \right] \times
\]

\[
\times \text{Im} \left\{ \frac{3}{2} G^{(0)}_C(0,0;E_W) + 2 \int_0^k dk \frac{G^{(0)}_C(0,0;E_W-k)}{[k]_{M_W+}} \right\}. \tag{6.20}
\]

Note that, to obtain this result, a term involving the integral \(\text{Im}\left[ \int_M^{\infty} dk G^{(0)}_C(0,0;E_W-k)/k \right]\) has been dropped. This is anyway suppressed by a power of \(E_W/M_W\) and corresponds to hard-collinear initial-state radiation.

Replacing (6.20) and (6.15) in (6.19) and choosing \(a = M_W\), we finally obtain

\[
\hat{\sigma}^{C\times[S+H]}_{LR}(s) = \frac{16\pi\alpha_e^2\alpha}{27sM_W^2} \left[ \left( \frac{9}{2} + \frac{\pi^2}{4} + \text{Re} c^{(1),\text{fin}}_{p,LR} \right) \text{Im} G^{(0)}_C(0,0;E_W) \right]
\]

\[
+ 2 \text{Im} \left[ \int_0^k dk \frac{G^{(0)}_C(0,0;E_W-k)}{[k]_{M_W+}} \right], \tag{6.21}
\]

where the dependence on the large logarithms \(\ln(2M_W/m_e)\) has cancelled out.

Substituting \(G^{(0)}_C(0,0;E_W)\) in (6.17) with the one-Coulomb exchange term, we get the \(\text{N}^3/\text{LO}\) “partonic” cross section to be convoluted with the electron structure functions in (4.50),

\[
\Delta\hat{\sigma}^{C\times[S+H]}_{LR} = -\frac{\alpha_e^2\alpha}{27s} \left\{ \left( 9 + \frac{\pi^2}{2} + 2 \text{Re} c^{(1),\text{fin}}_{p,LR} \right) \text{Im} \left[ \ln \left( -\frac{E_W}{M_W} \right) \right] + 2 \text{Im} \left[ \ln^2 \left( -\frac{E_W}{M_W} \right) \right] \right\}. \tag{6.22}
\]

6.2.5 Semi-soft and hard corrections to the Coulomb potential

Other \(\text{N}^3/\text{LO}\) contributions arise from semi-soft and hard corrections to the Coulomb potential between the two Ws. The hard corrections are closely related to the renormalisation of the electromagnetic coupling in the Coulomb potential. They were discussed in details in [48], and are reported in Appendix F.
The first class of semi-soft loop contributions is represented by diagrams containing semi-soft loops of light fermions. These include bubble insertions into a potential photon line (Figure 6.5a), vertex corrections (Figure 6.5b) and box corrections (Figure 6.5c). At $N^{3/2}\text{LO}$ the only non-vanishing contribution comes from the bubble insertion. Considering that a semi-soft fermion propagator $1/k_{ss}$ counts as $\delta^{-1/2}$ and the loop measure as $d^4 k_{ss} \sim \delta^2$ the fermion loop insertion scales as $\alpha \delta^1 \sim \delta^2$. Taking the additional potential photon propagator $\sim \delta^{-1}$ into account, we see that diagram (a) is suppressed by a factor $\delta$ compared to the pure single-Coulomb exchange, and is therefore a $N^{3/2}\text{LO}$ correction to the forward-scattering amplitude. From the same power-counting argument we find that diagrams (b) and (c) are suppressed by at least $\delta^2$ compared to the single-Coulomb exchange and do not have to be considered.

The other class of $N^{3/2}\text{LO}$ corrections arises from box diagrams with exchange of two semi-soft photons. A sample of such SM diagrams as shown in Figure 6.6a-c, along with a representative diagram in the effective theory (Figure 6.6d). After interacting with a semi-soft photon with momentum $k_{ss}$, the $W$ propagator turns into $1/(2M_W k_{ss,0}) \sim \delta^{-1/2}$, while photon propagators are given by $1/k_{ss}^2 \sim \delta^{-1}$. Using the same power-counting argument used above, the diagrams (a) and (b) in Figure (6.6) can be shown to have the correct scaling $\delta^{3/2}$ compared to the leading-order cross section. However, at leading order in the non-relativistic expansion the two diagrams cancel each other. For this it is essential that the width in the propagator can be treated perturbatively in the semi-soft region and is not resummed in the propagator, which is justified since $k_{ss,0} \sim M_W \sqrt{\delta} \gg \Gamma_W$. Contrary to diagrams (a) and (b), diagram (c) is suppressed by $\delta^2$ compared to the single Coulomb exchange diagram, and does not contribute at $N^{3/2}\text{LO}$. Therefore there are no contributions from semi-soft photons at this order. In QED this is well known from explicit calculations of the Coulomb potential [93, 94].

To obtain the $N^{3/2}\text{LO}$ correction to the Coulomb potential, one therefore has to compute only the semi-soft fermion bubble insertion in the single-Coulomb exchange diagram shown in Figure 6.5d and the hard corrections discussed in Appendix F. In [48] the calculation has been performed in the $\alpha(M_Z)$ scheme (see e.g. [78, 79]) and converted to the $G_\mu$ scheme afterwards. Since the hard corrections to the forward-scattering amplitude
are proportional to the single Coulomb exchange, the correction to the forward-scattering amplitude is
\[
\Delta A_{LR}^{NLO-C}|_{\alpha(M_Z)} = \Delta A_{LR}^{ss} + A_{LR}^{C1} \delta_{\text{hard}}^{a(M_Z)},
\]  
(6.23)
where \(\Delta A_{LR}^{ss}\) is the unrenormalised semi-soft amplitude corresponding to the diagram in Figure 6.5d, and \(A_{LR}^{C1}\) the single-Coulomb correction to the forward-scattering amplitude, i.e the logarithmic term in (4.21). The hard coefficient \(\delta_{\text{hard}}^{a(M_Z)}\) is given in (F.14). Thus, the total correction to the forward-scattering amplitude from the semi-soft bubble and the hard correction, at all orders in \(\epsilon\), reads
\[
\Delta A_{LR}^{NLO-C}|_{\alpha(M_Z)} = - \sum_f C_f Q_f^2 \alpha_e^2 \frac{\alpha_e^2}{2\pi} e^{3\gamma_E(1-\epsilon)} \frac{1}{\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)\Gamma(3/2-\epsilon)} \\
\times \left[ \left( \frac{M_W}{\mu} \right)^{-3\epsilon} \left( -\frac{\varepsilon_W}{\mu} \right)^{-3\epsilon} \frac{\Gamma(3\epsilon)\Gamma(1/2-2\epsilon)\Gamma^2(1/2+2\epsilon)}{\Gamma(4\epsilon)} \right] \\
-2 \left( \frac{M_W}{\mu} \right)^{-2\epsilon} \left( -\frac{\varepsilon_W}{\mu} \right)^{-2\epsilon} \text{Re} \left[ \left( -\frac{M_Z^2}{\mu^2} \right)^{-\epsilon} \right] \frac{\Gamma(1/2-\epsilon)\Gamma^2(1/2+\epsilon)}{\Gamma(4\epsilon)},
\]  
(6.24)
where \(C_f = 1\) for the leptons and \(C_f = N_c = 3\) for the quarks, and \(Q_f\) is the electric charge of \(f\) in units of \(e\), so that \(\sum_f C_f Q_f^2 = 20/3\). Expanding (6.24) in \(\epsilon\) one gets
\[
\Delta \sigma_{LR}^{NLO-C}|_{\alpha(M_Z)} = -\frac{20\alpha_e^2 \alpha^2}{243s} \left\{ 4\ln \left( \frac{2M_W}{M_Z} \right) \text{Im} \left[ \ln \left( -\frac{\varepsilon_W}{M_W} \right) \right] + \text{Im} \left[ \ln^2 \left( -\frac{\varepsilon_W}{M_W} \right) \right] \right\}.
\]  
(6.25)
To convert (6.25) to the \(G_\mu\) scheme, one has to add another term, according to the discussion in Appendix F:
\[
\Delta \sigma_{LR}^{NLO-C} = \Delta \sigma_{LR}^{NLO-C}|_{\alpha(M_Z)} + \delta_{\alpha(M_Z)\to G_\mu} \Delta \sigma_{LR}^{C1} \\
= \Delta \sigma_{LR}^{NLO-C}|_{\alpha(M_Z)} - \frac{2\pi\alpha_e^2}{27s} \delta_{\alpha(M_Z)\to G_\mu} \text{Im} \left[ \ln \left( -\frac{\varepsilon_W}{M_W} \right) \right],
\]  
(6.26)
where the single-Coulomb exchange cross-section $\Delta \sigma^{\text{C1}}_{\text{LR}}$ is the order $\alpha$ correction in (4.23). The result for the finite conversion factor $\delta_{\alpha(M_Z)\rightarrow G_\mu}$ is given in (F.19). The numerical value, for the same input parameters used in Chapter 5, is given by $\delta_{\alpha(M_Z)\rightarrow G_\mu} = 4.103\alpha$.

### 6.2.6 Decay corrections

The extraction of the flavour-specific cross section from the forward-scattering amplitude requires some attention. In the evaluation of the NLO results presented in Chapter 5 this was done by multiplying all resonant terms, including the first Coulomb correction, by a factor $\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}/\Gamma_W$ and non-resonant contributions by either $\Gamma^{(0)}_{\mu^-\bar{\nu}/\Gamma_W}$ or $\Gamma^{(0)}_{\mu^-\bar{\nu}/\Gamma_W}$, and adding the flavour-specific radiative correction (5.5).

At $N^{3/2}$LO these flavour-specific corrections are included in the EFT calculation by adding the term

$$
\Delta \sigma^{\text{C1}}_{\text{LR}} = \left( \frac{\Gamma^{(1,\text{ew})}_{\mu^-\bar{\nu}_\mu}}{\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}} + \frac{\Gamma^{(1,\text{ew})}_{\mu^-\bar{\nu}/\Gamma_W}}{\Gamma^{(0)}_{\mu^-\bar{\nu}/\Gamma_W}} \right) \Delta \sigma_{\text{LR}},
$$

with the one-loop electroweak corrections to the partial decay-widths, $\Gamma^{(1,\text{ew})}_{\mu^-\bar{\nu}_\mu}$ and $\Gamma^{(1,\text{ew})}_{\mu^-\bar{\nu}/\Gamma_W}$, given in Appendix C.2. Note that the numerical predictions presented in Chapter 5 already included the 2-loop QCD corrections to the hadronic decay-width multiplied by the full NLO electroweak cross section, including the single-Coulomb exchange. Therefore QCD corrections do not have to be considered in this section.

### 6.2.7 Residue corrections to the $W$-propagators

In a "correct" effective-theory treatment the hard matching coefficients should include a factor $\sqrt{2M_W(\varpi R_{hW})^{-1/2}}$ for each external $\Omega$, where $\varpi$, given in equation (2.41), accounts for the normalisation of non-relativistic fields, and $R_{hW} = 1 + \Pi^{(1,1)} + ...$ is the hard contribution to the LSZ residue of the $W$ propagator. As seen in Section 3.1, in the EFT calculation these factors reproduce the correct expansion of the full renormalised transverse $W$-propagator, equation (3.9). At order $N^{3/2}$LO, the proper factors of $\sqrt{R_{hW}}$ should be included in the NLO matching coefficient $C_{p,LR}^{(1)}$ of the production operator in the calculation in Section 6.2.2 and in the $WW\gamma$ vertex in Appendix F. However, in order to compare the effective-theory prediction with the Born cross section computed with a fixed-width resummation of the propagator, in Chapter 3 we departed from this standard procedure, and included the factors $\varpi R_{hW}$ in the EFT tree-level result. Therefore, we have to discuss them separately, paying some attention to isolate those contributions that are not included in a fixed-order NLO calculation in the complex-mass scheme.

The factor $\varpi$ is solely due to the use of a non-relativistic propagator. Since no kinematic expansion is performed in the calculation in the fixed-width or complex-mass scheme, all corrections of this kind are already included in [43,44] and do not have to be considered here. This is also true for corrections corresponding to insertions of higher-dimensional bilinear operators (second line of (3.9)). This leaves the correction from an insertion of the derivative of the self-energy, $\Pi^{(1,1)}$, that contributes in two different ways to the imaginary part of the forward-scattering amplitude, as shown in Figure 6.7.
Figure 6.7: Cut EFT diagrams originating from the residue corrections to the imaginary part of the forward-scattering amplitude. The Ω propagator with a dot indicates an insertion of the residue-factor correction Π\(^{(1,1)}\)(k\(^2\)), and, as usual, η\(_r\) = \(r_0 - \vec{r}_q^2/(2M_W) + i\Gamma_W(0)/2\). η\(_d\) = \(E - r_0 - \vec{r}_q^2/(2M_W) + i\Gamma_W(0)/2\). Analogous diagrams with insertions at the other three Ω propagators are not shown.

The first contribution in Figure 6.7a comes from a cut through the leading-order propagator, \(2 \text{Im}[1/\eta] = -\Gamma_W(0)/|\eta|^2\). This can be interpreted as a cut through a self-energy insertion implicitly contained in the resummed propagator. To select the flavour-specific final state, the total decay width in the numerator has to be replaced by the leading-order partial decay width \(\Gamma_W(0)\)µ̄νµ or \(\Gamma_W(0)\)ūd. In the full theory this diagram corresponds to a single-Coulomb photon exchange diagram, in which the decay matrix elements have been expanded at lowest order in \(\delta\), and with an additional insertion of a W self-energy, expanded to first order around the mass-shell. These contributions are not included in the fixed-order NLO calculation in the fixed-width or complex-mass scheme, where only the self-energy evaluated at \(M_W^2\) is resummed in the propagator [44], and the remaining momentum-dependent terms, \(\Pi^W(k^2) = \Pi^W(M_W^2) = \Pi^{(1,1)}(k^2 - M_W^2)\), are included perturbatively.

The diagram in Figure 6.7b includes a cut through a propagator modified by the factor of \(\Pi^{(1,1)} = -i\Gamma_W(0)/M_W\). The explicit expression of the cut reads

\[
2 \text{Im} \left[ \frac{\Pi^{(1,1)}}{\eta} \right] = -\frac{\Gamma_W(0)}{M_W} \left( r_0 - \frac{\vec{r}_q^2}{2M_W} \right) \frac{1}{|\eta|^2}. \tag{6.28}
\]

The interpretation of (6.28) is clear if one remembers that the integration of the decay squared matrix element over the two-body phase-space is \(\text{Im}\Pi^W_T(k^2)\). At leading order in \(\alpha\) this is simply

\[
\frac{\Gamma_W(0)}{M_W} k^2 \sim M_W \Gamma_W(0) \left( 1 + \frac{2M_W r_0 - \vec{r}_q^2}{M_W^2} \right), \tag{6.29}
\]

where we have used the usual parametrisation \(k = M_W v + \vec{r}\) and expanded to NLO in \(\delta\). Thus the contribution of cut (b) corresponds to a full-theory diagram with a single-Coulomb insertion, and kinematical corrections from the threshold expansion of a decay
matrix element around the mass-shell [48]. Since no kinematical approximation is applied in the fixed-order calculation, this term does not correspond to a genuine NNLO contribution and must not be included here.

Therefore, the only corrections that have to be included are four cut diagrams of the form of Figure 6.7a with the residue correction inserted at different $W$-propagators. In the following we use again the full on-shell width $\Gamma_W$ rather than the tree-expression $\Gamma_W^{(0)}$. Shifting the loop momenta in some of the diagrams, the sum of the four terms can be brought to the form

$$\Delta \sigma_{LR}^{\text{C\times res}} = -\frac{32(1 - \epsilon)^3 \alpha_e^2 \alpha \bar{\mu}^4 \epsilon}{27 M_W^2 s} \int \frac{d^d r}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} 2 \text{Im} \left[ \frac{1}{q_\epsilon} \right] 2 \text{Im} \left[ \frac{1}{q_\epsilon} \right] 2 \text{Re} \left[ \frac{2 \Pi^{(1,1)}}{q^2 - \eta_{r+q} \eta_{r+q}} \right]$$

$$= -\frac{32(1 - \epsilon)^3 \alpha_e^2 \alpha \bar{\mu}^4 \epsilon}{27 M_W^2 s} \times \left[ \text{Re} G_{C1}^{(0)}(0, 0, E_W) - 4\pi \alpha \bar{\mu}^4 \int \frac{d^d r}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - \eta_{r+q} \eta_{r+q}} \right].$$

(6.30)

Here $G_{C1}^{(0)}$ denotes the single-Coulomb exchange term in the Coulomb Green function (4.22). Terms that vanish upon performing the $q_0$ and $r_0$ integrations by closing the integration contour in the upper half-plane have been dropped. The second term in (6.30),

$$I = -4\pi \alpha \bar{\mu}^4 \int \frac{d^d r}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - \eta_{r+q} \eta_{r+q}} \right],$$

(6.31)

can be explicitly integrated in the following steps. We first perform both $r_0$ and $q_0$ integration by closing the contour in the upper half plane. This gives

$$I = \frac{4\pi \alpha \bar{\mu}^4}{(2\pi)^{d-4}} \int d^{d-1} r \int d^{d-1} q \frac{1}{|q|^2} \frac{1}{E - \frac{r^2}{M_W} + i\Gamma_W} \frac{1}{E - \frac{r^2 + q^2}{M_W} + i\Gamma_W}.$$

(6.32)

Then we combine the denominators $|q|^2$ and $E - r^2/q^2/M_W + i\Gamma_W$ with Feynman parameters, perform a standard loop-momentum shift and compute the $q$ integration:

$$I = -\frac{4\pi \alpha M_W^2 \bar{\mu}^4 |\gamma|^2}{(2\pi)^{d-4}} \Gamma \left( \frac{1}{2} + \epsilon \right) \int_0^1 dx [x(1 - x)]^{-1/2 - \epsilon} \int d^{d-1} r \frac{r^2 + m^2}{|r|^2 + m^2}$$

$$= -\frac{\alpha M_W^2}{8\pi^2} \left( e^\gamma \theta^2 \right)^{2\epsilon} \Gamma \left( 1/2 + \epsilon \right) \Gamma \left( 3/2 - \epsilon \right) \int_0^1 dx [x(1 - x)]^{-1/2 - \epsilon}$$

$$\times \int_0^\infty dy \frac{y^2 - 2\epsilon}{y^2 + m^2} \left[ y^2 + m^2/x \right]^{-1/2 - \epsilon}.$$

(6.33)

To lighten the notation, in (6.33) we have introduced the abbreviation $m^2 = -M_W \xi_W$ and $m^2* = -M_W \xi_W^*$, and in the second line we have explicitly performed the angular
integrations and renamed $|\vec{r}|$ to $y$. The two terms inside the $y$ integration cannot be further combined with the aid of Feynman parameters, because they have (finite) imaginary parts with opposite signs. We therefore use a Mellin-Barnes representation [69] to rewrite the $y$ integration in the following way,

$$
\int_0^\infty dy \frac{y^{2-2\epsilon}}{y^2 + m^2} \left[y^2 + m^2/\epsilon\right]^{-1/2-\epsilon} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \frac{\Gamma(1/2 + \epsilon + z)\Gamma(-z)}{\Gamma(1/2 + \epsilon)} \times \int_0^\infty dy \frac{y^{1-2z-4\epsilon}}{y^2 + m^2} \left(\frac{m^2}{\epsilon}\right)^z.
$$

(6.34)

Inserting (6.34) in (6.33), and performing the $x$ and $y$ integrations, one obtains

$$
I = -\frac{\alpha M_W^2}{8\pi^2} e^{2\gamma_E} \left(\frac{m^2}{\mu^2}\right)^{-2\epsilon} \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz (m^2)^z (m^2)^{-z} \Gamma(1/2 + \epsilon + z)\Gamma(2\epsilon + z)\Gamma(1/2 - \epsilon - z)\Gamma(-z).
$$

(6.35)

The constant $c$ has to be chosen such that the contour is at the right of all the poles of $\Gamma$s whose argument contains $z$ and to the left of the poles of $\Gamma$s whose argument contains $-z$. A sensible choice in this case is $c = -\epsilon$. The integral is then computed by closing the integration contour in the left half plane. This leads to

$$
I = -\frac{\alpha M_W^2}{8\pi} e^{2\gamma_E} \left(\frac{M_W\epsilon_W}{\mu^2}\right)^{-2\epsilon} \left\{1 + 2\epsilon \left[1 + \ln(-\epsilon_W) - \ln(-\epsilon_W^*)\right] \right. \\
- \ln \left(\frac{\epsilon_W - \epsilon_W^*}{\epsilon_W^*}\right) + \ln \left(\frac{\sqrt{\epsilon_W^*} + \sqrt{\epsilon_W}}{\sqrt{\epsilon_W^*} - \sqrt{\epsilon_W}}\right)\right\}.
$$

(6.36)

Equation (6.36) contains an $\epsilon$-pole that cancels against a pole in the unrenormalised green function $G_{C1}^{(0)}(0, 0, \epsilon_W)$. After inserting the result (6.36), with poles minimally subtracted, into (6.30) and rearranging the terms in (6.36) and $G_{C1}^{(0)}(0, 0, \epsilon_W)$, we obtain the final expression for the residue correction:

$$
\Delta \sigma^{C \times \text{res}}_{LR} = \frac{4\pi\alpha_s\alpha}{27s} \frac{\Gamma_W}{M_W} \ln \left[\frac{2|\epsilon_W|}{\Gamma_W^2} \frac{|\epsilon_W + |\epsilon_W^*|)}{\Gamma_W^2} \ln \left[\frac{2|\epsilon_W|}{|\epsilon_W^*|}\right] \right].
$$

(6.37)

6.3 Numerical analysis

We now combine the $N^{3/2}$LO contribution to the total cross section of the scattering process $e^-e^+ \rightarrow \mu^-\mu^+u\bar{d} + X$, given by the sum of corrections computed in Section 6.2,

$$
\hat{\sigma}^{(3/2)}_{LR} = \Delta \sigma^{C \times [S+H]}_{LR} + \Delta \sigma^{NLO-C}_{LR} + \Delta \sigma^{C \times \text{decay}}_{LR} + \Delta \sigma^{C \times \text{res}}_{LR} + \Delta \sigma^{C3}_{LR}.
$$

(6.38)

This has to be inserted into the convolution with the electron structure functions in (4.50). Recall that this correction refers to the $e_L e_R$ helicity state while there are no contributions
to the other helicity combinations from NNLO SM diagrams that contribute at $N^{3/2}\text{LO}$ in the EFT power counting. As input parameters we use the values given in Chapters 3 and 5.

The numerical results, for various centre-of-mass energies near the $W$-pair production threshold, are shown in table 6.1. More in details, the table contains corrections from interference of single-Coulomb exchange and soft/hard corrections from (6.22) (third column), from the NLO corrections to the Coulomb-potential, (6.26) (fourth column), from interference of single-Coulomb exchange with decay and residue correction in equations (6.27) and (6.37) (fifth and sixth columns) and from the triple-Coulomb correction in (6.3) (seventh column). The sum of the individual contributions, equation (6.38), is given in the second column. We also show results for the pure two-Coulomb exchange correction, given by the second term in (4.24) (eighth column), since this also arises from NNLO diagrams in the full Standard Model (but appears already at NLO in the EFT power counting). Figure 6.8 shows the size of the individual contributions and the combined correction relative to the full Born cross section from WHIZARD. We observe that individual corrections are comparable in magnitude to the second Coulomb correction but cancel to a certain extent, in particular in the immediate threshold region, where the total $N^{3/2}\text{LO}$ correction is about one per-mille of the Born result. Above threshold the total correction is dominated by the residue contribution, and rises to about three per-mille of the Born cross section.

Figure 6.8: Corrections relative to the Born cross section: Combined $N^{3/2}\text{LO}$ (solid/black), interference of Coulomb with soft and hard corrections (long-dashed/blue), correction to the Coulomb potential (dash-dotted/red), interference of Coulomb and decay corrections (short-dashed/green) and interference of residue correction and single-Coulomb exchange (dotted/magenta). For comparison the NLO correction from double-Coulomb exchange (light solid/cyan) is also shown.
\begin{table}[h]
\centering
\begin{tabular}{c|cccccccc}
\hline
$\sqrt{s}$ [GeV] & $\hat{\sigma}^{(3/2)}$ & $\Delta\hat{\sigma}^{C\times[S+H]}$ & $\Delta\hat{\sigma}^{NLO-C}$ & $\Delta\hat{\sigma}^{C\times\text{decay}}$ & $\Delta\hat{\sigma}^{C\times\text{res}}$ & $\Delta\sigma^{C3}$ & $\Delta\sigma^{C2}$ \\
\hline
158 & -0.001 & -0.116 & 0.104 & -0.037 & 0.044 & 0.004 & 0.151 \\
161 & 0.147 & -0.321 & 0.226 & -0.091 & 0.324 & 0.010 & 0.437 \\
164 & 0.811 & -0.417 & 0.393 & -0.134 & 0.965 & 0.003 & 0.399 \\
167 & 1.287 & -0.389 & 0.473 & -0.142 & 1.345 & 0.001 & 0.303 \\
170 & 1.577 & -0.354 & 0.511 & -0.142 & 1.561 & 0.000 & 0.246 \\
\hline
\end{tabular}
\caption{Combined $N^{3/2}$LO corrections (second column) and separate contributions from interference of single-Coulomb exchange with soft and hard corrections (third column), renormalisation of the Coulomb potential (fourth column), interference of decay correction and single-Coulomb exchange (fifth column), interference of residue correction and single-Coulomb exchange (sixth column) and triple-Coulomb exchange (all corrections are without ISR improvement). For comparison the NLO contribution from double-Coulomb exchange (C2, second column) are also shown.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{c|cccccc}
\hline
$\sqrt{s}$ [GeV] & Born & Born (ISR) & NLO & $\hat{\sigma}^{(3/2)}$ & $\sigma^{(3/2)}_{\text{ISR}}$ \\
\hline
158 & 61.67(2) & 45.64(2) & 49.19(2) & -0.001 & 0.000 \\
 & [-26.0\%] & [-20.2\%] & [-0.0\%e] & [+0.0\%e] \\
161 & 154.19(6) & 108.60(4) & 117.81(5) & 0.147 & 0.087 \\
 & [-29.6\%] & [-23.6\%] & [+1.0\%e] & [+0.6\%e] \\
164 & 303.0(1) & 219.7(1) & 234.9(1) & 0.811 & 0.544 \\
 & [-27.5\%] & [-22.5\%] & [+2.7\%e] & [+1.8\%e] \\
167 & 408.8(2) & 310.2(1) & 328.2(1) & 1.287 & 0.936 \\
 & [-24.1\%] & [-19.7\%] & [+3.1\%e] & [+2.3\%e] \\
170 & 481.7(2) & 378.4(2) & 398.0(2) & 1.577 & 1.207 \\
 & [-21.4\%] & [-17.4\%] & [+3.3\%e] & [+2.5\%e] \\
\hline
\end{tabular}
\caption{Two implementations of the $N^{3/2}$LO corrections, which differ by the treatment of initial-state radiation compared to the “exact” Born cross section without (second column) and with (third column) ISR improvement and the NLO EFT result including ISR (fourth column). The relative correction in brackets is given with respect to the Born cross section in the second column.}
\end{table}
The final results for the $N^{3/2}$LO corrections including ISR improvement, i.e. the result of inserting $\hat{\sigma}^{(3/2)}$ into the convolution with the electron structure functions in (4.50), is given in the last column of table 6.2, together with the previous result without ISR improvement, shown in the fifth column. For comparison, we also give the results for the Born cross section, with and without ISR improvement, and the result of the NLO calculation (including ISR improvement), as previously presented in Table 5.2. The ISR improvement is seen to reduce the $N^{3/2}$LO corrections by about 40% at threshold and 25% at 170 GeV. The effect is bigger than in the case of the NLO corrections presented in Table 6.2, that are reduced by ISR improvement by about 20% near threshold while there is almost no effect at 170 GeV. We note that if one wants to add the dominant NNLO electroweak corrections, computed in the effective theory, on the top of the full 1-loop calculation in the fixed-width or complex-mass scheme [43, 44] (as shown in Appendix B of [48], the $N^{3/2}$LO result can be added to the fixed-order NLO result in the complex-mass scheme without modification), both the $N^{3/2}$LO terms presented here and the double-Coulomb exchange terms C2 shown in Table 6.2 must be added, since this two-loop virtual effect is also not included in the fixed-order NLO calculation [43, 44].

In Section 5.4 we estimated the impact of the interference of single-Coulomb exchange with hard and soft corrections on the $W$-mass measurement as $[\delta M_W] \approx -5$ MeV. This expectation was based on the estimate based on equation (5.12), that corresponds to a correction to the cross section of $\Delta\sigma^{3/2}_{\text{val}}(161\text{GeV}) \sim -0.27\text{fb}$. We see from Table 6.1 (third column) that this correctly captures the order of magnitude of the contribution. However, this term is almost completely cancelled by the correction to the Coulomb potential, that was not considered in Section 5.4. Adopting the same procedure as in Section 5.4, but assigning a relative error to each energy point that scales as one over the square root of the expected number of events, we estimate the shift on the $M_W$, and find that the impact of the $N^{3/2}$LO corrections on the $W$-mass measurement is about 3 MeV (5 MeV if the $N^{3/2}$LO correction is not convoluted with ISR). Since other SM NNLO terms are expected to be even smaller, we may conclude that the (partonic) four-fermion cross section near the $W$-pair production threshold is known with sufficient precision.

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Chapter 7

Resummation of threshold logarithms

In order to reduce the total theoretical uncertainty on the $W$-pair cross section near threshold below the accuracy of $\sim 1 \, \%$ required for a precise determination of $M_W$ at the ILC, in Chapter 4 we have computed all the relevant next-to-leading order (in the EFT counting) contributions to the the process (1.2). Furthermore, in Chapter 6 we have presented $N^{3/2}$LO corrections not already included in the full NLO calculation [43, 44]. These corrections arise from the expansion in $\delta$ of NNLO Standard-Model diagrams, and turn out to be numerically in the range of few per-milles of the Born cross section. We thus concluded that the remaining contributions to the cross section, that are suppressed by at least $\sqrt{\delta}$ compared to the $N^{3/2}$LO corrections, are below the target theoretical accuracy for ILC, at least for the ”partonic” cross section. However, the results for hard and soft corrections, given in equations (4.6) and (4.42), contain terms proportional to double and single logarithms of ratios of scales, $\ln(-4M_W^2/\mu^2)$ and $\ln(-8(E+i\Gamma_W)/\mu)$ respectively. Since the renormalisation scale $\mu$ cannot be chosen such that both logarithms are small, and minimising either logarithm sets the other to $\sim \ln(2M_W/\Gamma_W)$, the question arises whether formally subleading corrections could be numerically enhanced by large logarithms, leading to a poor convergence of the perturbative expansion. Therefore, for the total theoretical uncertainty to be under control, and below the accuracy needed at ILC, the contribution of higher-order logarithmically-enhanced corrections must be assessed, and, if necessary, included to all orders in our predictions.

In this chapter we apply the formalism presented in [95–97] to the resummation of threshold logarithms $\sim \ln(2M_W/\Gamma_W)$, and we numerically estimate the impact of the resummation on the NLO result presented in Chapter 5. As we will see, the effect of resummation on the four-fermion production cross section is less than $1 \, \%$ of the Born result, and can be neglected. Nonetheless, the formulas derived in the following sections can be easily extended to describe other pair-production processes, where resummation of threshold logarithms might be important. Note that here we do not consider logarithms of $2M_W/m_e$, whose NLL resummation should however be pursued to achieve the required target accuracy, as pointed out in Section 5.4.
7.1 Factorisation and resummation in Drell-Yan

Here we briefly review the main features of the formalism introduced in [95] for resummation of threshold logarithms, and applied to deep-inelastic scattering (DIS) and Drell-Yan (DY) processes in [96, 97]. For definiteness, in the following we explicitly refer to the Drell-Yan case, discussed in [97], to which we refer the reader for details.

The Drell-Yan process, the production of a lepton pair in hadron-hadron collisions,

\[ N_1 + N_2 \rightarrow l^- l^+ + X, \quad (7.1) \]

plays a key role in testing our picture of hard interactions in QCD, and represents a promising channel for search of new heavy particles. Consequently, a lot of effort has been devoted to obtaining accurate theoretical predictions for this process. The NLO cross section and rapidity distributions have been known since many years [98], while NNLO corrections to the total cross section [99, 100], and to rapidity distributions [101, 102] and to the fully-differential cross section [103, 104] were added later.

After cancellation of virtual and real soft divergences, the radiative corrections to the DY process contain Sudakov logarithms of the form \( \ln(1 - \tau) \), where \( \tau = M^2 / s \), and \( M^2 \) and \( s \) represent the invariant mass squared of the lepton pair and the centre-of-mass energy, respectively. In the threshold region, \( \tau \rightarrow 1 \), these terms are large, and spoil the convergence of the perturbative expansion in \( \alpha_s \). Thus, for a reliable theoretical description of (7.1), they need to be resummed to all orders in perturbation theory. This was accomplished in [105, 106] for the inclusive cross section, by solving appropriate evolution equations in Mellin space. The formalism was extended to rapidity distributions in [107], and pursued to higher logarithmic accuracy in later works [108–111].

In a typical experiment, the phenomenologically relevant values of the variable \( \tau \) are never numerically close to the end point \( \tau = 1 \). One may then wonder why resummation of threshold logarithms is relevant, if at all. In [112] it was argued that, even for moderate values of \( \tau \approx 0.2 \), the rapid fall-off of the parton-luminosity functions at large \( x \) may lead to a dynamical enhancement of the partonic threshold region, \( z \rightarrow 1 \), where \( z = M^2 / \hat{s} \), and \( \hat{s} \) is the centre-of-mass energy of the short-distance partonic collision. If fact, also for small values of the variable \( \tau, \tau \lesssim 0.05 \), the dominant contributions to the cross section are represented by those terms that are enhanced in the partonic threshold region, \( z \sim 1 \) [97]. It may therefore be important to resum logarithms of the form \( \ln(1 - z) \) in the hard partonic cross section.

The new formalism introduced in [95–97] is based on techniques used in the context of soft-collinear effective theory (SCET), where an explicit separation of the effects associated with different scales is performed. Near the partonic threshold, where \( \hat{s} \sim M^2 \) and the quantity \( 1 - z = (\hat{s} - M^2) / \hat{s} \) is small, the momentum regions relevant for the short-distance process are given by an hard region, \( k^2 \sim M^2 \), a collinear region \( k_0 \sim M, k^2 \sim M^2(1 - z) \), and a soft region \( k_0 \sim \vec{k} \sim M(1 - z) \). Further contributions come from the very long-distance modes with virtuality \( \Lambda_{QCD}^2 \ll M^2(1 - z) \), associated with the non-perturbative regime of the strong interactions. The separation of the different scales leads...
to a factorisation of the cross section near the partonic threshold,

\[
\frac{d\sigma}{dM^2} = \frac{4\pi\alpha^2}{3N_cM^2s}|C_V(-M^2, \mu)|^2 \sum_q c_q^2 \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \left[ f_{q/N_1}(x_1, \mu)f_{\bar{q}/N_2}(x_2, \mu) + (q \leftrightarrow \bar{q}) \right] \times \sqrt{s}W_{DY}(\sqrt{s}(1-z), \mu),
\]

(7.2)

where \( z = x_1x_2 \), and \( f_{q/N}(x_1, \mu) \) denotes the parton distribution function of the quark \( q \) inside the nucleon \( N \). The coefficient function \( C_V(-M^2, \mu) \) contains contributions from hard modes with virtuality \( k^2 \sim M \) while the soft function \( W_{DY}(\sqrt{s}(1-z), \mu) \) encodes effects associated with the soft scale \( M(1-z) \). The parton distribution functions \( f_{q/N}(x, \mu) \) are related to matrix elements of collinear fields, and receive contributions from the non-perturbative momentum region \( k^2 \sim \Lambda_{QCD}^2 \). The factorisation formula (7.2) will be explicitly derived below for the \( W \)-pair production case.

Each of the quantities entering the factorisation formula contains (double) logarithms of ratios of scales, \( \ln(Q^2/\mu^2) \) where \( Q^2 \) is equal to \( M^2 \) for the hard coefficient \( C_V \), \( M^2(1-z) \) for the soft function \( W_{DY} \), and \( \Lambda_{QCD}^2 \) for \( f_{q/N} \), and \( \mu \) is the common renormalisation (factorisation) scale. The dependence of these three functions on the renormalisation scale \( \mu \) is governed by renormalisation-group (RG) equations, that follow from the field-theoretic definition of these quantities. For example [97]

\[
\frac{d}{d\ln\mu} C_V(-M^2, \mu) = \left[ \Gamma_{\text{cusp}}(\alpha_s) \left( \ln \frac{M^2}{\mu^2} - i\pi \right) + \gamma^V(\alpha_s) \right].
\]

(7.3)

The large logarithms are thus automatically resummed by the renormalisation-group evolution of each quantity from a scale \( \mu_Q \sim Q \), at which the function can be computed in standard perturbation theory, to the common scale \( \mu \). Note that, unlike the standard approach [105, 106] where the resummation is performed in Mellin-moment space, in this framework the resummation is obtained directly in momentum space.

Since SCET is one of the building units of the unstable-particle effective theory introduced in Chapter 2, the formalism of [95,96] represents the natural choice for studying the factorisation properties of \( e^-e^+ \rightarrow \mu^-\bar{\nu}_\mu u\bar{d}X \) near the \( W \)-pair production threshold, and to perform the resummation of logarithmically enhanced corrections. However, one has first to understand how instability-effects modify the picture given in [97], and whether additional logarithmically-enhanced terms are introduced when considering production of unstable particles.

To this end, we have studied the process of Drell-Yan production of a single \( W \) boson

\[
N_1 + N_2 \rightarrow W^+ + X
\]

(7.4)
in the partonic threshold region for the two different cases of a stable and unstable \( W \) boson. The calculation has been performed in the framework of the unstable-particle effective theory presented in Chapter 2, where obviously, in the case of a stable \( W \), all finite-width effects have been set to zero. The cross sections have been computed at leading-order in \( \alpha \) and to next-to-leading order in \( \alpha_s \).
The total partonic cross section for the production of a stable $W$ boson in the limit $z \to 1$ reads

$$\hat{\sigma}_W^s = \frac{\pi^2 \alpha}{3 \hat{s} s_w^s \hat{s}} C(z, M_W, \mu), \quad (7.5)$$

where $\hat{s}$ represents the partonic centre-of-mass energy and $z = M_W^2 / \hat{s}$. The short-distance function $C(z, M_W, \mu)$ is determined by hard and soft-gluon exchange, and to NLO in $\alpha_s$, is given by

$$C(z, M_W, \mu) = \delta(1 - z) + \frac{\alpha_s C_F}{\pi} \left\{ 4 \left[ \ln(1 - z) - 1 \right]_+ + \frac{2}{(1 - z)_+} \ln \left( \frac{M_W^2}{\mu^2} \right) 
+ \delta(1 - z) \left[ \frac{3}{2} \ln \left( \frac{M_W^2}{\mu^2} \right) - 4 + \frac{\pi^2}{3} \right] \right\}. \quad (7.6)$$

For an unstable $W$ the partonic cross section has the following expression

$$\hat{\sigma}_W^{us} = -\frac{\pi \alpha}{6 \hat{s} s_w^s \hat{s}} \text{Im} \left[ \frac{1}{\eta(z)} \right] \left\{ 1 + \frac{\alpha_s C_F}{\pi} \left[ 2 \ln^2[-2\eta(z)] + 2 \ln[-2\eta(z)] \ln \left( \frac{M_W^2}{\mu^2} \right) 
+ \frac{3}{2} \ln \left( \frac{M_W^2}{\mu^2} \right) - 4 + \frac{\pi^2}{3} \right] \right\}, \quad (7.7)$$

where we have introduced the quantity $\eta(z) = 1 - \sqrt{z} + i \frac{\Gamma_W}{2M_W}$. Note that $1/\eta(z)$ is nothing else than an unstable (non-relativistic) propagator, expressed in terms of the quantity $z = M_W^2 / \hat{s}$. We thus recognise in (7.7) terms analogous to the ones appearing in the expressions of the hard and soft contributions computed in Chapter 4, equations (4.6) and (4.42). To make the correspondence between (7.5) and (7.7) clear, we note that all functions in (7.7) can be written as a convolution of the distributions in (7.6) with a non-relativistic Breit-Wigner line shape. More precisely

$$\int_0^1 dx \delta(1 - x) \text{Im} \left[ \frac{1}{\eta(z/x)} \right] = \text{Im} \left[ \frac{1}{\eta(z)} \right],$$
$$\int_0^1 dx \left[ \frac{1}{1 - x} \right]_+ \text{Im} \left[ \frac{1}{\eta(z/x)} \right] = \text{Im} \left[ \frac{1}{\eta(z)} \ln[-2\eta(z)] \right],$$
$$\int_0^1 dx \left[ \frac{\ln(1 - x)}{1 - x} \right]_+ \text{Im} \left[ \frac{1}{\eta(z/x)} \right] = \text{Im} \left[ \frac{1}{\eta(z)} \left( \frac{1}{2} \ln^2[-2\eta(z)] + \frac{\pi^2}{6} \right) \right], \quad (7.8)$$

where to obtain the right-hand side of the equations the integrand has been expanded around $x = 1$, and some subleading terms in $\Gamma_W / M_W$ have been dropped. Hence equation (7.7) can be simply written as

$$\hat{\sigma}_W^{us} = \frac{\pi^2 \alpha}{3 \hat{s} s_w^s \hat{s}} \int_0^{1+} dx C(x, M_W, \mu) \left[ -\frac{1}{2\pi} \text{Im} \left( \frac{1}{\eta(z/x)} \right) \right], \quad (7.9)$$

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where the upper limit of the integral, \(1_+ = \lim_{\epsilon \to 0} (1 + \epsilon)\), ensuring that the contribution of the \(\delta\) functions at \(z = 1\) is completely included in the integral. We clearly see that, at this order, the effect of the instability is to introduce and additional convolution of the hard function \(C(z, M, \mu)\) with a (non-relativistic) resummed \(W\) propagator. Note that in the limit \(\Gamma_W/M_W \to 0\) the Breit-Wigner line shape reduces to

\[
-\frac{1}{2\pi} \text{Im} \frac{1}{\eta(z)} = \frac{\Gamma_W}{4\pi M_W} \frac{M_W^2}{(1 - \sqrt{z})^2 + \frac{\Gamma_W^2}{4M_W^2}} \to \delta(1 - z)
\]

and one recovers the result for stable-\(W\) production. We now want to understand whether the convolution (7.9) could introduce additional logarithms besides the ones already present in the cross section for a stable \(W\), equation (7.5). This is more easily seen by considering the problem in Mellin-moment space. We define the \(N\)th Mellin moment of the function (distribution) \(f(z)\) as

\[
f_N = \int_0^{1_+} dz \, z^{N-1} f(z).
\]

Note that the moment \(f_N\) probes regions where \((1 - z) \sim 1/N\). Thus, in Mellin-moment space the threshold limit \(z \to 1\) corresponds to the large-\(N\) limit, \(N \to \infty\). Keeping only the leading terms in \(N\), one explicitly finds the following results for the distributions in (7.6):

\[
\begin{align*}
\int_0^{1_+} dz \, z^{N-1} \delta(1 - z) &= 1, \\
\int_0^{1_+} dz \, \frac{z^{N-1}}{(1 - z)_+} &\sim -\ln N, \\
\int_0^{1_+} dz \, z^{N-1} \left[ \frac{\ln(1 - z)}{(1 - z)} \right]_+ &\sim \frac{1}{2} \ln^2 N.
\end{align*}
\]

The Mellin moments of a \(\delta\) function are constant, while \(f_N\) is proportional to a single or double logarithm for \(1/(1 - z)_+\) and \([\ln(1 - z)/(1 - z)]_+\) respectively. In the unstable case the analytic calculation of the Mellin moments is more complicated. Here we consider only the simplest case represented by the Mellin moment of a Breit-Wigner line-shape:

\[
\int_0^{1_+} dz \, z^{N-1} \left[ -\frac{1}{2\pi} \text{Im} \left[ \frac{1}{\eta(z)} \right] \right] = -\text{Im} \frac{2F_1(1; 2N; 2N + 1; \frac{1}{1 + i\Gamma_W/(2M_W)})}{\pi N(1 + i\Gamma_W/(2M_W))}.
\]

When \(N\) is large, but smaller than \(M_W/\Gamma_W, 1 \ll N \ll M_W/\Gamma_W\), the above expression can be approximated by

\[
\int_0^{1_+} dz \, z^{N-1} \left[ -\frac{1}{2\pi} \text{Im} \left[ \frac{1}{\eta(z)} \right] \right] \sim 1 - \frac{2N\Gamma_W}{\pi M_W} \left( 1 - \gamma_E - \ln \frac{N\Gamma_W}{M_W} \right).
\]

We see that the leading-order term correctly reproduces the result for a stable \(W\). Equation (7.14) contains an extra logarithm of \(N\), but this is suppressed by a power of \(\Gamma_W/M_W\),
and vanishes in the limit $\Gamma_W/M_W \to 0$. When $N$ is much larger than $M_W/\Gamma_W$, $1 \ll M_W/\Gamma_W \ll N$, the leading-order expansion of expression (7.13) is

$$\int_0^{1+} dz \frac{z^{N-1}}{2\pi} \left[ \frac{1}{\eta(z)} \right] \sim \frac{2M_W}{\pi N \Gamma_W}. \quad (7.15)$$

In this case the finite width of the $W$ boson induces an additional suppression by inverse powers of $N$ with respect to the stable case (7.12). This was to be expected, since for very large $N$ the moment receives its dominant contribution from regions with $1 - z \sim 1/N \ll \Gamma_W/M_W$, and only a small portion of the Breit-Wigner line shape, which is smeared over an interval $1 - z \sim \Gamma_W/M_W$, effectively contributes to the integration. This clearly does not happen in the stable case, where the integration measure is always concentrated at the end-point. We therefore get to the conclusion that, for the leading-order cross section, finite-width effects do not introduce further logarithmically-enhanced terms besides the ones already present in the stable case. While we do not have a rigorous proof that this conclusion can be extended to the NLO distributions $1/(1 - z) + [\ln(1 - z)/(1 - z)]_+$, numerical studies of the NLO cross sections (7.5) and (7.7) seem to suggest that there is a one-to-one correspondence between large logarithms in the stable and unstable case.

### 7.2 Factorisation of the $W$-pair production cross section near threshold

We are now ready to apply the resummation formalism to the process $e^-(p_1) + e^+(p_2) \to \mu^- \bar{\nu}_\mu u d + X$ near the $W$ pair-production threshold, $s \sim 4M_W^2$. Here we follow closely the calculation for the Drell-Yan cross section, and refers to [97] for details. In this section we present the derivation of the factorised expression for the four-fermion cross section, while the actual resummation of threshold logarithms will be discussed in the next section.

As already noted in Section 2.1.1, the inverse covariant derivative $(i n_+ \cdot D_{c1/c2} + i \epsilon)^{-1}$ appearing in the SCET Lagrangian, equation (2.19), contains interactions of fermion fields with any number of collinear photons. These can be expressed in terms of collinear Wilson lines

$$(in_+ \cdot D_{c1} + i \epsilon)^{-1} = W_{c1}(in_+ \cdot \partial + i \epsilon)^{-1} W_{c1}^\dagger, \quad (7.16)$$

where $W_{c1}$ is defined by

$$W_{c1}(x) = \exp \left(-i e \int_{-\infty}^{0} ds n_+ \cdot A_{c1}(x + sn_+) \right), \quad (7.17)$$

and analogously for the collinear direction $c2$. $A_{c1}$ clearly denotes the $c1$-collinear photon field. These interactions can be removed from the effective Lagrangian through a redefinition of the collinear fields $e_{c1}$ and $e_{c2}$:

$$e_{c1}(x) \to W_{c1}^\dagger(x)e_{c1}(x)$$

$$e_{c2}(x) \to W_{c2}^\dagger(x)e_{c2}(x). \quad (7.18)$$
This redefinition implies a modification of the effective production operators \( O^{(k)}_p \) (see equation (2.53))

\[
O^{(k)}_p(0) = \frac{C_p(4M^2_{W,R})}{M^2_{W}} \left( \bar{c}_{e_2,L/R}\mathcal{F}(\vec{n}, D)GW_{c_1,L/R}^\dagger \left( \Omega^i_+ \mathcal{G}(D)\Omega^{i\dagger}_+ \right) \right). \tag{7.19}
\]

Note that the effective four-fermion operators \( O^{(k)}_{4e} \), equation (2.49), are not modified by the redefinition (7.18), since the four collinear Wilson lines cancel pairwise,

\[
W_{c_1,r}^\dagger c_{1} W_{c_1,l} = 0.
\]

Analogously we note that the leading-order soft interactions of the effective Lagrangian, represented by the terms \( \bar{c}_1 \left[ \mathbf{i} n^{-} \cdot D_{s} \right] e_{1}/2c_{1} \) and \( \bar{c}_2 \left[ \mathbf{i} n^{+} \cdot D_{s} \right] e_{2}/2c_{2} \) in the SCET Lagrangian (2.19), and by the term \( \Omega^i_\mp [\mathbf{i} D_{s}]\Omega^i_\mp \) in the PNRQED Lagrangian (2.39), can as well be removed from \( \mathcal{L}_{EFT} \) with a field redefinition involving soft Wilson lines:

\[
\begin{align*}
W_{c_1}^\dagger c_{1} &= S_{c_1} W_{c_1}^{(0)i} c_{1}^i \\
\bar{c}_1 W_{c_1} &= \bar{c}_1 W_{c_1}^{(0)i} S_{c_1}^\dagger c_{1}^i \\
W_{c_2}^\dagger c_{2} &= \tilde{S}_{c_2} W_{c_2}^{(0)i} c_{2}^i \\
\bar{c}_2 W_{c_2} &= \bar{c}_2 W_{c_2}^{(0)i} \tilde{S}_{c_2}^\dagger c_{2}^i \\
\Omega^i_+ &= S_{v} \Omega^{i\dagger}_+ \\
\Omega^i_- &= \tilde{S}_{v} \Omega^{i\dagger}_-,
\end{align*}
\tag{7.20}
\]

where the soft Wilson lines \( S_{c_1}, \tilde{S}_{c_1} \) and \( S_v \) are defined by

\[
\begin{align*}
S_{c_1}(x) &= \mathcal{P} \exp \left[ \mathbf{i} c \int_{-\infty}^{0} dt n_\mp \cdot A_s(x + n_\mp t) \right] \\
\tilde{S}_{c_1}(x) &= \mathcal{P} \exp \left[ \mathbf{i} c \int_{0}^{\infty} dt n_\mp \cdot A_s(x + n_\mp t) \right] \\
S_v(x) &= \mathcal{P} \exp \left[ \mathbf{i} c \int_{-\infty}^{0} dt \mathbf{v} \cdot A_s(x + \mathbf{v} t) \right],
\end{align*}
\tag{7.21}
\]

with \( v = (1, \vec{0}) \) as usual. Wilson lines for the direction \( c_2 \) are obtained from (7.21) substituting \( n_\mp \) with \( n_\pm \). It is easy to verify that \( S_{c_1}[\mathbf{i} n_\mp \cdot D_{s}] S_{c_1} = [\mathbf{i} n_\mp \cdot \partial] \), and similarly for the other two covariant derivatives. The definition of different soft Wilson lines for incoming and outgoing particles and antiparticles is enforced by the \( \mathbf{i} e \) prescription of the propagator [113]. Again the field redefinition induces a modification of the effective operators entering the definition of (2.40). As before, the effective four-fermion operators

\footnote{Note that operators at an arbitrary point \( z \) contain an extra phase \( e^{2ikWz} \).}
are not affected by (7.20), while for the production operators we obtain

\[
\mathcal{O}_p^{(k)}(0) = \frac{C_p^{(k)}(4M_W^2, \mu)}{M_W^{2(1+k)}} (e_{c_2, L/R} \cdot \bar{W}^{(0)}_{c_2} W^{(0)}_{c_1} e_{c_1,L/R}) \cdot \langle \Omega_+^{(0)i} \bar{S}^+ \Gamma(D) S_\pm \Omega_0^{(0)j} \rangle .
\] (7.22)

We now consider how the expression of the forward-scattering amplitude (2.40) is modified by the redefinitions (7.18) and (7.20). We will consider only the contribution arising from the leading-order operator, \( \mathcal{O}_p^{(0)} \). This is justified, since non-resonant contributions are not modified by the resummation presented below, and, as we will see, the effect of resummation is too small to be of any relevance for higher-dimensional production operators.

In terms of the new fields, \( \mathcal{O}_p^{(0)} \) reads

\[
\mathcal{O}_p^{(0)}(0) = \frac{\pi \alpha \mu \nu C_p(4M_W^2, \mu)}{M_W^2} (\bar{e}_{c_2, L} W^{(0)}_{c_2} \gamma^{[n]} S_{c_2} W^{(0)}_{c_1} e_{c_1, L}) \cdot \langle \Omega_+^{(0)i} \Gamma(D) \Omega_-^{(0)j} \rangle ,
\] (7.23)

where \( C_p(4M_W^2, \mu) \) denotes the \( O(\alpha) \) corrected matching coefficient. Note that the soft Wilson lines along the direction \( u \) dropped out. This is consistent with the calculations presented in Section 4.3 and Appendix D, where all corrections associated with the leading order \( \gamma_s \Omega \Omega \) vertex cancelled. Inserting the above expression in (2.40) we obtain

\[
A_{LR} = \frac{i(\pi \alpha \mu \nu)^2}{M_W^4} |C_p(4M_W^2, \mu)|^2 \int d^4 z e^{-2iM_W z \cdot v} \langle 0 | T \left[ \bar{\Omega}_-^{(0)l} \Omega_+^{(0)m} \right] (z) \bar{\Omega}_-^{(0)i} \Omega_+^{(0)j} |0 \rangle \langle 0 | T \left[ \bar{S}_{c_1}^{\dagger} \bar{S}_{c_2} \right] (z) \bar{S}_{c_2} S_{c_1} |0 \rangle |0 \rangle |
\times |e_L e_R \rangle | T \left[ \bar{e}_{c_1, L} W^{(0)}_{c_1} \gamma^{[n]} S_{c_1} W^{(0)}_{c_2} \bar{e}_{c_2, L} |(z) \bar{e}_{c_2, L} W^{(0)}_{c_2} \gamma^{[n]} S_{c_2} W^{(0)}_{c_1} e_{c_1, L} |0 \rangle \right] |e_L e_R \rangle .
\] (7.24)

To write (7.24) we have used the fact that, after the redefinition (7.20), the fields \( e_{c_1}, e_{c_2} \) and \( \Omega_\pm \) do not interact with soft photons and with each others (at leading-order in \( \delta \)), and assumed that the Fock space factorises into a direct product \( |e_L e_R \rangle = |e_L^r e_R^r \rangle c \otimes |0 \rangle_p \otimes |0 \rangle_s \), where the three factors contain, respectively, collinear, potential and soft states. We are then allowed to factorise the matrix element as indicated in (7.24). To better disentangle the different components of the amplitude, we insert a dummy integral \( \int d^4 x \delta(4)(x - z) \).
in (7.24)

\[ A_{LR} = \frac{i(\pi \alpha_{em})^2}{M_W^4} |C_p(4M_W^2, \mu)|^2 \int d^4z \int d^4x \delta^{(4)}(x - z) \times e^{-2iMwv \cdot x} \langle 0|T \left[ [\Omega^{-}(0)_m \Omega^{+}(0)](x) [\Omega^{-}(0)_m \Omega^{+}(0)](0) \right] |0 \rangle \times \langle 0|T \left[ \left[ \tilde{S}^+_{c1} \tilde{S}^{-}_{c1} \right](z) [\tilde{S}^+_{c2} \tilde{S}^{-}_{c2}](0) \right] |0 \rangle \times \langle e_L e_R^+|T \left[ \left[ \epsilon^{(0)}_{c1,L} W^{(0)}_{c1} \gamma^i n^m W^{(0)}_{c2} \gamma^{i \dagger} \epsilon^{(0)}_{c2,L} \right](z) \left[ \epsilon^{(0)}_{c1,L} W^{(0)}_{c1} \gamma^{i \dagger} n^j W^{(0)}_{c1} \gamma^{i \dagger} \epsilon^{(0)}_{c2,L} \right](0) \right] |e_L e_R^+ \rangle , \]

(7.25)

and replace the \( \delta \) function with its integral representation \( \delta^{(4)}(X) = \int \frac{d^4q}{(2\pi)^4} e^{iq \cdot X} \),

\[ A_{LR} = \frac{i(\pi \alpha_{em})^2}{M_W^4} |C_p(4M_W^2, \mu)|^2 \int d^4z \int d^4q \frac{4}{(2\pi)^4} e^{-iq \cdot z} \times \int d^4x e^{iq \cdot (q - 2Mwv)} \langle 0|T \left[ [\Omega^{-}(0)_m \Omega^{+}(0)](x) [\Omega^{-}(0)_m \Omega^{+}(0)](0) \right] |0 \rangle \times \langle 0|T \left[ \left[ \tilde{S}^+_{c1} \tilde{S}^{-}_{c1} \right](z) [\tilde{S}^+_{c2} \tilde{S}^{-}_{c2}](0) \right] |0 \rangle \times \langle e_L e_R^+|T \left[ \left[ \epsilon^{(0)}_{c1,L} W^{(0)}_{c1} \gamma^i n^m W^{(0)}_{c2} \gamma^{i \dagger} \epsilon^{(0)}_{c2,L} \right](z) \left[ \epsilon^{(0)}_{c1,L} W^{(0)}_{c1} \gamma^{i \dagger} n^j W^{(0)}_{c1} \gamma^{i \dagger} \epsilon^{(0)}_{c2,L} \right](0) \right] |e_L e_R^+ \rangle . \]

(7.26)

We now define the scalar function \( \Omega(p, \mu) \),

\[ \Omega(p, \mu) = \frac{i}{9} \int d^4x e^{ix \cdot p} \langle 0|T \left[ [\Omega^{-}(0)_m \Omega^{+}(0)](x) [\Omega^{-}(0)_m \Omega^{+}(0)](0) \right] |0 \rangle . \]

(7.27)

and note that

\[ i \int d^4x e^{ix \cdot (q - 2Mwv)} \langle 0|T \left[ [\Omega^{-}(0)_m \Omega^{+}(0)](x) [\Omega^{-}(0)_m \Omega^{+}(0)](0) \right] |0 \rangle = \delta^{i \dagger \delta^{m \dagger}} \Omega(q - 2Mwv, \mu) . \]

(7.28)

The simple tensor structure of the matrix element \( \langle 0|T \left[ [\Omega^{-}(0)_m \Omega^{+}(0)](x) [\Omega^{-}(0)_m \Omega^{+}(0)](0) \right] |0 \rangle \) is a consequence of the spin-independent leading-order interaction term in the PNRQED Lagrangian (2.39). At higher orders the Lagrangian includes spin-dependent interactions, and the matrix element has also spin-1 and spin-2 components. Expressing (7.26) in terms
of $\Omega$ we obtain

$$A_{LR} = \frac{(\pi \alpha_{em})^2}{M_W^4} C_p(4M_W^2, \mu)^2 \int d^4z \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot z} \Omega(q - 2MWv, \mu)$$

$$\times \langle 0 | T \left[ [\tilde{S}_{c1}^\dagger \tilde{S}_{c2}^\dagger](z)[S_{c2}^\dagger S_{c1}^\dagger](0) \right] | 0 \rangle$$

$$\times \langle e_L^- e_R^+ | T \left[ [e_{c1,L}^0 W_{c1}^0 \gamma^i n^j](z)[e_{c2,L}^0 W_{c2}^0 \gamma^i n^j](z) [e_{c1,L}^0 W_{c1}^0] [e_{c2,L}^0 W_{c2}^0] \right] | e_L^- e_R^+ \rangle .$$

(7.29)

We can further simplify the collinear matrix element in (7.29) by noting that

$$(\gamma^i n^j) \otimes (\gamma^i n^j) = 2 \left[ \gamma^i \otimes \gamma^i + (n^i \gamma^i) \otimes (n^j \gamma^j) \right].$$

(7.30)

Furthermore, since $\not\!e_{c1,L} = \not\!e_{c2,L} = 0$, we have

$$e_{c2,L}^0 \gamma^i e_{c1,L}^0 = \frac{1}{2} e_{c2,L}^0 (\not\!e_+ - \not\!e_-) e_{c1,L}^0 = 0$$

and

$$e_{c2,L}^0 \gamma^0 e_{c1,L}^0 = \frac{1}{2} e_{c2,L}^0 (\not\!e_+ + \not\!e_-) e_{c1,L}^0 = 0 .$$

(7.31)

Thus the Dirac structure in (7.29) can be written as

$$(\gamma^i n^j) \otimes (\gamma^i n^j) = 2 \gamma^i \otimes \gamma^i = -2\gamma^\mu \otimes \gamma_\mu ,$$

(7.32)

and we can use Fierz identities to rearrange the matrix element of the collinear fields,

$$[e_{c1,L}^0 \gamma^\mu e_{c2,L}^0] [\bar{e}_{c2,L}^0 \gamma_\mu e_{c1,L}^0] = -2 [\bar{e}_c e_{c1,L}^0 \not\!e_+ e_{c2,L}^0] .$$

(7.33)

Consequently equation (7.29) simplifies to

$$A_{LR} = \frac{(2\pi \alpha_{em})^2}{M_W^4} C_p(4M_W^2, \mu)^2 \int d^4z \int \frac{d^4q}{(2\pi)^4} e^{-iq \cdot z} \Omega(q - 2MWv, \mu)$$

$$\times \langle 0 | T \left[ [\tilde{S}_{c1}^\dagger \tilde{S}_{c2}^\dagger](z)[S_{c2}^\dagger S_{c1}^\dagger](0) \right] | 0 \rangle$$

$$\times \langle e_L^- | T \left[ [e_{c1,L}^0 W_{c1}^0] (z) [W_{c1}^0 e_{c1,L}^0] | e_L^- \rangle \right]$$

$$\times \langle e_R^+ | T \left[ [e_{c2,L}^0 W_{c2}^0] (0) [W_{c2}^0 e_{c2,L}^0] | e_R^+ \rangle \right] ,$$

(7.34)

where, as before, we have used the fact that the redefined $c_1$ and $c_2$ collinear fields do not have leading-order interactions to split further the matrix element. Introducing the
This leads us to the final factorised expression of \( \hat{W}_{WW}(z, \mu) = \langle 0 \mid T \left[ \hat{S}_{c1}^\dagger \hat{S}_{c2} \right] (z) \left[ \hat{S}_{c2}^\dagger \hat{S}_{c1} \right] (0) \mid 0 \rangle \),

\[
\begin{align*}
    f_{ee}(x_1, \mu) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-ix_1 t} e^{p_1 \cdot q} |e_L^-| T \left[ e_c^{(0)} W^{(0)}(0) \right] (t n_+) \left[ \frac{W^{(0)}(0)}{2} \right] W^{(0)}(0) (0) \mid e_L^- \rangle, \\
    f_{ee}(x_2, \mu) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-ix_2 t} e^{p_2 \cdot q} (e_R^+ | T \left[ e_c^{(0)} W^{(0)}(0) \right] (0) \left[ \frac{W^{(0)}(0)}{2} \right] W^{(0)}(0) (0) \mid e_R^+ \rangle, \\
\end{align*}
\]

we can recast the amplitude in the form \(^2\)

\[
\hat{A}_{LR} = \frac{(4\pi \alpha_{ew})^2}{M_W^3} |C_p(4M_W^2, \mu)|^2 \int dx_1 dx_2 f_{ee}(x_1, \mu) f_{ee}(x_2, \mu) \\
\times \int \frac{d^4 q}{(2\pi)^4} \Omega(q - 2M_W v, \mu) \int d^4 z e^{i(x_1 p_1 + x_2 p_2 - q) z} \hat{W}_{WW}(z, \mu) \\
= \frac{(4\pi \alpha_{ew})^2}{M_W^3} |C_p(4M_W^2, \mu)|^2 \int dx_1 dx_2 f_{ee}(x_1, \mu) f_{ee}(x_2, \mu) \\
\times \int \frac{d^4 q}{(2\pi)^4} \Omega(x_1 p_1 + x_2 p_2 - 2M_W v - q, \mu) \int d^4 z e^{i q z} \hat{W}_{WW}(z, \mu). \quad (7.36)
\]

The last equality has been obtained by shifting the loop momentum \( q, x_1 p_1 + x_2 p_2 - q \rightarrow q \). The function \( W_{WW} \) contains only soft fields, that vary significantly only on distances \( z \sim 1/\Gamma_W \). Hence, the \( q \) integration receives a non-vanishing contribution only from soft-momentum regions where \( q \sim \Gamma_W \). The momentum \( x_1 p_1 + x_2 p_2 - 2M_W v - q \) must also be soft, since in the partonic centre-of-mass frame \( x_1 p_1 + x_2 p_2 - 2M_W v \) is soft by assumption. But the function \( \Omega \) contains potential fields \( (q_0 \sim \Gamma_W, q \sim \sqrt{M_W^2 - T_W} \) ), and it can thus depend only on the time-like component of \( x_1 p_1 + x_2 p_2 - 2M_W v - q \). This means that the \( \vec{q} \) integration in (7.36) can be performed, and gives

\[
\int \frac{d^4 q}{(2\pi)^4} e^{-i q z} = \delta(3)(\vec{p}). \quad (7.37)
\]

This leads us to the final factorised expression of \( \hat{A}_{LR} \), that reads

\[
\hat{A}_{LR} = \frac{(4\pi \alpha_{ew})^2}{M_W^3} |C_p(4M_W^2, \mu)|^2 \int dx_1 dx_2 f_{ee}(x_1, \mu) f_{ee}(x_2, \mu) \\
\times \int_{-\infty}^{+\infty} dq_0 \Omega(\vec{q} - q_0, \mu) \int_{-\infty}^{+\infty} \frac{dz_0}{(2\pi)} e^{i q_0 z_0} \hat{W}_{WW}(z_0, \vec{z} = 0, \mu), \quad (7.38)
\]

\(^2\)Applying an inverse Fourier transform to \( f_{ee}(x_1, \mu) \) one finds

\[
\langle e_L^- | T \left[ e_c^{(0)} W^{(0)}(0) \right] (z) \frac{W^{(0)}(0)}{2} e_c^{(0)} (0) \mid e_L^- \rangle = 2M_W \int dx_1 e^{i x_1 p_1} f_{ee}(x_1, \mu).
\]
where \( \tilde{E} = E - (1 - x_1)M_W - (1 - x_2)M_W \), and \( E \) is the centre-of-mass energy measured from the threshold, \( E = \sqrt{s} - 2M_W \). As it is clear from their definitions, each function in the convolution (7.38) receives contribution from a single momentum region. More precisely \( C_p \) is completely determined by hard modes, \( f_{ee} \) by c1 (c2) collinear modes, \( \Omega \) by potential modes and \( \hat{W}_{WW} \) by soft modes. Note that equation (7.38) has the same structure of the factorised expression for Drell-Yan production of a lepton pair, given in equation (34) of [97], the only difference being an additional convolution of the soft function with \( \Omega \). This is consistent with the remarks made at the end of Section 7.1.

\( f_{ee}, \hat{W}_{WW} \) and \( \Omega \) can be computed order-by-order in \( \alpha \) from their definitions, equations (7.17), (7.21), (7.35) and (7.27), while \( C_p \) was given in (4.6). At lowest order in \( \alpha \) the electron structure function \( f_{ee} \) is

\[
f_{ee}(x_1, \mu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-ix_1 t} p_1 |e_L^0\rangle^\dagger \langle e_L^0| (tn_+ \frac{\hat{t}^+}{2} e_{1,\dagger}^0 (0) |e_L^-\rangle
\]

while the \( O(\alpha^0) \) soft function and hard matching coefficient are respectively \( \hat{W}_{WW}(z, \mu) = 1 \) and \( C_p^{(0)} = 1 \). Thus the lowest-order forward-scattering amplitude is given by

\[
A_{LR} = \frac{(4\pi \alpha_{ew})^2}{M_W^4} \int d\omega_1 d\omega_2 \delta(1 - x_1)\delta(1 - x_2) \int dq_0 \Omega^{(0)}(\tilde{E} - q_0) \int \frac{dz_0}{2\pi} e^{ix_0 z_0}
\]

\[
= \frac{(4\pi \alpha_{ew})^2}{M_W^4} \Omega^{(0)}(E).
\]

The leading-order expression for \( \Omega(E, \mu) \) can be easily computed. Introducing the notation \( P = p_1 + p_2 \) for the total momentum of the \( \Omega^- \Omega^+ \) pair, we have:

\[
\Omega^{(0)}(E) = \frac{i}{9} \int d^4x e^{ix(P-2M_Wv)} \langle 0|T \left[ \Omega^{(0)i}\Omega^{(0)j}\right](x)\Omega^{(0)i}\Omega^{(0)j}\right](0) |0\rangle
\]

\[
= \frac{i}{9} \int d^4x e^{ix(P-2M_Wv)} D^{ij}_0(x) D_{ij}^{00}(x)
\]

\[
= \frac{i}{9} \int d^4x e^{ix(P-2M_Wv)} \int \frac{d^4r}{(2\pi)^4} \frac{i\delta^{ij}e^{-irx}}{r_0 - \frac{|r|^2}{2M_W} + i\Gamma_W/2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l_0 - \frac{|l|^2}{2M_W} + i\Gamma_W/2} \int d^4xe^{ix(P-2M_Wv-r-l)}
\]

\[
= -i \int \frac{d^4r}{(2\pi)^4} \frac{1}{r_0 - \frac{|r|^2}{2M_W} + i\Gamma_W/2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l_0 - \frac{|l|^2}{2M_W} + i\Gamma_W/2} \int d^4xe^{ix(P-2M_Wv-r-l)}
\]

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\[
\begin{align*}
&= -i \int \frac{d^4r}{(2\pi)^4} \frac{1}{r_0 - \frac{|r|^2}{2M_W} + i\frac{\Gamma_W}{2}} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l_0 - \frac{|l|^2}{2M_W} + i\frac{\Gamma_W}{2}} \\
&\quad \times (2\pi)^4 \delta^{(4)}(P - 2M_W v - r - l) \\
&= -i \int \frac{d^4r}{(2\pi)^4} \frac{1}{r_0 - \frac{|r|^2}{2M_W} + i\frac{\Gamma_W}{2}} E - r_0 - \frac{|r|^2}{2M_W} + i\frac{\Gamma_W}{2} \\
&\quad \times (2\pi)^4 \delta^{(4)}(P - 2M_W v - r - l) \\
&= -2\pi \int \frac{d^3r}{(2\pi)^3} \frac{1}{\frac{E}{M_W} - \frac{|r|^2}{M_W} + i\frac{\Gamma_W}{2}} \\
&\quad \times \frac{2\pi}{4\pi} \frac{\sqrt{E - i\frac{\Gamma_W}{2}}}{M_W},
\end{align*}
\]
which coincides with the leading-order contribution to the Coulomb Green function given in (4.22). The leading-order forward-scattering amplitude is then
\[
A_{LR}^{(0)} = -4\pi\alpha^2 \sqrt{-E + i\frac{\Gamma_W}{2}},
\]
which agrees with (2.46).

We now want to determine the expression of the four functions \( C_p, \Omega, \, f_{ee} \) and \( \hat{W}_{WW} \) to order \( \alpha \). The computation of the matching coefficient \( C_{p,LR} \) has been presented in Subsection 4.1.1, and will not be discussed here. The structure functions \( f_{ee} \) and \( f_{\bar{e}\bar{e}} \) do not receive corrections at \( O(\alpha) \),
\[
f_{ee}(x_1, \mu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-i\mu t n_+} \langle e_L^- | e_{c1,L}^{(0)}(tn_+) | e_L^- \rangle \left[ 1 + 4\pi\alpha \int_{-\infty}^{0} ds_1 ds_2 (0) \langle n_+ \cdot A_{c1}((t + s_1)n_+) n_+ \cdot A_{c1}(s_2 n_+) | 0 \rangle \right] + O(\alpha^2) \\
&= \delta(1 - x_1) + O(\alpha^2),
\]
since the \( O(\alpha) \) term is proportional to \( n_+^2 = 0 \) and thus vanishes. In fact, as long as one considers massless initial states, the structure functions vanish to all order in \( \alpha \) beyond the lowest-order expression \( \delta(1 - x) \). The \( O(\alpha) \) correction to \( \Omega \) is given by the \( O(\alpha) \) term in the expansion of the Green function (4.22). The soft function \( \hat{W}_{WW} \) is given, up to NLO in \( \alpha \), by
\[
\hat{W}_{WW}(z) = \langle 0 | T \left[ \hat{S}_{c1} S_{c2}(z) S_{c2}^\dagger S_{c1}(0) \right] | 0 \rangle \\
&= 1 + 4\pi\alpha \left\{ -\frac{1}{2} \langle 0 | T \left[ \int_{-\infty}^{\infty} dtn_+ \cdot A_s(z + n_- t) \right]^2 | 0 \rangle \\
&\quad -\frac{1}{2} \langle 0 | T \left[ \int_{-\infty}^{\infty} dtn_+ \cdot A_s(z + n_+ t) \right]^2 | 0 \rangle \right\}.
\]
\[ -\frac{1}{2} \langle 0 | T \left[ \int_0^\infty dt n_+ \cdot A_s(n_+t) \right]^2 | 0 \rangle \\
+ \langle 0 | T \left[ \int_0^\infty dt n_- \cdot A_s(n_-t) \right]^2 | 0 \rangle \\
+ (0 | T \left[ \int_0^\infty dt n_- \cdot A_s(z + n_-t) \int_0^\infty ds n_+ \cdot A_s(z + n_+s) \right] | 0 \rangle \\
+ (0 | T \left[ \int_0^\infty dt n_- \cdot A_s(z + n_-t) \int_0^0 ds n_+ \cdot A_s(n_+s) \right] | 0 \rangle \\
- (0 | T \left[ \int_0^\infty dt n_- \cdot A_s(z + n_-t) \int_0^\infty ds n_- \cdot A_s(n_-s) \right] | 0 \rangle \\
- (0 | T \left[ \int_0^\infty dt n_+ \cdot A_s(z + n_+t) \int_0^0 ds n_- \cdot A_s(n_-s) \right] | 0 \rangle \\
+ (0 | T \left[ \int_0^\infty dt n_+ \cdot A_s(z + n_+t) \int_0^0 ds n_- \cdot A_s(n_-s) \right] | 0 \rangle \right) + O(\alpha^2). \tag{7.44} \]

The matrix elements containing two soft photon fields contracted with the same collinear vector vanish, because \( n_\pm^2 = 0 \). The remaining contributions read

\[ \hat{W}_{WW}(z) = 1 - 8\pi\alpha \int_{-\infty}^0 dt ds \left[ D(-n_-t + n_+s) + D(z - n_-t - n_+s) \right. \]
\[ + D(z - n_+t - n_-s) + D(n_+t - n_-s) \left. \right] + O(\alpha^2), \tag{7.45} \]

where \( D(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ixk} \frac{i}{k^2 + i\epsilon} \). To obtain the expression (7.45) we have redefined the integration variables in such a way that all integrals run from \(-\infty\) to 0. To simplify (7.45) we use the identity

\[ \int_{-\infty}^{+\infty} dt e^{-i\omega t} \theta(-t) = \frac{i}{\omega + i\epsilon}, \tag{7.46} \]

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which allows us to rewrite the four terms in (7.45) in the following form:

\[
\int_{-\infty}^{0} dt ds D(-n_- t + n_+ s) = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{i}{n_+ \cdot k + i\epsilon}
\]

\[
\int_{-\infty}^{0} dt ds D(z - n_- t - n_+ s) = \int \frac{d^4 k}{(2\pi)^4} e^{-iz \cdot k} \frac{i}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{i}{n_+ \cdot k + i\epsilon}
\]

\[
\int_{-\infty}^{0} dt ds D(z - n_+ t - n_- s) = \int \frac{d^4 k}{(2\pi)^4} e^{-iz \cdot k} \frac{i}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{i}{n_+ \cdot k + i\epsilon}
\]

\[
\int_{-\infty}^{0} dt ds D(n_+ t - n_- s) = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{i}{n_+ \cdot k + i\epsilon} .
\]  

(7.47)

The above integrals contain singularities that are regularised by shifting the dimensions of the loop-momentum \(k\) from 4 to \(4 - 2\epsilon\). The first and fourth integral are scaleless integrals, and vanish in dimensional regularisation. Thus the NLO soft function is

\[
\tilde{W}_{WW}(z) = 1 + 16\pi i\alpha\mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^d} e^{-iz \cdot k} \frac{1}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{1}{n_+ \cdot k + i\epsilon} ,
\]

(7.48)

where we have introduced the usual dimensional-regularisation factor \(\tilde{\mu}^{2\epsilon} = e^{\gamma_E} \mu^{2\epsilon}/(4\pi)^\epsilon\). Setting \(f_{\text{ee}}(x, \mu) = f_{\text{ee}}(x, \mu) = \delta(1 - x)\) in (7.38), and keeping for the moment the function \(\Omega\) unspecified, the NLO forward-scattering amplitude reads

\[
A_{LR} = \frac{(4\pi\alpha_{\text{ew}})^2}{M_W^2} |C_p(4M_W^2, \mu)|^2 \int dq_0 \Omega(E - q_0, \mu)
\]

\[
\times \left[ \delta(q_0) + 16\pi i\alpha\mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^d} \left( \frac{1}{k^2 + i\epsilon - n_- \cdot k + i\epsilon} \frac{1}{n_+ \cdot k + i\epsilon} \right) \delta(q_0 - k_0) \right]
\]

\[
= \frac{(4\pi\alpha_{\text{ew}})^2}{M_W^2} |C_p(4M_W^2, \mu)|^2 \int dq_0 \Omega(E - q_0, \mu)
\]

\[
\times \left[ \delta(q_0) + \frac{2\alpha}{\pi} e^{\gamma_E} \Gamma(-\epsilon) \frac{1}{q_0} \left( \frac{2q_0}{\mu} \right)^{-2\epsilon} \theta(q_0) \right] .
\]  

(7.49)

The two \(\delta\) functions in the second line of (7.49) come from the trivial integral \(\int dz_0 e^{iz_0 z_0} = 2\pi \delta(z_0)\). Using the relation

\[
\frac{\Gamma(-\epsilon)4^{-\epsilon}}{\Gamma(1-2\epsilon)} = -\frac{\sqrt{\pi}}{\epsilon \Gamma(1/2 - \epsilon)},
\]

(7.50)

we explicitly see that equation (7.49) coincides with the results given in equation (6.4) and (6.6) for the soft correction to the Coulomb-corrected forward-scattering amplitude.
7.2.1 Comparison of $\hat{W}_{WW}$ with the DY soft function

As already noted below equation (7.21), the soft Wilson line $s$ along the direction $v$ of the two non-relativistic $W$s do not enter the definition of the soft function $\hat{W}_{WW}$, equation (7.35), since they cancel in the leading-order production operator. We thus expect the soft function $\hat{W}_{WW}$ to coincide, up to colour factors, with the Drell-Yan soft function given in [97], since they have the same expression in terms of the soft Wilson line $S_{c1}$ and $S_{c2}$ (see (27) of [97]). Here we prove explicitly this statement.

We consider the Fourier transform of the soft function (7.48) at time-like separation (as defined in equation (35) of [97]),

$$W_{WW}(\omega,\mu) = \int \frac{dz_0}{4\pi} e^{iz_0\omega/2} W_{WW}(z_0, \vec{0}) = \int \frac{dz_0}{4\pi} e^{iz_0\omega/2} \left[ 1 + \frac{16\pi i \alpha e^{\gamma_E} \mu^{2\epsilon}}{(4\pi)^e} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i z_0 k_0}}{k^2 + i\epsilon - n \cdot k + i\epsilon - n_+ \cdot k + i\epsilon} \right].$$

(7.51)

The result of the integration can be easily read off equation (7.49), and is

$$W_{WW}(\omega,\mu) = \delta(\omega) + \frac{2\alpha e^{\gamma_E} \Gamma(-\epsilon)}{\pi \Gamma(1-2\epsilon)} \frac{1}{\omega} \left( \frac{\omega}{\mu} \right)^{-2\epsilon} \theta(\omega).$$

(7.52)

We now introduce the Laplace transform $^3$ of $W_{WW}(\omega,\mu)$ with respect to the variable $s = 1/(e^{\gamma_E} \mu e^{L/2})$

$$\tilde{s}_{WW}(L,\mu) = \int_{0_-}^{\infty} d\omega e^{-s\omega} W_{WW}(\omega,\mu)$$

$$= 1 + \frac{2\alpha e^{\gamma_E} \Gamma(-\epsilon)}{\pi \Gamma(1-2\epsilon)} \int_0^{\infty} d\omega e^{-s\omega} \frac{1}{\omega} \left( \frac{\omega}{\mu} \right)^{-2\epsilon}$$

$$= 1 + \frac{2\alpha e^{\gamma_E} \Gamma(-\epsilon) \Gamma(-2\epsilon)}{\pi \Gamma(1-2\epsilon)} (s\mu)^{2\epsilon}$$

$$= 1 - \frac{\alpha \Gamma(-\epsilon)}{\pi} e^{-\epsilon(L+\gamma_E)}$$

$$= 1 + \frac{\alpha}{4\pi} \left[ \frac{4}{\epsilon^2} - \frac{4}{\epsilon} L + 2L^2 + \frac{\pi^2}{3} \right].$$

(7.53)

The $\overline{\text{MS}}$-renormalised quantity reads

$$\tilde{s}_{WW}(L,\mu) = 1 + \frac{\alpha}{4\pi} \left( 2L^2 + \frac{\pi^2}{3} \right),$$

(7.54)

^3The lower integration limit $0_-$ in the definition of the Laplace transform stands for $0_- = \lim_{\epsilon \to 0} -\epsilon$. This ensures that the contribution of the $\delta$ function in 0 is completely included in the integration.
and coincides with the Drell-Yan result given in [97] (Appendix B, equation (78)) at NLO in $\alpha$. Note that in terms of the Fourier transform of the soft function $W_{WW}(\omega, \mu)$ the forward-scattering amplitude can be written as

$$A_{LR} = \frac{(4\pi\alpha_{ew})^2}{M_W^2} |C_p(4M_W^2, \mu)|^2 \int dx_1 dx_2 f_{ee}(x_1, \mu) f_{\bar{e}\bar{e}}(x_2, \mu) \times \int d\omega \Omega(\hat{E} - \omega/2, \mu) W_{WW}(\omega, \mu). \quad (7.55)$$

### 7.3 Resummation of the $W$-pair production cross section near threshold

In the previous section we have derived a factorised expression for the $W$-pair production cross section near threshold in terms of the hard matching coefficient $C_p(4M_W^2, \mu)$, the electron (positron) structure function $f_{ee}(x, \mu)$ ($f_{\bar{e}\bar{e}}(x, \mu)$), the Coulomb Green function $\Omega(E, \mu)$ and the soft function $W_{WW}(\omega, \mu)$, and given a clear definition of these objects in terms of field operators. Each of these functions depends on a single physical scale ($M_W$, $m_e$, $\sqrt{M_W \Gamma_W}$, $\Gamma_W$ respectively) and on the scale $\mu$ at which the corresponding operators are renormalised. These quantities can be reliably calculated in standard fixed-order perturbation theory only if the scale $\mu$ is chosen is such a way that all the logarithms are small. Therefore, $C_p$ should renormalised at a scale $\mu_h \sim 2M_W$, $f_{ee}$ at a scale $\mu_c \sim m_e$, $\Omega$ at $\mu_p \sim \sqrt{M_W \Gamma_W}$ and $W_{WW}$ at $\mu_s \sim \Gamma_W$, and then evolved to a common scale $\mu$. This is done through the renormalisation-group equations obeyed by these quantities. As anticipated in the previous section, as long as the electron mass $m_e$ is set to 0, the electron structure functions are equal to $\delta(1-x)$ to all orders in perturbation theory. Furthermore, the total cross section of the four-fermion production process depends only on the $\mu$-independent imaginary part of the Green function $\Omega$. Hence, in the following we focus on the resummation of the short-distance matching coefficient $C_p(4M_W^2, \mu)$ and of the soft function $W_{WW}(\omega, \mu)$.

#### 7.3.1 Resummation of the hard matching coefficient

We start discussing the resummation of the short-distance coefficient $C_{p, LR}$. As explained in Subsection 4.1.1, $C_{p, LR}$ is determined from the one-shell matching of the one-loop SM matrix element for $e^-e^+ \rightarrow W^-W^+$ (consistently expanded in $\delta$) onto the corresponding quantity in the effective-theory. At $O(\alpha)$, and leading order in $\delta$, the matching equation reads

$$C_{p, LR}^{(1)} M_{eeWW,LR}^{\text{EFT},(0)} = M_{eeWW,LR}^{\text{SM},(1)} - M_{eeWW,LR}^{\text{EFT},(1)}, \quad (7.56)$$

where $M_{eeWW,LR}^{\text{SM},(1)}$, $M_{eeWW,LR}^{\text{EFT},(1)}$ are the $O(\alpha \delta^0)$ (compared to the leading-order matrix element) contributions to the SM and EFT matrix elements, and $M_{eeWW,LR}^{\text{EFT},(0)}$ is the leading-order EFT term. As seen in 4.1.1, $M_{eeWW,LR}^{\text{SM},(1)}$ contains uncancelled infrared poles. When
the matching is performed on-shell, $\mathcal{M}_{eeWW,LR}^{\text{EFT,(1)}}$ is given by scaleless diagrams that vanish in dimensional regularisation. Thus, at $O(\alpha \delta^0)$ we can identify $C_{p,LR}^{\text{(1)}}\mathcal{M}_{eeWW,LR}^{\text{EFT,(0)}}$ with $\mathcal{M}_{eeWW,LR}^{\text{SM,(1)}}$, but the infrared poles of $\mathcal{M}_{eeWW,LR}^{\text{SM,(1)}}$ are replaced in $C_{p,LR}^{\text{(1)}}$ by identical ultraviolet poles. This can be understood remembering that the vanishing of $\mathcal{M}_{eeWW,LR}^{\text{EFT,(1)}}$ is a consequence of the cancellation of infrared and ultraviolet divergences. Schematically we have

$$\mathcal{M}_{eeWW,LR}^{\text{EFT,(1)}} = \frac{1}{\epsilon_{IR}} - \frac{1}{\epsilon_{UV}},$$

where $1/\epsilon_{IR}$ ($1/\epsilon_{UV}$) denotes both double and single infrared (ultraviolet) poles. Similarly

$$\mathcal{M}_{eeWW,LR}^{\text{SM,(1)}} = \frac{1}{\epsilon_{IR}} + \text{finite terms} \ (7.58)$$

Thus the matching equation (7.56) reads

$$C_{p,LR}^{\text{(1)}}\mathcal{M}_{eeWW,LR}^{\text{EFT,(1)}} = \left(\frac{1}{\epsilon_{IR}} + \text{finite terms}\right) - \left(\frac{1}{\epsilon_{IR}} - \frac{1}{\epsilon_{UV}}\right) = \frac{1}{\epsilon_{UV}} + \text{finite terms}. \ (7.59)$$

The expression of the bare hard matching coefficient $\tilde{C}_{p,LR}(4M_W^2, \mu, \epsilon)$ has been given in Subsection 4.1.1:

$$\tilde{C}_{p,LR}(4M_W^2, \mu; \epsilon_{UV}) = 1 + \alpha \left[ \left(\frac{1}{2\epsilon_{UV}} - \frac{3}{2\epsilon_{UV}}\right) \left(-\frac{4M_W^2}{\mu^2}\right)^{-\epsilon_{UV}} + c_{p,LR}^{(1,\text{fin})}\right] + O(\alpha^2), \ (7.60)$$

where $c_{p,LR}^{(1,\text{fin})}$ is the finite, $\mu$-independent part. The corresponding $\overline{\text{MS}}$-renormalised quantity reads

$$C_{p,LR}(4M_W^2, \mu; \epsilon_{UV}) \equiv \lim_{\epsilon_{UV} \to 0} Z_{WW}^{-1}(4M_W^2, \mu; \epsilon_{UV}) \tilde{C}_{p,LR}(4M_W^2, \mu; \epsilon_{UV})$$

$$= 1 + \alpha \left[ \frac{1}{2} \ln^2 \left(-\frac{4M_W^2}{\mu^2}\right) + \frac{3}{2} \ln \left(-\frac{4M_W^2}{\mu^2}\right) + c_{p,LR}^{(1,\text{fin})}\right]. \ (7.61)$$

The function $Z_{WW}(4M_W^2, \mu; \epsilon_{UV})$ cancels the $\epsilon_{UV}$ poles, and up to $O(\alpha)$ has the following form

$$Z_{WW}(4M_W^2, \mu; \epsilon_{UV}) = 1 + \frac{\alpha}{2\pi} \left[ \frac{1}{\epsilon_{UV}^2} + \frac{1}{\epsilon_{UV}} \left( \ln \left(-\frac{4M_W^2}{\mu^2}\right) - \frac{3}{2} \right) \right] + O(\alpha^2). \ (7.62)$$

The renormalised matching coefficient, equation (7.61), obeys the renormalisation-group equation [97]

$$\frac{d}{d \ln \mu} C_{p,LR}(4M_W^2, \mu) = \left[ \Gamma_{\text{cusp}}(\alpha) \ln \left(-\frac{4M_W^2}{\mu^2}\right) + \gamma^{WW}(\alpha) \right] C_{p,LR}(4M_W^2, \mu), \ (7.63)$$

where the universal cusp anomalous dimension $\Gamma_{\text{cusp}}(\alpha) = \sum_n \Gamma_n \left(\frac{\alpha}{4\pi}\right)^{n+1}$ [114, 115], and the process-dependent anomalous dimension $\gamma^{WW}(\alpha) = \sum_n \gamma_n^{WW} \left(\frac{\alpha}{4\pi}\right)^{n+1}$ account for the
resummation of double and single logarithms of \(-4M_W^2/\mu^2\). They are determined from the residue \(Z_{WW}^{(1)}(4M_W^2, \mu)\) of the 1/\(\epsilon_{UV}\) pole of \(Z_{WW}(4M_W^2, \mu; \epsilon_{UV})\) [96],

\[
\Gamma_{\text{cusp}}(\alpha) \ln \left( \frac{-4M_W^2}{\mu^2} \right) + \gamma_{WW}(\alpha) \equiv 2\alpha \frac{\partial}{\partial\alpha} Z_{WW}^{(1)}(4M_W^2, \mu).
\]

(7.64)

From (7.64) and (7.62) we easily derive

\[
\Gamma_0 = 4 \quad \gamma_{WW}^0 = -6,
\]

(7.65)

which coincide, up to a trivial colour factor, with the results given in [97] for the Drell-Yan case. The solution of equation (7.63) has been derived in [96, 97], and reads

\[
C_{\text{res,LR}}(4M_W^2, \mu) = \exp \left[ 2S(\mu_h, \mu) - a_{\gamma,WW}(\mu_h, \mu) \right] \left( \frac{-4M_W^2}{\mu_h^2} \right)^{-a_\Gamma(\mu_h, \mu)} C_{\text{p,LR}}(4M_W^2, \mu_h),
\]

(7.66)

with the functions \(S, a_{\gamma,WW}\) and \(a_\Gamma\) defined as in [97],

\[
S(\mu_h, \mu) = -\int_{\alpha(\mu_h)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_{\text{cusp}}(\alpha) \int_{\alpha(\mu_h)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')},
\]

\[
a_\Gamma(\mu_h, \mu) = -\int_{\alpha(\mu_h)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} \Gamma_{\text{cusp}}(\alpha),
\]

\[
a_{\gamma,WW}(\mu_h, \mu) = -\int_{\alpha(\mu_h)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} \gamma_{WW}(\alpha).
\]

(7.67)

\(\alpha(\mu)\) represents the QED running coupling constant, and \(\beta(\alpha) = -2\alpha \sum \beta_n \left( \frac{\alpha}{4\pi} \right)^{n+1}\) is the corresponding \(\beta\)-function. The scale \(\mu_h\) is the scale at which the renormalisation of \(C_{\text{p,LR}}\) is performed, and should be chosen of order \(2M_W\). The truncation of the perturbative series of \(\Gamma_{\text{cusp}}, \gamma_{WW}\) and \(C_{\text{p,LR}}\) depends on the required accuracy of the resummed result \(C_{\text{res,LR}}(4M_W^2, \mu)\). Counting \(\ln(-4M_W^2/\mu^2)\) as \(\alpha^{-1}\), for a NLO resummation of the coefficient \(C_{\text{p,LR}}\) we need the 1, 2 and 3-loop cusp anomalous dimension \(\Gamma_0, \Gamma_1, \Gamma_2\), the 1 and 2-loop \(\gamma_{WW}\) anomalous dimension, \(\gamma_0^{WW}, \gamma_1^{WW}\), and the 1-loop fixed order matching coefficient \(C_{\text{p,LR}} = 1 + C_{\text{p,LR}}^{(1)}\) [96]. Explicit results for the expansion of the functions \(S(\nu, \mu), a_\Gamma(\nu, \mu)\) and \(a_{\gamma,WW}(\nu, \mu)\) in \(\alpha\) can be found in Appendix B.3 of [97]. Note that the all-order resummed expression (7.66) does not depend on \(\mu_h\), but truncating the perturbative expansion at a finite order in \(\alpha\) introduces a dependence on \(\mu_h\), which is however of next order in \(\alpha\).

Before presenting numerical results for the resummed matching coefficient, we can give a first estimate of the impact of resummation on \(C_{\text{p,LR}}\) by re-expanding the expression (7.66) to \(O(\alpha)\), except for the terms in the exponent containing the leading-order cusp anomalous dimension \(\Gamma_0\), and keeping the exponent unexpanded when it multiplies double logarithms. This corresponds to exponentiating only double logarithms,
\[ \alpha^n \ln^n (-4M_W^2/\mu^2). \]

In this approximation, the resummed coefficient (7.66) assumes the simple form

\[
C_{p, LR}^{\text{res}}(4M_W^2, \mu) = \exp \left[ -\frac{\alpha}{4\pi} \Gamma_0 \ln^2 \left( \frac{\mu}{\mu_h} \right) \left( -\frac{4M_W^2}{\mu^2} \right) \right. \\
+ \frac{\alpha}{2\pi} \left( \frac{3}{2} \ln \left( -\frac{4M_W^2}{\mu^2} \right) + c_{p, LR}^{(1, \text{fin})} \right) \\
\left. \times \left[ 1 - \frac{\alpha}{4\pi} \ln^2 \left( -\frac{4M_W^2}{\mu^2} \right) \right] + \frac{\alpha}{2\pi} \left( \frac{3}{2} \ln \left( -\frac{4M_W^2}{\mu^2} \right) + c_{p, LR}^{(1, \text{fin})} \right) \right],
\]

(7.68)

where, since the running of the coupling is a single-logarithm effect, \( \alpha(\mu) \) has been replaced by the usual fixed coupling constant \( \alpha \) in the \( G_\mu \)-scheme. If the exponential in (7.68) is completely expanded to \( O(\alpha) \), we obtain the correct \( \mu_h \)-independent result, equation (7.61). A measure of the effect due to resummation of the leading logarithms is given by the quantity

\[
\delta(\mu) = \frac{|C_{p, LR}^{\text{res}}(4M_W^2, \mu)|^2 - |C_{p, LR}(4M_W^2, \mu)|^2}{|C_{p, LR}(4M_W^2, \mu)|^2},
\]

(7.69)

where \( C_{p, LR}(4M_W^2, \mu) \) is given by (7.61). If we choose the renormalisation scale equal to the \( W \)-boson width, \( \Gamma_W = 2.09201 \) GeV, we obtain \( \delta(\Gamma_W) = 0.00045, 0.00041, 0.00042 \) for \( \mu_h = M_W, 2M_W, 3M_W \) respectively. As already pointed out, the mild dependence on the hard scale \( \mu_h \) is a consequence of the truncation of the perturbative series to a finite order in \( \alpha \). The effect of leading-log resummation of the hard matching coefficient appears to be quite small. The full renormalisation-improved NLO result is presented in Subsection 7.3.3

### 7.3.2 Resummation of the soft function

The definition of the \( W \)-pair soft function \( \hat{W}_{WW} \), equation (7.35), coincides with the definition of the Drell-Yan soft function \( \hat{W}_{DY} \), equation (27) of [97] (with the obvious substitution of gluon fields with photon fields, and up to trivial colour prefactors). Thus the renormalisation-group equation for the Fourier transform of \( \hat{W}_{WW} \) is [97]

\[
\frac{dW_{WW}(\omega, \mu)}{d\ln \mu} = -4\Gamma_{\text{cusp}}(\alpha) \ln \left( \frac{\omega}{\mu} \right) + 2\gamma^W(\alpha) \right] W_{WW}(\omega, \mu)
\\
-\Gamma_{\text{cusp}}(\alpha) \int_0^\omega d\omega' \frac{W_{WW}(\omega', \mu) - W_{WW}(\omega, \mu)}{\omega - \omega'},
\]

(7.70)

where \( \Gamma_{\text{cusp}} \) is the cusp anomalous dimension already introduced in (7.63), and the quantity \( \gamma^W \) is defined by

\[
\gamma^W(\alpha) = 2\gamma^\phi(\alpha) + \gamma^{WW}(\alpha).
\]

(7.71)
\( \gamma^\phi \) is extracted from the threshold limit \((z \to 1)\) of the Altarelli-Parisi electron splitting function,
\[
P_{e^{-e}}(z) = \frac{2\Gamma_{\text{cusp}}(\alpha)}{[1 - z]^+} + 2\gamma^\phi(\alpha)\delta(1 - z) + \ldots \tag{7.72}
\]
Since the anomalous dimensions \( \gamma^W \) and \( \gamma^\phi \) are identical for \( W \)-pair production and DY, it follows that also the quantity \( \gamma^{WW} \) introduced in (7.63) and \( \gamma^V \) given in equation (38) of [97] must be the same, up to trivial modifications that take into account the different couplings and colour factors of QED and QCD.

The solution of equation (7.70) can be read off equation (45) of [97], and is
\[
W^{\text{res}}_{WW}(\omega, \mu) = \exp[-4S(\mu_s, \mu) + 2a_{\gamma^W}(\mu_s, \mu)]\tilde{s}_{WW}(\partial_\eta, \mu_s)\frac{1}{\omega} \left( \frac{\omega}{\mu_s} \right)^{2\eta} \theta(\omega) \frac{e^{-2\gamma E\eta}}{\Gamma(2\eta)}, \tag{7.73}
\]
where \( \eta = 2a_{\gamma^W}(\mu_s, \mu) \), and \( a_{\gamma^W}(\mu_s, \mu) \) is defined as in (7.67). The function \( \tilde{s}_{WW}(\partial_\eta, \mu_s) = \tilde{s}_{WW}(L, \mu_s) \big|_{L \to \partial_\eta} \) is the Laplace transform of the function \( W_{WW}(\omega, \mu) \), equation (7.54), and \( \mu_s \) is the scale at which the fixed-order soft function is renormalised. To compute the forward-scattering amplitude (7.55), we have first to calculate the convolution of the resummed soft function \( W^{\text{res}}_{WW} \) with the Coulomb Green function \( \Omega \),
\[
\mathcal{F}(\hat{E}, \mu) = \int d\omega \Omega(\hat{E} - \omega/2, \mu)W^{\text{res}}_{WW}(\omega, \mu). \tag{7.74}
\]
Considering the estimate of the effects of resummation of logarithmically enhanced terms given in the previous subsection, we can limit ourselves to the convolution of the leading-order Green function, \( \Omega(\hat{E}) = -\frac{M_W^2}{4\pi} \sqrt{\frac{\hat{E} + i\Gamma_W}{M_W}} \). This can be easily performed, and leads to the result
\[
\mathcal{F}(\hat{E}, \mu) = \Omega(\hat{E}) \exp[-4S(\mu_s, \mu) + 2a_{\gamma^W}(\mu_s, \mu)] \times \tilde{s}_{WW}(\partial_\eta, \mu_s)e^{-2\gamma E\eta} \frac{\Gamma(-\frac{1}{2} - 2\eta)}{\Gamma(-\frac{1}{2})} \left( -\frac{2(\hat{E} + i\Gamma_W)}{\mu_s} \right)^{2\eta}. \tag{7.75}
\]
If we insert in (7.75) the one-loop result for \( \tilde{s}_{WW} \), equation (7.54), we obtain
\[
\mathcal{F}(\hat{E}, \mu) = \Omega(\hat{E}) \exp[-4S(\mu_s, \mu) + 2a_{\gamma^W}(\mu_s, \mu)]e^{-2\gamma E\eta} \frac{\Gamma(-\frac{1}{2} - 2\eta)}{\Gamma(-\frac{1}{2})} \left( -\frac{2(\hat{E} + i\Gamma_W)}{\mu_s} \right)^{2\eta} \times \left\{ 1 + \frac{\alpha(\mu_s)}{\pi} \left[ 2 \gamma_E + \psi' \left( -\frac{1}{2} - 2\eta \right) - \ln \left( -\frac{2(\hat{E} + i\Gamma_W)}{\mu_s} \right) \right]^2 + 2\psi' \left( -\frac{1}{2} - 2\eta \right) + \frac{\pi^2}{12} \right\} \right\}. \tag{7.76}
\]
Re-expanding the above equation to order $\alpha$, one recovers the fixed-order result, given by the sum of the first term in equation (4.22) and equation (4.42):

$$
\mathcal{F}(\hat{E}, \mu) = \Omega(\hat{E}) \left\{ 1 + \frac{2\alpha}{\pi} \ln^2 \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu} \right) - 4 \ln \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu} \right) + 8 + \frac{13}{24} \pi^2 \right\}.
$$

(7.77)

Note that equation (7.77) seems to suggest that the soft scale $\mu_s$ should be chosen of order $8\sqrt{E^2 + \Gamma_W^2}$ rather than $\Gamma_W$ in order to minimise the contribution of the logarithmically enhanced terms in the fixed-order result.

Analogously to the hard matching coefficient, we can estimate the leading-order effect originating from resummation of the double logarithms. Performing the leading-logarithmic substitutions $S(\mu_s, \mu) = -\frac{\alpha}{8\pi} \Gamma_0 \ln^2 \frac{\mu}{\mu_s}, \eta = 2\alpha \Gamma(\mu_s, \mu) = -\frac{\alpha}{\pi} \Gamma_0 \ln \frac{\mu}{\mu_s}$ and $a_{\gamma W}(\mu_s, \mu) = 0$, and re-expanding the subleading terms, equation (7.76) reduces to

$$
\mathcal{F}^{\text{res}}(\hat{E}, \mu) = \Omega(\hat{E}) \left\{ \exp \left[ \frac{2\alpha}{\pi} \ln^2 \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu} \right) - \frac{2\alpha}{\pi} \ln^2 \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu_s} \right) \right] 
\times \left( 1 + \frac{2\alpha}{\pi} \ln^2 \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu_s} \right) \right) \right.
+ \left. \frac{2\alpha}{\pi} \left[ -4 \ln \left( -\frac{8(\hat{E} + i\Gamma_W)}{\mu} \right) + 8 + \frac{13}{24} \pi^2 \right] \right\},
$$

(7.78)

where again, at the leading-logarithmic level, the running coupling $\alpha(\mu)$ is replaced by the fixed coupling in the $G_\mu$ scheme. In Figure 7.1 we plot the quantity

$$
\frac{\text{Im}\mathcal{F}^{\text{res}}(E, \mu) - \text{Im}\mathcal{F}(E, \mu)}{\text{Im}\mathcal{F}(E, \mu)},
$$

(7.79)

with $\mathcal{F}^{\text{res}}(E, \mu)$ and $\mathcal{F}(E, \mu)$ given respectively by equations (7.78) and (7.77), for different values of the centre-of-mass energy and for different choices of the soft scale $\mu_s$. In the following we will choose the default value of $\mu_s$ to be $8\sqrt{E^2 + M_W^2}$, and consider variations in the interval $4\sqrt{E^2 + M_W^2} < \mu_s < 16\sqrt{E^2 + M_W^2}$. The renormalisation scale $\mu$ is set to $2M_W$. From Figure 7.1 we see that, also in this case, the effect of leading resummation is below 1‰ near threshold.

7.3.3 Numerical results

We now present numerical results for the complete NLO threshold resummation of the four-fermion production cross section. As mentioned in the previous section, all the relevant anomalous dimensions can be obtained from the analogous Drell-Yan quantities by replacing the QCD coupling constant with $\alpha$ and adjusting the colour factors. This reduces to the replacements $C_F \rightarrow 1$, $C_A \rightarrow 0$, $T_f m_f \rightarrow Q^2 \equiv \sum_f Q_f^2$, where $Q_f$ represents the
electric charge of the flavour $f$, and the sum extends over the active flavours. Here we consider a flavour active when the scale $\mu$ is larger than the pair-production threshold for that flavour, $\mu > 2m_f$. Thus, since we are primarily interested in the region $\Gamma_W < \mu < 2M_W$, the active flavours are represented by 3 leptons and 5 light quarks \footnote{Strictly speaking, at $\mu \sim \Gamma_W$ the bottom quark cannot be considered massless. However the error of our approximation is well beyond the accuracy considered here.}, and $Q^2 = 20/3$.

From Appendix B.2 of [97] one can extract the cusp anomalous dimension, needed to the three-loop level:

\begin{align}
\Gamma_0 &= 4, \\
\Gamma_1 &= -\frac{80}{9} Q^2 \\
\Gamma_2 &= 4 \left[ \left( -\frac{55}{3} + 16\zeta_3 \right) Q^2 - \frac{16}{27} (Q^2)^2 \right].
\end{align}

The anomalous dimension $\gamma^{WW}$, to the two-loop level, is given by [97]

\begin{align}
\gamma_0^{WW} &= -6, \\
\gamma_1^{WW} &= -3 + 4\pi^2 - 48\zeta_3 + \left( \frac{260}{27} + \frac{4\pi^2}{3} \right) Q^2,
\end{align}

Figure 7.1: Effect of resummation of leading logarithms on the soft function for different centre-of-mass energies and $\mu_s = 4 \sqrt{E^2 + M_W^2}$ (dotted red), $\mu_s = 8 \sqrt{E^2 + M_W^2}$ (solid green) and $\mu_s = 16 \sqrt{E^2 + M_W^2}$ (dashed blue). The renormalisation scale $\mu$ is set to $2M_W$.\[\text{\Large $\delta \mathcal{F}/\mathcal{F}$ [\%]} \quad \sqrt{s} \text{ [GeV]}
\]
while \( \gamma^W \), again up to the two-loop level, is
\[
\gamma^W_0 = 0, \\
\gamma^W_1 = \left(\frac{224}{27} - \frac{4\pi^2}{9}\right) Q^2.
\] (7.82)

Finally the QED beta function, needed to the three-loop level, reads
\[
\begin{align*}
\beta_0 &= -\frac{4}{3} Q^2, \\
\beta_1 &= -\frac{4}{3} Q^2, \\
\beta_2 &= 2 Q^2 + \frac{44}{9} (Q^2)^2.
\end{align*}
\] (7.83)

\( \alpha(\mu) \) is obtained by numerically evolving the coupling constant with the differential equation
\[
\mu \frac{d\alpha(\mu)}{d\mu} = -2\alpha(\mu) \left[ \beta_0 \frac{\alpha(\mu)}{4\pi} + \beta_1 \left( \frac{\alpha(\mu)}{4\pi} \right)^2 + \beta_2 \left( \frac{\alpha(\mu)}{4\pi} \right)^3 \right]
\] (7.84)
from an initial value \( \alpha(\mu_i) \). Since in the limit \( \beta_i \to 0 \) we want to recover our fixed-order result in the \( G^\mu \) scheme, \( \mu_i \) has been chosen in such a way that \( \alpha(\mu_i) = \alpha \). Numerically one finds \( \mu_i \sim 9.2 \text{ GeV} \).

Before presenting numerical results, we will define more precisely our resummed NLO cross section. Our master result is given by equation (7.55), where the electron structure functions are set to \( \delta(1-x) \), and \( C_p \) and \( W_W \) are replaced by the corresponding resummed expressions. Taking the imaginary part, and adding the usual prefactor \( 1/(27s) \) we obtain
\[
\frac{1}{27s} \text{Im} \mathcal{A}_{LR}^{\text{res}} = \frac{(4\pi\alpha_{\text{ew}})^2}{27 M_{W}^2} |C_p^{\text{res}}(4M_{W}^2, \mu)|^2 \text{Im} \mathcal{F}^{\text{res}}(E, \mu),
\] (7.85)
where \( C_p^{\text{res}}(4M_{W}^2, \mu) \) and \( \mathcal{F}^{\text{res}}(E, \mu) \) were given in equations (7.66) and (7.76) respectively. If we re-expand equation (7.85) to order \( \alpha \), we obtain
\[
\frac{1}{27s} \text{Im} \mathcal{A}_{LR}^{\text{res}} = \sigma_{LR}^{(0)} + \Delta \sigma_{\text{hard}}^{(1)} + \Delta \sigma_{\text{soft}}^{(1)} + O(\alpha^2).
\] (7.86)
Thus the resummed expression does not include neither higher-order contributions to the Born cross-section, since in the derivation of the factorisation formula we have considered only the leading-order production operator, nor Coulomb-photon corrections, since in (7.75) we have set \( \Omega \) to the leading-order Green function. Furthermore, since we have set \( m_e = 0 \), the contributions from hard-collinear and soft-collinear modes are also missing.

We thus define our resummed “partonic” cross section as
\[
\hat{\sigma}_{LR}^{\text{res}}(s) = \sigma_{LR, \text{Born}} + \sigma_{LR, \text{conv}}^{(1)} + \Delta \sigma_{LR}^{\text{res}},
\] (7.87)
Figure 7.2: Correction to the unpolarised cross section from NLO resummation of threshold logarithms, for $\mu_s = 8\sqrt{E^2 + \Gamma_W^2}$ and different values of $\mu_h$: $\mu_h = M_W$ (dotted red), $\mu_h = 2M_W$ (solid green), $\mu_h = 4M_W$ (dashed blue). The correction is normalised to the Born cross section computed with WHIZARD.

Figure 7.3: Correction to the unpolarised cross section from NLO resummation of threshold logarithms, for $\mu_h = 2M_W$ and different values of $\mu_s$: $\mu_s = 4\sqrt{E^2 + \Gamma_W^2}$ (dotted red), $\mu_s = 8\sqrt{E^2 + \Gamma_W^2}$ (solid green), $\mu_s = 16\sqrt{E^2 + \Gamma_W^2}$ (dashed blue). The correction is normalised to the Born cross section computed with WHIZARD.

where $\sigma_{LR,\text{Born}}$ is the Born cross section, $\hat{\sigma}_{LR,\text{conv}}^{(1)}$ is the NLO result given in equation (4.72), and $\Delta\sigma_{LR}^{\text{res}}$ is defined as

$$\Delta\sigma_{LR}^{\text{res}} = \frac{(4\pi\alpha_{\text{ew}})^2}{27M_W^2s} \left| C_p^{\text{res}}(4M_W^2, \mu) \right|^2 \text{Im} F_{\text{res}}^{\text{res}}(E, \mu) - \left( \sigma_{LR}^{(0)} + \Delta\sigma_{\text{hard}}^{(1)} + \Delta\sigma_{\text{soft}}^{(1)} \right), \quad (7.88)$$
Figure 7.4: Correction to the unpolarised cross section from NLO resummation of threshold logarithms, for $\mu_h = 2M_W$ and $\mu_s = 8\sqrt{E^2 + \Gamma_W^2}$ (solid black line). The grey area corresponds to variations of the scales in the intervals $M_W < \mu_h < 4M_W$, $4\sqrt{E^2 + \Gamma_W^2} < \mu_s < 16\sqrt{E^2 + \Gamma_W^2}$. The correction is normalised to the Born cross section computed with WHIZARD.

In $\Delta\sigma_{LR}^{\text{res}}$ the renormalisation scale $\mu$ should be set to $\mu = \sqrt{s}$, since this is the choice made in the explicit evaluation of the convolution of the partonic cross section with the electron structure functions (see Section 4.5). In Figures 7.2 and 7.3 we present the correction from the NLO (unpolarised) resummation contribution $\Delta\sigma_{LR}^{\text{res}} = \Delta\sigma_{LR}^{\text{res}}/4$ (normalised to the Born cross section $\sigma_{\text{Born}}$ from WHIZARD) for different values of the hard and soft scales $\mu_h$ and $\mu_s$. In both cases the order of magnitude of the correction is consistent with the effect observed for the resummation of the leading double logarithms (see Figure 7.1). However we note that the shape of the correction is very different. This effect originates from the formally subleading contributions from the running of the coupling and the anomalous dimensions $\gamma_{WW}$ and $\gamma_W$, and it can be understood by noting that, for the choice $\mu = \sqrt{s} \sim 2M_W$, one has $|\ln(-8(E + i\Gamma_W)/\mu)| \sim |\ln(-4i\Gamma_W/M_W)| = 2.8$. Thus the assumption $|\ln^2(-8(E + i\Gamma_W)/\mu)| \gg |\ln(-8(E + i\Gamma_W)/\mu)| \gg 1$ is not completely justified, especially considering the large coefficient multiplying the single logarithm in equation 7.77. The numerical value of $|\ln(-8(E + i\Gamma_W)/\mu)|$ also explains the moderate impact of resummation on the four-fermion production cross section.

Our best estimate of the effect of NLO resummation on the total cross section, corresponding to $\mu_h = 2M_W$, $\mu_s = 8\sqrt{E^2 + \Gamma_W^2}$, is presented in Figure 7.4. The total uncertainty related to the choice of $\mu_h$ and $\mu_s$ is represented by the grey region in the plot. We clearly see that the effect is small near threshold ($\sim 1\%$), and of the same order of magnitude of the uncertainty related to the choice of $\mu_h$ and $\mu_s$, revealing that resummation of threshold logarithms is not relevant for the four-fermion production process.
Chapter 8

Conclusion

In this thesis we have presented a dedicated study of the process of four-fermion production near the $W$-pair production threshold. This was motivated by the importance of this process for an accurate determination of the $W$-boson mass, and by the necessity to reduce the total theoretical uncertainty to the level of $1\%$ needed at the planned International Linear Collider, where $M_W$ might be measured with an error of only $6\text{ MeV}$ from a scan of the threshold region [12].

The study was performed in the context of the recently developed unstable-particle effective theory [50, 51], that provides a framework for a consistent gauge-invariant treatment of finite-width effects, and for a straightforward inclusion of higher-order corrections. In this approach the computation is organised as a systematic expansion of the cross section in the couplings $\alpha, \alpha_s$ and the ratios $\Gamma_W/M_W$ and $(\sqrt{s} - 2M_W)/M_W$. Due to the simplifications allowed by the threshold kinematics, the calculation is much simpler than the full Standard Model computation and results in compact, analytic expressions. These have to be compared with the technically demanding numerical calculation of the cross section in the complex-mass scheme [43, 44]. The comparison of our predictions with numerical results for the full Born cross section and NLO electroweak and QCD corrections has shown a nice convergence of the effective-theory expansion, and very good agreement near threshold (once the first subleading term in each relevant region is included in the calculation), thus confirming the reliability of the method.

In view of the determination of the $W$-boson mass from the four-fermion cross section, we have estimated how remaining theoretical uncertainties on the NLO effective-theory result translate into an error on $M_W$. Our study led us to the following conclusions:

- A resummation of next-to-leading collinear logarithms from initial-state radiation is mandatory to reduce the error on $M_W$ below the $30 \text{ MeV}$ level.

- The NLO partonic cross-section calculation in the effective theory approach implies a residual error of about $10 - 15 \text{ MeV}$ on $M_W$.

The first uncertainty is common to all high-precision studies in high-energy $e^-e^+$ collision, and can be forseeably removed with further work on a next-to-leading-logarithmic ISR resummation. We thus do not consider it as a fundamental difficulty. Of the second
uncertainty, the largest part arises from terms included in the full NLO four-fermion calculation, and can thus be eliminated. The remaining part is associated with the dominant contribution of NNLO Standard Model diagrams, which represents a $\mathcal{N}^{3/2}$LO correction in the effective-theory counting. This has been explicitly calculated, and amounts to few-permilles of the Born cross section near threshold, while higher-order corrections have been estimated to be well below the 1% target accuracy. We therefore conclude that, combining the full NLO computation and results from the effective-theory calculation, a theoretical description of the four-fermion production process near threshold that matches the expected experimental accuracy at ILC is available.

The calculation presented here is the first NLO calculation of a realistic process in unstable-particle effective theory, since \cite{50,51} discussed the case of a single resonance in a gauged Yukawa model. The formalism is not limited to gauge bosons, but can be extended to describe arbitrary processes involving production and decay of massive particles. If LHC will discover heavy resonances (SUSY partners, Kaluza-Klein modes,...), their masses could be precisely measured at ILC using threshold scans, and the effective-theory formalism appears to be a convenient choice for a precise theoretical description of the experimental set-up. Also, effective-theory methods allow for a straightforward inclusion of enhanced logarithmic terms $\alpha^n \ln^m (\Gamma/M)$ to all order in perturbation theory, via a threshold resummation of the cross section \cite{95,96}. Though we have shown that the effect is small for the four-fermion production process, threshold resummation could be relevant for production of heavy coloured particles (top quarks, squarks, gluinos) at LHC. Work in this direction is in progress, and will appear in forthcoming publications.

Finally, we should mention that our calculation is restricted to the inclusive cross section, while a more flexible treatment of the final-state phase space is obviously desirable. As explained in the present thesis, this requires either applying effective field theory methods to four-fermion production amplitudes rather than the forward-scattering amplitude, or the consideration of specific cuts, such as corresponding to invariant-mass distributions, that allow for a semi-inclusive treatment. Interesting developments in this direction have recently been reported for top-quark pair production \cite{92}. 


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Appendix A

The expansion by regions

In this appendix we review the method known as “expansion by regions” and its application to the calculation of the asymptotic expansion of Feynman integrals, with particular emphasis on the case of diagrams with threshold kinematics. We refer the reader to [52] and [69] for more details and examples.

The method applies to Feynman diagrams involving different (hierarchically ordered) mass scales, $\Lambda_1 \ll \Lambda_2 \ll \ldots \ll \Lambda_n$, which are notoriously difficult to calculate in perturbation theory beyond the one-loop level, and gives prescriptions for the systematic expansion of the exact integral in ratios of the mass scales, $\delta_i = \Lambda_1/\Lambda_{1+i}$, with $i = 1, \ldots, n-1$. Each single term in the expansion is represented by a manifestly homogeneous integral, depending only on one scale, and simpler than the original integral. Thus the initial analytic complexity is replaced by the algebraic complexity of computing a large-enough number of terms in the series.

The procedure for the calculation of the expansion of a given integral $I_F(\Lambda_1, \ldots, \Lambda_n; \epsilon)$ is organised in four steps:

1. Identify the large and small scales in the problem, $\Lambda_1 \ll \Lambda_2 \ll \ldots \ll \Lambda_n$.

2. Divide the integration domain in regions in which the loop momentum is considered to be of order of one of the scales in the problem.

3. In every region expand the integrand in the parameters and momenta (internal and external) that are small in that region.

4. After the expansion restore the entire loop-integration domain in every region.

$I_F(\Lambda_1, \ldots, \Lambda_n; \epsilon)$ is then reproduced, order by order in $\delta_i$, by the sum of the contributions of the different regions. In order for the procedure to correctly reproduce the expansion of the original integral in the ratios of scales, dimensional regularisation must be used [52]. While single contributions may contain extra singularities in $\epsilon$, as a consequence of the expansion and integration at steps 3 and 4, the sum correctly reproduces the pole structure of $I_F$.
As a first simple example we consider the toy integral
\[ I_F(m, M; \epsilon) = \int_0^\infty dk \frac{k^{-\epsilon}}{(k + m)(k + M)}, \]  
(A.1)
where the two masses satisfy \( m \ll M \), and the term \( dk k^{-\epsilon} \) plays the role of the \( d \)-dimensional volume element, \( d^dk \), in dimensional regularisation. In this case the integral can be computed without any approximation, and the explicit result reads
\[ I_F(m, M; \epsilon) = -\frac{\pi}{\sin(\pi \epsilon)} \frac{m^{-\epsilon} - M^{-\epsilon}}{m - M} = 1 + \ln \left( \frac{M}{m} \right) + O(\epsilon). \]  
(A.2)

We now apply the expansion by regions to reproduce the expression (A.2) order-by-order in the ratio \( m/M \ll 1 \). The relevant loop-momentum regions are represented by a large-momentum region \((l)\), where \( k \sim M \), and a small-momentum region \((s)\), where \( k \sim m \). We will now consider the contribution of these two regions in turn.

The large-momentum region is identified by the condition \( m \ll k \sim M \) relating the loop-momentum \( k \) to the two masses \( m \) and \( M \). Since \( m \ll k \), we can Taylor-expand the term \( 1/(k + m) \) in the integrand of (A.1). Thus the all-order expansion in the large-momentum region is
\[ I_F^{(l)}(m, M; \epsilon) = \sum_{n=0}^{\infty} \int_0^\infty dk \frac{k^{-\epsilon}}{k(k + M)} \left( -\frac{m}{k} \right)^n. \]  
(A.3)

Note that now each integral in the series (A.3) depends only on the scale \( M \). Solving the integral explicitly we obtain
\[ I_F^{(l)}(m, M; \epsilon) = -\frac{\pi}{\sin(\pi \epsilon)} M^{-1-\epsilon} \sum_{n=0}^{\infty} \left( \frac{m}{M} \right)^n. \]  
(A.4)
The elements of the series scale as powers of the ratio of the two masses. Since \( m/M \ll 1 \), we could in principle retain only a finite number of terms, i.e. those necessary to achieve the required final accuracy. In this simple case the terms of the series can be easily resummed to all orders,
\[ I_F^{(l)}(m, M; \epsilon) = -\frac{\pi}{\sin(\pi \epsilon)} M^{-1-\epsilon} \sum_{n=0}^{\infty} \left( \frac{m}{M} \right)^n. \]  
(A.5)

Next we consider the contribution of the small-momentum region, where \( m \sim k \ll M \). Now the scaling of the momentum is such that we can expand the quantity \( 1/(k + M) \) in the ratio \( k/M \),
\[ I_F^{(s)}(m, M; \epsilon) = \frac{1}{M} \sum_{n=0}^{\infty} \int_0^\infty dk \frac{k^{-\epsilon}}{k + m} \left( -\frac{k}{M} \right)^n. \]  
(A.6)
Performing the integration, and resumming the terms \((m/M)^n\) to all orders as before, we straightforwardly obtain
\[ I_F^{(s)}(m, M; \epsilon) = \frac{\pi}{\sin(\pi \epsilon)} \frac{m^{-\epsilon}}{M - m} = 1 + \ln \left( \frac{M}{m} \right) + O(\epsilon). \]  
(A.7)
The sum of (A.5) and (A.7) manifestly coincides with the full result (A.2). As anticipated, separate contributions of different regions contain additional singularities in \( \epsilon \), as a consequence of the separation of scales, but the sum correctly reproduces the pole structure of (A.2).

We now turn to a more interesting example, the scalar integral shown in Figure A.1. The internal lines represent scalars with equal masses \( M^2 \) and total momentum \( P \). The loop momentum is chosen such that half of the total external momentum \( P \) flows in each internal line. Thus the corresponding Feynman integral reads

\[
I_F(P^2, \Delta; \epsilon) = \frac{1}{(2\pi)^d} \int \frac{d^d r}{(r^2 + P \cdot r + \Delta + i\epsilon)(r^2 - P \cdot r + \Delta + i\epsilon)}, \tag{A.8}
\]

where we have introduced the notation \( \Delta = P^2 - 4M^2 \), and \( d = 4 - 2\epsilon \) as usual. It is useful to define the dimensionless quantity \( \delta \equiv \Delta/P^2 \). Here we are interested in the threshold limit \( \Delta \ll P^2 \), or, equivalently, \( \delta \ll 1 \). The situation is very similar to that of \( W \)-pair production near threshold, where \( s - 4M_W^2 \ll s \), the major difference being that the mass parameters in the integral are complex rather than real, though this does not modify any of the considerations made below. Our aim is to compute \( I_F(P^2, \Delta; \epsilon) \) with the method of the expansion by regions. Also in this case the integral (A.8) can be computed without any approximation, and we will use the known expression [69],

\[
I_F(P^2, \Delta; \epsilon) = i(4\pi)^{-2}\Gamma(\epsilon)(-\Delta)^{-\epsilon} \frac{1}{2} \frac{3}{2} \frac{P^2}{4\Delta} F_1 \left( \frac{1}{2}, \epsilon; \frac{3}{2}; \frac{P^2}{4\Delta} \right), \tag{A.9}
\]

to check the result of the expansion by regions.

The first step in the calculation of \( I_F(P^2, \Delta; \epsilon) \) is the identification of the relevant loop-momentum regions. We work in the rest frame of the total momentum \( P^\mu \equiv (P, \vec{0}) \), where \( P \equiv \sqrt{P^2} \). In this reference frame, the regions contributing to the expansion of (A.8) can be shown to be [52,69]

\[
\begin{align*}
\text{hard (h)} : & \quad r_0 \sim |\vec{r}| \sim P, \\
\text{semi-soft (s-s)} : & \quad r_0 \sim |\vec{r}| \sim P\sqrt{\delta}, \\
\text{potential (p)} : & \quad r_0 \sim P\delta, \quad |\vec{r}| \sim P\sqrt{\delta}, \\
\text{soft (s)} : & \quad r_0 \sim |\vec{r}| \sim P\delta.
\end{align*}
\tag{A.10}
\]
We start with the calculation of the contribution from the hard region. Both invariants \( r^2 \) and \( P \cdot r \) in the propagators scale as \( P^2 \), while the term \( \Delta \) scales as \( P^2 \delta \) by assumption, and represents the only expansion parameter in this region. Thus the expansion of the integral reads

\[
I_F^{(h)}(P^2, \Delta; \epsilon) = \sum_{n=0}^{\infty} (-\Delta)^n \sum_{k=0}^{n} \int \frac{d^d r}{(2\pi)^d} \frac{1}{(r^2 + P \cdot r + i\epsilon)^{1+k}} (r^2 - P \cdot r + i\epsilon)^{1+n-k}. \tag{A.11}
\]

The integrals are easily computed using the tabulated expression

\[
\int \frac{1}{(2\pi)^d (r^2 + P \cdot r + i\epsilon)^{1+k}} (r^2 - P \cdot r + i\epsilon)^{1+n-k} = (-1)^n i(4\pi)^{d-2} \frac{\Gamma(n+\epsilon)\Gamma(1-n-k-2\epsilon)}{\Gamma(2-n-2\epsilon)\Gamma(1+n-k)} \left( \frac{P^2}{4} \right)^{-n-\epsilon}, \tag{A.12}
\]

which leads to

\[
I_F^{(h)}(P^2, \Delta; \epsilon) = i(4\pi)^{d-2} \left( \frac{4}{P^2} \right)^{\epsilon} \sum_{n=0}^{\infty} \frac{\Gamma(n+\epsilon)}{(1-n-2\epsilon)\Gamma(1+n-k)} \left( \frac{4\Delta}{P^2} \right)^n. \tag{A.13}
\]

Using the result

\[
\sum_{k=0}^{n} \frac{\Gamma(1-n-k-2\epsilon)}{\Gamma(1+n-k)} = \frac{\Gamma(2-n-2\epsilon)}{\Gamma(n+1)(1-2n-2\epsilon)} \tag{A.14}
\]

one obtains

\[
I_F^{(h)}(P^2, \Delta; \epsilon) = i(4\pi)^{d-2} \left( \frac{4}{P^2} \right)^{\epsilon} \sum_{n=0}^{\infty} \frac{\Gamma(n+\epsilon)}{(1-2n-2\epsilon)\Gamma(n+1)} \left( \frac{4\Delta}{P^2} \right)^n. \tag{A.15}
\]

In the semi-soft region \( P \cdot r = Pr_0 \) scales as \( P^2 \sqrt{\delta} \), while \( r^2 \sim \Delta \sim P^2 \delta \ll P \cdot r \). Thus the expansion of the integrand in this region reads

\[
I_F^{(s-s)}(P^2, \Delta; \epsilon) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int \frac{d^d r}{(2\pi)^d} \frac{(-r^2 - \Delta)^n}{(Pr_0 + i\epsilon)^{1+k}} (-Pr_0 + i\epsilon)^{1+n-k}. \tag{A.16}
\]

All the terms of the series are scaleless integrals and vanish in dimensional regularisation \(^1\). Hence the semi-soft region does not contribute to the expansion of (A.8): \( I_F^{(s-s)}(P^2, \Delta; \epsilon) = 0 \).

We now consider the potential region. In this case \( P \cdot r \sim \Delta \sim P^2 \delta \), but the quantity \( r^2 \) is not homogeneous, since the component \( r_0^2 \) scales as \( P^2 \delta^2 \), while \(-r^2 \sim P^2 \delta \). Thus

\(^1\)Strictly speaking the integral in \( r_0 \) is ill-defined because of pinching singularities. However the presence of scaleless integrals in the space-like component \( r \) shows that the integral is 0.
the only expansion parameter is \( r_0^2 \), and the contribution of the potential region to (A.8) is

\[
I_F^{(p)}(P^2, \Delta; \epsilon) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int \frac{d^d r}{(2\pi)^d} \frac{(-r_0^2)^n}{(P r_0 - r^2 + \Delta + i\epsilon)^{1+k}(-P r_0 - r^2 + \Delta + i\epsilon)^{1+n-k}}.
\]

(A.17)

The \( r_0 \) integration is performed by closing the contour in the upper half-plane, and picking up the residue at the pole \( r_0 = (\Delta - r^2 + i\epsilon)/P \) according to Cauchy theorem. The residue of the integrand is given by the rather cumbersome expression

\[
\frac{2^{-1-3n}}{P^3} \left( - \frac{\Delta - r^2 + i\epsilon}{P^2} \right)^{n-1} \frac{\sqrt{\pi} \Gamma(2n+1) \, {}_2F_1(k-n,-2n;-n;2)}{\Gamma(n+1/2)\Gamma(1+k)\Gamma(-k+n+1)}.
\]

(A.18)

Introducing the notation

\[
K(n) = \sum_{k=0}^{n} \frac{2^{-1-3n}}{P^3} \frac{\sqrt{\pi} \Gamma(2n+1) \, {}_2F_1(k-n,-2n;-n;2)}{\Gamma(n+1/2)\Gamma(1+k)\Gamma(-k+n+1)},
\]

(A.19)

equation (A.17) can be rewritten as

\[
I_F^{(p)}(P^2, \Delta; \epsilon) = i \sum_{n=0}^{\infty} \int \frac{d^{d-1} r}{(2\pi)^{d-1}} K(n) \left( - \frac{\Delta - r^2 + i\epsilon}{P^2} \right)^{n-1}.
\]

(A.20)

One sees immediately that for \( n > 0 \) the terms of the series are scaleless integrals. Thus the only non-vanishing contribution comes from the \( n = 0 \) term, and reads (\( K(0) = 1/(2P^3) \))

\[
I_F^{(p)}(P^2, \Delta; \epsilon) = i \int \frac{d^{d-1} r}{(2\pi)^{d-1}} \frac{1}{2P(r^2 - \Delta - i\epsilon)}
\]

(A.21)

The integration over the remaining \( d-1 \) space-like components is straightforward, and gives

\[
I_F^{(p)}(P^2, \Delta; \epsilon) = i(4\pi)^{\epsilon-2} \Gamma \left( \epsilon - \frac{1}{2} \right) \sqrt{\frac{\pi\Delta}{P^2}}(-\Delta)^{-\epsilon}.
\]

(A.22)

We finally consider the soft region. Both \( P \cdot r \) and \( \Delta \) scale homogeneously as \( P^2\delta \), while \( r^2 \sim M^2\delta^2 \). Hence we expand in powers of \( r^2 \) and find

\[
I_F^{(s)}(P^2, \Delta; \epsilon) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int \frac{d^d r}{(2\pi)^d} \frac{(-r^2)^n}{(P r_0 + \Delta + i\epsilon)^{1+k}(-P r_0 + \Delta + i\epsilon)^{1+n-k}}.
\]

(A.23)

As in the semi-soft case, the integration over the time-like component \( r_0 \) and the space-like components \( \vec{r} \) can be performed independently, and lead to scaleless integrals that vanish in dimensional regularisation. Thus also for the soft region \( I_F^{(s)}(P^2, \Delta; \epsilon) = 0 \) at all orders in the expansion in \( \delta \).
To summarise, the integral (A.8) receives contributions from four different loop-momentum regions. These contributions have the following expression

\[
I_F^{(h)}(P^2, \Delta; \epsilon) = i(4\pi)^{\epsilon - 2} \left( \frac{4}{P^2} \right)^\epsilon \Gamma(\epsilon) \frac{2 F_1 \left( \epsilon, \epsilon - \frac{1}{2}; \epsilon + \frac{1}{2}, \frac{4\Delta}{P^2} \right)}{1 - 2\epsilon}
\]

\[
I_F^{(p)}(P^2, \Delta; \epsilon) = 0
\]

\[
I_F^{(s)}(P^2, \Delta; \epsilon) = i(4\pi)^{\epsilon - 2} \Gamma \left( \epsilon - \frac{1}{2} \right) \sqrt{-\frac{\pi \Delta}{P^2}} (-\Delta)^{-\epsilon}
\]

\[
I_F^{(i)}(P^2, \Delta; \epsilon) = 0
\]

(A.24)

where the identity

\[
\sum_{n=0}^{\infty} \Gamma(n + \epsilon) (1 - 2n - 2\epsilon) \Gamma(n + 1) x^n = \frac{\Gamma(\epsilon) 2 F_1 \left( \epsilon, \epsilon - \frac{1}{2}; \epsilon + \frac{1}{2}, x \right)}{1 - 2\epsilon}
\]

(A.25)

has been used to simplify the hard-region contribution.

Now consider again the result obtained with usual Feynman parameterisation, equation (A.9). Using the identity

\[
2 F_1(a, b; c; z) = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} (-z)^{-a} 2 F_1 \left( a, 1 - c + a; 1 - b + a; \frac{1}{z} \right)
\]

\[
+ \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} (-z)^{-b} 2 F_1 \left( b, 1 - c + b; 1 - a + b; \frac{1}{z} \right)
\]

(A.26)

we can rewrite \( I_F(P^2, \Delta; \epsilon) \) as

\[
I_F(P^2, \Delta; \epsilon) = i(4\pi)^{\epsilon - 2} \Gamma \left( \epsilon - \frac{1}{2} \right) \sqrt{-\frac{\pi \Delta}{P^2}} (-\Delta)^{-\epsilon}
\]

\[
+ i(4\pi)^{\epsilon - 2} \left( \frac{4}{P^2} \right)^\epsilon \frac{\Gamma(\epsilon) 2 F_1 \left( \epsilon, \epsilon - \frac{1}{2}; \epsilon + \frac{1}{2}, \frac{4\Delta}{P^2} \right)}{1 - 2\epsilon},
\]

(A.27)

which coincides with the sum of \( I_F^{(h)}(P^2, \Delta; \epsilon) \) and \( I_F^{(p)}(P^2, \Delta; \epsilon) \).

We conclude the section with two remarks. In the two examples discussed here the expanded integrals could be computed to all orders in \( \delta \) and the elements of the series resummed after integration, thus fully reproducing the exact result to all orders in the expansion parameters. More typically, only a finite number of terms in the expansion is computed, that is, only those required to match a given target accuracy in \( \delta \), as in the case of the calculations of Chapters 3 and 4. The second remark concerns the contribution of different regions to the expansion of the full integral. As seen in the second example, only hard and potential loop momenta determine the expansion of the diagram shown in Figure 140.
A.1. This is analogous to the calculation of the $W$-pair production cross section presented in Chapter 3, where the expansion of the Born cross section is determined entirely by potential and hard contributions. The soft loop-momentum region becomes relevant when we attach an additional photon line to a diagram with massive lines, as seen in Section 4.3. Semi-soft loop-momenta contributes only in specific patterns, and become important at higher orders, as seen for example in Chapter 6.
Appendix B

Expansion of the Born cross section

This appendix contains details of the calculation of the threshold expansion of the Born cross section in the potential and hard regions.

B.1 Expansion in the potential region

Here we explicitly derive the expansion of the function \( \Phi(E, r) \) defined by equation 3.3. We will first give the full expression of \( \Phi \) in terms of the four invariants \( D_1 \equiv k_2^2 - M_W^2, \) \( D_2 \equiv k_2^2 - M_W^2, \) \( \tilde{r}^2 \) and \( p_1 \cdot \tilde{r} \), where \( \tilde{r} \) is related to the vector \( r \) introduced in Section 3.1 by \( \tilde{r} \equiv (r_0 - E/2, \vec{r}) \). In terms of \( \tilde{r} \) the momenta of the \( W_\mp \) pair are \( k_{1,2} = P/2 \pm \tilde{r} \).

Clearly the result (3.18) does not depend on the particular parametrisation we adopt, since the integration in (3.4) allows to redefine the loop momentum \( r \) by an amount of order \( M_W \delta \). However by choosing \( \tilde{r} \) as residual momentum, the function \( \Phi(E, r) \) assumes a more symmetric and compact form. Note that in complete generality we should compute the squared matrix elements in 3.3 in \( d = 4 - 2\epsilon \) dimensions, but here we can set \( d = 4 \), since the integration over \( r \) does not generate poles in \( \epsilon \).

We start with the t-channel contribution, given by the third diagram in Figure 2.6, which we will denote as \( \Phi_{tt} \). For the LR helicity combination the explicit computation yields

\[
\Phi_{tt,LR} = -64\pi^2\alpha^2_{\text{ew}} \left[ \frac{1}{4} + \frac{M_W^2}{M_W^2 + D_1} \left( \frac{1}{4} \cdot \frac{p_1 \cdot \tilde{r}}{M_W^2} + \frac{2D_2 - 5\tilde{r}^2}{8M_W^2} \right) \right. \\
\left. + \frac{M_W^2}{M_W^2 + D_2} \left( \frac{1}{4} + \frac{p_1 \cdot \tilde{r}}{M_W^2} + \frac{2D_1 - 5\tilde{r}^2}{8M_W^2} \right) - \frac{(p_1 \cdot \tilde{r})^2 - (\tilde{r}^2)^2}{4(M_W^2 + D_1)(M_W^2 + D_2)} \right. \\
\left. + \frac{M_W^2}{M_W^2 + D_2 + 2p_1 \cdot \tilde{r} - 2\tilde{r}^2} \left( \frac{1}{2} + \frac{D_1 + D_2 - 4\tilde{r}^2}{4M_W^2} \right) \right. \\
\left. - \frac{M_W^4}{(M_W^2 + D_2 + 2p_1 \cdot \tilde{r} - 2\tilde{r}^2)^2} \left( \frac{1}{4} + \frac{D_1 + D_2}{4M_W^2} + \frac{D_1D_2}{4M_W^4} \right) \right], \quad (B.1)
\]
while the $RL$, $LL$ and $RR$ combinations vanish. Near threshold the four invariants defined above have the following scaling: $p_1 \cdot \tilde{r} \sim M_W^2 \delta^{1/2}$, $D_1 \sim D_2 \sim \tilde{r}^2 \sim M_W^2 \delta$. Expanding (B.1) to $O(\delta)$ we obtain

$$
\Phi_{tt} = -64\pi^2 \alpha^2_{ew} \left[ 1 - \frac{9M_W^2 \tilde{r}^2 + 5(p_1 \cdot \tilde{r})^2}{4M_W^2} \right] + O(\delta^{3/2})
$$

$$
= -64\pi^2 \alpha^2_{ew} \left[ 1 + \frac{9\tilde{r}^2 - 5(\tilde{n} \cdot \tilde{r})^2}{4M_W^2} \right] + O(\delta^{3/2}),
$$

(B.2)

where in the last line we have used the fact that, at this accuracy, $\tilde{r}^2 \sim -\tilde{r}^2$ and $p_1 \cdot \tilde{r} \sim -M_W \tilde{n} \cdot \tilde{r}$.

We now turn to the $s$-channel contribution represented by the last diagram in Figure 2.6. The $LL$ and $RR$ helicity combinations vanish, while for $LR$ and $RL$ we obtain

$$
\Phi_{ss,LR} = \frac{-16}{9} \pi^2 \alpha^2_{ew} \xi(s)^2 \left[ \frac{M_W^2}{M_{LR}^4 + D_1} \left( 1 - \frac{p_1 \cdot \tilde{r}(D_2^2 - 4\tilde{r}^2)}{M_W^4} \frac{M_W^2}{M_W^4} \frac{6D_2 - 21\tilde{r}^2 - 2(p_1 \cdot \tilde{r})^2}{M_W^2} \right) \right.
$$

$$
- \frac{(p_1 \cdot \tilde{r})(D_2^2 - 8D_2\tilde{r} + 16(\tilde{r}^2)^2)}{2M_W^4} + \frac{2D_2^2 - 21D_2^2\tilde{r}^2 + 74D_2(\tilde{r}^2)^2 - 96(\tilde{r}^2)^3}{2M_W^4}
$$

$$
+ \frac{M_W^2}{M_W^4 + D_2} \left( 1 + \frac{p_1 \cdot \tilde{r}}{2M_W^4} + \frac{M_W^2}{M_W^4} \frac{6D_1 - 21\tilde{r}^2 - 2(p_1 \cdot \tilde{r})^2}{M_W^2} \right)
$$

$$
+ \frac{(p_1 \cdot \tilde{r})(D_1^2 - 8D_1\tilde{r} + 16(\tilde{r}^2)^2)}{2M_W^4} + \frac{2D_1^2 - 21D_1^2\tilde{r}^2 + 74D_1(\tilde{r}^2)^2 - 96(\tilde{r}^2)^3}{2M_W^4}
$$

$$
- 2 - \frac{M_W^2}{M_W^4} \left[ 2(D_1 + D_2) + 55\tilde{r}^2 \right] + \frac{10(p_1 \cdot \tilde{r})^2}{M_W^4} + \frac{9(D_1 - D_2)p_1 \cdot \tilde{r}}{2M_W^4}
$$

$$
+ \frac{16\tilde{r}^2 - p_1 \cdot \tilde{r}}{(M_W^4 + D_1)(M_W^4 + D_2)} \frac{4D_1D_2 + 55(D_1 + D_2)\tilde{r}^2 - 164(\tilde{r}^2)^2}{2M_W^4}
$$

$$
\Phi_{ss,RL} = \Phi_{ss,LR} \bigg|_{\xi(s) \rightarrow \chi(s)}.
$$

(B.3)

The NLO expansion in $\delta$ reads

$$
\Phi_{ss,LR} = -\frac{64}{9} \pi^2 \alpha^2_{ew} \xi(s)^2 \left[ - \frac{19M_W^4 \tilde{r}^2 + 3(p_1 \cdot \tilde{r})^2}{M_W^4} \right] + O(\delta^{3/2})
$$

$$
= -\frac{64}{9} \pi^2 \alpha^2_{ew} \xi(s)^2 \left[ \frac{19\tilde{r}^2 - 3(\tilde{n} \cdot \tilde{r})^2}{M_W^4} \right] + O(\delta^{3/2}),
$$

(B.4)
and similarly for $\Phi_{s,s,RL}$. Note that $\Phi_{s,s}$ vanishes at leading order in $\delta$. This is the on-shell analogue of the well-know result that on-shell $W$-pair production in the $s$-channel is suppressed by powers of the non-relativistic velocity $v$ near threshold.

Finally we come to the contribution of the first two diagrams in Figure 2.6, which represent interferences of $t$ and $s$-channel diagrams. As for the pure $t$-channel contributions, the helicity combinations $LL$, $RR$ and $RL$ vanish, whereas the $LR$ combination reads

$$
\Phi_{t,s,LR} = \frac{-32}{3} \pi^2 \alpha_{ew}^2 \xi(s) \left[ \frac{M_W^2}{M_W^2 + D_2 + 2p_1 \cdot \vec{r} - 2r^2 + D_1} \left(1 + \frac{3p_1 \cdot r}{2M_W^2}\right) \right. \\
+ \frac{M_W^2 (6D_2 - 17r^2)}{2M_W^5} - 4(p_1 \cdot \vec{r})^2 + (p_1 \cdot \vec{r}) \left[ M_W^2 (3D_2 - 10r^2) - 2(p_1 \cdot \vec{r})^2 \right] \\
+ \frac{M_W^2 (3D_2 - 17D_2 r^2 + 24(\vec{r}^2)^2)}{2M_W^5} - 2(p_1 \cdot \vec{r})^2 (D_2 - 5r^2) \\
+ \frac{p_1 \cdot \vec{r} (3D_2 - 20D_2 r^2 + 36(\vec{r}^2)^2)}{2M_W^5} + \frac{2D_2^3 - 17D_2^2 r^2 + 48D_2 (\vec{r}^2)^2 - 52(\vec{r}^2)^3}{2M_W^5} \\
+ \frac{2(p_1 \cdot \vec{r})^3}{M_W^5} + \frac{M_W^2 (-4D_1 r^2 + 13(\vec{r}^2)^2)}{M_W^5} + (p_1 \cdot \vec{r})^2 (D_1 - 2r^2) \\
+ \frac{p_1 \cdot \vec{r} (2D_1^2 - 14D_1 r^2 + 26(\vec{r}^2)^2)}{M_W^5} - \frac{2D_1^2 r^2 - 13D_1 (\vec{r}^2)^2 + 22(\vec{r}^2)^3}{M_W^5} \\
- \frac{1 + 25p_1 \cdot \vec{r}}{2M_W^5} + \frac{M_W^2 (6D_1 - 2D_2 + 55r^2)}{2M_W^5} + \frac{10(p_1 \cdot \vec{r})^2}{M_W^5} \\
+ \frac{p_1 \cdot \vec{r} (17D_1 + 8D_2 - 56r^2)}{2M_W^5} - \frac{2(D_1^2 - D_2^2 + D_1 D_2) + (2D_1 + 34D_2) r^2}{2M_W^5} \\
+ \frac{41(\vec{r})^2}{2M_W^5} + \frac{8 [(p_1 \cdot \vec{r})^3 r^2 - (p_1 \cdot \vec{r})^2 (\vec{r}^2)^2 - p_1 \cdot \vec{r} (\vec{r}^2)^3 + (\vec{r}^2)^4]}{M_W^5 (M_W^2 + D_1) (M_W^2 + D_2)} \right].
$$

The expansion of (B.5) up to $O(\delta)$ is

$$
\Phi_{t,s,LR} = \frac{-64}{3} \pi^2 \alpha_{ew}^2 \xi(s) \left[ \frac{8p_1 \cdot \vec{r}}{M_W^5} - \frac{M_W^2 (2(D_1 - D_2) + 19r^2) + 19(p_1 \cdot \vec{r})^2}{M_W^5} \right] + O(\delta^{3/2})
$$

$$
= \frac{-64}{3} \pi^2 \alpha_{ew}^2 \xi(s) \left[ \frac{-8\vec{n} \cdot \vec{r}}{M_W^2} + \frac{19(r^2 - (\vec{n} \cdot \vec{r})^2)}{M_W^2} \right] + O(\delta^{3/2}).
$$

The second line follows from the replacement of each term with the appropriate expansion in $\delta$ (remember that $|\vec{r}| \sim M_W^{1/2}$, $r_0 \sim \sim M_W \delta$): $D_1 \rightarrow 2M_W r_0 - \vec{r}^2$, $D_2 \rightarrow 2M_W (E - r_0) - \vec{r}^2$, $r^2 \rightarrow -\vec{r}^2$, $p_1 \cdot \vec{r} \rightarrow M_W (r_0 - E/2 - \vec{n} \cdot \vec{r})$, $(p_1 \cdot \vec{r})^2 \rightarrow M_W^2 (\vec{n} \cdot \vec{r})^2$. 

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Note that the first term in (B.6) counts as $\delta^{1/2}$, and represents a $N^{1/2}$LO correction to the leading-order squared matrix element, $\Phi_{t_t,LR}^{(0)} = -64\pi^2\alpha_{ew}^2$.

Unlike $\Phi_{tt}$ and $\Phi_{ss}$, the NLO expansion of $\Phi_{ts}$ explicitly contains the two invariants $D_1$ and $D_2$. Since the matching of the effective Lagrangian is performed at the complex pole $k^2 = \bar{s} \equiv M_W^2 + M_W\Delta$ rather than at $k^2 = M_W^2$, the matching coefficient of the production operators $O_{\mu}^{(k)}$ has in general a non-vanishing imaginary part. The complication can be ignored at NLO in $\delta$, where $D_1$ and $D_2$ appear in (B.6) in the combination $D_1 - D_2$, and imaginary contributions cancel pairwise. However the issue is relevant for the expansion of $\Phi$ beyond NLO, and should be taken into account.

Summing the contributions of the two channels, equations (B.2) and (B.4), and the interference terms (B.6), we obtain the following total results for the two helicity combinations $LR$ and $RL$:

$$
\Phi_{LR} = -64\pi^2\alpha_{ew}^2 \left[ 1 + \xi(s) \frac{8\vec{n} \cdot \vec{r}}{3M_W} + \frac{9r^2 - 5(\vec{n} \cdot \vec{r})^2}{4M_W^2} \right. \\
\left. + \xi(s)^2 \frac{19r^2 - 3(\vec{n} \cdot \vec{r})^2}{9M_W^2} \right] \\
\Phi_{RL} = -64\pi^2\alpha_{ew}^2 \frac{\xi(s)^2 (19r^2 - 3(\vec{n} \cdot \vec{r})^2)}{9M_W^2}.
$$

The expressions given in (3.13) are obtained by averaging equation (B.7) over the solid angle $\Omega$ in $d = 3$ dimensions. This corresponds to the obvious substitutions $(\vec{n} \cdot \vec{r}) \to 0$, $(\vec{n} \cdot \vec{r})^2 \to \vec{r}^2/3$, that lead us to the final result:

$$
\Phi_{LR} = -64\pi^2\alpha_{ew}^2 \left[ 1 + \left( \frac{11}{6} + 2\xi(s)^2 + \frac{38}{9} \xi(s) \right) \frac{\vec{r}^2}{M_W^2} \right] \\
\Phi_{RL} = -128\pi^2\alpha_{ew}^2 \xi(s)^2 \frac{\vec{r}^2}{M_W^2}.
$$

As anticipated in Section 3.1, the $N^{1/2}$LO contribution proportional to $\vec{n} \cdot \vec{r}$ has vanished upon angular integration.
B.2 Expansion in the hard region

In this section we compute the hard coefficient functions $K_{h1}$, $K_{h2}$ and $K_{h3}$ in equation (3.27), and give the explicit expressions of the remaining matching coefficients $C_{i,h}^I$ and $K_{h}^I$. We also discuss the inclusion of hard corrections beyond N$^{1/2}$LO in $\alpha \sim \delta$.

We start considering the first three two-loop cut diagrams in Figure 2.6. As in (3.20), the contribution of these cuts to the imaginary part of the forward-scattering has the following structure

$$2\text{Im} \mathcal{A} = \sum_{c_a,c_b} \int \frac{d^d k}{(2\pi)^d} \left[ iM_\mu^{\mu\nu}_{eeW_W,c_a}(k_1,k_2) \right]$$

$$\times \frac{\Gamma_{f_i f_j}^{(0)}}{M_W} \left[ \frac{2k_1^2 \Gamma_{f_i f_j}^{(0)} \theta(k_1^2)\theta(k_1^0)}{M_W} \left( -g_{\mu\nu} + \frac{k_{1\mu} k_{1\nu}}{k_1^2} \right) \right]$$

$$\times \frac{-i}{k_1^2 - M_W^2 - i\epsilon} \left( -g_{\mu\nu} + \frac{k_{2\mu} k_{2\nu}}{M_W^2} \right) \delta(k_2^2 - M_W^2) \theta(k_2^0) \left[ -iM_\mu^{\mu\nu^\prime}_{eeW_W,c_b}(k_1,k_2) \right]$$

$$= \frac{2\Gamma_{f_i f_j}^{(0)}}{M_W} \int \frac{d^d k}{(2\pi)^d} \Phi_{\text{hard}}(k_1,k_2) \frac{k_1^2 \theta(k_1^2)\theta(k_1^0)}{(k_1^2 - M_W^2)^2} \delta(k_2^2 - M_W^2) \theta(k_2^0).$$

(B.9)

The quantity in square brackets represents the imaginary part of the fermion-loop insertion on the upper $W$ line, with $\Gamma_{f_i f_j}^{(0)}$ being the tree-level on-shell partial $W$ decay width to the final state $f_i f_j$, $\Gamma_{f_i f_j}^{(0)} = Br(W \to f_i f_j) \Gamma_W^{(0)}$. Note that the expression (B.9) can be interpreted as the interference of the leading-order term in the expansion 3.19 with the subleading correction $\text{PV}[M_W \Gamma_W/(k^2 - M_W^2)^2]$. The principal-value prescription is redundant at N$^{1/2}$LO, where the singularity in the integrand is located at one of the integration limits, and is regularised by dimensional regularisation. Note also that in (B.9) one of the two unitary-gauge polarisation tensors is replaced by $-g_{\mu\nu} + k_{\mu} k_{\nu}/k^2$, because the massless-fermion loop is proportional to the transverse-polarisation projector. The integrand in (B.9), including the distributions $\delta(x)$ and $\theta(x)$, has to be expanded in $E$. At lowest order this corresponds to setting $s$ to $4M_W^2$. Since the integration over $k_1$ does not generate poles in $\epsilon$, as we will show below, the function $\Phi_{\text{hard}}$ is computed in $d = 4$ dimensions. The calculation of (B.9) is straightforward but tedious, and has been implemented in Mathematica. After performing the $k_1^0$ integration with Cauchy’s theorem, and solving the integral over the solid angle $\Omega$, we obtain the relatively simple result

$$2\text{Im} \mathcal{A}_h = \frac{8\alpha^3}{27s_w} \sum_{i=h_1}^{h_3} C_{i,h}(s) \int_0^\frac{4}{\alpha} dy f_{i,h}(y),$$

(B.10)

Note that each cut represents the sum of the diagram shown in the figure and the one with a self-energy insertion on the lower $W$ line, and the cut h2 includes also the complex conjugate of the diagram presented in figure.
where we have defined $k_1 = MW$. The sum extends over the three cuts, and $h$ indicates a specific helicity configuration. In (B.10) we have replaced $\Gamma_{f_i f_j}^{(0)}$ with $3eW/(4s_w^2)\text{Br}(W \to f_i f_j)$, and multiplied the right-hand side of the equation with the appropriate branching ratio for the flavour-specific decay of the lower $W$ line. (see Section 5.1). The functions $C_{i,h}$ are linear combinations of the photon and $Z$-boson propagators entering $\Phi^{\text{hard}}$. Instead of being expanded around $s = 4M^2_W$, their explicit $s$-dependence is kept, as for the potential contributions discussed in the previous section. In the limit of vanishing fermion masses only the helicity configurations $LR$ and $RL$ contribute to the matching of the coefficients $C_{i,h}$, and for these we explicitly find $C_{h1,LR} = 1$, $C_{h1,RL} = 0$, $C_{h2,LR} = \xi(s)$, $C_{h2,RL} = 0$, $C_{h3,LR} = \xi(s)^2$ and $C_{h3,RL} = \chi(s)^2$, where the functions $\xi(s)$ and $\chi(s)$ have been defined below equation (3.13). The last integration, $\int_{3/4}^{0} dy f_{i,h}(y)$, in (B.10) cannot be in general performed analytically. We thus compute the integral numerically. However, as anticipated in Section 3.2, the threshold expansion leads to the appearance of infrared singularities in (B.10). These are regularised and finite in dimensional regularisation, but to be able to perform the integral $\int_{0}^{3/4} dy f_{i,h}(y)$ numerically we have to isolate them.

The integrand corresponding to the cut h1, for the $LR$ helicity combination, reads

$$
\tilde{f}_{h1,LR}(y) = \frac{y^{1-2\epsilon}}{64\sqrt{1+y^2} \left(1 - \sqrt{1+y^2}\right)^2} \left[4y(3+y^2) + 3 \left(13 + 8y^2 - 14\sqrt{1+y^2}\right) \ln \left[\frac{2\sqrt{1+y^2} - 2y - 1}{2\sqrt{1+y^2} + 2y - 1}\right]\right]. \quad (B.11)
$$

The asymptotic behaviour of the function $f_{h1,LR}(y)$ in the limit $y \to 0$ is

$$
f_{h1,LR}(y) = \frac{3}{2} y^{-2-2\epsilon} + O(y^{-2\epsilon}), \quad (B.12)
$$

leading to a non-integrable singularity (in the sense of Riemann integration) when $\epsilon \to 0$. However in dimensional regularisation the integral

$$
\int_{0}^{3/4} dy \frac{3}{2} y^{-2-2\epsilon} = -2 + O(\epsilon) \quad (B.13)
$$

is finite, and the quantity $\tilde{f}_{h1,LR}(y) \equiv f_{h1,LR}(y) - 3/2 y^{-2-2\epsilon}$ does not contain non-integrable singularities for vanishing $\epsilon$. Hence we can integrate $\tilde{f}_{h1,LR}(y)$ numerically in the limit $\epsilon \to 0$:

$$
\int_{0}^{3/4} dy \lim_{\epsilon \to 0} \tilde{f}_{h1,LR}(y) = -0.35493. \quad (B.14)
$$

Combining the results (B.13) and (B.14), we obtain

$$
\int_{0}^{3/4} dy f_{h1,LR}(y) = -2.35493 + O(\epsilon). \quad (B.15)
$$

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The upper integration limit is set by the constraint encoded in the step-function $\theta(k_1^2)$.\footnote{The upper integration limit is set by the constraint encoded in the step-function $\theta(k_1^2)$.}
The functions \( f_{h^2,LR} \) and \( f_{h^3,LR} \) are given by

\[
f_{h^2,LR} = \frac{y^{1-2\epsilon}}{192\sqrt{1+y^2}(1-\sqrt{1+y^2})^2} \left[ 4y \left( -207 - 124y^2 + 2\sqrt{1+y^2}(117 + 4y^2) \right) \right. \\
+ 3 \left( 225 + 80y^2 - 216\sqrt{1+y^2} \right) \ln \left[ \frac{2\sqrt{1+y^2} - 2y - 1}{2\sqrt{1+y^2} + 2y - 1} \right] \\
\left. \right] ,
\]

\[
f_{h^3,LR} = \frac{y^{4-2\epsilon}}{36\sqrt{1+y^2}(1-\sqrt{1+y^2})^2} (87 + 4y^2 - 60\sqrt{1+y^2}) ,
\]

(B.16)

and for the only non-vanishing contribution to the \( RL \) helicity structure, \( f_{h^3,RL} \), we have \( f_{h^3,RL} = f_{h^3,LR} \). Both \( f_{h^2,LR} \) and \( f_{h^3,LR} \) are free of non-integrable singularities in the interval \([0,3/4] \) for vanishing \( \epsilon \), and can be integrated numerically. The results of the numerical integration read

\[
\int_0^{3/4} dy f_{h^2,LR}(y) = 3.86286 + O(\epsilon) ,
\]

\[
\int_0^{3/4} dy f_{h^3,LR}(y) = 1.88122 + O(\epsilon) ,
\]

(B.17)

which confirm the numbers given in Section 3.2 for the coefficients \( K_{h^1} \), \( K_{h^2} \) and \( K_{h^3} \).

Subleading terms in the threshold expansion of equation (B.9) generate higher-order hard corrections to the four-fermion cross section that are suppressed by powers of \( E \) with respect to the dominant term (B.10). The all-order (in \( E \) ) contribution to the total cross section of the cuts \( h_1-h_3 \) can be written as

\[
\sigma_{\text{hard,}LR,\text{Born}} = \frac{4\alpha^3}{27s^{3/2}w(s)} \sum_n \left( \frac{E}{M_W} \right)^n \left[ K_{h^1}^{(1/2+n)} + K_{h^2}^{(1/2+n)}(s) + K_{h^3}^{(1/2+n)}(s)^2 \right] \\
\sigma_{\text{hard,}RL,\text{Born}} = \frac{4\alpha^3}{27s^{3/2}w(s)} \sum_n \left( \frac{E}{M_W} \right)^n K_{h^3}^{(1/2+n)} ,
\]

(B.18)

where \( K_{h_1,h_2,h_3} \) obviously coincide with the coefficients \( K_{h_1,h_2,h_3} \) given in equation (3.28).

Terms proportional to \( E^n \) are suppressed by \( \delta^{1/2+n} \) compared to the leading-order EFT approximation (2.48), and count as \( N^{1/2+n}\text{LO} \) corrections. The coefficients \( K_{hi}^{(1/2+n)} \) are given by dimensional regularised integrals,

\[
K_{hi}^{(1/2+n)} = \int_0^{3/4} dy f_{hi}^{(1/2+n)}(y) .
\]

(B.19)

As before the singularities of the integrand must be isolated before performing the numerical integration of the finite part. As a rule, the degree of divergence (in a Riemann-integral
sense) grows with \( n \). For example for \( n = 1 \), in the limit \( y \to 0 \) we find the following asymptotic behaviour

\[
f^{(3/2)}_{h1}(y) = 3y^{-4-2\epsilon} + \frac{1}{2}y^{-2-2\epsilon} + O(y^{-2\epsilon})
\]

\[
f^{(3/2)}_{h2}(y) = \frac{38}{3}y^{-2-2\epsilon} + O(y^{-2\epsilon})
\]

\[
f^{(3/2)}_{h3}(y) = 6y^{-2-2\epsilon} + O(y^{-2\epsilon}).
\]

The calculation of coefficients of (B.18) has been programmed in Mathematica, and can be in principle pursued at any desired order in \( E \). For example, for the first three orders beyond the leading term (B.10), we obtain

\[
K^{(3/2)}_{h1} = -5.87912 \quad K^{(3/2)}_{h2} = -19.15095 \quad K^{(3/2)}_{h3} = -6.18662
\]

\[
K^{(5/2)}_{h1} = 4.00538 \quad K^{(5/2)}_{h2} = -6.35823 \quad K^{(5/2)}_{h3} = -13.13844
\]

\[
K^{(7/2)}_{h1} = -2.87173 \quad K^{(7/2)}_{h2} = -3.39521 \quad K^{(7/2)}_{h3} = -9.42367
\]

As shown in Section 3.3, the \( N^{3/2}\)LO correction corresponding to the coefficients in the first line are relevant for a numerical comparison of the EFT prediction with the full Born cross section at the per-mille level, but \( N^{5/2}\)LO and \( N^{7/2}\)LO corrections are already below this accuracy, at least above threshold.

We conclude this section with the (energy-independent) contribution of the cuts \( h4-h5 \) in Figure 2.6, corresponding to the interference of double and single-resonant tree diagrams, and give explicit results for the coefficients \( C_{i,h}^f \) and \( K_{i,h}^f \). The functions \( f_{i,h}^f \) are free of non-integrable singularities at leading order in \( E \), and can be integrated numerically after taking the limit \( \epsilon \to 0 \). The result given below for each cut includes the contribution of the complex-conjugate diagram, except for cut \( h6 \), where the complex conjugate is the diagram itself, and cut \( h7 \), where the symmetric diagram is automatically taken into account by summing over the four flavours.

Only the configuration \( e_L^+ e_R^- \) contributes to the cut diagram \( h4 \):

\[
C_{h4,L,R}^f = 3M_W^2 \frac{s_w^2}{s} \left( -\frac{Q_f}{s} + \frac{C_L^f C_L^f}{s - M_Z^2} \right),
\]

\[
K_{h4}^\nu = K_{h4}^{\nu\bar{\nu}} = -0.266477, \quad K_{h4}^d = K_{h4}^{\mu} = 0.190394,
\]  

(B.21)

where \( Q_f \) and \( C_L^f = \frac{f_{h4,L} - s_w^2 Q_f}{s_w c_w} \) are the couplings of left-handed fermions to \( \gamma \) and \( Z \). \( Q_f \) always denotes the charge of the particle (not the anti-particle) in units of the positron.
charge. For the cut diagram h5 we have

\[ C_{h5,h}^f = 9M_W^4s_w^4 \left( \frac{-Q_f}{s^2} + \frac{C_c^h C_L^h}{s(s - M_Z^2)} + \frac{c_w}{s_w} \frac{Q_f C_e^h}{s(s - M_Z^2)} - \frac{c_w}{s_w} \frac{C_{e}^{h2}C_{Lf}^h}{s(s - M_Z^2)^2} \right), \]

\[ K_{h5}^u = K_{h5}^d = 0.455244, \quad K_{h5}^\nu = -0.455244, \]

where \( C_c^{LR} = C_c^L \) and \( C_c^{RL} = C_c^R = -\frac{s_{ew}}{c_w} Q_e \). In this case both left-handed and right-handed incoming fermions contribute \((h = LR, RL)\), but only left-handed internal fermions.

The coefficients of h6 are

\[ C_{h6,h}^f = 9M_W^4s_w^4 \left( -\frac{Q_f}{s} + \frac{C_c^h C_L^h}{s - M_Z^2} \right)^2, \]

\[ K_{h6}^u = K_{h6}^d = K_{h6}^\nu = K_{h6}^{\nu u} = 0.0804075, \]

while for h7 we get

\[ C_{h7,h}^f = 9M_W^4s_w^4 \left( \frac{Q_f Q_f}{s^2} - \frac{Q_f C_h^h C_L^h}{s(s - M_Z^2)} - \frac{Q_f C_c^h C_L^h}{s(s - M_Z^2)^2} + \frac{C_{e}^{h2}C_{Lf}^h C_{Lf}^h}{s(s - M_Z^2)^2} \right), \]

\[ K_{h7}^u = K_{h7}^d = K_{h7}^\nu = K_{h7}^{\nu u} = 0.0213082, \]

where \( Q_f, Q_f \) and \( C_L^f, C_L^f \) are the couplings to \( \gamma \) and \( Z \) of the particles in the same SU(2) doublet (i.e. \( \mu, \nu, \mu \) and \( u, d \)). As pointed out in Section 3.2, the leading-order contribution from cuts h4-h7 is numerically much smaller than the corresponding correction from h1-h3. Therefore, for the comparison with the full Born result we can avoid the inclusion of higher-order energy-dependent contributions from cuts h4-h7.
Appendix C

One-loop electroweak hard corrections to production and decay

The appendix gives the explicit analytic results for the hard one-loop coefficients presented in Section 4.1.

C.1 Production vertices

The general $e^- e^+ \to W^- W^+$ production operator we are concerned with reads

$$O_p = \frac{\pi \alpha_{\text{ew}}}{M_W^2} C_p \left( \bar{e} \gamma^{[i} \nabla^{j]} e \right) \left( \Omega_i^{[j} \Omega_j^{i]} \right), \quad (C.1)$$

where $C_p = C_{p,h}$ is the hard matching coefficient and $h = LR, RL$ refers to the helicity of the incoming leptons ($e_L^{-} e_R^+$ or $e_R^{-} e_L^+$). The full set of diagrams contributing to the matching of the production operator (at leading order in $\vec{r}/M_W$) is given in Figures C.1, C.2, C.3 and C.4. Starting with $e_L^- e_R^+ \to W^- W^+$, the matching coefficient at tree level is equal to 1, as can be read off (2.42). At NLO we have

$$C_{p,LR} = 1 + \frac{\alpha}{2\pi} c_{p,LR}^{(1)} + \mathcal{O} (\alpha^2) \equiv 1 + \frac{\alpha}{2\pi} c_{p,LR}^{(1)} + \mathcal{O} (\alpha^2), \quad (C.2)$$

where $c_{p,LR}^{(1)}$ is the coefficient in (2.54). Before renormalisation the NLO short-distance coefficient reads

$$c_{p,LR}^{(1), \text{bare}} = -\frac{1}{\epsilon^2} \left( -\frac{4M_W^2}{\mu^2} \right)^{-\epsilon} + \frac{8c_w^4 + 10c_w^2 + 1}{8c_w^2 s_w^2 \epsilon} \left( -\frac{4M_W^2}{\mu^2} \right)^{-\epsilon} \left( \begin{array}{c}
\frac{(2c_w^2 - 1)(24c_w^4 + 16c_w^2 - 1)}{8c_w^2 s_w^2} M_W^2 \left( C_0 \left( 0, M_W^2, -M_W^2, 0, M_Z^2, M_W^2 \right) \right) \\
-\frac{(2c_w^2 - 1) M_W^2 C_0 \left( 0, 4M_W^2, 0, 0, M_Z^2, M_Z^2 \right)}{2c_w^4 s_w^2} \end{array} \right).$$
\[
- \frac{((c_w^4 + 17c_w^2 - 16) M_H^2 + M_W^2)}{4M_H^2} \cdot \frac{M_W^2}{s_w^2} C_0 (-M_W^2, M_W^2, 0, 0, 0, M_H^2) \\
+ \frac{(M_H^2 + M_W^2)}{4M_H^2} M_W^2 C_0 (-M_W^2, M_W^2, 0, 0, M_H^2, M_H^2) \\
- \frac{(2c_w^8 + 32c_w^6 + 32c_w^4 - 11c_w^2 - 16) M_W^2}{8c_w^2 s_w^2} C_0 (-M_W^2, M_W^2, 0, 0, M_Z^2, M_Z^2) \\
+ 3 \left(33 - 46c_w^2\right) \frac{M_W^2}{8s_w^4} C_0 (M_W^2, -M_W^2, 0, 0, M_W^2) \\
+ \frac{(4c_w^4 - 1) (14c_w^6 + 15c_w^4 - 2c_w^2 - 1)}{16c_w^2 s_w^4} M_W^2 C_0 (M_W^2, -M_W^2, 0, 0, M_Z^2) \\
- \frac{(1 - 2c_w^2)^2 (c_w^2 + 1) (4c_w^2 + 1)^2 M_W^2}{16c_w^8 s_w^2} C_0 (4M_W^2, 0, 0, 0, 0, M_Z^2) \\
- \frac{25M_W^2}{4s_w^2} C_0 (4M_W^2, 0, 0, 0, 0, M_Z^2) + \frac{M_W^2 \ell (M_W^2, M_Z^2, M_H^2)}{4M_H^2 s_w^2} \\
+ \frac{(-168c_w^6 - 214c_w^6 + 56c_w^4 + 32c_w^2 - 3)}{24c_w^2 (1 - 4c_w^2) s_w^2} \ell (M_W^2, M_W^2, M_Z^2) \\
+ \frac{(1 - 2c_w^2) (8c_w^4 + c_w^2 + 3)}{6c_w^2 s_w^2} \ell (4M_W^2, M_Z^2, M_Z^2) \\
+ \frac{3 (c_w^2 + 1) \ln \left(\frac{M_Z^2}{M_W^2} + 1\right)}{16c_w^6} + \frac{(1 - 2c_w^2) (64c_w^4 + 4c_w^2 + 1) \ln \left(\frac{M_Z^2}{M_W^2} - 1\right)}{24c_w^4} \\
+ \frac{(-512c_w^6 + 1536c_w^6 - 672c_w^4 + 44c_w^4 + 3c_w^2 - 3)}{48c_w^4 (1 - 4c_w^2) s_w^2} \ln \left(\frac{M_Z^2}{M_W^2}\right) \\
+ \frac{(-128c_w^6 + 304c_w^6 + 144c_w^6 - 38c_w^4 + 9c_w^2 + 3)}{24c_w^6 s_w^2} \ln 2 \\
+ \frac{96c_w^6 - (10 - 2s_w^2) c_w^4 - 9c_w^2 - 6}{24c_w^6 s_w^2} \\
- \frac{(128c_w^8 - 64c_w^6 + 4c_w^4 + 23c_w^2 + 5) i\pi}{48c_w^4 s_w^2},
\]

(C.3)

where all functions appearing in the above expression, \( C_0(p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2) \) and \( \ell(q^2, M_1^2, M_2^2) \), are known analytically and are given in Appendix C.3. The counterterms in the \( \mu \) scheme
are computed from (4.1) and are given by

\[
\begin{align*}
\gamma^{(1),\text{ct}}_{p,LR} &= \frac{4c_w^4 + 22c_w^2 - 1}{8c_w^2 s_w^2} \left(-\frac{4M_W^2}{\mu^2}\right)^\epsilon - \left(M_H^4 - 3M_W^2 M_H^2 + 6M_W^4\right) \ell (M_W^2, M_H^2, M_W^0) \\
&\quad - \left(\frac{M_H^4 - 5M_W^2}{12M_W^4 s_w^2}\right) \ell (0, M_H^2, M_W^2) - \left(\frac{8c_w^4 + 27c_w^2 - 5}{12c_w^2 s_w^2}\right) \ell (0, M_W^2, M_Z^2) \\
&\quad + \left(\frac{42c_w^4 - 11c_w^2 w^2}{12c_w^2 s_w^2}\right) \ell (M_H^2, M_W^2, M_Z^2) \\
&\quad - \left(\frac{M_H^4 - 4M_W^2 M_H^2 + 12M_W^4}{24M_W^4 s_w^2}\right) \partial B_0 (M_W^2, M_H^2, M_W^0) \\
&\quad + \left(\frac{2M_H^4 - 3M_W^2 M_H^2 + 2M_W^4}{24M_W^4 (M_H^2 - M_W^2)s_w^2}\right) \ln \left(\frac{M_H^2}{M_W^2}\right) + \left(\frac{M_H^4}{12M_W^4 s_w^2}\right) - \left(\frac{3M_W^2}{16M_W^4 s_w^2}\right) \\
&\quad - \left(\frac{3m_t^2 (m_t^4 - M_W^2)}{4M_W^4 s_w^2}\right) \ln \left(\frac{1 - M_Z^2}{m_t^2}\right) - \left(\frac{3m_t^2}{8M_W^4 s_w^2}\right) - \left(\frac{3m_t^4}{4M_W^4 s_w^2}\right) \\
&\quad - \left(\frac{12c_w^8 - 72c_w^6 + 26c_w^4 - 15c_w^2 - 2}{24c_w^4 s_w^4}\right) \ln \left(\frac{M_Z^2}{M_W^2}\right) + \left(\frac{4c_w^4 - 22c_w^2 - 1}{4c_w^2 s_w^2}\right) \ln 2 \\
&\quad + \left(\frac{2(35 - 6i\pi)c_w^6 + (-112 + 66i\pi)c_w^4 + (13 + 3i\pi)c_w^2 + 2}{24c_w^4 s_w^4}\right). \\
\end{align*}
\]

The full renormalised coefficient is obtained by adding bare result and counterterms

\[
\gamma^{(1)}_{p,LR} = \gamma^{(1),\text{bare}}_{p,LR} + \gamma^{(1),\text{ct}}_{p,LR}.
\]

The poles of \( \gamma^{(1)}_{p,LR} \) are given explicitly in (4.6) and cancel once one takes into account soft and initial-state collinear radiation.

Turning to the \( e_R e_L \rightarrow W^- W^+ \) case, the matching coefficient \( C_{p,RL} \) vanishes at tree level, as can be seen from (2.42). The NLO correction is therefore finite. We have

\[
C_{p,RL} = C^{(1)}_{p,RL} + \mathcal{O} (\alpha^2) = \frac{\alpha}{2\pi} C^{(1)}_{p,RL} + \mathcal{O} (\alpha^2),
\]

where \( C^{(1)}_{p,RL} \) is the coefficient in (2.54). We find

\[
\begin{align*}
C^{(1)}_{p,RL} &= \frac{4s_w^2 M_W^2}{c_w^2 (2c_w^2 - 1)} C_0 (0, M_W^2, -M_W^2, 0, M_2^2, M_W^0) \\
&\quad - \frac{2s_w^2 M_W^2}{c_w^4 (2c_w^2 - 1)} C_0 (0, 4M_W^2, 0, 0, M_2^2, M_Z^2) \\
&\quad + \left(\frac{24c_w^4 + 20c_w^2 s_w^2}{3c_w^2 (2c_w^2 - 1)(4c_w^2 - 1)} - 2\left(\frac{8c_w^4 + c_w^2 + 3}{3c_w^2 (2c_w^2 - 1)}\right) \ell (M_W^2, M_W^2, M_Z^2)\right).
\end{align*}
\]
The one-loop correction to the leptonic decay vertex reads

\[
\left. \frac{1}{2} \left( 1 + \frac{\alpha}{\pi} \right) + \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 \right ] e^{i \alpha \ln (\frac{M_Z^2}{M_W^2})} + \frac{\alpha}{\pi} \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{M_Z^2}{M_W^2} \right)^{i \alpha}.
\]

\( C.2 \) Virtual corrections to \( W \) decay

The decay of a \( W \) boson is implemented in the effective theory analogous to the production [61]. There are decay operators with collinear fields describing the decay products of the non-relativistic vector boson. For the flavour-specific decays under consideration we have up to NLO

\[
\mathcal{O}_d = -\frac{g_{ew}}{2\sqrt{M_W}} \left( C_{d,l} \Omega_{\mu \nu \alpha}^{\gamma} \bar{\mu} c_{\nu \alpha} + C_{d,h} \Omega_{\mu \nu \alpha}^{\gamma} \bar{d}_c c_{\nu \alpha} \right),
\]

where the subscripts \( c_i \) indicate the collinear directions of the final fermions. These operators would be needed for the calculation of the \( e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu u \bar{d} \) scattering amplitude in the effective theory. However, for the total cross section (or the forward scattering amplitude) the directions \( c_3, c_4 \) of the decay products are integrated over and, as indicated in (2.40), there is no need to introduce collinear fields \( \bar{\mu}_{c_3, L}, \nu_{c_3, L} \), \( \bar{u}_{c_3, L} \) and \( d_{c_4, L} \) in the effective theory. The matching coefficients of the decay operators enter only indirectly through \( \Delta^{(2)} \). The virtual correction to the \( W \) decay width is related to the coefficient functions of the decay operators. Ignoring QCD corrections, at NLO we have

\[
C_{d,l} = 1 + C_{d,l}^{(1)} \alpha^2 \equiv 1 + \frac{\alpha}{2\pi} c_{d,l}^{(1)} + \mathcal{O}(\alpha^2),
\]

\[
C_{d,h} = 1 + C_{d,h}^{(1)} \alpha^2 \equiv 1 + \frac{\alpha}{2\pi} c_{d,h}^{(1)} + \mathcal{O}(\alpha^2).
\]

We give here the explicit results for the electroweak corrections. The unrenormalised one-loop correction to the leptonic decay vertex reads

\[
c_{d,l}^{(1), \text{bare}} = -\frac{1}{2e^2} \left( \frac{M_W^2}{\mu^2} \right)^{-e} + \frac{8c_w^4 + 2c_w^2 + 1}{8c_w^2 s_w^2} \left( \frac{M_W^2}{\mu^2} \right)^{-e}
\]

\[
+ \frac{\left( c_w^2 + 1 \right)^2 (2c_w^2 - 1)}{4c_w s_w^2} \left( M_W^2 \right) C_0 \left( M_W^2, 0, 0, 0, M_Z^2 \right)
\]

\[
+ \frac{(c_w^2 + 2)}{s_w^2} M_W^2 C_0 \left( M_W^2, 0, 0, M_W^2, M_Z^2 \right)
\]

\[
+ \frac{\left( 2c_w^2 + 1 \right)}{2s_w^2} \ell \left( M_W^2, M_W^2, M_Z^2 \right)
\]

\[
- \frac{\left( 24 + \pi^2 \right)c_w^6 + (\pi^2 - 18i\pi) c_w^4 - 3i\pi c_w^2 + 6i\pi + 6}{24c_w s_w^2}, \quad \text{(C.10)}
\]
and the corresponding counterterms computed from (4.1) are
\[
C^{(1), \text{ct}}_{d,l} = \frac{C^{(1), \text{ct}}_{p,LR}}{2} - \frac{2c_w^2 + 1}{16c_w^2s_w^2} \epsilon \left( \frac{M_W^2}{\mu^2} \right)^{-\epsilon} + \frac{\ln \left( \frac{M_W^2}{M_0^2} \right)}{16c_w^2s_w^2} + \frac{2c_w^2 + 1}{32c_w^2s_w^2}.
\]
Similarly the NLO bare correction to the hadronic vertex is given by
\[
C^{(1), \text{bare}}_{d,h} = -\frac{1}{2\epsilon^2} \left( \frac{M_W^2}{\mu^2} \right)^{-\epsilon} + \frac{2}{9\epsilon^2} \left( -\frac{M_W^2}{\mu^2} \right)^{-\epsilon}
\]
\[
+ \frac{1 + 2c_w^2}{72s_w^2c_w^2} \epsilon \left( \frac{M_W^2}{\mu^2} \right)^{-\epsilon} + \frac{1}{3\epsilon} \left( -\frac{M_W^2}{\mu^2} \right)^{-\epsilon}
\]
\[
+ \frac{8c_w^8 + 18c_w^6 + 11c_w^4 - 1}{36c_w^2s_w^2} M_W^2 C_0 (M_W^2, 0, 0, 0, M_Z^2)
\]
\[
+ \frac{(c_w^2 + 2) M_W^2 C_0 (M_W^2, 0, 0, M_Z^2, 0)}{s_w^2}
\]
\[
+ \frac{2c_w^2 + 1}{2s_w^2} \ell (M_W^2, M_Z^2) - \frac{(20c_w^6 + 6c_w^4 + 1) \ln \left( \frac{M_Z^2}{M_W^2} \right)}{36c_w^2s_w^2}
\]
\[
+ \frac{120c_w^6 + (48 - 13s_w^2\pi^2) c_w^4 - 6}{216c_w^4s_w^2} - \frac{(24c_w^6 + 22c_w^4 + c_w^2 - 2) i\pi}{72c_w^2s_w^2},
\]
and the corresponding counterterms are
\[
C^{(1), \text{ct}}_{d,h} = \frac{C^{(1), \text{ct}}_{p,LR}}{2} + \frac{16c_w^4 - 50c_w^2 + 7}{144c_w^2s_w^2} \epsilon \left( \frac{M_W^2}{\mu^2} \right)^{-\epsilon}
\]
\[- \frac{(16c_w^4 - 32c_w^2 + 7)}{144c_w^2s_w^2} \ln \left( \frac{M_W^2}{M_0^2} \right) - \frac{16c_w^4 - 50c_w^2 + 7}{288c_w^2s_w^2}.
\]

C.3 Integrals and auxiliary functions

The results for the short-distance coefficients and their counterterms have been written such that all poles in \( \epsilon \) are apparent and the remaining functions are finite. We give here their analytic expressions. As usual the scalar two- and three-point functions are defined by
\[
B_0(k^2, m_1^2, m_2^2) \equiv \int \frac{[dl]}{(l^2 - m_1^2)((l + k)^2 - m_2^2)}, \quad [dl] \equiv \frac{(e\gamma \mu^2)\epsilon}{i\pi d/2},
\]
and
\[
C_0(k_1^2, k_2^2, (k_1 + k_2)^2, m_1^2, m_2^2, m_3^2) \equiv \int \frac{[dl]}{(l^2 - m_1^2)((l + k_1)^2 - m_2^2)((l + k_1 + k_2)^2 - m_3^2)}.
\]
\( \partial B_0(k^2, m_1^2, m_2^2) \) is then defined as
\[
\partial B_0(k^2, m_1^2, m_2^2) = \frac{\partial B_0(q^2, m_1^2, m_2^2)}{\partial q^2} \bigg|_{q^2 = k^2} .
\] (C.16)

The auxiliary function \( \ell(k^2, m_1^2, m_2^2) \) used in the expressions for the matching coefficients is related to the two-point function by
\[
B_0(k^2, m_1^2, m_2^2) = \frac{1}{\epsilon} \left( \frac{m_2^2}{\mu^2} \right)^{-\epsilon} + 2 - \ell(k^2, m_1^2, m_2^2) \tag{C.17}
\]
and satisfies \( \ell(k^2, m_1^2, m_2^2) = \ell(k^2, m_2^2, m_1^2) + \ln(m_2^2/m_1^2) \). It is sufficient to give this function for the following special arguments:
\[
\ell(0, M_W^2, M_Z^2) = 1 + \frac{M_Z^2}{M_W^2 - M_Z^2} \ln \left( \frac{M_W^2}{M_Z^2} \right) ,
\]
\[
\ell(M_Z^2, M_W^2, M_W^2) = \frac{M_Z^2 - M_{ZW}^2}{2 M_Z^2} \ln \left( 1 + \frac{M_{ZW}^2 - M_Z^2}{2 M_W^2} \right) + \frac{M_Z^2 + M_{ZW}^2}{2 M_W^2} \ln \left( 1 - \frac{M_{ZW}^2 + M_Z^2}{2 M_W^2} \right) ,
\]
\[
\ell(M_W^2, M_Z^2, M_W^2) = \frac{2 M_{ZW}^2 - M_Z^2 + M_{ZW}^2}{2 M_W^2} \ln \left( \frac{M_Z^2 - M_{ZW}^2}{2 M_Z^2} \right) + \frac{2 M_{ZW}^2 - M_Z^2 - M_{ZW}^2}{2 M_W^2} \ln \left( \frac{M_Z^2 + M_{ZW}^2}{2 M_Z^2} \right) ,
\] (C.18)

where we introduced \( M_{ZW}^2 = \sqrt{M_Z^4 - 4 M_Z^2 M_{ZW}^2} \). The explicit result for the derivative of the two-point function that is needed reads
\[
\partial B_0(M_W^2, M_W^2, M_Z^2) = \tag{C.19}
\]
\[
-\frac{1}{M_W^2} \left\{ 1 + \frac{M_W^2 - M_Z^2}{2 M_{ZW}^2} \ln \left( \frac{M_Z^2}{M_W^2} \right) + \frac{M_Z^2 (3 M_W^2 - M_Z^2)}{M_W^2 M_{ZW}^2} \ln \left( \frac{M_Z^2 - M_{ZW}^2}{2 M_W^2 M_Z^2} \right) \right\} .
\]

The analytic expressions of the finite three-point functions appearing in the results given in (C.3)–(C.13) can all be obtained from
\[
C_0(0, M_W^2, -M_W^2, 0, M_Z^2, M_W^2) = \tag{C.20}
\]
\[
\frac{1}{4 M_W^2} \left\{ 2 \text{Li}_2 \left( 1 - \frac{2 M_{ZW}^2}{M_Z^2} \right) + 2 \text{Li}_2 \left( \frac{2 M_{ZW}^2 - M_Z^2}{4 M_W^2 - M_Z^2} \right) - \text{Li}_2 \left( \frac{M_Z^2}{M_{ZW}^2} \right) \right. \\
- 2 \text{Li}_2 \left( \frac{2 M_{ZW}^2 - M_Z^2}{M_{ZW}^2} \right) - 2 \text{Li}_2 \left( \frac{M_Z^2 - 2 M_{ZW}^2}{M_{ZW}^2} \right) - \frac{\pi^2}{3} \right\} ,
\]

\text{Li}_2 \left( \frac{M_Z^2 - 2 M_{ZW}^2}{M_{ZW}^2} \right) - \frac{\pi^2}{3} \right\} ,
\]

\text{Li}_2 \left( \frac{M_Z^2 - 2 M_{ZW}^2}{M_{ZW}^2} \right) - \frac{\pi^2}{3} \right\} ,
\[ C_0(0, 4M_W^2, 0, 0, M_Z^2, M_Z^2) = \]
\[
\begin{align*}
&- \frac{1}{8M_W^2} \left\{ \ln^2 \left( \frac{-M_W^2}{M_Z^2} \right) + \ln^2 \left( \frac{M_W^2 + Z}{M_Z^2} \right) + 2 \text{Li}_2 \left( \frac{M_W^2}{M_Z^2} \right) + 2 \text{Li}_2 \left( \frac{M_W^2}{M_Z^2} \right) + \pi^2 \right\} , \\
&+ 2 \text{Li}_2 \left( \frac{4M_W^2 - M_Z^2}{M_W^2 - Z} \right) + 2 \text{Li}_2 \left( \frac{4M_W^2 - M_Z^2}{M_W^2 + Z} \right) + \pi^2 \right\} , \\
C_0(-M_W^2, M_W^2, 0, 0, M_Z^2, M_Z^2) =
\begin{align*}
&- \frac{1}{2M_W^2} \left\{ \text{Li}_2 \left( -\frac{M_W^2}{M_W^2 + 2M_Z^2} \right) - \text{Li}_2 \left( \frac{M_W^2}{M_W^2 + 2M_Z^2} \right) \\
&+ \text{Li}_2 \left( \frac{M_W^2}{M_W^2 - M_Z^2 + M_Z^2} \right) - \text{Li}_2 \left( \frac{M_W^2}{M_W^2 - M_Z^2 - M_Z^2} \right) \\
&+ \text{Li}_2 \left( \frac{M_W^2}{M_W^2 - M_Z^2 + M_Z^2} \right) - \text{Li}_2 \left( \frac{M_W^2}{M_W^2 - M_Z^2 - M_Z^2} \right) + \frac{\pi^2}{4} \right\} , \\
C_0(M_W^2, 0, 0, M_W^2, M_Z^2, 0) =
\begin{align*}
&\frac{1}{M_W^2} \left\{ \text{Li}_2 \left( \frac{2M_W^2}{M_Z^2 + M_W^2} \right) + \text{Li}_2 \left( \frac{2M_W^2 + Z}{2M_Z^2} \right) - \frac{\pi^2}{6} \right\} , \\
C_0(M_W^2, -M_W^2, 0, 0, 0, M_Z^2) =
\begin{align*}
&\frac{1}{4M_W^2} \left\{ \ln \left( \frac{2M_W^2}{M_Z^2} + 1 \right) \left( \ln \left( \frac{2M_W^2}{M_Z^2} + 1 \right) - 2i\pi \right) - \text{Li}_2 \left( \frac{M_Z^2}{(2M_W^2 + M_Z^2)^2} \right) \\
&+ 2 \text{Li}_2 \left( 1 - \frac{2M_W^2}{M_Z^2} \right) + 2 \text{Li}_2 \left( \frac{M_W^2}{M_Z^2} \right) - 2 \text{Li}_2 \left( \frac{M_Z^2 - 2M_W^2}{2M_W^2 + M_Z^2} \right) \\
&+ 6 \text{Li}_2 \left( \frac{M_Z^2}{2M_W^2 + M_Z^2} \right) - \frac{2\pi^2}{3} \right\} , \\
C_0(M_W^2, 0, 0, 0, 0, M_Z^2) =
\begin{align*}
&\frac{1}{M_W^2} \left\{ \frac{1}{2} \ln^2 \left( \frac{M_Z^2}{M_Z^2} \right) - i\pi \ln \left( \frac{M_Z^2}{M_Z^2} + M_Z^2 \right) + \text{Li}_2 \left( \frac{M_Z^2}{M_Z^2 + M_Z^2} \right) - \frac{\pi^2}{6} \right\} , \\
\end{align*}
\end{align*}
\]

where we introduced \( M_{W\pm Z} \equiv M_W \pm \sqrt{M_W^2 - M_Z^2} \).
Figure C.1: Non-vanishing self-energy diagrams for the production vertex $e_L e_R^+ \rightarrow W^- W^+$ at leading order in $\vec{T}/M_W$.

Figure C.2: Non-vanishing triangle diagrams for the production vertex $e_L e_R^+ \rightarrow W^- W^+$ at leading order in $\vec{T}/M_W$. 
Figure C.3: Four-point diagrams contributing to the production vertex $e_L^- e_R^+ \rightarrow W^- W^+$ and $e_R^- e_L^+ \rightarrow W^- W^+$ at leading-order in $\vec{r}/M_W$. 

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Figure C.4: Four-point diagrams contributing only to the production vertex $e_L^+ e_R^- \to W^- W^+$ at leading order in $\tilde{t}/M_W$. 
In this appendix we explicitly prove the equivalence of the soft-photon contribution to the total cross section extracted from the forward-scattering amplitude $e^- e^+ \to WW \to e^- e^+$, as presented in Section 4.3, and the result of a direct calculation of the $2 \to 4$ process $e^- e^+ \to \mu^- \bar{\nu}_\mu u \bar{d}X$. We therefore explicitly include collinear fields describing the final-state fermions in the effective Lagrangian (2.34). The new modes couple to the non-relativistic fields $\Omega_\pm$ through the (leading-order) interaction term \cite{Footnote1}

$$\delta \mathcal{L}_{\text{decay}} = -\frac{g}{2\sqrt{M_W}} \left( \bar{\mu} c_3 \gamma^i P_L \nu c_4 \Omega^i_\pm + \bar{u} c_3 \gamma^i P_L d c_4 \Omega^i_\pm \right), \quad (D.1)$$

with the labels $c_3$ and $c_4$ indicating the collinear directions of the final fermions, while the interaction with soft photons is encoded in the SCET Lagrangian (2.22), and corresponds to the eikonal approximation (4.26). Before presenting results for the $O(\alpha)$ soft corrections, we show how the calculation of the leading-order cross section is modified by (D.1).

The leading-order EFT matrix element for the process

$$e^-_L(p_1)e^+_R(p_2) \to \mu^- (p_3)\bar{\nu}_\mu (p_4)u(p_5)d(p_6) \quad (D.2)$$

reads

$$\mathcal{M}^{(0)\delta L,LR}_{4L} = i \frac{\pi \alpha}{s^2_w M_W^2} \bar{e} \gamma^{[i} n j] P_L e \frac{i \delta^{ik} \delta^{lh}}{r_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2} l_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2}} \times \frac{-i g}{2 \sqrt{M_W}} \bar{\mu} \gamma^k P_L \nu \frac{-i g}{2 \sqrt{M_W}} \bar{u} \gamma^h P_L d$$

$$= i \frac{\pi \alpha^2}{s^2_w M_W^2} \bar{e} \gamma^{[i} n j] P_L e \frac{1}{r_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2} l_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2}} \times \frac{1}{r_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2} l_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2}} \frac{1}{r_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2} l_0 - \frac{r_1^2}{2 M_W} + i \frac{r_1^{0j}}{2}} \times \bar{\mu} \gamma^k P_L \nu \bar{u} \gamma^h P_L d. \quad (D.3)$$

In order to lighten the notation, we have introduced in (D.3) the two shortcuts $r = p_3 + p_4 - 2 M_W v$, $l = p_5 + p_6 - 2 M_W v$, with $v^\mu = (1, \vec{0})$, and dropped sums over repeated
indices (note that the sum runs over $i = 1, 2, 3$ when Latin indices are used). The $RL$ helicity combination vanishes at lowest order in $\delta$. The total polarised cross section is obtained by summing $|\mathcal{M}_{4f,LR}|^2$ over final state polarisations and colours, and integrating over the phase-space of the four final fermions:

$$
\sigma^{(0)}_{LR} = \frac{1}{2s} \prod_{i=3}^{6} \int d\Phi_i \sum_{\text{pol}} \sum_{\text{col}} |\mathcal{M}^{(0)}_{4f,LR}|^2 (2\pi)^4 \delta^{(4)}(\sum p_i - P)
$$

$$
= \frac{3\pi^4 \alpha^4}{2s_w^8 M_W^6} \prod_{i=3}^{6} \int d\Phi_i \frac{1}{(r_0 - \frac{|\vec{r}|^2}{2M_W})^2} \frac{1}{(l_0 - \frac{|\vec{l}|^2}{2M_W})^2} + \frac{\Gamma^{(0)}_W}{4}
$$

$$
\times \text{Tr}[\gamma^i \gamma^j \gamma^k n^h P_L] \text{Tr}[\gamma^i \gamma^j \gamma^k P_L] \text{Tr}[\gamma^i \gamma^j \gamma^h P_L] (2\pi)^4 \delta^{(4)}(\sum p_i - P).
$$

To disentangle the production and decay subprocesses, we insert two dummy integrations over $r$ and $l$ in (D.4):

$$
\sigma^{(0)}_{LR} = \frac{3\pi^4 \alpha^4}{2s_w^6 M_W^6} \int \frac{d^4r}{(2\pi)^4} \frac{d^4l}{(2\pi)^4}
$$

$$
= \frac{1}{(r_0 - \frac{|\vec{r}|^2}{2M_W})^2} \frac{1}{(l_0 - \frac{|\vec{l}|^2}{2M_W})^2} + \frac{\Gamma^{(0)}_W}{4}
$$

$$
\times \text{Tr}[\gamma^i \gamma^j \gamma^k n^h P_L] \text{Tr}[\gamma^i \gamma^j \gamma^h P_L] (2\pi)^4 \delta^{(4)}(2M_W v - P + r + l)
$$

$$
\times \int d\Phi_3 d\Phi_4 \text{Tr}[\gamma^i \gamma^j \gamma^k P_L] (2\pi)^4 \delta^{(4)}(p_3 + p_4 - M_W v - r)
$$

$$
\times \int d\Phi_5 d\Phi_6 \text{Tr}[\gamma^i \gamma^j \gamma^h P_L] (2\pi)^4 \delta^{(4)}(p_5 + p_6 - M_W v - l).
$$

Since, by assumptions, the components of $r, l$ are much smaller than the components of the external-state momenta $p_i$, we neglect them inside the phase-space integrations,

$$
\delta^{(4)}(p_3 + p_4 - M_W v - r) \sim \delta^{(4)}(p_3 + p_4 - M_W v),
$$

and perform the phase-space integrals independently from the $r$ and $l$ integrations,

$$
\int d\Phi_3 d\Phi_4 \text{Tr}[\gamma^i \gamma^j \gamma^k P_L] (2\pi)^4 \delta^{(4)}(p_3 + p_4 - M_W v) = \frac{M_W^2}{12\pi} \delta^{ik}.
$$
Inserting (D.7) into (D.5) we obtain the much simpler expression

\[ \sigma^{(0)}_{LR} = \frac{\pi^2 \alpha^4}{96 s_w M_W^2} \text{Tr}[\bar{p}_1 \gamma^\gamma [i \gamma^\gamma \bar{p}_2] [i \gamma^\gamma P_L] \times \int \frac{d^4 r}{(2\pi)^4} \left( r_0^2 - \frac{|r|^2}{2M_W} \right)^2 + \frac{\Gamma_W^{(0)} \gamma^2}{4} \left[ (E - r_0 - \frac{|r|^2}{2M_W})^2 + \frac{\Gamma_W^{(0)} \gamma^2}{4} \right]. \] (D.8)

Performing the \( r_0 \) integration, equation (D.5) can be recast into

\[ \sigma^{(0)}_{LR} = \frac{\pi^2 \alpha^4}{6s_w} \int \frac{d^4 r}{(2\pi)^4} \frac{1}{(r_0 - \frac{|r|^2}{2M_W})^2} \frac{1}{\left( E - r_0 - \frac{|r|^2}{2M_W} \right)^2 + \frac{\Gamma_W^{(0)} \gamma^2}{4}}. \] (D.9)

which coincides with equation (2.48) once we have replaced \( \Gamma_W^{(0)} \) with its explicit value \( \Gamma_W^{(0)} = \frac{3\alpha^3}{4\pi} M_W \).

We now turn to the calculation of soft-photon corrections to the total cross section \( e^- e^+ \rightarrow \mu^- \bar{\nu}_\mu \bar{u}d \). Both virtual and real-photon interferences must be taken into account, with the relevant topologies represented by the cut diagrams shown in Figure D.1. As in the calculation discussed in Section 4.3, interferences of photons attached to external legs with same momentum are proportional to \( p_i^2 = 0 \), and vanish. In the soft limit virtual and real corrections to the total cross section can be respectively written as

\[ \Delta \sigma^{(1)}_{V,X} = \frac{1}{2s} \int d\Phi_{4f} \langle |\mathcal{M}_{4f,LR}^{(0)}|^2 \rangle 2\text{Re} \left[ V_X(p_i; \epsilon) \right], \]
\[ \Delta \sigma^{(1)}_{R,X} = \frac{1}{2s} \int d\Phi_{4f} \langle |\mathcal{M}_{4f,LR}^{(0)}|^2 \rangle 2\text{Re} \left[ R_X(p_i; \epsilon) \right], \] (D.10)

where \( V_X \) and \( R_X \) represents scalar functions encoding the contribution of the different topologies shown in Figure D.1, and \( d\Phi_{4f} \) and \( \langle |\mathcal{M}_{4f,LR}^{(0)}|^2 \rangle \) are defined as

\[ \int d\Phi_{4f} = \prod_{i=1}^4 \int d\Phi_i (2\pi)^4 \delta^{(4)}(\sum_i p_i - P) \]
\[ \langle |\mathcal{M}_{4f,LR}^{(0)}|^2 \rangle = \sum_{\text{pol}} \sum_{\text{col}} |\mathcal{M}_{4f,LR}^{(0)}|^2. \] (D.11)
Figure D.1: Soft-photon interferences.
The term $2 \text{Re}$ in (D.10) comes from considering the diagrams in figure and the ones symmetric with respect to the central cut. We will now discuss virtual and real corrections in turn.

### D.1 Virtual corrections

We start listing the virtual corrections corresponding to the contributions of the first nine interference diagrams in Figure D.1. The integrals are computed with the same techniques and approximations used in Section 4.3. For simplicity we introduce the abbreviation $[d^d q] = \frac{\epsilon^{\alpha \beta \mu \nu} \delta^{\mu \nu}}{(2\pi)^d}$, and the vectors $v_i = p_i/M_W$ and $v_f = p_f/M_W$ for each initial and final momentum.

#### Initial-initial state interference

\[ V_{e^- e^+} = 8\pi i \alpha \int [d^d q] \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_i \cdot q + i\epsilon)} \frac{1}{(-v_j \cdot q + i\epsilon)} = 0. \]  

#### Initial-intermediate state interference

\[ V_{i \Omega} = -4\pi i \alpha Q_i Q_\Omega \int [d^d q] \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_i \cdot q + i\epsilon)} \frac{1}{(-q^0 + \eta_\Omega)} = \frac{\alpha}{2\pi} Q_i Q_\Omega \left\{ \frac{1}{2e^2} - \frac{1}{\epsilon} \ln \left( -\frac{2\eta_\Omega}{\mu} \right) + \ln^2 \left( -\frac{2\eta_\Omega}{\mu} \right) + \frac{5}{24} \pi^2 \right\}, \]  

with $i = e^\pm$, $\Omega = \Omega_\pm$ and $\eta_\Omega = \eta_\pm$ for $\Omega = \Omega_\pm$ respectively.

#### Initial-final state interference

\[ V_{i f} = -4\pi i \alpha Q_i Q_f (v_i \cdot v_f) \int [d^d q] \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_i \cdot q + i\epsilon)} \frac{1}{(-v_f \cdot q + i\epsilon)} \frac{1}{(-q^0 + \eta_f)} = \frac{\alpha}{2\pi} Q_i Q_f \left\{ -\frac{1}{e^2} + \frac{1}{\epsilon} \left[ 2 \ln \left( -\frac{2\eta_f}{\mu} \right) - \ln \left( \frac{2v^0_f v^0_i}{v_f \cdot v_i} \right) \right] - 2 \ln^2 \left( -\frac{2\eta_f}{\mu} \right) \\ + 2 \ln \left( -\frac{2\eta_f}{\mu} \right) \ln \left( \frac{2v^0_f v^0_i}{v_f \cdot v_i} \right) + \text{Li}_2 \left( -\frac{2v^0_f v^0_i - v_f \cdot v_i}{v_f \cdot v_i} \right) - \frac{5}{12} \pi^2 \right\}, \]  

where $i = e^\pm$, $f = \mu^-$, $\bar{\nu}_\mu$, $u$, $\bar{d}$, and $\eta_f = \eta_-$ for $f = \mu^-$, $\bar{\nu}_\mu$ and $\eta_f = \eta_+$ for $f = u$, $\bar{d}$.  

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Intermediate-intermediate state interference

\[
V_{\Omega+\Omega\pm} = -\frac{4\pi i\alpha}{\eta_\pm} \int [d^d q] \frac{1}{(q^2 + io) (-q^0 + \eta_\pm)} \\
= \frac{\alpha}{2\pi} \left\{ \frac{1}{\epsilon} + 2 - 2\ln \left( -\frac{2\eta_\pm}{\mu} \right) \right\}, \quad \text{(D.15)}
\]

\[
V_{\Omega_-\Omega_+} = 4\pi i\alpha \int [d^d q] \frac{1}{(q^2 + io) (q^0 + \eta_-) (-q^0 + \eta_+)} \\
= \frac{\alpha}{2\pi} \left\{ \frac{1}{\epsilon} - 2 + \frac{2\eta_-}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_-}{\mu} \right) + \frac{2\eta_+}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_+}{\mu} \right) \right\}. \quad \text{(D.16)}
\]

Intermediate-final state interference

\[
V_{\Omega f}^{(1)} = -2\pi i\alpha Q_\Omega Q_f \int [d^d q] \frac{1}{(q^0 + \eta_\Omega)(-q^0 + \eta_f)} \\
= \frac{\alpha}{2\pi} Q_\Omega Q_f \left\{ \frac{1}{2\epsilon^2} - \frac{1}{\epsilon} \ln \left( -\frac{2\eta_\Omega}{\mu} \right) + \ln^2 \left( -\frac{2\eta_\Omega}{\mu} \right) + \frac{5}{24} \pi^2 \right\},
\]

\[
V_{\Omega f}^{(2)} = -2\pi i\alpha Q_\Omega Q_f \int [d^d q] \frac{1}{(q^0 + \eta_\Omega)(-q^0 + \eta_f)} \\
= \frac{\alpha}{2\pi} Q_\Omega Q_f \frac{\eta_f}{\eta_\Omega + \eta_f} \left\{ \frac{1}{\epsilon} \left[ \ln \left( -\frac{2\eta_\Omega}{\mu} \right) - \ln \left( -\frac{2\eta_f}{\mu} \right) \right] \right. \\
\left. - \ln^2 \left( -\frac{2\eta_\Omega}{\mu} \right) + \ln^2 \left( -\frac{2\eta_f}{\mu} \right) \right\}, \quad \text{(D.17)}
\]

where the possible combinations \((\Omega, f)\) are \((\Omega_-, \mu^-), (\Omega_-, \bar{\nu}_\mu), (\Omega_+, u), (\Omega_+, \bar{d})\) for \(V_{\Omega f}^{(1)}\) and \((\Omega_+, \mu^-), (\Omega_+, \bar{\nu}_\mu), (\Omega_-, u), (\Omega_-, \bar{d})\) for \(V_{\Omega f}^{(2)}\).
Final-final state interference

\[ V^{(1)}_{f g} = -4\pi i \alpha Q_f Q_g (v_f \cdot v_g) \int [d^4q] \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_g \cdot q + i\epsilon)} \frac{1}{(v_f \cdot q + i\epsilon)} = 0, \]

\[ V^{(2)}_{f g} = -4\pi i \alpha Q_f Q_g (v_f \cdot v_g) \int [d^4q] \times \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_g \cdot q + i\epsilon)} \frac{\eta_-}{(-v_g \cdot q + i\epsilon) (q^0 + \eta_-)} \frac{\eta_+}{(-q^0 + \eta_+)} \]

\[ = \alpha \frac{Q_f Q_g}{2\pi} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \left[ \frac{2\eta_-}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_+}{\mu} \right) + \frac{2\eta_+}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_-}{\mu} \right) \right] \right. \]

\[ - \ln \left( \frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right) + \frac{2\eta_-}{\eta_- + \eta_+} \ln^2 \left( -\frac{2\eta_+}{\mu} \right) + \frac{2\eta_+}{\eta_- + \eta_+} \ln^2 \left( -\frac{2\eta_-}{\mu} \right) \]

\[ - \left[ \frac{2\eta_-}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_+}{\mu} \right) + \frac{2\eta_+}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_-}{\mu} \right) \right] \ln \left( \frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right) \]

\[ - \text{Li}_2 \left[ -\frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right] + \frac{5}{12} \pi^2 \right\}, \quad (D.18) \]

with the possible combinations \((f, g)\) given by \((\mu^-, \bar{\nu}_\mu)\), \((u, \bar{d})\) for \(V^{(1)}_{f g}\) and \((\mu^-, u)\), \((\mu^-, \bar{d})\), \((\bar{\nu}_\mu, u)\), \((\bar{\nu}_\mu, \bar{d})\) for \(V^{(2)}_{f g}\).

### D.2 Real Corrections

Here we give results for the contributions of the real-emission diagrams \(R_X\) shown in Figure D.1. As before \([d^4q] = \frac{e^{+3\epsilon\mu^2}}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}\) and \(v_i = p_i/M_W, v_f = p_f/M_W\). In deriving the starting integrals \(R_X\), we used the freedom of shifting the final-state momenta by a small soft momentum, \(p_f \rightarrow p_f \pm q\), justified in the calculation of the total cross section, to make cancellations between virtual and real corrections as explicit as possible.

Initial-initial state interference

\[ R_{e^- e^+} = -8\pi \alpha \int [d^{d-1}q] \frac{1}{2|q|} \frac{1}{(-v_i \cdot q + i\epsilon)} \frac{1}{(v_j \cdot q + i\epsilon)} \frac{1}{(-|q| + \eta_-)} \frac{\eta^+}{(-|q| + \eta^+)} \]

\[ = \alpha \frac{1}{2\pi} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \left[ \frac{2\eta_-}{\eta_- - \eta^+} \ln \left( -\frac{2\eta^+}{\mu} \right) - \frac{2\eta^+}{\eta_- - \eta^+} \ln \left( -\frac{2\eta_-}{\mu} \right) \right] \right. \]

\[ + \frac{5}{12} \pi^2 + \frac{2\eta_-}{\eta_- - \eta^+} \ln^2 \left( -\frac{2\eta^+}{\mu} \right) - \frac{2\eta^+}{\eta_- - \eta^+} \ln^2 \left( -\frac{2\eta_-}{\mu} \right) \right\}. \quad (D.19) \]
Initial-intermediate state interference
\[ R_{i\Omega} = -4\pi \alpha Q_i Q\Omega \int [d^{d-1}q] \frac{1}{2|q|} \frac{1}{(-v_i \cdot q + i\alpha)} \frac{1}{(-|q| + \eta_\Omega)} \frac{1}{(-|q| + \eta^*_\Omega)} \]
\[ = \frac{\alpha}{2\pi} Q_i Q\Omega \eta_\Omega \frac{\eta^*_\Omega}{\eta^*_\Omega - \eta_\Omega} \left\{ \frac{1}{\epsilon} \left[ \ln \left( -2\eta_\Omega \mu \right) - \ln \left( -2\eta^*_\Omega \mu \right) \right] - \ln^2 \left( -2\eta_\Omega \mu \right) + \ln^2 \left( -2\eta^*_\Omega \mu \right) \right\}, \quad (D.20) \]

where \( i = e^\pm, \Omega = \Omega^\pm \).

Initial-final state interference
\[ R_{i,f} = 4\pi \alpha Q_i Q_f (v_i \cdot v_f) \int [d^{d-1}q] \frac{1}{2|q|} \frac{1}{(-v_i \cdot q + i\alpha)} \frac{1}{(-v_f \cdot q + i\alpha)} \frac{\eta_f}{(-|q| + \eta_f)} \]
\[ = \frac{\alpha}{2\pi} Q_i Q_f \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \left[ 2 \ln \left( -\frac{2\eta_f}{\mu} \right) - \ln \left( \frac{2v_f \cdot v_i}{v_f \cdot v_i} \right) \right] + 2\ln^2 \left( -\frac{2\eta_f}{\mu} \right) \right. \]
\[ - 2\ln \left( -2\eta_f \right) \ln \left( 2v_f^0 \right) - \text{Li}_2 \left[ -2v_f^0 \cdot v_f - v_f \cdot v_i \right] + \frac{5}{12} \pi^2 \left\}, \quad (D.21) \]

with \( i = e^\pm, f = \mu^-, \bar{\nu}_\mu, u, d \).

Intermediate-intermediate state interference \(^1\)
\[ R_{\Omega^\pm \Omega^\pm} = -2\pi \alpha \int [d^{d-1}q] \frac{1}{2|q|} \frac{1}{(-|q| + \eta^\pm)} \frac{1}{(-|q| + \eta^*_\pm)} = \]
\[ = \frac{\alpha}{4\pi} \left\{ -\frac{1}{\epsilon} - 2 + \frac{2\eta^\pm}{\eta^\pm - \eta^*_\pm} \ln \left( -\frac{2\eta^\pm}{\mu} \right) - \frac{2\eta^*_\pm}{\eta^\pm - \eta^*_\pm} \ln \left( -\frac{2\eta^*_\pm}{\mu} \right) \right\}, \quad (D.22) \]
\[ R_{\Omega^- \Omega^+} = 4\pi \alpha \eta^- \int [d^{d-1}q] \frac{1}{2|q|} \frac{1}{(-|q| + \eta^-)} \frac{1}{(-|q| + \eta^*_+)} \frac{1}{(|q| + \eta^*_+)} = \]
\[ = \frac{\alpha}{2\pi} \eta^- \frac{\eta^*_+}{\eta^- - \eta^*_+} \left\{ -\frac{2\eta^-}{\eta^- + \eta^*_+} \ln \left( -\frac{2\eta^-}{\mu} \right) + \frac{2\eta^*_+}{\eta^- + \eta^*_+} \ln \left( -\frac{2\eta^*_+}{\mu} \right) \right. \]
\[ - \frac{2\eta^*_+}{\eta^- + \eta^*_+} \ln \left( \frac{2\eta^*_+}{\eta^- + \eta^*_+} \right) + \frac{2\eta^*_+}{\eta^- + \eta^*_+} \ln \left( \frac{2\eta^*_+}{\eta^- + \eta^*_+} \right) \right\}. \quad (D.23) \]

\(^1\)Note that \( R_{\Omega^\pm \Omega^\pm} \) is half the contribution coming from the diagram in Figure D.1, because a factor 2 has already been included in D.10.

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Intermediate-final state interference

\[ R_{\Omega f}^{(1)} = 2\pi \alpha Q_\Omega Q_f \int [d^{d-1} q] \frac{1}{2|q|} \frac{1}{(-v_f \cdot q + i\epsilon)} \frac{1}{(-|q| + \eta)} \]

\[ = \frac{\alpha}{2\pi} Q_\Omega Q_f \left\{ -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \ln \left( -\frac{2\eta}{\mu} \right) - \ln^2 \left( -\frac{2\eta}{\mu} \right) - \frac{5}{24\pi^2} \right\} , \]

\[ R_{\Omega f}^{(2)} = 2\pi \alpha Q_\Omega Q_f \int [d^{d-1} q] \frac{1}{2|q|} \frac{1}{(-v_f \cdot q + i\epsilon)} \frac{1}{(-|q| + \eta)} \frac{\eta_f}{\eta + \eta_f} \]

\[ = \frac{\alpha}{2\pi} Q_\Omega Q_f \left\{ -\frac{1}{\epsilon} \ln \left( \frac{2\eta}{\mu} \right) - \ln \left( -\frac{2\eta}{\mu} \right) \right\} + \ln^2 \left( \frac{2\eta}{\mu} \right) - \ln^2 \left( -\frac{2\eta}{\mu} \right) , \quad (D.24) \]

where, as before, the possible combinations \((\Omega, f)\) are \((\Omega_-, \mu^-), (\Omega_-, \bar{\nu}_\mu), (\Omega_+, u), (\Omega_+, \bar{d})\) for \(R_{\Omega f}^{(1)}\) and \((\Omega_+, \mu^-), (\Omega_+, \bar{\nu}_\mu), (\Omega_, u), (\Omega_, \bar{d})\) for \(R_{\Omega f}^{(2)}\).

Final-final state interference

\[ R_{fg}^{(1)} = 4\pi i \alpha Q_f Q_g (v_f \cdot v_g) \int [d^{d} q] \frac{1}{(q^2 + i\epsilon)} \frac{1}{(-v_g \cdot q + i\epsilon)} \frac{1}{(-v_f \cdot q + i\epsilon)} = 0 , \]

\[ R_{fg}^{(2)} = 4\pi \alpha Q_f Q_g (v_f \cdot v_g) \int [d^{d-1} q] \]

\[ \times \frac{1}{2|q|} \frac{1}{(-v_f \cdot q + i\epsilon)} \frac{1}{(-v_g \cdot q + i\epsilon)} \frac{\eta_-}{\eta_- + \eta_+} \frac{\eta_+}{\eta_- + \eta_+} \]

\[ = \frac{\alpha}{2\pi} Q_f Q_g \left\{ -\frac{1}{2} + \frac{2\eta_-}{\eta_- + \eta_+} \ln \left( -\frac{2\eta_+}{\mu} \right) + \frac{2\eta_+}{\eta_- + \eta_+} \ln \left( 2\eta_- \right) \right\} \]

\[ - \ln \left( \frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right) - \frac{2\eta_-}{\eta_- + \eta_+} \ln^2 \left( -\frac{2\eta_+}{\mu} \right) - \frac{2\eta_+}{\eta_- + \eta_+} \ln^2 \left( 2\eta_- \right) + \ln \left( \frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right) \]

\[ + \text{Li}_2 \left[ -\frac{2v_g^0 v_f^0}{v_g \cdot v_f} \right] - \frac{5}{12\pi^2} \right\} , \quad (D.25) \]

where again the possible combinations \((f, g)\) are represented by \((\mu^-, \bar{\nu}_\mu), (u, \bar{d})\) for \(V_{fg}^{(1)}\) and \((\mu^-, u), (\bar{\mu}^-, \bar{d}), (\bar{\nu}_\mu, u), (\bar{\nu}_\mu, \bar{d})\) for \(V_{fg}^{(2)}\).
D.3 Cancellation between real and virtual corrections

We now show how most of the virtual and real corrections found in the two previous subsections actually cancel. We first note that the sum of all contributions corresponding to initial-intermediate state interferences vanish, as already seen for the forward-scattering calculation:

$$\sum_{i=e^\pm} \sum_{\Omega=\Omega_\pm} V_{i\Omega} = \sum_{i=e^\pm} \sum_{\Omega=\Omega_\mp} R_{i\Omega} = 0. \quad (D.26)$$

Other cancellations take place pairwise between virtual corrections and their real counterparts.

The cancellation of the initial-final state interferences is manifest. Summing the contributions of virtual and real interferences we obtain

$$\Delta \sigma_{V,if}^{(1)} + \Delta \sigma_{R,if}^{(1)} = \frac{1}{2s} \sum_{i,f} \int d\Phi_4 \langle |M^{(0)}_{4i,LR}|^2 \rangle 2 \text{Re} [V_{if} + R_{if}] = 0. \quad (D.27)$$

Analogously, for the intermediate-final state interferences $V_{\Omega f}^{(1)}$ and $R_{\Omega f}^{(1)}$ one has

$$\Delta \sigma_{V,\Omega f}^{(1)} + \Delta \sigma_{R,\Omega f}^{(1)} = \frac{1}{2s} \sum_{i,f} \int d\Phi_4 \langle |M^{(0)}_{4i,LR}|^2 \rangle 2 \text{Re} [V_{\Omega f}^{(1)} + R_{\Omega f}^{(1)}] = 0. \quad (D.28)$$

The cancellation among intermediate-intermediate state interferences is more subtle. First note that, since $V_{WW}$ and $R_{WW}$ only depend on $\eta_\pm$, we can rewrite D.10 as

$$\Delta \sigma_{V,\Omega_\Omega}^{(1)} = \frac{\pi^2 \alpha^4}{6s^8 W s} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{|\eta_-|^2 |\eta_+|^2} 2 \text{Re} \left[ V_{\Omega_- \Omega_-} + V_{\Omega_+ \Omega_+} + V_{\Omega_- \Omega_+} \right],$$

$$\Delta \sigma_{R,\Omega_\Omega}^{(1)} = \frac{\pi^2 \alpha^4}{6s^8 W s} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{|\eta_-|^2 |\eta_+|^2} 2 \text{Re} \left[ R_{\Omega_- \Omega_-} + R_{\Omega_+ \Omega_+} + R_{\Omega_- \Omega_+} \right]. \quad (D.29)$$

The sum of the virtual corrections reads

$$V_{\Omega_- \Omega_-} + V_{\Omega_+ \Omega_+} + V_{\Omega_- \Omega_+} = \frac{\alpha}{2\pi} \left\{ \frac{1}{\epsilon} + 2 + \frac{2\eta_+}{\eta_- + \eta_+} \ln \left( \frac{-2\eta_-}{\eta_- + \eta_+} \right) + \frac{2\eta_-}{\eta_- + \eta_+} \ln \left( \frac{-2\eta_+}{\eta_- + \eta_+} \right) \right\}. \quad (D.30)$$

Performing the $r^0$ integration in $\Delta \sigma_{V,\Omega_\Omega}^{(1)}$ with the aid of Cauchy theorem we obtain

$$\Delta \sigma_{V,\Omega_\Omega}^{(1)} = \frac{\pi^2 \alpha^4}{3s^8 W s} \Gamma^{(0)}_W \int \frac{d^3 r}{(2\pi)^3} \frac{1}{\left( E - \frac{|\eta|^2}{M_W} \right)^2 + \Gamma^{(0)}_W^2} \frac{\alpha}{\pi} \left[ \frac{1}{\epsilon} + 2 - 2 \ln \left( \frac{2\Gamma^{(0)}_W}{\mu} \right) \right], \quad (D.31)$$

while the corresponding expression for $\Delta \sigma_{R,\Omega_\Omega}^{(1)}$ reads

$$\Delta \sigma_{R,\Omega_\Omega}^{(1)} = -\frac{\pi^2 \alpha^4}{3s^8 W s} \Gamma^{(0)}_W \int \frac{d^3 r}{(2\pi)^3} \frac{1}{\left( E - \frac{|\eta|^2}{M_W} \right)^2 + \Gamma^{(0)}_W^2} \frac{\alpha}{\pi} \left[ \frac{1}{\epsilon} + 2 - 2 \ln \left( \frac{2\Gamma^{(0)}_W}{\mu} \right) \right]. \quad (D.32)$$
Thus the two contributions cancel exactly. The same happens for $V_{\Omega f}^{(2)}$ and $R_{\Omega f}^{(2)}$. The terms proportional to $\ln \left(-\frac{2\eta}{\mu}\right)$ manifestly drop out in the sum, and we ignore them in the following. Summing explicitly over the possible values of $(\Omega, f)$ we obtain for the remaining terms

$$
\Delta \sigma_{V,\Omega f}^{(1)} = \frac{\pi^2 \alpha^4}{6s_w^6 s_f^8 s} \int \frac{d^3 r}{(2\pi)^3} \frac{1}{|\eta_-| \eta_+} \left( \frac{\alpha}{\pi} \right) \Re \left[ \frac{\eta_-}{\eta_- + \eta_+} \left( \frac{1}{\epsilon} \ln \left( -\frac{2\eta_-}{\mu} \right) - \ln^2 \left( -\frac{2\eta_-}{\mu} \right) \right) \right].
$$

Performing the $r_0$ integration yields the expression

$$
\Delta \sigma_{V,\Omega f}^{(1)} = -\frac{\pi^2 \alpha^4}{3s_w^8 \Gamma_W^{(0)}} \frac{1}{s^2 \Gamma_W^{(0)} + \Gamma_W^{(0)}} \times \frac{\alpha}{\pi} \left[ \frac{1}{\epsilon} \ln \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) - \ln^2 \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) + \frac{\pi^2}{4} \right].
$$

Analogously we obtain

$$
\Delta \sigma_{R,\Omega f}^{(1)} = \frac{\pi^2 \alpha^4}{12s_w^8 \Gamma_W^{(0)}} \frac{1}{s^2 \Gamma_W^{(0)} + \Gamma_W^{(0)}} \times \frac{\alpha}{\pi} \left[ \frac{1}{\epsilon} \ln \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) - \ln^2 \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) + \frac{\pi^2}{4} \right],
$$

which exactly cancels (D.34).

We now come to the final-final state interferences. Most of the terms in $V_{f g}^{(2)}$ and $R_{f g}^{(2)}$ manifestly cancel when the two contributions are summed, and we drop them in the following equations. The surviving contributions read

$$
\Delta \sigma_{V,f g}^{(1)} = \frac{1}{2s} \sum_{f,g} \int d\Phi_4 \langle |M_{1f}^{(0)}|^2 \rangle \frac{\alpha}{\pi} Q_f Q_g \Re \left[ \frac{2\eta_+}{\eta_+ + \eta_-} \left( -\frac{1}{\epsilon} \ln \left( -\frac{2\eta_-}{\mu} \right) + \ln^2 \left( -\frac{2\eta_-}{\mu} \right) \right) \right]
$$

$$
= \frac{3\pi^4 \alpha^4}{2s_w^8 M_w^{(0)}} \text{Tr}\left[p_1 \gamma^i n_f p_2 \gamma^k n_h P_L\right] \times \int d\Phi_3 d\Phi_4 \text{Tr}\left[p_3 \gamma^i p_4 \gamma^k P_L\right](2\pi)^4 \delta^4(p_3 + p_4 - M_W v)
$$

$$
\times \int d\Phi_5 d\Phi_6 \text{Tr}\left[p_5 \gamma^j p_6 \gamma^h P_L\right](2\pi)^4 \delta^4(p_5 + p_6 - M_W v)
$$

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\[ \Delta \sigma_{V;fg}^{(1)} = \frac{3\pi^4\alpha^4}{s_w M_W^2 \Gamma_W^{(0)}} \text{Tr}[\hat{p}_1 \gamma^1 \hat{p}_2 \gamma^2 \eta^h] P_L \]
\[ \times \int \frac{d^4r}{(2\pi)^4} \frac{1}{|\eta_+|^2 |\eta_-|^2} \sum_{f,g} \frac{\alpha}{\pi} Q_f Q_g \text{Re} \left[ \frac{2\eta_+}{\eta_+ + \eta_-} \left( -\frac{1}{\epsilon} \ln \left( \frac{2\eta_-}{\mu} \right) \right) \right] \]
\[ \times \ln^2 \left( \frac{2\eta_-}{\mu} \right) + \ln \left( \frac{2\eta_-}{\mu} \right) \ln \left( \frac{2\eta_+}{v_g \cdot v_f} \right) \right] \] .

(D.36)

In this case we cannot easily perform the integration over the phase-space of the final-state particles, because the function \( V^{(2)} \) depends non-trivially on the velocity vectors \( v_f^\mu \) and \( v_g^\mu \). After solving the \( r_0 \) integration, and extracting the real part of the expression in brackets, we are left with

\[ \Delta \sigma_{V;fg}^{(1)} = \frac{3\pi^4\alpha^4}{s_w M_W^2 \Gamma_W^{(0)}} \text{Tr}[\hat{p}_1 \gamma^1 \hat{p}_2 \gamma^2 \eta^h] P_L \]
\[ \times \int d\Phi_4 d\Phi_6 \text{Tr}[\hat{p}_3 \gamma^3 \hat{p}_4 \gamma^4 \eta^h] P_L (2\pi)^4 \delta^{(4)} (p_3 + p_4 - M_W v) \]
\[ \times \int \frac{d^4r}{(2\pi)^4} \frac{1}{(E + \frac{m^2}{p_W})^2 + \Gamma_W^{(0)}} \sum_{f,g} \frac{\alpha}{\pi} Q_f Q_g \text{Re} \left[ -\frac{1}{\epsilon} \ln \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) \right] \]
\[ \times \ln^2 \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) + \ln \left( \frac{2\Gamma_W^{(0)}}{\mu} \right) \ln \left( \frac{2\eta_+}{v_g \cdot v_f} \right) - \frac{\pi^2}{4} \right] .

(D.37)

In the same way, for \( R_{fg}^{(2)} \) we find

\[ \Delta \sigma_{R;fg}^{(1)} = -\Delta \sigma_{V;fg}^{(1)} . \] (D.38)

Equations D.27), (D.28), (D.37) and (D.38) confirm the well known result that all soft corrections related to final-state interferences vanish for the totally-inclusive cross section [86,87].

The only non-vanishing soft contribution to the four-fermion cross section is represented by the real initial-final state interference diagram, and reads

\[ \Delta \sigma_{R;e-e^+}^{(1)} = \frac{\pi^5\alpha^4}{6s_w^2 s} \int \frac{d^4r}{(2\pi)^4} \frac{1}{|\eta_-|^2 |\eta_+|^2} \sum_{f,g} \frac{\alpha}{\pi} \text{Re} \left[ \frac{5}{12} \pi^2 \right] \]
\[ + \frac{2\eta_-}{\eta_- - \eta_+} \ln^2 \left( \frac{2\eta_-}{\mu} \right) - \frac{2\eta_+}{\eta_- - \eta_+} \ln^2 \left( \frac{2\eta_-}{\mu} \right) \]
\[ = \frac{\pi\alpha^5}{6s_w^2 s} \text{Re} \int \frac{d^4r}{(2\pi)^4} \frac{1}{|\eta_-|^2 |\eta_+|^2} \left[ \frac{5}{12} \pi^2 - \frac{4\eta_-}{\eta_- - \eta_+} \ln^2 \left( \frac{2\eta_-}{\mu} \right) \right] , \] (D.39)

where we have subtracted minimally the double and single \( \epsilon \) poles of \( R_{fg}^{(2)} \), as done in (4.42), and used the fact that \( 1/(|\eta_-|^2 |\eta_+|^2) \) is a purely-real quantity to extract \( \text{Re} \) from
the integral. We have also reintroduced a $d$-dimensional integration to regularise formally divergent terms. It is easy to show that (D.39) is equivalent to (4.42). The constant term is just proportional to the leading-order cross section (D.9). Thus, after the $r_0$ integration, equation (D.39) simplifies to

$$
\Delta \sigma^{(1)}_{R,e^+e^-} = \frac{5\pi^2\alpha^3}{81s_w^4s} \Im \left[ -\sqrt{-\frac{E + i\Gamma_W^{(0)}}{M_W}} \right] 
$$

$$
- \frac{32\pi\alpha^3}{27s_w^4M_W^4s} \Im \left[ \int \frac{d^{d-1}r}{(2\pi)^{d-1}} \frac{1}{E - \frac{r^2}{M_W} + i\Gamma_W^{(0)}} \ln^2 \left( -2E - \frac{r^2}{M_W} + i\Gamma_W^{(0)} \right) \right].
$$

(D.40)

The formally divergent integral in (D.40) is finite in dimensional regularisation. Taking the limit $\epsilon \to 0$ after performing the $\vec{r}$-integration, we obtain the final result

$$
\Delta \sigma^{(1)}_{R,e^+e^-} = \frac{5\pi^2\alpha^3}{81s_w^4s} \Im \left[ -\sqrt{-\frac{E + i\Gamma_W^{(0)}}{M_W}} \right] + \frac{4\alpha^3}{27s_w^4s} \Im \left[ -\sqrt{-\frac{E + i\Gamma_W^{(0)}}{M_W}} \right] 
$$

$$
\times \left[ 2\ln^2 \left( -\frac{8(\frac{E + i\Gamma_W^{(0)}}{\mu})}{\frac{8(E + i\Gamma_W^{(0)})}{\mu}} \right) - 8\ln \left( -\frac{8(E + i\Gamma_W^{(0)})}{\mu} \right) + 16 + \frac{2}{3\pi^2} \right]
$$

$$
= \frac{8\alpha^3}{27s_w^4s} \Im \left[ -\sqrt{-\frac{E + i\Gamma_W^{(0)}}{M_W}} \right] \ln^2 \left( -\frac{8(E + i\Gamma_W^{(0)})}{\mu} \right) 
$$

$$
- 4\ln \left( -\frac{8(E + i\Gamma_W^{(0)})}{\mu} \right) + 8 + \frac{13}{24\pi^2} \right].
$$

(D.41)

This coincides with $\Im[\Delta A^{(1,\text{fin})}_{\text{soft}}]/(27s)$, with $\Delta A^{(1,\text{fin})}_{\text{soft}}$ given by equation (4.42).
Appendix E

Hard-collinear corrections

In this appendix we compute the coefficient $C^{(1)}_{h-c}$ encoding the contribution of hard-collinear photons to the forward-scattering amplitude, equation (4.52). Even though the coefficient $C^{(1)}_{h-c}$ can be formally seen as a (purely-real) correction to the hard matching coefficient $C_{p,LR}$, it actually contains both the contribution of what we would define a NLO hard-collinear matching coefficient $C^{(1)}_{p,h-c}$ and of effective-theory diagrams with the same topologies shown in Figure 4.6, that do not vanish in the hard-collinear approximation. Since both contributions have the factorised structure (4.52), we find more practical to directly give the sum of the two terms. Note that there are no diagrams with hard-collinear photons connecting different vertices, since this would set the intermediate $W$-propagators off-shell.

The coefficient $C^{(1)}_{h-c}$ is extracted from the threshold expansion of full SM diagrams for the on-shell process $e^- e^+ \rightarrow W^- W^+$. The one-loop diagrams relevant for the determination of the contribution of hard-collinear modes with direction $n_-$ are shown in Figure E.1. The diagrams for the $n_+$ hard-collinear region are obtained by exchanging electron and positron legs.

For the calculation we set the external $W$ momenta to $k_1 = k_2 = M_W (1, \vec{0})$, as in the hard matching-coefficient calculation, and the incoming electron and positron momenta to $p_{1,2} = M_W n_\pm + m_e^2 / (2M_W) n_\pm$, as in the soft-collinear calculation. The definition of a hard-collinear momentum $(q_0 \sim M_W, q^2 \sim m_e^2)$ implies the following scaling for the momentum components in the parameterisation (4.55):

$$q_+ \sim M_W, \quad q_\perp \sim m_e, \quad q_- \sim m_e^2.$$

(E.1)

Accordingly, the relevant propagators entering the expression of the non-vanishing hard-collinear loops are given, at leading order in $\delta$ and $m_e^2/M_W^2$, by

$$q^2 + i\epsilon \rightarrow q_+ q_- + q_\perp^2 + i\epsilon,$$

$$(p_1 - q)^2 - m_e^2 + i\epsilon \rightarrow -2M_W \left( q_- + \frac{m_e^2}{4M_W^2} q_+ \right) + q_+ q_- + q_\perp^2 + i\epsilon,$$

$$(p_2 \pm q)^2 - m_e^2 + i\epsilon \rightarrow \pm 2M_W q_+ + i\epsilon.$$
Figure E.1: Non-vanishing diagrams contributing to the non-hard-collinear region in unitary gauge.

\[
(p_1 - k_1 - q)^2 + i\epsilon \to -2M^2_W \left(1 - \frac{q_+}{2M_W}\right) + M^2_W + i\epsilon, \\
(p_1 + p_2 - q)^2 + i\epsilon \to s \left(1 - \frac{q_+}{2M_W}\right) + i\epsilon, \\
(p_1 + p_2 - q)^2 - M^2_Z + i\epsilon \to s \left(1 - \frac{q_+}{2M_W}\right) - M^2_Z + i\epsilon, \\
(k_1 - q)^2 - M^2_W + i\epsilon \to -q_+ M_W + i\epsilon. 
\]  
(E.2)

Note that there is no need of self-energy resummation, since the interaction of a non-hard-collinear photon with an external on-shell $W$ leads to a configuration with virtuality $(k_1 - q)^2 - M^2_W \sim -2M_W q_0 \sim M^2_W$. In the following we use the unitary gauge to avoid unphysical degrees of freedom. Note however that $C^{(1)}_{h-c}$ is a gauge-invariant quantity, as already seen for the hard matching coefficient $C^{(1)}_p$. 

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The contribution of diagram h-c2 to the $C_{h-c}^{(1)}$ reads (before performing the threshold expansion)

$$C_{h-c}^{(1)} = -\frac{i\pi\alpha\bar{\mu}^{2\epsilon}}{4M_W^4(1 - \epsilon)} \int \frac{d^d q}{(2\pi)^d} \text{Tr} [\gamma^\alpha(p_2)\gamma^\mu(p_1 - q - k_1)]$$

$$\times \gamma^{\nu}(p_1 - q)(\gamma^\alpha\gamma^\nu(p_1 - k_1)) P_{\nu\nu}(k_1) P_{\mu\nu}(k_2)$$

$$\times \frac{1}{(q^2 + i\epsilon)((p_1 - q)^2 - m_c^2 + i\epsilon)((p_2 + q)^2 - m_c^2 + i\epsilon)((p_1 - q - k_1)^2 + i\epsilon)}.$$

(E.3)

where $P_{\mu\nu}(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}$ is the $W$-boson polarisation tensor in unitary gauge. Expanding expression (E.3) according to (E.1), and using (E.2), we can rewrite (E.3) as

$$C_{h-c}^{(1)} = -i16\pi\alphaM_W^4\bar{\mu}^{2\epsilon} \int_{-\infty}^{\infty} dq_+ \int_{-\infty}^{\infty} dq_- \int \frac{d^{d-2}q_\perp}{(2\pi)^{d-2}} \frac{(1 - \frac{q_+}{2M_W})^2}{(q_+^2 + q_+q_- + i\epsilon)(2M_Wq_+ + i\epsilon)}$$

$$\times \left(\frac{q_+^2}{2M_W^2} q_- - \frac{m_c^2}{2M_W^2} q_+ + \frac{q_+q_- + i\epsilon}{2M_W^2} \right) \left(-M_W^2 + M_Wq_+ + i\epsilon\right).$$

(E.4)

The trace of Dirac matrices has been performed with the Mathematica package Tracer [116], and evaluates to $64M_W^6(1 - \epsilon)(1 - q_+/2M_W)^2$. As in the soft-collinear case we define $x = \frac{q_+}{2M_W}$, $y = \frac{q_+}{2M_W}$, and $l^2 = -q_+^2$, and recast (E.4) into

$$C_{h-c}^{(1)} = -\frac{2\alpha}{\pi^2}M_W^2 \frac{e^{\gamma\epsilon}\mu^{2\epsilon}}{\Gamma(1 - \epsilon)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int dl l^{1-2\epsilon} \frac{(1 - x)^2}{(-l^2 + 4M_W^2 xy + i\epsilon)(x + i\epsilon)}$$

$$\times \frac{1}{(-m_c^2 x - 4M_W^2 y(1 - x) - l^2 + i\epsilon)(-1 + 2x + i\epsilon)}.$$

(E.5)

We start performing the $y$ integration by means of Cauchy theorem. Here, contrary to the soft-collinear case, the integral is non-vanishing if $x(x - 1) < 0$, i.e. for $0 < x < 1$, since for $x(x - 1) > 0$ the poles of the propagators lay in the same half-plane. For $0 < x < 1$ we extract the residue in the lower half-plane, $y = \frac{l^2}{2x} - i\epsilon$:

$$C_{h-c}^{(1)} = -\frac{\alpha e^{\gamma\epsilon}\mu^{2\epsilon}}{\pi \Gamma(1 - \epsilon)} \int_0^1 dx \int dl l^{1-2\epsilon} \frac{(1 - x)^2}{(x + i\epsilon)(-1 + 2x + i\epsilon)(-x^2m_c^2 - l^2 + i\epsilon)}.$$

(E.6)

The integration over $l$ is straightforward and results in

$$C_{h-c}^{(1)} = \frac{\alpha}{2\pi} e^{\gamma\epsilon}\Gamma(\epsilon) \left(\frac{m_c^2}{\mu^2}\right)^{-\epsilon} \int_0^1 dx x^{1-2\epsilon} \frac{(1 - x)^2}{-1 + 2x + i\epsilon}.$$

(E.7)

To compute (E.7) we first expand the integrand to the required order in $\epsilon$, and then integrate the individual terms. Note that the expansion generates distributions in $x,$
rather than simple functions:
\[
\int_0^1 dx \frac{(1-x)^2 x^{-1-2\epsilon}}{1 + 2x + i\epsilon} = \int_0^1 dx \frac{x^{-1-2\epsilon}}{1 + 2x + i\epsilon} - 2 \int_0^1 dx \frac{x^{-2\epsilon}}{1 + 2x + i\epsilon} \\
+ \int_0^1 dx \frac{x^{1-2\epsilon}}{1 + 2x + i\epsilon} \\
= -\frac{1}{2\epsilon} \int_0^1 dx \frac{\delta(x)}{1 + 2x + i\epsilon} + \int_0^1 dx \frac{1}{x + 1 + 2x + i\epsilon} \\
- 2\epsilon \int_0^1 dx \frac{1}{x + 1 + 2x + i\epsilon} + 4\epsilon \int_0^1 dx \frac{\log x}{1 + 2x + i\epsilon} \\
+ \int_0^1 dx \frac{x \log x}{1 + 2x + i\epsilon} \\
= \frac{1}{2\epsilon} + \frac{1}{2} - \frac{i\pi}{4} + \epsilon \left( 1 - \frac{\pi^2}{8} - \frac{i\pi}{2} \log 2 \right) + O(\epsilon^2). \tag{E.8}
\]

Combining equations (E.7) and (E.8) we finally obtain
\[
C^{(1)}_{\rho, h-c2} = \frac{\alpha}{4\pi} e^{\gamma\epsilon} \Gamma(\epsilon) \left( \frac{m^2}{\mu^2} \right)^{-\epsilon} \left[ \frac{1}{\epsilon} + 1 - \frac{i\pi}{2} + 2\epsilon \left( 1 - \frac{\pi^2}{8} - \frac{i\pi}{2} \log 2 \right) \right]. \tag{E.9}
\]

The remaining diagrams are computed with analogous methods. For h-c3 we start with the expression
\[
C^{(1)}_{h-c3} = -\frac{i\pi\alpha \tilde{\mu}^{2\epsilon}}{4M_W^2 (1 - \epsilon)} \int \frac{d^4q}{(2\pi)^d} Tr \left[ p_2 \gamma^\mu (p_1 - q - k) \gamma^\nu (p_1 - q) \gamma^\alpha \gamma^\nu (p_1 - k) \gamma^\mu P_L \right] \\
\times P_{\nu\nu'}(k_1) P_{\sigma\mu'}(k_2) P_{\mu\rho}(k_2 - q) \left( g_5^\rho (k_2 + q) + g_5^\sigma (q - 2k_2) + g_5^\rho (k_2 - 2q) \right) \\
\times \frac{1}{(q^2 + i\epsilon) ((p_1 - q)^2 - m_c^2 + i\epsilon) ((k_2 - q)^2 - M_W^2 + i\epsilon) ((p_1 - q - k_1)^2 + i\epsilon)}, \tag{E.10}
\]

which after expansion, simplification of the Dirac algebra, and integration over \( y \) and \( l \) reduces to
\[
C^{(1)}_{\rho, h-c3} = -\frac{\alpha}{2\pi} e^{\gamma\epsilon} \Gamma(\epsilon) \left( \frac{m^2}{\mu^2} \right)^{-\epsilon} \int_0^1 dx x^{-1-2\epsilon} \left( 1 - x \right)^2 - 1 + 2x + i\epsilon, \tag{E.11}
\]
thus exactly cancelling the contribution of diagram h-c2, equation (E.7).
The propagator celled by similar terms in the numerator, and equations (E.14) and (E.15) simplifies to

\[ C^{(1)}_{\text{h-c4}} = -\frac{i\alpha s^2_{\nu} \mu^2}{2M_W^2 (1 - \epsilon)} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ p_2^\alpha (\not{p}_1 - \not{q}) \gamma^\beta \gamma^\nu (\not{p}_1 - \not{q}_1) \gamma^\mu P_L \right] \]

\[ \times P_{\nu\nu'}(k_1) P_{\mu\nu'}(k_2) \left( 2g^{\mu\nu} g_{\alpha\beta} - g^{\mu}_{\alpha} g^{\nu}_{\beta} - g^{\mu}_{\beta} g^{\nu}_{\alpha} \right) \]

\[ \times \frac{1}{(q^2 + i\epsilon) ((p_1 - q)^2 - m_\tau^2 + i\epsilon) ((p_1 + p_2 - q)^2 + i\epsilon) ((p_1 - q_1)^2 - M_W^2 + i\epsilon)} \cdot \] (E.12)

The propagator \((p_1 + p_2 - q)^2 + i\epsilon)\) is cancelled by a factor \((1 - \frac{q}{2M_W})\) originating from the trace over the Dirac matrices. After expanding (E.12) and performing the \(y\) and \(l\) integrations we are led to the final simple expression

\[ C^{(1)}_{\text{h-c4}} = -\frac{\alpha}{4\pi} s^2_{\nu} e^{\gamma_E} \Gamma(e) \left( \frac{m_\tau^2}{\mu^2} \right)^{\epsilon - \epsilon} \int_0^1 dx x^{-\epsilon} \]

\[ = -\frac{\alpha}{4\pi} s^2_{\nu} e^{\gamma_E} \Gamma(e) \left( \frac{m_\tau^2}{\mu^2} \right)^{\epsilon - \epsilon} \cdot \] (E.13)

Diagrams h-c5 and h-c6 give respectively the following (unexpanded) contributions

\[ C^{(1)}_{\text{h-c5}} = -\frac{i\alpha s^2_{\nu} \mu^2}{2M_W^2 (1 - \epsilon)} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ p_2^\alpha (\not{p}_1 - \not{q}) \gamma^\beta \gamma^\nu (\not{p}_1 - \not{q}_1) \gamma^\mu P_L \right] \]

\[ \times P_{\rho\sigma}(q - k_1) P_{\nu\nu'}(k_1) P_{\mu\nu'}(k_2) \left( g_{\beta\sigma}(q + k_1)\rho + g_{\rho}^\sigma(q - 2k_1)\beta + g_{\beta}^\rho(k_1 - 2q)_\nu \right) \]

\[ \times (g^\sigma_{\mu}(k_2 + 2k_1 - 2q)_\mu + g^\mu_{\sigma}(q - k_1 + k_2)_\sigma + g_{\mu\alpha}(-2k_2 - k_1 + q)^\sigma) \]

\[ \times \frac{1}{(q^2 + i\epsilon) ((p_1 - q)^2 - m_\tau^2 + i\epsilon) ((p_1 + p_2 - q)^2 + i\epsilon) ((q - k_1)^2 - M_W^2 + i\epsilon)} \]

(E.14)

\[ C^{(1)}_{\text{h-c6}} = -\frac{i\alpha s^2_{\nu} \mu^2}{2M_W^2 (1 - \epsilon)} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ p_2^\alpha (\not{p}_1 - \not{q}) \gamma^\beta \gamma^\nu (\not{p}_1 - \not{q}_1) \gamma^\mu P_L \right] \]

\[ \times (g_{\alpha\nu}(-q + 2k_1 + k_2)\rho + g_{\rho}^\nu(-q - k_1 + k_2)_\alpha + g_{\nu}^\rho(2q - k_1 - 2k_2)_\rho \)

\[ \times P_{\rho\sigma}(k_2 - q) P_{\nu\nu'}(k_1) P_{\mu\nu'}(k_2) \left( g^\sigma_{\mu}(2q - k_2)_\mu + g^\mu_{\sigma}(2k_2 - q)_\beta + g_{\mu\beta}(-k_2 - q)^\sigma \right) \]

\[ \times \frac{1}{(q^2 + i\epsilon) ((p_1 - q)^2 - m_\tau^2 + i\epsilon) ((p_1 + p_2 - q)^2 + i\epsilon) ((k_2 - q)^2 - M_W^2 + i\epsilon)} \]

(E.15)

In both cases the propagators \((p_1 + p_2 - q)^2 + i\epsilon)\) and \((q - k_{1,2})^2 - M_W^2 + i\epsilon)\) are cancelled by similar terms in the numerator, and equations (E.14) and (E.15) simplifies to

\[ C^{(1)}_{\text{h-c5}} = C^{(1)}_{\text{h-c6}} = -\frac{\alpha}{8\pi} s^2_{\nu} e^{\gamma_E} \Gamma(e) \left( \frac{m_\tau^2}{\mu^2} \right)^{\epsilon - \epsilon} . \] (E.16)
Thus the sum of diagrams h-c4, h-c5 and h-c6 vanishes.

The contributions of diagrams h-c7, h-c8 and h-c9 can be easily derived from h-c4, h-c5 and h-c6 by replacing one photon propagator with a $Z$-boson propagator, and substituting the $\gamma W^- W^+$ coupling $ie$ with the $ZW^- W^+$ coupling $-igc_w$. For h-c7 one obtains

$$C_{(1)}^{(1)} = \frac{\alpha}{4\pi} s_w c_w e^{\gamma E} \Gamma(\epsilon) \left( \frac{m_e^2}{\mu^2} \right)^{-\epsilon} \int_0^1 dx \frac{(1-x)x^{-2\epsilon}}{1-x - \frac{M_Z^2}{4M_W^2} + i\epsilon}$$

$$= \frac{\alpha}{4\pi} s_w c_w e^{\gamma E} \Gamma(\epsilon) \left( \frac{m_e^2}{\mu^2} \right)^{-\epsilon} \left\{ 1 + \frac{M_Z^2}{4M_W^2} \left( \log \left( \frac{4M_W^2 - M_Z^2}{M_Z^2} \right) - i\pi \right) + 2\epsilon \left[ 1 + \frac{M_Z^2}{4M_W^2} \text{Li}_2 \left( \frac{4M_W^2 - M_Z^2 + i\epsilon}{4M_W^2 - M_Z^2} \right) \right] + O(\epsilon^2) \right\}, \quad (E.17)$$

and for h-c8 and h-c9

$$C_{(1)}^{(1)} = C_{(1)}^{(1)} = -\frac{C_{(1)}^{(1)}}{2}. \quad (E.18)$$

As for h-c4, h-c5 and h-c6, the sum of the three diagrams vanishes.

We therefore conclude that the only non-vanishing correction originates from diagram h-c1. This contribution is

$$C_{(1)}^{(1)} = -\frac{i\pi s_w c_w e^{\gamma E} \Gamma(\epsilon)}{4M_W^2 (1 - \epsilon)} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ \gamma^\mu (p_1 - k_1) \gamma^\nu (p_1 - q) \gamma^\alpha p_1 \gamma^\nu (p_1 - k_1) \gamma^\mu P_L \right]$$

$$\times \delta_{\nu \rho} (k_1 - q) P_{\sigma \nu'} (k_1 P_{\mu \nu'} (k_2) (g_\alpha (q + k_1) \rho + g_\sigma (q - k_1) \alpha + g^\sigma (k_1 - 2q) \sigma)$$

$$\times \frac{1}{(q^2 + i\epsilon) ((p_1 - q)^2 - m_e^2 + i\epsilon) ((q - k_1)^2 - M_W^2 + i\epsilon)}, \quad (E.19)$$

and can be simplified to

$$C_{(1)}^{(1)} = -\frac{\alpha}{2\pi} e^{\gamma E} \Gamma(\epsilon) \left( \frac{m_e^2}{\mu^2} \right)^{-\epsilon} \int_0^1 dx x^{-1-2\epsilon} (1 - x)$$

$$= \frac{\alpha}{4\pi} e^{\gamma E} \Gamma(\epsilon) \left( \frac{m_e^2}{\mu^2} \right)^{-\epsilon} \frac{1}{\epsilon(1 - 2\epsilon)}. \quad (E.20)$$

To equation (E.20) we have to add the hard-collinear contribution to the external-field renormalisation, $\delta Z_{e,h-c}$. This is extracted from the hard-collinear expansion of the
electron self-energy

\[-i \Sigma(p) = \tilde{\mu}^2 \int \frac{d^4q}{(2\pi)^4} \frac{ie\gamma^\mu(i\not{p} - i\not{q})ie\gamma^\nu}{((p-q)^2 - m_e^2 + i\epsilon)(q^2 + i\epsilon)} - ig_{\mu\nu}\]

\[= -e^2 \tilde{\mu}^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma^\mu(p-q)\gamma^\nu}{((p-q)^2 - m_e^2 + i\epsilon)(q^2 + i\epsilon)} \]

\[= 2(1 - \epsilon)e^2 \tilde{\mu}^2 \int \frac{d^4q}{(2\pi)^4} \frac{(p-q)\gamma^\mu((p-q)^2 - m_e^2 + i\epsilon)}{(q^2 + i\epsilon)} \]

\[= (1 - \epsilon)e^2 \tilde{\mu}^2 \int_{-\infty}^{\infty} dq_+ \int_{-\infty}^{\infty} dq_- \int \frac{d^4q}{(2\pi)^4} \]

\[\times \left(1 - \frac{q_+}{2M_W}\right) \frac{m_e^2}{2M_W q_+ - 2M_W q_- + q_+^2 + q_-^2 + i\epsilon} \frac{1}{(q_+^2 + q_-^2 + i\epsilon)} \right). \tag{E.21}\]

This expression is straightforwardly integrated, and leads to

\[-i \Sigma(p) = i\frac{\alpha}{4\pi} e^{\epsilon\gamma_E} \frac{\Gamma(\epsilon)}{1 - 2\epsilon} \left(\frac{m_e^2}{\mu^2}\right)^{-\epsilon}. \tag{E.22}\]

Thus we have

\[\delta Z_{\varepsilon,bc} \equiv \frac{d\Sigma(p)}{dy\mid_{p^2=0}} = -\frac{\gamma}{4\pi} e^{\epsilon\gamma_E} \frac{\Gamma(\epsilon)}{1 - 2\epsilon} \left(\frac{m_e^2}{\mu^2}\right)^{-\epsilon}, \tag{E.23}\]

which translates in the following correction to the coefficient \(C_{b,c}^{(1)}\):

\[C_{\delta Z_{\varepsilon,bc}}^{(1)} = -\frac{\gamma}{8\pi} e^{\epsilon\gamma_E} \frac{\Gamma(\epsilon)}{1 - 2\epsilon} \left(\frac{m_e^2}{\mu^2}\right)^{-\epsilon}. \tag{E.24}\]

The total coefficient \(C_{b,c}^{(1)}\), including contribution from hard-collinear photons along both directions \(n_-\) and \(n_+\), is equal to twice the sum of the results (E.20) and (E.24),

\[C_{b,c}^{(1)} = \frac{\alpha}{4\pi} e^{\epsilon\gamma_E} \Gamma(\epsilon) \frac{2 - \epsilon}{\epsilon(1 - 2\epsilon)} \left(\frac{m_e^2}{\mu^2}\right)^{-\epsilon} \]

\[= \frac{\alpha}{2\pi} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[-2 \ln \left(\frac{m_e}{\mu}\right) + \frac{3}{2}\right] + 2 \ln^2 \left(\frac{m_e}{\mu}\right) - 3 \ln \left(\frac{m_e}{\mu}\right) + \frac{\pi^2}{12} + 3 \right]. \tag{E.25}\]
Appendix F

Renormalisation of the Coulomb potential by hard corrections

In this appendix we discuss several technical aspects related to the hard corrections to the Coulomb potential used in Section 6.2.5 and their dependence on the renormalisation scheme. The calculation of the corrections to the single-Coulomb exchange requires a matching calculation where one computes the renormalised $W^+W^- \rightarrow W^+W^-$ NLO scattering amplitude in the full theory for $(p_1 + p_2)^2 = 4M_W^2$ and compares to the one-loop $\Omega^+\Omega^- \rightarrow \Omega^+\Omega^-$ amplitude in the effective theory. Equivalently, the full-theory calculation can be split into contributions from different momentum regions. The relevant regions are the hard, potential, soft and semi-soft regions. The only contribution that will not be reproduced by diagrams in the EFT is that from the hard region, so for the matching calculation it is sufficient to calculate the hard corrections. At leading order in the non-relativistic expansion only the corrections to the single-Coulomb exchange diagram contribute. In Section F.1 we define our renormalisation conventions, the hard corrections to the process $W^+W^- \rightarrow W^+W^-$ are discussed in Section F.2. The relevant results in the $\alpha(M_Z)$ and $G_\mu$ input parameter schemes are collected in Section F.3.

F.1 Charge renormalisation

The lowest perturbative scale relevant near the $W$-pair production threshold is the $W$ width $\Gamma_W$, so we will employ a renormalisation scheme $S$ for the electric charge that is not sensitive to smaller scales, in particular not to the light-fermion masses. In practice we will use $\alpha(M_Z)$ or the Fermi constant $G_\mu$ as input parameter (see e.g. [78]), but for the moment we will leave the renormalisation scheme unspecified. Following the renormalisation conventions of [47] the overall one-loop counterterm of the $W^+W^- \rightarrow W^+W^-$ amplitude in the full theory in a given charge-renormalisation scheme $S$ is given by

$$\Delta_{\text{counter}}^S = [\text{tree}] \times (2\delta Z_e^S + 2\delta Z_W),$$

(F.1)

since the tree-level single photon exchange diagram (denoted by [tree]) is proportional to $e^2 = (4\pi\alpha)^2$ and has four external $W$-legs.
In the following we write the charge counterterm in a given scheme $S$ as the counterterm in the $\alpha(0)$ scheme and a finite scheme-dependent shift

$$\delta Z^S_e = \delta Z^{\alpha(0)}_e - \frac{1}{2} \Delta \alpha^S,$$  

with the explicit expressions for the charge counterterm in the $\alpha(0)$ scheme [79]\(^1\)

$$\delta Z^{\alpha(0)}_e = -\frac{1}{2} \delta Z_{AA} - \frac{s_w}{c_w} \frac{1}{2} \delta Z_{ZA} = -\frac{1}{2} \frac{\partial \Pi^{AA}(k^2)}{\partial k^2} \bigg|_{k^2=0} + \frac{s_w}{c_w} \frac{\Pi^{AZ}(0)}{M_Z^2},$$  

where the transverse self-energies are defined by the decomposition

$$\Pi^{VV}_{\mu\nu}(q) = \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \Pi^{VV}(q^2) + \frac{q_\mu q_\nu}{q^2} \Pi^{VV}_L(q^2).$$  

For the argument given below we assume that light-fermion masses are used as IR regulators in the $\alpha(0)$ scheme, but this will only be used in intermediate steps and the dependence on the light-fermion masses drops out in the end.

### F.2 Hard corrections

As discussed at the beginning of this appendix we need to calculate the hard corrections to the $W^+W^- \rightarrow W^+W^-$ amplitude for external momenta directly at threshold. The contributing diagrams are of the form of box corrections (figure F.1a), vertex corrections (figure F.1b) and self-energy insertions (figure F.1c,d). The box corrections with a hard loop momentum do not contribute at the order we are considering since near threshold they are suppressed by a factor $v$ compared to the diagrams with a Coulomb photon. This leaves the vertex and the bubble corrections.

Figure F.1: Sample diagrams in the full SM contributing to the hard corrections to the $WW \rightarrow WW$ subprocess

\(^1\)Here all conventions are the same as in [79] apart from replacing $\Sigma \rightarrow -\Pi$. Note that [79] defines the vacuum polarisation $\Pi^{AA}(k^2) = \Sigma^{AA}(k^2)/k^2$ which we don’t use in the following.
Expanding the vertex-correction diagrams of the form of Figure F.1b in the region where the photon and the W's attached to the vertex sub-loop are potential and the momentum running in the vertex loop is hard, one obtains the single-Coulomb exchange diagram with an insertion of the one-loop WWγ vertex function evaluated with on-shell external momenta. The renormalised on-shell vertex function vanishes in the conventional on-shell renormalisation scheme where α(0) is used as input parameter.\(^2\) Therefore the diagrams with insertion of a one-loop unrenormalised WWγ vertex are given by the negative of the corresponding counterterm in the on-shell renormalisation scheme:

\[
\Delta_{\text{vertex}} = \text{[tree]} \times (-2\delta_{\text{os}}^{\text{ext}}_{WW\gamma}) = \text{[tree]} \times (-2) \left( \delta Z_{e}^{(0)}(k^2) + \delta Z_{W} + \frac{1}{2} \delta Z_{AA} - \frac{1}{2} \frac{c_w}{s_w} \delta Z_{ZA} \right).
\]  

(F.5)

A similar argument shows that in the hard region the self-energies in Figures F.1c/d are evaluated at zero external momentum. Since the (unrenormalised) one-loop photon self-energy vanishes at zero external momentum the first non-vanishing contributions of diagrams of the form F.1c comes from expanding the self-energy to first order:

\[
\Delta_{\gamma} = \text{[tree]} \times (i\Pi_{\text{hard}}^{AA}(k^2))^{-i} \left( \frac{\partial \Pi_{\text{heavy}}^{AA}(k^2)}{\partial k^2} \right)_{k^2=0} = \text{[tree]} \times \frac{\partial \Pi_{\text{AA}}^{\text{Z}}(k^2)}{\partial k^2} \left|_{k^2=0} \right) \right) \right),
\]  

(F.6)

where \(\Pi_{\text{heavy}}^{AA}\) includes all particles except the light fermions. Here it was used that for a hard loop momentum the light-fermion masses must be set to zero so the loop integral is scaleless and only the heavy particles contribute to the hard part of the self-energy at zero external momentum. The unrenormalised photon-Z mixing diagrams give (suppressing the transverse projectors)

\[
\Delta_{\gamma/Z} = \text{[tree]} \times 2 \left( i\Pi_{T}^{AA}(0) \right) \left( \frac{-i}{k^2 - M_{Z}^2} \right) \left( \frac{g_{ZWW}}{g_{WW}} \right) = \text{[tree]} \times \left( \delta Z_{ZA} \right) \left( \frac{-c_w}{s_w} \right),
\]  

(F.7)

since for a potential momentum \(k^2 \ll M_{Z}^2\). Here the on-shell scheme definition \(\delta Z_{ZA} = -2\Pi_{T}^{AA}(0)/M_{Z}^2\) has been used. We also used that massless fermion loops do not contribute to the \(\gamma - Z\) mixing at zero external momenta, as can be seen from the explicit one-loop result in \([79]\). Therefore \(\Pi_{T,\text{heavy}}^{AA}(0) = \Pi_{T}^{AA}(0)\).

Adding the vertex correction, the \(\gamma/Z\) mixing and the counterterm (F.1) (where ‘tree’ is again only the photon exchange diagram) one obtains the hard correction in the renormalisation scheme \(S\):

\[
\Delta_{\text{hard}}^{S} = \Delta_{\text{vertex}} + \Delta_{\gamma} + \Delta_{\gamma/Z} + \Delta_{\text{counter}}^{S} \equiv \text{[tree]} \times \delta_{\text{hard}}^{S},
\]  

(F.8)

where we defined the correction factor

\[
\delta_{\text{hard}}^{S} = 2(\delta Z_{e}^{S} - \delta Z_{e}^{(0)}) - \delta Z_{AA} + \frac{\partial \Pi_{\text{heavy}}^{AA}(k^2)}{\partial k^2} \bigg|_{k^2=0} = -\Delta^{S} - \frac{\partial \Pi_{f}^{AA}(k^2)}{\partial k^2} \bigg|_{k^2=0}.
\]  

(F.9)

\(^2\)This relies on ‘charge universality’ in the standard model, i.e. on the fact that the on-shell electron-photon vertex and the on-shell W-photon vertex receive the same radiative corrections \([117]\).
In this expression the derivative of the light-fermion contribution to the photon self-energy depends on light-fermion masses used as regulators in the $\alpha(0)$ scheme. This dependence will be cancelled by a similar term in the conversion factor $\Delta\alpha^S$ for a scheme $S$ that is not sensitive to scales below $\Gamma_W$.

Alternatively, the result (F.9) is obtained by considering the individual renormalised contributions instead of applying the overall counterterm (F.1). In this case

$$\Delta_{\text{hard}}^S = \Delta_{\text{vertex}}^r + \Delta_{\gamma/Z}^r .$$

Since the on-shell renormalised $Z - \gamma$ mixing two-point function vanishes at zero momentum we have $\Delta_{\gamma/Z}^r = 0$. The renormalised one-loop correction to the $WW\gamma$ vertex vanishes in the on-shell scheme so the only non-vanishing contribution comes from the change in the charge-counterterm:

$$\Delta_{\text{vertex}}^r = 2 \text{[tree]} \times (\delta Z_e^S - \delta Z_e^{\alpha(0)}).$$

The renormalised self-energy correction to the photon-exchange is given by (F.6) and the corresponding counterterm:

$$\Delta_{\gamma}^r = [\text{tree}] \times \left. \left( \frac{\partial \Pi_{\text{heavy}}^{AA}(k^2)}{\partial k^2} \right) \right|_{k^2 = 0} - \delta Z_{AA}.$$

### F.3 Formulas for the $\alpha(M_Z)$ and $G_\mu$ schemes

We now specialise the result (F.9) to the two schemes used in the main text. In the $\alpha(M_Z)$ scheme the finite shift of the charge counterterm is given by (see e.g. [79])

$$\Delta\alpha_{M_Z} = - \frac{\partial \Pi_{\vec{f}\neq t}^{AA}(k^2)}{\partial k^2} \bigg|_{k^2 = 0} + \frac{\text{Re} \, \Pi_{\vec{f}\neq t}^{AA}(M_Z^2)}{M_Z^2}.$$

Inserting this definition into (F.9), the sensitivity on the light-fermion masses drops out and one obtains the final result for the hard corrections to the Coulomb potential (charge counterterm) in the $\alpha(M_Z)$ scheme:

$$\delta_{\text{hard}}^{\alpha(M_Z)} = - \frac{\text{Re} \, \Pi_{\vec{f}\neq t}^{AA}(M_Z^2)}{M_Z^2} = \sum_f C_f Q_f^2 \left( \frac{\alpha}{\pi} \right) e^{\gamma_E} \Gamma(\epsilon) \frac{\Gamma^2(2 - \epsilon)}{\Gamma(4 - 2\epsilon)} \left( \frac{M_Z^2}{\mu^2} \right)^{-\epsilon}.$$

Here $C_f = 1$ for the leptons and $C_f = N_c = 3$ for the quarks, and $Q_f$ is the electric charge of $f$ in units of $e$, such that $\sum_f C_f Q_f^2 = 20/3$.

In the $G_\mu$ scheme the shift in the counterterm is instead given by the correction to
Muon decay, $\Delta r$,

$$\Delta \alpha_{\mu} = \Delta r = -\frac{\partial \Pi_{f}^{AA}(k^2)}{\partial k^2}|_{k^2=0} - 2 \frac{\delta s_w}{s_w} \frac{c_w}{s_w} \frac{\Pi_{f}^{AZ}(0)}{M^2_Z} - 2 \frac{\Pi_{f}^{W}(0) - \Re \Pi_{f}^{W}(M^2_W)}{M^2_W} + \delta r,$$

(F.15)

$$\delta s_w = \frac{1}{2} \frac{c_w^2}{s_w^2} \left( \frac{\Re \Pi_{f}^{W}(M^2_W)}{M^2_W} - \frac{\Re \Pi_{f}^{Z}(M^2_Z)}{M^2_Z} \right),$$

(F.16)

$$\delta r = \frac{\alpha}{4 \pi s^2_w} \left( 6 + \frac{7 - 4 s^2_w}{2 s^2_w \ln c^2_w} \right).$$

(F.17)

The explicit result for the hard corrections in the $\alpha_{\mu}$ scheme from (F.9) reads

$$\delta_{G\mu}^{\text{hard}} = -\Delta r - \frac{\partial \Pi_{f}^{AA}(k^2)}{\partial k^2}|_{k^2=0}$$

$$= \frac{\partial \Pi_{\text{heavy}}^{AA}(k^2)}{\partial k^2}|_{k^2=0} + 2 \frac{\delta s_w}{s_w} \frac{c_w}{s_w} \frac{\Pi_{f}^{AZ}(0)}{M^2_Z} + \frac{\Pi_{f}^{W}(0) - \Re \Pi_{f}^{W}(M^2_W)}{M^2_W} - \delta r.$$ 

(F.18)

To convert the result from the $\alpha(M_Z)$ scheme to the $G\mu$ scheme we have to add

$$\delta_{\alpha(M_Z)\rightarrow G\mu} = \delta_{G\mu}^{\text{hard}} - \delta_{\text{hard}}^{\alpha(M_Z)} = \frac{\Re \Pi_{f}^{AA}(M^2_Z)}{M^2_Z} + \frac{\partial \Pi_{\text{heavy}}^{AA}(k^2)}{\partial k^2}|_{k^2=0}$$

$$+ 2 \frac{\delta s_w}{s_w} \frac{c_w}{s_w} \frac{\Pi_{f}^{AZ}(0)}{M^2_Z} + \frac{\Pi_{f}^{W}(0) - \Re \Pi_{f}^{W}(M^2_W)}{M^2_W} - \delta r.$$ 

(F.19)

Explicit expressions for the self-energies appearing in these quantities can be found for instance in [79].
Bibliography


Acknowledgements

First of all, I would like to thank my advisor Martin Beneke for supervision in the last three years, and for encouraging and helping me to widen my scientific horizons. I also thank Stefano Actis, Christian Schwinn, Adrian Signer and Giulia Zanderighi for fruitful collaboration on the projects that are at the core of this thesis, and I am particularly grateful to Christian Schwinn for helpful comments on the manuscript. Furthermore, I would like to thank all the members of the Institute for Theoretical Particle Physics in Aachen, for useful discussions and suggestions, and for establishing a friendly and encouraging atmosphere. I am greatly indebted to Anke Bachmich for her invaluable help in many occasions. Finally, a big thanks goes to my family for their moral support in these three years.
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