Blow-up in a degenerate parabolic equation
with gradient nonlinearity

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Introduction

We study positive classical solutions of the degenerate parabolic problem

\[
\begin{align*}
  u_t & = u^p \Delta u + u^q \kappa |\nabla u|^2 & \text{in } \Omega \times (0, T), \\
  u|_{\partial \Omega} & = 0, \\
  u|_{t=0} & = u_0,
\end{align*}
\]

where \( p > 0, \quad q > 1, \quad r > -1 \quad \text{and} \quad \kappa \in \mathbb{R} \)

are fixed real parameters,

\[ \Omega \subset \mathbb{R}^n \quad \text{is a bounded domain of class } C^3 \]

(with \( n \in \mathbb{N} \)), \( T \in (0, \infty] \) is a positive time and \( u_0 \) is a given initial function fulfilling

\[ u_0 \in C^0(\bar{\Omega}) \quad \text{with} \quad u_0 > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0|_{\partial \Omega} = 0. \]

Applications of equations with this type of degeneracy, which degenerate at points where the solution \( u \) is zero, can be found in biology and physics. In [All] the principal part \( u^p \Delta u \) in the special case \( p = 1 \) is used to model the biased diffusion processes in the evolution of epidemics. One equation arising in this context is

\[ u_t = u \Delta u + \alpha u(1 - \beta u) \]

with \( \alpha, \beta > 0 \), where \( u \) denotes the respective pathogen density.

Moreover, in [GuMacC] a model for the spatial diffusion of biological populations is developed, where the population density \( u \) satisfies the equation

\[ u_t = K \Delta (u^\alpha) + \mu u \equiv K \alpha u^{\alpha-1} \Delta u + \mu u + K \alpha (\alpha - 1) u^{\alpha-2} |\nabla u|^2 \]

with \( K, \mu > 0 \) and \( \alpha \geq 2 \).

In addition, in [Low] the equation

\[ u_t = u^2 \Delta u + u^3 \]

for dimension \( n = 1 \), which covers the case \( \kappa = 0 \), describes a model for the resistive diffusion process of a force-free magnetic field in a passive medium. Furthermore, in [BBCP] the filtration-absorption equation

\[ u_t = u \Delta u - \mu |\nabla u|^2 \]
for \( n = 1 \) and \( \mu > 0 \) models the groundwater flow in a water-absorbing fissurized porous rock.

Apart from that, (0.1) is a generalization of the forced porous medium equation

\[
\frac{\partial u}{\partial t} = \Delta u^m + u^{q-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2
\]  

(0.5)

with \( m > 1 \), which can be transformed into (0.1) with \( p = \kappa = m - 1 \) and \( r = m - 2 \). This equation has intensively been studied during the last three decades. Results can for example be found in the book [SGKM] and the references therein.

Another similar equation is

\[
\frac{\partial u}{\partial t} = u^p \Delta u^q,
\]  

(0.6)

with \( p > 0 \) and \( q > 0 \), where the gradient term is absent. In case of \( p < 1 \), by the substitution \( v(x,t) := u^{1-p}(x, \frac{t}{1-p}) \) the solution of (0.6) is transformed into a solution of the forced porous medium equation \( \frac{\partial v}{\partial t} = \Delta v^{\frac{1}{1-p}} + v^{\frac{q-p}{1-p}} \). To the best of our knowledge, the first results for \( p \geq 1 \) concerning the question, whether the solutions are global in time or blow up in finite time (in the \( L^\infty \) norm), were found in [FriMcL2]. Especially it was shown that \( q = p + 1 \) is the critical exponent with respect to blow-up (see [SGKM], [Wie1], [Wie2] and [Win3]). Furthermore, the behavior in case of blow-up like blow-up set, blow-up rate and blow-up profile was studied (see [SGKM], [Wie3], [Win4] and [Win5]) and the asymptotic behavior of global solutions was described (see [SGKM], [Wie2], [Win6] and [Win7]).

One main aspect of this thesis is to show the influence of the additional gradient term in (0.1) with respect to blow-up in finite time. In case of \( \kappa > 0 \) this gradient term is a source and can possibly enforce blow-up, whereas for \( \kappa < 0 \) it is an absorption term and can possibly prevent blow-up. In context of diffusion equations, the phenomenon of finite-time blow-up has been studied extensively. In particular, results concerning the question whether a negative gradient term can prevent blow-up and how this term influences the properties of the solutions have been established. Especially the Chipot-Weissler equation

\[
\frac{\partial u}{\partial t} = \Delta u + u^q - \mu |\nabla u|^s,
\]

with \( q > 1 \), \( s \geq 1 \) and \( \mu > 0 \), has raised attention. It was introduced in [ChiWei]. For an overview we refer to the survey paper [Sou2] and the references given there (see e.g. [CFQ], [Fil], [KawPel], [Sou1]). In particular, the exponent \( s = q \) is critical with respect to finite-time blow-up, because blow-up in finite time only occurs in case of \( s > q \) (see [SouWei1], [SouWei2]). Furthermore, in [Bar] the equation

\[
\frac{\partial u}{\partial t} = \Delta u + u^q - \mu u^r |\nabla u|^s
\]

with \( r, s > 0 \), \( r + s \geq 1 \), \( q > 1 \) and \( \mu > 0 \) is considered and it is shown that the exponent \( q = r + s \) is critical with respect to blow-up in finite time, in the sense that it is important whether the difference \( q - r - s \) is positive or nonpositive. The
same equation with $r < 0$ and $s = 2$ is studied in [Sou3] (among other equations of a more general class) with respect to the influence of gradient perturbations on blow-up asymptotics.

The influence of a positive gradient term with respect to blow-up has raised less attention. For the equation

$$u_t = \Delta u + a u^q + b|\nabla u|^s$$

with $q, s > 1$ and $a, b > 0$ the existence of nonnegative global solutions for small initial data is shown in [STW] and finite-time blow-up for large initial data is proved in [HesMoa]. Similar results have been shown in [Che] for another class of equations, where especially the equation

$$u_t = \Delta u + u^q + \kappa u^r |\nabla u|^2$$

with $q > 1, r > 0$ and $\kappa > 0$ is covered.

In view of degenerate diffusion, the equation

$$u_t = \Delta u^m + u^q - |\nabla u^\alpha|^s$$

is considered in [AMST] and the existence of global weak solutions for sufficiently regular initial data is shown in case of $m \geq 1, \alpha > \frac{m}{2}, 1 \leq s < 2$ and $1 \leq q < \alpha s$. Moreover, in [SouWei1] it was shown that nonnegative solutions of the equation

$$u_t = u^p \Delta u + u^q - \mu u^r |\nabla u|^s$$

with $q > p + 1 \geq 2, r \geq 1, s \geq 1, r + s < q$ and $\mu \in \mathbb{R}$ blow up in finite time for large initial data.

This thesis is structured in the following way:

In Chapter I, we prove the existence of a maximal classical solution of (0.1) by approximating this solution with solutions of strictly parabolic problems. Moreover, we give a partial result concerning uniqueness of classical solutions of (0.1) and show some properties of the maximal solution.

One main purpose of this thesis is to study the question whether the maximal solution is global in time or blows up in finite time. In Chapter II we give the results in case of $\kappa > 0$, where the gradient term acts like a source. We especially prove that besides the exponent $q = p + 1$, which is the critical exponent for (0.6), $r = 2p - q$ is another critical exponent. Furthermore, the size of the domain plays an important role with respect to blow-up, in contrast to most constellations for (0.6).

In Chapter III we deal with the influence of a negative gradient term ($\kappa < 0$) which acts like an absorption term. We prove that besides $q = p + 1$ the exponent $r = q - 2$ is critical with respect to blow-up in this case. Moreover, we study the question
whether the global solutions of (0.1) converge to 0 as $t \to \infty$. In particular, we have discovered that the exponent $r = q - 2$ is critical with respect to this question. Finally, we study the size of the blow-up set for blowing up solutions for $\kappa > 0$, in the case that the domain $\Omega$ is a ball centered at 0 and the initial data are radially symmetric and nonincreasing with respect to $|x|$. It is proved in Chapter IV that the solutions blow up in a single point, if $q > \max\{p+1, r+2\}$ is fulfilled. Otherwise the blow-up set is shown to have a positive Lebesgue measure.

The results of Chapter 2 (including Lemma 1.3.4) are published in [StiWin1] and they were partly discovered by M. Winkler (see the beginning of Chapter 2 for more details).

At this point, I would like to take the opportunity to express my gratitude to Prof. Dr. Michael Wiegner for supervising my thesis and supporting me. His comments have helped to improve part of the results. Furthermore, I would like to thank Dr. Michael Winkler for many discussions touching various topics and questions that are investigated here. Moreover, thanks to Ellen Behnke, Tatjana Gerzen, Dr. Hans Jürgen Heep and, in particular, Kianhwa Djie for creating a pleasant atmosphere in which I really enjoyed working.

**Notation**

Let $\mathbb{R}$ denote the field of real numbers and define $\mathbb{N} := \{1, 2, 3, \ldots\}$ to be the set of naturals as well as $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, for $n \in \mathbb{N}$, $\mathbb{R}^n := \mathbb{R} \times \cdots \times \mathbb{R}$ stands for the cartesian product with $n$ factors. The Euclidean scalar products of the vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ will be written as $x \cdot y := \sum_{i=1}^n x_i y_i$ and the corresponding norm is defined as $|x| := \sqrt{x \cdot x}$. An open ball in $\mathbb{R}^n$ with radius $R > 0$ and center $x$ is denoted by $B_R(x) := \{y| |y - x| < R\}$. For a nonempty set $G \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we let $\text{dist}(x, G) := \inf_{y \in G} |x - y|$ denote the distance between $x$ and $G$. If $G \subset \mathbb{R}^n$ is a domain, then $\text{diam}(G) := \sup_{x,y \in G} |x - y|$ stands for the diameter of $G$. For $G \subset \mathbb{R}^n$ we let $\overline{G}, G^\circ$ and $\partial G$ denote the closure, interior and boundary of $G$, respectively. If $F, G \subset \mathbb{R}^n$ are such that $\overline{F} \subsetneq G$, this will be written as $F \subsetneq G$.

For a Lebesgue-measurable set $G \subset \mathbb{R}^n$, the $n$-dimensional Lebesgue measure of $G$ is labelled $|G|$ and

\[
L^\infty(G) := \{u : G \to \mathbb{R} | \text{u measurable}, \|u\|_{L^\infty(G)} < \infty\}
\]

denotes the Lebesgue space with the norm $\|u\|_{L^\infty(G)} := \text{esssup}_{x \in G} |u(x)|$.

For a given $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and a real-valued function $u(x) = u(x_1, \ldots, x_n)$ (defined on a subset $G$ of $\mathbb{R}^n$), we define $|\beta| := \sum_{i=1}^n \beta_i$ and write $D^\beta u(x) := \frac{\partial^{|eta|} u(x)}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$.
for the classical partial derivatives. Moreover, we abbreviate \( u_{x_i} := \frac{\partial u}{\partial x_i} \) and let \( \nabla u := (u_{x_1}, \ldots, u_{x_n}) \) denote the gradient and \( \Delta u := \sum_{i=1}^nu_{x_ix_i} \) the Laplacian of the function \( u \). If \( u \) depends on a "spatial" variable \( x \in G \subset \mathbb{R}^n \) and a "time" variable \( t \in I \subset \mathbb{R} \), the expressions \( \nabla u, \Delta u \) and \( D^\alpha_u \) only refer to differentiation with respect to \( x \). Given an open set \( G \subset \mathbb{R}^n \), a point \( x \in \partial G \) and a function \( u : G \cup \{x\} \rightarrow \mathbb{R} \), then let \( \frac{\partial}{\partial n}u(x) \) be the directional derivative along the outward unit normal \( N \) at \( x \), if it exists.

For an open bounded set \( G \subset \mathbb{R}^n \), \( k \in \mathbb{N}_0 \) and \( \alpha \in (0,1) \) we define

\[
C^k(G) := \{ u : G \rightarrow \mathbb{R} \mid D^\beta u \text{ exists and is continuous } \forall \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq k \},
\]

which is equipped with the norm

\[
\|u\|_{C^k(G)} := \sum_{|\beta| \leq k} \sup_{x \in G} |D^\beta u(x)|
\]

and hence becomes a Banach space. For arbitrary sets \( G \subset \mathbb{R}^n \), we denote

\[
C^k(G) := \bigcap_{K \subset \subset G, K \text{ open and bounded}} C^k(K).
\]

This set becomes a Fréchet space, when it is equipped with the family of local seminorms \( \|u\|_{C^k(K)} \), where \( K \subset \subset G \) is open and bounded. The Fréchet space is defined to be \( C^\alpha_{loc}(G) \). Furthermore, we set

\[
C^\infty(G) := \bigcap_{k \in \mathbb{N}} C^k(G) \text{ and } C^\infty_0(G) := \{ u \in C^\infty(G) \mid \text{supp } u \subset \subset G \}
\]

for arbitrary \( G \subset \mathbb{R}^n \), where we denote the support of a function \( u : G \rightarrow \mathbb{R} \) by \( \text{supp } u := \{ x \in G \mid u(x) \neq 0 \} \).

Moreover, for open and bounded subsets \( G \subset \mathbb{R}^n \) and \( I \subset \mathbb{R} \) as well as \( \alpha \in (0,1) \) we define the classical parabolic function spaces

\[
C^{2,1}(\bar{G} \times \bar{I}) := \{ u : G \times I \rightarrow \mathbb{R} \mid D_x^\beta \partial_t^j u \text{ exists and } D_x^\beta \partial_t^j u \in C^0(\bar{G} \times \bar{I}) \}
\]

\[
\forall \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0 \text{ such that } |\beta| + 2j \leq 2,
\]

\[
C^{\alpha,2}(\bar{G} \times \bar{I}) := \{ u \in C^0(\bar{G} \times \bar{I}) \mid \|u\|_{C^{\alpha,2}(\bar{G} \times \bar{I})} < \infty \},
\]

\[
C^{2+\alpha,1+2}(\bar{G} \times \bar{I}) := \{ u \in C^{2,1}(\bar{G} \times \bar{I}) \mid D_x^\beta \partial_t^j u \in C^{\alpha,2}(\bar{G} \times \bar{I}) \forall \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0 \text{ such that } |\beta| + 2j \leq 2 \},
\]

equipped with the norms

\[
\|u\|_{C^{2,1}(G \times I)} := \sum_{|\beta| + 2j \leq 2} \|D_x^\beta \partial_t^j u\|_{C^0(G \times I)},
\]

\[
\|u\|_{C^{\alpha,2}(G \times I)} := \|u\|_{C^0(G \times I)} + \sup_{x,y \in G, x \neq y} \sup_{t,s \in I, t \neq s} \frac{|u(x,t) - u(y,s)|}{|x-y|^{\alpha} + |t-s|^{2}},
\]

\[
\|u\|_{C^{2+\alpha,1+2}(G \times I)} := \sum_{|\beta| + 2j \leq 2} \|D_x^\beta \partial_t^j u\|_{C^{\alpha,2}(G \times I)}.
\]
respectively. These function spaces are again Banach spaces and the corresponding set
\[ C^{2,1}(G \times I) := \bigcap_{K \subseteq G, J \subseteq I; \text{K,J open and bounded}} C^{2,1}(\bar{K} \times \bar{J}), \]
for general \( G \subset \mathbb{R}^n \) and \( I \subset \mathbb{R} \), becomes the Fréchet space \( C^{2,1}_{loc}(G \times I) \) by equipping it with the corresponding local seminorms.

The boundary \( \partial G \) of an open set \( G \subset \mathbb{R}^n \) is of class \( C^k \) (\( k \in \mathbb{N} \cup \{\infty\} \)), if for every \( x \in \partial G \) there is a neighborhood \( U(x) \) of \( x \) and a \( C^k \)-diffeomorphism \( F : U(x) \to B_1(0) \), \( B_1(0) \subset \mathbb{R}^n \), such that \( F(U(x) \cap G) = \{ y = (y_1, \ldots, y_n) \in B_1(0) \mid y_n > 0 \} \) and \( F(U(x) \cap \partial G) = \{ y = (y_1, \ldots, y_n) \in B_1(0) \mid y_n = 0 \} \). A bounded domain \( G \) is said to be of class \( C^k \), if its boundary \( \partial G \) is of class \( C^k \). Moreover, a bounded domain of class \( C^3 \) is called a smoothly bounded domain.

Furthermore, for an arbitrary smoothly bounded domain \( G \subset \mathbb{R}^n \) we let \( \lambda_1(G) \) denote the first Dirichlet eigenvalue of \(-\Delta\) in \( G \), corresponding to the normalized principal eigenfunction \( \Theta = \Theta(x; G) \) satisfying \( \max_{x \in \bar{G}} \Theta(x; G) = 1 \).

If the function \( u : G \times I \to \mathbb{R} \), where \( G \subset \mathbb{R}^n \) and \( I \subset \mathbb{R} \) is radially symmetric, we define \( \rho := |x| \) and we will switch between the notation \( u = u(x,t) \) and \( u = u(\rho,t) \), if this is convenient.
Chapter 1

The maximal solution

We prove in this chapter that positive classical solutions of the degenerate parabolic problem (0.1) indeed exist and study the question of uniqueness of these solutions. Similar to other degenerate parabolic problems (see e.g. [Wie2]), one classical solution of (0.1) can be approximated by solutions of certain strictly parabolic problems. More precisely, by parabolic standard arguments (see e.g. [LSU]) there is for any \( \varepsilon > 0 \) a unique solution \( u_\varepsilon \) of the problem

\[
\begin{align*}
    u_{\varepsilon t} &= u_\varepsilon^p \Delta u_\varepsilon + u_\varepsilon^q + \kappa u_\varepsilon^r |\nabla u_\varepsilon|^2 \quad \text{in } \Omega \times (0, T_\varepsilon), \\
    u_\varepsilon|_{\partial \Omega} &= \varepsilon, \\
    u_\varepsilon|_{t=0} &= u_{0\varepsilon},
\end{align*}
\]

(1.1)

where, for a given function \( u_0 \) satisfying (0.4), we choose \( u_{0\varepsilon} \in C^3(\bar{\Omega}) \) such that

\[ u_0 + \varepsilon \leq u_{0\varepsilon} \leq u_0 + 2\varepsilon \quad \text{in } \Omega \]

is fulfilled. Since \( u_0 \) is positive in the whole domain \( \Omega \), we can adapt standard arguments from related degenerate problems to show that these functions \( u_\varepsilon \) decrease, as \( \varepsilon \searrow 0 \), to a classical solution \( u \) of (0.1). This solution \( u \) is in fact a maximal solution of (0.1), in the sense that any positive classical solution \( v \) of (0.1) satisfies \( v \leq u \) as long as both solutions exist.

The question of uniqueness of positive classical solutions seems to be delicate. In the case of the related equation (0.6) without the gradient term uniqueness of classical solutions has been shown in [Wie2]. However, in (0.1) the additional gradient term causes some problems. We prove the uniqueness only in case of \( r \geq p - q \) and \( \kappa > 0 \). This is done by transforming the solutions of (0.1) into solutions of a problem without gradient term, for which uniqueness has already been shown (see e.g. [Win1]). In the other cases we are neither able to show the uniqueness of positive classical solutions nor to prove the existence of another positive classical solution apart from the maximal solution. Concerning the latter aspect, a result of nonuniqueness of certain weak solutions has been obtained e.g. in [DalPLu] for a degenerate problem without gradient term.
CHAPTER 1. THE MAXIMAL SOLUTION

Since the question of uniqueness is not completely solved, we will mainly focus on the maximal solution during the following chapters. For this solution we can use the approximation by solutions of (1.1) to overcome for example the difficulties of the comparison principle which arise for solutions of (0.1). Therefore, we state some properties of the maximal solution which will be used in the following chapters. We show that the maximal solution is smooth in $\Omega \times (0, T)$ and that it remains radially symmetric and radially nonincreasing, if this is satisfied by the initial function $u_0$. Furthermore, we give a condition on the initial data $u_0$ to ensure that the maximal solution $u$ fulfills $u_t \geq 0$ in $\Omega \times (0, T)$. The latter property will be used to prove blow-up in Chapter 2 as well as to study the size of the blow-up set in Chapter 4.

1.1 Existence of a maximal solution

First we give a suitable comparison principle, which is proved e.g. in [Sti]. We remark that this comparison principle does not enable us to conclude the uniqueness of positive classical solutions of (0.1). Even in case of $p \geq 1$ and $r \geq 1$, when all terms at the right hand side of the differential equation in (0.1) are locally Lipschitz continuous with respect to $u$, the spatial derivatives $u_{x_i}$ of the solutions can become unbounded near the boundary $\partial \Omega$. In particular it has been shown in [StiWin2] that in case of $1 \leq q \leq p - 1$, $1 \leq r < p - 1$ and $\kappa \geq 0$ for suitably chosen initial data any positive classical solution $u$ of (0.1) is bounded in $\Omega \times (0, \infty)$ and fulfills $\sup_{x \in \Omega} |\nabla u(x, t_0)| = \infty$ with some finite $t_0 \in (0, \infty)$. Hence, especially in this case the comparison principle fails to provide uniqueness of classical solutions for (0.1).

Lemma 1.1.1 (Comparison Principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and let

$$ F(u) := u_t - \sum_{i,j=1}^{n} a_{ij}(x, t, u, \nabla u)u_{x_i x_j} - f(x, t, u, \nabla u) $$

 denote a parabolic differential operator with continuous functions $f$ and $a_{ij}$ such that

$$ \sum_{i,j=1}^{n} a_{ij}(x, t, u, \nabla u)\xi_i \xi_j \geq 0 $$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Assume that, for $l \in \{1, 2\}$, $u_l \in C^0(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T])$ and $F(u_l)$ is defined in $\Omega \times (0, T]$ with

$$ F(u_1) \leq F(u_2) \quad \text{in} \quad \Omega \times (0, T], $$

$$ u_1 \leq u_2 \quad \text{on} \quad (\partial \Omega \times [0, T]) \cup (\Omega \times \{0\}). $$

Then

$$ u_1 \leq u_2 \quad \text{in} \quad \Omega \times [0, T], $$

provided that at least for one $l_0 \in \{1, 2\}$ additionally either
1.1. EXISTENCE OF A MAXIMAL SOLUTION

(a) \( a_{ij}(x,t,u,p) \) and \( f(x,t,u,p) \) are Lipschitz continuous with respect to \( u \) and \( p \) in a neighborhood of \( u_{0e}(\bar{\Omega} \times [0,T]) \times \nabla u_{0e}(\Omega \times (0,T)) \) and

\[
\| \nabla u_{0e} \|_{L^\infty(\Omega \times (0,T))} + \sum_{i,j=1}^{n} \| (u_{0e})_{x_{i}x_{j}} \|_{L^\infty(\Omega \times (0,T))} < \infty,
\]

or

(b) \( a_{ij}(x,t,u,p) \) and \( f(x,t,u,p) \) are Lipschitz continuous with respect to \( u \) and \( p \) in a neighborhood of \( u_{0e}(M) \times \nabla u_{0e}(M) \) for any \( M \subset \subset \Omega \times (0,T) \) and

\[
u_{1} < u_{2} \quad \text{on} \quad (\partial \Omega \times [0,T]) \cup (\Omega \times \{0\}).
\]

Now we show the existence of a positive classical solution of (0.1) by using the approximation procedure which is described at the beginning of this chapter. Moreover, we prove that this solution is in fact the maximal solution of (0.1).

**Theorem 1.1.2** Let conditions (0.2), (0.3) and (0.4) be fulfilled.

(i) For every \( \varepsilon > 0 \) there exists a function \( u_{0\varepsilon} \in C^{3}(\bar{\Omega}) \) with \( u_{0} + \frac{\varepsilon}{2} \leq u_{0\varepsilon} \leq u_{0} + 2\varepsilon \) in \( \Omega \) and \( u_{0\varepsilon}|_{\partial\Omega} = \varepsilon \) and a unique solution \( u_{\varepsilon} \in C^{0}(\bar{\Omega} \times [0,T_{\varepsilon}]) \cap C^{2,1}(\Omega \times (0,T_{\varepsilon})) \) of (1.1) with maximal existence time \( T_{\varepsilon} \in (0,\infty) \). Moreover, for any positive decreasing sequence \( (\varepsilon_{k})_{k \in \mathbb{N}} \) it is possible to choose \( u_{0\varepsilon_{k}} \) such that additionally \( (u_{0\varepsilon_{k}})_{k \in \mathbb{N}} \) is a decreasing sequence.

(ii) Furthermore, there exists a solution \( u \in C^{0}(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T)) \) of (0.1) with \( T := \lim_{k \to \infty} T_{\varepsilon_{k}} \in (0,\infty] \), which is positive in \( \Omega \times (0,T) \), such that \( u_{\varepsilon_{k}}(x,t) \searrow u(x,t) \) as \( k \to \infty \) for every \( (x,t) \in \bar{\Omega} \times [0,T) \) and \( u_{\varepsilon_{k}} \to u \) in \( C^{0}_{loc}(\bar{\Omega} \times [0,T)) \cap C^{2,1}_{loc}(\Omega \times (0,T)) \) as \( k \to \infty \), where \( (\varepsilon_{k})_{k \in \mathbb{N}} \) is a decreasing sequence satisfying \( \varepsilon_{k} \searrow 0 \) as \( k \to \infty \).

(iii) Moreover, \( T \) is the maximal existence time of \( u \) and additionally, in case of \( T < \infty \), \( \limsup_{t < T} \| u(\cdot,t) \|_{L^\infty(\Omega)} = \infty \) is fulfilled.

(iv) \( u \) is the maximal solution of (0.1) in the sense that any positive solution \( v \in C^{0}(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T)) \) of (0.1) with \( \tau \in (0,\infty] \) satisfies \( v \leq u \) in \( \Omega \times [0,\min\{T,\tau\}) \).

**Proof.**

(i) We fix \( \varepsilon > 0 \). Then the function \( v_{\varepsilon} := (u_{0} - \frac{\varepsilon}{4})_{+} \) fulfills \( v_{\varepsilon} \in C^{0}(\bar{\Omega}) \) with \( \text{supp}(v_{\varepsilon}) \subset \subset \Omega \) due to the properties of \( u_{0} \). Hence, there is \( w_{\varepsilon} \in C^{0}_{0}(\Omega) \) with \( \|w_{\varepsilon} - v_{\varepsilon}\|_{C^{0}(\bar{\Omega})} \leq \frac{\varepsilon}{4} \). Thus \( u_{0\varepsilon} := w_{\varepsilon} + \varepsilon \in C^{3}(\bar{\Omega}) \) satisfies \( u_{0} + \frac{\varepsilon}{2} \leq u_{0\varepsilon} \leq u_{0} + 2\varepsilon \) in \( \Omega \) and \( u_{0\varepsilon}|_{\partial\Omega} = \varepsilon \). For \( u_{0} \in C^{3}(\bar{\Omega}) \) we can choose \( u_{0\varepsilon} = u_{0} + \varepsilon \).
(Furthermore, for any given positive decreasing sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) we can choose \(u_{0k}\) such that additionally \(u_0 + \frac{\varepsilon_k + \varepsilon_{k+1}}{2} \leq u_{0k} \leq u_0 + \frac{\varepsilon_k + \varepsilon_{k-1}}{2}\) is fulfilled for every \(k \geq 2\), which is possible with an appropriate choice of \(v_{\varepsilon_k}\) and \(w_{\varepsilon_k}\). Then \((u_{0k})_{k \in \mathbb{N}}\) is a decreasing sequence.)

Moreover, we choose \(\rho_\varepsilon \in C^\infty(\mathbb{R})\) such that
\[
\rho_\varepsilon(z) := \begin{cases} 
    z^p, & \text{for } z \geq \frac{\varepsilon}{2} \\
    z^q, & \text{for } z \leq \frac{\varepsilon}{4} 
\end{cases}
\]
and \(\rho_\varepsilon(z) \geq \frac{\varepsilon}{4}\) for \(z \in \mathbb{R}\).

For \(s > -1\) and \(M \geq \varepsilon + 1\) we choose \(\varphi_{s,M,\varepsilon} \in C^\infty(\mathbb{R})\) such that
\[
\varphi_{s,M,\varepsilon}(z) := \begin{cases} 
    z^s + 1, & \text{for } z \geq M + 1 \\
    z^s, & \text{for } z \in [\frac{\varepsilon}{2}, M] \\
    0, & \text{for } z \leq 0 
\end{cases}
\]
and \(\varphi_{s,M,\varepsilon}(z) \geq 0\) for \(z \in \mathbb{R}\).

By Theorem VI.4.4 in [LSU] there is \(T_{M,\varepsilon} > 0\) such that for \(M := \|u_{0\varepsilon}\|_{C^0(\Omega)} + 1\) the problem
\[
\begin{align*}
(u_{M,\varepsilon})_t &= \rho_\varepsilon(u_{M,\varepsilon}) \Delta u_{M,\varepsilon} + \varphi_{q,M,\varepsilon}(u_{M,\varepsilon}) + \kappa \varphi_{r,M,\varepsilon}(u_{M,\varepsilon}) |\nabla u_{M,\varepsilon}|^2 \\
(u_{M,\varepsilon})_\Omega &= \varepsilon, \\
(u_{M,\varepsilon})_{t=0} &= u_{0\varepsilon}
\end{align*}
\]
in \(\Omega \times (0, T_{M,\varepsilon})\),

has a solution \(u_{M,\varepsilon} \in C^0(\Omega \times [0, T_{M,\varepsilon}]) \cap C^{2,1}(\Omega \times (0, T_{M,\varepsilon}))\) with \(u_{M,\varepsilon} \leq M\) in \(\Omega \times [0, T_{M,\varepsilon})\). By comparison we have \(u_{M,\varepsilon} \geq \frac{\varepsilon}{2}\) in \(\Omega \times (0, T_{M,\varepsilon})\), so that \(u_{M,\varepsilon}\) is a local solution of (1.1). Hence, we can extend the local solution \(u_{M,\varepsilon}\) to a solution \(u_\varepsilon \in C^0(\Omega \times [0, T_{\varepsilon}]) \cap C^{2,1}(\Omega \times (0, T_{\varepsilon}))\) of (1.1) with maximal existence time \(T_\varepsilon \in (0, \infty]\).

Furthermore, the solution of (1.1) is unique by comparison, because \(|\nabla u_\varepsilon|\) and \(\Delta u_\varepsilon\) are bounded in \(\Omega \times (0, T)\) for every \(T \in (0, T_\varepsilon]\) by some constant depending on \(\varepsilon\) by Lemma VI.3.1, Theorem V.4.2, Theorem V.5.1 and Remark V.5.2 in [LSU].

(ii) Fixing a decreasing sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) with \(\varepsilon_k \searrow 0\) as \(k \to \infty\) and choosing \(u_{0k}\) such that \((u_{0k})_{k \in \mathbb{N}}\) is a decreasing sequence, we conclude \(u_{\varepsilon_k} \geq u_{\varepsilon_j} \geq 0\) in \(\Omega \times [0, T_{\varepsilon_k}]\) by comparison and hence \(T_{\varepsilon_k} \leq T_{\varepsilon_j}\) for \(k, j \in \mathbb{N}\) with \(k < j\).

Thus there exists \(T := \lim_{k \to \infty} T_{\varepsilon_k} \in (0, \infty]\) and for every \((x, t) \in \Omega \times [0, T)\) there exists \(u(x, t) \geq 0\) such that \(u_{\varepsilon_k}(x, t) \searrow u(x, t)\) as \(k \to \infty\).
1.1. EXISTENCE OF A MAXIMAL SOLUTION

Let \( x_0 \in \Omega \) and \( R > 0 \) be chosen such that \( B := B_R(x_0) \subset \subset \Omega \) is satisfied. Since \( p > 0 \) and \( r > -1 \), we have

\[
M_1 := \sup_{x \in B} e^{-p \frac{|x-x_0|^2}{R^2 - |x-x_0|^2}} \frac{2(n+4)R^6}{(R^2 - |x-x_0|^2)^4} < \infty
\]

and

\[
M_2 := \sup_{x \in B} e^{-(r+1) \frac{|x-x_0|^2}{R^2 - |x-x_0|^2}} \frac{4R^6}{(R^2 - |x-x_0|^2)^4} < \infty.
\]

Moreover, there is some small \( c_0 > 0 \) such that \( u_0 \geq c_0 \) in \( \bar{B} \) because \( u_0 \) is positive in \( \Omega \). Choosing \( \gamma := c_0^pM_1 + |\kappa|c_0^{r+1}M_2 \), we define

\[
v_B(x,t) := c_0e^{-\gamma t}e^{-\frac{|x-x_0|^2}{R^2 - |x-x_0|^2}} \quad \text{for} \ (x,t) \in B \times [0, \infty)
\]

and \( v_B := 0 \) on \( \partial B \times [0, \infty) \). Then we have \( u_{\varepsilon k} \geq v_B \) on the parabolic boundary of \( B \times (0, T_{\varepsilon k}) \) and (writing \( y := \frac{|x-x_0|^2}{R^2 - |x-x_0|^2} \))

\[
(v_B)_t - v_B^p \Delta v_B - v_B^r - \kappa v_B^r|\nabla v_B|^2 = -\gamma c_0 e^{-\gamma t} e^{-y}
\]

\[
+ c_0^{p+1} e^{-(p+1)\gamma t} e^{-(p+1)y} \frac{2R^2}{(R^2 - |x-x_0|^2)^4} \left\{ -2R^2 |x-x_0|^2 \right\}
\]

\[
+ n(R^2 - |x-x_0|^2)^2 + 4|x-x_0|^2(R^2 - |x-x_0|^2) \right\}
\]

\[
- c_0^q e^{-q\gamma t} e^{-qy} - \kappa c_0^{r+2} e^{-(r+2)\gamma t} e^{-(r+2)y} \frac{4R^6|x-x_0|^2}{(R^2 - |x-x_0|^2)^4}
\]

\[
\leq c_0 e^{-\gamma t} e^{-y} \left[ -\gamma + c_0^{p+1} e^{-qy} \frac{2(n+4)R^6}{(R^2 - |x-x_0|^2)^4}
\right.
\]

\[
+ |\kappa| c_0^{r+1} e^{-(r+1)y} \frac{4R^6}{(R^2 - |x-x_0|^2)^4} \right]\]

\[
\leq c_0 e^{-\gamma t} e^{-y} \left[ -\gamma + c_0^p M_1 + |\kappa| c_0^{r+1} M_2 \right]
\]

\[
= 0 \quad \text{in} \ B \times (0, T_{\varepsilon k})
\]

for all \( k \in \mathbb{N} \) due to the choice of \( \gamma \) and since \( p > 0 \) and \( r > -1 \). This implies \( u_{\varepsilon k} \geq v_B \) in \( B \times [0, T_{\varepsilon k}) \) for all \( k \in \mathbb{N} \) by comparison. Hence, for any \( \bar{B} \subset \subset B \) and \( T_0 \in (0, T_{\varepsilon k}) \) there is a positive constant \( c > 0 \), which depends on \( \bar{B} \) and \( T_0 \), such that \( u_{\varepsilon k} \geq v_B \geq c \) in \( \bar{B} \times [0, T_0) \).

Now let \( K \subset \subset \Omega \) and \( 0 < t_0 < T_0 < T \). Then there is \( k_0 \in \mathbb{N} \) such that \( T_{\varepsilon k} > T_0 \) for all \( k \geq k_0 \). Thus there is a positive constant \( c \) depending on \( K \) and \( T_0 \) such that \( u_{\varepsilon k} \geq c \) in \( K \times [0, T_0] \) for every \( k \geq k_0 \), because \( K \) can be covered with a finite number of balls. Hence, we get constants \( C > 0 \) and \( \beta \in (0, 1) \) depending on \( K, t_0 \) and \( T_0 \) such that \( \|u_{\varepsilon k}\|_{C^{2+\beta,1+\beta,2}(K \times [t_0,T_0])} \leq C \) for
all \( k \geq k_0 \) by Theorem V.1.1, Theorem VI.3.4 and Theorem VII.5.1 in [LSU]. Therefore, we have \( u_{\varepsilon_k} \rightarrow u \) in \( C^2(\bar{K} \times [t_0, T_0]) \) for \( k \rightarrow \infty \) and, moreover, \( u \in C^2(\Omega \times (0, T)) \) fulfills the differential equation of (0.1) in \( \Omega \times (0, T) \) because \( K, t_0 \) and \( T_0 \) are arbitrary.

Next we show that \( u_{\varepsilon_k} \rightarrow u \) in \( C^0(\bar{\Omega} \times [0, T_0]) \) as \( k \rightarrow \infty \) is fulfilled for all \( T_0 \in (0, T) \). Therefore, we fix \( T_0 \in (0, T) \) and \( \delta > 0 \). Then there is \( k_0 \in \mathbb{N} \) such that \( T_{\varepsilon_k} > T_0 \) and \( \varepsilon_k < \delta \) for \( k \geq k_0 \). Since \( u_{\varepsilon_{k_0}} \) is continuous in \( \bar{\Omega} \times [0, T_0] \) with \( u_{\varepsilon_{k_0}} = \varepsilon_{k_0} \) on \( \partial \Omega \) there is a neighborhood \( U \subset \bar{\Omega} \) of \( \partial \Omega \) with \( u_{\varepsilon_{k_0}} \leq 2\delta \) in \( U \times [0, T_0] \). Hence, we have

\[
0 \leq u \leq u_{\varepsilon_k} \leq 2\delta \quad \text{in } U \times [0, T_0]
\]

for \( k \geq k_0 \), because \( u \geq 0 \) and \( u_{\varepsilon_k}(x, t) \searrow u(x, t) \) for \( (x, t) \in \bar{\Omega} \times (0, T) \) as \( k \rightarrow \infty \).

Furthermore, we fix a smoothly bounded domain \( G \subset \subset \Omega \) such that \( G \cup U = \bar{\Omega} \) and \( \Omega^0 \cap U^0 \neq \emptyset \). Then there is \( \eta \in (0, \delta) \) with \( \eta < \text{dist}(G, \partial \Omega) \) and \( u_0 \geq 2\eta \) in \( \bar{G} \). Moreover, we choose \( \mu \in (0, \eta) \) such that \( |u_0(x) - u_0(y)| \leq \eta \) for all \( x, y \in \Omega \) and \( x_0 \in \bar{G} \) with \( x, y \in B_\mu(x_0) \). For \( x_0 \in \bar{G} \) fixed, we define \( B := B_\mu(x_0) \) and choose \( c_0 := \min_{x \in B} u_0(x) > 0 \). Then we can show like above that \( u_{\varepsilon_k}(x, t) \geq u_B(x, t) \) for \( (x, t) \in \bar{B} \times [0, T_0] \) is fulfilled by comparison for all \( k \in \mathbb{N} \). Hence, there is \( \alpha(x_0) \in (0, \mu) \) with \( u_{\varepsilon_k} \geq c_0 - \eta \) in \( B_{\alpha(x_0)}(x_0) \times [0, \alpha(x_0)) \) for all \( k \in \mathbb{N} \) (since \( u_B(x_0, 0) = c_0 \)). This implies

\[
u_{\varepsilon_k}(x, t) \geq u_0(x) - 2\eta \quad \text{in } B_{\alpha(x_0)}(x_0) \times [0, \alpha(x_0))
\]

for all \( k \in \mathbb{N} \) according to the choice of \( c_0 \) and \( \mu \). Since \( \bar{G} \) is compact, there is \( \alpha > 0 \) such that

\[
u_{\varepsilon_k}(x, t) \geq u_0(x) - 2\eta \quad \text{in } \bar{G} \times [0, \alpha)
\]

for all \( k \in \mathbb{N} \) is fulfilled. Moreover, there is \( \beta \in (0, \alpha) \) with \( u_{\varepsilon_{k_0}} \leq u_0 + 3\delta \) in \( \bar{G} \times [0, \beta) \), because \( k_0 \) is chosen suitably and \( u_{\varepsilon_k} \) is continuous with \( u_{0\varepsilon_k} \leq u_0 + 2\varepsilon_k \). Since \( u_{\varepsilon_k} \searrow u \) for \( k \rightarrow \infty \), this and (1.3) imply

\[
u_0 - 2\delta \leq u \leq u_{\varepsilon_k} \leq u_0 + 3\delta \quad \text{in } \bar{G} \times [0, \beta)
\]

for all \( k \geq k_0 \) (due to \( \eta \leq \delta \)). Moreover, there is \( k_1 \geq k_0 \) with \( |u_{\varepsilon_k} - u| \leq \delta \) in \( \bar{G} \times [\frac{\beta}{2}, T_0] \), because \( u_{\varepsilon_k} \rightarrow u \) in \( C^2(\bar{K} \times [t_0, T_0]) \) for \( k \rightarrow \infty \) for all \( K \subset \subset \Omega \) and all \( t_0 \in (0, T_0) \). Hence we have \( |u_{\varepsilon_k} - u| \leq 5\delta \) in \( \bar{\Omega} \times [0, T_0] \) by (1.2) and (1.4). Thus \( u \in C^0(\bar{\Omega} \times [0, T]) \) with \( u|_{t=0} = u_0 \) and \( u|_{\partial \Omega} = 0 \) is fulfilled since \( T_0 \) and \( \delta \) are arbitrary.
(iii) Moreover, \( T \) is the maximal existence time of \( u \). If this was false, especially \( T < \infty \) and \(|u| \leq M \) in \( \bar{\Omega} \times [0, T) \) would be satisfied for some \( M \in (0, \infty) \). Let \( t_0 \in (0, T) \) be chosen such that there is \( y \in C^2([0, t_0]) \) satisfying \( y'(t) = y''(t) \) for \( t \in [0, t_0] \) and \( y(0) = M + 1 \). Since \( T_{\varepsilon_k} \neq T \) as \( k \to \infty \), there is some \( k_0 \in \mathbb{N} \) such that \( T_{\varepsilon_k} > T := T - \frac{t_0}{2} \) and \(|u_{\varepsilon_k}| < M + 1 \) in \( \bar{\Omega} \times [0, \tilde{T}] \) hold for all \( k \geq k_0 \) by part (ii) of this proof.

We fix \( k \geq k_0 \). Hence, \( w_{0\varepsilon_k}(x) := u_{\varepsilon_k}(x, \tilde{T}) \) for \( x \in \bar{\Omega} \) satisfies \( w_{0\varepsilon_k} \in C^0(\bar{\Omega}) \) with \( w_{0\varepsilon_k} \geq \frac{\varepsilon_k}{2} \) in \( \Omega \) and \( w_{0\varepsilon_k} = \varepsilon_k \) on \( \partial \Omega \). We show that there is a local solution \( w_{\varepsilon_k} \in C^0(\bar{\Omega} \times [0, t_k)) \cap C^{2,1}(\bar{\Omega} \times (0, t_k)) \) of (1.1) with \( w_{\varepsilon_k}|_{t=0} = w_{0\varepsilon_k} \) \( w_{\varepsilon_k} \) can be approximated by solutions \( v_{\delta} \) of (1.1) satisfying \( v_{\delta} = \varepsilon_k + \delta \) on \( \partial \Omega \) and \( v_{\delta}|_{t=0} = v_{0\delta} \in C^3(\bar{\Omega}) \) with \( v_{0\delta} \) uniformly in \( \bar{\Omega} \); for details we refer to part (i) and (ii) of this proof, where a similar approximation has been done for \( u \). Since \( \varepsilon_k \leq w_{\varepsilon_k}(x, t) \leq y(t) \) is fulfilled for \( (x, t) \in \bar{\Omega} \times [0, \min\{t_k, t_0\}) \) by comparison, \( w_{\varepsilon_k} \) is in fact a classical solution of (1.1) in \( \bar{\Omega} \times (0, t_0) \). Hence, \( w_{\varepsilon_k}(x, t) = u_{\varepsilon_k}(x, t + \tilde{T}) \) is satisfied for \( (x, t) \in \bar{\Omega} \times [0, t_0) \), because the solution of (1.1) is unique. Since \( T_{\varepsilon_k} \) is the maximal existence time of \( u_{\varepsilon_k} \), we have in fact \( T_{\varepsilon_k} \geq \tilde{T} + t_0 > T \), which contradicts the definition of \( T \). Thus \( T \) is the maximal existence time of \( u \). Furthermore, this part of the proof shows that \( \limsup_{t \downarrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \) has to be satisfied in case of \( T < \infty \).

(iv) Finally we show that \( u \) is a maximal solution of (0.1). If \( v \in C^0(\bar{\Omega} \times [0, \tau)) \cap C^{2,1}(\bar{\Omega} \times (0, \tau)) \) is a solution of (0.1) for some \( \tau \in (0, \infty) \), then \( v \leq u \) in \( \Omega \times (0, \min\{T, \tau\}) \) is fulfilled by comparison for every \( \varepsilon > 0 \). This implies \( v \leq u \) in \( \Omega \times (0, \min\{T, \tau\}) \) and hence \( u \) is the maximal solution of (0.1).

\[ \text{Remark.} \quad \text{In case of } \kappa > 0, \text{ the existence of a maximal solution of (0.1) can be shown for arbitrary } r \in \mathbb{R}. \text{ The only difference of the proof is that in part (ii) we use } v_B(x, t) := c_0 e^{-\gamma t} \Theta(x) \text{ for } (x, t) \in \bar{B} \times [0, \infty), \text{ with } \Theta := \Theta(\cdot ; B) \text{ and } \gamma := c_0^p \lambda_1(B). \]

Then it can easily be shown that

\[ (v_B)_t - v_B^p \Delta v_B - v_B^q - \kappa v_B^p |\nabla v_B|^2 \leq (v_B)_t - v_B^p \Delta v_B \leq 0 \quad \text{in } B \times (0, \infty) \]

is fulfilled due to \( \kappa > 0 \).

### 1.2 The question of uniqueness

In this section we show that the positive classical solution of (0.1) is unique in case of \( r \geq p - q \) and \( \kappa > 0 \). Therefore we transform the solutions of (0.1) into solutions of a degenerate parabolic problem without gradient term, for which uniqueness has
already been shown. We have to leave open, if in the other cases the positive classical solutions of (0.1) are unique or not.

First we give a Lemma from [Win1], which states that two solutions \( v_1 \) and \( v_2 \) of a suitable problem without gradient term are equal if \( v_1 \) is smaller than \( v_2 \).

**Lemma 1.2.1** Suppose \( \phi \in C^0([0, \infty)) \cap C^1((0, \infty)) \) is positive in \((0, \infty)\) with \( \phi(0) = 0 \), \( \psi \in C^1((0, \infty)) \) such that \( \phi \psi \in C^0([0, \infty)) \) with \( \phi \psi(0) = 0 \). Furthermore assume that for any \( M > 0 \) there is \( C(M) \in (0, \infty) \) such that

\[
|\phi(s)\psi'(s)| + |(\phi \psi)'(s)| \leq C(M) \quad \text{for all } s \in (0, M].
\]

If \( \Omega \) is a smoothly bounded domain, \( v_0 \in C^0(\bar{\Omega}) \) is positive in \( \Omega \) with \( v_0 = 0 \) on \( \partial \Omega \), \( t_0 > 0 \) and \( v_1, v_2 \in C^0(\bar{\Omega} \times [0, t_0]) \cap C^{2,1}(\Omega \times (0, t_0]) \) are positive solutions of the problem

\[
\begin{aligned}
v_t &= \phi(v)(\Delta v + \psi(v)) \quad \text{in } \Omega \times (0, t_0], \\
v|_{\partial \Omega} &= 0, \\
v|_{t=0} &= v_0,
\end{aligned}
\]

satisfying \( v_1 \leq v_2 \) in \( \Omega \times [0, t_0] \), then \( v_1 = v_2 \) in \( \Omega \times [0, t_0] \) is fulfilled.

For a proof we refer to Lemma 1.1.2 in [Win1] and we remark that the condition \( \int_0^\infty \frac{dr}{\phi(r)} = \infty \) from [Win1] is not needed for this proof. Furthermore, it is sufficient for \( \phi \psi \) being locally Lipschitz on \([0, \infty)\) instead of being globally Lipschitz on \([0, \infty)\), since \( v_1 \) and \( v_2 \) are bounded in \( \bar{\Omega} \times [0, t_0] \).

Now we are able to prove that in case of \( r \geq p - q \) and \( \kappa > 0 \) there is a unique positive classical solution of (0.1). The transformation of (0.1), which we use in the following proof, will furthermore be used to prove two particular blow-up results in Chapter 2.

**Theorem 1.2.2** Let conditions (0.2), (0.3) and (0.4) be fulfilled with \( r \geq p - q \) and \( \kappa > 0 \). Then (0.1) has a unique positive solution \( u \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \), defined up to its maximal existence time \( T \in (0, \infty] \).

**Proof.** By Theorem 1.1.2 there is a maximal solution \( u \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T)) \) of (0.1) with maximal existence time \( T \in (0, \infty] \). Suppose \( \tilde{u} \in C^0(\bar{\Omega} \times [0, t_0]) \cap C^{2,1}(\Omega \times (0, t_0]) \) is another positive classical solution of (0.1) for some \( t_0 \in (0, T) \). Then \( u \geq \tilde{u} \) in \( \bar{\Omega} \times [0, t_0] \) holds by Theorem 1.1.2 (iv).

In case of \( r > p - 1 \) let \( f \in C^0([0, \infty)) \cap C^2((0, \infty)) \) denote a solution of the initial value problem

\[
\begin{aligned}
f'(s) &= e^{-\kappa (f(s))^{r-q+1} \over r-p+1} \quad \text{in } (0, \infty), \\
f(0) &= 0.
\end{aligned}
\]
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In case of \( r = p - 1 \), we define \( f(s) := s^{\frac{1}{r-p+1}} \) for \( s \geq 0 \).

In case of \( r < p - 1 \), for \( \beta := r - p + 1 < 0 \) and \( \varepsilon > 0 \) let \( f_{\varepsilon} \in C^0([0,\infty)) \cap C^2((0,\infty)) \) be the unique solution of the initial value problem

\[
\begin{cases}
 f_{\varepsilon}'(s) = e^{-\frac{\varepsilon f_{\varepsilon}(s)}{\beta}} & \text{in } (0,\infty), \\
 f_{\varepsilon}(0) = \varepsilon.
\end{cases}
\]

We now derive estimates for the function \( f_{\varepsilon}(v) \) for \( v > 0 \) which are independent of \( \varepsilon > 0 \) to show that \( f_{\varepsilon} \) converges to a positive solution of (1.5) as \( \varepsilon \searrow 0 \).

Therefore we choose \( c := \max\{\ln\left(\left(\frac{-\beta}{2}\right)^{\frac{1-\beta}{\beta}}\kappa^{-\frac{1}{\beta}}\right), 0\} \) and define

\[ g_1(v) := \left(\frac{\beta}{\kappa}\left[\ln(v) + \frac{\beta - 1}{\beta} \ln(-\ln(v)) - c\right]\right)^{\frac{1}{\beta}} \text{ with } v \in (0, v_0), \]

where \( v_0 \in (0, 1) \) is chosen small enough such that \( \left(\frac{2\beta}{\kappa} \ln(v)\right)^{\frac{1}{\beta}} \leq g_1(v) \leq \left(\frac{\beta}{2\kappa} \ln(v)\right)^{\frac{1}{\beta}} \) holds for \( v \in (0, v_0) \). Then we have

\[
\frac{g_1'(v)}{e^{-\frac{\kappa g_1(v)}{\beta}}} = \frac{\left(\frac{\beta}{\kappa}\left[\ln(v) + \frac{\beta - 1}{\beta} \ln(-\ln(v)) - c\right]\right)^{\frac{1-\beta}{\beta}}}{v^{-1}(-\ln(v))^{\frac{\beta - 1}{\beta}}} e^c \leq \left(\frac{-\beta}{2}\right)^{\frac{1-\beta}{\beta}} \kappa^{-\frac{1}{\beta}} e^{-c} \leq 1
\]

if \( v \in (0, v_0) \), according to the choice of \( v_0 \) and \( c \). As \( f_1(0) = \varepsilon > 0 = g_1(0) \), there exists \( \delta_\varepsilon \in (0, v_0) \) with \( f_\varepsilon(v) \geq g_1(v) \) for \( v \in (0, \delta_\varepsilon) \). Hence we obtain \( f_{\varepsilon}(v) \geq g_1(v) \) for \( v \in (0, v_0) \) by standard comparison results of ordinary differential equations. We remark that the choice of \( v_0 \) only depends on \( \beta \) and \( \kappa \) but not on \( \varepsilon \). Moreover, we have \( f_{\varepsilon}(v) \geq v \) if \( v > 0 \), since \( f_{\varepsilon}'(v) \geq 1 \) for \( v > 0 \).

Since \( f_{\varepsilon}(v) \geq f_{\delta}(v) \) for \( v \geq 0 \) and \( 0 < \delta < \varepsilon \) by comparison, we can define \( f(v) := \lim_{\varepsilon \searrow 0} f_{\varepsilon}(v) \) for \( v \geq 0 \). Due to our estimates for \( f_{\varepsilon} \) we have

\[
\int_0^v |f_{\varepsilon}'(s)| ds = \int_0^v e^{-\frac{\kappa (f_{\varepsilon}(s))}{\beta}} ds \leq \int_0^v s^{-1}(-\ln(s))^{\frac{1-\beta}{\beta}} e^c ds < \infty
\]

with \( v \in (0, v_0) \) and \( \varepsilon > 0 \). Since \( 1 \leq f_{\varepsilon}'(v) \leq e^{-\kappa \frac{v}{\beta}} \) for \( v \geq v_0 \) and \( \varepsilon > 0 \) due to \( f_{\varepsilon}(v) \geq v \) if \( v > 0 \) and \( \varepsilon > 0 \), we see that

\[
\int_0^v f_{\varepsilon}'(s) ds \to \int_0^v e^{-\frac{\kappa (f_{\varepsilon}(s))}{\beta}} ds \text{ as } \varepsilon \searrow 0
\]
for every $v > 0$ by the dominated convergence theorem.

Hence, we have $f(v) = \lim_{\varepsilon \searrow 0} \left( \int_0^v f'(s) ds + \varepsilon \right)$ if $v > 0$ and thus $f \in C^0(0, \infty) \cap C^2((0, \infty))$ is a solution of (1.5). Moreover, $f(v) \geq g_1(v)$ for $v \in (0, v_0)$ and $f(v) \geq v$ is fulfilled for $v > 0$ due to our estimates of $f_v$. (We remark that this is also valid for any $r < p - 1$ (without the restriction $r \geq p - q$), because this will be used in Chapter 2.)

Thus, for any $r \geq p - q$, we have defined a function $f$ fulfilling $f'' = -\kappa f^{r-p}(f')^2$ in $(0, \infty)$. Moreover, $f'(s) > 0$ for $s \in (0, \infty)$, $f$ is positive in $(0, \infty)$ and strictly increasing in $[0, \infty)$. Hence, $v_0 := f^{-1}(u_0) \in C^0(\bar{\Omega})$ is positive in $\Omega$ with $v_0|_{\partial \Omega} = 0$. Moreover, $v, \tilde{v} \in C^0(\bar{\Omega} \times [0, t_0]) \cap C^{2,1}(\Omega \times (0, t_0))$, defined by $u = f(v)$ and $\tilde{u} = f(\tilde{v})$ satisfy

$$f'(v)v_t = (f(v))^p f'(v) \Delta v + (f(v))^q + [\kappa(f(v))'](f'(v))^2 + (f(v))^p f''(v)|\nabla v|^2$$

in $\Omega \times (0, t_0]$ and thus are solutions of

$$\begin{cases}
v_t = (f(v))^p \left( \Delta v + \frac{(f(v))^q}{f'(v)} \right) & \text{in } \Omega \times (0, t_0], \\
v|_{\partial \Omega} = 0, \\
v|_{t=0} = v_0,
\end{cases}$$

because $f'' = -\kappa f^{r-p}(f')^2$ in $(0, \infty)$. Furthermore, $v \geq \tilde{v}$ in $\Omega \times [0, t_0]$ is fulfilled, since $f$ is strictly increasing in $[0, \infty)$.

Defining $\phi(s) := (f(s))^p$ for $s \geq 0$ and $\psi(s) := \frac{(f(s))^{q-p}}{f'(s)}$ if $s > 0$, $\phi$ is positive in $(0, \infty)$ with $\phi(0) = 0$ due to $p > 0$ and $\kappa > 0$. Moreover, $\phi \psi = \frac{\phi'}{f'} \in C^0([0, \infty))$ with $(\phi \psi)(0) = 0$ is fulfilled, because $f'(0) = 1$ in case of $r > p - 1$ and $\frac{1}{f'(s)} = e^{\int_0^s (\frac{f(s)^{r-p+1}}{r-p+1})} \to 0$ as $s \searrow 0$ in case of $r < p - 1$ due to $\kappa > 0$.

In all cases, if $M > 0$ and $s \in (0, M]$ we have

$$|\phi(s)\psi'(s)| = (f(s))^p \left| (q-p)(f(s))^{q-p-1} - (f(s))^{q-p} \frac{f''(s)}{(f'(s))^2} \right|$$

$$\leq |q-p|(f(M))^{q-1} + \kappa(f(M))^{q+r-p}$$

and

$$|\phi \psi)'(s)| = |p(f(s))^{q-1} + (q-p)(f(s))^{q-1} + \kappa(f(s))^{q+r-p}|$$

$$\leq q(f(s))^{q-1} + \kappa(f(M))^{q+r-p} \leq q(f(M))^{q-1} + \kappa(f(M))^{q+r-p}$$

due to $q \geq 1$ and $q + r - p \geq 0$. Hence we can apply Lemma 1.2.1 and get $v = \tilde{v}$ in $\Omega \times [0, t_0]$. This implies $u = \tilde{u}$ in $\Omega \times [0, t_0]$, which yields the claim. \qed
1.3 Some properties of the maximal solution

In this section we state properties of the maximal solution that will be used during the following chapters. Similar results are well-known for many parabolic and degenerate parabolic equations.

The first lemma shows that the maximal solution of (0.1) has in fact more regularity inside Ω. This result is proved in Lemma 2.4 of [Sti] in case of Ω ⊂ \( \mathbb{R} \). The proof of the general case Ω ⊂ \( \mathbb{R}^n \) is analogous, because the results from [LSU], which are used in this proof, are valid for any dimension \( n \in \mathbb{N} \).

**Lemma 1.3.1** Let assumptions (0.2), (0.3) and (0.4) be fulfilled and let \( u \in C^0(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times (0,T)) \) denote the maximal solution of (0.1). Then we have \( u \in C^\infty(\Omega \times (0,T)) \) and, furthermore, in case of \( u_0 \in C^3(\bar{\Omega}) \), \( u_x \in C^0(\Omega \times (0,T)) \) for \( i \in \{1, \ldots, n\} \) is satisfied.

Next we show that the maximal solution of (0.1) is radially symmetric and nonincreasing with respect to \( |x| \), if the same holds for the initial data and Ω is a ball with center 0.

To prove this, we give a suitable comparison principle for parabolic problems with Dirichlet and Neumann data on the boundary.

**Lemma 1.3.2** Let \( \Omega := (b,c) \subset \mathbb{R} \) be a bounded interval, \( T > 0 \) and let

\[
F(u) := u_t - a(\rho,t,u,u_\rho)u_{\rho\rho} - f(\rho,t,u,u_\rho)
\]

denote a parabolic differential operator with continuous functions \( f \) and \( a \). Assume that, for \( l \in \{1,2\} \), \( u_l \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\Omega \times (0,T)) \) such that there is \( M > 0 \) with \( 0 \leq a(\rho,t,u_l(\rho,t),u_l(\rho,t)) \leq M \) for \( (\rho,t) \in \Omega \times (0,T) \). Furthermore, suppose \( F(u_l) \) is defined in \( \Omega \times (0,T) \) with

\[
F(u_1) \leq F(u_2) \quad \text{in } \Omega \times (0,T),
\]

\[
u_1 \leq u_2 \quad \text{on } (\Omega \times \{0\}) \cup (\{c\} \times [0,T)),
\]

\[
\frac{\partial}{\partial N} u_1 \leq \frac{\partial}{\partial N} u_2 \quad \text{on } \{b\} \times [0,T).
\]

Then

\[
u_1 \leq u_2 \quad \text{in } \bar{\Omega} \times [0,T],
\]

provided that additionally, for at least one \( l_0 \in \{1,2\} \), \( a(\rho,t,u,p) \) and \( f(\rho,t,u,p) \) are Lipschitz continuous with respect to \( u \) and \( p \) in a neighborhood of \( K := u_{l_0}(\bar{\Omega} \times [0,T]) \times \nabla u_{l_0}(\Omega \times (0,T)) \) and

\[
\| (u_{l_0})_\rho \|_{L^\infty(\Omega \times (0,T))} + \sum_{i,j=1}^n \| (u_{l_0})_{\rho \rho} \|_{L^\infty(\Omega \times (0,T))} \leq C_1 < \infty
\]

is fulfilled with some positive constant \( C_1 \).
Proof. We assume \( t_0 = 2 \) throughout this proof. The proof of the other case is analogous.

Let \( h(\rho) \) denote the solution of \(-h_{\rho\rho} = 1\) in \( \Omega \) with \( h = 1 \) on \( \partial \Omega \). Then we define \( d(\rho, t) := u(\rho, t) - w(\rho, t) - \varepsilon e^{\gamma t} g(\rho) \) for \( \gamma > 0 \) with \( \gamma := M + L_1 C_1 + L_2 + 1 \) and \( \varepsilon \in (0, \varepsilon_0) \), choosing \( \varepsilon_0 := (e^{\gamma T} \| h_0 \|_{L^\infty(\Omega)})^{-1} \), where \( L_1 \) and \( L_2 \) are the Lipschitz-constants of \( a \) and \( f \) in \( K \), \( u := u_1 \), \( w := u_2 \) and \( g(\rho) := \| h \|_{L^\infty(\Omega)} + 1 - h(\rho) \).

Then we have \( d < 0 \) on \((\Omega \times \{0\}) \cup \{(c) \times [0, T)\})\) and \( \frac{\partial}{\partial \nu} d < 0 \) on \( \{b\} \times [0, T)\) because \( \frac{\partial}{\partial \nu} h < 0 \) on \( \partial \Omega \).

We will show that \( d \leq 0 \) holds in \( \tilde{\Omega} \times [0, T] \).

If this was false, there would be \( t_0 \in (0, T) \) and \( \rho_0 \in \tilde{\Omega} \) with \( d(\rho_0, t_0) = \max_{\rho \in \tilde{\Omega}} d(\rho, t) = 0 \) and \( \max_{\rho \in \tilde{\Omega}} d(\rho, t) < 0 \) for all \( t \in (0, t_0) \) because \( d \) is continuous.

Thus, \( d(\rho, t_0) \leq 0 \) for all \( \rho \in \tilde{\Omega} \) is fulfilled and, therefore, \( \rho_0 \in \Omega \) due to the properties of \( d \) on the parabolic boundary of \( \Omega \times (0, T) \) which are stated above.

Hence, \( d(t(\rho_0, t), t) \geq 0 \), \( d_p(\rho_0, t_0) = 0 \), \( d_{pp}(\rho_0, t_0) \leq 0 \) and \( |u_p(\rho_0, t_0) - w_p(\rho_0, t_0)| = \varepsilon e^{\gamma t_0} |h_p(\rho_0)| \leq 1 \) (due to the choice of \( \varepsilon_0 \)) are satisfied.

Therefore, we compute:

\[
\begin{align*}
0 \geq & \quad F(u(\rho_0, t_0)) - F(w(\rho_0, t_0)) \\
= & \quad d_t(\rho_0, t_0) + \gamma \varepsilon e^{\gamma t_0} g(\rho_0) - [a(\rho_0, t_0, u(\rho_0, t_0), u_p(\rho_0, t_0))] w_{\rho p}(\rho_0, t_0) \\
& - a(\rho_0, t_0, w(\rho_0, t_0), w_p(\rho_0, t_0)) w_{\rho p}(\rho_0, t_0) \\
& - a(\rho_0, t_0, u(\rho_0, t_0), u_p(\rho_0, t_0))(d_{pp}(\rho_0, t_0) + \varepsilon e^{\gamma t_0}) \\
& - f(\rho_0, t_0, u(\rho_0, t_0), u_p(\rho_0, t_0)) + f(\rho_0, t_0, w(\rho_0, t_0), w_p(\rho_0, t_0)) \\
\geq & \quad d_t(\rho_0, t_0) + \gamma \varepsilon e^{\gamma t_0} g(\rho_0) - [a(\rho_0, t_0, u(\rho_0, t_0), u_p(\rho_0, t_0))] w_{\rho p}(\rho_0, t_0) - M \varepsilon e^{\gamma t_0} g(\rho_0) \\
& - f(\rho_0, t_0, u(\rho_0, t_0), u_p(\rho_0, t_0)) + f(\rho_0, t_0, w(\rho_0, t_0), w_p(\rho_0, t_0)) \\
\geq & \quad d_t(\rho_0, t_0) + (\gamma - M) \varepsilon e^{\gamma t_0} g(\rho_0) \\
& - (L_1 C_1 + L_2) |u(\rho_0, t_0) - w(\rho_0, t_0)| \cdot |u_p(\rho_0, t_0) - w_p(\rho_0, t_0)| \\
\geq & \quad \varepsilon e^{\gamma t_0} g(\rho_0)(\gamma - M - L_1 C_1 - L_2) \\
> & \quad 0,
\end{align*}
\]

which is a contradiction. Hence, \( d \leq 0 \) in \( \tilde{\Omega} \times [0, T] \) holds and the claim follows with \( \varepsilon \searrow 0 \).

Now we are able to prove the announced result.

**Lemma 1.3.3** Suppose \( a > 0 \), \( \Omega := B_a(0) \subset \mathbb{R}^n \) and assumptions (0.2) and (0.4) are fulfilled such that \( u_0 \) is radially symmetric and nonincreasing with respect to \( |x| \). Moreover, let \( u \in C^0(\tilde{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T)) \) denote the maximal solution of
Moreover, let \( u \) like above that \( u \) function \( T \) evolves from \( u_0 \). Then \( u(\cdot, t) \) is radially symmetric and nonincreasing with respect to \(|x|\) in \( \bar{\Omega} \) for every \( t \in (0, T) \).

**Proof.** The function \( v(x, t) := u(Ax, t) \) for \((x, t) \in \bar{\Omega} \times [0, T)\) is a maximal solution of (0.1) with \( v(x, 0) = u_0(x) \) in \( \bar{\Omega} \), where \( A \in \mathbb{R}^{n \times n} \) is an orthogonal matrix. Hence we have \( v = u \) because the maximal solution of (0.1) is unique. So \( u(\cdot, t) \) is radially symmetric in \( \bar{\Omega} \) for every \( t \in (0, T) \).

We define \( \rho := |x| \) and fix \( \varepsilon \in (0, 1) \) and \( \eta > 0 \). Then we are able to choose \( w_\varepsilon \in C_0^\infty(\Omega) \) such that \( w_\varepsilon \) is radially symmetric and nonincreasing with respect to \(|x|\) and, furthermore, fulfills \( 0 \leq w_\varepsilon - (u_0 - \frac{\varepsilon}{2})_+ \leq \frac{\varepsilon}{2} \) in \( \Omega \). Thus, \( u_{0\varepsilon} := w_\varepsilon + \varepsilon \in C^3(\Omega) \) is radially symmetric, nonincreasing with respect to \(|x|\) and satisfies \( u_0 + \frac{\varepsilon}{2} \leq u_{0\varepsilon} \leq u_0 + 2\varepsilon \) in \( \Omega \) with \( u_{0\varepsilon}|_{\partial\Omega} = \varepsilon \). Let \( u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_\varepsilon)) \cap C^{2,1}(\Omega \times (0, T_\varepsilon)) \) denote the solution of (1.1), where \( T_\varepsilon \in (0, \infty) \) is the maximal existence time of \( u_\varepsilon \).

The function \( u_\varepsilon \) is unique by comparison (see Theorem 1.1.2 (i)) and hence we can show like above that \( u_\varepsilon(\cdot, t) \) is radially symmetric in \( \bar{\Omega} \) for every \( t \in (0, T_\varepsilon) \).

Moreover, let \( v_\eta \in C^0([0, a] \times [0, t_\eta)) \cap C^{2,1}((0, a) \times (0, t_\eta)) \) denote a solution of

\[
\begin{align*}
(v_\eta)_t &= v_\eta^p((v_\eta)_\rho + n + 1)(v_\eta)_\rho + v_\eta^q + \kappa v_\eta^\rho w_\eta + v_\eta^\rho w - \frac{n-1}{\rho + \eta} v_\eta^\rho w + \frac{n-1}{\rho + \eta} v_\eta^{p(q-1)} w + \kappa v_\eta^{\rho(q-1)} w + 2\kappa v_\eta^{\rho(q-1)} w,
\end{align*}
\]

where \( t_\eta \in (0, \infty) \) denotes the maximal existence time of \( v_\eta \). By Lemma 1.3.2 we have \( v_\eta \geq \frac{\varepsilon}{2} \) in \([0, a] \times [0, t_\eta)\). Hence, \( w := (v_\eta)_\rho \) fulfills \( w \leq 0 \) on the parabolic boundary of \((0, a) \times (0, t_\eta)\) because \((u_{0\varepsilon})_\rho \leq 0 \) in \([0, a]\). Like in Lemma 1.3.1 we can show that \( v_\eta \in C^\infty((0, a) \times (0, t_\eta)) \) and \( w \in C^0([0, a] \times [0, t_\eta)) \) is satisfied. Moreover, the following equation holds in \((0, a) \times (0, t_\eta)\):

\[
\begin{align*}
\frac{\partial}{\partial \eta} v_\eta = v_\eta^p((v_\eta)_\rho + n + 1)(v_\eta)_\rho + v_\eta^q + \kappa v_\eta^\rho w_\eta + v_\eta^\rho w - \frac{n-1}{\rho + \eta} v_\eta^\rho w + \frac{n-1}{\rho + \eta} v_\eta^{p(q-1)} w + \kappa v_\eta^{\rho(q-1)} w + 2\kappa v_\eta^{\rho(q-1)} w.
\end{align*}
\]

Hence, \( w \leq 0 \) is fulfilled by Lemma 1.3.2. Therefore, we have \( t_\eta \geq t_{\eta_2} \) and \( v_{\eta_1} \leq v_{\eta_2} \) in \([0, a] \times [0, t_{\eta_2})\) for \( 0 < \eta_1 < \eta_2 \) by Lemma 1.3.2. Thus, there is a function \( v \) such that \( v_\eta(x, t) \downarrow v(x, t) \) for \((x, t) \in [0, a] \times [0, t_0)\) with \( t_0 := \lim_{\eta \to 0} t_\eta \). Furthermore, we can show like in the proof of Theorem 2.1 of [Sti] that \( v_\eta \to v \) in \( C^{2,1}(K \times [0, t_1]) \) is fulfilled for all \( K \subset\subset (0, a) \) and any \( t_1 \in (0, t_0) \) by Theorem 1.6, Theorem 1.7 and Theorem 1.8 of [Sti] (where results from [LSU] are summarized). Moreover, \( v_{\eta} \to v \) in \( C^0([0, a] \times [0, t_1]) \) is satisfied for all \( t_1 \in (0, t_0) \) because \( v_\eta(x, t) \downarrow v(x, t) \) as \( \eta \to 0 \) and since \( v_\eta(\cdot, t) \) and \( v(\cdot, t) \) are nonincreasing functions on \([0, a]\) for all \( t \in (0, t_\eta) \).

Thus \( \hat{v} \in C^0(B_0(0) \times [0, t_0)) \cap C^{2,1}(B_0(0) \times (0, t_0)) \), defined by \( \hat{v}(x, t) := v(|x|, t) \), is a radially symmetric solution of (1.1) which is nonincreasing with respect to \(|x|\) in (0.1) evolving from \( u_0 \).
$B_0(0)$ with maximal existence time $t_0$. Hence we have $\tilde{v} = u_\varepsilon$ and $t_0 = T_\varepsilon$ since the solution of (1.1) is unique.

Finally, $u(\cdot, t)$ is nonincreasing with respect to $|x|$ for all $t \in (0, T)$ because $u_\varepsilon \to u$ in $C^0(\bar{\Omega} \times [0, \bar{T}])$ as $\varepsilon \to 0$ for any $\bar{T} \in (0, T)$ and since $u_\varepsilon(\cdot, t)$ is nonincreasing with respect to $|x|$ for any $t \in (0, T_\varepsilon)$ and any $\varepsilon \in (0, 1)$. This implies the claim.  

Finally, we state that the solutions of (1.1) are nondecreasing with time, if the initial data satisfy a condition, which appears to be natural in this sense. This result was proved by M. Winkler in [StiWin1], but for completeness it is given here, too.

**Lemma 1.3.4** Suppose $\varepsilon > 0$, assumptions (0.2) and (0.3) are fulfilled, $u_{0\varepsilon} \in C^3(\bar{\Omega})$ satisfies $u_{0\varepsilon} \geq \frac{\varepsilon}{2}$ in $\Omega$ with $u_{0\varepsilon}|_{\partial\Omega} = \varepsilon$ and let $u_\varepsilon$ denote the corresponding solution of (1.1). If

$$\Delta u_{0\varepsilon} + u_{0\varepsilon}^{r-p} + \kappa u_{0\varepsilon}^{r-p} |\nabla u_{0\varepsilon}|^2 \geq 0 \quad \text{in } \Omega$$

then

$$u_{\varepsilon t} \geq 0 \quad \text{in } \Omega \times (0, T_\varepsilon).$$

**Remark.** The point to be noted here is that no higher order compatibility condition is required (ensuring, for instance, that $u_\varepsilon$ is continuous).

**Proof.** Let $v := u_{\varepsilon t}$. Then $v \in C^0(\bar{\Omega} \times (0, T_\varepsilon)) \cap C^{2,1}(\bar{\Omega} \times (0, T_\varepsilon))$ and, apart from that, $v \in L^\infty(\bar{\Omega} \times (0, T_\varepsilon - \delta))$ for any $\delta > 0$ by Lemma VI.3.1, Theorem V.4.2, Theorem V.5.1 and Remark V.5.2 in [LSU]. Differentiating (1.1), we see that

$$v_t = a(x, t)\Delta v + b(x, t) \cdot \nabla v + c(x, t) v \quad \text{in } \Omega \times (0, T_\varepsilon), \quad (1.7)$$

where the functions $a := u_\varepsilon^{p-1}$, $b := 2\kappa u_\varepsilon \nabla u_\varepsilon$ and $c := pu_\varepsilon^{p-1} \Delta u_\varepsilon + qu_\varepsilon^{q-1} + \kappa r u_\varepsilon^{r-1} |\nabla u_\varepsilon|^2$ as well as $\nabla a = pu_\varepsilon^{p-1} \nabla u_\varepsilon$ are bounded in $\Omega \times (0, t_0)$ by some constant depending on $\varepsilon$ and $0 < t_0 < T_\varepsilon$ ([LSU]). Moreover, $a(x, t) \geq (\frac{\varepsilon}{2})^p$ in $\Omega \times (0, T_\varepsilon)$. Since $v|_{\partial\Omega} = 0$ if $t \in (0, T_\varepsilon)$, we can multiply (1.7) by $v_- := \max\{0, -v\}$, integrate over $\Omega \times (\tau, t)$ with $0 < \tau < t < t_0$ to obtain

$$\frac{1}{2} \int_\Omega v_-^2(t) - \frac{1}{2} \int_\Omega v_-^2(\tau) = \int_\tau^t \int_\Omega a \Delta v_- \cdot v_- + \int_\tau^t \int_\Omega v_- b \cdot \nabla v_- + \int_\tau^t \int_\Omega c v_-^2$$

$$= -\int_\tau^t \int_\Omega a |\nabla v_-|^2 + \int_\tau^t \int_\Omega v_- (b - \nabla a) \cdot \nabla v_-$$

$$+ \int_\tau^t \int_\Omega c v_-^2$$

$$\leq -\frac{1}{2} \varepsilon^p \int_\tau^t \int_\Omega |\nabla v_-|^2 + c(\varepsilon, t_0) \int_\tau^t \int_\Omega v_-^2$$

using Young’s inequality. By Gronwall’s lemma,

$$\int_\Omega v_-^2(t) \leq \left( \int_\Omega v_-^2(\tau) \right) \cdot e^{2c(\varepsilon, t_0)(t-\tau)} \quad \forall t \in (\tau, t_0).$$
Since \( \int_{\Omega} v^2(\tau) \to 0 \) as \( \tau \to 0 \) due to the dominated convergence theorem and the regularity properties of \( v \), this implies \( v_- \equiv 0 \) in \( \Omega \times (0, t_0) \) and thereby yields the claim, because \( t_0 \in (0, T_\varepsilon) \) was arbitrary.

A similar condition ensures that the solutions of (1.1) are nonincreasing with time. This result will be used in Chapter 3 to show that some global solutions of (0.1) converge to 0 as \( t \to \infty \).

**Lemma 1.3.5** Suppose \( \varepsilon > 0 \), assumptions (0.2) and (0.3) are fulfilled, \( u_{0e} \in C^3(\overline{\Omega}) \) satisfies \( u_{0e} \geq \frac{\varepsilon}{2} \) in \( \Omega \), with \( u_{0e}|_{\partial\Omega} = \varepsilon \) and let \( u_\varepsilon \) denote the corresponding solution of (1.1). If

\[
\Delta u_{0e} + u_{0e}^{q-p} \varepsilon u_{0e}^{r-p} |\nabla u_{0e}|^2 \leq 0 \quad \text{in } \Omega
\]

then

\[
u_{st} \leq 0 \quad \text{in } \Omega \times (0, T_\varepsilon).
\]

**Proof.** Defining \( v := u_{st} \) and \( v_+ := \max\{0, v\} \), we obtain \( v_+ \equiv 0 \) in \( \Omega \times (0, t_0) \) for any \( t_0 \in (0, T_\varepsilon) \). The proof is analogous to the one of Lemma 1.3.4 by replacing \( v_- \) with \( v_+ \) in the latter proof. Altogether, this implies the claim.
Chapter 2

Boundedness versus blow-up in case of a gradient source term

In this chapter we consider the equation (0.1) in case of $\kappa > 0$, where the gradient term is a source. We show the influence of the exponents $p$, $q$ and $r$ and the factor $\kappa$ with respect to finite-time blow-up of solutions. Loosely speaking, at points where a given solution is large, high values of $p$ should enhance the damping effects of diffusion, whereas large $q$ and $r$ will benefit reaction and thereby push the solution up; converse effects can be expected where $u$ attains small values.

A similar antagonism can be observed in the equation

$$u_t = u^p \Delta u + u^q,$$

without gradient term. This problem has extensively been studied especially in case of $p < 1$, when (2.1) can be transformed into the corresponding forced porous medium equation $v_t = \Delta v^m + v^s$ (for details we refer to the introduction). The results that are known for the latter equation allow us to conclude that whenever $p > 0$, the difference $q - p$ is critical in (2.1):

- If $q < p + 1$ then all solutions of the corresponding Dirichlet problem are global in time and bounded (see [SGKM] for $p < 1$ and [Wie2], [Win7] for $p \geq 1$).

- If $q > p + 1$, however, then there exist both global bounded solutions (for small initial data) and blow-up solutions (emanating from large initial values); in other words, the picture is then quite similar to that obtained for the heat equation $u_t = \Delta u + u^\alpha$ with superlinear source $u^\alpha$, $\alpha > 1$ ([SGKM], [Win3]).

- In the critical case $q = p + 1$, the size of the domain – rather than the initial data – decides on blow-up: In large domains satisfying $\lambda_1(\Omega) < 1$, all positive solutions blow up, while in small domains with $\lambda_1(\Omega) > 1$ all solutions are global and bounded ([SGKM], [Wie1]).

Taking into account these results, we raise the question whether the additional gradient term in (0.1) can enforce blow-up in some of the cases where this is impossible in
CHAPTER 2. BOUNDEDNESS VS. BLOW-UP, PART I

(2.1). In this chapter we show that the answer to this question depends on whether or not the exponent \( r \) exceeds the critical value \( r = 2p - q \).

The results of this chapter are published in [StiWin1]. A part of them was found by M. Winkler (Theorem 2.1.3, Lemma 2.1.6, Lemma 2.2.2, Lemma 2.2.3, Theorem 2.2.4), but to achieve completeness they are presented here, too.

More precisely, a summary of our results gives the following rather complete classification in this respect.

- Let \( q < p + 1 \).
  - If \( r < 2p - q \) then all solutions are global and bounded (Theorem 2.1.2).
  - If \( r > 2p - q \) and
    * if \( \Omega \) contains a ball with sufficiently large radius then all solutions blow up (Theorem 2.1.3);
    * if \( u_0 \) is large enough then \( u \) blows up (Theorem 2.1.4); on the other hand,
    * if \( \Omega \) has small diameter and \( \|u_0\|_{L^\infty(\Omega)} \) is small enough then \( u \) is global and bounded (Theorem 2.1.7).

- If \( r = 2p - q \) and
  * if \( \Omega \) contains a large ball then all solutions blow up (Theorem 2.1.5),
  while
  * if \( \Omega \) has small diameter then all solutions are global and bounded (Corollary 2.1.8).

But also for \( q > p + 1 \) the gradient term may exert a significant influence. Surprisingly, the exponent \( r = 2p - q \) still remains critical, albeit with a slight change in meaning:

- Assume \( q > p + 1 \).
  - If \( r > 2p - q \) and \( \|u_0\|_{L^\infty(\Omega)} \) is small then \( u \) is global and bounded (Lemma 2.2.1).
  - If \( r \leq 2p - q \) and
    * if \( \Omega \) contains a large ball then all solutions blow up (Theorems 2.2.4 and 2.2.5), while
    * if both \( \text{diam}(\Omega) \) and \( \|u_0\|_{L^\infty(\Omega)} \) are small then \( u \) is global and bounded (Lemma 2.2.6).

Finally, also in case of \( q = p + 1 \) the value \( r = 2p - q \equiv p - 1 \) separates regimes with different types of behavior:

- Suppose \( q = p + 1 \).
2.1. THE CASE $Q < P + 1$

- If $r > p - 1$ and $\lambda_1(\Omega) > 1$ then both global bounded and blow-up solutions exist; if $\lambda_1(\Omega) < 1$ then all solutions blow up (Lemma 2.3.2 and Corollaries 2.3.1 and 2.3.3).

- If $r < p - 1$ and $\lambda_1(\Omega) > 1$ then all solutions are global and bounded, while if $\lambda_1(\Omega) < 1$ then all solutions blow up (Lemma 2.3.2 and Corollary 2.3.3).

- In the case $r = p - 1$ the latter statements remain true under the modified conditions $\lambda_1(\Omega) > \kappa + 1$ and $\lambda_1(\Omega) < \kappa + 1$, respectively (Corollary 2.3.3).

We remark that the critical ‘size’ of $\Omega$ – measured in terms of $\lambda_1(\Omega)$ – remains the same as for (2.1), except for the particular case $r = p - 1$. In this case, where all three terms on the right hand side of (0.1) have the same order $p + 1$, this critical size depends on $\kappa$.

In all other cases the factor $\kappa$ is less important than the exponents $p$, $q$ and $r$. In case of $r \neq p - 1$, this can be seen in the following way. The substitution $v(x, t) = a u(bx, ct)$ with $a := \kappa^{\frac{1}{p + 1 - r}}$, $b := \kappa^{\frac{q + 1}{p + 1 - r}}$ and $c := \kappa^{\frac{1}{r + 1 - p}}$ transforms (0.1) into the problem $v_t = v^p \Delta v + v^q + v^r |\nabla v|^2$ in the spatial domain $G := \{b^{-1} x | x \in \Omega\}$. Hence for $r \neq p - 1$ all informations can be gained from the case $\kappa = 1$, but in the special case $q = p + 1$ and $r = p - 1$ the important role of $\kappa$ should be noticed.

Since the question of uniqueness is not answered yet in case of $r < p - q$, we mainly focus on the maximal classical solution of (0.1) where we can use the approximation presented in Chapter 1. Throughout this chapter, by $u$ we exclusively mean a positive classical solution of (0.1) and solution stands for classical solution.

2.1 The case $q < p + 1$

As to the problem $u_t = u^p \Delta u + u^q$, it was already mentioned above that if $q < p + 1$ then all solutions of the corresponding Dirichlet problem are global in time and uniformly bounded. Even the solutions of the corresponding Cauchy problem in $\mathbb{R}^n$ are global in time, provided that their initial data decay sufficiently fast in space; in the latter case, however, all these solutions are unbounded as $t \to \infty$ ([Win3]).

Now if the positive gradient term $\kappa u^r |\nabla u|^2$ is added to the equation, the solutions can be expected to be bounded if $r$ is sufficiently small. Our goal is to show that the critical borderline in this respect is marked by $r = 2p - q$.

2.1.1 Boundedness of all solutions for $r < 2p - q$

We first consider the subcritical case $r < 2p - q$ and derive uniform upper bounds by constructing arbitrarily large stationary supersolutions.
Lemma 2.1.1 Suppose (0.2) and (0.3) are fulfilled with $q < p + 1$, $r < 2p - q$ and $\kappa > 0$. Then all positive solutions of (1.1) are global and we have

$$\|u_\varepsilon\|_{L^\infty(\Omega \times (0, \infty))} \leq c(1 + \|u_0\|_{L^\infty(\Omega)})$$

with some constant $c > 0$.

**Proof.** Let $e \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ denote the solution of $-\Delta e = 1$ in $\Omega$ with $e|_{\partial \Omega} = 1$. Since $r < 2p - q$ and $q - p < 1$, it is possible to fix a number $\gamma < 1$ such that $q - p < \gamma < p - r$, whence in particular

$$M \leq M + M^\gamma e \leq (1 + \|e\|_{L^\infty(\Omega)}) \cdot M =: CM \quad \forall M \geq 1.$$

Thus, for any $M \geq 1$ the function

$$v(x, t) := M + M^\gamma e(x), \quad (x, t) \in \Omega \times (0, \infty),$$

satisfies

$$v_t - v^p \Delta v - v^q - \kappa v^r |\nabla v|^2 = v^p \cdot \left[ M^\gamma - (M + M^\gamma e)^{q-p} - \kappa(M + M^\gamma e)^{r-p}M^{2\gamma}|\nabla e|^2 \right] \geq v^p \cdot \left[ M^\gamma - c_1(M^{q-p} + M^{2\gamma+r-p}) \right] \quad \text{in } \Omega \times (0, \infty)$$

with some $c_1 > 0$. We now fix $M \geq 1$ large enough such that $u_{0\varepsilon} < M$ in $\Omega$ and $M^\gamma - c_1(M^{q-p} + M^{2\gamma+r-p}) \geq 0$, where the latter is possible since our choice of $\gamma$ implies $q - p < \gamma$ and $2\gamma + r - p < \gamma$. Therefore, the comparison principle yields $u_\varepsilon \leq v$ in $\Omega \times (0, T_\varepsilon)$, whence, due to standard parabolic estimates, $u_\varepsilon$ cannot blow up in finite time and, moreover, must obey the claimed estimate. 

Letting $\varepsilon \searrow 0$, we easily obtain (by using Theorem 1.1.2)

**Theorem 2.1.2** Let (0.2), (0.3) and (0.4) be satisfied with $q < p + 1$, $r < 2p - q$ and $\kappa > 0$. Then all positive solutions of (0.1) are global and bounded. More precisely, there exists $c > 0$ such that the a priori estimate

$$\|u\|_{L^\infty(\Omega \times (0, \infty))} \leq c(1 + \|u_0\|_{L^\infty(\Omega)})$$

holds.

### 2.1.2 Blow-up results for $r \geq 2p - q$

In contrast to the above result, the situation changes significantly if $r > 2p - q$ and $\Omega$ is large in the following sense:
Theorem 2.1.3  Suppose (0.2), (0.3) and (0.4) are fulfilled with \( q < p + 1 \), \( r > 2p - q \) and \( \kappa > 0 \). Then there exists \( R > 0 \) such that if \( \Omega \) contains a ball with radius \( R \) then all maximal solutions of (0.1) blow up in finite time.

Proof. We fix \( m \in \mathbb{N} \) with \( m > \frac{q}{2p - r} \) and \( m > 2 \). Writing \( B := B_1(0) \), we choose \( 0 \leq \varphi \in C_0^\infty(B) \) such that \( \int_B \varphi^{2m} = 1 \) and set

\[
C_0 := \begin{cases} 
\frac{1}{2} \cdot \frac{q + r - 2p}{2p - r} \left( \frac{4(2p - r)m^2}{\kappa q} \right) \cdot \int_B \varphi^{2(m - \frac{r}{2p - q})} \cdot |\nabla \varphi|^{\frac{2q}{2p - q}} & \text{if } r < 2p, \\
\frac{2m^2}{\kappa} \cdot \int_B \varphi^{2m - 2} \cdot |\nabla \varphi|^2 & \text{if } r \geq 2p.
\end{cases}
\]

Then we can choose \( M > 1 \) large such that

\[
M \geq \left( \frac{2p}{\kappa} \right)^{\frac{1}{r+1-q}}
\]

and

\[
M > (2C_0)^{\frac{1}{r}}.
\]

Next, we pick \( R > 0 \) large enough fulfilling

\[
\frac{1}{2} y_R \Theta_R \geq M \quad \text{in } B,
\]

where

\[
y_R := \frac{1}{2 \cdot (2\lambda_R)^{\frac{1}{r+1-q}}},
\]

\( \Theta_R := \Theta(\cdot; B_R(0)) \) and \( \lambda_R := \lambda_1(B_R(0)) \). Note that \( R \) fulfilling (2.4) exists since it is well-known that \( \lambda_R \to 0 \) as \( R \to \infty \).

Now suppose \( \Omega \) contains a closed ball with radius \( R \); without loss of generality we may assume \( B_R(0) \subset \Omega \). To prove that all maximal solutions in \( \Omega \) blow up in finite time, suppose on the contrary that (0.1) has a global maximal solution \( u \) with some initial data \( u_0 \).

Let \( R' > R \) be such that still \( B_{R'}(0) \subset \Omega \). Then, since \( u_0 > 0 \) in \( \Omega \), we have \( u_0 \geq c_0 \Theta_{R'} \) in \( B_{R'} \) for some \( c_0 \in (0, \lambda_{R'}^{\frac{1}{r+1-q}}) \). Thus, \( u \geq c_0 \Theta_{R'} \) in \( B_{R'}(0) \times (0, \infty) \) by comparison of \( u_\varepsilon \) with \( z(x,t) := c_0 \Theta_{R'} \) and letting \( \varepsilon \searrow 0 \) – observe that \( z \) satisfies

\[
z_t - z^p \Delta z - z^q - \kappa z^r |\nabla z|^2 \leq z_t - z^p \Delta z - z^q = \lambda_{R'} c_0^{p+1} \Theta_{R'}^{p+1} - c_0^q \Theta_{R'}^q \leq 0
\]

in \( B_{R'}(0) \times (0, \infty) \). Consequently, \( u \geq c_1 \) in \( B_R(0) \times (0, \infty) \) holds with suitably small \( c_1 > 0 \). Therefore, we can pick positive numbers \( y_0 \) and \( \delta \) such that \( y_0 \leq \frac{\delta}{2} \) and \( \delta \leq \min \left\{ \frac{CA}{y_0}, 1 \right\} \) and a nondecreasing positive function \( y \) on \([0, \infty)\) with \( y' \leq \frac{\delta y}{2} y'' \), \( y(0) = y_0 \) and \( y(t) \nearrow y_R \) as \( t \to \infty \). Defining

\[
v(x,t) := y(t) \cdot (\Theta_R(x) + \delta), \quad (x,t) \in B_R(0) \times (0, \infty),
\]
we have \( v \leq u_\varepsilon \) at \( t = 0 \) and on \( \partial B_R(0) \) according to our choice of \( y_0 \) and \( \delta \). Moreover,

\[
v_t - v^p \Delta v - v^q - \kappa v^r |\nabla v|^2 \leq y'(\Theta_R + \delta) + \lambda_R y^{p+1}(\Theta_R + \delta)^p \Theta_R - y^q(\Theta_R + \delta)^q =: I_1 + I_2 - I_3 \quad \text{in } B_R(0) \times (0, \infty),
\]

where

\[
\frac{I_1}{I_3} = \frac{y'}{y^q}(\Theta_R + \delta)^{-(q-1)} \leq \frac{y'}{y^q} \cdot \delta^{-(q-1)} \leq 1
\]

and

\[
\frac{I_2}{I_3} = 2\lambda_R y^{p+1-q}(\Theta_R + \delta)^{p-q} \Theta_R
\]

\[
\leq 2\lambda_R \cdot (2y)^{p+1-q}
\]

\[
\leq 2\lambda_R \cdot (2y_R)^{p+1-q}
\]

\[
= 1.
\]

Thus, due to the comparison principle, \( u_\varepsilon \geq v \) and hence \( u \geq v \) in \( B_R(0) \times (0, \infty) \).

In particular, if we choose \( t_0 > 0 \) such that \( y(t_0) \geq \frac{1}{2}y_R \) then from (2.4) we obtain

\[
u(x,t) \geq v(x,t) \geq \frac{1}{2}y_R \Theta_R(x) \geq M \quad \forall (x,t) \in B \times (t_0, \infty).
\]

(2.5)

We now multiply (0.1) by \( \varphi^{2m} \) and integrate over \( B \) to see that

\[
\frac{d}{dt} \int_B \varphi^{2m} u = -p \int_B \varphi^{2m} u^{p-1} |\nabla u|^2 - 2m \int_B \varphi^{2m-1} u^p \nabla u \cdot \nabla \varphi
\]

\[
+ \int_B \varphi^{2m} u^q + \kappa \int_B \varphi^{2m} u^r |\nabla u|^2
\]

\[
=: -J_1 - J_2 + J_3 + J_4 \quad \forall t > 0.
\]

(2.6)

Here, by (2.2),

\[
J_1 \leq p M^{-(r+1-p)} \int_B \varphi^{2m} u^r |\nabla u|^2 \leq \frac{1}{2} J_4 \quad \forall t > t_0
\]

(2.7)

and

\[
J_2 \leq 2m \cdot \frac{\eta}{2} \int_B \varphi^{2m} u^r |\nabla u|^2 + 2m \cdot \frac{1}{2\eta} \int_B \varphi^{2m-2} |\nabla \varphi|^2 u^{2p-r}
\]

for any \( \eta > 0 \) by Young’s inequality. Choosing \( \eta := \frac{\eta}{2m} \) we obtain

\[
J_2 \leq \frac{1}{2} J_4 \cdot \frac{2m^2}{\kappa} \int_B \varphi^{2m-2} |\nabla \varphi|^2 u^{2p-r}.
\]

(2.8)
Since if \( r < 2p \), again due to Young’s inequality,

\[
\int_B \varphi^{2m-2} |\nabla \varphi|^2 u^{2p-r} \leq \frac{\mu}{s} \int_B \varphi^{2m} u^q + \frac{1}{s'} \mu \int_B \varphi^{2(m-s')} |\nabla \varphi|^{2s'}
\]

holds with \( s := \frac{q}{2p-r} > 1 \), \( s' = \frac{s}{s-1} = \frac{q}{r-2p+q} \) and \( \mu := \frac{\kappa s}{4m^2} \), we deduce the estimate

\[
J_2 \leq \frac{1}{2} J_4 + \frac{1}{2} J_3 + r - 2p + q \cdot \left( \frac{4(2p-r)m^2}{\kappa q} \right)^{-\frac{q}{2p+q}} \int_B \varphi^{2(m-r-2p+q)} |\nabla \varphi|^{\frac{2p}{2p+q}}. \quad (2.9)
\]

In the case \( r \geq 2p \), (2.8) and (2.5) imply

\[
J_2 \leq \frac{1}{2} J_1 + \frac{2m^2}{\kappa} \int_B \varphi^{2m-2} |\nabla \varphi|^2 \quad \forall \ t > t_0, \quad (2.10)
\]

because \( M > 1 \). Combining (2.9) and (2.10) with (2.6), (2.7) and our definition of \( C_0 \), we arrive at

\[
\frac{d}{dt} \int_B \varphi^{2m} u \geq \frac{1}{2} \int_B \varphi^{2m} u^q - C_0 \geq \frac{1}{2} \left( \int_B \varphi^{2m} u \right)^q - C_0 \quad \forall \ t > t_0, \quad (2.11)
\]

where we have used Hölder’s inequality and the normalization \( \int_B \varphi^{2m} = 1 \). Since (2.5) and (2.3) entail

\[
\frac{1}{2} \left( \int_B \varphi^{2m} u(\cdot, t_0) \right)^q \geq \frac{1}{2} \left( M \cdot \int_B \varphi^{2m} \right)^q = \frac{1}{2} M^q > C_0,
\]

(2.11) implies that \( \int_B \varphi^{2m} u(\cdot, t) \) must blow up in finite time, contradicting \( u \) being global.

In addition, we can show that in case of \( r > 2p - q \) blow-up occurs in arbitrary domains for large initial data. For later reference, we state this result for the slightly larger regime \( q \leq p + 1 \).

**Theorem 2.1.4** Assume (0.2), (0.3) and (0.4) with \( q \leq p + 1 \), \( r > 2p - q \) and \( \kappa > 0 \). Then for every \( w \in C^0(\bar{\Omega}) \) which is positive in \( \Omega \) with \( w = 0 \) on \( \partial \Omega \) there is \( b_0 > 0 \) such that the maximal classical solution of (0.1) evolving from \( u_0 := bw \) with \( b \geq b_0 \) blows up in finite time.
Proof. We again pick \( m \in \mathbb{N} \) large enough such that \( m > \frac{q}{r-2p+q} \) and \( m > 2 \). Then we choose \( R > 0 \) such that \( \Omega \) contains a ball with radius \( 2R \), where we may assume \( B_{2R}(0) \subset \Omega \). Letting \( B := B_{R}(0) \), we fix a nonnegative \( \varphi \in C_0^\infty(B) \) such that \( \int_B \varphi^{2m} = 1 \) and define \( C_0 > 0 \) as in the proof of Theorem 2.1. Moreover, we choose \( M > 1 \) large such that (2.2) and (2.3) are fulfilled and fix \( \gamma > \max\{4, 2 \frac{r+1-p}{r+q-2p}\} \). We now set

\[
v(x) := \beta e^{-\frac{|x|^\gamma}{R^\gamma - |x|^\gamma}} \quad \text{for} \quad x \in B_R(0)
\]

with \( \beta > 0 \) to be specified soon. Writing \( y := \frac{|x|^\gamma}{R^\gamma - |x|^\gamma} \), for \( x \in B_R(0) \) we compute

\[
v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = \beta^{p+1} e^{-(p+1)y} \frac{|x|^\gamma-2}{(R^\gamma - |x|^\gamma)^4} \left\{ |x|^\gamma((\gamma-2)|x|^2 - \gamma R^2)^2 \right. \\
\quad + n((\gamma-2)|x|^2 - \gamma R^2)(R^\gamma - |x|^\gamma)^2 \\
\quad - \gamma((\gamma-2)(R^\gamma - |x|^\gamma)^3 \\
\quad + 4|x|^2(R^\gamma - |x|^\gamma)((\gamma-2)|x|^2 - \gamma R^2) \\
\quad + \beta^q e^{-qy} + \kappa \beta^{r+2} e^{-(r+2)y} \frac{|x|^2r^2((\gamma-2)|x|^2 - \gamma R^2)^2}{(R^\gamma - |x|^\gamma)^4}.
\]

Furthermore, we set

\[
p(t) := t^\gamma ((\gamma-2)t^2 - \gamma R^2)^2 + n((\gamma-2)t^2 - \gamma R^2)(R^\gamma - t^2)^2 \\
\quad - \gamma((\gamma-2)(R^\gamma - t^2)^3 + 4t^2(R^\gamma - t^2)((\gamma-2)t^2 - \gamma R^2) \quad \text{with} \quad t \in [0, R].
\]

Since \( p(R) = 4R^\gamma+4 > 0 \) and \( p \) is continuous in \([0, R]\), there is \( c \in (\frac{1}{2}, 1) \) such that \( p(t) \geq 0 \) in \([cR, R]\). We therefore have \( v^p \Delta v \geq 0 \) for \( |x| \in [cR, R] \), and, moreover, we obtain \( |p(t)| \leq \gamma^2 R^{\gamma+4} + \gamma(\gamma + n + 2)R^6 \) for \( t \in [0, R] \).

Next we fix

\[
k \in \left( \frac{p + 1 - q}{\gamma - 2}, \frac{r + 1 - p}{\gamma} \right),
\]

which is possible due to the choice of \( \gamma \), and then take some \( \beta_0 \geq 1 \) such that \( \delta := \beta^{-k} \in (0, \frac{1}{2}) \) is fulfilled for all \( \beta \geq \beta_0 \). Upon these choices, it is now possible to pick \( \beta_1 \geq \beta_0 \) such that

\[
v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq \frac{-\beta^{p+1}}{(1 - \delta^2)^4} \frac{\delta^\gamma - 2 R^\gamma - 10}{\gamma^2 R^\gamma + 4 + \gamma(\gamma + n + 2)R^6} \]

\[
\quad + \beta^q e^{-qy} \frac{\delta^\gamma - 2 R^\gamma - 10}{1 - \delta^2} \frac{\gamma^2 R^\gamma + 4 + \gamma(\gamma + n + 2)R^6}{(\frac{3}{4})^4} + \beta^q e^{-\frac{3}{4}(\frac{2}{3})^{-2}} \quad \text{in} \quad B_{\delta R}(0)
\]

\[
\geq 0
\]
holds for all $\beta \geq \beta_1$. Furthermore, there is $\beta_2 \geq \beta_1$ such that
\[
v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq \frac{|x|^{-2}}{(R^2 - |x|^2)^4} \left\{ - \beta^{p+1} \left( \gamma^2 R^{\gamma+4} + \gamma(n+2)R^6 \right) 
+ \kappa \beta^{p+2} \gamma^4 4R^{\gamma+4}e^{-(r+2) \frac{\gamma(R)}{(1-\sigma^2)}R^2} \right\}
\geq \frac{|x|^{-2}}{(R^2 - |x|^2)^4} \left\{ - \beta^{p+1} \left( \gamma^2 R^{\gamma+4} + \gamma(n+2)R^6 \right) 
+ \kappa \beta^{p+2} \gamma^4 4R^{\gamma+4}e^{-(r+2) \frac{\gamma(R)}{(1-\sigma^2)}R^2} \right\}
\geq 0 \quad \text{for } |x| \in [\delta R, cR]
\]
is fulfilled for every $\beta \geq \beta_2$ due to our selection of $\delta$ and $k$. Now we choose $\beta \geq \beta_2$ such that $v(x) \geq M$ in $B_{\frac{R}{2}}(0)$. Hence we obtain $v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq 0$ in $B_R(0)$ and $v \geq M$ in $B$.

Now given any positive $w \in C^0(\bar{\Omega})$ with $w = 0$ on $\partial\Omega$, we can choose $b_0 > 0$ such that $b_0 \delta v \geq v$ in $B_R(0)$. We then claim that the maximal solution $u$ of (0.1) with initial data $u_0 = bw$ cannot be global whenever $b \geq b_0$. In fact, if this was false then we would have $u_\varepsilon \geq v$ in $B_R(0) \times (0, \infty)$ for all $\varepsilon > 0$ by comparison, because $u_0 \geq v$ in $B_R(0)$ and $u(x, t) > 0 = v(x)$ for $|x| = R$ and $t \geq 0$. Hence $u(x, t) \geq v(x) \geq M$ in $B \times (0, \infty)$ is fulfilled due to our choice of $v$. This implies that $u$ fulfills (2.5) with $t_0 = 0$. Now we can continue as in the second part of the proof of Theorem 2.1.3 beginning at (2.5) to show that $\int_B \varphi^{2m}u(\cdot, t)$ (and hence $u$ itself) must blow up in finite time.

Let us now have a closer look at the critical case $r = 2p - q$. Here we first focus on sufficiently large domains and show that then the critical plane belongs to the blow-up regime. Small domains will be considered in the next section where the case $r = 2p - q$ can be embedded into a more general setting.

The above methods for the proof of blow-up cease to work for $r = 2p - q$. Fortunately, however, this special case allows for a transformation of (0.1) into a ‘nice’ problem. Using this, we can establish a result similar to the one obtained in Theorem 2.1.3.

**Theorem 2.1.5** Suppose (0.2), (0.3) and (0.4) are satisfied with $q < p + 1$, $r = 2p - q$ and $\kappa > 0$. Then there exists $R > 0$ such that if $\Omega$ contains a ball with radius $R$ then all maximal solutions of (0.1) blow up in finite time.

**Proof.**

i) The substitution $u(x, t) = f(v(x, t))$ transforms (0.1) into the equation

\[
f'(v) v_t = (f(v))^p f'(v) \Delta v + (f(v))^q + [\kappa (f(v))^r (f'(v))^2 + (f(v))^p f''(v)] |\nabla v|^2.
\]

Now we choose $\beta := r - p + 1$ (with $\beta > 0$ due to our choice of $p$, $q$ and $r$) and define $f \in C^0([0, \infty)) \cap C^2((0, \infty))$ to be a solution of the initial value problem

\[
\begin{align*}
  f(s) &= e^{-\frac{\gamma(R)}{\beta} s} && \text{in } (0, \infty), \\
  f(0) &= 0. 
\end{align*}
\tag{2.12}
\]
Hence, upon this choice of \( f, v \) fulfills
\[
v_t = (f(v))^p \left( \Delta v + (f(v))^{1-\beta} e^{\kappa (f(v))^{\frac{\beta}{\gamma}}} \right) \quad \text{in } \Omega \times (0, T),
\] (2.13)

because \( q - p = p - r = 1 - \beta \). For brevity we write \( h(v) := (f(v))^{1-\beta} e^{\kappa (f(v))^{\frac{\beta}{\gamma}}} \).

Now we consider the case \( \beta \in (0, \infty) \setminus \{1\} \) and derive estimates of the functions \( f(v) \) and \( h(v) \) for large values of \( v \).

For this purpose, choosing
\[
c_1 := \left\{ \begin{array}{ll}
\ln \left( 4 \left( \frac{\beta}{2} \right)^{1-\beta} \kappa^{-\frac{1}{\beta}} \right) & \text{if } \beta > 1, \\
\ln \left( 2 \beta^{-\frac{1}{\beta}} \kappa^{-\frac{1}{\beta}} \right) & \text{if } \beta \in (0, 1),
\end{array} \right.
\]
we define
\[
g_1(v) := \left( \frac{\beta}{\kappa} \left[ \ln(v + 1) + \frac{\beta-1}{\beta} \ln(\ln(v + 1)) - c_1 \right] \right)^{\frac{1}{\beta}} \quad \text{with } v \geq v_1,
\]
where \( v_1 \geq 2 \) is chosen large enough such that \( g_1(v) \geq \left( \frac{\beta}{2\kappa} \ln(v + 1) \right)^{\frac{1}{\beta}} \) holds for \( v \geq v_1 \). Then there is \( v_2 \geq v_1 \) such that
\[
\frac{g_1'(v)}{e^{-\kappa (g_1(v))^{\frac{\beta}{\gamma}}}} = \left( \frac{\beta}{\kappa} \left[ \ln(v + 1) + \frac{\beta-1}{\beta} \ln(\ln(v + 1)) - c_1 \right] \right)^{\frac{1-\beta}{\beta}} \left( 1 + \frac{\beta-1}{\beta \ln(v+1)} \kappa^{-\frac{1}{\beta}} e^{-c_1} \right)
\]
\[
= \left( \beta + (\beta-1) \frac{\ln(\ln(v+1))}{\ln(v+1)} - \frac{\beta c_1}{\ln(v+1)} \right)^{\frac{1-\beta}{\beta}} \left( 1 + \frac{\beta-1}{\beta \ln(v+1)} \kappa^{-\frac{1}{\beta}} e^{-c_1} \right)
\]
\[
\leq \left\{ \begin{array}{ll}
(\frac{\beta}{2})^{\frac{1-\beta}{\beta}} 2^{\frac{1}{\beta}} e^{-c_1} \leq \frac{1}{2} & \text{if } \beta > 1 \\
\beta^{\frac{1-\beta}{\beta}} 2^{\frac{1}{\beta}} e^{-c_1} \leq \frac{1}{2} & \text{if } \beta \in (0, 1)
\end{array} \right.
\]
for \( v \geq v_2 \) due to the choice of \( c_1 \). If we had \( g_1(v) \geq f(v) \) for all \( v \geq v_2 \), then
\[
0 \geq f(v) - g_1(v) \geq f(v_2) - g_1(v_2) + \int_{v_2}^{v} \left( e^{-\kappa (g_1(v))^\frac{\beta}{\gamma}} - \frac{1}{2} e^{-\kappa (g_1(v))^\frac{\beta}{\gamma}} \right) ds
\]
\[
\geq f(v_2) - g_1(v_2) + \frac{1}{2} \int_{v_2}^{v} \left( s + 1 \right)^{\frac{1-\beta}{\beta}} e^{c_1} ds
\]
\[
= f(v_2) - g_1(v_2) + \frac{1}{2} e^{c_1} \left( \beta \ln(v+1)^{\frac{1}{\beta}} - \beta \ln(v_2+1)^{\frac{1}{\beta}} \right)
\]
would hold for all \( v \geq v_2 \), which is a contradiction for large \( v \). Hence, there is \( v_3 \geq v_2 \) with \( g_1(v_3) < f(v_3) \). Therefore, we conclude \( f(v) \geq g_1(v) \) for all \( v \geq v_3 \) by standard comparison results for ordinary differential equations, since \( g_1 \) is a subsolution of
the differential equation in (2.12) with $v \geq v_2$.
Next we choose
\[
c_2 := \begin{cases} 
\ln \left( 2(3\beta) \frac{\beta-1}{\beta} \kappa^\frac{1}{\beta} \right) & \text{if } \beta > 1, \\
\ln \left( 4(\frac{\beta}{2}) \frac{\beta-1}{\beta} \kappa^\frac{1}{\beta} \right) & \text{if } \beta \in (0,1).
\end{cases}
\]
Then we define $g_2(v) := \left( \frac{2}{\kappa} \ln(v+1) + \frac{\beta-1}{\beta} \ln(\ln(v+1)) + c_2 \right)^{\frac{1}{\beta}}$ with $v \geq v_1$. Hence, there is $v_4 \geq v_1$ such that
\[
\frac{g_2'(v)}{e^{-\kappa g_2(v)^{\beta}}} = \left( \frac{2}{\kappa} \ln(v+1) + \frac{\beta-1}{\beta} \ln(\ln(v+1)) + c_2 \right)^{\frac{1-\beta}{\beta}} (1 + \frac{\beta-1}{\beta \ln(v+1)}) \frac{1}{\kappa(v+1)}
\]
\[
= \left( \beta + (\beta-1) \frac{\ln(\ln(v+1))}{\ln(v+1)} + \frac{\beta c_2}{\ln(v+1)} \right) \frac{1-\beta}{\beta} (1 + \frac{\beta-1}{\beta \ln(v+1)}) \kappa^{-\frac{1}{\beta}} e^{-c_2}
\]
\[
\geq \begin{cases} 
(3\beta) \frac{1-\beta}{\beta} \kappa^{-\frac{1}{\beta}} e^{c_2} & \text{if } \beta > 1 \\
\left( \frac{\beta}{2} \right) \frac{1-\beta}{\beta} \kappa^{-\frac{1}{\beta}} e^{c_2} & \text{if } \beta \in (0,1)
\end{cases}
\]
with $v \geq v_4$, according to the choice of $c_2$. Moreover, there exists $v_5 \geq v_4$ with $g_2(v_5) > f(v_5)$, because otherwise we could derive a contradiction in a similar way as above for $g_1$. Hence, we obtain $f(v) \leq g_2(v)$ for $v \geq v_5$, because $g_2$ is a supersolution of the differential equation in (2.12) for $v \geq v_4$.

Now we define $v_0 := \max\{v_3,v_5\} \geq v_1$ and remark that $v_0$ only depends on $\beta$ and $\kappa$. Since $g_1(v) \leq f(v) \leq g_2(v)$ for $v \geq v_0$, we have in case of $\beta > 1$
\[
h(v) = (f(v))^{1-\beta} e^{F(v)} \leq \left( \frac{\beta}{\kappa} \ln(v+1) + \frac{\beta-1}{\beta} \ln(\ln(v+1)) + c_2 \right)^{\frac{1-\beta}{\beta}} (1 + \frac{\beta-1}{\beta \ln(v+1)}) \frac{1}{\kappa(v+1)}
\]
\[
= \left( \frac{\beta}{\kappa} + (\beta-1) \frac{\ln(\ln(v+1))}{\ln(v+1)} + \frac{\beta c_2}{\kappa \ln(v+1)} \right) \frac{1-\beta}{\beta} e^{-c_1}(v+1)
\]
\[
\geq c_0(v+1)
\]
with $v \geq v_0$, where $c_0$ is a positive constant that only depends on $\beta$ and $\kappa$.
In case of $\beta \in (0,1)$ we get
\[
h(v) = (f(v))^{1-\beta} e^{F(v)} \leq \left( \frac{\beta}{\kappa} \ln(v+1) + \frac{\beta-1}{\beta} \ln(\ln(v+1)) - c_1 \right)^{\frac{1-\beta}{\beta}} (1 + \frac{\beta-1}{\beta \ln(v+1)}) \frac{1}{\kappa(v+1)}
\]
\[
= \left( \frac{\beta}{2\kappa} \ln(v+1) \right) \frac{1-\beta}{\beta} e^{-c_1}(v+1)
\]
\[
\geq \left( \frac{\beta}{2\kappa} \right) \frac{1-\beta}{\beta} e^{-c_1}(v+1)
\]
with \( v \geq v_0 \), because \( v_0 \geq v_1 \).

If \( \beta = 1 \) we have \( f(v) = \frac{1}{\kappa} \ln(\kappa v + 1) \) and \( h(v) = \kappa v + 1 \) with \( v \geq 0 \). Hence, for all \( \beta > 0 \) we can find constants \( v_0 \geq 2 \) and \( c_0 > 0 \), only depending on \( \beta \) and \( \kappa \), such that \( f(v) \geq \left( \frac{\beta}{2\kappa} \ln(v + 1) \right)^\frac{3}{2} \) with \( v \geq v_0 \), and such that \( v \) is a supersolution of

\[
    w_t = (f(w))^p(\Delta w + c_0 w) \tag{2.14}
\]

in \( \Omega \times (t_0, T) \) whenever \( v \geq v_0 \) in \( \Omega \times (t_0, T) \).

ii) Now let \( q < p + 1 < r \), \( \kappa > 0 \) and \( \beta \equiv r - p + 1 \). Moreover, let \( v_0 \geq 2 \) and \( c_0 > 0 \) be the constants from part i) that only depend on \( \beta \) and \( \kappa \). For \( R > 0 \), let \( \Theta_R := \Theta(\cdot, B_R(0)) \) and \( \lambda_R := \lambda_1(B_R(0)) \). We fix \( R_0 > 0 \) large enough such that \( \lambda_{R_0} < c_0 \) is fulfilled. Furthermore, we choose \( M > f(v_0) \), where \( f \) is the solution of (2.12) like in part i). Next we pick \( R > R_0 \) large enough such that \( \frac{1}{2} y_R \Theta_R \geq M \) holds in \( B := B_{R_0}(0) \) with \( y_R := \frac{1}{2(2\lambda_{R_0})^{p+q}} \), which is possible because \( \lambda_R \to 0 \) as \( R \to \infty \).

We proceed to prove the claim of the theorem by assuming that \( \Omega \) contains a closed ball with radius \( R \), where without loss of generality \( B_R(0) \subset \Omega \). We assume that there are some initial data \( u_0 \) such that the corresponding maximal solution is global in time. Then we can show in a way similar to the proof of Theorem 2.1.3 that there is \( t_0 > 0 \) such that \( u(x, t) \geq \frac{1}{2} y_R \Theta_R(x) \geq M \) is fulfilled for \( (x, t) \in B \times (t_0, \infty) \). Hence we obtain \( v \geq v_0 \) in \( B \times (t_0, \infty) \), where \( v \) is defined like in part i). Therefore, \( v \) is a supersolution of (2.14) in \( B \times (t_0, \infty) \) by part i). Since

\[
    \int_1^\infty \frac{ds}{s(f(s))^p} < \infty
\]

is fulfilled due to \( f(v) \geq \left( \frac{\beta}{2\kappa} \ln(v + 1) \right)^\frac{3}{2} \) with \( v \geq v_0 \) and \( p > p - q + 1 = r - p + 1 = \beta \) (which is implied by \( q > 1 \)), \( v \) blows up in finite time by comparison. This is because every positive solution \( w \) of (2.14) in \( B \times (t_0, \infty) \) blows up in finite time by Theorem 1 in [Win2] – note that \( \lambda_{R_0} < c_0 \). Hence \( u \) blows up in finite time contradicting \( u \) being global.

\[\blacksquare\]

### 2.1.3 Bounded solutions in small domains

In domains with sufficiently small diameter, we shall use another family of time-independent functions to obtain stationary supersolutions that serve as upper bounds for the solutions of (1.1). This method applies to arbitrarily large values of \( r \).

**Lemma 2.1.6** Let (0.2) and (0.3) be fulfilled with \( q < p + 1 < r > p - 1 \) and \( \kappa > 0 \). Then there exist \( d_0 > 0 \) and \( M > 0 \) such that if \( \text{diam}(\Omega) < d_0 \) then all positive solutions of (1.1) with \( \|u_0\|_{L^\infty(\Omega)} \leq M \) are global and bounded.
2.1. THE CASE $Q < P + 1$

Proof. Without loss of generality we may assume $\Omega \subset \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid d < x_1 < 2d \}$ with $d := \text{diam} (\Omega)$. We fix $\gamma \in (0, 1)$ such that $\gamma \leq \frac{2}{p+1-q}$ and let

$$A := \left( \frac{1 - \gamma}{2\kappa \gamma} \right)^{\frac{1}{p+1-q}} \cdot (2d)^{-\gamma}.$$  \hfill (2.15)

Then there exists $d_0 > 0$ such that whenever $d < d_0$ then

$$A \geq \left( \frac{2}{\gamma(1 - \gamma)} \right)^{\frac{1}{p+1-q}} \cdot (2d)^{\frac{2}{p+1-q} - \gamma}.$$  \hfill (2.16)

We set

$$v(x,t) := Ax_1^\gamma, \quad (x,t) \in \Omega \times (0, \infty),$$  \hfill (2.17)

and calculate

$$v_t - v^p \Delta v - v^q - \kappa v^r \lvert \nabla v \rvert^2 = \gamma(1 - \gamma) A^{-(p+1-q)} x_1^{2-(p+1-q)\gamma} - A^q x_1^{q\gamma} - \kappa \gamma^2 A^{r+2} x_1^{(r+2)\gamma-2} =: I_1 - I_2 - I_3.$$

Here,

$$\frac{I_2}{I_1} = \frac{2}{\gamma(1 - \gamma)} A^{-(p+1-q)} x_1^{2-(p+1-q)\gamma} \leq \frac{2}{\gamma(1 - \gamma)} A^{-(p+1-q)} \cdot (2d)^{2-(p+1-q)\gamma} \leq 1$$

due to (2.16) and the fact that $\gamma \leq \frac{2}{p+1-q}$. Moreover,

$$\frac{I_3}{I_1} = \frac{2\kappa \gamma}{1 - \gamma} A^{r+1-p} x_1^{(r+1-p)\gamma} \leq \frac{2\kappa \gamma}{1 - \gamma} A^{r+1-p} \cdot (2d)^{(r+1-p)\gamma} = 1$$

in view of (2.15) and the assumption on $r$. Therefore, $v$ is a supersolution of (1.1) with $v \geq M := A \cdot d^\gamma$ on the parabolic boundary of $\Omega \times (0, \infty)$. Now the comparison principle yields the claim. \hfill $\blacksquare$

Let us summarize the above results.

**Theorem 2.1.7** Assume that (0.2), (0.3) and (0.4) are satisfied, $q < p + 1$, $\kappa > 0$ and $r > -1$ is arbitrary. Then there are $d_0 > 0$ and $M > 0$ such that $\text{diam} (\Omega) < d_0$ and $\| u_0 \|_{L^\infty(\Omega)} \leq M$ imply that any corresponding positive solution $u$ of (0.1) is global and bounded.
Proof. The case $r < 2p - q$ is covered by Theorem 2.1.2, whereas otherwise we have $r > p - 1$ and hence Lemma 2.1.6 and Theorem 1.1.2 apply.

For $r = 2p - q$ we can improve the reasoning of Lemma 2.1.6 in order to cover large initial data as well.

Corollary 2.1.8 Let (0.2), (0.3) and (0.4) be fulfilled with $q < p + 1$, $r = 2p - q$ and $\kappa > 0$. Then there exists $d_0 \leq \frac{1}{4\sqrt{\kappa}}$ such that $\text{diam}(\Omega) < d_0$ implies that all positive solutions of (0.1) are global and bounded.

Proof. We assume that $d := \text{diam}(\Omega) < \frac{1}{4\sqrt{\kappa}}$, and without loss of generality we suppose that $\Omega \subset \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid d < x_1 < 2d\}$. We choose $\gamma \in (0, 1)$ such that $\gamma \leq \frac{1}{p+1-q}$, $\gamma \leq 1 - 4d\sqrt{\kappa}$ and $(\frac{1-\gamma}{4\kappa\gamma})^{\frac{r+1-q}{p+1-q}} \geq M := \|u_0\|_{L^\infty(\Omega)} + 2$ and define $A > 0$ through (2.15) and $v$ as in (2.17). Then inequality (2.16) holds due to $r + 1 - p = p + 1 - q > 0$ and $2d \leq \frac{1-\gamma}{4\sqrt{\kappa}}$. Hence, we can show as in the proof of Lemma 2.1.6 that $v$ is a supersolution of (1.1) in $\Omega \times (0, \infty)$ with

$$v \geq A \cdot d^\gamma \geq \left(\frac{1-\gamma}{4\kappa\gamma}\right)^{\frac{r+1-q}{p+1-q}} \geq M$$

on the parabolic boundary of $\Omega \times (0, \infty)$, because $\gamma \leq \frac{1}{p+1-q}$. Thus, due to the choice of $M$ we find that the solution $u_\varepsilon$ of (1.1) satisfies $u_\varepsilon \leq v$ in $\Omega \times (0, T_\varepsilon)$ for all $\varepsilon < 1$ by the comparison principle, which implies the claim due to Theorem 1.1.2.

2.2 The case $q > p + 1$

2.2.1 Bounded solutions for $r > 2p - q$

If $q > p + 1$ and $r$ is arbitrary then there always exist solutions blowing up in finite time. This is an easy consequence of the comparison principle and the fact that the same is true for equation (2.1) without gradient term ([Win3]). Therefore, one nontrivial question here is whether there exist (small-data) global solutions at all. As to (2.1), the answer is yes ([Win3]), while here it will depend on $r$. Again, $r = 2p - q$ turns out to be critical.

Lemma 2.2.1 Suppose (0.2), (0.3) and (0.4) are satisfied with $q > p + 1$, $r > 2p - q$ and $\kappa > 0$. Then there exists $C > 0$ such that whenever $\|u_0\|_{L^\infty(\Omega)} \leq C$ then any corresponding positive solution $u$ of (0.1) is global and bounded.

Proof. We fix a number $\alpha \in (\frac{1}{q-p}, 1)$ such that $(p - r)\alpha < 1$ – this is possible since $q > p + 1$ and $r > 2p - q$. Set

$$v(x, t) := \delta^\alpha + \delta e(x), \quad (x, t) \in \Omega \times (0, \infty),$$

where $\delta$ is a positive number.

...
with \( \delta \in (0, 1) \) and the solution \( e \) of \(-\Delta e = 1 \) in \( \Omega \), \( e|_{\partial \Omega} = 1 \). Then

\[
v_t - v^p \Delta v - v^q - \kappa v^r|\nabla v|^2 = \delta(\delta^\alpha + \delta e)^p - (\delta^\alpha + \delta e)^q - \kappa \delta^2(\delta^\alpha + \delta e)^r|\nabla e|^2
\]

\[
=: I_1 - I_2 - I_3.
\]

Here,

\[
\frac{I_2}{I_1} = 2(\delta^\alpha + \delta e)^{q-p}\delta^{-1}
\]

\[
\leq 2(1 + \|e\|_{L^\infty(\Omega)})^{q-p} \cdot \delta^{(q-p)\alpha-1}
\]

\[
\leq 1 \quad \text{in } \Omega \times (0, \infty)
\]

for small \( \delta \), because \((q-p)\alpha - 1 > 0\). Moreover, in the case \( r \leq p \) we have

\[
\frac{I_3}{I_1} = 2\kappa \delta(\delta^\alpha + \delta e)^{r-p}|\nabla e|^2
\]

\[
\leq 2\kappa \delta^{1-(p-r)\alpha}|\nabla e|^2
\]

\[
\leq 1 \quad \text{in } \Omega \times (0, \infty)
\]

for small \( \delta \) due to the fact that \( 1 - (p-r)\alpha > 0 \). However, if \( r > p \) then similarly

\[
\frac{I_3}{I_1} \leq 2\kappa(1 + \|e\|_{L^\infty(\Omega)})^{r-p} \cdot \delta^{1+(r-p)\alpha} \cdot |\nabla e|^2
\]

\[
\leq 1 \quad \text{in } \Omega \times (0, \infty)
\]

for small \( \delta \). Altogether, this shows that if we fix \( \delta > 0 \) sufficiently small then \( v \) is a supersolution of (1.1) for suitably small \( \varepsilon < \varepsilon_0(\delta) \). Therefore, all solutions \( u \) of (0.1) with initial data \( u_0 \) fulfilling \( u_0 < \delta^\alpha + \delta e \) in \( \Omega \) must be global in time and uniformly bounded due to the comparison principle and Theorem 1.1.2.

2.2.2 Unconditional blow-up in large domains for \( r \leq 2p - q \)

Going back to our original question whether the gradient term can enforce blow-up, we now ask if there are circumstances under which all solutions blow up. Contrary to (2.1), the problem (0.1) turns out to have this property for \( r \leq 2p - q \) and sufficiently large domains.

To see this, we first assume that the strict inequality \( r < 2p - q \) holds. Our strategy will be to construct solutions of (0.1) that are small at \( t = 0 \) and increase with time. We shall show in the proof of Theorem 2.2.4 that if these solutions were global then they should approach certain (possibly unbounded) steady state as \( t \to \infty \). But such equilibria do not exist, as Lemma 2.2.3 will reveal.

We start with the explicit definition of arbitrarily small stationary subsolutions to (1.1).
Lemma 2.2.2 Assume (0.2), \( q > p + 1 \), \( r < 2p - q \), \( \kappa > 0 \) and \( R > 0 \). Then there are \( m \in \mathbb{N} \) and \( \delta_0 > 0 \) such that for all \( \delta < \delta_0 \) and any \( \varepsilon \in (0, 1) \) the function
\[
\varphi_\varepsilon(x) := \delta \left( 1 - \left( \frac{|x|}{R} \right)^2 \right)^m + \varepsilon, \quad x \in \bar{B}_R(0),
\]
satisfies
\[
\varphi_\varepsilon^p \Delta \varphi_\varepsilon + \varphi_\varepsilon^q + \kappa \varphi_\varepsilon^r |\nabla \varphi_\varepsilon|^2 \geq 0 \quad \text{in} \ B_R(0). \tag{2.18}
\]
PROOF. Fix \( m \in \mathbb{N} \) such that \( m \geq \frac{p-r}{2p-q-r} \). Switching to polar coordinates and writing \( \rho = |x| \), we compute
\[
\begin{align*}
\varphi_\varepsilon^p \Delta \varphi_\varepsilon + \varphi_\varepsilon^q + \kappa \varphi_\varepsilon^r |\nabla \varphi_\varepsilon|^2 &= -2m(2m + n - 2)\delta R^{-2m} \rho^{-2m-2} \cdot \varphi_\varepsilon^p \\
&\quad + \varphi_\varepsilon^q + 4\kappa m^2 \delta^2 R^{-4m} \rho^{4m-2} \cdot \varphi_\varepsilon^r \\
&= : -I_1 + I_2 + I_3.
\end{align*}
\]
Here,
\[
\frac{I_1}{I_3} = \frac{2m + n - 2}{2km} \frac{\delta^{-1} R^{-2m} \rho^{-2m} \cdot \varphi_\varepsilon^{p-r}}{\leq 1}
\]
is valid at those points for which
\[
\varphi_\varepsilon \leq \left( \frac{2km}{2m + n - 2} \right)^\frac{1}{p-r} \delta^\frac{1}{p-r} R^{\frac{2m}{p-r}} \rho^{\frac{2m}{p-r}} \tag{2.19}
\]
holds – note here that \( r < 2p - q \) and \( q > p + 1 \) imply \( p-r > 0 \). On the other hand, if (2.19) is violated then our choice of \( m \) yields
\[
\frac{I_1}{I_2} = \frac{2m(2m + n - 2)\delta R^{-2m} \rho^{-2m-2} \cdot \varphi_\varepsilon^{-(q-p)}}{< 2m(2m + n - 2) \cdot \left( \frac{2m + n - 2}{2km} \right)^\frac{2m}{2p-q-r} \delta R^{-2m} \rho^{-2m-2} \cdot \varphi_\varepsilon^{-(q-p)} \\
\leq c(m, n, \kappa) R^{-2m} \rho^{-2m-2} \cdot \varphi_\varepsilon^{-(q-p)} \leq c(m, n, \kappa) R^{-2m} \rho^{-2m-2} \cdot \varphi_\varepsilon^{-(q-p)} \leq 1,
\]
provided that \( \delta < \delta_0 \) and \( \delta_0 \) is sufficiently small. This proves (2.18).

As a final preparation, we proceed to exclude the existence of radially decreasing steady states with mild singularities at the origin for \( r < 2p - q \).

Lemma 2.2.3 Let (0.2) be fulfilled with \( q > p + 1 \), \( r < 2p - q \) and \( \kappa > 0 \). Then there exists \( R_0 > 0 \) such that for all \( R > R_0 \) the problem
\[
\Delta w + w^{q-p} + \kappa w^{r-p} |\nabla w|^2 = 0 \quad \text{in} \ B_R(0) \setminus \{0\} \tag{2.20}
\]
has no positive radially symmetric solution \( w \in L^{q-p}(B_R(0)) \) that is nonincreasing with respect to \( |x| \).
Remark. Observe that the lemma in particular excludes solutions which are regular at the origin.

Proof. Let $a_0 > 0$ be defined by

$$a_0 := \begin{cases} 
\left(\frac{2(1-2^{-n})\kappa}{n(n-1)}\right)^{\frac{1}{2p-q-r}} & \text{if } n > 1, \\
\kappa^{\frac{1}{2p-q-r}} & \text{if } n = 1,
\end{cases}$$

and let $\rho_1 \geq 1$ be large such that

$$\left(\frac{\lambda_{\rho_1}}{q - p}\right)^{\frac{1}{q-p-1}} \leq a_0.$$

Here again, $\lambda_{\rho}$ denotes the principal Dirichlet eigenvalue of $-\Delta$ in $B_{\rho} := B_{\rho}(0)$.
Assume that $R > 2\rho_1$ is such that (2.20) has a radially decreasing solution $w$ in $B_R$.
Our goal is to derive an upper bound for $R$. In polar coordinates, $w = w(\rho)$ solves

$$w_{\rho\rho} + \frac{n - 1}{\rho} w_\rho = -w^{q-p} - \kappa w^{r-p} w^2, \quad \rho \in (0, R). \quad (2.21)$$

We first claim that

$$w(\rho) \leq \left(\frac{\lambda_{\rho}}{q - p}\right)^{\frac{1}{q-p-1}} \forall \rho > 0, \quad (2.22)$$
whence particularly

$$w(\rho_1) \leq a_0. \quad (2.23)$$

To this end, we multiply (2.20) by $\Theta := \Theta(\cdot, B_{\rho})$ and integrate to obtain

$$\lambda_{\rho} \int_{B_{\rho}} w\Theta + \int_{\partial B_{\rho}} w\partial_{\nu}\Theta = \int_{B_{\rho}} w^{q-p}\Theta + \kappa \int_{B_{\rho}} w^{r-p}\left|\nabla w\right|^2\Theta \geq \int_{B_{\rho}} w^{q-p}\Theta.$$ 

Thus, since $\int_{\partial B_{\rho}} \partial_{\nu}\Theta = -\lambda_{\rho} \cdot \int_{B_{\rho}} \Theta$,

$$\lambda_{\rho} w(\rho) \leq \lambda_{\rho} \frac{\int_{B_{\rho}} w\Theta}{\int_{B_{\rho}} \Theta} - \frac{\int_{B_{\rho}} w^{q-p}\Theta}{\int_{B_{\rho}} \Theta} \leq \lambda_{\rho} \left(\frac{\int_{B_{\rho}} w\Theta}{\int_{B_{\rho}} \Theta}\right)^{\frac{q-p}{q-p-1}}$$
due to Hölder’s inequality. Since $\lambda_{\rho} z - z^{q-p} \leq (q - p)^{-\frac{1}{q-p-1}} \lambda_{\rho}^{1+\frac{1}{q-p-1}}$ for all $z > 0$
by an elementary optimization, this yields (2.22).
Next, if we set $a := w(2\rho_1)$ then from (2.21) we deduce

$$\frac{1}{\rho^{n-1}}(\rho^{n-1} w)_{\rho} \leq -a^{q-p} \quad \forall \rho \in [\rho_1, 2\rho_1]$$
Consequently, as \( \psi \) follows that \( e \) since the term on the right hand side becomes negative when 

\[
\frac{1}{2}w_\rho^2 \geq -w_\rho^{q-p}w_\rho - \kappa w_\rho^{r-p}w_\rho^3 - \frac{n-1}{\rho}w_\rho^2 \geq -\kappa a^{r-p}w_\rho^3 - \frac{n-1}{2\rho_1}w_\rho^2 \quad \forall \rho \in (2\rho_1, R),
\]

since \( w_\rho(\rho_1) \leq 0 \). Multiplying (2.21) by \( w_\rho \leq 0 \), we furthermore obtain

\[
\left( \frac{1}{2}w_\rho^2 \right)_\rho = -w_\rho^{q-p}w_\rho - \kappa w_\rho^{r-p}w_\rho^3 - \frac{n-1}{\rho}w_\rho^2 \\
= -\kappa a^{r-p}w_\rho^3 - \frac{n-1}{2\rho_1}w_\rho^2 \quad \forall \rho \in (2\rho_1, R),
\]

because \( r < 2p - q < p - 1 < p \) and \( w(\rho) \leq w(2\rho_1) = a \) for \( \rho \geq 2\rho_1 \). Therefore, 
\[
\psi(\rho) := \frac{1}{2}e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)}w_\rho^2 \quad \forall \rho \in (2\rho_1, R).
\]

Thus, if \( n > 1 \),

\[
\left. \frac{1}{2} \psi(\rho) - \frac{1}{2}(2\rho_1) \right| \geq 2^\frac{3}{2} \kappa a^{r-p} \cdot \int_{2\rho_1}^\rho e^{-\frac{n-1}{\rho_1}(s-2\rho_1)} ds \\
= \frac{2^\frac{3}{2} \kappa \rho_1}{n-1} a^{r-p} \cdot \left( 1 - e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)} \right) \forall \rho \in (2\rho_1, R). (2.26)
\]

Consequently, as \( \psi^{-\frac{1}{2}}(2\rho_1) \leq \frac{n}{2^\frac{1}{2}(1-2^{-n})\rho_1}a^{p-q} \) by (2.24),

\[
\psi^{-\frac{1}{2}}(\rho) \leq \frac{n}{2^\frac{1}{2}(1-2^{-n})\rho_1}a^{p-q} - \frac{2^\frac{3}{2} \kappa \rho_1}{n-1} a^{r-p} \left( 1 - e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)} \right) \\
= \frac{2^\frac{3}{2} \kappa \rho_1}{n-1} a^{r-p} \cdot \left[ \frac{1}{2\rho_1^2} \left( \frac{a}{a_0} \right)^{2p-q-r} \left( 1 - e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)} \right) \right] \\
\leq \frac{2^\frac{3}{2} \kappa \rho_1}{n-1} a^{r-p} \cdot \left[ \frac{1}{2} - \left( 1 - e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)} \right) \right] \quad \forall \rho \in (2\rho_1, R).
\]

Since the term on the right hand side becomes negative when \( e^{-\frac{n-1}{\rho_1}(\rho-2\rho_1)} < \frac{1}{2} \), it follows that

\[
R \leq 2\rho_1 + \frac{2\rho_1}{n-1} \ln 2,
\]
which proves the lemma in the case \( n > 1 \). In the one-dimensional setting, however, (2.25) even takes a more convenient form and the rest of the proof is even simpler: Instead of (2.26) we then obtain

\[
\psi^{-\frac{1}{2}}(\rho) - \psi^{-\frac{1}{2}}(2\rho_1) \geq 2^\frac{3}{2}\kappa a^{r-p}(\rho - 2\rho_1)
\]

and, again using (2.24), we end up with the conclusion that

\[
R \leq 2\rho_1 + 1
\]

in the case of \( n = 1 \).

\[\text{Remark.}\]

Note that the eigenvalues \( \lambda_\rho \) in the above proof satisfy \( \lambda_\rho < \frac{c_n}{\rho^2} \) with some constant \( c_n \) – for instance, one may pick \( c_n = \frac{n^2\pi^2}{4} \) which is gained by explicitly calculating the corresponding eigenvalue of the largest cube contained in \( B_\rho \). Now a consequent inspection of the constants in use shows that in the case \( n > 1 \) we may take

\[
R_0 = 2\left(\frac{c_n}{q-p}\right)^{\frac{1}{2}} \cdot \left(\frac{n(n-1)}{2(1-2^{-n})\kappa}\right)^{\frac{n-p-1}{2n(p-q-1)}} \cdot \left(1 + \frac{1}{n-1} \cdot \ln 2\right).
\]

Now we are ready for the proof of the result mentioned above.

**Theorem 2.2.4** Suppose that (0.2), (0.3) and (0.4) are satisfied with \( q > p + 1 \), \( r < 2p - q \) and \( \kappa > 0 \). Then there exists \( R_0 > 0 \) such that whenever \( \Omega \) contains a ball with radius \( R > R_0 \) then any maximal solution of (0.1) in \( \Omega \) blows up in finite time.

**Proof.** Let \( R_0 \) be as provided by Lemma 2.2.3 and assume without loss of generality that \( \bar{B}_R \subset \Omega \) with some \( R > R_0 \), where again \( B_\rho := B_\rho(0) \). Suppose that \( u \) is a global maximal solution of (0.1) with positive initial data \( u_0 \). Then there exist \( \delta > 0 \) and \( m \in \mathbb{N} \) such that

\[
u_0(x) > \delta \left(1 - \left(\frac{|x|}{R}\right)^{2m}\right) + \varepsilon =: \varphi_\varepsilon(x) \quad \forall x \in B_R
\]

holds for all sufficiently small \( \varepsilon > 0 \). Let \( v_\varepsilon \) denote the maximally extended solution of (1.1) in \( B_R \times (0, T_\varepsilon) \) with initial data \( v_{\varepsilon}|_{t=0} = \varphi_\varepsilon \). According to Lemma 1.3.4 and Lemma 2.2.2 we have \( v_{\varepsilon t} \geq 0 \) in \( B_R \times (0, T_\varepsilon) \). Thus, \( v := \lim_{\varepsilon \searrow 0} v_\varepsilon \) satisfies \( v_t \geq 0 \) for \( x \in B_R \) and \( 0 < t < T := \lim_{\varepsilon \searrow 0} T_\varepsilon \) and, by standard arguments, \( v \) is a radially symmetric solution of (0.1) in \( B_R \times (0, T) \) which is nonincreasing with respect to \( |x| \). Our goal is to show that \( T < \infty \); this implies that \( u \) blows up in finite time, because \( u \geq v \) in \( B_R \times (0, T) \) by the comparison principle.
In fact, suppose on the contrary that \( v \) exists globally, that is, \( T = \infty \). Since \( v_t \geq 0 \), we have \( v(t) \not\to w \) in \( B_R \) as \( t \to \infty \), where \( w : B_R \to (0, \infty) \) is radially symmetric and nonincreasing with respect to \( |x| \). We first claim that

\[
\int_{B_R} w^{q-p} < \infty. \tag{2.27}
\]

For this purpose let \( \lambda := \lambda_1(B_{R/2}) \) and \( \tilde{\Theta} := \alpha \Theta(\cdot, B_R) \) with \( \alpha > 0 \) such that \( \int_{B_R} \tilde{\Theta} = 1 \). We multiply the inequality \( v_t \geq v^p \Delta v + v^q \) by \( v^{-p} \tilde{\Theta} \), integrate over \( B_{R/2} \) and use the estimate \( z - \frac{1}{2\lambda} z^{q-p} \leq \left( \frac{2\lambda}{q-p} \right)^{\frac{q-p}{q-1}} =: c_0 \quad \forall z > 0 \) to see that

\[
\frac{d}{dt} \int_{B_{R/2}} H(v) \tilde{\Theta} \geq \int_{B_{R/2}} \Delta v \cdot \tilde{\Theta} + \int_{B_{R/2}} v^{q-p} \tilde{\Theta}
\geq -\lambda \int_{B_{R/2}} v \tilde{\Theta} + \int_{B_{R/2}} v^{q-p} \tilde{\Theta}
\geq -\lambda c_0 + \frac{1}{2} \int_{B_{R/2}} v^{q-p} \tilde{\Theta} \tag{2.28}
\]

for \( t > 0 \), where we have set

\[
H(s) := \begin{cases} 
\frac{1}{1-p} s^{1-p} & \text{if } p \neq 1, \\
\ln s & \text{if } p = 1
\end{cases}
\]

for \( s > 0 \). It is easy to check that \( \Phi(s) := H(s^{\frac{1}{q-p}}) \) is increasing and concave on \((0, \infty)\). Therefore, Jensen’s inequality can be applied to yield

\[
\int_{B_{R/2}} \Phi(v^{q-p} \tilde{\Theta}) \leq \Phi \left( \int_{B_{R/2}} v^{q-p} \tilde{\Theta} \right)
\]

in view of our normalization \( \int_{B_{R/2}} \tilde{\Theta} = 1 \), and thus

\[
\int_{B_{R/2}} v^{q-p} \tilde{\Theta} \geq \Phi^{-1} \left( \int_{B_{R/2}} \Phi(v^{q-p} \tilde{\Theta}) \right)
= \Phi^{-1} \left( \int_{B_{R/2}} H(v) \tilde{\Theta} \right).
\]

Inserted into (2.28), this shows that \( y(t) := \int_{B_{R/2}} H(v(\cdot, t)) \tilde{\Theta} \) satisfies the differential inequality

\[
y'(t) \geq -c_0 \lambda + \frac{1}{2} \Phi^{-1}(y), \quad t > 0. \tag{2.29}
\]
Evaluating $\Phi^{-1}$ explicitly, we obtain

$$y'(t) \geq \begin{cases} -c_0\lambda + \frac{1-p}{2} \frac{y^{\frac{q}{p}}}{y^{\frac{q}{p}}} & \text{if } p < 1, \\ -c_0\lambda + \frac{1}{2} e^{(q-p)y} & \text{if } p = 1, \\ -c_0\lambda + \frac{1}{2(p-1)^{p-1}} (-y)^{-\frac{q-p}{p-1}} & \text{if } p > 1. \end{cases}$$

Consequently, $y(t) \leq C_0$ holds for all $t > 0$ with an appropriate constant

$$\begin{cases} C_0 > 0 & \text{if } p < 1, \\ C_0 \in \mathbb{R} & \text{if } p = 1, \\ C_0 < 0 & \text{if } p > 1, \end{cases}$$

for otherwise $y$ should either blow up (if $p \leq 1$) or become zero (if $p > 1$) in finite time. Now integrating (2.28) over the time interval $(t, t+1)$ for $t > 0$ shows that

$$\frac{1}{2} \int_t^{t+1} \int_{B_{\frac{R}{2}}} v^{q-p} \tilde{\Theta} \leq c_0\lambda + y(t+1) - y(t)$$

$$= c_0\lambda + \int_{B_{\frac{R}{2}}} H(v(\cdot, t+1)) \tilde{\Theta} - \int_{B_{\frac{R}{2}}} H(v(\cdot, t)) \tilde{\Theta}.$$

Here,

$$y(t+1) - y(t) \leq \begin{cases} C_0 & \text{if } p < 1, \\ C_0 - \int_{B_{\frac{R}{2}}} \ln v(\cdot, t) \tilde{\Theta} \leq C_0 - \int_{B_{\frac{R}{2}}} \ln v(\cdot, 0) \tilde{\Theta} & \text{if } p = 1, \\ \frac{1}{p-1} \int_{B_{\frac{R}{2}}} v^{1-p}(\cdot, t) \tilde{\Theta} \leq \frac{1}{p-1} \int_{B_{\frac{R}{2}}} v^{1-p}(\cdot, 0) \tilde{\Theta} & \text{if } p > 1, \end{cases}$$

where we have used that $y$ is positive if $p < 1$ and negative if $p > 1$, and that $v_t \geq 0$ in the case $p \geq 1$. We thereby have shown that $\int_t^{t+1} \int_{B_{\frac{R}{2}}} v^{q-p} \tilde{\Theta} \leq c_1$ for all $t > 0$ and some $c_1 > 0$. But $v_t \geq 0$ then implies

$$c_1 \geq \int_{B_{\frac{R}{2}}} \left( \int_t^{t+1} v^{q-p}(\cdot, s) ds \right) \tilde{\Theta}$$

$$\nearrow \int_{B_{\frac{R}{2}}} w^{q-p} \tilde{\Theta} \quad \text{as } t \to \infty$$

by the monotone convergence theorem. As $\tilde{\Theta}$ is uniformly positive in $B_{\frac{R}{2}}$ and $w$ decreases with $|x|$, (2.27) has thus been established.

In particular, we now know that for all $\eta > 0$ the two-sided estimate $c_\eta^{-1} \leq v \leq c_\eta$ holds in $(B_{R-\eta} \setminus B_\eta) \times (0, \infty)$ with some positive constant $c_\eta$. Therefore, parabolic Schauder theory ensures that the convergence $v(\cdot, t) \to w$ takes place in
$C^2_{loc}(B_R \setminus \{0\})$. Since, furthermore, $v_i \geq 0$ and $\lim_{t \to \infty} v(x, t) < \infty \quad \forall 0 < |x| < R$ imply that for each $0 < |x| < R$ we have $\limsup_{t \to \infty} v_i(x, t) = 0$, we altogether obtain that $w$ is a radially nonincreasing positive solution of

$$\Delta w + w^{q-p} + \kappa w^{r-p} |\nabla w|^2 = 0 \quad \text{in } B_R \setminus \{0\}.$$ 

According to Lemma 2.2.3, however, such a solution with the additional property (2.27) cannot exist. Therefore, our assumption $T = \infty$ must fail and thus both $v$ and $u$ blow up in finite time.

In the borderline case $r = 2p - q$, we again invoke a suitable transformation of (0.1) in order to circumvent the inconvenience that the above technique appears to fail. Using this substitution, we can establish a result similar to the one just obtained for $r < 2p - q$.

**Theorem 2.2.5** Assume that (0.2), (0.3) and (0.4) are fulfilled with $q > p + 1$, $r = 2p - q$ and $\kappa > 0$. Then there exists $R_0 > 0$ such that whenever $\Omega$ contains a ball with radius $R > R_0$ then any maximal solution of (0.1) in $\Omega$ blows up in finite time.

**Proof.** i) For $\beta < 0$ we choose $c := \max\{\ln\left(\left(\frac{-\beta}{\kappa}\right) \frac{e^{\kappa \beta}}{\kappa}\right), 0\}$ and define

$$g_1(v) := \left(\frac{\beta}{\kappa} \left[\ln(v) + \frac{\beta - 1}{\beta} \ln(-\ln(v)) - c\right]\right)^{\frac{1}{\beta}} \quad \text{for } v \in (0, v_0),$$

where $v_0 \in (0, 1)$ is chosen small enough such that $\left(\frac{2\beta}{\kappa} \ln(v)\right)^{\frac{1}{\beta}} \leq g_1(v) \leq \left(\frac{\beta}{2\kappa} \ln(v)\right)^{\frac{1}{\beta}}$ holds for $v \in (0, v_0)$. We have already shown in the proof of Theorem 1.2.2 that there is a solution $f \in C^0([0, \infty)) \cap C^1((0, \infty))$ of (2.12), such that $f(v) \geq g_1(v)$ for $v \in (0, v_0)$ and $f(v) \geq v$ for $v > 0$ is fulfilled.

Furthermore, we define $c_0 := \min\left\{(-\frac{2\beta}{\kappa}) \frac{e^{-c}}{\kappa^{\frac{1}{\beta}}}, v_0^{-\beta} e^{\kappa^{\frac{1}{\beta}}}\right\} > 0$. Then by the choice of $v_0$ we have

$$h(v) := \left(f(v)\right)^{1-\beta} e^{\kappa \left(f(v)\right)^{\beta}} = \left(\frac{\beta}{\kappa} \left[\ln(v) + \frac{\beta - 1}{\beta} \ln(-\ln(v)) - c\right]\right)^{\frac{1}{\beta}} \left(-\ln(v)\right)^{\frac{2\beta - 1}{\beta}} e^{-c}$$

$$= \left(\frac{-\beta}{\kappa} + \frac{\beta - 1}{\kappa} \ln(-\ln(v)) + \frac{\beta e}{\kappa \ln(v)}\right)^{\frac{1-\beta}{\beta}} e^{-c} v \geq c_0 v$$

for $v \in (0, v_0)$ and

$$h(v) := \left(f(v)\right)^{1-\beta} e^{\kappa \left(f(v)\right)^{\beta}} \geq v^{1-\beta} e^{\kappa^{\frac{1}{\beta}}} \geq \left(v_0^{-\beta} e^{\kappa^{\frac{1}{\beta}}}\right) v \geq c_0 v.$$
if \( v \geq v_0 \).

Summarizing, we have found a constant \( c_0 > 0 \) which only depends on \( \beta \) and \( \kappa \) such that \( f(v) \geq v \) and \( h(v) \geq c_0 v \) hold for all \( v > 0 \).

ii) Now suppose \( q > p + 1, r = 2p - q \) and \( \kappa > 0 \). Then we define \( \beta := r - p + 1 < 0 \) and, furthermore, we can choose \( c_0 > 0 \) as in part i), observing that this choice depends on \( \beta \) and \( \kappa \). For \( R > 0 \) let \( \lambda_R := \lambda_1(B_R(0)) \) and fix \( R_0 > 0 \) large enough such that \( \lambda_R < c_0 \) holds.

The rest of the proof is analogous to that of Theorem 2.1.5: Assuming that \( B := B_{R_0}(0) \subset \Omega \) and that \( u \) is a global maximal solution of (0.1) corresponding to some initial data \( u_0 \), we substitute \( u(x,t) = f(v(x,t)) \) for \( (x,t) \in \bar{\Omega} \times [0,\infty) \), where \( f \) is defined as in part i). Then we can proceed as in the proof of Theorem 2.1.5 to see that \( v \) fulfills (2.13) in \( \Omega \times [0,\infty) \). Hence, \( v \) is a positive supersolution of (2.14) in \( B \times [0,\infty) \) by part i). Since \( \int_1^{\infty} \frac{ds}{s(f(s))^{p}} < \infty \) because \( f(v) \geq v \) if \( v > 0 \), \( v \) blows up in finite time by Theorem 1 in [Win2] and the maximum principle.

\[ \blacksquare \]

### 2.2.3 Bounded solutions in small domains

For \( r \leq 2p - q \), boundedness in small domains can be seen in a way quite similar to the one used to prove Lemma 2.1.6.

**Lemma 2.2.6** Let (0.2), (0.3) and (0.4) be satisfied with \( q > p + 1, r \leq 2p - q \) and \( \kappa > 0 \). Then there are \( d_0 > 0 \) and \( M > 0 \) such that \( \text{diam}(\Omega) < d_0 \) implies that each positive solution of (0.1) with \( \|u_0\|_{L^\infty(\Omega)} \leq M \) is global and bounded.

**Proof.** Let \( d := \text{diam}(\Omega) \) and \( \gamma \in (0,1) \), and assume that after a change of coordinates we have \( \Omega \subset \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n \mid d < x_1 < 2d\} \). We choose \( A := \left(\frac{1}{2\kappa \gamma}\right)^{1-\frac{1}{p-1}} d^{-\gamma} \) and \( d_0 > 0 \) small so that \( A \leq \left(\frac{2}{\gamma(1-\gamma)}\right)^{\frac{1}{q(p-1)}} (2d)^{-\frac{2}{q(p-1)}-\gamma} \) for all \( 0 < d < d_0 \).

Then we define \( v(x,t) := Ax_1^\gamma \) for \( (x,t) \in \Omega \times (0,\infty) \) and obtain

\[
v_t - v^p \Delta v - v^q - \kappa v |\nabla v|^2 = \gamma(1-\gamma)Ap^{p+1}x_1^{p(1+\gamma)-2} - Ap^2x_1^{\gamma} - \kappa \gamma^2 A^{r+2}x_1^{r+2}\gamma^{-2} =: I_1 - I_2 - I_3.
\]

Due to our choice of \( A, \gamma \) and \( d_0 \) we have \( I_2 \leq \frac{1}{2}I_1 \) and \( I_3 \leq \frac{1}{2}I_1 \) for \( 0 < d < d_0 \) (which can be seen as in the proof of Lemma 2.1.6). Hence, \( v \) is a supersolution of (0.1) with \( v \geq M := A \cdot d^{\gamma} \) on the parabolic boundary of \( \Omega \times (0,\infty) \) and the claim follows upon applying the comparison principle to \( v \) and \( u_\varepsilon \) and then letting \( \varepsilon \searrow 0 \) and using Theorem 1.1.2.

\[ \blacksquare \]

Altogether, as a consequence of Lemma 2.2.1 and Lemma 2.2.6 we obtain global bounded solutions in sufficiently small domains for any \( r > -1 \).
CHAPTER 2. BOUNDEDNESS VS. BLOW-UP, PART I

Corollary 2.2.7  Assume that (0.2), (0.3) and (0.4) are satisfied, \( q > p + 1 \), \( \kappa > 0 \) and that \( r > -1 \) is arbitrary. Then there exist \( d_0 > 0 \) and \( M > 0 \) such that \( \text{diam}(\Omega) < d_0 \) and \( \| u_0 \|_{L^\infty(\Omega)} \leq M \) imply that any corresponding positive solution \( u \) of (0.1) is global and bounded.

2.3 The case \( q = p + 1 \)

Let us recall that all positive solutions of the simpler equation \( u_t = u^p \Delta u + u^{p+1} \) blow up in finite time if and only if \( \Omega \) is large in the sense that \( \lambda_1(\Omega) < 1 \) ([SGKM], [Wie1]). In contrast to this, the additional gradient term can enforce blow-up of large-data solutions in arbitrarily small domains, provided that \( r > p - 1 \).

Corollary 2.3.1  Let (0.2), (0.3) and (0.4) be fulfilled with \( q = p + 1 \), \( r > p - 1 \) and \( \kappa > 0 \). Then for every \( w \in C^0(\bar{\Omega}) \) which is positive in \( \Omega \) with \( w = 0 \) on \( \partial \Omega \) there is \( b_0 > 0 \) such that the maximal solution of (0.1) having \( u|_{t=0} = bw \) with \( b \geq b_0 \) blows up in finite time.

Proof. The claim immediately follows from Theorem 2.1.4, because we have \( r > p - 1 \equiv 2p - q \).

However, for \( r \neq p - 1 \) the value \( \lambda_1(\Omega) = 1 \) remains a critical first eigenvalue in respect of existence of bounded solutions. Namely, we first have the following lemma.

Lemma 2.3.2  Assume (0.2), (0.3) and (0.4) are satisfied, \( q = p + 1 \), \( \kappa > 0 \) and \( \Omega \) is small such that \( \lambda_1(\Omega) > 1 \) is fulfilled. Then for \( r < p - 1 \) all positive solutions of (0.1) are global and bounded, whereas in case of \( r > p - 1 \) there exists \( M > 0 \) such that \( \| u_0 \|_{L^\infty(\Omega)} \leq M \) implies that each corresponding positive solution of (0.1) is global and bounded.

Proof. Since \( \lambda_1(\Omega) > 1 \), we can fix a smoothly bounded domain \( G \) with \( \Omega \subset \subset G \) and \( \lambda := \lambda_1(G) > 1 \). Moreover, we define \( \Theta := \Theta(\cdot, G) \) and due to the choice of \( G \) there is \( \mu > 0 \) with \( \Theta \geq \mu \) in \( \bar{\Omega} \). Furthermore, we pick \( \alpha \in (\frac{1}{\chi}, 1) \).

First, let us assume \( r > p - 1 \) and define \( \delta := \left( \frac{1 - \alpha}{\mu \alpha} \right)^{\frac{1}{r - p + 1}} \). Then \( v := \delta \Theta^\alpha \) satisfies

\[
\Delta v + v + \kappa v^{r-p}|\nabla v|^2 = -\delta \alpha \lambda \Theta^\alpha + \delta \alpha (\alpha - 1) \Theta^{\alpha - 2} |\nabla \Theta|^2 + \delta \Theta^\alpha + \kappa \delta^{r-p+2} \alpha^2 \Theta^{\alpha(r-p)+2(\alpha-1)} |\nabla \Theta|^2 \\
= \delta \Theta^\alpha (\alpha \lambda + 1) + \delta \alpha \Theta^{\alpha - 2} |\nabla \Theta|^2 \left( \alpha - 1 + \kappa \delta^{r-p+1} \alpha \Theta^{\alpha(r-p+1)} \right) \\
\leq 0 \quad \text{in } \Omega
\]
due to the choice of \( \alpha \) and \( \delta \), so that \( v \) is a supersolution of (0.1) in \( \Omega \times (0, \infty) \). Thus, if \( u_0 \) is small enough such that \( \| u_0 \|_{L^\infty(\Omega)} < C := \delta \mu^{\alpha} \) and \( u_\epsilon \) is the corresponding solution of (1.1), then \( u_\epsilon \leq v \) holds on the parabolic boundary of \( \Omega \times (0, T_\epsilon) \) for small
\[ \varepsilon \in (0, 1) \] thanks to the choice of \( \mu \). Therefore, \( u_\varepsilon(x, t) \leq v(x) \) for \( (x, t) \in \Omega \times (0, T_\varepsilon) \) by comparison, which according to Theorem 1.1.2 implies that a corresponding positive solution \( u \) of (0.1) is global and bounded.

The procedure in case of \( r < p - 1 \) is quite similar: Given any \( u_0 \), we now pick \( \delta \) in such a way that \( \delta > \mu^{-\alpha}\|u_0\|_{L^\infty(\Omega)} \) as well as \( \delta \geq \left( \frac{1}{\kappa^{\alpha(p-1-r)}} \right)^{\frac{1}{p-1-r}} \). Upon this choice, (2.30) remains valid line by line (since \( \Theta^{\alpha(r-p+1)} \leq \mu^{\alpha(r-p+1)} \) in \( \Omega \)), so that the rest of the argument is the same as before.

Secondly, to complete the picture, we state another easy consequence of the known blow-up results for \( u_t = u^p \Delta u + u^{p+1} \).

**Corollary 2.3.3** Let (0.2), (0.3) and (0.4) be fulfilled, \( q = p + 1 \), \( \kappa > 0 \) and \( r > -1 \) be arbitrary. If \( \lambda_1(\Omega) < 1 \) then all positive solutions of (0.1) blow up in finite time. Moreover, if \( r = p - 1 \) then the same is true even under the weaker assumption \( \lambda_1(\Omega) < \kappa + 1 \), whereas if \( \lambda_1(\Omega) > \kappa + 1 \) then all positive solutions are global and bounded.

**Proof.** While the first part is obvious from the comparison principle, the assertions concerning \( r = p - 1 \) can be seen upon the substitution \( v(x, t) = u^{\kappa+1}\left( \frac{\varepsilon^{\frac{1}{\kappa+1}}}{\kappa+1} \right) \) which transforms (0.1) into the Dirichlet problem for \( v_t = v^{\kappa+1}\Delta v + v^{\kappa+1+1} \) in \( G \times (0, (\kappa + 1)T) \), where \( G := \{\varepsilon^{\frac{1}{\kappa+1}}x \mid x \in \Omega \} \). As to the latter, however, it is known that if \( \lambda_1(G) < 1 \) then all solutions blow up, while in case of \( \lambda_1(G) > 1 \) all positive solutions are global and bounded (cf. [SGKM], [Wie1] and [Wie2], for instance; more results based on this transformation can be found in [Sti]).
Chapter 3

Boundedness versus blow-up in case of gradient absorption

In this chapter, we study the behavior of the maximal solution of (0.1) in case of $\kappa < 0$. Since the gradient term is now absorbing, blow-up in finite time can only occur if the influence of the source term $u^q$ is stronger than the influence of the gradient term $|\kappa| u^r|\nabla u|^2$ and of the diffusion term $u^p \Delta u$. It turns out that this phenomenon only arises in case of $q \geq \max\{p + 1, r + 2\}$, when the source term has the highest order compared to the gradient term and the diffusion term. More precisely, we show that $r = q - 2$ is the critical exponent for (0.1) concerning global existence versus blow-up in finite time in the following sense:

- In case of $r > q - 2$, any maximal solution is global and bounded (Theorem 3.1.1).

- In case of $r = q - 2$, all maximal solutions blow up for $q = p + 1$ and $\kappa > -1$ in large domains $\Omega$ fulfilling $\lambda_1(\Omega) \in (0, \kappa + 1)$, whereas in all other constellations of the parameters the maximal solution is global and bounded (Theorem 3.2.1).

- In case of $r < q - 2$, the maximal solution blows up for $q > p + 1$ and large initial data, whereas in the other cases (apart from the case $q = p + 1$ in large domains and with large initial data, which remains open) every maximal solution is global and bounded (Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.3).

Compared to the equation (0.6) without the gradient term (see the beginning of Chapter 2), this shows that the gradient term can prevent blow-up of any solution if $r$ exceeds the critical value $q - 2$. However, if $r$ is small, then the gradient term is able to prevent blow-up only for small initial data. Moreover, similar to the case $\kappa > 0$ (see Chapter 2), the factor $\kappa$ plays an important role in case of $q = p + 1$ and $r = q - 2(= p - 1)$, whereas in the other cases only the sign of $\kappa$ seems to be important.

A similar behavior has already been observed for a number of diffusion equations with gradient absorption. Especially for the Chipot-Weissler equation $u_t = \Delta u + \ldots$
$u^q - \mu|\nabla u|^r$, where $q > 1$, $s \geq 1$ and $\mu > 0$, it has been shown in [SouWei1] and [SouWei2] that blow-up only occurs in case of $s < q$, where the source term has a higher order than the gradient term. In [SouWei1], the blow-up result for (0.1) in case of $q > p + 1 \geq 2$, $1 \leq r < q - 2$ and $\kappa < 0$ is contained, too. Similar results for other diffusion equations can be found for example in [AMST], [Bar] and [Sou3]. Additionally, we deal with the question whether the global solutions of (0.1) converge to 0 as $t \to \infty$. In case of the Dirichlet problem corresponding to equation (0.6) without the gradient term, it has been proved that for $q > p + 1$ with small initial data and for $q = p + 1$ in small domains $\Omega$ satisfying $\lambda_1(\Omega) > 1$ the solutions converge to 0 uniformly as $t \to \infty$. Furthermore, in case of $q < p + 1$, for any positive solution $u$ of (0.6) and any $K \subset \subset \Omega$ there is a positive constant $c_K$ such that $u \geq c_K$ in $K \times [0, \infty)$ is fulfilled. For more details we refer to [SGKM], [Wie2] and [Win3].

Concerning the equation (0.1), we show that again the exponent $r = q - 2$ is critical. Especially in case of $q < p + 1$ we prove that for $r > q - 2$ any maximal solution remains uniformly positive in $K \times [0, \infty)$ for any $K \subset \subset \Omega$, whereas for $r \leq q - 2$ every maximal solution converges to 0 as $t \to \infty$.

### 3.1 The case $r > q - 2$

In case of $r > q - 2$ all maximal solutions of (0.1) are global and bounded. This is shown in the next theorem.

**Theorem 3.1.1** Assume that (0.2), (0.3) and (0.4) are satisfied with $r > q - 2$ and $\kappa < 0$. Then every maximal solution of (0.1) is global in time and bounded.

**Proof.** In case of $q < p + 1$, for a given initial function $u_0$, we fix a smoothly bounded domain $G \subset \mathbb{R}^n$ and $v_0 \in C^0(\bar{G})$ such that $\Omega \subset \subset G$, $v_0 > 0$ in $G$ with $v_0 = 0$ on $\partial G$ and $v_0 \geq u_0 + 2$ in $\bar{G}$. Then, by [Wie2], the solution $v$ of (0.6) in $G \times (0, \infty)$, fulfilling $v|_{\partial G} = 0$ and $v|_{t=0} = v_0$, is global in time and bounded. Moreover, for any $t_0 \in (0, \infty)$ there is a positive constant $c_{t_0}$ such that $v \geq c_{t_0} > 0$ in $\bar{\Omega} \times [0, t_0]$ is satisfied (this is contained in the proof of Theorem 1.1.2 in case of $\kappa = 0$). Hence, for any $\varepsilon \in (0, 1)$ such that $\varepsilon < c_{t_0}$ the solution $u_\varepsilon$ of (1.1) satisfies $u_\varepsilon \leq v$ in $\Omega \times (0, t_0)$ by comparison. Thus, as $\varepsilon \searrow 0$, Theorem 1.1.2 implies that $u \leq v$ in $\Omega \times (0, t_0)$ holds for the maximal solution $u$ of (0.1). The claim now follows since $t_0$ is arbitrary and $v$ is bounded in $G \times (0, \infty)$.

In case of $q \geq p + 1$ we choose $\alpha := \min\{x_1 \mid x = (x_1, \ldots, x_n) \in \bar{\Omega}\}$ and $c := \max\{\|u_0\|_{L^\infty(\Omega)} + 2, (\frac{2}{|\alpha|})^{\frac{1}{r-2}}\}$, which is possible since $\Omega$ is bounded. With

$$v(x) := c e^{\alpha x_1 - \alpha}, \quad x = (x_1, \ldots, x_n) \in \bar{\Omega},$$

we get

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = e^{p+1}(p+1)(x_1-\alpha) + c^q e^q(x_1-\alpha) - |\kappa| c^{r+2} e^{(r+2)(x_1-\alpha)}$$
3.1. THE CASE $R > Q - 2$

Theorem 3.1.2 Suppose that $(0.2)$, $(0.3)$ and $(0.4)$ are fulfilled with equation $(0.6)$ without the gradient term.

Concerning the question whether these solutions converge to 0 as $t \to \infty$, we first study the case $q < p + 1$. In this case all solutions remain uniformly positive in the interior of $\Omega$, which corresponds to the behavior that has been observed for the equation $(0.6)$ without the gradient term.

Theorem 3.1.2 Suppose that $(0.2)$, $(0.3)$ and $(0.4)$ are fulfilled with $r > q - 2$, $q < p + 1$ and $\kappa < 0$. Then for any maximal solution $u$ of $(0.1)$ and any $K \subset \subset \Omega$ there is a positive constant $c_K$ such that $u \geq c_K > 0$ holds in $K \times (0, \infty)$.

Proof. Let $R > 0$ and $x_0 \in \Omega$ be chosen such that $B_R(x_0) \subset \subset \Omega$. Without loss of generality we assume that $x_0 = 0$. We define

$$v(x) := c e^{-\frac{|x|^2}{R^2-|x|^2}}, \quad x \in B_R(0),$$

where $c > 0$ will be fixed later. Then, due to $q < p + 1$ and $r > q - 2$, there are positive constants $M_1$ and $M_2$ such that (with $y := \frac{|x|^2}{R^2-|x|^2}$)

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = c^{p+1} e^{-(p+1)q} \frac{2R^2}{(R^2 - |x|^2)^4} \left\{ 2R^2 |x|^2 - n(R^2 - |x|^2)^2 \right\} - 4|x|^2 (R^2 - |x|^2) + c^q e^{-qy} - \kappa |c^{r+2} e^{-(r+2)q}y a \left\{ 2R^2 |x|^2 - n(R^2 - |x|^2)^2 \right\} - 4|x|^2 (R^2 - |x|^2) + 1 - |\kappa| |c^{r+2-q} e^{-(r+2-q)y} | \frac{4R^2 |x|^2}{(R^2 - |x|^2)^4} \right\} \geq c^q e^{-qy} \left[ c^{p+1-q} M_1 + 1 - |\kappa| |c^{r+2-q} M_2 \right] \geq 0 \quad \text{in } \Omega$$

due to the choice of $\alpha$ and $c$ and since $p + 1 \leq q < r + 2$. Hence, for any $\varepsilon \in (0, 1)$, the solution $u_\varepsilon$ of $(1.1)$ satisfies $u_\varepsilon(x, t) \leq v(x)$ in $\Omega \times (0, \infty)$ by comparison. Thus, Theorem 1.1.2 implies the claim.

Concerning the question whether these solutions converge to 0 as $t \to \infty$, we first study the case $q < p + 1$. In this case all solutions remain uniformly positive in the interior of $\Omega$, which corresponds to the behavior that has been observed for the equation $(0.6)$ without the gradient term.

Thus, if $u$ is a maximal solution of $(0.1)$ evolving from $u_0$, then we can fix $c \in (0, c_0]$ small enough such that $u_0 \geq c$ in $B_R(0)$ holds. By comparison we have $u_\varepsilon \geq v$ in $B_R(0) \times [0, \infty)$ for the solution $u_\varepsilon$ of $(1.1)$ and any $\varepsilon > 0$. Hence, $u \geq v$ in $B_R(0) \times [0, \infty)$ holds due to Theorem 1.1.2. This implies $u \geq c_B > 0$ in $B_{\frac{R}{2}}(0) \times [0, \infty)$ and the claim follows.
In the critical case \( q = p + 1 \) the size of the domain is important. Again, the value \( \lambda_1(\Omega) = 1 \) plays a critical role.

**Theorem 3.1.3** Assume that (0.2), (0.3) and (0.4) are satisfied with \( r > q - 2 \), \( q = p + 1 \) and \( \kappa < 0 \).

(i) In case of \( \lambda_1(\Omega) > 1 \), every maximal solution of (0.1) converges to 0 uniformly as \( t \to \infty \).

(ii) In case of \( \lambda_1(\Omega) < 1 \), for any maximal solution \( u \) of (0.1) and each \( K \subset \Omega \) there is \( c_K > 0 \) such that \( u \geq c_K > 0 \) in \( K \times [0, \infty) \) is fulfilled.

**Proof.**

(i) It was shown in [Wie2] that in case of \( \lambda_1(\Omega) > 1 \) any solution of (0.1) with \( \kappa = 0 \) converges to 0 uniformly as \( t \to \infty \). Hence, the claim of (i) follows by comparison. For further details we refer to the first part of the proof of Theorem 3.1.1. There, \( G \) has to be chosen such that additionally \( \lambda_1(G) > 1 \) is satisfied.

(ii) In case of \( \lambda_1(\Omega) < 1 \), suppose that \( u \) is a maximal solution of (0.1) evolving from \( u_0 \) and let \( K \subset \Omega \). Thus, we can fix a smoothly bounded domain \( G \) with \( K \subset G \subset \Omega \) such that \( \lambda := \lambda_1(G) < 1 \) is satisfied. Moreover, we pick \( \alpha \in (1, \frac{1}{\lambda}) \) and define \( \Theta := \Theta(\cdot; G) \). Since \( u_0 \) is positive in \( \Omega \), we can fix \( \delta \in \left(0, \frac{1}{r(q-2)}\right) \) such that \( u_0 \geq \delta \) in \( \bar{G} \). Then

\[
v(x) := \delta \Theta^\alpha(x), \ x \in G,
\]

satisfies

\[
v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = -\delta^{p+1} \alpha \lambda \Theta^\alpha(p+1) + \delta^{p+1} \alpha (\alpha - 1) \Theta^\alpha(p+1-2) |
\nabla \Theta|^2 + \delta \kappa |\nabla \Theta|^2
\]

\[
= \delta^s \Theta^q (1 - \alpha \lambda) + \delta^s \alpha \Theta^\alpha - 2 |\nabla \Theta|^2 (\alpha - 1)
\]

\[
- |\kappa| \delta^{r+2-q} \alpha \Theta^\alpha (r+2-q) \Theta^\alpha
\]

\[
\geq 0 \quad \text{in} \ G
\]

due to the choice of \( \alpha \) and \( \delta \). Hence, by comparison \( u_\varepsilon \geq v \) in \( G \times [0, \infty) \) holds for the solution \( u_\varepsilon \) of (1.1) and any \( \varepsilon > 0 \). Theorem 1.1.2 then implies \( u \geq v \) in \( G \times [0, \infty) \). Thus, the claim follows since \( v > 0 \) in \( \bar{K} \) and \( \bar{K} \) is compact. ■
3.1. THE CASE $R > Q - 2$

In case of $q > p + 1$, the behavior of the solutions is again different. Here, it depends on the size of the domain and on the size of the initial data whether the solutions converge to 0.

**Theorem 3.1.4** Let (0.2), (0.3) and (0.4) be fulfilled with $r > q - 2$, $q > p + 1$ and $\kappa < 0$.

(i) There is $C > 0$, depending on $p, q, r, \kappa$ and $\Omega$, such that $\|u_0\|_{L^\infty(\Omega)} < C$ implies that the maximal solution of (0.1) converges to 0 uniformly as $t \to \infty$.

(ii) There is $\delta > 0$, depending on $p, q, r$ and $\kappa$, such that if $\text{diam}(\Omega) \leq \delta$ then any maximal solution of (0.1) converges to 0 uniformly as $t \to \infty$.

(iii) There is $R_0 > 0$, depending on $p$, $q$, $r$, $\kappa$ and $n$, with the following property: If $\Omega$ contains a closed ball with radius $R_0$, then for every $w \in C^0(\bar{\Omega})$ with $w > 0$ in $\Omega$ and $w|_{\partial\Omega} = 0$ there exist $b_0 > 0$, $K \subset \subset \Omega$ and $c_K > 0$ such that the maximal solution $u$ of (0.1) evolving from $u_0 = bw$ with $b \geq b_0$ fulfills $u \geq c_K > 0$ in $K \times [0, \infty)$.

**Proof.**

(i) We define $m := \min\{x_1 | x = (x_1, \ldots, x_n) \in \bar{\Omega}\}$ and $M := \max\{x_1 | x = (x_1, \ldots, x_n) \in \bar{\Omega}\}$. Moreover, let $\alpha, \gamma \in (0,1)$, $\beta > -m$, $y_0 > 0$ and $y(t) := \left(p(1 - \alpha)\gamma(1 - \gamma)(M + \beta)^{-\gamma - 2}(m + \beta)^{(p+1)\gamma}t + y_0^p\right)^{-\frac{1}{p}}$ for $t \geq 0$. Defining

$$v(x,t) := y(t)(x_1 + \beta)^\gamma, \ (x,t) \in \bar{\Omega} \times [0, \infty),$$

we have

$$v_t - v^p \Delta v - v^q - \kappa v^p\|\nabla v\|^2 = y'(x_1 + \beta)^\gamma + \gamma(1 - \gamma)y^{p+1}(x_1 + \beta)^{(p+1)\gamma - 2} - v^q + |\kappa|\gamma^2 y^2(v + (1 + \beta)^{(1 - \gamma)\alpha^2} - v^q + |\kappa|\gamma^2 \gamma^2(v + (1 + \beta)^{\gamma - 2} - v^q + |\kappa|\gamma^2 (M + \beta)^{-2}(v + (1 + \beta)^{\gamma - 2} - v^q + |\kappa|\gamma^2 (M + \beta)^{-2}(v + (1 + \beta)^{\gamma - 2} - v^q + |\kappa|\gamma^2 (M + \beta)^{-2}(v + (1 + \beta)^{\gamma - 2} - v^q + |\kappa|\gamma^2 (M + \beta)^{-2}(v + (1 + \beta)^{\gamma - 2})$$

Hence, we choose $y_0 > 0$ such that $y_0(M + \beta)^\gamma \leq \left(\frac{\alpha\gamma(1 - \gamma)}{(M + \beta)^{(p+1)\gamma}}\right)^{-\frac{q}{p-1}}$. This implies $v_t - v^p \Delta v - v^q - \kappa v^p\|\nabla v\|^2 \geq 0$ in $\Omega \times [0, \infty)$, since $0 \leq v \leq y_0(M + \beta)^\gamma$ in $\Omega \times [0, \infty)$. Hence, if $\|u_0\|_{L^\infty(\Omega)} < C := y_0(M + \beta)^\gamma$, then, for $t_0 \in (0, \infty)$ and
\( \varepsilon > 0 \) sufficiently small, \( u_\varepsilon \leq v \) in \( \Omega \times [0, t_0) \) holds for the solution \( u_\varepsilon \) of (1.1) by comparison. Theorem 1.1.2 implies that \( u \leq v \) in \( \Omega \times [0, \infty) \) is fulfilled, since \( t_0 \) is arbitrary. Thus, the claim is proved.

(ii) With the notation of part (i), we can choose \( \beta > -m \) such that \( M + \beta \leq 2 \text{diam}(\Omega) \), since \( M - m \leq \text{diam}(\Omega) \). Furthermore, there is \( \delta > 0 \) such that
\[
\left( \frac{(M + \beta)^2}{|\kappa| \gamma^2} \right)^{\frac{1}{p+q-2}} \leq \left( \frac{(\text{diam}(\Omega))^2}{|\kappa| \gamma^2} \right)^{\frac{1}{p+q-2}}
\]
holds, if \( \text{diam}(\Omega) \leq \delta \) and \( M + \beta \leq 2 \text{diam}(\Omega) \). Thus, we have \( u \leq v \) in \( \Omega \times [0, \infty) \) of (0.1) evolving from \( u_0 \) by comparison. Theorem 1.1.2 implies that \( u \) is arbitrary. Thus, the claim is proved.

(iii) We define \( q(\alpha) := 2\alpha^2 - n(1 - \alpha^2)^2 - 4\alpha^2(1 - \alpha^2) \). Since \( q(1) = 2 \), we can fix \( \alpha_0 \in (0, 1) \) such that \( q(\alpha) \geq 1 \) for all \( \alpha \in [\alpha_0, 1] \). Moreover, we can choose \( \alpha_1 \in [\alpha_0, 1) \) such that \( 2|\kappa| e^{-(r-p+1)\frac{\alpha_1^2}{1-\alpha_1^2}} \leq 1 \) (since \( r > p - 1 \)). Furthermore, there is \( R_0 > 0 \) with
\[
\frac{2(n + 6) + 4|\kappa|}{(1 - \alpha_1^2)^4 R_0^2} \leq e^{-q_1 \frac{\alpha_1^2}{1-\alpha_1^2}}.
\]
Suppose that \( \Omega \) contains a closed ball with radius \( R_0 \). Hence, without loss of generality, we can assume \( B_R(0) \subset \subset \Omega \) for some \( R \geq R_0 \). Defining
\[
v(x) := e^{-\frac{|x|^2}{R^2 - |x|^2}}, \quad x \in B_R(0),
\]
and writing \( y := \frac{|x|^2}{R^2 - |x|^2} \), we compute
\[
v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 =\]
\[
e^{-(p+1)y} \frac{2R^2}{(R^2 - |x|^2)^4} \left\{ 2R^2 |x|^2 - n(R^2 - |x|^2)^2 \right. \]
\[
- 4|x|^2(R^2 - |x|^2) \left. \right\} + e^{-qy} \]
\[
- |\kappa| e^{-(r+2)y} \frac{4R^4|x|^2}{(R^2 - |x|^2)^4} \quad \text{in } B_R(0).
\]
3.2. THE CASE R = Q - 2

With \( p(t) := 2R^2t^2 - n(R^2 - t^2)^2 - 4t^2(R^2 - t^2) \) for \( t \in [0, R] \), we get \( p(\alpha R) = R^4q(\alpha) \) for \( \alpha \in [0, 1] \). This implies \( p(\alpha R) \geq R^4 \) for \( \alpha \in [\alpha_0, 1] \) and \( |p(\alpha R)| \leq (n + 6)R^4 \) for \( \alpha \in [0, 1] \). Thus,

\[
v^p\Delta v + v^q + \kappa v^r|\nabla v|^2 \geq e^{-(p+1)q} \frac{2R^6}{(R^2 - |x|^2)^4} - |\kappa|e^{-(r+2)q} \frac{4R^6}{(R^2 - |x|^2)^4} = e^{-(p+1)q} \frac{2R^6}{(R^2 - |x|^2)^4} \left( 1 - 2|\kappa|e^{-(r-p+1)q} \right)
\]

\[
\geq e^{-(p+1)q} \frac{2R^6}{(R^2 - |x|^2)^4} \left( 1 - 2|\kappa|e^{-(r-p+1)\frac{\alpha_1^2}{1-\alpha_1^2}} \right)
\]

\[
\geq 0 \quad \text{for } |x| \in [\alpha_1R, R)
\]

is fulfilled due to the choice of \( \alpha_1 \). Furthermore, we have

\[
v^p\Delta v + v^q + \kappa v^r|\nabla v|^2 \geq - \frac{2(n + 6)R^6}{(1 - \alpha_1^2)^4R^8} + e^{-\frac{\alpha_1^2}{1-\alpha_1^2}} - \frac{4|\kappa|R^6}{(1 - \alpha_1^2)^4R^8}
\]

\[
= - \frac{2(n + 6)}{(1 - \alpha_1^2)^4R^2} + e^{-\frac{\alpha_1^2}{1-\alpha_1^2}} - \frac{4|\kappa|}{(1 - \alpha_1^2)^4R^2}
\]

\[
\geq 0 \quad \text{for } x \in B_{\alpha_1R}(0),
\]

because \( R \geq R_0 \). Thus, \( v^p\Delta v + v^q + \kappa v^r|\nabla v|^2 \geq 0 \) in \( B_R(0) \) is satisfied.

Now let \( w \in C^0(\bar{\Omega}) \) with \( w > 0 \) in \( \Omega \) and \( w|_{\partial \Omega} = 0 \). Hence, there is \( b_0 > 0 \) such that \( b_0w \geq v \) in \( B_R(0) \) since \( B_R(0) \subset \subset \Omega \). If \( u \) is the maximal solution of (0.1) evolving from \( u_0 := bw \) with \( b \geq b_0 \), we get \( u \geq v \) in \( B_R(0) \times [0, \infty) \) by comparison (\( u \) is global in time by Theorem 3.1.1). Thus, the claim follows with \( K := B_R(0) \).

3.2 The case \( r = q - 2 \)

In the critical case \( r = q - 2 \) blow-up in finite can only occur if \( q = p + 1 \), \( |\kappa| \) is sufficiently small and \( \Omega \) is large enough. We remark that for \( q = p+1 \) and \( r = q-2 \), similar to the case \( \kappa > 0 \), the value \( \lambda_1(\Omega) = \kappa + 1 \) again denotes the critical size of \( \Omega \) with respect to global existence versus blow-up.

**Theorem 3.2.1** Assume that (0.2), (0.3) and (0.4) are fulfilled with \( r = q - 2 \) and \( \kappa < 0 \).

(i) In case of \( q \neq p + 1 \) and in case of \( q = p + 1 \) and \( \lambda_1(\Omega) > \kappa + 1 \) every maximal solution of (0.1) is global in time and bounded.
(ii) In case of $q = p + 1$ and $0 < \lambda_1(\Omega) < \kappa + 1$ any maximal solution of (0.1) blows up in finite time.

**Proof.** In case of $q < p + 1$ it was shown in [Wie2] that every solution of (0.1) with $\kappa = 0$ is global in time and bounded. This implies the claim for $q < p + 1$ by comparison (for details we refer to the proof of Theorem 3.1.1).

In case of $q > p + 1$ we set $\alpha := \min\{x_1 \mid x = (x_1, \ldots, x_n) \in \bar{\Omega}\}$, $\beta := \sqrt{\frac{2}{|\kappa|}}$ and $c := \max\{\|u_0\|_{L^\infty(\Omega)} + 1, \beta^{\frac{2}{q-p-1}}\}$. Defining

$$v(x) := ce^{\beta(x_1-\alpha)}, x = (x_1, \ldots, x_n) \in \bar{\Omega},$$

we have

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = e^{p+1} \beta^2 e^{(p+1)\beta(x_1-\alpha)} + c^q e^{\beta(x_1-\alpha)} - |\kappa|^2 c^q e^{\beta(x_1-\alpha)} \leq e^{\beta(x_1-\alpha)} \left(e^{p+1}\beta^2 - c^q\right) \leq 0 \quad \text{in } \Omega$$

due to the choice of $\alpha$, $\beta$ and $c$. Hence, by comparison $u \leq v$ in $\Omega \times [0, \infty)$ is fulfilled for the maximal solution $u$ of (0.1) and the claim follows in this case.

In case of $q = p + 1$ and $\kappa \leq -1$ we fix $c > 0$ such that $e^c \geq \|u_0\|_{L^\infty(\Omega)} + 1$. Let $f$ denote the solution of $-\Delta f = 1$ in $\Omega$ with $f|_{\partial\Omega} = c$. Then $w(x) := e^{f(x)}$, $x \in \tilde{\Omega}$, satisfies

$$-\Delta w = w - \frac{|\nabla w|^2}{w} \quad \text{in } \Omega,$$

$$w|_{\partial\Omega} = e^c.$$

Thus, the maximal solution $u$ of (0.1) evolving from $u_0$ fulfills $u \leq w$ in $\Omega \times [0, \infty)$ by comparison. Hence, $u$ is global in time and bounded.

In case of $q = p + 1$ and $\kappa \in (-1, 0)$ let $u$ be a maximal solution of (0.1). Then

$$v(x, t) := u^{\kappa+1}\left(\frac{1}{\sqrt{\kappa+1}} x, \frac{1}{\kappa+1} t\right) \quad (3.3)$$

is a solution of

$$\begin{cases}
  v_t = v^{\frac{p}{\kappa+1}} \Delta v + v^{\frac{r}{\kappa+1}+1}, & \text{in } G \times (0, (\kappa+1)T), \\
  v|_{\partial G} = 0, \\
  v|_{t=0} = v_0,
\end{cases} \quad (3.4)$$

where $G := \{\sqrt{\kappa+1} x \mid x \in \Omega\}$ and $v_0(x) := u_0^{\kappa+1}\left(\frac{1}{\sqrt{\kappa+1}} x\right)$ for $x \in \bar{G}$. The solutions $v$ of (3.4) are global and bounded for $\lambda_1(G) > 1$ and blow up in finite time for $\lambda_1(G) < 1$ (see [SGKM], [Wie1], [Wie2]). This implies the claim due to $\lambda_1(G) = \frac{1}{\kappa+1} \lambda_1(\Omega)$. \qed
Concerning the question whether global solutions of (0.1) converge to 0 as \( t \to \infty \), in case of \( q > p + 1 \) the result is similar to the result in the case of \( r > q - 2 \) (see Theorem 3.1.4).

**Theorem 3.2.2** Let (0.2), (0.3) and (0.4) be satisfied with \( r = q - 2, q > p + 1 \) and \( \kappa < 0 \).

(i) There is \( C > 0 \), depending on \( p, q, r, \kappa \) and \( \Omega \), such that \( \| u_0 \|_{L^\infty(\Omega)} < C \) implies that the maximal solution of (0.1) converges to 0 uniformly as \( t \to \infty \).

(ii) If \( \text{diam} \ (\Omega) < \sqrt{|\kappa|} \) then any maximal solution of (0.1) converges to 0 uniformly as \( t \to \infty \).

(iii) There is \( R_0 > 0 \), depending on \( p, q, r, \kappa \) and \( \Omega \), with the following property: If \( \Omega \) contains a closed ball with radius \( R_0 \), then for every \( w \in C^0(\bar{\Omega}) \) with \( w > 0 \) in \( \Omega \) and \( w|_{\partial \Omega} = 0 \) there exist \( b_0 > 0, K \subset \subset \Omega \) and \( c_K > 0 \) such that the maximal solution \( u \) of (0.1) evolving from \( u_0 = bw \) with \( b \geq b_0 \) fulfills \( u \geq c_K > 0 \) in \( K \times [0, \infty) \).

**Proof.** The proof of (i) and (iii) is just the same as in Theorem 3.1.4 (i) and (iii). To prove (ii), we assume \( \text{diam} \ (\Omega) < \sqrt{|\kappa|} \). Then, with the notation of the proof of Theorem 3.1.4 (i), we can choose \( \beta > -m \) and \( \gamma \in (0, 1) \) such that \( \frac{|\kappa|^2}{(M + \beta)^2} \geq 1 \) due to \( M - m \leq \text{diam} \ (\Omega) \). Moreover, we fix \( y_0 := (\| u_0 \|_{L^\infty(\Omega)} + 1)\mu - \beta \) and \( y(t) := (p\mu^p(\alpha\lambda - 1)t + y_0^p)^{1 \over p} \), we define

\[ v(x, t) := y(t)\Theta^\alpha(x), \ (x, t) \in \bar{\Omega} \times [0, \infty). \]

In case of \( q = p + 1 \), we have just seen that it depends on the size of \( \kappa \) and \( \Omega \) whether the solutions are global in time or blow up in finite time. Moreover, the global solutions that we got in Theorem 3.2.1 all converge to 0 as \( t \to \infty \).

**Theorem 3.2.3** Suppose that (0.2), (0.3) and (0.4) are fulfilled with \( r = q - 2, q = p + 1, \kappa < 0 \) and \( \lambda_1(\Omega) > \kappa + 1 \). Then any maximal solution of (0.1) converges to 0 uniformly as \( t \to \infty \).

**Proof.** Since \( \lambda_1(\Omega) > \kappa + 1 \), we can fix a smoothly bounded domain \( G \) with \( \Omega \subset \subset G \) and \( \lambda := \lambda_1(G) > \kappa + 1 \). Setting \( \Theta := \Theta(\cdot; G) \), there is \( \mu > 0 \) such that \( \Theta \geq \mu \) in \( \bar{\Omega} \) holds. Moreover, we can fix \( \alpha > {1 \over \lambda} \) such that \( \alpha(\kappa + 1) \leq 1 \). With \( y_0 := (\| u_0 \|_{L^\infty(\Omega)} + 1)\mu^{-\alpha} \) and \( y(t) := (p\mu^p(\alpha\lambda - 1)t + y_0^p)^{1 \over p} \), we define

\[ v(x, t) := y(t)\Theta^\alpha(x), \ (x, t) \in \bar{\Omega} \times [0, \infty). \]
Hence, we have

\[ v_t - v^p \Delta v - v^q - \kappa v' |\nabla v|^2 = \]

\[ v^r \Theta^\alpha + y^{p+1} \alpha \lambda \Theta^{\alpha(p+1)} - y^{p+1} \alpha \Theta^{\alpha(p+1)-2} |\nabla \Theta|^2 - y^{p+1} \Theta^{\alpha(p+1)} - y^{p+1} \Theta^{\alpha} \]

\[ - \kappa y^{p+1} \alpha \Theta^{\alpha(p+1)-2} |\nabla \Theta|^2 = \]

\[ v^r \Theta^\alpha + y^{p+1} \Theta^{\alpha(p+1)} (\alpha \lambda - 1) + y^{p+1} \alpha \Theta^{\alpha(p+1)-2} |\nabla \Theta|^2 (1 - \alpha - \kappa \alpha) \]

\[ \geq \Theta^\alpha (y' + y^{p+1} \mu^p (\alpha \lambda - 1)) + y^{p+1} \alpha \Theta^{\alpha(p+1)-2} |\nabla \Theta|^2 (1 - \alpha (1 + \kappa)) \]

\[ \geq 0 \quad \text{in } \Omega \times (0, \infty) \]

due to the choice of \( \alpha \) and \( y \). Thus, the maximal solution \( u \) of (0.1) evolving from \( u_0 \) satisfies \( u \leq v \) in \( \Omega \times (0, \infty) \) by comparison and Theorem 1.1.2, which implies the claim.

In case of \( q < p+1 \), all solutions converge to 0 as \( t \to \infty \). One main step in the proof of this claim is to show that, in a fixed domain \( G \subset \mathbb{R} \), solutions \( w_\varepsilon \) of the elliptic equation corresponding to (1.1), satisfying \( w_\varepsilon |_{\partial G} = \varepsilon \), converge to 0 uniformly in \( \bar{G} \) as \( \varepsilon \downarrow 0 \).

**Lemma 3.2.4** Let \( R > 0 \), \( G := (-R, R) \subset \mathbb{R} \) and let \( \alpha, \gamma, \kappa \in \mathbb{R} \) such that \( \alpha < 1 \), \( \gamma \leq \alpha - 2 \) and \( \kappa < 0 \) is fulfilled. Moreover, for any \( \varepsilon \in (0, 1] \), let \( w_\varepsilon \in C^2(G) \cap C^0(\bar{G}) \) denote a solution of

\[
\begin{cases}
  w''_\varepsilon + (w_\varepsilon)^\alpha + \kappa (w_\varepsilon)^\gamma (w'_\varepsilon)^2 = 0 & \text{in } G, \\
  w_\varepsilon |_{\partial G} = \varepsilon,
\end{cases}
\]

(3.5)

satisfying \( w_\varepsilon(x) = w_\varepsilon(-x) \) for all \( x \in G \), \( w'_\varepsilon(0) = 0 \) and \( w'_\varepsilon \leq 0 \) in \( (0, R) \). Furthermore, suppose that \( w_{\varepsilon_1} \leq w_{\varepsilon_2} \) in \( \bar{G} \) holds for \( 0 < \varepsilon_1 \leq \varepsilon_2 \leq 1 \). Then \( \|w_\varepsilon\|_{C^0(\bar{G})} \to 0 \) as \( \varepsilon \downarrow 0 \) is fulfilled.

**Proof.** Setting \( a := w_1(0) > 0 \) and \( \beta := \gamma + 1 < 0 \), we let \( f \in C^2([0, \infty)) \) denote the solution of

\[
\begin{cases}
  f'(s) = -e^{[\beta/(1-\alpha)]s} & \text{in } (0, \infty), \\
  f(0) = a.
\end{cases}
\]

(3.6)

We derive estimates for the function \( f \) which will particularly show that \( f \) exists on the whole interval \([0, \infty)\) with \( f(s) > 0 \) for all \( s > 0 \). Then we will use \( f \) to transform the solutions \( w_\varepsilon \) of (3.5) into solutions of an elliptic equation without a gradient term.
Therefore, we fix $c_1 > 0$ such that \( \left( \frac{-\beta}{|\kappa|} \right)^{\frac{1-\beta}{\beta}} \left( 1 + \frac{\beta - 1}{\beta \ln 3} \right) e^{-c_1} \leq 1 \) is satisfied. Moreover, we choose $s_0 \geq 3$ such that $\ln s_0 \geq 2c_1$ is fulfilled and define

\[
g_1(s) := \left( -\frac{\beta}{|\kappa|} \left[ \ln(s + s_0) + \frac{\beta - 1}{\beta} \ln(\ln(s + s_0)) - c_1 \right] \right)^{\frac{1-\beta}{\beta}}, \quad \text{with } s \geq 0.
\]

Then, we obtain

\[
g_1'(s) \left/ e^{\frac{|\kappa|}{\beta} (g_1(s))^\beta} \right. = \left( \frac{-\beta}{|\kappa|} \left[ \ln(s + s_0) + \frac{\beta - 1}{\beta} \ln(\ln(s + s_0)) - c_1 \right] \right)^{\frac{1-\beta}{\beta}} \times \left( 1 + \frac{\beta - 1}{\beta \ln 3} \right) e^{-c_1} \leq -1 \quad \text{for } s \geq 0,
\]

since $c_1$ and $s_0$ are chosen suitably and $\beta < 0$ is satisfied.

Furthermore, we define

\[
g_2(s) := \left( -\frac{\beta}{|\kappa|} \left[ \ln(s + s_0) + \frac{\beta - 1}{\beta} \ln(\ln(s + s_0)) + c_2 \right] \right)^{\frac{1-\beta}{\beta}}, \quad \text{with } s \geq 0,
\]

where $c_2 > 0$ is chosen large enough such that \( \left( \frac{1-2\beta}{|\kappa|} + \frac{\beta c_2}{\ln 3} \right)^{\frac{1-\beta}{\beta}} e^{c_2} \geq 1 \) is fulfilled and $g_2(0) \in (0, a)$ holds. Then, we compute

\[
g_2'(s) \left/ e^{\frac{|\kappa|}{\beta} (g_2(s))^\beta} \right. = \left( \frac{-\beta}{|\kappa|} \left[ \ln(s + s_0) + \frac{\beta - 1}{\beta} \ln(\ln(s + s_0)) + c_2 \right] \right)^{\frac{1-\beta}{\beta}} \times \left( 1 + \frac{\beta - 1}{\beta \ln 3} \right) e^{-c_2} \leq -1 \quad \text{for } s \geq 0,
\]

due to $\beta < 0$, $s_0 \geq 3$ and because $c_2$ is chosen suitably.
Moreover, we have $g_2(0) < a = f(0)$, $g_2 > 0$ in $[0, \infty)$ and $f' \leq 0$, where $f$ is the solution of (3.6). Hence, (3.8) and standard comparison results for ordinary differential equations imply that $f$ exists in $[0, \infty)$ and satisfies
\begin{equation}
0 < g_2(s) \leq f(s) \leq a \quad \text{for all } s \geq 0.
\end{equation}
If $f(s) \geq g_1(0)$ was satisfied for all $s \geq 0$, $f(s) \leq a - s e^{[\kappa/(g_2(0))]\beta}$ would be fulfilled for all $s \geq 0$. However, this is a contradiction since $f$ is positive in $[0, \infty)$. Hence, there is $s_1 \geq 0$ with $g_1(0) > f(s_1)$. Furthermore, we get
\begin{equation}
f(s + s_1) \leq g_1(s) \quad \text{for all } s \geq 0
\end{equation}
by (3.7) and standard comparison results for ordinary differential equations. In particular, this implies $\lim_{s \to \infty} f(s) = 0$.

Next, we fix $\varepsilon \in (0, 1]$. Thus, $\varepsilon \leq w_\varepsilon(x) \leq w_\varepsilon(0) \leq w_1(0) = a$ for all $x \in \bar{G}$ is fulfilled due to our assumptions. Moreover, there is a unique $b_\varepsilon \in [0, \infty)$ such that $f(b_\varepsilon) = w_\varepsilon(0)$ holds, because $f' < 0$ in $(0, \infty)$, $f(0) = a$ and $\lim_{s \to \infty} f(s) = 0$. Furthermore, these properties of $f$ imply the existence of a unique function $y_\varepsilon \in C^2(G) \cap C^0(\bar{G})$ satisfying
\[ w_\varepsilon(x) = f(y_\varepsilon(x) + b_\varepsilon) \quad \text{for } x \in \bar{G}. \]
Hence, we have $y_\varepsilon(0) = 0$ and $y_\varepsilon'(0) = 0$, since $b_\varepsilon$ is chosen suitably, $w_\varepsilon'(0) = 0$ and $f'(b_\varepsilon) < 0$. Moreover, with $z_\varepsilon := y_\varepsilon + b_\varepsilon$,
\[ f'(z_\varepsilon)y_\varepsilon'' + f''(z_\varepsilon)(y_\varepsilon')^2 + (f(z_\varepsilon))^\alpha - |\kappa| (f(z_\varepsilon))^{-\gamma}(f'(z_\varepsilon))^2(y_\varepsilon')^2 = 0 \quad \text{in } G \]
holds because $w_\varepsilon$ is a solution of (3.5). Using now $f'' = |\kappa| f^\gamma (f')^2$, we get
\begin{equation}
y_\varepsilon'' = \frac{(f(y_\varepsilon + b_\varepsilon))^\alpha}{f'(y_\varepsilon + b_\varepsilon)} \quad \text{in } G.
\end{equation}
Defining
\[ h(s) := - \frac{(f(s))^\alpha}{f'(s)} = (f(s))^\alpha e^{[\kappa/(f(s))]\beta}, \quad \text{with } s \geq 0, \]
we get $h > 0$ in $[0, \infty)$. Moreover, in case of $\alpha \leq 0$, (3.9) implies
\begin{align*}
h(s) & \leq (g_2(s))^\alpha e^{[\kappa/(g_2(s))]\beta} \\
& = \left( -\frac{\beta}{|\kappa|} \left[ \ln(s + s_0) + \frac{\beta - 1}{\beta} \ln(\ln(s + s_0)) + c_2 \right] \right)^\frac{\alpha}{\beta} \times \\
& \quad (s + s_0)(\ln(s + s_0))^{\frac{\alpha - 1}{\beta}} e^{c_2} \\
& \leq \left( -\frac{\beta}{|\kappa|} \left( 1 + \frac{\beta - 1}{\beta} + c_2 \right) \right)^\frac{\alpha}{\beta} e^{c_2}(s + s_0)(\ln(s + s_0))^{\frac{\alpha + \beta - 1}{\beta}} \\
& \leq \left( -\frac{\beta}{|\kappa|} \left( 1 + \frac{\beta - 1}{\beta} + c_2 \right) \right)^\frac{\alpha}{\beta} e^{c_2}(s + s_0)(\ln(s + s_0))^2 \quad \text{for } s \geq 0, \quad (3.12)
\end{align*}
3.2. THE CASE $R = Q - 2$

because $\alpha + \beta - 1 \geq \gamma + 2 + \beta - 1 = \beta - 1 + 2 + \beta - 1 = 2\beta$, $\beta < 0$ and $s_0 \geq 3$ is satisfied. In case of $\alpha \in (0, 1)$, we fix $s_2 \geq \max\{s_0, s_1\}$ such that $s_1 \leq s + s_0 - \sqrt{s + s_0}$ for all $s \geq s_2$ is satisfied. Thus, by (3.9) and (3.10), we have

$$h(s) \leq (g_1(s - s_1))^{\alpha} e^{\kappa |f(x)|^{(g_1(s_2))^{\beta}}}
= \left(\frac{-\beta}{\kappa} \ln(s - s_1 + s_0) + \beta - 1 \ln(\ln(s - s_1 + s_0)) - c_1\right)^{\tilde{\gamma} \frac{\alpha}{\beta}}
\leq \left(\frac{-\beta}{2|\kappa|} \ln(s - s_1 + s_0)\right)^{\tilde{\gamma} \frac{\alpha}{\beta}}
\leq \left(\frac{-\beta}{2|\kappa|} \ln(\sqrt{s + s_0})\right)^{\tilde{\gamma} \frac{\alpha}{\beta}}
\leq \left(\frac{-\beta}{4|\kappa|} \ln(s + s_0)\right)^{\tilde{\gamma} \frac{\alpha + \beta - 1}{\beta}}
\leq \left(\frac{-\beta}{4|\kappa|} \ln(s + s_0)\right)^{\tilde{\gamma} e^{c_2}(s + s_0)\ln(s + s_0)^2}
\quad \text{for } s \geq s_2,
(3.13)$$
due to $c_1 \leq \frac{1}{2} \ln s_0$, $\alpha + \beta - 1 \geq 2\beta$, $\beta < 0$ and $s_0 \geq 3$. Furthermore, for $\alpha \in (0, 1)$, the properties of $f$ imply

$$h(s) \leq (f(0))^{\alpha} e^{\kappa |f(x)|^{(g_1(s_2))^{\beta}}}
\leq a^{\alpha} e^{\kappa |f(x)|^{(g_1(s_2))^{\beta}}} (s + s_0)^2 \ln(s + s_0)^2
\quad \text{for } s \in [0, s_2]
(3.14)$$
due to $s_0 \geq 3$. Altogether, (3.12), (3.13) and (3.14) imply that for any $\alpha < 1$ there is a constant $C > 0$ such that

$$h(s) \leq C(s + s_0)\ln(s + s_0)^2
\quad \text{for } s \geq 0
(3.15)$$
is fulfilled.

Moreover, for any $\varepsilon \in (0, 1]$, we have $y''_\varepsilon = h(y_\varepsilon + b_\varepsilon) > 0$ in $[0, R]$ by (3.11). Thus, $y'_\varepsilon > 0$ and $y_\varepsilon > 0$ is satisfied in $(0, R)$ due to $y'_\varepsilon(0) = y_\varepsilon(0)$. Therefore, (3.11) and (3.15) imply

$$\frac{1}{2}(y'_\varepsilon)^2(x) = \int_0^x y''_\varepsilon(s)y'_\varepsilon(s)ds = \int_0^x h(y_\varepsilon(s) + b_\varepsilon)y'_\varepsilon(s)ds
\leq C \int_0^x (y_\varepsilon(s) + b_\varepsilon + s_0)(\ln(y_\varepsilon(s) + b_\varepsilon + s_0))^2 y'_\varepsilon(s)ds
= C \frac{1}{2} (y_\varepsilon(x) + b_\varepsilon + s_0)^2 \left((\ln(y_\varepsilon(x) + b_\varepsilon + s_0))^2 - \ln(y_\varepsilon(x) + b_\varepsilon + s_0)\right)$$
In case of $\alpha < 1$, then $w$ is positive in $(\bar{\Omega} \cap \{0 < w \})$ and define

$$\alpha < 1$$

for $x \in [0, R)$, due to $s_0 \geq 3$. Hence, we have

$$y'_\varepsilon(x) \leq \sqrt{2C} (y_\varepsilon(x) + b_\varepsilon + s_0) \ln(y_\varepsilon(x) + b_\varepsilon + s_0) \quad \text{for } x \in [0, R),$$

because $y'_\varepsilon \geq 0$ in $[0, R)$. By integration, we conclude

$$\sqrt{2C} x \geq \int_0^x \frac{y'_\varepsilon(s)}{(y_\varepsilon(s) + b_\varepsilon + s_0) \ln(y_\varepsilon(s) + b_\varepsilon + s_0)} \, ds$$

$$= \ln(\ln(y_\varepsilon(x) + b_\varepsilon + s_0)) - \ln(\ln(b_\varepsilon + s_0)) \quad \text{for } x \in [0, R].$$

In particular, for any $\varepsilon \in (0, 1)$, this implies

$$f^{-1}(\varepsilon) = f^{-1}(w_\varepsilon(R)) = y_\varepsilon(R) + b_\varepsilon \leq e^{\varepsilon \sqrt{2C} R + \ln(\ln(b_\varepsilon + s_0))} - s_0.$$

Using $f^{-1}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$, we have $b_\varepsilon \to \infty$ as $\varepsilon \searrow 0$. Finally, we conclude

$$\lim_{\varepsilon \searrow 0} \|w_\varepsilon\|_{C^0(\bar{\Omega})} = \lim_{\varepsilon \searrow 0} w_\varepsilon(0) = \lim_{\varepsilon \searrow 0} f(b_\varepsilon) = \lim_{s \to \infty} f(s) = 0,$$

by which the claim is proved.

Moreover, to prepare the proof of the result in case of $q < p + 1$, we give a comparison principle for the elliptic equation corresponding to $(0.1)$ with $\kappa = 0$.

**Lemma 3.2.5** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha < 1$. Furthermore, for $l \in \{1, 2\}$, assume that $w_l \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is positive in $\Omega$ such that

$$-\Delta w_1 \leq w_1^\alpha \quad \text{in } \Omega,$$

$$-\Delta w_2 \geq w_2^\alpha \quad \text{in } \Omega,$$

$$w_1 \leq w_2 \quad \text{on } \partial \Omega.$$

Then $w_1 \leq w_2$ in $\bar{\Omega}$ is fulfilled.

**Proof.** In case of $\alpha < 0$, the claim is proved in comparison principle 1.1 of [Wie2]. In case of $\alpha \in [0, 1)$, we fix $\gamma \in (0, 1 - \alpha)$ and define $v_l(x) := (w_l(x))^\gamma$ for $x \in \bar{\Omega}$ and $l \in \{1, 2\}$. Thus, we get

$$-\Delta v_1 = -\gamma w_1^{\gamma - 1} \Delta w_1 - \gamma(\gamma - 1)w_1^{\gamma - 2} |\nabla w_1|^2 \leq \gamma w_1^{\gamma - 1 + \alpha} - \gamma(\gamma - 1)w_1^{\gamma - 2} |\nabla w_1|^2$$

$$= \gamma v_1^{\frac{1}{\gamma} - \frac{\alpha}{\gamma}} - \gamma v_1^{\frac{1}{\gamma} - \frac{\alpha}{\gamma}} |\nabla v_1|^2 = \gamma v_1^{\frac{2 - \alpha}{\gamma}} + 1 - \gamma v_1^{\frac{2 - \alpha}{\gamma}} |\nabla v_1|^2 \quad \text{in } \Omega.$$
Similarly, we conclude that \(-\Delta v_2 \geq \gamma v_2^{\frac{2-1+\alpha}{\gamma}} + \frac{1-\gamma}{\gamma} v_2^{-1}|\nabla v_2|^2\) in \(\Omega\) is satisfied. Defining \(d(x) := v_1(x) - v_2(x)\) for \(x \in \Omega\), we assume that there is \(x_0 \in \Omega\) with \(d(x_0) > 0\). Hence, there is \(x_1 \in \Omega\) such that \(d(x_1) = \max d(x)\), because \(d \leq 0\) on \(\partial \Omega\) holds due to \(\gamma > 0\). Thus, \(d(x_1) > 0\), \(|\nabla d|(x_1) = 0\) and \(\Delta d(x_1) \leq 0\) is fulfilled. This implies

\[
0 \leq \left[ \Delta v_1 + \gamma v_1^{\frac{2-1+\alpha}{\gamma}} + \frac{1-\gamma}{\gamma} v_1^{-1}|\nabla v_1|^2 - \Delta v_2 - \gamma v_2^{\frac{2-1+\alpha}{\gamma}} - \frac{1-\gamma}{\gamma} v_2^{-1}|\nabla v_2|^2 \right] (x_1)
\]

\[
= \left[ \Delta d + \gamma \left( v_1^{\frac{2-1+\alpha}{\gamma}} - v_2^{\frac{2-1+\alpha}{\gamma}} \right) + \frac{1-\gamma}{\gamma} |\nabla v_1|^2 \left( v_1^{-1} - v_2^{-1} \right) \right] (x_1)
\]

\[
\leq \left[ \gamma \left( v_1^{-\frac{2-1+\alpha}{\gamma}} - v_2^{-\frac{2-1+\alpha}{\gamma}} \right) \right] (x_1) < 0,
\]

because \(\gamma - 1 + \alpha < 0\), \(\gamma \in (0, 1)\) and \(v_1(x_1) > v_2(x_1)\). This is a contradiction and, hence, we conclude that \(d \leq 0\) in \(\Omega\) is satisfied. Thus, due to \(\gamma > 0\), the claim is proved. 

Now we show that in case of \(q < p + 1\) any solution converges to 0 as \(t \to \infty\). For referencing in the following section, we state this result for \(r \leq q - 2\). Hence, if \(q < p + 1\) and \(\kappa < 0\), all solutions of (0.1) converge to 0 as \(t \to \infty\) for \(r \leq q - 2\), whereas, in case of \(r > q - 2\), any solution remains uniformly positive in \(K \times [0, \infty)\) for every \(K \subset \Omega\) (see Theorem 3.1.2).

**Theorem 3.2.6** Assume that (0.2), (0.3) and (0.4) are fulfilled with \(r \leq q - 2\), \(q < p + 1\) and \(\kappa < 0\). Then any maximal solution of (0.1) converges to 0 uniformly as \(t \to \infty\).

**Proof.** Choosing \(d := \frac{1}{2} \text{diam}(\Omega)\), we may assume without loss of generality that \(\Omega \subset \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_1| < d \}\) is satisfied. Furthermore, let \(u\) denote the maximal solution of (0.1) with \(u|_{t=0} = u_0\). Moreover, we set \(\alpha := q - p < 1\), \(\mu_R := \frac{\pi}{2R}\) and \(\varphi_R(z) := \frac{1}{z^2} \cos(\mu_R z)\) with \(R > 0\) and \(z \in [-R, R]\). Then, there is \(R_0 > d\) such that \(\varphi_{R_0} \geq ||u_0||_{L^\infty(\Omega)} + 1\) in \([-d, d]\) is fulfilled, because \(\mu_R \to \infty\) as \(R \to \infty\). Furthermore, fixing \(G := (-R_0, R_0)\), we have

\[
-(\varphi_{R_0})'' = (\mu_{R_0})^2 \varphi_{R_0} \leq (\mu_{R_0})^2 (\mu_{R_0})^{-2(1-\alpha)} (\varphi_{R_0})^\alpha = (\varphi_{R_0})^\alpha \quad \text{in} \ G. \tag{3.16}
\]

Let \(\varepsilon \in (0, 1)\) be arbitrary. Then, by Proposition 1.2 and Theorem 2.1 in [Wie2], there exists a solution \(\psi_\varepsilon \in C^2(\Omega) \cap C^0(\bar{\Omega})\) of the problem

\[
\begin{cases}
\psi''_\varepsilon + (\psi_\varepsilon)^\alpha = 0 & \text{in} \ G, \\
\psi_\varepsilon|_{\partial G} = \varepsilon.
\end{cases} \tag{3.17}
\]
Then, $\psi_\varepsilon \geq \varepsilon$ in $\tilde{G}$ holds since $\psi_\varepsilon'' < 0$ in $G$. Moreover, $\tilde{\psi}_\varepsilon(z) := \psi_\varepsilon(-z)$ with $z \in \tilde{G}$ is a solution of (3.17). Hence, we conclude $\psi_\varepsilon = \tilde{\psi}_\varepsilon$ in $\tilde{G}$, since the solution of (3.17) is unique by Lemma 3.2.5. This implies $\psi_\varepsilon(z) = \tilde{\psi}_\varepsilon(-z)$ for $z \in \tilde{G}$ and $\tilde{\psi}_\varepsilon'(0) = 0$. Furthermore, $\psi_\varepsilon' < 0$ in $(0, R)$ is fulfilled because $\psi_\varepsilon'' < 0$ in $G$. Moreover, we have $\psi_\varepsilon \geq \varphi_{R_0}$ in $\tilde{G}$ by (3.16) and Lemma 3.2.5. Hence,

$$
\psi_\varepsilon(z) \geq \|u_0\|_{L^\infty(\Omega)} + 1 \quad \text{for } z \in [-d, d]
$$

(3.18)
is satisfied due to the choice of $\varphi_{R_0}$.

Furthermore, $\psi_\varepsilon \in C^3(\tilde{G})$ is satisfied. Thus, we let $v_\varepsilon \in C^{2,1}(G \times (0, T_\varepsilon)) \cap C^0(\tilde{G} \times [0, T_\varepsilon])$ denote the solution of (1.1) satisfying $v_\varepsilon|_{t=0} = \tilde{\psi}_\varepsilon$. Then, $v_\varepsilon \geq \varepsilon$ in $G \times (0, T_\varepsilon)$ holds by comparison and $(v_\varepsilon)_t \leq 0$ in $G \times (0, T_\varepsilon)$ is fulfilled by Lemma 1.3.5 due to $\kappa < 0$. This implies $T_\varepsilon = \infty$ and the existence of a function $w_\varepsilon : \tilde{G} \to \mathbb{R}$ such that $v_\varepsilon(z, t) \leq w_\varepsilon(z)$ as $t \to \infty$ is satisfied for any $z \in \tilde{G}$. In particular, we have $w_\varepsilon \geq \varepsilon$ in $\tilde{G}$ and $w_\varepsilon|_{\partial G} = \varepsilon$.

Moreover, we fix $K \subset \subset G$ and define $v_{\varepsilon,n}(z, t) := v_\varepsilon(z, t + n)$ with $(z, t) \in \tilde{G} \times [0, \infty)$ and $n \in \mathbb{N}$. Thus, due to $\varepsilon \leq v_\varepsilon \leq \psi_\varepsilon(0)$ in $G \times [0, \infty)$ and parabolic Schauder estimates, there are constants $\nu \in (0, 1)$ and $c_K > 0$ such that $\|v_{\varepsilon,n}\|_{C^{\nu,v+1,\frac{2}{\nu}}(K \times [1, 2])} \leq c_K$ for all $n \in \mathbb{N}$ is fulfilled (see Theorem V.1.1, Theorem VI.3.4 and Theorem VII.5.1 in [LSU]). This implies $v_{\varepsilon,n}(z, t) \to w_\varepsilon(z)$ in $C^{2,1}(K \times [1, 2])$ as $n \to \infty$. Hence, $\limsup_{t \to \infty} v_{\varepsilon,t} = 0$ for all $z \in G$, and, therefore, $w \in C^2(G)$ satisfies

$$
w_\varepsilon^p w_\varepsilon^q + w_\varepsilon^q + \kappa w_\varepsilon^p (w_\varepsilon')^2 = 0 \quad \text{in } G.
$$

(3.19)

Moreover, for any $\delta > 0$, there is a neighborhood $U_\delta$ of $\partial G$ such that $\varepsilon \leq \psi_\varepsilon(z) \leq \varepsilon + \delta$ for $z \in U_\delta$ is fulfilled. This implies $\varepsilon \leq w_\varepsilon(z) \leq v_\varepsilon(z, t) \leq \tilde{\psi}_\varepsilon(z) \leq \varepsilon + \delta$ for $(z, t) \in U_\delta \times [0, \infty)$ due to $(v_\varepsilon)_t \leq 0$ in $G \times (0, \infty)$. Hence, we conclude that $v_\varepsilon(\cdot, t) \to w_\varepsilon(\cdot)$ in $C^0(\tilde{G})$ is satisfied, because $\delta > 0$ is arbitrary and $v_\varepsilon(\cdot, t) \to w_\varepsilon(\cdot)$ in $C^0(\tilde{K})$ holds for any $K \subset \subset G$. In particular, by (3.19), $w_\varepsilon \in C^2(G) \cap C^0(\tilde{G})$ is a solution of (3.5) with $\gamma := r - p$. Furthermore, the properties of $\psi_\varepsilon$ imply that $v_\varepsilon(z, t) = v_\varepsilon(-z, t)$ for $(z, t) \in \tilde{G} \times [0, \infty)$, $(v_\varepsilon)_x(0) = 0$ and $(v_\varepsilon)_x(z, t) \leq 0$ for $(z, t) \in (0, R) \times (0, \infty)$ is satisfied. The proof of these claims is completely analogous to the one of Lemma 1.3.3. Thus, we have $w_\varepsilon(z) = w_\varepsilon(-z)$ for $z \in \tilde{G}$, $w_\varepsilon'(0) = 0$ and $w_\varepsilon'(z) \leq 0$ for $z \in (0, R)$, because $v_\varepsilon(z, t) \to w_\varepsilon(z)$ as $t \to \infty$ for any $z \in \tilde{G}$. Moreover, if $0 < \varepsilon_1 < \varepsilon_2 \leq 1$, $v_{\varepsilon_1} \leq v_{\varepsilon_2}$ in $\tilde{G} \times [0, \infty)$ is satisfied by comparison, since we have $\psi_{\varepsilon_1} \leq \psi_{\varepsilon_2}$ in $\tilde{G}$ by (3.17) and Lemma 3.2.5. Thus, we conclude that $w_{\varepsilon_1} \leq w_{\varepsilon_2}$ in $\tilde{G}$ is fulfilled.

Let $\delta > 0$ be arbitrary. Then, by Lemma 3.2.4, there is $\varepsilon \in (0, 1]$ such that $w_\varepsilon \leq \delta$ in $\tilde{G}$ is satisfied, because $\gamma + 2 = r - p + 2 \leq q - p = \alpha < 1$ holds. Moreover, there is $t_0 \in (0, \infty)$ such that $v_\varepsilon(z, t) \leq w_\varepsilon(z) + \delta$ holds for $(z, t) \in \tilde{G} \times [t_0, \infty)$, because $v_\varepsilon(\cdot, t) \to w_\varepsilon(\cdot)$ in $C^0(\tilde{G})$ as $t \to \infty$. Altogether, $v_\varepsilon \leq 2\delta$ in $\tilde{G} \times [t_0, \infty)$. 

Let us recall that $\Omega \subset \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_1| < d \}$ holds and $u$ denotes the maximal solution of (0.1) with $u|_{t=0} = u_0$. Defining now $\tilde{v}_\varepsilon(x, t) := v_\varepsilon(x_1, t)$
with \( x = (x_1, \ldots, x_n) \in \bar{\Omega} \) and \( t \in [0, \infty) \), we conclude that \( \tilde{v}_\varepsilon \in C^{2,1}(\Omega \times (0, \infty)) \cap C^0(\bar{\Omega} \times [0, \infty)) \) is a solution of the differential equation in (0.1). Furthermore, \( \tilde{v}_\varepsilon \geq \varepsilon \) in \( \bar{\Omega} \times [0, \infty) \) and \( \tilde{v}_\varepsilon(x,0) = \psi_\varepsilon(x_1) \geq \|u_0\|_{L^\infty(\Omega)} + 1 \) for \( x \in \bar{\Omega} \) holds due to (3.18). Thus, \( u \leq \tilde{v}_\varepsilon \) in \( \bar{\Omega} \times [0, \infty) \) is satisfied by Lemma 1.1.1. Hence,

\[
0 \leq u(x,t) \leq \tilde{v}_\varepsilon(x,t) = v_\varepsilon(x_1,t) \leq 2\delta \quad \text{in} \quad \bar{\Omega} \times [t_0, \infty)
\]
is fulfilled. Hence, the claim is proved because \( \delta > 0 \) is arbitrary. 

### 3.3 The case \( r < q - 2 \)

In case of \( r < q - 2 \), for any bounded domain there are initial data which are sufficiently small such that the corresponding maximal solution is global and bounded. However, blow-up in finite time additionally occurs for \( q > p + 1 \) and large initial data.

We first state the results in case of \( q < p + 1 \), which are similar to the case \( r = q - 2 \) (see Theorem 3.2.1 and Theorem 3.2.6). Again, all global solutions converge to 0 as \( t \to \infty \).

**Theorem 3.3.1** Let (0.2), (0.3) and (0.4) be fulfilled with \( r < q - 2 \), \( q < p + 1 \) and \( \kappa < 0 \). Then any maximal solution of (0.1) is global and bounded and, furthermore, converges to 0 uniformly as \( t \to \infty \).

**Proof.** For \( q < p + 1 \) every solution of (0.1) with \( \kappa = 0 \) is global and bounded (see [Wie2]). Thus, in case of \( r < q - 2 \) and \( q < p + 1 \), all maximal solutions of (0.1) are global and bounded by comparison (for details we refer to the proof of Theorem 3.1.1). Moreover, any maximal solution converges to 0 uniformly as \( t \to \infty \) by Theorem 3.2.6.

In case of \( q = p + 1 \), it depends not only on the size of the domain (like in case of \( r = q - 2 \)), but also on the size of the initial data if blow-up in finite time occurs. It remains open if the maximal solution blows up in case of \( \lambda_1(\Omega) < 1 \) for large initial data.

**Theorem 3.3.2** Assume that (0.2), (0.3) and (0.4) are satisfied with \( r < q - 2 \), \( q = p + 1 \) and \( \kappa < 0 \).

(i) There is \( C > 0 \), depending on \( p,q,r,\kappa \) and \( \Omega \), such that \( \|u_0\|_{L^\infty(\Omega)} < C \) implies that the maximal solution of (0.1) is global in time and bounded and converges to 0 uniformly as \( t \to \infty \).
(ii) If \( \lambda_1(\Omega) > 1 \) then any maximal solution of (0.1) is global in time and bounded and converges to 0 uniformly as \( t \to \infty \).

**Proof.**

(i) We fix a smoothly bounded domain \( G \) with \( \Omega \subset \subset G \), set \( \lambda := \lambda_1(G) > 0 \) and choose \( \alpha > \frac{1}{\lambda} \). Setting \( \Theta := \Theta(\cdot; G) \), there is \( \mu > 0 \) such that \( \Theta \ge \mu \) in \( \bar{\Omega} \) holds. Moreover, due to \( r < p - 1 \), we are able to fix \( y_0 > 0 \) small enough such that

\[
1 - \alpha + |\kappa y_0^{-(p-1-r)}\alpha| \ge 0.
\]

(3.20)

With \( y(t) := (p\mu^\alpha (\alpha - 1) + y_0^{-p})^{-\frac{1}{\beta}} \), we define

\[
v(x,t) := y(t)\Theta^\alpha(x), \quad (x,t) \in \bar{\Omega} \times [0, \infty).\]

Hence, we have

\[
v_t - v^p \Delta v - v^q - \kappa v^r |\nabla v|^2 = y' \Theta^\alpha + y^{p+1} \alpha \lambda \Theta^{a(p+1)}
\]

\[
- y^{p+1} (\alpha - 1) \Theta^{a(p+1)-2} |\nabla \Theta|^2
\]

\[
- y^{p+1} \Theta^{a(p+1)} + |\kappa| y^{r+2} \alpha^2 \Theta^{a(r+2)-2} |\nabla \Theta|^2
\]

\[
= y' \Theta^\alpha + y^{p+1} \Theta^{a(p+1)} (\alpha - 1)
\]

\[
+ y^{p+1} \alpha \Theta^{a(p+1)-2} |\nabla \Theta|^2 \left( 1 - \alpha + |\kappa| y_0^{-(p-1-r)} \alpha \right)
\]

\[
\ge \Theta^\alpha \left( y' + y^{p+1} \mu^\alpha (\alpha - 1) \right)
\]

\[
+ y^{p+1} \alpha \Theta^{a(p+1)-2} |\nabla \Theta|^2 \left( 1 - \alpha + |\kappa| y_0^{-(p-1-r)} \alpha \right)
\]

\[
\ge 0 \quad \text{in} \ \Omega \times (0, \infty)
\]

due to the choice of \( \alpha, y_0 \) and \( y \) and since \( y(t) \in (0, y_0] \) for \( t \ge 0 \) and \( \Theta(x) \in [\mu, 1] \) for \( x \in \bar{\Omega} \). Thus, for \( \|u_0\|_{L^\infty(\Omega)} < C := y_0 \mu^\alpha \), the maximal solution \( u \) of (0.1) evolving from \( u_0 \) satisfies \( u \le v \) in \( \Omega \times (0, \infty) \) by comparison and Theorem 1.1.2, which implies the claim.

(ii) We assume \( \lambda_1(\Omega) > 1 \). Using the notation of part (i), it is now possible to choose \( G \) such that additionally \( \lambda := \lambda_1(G) > 1 \) is satisfied. Moreover, we fix \( \alpha \in (\frac{1}{\lambda^2}, 1) \). Since (3.20) is now fulfilled for every \( y_0 > 0 \) (due to \( \alpha \in (0, 1) \)), we can choose \( y_0 := (\|u_0\|_{L^\infty(\Omega)} + 1) \mu^{-\alpha} \). Then we show just like in part (i) that the maximal solution \( u \) of (0.1) satisfies \( u \le v \) in \( \Omega \times (0, \infty) \) by comparison and Theorem 1.1.2, whereby the claim is proved. ■
3.3. THE CASE $R < Q - 2$

In case of $q > p + 1$, blow-up in finite time occurs for large initial data, while for small initial data the solution is global in time and converges to 0 as $t \to \infty$. A similar behavior has been observed for the equation (0.6) without the gradient term. The second part of the following theorem was shown in [SouWei1] (Theorem 4) for $p \geq 1$ and $r \geq 1$. The proof given there can easily be extended to the case $p > 0$ and $r \geq 0$. But since the subsolution which is used changes its sign, there are some problems for $r < 0$. Our proof can directly be used for all values of the parameters.

**Theorem 3.3.3** Suppose that (0.2), (0.3) and (0.4) are fulfilled with $r < q - 2$, $q > p + 1$ and $\kappa < 0$.

(i) There is $C > 0$, depending on $p, q, r, \kappa$ and $\Omega$, such that $\|u_0\|_{L^\infty(\Omega)} < C$ implies that the maximal solution of (0.1) is global in time and bounded and converges to 0 uniformly as $t \to \infty$.

(ii) For every $w \in C^0(\overline{\Omega})$ with $w > 0$ in $\Omega$ and $w|_{\partial\Omega} = 0$ there exists $b_0 > 0$ such that the maximal solution $u$ of (0.1) evolving from $u_0 = bw$ with $b \geq b_0$ blows up in finite time.

**Proof.**

(i) This part can be proved just like Theorem 3.1.4 (i).

(ii) We first fix $\alpha \in (\max\{p + 1, r + 2\}, q)$ and then choose $\delta > 0$ small enough such that $r + 2 + 2\delta < \alpha$ and $p + 1 + \delta < \alpha$ is satisfied. Furthermore, we fix $R > 0$ such that $B_R(x_0) \subset \subset \Omega$ holds for some $x_0 \in \Omega$. Without loss of generality we assume $x_0 = 0$. Moreover, let $\beta(t)$ denote the solution of

$$
\begin{cases}
\beta' = \beta^\alpha, & t \in (0, t_0), \\
\beta(0) = \beta_0,
\end{cases}
$$

with maximal existence time $t_0 \in (0, \infty)$, where $\beta_0 \geq b := \max \left\{(6y_0)^\frac{1}{2}, \left(\frac{4}{R^2}\right)^\frac{1}{2}\right\}$ will be fixed below and $y_0 := \max \left\{\frac{2}{3}, (\frac{3}{4}R^2)^{-1}\right\}$.

Defining

$$v(x, t) := \beta(t)e^{-\beta^\alpha(t)\frac{|x|^2}{R^2-|x|^2}} \text{ for } (x, t) \in B_R(0) \times [0, t_0),$$

we compute (abbreviating $y := \beta^\alpha(t)\frac{|x|^2}{R^2-|x|^2}$)

$$v_t - v^p \Delta v - v^q - \kappa v|\nabla v|^2 = \beta^\alpha e^{-y} - \delta \beta^\alpha ye^{-y} + \beta^{p+1}e^{-(p+1)y} \frac{2\beta^\alpha R^2}{(R^2 - |x|^2)^2} \left\{ -2\beta^\alpha R^2 |x|^2 + n(R^2 - |x|^2)^2 + 4|x|^2(R^2 - |x|^2) \right\}.$$
\[-\beta^a e^{-qy} + |\kappa| \beta^{r+2} e^{-(r+2)y} \frac{4\beta^d R^4|x|^2}{(R^2 - |x|^2)^4} \leq \beta^a e^{-y} - \delta \beta^a ye^{-y} + \beta^p+1+\delta e^{-(p+1)y} \frac{2(n + 4)R^6}{(R^2 - |x|^2)^4} - \beta^a e^{-qy} + |\kappa| \beta^{r+2+2d} e^{-(r+2)y} \frac{4R^6}{(R^2 - |x|^2)^4} \]

in $B_R(0) \times (0, t_0)$. (3.21)

Since $y \geq \frac{1}{3} b^d \geq 2y_0$ is satisfied for $|x| \geq \frac{B}{2}$, $y \leq y_0$ implies $|x| \leq \frac{R}{2}$. Hence by (3.21) there is $\beta_1 \geq b$ such that

\[ v_t - v^\rho \Delta v - v^q - \kappa v^r |\nabla v|^2 \leq \beta^a + \beta^p+1+\delta \frac{2(n + 4)R^6}{(\frac{3}{4}R^2)^4} - \beta^a e^{-qy_0} + |\kappa| \beta^{r+2+2d} \frac{4R^6}{(\frac{3}{4}R^2)^4} \leq 0 \quad \text{for } y \leq y_0 \]

is fulfilled due to the choice of $\alpha$ and $\delta$, if $\beta_0 \geq \beta_1$.

Moreover, $y \geq y_0$ implies $1 \leq y_0 (\frac{3}{4}R^2)^4 \leq y (\frac{3}{4}R^2)^4 \leq \beta^a |x|^2$ in case of $|x| \leq \frac{B}{2}$, whereas $\beta^a |x|^2 \geq b^d R^2 = 1$ is satisfied for $|x| \geq \frac{B}{2}$. Hence, $\beta^a |x|^2 \geq 1$ is fulfilled for $y \geq y_0$ and $x \in B_R(0)$. Thus, according to (3.21) and since $y_0 \geq \frac{2}{5}$, we can choose $\beta_0 \geq \beta_1$ large enough such that

\[ v_t - v^\rho \Delta v - v^q - \kappa v^r |\nabla v|^2 \leq \frac{\delta}{2} y \beta^a e^{-y} - \delta \beta^a ye^{-y} + \beta^p+1+\delta e^{-(p+1)y} y^4 (n + 4)R^6 \]

\[ + |\kappa| \beta^{r+2+2d} e^{-(r+2)y} y^4 R^6 \leq ye^{-y} \left[ - \frac{\delta}{2} \beta^a + \beta^p+1+\delta e^{-p y} y^3 (n + 4)R^6 \right] + |\kappa| \beta^{r+2+2d} e^{-(r+1)y} y^3 R^6 \leq 0 \quad \text{for } y \geq y_0 \]

is satisfied, according to the choice of $\alpha$ and $\delta$ and since $p > 0$ and $r > -1$. Hence, $v_t \leq v^\rho \Delta v + v^q + \kappa v^r |\nabla v|^2$ holds in $B_R(0) \times (0, t_0)$.

Let $w \in C^0(\Omega)$ be positive in $\Omega$ with $w = 0$ on $\partial \Omega$. Since $B_R(0) \subset \subset \Omega$, we can choose $b_0 > 0$ large enough such that $b_0 w(x) \geq v(x, 0)$ in $B_R(0)$. Thus, $u_0 := bw$ with some $b \geq b_0$ fulfills $u_0(x) \geq v(x, 0)$ in $B_R(0)$. Hence, $u_\varepsilon \geq v$ in $B_R(0) \times (0, T_\varepsilon)$ for $\varepsilon > 0$ by comparison, where $T_\varepsilon$ is the maximal existence time of $u_\varepsilon$. As $\varepsilon \searrow 0$, we get $u \geq v$ in $B_R(0) \times (0, T)$ and hence $u$ has to blow up in finite time because $v_0(0, t) = \beta(t) \to \infty$ as $t \nearrow t_0$. This implies the claim.
Chapter 4

The size of the blow-up set

In Chapter 2 we have presented in detail which conditions lead to finite-time blow-up in case of $\kappa > 0$. Hence throughout this chapter we require that $u$ is the maximal solution of (0.1) with $\kappa > 0$ such that $u$ blows up in finite time with blow-up time $T \in (0, \infty)$. Then, we analyze the size of the blow-up set

$$S := \{ x \in \bar{\Omega} \mid \exists ((x_k, t_k))_{k \in \mathbb{N}} \subset \Omega \times (0, T) \text{ such that } x_k \to x \text{ and } u(x_k, t_k) \to \infty \text{ as } k \to \infty \}$$

of $u$ and distinguish if $|S| = 0$, which is called single point blow-up, or if $|S| > 0$, which is called regional blow-up, is fulfilled.

Moreover, we assume that

$$\Omega = B_a(0) \subset \mathbb{R}^n \text{ is a ball} \quad (4.1)$$

with $a > 0$ and $u_0$ is a smooth function fulfilling

$$u_0 \in C^3(\bar{\Omega}) \text{ with } u_0 > 0 \text{ in } \Omega \text{ and } u_0|_{\partial \Omega} = 0, \text{ such that }$$

$$u_0 \text{ is radially symmetric and nonincreasing with respect to } |x|. \quad (4.2)$$

One aspect of this chapter is to show how the gradient term $\kappa u^r|\nabla u|^2$ in (0.1) influences the size of the blow-up set as compared to the equation (0.6) without gradient term. For the latter equation it has been proved that in case of $q > p + 1$ for large initial data single point blow-up occurs (see e.g. [SGKM]), whereas in case of $q = p + 1$ in large domains $\Omega$ satisfying $\lambda_1(\Omega) < 1$ we have regional blow-up (see e.g. [SGKM], [Win4], [Win5]).

For (0.1) in case of dimension $n = 1$, it was already shown in [Sti] that the additional positive gradient term can indeed enforce regional blow-up. In particular, it has been shown that in case of $q = p + 1$ with $r > 1$ and in case of $q > p + 1$ with $r > q - 2$ regional blow-up occurs, whereas in case of $q > p + 1$ with $r < q - 2$ single point blow-up occurs, if $u_0$ is chosen such that the maximal solution $u$ of (0.1) evolving from $u_0$ blows up in finite time $T$ and $u_t \geq 0$ in $\Omega \times (0, T)$ is fulfilled.
Here we extend these results to arbitrary dimensions $n$ and furthermore, we show that single point blow-up only occurs in case of $q > \max\{p + 1, r + 2\}$, whereas in case of $q \leq \max\{p + 1, r + 2\}$ regional blow-up occurs. To explain this phenomenon, we state that the source term $u^q$ enforces blow-up in a single point in contrast to the diffusion term and the gradient term which both enforce regional blow-up. Thus, single point blow-up only occurs if the source term $u^q$ has the highest order of these three terms.

Moreover, we get these results in case of regional blow-up for another class of initial data. If we require that the initial data should be large enough, then the condition $u_t \geq 0$ is not needed any more.

Apart from the results for (0.6) and the related forced porous medium equation $u_t = \Delta u^m + u^p$, to the best of our knowledge there are only few results concerning regional blow-up for diffusion equations. In case of an absorbing gradient term it has been shown in [CFQ] that single point blow-up occurs for solutions of the Chipot-Weissler equation $u_t = \Delta u + u^q - \mu |\nabla u|^s$ for $q > s > 1$, $\mu > 0$. A similar behavior has been observed in case of the semilinear heat equation $u_t = \Delta u + u^q$, $q > 1$. But if a positive gradient term occurs, it has been proved in [KawPel] for the equation $u_t = \Delta u + u^q + |\nabla u|^2$, $q > 1$, that single point blow-up occurs in case of $q > 2$, whereas regional blow-up occurs in case of $1 < q \leq 2$. This behavior corresponds to our results for (0.1), because in both equations single point blow-up only occurs if the source term $u^q$ has the highest order as compared to the diffusion term and the gradient term. Furthermore, there are some results concerning the blow-up set of semilinear diffusion equations without gradient terms of the form $u_t = \Delta u + f(u)$ (see e.g. [FriMcL1], [Lac]).

### 4.1 Regional blow-up for $q \leq \max\{p + 1, r + 2\}$

In our proofs of regional blow-up we adapt an idea which was demonstrated in [FriMcL1]. A suitable function $J$, dependent on both the solution $u$ and its spatial derivatives, is shown to be nonnegative and this enables us to conclude that a suitable ball has to be contained in the blow-up set.

First we consider the case $q < p + 1$. Since we need the condition $u_t \geq 0$ in $\Omega \times (0, T)$ for obtaining the result of regional blow-up, we first show that there exist suitable initial data which can be used in the next theorem.

**Remark 4.1.1** Let $a > 0$, $\Omega = B_a(0) \subset \mathbb{R}^n$ and suppose that assumption (0.2) is fulfilled with $q < p + 1$ and $\kappa > 0$. Then there exists $u_0 \in C^3(\overline{\Omega})$ that fulfills (4.2) with $\Delta u_0 + (u_0 + \varepsilon)^{q-p} + \kappa(u_0 + \varepsilon)^{r-p}|\nabla u_0|^2 \geq 0$ in $\overline{\Omega}$ for all $\varepsilon \in (0, 1)$.

**Proof.** Let $\Theta := \Theta(\cdot; \Omega)$ corresponding to the principal eigenvalue $\lambda_1 := \lambda_1(\Omega) > 0$. Furthermore, we choose $\delta \in (0, 1)$ such that $\delta \leq \lambda_1^{\frac{1}{p+1-q}}$ in case of
4.1. REGIONAL BLOW-UP FOR $Q \leq \text{MAX}\{P + 1, R + 2\}$

$q > p$ and $\delta \leq 2^{q-p} \lambda_1^{-1}$ in case of $q \leq p$ is satisfied. Defining $u_0(x) := \delta \Theta(x)$ for $x \in \bar{\Omega}$, we get for all $\varepsilon \in (0, 1)$

$$
\Delta u_0 + (u_0 + \varepsilon)^{q-p} + \kappa(u_0 + \varepsilon)^r - p |\nabla u_0|^2 \geq -\lambda_1 \delta \Theta + (\delta \Theta + \varepsilon)^{q-p} \\
\geq 0 \text{ in } \bar{\Omega}.
$$

Since it can easily be verified that $u_0$ fulfills the other conditions that are given in the claim, the remark is proved.

Now we are able to prove the result in case of $q < p + 1$ mentioned above. We only study the case $r \geq 2p - q$, since it is shown in Chapter 2 that in case of $q < p + 1$ all solutions of (0.1) are global and bounded for $r < 2p - q$. In particular, we know from Chapter 2 that in large domains all solutions blow up in finite time for $q < p + 1$ and $r \geq 2p - q$.

**Theorem 4.1.2** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q < p + 1$, $r \geq 2p - q$ and $\kappa > 0$. Furthermore, assume that $\Delta u_0 + (u_0 + \varepsilon)^{q-p} + \kappa(u_0 + \varepsilon)^r - p |\nabla u_0|^2 \geq 0$ in $\bar{\Omega}$ is satisfied for all $\varepsilon \in (0, \varepsilon_0)$ with a suitable $\varepsilon_0 \in (0, 1)$. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose that $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. Then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** By Lemma 1.3.3 we know that $u$ is radially symmetric with $u(0, t) = \max_{x\in\Omega} u(x, t)$ for $t \in (0, T)$, where $T$ shall denote the finite blow-up time of $u$. Thus $x = 0$ is a blow-up point since $u \geq 0$ in $\bar{\Omega} \times [0, T)$ by Theorem 1.1.2.

For $\varepsilon \in (0, \varepsilon_0)$ let $u_\varepsilon$ denote the solution of (1.1) with $u_{\varepsilon, t} := u_\varepsilon + \varepsilon$, satisfying $(u_{\varepsilon, t}) \geq 0$ in $\Omega \times (0, T_\varepsilon)$ by Lemma 1.3.4. Hence, $u_t \geq 0$ in $\Omega \times (0, T)$ is fulfilled, because $u_{\varepsilon}(x, t) \rightarrow u(x, t)$ as $\varepsilon \searrow 0$ for $(x, t) \in \bar{\Omega} \times [0, T)$ by Theorem 1.1.2.

Since $r \geq 2p - q > p - 1$ holds, we can choose $\alpha \in (0, 1)$ such that $r > p - \alpha$. This implies $\alpha + r - p > q - 1 - p$ due to $q < p + 1$.

Thus, we can choose $M \geq 1$ large enough, such that the following conditions are fulfilled:

\begin{align*}
-\frac{p^2}{2} + \frac{p}{2} (2\alpha + p) u^{\alpha-1} + p\kappa u^{\alpha+r-p} - \frac{(n-1)p}{2} \leq 0 \text{ for } u \geq M, \\
(\alpha - q) u^{q-1} - p + (2\alpha + p) u^{\alpha-1} \leq 0 \text{ for } u \geq M, \\
(\alpha + r) u^{\alpha+r-1-p} \geq \frac{2}{4p} \alpha(p + \alpha - 1) u^{2\alpha-2} + 2\kappa u^{\alpha+r-p} \geq 0 \text{ for } u \geq M, \beta \geq 1.
\end{align*}
Thus, we can fix $\beta$ such that $u$ blow-up point and $u$ is continuous. Moreover, we can choose $t_0 \in (0, T)$ such that $u(0, t_0) \geq 2M$, because $x = 0$ is a blow-up point and $u(0, t) = \max u(x, t)$ for $t \in (0, T)$. Furthermore, there is $\delta > 0$ such that $u(\rho, t_0) \geq M$ for $\rho \in [0, \delta]$ since $u$ is continuous. Thus, we can fix $\beta \geq 1$ large enough, such that $u_{\rho\rho}(\rho, t_0) \geq -\beta$ and $u(\rho, t_0) \geq M$ for $\rho \in [0, \rho_\infty]$ with $\rho_\infty := \frac{\pi}{2} \sqrt{\frac{2}{\beta}}$ is satisfied.

Furthermore, we define

$$J(\rho, t) := u_\rho(\rho, t) + (c(\rho) + \varepsilon)u^\alpha(\rho, t) \text{ for } (\rho, t) \in [0, \rho_\infty) \times [t_0, T),$$

where $c(\rho) := \sqrt{\frac{2\beta}{p}} \tan \left(\sqrt{\frac{p^2}{2\rho}}\right)$ and $\varepsilon := \frac{\pi}{4} \sqrt{\frac{2\beta}{p}}$.

We will show $J \geq 0$ in $(0, \rho_\infty) \times (t_0, T)$.

First $u \geq M$ in $(0, \rho_\infty) \times (t_0, T)$ holds due to the choice of $t_0$ and $\rho_\infty$ and because $u_t \geq 0$ in $\Omega \times (0, T)$. Furthermore, we get with $\rho_0 := \frac{\rho_\infty}{2}$

$$\frac{\varepsilon}{\rho} - \frac{p}{2} c(\rho)^2 \geq \frac{\varepsilon}{\rho_0} - \beta \tan \left(\sqrt{\frac{p\beta}{2\rho_0}}\right) = \beta - \beta \tan \left(\frac{\pi}{4}\right) = 0 \text{ for } \rho \in (0, \rho_0) \quad (4.5)$$

and

$$\frac{n-1}{\rho} u^{p+\alpha} \left(\frac{\varepsilon}{\rho} - \frac{p}{2} c^2\right) \geq -\frac{(n-1)p}{2} u^{p+\alpha} (c + \varepsilon) \frac{c}{\rho_0} \geq - (c + \varepsilon) u^{p+\alpha} \frac{(n-1)p}{2} \left(c^2 + \frac{1}{\rho_0^2}\right) = (c + \varepsilon) u^{p+\alpha} \left(-\frac{(n-1)p}{2} c^2 \frac{1}{\rho_0^2}\right) \text{ in } [\rho_0, \rho_\infty) \times (t_0, T). \quad (4.6)$$

We now use $c'(\rho) = \frac{p}{2} c(\rho)^2 + \beta$, $c''(\rho) = p c(\rho) c'(\rho)$, $c(\rho) \geq \beta \rho$ and $(c(\rho) + \varepsilon)^2 \leq c(\rho)^2 + 2\varepsilon c(\rho) + \varepsilon^2 \leq 2(c(\rho)^2 + \varepsilon^2)$ for $\rho \in (0, \rho_\infty)$.

Thus, we compute

$$J_t = u_{\rho\rho} + (c + \varepsilon)\alpha u^{\alpha-1} u_t$$

$$= \left[u^p u_{\rho\rho} + \frac{n-1}{\rho} u^p u_{\rho} + u^q + \kappa u'(u_\rho)^2\right]_{\rho} + (c + \varepsilon)\alpha u^{\alpha-1} u_t$$

$$= u^p u_{\rho\rho\rho} + pu^{p-1} u_{\rho\rho} + \frac{n-1}{\rho} u^p u_{\rho\rho} + p \frac{n-1}{\rho} u^{p-1} (u_\rho)^2 - \frac{n-1}{\rho^2} u^p u_{\rho}\right.$$  

$$+ qu^{p-1} u_\rho + \kappa u^{p-1} (u_\rho)^3 + 2\kappa u' u_{\rho} u_\rho + (c + \varepsilon)\alpha u^{\alpha-1} \left[u^p u_{\rho\rho} + \frac{n-1}{\rho} u^p u_{\rho}\right]$$

$$+ u^q + \kappa u'(u_\rho)^2\right)$$
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$$
= u^p J_{\rho \rho} + pu^{p-1}u_{\rho\rho} + \frac{n-1}{\rho} u^p u_{\rho\rho} + \frac{n-1}{\rho^2} u^{p-1}(u_{\rho})^2 - \frac{n-1}{\rho^2} u^p u_{\rho}
+ qu^{q-1}u_{\rho} + \kappa u^{r-1}(u_{\rho})^3 + 2\kappa u^r u_{\rho\rho} + (c + \varepsilon)\alpha u^{a-1}\left(\frac{n-1}{\rho} u^p u_{\rho} + u^q\right)
+ \kappa u^r(u_{\rho})^2 - u^p(c' u^a + 2\epsilon' \alpha u^{a-1} u_{\rho} + (c + \varepsilon)\alpha (a-1) u^{a-2}(u_{\rho})^2)
= u^p J_{\rho \rho} + \left(pu^{p-1}u_{\rho} + \frac{n-1}{\rho} u^p + 2\kappa u^r u_{\rho}\right) J_{\rho} + \frac{n-1}{\rho} u^{p-1}(u_{\rho})^2
- \frac{n-1}{\rho^2} u^p u_{\rho} + qu^{q-1}u_{\rho} + \kappa u^{r-1}(u_{\rho})^3 + (c + \varepsilon)\alpha u^{a-1}\left(\frac{n-1}{\rho} u^p u_{\rho} + u^q\right)
+ \kappa u^r(u_{\rho})^2 - u^p(c' u^a + 2\epsilon' \alpha u^{a-1} u_{\rho} + (c + \varepsilon)\alpha (a-1) u^{a-2}(u_{\rho})^2)
- \left(pu^{p-1}u_{\rho} + \frac{n-1}{\rho} u^p + 2\kappa u^r u_{\rho}\right)(c' u^a + (c + \varepsilon)\alpha u^{a-1} u_{\rho})
= u^p J_{\rho \rho} + \left(pu^{p-1}u_{\rho} + \frac{n-1}{\rho} u^p + 2\kappa u^r u_{\rho}\right) J_{\rho} + \left[\frac{n-1}{\rho} u^{p-1}u_{\rho} - \frac{n-1}{\rho^2} u^p\right]
+ qu^{q-1} + \kappa u^{r-1}(u_{\rho})^2 + (c + \varepsilon)\alpha \frac{n-1}{\rho} u^{p+a-1} + \kappa (c + \varepsilon)\alpha u^{a-1+r} u_{\rho}
- 2\epsilon' \alpha u^{a-1+p} - (c + \varepsilon)\alpha (a-1) u^{a-2+p} u_{\rho} - pc' u^{a+p-1} - 2\kappa c' u^{a+r}
- p(c + \varepsilon)\alpha u^{a+p-2} u_{\rho} - (c + \varepsilon)\alpha \frac{n-1}{\rho} u^{a+p-1} - 2\kappa (c + \varepsilon)\alpha u^{a-1+r} u_{\rho}
+ (c + \varepsilon)\alpha u^{a-1+r} - c' u^{p+a} - c' \frac{n-1}{\rho} u^{p+a} - (c + \varepsilon)\alpha \left[\frac{n-1}{\rho} u^{p-1}u_{\rho}\right]
- \frac{n-1}{\rho^2} u^p + qu^{q-1} + \kappa u^{r-1}(u_{\rho})^2 + (c + \varepsilon)\alpha \frac{n-1}{\rho} u^{p+a-1}
+ \kappa (c + \varepsilon)\alpha u^{a-1+r} u_{\rho} - 2\epsilon' \alpha u^{a-1+p} - (c + \varepsilon)\alpha (a-1) u^{a-2+p} u_{\rho} - pc' u^{a+p-1} - 2\kappa c' u^{a+r}
- 2\kappa c' u^{a+r} - p(c + \varepsilon)\alpha u^{a+p-2} u_{\rho} - (c + \varepsilon)\alpha \frac{n-1}{\rho} u^{a+p-1}
- 2\kappa (c + \varepsilon)\alpha u^{a-1+r} u_{\rho}$$
\[-pc'u^{a-1+p} - 2\kappa c'u^{a-2+p} \]
\[(c + \varepsilon)\alpha \frac{n - 1}{\rho} - u^{p+\alpha - 1}] + (c + \varepsilon)^2 u^{2\alpha} \left[ \frac{n - 1}{\rho} - u^{p-1} \right] + \kappa(c + \varepsilon)u^{a-1+r} - (c + \varepsilon)(\alpha - 1)u^{a-2+p} - p(c + \varepsilon)\alpha u^{a-2+p} \]
\[2\kappa(c + \varepsilon)u^{a-1+r} - (c + \varepsilon)^3 u^{3}\alpha \kappa u^{r-1} \]

\[\leq u^p J_{\rho_0} + B(p, t)J_{\rho} + C(\rho, t)J + (c + \varepsilon)u^{p+\alpha} \left[ (\alpha - q)u^{q-1-p} - \frac{\varepsilon'}{\varepsilon'} \right] + \varepsilon'(2\alpha + p)u^{a-1} + 2\kappa u^{a+r-p}) - (c + \varepsilon)^2(\alpha(p + \alpha - 1)u^{2a-2} + \kappa(r + \alpha)u^{2a+r-1-p}) + \frac{n - 1}{\rho} u^{p+\alpha} \left[ - c' + \frac{c + \varepsilon}{\rho} + p(c + \varepsilon)^2 u^{a-1} \right] \]

\[\leq u^p J_{\rho_0} + B(p, t)J_{\rho} + C(\rho, t)J + (c + \varepsilon)u^{p+\alpha} \left[ (\alpha - q)u^{q-1-p} + \beta(-p) + (2\alpha + p)u^{a-1} + 2\kappa u^{a+r-p}) + c^2 \left( - \frac{p^2}{2} + \frac{p}{2}(2\alpha + p)u^{a-1} + pk u^{a+r-p}) - 2(c^2 + \varepsilon^2)(\alpha(p + \alpha - 1)u^{2a-2} + \kappa(r + \alpha)u^{2a+r-1-p}) \right] + \frac{n - 1}{\rho} u^{p+\alpha} \left[ - p c^2 - \beta + \frac{\varepsilon}{\rho} + p(c + \varepsilon)^2 u^{a-1} \right] \]

\[\geq u^p J_{\rho_0} + B(p, t)J_{\rho} + C(\rho, t)J + (c + \varepsilon)u^{p+\alpha} \left[ (\alpha - q)u^{q-1-p} + \beta(-p) + (2\alpha + p)u^{a-1} + 2\kappa u^{a+r-p}) - 2\varepsilon^2(\alpha(p + \alpha - 1)u^{2a-2} + \kappa(r + \alpha)u^{2a+r-1-p}) \right] + c^2 \left( - \frac{p^2}{2} + \frac{p}{2}(2\alpha + p)u^{a-1} + pk u^{a+r-p}) - 2(\alpha(p + \alpha - 1)u^{2a-2} + \kappa(r + \alpha)u^{2a+r-1-p}) \right] + \frac{n - 1}{\rho} u^{p+\alpha} \left[ - \frac{p c^2}{2} + \frac{\varepsilon}{\rho} \right] \]

\[\geq u^p J_{\rho_0} + B(p, t)J_{\rho} + C(\rho, t)J \text{ in } (0, \rho_\infty) \times (t_0, T) \] (4.7)

because (4.3), (4.4), (4.5) and (4.6) are fulfilled and \(2\varepsilon^2 = \frac{\varepsilon^2}{q\rho} \beta\), where \(u^p, B \) and \(C \) are continuous functions in \((0, \rho_\infty) \times (t_0, T) \) since \(u \in C^\infty(\Omega \times (0, T)) \) by Lemma 1.3.1. According to \(\tan(y) \geq y \) for \(y \in (0, \frac{\pi}{2}) \), \(M \geq 1 \) and \(\alpha > 0 \),

\[J(\rho, t_0) = u_\rho(\rho, t_0) + (c(\rho) + \varepsilon)u^\alpha(\rho, t_0) \geq -\beta \rho + (\beta \rho + \varepsilon)M^\alpha \geq \varepsilon > 0 \]

is satisfied for \(\rho \in (0, \rho_\infty) \). Moreover, as \(u_\rho(0, t) = 0 \) for \(t \in (0, T) \) by Lemma 1.3.3, \(J(0, t) \geq \varepsilon M^\alpha \geq \varepsilon > 0 \) holds for \(t \in [t_0, T) \).
Next let $T_1 \in (t_0, T)$. Since $u_\rho \in C^0([0, \rho_\infty] \times [t_0, T_1])$, there exists $d > 0$ with $|u_\rho| \leq d$ in $[0, \rho_\infty] \times [t_0, T_1]$. Thus, we are able to choose $\rho_0 \in (0, \rho_\infty)$ such that $M^\alpha c(\rho_0) \geq d$. This implies $J(x, t) \geq -d + (c(\rho_0) + \varepsilon)M^\alpha \geq \varepsilon M^\alpha \geq \varepsilon > 0$ for $(\rho, t) \in [\rho_0, \rho_\infty] \times [t_0, T_1]$.

Hence, we conclude that $J \geq \varepsilon > 0$ on the parabolic boundary of $[0, \rho] \times [t_0, T_1]$ is satisfied for all $\rho \in [\rho_0, \rho_\infty)$. Thus, we have $J \geq 0$ in $[0, \rho] \times [t_0, T_1]$ for all $\rho \in [\rho_0, \rho_\infty]$ by (4.7) and Lemma 1.1.1 and hence $J \geq 0$ in $(0, \rho_\infty) \times [t_0, T_1]$.

Furthermore, we deduce $J \geq 0$ in $(0, \rho_\infty) \times (t_0, T)$ because $T_1 \in (t_0, T)$ was arbitrary.

Fixing $\rho_0 \in (0, \rho_\infty)$, $u_\rho \geq -(c(\rho_0) + \varepsilon)u^\alpha$ in $(0, \rho_0) \times (t_0, T)$ is fulfilled. Since $\alpha \in (0, 1)$, $x = 0$ is a blow-up point and $u(0, t) = \max_{x \in \Omega} u(x, t)$ for $t \in (0, T)$, we can choose a sequence $(t_k)_{k \in \mathbb{N}} \subset (t_0, T)$ such that $u(0, t_k)_{1-\alpha} - (1 - \alpha)(c(\rho_0) + \varepsilon)\rho_0 > 0$ for all $k \in \mathbb{N}$ and $u(0, t_k) \to \infty$ for $k \to \infty$. Hence, we have

$$u(\rho, t_k) \geq (1 - \alpha)(c(\rho_0) + \varepsilon)\rho + u(0, t_k)_{1-\alpha} \frac{1}{1-\alpha} \geq (1 - \alpha)(c(\rho_0) + \varepsilon)\rho + u(0, t_k)_{1-\alpha} \frac{1}{1-\alpha}$$

for $\rho \in (0, \rho_0)$ and $k \in \mathbb{N}$. Thus, the interval $(0, \rho_0)$ is contained in the blow-up set, since $u(0, t_k) \to \infty$ for $k \to \infty$ and $\alpha \in (0, 1)$ is fulfilled. Hence, due to the symmetry of $u$ (by Lemma 1.3.3), the ball $B_{\rho_\infty}(0)$ is contained in the blow-up set and the claim follows.

As the condition $u_t \geq 0$ in $\Omega \times (0, T)$ is only fulfilled for a rather small class of initial data, we present another result similar to the last one. But now we require that the initial data should be large in a certain sense instead of assuming $u_t \geq 0$. This enables us to get the result of regional blow-up for slightly more initial data.

**Corollary 4.1.3** Suppose that $a > 0$, $\Omega := B_a(0)$ and assumptions (0.2) and (4.2) are fulfilled with $q < p + 1$, $r \geq 2p - q$ and $\kappa > 0$ and choose $\alpha \in (0, 1)$ with $r > p - \alpha$ and $M \geq 1$ such that (4.3) and (4.4) are fulfilled. Furthermore, for $R > 0$ let $\Theta_R$ denote the principal eigenfunction of $-\Delta$ in $B_R := B_R(0)$ with $\max_{x \in B_R} \Theta_R(x) = 1$, corresponding to the first eigenvalue $\lambda_R$. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. If there are $R \in (0, a]$ such that $\lambda_R^{p+1-q} > M$ is fulfilled and $c_0 \in (M, \lambda_R^{\frac{1}{1-q}})$ such that $u_0 \geq c_0 \Theta_R$ in $B_R$ is satisfied, then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** The function $z(x, t) := c_0 \Theta_R(x)$ for $(x, t) \in \bar{B}_R \times [0, T]$ satisfies

$$z_t - z^p \Delta z - z^q - \kappa z^r |\nabla z|^2 \leq z_t - z^p \Delta z - z^q \leq \lambda_R^{p+1} \Theta_R^{p+1} - c_0^p \Theta_R^p \leq 0$$

in $B_R \times (0, T)$ due to the choice of $c_0$. Moreover, we have $u \geq z$ on the parabolic boundary of $B_R \times (0, T)$ due to the choice of $c_0$ and $u_0$ and hence $u \geq z$ is fulfilled.
in $B_R \times (0, T)$ by comparison. Furthermore, with $\rho := |x|$, there is $\delta > 0$ such that $u(\rho, t) \geq z(\rho, t) \geq M$ for $(\rho, t) \in [0, \delta) \times [0, T)$ due to $c_0 > M$.

Hence, we can choose $t_0 = 0$ and the claim can be proved like in the proof of Theorem 4.1.2 because $u_{\rho} \in C^0(\Omega \times [0, T))$ by Lemma 1.3.1.

Before presenting the results in case of $q \geq p + 1$ we show in the following that, in case of $q = p + 1$ and $r > p - 1$ and in case of $q > p + 1$ with $r > -1$, there are arbitrarily large stationary subsolutions of (0.1). These functions will be used in the following proofs.

**Lemma 4.1.4** Suppose that assumption (0.2) holds with $\kappa > 0$ and either $q = p + 1$ and $r > p - 1$ or $q > p + 1$ and $r > -1$ is fulfilled. Furthermore, let $R > 0$ and $c_R \in (0, 1]$ such that additionally $c_R \leq \left( \frac{3}{4} \right)^4 \frac{R^2}{2(n+6)} e^{-\frac{4}{3}}$ is satisfied in case of $q = p + 1$. Then there is a constant $b_R > 0$ such that for every $b \geq b_R$ the function $v(x) := be^{-\frac{c_R|v|^2}{R^2-|v|^2}}, \ x \in B_R(0) \subset \mathbb{R}^n,$ fulfills $v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq 0$ in $B_R(0)$ and, in case of $q > p + 1$, $v^p \Delta v + v^q \geq 0$ in $B_R(0)$.

**Proof.** Writing $y := \frac{c_R|v|^2}{R^2-|v|^2}$, the function $v$ satisfies for $x \in B_R(0)$

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 = b^{p+1}e^{-(p+1)y} \left( \frac{2c_R R^2}{R^2-|x|^2} \right)^2 \left( 2c_R R^2 |x|^2 - n(R^2 - |x|^2)^2 \right) - 4|x|^2(R^2 - |x|^2) + b^q e^{-qy} + \kappa b^{r+2}e^{-(r+2)y} \frac{4c_R^2 R^4 |x|^2}{(R^2 - |x|^2)^4}.$$}

Furthermore, we set $p(t) := 2c_R R^2 t^2 - n(R^2 - t^2)^2 - 4t^2(R^2 - t^2)$ for $t \in [0, R]$. Due to $c_R \in (0, 1]$ we have $|p(t)| \leq (2c_R + n + 4)R^4 \leq (n + 6)R^4$ for $t \in [0, R]$. Since $p(R) = 2c_R R^4 > 0$ and $p$ is continuous in $[0, R]$, there is $\gamma \in (\frac{1}{2}, 1)$ such that $p(t) \geq 0$ in $[\gamma R, R]$. Thus, $v^p \Delta v \geq 0$ holds for $|x| \in [\gamma R, R]$. Moreover, we obtain in case of $q = p + 1$

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq b^{p+1} \left\{ \frac{2(n + 6)c_R}{(1 - \frac{1}{4})^4 R^2} + e^{-q} \right\} \geq 0 \quad \text{in } B_{\frac{R}{2}}(0),$$

due to the choice of $c_R$ and because $b > 0$. Furthermore, there is $b_R > 1$ such that

$$v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq -b^{p+1} \frac{2(n + 6)c_R}{(1 - \frac{1}{4})^4 R^2} + \kappa b^{r+2}e^{-(r+2)y} \frac{c_R^2}{(1 - \frac{1}{4})^4 R^2} \geq 0 \quad \text{for } |x| \in \left[ \frac{R}{2}, \gamma R \right]$$

is fulfilled for every $b \geq b_R$ due to $r > p - 1$. Hence, we deduce $v^p \Delta v + v^q + \kappa v^r |\nabla v|^2 \geq 0$ in $B_R(0)$ due to $b \geq b_R$ (in case of $q = p + 1$). In case of $q > p + 1$ there is $b_R > 1$ such that

$$v^p \Delta v + v^q \geq b^{p+1} \left\{ \frac{2(n + 6)c_R}{(1 - \gamma^2)^4 R^2} + b^q - 1 - e^{-q} \frac{c_R^2}{(1 - \frac{1}{4})^4 R^2} \right\} \geq 0 \quad \text{in } B_{\gamma R}(0).$$
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is satisfied for every $b \geq b_R$ due to $q > p + 1$. Thus, we conclude $v^p \Delta v + v^q \geq 0$ in $B_R(0)$ for $q > p + 1$ due to $b \geq b_R$ and we have proved the claim. 

In case of $q = p + 1$ and $r > -1$ regional blow-up occurs, too. This corresponds to the behavior for the equation $u_t = u^p \Delta u + u^q$. The following proofs use ideas that are similar to those from the former proofs for $q < p + 1$.

Since there are different proofs for $r > p - 1$ and for $r \leq p - 1$, first we give the results in case of $q = p + 1$ and $r > p - 1$ that are the extensions of Theorem 5.9 of [Sti]. The initial data $u_0 := \Theta(\cdot ; \Omega)$ fulfill the conditions of the following theorem, if $\Omega$ is chosen such that $\lambda_1(\Omega) \leq 1$ holds. In particular, we use again the condition $u_t \geq 0$ in $\Omega \times (0, T)$.

**Theorem 4.1.5** Assume that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q = p + 1$, $r > p - 1$ and $\kappa > 0$. Furthermore, let $\Delta u_0 + u_0 \geq 0$ in $\Omega$ be satisfied. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_t$ is chosen such that $u$ blows up in finite time with blow-up time $T$.

Then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** For $\varepsilon \in (0, 1)$ let $u_{\varepsilon}$ denote the solution of (1.1) with $u_{0\varepsilon} := u_0 + \varepsilon$. Then we have $(u_{0\varepsilon} + \varepsilon)^p \Delta (u_{0\varepsilon} + \varepsilon) + (u_{0\varepsilon} + \varepsilon)^q \geq (u_{0\varepsilon} + \varepsilon)^p (\Delta u_{0\varepsilon} + u_{0\varepsilon}) \geq 0$ in $\Omega$ due to $q > p$ and thus $(u_{0\varepsilon})_t \geq 0$ in $\Omega \times (0, T)$ is fulfilled by Lemma 1.3.4. This implies $u_t \geq 0$ in $\Omega \times (0, T)$, because $u_{\varepsilon}(x, t) \rightarrow u(x, t)$ for $\varepsilon \downarrow 0$ and $(x, t) \in \Omega \times [0, T)$ by Theorem 1.1.2.

Since $r > p - 1$ is satisfied, we can choose $\alpha \in (0, 1)$ such that $r + \alpha > p$.

With this choice of $\alpha$ we can prove the claim just like Theorem 4.1.2, because $\alpha + r - p > 0 = q - 1 - p$ is fulfilled.

The condition $u_t \geq 0$ in $\Omega \times (0, T)$ is again not necessary, if the initial data are large in a certain sense.

**Corollary 4.1.6** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q = p + 1$, $r > p - 1$ and $\kappa > 0$ and choose $\alpha \in (0, 1)$ with $r > p - \alpha$ and $M \geq 1$ such that (4.3) and (4.4) are satisfied. Furthermore, let $v_R(x) := e^{-\frac{c_R^{p+q}}{R^{p+q}}} \text{ for } x \in B_R := B_R(0)$ and $R > 0$ with $c_R \in (0, 1]$ such that $c_R \leq \left(\frac{3}{4}\right)^4 \frac{R^2}{2(n+6)} e^{-\frac{3}{4}}$. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. If there are $R \in (0, a]$ and $b > M$ such that $b \geq b_R$ ($b_R > 1$ is defined like in Lemma 4.1.4) and $u_0 \geq b v_R$ in $B_R$ is satisfied, then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** We define $z(x, t) := b v_R(x)$ for $(x, t) \in \tilde{B}_R \times [0, T)$ and can show that $u \geq z$ is fulfilled in $B_R \times (0, T)$ by Lemma 4.1.4 and the comparison principle.
Furthermore, with \( \rho := |x| \), there is \( \delta > 0 \) such that \( u(\rho, t) \geq z(\rho, t) \geq M \) for \((\rho, t) \in [0, \delta) \times [0, T)\) due to \( b > M \).

Hence, we can choose \( \tau_0 = 0 \) and the claim can be proved like in the proof of Theorem 4.1.5, because \( u_\rho \in C^0(\Omega \times [0, T)) \) by Lemma 1.3.1.

The following result in case of \( q = p+1 \) and \( r \leq p-1 \) is the extension of Theorem 5.7 of [Sti]. It differs from the other results of regional blow-up that are given in this section because we do not need any further assumption on the initial data \( u_0 \) apart from (4.2). Moreover, we remark that in contrast to the case \( q = p+1 \) and \( r > p-1 \) we cannot show that a ball with a radius which is larger than \( \frac{\pi}{p} \) is contained in the blow-up set - independent of the size of \( \Omega \). In case of \( r > p-1 \) there is no such bound on the radius of the ball.

Theorem 4.1.7 Suppose that \( a > 0 \), \( \Omega := B_a(0) \subset \mathbb{R}^n \) and assumptions (0.2) and (4.2) are fulfilled with \( q = p+1 \), \( r \leq p-1 \) and \( \kappa > 0 \). Moreover, let \( u \) be the maximal solution of (0.1) evolving from \( u_0 \) and suppose \( u_0 \) is chosen such that \( u \) blows up in finite time with blow-up time \( T \). Then the blow-up set \( S \) of \( u \) fulfills \( |S| > 0 \).

Proof. By Lemma 1.3.3 we know that \( u \) is radially symmetric with \( u(0, t) = \max_{x \in \mathbb{R}^n} u(x, t) \) for \( t \in (0, T) \), where \( T \) shall denote the finite blow-up time of \( u \). Thus, \( x = 0 \) is a blow-up point since \( u \geq 0 \) in \( \bar{\Omega} \times [0, T) \) by Theorem 1.1.2.

Using that \( u \) is radially symmetric, we can define \( \rho := |x| \) and \( u(\rho, t) := u(x, t) \) for \((x, t) \in \bar{\Omega} \times [0, T) \). Hence, \( u(\rho, t) \) is defined in \([0, a] \times [0, T)\) and we define \( u_\rho(\rho, t) := \frac{\partial}{\partial \rho} u(\rho, t) \) for \((\rho, t) \in [0, a] \times [0, T)\).

Furthermore, we can choose \( \beta \geq \frac{p}{2} \) such that \( \rho_1 := \frac{n}{p} \sqrt{2 \rho^2 - \rho} < a \) and \( \rho_{p\rho} \geq -\gamma \) in \([0, \rho_1]\) is fulfilled with \( \gamma := \frac{\beta a_0(0) \cdot 1}{1 + \frac{n}{p} \beta} \). Due to our choice of \( \gamma \) and \( \rho_1 \) we have \( \gamma = \frac{\beta a_0(0) \cdot 1}{1 + \frac{n}{p} \beta} \).

Moreover, we choose \( \varepsilon \in (0, \rho_1) \) and set \( \rho_\infty := \rho_1 - \varepsilon > 0 \).

Furthermore, we define

\[
J(\rho, t) := u_\rho(\rho, t) + c_\varepsilon(\rho) u(\rho, t) \quad \text{for} \quad (\rho, t) \in [0, \rho_\infty) \times [0, T),
\]

where \( c_\varepsilon(\rho) := \sqrt{\frac{2\pi}{p}} \tan \left( \sqrt{\frac{\pi}{2}} (\rho + \varepsilon) \right) \).

We will show \( J \geq 0 \) in \((0, \rho_\infty) \times (0, T)\).

Using \( c_\varepsilon'(\rho) = \frac{\pi}{2} c_\varepsilon(\rho)^2 + \beta \), \( c_\varepsilon''(\rho) = p c_\varepsilon(\rho) c_\varepsilon'(\rho) \) and \( c_\varepsilon(\rho) \geq \beta \rho \) for \( \rho \in (0, \rho_\infty) \), we have (like in (4.7) with \( \alpha = 1 \), \( q = p + 1 \) and \( c_\varepsilon \) instead of \( c + \varepsilon \))

\[
J_t = u^p J_{pp} + \left( pu^{p-1} u_\rho + \frac{n-1}{p} u^p + 2\kappa u^{q-1} u_\rho \right) J_\rho + \left[ \frac{n-1}{\rho} u^{p-1} u_\rho - \frac{n-1}{\rho^2} u^p \right. \\
+ (p + 1) u^p + \kappa u^{q-1} u_\rho \frac{n-1}{\rho} u^p + \kappa c_\varepsilon u^{q-1} u_\rho - 2 c_\varepsilon' u^p - p c_\varepsilon' u^p \\
\left. - 2\kappa c_\varepsilon u^{q-1} u_\rho - c_\varepsilon \frac{n-1}{\rho} u^p - 2 \kappa c_\varepsilon u^{q-1} u_\rho - c_\varepsilon u \left( \frac{n-1}{\rho} u^{p-1} \right) \right].
\]
4.1. REGIONAL BLOW-UP FOR $Q \leq \max\{P + 1, R + 2\}$

$$+kr u^{r-1}u_p + \kappa c u' - c p u^{r-1} - 2\kappa c u' - \kappa c u' u_t\right] J + c u^{p+1} - c u^{p+1}$$

$$-c_p n - 1 \rho u^{p+1} - c u\left[ - \frac{n - 1}{\rho^2} u^p + (p + 1)u^p + c \frac{n - 1}{\rho} u^p - 2c' u^p$$

$$-pc_p u^p - 2\kappa c u' + c \frac{n - 1}{\rho} u^p\right] + c^2 u^2 \left[ \frac{n - 1}{\rho} u^{p+1} + \kappa c u'\right]$$

$$-pc_p u^{p+1} - 2\kappa c u' - c^3 u^2 kr u^{r-1}$$

$$= u^p J_{pp} + B(p, t) J_p + C(p, t) J + c u^{p+1}\left[ - p - \frac{c_p}{c} + c'(2 + 2\kappa u^{1+r-p})\right]$$

$$-c^2(p + \kappa(r + 1) u^{1+r-p}) + n - 1 \rho u^{p+1}\left( - c' + \frac{c_p}{\rho} + pc^2\right)$$

$$= u^p J_{pp} + B(p, t) J_p + C(p, t) J + c u^{p+1}\left[ - p - \frac{pc_p}{pc} + c'(2 + 2\kappa u^{1+r-p})\right]$$

$$+2\kappa u^{1+r-p} - c^2(p + \kappa(r + 1) u^{1+r-p}) + n - 1 \rho u^{p+1}\left( - c' + \frac{c_p}{\rho} + pc^2\right)$$

$$\geq u^p J_{pp} + B(p, t) J_p + C(p, t) J$$

$$+c^2 u^{p+1}\left[ - p + 2\beta + \kappa u^{1+r-p}(p - r - 1)c^2 + 2\beta)\right]$$

$$\geq u^p J_{pp} + B(p, t) J_p + C(p, t) J$$

in $(0, \rho_\infty) \times (0, T)$

because $\beta \geq \frac{p}{2}$, $r \leq p - 1$ and $u \geq 0$, $c_e \geq 0$ in $(0, \rho_\infty) \times (0, T)$. $u^p$, $B$ and $C$ are continuous functions in $(0, \rho_\infty) \times [0, T)$ since $u \in C^\infty(\Omega \times (0, T))$ and $u_\rho \in C^0(\Omega \times (0, T))$ by Lemma 1.3.1.

Using that $u_0$ is radially decreasing, $\gamma$ is chosen suitably and $\tan(y) \geq y$ for $y \in (0, \frac{\pi}{2})$, we get

$$J(p, 0) = (u_0)_p(c) u_0(c) \geq -\gamma p + \beta(p + c) u_0(\rho_\infty)$$

$$\geq -\gamma p + \beta p \left( u_0(0) - \frac{\gamma}{2}\rho_\infty^2\right) + \beta \varepsilon u_0(\rho_\infty)$$

$$= \rho \left[ \beta u_0(0) - \gamma \left(1 + \frac{\rho_\infty^2}{2}\beta\right)\right] + \beta \varepsilon u_0(\rho_\infty) = \beta \varepsilon u_0(\rho_\infty) > 0$$

for $\rho \in (0, \rho_\infty)$ due to $\rho_\infty < \rho_1 < a$.

Fixing $T_1 \in (0, T)$, there is $\delta > 0$ such that $u \geq \delta$ in $[0, \rho_\infty] \times [0, T_1]$ by Theorem 1.1.2 due to $\rho_\infty < a$. As $u_\rho(0, t) = 0$ for $t \in [0, T)$ by Lemma 1.3.3, we deduce $J(0, t) \geq c_e(0) \delta > 0$ for $t \in [0, T_1]$.

Hence, we can show $J \geq 0$ in $[0, \rho_\infty) \times [0, T)$ like in the proof of Theorem 4.1.2. Thus, for $\rho_0 \in (0, \rho_\infty)$ we have $u_\rho \geq -c_e(\rho_0) u$ in $(0, \rho_0) \times (0, T)$ and hence $u(\rho, t) \geq e^{-c_e(\rho_0)\rho} u(0, t)$ is fulfilled in $(0, \rho_0) \times (0, T)$. 

Thus, the ball $B_{\rho_{\infty}}(0)$ is contained in the blow-up set of $u$, because $u$ is radially symmetric by Lemma 1.3.3 and $x = 0$ is a blow-up point.  

Finally we consider the case $q > p + 1$. We show that regional blow-up occurs for $r \geq q - 2$, which, in case of $q > p + 1$, is not possible in the equation (0.6) without the gradient term. First we give the results for $r > q - 2$ that are extensions of Theorem 5.5 of [Sti]. By Lemma 4.1.4 there are initial data that fulfill the conditions of the following theorem. In particular, it is ensured that $u_t \geq 0$ in $\Omega \times (0, T)$ is satisfied.

**Theorem 4.1.8** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q > p + 1$, $r > q - 2$ and $\kappa > 0$. Furthermore, let
\[
\Delta u_0 + (u_0)^{q-p} \geq 0 \quad \text{in } \bar{\Omega}.
\]
Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. Then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** For $\varepsilon \in (0, 1)$ let $u_\varepsilon$ denote the solution of (1.1) with $u_{0\varepsilon} := u_0 + \varepsilon$. Then we have $(u_0 + \varepsilon)^p \Delta (u_0 + \varepsilon) + (u_0 + \varepsilon)^q \geq (u_0 + \varepsilon)^p (\Delta u_0 + (u_0)^{q-p}) \geq 0$ in $\Omega$ due to $q > p$ and thus $(u_\varepsilon)_t \geq 0$ in $\Omega \times (0, T_\varepsilon)$ is satisfied by Lemma 1.3.4. This implies $u_\varepsilon \geq 0$ in $\Omega \times (0, T)$, because $u_\varepsilon(x, t) \to u(x, t)$ for $\varepsilon \searrow 0$ and $(x, t) \in \Omega \times [0, T)$ by Theorem 1.1.2.

Since $r > q - 2$ is fulfilled, we can choose $\alpha \in (0, 1)$ such that $r > q - 1 - \alpha$. With this choice of $\alpha$ we can prove the claim just like Theorem 4.1.2, because $\alpha + r - p > q - 1 - p > 0$ is satisfied.

Once more the condition $u_t \geq 0$ is not necessary for suitably large initial data.

**Corollary 4.1.9** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q > p + 1$, $r > q - 2$ and $\kappa > 0$ and choose $\alpha \in (0, 1)$ with $r > q - 1 - \alpha$ and $M \geq 1$ such that (4.3) and (4.4) are satisfied. Furthermore, let $v_R(x) := e^{-\frac{c_{\text{pl}} x^2}{R^2 - |x|^2}}$ for $x \in B_R := B_R(0)$ and $R > 0$ with $c_R \in (0, 1]$. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. If there are $R \in (0, a]$ and $b > M$ such that $b \geq b_R$ ($b_R > 1$ is defined like in Lemma 4.1.4) and $u_0 \geq b v_R$ in $B_R$ is fulfilled, then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** We define $z(x, t) := b v_R(x)$ for $(x, t) \in \bar{B}_R \times [0, T)$ and can show that $u \geq z$ is fulfilled in $B_R \times (0, T)$ by Lemma 4.1.4 and the comparison principle. Furthermore, with $\rho := |x|$, there is $\delta > 0$ such that $u(\rho, t) \geq z(\rho, t) \geq M$ for $(\rho, t) \in [0, \delta) \times [0, T)$ due to $b > M$.

Hence, we can choose $t_0 = 0$ and the claim can be proved like in the proof of Theorem 4.1.8 because $u_\rho \in C^0(\Omega \times [0, T))$ by Lemma 1.3.1.
Furthermore, we show that for \( q > p + 1 \) regional blow-up occurs in the borderline case \( r = q - 2 \), too. Again we can only show that balls with a radius that does not exceed a fixed bound, which is independent of the domain \( \Omega \), are contained in the blow-up set. This behavior is different to the case \( q > p + 1 \) and \( r > q - 2 \). However, here we need the condition \( u_t \geq 0 \) in \( \Omega \times (0, T) \) again.

**Theorem 4.1.10** Suppose that \( a > 0 \), \( \Omega := B_a(0) \subset \mathbb{R}^n \) and assumptions (0.2) and (4.2) are fulfilled with \( q > p + 1 \), \( r = q - 2 \) and \( \kappa > 0 \). Furthermore, let \( \Delta u_0 + (u_0)^q - p \geq 0 \) in \( \bar{\Omega} \) be satisfied. Moreover, let \( u \) be the maximal solution of (0.1) evolving from \( u_0 \) and suppose \( u_0 \) is chosen such that \( u \) blows up in finite time with blow-up time \( T \). Then the blow-up set \( S \) of \( u \) fulfills \( |S| > 0 \).

**Proof.** By Lemma 1.3.3 we know that \( u \) is radially symmetric with \( u(0, t) = \max_{x \in \Omega} u(x, t) \) for \( t \in (0, T) \), where \( T \) shall denote the finite blow-up time of \( u \). Thus, \( x = 0 \) is a blow-up point since \( u \geq 0 \) in \( \Omega \times [0, T) \) by Theorem 1.1.2.

We can show like in the proof of Theorem 4.1.8 that \( u_t \geq 0 \) in \( \Omega \times (0, T) \) is fulfilled. Next we define \( \beta_0 := \frac{q}{(2 - \frac{\pi^2}{8q})\kappa} > 0 \). Due to \( q - 1 - p > 0 \), we can choose \( M \geq 1 \) large enough, such that the following conditions are satisfied:

\[
\begin{align*}
q(-2q + p + 2) + 2\kappa u^{q-1-p} - 2p - (n-1)q &\geq 0 \quad \text{for } u \geq M, \\
(1 - q)u^{q-1-p} + \beta\left[-2q + p + 2 - \frac{\pi^2 p}{8q}\right] &\geq 0 \quad \text{for } u \geq M, \beta \geq \beta_0.
\end{align*}
\]

Since \( u \) is radially symmetric, we can define \( \rho := |x| \) and \( u(\rho, t) := u(x, t) \) for \( (x, t) \in \Omega \times [0, T) \). Hence, \( u(\rho, t) \) is defined in \( [0, a] \times [0, T) \) and we define \( u(\rho, t) := \frac{\partial}{\partial \rho} u(\rho, t) \) for \( (\rho, t) \in [0, a] \times [0, T) \).

As \( x = 0 \) is a blow-up point and \( u(0, t) = \max_{x \in \Omega} u(x, t) \) for \( t \in (0, T) \), we can choose \( t_0 \in (0, T) \) such that \( u(0, t_0) \geq 2M \). Furthermore, there is \( \delta > 0 \) so that \( u(\rho, t_0) \geq M \) for \( \rho \in [0, \delta] \) because \( u \) is continuous.

Moreover, we can choose \( \beta \geq \beta_0 \) large enough, such that \( u(\rho, t_0) \geq -\beta \) and \( u(\rho, t_0) \geq M \) for \( \rho \in [0, \rho_{\infty}] \) with \( \rho_{\infty} := \frac{\pi}{2} \sqrt{\frac{1}{q^3}} \) is fulfilled.

With the definition

\[
J(\rho, t) := u_{\rho}(\rho, t) + (c(\rho) + \varepsilon)u(\rho, t) \quad \text{for } (\rho, t) \in [0, \rho_{\infty}] \times [t_0, T),
\]

where \( c(\rho) := \sqrt{\frac{\alpha}{q}} \tan \left( \sqrt{\frac{q}{3}} \rho \right) \) and \( \varepsilon := \frac{\pi}{4} \sqrt{\frac{\beta}{q}} \), we will show \( J \geq 0 \) in \( (0, \rho_{\infty}) \times (t_0, T) \).

First we have \( u \geq M \) in \( (0, \rho_{\infty}) \times (t_0, T) \) due to the choice of \( t_0 \) and \( \rho_{\infty} \) and because \( u_t \geq 0 \) in \( \Omega \times (0, T) \). Furthermore, we get with \( \rho_0 := \frac{\rho_{\infty}}{2} \)

\[
\frac{\varepsilon}{\rho} - qc(\rho)^2 \geq \frac{\varepsilon}{\rho_0} - \beta \tan \left( \sqrt{\frac{q}{3}} \rho_0 \right) = \beta - \beta \tan \left( \frac{\pi}{4} \right) = 0 \quad \text{for } \rho \in (0, \rho_0)
\]

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and

\[
\frac{n-1}{\rho} u^{p+1} \left( \frac{\varepsilon}{\rho} - qc^2 \right) \geq - (n-1) q u^{p+1} (c + \varepsilon) \frac{c}{\rho_0} \\
\geq - (c + \varepsilon) u^{p+1} (n-1) q \left( c^2 + \frac{1}{\rho_0^2} \right) \\
= (c + \varepsilon) u^{p+1} \left( - (n-1) q c^2 - \frac{(n-1) 16 q^2}{\pi^2} \right) \in [\rho_0, \rho_\infty) \times (t_0, T). \quad (4.11)
\]

Using \( c'(\rho) = q \, c(\rho)^2 + \beta, \ c''(\rho) = 2q \, c(\rho)c'(\rho), \) \( c(\rho) \geq \beta \rho \) and \( c(\rho) + \varepsilon^2 \leq c(\rho)^2 + 2\varepsilon c(\rho) + \varepsilon^2 \leq 2(c(\rho)^2 + \varepsilon^2) \) for \( \rho \in (0, \rho_\infty) \), we get (like in \( (4.7) \) with \( r = q - 2 \) and \( \alpha = 1 \))

\[
J_t = u^p J_{\rho\rho} + \left( pu^{p-1} u_\rho + \frac{n-1}{\rho} u^{p} + 2 \kappa u^{q-2} u_\rho \right) J_\rho + \left[ \frac{n-1}{\rho} u^{p-1} u_\rho - \frac{n-1}{\rho^2} u^p \right. \\
\left. + q u^{q-1} + \kappa (q - 2) u^{q-3} u_\rho \right]^2 + (c + \varepsilon) u^{p+1} - (c + \varepsilon) u^{p-1} u_\rho - (c + \varepsilon) \frac{n-1}{\rho} u^p + \kappa (c + \varepsilon) u^{q-2} u_\rho - 2c' u^p \\
- p c' u^p - 2 \kappa c' u^{q-1} - p (c + \varepsilon) u^{p+1} - (c + \varepsilon) \frac{n-1}{\rho} u^p - 2 \kappa (c + \varepsilon) u^{q-2} u_\rho \\
+ (c + \varepsilon) \left( \frac{n-1}{\rho} u^{p-1} + \kappa (q - 2) u^{q-3} u_\rho + \kappa (c + \varepsilon) u^{q-2} - (c + \varepsilon) pu^{p-1} \right) - 2 \kappa (c + \varepsilon) u^{q-2} - \kappa (c + \varepsilon) (q - 2) u^{q-2} \right] J + (c + \varepsilon) u^q - c'' u^{p+1} \\
- c' \frac{n-1}{\rho} u^{p+1} - (c + \varepsilon) u \left[ - \frac{n-1}{\rho^2} u^p + q u^{q-1} + (c + \varepsilon) \frac{n-1}{\rho} u^p \right] \\
- 2c' u^p - pc' u^p - 2 \kappa c' u^{q-1} - (c + \varepsilon) \frac{n-1}{\rho} u^p + (c + \varepsilon)^2 u^q \left[ \frac{n-1}{\rho} u^{p-1} \right. \\
\left. + \kappa (c + \varepsilon) u^{q-2} - p (c + \varepsilon) u^{p-1} - 2 \kappa (c + \varepsilon) u^{q-2} \right] - (c + \varepsilon)^3 u^3 \kappa (q - 2) u^{q-3} \]
\[ \geq u^p J_{pp} + B(\rho, t)J_\rho + C(\rho, t)J + (c + \varepsilon)u^{p+1} \left[ (1 - q)u^{q-1-p} + \beta(-2q + p + 2) \right] + c^2 \left( q(-2q + p + 2) + 2k\kappa u^{q-1-p} \right) - 2(c^2 + \varepsilon^2)(p + \kappa(q - 1)u^{q-1-p}) \]

\[ \geq u^p J_{pp} + B(\rho, t)J_\rho + C(\rho, t)J + (c + \varepsilon)u^{p+1} \left[ (1 - q)u^{q-1-p} + \beta(-2q + p + 2) \right] + c^2 \left( q(-2q + p + 2) + 2k\kappa u^{q-1-p} \right) - 2(c^2 + \varepsilon^2)(p + \kappa(q - 1)u^{q-1-p}) \]

\[ \geq u^p J_{pp} + B(\rho, t)J_\rho + C(\rho, t)J + (c + \varepsilon)u^{p+1} \left[ (1 - q)u^{q-1-p} + \beta(-2q + p + 2) \right] + c^2 \left( q(-2q + p + 2) + 2k\kappa u^{q-1-p} \right) - 2(c^2 + \varepsilon^2)(p + \kappa(q - 1)u^{q-1-p}) \]

\[ \geq u^p J_{pp} + B(\rho, t)J_\rho + C(\rho, t)J \quad \text{in } (0, \rho) \times (t_0, T) \] (4.12)

because (4.8), (4.9), (4.10) and (4.11) are fulfilled and \( 2\varepsilon^2 = \frac{\pi^2}{8q}\beta \), where \( u^p, B \) and \( C \) are continuous functions in \( (0, \rho) \times [t_0, T) \) since \( u \in C^\infty(\Omega \times (0, T)) \) by Lemma 1.3.1.

According to \( \tan(y) \geq y \) for \( y \in (0, \frac{\pi}{2}) \) and \( M \geq 1 \), we deduce

\[ J(\rho, t_0) = u_\rho(\rho, t_0) + (c(\rho) + \varepsilon)u(\rho, t_0) \geq -\beta\rho + (\beta \rho + \varepsilon)M \geq \varepsilon > 0 \]

for \( \rho \in (0, \rho) \).

As \( u_\rho(0, t) = 0 \) for \( t \in (0, T) \) by Lemma 1.3.3, we get \( J(0, t) \geq \varepsilon M \geq \varepsilon > 0 \) for \( t \in [t_0, T) \).

Furthermore, we fix \( T_1 \in (t_0, T) \). Since \( u_\rho \in C^0([0, \rho] \times [t_0, T_1]) \), there exists \( d > 0 \) with \( |u_\rho| \leq d \) in \( [0, \rho] \times [t_0, T_1] \). Now we choose \( \rho_0 \in (0, \rho) \) such that \( M\kappa(\rho_0) \geq d \). Then we have \( J(x, t) \geq -d + (c(\rho_0) + \varepsilon)M \geq \varepsilon M \geq \varepsilon > 0 \) for \( \rho \in [0, \rho_0] \times [t_0, T_1] \).

Hence, we conclude \( J \geq \varepsilon > 0 \) on the parabolic boundary of \( [0, \rho] \times [t_0, T_1] \) for all \( \rho \in [\rho_0, \rho) \). This implies \( J \geq 0 \) in \( [0, \rho] \times [t_0, T_1] \) for all \( \rho \in [\rho_0, \rho) \) by (4.12) and Lemma 1.1.1 and thus \( J \geq 0 \) in \( (0, \rho) \times [t_0, T_1] \). Therefore, we deduce that \( J \geq 0 \) holds in \( (0, \rho) \times (t_0, T) \) because \( T_1 \in (t_0, T) \) was arbitrary.

Let \( \rho_0 \in (0, \rho) \). Then we have \( u_\rho \geq -(c(\rho_0) + \varepsilon)u \) in \( (0, \rho_0) \times (t_0, T) \). Since \( x = 0 \) is a blow-up point and \( u(0, t) = \max_{x \in \Omega} u(x, t) \) for \( t \in (0, T) \), we can choose a sequence \( (t_k)_{k \in \mathbb{N}} \subset (t_0, T) \) such that \( u(0, t_k) \to \infty \) for \( k \to \infty \). Hence, we conclude

\[ u(\rho, t_k) \geq e^{-(c(\rho_0) + \varepsilon)\rho}u(0, t_k) \geq e^{-(c(\rho_0) + \varepsilon)\rho_0}u(0, t_k) \]
for $\rho \in (0, \rho_0)$ and $k \in \mathbb{N}$. Thus, the interval $(0, \rho_0)$ is contained in the blow-up set since $u(0, t_k) \to \infty$ for $k \to \infty$. Hence, by the symmetry of $u$ (Lemma 1.3.3), the ball $B_{\rho_\infty}(0)$ is contained in the blow-up set and the claim follows.

Once more the last result is valid without the condition $u_t \geq 0$, if the initial data are large.

**Corollary 4.1.11** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q > p + 1$, $r = q - 2$ and $\kappa > 0$ and choose $M \geq 1$ such that (4.8) and (4.9) are satisfied. Furthermore, let $v_R(x) := e^{-\frac{c_R|x|^2}{R^2-|x|^2}}$ for $x \in B_R := B_R(0)$ and $R > 0$ with $c_R \in (0,1]$. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. If there are $R \in (0, a]$ and $b > M$ such that $b \geq b_R$ (where $b_R > 1$ is defined like in Lemma 4.1.4) and $u_0 \geq bv_R$ in $B_R$ is fulfilled, then the blow-up set $S$ of $u$ fulfills $|S| > 0$.

**Proof.** We define $z(x, t) := bv_R(x)$ for $(x, t) \in \bar{B}_R \times [0, T)$ and can show that $u \geq z$ is satisfied in $B_R \times (0, T)$ by Lemma 4.1.4 and the comparison principle. Furthermore, with $\rho := |x|$, there is $\delta > 0$ such that $u(\rho, t) \geq z(\rho, t) \geq M$ for $(\rho, t) \in [0, \delta) \times [0, T)$ due to $b > M$.

Hence, we can choose $t_0 = 0$ and the claim can be proved like in the proof of Theorem 4.1.10 because $u_\rho \in C^0(\Omega \times [0, T))$ by Lemma 1.3.1.

### 4.2 Single point blow-up for $q > \max\{p + 1, r + 2\}$

In this section we prove that single point blow-up occurs in case of $q > \max\{p + 1, r + 2\}$ and $\kappa > 0$. This result is an extension of Theorem 5.3 of [Sti]. In the proof we adapt ideas which are similar to those of the previous section and which were presented in [FriMcL1]. But in contrast to the results for regional blow-up we are not able to eliminate the condition $u_t \geq 0$ in $\Omega \times (0, T)$ with the help of large initial data in the present case.

It is ensured by Lemma 4.1.4 that there are initial data that fulfill the requirements of the following theorem.

**Theorem 4.2.1** Suppose that $a > 0$, $\Omega := B_a(0) \subset \mathbb{R}^n$ and assumptions (0.2) and (4.2) are fulfilled with $q > p + 1$, $r < q - 2$ and $\kappa > 0$. Furthermore, let $\Delta u_0 + (u_0)^\eta - p \geq 0$ in $\bar{\Omega}$ be satisfied. Moreover, let $u$ be the maximal solution of (0.1) evolving from $u_0$ and suppose $u_0$ is chosen such that $u$ blows up in finite time with blow-up time $T$. Then $x = 0$ is the only blow-up point of $u$. 

4.2. SINGLE POINT BLOW-UP FOR $Q > \max\{P + 1, R + 2\}$

PROOF. By Lemma 1.3.3 we know that $u$ is radially symmetric with $u(0, t) = \max_{x \in \Omega} u(x, t)$ for $t \in (0, T)$, where $T$ shall denote the finite blow-up time of $u$. Thus, $x = 0$ is a blow-up point since $u \geq 0$ in $\Omega \times [0, T)$ by Theorem 1.1.2.

We can show like in the proof of Theorem 4.1.8 that $u_t \geq 0$ in $\Omega \times (0, T)$ is fulfilled. Using that $u$ is radially symmetric, we can define $\rho := |x|$ and $u(\rho, t) := u(x, t)$ for $(x, t) \in \bar{\Omega} \times [0, T)$. Hence, $u(\rho, t)$ is defined in $[0, a] \times [0, T)$ and we define $u(\rho, t) := \frac{\partial}{\partial \rho} u(\rho, t)$ for $(\rho, t) \in [0, a] \times [0, T)$.

Throughout the rest of this proof, we shall identify ‘$B_\alpha'0' with ‘$[0, a]$ etc. and switch between ‘$x$-notation’ and ‘$p$-notation’ whenever this is convenient and they cannot be confused for one another.

For $\tau \in (0, T)$ and $\mu \in (0, a)$ let $Q := (\mu, a - \mu) \times (\tau, T)$. Then $w := u_\rho \in C^\infty(Q)$ fulfills in $Q$ the following linear strict parabolic equation with smooth coefficients by Lemma 1.3.1:

$$w_t = u^p w_{\rho \rho} + \left(\nu u^{p-1} u_\rho + \frac{n-1}{\rho} u^p + 2\kappa u^{\rho} u_\rho\right) w_\rho + \left(\frac{n-1}{\rho^2} u^{p-1} u_\rho - \frac{n-1}{\rho^2} u^p + q u^{\rho-1} + \kappa u^{\rho-1}(u_\rho)^2\right) w$$

Since $w \leq 0$ on the parabolic boundary of $Q$ by Lemma 1.3.3, we get $w < 0$ in $Q$ by the strong maximum principle. Hence, $u_\rho < 0$ is satisfied in $(0, a) \times (0, T)$ because $\tau \in (0, T)$ and $\mu \in (0, a)$ are arbitrary.

Now we assume that there exists $x \in B_\alpha'(0) \setminus \{0\}$ which is a blow-up point. Thus, there exist $(x_n)_{n \in \mathbb{N}} \subset \Omega$ and $(t_n)_{n \in \mathbb{N}} \subset (0, T)$ with $u(x_n, t_n) \to \infty$, $x_n \to x$ and $t_n \to T$ for $n \to \infty$. With $\rho_n := |x_n|$ and $\rho := |x|$ we have $u(\rho_n, t_n) \to \infty$ and $\rho_n \to \rho \in (0, a)$. Hence, there are $y \in (0, a)$ and $n_0 \in \mathbb{N}$ with $y \leq \rho_n$ for $n \geq n_0$. Thus, we have $u(y, t_n) \to \infty$ for $n \to \infty$, since $u(y, t_n) \geq u(\rho_n, t_n)$ for $n \geq n_0$ by Lemma 1.3.3.

Next we fix $\delta \in (0, y)$ and $\alpha > 1$ such that $\alpha < \min\{q - p, q - 1 - r\}$. Moreover, we set $\Theta := \Theta(\cdot; G)$ and $\lambda := \lambda_1(G) > 0$, where $G := B_\delta(0) \setminus B_\delta(0)$. Hence, $\Theta \in C^2(\bar{G})$ is radially symmetric such that $C_0 := \max_{x \in \partial G} |\nabla \Theta(x)| < \infty$. Furthermore, we can choose $M \geq 1$ such that

$$(\alpha - q)u^{\alpha-1-p} + C_0((2\alpha + p)u^{\alpha-1} + 2\kappa u^{\alpha+r-p})$$

$$+ \frac{n-1}{\delta^2} + p \frac{n-1}{\delta} u^{\alpha-1} + \lambda \leq 0 \quad \text{for} \quad u \geq M, \quad (4.13)$$

which is possible because $\alpha$ was chosen suitably.

Moreover, we fix $t_0 \in (0, T)$ with $u(y, t) \geq M \forall t \in [t_0, T)$, which is possible because of our choice of $y$ and since $u_t \geq 0$ in $\Omega \times (0, T)$. Then we have $u \geq M$ and $u_\rho < 0$ in $[\delta, y] \times [t_0, T)$ because $u_\rho < 0$ in $(0, a) \times (0, T)$.

Choosing $c_\varepsilon(\rho) := \varepsilon \Theta(\rho)$ for $\rho \in [\delta, y]$, we define

$$J(\rho, t) := u_\rho(\rho, t) + c_\varepsilon(\rho) u^\alpha(\rho, t) \quad \text{for} \quad (\rho, t) \in [\delta, y] \times [t_0, T),$$

$$\frac{\partial J}{\partial \rho} \leq 0$$

whenever $\rho \geq M$. Hence, $u_\rho < 0$ in $[\delta, y] \times [t_0, T)$ since $J(\rho, t) \to \infty$ for $\rho \to \infty$ in $[\delta, y] \times [t_0, T)$.
Thus, we get (like in (4.7) with $c_\varepsilon$ instead of $(c + \varepsilon)$, where we use $\alpha > 1$ and $\varepsilon \in (0, 1)$)

$$J_t = u^p J_{pp} + \left( p u^{p-1} u_p + \frac{n-1}{\rho} u^p + 2\kappa u^r u_p \right) J_p + \left( \frac{n-1}{\rho} u^{p-1} u_p - \frac{n-1}{\rho^2} u^p \right) + q u^{q-1} + \kappa r u^{r-1} (u_p)^2 + c_\varepsilon \alpha \left( \frac{n-1}{\rho} u^{p+1-1} + \kappa c_\varepsilon \alpha u^{a+1+r} u_p - 2c'_\varepsilon \alpha u^{a+1} \right)$$

$$- c_\varepsilon \alpha (\alpha - 1) u^{a-2+p} \rho u_p - pc'_\varepsilon u^{a+p-1} - 2\kappa c'_\varepsilon u^{a+r} - pc_\varepsilon \alpha u^{a-2+p} +pc_\varepsilon \alpha u^{a-2+p}$$

$$- c_\varepsilon \alpha \left( \frac{n-1}{\rho} u^{p+1-1} - 2\kappa c_\varepsilon \alpha u^{a+1+r} u_p - c_\varepsilon \alpha \left( \frac{n-1}{\rho} u^{p-1} \right) \right) + \kappa ru^{r-1} u_p + \kappa c_\varepsilon \alpha u^{a+1+r} - c_\varepsilon \alpha (\alpha - 1) u^{a-2+p} - \kappa c_\varepsilon \alpha u_{\alpha+1} - c_\varepsilon \alpha u^{a+1+r} - c_\varepsilon \alpha u^{a+1}$$

$$- c_\varepsilon \alpha u^{a-2+p} - pc'_\varepsilon u^{a+1+p} - 2\kappa c'_\varepsilon u^{a+r} - c_\varepsilon \alpha \frac{n-1}{\rho} u^{p+1-1}$$

$$+ c^2 \alpha \left( \frac{n-1}{\rho} u^{p-1} + \kappa c_\varepsilon \alpha u^{a+1+r} - c_\varepsilon \alpha (\alpha - 1) u^{a-2+p} \right)$$

$$- c_\varepsilon \alpha u^{a-2+p} - 2\kappa c_\varepsilon \alpha u^{a+1+r} - c_\varepsilon \alpha u^{a+1}$$

by (4.13) due to the choice of $M$ and $t_0$, where $u^\rho$, $B$ and $C$ are continuous functions in $[\delta, y] \times [t_0, T]$ since $u \in C^\infty(\Omega \times (0, T))$ by Lemma 1.3.1.

According to $c_\varepsilon(\delta) = c_\varepsilon(y) = 0$, we conclude $J \leq 0$ on the parabolic boundary of
(δ, y) × [t₀, T) due to the choice of ε. Thus by (4.14) and Lemma 1.1.1 we deduce $J \leq 0$ in $[\delta, y] \times [t₀, T)$, which is equivalent to $-u_\rho \geq c_\varepsilon u^\alpha$ in $[\delta, y] \times [t₀, T)$. By integration, we get for $t ∈ [t₀, T)$ with $G(s) := \frac{1}{1-\alpha}s^{1-\alpha}$ for $s \geq 0$

$$G(u(y, t)) - G(u(\delta, t)) = \frac{-1}{1-\alpha}((u(y, t))^{1-\alpha} - (u(\delta, t))^{1-\alpha}) = \int_\delta^y -\frac{u_\rho(s, t)}{u^\alpha(s, t)} ds$$

$$\geq \int_\delta^y c_\varepsilon(s) ds > 0,$$

which is a contradiction to our assumption since $G(u(y, t)) \to 0$ as $t \uparrow T$, because $u(y, t) \to \infty$ as $t \uparrow T$, $\alpha > 1$ and $G(u(\delta, t)) \geq 0$ for $t ∈ (t₀, T)$. The claim now follows since $x ∈ B_\alpha(0) \setminus \{0\}$ was arbitrary. ■
Bibliography


BIBLIOGRAPHY


Lebenslauf

Am 28. April 1980 wurde ich als Sohn der Eheleute Bernhard Stinner und Dr. Doris Stinner in Aachen geboren.


