# Embedding Theorems for Decomposition Spaces with Applications to Wavelet Coorbit Spaces

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

vorgelegt von

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aus Aachen, Deutschland

Berichter:
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## ABSTRACT

The main topic of this thesis is the development of criteria for the (non)-existence of *embeddings between decomposition spaces*.

A decomposition space is defined in terms of

- a covering $\mathcal{Q} = (Q_i)_{i \in I}$ of (a subset) of the frequency space $\mathbb{R}^d$,
- an integrability exponent $p$ and
- a certain discrete sequence space $Y$ on the index set $I$.

The decomposition space norm of a distribution $f$ is then computed by decomposing *the frequency content* of $f$ according to the covering $\mathcal{Q}$, using a suitable partition of unity. Each of the localized pieces is measured in the Lebesgue space $L^p$ and the contributions of the individual pieces are aggregated using the discrete sequence space norm $\|\cdot\|_Y$.

Given two decomposition spaces, it is of interest to know whether there is an embedding between these two spaces. Since both decomposition spaces are defined only in terms of the respective coverings, weights and discrete sequence spaces, it should be possible to decide the existence of the embedding only based on these quantities.

Our findings will show that this is not only possible, but that the resulting criteria only involve discrete combinatorial considerations. In particular, no knowledge of Fourier analysis is needed for *the application* of these criteria. Finally, our results *completely characterize* the existence of the desired embedding under mild assumptions on the two coverings and sequence spaces.

We apply our findings to a large number of concrete examples. Among others, we consider embeddings between

- $\alpha$-modulation spaces,
- homogeneous and inhomogeneous Besov spaces and
- shearlet-type coorbit spaces.

In all cases, the known results for embeddings between these spaces turn out to be special cases of our criteria; in some cases, our new approach even yields stronger results than those previously known.

For the discussion of shearlet-type coorbit spaces, we employ the second main result of this thesis which shows that the Fourier transform induces a natural isomorphism between a large class of wavelet coorbit spaces and certain decomposition spaces. This further emphasizes the scope of our embedding results for decomposition spaces.

# ZUSAMMENFASSUNG

Das Hauptthema dieser Arbeit ist die Entwicklung von Kriterien für die (nicht)-Existenz von *Einbettungen zwischen Dekompositionsräumen*.

Ein Dekompositionsraum ist hierbei definiert über

- eine Überdeckung $\mathcal{Q} = (Q_i)_{i \in I}$ des Frequenzraumes $\mathbb{R}^d$,
- einen Integrabilitätsexponenten $p$ und
- einen diskreten Folgenraum $Y$ auf der Indexmenge $I$.

Zur Berechnung der Dekompositionsraum-Norm einer Distribution $f$ zerlegt man $f$ *auf der Fourierseite* gemäß der Überdeckung $\mathcal{Q}$ (mittels einer zugehörigen Zerlegung der Eins). Die einzelnen Teile werden in $L^p$ gemessen und die Gesamt-Norm ergibt sich durch Zusammenfassen aller einzelnen Normen mittels des Folgenraumes $Y$.

Falls zwei verschiedene Dekompositionsräume gegeben sind, stellt sich die Frage, ob eine Einbettung zwischen diesen existiert. Da beide Räume nur über die zugehörigen Überdeckungen, Gewichte und Integrabilitätsexponenten definiert sind, sollte es möglich sein, die Existenz der Einbettung nur anhand dieser Größen zu entscheiden.

Wir werden nicht nur sehen, dass dies tatsächlich möglich ist, sondern auch, dass *für die Anwendung* der sich ergebenden Kriterien nur diskrete, kombinatorische Überlegungen nötig sind; insbesondere benötigt man keinerlei Wissen über Fourieranalysis. Weiterhin bemerken wir, dass unsere Resultate – unter milden Annahmen an die Überdeckungen und Folgenräume – eine *äquivalente* Bedingung für die Existenz der jeweiligen Einbettung geben.

Als Anwendung diskutieren wir unter anderem die Existenz von Einbettungen zwischen

- $\alpha$-Modulations Räumen,
- homogenen und inhomogenen Besovräumen und
- Coorbit-Räumen vom Shearlet-Typ.

Wir werden sehen, dass in jedem dieser Fälle die existierenden Resultate Spezialfälle unserer neuen Kriterien sind. In vielen Fällen liefern die neuen Kriterien sogar stärkere Aussagen als die bisher bekannten.

Die Behandlung der Einbettungen für Coorbit-Räume vom Shearlet-Typ wird durch das zweite Hauptergebnis der Arbeit möglich. Konkret werden wir sehen, dass die Fouriertransformation einen Isomorphismus zwischen einer großen Klasse von Wavelet Coorbit-Räumen und gewissen Dekompositionsräumen liefert. Damit sind unsere Kriterien für Einbettungen zwischen Dekompositionsräumen auch auf Wavelet Coorbit-Räume anwendbar.

*I have made this letter longer than usual,*
*because I lack the time to make it short.*

— Blaise Pascal

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# CONTENTS


1 INTRODUCTION 1
- 1.1 Coorbit theory (for Quasi-Banach spaces) 2
  - 1.1.1 Examples 2
  - 1.1.2 Features of coorbit theory 4
  - 1.1.3 Coorbit theory for Quasi-Banach spaces 5
  - 1.1.4 New contributions 6
- 1.2 (Fourier-side) decomposition spaces 7
  - 1.2.1 General decomposition spaces 7
  - 1.2.2 Fourier-side decomposition spaces 8
  - 1.2.3 Related work 9
  - 1.2.4 New contributions 10
- 1.3 Wavelet coorbit spaces as decomposition spaces 10
  - 1.3.1 Wavelet coorbit spaces 10
  - 1.3.2 Isomorphism between wavelet coorbit spaces and certain decomposition spaces 11
  - 1.3.3 Applications of the decomposition space view of coorbit spaces 12
  - 1.3.4 New contributions 13
- 1.4 Embeddings between decomposition spaces 13
  - 1.4.1 New contributions 16
- 1.5 Applications 16

2 COORBIT THEORY FOR QUASI-BANACH SPACES 21
- 2.1 A crash course on Quasi-Banach spaces 22
  - 2.1.1 Properties of Quasi-Banach spaces 22
  - 2.1.2 Pathological examples of Quasi-Banach spaces * 32
- 2.2 Solid Quasi-Banach spaces of functions 34
  - 2.2.1 Definition and basic properties 35
  - 2.2.2 Pointwise convergence and convergence in solid function spaces 38
  - 2.2.3 Completeness of function spaces and the Riesz-Fischer property 43
  - 2.2.4 Weighted spaces and translation invariance 48
  - 2.2.5 Completions of solid function spaces and the Fatou property * 55
- 2.3 Wiener amalgam spaces for Quasi-Banach spaces 59
  - 2.3.1 Definition and elementary properties of Wiener amalgam spaces 60
  - 2.3.2 A discrete characterization of Wiener amalgam spaces 68
  - 2.3.3 Convolution relations for Wiener amalgam spaces 84
  - 2.3.4 Further properties of Wiener amalgam spaces 106

2.4 Coorbit theory for Quasi-Banach spaces 110
2.4.1 Assumptions for coorbit theory 111
2.4.2 The reproducing formula for square integrable representations 113
2.4.3 Construction of the reservoir $\mathcal{R} = \left(\mathcal{H}_v^1\right)^\urcorner$ 119
2.4.4 Definition of coorbit spaces with respect to Quasi-Banach spaces 130
2.4.5 Atomic decomposition results 138
3 (FOURIER SIDE) DECOMPOSITION SPACES 159
3.1 Convolution relations for $L^p(\mathbb{R}^d)$, $0 < p < 1$ 160
3.2 Different types of coverings and their properties 168
3.3 Relations between coverings 186
3.4 (Fourier side) decomposition spaces 194
3.5 Dilations of decomposition spaces 210
4 WAVELET COORBIT SPACES AS DECOMPOSITION SPACES 217
4.1 Admissible dilation groups and the dual action 219
4.2 Applicability of coorbit theory for the spaces $L_m^{p,q}$ 225
4.3 Definition of $\mathcal{F}_\mathcal{R} : \mathcal{R}_v \to \mathcal{D}'(\mathcal{O})$ 246
4.4 The induced covering and a special BAPU 258
4.4.1 The induced covering $\mathcal{Q}$ 258
4.4.2 A special BAPU 264
4.5 Continuity of $\mathcal{F}_\mathcal{R} : \mathrm{Co}(L_m^{p,q}) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$ 271
4.6 Continuity of $\mathcal{F}_\mathcal{R}^{-1} : \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q) \to \mathrm{Co}(L_m^{p,q})$ 284
4.7 Dilation invariance of coorbit spaces 298
5 EMBEDDINGS BETWEEN DECOMPOSITION SPACES 309
5.1 Sufficient conditions 310
5.2 Simplification of the sufficient conditions 347
5.3 Necessary conditions 376
5.3.1 The general case 376
5.3.2 The relatively moderate case 404
5.4 Summary 421
5.5 Decomposition spaces as spaces of tempered distributions 439
6 APPLICATIONS 445
6.1 Embeddings between $\alpha$-modulation spaces 446
6.2 Embeddings for Besov spaces 458
6.3 Embeddings for shearlet type coorbit spaces 487
6.3.1 Shearlet type coorbit spaces as decomposition spaces 488
6.3.2 Embeddings between shearlet type coorbit spaces 496
6.3.3 Shearlet type coorbit spaces and Besov spaces 510
6.4 Embeddings for shearlet smoothness spaces 535
6.5 Dilations of decomposition spaces and coorbit spaces 559
6.5.1 Isotropic dilations of shearlet-type decomposition spaces 560
6.5.2 Universal dilations for shearlet-type coorbit spaces 569

6.5.3 Isotropic dilations of $\alpha$-modulation spaces 582
7 CONCLUSION AND OUTLOOK 611

BIBLIOGRAPHY 617

Here, we fix commonly used notations. More specific notations are introduced in the text and are not listed here.

- The letters $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ have their usual meanings. We define the natural numbers as $\mathbb{N} := \{n \in \mathbb{Z} \mid n \geq 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For convenience, we also define $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $\mathbb{R}_+^* := (0, \infty)$ and $\mathbb{R}_+ := [0, \infty)$.
- We write $A \subset B$ if $A$ is a subset of $B$, where $A = B$ is also allowed. For a strict inclusion, we use the notation $A \subsetneq B$.
- If $A, B$ are sets, then $B^A = \{f : A \to B\}$ is the set of all functions from $A$ to $B$.
- The notation $M \Subset U$ means that $U \subset \mathbb{R}^d$ is open and that $M \subset U$ is relatively compact, with $\overline{M} \subset U$, where $\overline{M}$ denotes the closure of $M$.
- For a finite set $M$, we write $|M| \in \mathbb{N}_0$ for the number of elements of $M$. If $M$ is an infinite set, we write $|M| = \infty$.
- If $X$ is a vector space, we write $Y \leq X$ if $Y \subset X$ is a subspace of $X$.
- If $X$ is a vector space and $M \subset X$, then $\langle M \rangle$ denotes the span of $M$.
- The open/closed Euclidean balls around $x \in \mathbb{R}^d$ of radius $r \geq 0$ are denoted by $B_r(x)$ and $\overline{B_r}(x)$, respectively.
- We denote the $d$-dimensional Lebesgue measure of a (measurable) set $M \subset \mathbb{R}^d$ by $\lambda_d(M)$ or simply by $\lambda(M)$. Furthermore, we define
$$v_d := \lambda_d(B_1(0))$$
as the measure of the Euclidean unit ball.
- For $n \in \mathbb{N}_0$, we set $\underline{n} := \{1, \ldots, n\}$. In particular, $\underline{0} = \varnothing$.
- For any set $X$ and any element $x \in X$, we define the Dirac delta $\delta_x : X \to \{0, 1\}, y \mapsto \delta_{x,y}$ with $\delta_{x,y} = 1$ for $x = y$ and $\delta_{x,y} = 0$ for $x \neq y$.
- For $x \in \mathbb{R}^d$, we write $|x| := \|x\|_2$ for the usual Euclidean norm of $x$. We will also use the "chinese bracket"
$$\langle x \rangle := \sqrt{1 + |x|^2}.$$

- If $f : X \to \mathbb{C}$ is a function, we denote its supremum norm by

$$\|f\|_{\sup} := \sup_{x \in X} |f(x)| \in [0, \infty].$$

Further, we set $\|f\|_{\sup,M} := \|f|_M\|_{\sup}$ for each subset $M \subset X$, where $f|_M$ denotes the restriction of $f$ to $M$.

- For a set $X$ and any subset $A \subset X$, we define the indicator function $\chi_A$ of $A$ by

$$\chi_A : X \to \{0,1\}, x \mapsto \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

- If $G$ is a group and $f : G \to X$ is a function, where $X \neq \varnothing$ is any (nonempty) set, we define the left- and right translations of $f$ by

$$\begin{aligned} L_x f &: G \to X, y \mapsto f(x^{-1}y), \\ R_x f &: G \to X, y \mapsto f(yx). \end{aligned}$$

Furthermore, we let

$$f^\vee : G \to X, x \mapsto f(x^{-1}).$$

For $X = \mathbb{C}$, we also define

$$f^\triangledown : G \to \mathbb{C}, x \mapsto \overline{f(x^{-1})}.$$

- For $d \in \mathbb{N}$ and $f : \mathbb{R}^d \to \mathbb{C}$, we define the modulation of $f$ by $\xi \in \mathbb{R}^d$ as

$$M_\xi f := e_\xi \cdot f,$$

with

$$e_\xi : \mathbb{R}^d \to \mathbb{C}, x \mapsto e^{2\pi i \langle x, \xi \rangle}.$$

- For $d \in \mathbb{N}$, $f : \mathbb{R}^d \to \mathbb{C}$ and $T \in \mathrm{GL}(\mathbb{R}^d)$, we define

$$\begin{aligned} D_T f &: \mathbb{R}^d \to \mathbb{C}, x \mapsto |\det(T)|^{-\frac{1}{2}} \cdot f(T^{-1}x), \\ \Delta_T f &: \mathbb{R}^d \to \mathbb{C}, x \mapsto f(T^T x), \end{aligned}$$

where $T^T$ denotes the transpose of $T$.

- A dual pairing of the form $\langle \cdot, \cdot \rangle$ is always supposed to be bilinear, whereas a dual pairing of the form $\langle \cdot \mid \cdot \rangle$ is linear in the first and antilinear in the second component.

- We use the following version of the Fourier transform: We set

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \cdot e^{-2\pi i \langle x,\xi \rangle} \, \mathrm{d}x$$

for every $f \in L^1(\mathbb{R}^d)$. The inverse Fourier transform is likewise given by

$$\mathcal{F}^{-1}f(x) = \widehat{f}(-x) = \int_{\mathbb{R}^d} f(\xi) \cdot e^{2\pi i \langle x,\xi \rangle} \, \mathrm{d}\xi.$$

We note that $\mathcal{F}$ extends to a unitary automorphism of $L^2(\mathbb{R}^d)$.

- If $X$ is a topological space, then $C_c(X)$ denotes the space of all continuous functions $f : X \to \mathbb{C}$ with compact support.

- We set

$$C_0(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \;\middle|\; \lim_{|x| \to \infty} |f(x)| = 0 \right\}.$$

- We write $A \lesssim B$ if there is a constant $C$ with $A \leq C \cdot B$. Usually, the (implied) constant $C$ is only allowed to depend on certain quantities which are either implicitly clear or mentioned explicitly. In many cases, we will write something like

$$A \lesssim_{r,s} B,$$

to indicate that the implied constant is only allowed to depend on $r$ and $s$.

- Finally, we write $A \asymp B$ if we have $A \lesssim B$ and $B \lesssim A$. The same restrictions as above apply.

## ACRONYMS

LCH GROUP locally compact Hausdorff group

BAPU bounded admissible partition of unity

BUPU bounded uniform partition of unity

# 1 INTRODUCTION

*The author discusses valueless measures in pointless spaces.*

— Paul R. Halmos [48]

The purpose of this thesis is the study of two different classes of (function) spaces, the COORBIT SPACES and the DECOMPOSITION SPACES. We will discover deep and novel connections between these two classes of spaces, but also between pairs of spaces belonging to one of the two classes:

Connecting these two classes of spaces is the Fourier transform. More precisely, if $\mathrm{Co}\left(L^{p,q}_m\right)$ is a WAVELET COORBIT SPACE (see below for definitions), then one can naturally extend the usual Fourier transform $\mathcal{F} : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ to an *isomorphism*

$$\mathcal{F} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$$

between this wavelet coorbit space and a certain decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ (also defined below).

The connection between different spaces of the *same* class is given by EMBEDDINGS. In view of the isomorphism between wavelet coorbit spaces and decomposition spaces obtained above, formulating these embeddings for decomposition spaces is sufficient. We will provide readily verifiable criteria for the existence of an embedding

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right)$$

between two different decomposition spaces. Our criteria will be based on the relation between the geometries of the two coverings. Under restrictive, but commonly satisfied assumptions on the two coverings $\mathcal{Q}, \mathcal{P}$, our criteria *completely characterize* the existence of such an embedding.

We now describe the spaces under consideration and our results in more detail. We remark that the structure of this introduction parallels the structure of the entire thesis, i.e., the different topics are considered in the same order and the sections in the introduction correspond to the different chapters of the thesis.

The first part of the thesis is concerned with the two types of spaces mentioned above. We start with motivation for and with an introduction to the general theory of COORBIT SPACES. Subsequently, we introduce the (FOURIER-SIDE) DECOMPOSITION SPACES which we will consider in the remainder of the thesis.

In the second part, we develop the connection between these two classes of spaces. We first introduce the special class of WAVELET COORBIT SPACES. We then go on to show that the Fourier transform induces an isomorphism between these spaces and certain Fourier-side decomposition spaces.

In the final part, we study embeddings between different decomposition spaces. We then apply these abstract results to prove (or disprove) the existence of embeddings for a large class of concrete examples. In particular, we consider embeddings between

- ($\alpha$)-modulation spaces,
- Besov spaces,
- Coorbit spaces of shearlet-type groups,
- Shearlet smoothness spaces as defined in [53].

We now decsribe these different topics in greater detail.

## 1.1 COORBIT THEORY (FOR QUASI-BANACH SPACES)

Many function spaces on $\mathbb{R}^d$ can be constructed using the following procedure: One fixes a certain family of functions $\Gamma = (\gamma_i)_{i \in I}$ on $\mathbb{R}^d$ and correlates each suitable function or (tempered) distribution $f$ with all of the $\gamma_i$ to obtain a *transformed version* of $f$, i.e.,

$$W_\Gamma f : I \to \mathbb{C}, i \mapsto \langle f \,|\, \gamma_i \rangle .$$

By imposing suitable conditions on these coefficients, one obtains the actual function space, for example by requiring $W_\Gamma f \in L^p(\mu)$ for a certain measure $\mu$ on the set $I$. The resulting (coorbit) space is thus

$$\mathrm{Co}(\Gamma, Y) := \{ f \in \mathcal{R} \,|\, W_\Gamma f \in Y \},$$

where $\mathcal{R}$ is a suitable RESERVOIR of functions or distributions, for example $\mathcal{R} = \mathcal{S}'(\mathbb{R}^d)$, and $Y$ is a certain space of functions on $I$, for example $Y = L^p(\mu)$.

In many cases, the index set $I$ is a locally compact Hausdorff group (LCH group) $I = G$ and the family $\Gamma$ is given by $\gamma_x = \pi(x)\, g$ for $x \in G$ and some fixed function $g$ (called the ANALYZING WINDOW) and a suitable representation $\pi$ of $G$. The measure $\mu$ is then usually taken to be the Haar measure on $G$. The spaces obtained from such a family can be considered as coorbit spaces *in the strict sense*. In this thesis, we will only consider coorbit spaces that arise in this way.

### 1.1.1 *Examples*

To make this abstract approach more concrete, we mention the following two well-known examples:

A. In (one-dimensional) WAVELET ANALYSIS, one considers families of the form $\Gamma = (L_x D_a g)_{x\in\mathbb{R}, a\in\mathbb{R}^*}$, where the dilation operator $D_a$ is defined by

$$D_a g(x) = |a|^{-1/2} \cdot g(x/a).$$

The intuition here is that the different values of the dilation parameter $a$ lead to an analysis of the signal/function $f$ at different *scales*.

In the background of this construction is the (non-connected) "$ax+b$"-group $G = \mathbb{R}\times\mathbb{R}^*$ with multiplication given by

$$(x,a)\cdot(y,b) = (x+ay, ab).$$

The family $\Gamma$ results from a fixed window $g$ by applying all values of the QUASI-REGULAR REPRESENTATION

$$G \to \mathcal{U}\left(L^2(\mathbb{R})\right), (x,a) \mapsto L_x D_a.$$

One can show that, for suitable analyzing windows $g$, the resulting family of coorbit spaces $\mathrm{Co}\left(\Gamma, L^p_{w_s}(G)\right)$, with weight given by $w_s(x,a) = |a|^{-s}$, yields precisely the class of HOMOGENEOUS BESOV SPACES $\dot{B}^p_{r,p}$, see [28, Section 7.2].

Since the homogeneous Besov spaces are special decomposition spaces, the above characterization of the spaces $\mathrm{Co}\left(\Gamma, L^p_{w_s}(G)\right)$ as homogeneous Besov spaces is also a special case of – as well as motivation for – the isomorphism between coorbit spaces and decomposition spaces which is obtained later in this thesis. See Chapters 4 and 6 for more details.

B. In the context of TIME FREQUENCY ANALYSIS, the class of MODULATION SPACES plays an important role. Here, the family $\Gamma = (L_x M_\omega g)_{x,\omega\in\mathbb{R}^d}$ consists of all time-frequency shifts of a fixed analyzing window $g$. One also says that $\Gamma$ is a GABOR FAMILY.

More group theoretically, the modulation spaces can be interpreted as coorbit spaces of the SCHRÖDINGER REPRESENTATION

$$\pi : \mathbb{H}_r \to \mathcal{U}\left(L^2(\mathbb{R}^d)\right), (x,\omega,e^{2\pi i\tau}) \mapsto e^{2\pi i\tau}\cdot e^{\pi i\langle x,\omega\rangle}\cdot L_x M_\omega$$

of the (reduced) HEISENBERG GROUP $\mathbb{H}_r = \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{T}$, see [28, Section 7.1]. An extensive introduction to modulation spaces is given in Gröchenig's book [45], especially in chapters 11 and 12.

*We remark that the group structure of $\mathbb{H}_r = \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{T}$ is of course* not *that of the direct product of the three factors.*

More recently, the discovery of the group theoretical background of SHEARLETS led to an investigation of the associated coorbit spaces,

cf. [13, 15]. Certain coorbit spaces of the Blaschke group that give rise to spaces of holomorphic functions are discussed in [31].

### 1.1.2 *Features of coorbit theory*

One question that arises in each of the above examples is whether the resulting spaces differ if different analyzing windows $g$ are chosen. Thus, it would be preferable to have a general framework at hand which shows (among other things) this independence of the analyzing window $g$, at least for "reasonable" windows $g$.

Such a framework, which exhibits the similarities between the concrete examples mentioned above, is provided by the theory of COORBIT SPACES. This theory was originally developed by Feichtinger and Gröchenig in the 80s in a series of papers [28, 29, 30]. They considered the setting of unitary, irreducible, (square) integrable representations $\pi : G \to \mathcal{U}(\mathcal{H})$ of a locally compact group $G$ and of a solid Banach function space $Y$ on $G$. In this case, the (GENERALIZED) WAVELET TRANSFORM of a function/distribution $f$ is given by

$$W_g f : G \to \mathbb{C}, x \mapsto \langle f \mid \pi(x)\, g \rangle$$

and the resulting coorbit space is defined as

$$\mathrm{Co}(Y) := \mathrm{Co}_g(Y) := \left\{ f \in \mathcal{R} \,\middle|\, W_g f \in Y \right\}$$

with norm $\|f\|_{\mathrm{Co}(Y)} := \|W_g f\|_Y$, where $Y$ is a solid function space on $G$.

The main building blocks of coorbit theory are the following:

*Because of the generality of the approach, one cannot simply use a space like $\mathcal{S}'(\mathbb{R}^d)$ as the reservoir $\mathcal{R}$, since we need to define the wavelet transform $W_g f(x) = \langle f \mid \pi(x)\, g \rangle$ for elements $f \in \mathcal{R}$. But the representation $\pi$ only acts on $\mathcal{H}$, so that $\mathcal{R}$ needs to be related to $\mathcal{H}$ in some way. But $\mathcal{H}$ is a general Hilbert space, not necessarily a space of functions on $\mathbb{R}^d$.*

1. Construction of a suitable RESERVOIR $\mathcal{R} = \mathcal{R}_Y$ such that $\mathrm{Co}_g(Y)$ becomes a well-defined Banach space.
2. Identification of a class of (GOOD) ANALYZING WINDOWS $\mathcal{A}_Y$ such that $\mathrm{Co}_g(Y) = \mathrm{Co}_h(Y)$ holds for all $g, h \in \mathcal{A}_Y$.

Here, it is relevant that the spaces $\mathcal{R}_Y, \mathcal{A}_Y$ actually do not depend on *all* properties of the space $Y$, but only on the norms $\|\|L_x\|\|_{Y \to Y}$ and $\|\|R_x\|\|_{Y \to Y}$ of the left/right translation operators $L_x, R_x$. Thus, one can often choose the same reservoir and the same class of analyzing vectors for a huge class of spaces $Y$.

Among the advantages of identifying a certain class of Banach spaces as a class of coorbit spaces are the following:

1. Coorbit theory provides ATOMIC DECOMPOSITIONS, cf. [29, Theorem 6.1]. This means that one can fix a suitable family $(x_i)_{i \in I}$ in $G$, as well as (bounded) linear functionals $(\lambda_i)_{i \in I}$ on $\mathrm{Co}(Y)$ such that each $f \in \mathrm{Co}(Y)$ can be written as

$$f = \sum_{i \in I} \lambda_i(f) \cdot \pi(x_i)\, g,$$

with a norm equivalence $\|f\|_{\mathrm{Co}(Y)} \asymp \left\|(\lambda_i(f))_{i\in I}\right\|_{Y_d}$. Here, $Y_d$ is a certain sequence space associated to $Y$.

The different $\pi(x)\, g$ are called ATOMS, which gives rise to the name ATOMIC DECOMPOSITION.

2. The question of inclusions or *embeddings between coorbit spaces* is greatly simplified. In fact, if $Y, Z$ are solid Banach function spaces *on a common group* $G$, then $\mathrm{Co}Y \hookrightarrow \mathrm{Co}Z$ holds if and only if the embedding $Y_d \hookrightarrow Z_d$ is true, cf. [30, Theorem 8.4]. Even compactness of the embedding between the coorbit spaces is equivalent to compactness of the embedding between the discrete sequence spaces, cf. [30, Theorem 9.4].

   Here, the two coorbit spaces of course have to use the same representation $\pi : G \to \mathcal{U}(\mathcal{H})$.

### 1.1.3 *Coorbit theory for Quasi-Banach spaces*

Several generalizations of the construction of coorbit spaces have been considered: In [18], the authors develop a form of coorbit spaces in cases where the representation is not necessarily irreducible or integrable. A construction of coorbit spaces on homogeneous spaces is developed in [11]. Here, the assumption is that the representation is square integrable modulo a certain subgroup of $G$.

Finally, in [58], the theory of coorbit spaces is extended to cover the case of *Quasi*-Banach spaces instead of genuine Banach spaces.

*For a Quasi-Banach space, the usual triangle inequality is replaced by* $\|x+y\| \leq C \cdot [\|x\| + \|y\|]$.

This generalization is important in connection with NONLINEAR APPROXIMATION. Here, one is given a (possibly redundant) DICTIONARY $(f_i)_{i\in I}$ of elements $f_i \in Y$ and seeks for arbitrary $f \in Y$ an approximation $\tilde{f} = \sum_{i\in I_f} \alpha_i \cdot f_i$ to $f$ under the restriction $\left|I_f\right| \leq K$, i.e., using only a fixed number of dictionary elements. The term NONLINEAR APPROXIMATION is used since the map $f \mapsto \tilde{f}$ is not required to be linear.

In the context of "usual" norms like the $\ell^p$-norms on $\mathbb{C}^N$ and with the dictionary given by the standard basis $(e_1, \ldots, e_N)$, the best approximation is obtained by $\tilde{x} = x \cdot \chi_{I_x}$, where the set $I_x \subset \underline{N}$ contains the (indices of) the $K$ largest entries of $x$. A vector $x \in \mathbb{C}^N$ is called $K$-SPARSE[35, Definition 2.1], if at most $K$ entries of $x$ are nonzero and $x$ is called COMPRESSIBLE if $x$ can be well approximated by sparse vectors. The minimal approximation error is denoted by

$$\sigma_K(x)_p := \min\left\{\|x-z\|_p \,\middle|\, z \in \mathbb{C}^N \text{ is } K\text{-sparse}\right\}.$$

A rigorous measure of the compressibility of a vector is given by the decay of $\sigma_K(x)_p$ for large $K$ (but of course $K \ll N$). In [35, Proposition 2.3] it is shown that this error obeys the bound

$$\sigma_K(x)_q \leq K^{\frac{1}{q}-\frac{1}{p}} \cdot \|x\|_p. \tag{1.1.1}$$

Thus, the decay is faster for large values of $1/p$, i.e., one wants to control $\|x\|_p$ for *small* $p > 0$. But the $\ell^p$- and $L^p$-norms are only genuine norms for $p \geq 1$ and QUASI-NORMS for $0 < p < 1$. Thus, it is worthwhile to also consider QUASI-BANACH SPACES, because the above considerations show that these are natural spaces for nonlinear approximation.

In the setting of coorbit spaces, the dictionary can be taken as $(\pi(x_i)\, g)_{i \in I}$, where the family $(x_i)_{i \in I}$ is as in the discussion of the atomic decomposition results for coorbit spaces above. These results show that in order to approximate $f$ by finite sums $\sum_{i \in J} \alpha_i \pi(x_i)\, g$, it is sufficient to approximate the coefficient sequence $(\lambda_i(f))_{i \in I}$ by a finitely supported sequence, with respect to the (quasi)-norm $Y_d$. For $Y = L^q(G)$, one has $Y_d = \ell^q(I)$. Thus, the decay in equation (1.1.1) implies that elements $f \in \mathrm{Co}(L^p)$ for *small* $p > 0$ can be well approximated *in the norm of* $\mathrm{Co}(L^q)$ for $q \gg p$ by finite linear combinations of the dictionary $(\pi(x_i)\, g)_{i \in I}$. Again, this shows that elements in $\mathrm{Co}(L^p)$ with small $p > 0$ are compressible as elements of $\mathrm{Co}(L^q)$ for $q \gg p$, so that it is desirable to be able to consider the space $\mathrm{Co}(L^p)$ for $0 < p < 1$.

### 1.1.4 *New contributions*

For the reasons outlined above, we will develop coorbit theory for Quasi-Banach spaces, following [58]. This is done mostly to make the thesis more self-contained. But we will see that the construction in [58] is based on an incorrect convolution relation for Wiener amalgam spaces given in [59]. Thus, the new contribution of our treatment is the realization that this convolution relation indeed fails in general, as well as the development of an alternative convolution relation for which we then verify that a modified version of the construction in [58] indeed yields well-defined coorbit spaces.

As a byproduct, we slightly reduce the assumptions that the space $Y$ has to fulfill: In [58], the space $W(L^\infty, Y)$ is required to be left- and right-invariant, whereas we only require right-invariance.

Furthermore, we give an introduction to the theory of solid quasi-normed function spaces. Most of the results here are well-known for Banach spaces, see e.g. [69]. For the case of quasi-normed spaces they are probably folklore, but I am not aware of a publication in which comparable results are discussed.

We finally remark that the chapter about coorbit theory for Quasi-Banach spaces contains two subsections marked with an asterisk ($*$). These subsections are not needed for the remainder of the thesis, but contain instructive examples indicating possibly pathological behaviour of Quasi-Banach spaces and solid function spaces.

## 1.2 (FOURIER-SIDE) DECOMPOSITION SPACES

Decomposition spaces were first introduced in 1985 by Feichtinger and Gröbner[27, 26] in a very general setting. The underlying idea (which should also be compared to the concept of general Wiener amalgam spaces[24, 23]) is to introduce a space $\mathcal{D}(\mathcal{Q}, B, Y)$ of functions or distributions, whose *local* properties are measured in the Banach space $B$ – which is a Banach space of functions/distributions, for example $B = L^p$ or $B = \mathcal{F}L^p$ with $p \geq 1$ – and the *global* behaviour of these local properties is measured using the discrete sequence space $Y$, e.g. $Y = \ell^q$ with $q \geq 1$.

### 1.2.1 *General decomposition spaces*

More precisely, $\mathcal{Q} = (Q_i)_{i \in I}$ is required to be a (well-behaved) covering of the (typically locally compact) space $X$ and $\Psi = (\psi_i)_{i \in I}$ is a suitable partition of unity subordinate to $\mathcal{Q}$. The general idea is to define the norm on the decomposition space $\mathcal{D}(\mathcal{Q}, B, Y)$ by

$$\|f\|_{\mathcal{D}(\mathcal{Q},B,Y)} := \left\| (\|\psi_i \cdot f\|_B)_{i \in I} \right\|_Y .$$

Well-known special cases of this construction include the class of (homogeneous and inhomogeneous) BESOV-SPACES[57, 66], as well as the class of ($\alpha$-)MODULATION SPACES[25, 43, 45]. In both cases, one uses certain weighted $\ell^q$ spaces as the global component $Y$ and certain $\mathcal{F}L^p$-spaces as the local component $B$. The main difference between the different spaces lies in the chosen covering $\mathcal{Q}$. For the Besov spaces, one uses a *dyadic* covering of the frequency space, whereas the usual modulation spaces use a *uniform* covering. The $\alpha$-modulation spaces are defined using a family of intermediate coverings[43] between these two extreme cases.

For the definition above to make sense, one has to impose suitable assumptions on the covering $\mathcal{Q}$, the spaces $B$ and $Y$ and on the partition of unity $\Psi$ in order to ensure that this indeed yields a *well-defined* Banach space. Well-defined here means (among other things) that the resulting space is independent of the chosen partition of unity $\Psi$, with equivalent norms for different choices.

To see that this can indeed pose a problem, we remark that for $B = \mathcal{F}L^1(\mathbb{R}^d)$, one can not simply take a partition of unity $(\psi_i)_{i \in I}$ consisting of characteristic functions, because this will lead to

$$\|\psi_i \cdot f\|_B = \left\| \mathcal{F}^{-1}(\psi_i \cdot f) \right\|_{L^1} = \infty,$$

even for very well behaved functions like a smooth bump function $f$, because the crude localization by $\psi_i$ will make $\psi_i \cdot f$ discontinuous, which implies $\mathcal{F}^{-1}(\psi_i \cdot f) \notin L^1$.

The assumption made in [27] is to take $B$ as a Banach module over a Banach algebra $A$ and to assume that the partition of unity $\Psi$ is uniformly bounded in $A$.

### 1.2.2 *Fourier-side decomposition spaces*

In this thesis, we will consider a more concrete class of decomposition spaces which differs slightly from the general setting introduced by Feichtinger and Gröbner. The main reason for this is that we allow the choice of the Quasi-Banach spaces $B = \mathcal{F}L^p$ for $0 < p < 1$ for the local component $B$. We will see later that these spaces are *not* Banach modules of the Banach algebra $\mathcal{F}L^1$ which is usually used as the space of multipliers for the choice $B = \mathcal{F}L^p$ with $p \geq 1$. In a certain sense, one can show that $\mathcal{F}L^p$ is a "Quasi-Banach module" of $\mathcal{F}L^p$. More precisely, it turns out that an estimate of the form

$$\|\mathcal{F}^{-1}(f \cdot g)\|_{L^p} \leq C \cdot \|\mathcal{F}^{-1} f\|_{L^p} \cdot \|\mathcal{F}^{-1} g\|_{L^p}$$

is true, but *only* if both $f$ and $g$ are compactly supported. Furthermore, the constant $C$ depends in an unexpected way on the supports of $f, g$, so that the Fourier multiplier $f$ does *not* yield a bounded operator on $\mathcal{F}L^p$, but only on $\mathcal{F}_K L^p$ for each compact set $K \subset \mathbb{R}^d$, where $\mathcal{F}_K L^p$ is the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{f} \subset K$.

Thus, this situation is not covered by the abstract setting considered in [27, 26]. Therefore, instead of trying to generalize the conditions considered by Feichtinger and Gröbner to match our present setting, we will stick to the less abstract (but less general) approach of Borup and Nielsen[5], which we will modify slightly. Nevertheless, we will make use of many of the ideas and results from [27], which mostly remain valid in the present setting.

More precisely, a FOURIER-SIDE DECOMPOSITION SPACES is defined as follows: It forms a space of distributions on an open set $\mathcal{O} \subset \mathbb{R}^d$ (thought of as a subset of the *frequency domain* $\mathbb{R}^d$) for which a covering $\mathcal{Q} = (Q_i)_{i \in I}$, together with a partition of unity $(\varphi_i)_{i \in I}$ is given (both have to satisfy certain admissibility conditions). The decomposition space norm is then defined by localising $f \in \mathcal{D}'(\mathcal{O})$ to $\varphi_i f$, by taking the $L^p$-norm of the individual pieces *on the space side* and by finally measuring the norms of the pieces in a suitable sequence space $Y$, that is

$$\|f\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} = \left\| \left( \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p} \right)_{i \in I} \right\|_Y .$$

We remark that this definition seems to differ significantly from the usual definition of these spaces, since these are defined *on the space*

*side*, i.e. for a given $f \in \mathcal{S}'(\mathbb{R}^d)$, the quantity $\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$ used above is replaced by

$$\|\mathcal{F}^{-1}(\varphi_i \widehat{f})\|_{L^p}$$

in the usual definition. But if one is willing to replace the reservoir $\mathcal{S}'(\mathbb{R}^d)$ by the space $[\mathcal{F}(C_c^\infty(\mathcal{O}))]'$, then it is easy to see that the Fourier transform yields an isomorphism of the "classical" space-side decomposition spaces to the Fourier-side decomposition spaces as defined above. Since the space $\mathcal{F}(C_c^\infty(\mathcal{O}))$ and its dual space appear less natural than the spaces $C_c^\infty(\mathcal{O})$ and $\mathcal{D}'(\mathcal{O})$, we prefer to work on the Fourier-side directly.

In section Section 5.5, we will give a natural criterion which ensures $\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, so that $\mathcal{S}'(\mathbb{R}^d)$ can be used as the reservoir for the "space-side" versions of the decomposition spaces.

In any case, it is crucial to note that the partition of unity $(\varphi_i)_{i\in I}$ in *both* cases partitions the frequency domain.

### 1.2.3 *Related work*

Recently, (the space-side versions of) these decomposition spaces received increased attention. This began with the work of Borup and Nielsen [5], who introduced the class of STRUCTURED ADMISSIBLE COVERINGS (but only in the case $\mathcal{O} = \mathbb{R}^d$). They showed that these coverings always admit a suitable partition of unity $(\varphi_i)_{i\in I}$ (called a BOUNDED ADMISSIBLE PARTITION OF UNITY) and constructed tight frames $(\eta_{n,T})_{(n,T)\in\mathbb{Z}^d\times\mathcal{T}}$ for $L^2(\mathbb{R}^d)$ which have the additional property that the decomposition space norm with respect to the structured admissible covering $\mathcal{Q} = (TQ)_{T\in\mathcal{T}}$ of $\mathbb{R}^d$ can be characterized using the frame coefficients (cf. [5, Proposition 3]), to wit

$$\|f\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_u^q)} \asymp \left\| \left( \left\| \left( \langle f, \eta_{n,T}^{(p)} \rangle \right)_{n\in\mathbb{Z}^d} \right\|_{\ell^p} \right)_{T\in\mathcal{T}} \right\|_{\ell_u^q}$$

with $\eta_{n,T}^{(p)} = |T|^{\frac{1}{2}-\frac{1}{p}} \cdot \eta_{n,T}$. Here, each $T \in \mathcal{T}$ is an affine map of the form $Tx = A_T x + b_T$ and $|T| := |\det(A_T)|$. The family $(\eta_{n,T})_{(n,T)\in\mathbb{Z}^d\times\mathcal{T}}$ even yields an atomic decomposition for the decomposition space (cf. [5, Theorem 2]).

Further recent publications connected to the theory of decomposition spaces are the papers of Toft and Wahlberg[64] and by Han and Wang[49]. They considered $\alpha$-MODULATION SPACES and their dilation- and embedding properties. Finally, Labate et al.[53] defined certain SHEARLET SMOOTHNESS SPACES as special decomposition spaces. They also showed that these spaces coincide with the curvelet smoothness spaces (cf. [53, Proposition 4.4]) and established embeddings between shearlet smoothness spaces and Besov spaces (cf. [53, Proposition 4.3]).

### 1.2.4 *New contributions*

Chapter 3 does not contain striking new results, but rather many small contributions, since it serves mainly to fix notation for the rest of the thesis.

Among these contributions are the following: We introduce a new class of coverings, the SEMI-STRUCTURED ADMISSIBLE COVERINGS. Furthermore, we give a precise definition of the Fourier-side decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ and show its well-definedness and completeness. In the main source[5] for this type of decomposition spaces, this is neglected by referring to the classical papers by Feichtinger and Gröbner[27, 26], although these are strictly speaking not applicable in the setting of $\mathcal{F}L^p$ for $p < 1$, or if the space $\mathcal{S}'\left(\mathbb{R}^d\right)$ is used as the reservoir. This leads in particular to the problem that the decomposition spaces as defined in [5, Definition 3] are *not* complete in general, as we will see.

As a final result in this section, we show that a decomposition space is invariant under dilation with a matrix $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$ if and only if there is an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(g\mathcal{Q}, L^p, \ell^q_u\right). \tag{1.2.1}$$

Although this is a rather simple observation, it seems to be new. Furthermore, we will see in Chapter 6 that this simple reformulation of dilation invariance using embeddings for decomposition spaces will allow us to asymptotically determine the operator norm of the isotropic dilation operators on $\alpha$-modulation spaces and shearlet-type coorbit spaces.

## 1.3 WAVELET COORBIT SPACES AS DECOMPOSITION SPACES

As indicated at the beginning of this introduction, the Fourier transform induces an isomorphism

$$\mathcal{F} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$$

between certain coorbit spaces and suitable decomposition spaces.

### 1.3.1 *Wavelet coorbit spaces*

More precisely, the coorbit spaces for which we will prove this result are the WAVELET COORBIT SPACES. These are formed as follows: We fix some closed matrix group $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$, which we call the DILATION GROUP. The group $G$ which is used to define the wavelet coorbit spaces is $G = \mathbb{R}^d \rtimes H$, which can be interpreted as the group of all affine mappings generated by arbitrary translations $x \in \mathbb{R}^d$ and by all dilations $h \in H$.

On this group, we consider the QUASI-REGULAR REPRESENTATION, given as

$$\pi : G \to \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right), (x,h) \mapsto L_x D_h.$$

Here, $D_h f\,(y) := \left|\det(h)\right|^{-1/2} \cdot f\left(h^{-1}y\right)$ is the ($L^2$-normalized) dilation of $f$ by $h$.

Finally, we only consider spaces $Y$ of the form $Y = L_m^{p,q}(G)$, where the (weighted) MIXED LEBESGUE SPACE $L_m^{p,q} = L_m^{p,q}(G)$ is defined for $p, q \in (0,\infty]$ as

$$L_m^{p,q}(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; f \text{ measurable and } \|f\|_{L_m^{p,q}} < \infty \right\},$$

with

$$\|f\|_{L_m^{p,q}} := \left[ \int_H \|m(\cdot,h) \cdot f(\cdot,h)\|_{L^p(\mathbb{R}^d)}^q \frac{\mathrm{d}h}{\left|\det(h)\right|} \right]^{1/q}$$

for $q \in (0,\infty)$ and with

$$\|f\|_{L_m^{p,q}} := \operatorname*{ess\,sup}_{h \in H} \|m(\cdot,h) \cdot f(\cdot,h)\|_{L^p(\mathbb{R}^d)}$$

for $q = \infty$. Here, $m : G \to (0,\infty)$ is a (measurable) weight.

The results in [38, 36] precisely characterize those dilation groups $H$ for which the quasi-regular representation is irreducible and square integrable; we call such groups ADMISSIBLE. This characterization is expressed in terms of the DUAL ACTION

$$H \times \mathbb{R}^d \to \mathbb{R}^d, (h,\xi) \mapsto h^{-T}\xi$$

of $H$ on the frequency space $\mathbb{R}^d$. The dilation group $H$ is admissible if and only if the dual action has a single open orbit $\mathcal{O} = H^T\xi_0$ of full measure, called the DUAL ORBIT, such that the isotropy group

$$H_{\xi_0} = \left\{ h \in H \;\middle|\; h^{-T}\xi_0 = \xi_0 \right\}$$

is compact.

We will see that for any such group, all assumptions for the application of coorbit theory to the spaces $L_m^{p,q}(G)$ are satisfied, at least under mild growth conditions on the weight $m$.

### 1.3.2 *Isomorphism between wavelet coorbit spaces and certain decomposition spaces*

By virtue of the dual action, the dilation group $H$ induces a covering of the frequency space $\mathbb{R}^d$, or more precisely of the dual orbit $\mathcal{O}$. To wit, if $(h_i)_{i\in I}$ is a well-spread family in $H$, we will see that there is always a set $Q \Subset \mathcal{O}$ such that $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ is a structured

*Here, $Q \Subset \mathcal{O}$ means that $Q$ is compactly contained in $\mathcal{O}$.*

admissible covering of $\mathcal{O}$. We call each such covering $\mathcal{Q}$ A COVERING OF $\mathcal{O}$ INDUCED BY $H$.

We will see that if the weight $m = m(h)$ only depends on the second factor, one can define a form of the FOURIER TRANSFORM which restricts to an isomorphism

$$\mathcal{F} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$$

between the wavelet coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$ and the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$. Here, $\mathcal{Q}$ is an induced covering of $\mathcal{O}$ as above and the weight $u$ depends on $H$, $m$ and the exponent $q$.

Thus, wavelet coorbit theory becomes a branch of decomposition space theory.

Furthermore, this isomorphism opens the door to a Fourier-analytic understanding of wavelet coorbit spaces. More conceptually, the existence of this isomorphism shows that a large part of the structure of the dilation group $H$ is actually not relevant for the approximation theoretic properties of the coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\right)$. More precisely, the only relevant property of $H$ is the way in which the dual action $h \mapsto h^{-T}\xi$ covers the dual orbit $\mathcal{O}$. Once we have developed embedding results for decomposition spaces, we will see in Chapter 6 how the geometry of the induced covering determines the existence of embeddings of the coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$ into other coorbit spaces or into more classical smoothness spaces like Besov spaces. The next subsection discusses these and other applications in more detail.

### 1.3.3 *Applications of the decomposition space view of coorbit spaces*

It turns out that many properties of the coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\right)$ are more apparent from the decomposition space point of view than from the coorbit space point of view. One example is the property of DILATION INVARIANCE of the space $\mathrm{Co}\left(L^{p,q}_m\right)$. Here, we ask whether for a given dilation $\gamma \in \mathrm{GL}\left(\mathbb{R}^d\right)$ and a function $f \in L^2\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p,q}_m\right)$, it automatically follows that $f \circ \gamma \in \mathrm{Co}\left(L^{p,q}_m\right)$, together with a norm estimate $\|f \circ \gamma\|_{\mathrm{Co}\left(L^{p,q}_m\right)} \lesssim \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)}$. As long as the space $L^{p,q}_m(G)$ is left-invariant, it is not hard to see that this is the case for every dilation $\gamma \in H$. But for dilations $\gamma \in \mathrm{GL}\left(\mathbb{R}^d\right) \setminus H$, this is not clear at all and not easy to answer from the point of view of coorbit theory.

As an application of our results, we will show that for the similitude group $H = \mathbb{R}^* \cdot O\left(\mathbb{R}^d\right)$, the resulting coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\right)$ are invariant under *arbitrary* dilations $\gamma \in \mathrm{GL}\left(\mathbb{R}^d\right)$, while this is not true for the shearlet group in dimension two, as we will see in Chapter 6.

More generally, the benefit of the decomposition space point of view is that it becomes possible to compare coorbit spaces that are defined using *different* dilation groups $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$. From the point

of view of coorbit spaces, it is not even clear what we mean by this: The coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is a subspace of the reservoir $\mathcal{R}_v$, which depends heavily on the group $\mathbb{R}^d \rtimes H$, so that it is not clear how two coorbit spaces "living" on different groups $\mathbb{R}^d \rtimes H_1$ and $\mathbb{R}^d \rtimes H_2$ can be compared.

But in the decomposition space point of view, this is possible: The most transparent case is if the two dilation groups $H_1, H_2$ have the same dual orbits $\mathcal{O}_1 = \mathcal{O} = \mathcal{O}_2$. In this case, the decomposition spaces $\mathcal{D}_1 := \mathcal{D}\left(\mathcal{Q}_1, L^{p_1}, \ell^{q_1}_{u_1}\right)$ and $\mathcal{D}_2 := \mathcal{D}\left(\mathcal{Q}_2, L^{p_2}, \ell^{q_2}_{u_2}\right)$ are both subspaces of the space $\mathcal{D}'(\mathcal{O})$ of distributions on $\mathcal{O}$ and one can thus ask whether an embedding $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2$ (or vice versa) is true. By employing the isomorphism of $\mathcal{D}_1$ and $\mathcal{D}_2$ with the respective coorbit spaces, this leads to a natural form of embedding between the coorbit spaces $\mathrm{Co}\left(L^{p_1,q_1}_{m_1}\left(\mathbb{R}^d \rtimes H_1\right)\right)$ and $\mathrm{Co}\left(L^{p_2,q_2}_{m_2}\left(\mathbb{R}^d \rtimes H_2\right)\right)$.

Thus, using the above isomorphism, the question of embeddings between coorbit spaces "living" on different groups can be reduced to the question of embeddings between decomposition spaces. The study of this question, and a multitude of applications, is taken on in Chapters 5 and 6. We will see that one can obtain reasonable forms of embeddings, even if the two sets $\mathcal{O}_1, \mathcal{O}_2$ do not coincide.

### 1.3.4 *New contributions*

The findings in this chapter are new contributions of this thesis, but some of the results (only Banach spaces, i.e., for $p, q \in [1, \infty]$) were already submitted for publication in the paper [41], written by Hartmut Führ and the present author.

Furthermore, the applicability of coorbit theory (for $p, q \in [1, \infty]$) for admissible dilation groups was already studied in [39, 40].

## 1.4 EMBEDDINGS BETWEEN DECOMPOSITION SPACES

In the description of decomposition spaces above, we already mentioned the papers of Toft and Wahlberg[64] and by Han and Wang[49] in which embeddings between $\alpha$-MODULATION SPACES for different values of $\alpha$ are studied. Furthermore, Labate et al.[53] consider embeddings of the newly defined SHEARLET SMOOTHNESS SPACES into the more classical Besov spaces (cf. [53, Proposition 4.3]).

As these results show, there is great interest in establishing embeddings between decomposition spaces that are defined using *different* coverings. Furthermore, an important application of the decomposition space view of wavelet coorbit spaces will be that embedding results for decomposition spaces will yield embeddings between coorbit spaces that "live" on different groups.

In Chapter 5, we solve the problem of existence of embeddings between different decomposition spaces in a quite general setting. We

will see that all previously known embedding results as mentioned above are special cases of the results in this chapter. In a lot of cases, our approach even yields stronger results. In fact, our criteria are quite often sharp, i.e., they characterize the existence of the respective embedding completely.

The precise problem that we want to solve is the following: Given two coverings $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = \left(P_j\right)_{j\in J}$, we want to find sufficient, as well as necessary criteria for the existence of an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right). \tag{1.4.1}$$

Since these spaces are defined solely in terms of the coverings $\mathcal{Q}, \mathcal{P}$, the weights $u, v$ and the exponents $p_1, p_2, q_1, q_2$, it should be possible to decide the existence of the embedding only in terms of these quantities.

We will show that this intuition is justified by proving that the existence of the embedding for decomposition spaces can be derived as a consequence of embeddings between certain nested sequence spaces. These sequence spaces *encode the geometric relation* between the two coverings $\mathcal{Q}, \mathcal{P}$.

For a more concrete description of our criteria, let us assume for simplicity that $\mathcal{Q}$ is subordinate to $\mathcal{P}$; this means $Q_i \subset P_{j_i}$ for each $i \in I$ and a suitable $j_i \in J$. In this case, the geometric relationship between the two coverings is predominantly determined by the family of INTERSECTION SETS

$$I_j := \left\{ i \in I \,\middle|\, Q_i \cap P_j \neq \varnothing \right\},$$

for $j \in J$.

Our results will show that the embedding (1.4.1) holds, if we have $p_1 \leq p_2$ and if we have an embedding

$$\left\| \left( \left\| (c_i)_{i\in I_j} \right\|_{\ell^{p_2^\nabla}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \lesssim \left\| \left( |\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}} c_i \right)_{i\in I} \right\|_{\ell_u^{q_1}} \tag{1.4.2}$$

*The exponent $p_2^\nabla$ used here is called the* LOWER CONJUGATE EXPONENT *for $p_2$, defined by $p_2^\nabla = \min\left\{p_2, p_2'\right\}$.*

of certain (nested) sequence spaces which are defined using the intersection sets $I_j$. We observe that it does not suffice to only consider the family of sets $I_j$; in case of $p_1 \neq p_2$ the geometry of $\mathcal{Q}$ also enters through the term $|\det(T_i)|$, which corresponds to the measure of the sets $Q_i = T_i Q_i' + b_i$, where we assume $Q_i' \subset B_R(0)$ for all $i \in I$ and some fixed $R > 0$.

The condition given above is chosen as to make the proof of our result more streamlined. If one actually wants to apply this criterion to concrete examples, checking estimate (1.4.2) for *all* sequences $(c_i)_{i\in I}$ (for which the right-hand side is finite) seems overly complicated. Luckily, the criterion given in equation (1.4.2) can be simplified

considerably. In fact, we will show that validity of this embedding of nested sequence spaces is equivalent to finiteness of the nested norm

$$\left\| \left( \left\| \left( u_i^{-1} \left| \det\left(T_i\right) \right|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I_j} \right\|_{\ell^{p_2^\nabla \cdot (q_1/p_2^\nabla)'}} \right)_{j \in J} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}}.$$

In other words, we have reduced validity of an *infinite* number of estimates to finiteness of the quasi-norm of a *single* sequence.

We have thus found a readily verifiable sufficient criterion for the existence of an embedding for decomposition spaces. We remark that the resulting criterion only involves discrete combinatorial considerations and can thus be applied without having to know anything about Fourier analysis. Furthermore, it is worth noting that the geometry of the coverings $\mathcal{Q}, \mathcal{P}$ only enters through three factors:

- We assumed that $\mathcal{Q}$ is subordinate to $\mathcal{P}$,
- the size/measure of the sets $Q_i$ enters through the term $|\det(T_i)|$,
- the *relative* geometry of the coverings $\mathcal{Q}, \mathcal{P}$ enters through the sets $I_j$.

In particular, the absolute position of the sets $Q_i, P_j$ is not relevant.

Given these sufficient conditions for the existence of an embedding for decomposition spaces, it is natural to study *sharpness* of these conditions. We will see that the condition $p_1 \leq p_2$ is always necessary. Furthermore, for the range $p_2 \in (0, 2]$ and under very mild conditions on the covering $\mathcal{Q}$, we will be able to show that existence of the embedding (1.4.1) already implies validity of the embedding for discrete sequence spaces given in equation (1.4.2). Thus, for $p \in (0, 2]$, the condition given above achieves a complete *characterization* of the existence of an embedding for decomposition spaces.

For the range $p_2 \in (2, \infty]$, we will also show that the embedding in equation (1.4.2) is necessary for the existence of an embedding between the decomposition spaces. But in this case, we will need to impose much stronger assumptions on the relation between the coverings $\mathcal{Q}, \mathcal{P}$. Precisely, we will assume that the covering $\mathcal{Q}$ and the weight $u$ are RELATIVELY $\mathcal{P}$-MODERATE; this means that the weights $u$ and $(|\det(T_i)|)_{i \in I}$ are almost constant on each of the sets $I_j$. Thus, we also achieve a characterization of the existence of an embedding for decomposition spaces for the range $p_2 \in (2, \infty]$, but only under more restrictive assumptions. Nevertheless, these assumptions are fulfilled for many concrete examples, especially if $\mathcal{Q}, \mathcal{P}$ are both coverings corresponding to $\alpha$-modulation spaces, even for different values of $\alpha$.

Finally, we remark that we only assumed $\mathcal{Q}$ to be subordinate to $\mathcal{P}$ for simplicity. In Chapter 5, we will consider the more general setting in which $\mathcal{Q}$ is *almost* subordinate to $\mathcal{P}$. In this case, we do not

only study the embedding in equation (1.4.1), but also the "reverse" embedding. Even more generally, we will consider the case of two coverings $\mathcal{R}, \mathcal{S}$ for which there are certain sets $A, B$ so that $\mathcal{R}$ is almost subordinate to $\mathcal{S}$ "near $A$" and $\mathcal{S}$ is almost subordinate to $\mathcal{R}$ "near $B$".

### 1.4.1 *New contributions*

The embedding results derived in Chapter 5 are new results of this thesis. To my knowledge, similar results have only been established for the special case of $\alpha$-modulation spaces in [43], [64] and [49] and furthermore in [53], where the authors consider embeddings between Besov spaces and the newly defined class of shearlet smoothness spaces.

Our approach also seems to be new: The proof idea used in [49] is to establish certain special cases of the embedding and then to derive the general case by interpolation. In contrast to this, we only use interpolation to derive certain "local" results. Once these have been obtained, we use embedding results for nested sequence spaces (as in equation (1.4.2) above), to establish the global embedding.

Furthermore, it should be observed that the approach in this thesis is applicable in the following cases which have (to my knowledge) not been treated elsewhere:

1. The case of different "integrability exponents" $(p_1, q_1) \neq (p_2, q_2)$ (for nonequivalent coverings $\mathcal{Q} \not\approx \mathcal{P}$) and

2. the case in which the two coverings $\mathcal{P}, \mathcal{Q}$ cover different sets $\mathcal{O}' \neq \mathcal{O}$. A special case of this are embeddings between homogeneous and inhomogeneous Besov spaces.

The first of these cases has of course been studied for concrete examples in the case $\mathcal{Q} = \mathcal{P}$, but apparently not for $\mathcal{Q} \neq \mathcal{P}$. Even in the (relatively) well-understood case of $\alpha$-modulation spaces, only the case $(p_1, q_1) = (p_2, q_2)$ has been considered, at least in [64] and [49].

## 1.5 APPLICATIONS

Finally, we illustrate the power and generality of our approach to embeddings for decomposition spaces by considering a large number of concrete examples.

As a litmus test, we first consider the well-studied case of embeddings between $\alpha$-MODULATION SPACES for different values of $\alpha \in [0, 1)$, i.e., we characterize the existence of embeddings

$$M^{\gamma_1,\alpha_1}_{p_1,q_1} \hookrightarrow M^{\gamma_2,\alpha_2}_{p_2,q_2}.$$

For the case $(p_1, q_1) = (p_2, q_2)$ a complete solution to this problem was found by Han and Wang[49]. We will see that their findings can be established conveniently using our general approach. Furthermore, we even achieve a complete characterization in case of $(p_1, q_1) \neq (p_2, q_2)$, thereby extending the known results.

Even more importantly, our unified approach indicates why the results in [49] actually hold. The main geometric fact that is used is that the $\alpha$-covering $\mathcal{O}^{(\alpha)}$ of $\mathbb{R}^d$, which is used to define the $\alpha$-modulation spaces, is almost subordinate to the $\beta$-covering $\mathcal{O}^{(\beta)}$ for $\alpha \leq \beta$. Furthermore, $\mathcal{O}^{(\alpha)}$ is relatively moderate with respect to $\mathcal{O}^{(\beta)}$ and the same is true of the weights used to define the $\alpha$-modulation spaces.

The next example that we consider is an even more classical class of spaces, the BESOV SPACES. These come in two variants, the homogeneous and the inhomogeneous Besov spaces. But although the properties of each individual of the two types of Besov spaces are very well studied, it seems that not much is known about the relation between homogeneous and inhomogeneous Besov spaces.

The only result in this direction of which I am aware is the fact (cf. [67, Theorem in §2.3.3]) that for $\alpha > \sigma_p = d\left(\frac{1}{p} - 1\right)_+$, the norm

$$f \mapsto \|f\|_{L^p} + \|f\|_{\dot{B}^p_{\alpha,q}}$$

defines an equivalent quasi-norm on the inhomogeneous Besov space $B^p_{\alpha,q}\left(\mathbb{R}^d\right)$. In particular, this yields an embedding

$$B^p_{\alpha,q}\left(\mathbb{R}^d\right) \hookrightarrow \dot{B}^p_{\alpha,q}\left(\mathbb{R}^d\right)$$

of the inhomogeneous Besov space $B^p_{\alpha,q}\left(\mathbb{R}^d\right)$ into the homogeneous Besov space $\dot{B}^p_{\alpha,q}\left(\mathbb{R}^d\right)$. Note that this is only a sufficient criterion, not a necessary one. Furthermore, nothing seems to be known about the reverse embedding.

Using our general embedding results, we change this situation completely. We will derive sufficient and necessary conditions for the existence of the embedding

$$B^{p_1}_{\alpha_1,q_1}\left(\mathbb{R}^d\right) \hookrightarrow \dot{B}^{p_2}_{\alpha_2,q_2}\left(\mathbb{R}^d\right)$$

and the reverse embedding. Although in some cases there is still a gap between the necessary and the sufficient conditions, this gap is very small: The only thing that can happen is that the sufficient condition requires a *strict* inequality between certain of the quantities $\alpha_1, \alpha_2, p_1, p_2, q_1, q_2$, whereas the necessary condition only yields a *non-strict* inequality.

Finally, we also consider embeddings between $\alpha$-modulation spaces and inhomogeneous Besov spaces. Again, for $(p_1, q_1) = (p_2, q_2)$, these results coincide with the characterization given by Han and Wang[49], but we also handle the case $(p_1, q_1) \neq (p_2, q_2)$.

A completely new class of decomposition spaces is given by the next example. Here, we study decomposition spaces that are obtained from the covering $\mathcal{S}^{(c)}$, which is an induced covering of the shearlet-type group

$$H_c = \left\{ \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \middle| \, \varepsilon \in \{\pm 1\}, \, a \in (0, \infty), \, b \in \mathbb{R} \right\}$$

for arbitrary $c \in \mathbb{R}$. For $c = \frac{1}{2}$, this yields the usual shearlet group as considered in [13, 15]. By our results on the isomorphism of wavelet coorbit spaces and decomposition spaces, we know that the Fourier transform yields an isomorphism between the SHEARLET-TYPE COORBIT SPACE $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ and the decomposition space $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)$ for a suitable weight $u$. We will use this isomorphism in conjunction with our embedding results for decomposition spaces to characterize (for a certain class of weights) the existence of embeddings of coorbit spaces

$$\mathrm{Co}\left(L^{p_1,q_1}_{m_1}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right) \to \mathrm{Co}\left(L^{p_2,q_2}_{m_2}\left(\mathbb{R}^2 \rtimes H_{c_2}\right)\right).$$

As promised above, these results indicate that the decomposition space point of view can indeed be used to derive embeddings between coorbit spaces "living" on different groups.

We remark that the coverings $\mathcal{S}^{(c)}$ are a challenging example, since $\mathcal{S}^{(c_1)}$ is *not* almost subordinate to $\mathcal{S}^{(c_2)}$ for $c_1 \neq c_2$. Nevertheless, for $c_1 < c_2$, one can show that $\mathcal{S}^{(c_1)}$ is almost subordinate to $\mathcal{S}^{(c_2)}$ "away from the $y$-axis", whereas $\mathcal{S}^{(c_2)}$ is almost subordinate to $\mathcal{S}^{(c_1)}$ "near the $y$-axis". Thus, our results from Chapter 5 will suffice to characterize the existence of the desired embeddings.

Finally, we use the isomorphism between wavelet coorbit spaces and decomposition spaces, as well as our embedding results for decomposition spaces, to derive necessary and sufficient criteria for the existence of embeddings between inhomogeneous Besov spaces and shearlet-type coorbit spaces. Since the geometries of the two coverings are very different (i.e. $\mathcal{S}^{(c)}$ is almost subordinate to the inhomogeneous Besov covering $\mathcal{B}$, but *not* relatively moderate), our criteria do not achieve a complete characterization in all cases. Nevertheless, our results are an improvement of the known results, since to my knowledge no embeddings of this type have been known. The only result which is slightly comparable to the embeddings derived here is [15, Theorem 4.7], where the authors consider embeddings of a *strict subspace* of $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_{1/2}\right)\right)$ into a *sum of two inhomogeneous* Besov spaces.

As our final example of embeddings for decomposition spaces, we consider the newly introduced class of SHEARLET SMOOTHNESS SPACES[53]. In [53, Proposition 4.3], the authors already give *sufficient* criteria for the existence of embeddings between inhomogeneous

Besov spaces and shearlet smoothness spaces, at least for the special case $(p_1, q_1) = (p_2, q_2)$. As a strong improvement, we will be able to completely characterize the existence of these embeddings, even for $(p_1, q_1) \neq (p_2, q_2)$. Furthermore, we are able to give a similar characterization for $\alpha$-modulation spaces instead of Besov spaces, at least in the range $\alpha \in \left[0, \frac{1}{2}\right]$. This seems to be a completely new result.

In the last section of Chapter 6, we show that embedding results for different decomposition spaces can also be used to study operators which act on a *single* fixed decomposition space or coorbit space. More precisely, we consider the question of dilation invariance of shearlet coorbit spaces and $\alpha$-modulation spaces. As we saw in equation (1.2.1) above, invariance of a decomposition space under dilation with $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$ can be reformulated as an embedding statement between the "original" decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ and the "dilated" decomposition space $\mathcal{D}\left(g\mathcal{Q}, L^p, \ell^q_u\right)$. In view of the isomorphism of wavelet coorbit spaces and decomposition spaces, a similar statement also holds for coorbit spaces.

These observations will allow us to asymptotically compute the norm of the isotropic dilation operator $D_\lambda : f \mapsto f \circ \lambda \mathrm{id}$ on the shearlet-type coorbit spaces $\mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. More precisely, we consider weights of the form

$$v_\alpha : H_c \to (0, \infty), \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mapsto a^\alpha.$$

We will then show

$$|||D_\lambda||| \asymp \lambda^{-\alpha - \frac{2}{p}} \cdot \lambda^{(1+c)\left(\frac{1}{q} - \frac{1}{2}\right) + (1-c)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+}$$

*Here, we have* $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$.

for $\lambda \in (1, \infty)$ and $c \in (-\infty, 1]$. A similar estimate also holds for $\lambda \in (0, 1)$.

For the shearlet-type coorbit spaces, our results are completely new. In contrast, the asymptotic norm of the isotropic dilation operator acting on the $\alpha$-modulation space $M^{s,\alpha}_{p,q}$ has been computed for almost all cases in [49]. As a further application, we show that almost the same results can be derived using the criteria developed in Chapter 5. For the range $p \in [1, \infty]$, our findings are identical to those in [49], but for $p \in (0, 1)$, $\lambda > 1$ and certain values of $s$, the results in [49] turn out to be slightly incorrect. This indicates that checking known results using our new, systematic approach to embeddings can still be worthwhile.

Returning to the setting of shearlet coorbit spaces, we also characterize the class of transformations $g \in \mathrm{GL}\left(\mathbb{R}^2\right)$ which leave *all* shearlet-type coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ invariant under conjugation. We will see that this holds precisely for upper triangular matrices with positive determinant. This is in sharp contrast to the

case of the similitude group described above. There, all coorbit spaces turned out to be invariant under *arbitrary* dilations. This is a further case in point where the decomposition space point of view, together with our criteria for embeddings between decomposition spaces, easily yields results which are not obvious from the point of view of coorbit spaces.

# 2

# COORBIT THEORY FOR QUASI-BANACH SPACES

In this chapter, we discuss the general theory of coorbit spaces. This is a preparation for the first main result of the thesis, the isomorphism between wavelet coorbit spaces and certain decomposition spaces on $\mathbb{R}^d$, which is presented in Chapter 4.

We consider the coorbit spaces directly in the setting of Quasi-Banach spaces. We remark that this is strictly speaking *not* a new contribution of this thesis, but was established before by Rauhut[58]. The main problem with the generalization to Quasi-Banach spaces is that the whole theory of coorbit spaces is based on REPRODUCING KERNEL BANACH SPACES, which are Banach spaces $Y$ of functions for which $f = f * g$ for all $f \in Y$ and some specific function $g$ is true. Hence, the theory is based largely on certain convolution relations. The problem is that many Quasi-Banach spaces like $L^p(G)$ (with $0 < p < 1$) do not satisfy any meaningful convolution relations, although they are invariant under left- and right translations. The reason for this is that the embedding $L^p(G) \hookrightarrow L^1_{\mathrm{loc}}(G)$ does *not* hold (for $0 < p < 1$ if $G$ is not discrete), so that a convolution $f * g$ is not defined in general, even for $g \in C_c(G)$. To circumvent this problem, Rauhut uses the WIENER AMALGAM SPACE $W(L^\infty, Y)$ instead of using the Quasi-Banach space $Y$ directly. For the space $W(L^\infty, Y)$, some useful convolution relations are indeed true.

But one problem of [58] is that it is based (in part) on the convolution relation

$$W\left(L^\infty, L^p_w\right) * W\left(L^\infty, L^p_{w^*}\right)^\vee \hookrightarrow W\left(L^\infty, L^p_w\right), \tag{2.0.1}$$

where $w : G \to (0, \infty)$ is a submultiplicative weight and $0 < p \leq 1$, see [58, Theorem 3.7]. As we will see below (cf. Example 2.3.26), this convolution relation is *false* in general. To circumvent this problem, we develop a slight variant of this convolution relation and check that the rest of the paper remains valid with suitable changes.

The chapter is structured as follows: In Section 2.1, we start with a brief introduction to the main properties of general Quasi-Banach spaces. In Section 2.2, we specialize this to the case of SOLID QUASI-BANACH SPACES OF FUNCTIONS, of which the usual $L^p$-spaces are the most prominent examples. In particular, we study the connection between pointwise convergence and convergence in the function space under consideration.

Section 2.3 is devoted to studying WIENER AMALGAM SPACES with respect to Quasi-Banach spaces, based on [59]. The general idea of these spaces is to "combine" two (Quasi-)Banach spaces, a LOCAL

COMPONENT $B$ and a GLOBAL COMPONENT $Y$ to obtain the space $W(B, Y)$ of functions/distributions whose local properties are determined by $B$, but whose global properties are determined by $Y$. In this thesis, we will only consider $B = L^\infty(G)$ as the local component to avoid some technicalities. In particular, this section contains an adjusted version of the incorrect convolution relation (2.0.1) above.

Finally, Section 2.4 is concerned with coorbit theory for Quasi-Banach spaces, as originally developed by Rauhut[58]. Our contribution here is to check that most of the paper remains valid, even though it originally uses the incorrect convolution relation (2.0.1).

We also remark that Rauhut assumes the space $W(L^\infty, Y)$ to be invariant under left-, as well as right translations, cf. [58, Definition 4.1]. In our modified approach, only right invariance of $W(L^\infty, Y)$ is needed.

## 2.1 A CRASH COURSE ON QUASI-BANACH SPACES

In this section, we will cover the most important properties of QUASI-NORMED SPACES and QUASI-BANACH SPACES. In the first subsection, we develop the most essential properties of these spaces, which will be used again and again in this chapter. The second subsection – which is not required for the rest of the thesis – illustrates some of the pathological properties of Quasi-Banach spaces, e.g. the (possible) failure of the Hahn-Banach theorem as well as the (possible) discontinuity of the quasi-norm.

### 2.1.1 *Properties of Quasi-Banach spaces*

The difference of quasi-normed spaces to ordinary normed vector spaces is that the triangle inequality only holds up to a constant:

**Definition 2.1.1.** (cf. [19, Chapter 2, §1] and [1, Definition H.1]) Let $X$ be a vector space over the field $\mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say that $\|\cdot\| : X \to [0, \infty)$ is a QUASI-NORM if the following properties are satisfied:

1. For $x \in X$, we have $\|x\| = 0$ if and only if $x = 0$,
2. We have $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
3. There is a constant $C \geq 0$ with $\|x + y\| \leq C \cdot [\|x\| + \|y\|]$ for all $x, y \in X$. We call each such constant $C \geq 0$ a TRIANGLE CONSTANT for $\|\cdot\|$.

In this case, we call $X$ (or more precisely $(X, \|\cdot\|)$) a QUASI-NORMED SPACE. Two quasi-norms $\|\cdot\|$ and $|||\cdot|||$ are called *equivalent* if there are constants $C_1, C_2 > 0$ with

$$C_1 \cdot \|x\| \leq |||x||| \leq C_2 \cdot \|x\| \qquad \forall x \in X.$$

Finally, for $p \in (0,1]$ we say that $|||\cdot||| : X \to [0,\infty)$ is a $p$-NORM if we have

$$|||x+y|||^p \leq |||x|||^p + |||y|||^p \qquad \forall x, y \in X.$$

*Remark* 2.1.2. Every $p$-norm $|||\cdot||| : X \to [0,\infty)$ is a quasi-norm with constant $C = 2^{\frac{1}{p}-1}$, because of

$$\begin{aligned} |||x+y||| &\leq \left[|||x|||^p + |||y|||^p\right]^{1/p} \\ &= 2^{1/p} \cdot \left[\frac{|||x|||^p}{2} + \frac{|||y|||^p}{2}\right]^{1/p} \\ &\leq 2^{1/p} \cdot \left[\frac{|||x|||}{2} + \frac{|||y|||}{2}\right] \\ &= 2^{\frac{1}{p}-1} \cdot \left[|||x||| + |||y|||\right]. \end{aligned}$$

Here, we employed convexity of $[0,\infty) \to [0,\infty), x \mapsto x^{1/p}$.

We also observe that a $p$-norm is also a $q$-norm for each $q \in (0,p)$. To see this, first note that for $r \in (0,1)$, the maps

$$\begin{aligned} f &: [0,\infty) \to \mathbb{R}, x \mapsto (1+x)^r, \\ g &: [0,\infty) \to \mathbb{R}, x \mapsto 1 + x^r, \end{aligned}$$

are both continuous with $f(0) = 1 = g(0)$ and differentiable on $(0,\infty)$ with

$$f'(x) = r \cdot (1+x)^{r-1} < r \cdot x^{r-1} = g'(x),$$

since $r - 1 < 0$. This yields $f \leq g$ on $[0,\infty)$ and hence

$$(x+y)^r = x^r \cdot \left(1 + \frac{y}{x}\right)^r \leq x^r \cdot \left(1 + \left(\frac{y}{x}\right)^r\right) = x^r + y^r \tag{2.1.1}$$

for all $x, y > 0$. In case of $x = 0$ or $y = 0$, this estimate is trivially satisfied.

Finally, since $r = q/p \in (0,1)$, we conclude

$$\begin{aligned} |||f+g|||^q = \left(|||f+g|||^p\right)^{q/p} &\leq \left(|||f|||^p + |||g|||^p\right)^{q/p} \\ &\leq |||f|||^{p\cdot\frac{q}{p}} + |||g|||^{p\cdot\frac{q}{p}} = |||f|||^q + |||g|||^q, \end{aligned}$$

so that $|||\cdot|||$ is a $q$-norm if it is a $p$-norm. ◀

**Example 2.1.3.** An important example of Quasi-Banach spaces are the Lebesgue spaces $L^p(\mu)$ for $0 < p < 1$ on some measure space $(X, \mathcal{A}, \mu)$ with the quasi-norm

$$\|f\|_p := \left(\int |f|^p \, \mathrm{d}\mu(x)\right)^{1/p}.$$

To see that this is indeed a quasi-norm, we show that it is a $p$-norm. Indeed, by equation (2.1.1) above, we have

$$\begin{aligned}\|f+g\|_p^p &= \int_X |f+g|^p \,\mathrm{d}\mu(x) \leq \int_X (|f|+|g|)^p \,\mathrm{d}\mu(x) \\ &\leq \int_X |f|^p + |g|^p \,\mathrm{d}\mu(x) = \|f\|_p^p + \|g\|_p^p .\end{aligned}$$

The remaining properties of a quasi-norm are easy to verify. ◀

The aim of this section is to establish just enough of the theory of quasi-normed spaces to be able to develop the theory of Wiener amalgam spaces and of coorbit spaces in the context of Quasi-Banach spaces in the rest of the chapter. All properties in this section are well-known; but to make the thesis more self-contained, we nevertheless provide proofs.

The first and most important result is that for each quasi-norm $\|\cdot\|$, there is always an equivalent $p$-norm $|||\cdot|||$ for some $p \in (0,1]$. This is (a slight variation of) the AOKI-ROLEWICZ THEOREM, cf. [42, Exercise 1.4.6].

The existence of such an equivalent $p$-norm is an advantage, since each $p$-norm $|||\cdot|||$ is continuous and can be used to define a metric on $X$ (cf. Lemma 2.1.5). Furthermore, the existence of an equivalent $p$-norm will allow us to prove a completeness criterion based on the convergence of (suitable) absolutely convergent series, cf. Lemma 2.1.8. This criterion is usually much more convenient than trying to show directly that arbitrary Cauchy sequences are convergent.

Finally, we will use the explicit form of the $p$-norm $|||\cdot|||$ constructed below to show that if $(X, \|\cdot\|)$ is a solid function space (cf. Definition 2.2.1), then the same is true of $(X, |||\cdot|||)$, see Corollary 2.2.12.

**Theorem 2.1.4** (Aoki-Rolewicz Theorem)**.** *(cf. [42, Exercise 1.4.6] and [19, Chapter 2, Theorem 1.1])*

*Let $(X, \|\cdot\|)$ be a quasi-normed space. Then there is some $p \in (0,1]$ such that $|||\cdot||| : X \to [0,\infty)$, defined by*

$$|||x||| := \inf\left\{ \left[\sum_{i=1}^n \|x_i\|^p\right]^{\frac{1}{p}} \,\middle|\, n \in \mathbb{N} \text{ and } x_1,\ldots,x_n \in X \text{ with } x = \sum_{i=1}^n x_i \right\}$$

*for $x \in X$, is a $p$-norm on $X$ which is equivalent to $\|\cdot\|$.*

*If $\|x+y\| \leq C \cdot [\|x\| + \|y\|]$ holds for all $x, y \in X$ with $C \geq 1$, we can choose*

$$p = \frac{\ln 2}{\ln(2C)} = \frac{1}{\log_2(2C)} = \frac{1}{1+\log_2 C} \in (0,1] .$$

*Remark.* With Remark 2.1.2 above, we see that this theorem is reasonably sharp: If $\|\cdot\|$ is a $q$-norm to begin with, we can take $C = 2^{\frac{1}{q}-1}$, which yields $p = \frac{1}{\log_2(2C)} = \frac{1}{\log_2 2^{1/q}} = q$, so that we get back a $p$-norm with the same exponent $p = q$.

It is worth noting that the proof uses only the semigroup properties of $(X,+)$ as well as the assumption $\|x+y\| \leq C\cdot[\|x\|+\|y\|]$ for the construction of $|||\cdot|||$. The remaining properties of a quasi-norm (positive definiteness and homogeneity) are *only* used to show that $|||\cdot|||$ inherits these properties. Hence, the same proof would work for a general nonnegative "quasi-subadditive" function on an abelian semigroup $G$.

*Proof.* We first set $C_0 := 2C$, define $p := \frac{\ln 2}{\ln(2C)} = \log_{C_0} 2$ and choose $|||\cdot|||$ as in the statement of the theorem.

Using $x_1 := x$, we get $0 \leq |||x||| \leq \|x\| < \infty$ for each $x \in X$.

Now, if $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{m} y_i$, then $x+y = \sum_{i=1}^{n+m} z_i$ with $z_i := x_i$ for $1 \leq i \leq n$ and $z_i := y_{i-n}$ for $n+1 \leq i \leq n+m$. Hence,

$$|||x+y|||^p \leq \sum_{i=1}^{n+m} \|z_i\|^p = \sum_{i=1}^{n} \|x_i\|^p + \sum_{i=1}^{m} \|y_i\|^p .$$

Since the decompositions $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{m} y_i$ where arbitrary, we conclude $|||x+y|||^p \leq |||x|||^p + |||y|||^p$ as desired.

Below, we will prove $\|\cdot\| \lesssim |||\cdot|||$. In particular, this will imply that $|||\cdot|||$ is positive definite, since $\|\cdot\|$ is. Furthermore, using the equivalence $x = \sum_{i=1}^{n} x_i \iff \alpha x = \sum_{i=1}^{n} \alpha x_i$ for $\alpha \in \mathbb{K}\setminus\{0\}$, it is not hard to see that $|||\cdot|||$ is homogeneous, since $\|\cdot\|$ is. Hence, $|||\cdot|||$ is indeed recognized as a $p$-norm, once we show $\|\cdot\| \lesssim |||\cdot|||$.

*Note that this is the only place in the proof where homogeneity and definiteness of $\|\cdot\|$ are used.*

To this end, observe that

$$\|x+y\| \leq C\cdot[\|x\|+\|y\|] \leq C_0 \cdot \max\{\|x\|, \|y\|\} \tag{2.1.2}$$

holds for all $x, y \in X$. By induction, this implies

$$\|x_1+\cdots+x_n\| \leq \max_{1\leq j\leq n} C_0^j \|x_j\| \text{ for all } x_1,\ldots,x_n \in X, \tag{2.1.3}$$

since the induction hypothesis implies (with $y_i := x_{i+1}$ for $1 \leq i \leq n$)

$$\begin{aligned}
\|x_1+\cdots+x_{n+1}\| &\leq C_0 \cdot \max\{\|x_1\|, \|x_2+\cdots+x_{n+1}\|\} \\
&= C_0 \cdot \max\{\|x_1\|, \|y_1+\cdots+y_n\|\} \\
&\leq C_0 \cdot \max\left\{\|x_1\|, \max_{1\leq j\leq n} C_0^j \|y_j\|\right\} \\
&= \max\left\{C_0^1 \|x_1\|, \max_{1\leq j\leq n} C_0^{j+1} \|x_{j+1}\|\right\} \\
&= \max_{1\leq j\leq n+1} C_0^j \|x_j\| .
\end{aligned}$$

We observe $(0,\infty) = \biguplus_{k\in\mathbb{Z}} (C_0^{k-1}, C_0^k]$, since $C_0 \geq 2 > 1$. This shows that for each $x \in X$ with $\|x\| \neq 0$, there is a unique $k_x \in \mathbb{Z}$ with $\|x\| \in (C_0^{k_x-1}, C_0^{k_x}]$. If we define

$$N : X \to \{C_0^k \mid k \in \mathbb{Z}\} \cup \{0\}, x \mapsto \begin{cases} 0, & \text{if } \|x\| = 0, \\ C_0^{k_x}, & \text{if } \|x\| \neq 0, \end{cases}$$

*It is worth noting that the proof also works without positive definiteness of $\|\cdot\|$, since we write "if $\|x\| = 0$" instead of "if $x = 0$".*

we easily see

$$\frac{N(x)}{C_0} = C_0^{k_x-1} \leq \|x\| \leq C_0^{k_x} = N(x) \tag{2.1.4}$$

for each $x \in X$ with $\|x\| \neq 0$. For $\|x\| = 0$, the inequality is trivially true.

We furthermore observe that "$N$ is monotonic with respect to $\|\cdot\|$", i.e. $\|x\| \leq \|y\|$ implies $N(x) \leq N(y)$.

Below, we will show

$$\|x_1 + \cdots + x_n\| \leq C_0 \cdot \left[(N(x_1))^p + \cdots + (N(x_n))^p\right]^{1/p} \tag{2.1.5}$$

for all $x_1, \ldots, x_n \in X$. For arbitrary $x \in X$ and $x_1, \ldots, x_n \in X$ with $x = \sum_{i=1}^n x_i$, this yields (together with equation (2.1.4)) the estimate

$$\begin{aligned} \|x\| = \|x_1 + \cdots + x_n\| &\leq C_0 \cdot \left[(N(x_1))^p + \cdots + (N(x_n))^p\right]^{1/p} \\ &\leq C_0 \cdot \left[C_0^p \|x_1\|^p + \cdots + C_0^p \|x_n\|^p\right]^{1/p} \\ &= C_0^2 \cdot \left[\|x_1\|^p + \cdots + \|x_n\|^p\right]^{1/p} \end{aligned}$$

and hence $\|x\| \leq C_0^2 \cdot |||x|||$, so that $\|\cdot\|$ and $|||\cdot|||$ are indeed equivalent, since $|||x||| \leq \|x\|$ was established already at the beginning of the proof.

We will prove the estimate (2.1.5) by induction on $n \in \mathbb{N}$. The case $n = 1$ follows from (2.1.4), since $C_0 \geq 1$. Now assume that (2.1.5) is true for some $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$ and let $x_1, \ldots, x_{n+1} \in X$ be arbitrary. Both sides of (2.1.5) are invariant under permutations of the $(x_i)_{i\in\underline{n+1}}$, so we can assume $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_{n+1}\|$. Since $N$ is "monotonic with respect to $\|\cdot\|$", this yields $N(x_1) \geq \cdots \geq N(x_{n+1})$. Let us now distinguish three cases:

A. We have $\|x_n\| = 0$. Because of $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_{n+1}\|$, this yields $\|x_n\| = \|x_{n+1}\| = 0$. Using equation (2.1.2), we see $\|x_n + x_{n+1}\| = 0$ and hence $N(x_n + x_{n+1}) = 0$. Applying the induction hypothesis to $(y_i)_{i\in\underline{n}}$ with $y_i := x_i$ for $i \in \underline{n-1}$ and $y_n := x_n + x_{n+1}$, we deduce

$$\begin{aligned} &\|x_1 + \cdots + x_{n+1}\| = \|y_1 + \cdots + y_n\| \\ \leq &C_0 \left[(N(y_1))^p + \cdots + (N(y_n))^p\right]^{1/p} \end{aligned}$$

$$= C_0 \left[ (N(x_1))^p + \cdots + (N(x_{n-1}))^p + (N(x_n + x_{n+1}))^p \right]^{1/p}$$
$$= C_0 \left[ (N(x_1))^p + \cdots + (N(x_{n-1}))^p + (N(x_n))^p + (N(x_{n+1}))^p \right]^{1/p}.$$

B. We have $\|x_n\| > 0$ and all values $(N(x_i))_{i \in \underline{n+1}}$ are distinct, i.e. $N(x_1) > \cdots > N(x_{n+1})$. Let

$$M := \begin{cases} n, & \text{if } \|x_{n+1}\| = 0, \\ n+1, & \text{otherwise.} \end{cases}$$

Using of $\|x_i\| \geq \|x_M\| > 0$ for $i \in \underline{M}$, we get $C_0^{k_{x_1}} > \cdots > C_0^{k_{x_M}}$ and hence $k_{x_i} > k_{x_{i+1}}$ for all $i \in \underline{M-1}$. Because these are integer quantities, we see $k_{x_i} + i \leq k_{x_1} + 1$ for $i \in \underline{M}$.

Hence, equation (2.1.4) implies

$$\begin{aligned} C_0^i \cdot \|x_i\| \leq C_0^i \cdot N(x_i) = C_0^{i+k_{x_i}} \leq C_0^{k_{x_1}+1} &= C_0 \cdot N(x_1) \\ &\leq C_0 \cdot \left[ (N(x_1))^p + \cdots + (N(x_{n+1}))^p \right]^{1/p} \end{aligned}$$

for all $i \in \underline{M}$. In the case $M = n$, we have $\|x_{n+1}\| = 0$, so that $C_0^{n+1} \cdot \|x_{n+1}\| = 0 \leq C_0 \cdot \left[ (N(x_1))^p + \cdots + (N(x_{n+1}))^p \right]^{1/p}$ is trivial. Otherwise, $M = n+1$.

All in all, we have shown that

$$C_0^i \cdot \|x_i\| \leq C_0 \cdot \left[ (N(x_1))^p + \cdots + (N(x_{n+1}))^p \right]^{1/p}$$

holds for all $i \in \underline{n+1}$, so that estimate (2.1.3) yields

$$\begin{aligned} \|x_1 + \cdots + x_{n+1}\| &\leq \max_{1 \leq i \leq n+1} C_0^i \cdot \|x_i\| \\ &\leq C_0 \cdot \left[ (N(x_1))^p + \cdots + (N(x_{n+1}))^p \right]^{1/p}. \end{aligned}$$

C. We have $\|x_n\| > 0$ and there is $i \in \underline{n}$ with $N(x_i) = N(x_{i+1})$. Note that $N(x_i) \geq N(x_n) > 0$, which yields $N(x_{i+1}) > 0$ as well as $k_{x_i} = k_{x_{i+1}}$. Using equations (2.1.2) and (2.1.4), we derive

$$\begin{aligned} \|x_i + x_{i+1}\| &\leq C_0 \cdot \max\{\|x_i\|, \|x_{i+1}\|\} \\ &\leq C_0 \cdot \max\{N(x_i), N(x_{i+1})\} = C_0^{k_{x_i}+1} \end{aligned}$$

and hence

$$N(x_i + x_{i+1}) \leq C_0^{k_{x_i}+1} = N(x_i) + N(x_{i+1}).$$

Since $0 < p \leq 1$, we can use equation (2.1.1) to derive

$$(N(x_i + x_{i+1}))^p \leq (N(x_i))^p + (N(x_{i+1}))^p.$$

Applying the induction hypothesis to the family $(y_j)_{j\in\underline{n}}$ defined by $y_j := x_j$ for $j < i$, $y_j := x_i + x_{i+1}$ for $j = i$ and $y_j := x_{j+1}$ for $i+1 \leq j \leq n$, we finally see

$$\begin{aligned}\|x_1 + \cdots + x_{n+1}\| &= \|y_1 + \cdots + y_n\| \\ &\leq C_0 \cdot \left[(N(y_1))^p + \cdots + (N(y_n))^p\right]^{1/p} \\ &\leq C_0 \cdot \left[(N(x_1))^p + \cdots + (N(x_{n+1}))^p\right]^{1/p}.\end{aligned}$$

Thus, we have established the estimate (2.1.5) with $n+1$ instead of $n$ in all three cases. As observed above, this completes the proof. □

We can now show that a quasi-norm always defines a metrizable topology on the vector space $X$.

**Lemma 2.1.5.** *Let $(X, \|\cdot\|)$ be a quasi-normed space. We say that $U \subset X$ is* OPEN *if*

$$\forall x \in U\, \exists r > 0 :\ B_r(x) \subset U,$$

*where*

$$B_r(x) := B_r^{\|\cdot\|}(x) := \{y \in X \mid \|x - y\| < \varepsilon\}$$

*is the* BALL OF RADIUS $r$ AROUND $x$.

*Then*

$$\mathcal{T} := \{U \subset X \mid U \text{ open}\}$$

*is a topology on $X$, called the* TOPOLOGY INDUCED BY $\|\cdot\|$. *Equipped with this topology, $X$ is a topological vector space.*

*Equivalent quasi-norms induce the same topology. Finally, if $|||\cdot|||$ is an equivalent $p$-norm on $X$ for some $p \in (0,1]$, then*

$$d : X \times X \to [0,\infty),\ (x,y) \mapsto |||x - y|||^p$$

*is a (translation invariant) metric on $X$ which induces the topology $\mathcal{T}$.*

*In particular, since $|||x||| = [d(x,0)]^{1/p}$, any $p$-norm is continuous with respect to the topology it generates.*

*Remark.* The above lemma shows in particular that the topology of a quasi-normed space is induced by a translation invariant metric. If $(X, \|\cdot\|)$ is complete (cf. Definition 2.1.7), then so is $(X, |||\cdot|||)$, where $|||\cdot|||$ is an equivalent $p$-norm as given by Theorem 2.1.4. It is easy to see that this implies that $(X, d)$ with $d$ as above is a complete metric space.

Hence, each complete quasi-normed vector space $(X, \|\cdot\|)$ is an *F*-SPACE in the sense of [61, Section 1.8].

*Proof.* It is straightforward to verify that unions and finite intersections of open sets are again open and that $\varnothing, X$ are open sets, so that $\mathcal{T}$ is a topology.

If $C_1 \cdot \|x\| \leq |||x||| \leq C_2 \cdot \|x\|$ holds for all $x \in X$ with $C_1, C_2 > 0$, then

$$B^{\|\cdot\|}_{r/C_2}(x) \subset B^{|||\cdot|||}_r(x) \subset B^{\|\cdot\|}_{r/C_1}(x),$$

which implies that a subset $U \subset X$ is open with respect to $\|\cdot\|$ iff it is open with respect to $|||\cdot|||$.

If $|||\cdot|||$ is a $p$-norm, then $d$ satisfies $d(x,y) = |||x-y|||^p > 0$ for $x \neq y$ as well as $d(x,y) = |||x-y|||^p = |||y-x|||^p = d(y,x)$ for all $x, y \in X$. For the triangle inequality, we have

$$\begin{aligned} d(x,z) &= |||(x-y)+(y-z)|||^p \\ &\leq |||x-y|||^p + |||y-z|||^p = d(x,y) + d(y,z). \end{aligned}$$

To see that $d$ and $|||\cdot|||$ induce the same topologies, we observe

$$\begin{aligned} B^d_r(x) &= \left\{ y \in X \,\middle|\, |||x-y|||^p = d(x,y) < r \right\} \\ &= \left\{ y \in X \,\middle|\, |||x-y||| < r^{1/p} \right\} = B^{|||\cdot|||}_{r^{1/p}}(x). \end{aligned}$$

This shows that $|||\cdot|||$ and $d$ generate the same families of balls and hence also the same topologies.

Finally, we show that $(X, \mathcal{T})$ is a topological vector space. To see this, we may assume (by Theorem 2.1.4) that $\|\cdot\| = |||\cdot|||$ is a $p$-norm, so that $\mathcal{T}$ is induced by the metric $d$. Now, addition is continuous since $x_n \to x$ and $y_n \to y$ implies

$$\|(x_n + y_n) - (x+y)\| \leq C \cdot [\|x_n - x\| + \|y_n - y\|] \xrightarrow[n\to\infty]{} 0$$

and hence $d(x_n + y_n, x+y) \to 0$. Furthermore, if $\alpha_n \to \alpha$ and $x_n \to x$, then

$$\begin{aligned} \|\alpha_n x_n - \alpha x\| &\leq C \cdot [\|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|] \\ &= C \cdot [|\alpha_n| \, \|x_n - x\| + |\alpha_n - \alpha| \, \|x\|] \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

which yields $d(\alpha_n x_n, \alpha x) \to 0$. □

A standard result for *normed* vector spaces is the characterization of continuous linear maps as bounded linear maps. Since the ball $B^{\|\cdot\|_Y}_1(0)$ is in general not open if $Y$ is merely a quasi-normed vector space (see Remark 2.1.9), the usual proof can not be applied verbatim. Nevertheless, the equivalence is still true, as can be seen by changing to an equivalent $p$-norm.

**Lemma 2.1.6.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be quasi-normed vector spaces. A linear map $T : X \to Y$ is continuous if and only if it is bounded, i.e. if the following expression is finite:*

$$|||T|||_{X\to Y} := \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|Tx\|_Y.$$

*Proof.* Let $|||\cdot|||_X$ be an equivalent $p$-norm on $X$ and let $|||\cdot|||_Y$ be an equivalent $q$-norm on $Y$ for suitable $p, q \in (0,1]$. By Lemma 2.1.5 it follows that $|||\cdot|||_X, |||\cdot|||_Y$ are continuous.

"$\Rightarrow$": Since $U := B_1^{|||\cdot|||_Y}(0) \subset Y$ is open, and because $T$ is continuous with $T(0) = 0 \in U$, there is some $r > 0$ with $B_r(0) \subset T^{-1}(U)$. For $x \in X$ with $\|x\|_X \leq 1$, we have $\left\|\frac{r}{2}x\right\|_X < r$ and hence $T\left(\frac{r}{2}x\right) \in U$, i.e. $\frac{r}{2} \cdot \|Tx\|_Y \leq C \cdot \left|\left|\left|T\left(\frac{r}{2}x\right)\right|\right|\right|_Y < C$, where the constant $C > 0$ is independent of $x$ and comes from the equivalence of $\|\cdot\|_Y$ and $|||\cdot|||_Y$. Hence, $|||T|||_{X\to Y} \leq \frac{2C}{r} < \infty$.

"$\Leftarrow$": Assume that $|||T|||_{X\to Y}$ is finite. If $|||T|||_{X\to Y} = 0$, then $T \equiv 0$ is continuous. Hence, we can assume $|||T|||_{X\to Y} > 0$.

Since $\|\cdot\|_X$ and $|||\cdot|||_X$ are equivalent, there is some $C > 0$ with $\|x\|_X \leq C \cdot |||x|||_X$ for all $x \in X$. Choose $x \in X$ and let $U \subset Y$ be open with $Tx \in U$. Then there is some $r > 0$ with $B_r^{\|\cdot\|_Y}(Tx) \subset U$. For $x' \in B^{|||\cdot|||_X}_{r/(2C|||T|||_{X\to Y})}(x)$, we get

$$\left\|\frac{2\,|||T|||_{X\to Y}}{r}\left(x - x'\right)\right\|_X \leq \frac{2C\,|||T|||_{X\to Y}}{r} \cdot |||x - x'|||_X < 1$$

and hence

$$\begin{aligned}\left\|Tx - Tx'\right\|_Y &= \frac{r/2}{|||T|||_{X\to Y}} \cdot \left\|T\left(\frac{2\,|||T|||_{X\to Y}}{r}\left(x - x'\right)\right)\right\|_Y \\ &\leq \frac{r/2}{|||T|||_{X\to Y}} \cdot |||T|||_{X\to Y} = \frac{r}{2} < r,\end{aligned}$$

i.e. $Tx' \in U$. This shows that $T^{-1}U \supset B^{|||\cdot|||_X}_{r/(2C|||T|||_{X\to Y})}(x)$ is a neighborhood of $x$. Since $x \in T^{-1}U$ was arbitrary, $T^{-1}U \subset X$ is open. □

Up to now, we have only considered quasi-normed spaces. Quasi-Banach spaces are formally introduced in the following definition.

**Definition 2.1.7.** Let $(X, \|\cdot\|)$ be a quasi-normed space. A sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ is called a CAUCHY SEQUENCE, if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall n, m \geq N : \|x_n - x_m\| < \varepsilon.$$

We say that $(X, \|\cdot\|)$ is a QUASI-BANACH SPACE if each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ converges to some $x \in X$.

*Remark.* It is clear that the notion of a Cauchy sequence is invariant under change to an equivalent quasi-norm. Since the induced topologies for equivalent quasi-norms are the same (cf. Lemma 2.1.5), the notion of a sequence being convergent to some $x \in X$ is also not affected by switching to an equivalent quasi-norm.

Thus, if $\|\cdot\|, |||\cdot|||$ are two equivalent quasi-norms on a vector space $X$, then $(X, \|\cdot\|)$ is a Quasi-Banach space iff $(X, |||\cdot|||)$ is one.

As our final result in this section, we use the existence of an equivalent $p$-norm to derive a convenient criterion for completeness of a quasi-normed space in terms of (suitable) absolutely convergent series. This should be compared to the classical version for normed vector spaces, cf. [33, Theorem 5.1].

**Lemma 2.1.8.** *Let $(X, \|\cdot\|)$ be a quasi-normed vector space which admits an equivalent $p$-norm for some $p \in (0,1]$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of* positive *numbers satisfying*

$$\sum_{n\in\mathbb{N}} \alpha_n^p < \infty.$$

*Then $(X, \|\cdot\|)$ is complete if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ with $\|x_n\| \leq \alpha_n$ for all $n \in \mathbb{N}$ there is some $x \in X$ with*

$$\sum_{n=1}^{N} x_n \xrightarrow[N\to\infty]{} x.$$

*Remark.* We emphasize that absolute convergence – in the usual formulation – does *not* imply convergence for Quasi-Banach spaces. For example in $\ell^{1/2}(\mathbb{N})$, set $x_n := \frac{1}{n^2} \cdot \delta_n$. This implies

$$\sum_{n\in\mathbb{N}} \|x_n\|_{\ell^{1/2}} = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty,$$

but the only possible limit would be $x = \left(1/n^2\right)_{n\in\mathbb{N}}$ (since convergence in $\ell^{1/2}(\mathbb{N})$ implies coordinatewise convergence), which is *not* in $\ell^{1/2}(\mathbb{N})$.

The above lemma shows that the adequate formulation of "absolute convergence" in the setting of a ($p$-normable) Quasi-Banach space is absolute convergence to the $p$-th power.

*Proof.* Let $|||\cdot||| : X \to [0,\infty)$ be an equivalent $p$-norm on $X$.

"$\Rightarrow$": Since $(X, |||\cdot|||)$ is complete, it suffices to show that the sequence $(y_N)_{N\in\mathbb{N}} := \left(\sum_{n=1}^{N} x_n\right)_{N\in\mathbb{N}}$ is Cauchy.

To this end, let $M \geq N \geq N_0$ and estimate

$$\begin{aligned}
|||y_M - y_N|||^p &= \left|\left|\left| \sum_{n=N+1}^{M} x_n \right|\right|\right|^p \\
&\leq \sum_{n=N+1}^{M} |||x_n|||^p \\
&\leq C^p \cdot \sum_{n=N+1}^{M} \|x_n\|^p \\
&\leq C^p \cdot \sum_{n=N_0+1}^{\infty} \alpha_n^p \xrightarrow[N_0\to\infty]{} 0.
\end{aligned}$$

Because of $p > 0$, this yields $|||y_M - y_N||| \xrightarrow[N_0 \to \infty]{} 0$, so that $(y_N)_{N \in \mathbb{N}}$ is indeed Cauchy and hence convergent to some $x \in X$.

"$\Leftarrow$": Let $(y_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $X$. It suffices to show that some subsequence $(y_{n_k})_{k \in \mathbb{N}}$ converges to some $y \in X$, since this implies

$$|||y_m - y|||^p \leq |||y_m - y_{n_m}|||^p + |||y_{n_m} - y|||^p \xrightarrow[m \to \infty]{} 0.$$

The first term vanishes for $m \to \infty$ because $(y_n)_{n \in \mathbb{N}}$ is Cauchy.

Since $(y_n)_{n \in \mathbb{N}}$ is Cauchy, we can construct a strictly increasing index sequence $(n_k)_{k \in \mathbb{N}}$ with $\|y_n - y_m\| \leq \alpha_k$ for all $n, m \geq n_k$. In particular, if we set $x_k := y_{n_{k+1}} - y_{n_k}$, then

$$\|x_k\| = \left\|y_{n_k} - y_{n_{k+1}}\right\| \leq \alpha_k \text{ since } n_{k+1} \geq n_k.$$

By assumption, there is some $x \in X$ satisfying

$$y_{n_{N+1}} - y_{n_1} = \sum_{k=1}^{N} \left(y_{n_{k+1}} - y_{n_k}\right) = \sum_{k=1}^{N} x_k \xrightarrow[N \to \infty]{} x.$$

This shows that the subsequence $(y_{n_{N+1}})_{N \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ converges to $x + y_{n_1} \in X$. As observed above, this suffices. □

### 2.1.2 *Pathological examples of Quasi-Banach spaces* ∗

In view of Lemma 2.1.5, which shows that quasi-normed spaces are still well-behaved as topological vector spaces, one could wonder why one usually insists on considering norms instead of quasi-norms. To answer this question, we recall from the (topological) theory of normed vector spaces that the triangle inequality is used to prove that the "open" balls $B_r(x)$ are indeed open. We now show that this can fail for quasi-norms.

*Remark* 2.1.9. We remark that in general, the ball $B_r(x)$ is *not* open with respect to the topology $\mathcal{T}$. In particular, the quasi-norm $\|\cdot\|$ is in general *not* continuous. One can even construct pathological examples where $B_r(x)$ is not Borel measurable with respect to the topology $\mathcal{T}$, so that $\|\cdot\|$ is not even Borel measurable:

The idea for the following construction is that *any* homogeneous function $g : X \to [0, \infty)$ which is equivalent to a (quasi-)norm in the sense

$$C_0 \cdot g(x) \leq \|x\| \leq C_1 \cdot g(x)$$

is again a quasi-norm because of

$$\begin{aligned} g(x+y) &\leq C_0^{-1} \|x+y\| \leq C C_0^{-1} [\|x\| + \|y\|] \\ &\leq C_1 C C_0^{-1} [g(x) + g(y)]. \end{aligned}$$

This will give us enough freedom to construct a nonmeasurable quasi-norm. For the precise construction, let

$$M_0 \subset \partial B_1(0) \cap \left[\mathbb{R}^{d-1} \times (0,\infty)\right]$$

be an arbitrary, non Borel-measurable subset (which is possible precisely for $d > 1$) and let $M := M_0 \uplus (-M_0)$. Using the identity $M_0 = M \cap \left[\mathbb{R}^{d-1} \times (0,\infty)\right]$, we see that $M$ is not Borel-measurable.

Now, let

$$f : X \to \{0,1\}, x \mapsto \begin{cases} \chi_M\left(\frac{x}{|x|}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

as well as

$$\|x\| := (1 + f(x)) \cdot |x| \text{ for } x \in \mathbb{R}^d.$$

This implies $|x| \leq \|x\| \leq 2|x|$ for all $x \in \mathbb{R}^d$ and thus $\|x\| = 0$ iff $x = 0$. From the general remark above, we see that $\|\cdot\|$ satisfies the quasi triangle inequality.

Finally, we clearly have $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha = 0$ or $x = 0$. For $\alpha \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^d \setminus \{0\}$, we observe $M = -M$, which implies $\chi_M(-x) = \chi_M(x)$. Hence,

$$\begin{aligned} \|\alpha x\| &= \left(1 + \chi_M\left(\frac{\alpha x}{|\alpha x|}\right)\right)|\alpha x| \\ &= |\alpha| \left(1 + \chi_M\left(\frac{x}{|x|}\right)\right)|x| = |\alpha| \|x\|. \end{aligned}$$

All in all, we have shown that $\|\cdot\|$ is a quasi-norm.

But for $x \in \mathbb{R}^d$ we have the equivalence

$$\|x\| = 2 \iff (|x| = 2 \text{ and } f(x) = 0) \text{ or } (|x| = 1 \text{ and } f(x) = 1),$$

which implies

$$\begin{aligned} &\left\{x \in \mathbb{R}^d \,\middle|\, \|x\| = 2\right\} \cap \partial B_1(0) \\ = &\left\{x \in \partial B_1(0) \,\middle|\, 1 = f(x) = \chi_M\left(\frac{x}{|x|}\right) = \chi_M(x)\right\} = M. \end{aligned}$$

This shows that $\left\{x \in \mathbb{R}^d \,\middle|\, \|x\| = 2\right\}$ is not Borel-measurable, which also implies that

$$\|\cdot\|^{-1}((-\infty, r)) = B_r^{\|\cdot\|}(0)$$

is not measurable for some (and then all) $r > 0$. ◀

It is important to note that the effect observed in the remark above simply results from having chosen the "wrong" quasi-norm. If we

switch to an equivalent $p$-norm $|||\cdot|||$ on $X$, then $|||\cdot|||$ *is* continuous, as seen in Lemma 2.1.5.

The following example shows that there are other huge differences between norms and quasi-norms. In fact, the Hahn-Banach theorem can fail completely for quasi-normed spaces. This implies that quasi-normed spaces (in contrast to normed spaces) are in general *not* LOCALLY CONVEX (cf. [61, Definition 1.8 and the corollary to Theorem 3.4]). This pathological behaviour can *not* be removed by simply choosing a different (equivalent) quasi-norm.

**Example 2.1.10.** (cf. [42, Theorem 1.4.1(i)]) Let $0 < p < 1$ and assume that $\varphi : L^p(\mathbb{R}) \to \mathbb{C}$ is a continuous linear functional. Then $\varphi \equiv 0$.

To see this, it suffices to show $\varphi(\chi_A) = 0$ for each set $A \subset \mathbb{R}$ of finite Lebesgue measure $0 < \lambda(A) < \infty$, since simple functions with "steps" of finite measure are dense in $L^p(\mathbb{R})$.

*We remark that one can show (cf. [42, Theorem 1.4.1(i)]) that the dual space of $L^p(\mu)$ for $0 < p < 1$ is trivial for any $\sigma$-finite,* NON-ATOMIC *measure $\mu$. The case of the Lebesgue measure $\mu = \lambda$ considered here is a special case of this general result.*

But for such a set $A$, the map $f : \mathbb{R} \to [0, \mu(A)], t \mapsto \lambda(A \cap (-\infty, t))$ is nondecreasing and (Lipschitz-)continuous with $f(x) \xrightarrow[x\to\infty]{} \mu(A)$ and $f(x) \xrightarrow[x\to-\infty]{} 0$. By the intermediate value theorem, we deduce that there is some $B \subset A$ with $\lambda(B) = \lambda(A)/2$. Thus,

$$A = B \uplus (A \setminus B)$$

with $\lambda(B) = \lambda(A)/2 = \lambda(A \setminus B)$.

Inductively, we can write $A = \biguplus_{m=1}^{2^n} A_m^{(n)}$ with $\lambda(A_m^{(n)}) = \lambda(A)/2^n$ for each $n \in \mathbb{N}$ and $m \in \underline{2^n}$. By linearity and boundedness of $\varphi$, we get

$$\begin{aligned} |\varphi(\chi_A)| &\leq \sum_{m=1}^{2^n} |\varphi(\chi_{A_m^{(n)}})| \leq C \cdot \sum_{m=1}^{2^n} \left\|\chi_{A_m^{(n)}}\right\|_{L^p(\mathbb{R})} \\ &= C \cdot \sum_{m=1}^{2^n} [\mu(A_m^{(n)})]^{1/p} \\ &= C \cdot [\lambda(A)]^{1/p} \cdot \frac{2^n}{2^{n/p}} \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

because of $p < 1$.

It is important to observe that we used the fact that the measure space $(\mathbb{R}, \lambda)$ is NONATOMIC. For atomic measures, the dual space of $L^p$ need not be trivial. For example, $\ell^p(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$ holds for $0 < p \leq 1$, so that each sequence $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ defines a bounded linear functional on $\ell^p(\mathbb{N})$. ◀

## 2.2 SOLID QUASI-BANACH SPACES OF FUNCTIONS

In this section, we study SOLID spaces of (measurable) functions. These spaces can be seen as generalizations of the usual $L^p$ spaces.

The section is structured as follows: In the first subsection, we introduce the definition and discuss elementary properties. The next sub-

section is concerned with the connection between pointwise convergence and convergence in the solid function space. In the third subsection, we use these properties to characterize the completeness of solid function spaces using (a variant of) the RIESZ-FISCHER PROPERTY. After that, we specialize from general measure spaces to the setting of (locally compact) groups, which will allow us to discuss translation invariance of (weighted) solid spaces. Most of these properties will be useful in the next section for studying WIENER AMALGAM SPACES.

In the last subsection, which is not required for understanding the remainder of the thesis, we discuss some (pathological) examples of solid function spaces. In particular, we give an example of a solid function space for which the completion is not a solid function space and we consider several ways in which the FATOU PROPERTY can fail.

### 2.2.1 *Definition and basic properties*

Solidity of a space/norm intuitively expresses that (the norm on) the space is based only on the magnitude of the values $f(x)$ for each individual $x \in X$, but not on properties like smoothness which depend on the relations between the values $f(x)$ for different "locations" $x \in X$. The precise definition is as follows:

**Definition 2.2.1.** (cf. [28, §2.3]) Let $(X, \mathcal{A}, \mu)$ a measure space. Let

$$\mathcal{L}^0(\mu) := \{f : X \to \mathbb{C} \mid f \text{ measurable}\}$$

and

$$\mathcal{N}(\mu) := \left\{f \in \mathcal{L}^0(\mu) \,\middle|\, f \equiv 0 \text{ almost everywhere}\right\} \leq \mathcal{L}^0(\mu),$$

as well as $L^0(\mu) := \mathcal{L}^0(\mu)/\mathcal{N}(\mu)$. By the usual abuse of notation, we interpret $L^0(\mu)$ as the space of all measurable complex valued functions on $X$, where two functions are identified if they coincide almost everywhere.

We say that a quasi-normed space $(Y, \|\cdot\|_Y)$ is a FUNCTION SPACE on $X$ if $Y$ is a subspace of $L^0(\mu)$. A function space $(Y, \|\cdot\|_Y)$ is called SOLID if for each measurable function $f : X \to \mathbb{C}$ which satisfies $|f| \leq |g|$ almost everywhere for some $g \in Y$, we already have $f \in Y$ with $\|f\|_Y \leq \|g\|_Y$.

Any function space $(Y, \|\cdot\|_Y)$ which is also complete is called a QUASI-BANACH FUNCTION SPACE (QBF SPACE). If the function space $(Y, \|\cdot\|_Y)$ is even a Banach space, we call it a BANACH FUNCTION SPACE (BF SPACE).

*Remark* 2.2.2. We remark that in the usual definitions (for example in [28, §2.3]), it is assumed that $Y$ embeds continuously into $L^1_{\text{loc}}(\mu)$.

For a general measure space, this means that for each $E \in \mathcal{A}$ of finite measure, we have $\|f \cdot \chi_E\|_{L^1} \leq C_E \cdot \|f\|_Y$ for all $f \in Y$ with a suitable constant $C_E > 0$. In the context of a locally compact group, this definition is sometimes changed so that $\|f \cdot \chi_K\|_{L^1} \leq C_K \cdot \|f\|_Y$ is only required for each compact set $K \subset G$ instead of for all $E \subset G$ of finite measure.

Since our main application will be to consider (weighted) $L^p$-spaces, also for $0 < p < 1$, we can not impose this requirement, since for example $L^p(\mathbb{R}) \not\subseteq L^1_{\text{loc}}(\mathbb{R})$, as can be seen by considering the function $x \mapsto x^{-\frac{1}{p+\varepsilon}} \cdot \chi_{(0,1)}$ for $\varepsilon \in (0, 1-p)$.

Instead, one could require $Y \hookrightarrow L^p_{\text{loc}}(X)$ for *some* $p \in (0,1]$. But we will see below (cf. Lemma 2.2.8) that each solid function space $Y$ automatically embeds into $L^0(\mu)$, equipped with the topology of (local) convergence in measure. This will suffice for our purposes, so that we will not require any further embedding properties.

We finally remark that for measurable $f, g : X \to \mathbb{C}$ with $|f| = |g|$, we have $f \in Y$ iff $g \in Y$ and in this case $\|f\|_Y = \|g\|_Y$. In particular, $f \in Y$ iff $|f| \in Y$, in which case $\|f\|_Y = \||f|\|_Y$ is true. ◀

Most of the theory of solid BF spaces was developed by Luxemburg and Zaanen, see in particular [69, Chapter 15]. Zaanen chooses as a starting point a FUNCTION NORM $\varrho : L^0_+(\mu) \to [0,\infty]$ which is defined on the set of *all* nonnegative measurable functions and is required to be positive definite, homogeneous with respect to nonnegative scalars and subadditive, as well as monotonic, i.e. $\varrho(f) \leq \varrho(g)$ is required if $0 \leq f \leq g$ holds almost everywhere. Zaanen then defines the space

$$L_\varrho = \left\{ f \in L^0(\mu) \,\middle|\, \varrho(|f|) < \infty \right\}$$

which is easily seen to be a solid (normed) function space in our terminology.

Conversely, if one starts with a solid function space $(Y, \|\cdot\|_Y)$ in which $\|\cdot\|_Y$ is a norm (instead of a quasi-norm), and defines

$$\varrho : \left\{ f \in L^0(\mu) \,\middle|\, f \geq 0 \right\} \to [0,\infty], f \mapsto \begin{cases} \|f\|_Y, & \text{if } f \in Y, \\ \infty, & \text{otherwise,} \end{cases}$$

then $\varrho$ is a function norm in the sense of Zaanen and one can easily check $L_\varrho = Y$. Thus, our definition and the definition used by Zaanen are equivalent, as long as one restricts to norms instead of quasi-norms.

Most of the results in this section are probably well-known (at least for Banach spaces instead of Quasi-Banach spaces) and thus no originality is claimed, although I am not aware of literature for those statements for which no source is indicated. In particular, this is true for Corollary 2.2.12.

Important examples of solid function spaces are given by (weighted) Lebesgue spaces. The purpose of the following definition is mainly to fix notation, i.e. to clarify if the weight is multiplied with the individual functions or with the measure $\mu$.

*We remark that both conventions for placing the weight are commonly used. For example, Feichtinger[28, §2.3] uses the convention of multiplying the individual functions with the weight, whereas Bergh and Löfström prefer to change the measure[2, Theoren 5.4.1].*

**Definition 2.2.3.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Each measurable function $w : X \to (0, \infty)$ is called a WEIGHT (FUNCTION) on $X$.

If $w : X \to (0, \infty)$ is a weight and if $0 < p \leq \infty$, we define the WEIGHTED LEBESGUE SPACE $L^p_w(\mu)$ by

$$L^p_w(\mu) := \left\{ f \in L^0(\mu) \,\middle|\, w \cdot f \in L^p(\mu) \right\}.$$

We equip this space with the (quasi) norm

$$\|f\|_{L^p_w(\mu)} := \|f \cdot w\|_{L^p(\mu)} \qquad \text{for } f \in L^p_w(\mu).$$

*Remark.* It is easy to verify that the spaces $L^p_w(\mu)$ are Quasi-Banach spaces as well as solid function spaces.

One special case of function spaces are SEQUENCE SPACES. These correspond to the case when $\mu$ is the counting measure on $X$. For these, we have the following characterization of the density of the space of finitely supported sequences:

UNCONDITIONAL CONVERGENCE *of a series $\sum_{i \in I} x_i$ with $x_i \in X$ for some quasi-normed vector space $(X, \|\cdot\|)$ means that there is some $x \in X$ such that for each $\varepsilon > 0$, there is a* finite *subset $I_\varepsilon \subset I$ satisfying $\left\| x - \sum_{i \in J} x_i \right\| < \varepsilon$ for each finite subset $J \subset I$ with $I_\varepsilon \subset J$.*

**Lemma 2.2.4.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, where $\mu$ is the counting measure on $X$ with $\mathcal{A} = \mathcal{P}(X)$. A (solid) function space $(Y, \|\cdot\|_Y)$ on $X$ is called a (solid)* (DISCRETE) SEQUENCE SPACE OVER $X$.

*If $(Y, \|\cdot\|_Y)$ is a solid sequence space, then the space $\langle \delta_x \mid x \in X \rangle \cap Y$ of finitely supported sequences in $Y$ is dense in $Y$ if and only if*

$$f = \sum_{x \in X} f(x) \cdot \delta_x$$

*holds for every $f \in Y$, with unconditional convergence of the series.*

*Remark.* We observe that $g := \sum_{x \in X_0} f(x) \cdot \delta_x \in \langle \delta_x \mid x \in X \rangle \cap Y$ holds for every $f \in Y$ and every finite subset $X_0 \subset X$, since we have $|g(x)| = |f(x)| \cdot \chi_{X_0}(x) \leq |f(x)|$ for all $x \in X$, which implies $g \in Y$ by solidity of $Y$.

*Proof.* "⇒": Let $f \in Y$ be arbitrary and $\varepsilon > 0$. Since $\langle \delta_x \mid x \in X \rangle \cap Y$ is dense in $Y$, there is some finitely supported sequence $g \in Y$ with $\|f - g\|_Y < \varepsilon$. Let $X_\varepsilon := \operatorname{supp} g$. For each finite subset $X_0 \subset X$ with $X_\varepsilon \subset X_0 \subset X$, let $f_{X_0} := f \cdot \chi_{X_0} = \sum_{x \in X_0} f(x) \delta_x$. For arbitrary $x \in X$, this implies

$$|f_{X_0}(x) - f(x)| = \begin{cases} 0, & \text{if } x \in X_0, \\ |f(x)|, & \text{if } x \notin X_0 \end{cases}$$

$$\overset{(*)}{\leq} \begin{cases} 0, & \text{if } x \in X_\varepsilon, \\ |f(x)| = |f(x) - g(x)|, & \text{if } x \notin X_\varepsilon \end{cases}$$
$$\leq |f(x) - g(x)|.$$

In the step marked with $(*)$, we used the inclusion $X_\varepsilon \subset X_0$. Since $Y$ is solid, we get $f_{X_0} - f \in Y$ with $\|f_{X_0} - f\|_Y \leq \|f - g\|_Y < \varepsilon$. Because $X_0 \subset X$ was an arbitrary finite subset with $X_\varepsilon \subset X_0 \subset X$, this proves the desired convergence.

"$\Leftarrow$": Let $f \in Y$ and $\varepsilon > 0$. By assumption, $f = \sum_{x \in X} f(x) \cdot \delta_x$ with unconditional convergence. Thus, there is some finite set $X_\varepsilon \subset X$ with $\left\| f - \sum_{x \in X_\varepsilon} f(x) \cdot \delta_x \right\|_Y < \varepsilon$. Since $\sum_{x \in X_\varepsilon} f(x) \cdot \delta_x \in Y \cap \langle \delta_x \mid x \in X \rangle$ by the remark above, this establishes the desired density. □

*A measure space $(X, \mathcal{A}, \mu)$ is called* SEMIFINITE *if for every $E \in \mathcal{A}$ with $\mu(E) = \infty$, there is some $F \in \mathcal{A}$ with $F \subset E$ and $0 < \mu(F) < \infty$, cf. [33, Section 1.3]. Every $\sigma$-finite measure space $(X, \mathcal{A}, \mu)$ is also semifinite, because if $X = \bigcup_{n \in \mathbb{N}} X_n$ with $\mu(X_n) < \infty$ for each $n$ and $E \in \mathcal{A}$ with $\mu(E) = \infty$, then there is some $n \in \mathbb{N}$ with $0 < \mu(E \cap X_n) \leq \mu(X_n) < \infty$, since otherwise $E = \bigcup_{n \in \mathbb{N}} (E \cap X_n)$ would be a null-set.*

For the usual $L^p$-spaces, it is well-known that each $L^p$ Cauchy sequence admits a subsequence which converges almost everywhere (to the $L^p$-limit of the sequence). Our next goal is to show (at least for SEMIFINITE measure spaces) that the same is true in general for solid function spaces. This may be somewhat surprising since we do not impose the requirement $Y \hookrightarrow L^1_{\text{loc}}(X)$ or something similar.

### 2.2.2 *Pointwise convergence and convergence in solid function spaces*

We begin our analysis with the following variant of Egoroff's theorem:

**Theorem 2.2.5** (Egoroff's theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $E \in \mathcal{A}$ with $\mu(E) < \infty$. Furthermore, let $f_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$ and $f : X \to \mathbb{C}$ be measurable functions. Then the following are true:*

1. *If $f_n(x) \xrightarrow[n \to \infty]{} f(x)$ holds for almost all $x \in E$, then the convergence is* almost uniform*. This means that for each $\varepsilon > 0$, there is some $E_\varepsilon \in \mathcal{A}$ with $\mu(E_\varepsilon) < \varepsilon$ such that $f_n \to f$ uniformly on $E \setminus E_\varepsilon$.*
2. *If $|f_n(x)| \xrightarrow[n \to \infty]{} \infty$ holds for almost all $x \in E$, then the divergence is almost uniform. This means that for each $\varepsilon > 0$ there is some $E_\varepsilon \in \mathcal{A}$ with $\mu(E_\varepsilon) < \varepsilon$ such that for each $M \in \mathbb{N}$ there is some $n_M \in \mathbb{N}$ with $|f_n(x)| \geq M$ for all $x \in E \setminus E_\varepsilon$ and all $n \geq n_M$.*

*Proof.* The first part is well-known, see e.g. [33, Theorem 2.33]. We now deduce the second part from the first. To this end, let

$$\varphi : [0, \infty) \to [0, 1), x \mapsto \frac{x}{1 + x}.$$

Then $\varphi$ is continuous with $\varphi'(x) = \frac{1}{(1+x)^2} > 0$, so that $\varphi$ is strictly increasing. It is easy to see $\varphi(x) \xrightarrow[x \to \infty]{} 1$.

Hence, the modified sequence $g_n : X \to \mathbb{C}, g_n := \varphi \circ |f_n|$ is a sequence of measurable function satisfying $g_n(x) \to g(x)$ for almost all $x \in E$ with $g \equiv 1$.

By the first part, we see that for each $\varepsilon > 0$, there is some $E_\varepsilon \in \mathcal{A}$ with $\mu(E_\varepsilon) < \varepsilon$ such that the convergence $g_n \to g$ is uniform on $E \setminus E_\varepsilon$. For arbitrary $M \in \mathbb{N}$, this implies the existence of some $n_M \in \mathbb{N}$ with

$$1 - g_n(x) \leq |1 - g_n(x)| < 1 - \varphi(M) \qquad \text{for } n \geq n_M \text{ and } x \in E \setminus E_\varepsilon.$$

But this means $\varphi(M) \leq g_n(x) = \varphi(|f_n(x)|)$ and thus $|f_n(x)| \geq M$ for $n \geq n_M$ and $x \in E \setminus E_\varepsilon$, since $\varphi$ is strictly increasing. □

As a consequence of this version of Egoroff's theorem, we deduce a first version of the desired pointwise convergence result, for "absolutely convergent" sequences. A similar statement is given (without proof) in [69, Exercise 64.1], but only under an additional completeness assumption.

**Lemma 2.2.6.** *Let $(Y, \|\cdot\|_Y)$ be a solid function space on a measure space $(X, \mathcal{A}, \mu)$. Let $p \in (0, 1]$ such that $Y$ admits an equivalent $p$-norm. Finally, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $Y$ with $\sum_{n=1}^{\infty} \|f_n\|_Y^p < \infty$.*

*Then, for each set $E \in \mathcal{A}$ of finite measure, $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ holds for almost all $x \in E$. If $(X, \mathcal{A}, \mu)$ is semifinite, then $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ holds for almost all $x \in X$.*

*Proof.* Let $E \in \mathcal{A}$ with $\mu(E) < \infty$ and let $|||\cdot|||$ be an equivalent $p$-norm on $Y$. Since each function $f_n$ is measurable, the same is true of $\sum_{n=1}^{\infty} |f_n| : X \to [0, \infty]$. Thus,

$$F := \{x \in E \,|\, \textstyle\sum_{n=1}^{\infty} |f_n(x)| = \infty\}$$

is measurable with $\mu(F) \leq \mu(E) < \infty$. Let us assume towards a contradiction that the statement of the lemma fails for the set $E$. Hence, $\mu(F) > 0$.

Set $\varepsilon := \frac{\mu(F)}{2} \in (0, \infty)$. The above variant of Egoroff's theorem (Theorem 2.2.5) yields a subset $G \subset F$ with $\mu(G) < \varepsilon$ such that the divergence $\sum_{n=1}^{N} |f_n| \to \infty$ is uniform on $M := F \setminus G$. Note that $\mu(M) \geq \mu(F) - \mu(G) > \frac{\mu(F)}{2} > 0$.

For arbitrary $L > 0$, there is some $N_L \in \mathbb{N}$ with $\sum_{n=1}^{N_L} |f_n| \geq L$ on $M$, i.e. with $\frac{1}{L} \sum_{n=1}^{N_L} |f_n| \geq \chi_M$. By solidity of $Y$, this yields $\chi_M \in Y$ with

$$\begin{aligned}
\|\chi_M\|_Y^p &\leq \frac{1}{L^p} \cdot \left\| \sum_{n=1}^{N_L} |f_n| \right\|_Y^p \\
&\leq \frac{C^p}{L^p} \cdot \left| \left| \left| \sum_{n=1}^{N_L} |f_n| \right| \right| \right|^p \\
&\leq \frac{C^p}{L^p} \cdot \sum_{n=1}^{N_L} |||f_n|||^p
\end{aligned}$$

$$\leq \frac{C^{2p}}{L^p} \cdot \sum_{n=1}^{N_L} |||f_n|||_Y^p$$
$$\leq \frac{C^{2p}}{L^p} \cdot \sum_{n=1}^{\infty} \|f_n\|_Y^p \xrightarrow[L\to\infty]{} 0,$$

where $C > 0$ is chosen such that $\|\cdot\| \leq C \cdot |||\cdot||| \leq C^2 \cdot \|\cdot\|$.

Thus, $\chi_M = 0$ as an element of $Y$. Because of $Y \leq L^0(\mu)$, this means $\chi_M = 0$ almost everywhere and thus $\mu(M) = 0$, a contradiction.

Finally, assume that $(X, \mathcal{A}, \mu)$ is semifinite. As above, we see that the set

$$N := \{x \in X \mid \textstyle\sum_{n=1}^{\infty} |f_n(x)| = \infty\}$$

is measurable. We want to show $\mu(N) = 0$. The case $0 < \mu(N) < \infty$ is excluded by the former part of the proof. Hence, if $N$ is *not* a null-set, then $\mu(N) = \infty$. But since $(X, \mathcal{A}, \mu)$ is semifinite, this yields some $E \subset N$ with $0 < \mu(E) < \infty$, which contradicts the former part of the proof. □

As a corollary, we see that every Cauchy-sequence in a solid function space admits a subsequence which converges "locally almost everywhere".

**Corollary 2.2.7.** *Let $(Y, \|\cdot\|_Y)$ be a solid function space on a measure space $(X, \mathcal{A}, \mu)$ and assume that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Y$.*

*Then there is a measurable function $f : X \to \mathbb{C}$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges almost everywhere to $f$ on sets of finite measure.*

*This means that for each $E \in \mathcal{A}$ of finite measure, $f_{n_k}(x) \xrightarrow[k\to\infty]{} f(x)$ holds for almost all $x \in E$.*

*If $(X, \mathcal{A}, \mu)$ is semifinite, the convergence holds almost everywhere.*

*Proof.* By Theorem 2.1.4, there is some $p \in (0, 1]$, such that $Y$ admits an equivalent $p$-norm. Since $(f_n)_{n\in\mathbb{N}}$ is Cauchy, we can inductively choose a strictly increasing index sequence $(n_k)_{k\in\mathbb{N}}$ with

$$\|f_n - f_\ell\|_Y \leq 2^{-k} \text{ for all } n, \ell \geq n_k.$$

In particular, $\|f_{n_{k+1}} - f_{n_k}\|_Y \leq 2^{-k}$ for all $k \in \mathbb{N}$, because of $n_{k+1} \geq n_k$. For

$$g_k := f_{n_{k+1}} - f_{n_k},$$

this yields $\sum_{k=1}^{\infty} \|g_k\|_Y^p \leq \sum_{k=1}^{\infty} 2^{-kp} < \infty$, so that Lemma 2.2.6 is applicable.

Since each function $g_k : X \to \mathbb{C}$ is measurable, we see that also $g : X \to [0, \infty], x \mapsto \sum_{k=1}^{\infty} |g_k(x)|$ and

$$N := \{x \in X \mid \textstyle\sum_{k=1}^{\infty} |g_k(x)| = \infty\}$$

are measurable. The series $\sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| = \sum_{k=1}^{\infty} |g_k(x)|$ is (absolutely) convergent on $N^c$, so that also the sequence

$$f_{n_{M+1}}(x) = f_{n_1}(x) + \sum_{k=1}^{M} \left( f_{n_{k+1}}(x) - f_{n_k}(x) \right)$$

is convergent on $N^c$. Thus,

$$f : X \to \mathbb{C}, x \mapsto \begin{cases} \lim_{M\to\infty} f_{n_{M+1}}(x), & \text{if } x \in N^c, \\ 0, & \text{if } x \in N \end{cases}$$

is well-defined and measurable.

Finally, Lemma 2.2.6 shows that $N \cap E$ is a null-set for each $E \subset X$ of finite measure. But we have $f_{n_{N+1}}(x) \xrightarrow[N\to\infty]{} f(x)$ on $E \setminus (N \cap E)$ and hence almost everywhere on $E$. If $(X, \mathcal{A}, \mu)$ is semifinite, then Lemma 2.2.6 even yields $\mu(N) = 0$ and hence $f_{n_{N+1}}(x) \xrightarrow[N\to\infty]{} f(x)$ for all $x \in X \setminus N$, i.e. almost everywhere. □

Our results up to now imply that each Cauchy sequence in a solid function space admits a subsequence which converges (locally) almost everywhere. What is still missing is the proof that the two convergences $f_n \xrightarrow[n\to\infty]{\|\cdot\|_Y} f$ and $f_n \xrightarrow[n\to\infty]{\text{a.e.}} g$ together imply $f = g$ almost everywhere. This is (at least in the semifinite case) a consequence of the following lemma, as we will see in the ensuing corollary.

**Lemma 2.2.8.** *Let $(Y, \|\cdot\|_Y)$ be a solid function space on a measure space $(X, \mathcal{A}, \mu)$. Then $Y$ embeds continuously into the space of measurable functions $L^0(\mu)$, equipped with the topology of local convergence in measure.*

*More precisely, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $Y$ with $f_n \xrightarrow[n\to\infty]{\|\cdot\|_Y} f$, then*

$$\mu\left(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}\right) \xrightarrow[n\to\infty]{} 0$$

*holds for each set $E \subset X$ of finite measure and each $\varepsilon > 0$.*

*We remark that it suffices to consider sequences in the proof of the lemma, since the topology on $(Y, \|\cdot\|_Y)$ is first countable, cf. Lemma 2.1.5. For this, it does not matter whether the topology of local convergence in measure is first countable or not.*

*Proof.* Assume towards a contradiction that the claim is false. This yields some set $E \subset X$ of finite measure, some $\varepsilon > 0$, as well as a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and some $\delta > 0$ with

$$\mu\left(\{x \in E \mid |f_{n_k}(x) - f(x)| > \varepsilon\}\right) > \delta$$

for all $k \in \mathbb{N}$.

Let us set $g_k := f_{n_k} - f \in Y$ as well as

$$M_k := \{x \in E \mid |g_k(x)| > \varepsilon\} = \{x \in E \mid |f_{n_k}(x) - f(x)| > \varepsilon\}.$$

It is easy to see $\varepsilon \cdot \chi_{M_k} \leq |g_k|$. Since $Y$ is solid, we conclude

$$\varepsilon \cdot \|\chi_{M_k}\|_Y \leq \|g_k\|_Y \xrightarrow[k\to\infty]{} 0.$$

In particular, $(\chi_{M_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $Y$.

Hence, Corollary 2.2.7 yields a subsequence $(\chi_{M_{k_\ell}})_{\ell\in\mathbb{N}}$ which converges locally almost everywhere (and hence almost everywhere on $E$) to some measurable function $f : X \to \mathbb{C}$. Egoroff's theorem (Theorem 2.2.5) yields a subset $F \subset E$ with $\mu(F) < \frac{\delta}{2}$ such that $\chi_{M_{k_\ell}}$ converges uniformly to $f$ on $E \setminus F$. Since the sequence $(\chi_{M_{k_\ell}}(x))_{\ell\in\mathbb{N}}$ is $\{0,1\}$-valued for every $x \in X$, we see that there is some $N \in \mathbb{N}$ with $\chi_{M_{k_\ell}}|_{E\setminus F} = f|_{E\setminus F}$ for all $\ell \geq N$.

Finally, this implies

$$\left\|\chi_{M_{k_N}\cap(E\setminus F)}\right\|_Y = \left\|\chi_{M_{k_\ell}\cap(E\setminus F)}\right\|_Y \leq \left\|\chi_{M_{k_\ell}}\right\|_Y \xrightarrow[\ell\to\infty]{} 0$$

and hence $\chi_{M_{k_N}\cap(E\setminus F)} \equiv 0$ almost everywhere. But this means

$$\begin{aligned} 0 &= \mu(M_{k_N} \cap (E \setminus F)) = \mu(M_{k_N} \setminus F) \\ &\geq \mu(M_{k_N}) - \mu(F) > \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0, \end{aligned}$$

a contradiction. □

The following corollary summarizes our results.

**Corollary 2.2.9.** *Let $(Y, \|\cdot\|_Y)$ be a solid function space on a measure space $(X, \mathcal{A}, \mu)$. Every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $Y$ admits a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges almost everywhere on sets of finite measure to a measurable function $f : X \to \mathbb{C}$. If $(X, \mathcal{A}, \mu)$ is semifinite, the convergence holds almost everywhere.*

*If also $\|f_n - g\|_Y \xrightarrow[n\to\infty]{} 0$ holds for some $g \in Y$, then $f = g$ almost everywhere on sets of finite measure (almost everywhere if $(X, \mathcal{A}, \mu)$ is semifinite).*

*In particular, if $Y$ is complete and if $\mu$ is semifinite, then every Cauchy-sequence admits a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges pointwise almost everywhere to the $\|\cdot\|_Y$-limit $f \in Y$ of the sequence $(f_n)_{n\in\mathbb{N}}$.*

*Proof.* Existence of a "locally almost everywhere" (almost everywhere if $(X, \mathcal{A}, \mu)$ is semifinite) convergent subsequence was already established in Corollary 2.2.7.

Now, assume $\|f_n - g\|_Y \xrightarrow[n\to\infty]{} 0$ for some $g \in Y$ and let $E \in \mathcal{A}$ be arbitrary with $\mu(E) < \infty$. For brevity, let us write

$$\mu_E(P(x)) := \mu(\{x \in E \mid P(x) \text{ is true}\})$$

for arbitrary predicates $P(x)$.

As a consequence of Egoroff's theorem, almost everywhere convergence on a set of finite measure implies convergence in measure on that set. Hence,

$$\mu_E\left(|f_{n_k}(x) - f(x)| > \varepsilon\right) \xrightarrow[k\to\infty]{} 0 \text{ for all } \varepsilon > 0.$$

But Lemma 2.2.8 also implies

$$\mu_E\left(|f_n(x) - g(x)| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0 \text{ for all } \varepsilon > 0.$$

Thus,

$$\begin{aligned}
&\mu_E\left(|f(x) - g(x)| > \varepsilon\right)\\
&\leq \mu_E\left(|f_{n_k}(x) - g(x)| > \frac{\varepsilon}{2} \text{ or } |f_{n_k}(x) - f(x)| > \frac{\varepsilon}{2}\right)\\
&\leq \mu_E\left(|f_{n_k}(x) - g(x)| > \frac{\varepsilon}{2}\right) + \mu_E\left(|f_{n_k}(x) - f(x)| > \frac{\varepsilon}{2}\right) \xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

which implies $f = g$ almost everywhere on $E$.

Finally, if $(X, \mathcal{A}, \mu)$ is semifinite, set $N := \{x \in X \mid f(x) \neq g(x)\}$. Since $f, g$ are measurable, so is $N$. We want to show $\mu(N) = 0$. The case $0 < \mu(N) < \infty$ is excluded by what was shown before (take $E = N$). In the case $\mu(N) = \infty$, semifiniteness of $\mu$ would yield some $E \subset N$ with $0 < \mu(E) < \infty$, in contradiction to what was shown above. Hence, $\mu(N) = 0$. □

### 2.2.3 *Completeness of function spaces and the Riesz-Fischer property*

Using our results regarding pointwise convergence of Cauchy sequences in function space, we will now develop a very convenient completeness criterion for solid function spaces. This should be compared to [69, §64, Theorem 2], where the case of genuine norms (instead of $p$-norms) is considered.

**Theorem 2.2.10.** *Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space and assume that $(Y, \|\cdot\|_Y)$ is a solid function space on $X$ such that $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0, 1]$.*

*Then $(Y, \|\cdot\|_Y)$ is complete if and only if it satisfies the* $p$-RIESZ-FISCHER PROPERTY, *i.e. if for each sequence $(f_n)_{n\in\mathbb{N}}$ of* nonnegative *functions $f_n \in Y$ with $\sum_{n=1}^{\infty} \|f_n\|_Y^p < \infty$, we have $f \in Y$ for the (necessarily almost everywhere convergent) series $f := \sum_{n=1}^{\infty} f_n$.*

*In this case, $\|f\|_Y^p \leq \sum_{n=1}^{\infty} \|f_n\|_Y^p$.*

*For completeness of $(Y, \|\cdot\|_Y)$, it even suffices if $\sum_{n=1}^{\infty} f_n \in Y$ holds for each sequence $(f_n)_{n\in\mathbb{N}}$ of nonnegative functions $f_n \in Y$ with $\|f_n\|_Y \leq 2^{-n}$ for all $n \in \mathbb{N}$.*

*It is important to observe that the condition $\|f_n\|_Y \leq 2^{-n}$ does not involve the exponent $p \in (0, 1]$. Thus, we have a characterization of completeness which is independent of $p$. This will allow us later (see Corollary 2.2.13) to prove the same characterization if $\|\cdot\|_Y$ is not necessarily a $p$-norm.*

*Proof.* (based on ideas in [69, §64, Theorems 1 and 2]) We show both implications separately.

"⇒": If $(Y, \|\cdot\|_Y)$ is complete, Lemma 2.1.8 implies that there is some function $g \in Y$ with $\left\|g - \sum_{n=1}^{N} f_n\right\|_Y \xrightarrow[N\to\infty]{} 0$. By Corollary 2.2.9, there is a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ with $\sum_{n=1}^{N_k} f_n \to g$ almost everywhere as $k \to \infty$. Since the sequence of partial sums $\left(\sum_{n=1}^{N} f_n\right)_{N\in\mathbb{N}}$ is increasing, we conclude

$$f = \sum_{n=1}^{\infty} f_n = \lim_{k\to\infty} \sum_{n=1}^{N_k} f_n = g \in Y,$$

where the equality holds almost everywhere. Further, Lemma 2.1.5 implies that $\|\cdot\|_Y$ is continuous, since it is a $p$-norm. Hence,

$$\|f\|_Y^p = \lim_{N\to\infty} \left\|\sum_{n=1}^{N} f_n\right\|_Y^p \le \lim_{N\to\infty} \sum_{n=1}^{N} \|f_n\|_Y^p = \sum_{n=1}^{\infty} \|f_n\|_Y^p .$$

"⇐": We first remark that for each sequence $(f_n)_{n\in\mathbb{N}}$ in $Y$ with $\sum_{n=1}^{\infty} \|f_n\|_Y^p < \infty$, Lemma 2.2.6 implies that the series $\sum_{n=1}^{\infty} f_n$ converges almost everywhere on $X$, since $(X, \mathcal{A}, \mu)$ is semifinite.

Our first aim is to show that each sequence $(f_n)_{n\in\mathbb{N}}$ of nonnegative functions in $Y$ with $\|f_n\|_Y \le 2^{-4^n}$ for all $n \in \mathbb{N}$ already satisfies $\|f\|_Y^p \le \sum_{n=1}^{\infty} \|f_n\|_Y^p$ for $f = \sum_{n=1}^{\infty} f_n$.

Assume towards a contradiction that this is false. Thus, there is some $\varepsilon > 0$ with $\|f\|_Y^p \ge \varepsilon + \sum_{n=1}^{\infty} \|f_n\|_Y^p$. Recall from Lemma 2.1.5 that $d(x,y) := \|x-y\|_Y^p$ defines a metric, since $\|\cdot\|_Y$ is a $p$-norm. Using $\|\sum_{n=1}^{r} g_n\|_Y^p \le \sum_{n=1}^{r} \|g_n\|_Y^p$, as well as the second triangle inequality for the metric $d$, we conclude for all $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ that

$$\begin{aligned} k^p\varepsilon &\le k^p\varepsilon + \sum_{n=r}^{\infty} \|kf_n\|_Y^p \le \|kf\|_Y^p - \sum_{n=1}^{r-1} \|kf_n\|_Y^p \\ &\le \|kf\|_Y^p - \left\|\sum_{n=1}^{r-1} kf_n\right\|_Y^p \le \left| d(kf, 0) - d\left(0, \sum_{n=1}^{r-1} kf_n\right)\right| \\ &\le d\left(kf, \sum_{n=1}^{r-1} kf_n\right) = \left\|\sum_{n=r}^{\infty} kf_n\right\|_Y^p . \end{aligned} \tag{2.2.1}$$

For each $\ell \in \mathbb{N}$, set

$$h_\ell := \begin{cases} k \cdot f_{n+k}, & \text{if } \ell = 2^n 3^k \text{ for some } n,k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

*We remark that $h_\ell$ is indeed well-defined, since $n, k$ are uniquely determined by the equation $\ell = 2^n 3^k$.*

For $n, k \in \mathbb{N}$, we have $k \le 2^k \le 2^{3^k}$ and $2^n + 1 \le 2 \cdot 2^n \le 2^n 2^n = 4^n$. For $\ell = 2^n 3^k$, these estimates and our assumption $\|f_n\|_Y \le 2^{-4^n}$ imply

$$2^\ell \|h_\ell\|_Y = 2^{2^n 3^k} \cdot k \cdot \|f_{n+k}\|_Y \le 2^{2^n 3^k} \cdot 2^{3^k} \cdot 2^{-4^{n+k}}$$

$$= 2^{(2^n+1)3^k} \cdot 2^{-4^n 4^k} \leq 2^{4^n\left(3^k-4^k\right)} \leq 1.$$

Since $h_\ell = 0$ if $\ell$ is not of the form $\ell = 2^n 3^k$ for suitable $k, n \in \mathbb{N}$, we have shown $\|h_\ell\|_Y \leq 2^{-\ell}$ for all $\ell \in \mathbb{N}$.

By assumption, this implies $h := \sum_{\ell\in\mathbb{N}} h_\ell \in Y$. But since each summand of this series is nonnegative, we have

$$h \geq \sum_{n\in\mathbb{N}} h_{2^n 3^k} = k \cdot \sum_{n=1}^{\infty} f_{n+k} = k \cdot \sum_{\ell=k+1}^{\infty} f_\ell$$

for each $k \in \mathbb{N}$. By solidity of $Y$ and by employing the estimate (2.2.1), we conclude

$$\infty > \|h\|_Y^p \geq \left\| k \cdot \sum_{\ell=k+1}^{\infty} f_\ell \right\|_Y^p \geq k^p \varepsilon.$$

But this is impossible, since $k \in \mathbb{N}$ can be chosen arbitraryly.

It remains to show that $Y$ is indeed complete. By Lemma 2.1.8, it suffices to show that the sequence $(h_N)_{N\in\mathbb{N}}$ defined by $h_N := \sum_{n=1}^{N} g_n$ converges to some $h \in Y$, where $(g_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $Y$ with $\|g_n\|_Y \leq 2^{-4^n}$ for each $n \in \mathbb{N}$. Set $f_n^{(N)} := |g_{N+n}|$ for $N \in \mathbb{N}_0$. Then $\left(f_n^{(N)}\right)_{n\in\mathbb{N}}$ is a sequence of nonnegative functions in $Y$ with $\left\|f_n^{(N)}\right\|_Y = \|g_{N+n}\|_Y \leq 2^{-4^{N+n}} \leq 2^{-4^n}$ for each $n \in \mathbb{N}$. As seen above, this yields

$$F_N := \sum_{n=N+1}^{\infty} |g_n| = \sum_{n=1}^{\infty} f_n^{(N)} \in Y$$

with

$$\begin{aligned} \|F_N\|_Y^p &\leq \sum_{n=1}^{\infty} \left\|f_n^{(N)}\right\|_Y^p = \sum_{n=N+1}^{\infty} \|f_n\|_Y^p \\ &\leq \sum_{n=N+1}^{\infty} \left(2^{-4^n}\right)^p \leq \sum_{n=N+1}^{\infty} 2^{-np} \xrightarrow[N\to\infty]{} 0. \end{aligned}$$

For $N = 0$, we see in particular $\sum_{n=1}^{\infty} |g_n| = \sum_{n=1}^{\infty} |f_n^{(0)}| < \infty$ almost everywhere, so that the series defining $h := \sum_{n=1}^{\infty} g_n$ and $G_N := \sum_{n=N+1}^{\infty} g_n$ converge almost everywhere. But $|G_N| \leq F_N$, so that the solidity of $Y$ implies

$$\|h - h_N\|_Y^p = \|G_N\|_Y^p \leq \|F_N\|_Y^p \xrightarrow[N\to\infty]{} 0.$$

Hence, $h \in Y$ with $\|h - h_N\|_Y \to 0$. As mentioned above, this shows that $Y$ is complete. □

In Theorem 2.1.4, we saw that every quasi-normed space admits an equivalent $p$-norm for some $p \in (0, 1]$. This is of course still true if $(Y, \|\cdot\|)$ is a solid function space, but it is not clear that the modified space $(Y, |||\cdot|||)$ is still a *solid* function space, because the condition

$\|f\|_Y \leq \|g\|_Y$ for $|f| \leq |g|$ is *not* invariant under change to an equivalent quasi-norm. This point is somewhat neglected in [59].

In view of the completeness criterion above, it would be very convenient if $(Y, |||\cdot|||)$ was again solid, since this would allow us to drop the requirement that $\|\cdot\|_Y$ is a $p$-norm in the theorem above. Fortunately, using the explicit form of the $p$-norm $|||\cdot|||$ constructed in Theorem 2.1.4, one can show that this is indeed the case. For the proof, the following reformulation of solidity is more convenient:

**Lemma 2.2.11.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(Y, \|\cdot\|_Y)$ be a function space on $X$. Then $Y$ is solid if and only if $Y$ is an $L^\infty$-*MODULE.

*Here, we say that $Y$ is an $L^\infty$-module if the map*

$$\Phi : L^\infty(\mu) \times Y \to Y, (f, g) \mapsto f \cdot g$$

*is well-defined with $\|\Phi(f, g)\|_Y \leq \|f\|_{L^\infty} \cdot \|g\|_Y$ for all $f \in L^\infty(\mu)$ and $g \in Y$.*

*Proof.* "⇒": If $Y$ is solid, $f \in L^\infty(\mu)$ and $g \in Y$, then $f \cdot g$ is measurable and $|f \cdot g| \leq \|f\|_{L^\infty} \cdot |g| = |\|f\|_{L^\infty} \cdot g|$ almost everywhere. By solidity, this yields $f \cdot g \in Y$ with

$$\|f \cdot g\|_Y \leq \|\|f\|_{L^\infty} \cdot g\|_Y = \|f\|_{L^\infty} \cdot \|g\|_Y,$$

so that $Y$ is an $L^\infty$-module.

"⇐": Let $f, g : X \to \mathbb{C}$ be measurable with $|f| \leq |g|$ almost everywhere and define $M := \{x \in X \mid g(x) \neq 0\}$. On $M$, the function $f/g$ is well-defined with $|f/g| \leq 1$ almost everywhere. Thus,

$$h : X \to \mathbb{C}, x \mapsto \begin{cases} 0, & \text{if } x \notin M, \\ \frac{f(x)}{g(x)}, & \text{if } x \in M, \end{cases}$$

is well-defined and measurable with $\|h\|_{L^\infty} \leq 1$.

For $x \in M$, we clearly have $f(x) = h(x) \cdot g(x)$. For almost all $x \in X \setminus M$, we have $0 \leq |f(x)| \leq |g(x)| \leq 0$ and hence also

$$f(x) = 0 = h(x) \cdot g(x).$$

Thus, $f = h \cdot g$ almost everywhere. This yields $f = \Phi(h, g) \in Y$ with

$$\|f\|_Y = \|\Phi(h, g)\|_Y \leq \|h\|_{L^\infty} \cdot \|g\|_Y \leq \|g\|_Y,$$

which completes the proof. □

**Corollary 2.2.12.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(Y, \|\cdot\|_Y)$ be a solid function space on $X$. Let $|||\cdot|||$ denote the $p$-norm constructed in Theorem 2.1.4. Then $(Y, |||\cdot|||)$ is again a solid function space.*

*Remark.* By this corollary, we can always assume in the following that our solid function spaces are equipped with a $p$-norm instead of a generic quasi-norm.

*Proof.* By Lemma 2.2.11, it suffices to show that

$$\Phi : L^\infty(\mu) \times Y \to Y, (f,g) \mapsto f \cdot g$$

is well-defined with $|||f \cdot g||| \leq \|f\|_{L^\infty} \cdot |||g|||$ for $f \in L^\infty(\mu)$ and $g \in Y$. Since $(Y, \|\cdot\|_Y)$ is solid, Lemma 2.2.11 implies that $\Phi$ is well-defined.

To prove the estimate $|||f \cdot g||| \leq \|f\|_{L^\infty} \cdot |||g|||$, we use the definition of $|||\cdot|||$ from Theorem 2.1.4: If $g = g_1 + \cdots + g_n$ with $g_1, \ldots, g_n \in Y$, then $f \cdot g = (fg_1) + \cdots + (fg_n)$ with $fg_i \in Y$ for $i \in \underline{n}$. Hence,

$$|||f \cdot g|||^p \leq \sum_{i=1}^n \|fg_i\|_Y^p \leq \|f\|_{L^\infty}^p \cdot \sum_{i=1}^n \|g_i\|_Y^p.$$

Since this holds for every decomposition $g = g_1 + \cdots + g_n$, we conclude $|||f \cdot g|||^p \leq \|f\|_{L^\infty}^p \cdot |||g|||^p$. This completes the proof. □

Using the fact that solidity is inherited by the $p$-norm $|||\cdot|||$ from Theorem 2.1.4, we can now show that the completeness characterization of solid spaces using (a form of) the RIESZ-FISCHER PROPERTY is also true if the norm is not necessarily a $p$-norm:

**Corollary 2.2.13.** *Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space and assume that $(Y, \|\cdot\|_Y)$ is a solid function space on $X$.*

*Then $Y$ is complete if and only if $\sum_{n=1}^\infty f_n \in Y$ holds for each sequence $(f_n)_{n\in\mathbb{N}}$ of* nonnegative *functions in $Y$ satisfying $\|f_n\|_Y \leq C \cdot 2^{-n}$ for all $n \in \mathbb{N}$ and some $C > 0$.*

*Proof.* By Corollary 2.2.12, we know that there is some $p \in (0,1]$, as well as an equivalent $p$-norm $|||\cdot|||$ on $Y$, so that $(Y, |||\cdot|||)$ is also a solid function space. Since $\|\cdot\|_Y$ and $|||\cdot|||$ are equivalent, $(Y, \|\cdot\|_Y)$ is complete iff $(Y, |||\cdot|||)$ is. Let us now show both implications.

"⇒": If $(Y, |||\cdot|||)$ is complete, Theorem 2.2.10 yields $\sum_{n=1}^\infty f_n \in Y$ for each sequence $(f_n)_{n\in\mathbb{N}}$ of nonnegative functions in $Y$ satisfying $\|f_n\|_Y \leq C \cdot 2^{-n}$, since this easily implies $\sum_{n=1}^\infty |||f_n|||^p < \infty$.

"⇐": Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions in $Y$ with $|||f_n||| \leq 2^{-n}$ for all $n \in \mathbb{N}$. This yields $\|f_n\|_Y \leq C \cdot 2^{-n}$ for all $n \in \mathbb{N}$ and some $C > 0$, since $\|\cdot\|_Y$ and $|||\cdot|||$ are equivalent. By assumption, this implies $\sum_{n=1}^\infty f_n \in Y$. Now Theorem 2.2.10 shows that $(Y, |||\cdot|||)$ is complete. □

A further important property of function spaces is the FATOU PROPERTY:

**Definition 2.2.14.** (cf. [69, §65]) Let $(X, \mathcal{A}, \mu)$ be a measure space and assume that $(Y, \|\cdot\|_Y)$ is a function space on $X$.

We say that $Y$ has the FATOU PROPERTY, if for each sequence $(f_n)_{n\in\mathbb{N}}$ in $Y$ with $\liminf_{n\to\infty} \|f_n\|_Y < \infty$ and for each measurable $f : X \to \mathbb{C}$ with $|f(x)| \leq \liminf_{n\to\infty} |f_n(x)|$ for almost all $x \in X$, it follows that $f \in Y$ with

$$\|f\|_Y \leq \liminf_{n\to\infty} \|f_n\|_Y .$$

*Remark.* For $f, g : X \to \mathbb{C}$ measurable with $g \in Y$ and $|f| \leq |g|$ almost everywhere, we can apply the Fatou property to the constant sequence $(f_n)_{n\in\mathbb{N}} := (g)_{n\in\mathbb{N}}$ to deduce $f \in Y$ with $\|f\|_Y \leq \|g\|_Y$. Thus, every quasi-normed space with the Fatou property is automatically solid.

We remark that the Fatou property is equivalent to a form of the monotone convergence theorem, cf. [69, §65, Theorem 3]. Furthermore, one can also introduce the WEAK FATOU PROPERTY (cf. [69, §65, Theorem 3] again), which states that

$$\|f\|_Y \leq C \cdot \liminf_{n\to\infty} \|f_n\|_Y$$

is true if $|f(x)| \leq \liminf_{n\to\infty} |f_n(x)|$ holds almost everywhere.

An interesting feature of the Fatou property is that it ensures that the space under consideration is complete, at least if $X$ is semifinite.

**Lemma 2.2.15.** *(cf. [69, §65, Theorem 1]) Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space. Assume that $(Y, \|\cdot\|_Y)$ is a function space on $X$ that satisfies the Fatou property. Then $(Y, \|\cdot\|_Y)$ is a solid QBF space.*

*Proof.* As seen in the remark after Definition 2.2.14, $(Y, \|\cdot\|_Y)$ is solid. To prove completeness, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $Y$. By Corollary 2.2.9, there is a measurable function $f : X \to \mathbb{C}$ as well as a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k} \xrightarrow[k\to\infty]{} f$ almost everywhere.

In particular, $|f(x) - f_{n_\ell}(x)| \leq \liminf_{k\to\infty} |(f_{n_k} - f_{n_\ell})(x)|$ holds almost everywhere. By the Fatou property, this implies $f - f_{n_\ell} \in Y$ with

$$\|f - f_{n_\ell}\|_Y \leq \liminf_{k\to\infty} \|f_{n_k} - f_{n_\ell}\|_Y \leq \sup_{k\geq\ell} \|f_{n_k} - f_{n_\ell}\|_Y \xrightarrow[\ell\to\infty]{} 0,$$

since $(f_{n_k})_{k\in\mathbb{N}}$ is Cauchy. This shows that the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ converges to $f$ with respect to $\|\cdot\|_Y$. As in the proof of Lemma 2.1.8, this also entails $\|f_n - f\|_Y \xrightarrow[n\to\infty]{} 0$, since $(f_n)_{n\in\mathbb{N}}$ is Cauchy. □

### 2.2.4 *Weighted spaces and translation invariance*

In this subsection, we consider locally compact groups as our measure spaces. In this case, one can translate functions. This raises the question of TRANSLATION INVARIANCE of function spaces:

**Definition 2.2.16.** Let $G$ be an LCH group. For each set $Z$, each function $f : G \to Z$ and arbitrary $x \in G$, we define the LEFT- AND RIGHT TRANSLATES OF $f$ by

$$L_x f : G \to Z, y \mapsto f\left(x^{-1}y\right),$$
$$R_x f : G \to Z, y \mapsto f\left(yx\right).$$

*The definition of $L_x$ and $R_x$ ensures the identities $L_x L_y = L_{xy}$ and $R_x R_y = R_{xy}$ as well as $L_x \chi_M = L_{xM}$ and $R_x \chi_M = \chi_{Mx^{-1}}$.*

If $(Y, \|\cdot\|_Y)$ is a function space on $G$, we say that $Y$ is LEFT INVARIANT or INVARIANT UNDER LEFT TRANSLATIONS if each map $L_x : Y \to Y, f \mapsto L_x f$ is well-defined and bounded. Invariance under right translations is defined analogously.

We are in particular interested in the question under which conditions translation invariance of the space $Y$ implies translation invariance of the weighted space $Y_w$.

**Definition 2.2.17.** (cf. [50, Definition 2.1.1]) Let $(X, \mathcal{A}, \mu)$ a measure space. Each measurable function $w : X \to (0, \infty)$ is called a WEIGHT on $X$. If $(Y, \|\cdot\|_Y)$ is a function space on $X$, we define the WEIGHTED VERSION OF $Y$ as

$$Y_w := \{f : X \to \mathbb{C} \mid w \cdot f \in Y\}$$

with quasi-norm

$$\|f\|_{Y_w} := \|w \cdot f\|_Y \text{ for } f \in Y_w.$$

If $G$ is a group and $m, w : G \to (0, \infty)$ are weights, then $m$ is called LEFT-MODERATE with respect to $w$ if $m(xy) \leq w(x)\, m(y)$ holds for all $x, y \in G$. The weight $m$ is called RIGHT-MODERATE with respect to $w$, if $m(xy) \leq m(x)\, w(y)$ holds for all $x, y \in G$.

Finally, we call the weight $w : G \to (0, \infty)$ SUMBULTIPLICATIVE if $w(xy) \leq w(x)\, w(y)$ holds for all $x, y \in G$.

*Remark.* We remark that the weighted Lebesgue spaces which were introduced in Definition 2.2.3 are a special case of this definition.

Furthermore, it is worth noting that a submultiplicative weight is left- and right moderate with respect to itself.

The following lemma summarizes the most important properties of weighted spaces.

**Lemma 2.2.18.** *Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space. Assume that $(Y, \|\cdot\|_Y)$ is a function space on $X$ and that $w : X \to (0, \infty)$ is a weight. Then the following properties hold:*

1. *$Y_w$ is a function space and $Y_w$ is solid if $Y$ is.*
2. *Each triangle constant $C \geq 0$ for $\|\cdot\|_Y$ is also valid for $\|\cdot\|_{Y_w}$.*
3. *If $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0, 1]$, the same is true for $\|\cdot\|_{Y_w}$.*

4. *If $Y$ is solid and complete, so is $Y_w$.*

5. *If $Y$ has the Fatou property, the same is true of $Y_w$.*

*Proof.* All statements except for the completeness of $Y_w$ are straightforward verifications and hence omitted. The completeness of $Y_w$ is a direct consequence of the completeness criterion using the Riesz-Fischer property (Corollary 2.2.13), since we are assuming that $Y$ is solid and that $(X, \mathcal{A}, \mu)$ is semifinite. □

If $m : G \to (0, \infty)$ is left/right moderate with respect to some other weight $w$, this guarantees that the weighted space $Y_m$ is invariant under left/right translations, as the following lemma shows:

**Lemma 2.2.19.** *(cf. [50, Theorem 2.1.6]) Let $G$ be an LCH group and assume that $(Y, \|\cdot\|_Y)$ is a left/right invariant,* solid *function space on $Y$. Furthermore, let $m : G \to (0, \infty)$ be a measurable weight that is left/right moderate with respect to $w : G \to (0, \infty)$.*

*Then $Y_m$ is left/right invariant with*

$$\|\|L_x\|\|_{Y_m \to Y_m} \leq \|\|L_x\|\|_{Y \to Y} \cdot w(x)$$

*or*

$$\|\|R_x\|\|_{Y_m \to Y_m} \leq \|\|R_x\|\|_{Y \to Y} \cdot w(x^{-1})$$

*for all $x \in G$.*

*Proof.* Let $f \in Y_m$. Assume that $Y$ is left-invariant and that $m$ is left-moderate with respect to $w$. For $x, y \in G$, we have

$$\begin{aligned} |m(y) \cdot L_x f(y)| &= |m(xx^{-1}y) \cdot f(x^{-1}y)| \\ &\leq w(x) \cdot |m(x^{-1}y) \cdot f(x^{-1}y)| \\ &= w(x) \cdot L_x[mf](y). \end{aligned}$$

By assumption, $Y$ is left invariant. Since $f \in Y_m$, we have $mf \in Y$ and thus also $L_x[mf] \in Y$. Since both sides of the above estimate are measurable as functions of $y$, solidity of $Y$ implies $m \cdot L_x f \in Y$, i.e. $L_x f \in Y_m$ with

$$\begin{aligned} \|L_x f\|_{Y_m} = \|m \cdot L_x f\|_Y &\leq w(x) \cdot \|L_x[mf]\|_Y \\ &\leq w(x) \cdot \|\|L_x\|\|_{Y \to Y} \cdot \|m \cdot f\|_Y \\ &= w(x) \cdot \|\|L_x\|\|_{Y \to Y} \cdot \|f\|_{Y_m}. \end{aligned}$$

The proof for right translations is analogous: Here, we have

$$\begin{aligned} |m(y) \cdot R_x f(y)| &= |m(yxx^{-1}) \cdot f(yx)| \\ &\leq w(x^{-1}) \cdot |m(yx) \cdot f(yx)| \\ &= w(x^{-1}) \cdot (R_x[mf])(y), \end{aligned}$$

since $m$ is right-moderate with respect to $w$. The rest of the proof is as before. □

Another way to obtain new (solid) function spaces from existing ones is given by inversion. This will play a role in the next section, since we will be able to deduce properties of the right-sided Wiener amalgam spaces from those of the (usual) left-sided Wiener amalgam spaces, cf. Definition 2.3.27 and Lemma 2.3.29.

**Definition 2.2.20.** Let $G$ be an LCH group and let $(Y, \|\cdot\|_Y)$ be a function space on $G$. We define the INVERSE SPACE $Y^\vee$ OF $Y$ by

$$Y^\vee := \left\{ f : G \to \mathbb{C} \,\middle|\, f^\vee \in Y \right\},$$

where $f^\vee$ is defined as $f^\vee : G \to \mathbb{C}, x \mapsto f\left(x^{-1}\right)$. We equip $Y^\vee$ with the quasi-norm

$$\|f\|_{Y^\vee} := \left\|f^\vee\right\|_Y \text{ for } f \in Y^\vee.$$

The most important elementary properties of this construction are summarized in the following lemma:

**Lemma 2.2.21.** *Let $G$ be an LCH group and let $(Y, \|\cdot\|_Y)$ be a function space on $G$. Then*

1. *$Y^\vee$ is a function space and $Y^\vee$ is solid iff $Y$ is.*
2. *Each triangle constant $C \geq 0$ for $\|\cdot\|_Y$ is also valid for $\|\cdot\|_{Y^\vee}$.*
3. *If $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0,1]$, the same is true for $\|\cdot\|_{Y^\vee}$.*
4. *$Y$ is complete if and only if $Y^\vee$ is.*
5. *$Y$ has the Fatou property if and only if $Y^\vee$ does.*
6. *$Y$ is left/right invariant if and only if $Y^\vee$ is right/left invariant. In this case, we have*

$$\left|\left|\left|L_x\right|\right|\right|_{Y^\vee \to Y^\vee} = \left|\left|\left|R_x\right|\right|\right|_{Y \to Y} \text{ and } \left|\left|\left|R_x\right|\right|\right|_{Y^\vee \to Y^\vee} = \left|\left|\left|L_x\right|\right|\right|_{Y \to Y}.$$

7. *We have $\left(L^p_w\right)^\vee = L_{w_{\vee,p}}$ for each weight $w : G \to (0,\infty)$ and each $p \in (0,\infty)$, with*

$$w_{\vee,p} : G \to (0,\infty), x \mapsto w\left(x^{-1}\right) \cdot \left[\Delta\left(x^{-1}\right)\right]^{1/p},$$

*where $\Delta : G \to (0,\infty)$ is the* MODULAR FUNCTION *of $G$.*

*In [59, 58] the notation $F^*(x) = \overline{\Delta\left(x^{-1}\right) \cdot F\left(x^{-1}\right)}$ is used. Further, $L^p_{w^*}$ is used in a context where $(L^p_w)^\vee$ is needed, cf. [59, Corollary 5.4]. Thus, it seems worthwile to point out that one has to use the $p$-specific weight $w_{\vee,p}$ instead.*

*Proof.* We first note that [32, Proposition (2.31)] shows that the left Haar measure $\mu$ and the right Haar measure $\varrho(A) := \mu\left(A^{-1}\right)$ are strongly equivalent and thus have the same nullsets. This implies that $N \subset G$ is a nullset if and only if $N^{-1}$ is one. For measurable $f, g : G \to \mathbb{C}$, this implies that $f = g$ holds almost everywhere if and

only if $f^\vee = g^\vee$ almost everywhere. This shows that $Y^\vee \leq L^0(\mu)$ is well-defined and that $\|\cdot\|_{Y^\vee}$ is positive definite.

Since the inversion $Y \to Y^\vee, f \mapsto f^\vee$ is a linear isometry which respects orderings (i.e. $|f| \leq |g|$ almost everywhere if and only if $|f^\vee| \leq |g^\vee|$ almost everywhere), properties (1)–(5) are immediate.

Regarding property (6) a direct calculation yields

$$\begin{aligned}(L_x f)^\vee(y) &= L_x f(y^{-1}) = f(x^{-1}y^{-1}) = f^\vee(yx) = \left(R_x f^\vee\right)(y),\\ (R_x f)^\vee(y) &= R_x f(y^{-1}) = f(y^{-1}x) = f^\vee(x^{-1}y) = \left(L_x f^\vee\right)(y)\end{aligned}$$

and thus

$$\begin{aligned}\|L_x f\|_{Y^\vee} &= \left\|(L_x f)^\vee\right\|_Y = \left\|R_x f^\vee\right\|_Y \leq |||R_x|||_{Y\to Y} \left\|f^\vee\right\|_Y,\\ \|R_x f\|_{Y^\vee} &= \left\|(R_x f)^\vee\right\|_Y = \left\|L_x f^\vee\right\|_Y \leq |||L_x|||_{Y\to Y} \left\|f^\vee\right\|_Y,\end{aligned}$$

which yields one implication and the "$\leq$" part of the identity for the operator norms. The "$\geq$" part and the other implication follow by symmetry, since $Y = (Y^\vee)^\vee$.

*The result quoted from [32] states $\int_G f(x^{-1})\,\mathrm{d}x = \int_G f(x)\,\Delta(x^{-1})\,\mathrm{d}x$.*

For the last property, we use [32, Proposition (2.31)] to calculate, for each measurable $f : G \to \mathbb{C}$,

$$\begin{aligned}\|f\|^p_{\left(L^p_w\right)^\vee} = \left\|f^\vee\right\|^p_{L^p_w} &= \int_G |f(x^{-1})|^p \cdot (w(x))^p \,\mathrm{d}\mu(x)\\ &= \int_G |f(y)|^p \cdot (w(y^{-1}))^p \cdot \Delta(y^{-1})\,\mathrm{d}\mu(y)\\ &= \|f\|^p_{w_\vee,p},\end{aligned}$$

where $\mu$ is the Haar measure on $G$. This easily implies the claim. □

*In the proof of [50, Theorem 2.1.4], the implication $X \setminus Y \supset X \setminus Z \implies X \cap Z \supset X \setminus Y$ is used. But this implication is incorrect as can be seen by considering $Y = \varnothing$ and $Z \subsetneq X$. In this case, $X \setminus Y = X \supset X \setminus Z$ is trivially true, but the conclusion $X = X \setminus Y \subset X \cap Z = Z$ is false, because of $Z \subsetneq X$.*

For later use, we will almost always want our weights to be locally bounded. Since we are working on a locally compact group, this is equivalent to being bounded on every compact set. One possibility would be to always impose this extra condition; but the next result shows that submultiplicative *measurable* weights automatically have this property. The proof below is a slightly corrected form of the proof of Theorem 2.1.4 from Christopher Heil's PhD thesis [50].

**Theorem 2.2.22.** *([50, Theorem 2.1.4]) Let $G$ be an LCH group and assume that $w : G \to (0,\infty)$ is a (measurable) submultiplicative weight. Then $w$ is bounded on compact sets.*

*Proof.* For technical reasons, it is preferable to work with a symmetric weight. Thus, we define

$$m : G \to (0,\infty), x \mapsto w(x) + w(x^{-1}).$$

Because of

$$m(xy) = w(xy) + w(y^{-1}x^{-1})$$

$$\begin{aligned}&\leq w(x)\, w(y) + w(y^{-1})\, w(x^{-1})\\ &\leq (w(x) + w(x^{-1})) \cdot (w(y) + w(y^{-1}))\\ &= m(x) \cdot m(y),\end{aligned}$$

this is a measurable, submultiplicative weight with $m(x^{-1}) = m(x)$ for all $x \in G$. Because of $0 \leq w(x) \leq m(x)$ for all $x \in G$, it suffices to show that $m$ is bounded on each compact subset of $G$.

To this end, it suffices to show that $m$ is bounded on *some* open neighborhood $U \subset G$ of the identity $1_G$, because in this case, every compact subset $K \subset G$ can be covered by finitely many translates $x_i U, i \in \underline{n}$ of $U$ for suitable $x_1, \ldots, x_n \in K$. For $x \in K$, this implies $x = x_i u$ for some $i \in \underline{n}$ and $u \in U$ and hence

$$0 \leq m(x) \leq m(x_i) \cdot m(u) \leq C_U \cdot \max_{j \in \underline{n}} m\left(x_j\right),$$

where $C_U$ is chosen with $m(v) \leq C_U$ for all $v \in U$.

Let us assume towards a contradiction that $m$ is unbounded on *every* open unit neighborhood $U \subset G$. Let $\mu$ denote a (left) Haar measure on $G$ and pick an open, precompact, symmetric unit neighborhood $U \subset G$. Since $U$ is precompact, $\chi_U \in L^1(G)$. Hence,

$$\begin{aligned}\mu(aU \setminus U) &\leq \mu(aU \Delta U)\\ &= \int_G |\chi_U - \chi_{aU}| \, \mathrm{d}\mu\\ &= \|\chi_U - L_a \chi_U\|_{L^1(G)} \xrightarrow[a \to 1_G]{} 0,\end{aligned}$$

cf. [32, Proposition (2.41)]. Here, $A \Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the SYMMETRIC DIFFERENCE of $A$ and $B$.

The above ensures existence of an open unit neighborhood $V \subset G$ with $\mu(aU \setminus U) < \frac{1}{2}\mu(U)$ for all $a \in V$.

By assumption, $m$ is unbounded on $V$, so that for each $N \in \mathbb{N}$, there is some $x_N \in V$ with $m(x_N) \geq N^2$. If we set

$$A_N := \{x \in U \mid m(x) \geq N\},$$

we get

$$\left[x_N (U \setminus A_N)^{-1}\right] \setminus (x_N U \setminus U) \subset A_N, \tag{2.2.2}$$

because $x \in \left[x_N (U \setminus A_N)^{-1}\right] \setminus (x_N U \setminus U)$ yields

$$x \in x_N (U \setminus A_N)^{-1} \subset x_N U^{-1} = x_N U$$

which – together with $x \notin (x_N U) \setminus U$ – implies $x \in U$. Furthermore, $x_N^{-1} x \in (U \setminus A_N)^{-1}$ and hence $x^{-1} x_N \in U \setminus A_N$. By definition of $A_N$, this means $m(x^{-1} x_N) < N$ and hence

$$N^2 \leq m(x_N) = m(x x^{-1} x_N) \leq m(x) \cdot m(x^{-1} x_N) < N \cdot m(x).$$

But this yields $m(x) > N$. Since we already established $x \in U$ above, we conclude $x \in A_N$.

Since $m$ is measurable, so is $A_N$. Thus, the inclusion (2.2.2) implies

$$\begin{aligned} \mu(A_N) &\geq \mu\left(\left[x_N (U \setminus A_N)^{-1}\right] \setminus (x_N U \setminus U)\right) \\ &\geq \mu\left(x_N (U \setminus A_N)^{-1}\right) - \mu(x_N U \setminus U) \\ &= \mu\left((U \setminus A_N)^{-1}\right) - \mu(x_N U \setminus U). \end{aligned}$$

But since $U$ and $m$ are symmetric, so is $A_N$, i.e. $(U \setminus A_N)^{-1} = U \setminus A_N$. Further, the choice of $V$ and $x_N \in V$ implies $\mu(x_N U \setminus U) \leq \frac{1}{2}\mu(U)$. Hence,

$$\begin{aligned} \mu(A_N) &\geq \mu\left((U \setminus A_N)^{-1}\right) - \mu(x_N U \setminus U) \\ &\geq \mu(U \setminus A_N) - \frac{1}{2}\mu(U) \\ &\geq \mu(U) - \mu(A_N) - \frac{1}{2}\mu(U). \end{aligned}$$

Rearranging yields $\mu(A_N) \geq \frac{1}{4}\mu(U)$. But we have $A_{N+1} \subset A_N$ for all $N \in \mathbb{N}$ as well as $A_N \subset U$ and hence $\mu(A_N) < \infty$. Using the continuity of $\mu$ from above (cf. [33, Theorem 1.8]), we conclude

$$\mu\left(\bigcap_{N \in \mathbb{N}} A_N\right) = \lim_{N \to \infty} \mu(A_N) \geq \frac{1}{4}\mu(U) > 0.$$

In particular, there is some $x \in \bigcap_{N \in \mathbb{N}} A_N$. But this implies $m(x) \geq N$ for all $N \in \mathbb{N}$ which is impossible, since $m(x) < \infty$. □

*One can even show that every left/right moderate weight is equivalent to a continuous weight, cf. [21, Satz 2.7], but we will not need this fact.*

Using the above theorem, we can now show that left/right moderate weights are "almost locally constant". This will be useful for identifying the associated sequence spaces to weighted $L^p$ spaces in the next section, cf. Definition 2.3.13 and Lemma 2.3.21.

**Corollary 2.2.23.** *Let $G$ be an LCH group, let $w : G \to (0, \infty)$ be a locally bounded (measurable) weight and assume that $m : G \to (0, \infty)$ is left moderate with respect to $w$.*

*Then, for each compact set $K \subset G$, there is a constant $C = C(K, w) > 0$ with*

$$\frac{1}{C} \cdot m(kx) \leq m(x) \leq C \cdot m(kx)$$

*for all $k \in K$ and $x \in G$.*

*If $m : G \to (0, \infty)$ is right moderate with respect to $w$, then*

$$\frac{1}{C} \cdot m(xk) \leq m(x) \leq C \cdot m(xk)$$

*holds for all $k \in K$ and $x \in G$.*

*In particular, left/right moderate weights are bounded from above and below on compact sets, at least if the "moderating" weight is locally bounded. Especially, the above applies if $w$ is submultiplicative, cf. Theorem 2.2.22.*

*Proof.* By assumption, $C := \sup_{k \in K \cup K^{-1}} w(k)$ is a finite, positive constant. If $m$ is left moderate with respect to $w$, then

$$m(kx) \leq w(k)\, m(x) \leq C \cdot m(x)$$

holds for all $x \in G$ and $k \in K$. Likewise,

$$m(x) = m(k^{-1}kx) \leq w(k^{-1}) \cdot m(kx) \leq C \cdot m(kx).$$

If $m$ is right moderate with respect to $w$, then

$$m(xk) \leq m(x)\, w(k) \leq C \cdot m(x)$$

and

$$m(x) = m(xkk^{-1}) \leq m(xk)\, w(k^{-1}) \leq C \cdot m(xk)$$

hold for all $x \in G$ and $k \in K$.

For the boundedness on compact sets, let $K \subset G$ compact and choose $C := C\left(K^{-1}, w\right)$ as above. For $x \in K$ and $k := x^{-1} \in K^{-1}$, this yields

$$\frac{1}{C} \cdot m(1_G) = \frac{1}{C} \cdot m(kx) \leq m(x) \leq C \cdot m(kx) = C \cdot m(1_G)$$

if $m$ is left-moderate with respect to $w$ and

$$\frac{1}{C} \cdot m(1_G) = \frac{1}{C} \cdot m(xk) \leq m(x) \leq C \cdot m(xk) = C \cdot m(1_G)$$

if $m$ is right-moderate with respect to $w$. □

### 2.2.5 *Completions of solid function spaces and the Fatou property* ∗

It is a natural question whether the completion of a solid function space $Y$ is again a solid function space. More concretely, one could conjecture that the completion can be obtained similarly to the construction of $L^1(\mu)$ as a completion of the space of step functions.

For example in [54, Chapter VI, §3], Serge Lang defines the space $\mathcal{L}^1(\mu)$ as

$$\mathcal{L}^1(\mu) := \left\{ f : X \to \mathbb{C} \;\middle|\; \begin{array}{c} \exists L^1 \text{ Cauchy seq. } (f_n)_{n\in\mathbb{N}} \text{ of step fct.} \\ \text{with } f_n \to f \text{ a.e.} \end{array} \right\}.$$

*We remark that Lang uses the notation $\mathcal{L}^1(\mu)$ for the space in which no identifications are made, i.e. two functions $f, g \in \mathcal{L}^1(\mu)$ are considered unequal if $f(x) \neq g(x)$ for some $x \in X$, even if $f = g$ almost everywhere. The notation $L^1(\mu)$ is used to denote the usual space where two functions are identified if they coincide almost everywhere.*

One then has to show that $\|f\|_{L^1} := \lim_{n\to\infty} \|f_n\|_{L^1}$ is independent of the choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$ of step functions, see [54, Chapter VI, Lemma 3.2]. Sadly, this is not the case for general solid function spaces, as the following example shows:

**Example 2.2.24.** Let $Y := L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and define

$$\|f\|_Y := \|f\|_{L^1} + \lim_{\varepsilon\downarrow 0} \left\| f \cdot \chi_{(-\varepsilon,\varepsilon)} \right\|_{L^\infty} \text{ for } f \in Y.$$

Note that the limit exists, since $\varepsilon \mapsto \|f \cdot \chi_{(-\varepsilon,\varepsilon)}\|_{L^\infty}$ is a nondecreasing function. It is easy to see that $\|\cdot\|_Y$ is indeed a norm on $Y$.

Now, consider the sequence $f_n \in Y$ defined by $f_n := \chi_{(0,1/n)}$. For $n \geq k \geq n_0$, we have

$$\begin{aligned} \|f_k - f_n\|_Y = \left\| \chi_{\left[\frac{1}{n},\frac{1}{k}\right)} \right\|_Y &= \left\| \chi_{\left[\frac{1}{n},\frac{1}{k}\right)} \right\|_{L^1} + \lim_{\varepsilon\downarrow 0} \left\| \chi_{\left[\frac{1}{n},\frac{1}{k}\right)} \cdot \chi_{(-\varepsilon,\varepsilon)} \right\|_{L^\infty} \\ &= \left\| \chi_{\left[\frac{1}{n},\frac{1}{k}\right)} \right\|_{L^1} = \frac{1}{k} - \frac{1}{n} \leq \frac{1}{k} \leq \frac{1}{n_0}, \end{aligned}$$

because of $\chi_{\left[\frac{1}{n},\frac{1}{k}\right)} \cdot \chi_{(-\varepsilon,\varepsilon)} = 0$ for $\varepsilon > 0$ small enough. Thus, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Y$ which also satisfies $f_n(x) \xrightarrow[n\to\infty]{} 0$ for almost every $x \in \mathbb{R}$.

But we also have

$$\|f_n\|_Y \geq \lim_{\varepsilon\downarrow 0} \left\| \chi_{\left(0,\frac{1}{n}\right)} \chi_{(-\varepsilon,\varepsilon)} \right\|_{L^\infty} = 1$$

for all $n \in \mathbb{N}$. By choosing $g_n := 0 \in Y$, we see that the limit $\lim_{n\to\infty} \|f_n\|_Y$ of a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $Y$ with $f_n \to f$ almost everywhere is in general *not* independent of the chosen sequence. Thus, the completion of $Y$ can not be constructed as in the case of $L^1$ above.

Finally, we see that the completion $\overline{Y}$ of $Y$ is not a solid function space. Because as seen above, the sequence $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $Y$ and hence in $\overline{Y}$. Thus, there is some $f \in \overline{Y}$ with $\|f_n - f\|_{\overline{Y}} \to 0$. If $\overline{Y}$ was a solid function space, then Corollary 2.2.9 would yield a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k}(x) \xrightarrow[k\to\infty]{} f(x)$ almost everywhere. But as seen above, this would entail $f = 0$ almost everywhere and thus

$$0 = \|f\|_{\overline{Y}} = \lim_{n\to\infty} \|f_n\|_{\overline{Y}} = \lim_{n\to\infty} \|f_n\|_Y \geq 1,$$

a contradiction. ◀

Since every function space satisfying the Fatou property is solid and complete by Lemma 2.2.15, the question arises whether the converse is also true, i.e. if completeness of a solid function space $(Y, \|\cdot\|_Y)$ already implies the (weak) Fatou property. Hence, the following example discusses different ways in which the Fatou property can fail. In particular, this shows that completeness of $(Y, \|\cdot\|_Y)$ is *not* enough to ensure the Fatou property.

**Example 2.2.25.** As observed in the remark after Definition 2.2.14, the Fatou property implies solidity. Hence, one way in which the Fatou property can fail is for a non-solid space. To construct such a space, consider

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \binom{x}{y} \mapsto \binom{x-y}{y}.$$

Since this is a linear bijection of $\mathbb{C}^2$, we can define a norm on $\mathbb{C}^2$ by

$$\left\|\binom{x}{y}\right\| := \left|\Phi\binom{x}{y}\right|.$$

If we consider $\mathbb{C}^2$ as $\ell^2(\{1,2\})$ with the counting measure, then $|(0,1)| \leq |(1,1)|$ (almost) everywhere, but

$$\left\|\binom{0}{1}\right\| = \left|\binom{-1}{1}\right| = \sqrt{2} \not\leq 1 = \left|\binom{0}{1}\right| = \left\|\binom{1}{1}\right\|,$$

so that $\left(\ell^2(\{1,2\}), \|\cdot\|\right)$ does not have the Fatou property, since it is not solid. But of course, $Y$ still satisfies the weak Fatou property.

One of the easiest examples of a *solid* space for which the Fatou property fails is

$$Y = c_0(\mathbb{Z}) = \left\{ (x_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \,\middle|\, |x_n| \xrightarrow[n\to\pm\infty]{} 0 \right\}.$$

It is not hard to see that $Y$ equipped with the $\ell^\infty$-norm is indeed a solid BF space on $\mathbb{Z}$ (equipped with the counting measure). But if we consider the sequence $(x^{(m)})_{m\in\mathbb{N}}$ defined by $x^{(m)} := \chi_{\underline{m}} \in Y$, then $x^{(m)} \to x$ pointwise for $x := \chi_{\mathbb{N}}$ and $\|x^{(m)}\|_{\ell^\infty} = 1$ for all $m \in \mathbb{N}$, but $x \notin Y$.

The above example is slightly unsatisfactory, since the space $c_0$ is just "too small". If $x \in c_0$ satisfies $|x| \leq \liminf_{n\to\infty} \left|x^{(n)}\right|$, then the condition

$$\|x\|_{\ell^\infty} \leq \liminf_{n\to\infty} \left\|x^{(n)}\right\|_{\ell^\infty}$$

*does* hold.

As our final example, we thus construct an example of a solid BF space for which even the *weak* Fatou property fails spectacularly: Even if we only consider elements $x \in Y$ and sequences $\left(x^{(n)}\right)_{n\in\mathbb{N}}$

in $Y$ with $|x| \leq \liminf_{n\to\infty} \left|x^{(n)}\right|$ everywhere, there is no absolute constant $C > 0$ with

$$\|x\|_Y \leq C \cdot \liminf_{n\to\infty} \left\|x^{(n)}\right\|_Y .$$

The mean idea for the construction is taken from [69, §64]). For each sequence $x = (x_{n,\ell})_{n,\ell\in\mathbb{Z}}$, we define

$$\|x\| := \|x\|_{\ell^\infty} + \sup_{\ell\in\mathbb{N}} \left[\ell \cdot \limsup_{n\to\infty} |x_{n,\ell}|\right] \in [0,\infty]$$

as well as

$$Y := \left\{x \in \mathbb{C}^{\mathbb{Z}^2} \;\middle|\; \|x\| < \infty\right\}.$$

It is not hard to see that $\|x\| = 0$ yields $\|x\|_{\ell^\infty} = 0$ and hence $x \equiv 0$. Furthermore, $\|\alpha x\| = |\alpha| \cdot \|x\|$ is trivial. For the subadditivity of $\|\cdot\|$, we observe

$$\begin{aligned}
\limsup_{n\to\infty} |x_{n,\ell} + y_{n,\ell}| &= \lim_{n\to\infty} \sup_{k\geq n} |x_{k,\ell} + y_{k,\ell}| \\
&\leq \lim_{n\to\infty} \left[\sup_{k\geq n} |x_{k,\ell}| + \sup_{k\geq \ell} |y_{k,\ell}|\right] \\
&= \limsup_{n\to\infty} |x_{n,\ell}| + \limsup_{n\to\infty} |y_{n,\ell}| \qquad (2.2.3)
\end{aligned}$$

for arbitrary sequences $x = (x_{n,\ell})_{n,\ell\in\mathbb{Z}}$ and $y = (y_{n,\ell})_{n,\ell\in\mathbb{Z}}$. This easily entails $\|x + y\| \leq \|x\| + \|y\|$. Thus, $(Y, \|\cdot\|)$ is a normed vector space with $Y \hookrightarrow \ell^\infty(\mathbb{Z}^2) \hookrightarrow \ell^1_{\mathrm{loc}}(\mathbb{Z}^2)$. Solidity of $Y$ is also immediate.

Furthermore, $Y$ is complete: Since $\|\cdot\|$ is a norm (i.e. a 1-norm), Theorem 2.2.10 shows that it suffices to show $x := \sum_{n=1}^\infty x^{(n)} \in Y$ for each sequence $(x^{(n)})_{n\in\mathbb{N}}$ in $Y$ which satisfies $\sum_{n=1}^\infty \|x^{(n)}\| < \infty$. To this end, note that $\|\cdot\|_{\ell^\infty} \leq \|\cdot\|$ implies that $x \in \ell^\infty(\mathbb{Z}^2)$ is well-defined with uniform convergence of the series defining $x$.

Furthermore, for each $\ell, M \in \mathbb{N}$, the (finite) subadditivity of the Limes superior shown in equation (2.2.3) implies

$$\begin{aligned}
\ell \cdot \limsup_{n\to\infty} |x_{n,\ell}| &= \ell \cdot \limsup_{n\to\infty} \left|\sum_{m=1}^M x_{n,\ell}^{(m)} + \sum_{m=M+1}^\infty x_{n,\ell}^{(m)}\right| \\
&\leq \ell \cdot \left[\sum_{m=1}^M \limsup_{n\to\infty} \left|x_{n,\ell}^{(m)}\right| + \limsup_{n\to\infty} \left|\sum_{m=M+1}^\infty x_{n,\ell}^{(m)}\right|\right] \\
&\leq \ell \cdot \left[\sum_{m=1}^M \limsup_{n\to\infty} \left|x_{n,\ell}^{(m)}\right| + \sum_{m=M+1}^\infty \left\|x^{(m)}\right\|_{\ell^\infty}\right] \\
&\leq \sum_{m=1}^M \|x^{(m)}\| + \ell \cdot \sum_{m=M+1}^\infty \|x^{(m)}\| .
\end{aligned}$$

Since the series $\sum_{m=1}^{\infty} \|x^{(m)}\|$ is convergent, we can let $M \to \infty$ to conclude

$$\ell \cdot \limsup_{n\to\infty} |x_{n,\ell}| \leq \sum_{m=1}^{\infty} \|x^{(m)}\| < \infty.$$

Since this holds for all $\ell \in \mathbb{N}$, with a uniform bound, we get $x \in Y$. As mentioned above, this suffices.

But $Y$ does *not* satisfy the weak Fatou property

$$\|f\|_Y \leq C \cdot \liminf_{n\to\infty} \|f_n\|_Y .$$

To see this, let $\ell_0 \in \mathbb{N}$ be arbitrary and set $x^{(m)} := \chi_{\underline{m}\times\{\ell_0\}}$. It is easy to see $x^{(m)} \in Y$ for all $m \in \mathbb{N}$ with

$$\|x^{(m)}\| = \|x^{(m)}\|_{\ell^\infty} = 1,$$

since $\limsup_{n\to\infty} \left|x^{(m)}_{n,\ell}\right| = 0$ holds for all $n, \ell \in \mathbb{Z}$. But $(x^{(m)})_{m\in\mathbb{N}}$ converges pointwise to $x := \chi_{\mathbb{N}\times\{\ell_0\}}$ with

$$\|x\| \geq \ell_0 \cdot \limsup_{n\to\infty} |x_{n,\ell_0}| = \ell_0.$$

We have thus shown that the (finite) constant $C$ above would have to satisfy $C \geq \ell_0$ for all $\ell_0 \in \mathbb{N}$, which is impossible. ◀

## 2.3 WIENER AMALGAM SPACES FOR QUASI-BANACH SPACES

In this section, we will study (based on [59]) the WIENER AMALGAM SPACES $W(L^\infty, Y)$, where $Y$ is a SOLID QUASI-BANACH SPACE OF FUNCTIONS on a locally compact group $G$. The idea behind the construction of these spaces is to consider the MAXIMAL FUNCTION $M_Q f(x) := \|f \cdot \chi_{xQ}\|_{L^\infty}$ of a function $f$ and to use the quasi-norm $\|f\|_{W_Q(L^\infty,Y)} := \|M_Q f\|_Y$ to measure the "size" of $f$. Intuitively, the space $W(L^\infty, Y)$ will thus contain functions whose global (decay) properties are determined by $Y$, but which are locally bounded.

The present section is structured as follows: In the first subsection, we provide the precise definition of the Wiener amalgam spaces and derive their elementary properties. In particular, we show that the Wiener amalgam space $W_Q(L^\infty, Y)$ is a solid function space which is complete if $Y$ is. Finally, we prove $W_Q(L^\infty, Y) \hookrightarrow L^\infty_{\text{loc}}(G)$.

In the second subsection, we derive an alternative (discrete) description of Wiener amalgam spaces. More precisely, we introduce the DISCRETE SEQUENCE SPACE $Y_d$ ASSOCIATED TO $Y$ and we show that for a suitable partition of unity $(\varphi_i)_{i\in I}$ on $G$, an equivalent quasi-norm on $W_Q(L^\infty, Y)$ is obtained by measuring the sequence $(\|\varphi_i f\|_{L^\infty})_{i\in I}$ of $L^\infty$-norms of the individual "pieces" of a function $f$ with respect to the sequence space norm $\|\cdot\|_{Y_d}$. Furthermore, we will show in this subsection that the space $W_Q(L^\infty, Y)$ is independent of

the choice of the unit neighborhood $Q \subset G$, at least under mild assumptions on the space $Y$.

In the next subsection, we study convolution relations for Wiener amalgam spaces. This is the most important part of this section on Wiener amalgam spaces, because the validity of these convolution relations is one of the main reasons for considering Wiener amalgam spaces in connection to the coorbit theory for Quasi-Banach spaces: Since the spaces $L^p(G)$ for $0 < p < 1$ do not embedd into $L^1_{\text{loc}}(G)$ in general, one can not define the convolution $f * g$ in a sensible way for $f \in L^p(G)$, even for $g \in C_c(G)$. But functions that belong to the Wiener amalgam space $W(L^\infty, L^p)$ are locally bounded and thus are not affected by this problem; in fact, we will see that the Wiener amalgam spaces – also for *Quasi*-Banach spaces $Y$ – admit strong enough convolution relations to generalize the construction of coorbit spaces to the setting of Quasi-Banach spaces. This is done in the next section.

In the final subsection, we study miscellaneous properties of Wiener amalgam spaces which are relevant for the construction of coorbit spaces in the next section. In particular, we show the embedding $W(L^\infty, Y) \hookrightarrow L^\infty_{1/w}(G)$ for a suitable weight $w : G \to (0, \infty)$.

The main new contribution of the treatment of Wiener amalgam spaces in this thesis is the realization that the convolution relation proposed in [59, Theorem 5.2] is false in general, but can be replaced by a similar relation which still allows to perform the construction of coorbit spaces with respect to Quasi-Banach spaces similar to [58]. This new convolution relation (and a counterexample to [59, Theorem 5.2]) is given in subsection 2.3.3.

### 2.3.1 *Definition and elementary properties of Wiener amalgam spaces*

The definition of the Wiener amalgam space is based on the notion of a certain maximal function which we now introduce.

*The* MAXIMAL FUNCTION $M_Q f$ *is often also called* CONTROL FUNCTION, *see e.g. the discussion before [29, Definition 3.1].*

**Definition 2.3.1.** (cf. [24], [50, Definition 2.2.2] and [59])

Let $G$ be an LCH group, let $f : G \to \mathbb{C}$ or $f : G \to [0, \infty]$ be measurable and assume that $Q \subset G$ is measurable and precompact. Define the MAXIMAL FUNCTION OF $f$ (WITH RESPECT TO $Q$) as

$$M_Q f : G \to [0, \infty], x \mapsto \operatorname*{ess\,sup}_{y \in xQ} |f(y)| .$$

*In this thesis, it is safe to use the notations $W_Q(Y)$ instead of $W(L^\infty, Y)$, since we will never consider Wiener amalgam spaces with local components other than $L^\infty$ or $C^0$.*

If $(Y, \|\cdot\|_Y)$ is a solid function space, we define the WIENER AMALGAM SPACE with window $Q$, local component $L^\infty$ and global component $Y$ as

$$W_Q(Y) := W_Q(L^\infty, Y) := \{f \in L^\infty_{\text{loc}}(G) \mid M_Q f \in Y\},$$

with

$$\|f\|_{W_Q(Y)} := \|f\|_{W_Q(L^\infty, Y)} := \|M_Q f\|_Y \text{ for } f \in W_Q(L^\infty, Y).$$

Here, the space $L^\infty_{\mathrm{loc}}(G)$ is given by

$$L^\infty_{\mathrm{loc}}(G) := \left\{ f \in L^0(\mu) \,\middle|\, f\chi_K \in L^\infty(G) \ \forall \,\text{compact } K \subset G \right\},$$

where $\mu$ denotes the Haar measure on $G$. Finally, we denote the subspace of continuous functions in $W_Q(L^\infty, Y)$ by

$$W_Q\left(C^0, Y\right) := C^0(G) \cap W_Q(L^\infty, Y).$$

We will see later (in Theorem 2.3.17) that under certain assumptions on the space $Y$, the resulting space $W_Q(L^\infty, Y)$ is actually independent of the choice of $Q$, with equivalent quasi-norms for different choices.

*Remark* 2.3.2. For "large" groups $G$, the above definition of the space $W_Q(L^\infty, Y)$ is slightly problematic: In [32, Discussion after Proposition (2.22)] it is shown that if we take $G = \mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}_d$ denotes the group of real numbers, but endowed with the discrete topology, then $Z := \{0\} \times \mathbb{R}_d$ has infinite Haar measure $\mu(Z) = \infty$, but for each compact set $K \subset G$, we have $\mu(Z \cap K) = 0$.

For $f := \chi_Z$, this implies $f \in L^\infty_{\mathrm{loc}}(G)$ with $\|f \cdot \chi_K\|_{L^\infty(G)} = 0$ for all compact sets $K \subset G$, although it is *not* true that $f \equiv 0$ almost everywhere. This means that the family of semi norms

$$\left(f \mapsto \|f \cdot \chi_K\|_{L^\infty(G)}\right)_{K \subset G \text{ compact}}$$

is *not* separating if we consider elements of $L^\infty_{\mathrm{loc}}(G)$ as different if they do not coincide almost everywhere.

Similarly, with $f = \chi_Z$ as above, we have $M_Q f \equiv 0$ for each precompact set $Q \subset G$ and thus $\|f\|_{W_Q(L^\infty, Y)} = \|M_Q f\|_Y = 0$, although again $f$ does *not* vanish almost everywhere. Thus, $\|\cdot\|_{W_Q(L^\infty, Y)}$ is *not* definite if elements of $W_Q(L^\infty, Y)$ are considered as distinct if they do not coincide almost everywhere.

There are basically two ways out of this problem:

1. One can introduce the $\sigma$-algebra of LOCALLY MEASURABLE SETS $M \subset G$, i.e. of those sets $M \subset G$ for which $M \cap F$ is Borel measurable for each Borel set $F \subset G$ of finite measure. Then, one defines the notions of LOCALLY MEASURABLE FUNCTIONS and of LOCAL NULL-SETS.

   By considering two functions $f, g : G \to \mathbb{C}$ as equal if they coincide *locally* almost everywhere, one obtains a separating family of semi norms $\left(f \mapsto \|f \cdot \chi_K\|_{L^\infty(G)}\right)$ as well as a genuine quasi-norm $\|\cdot\|_{W_Q(L^\infty, Y)}$, if $Q$ has nonempty interior. These notions are explained in more detail in [32, Section 2.3].

2. One can restrict the class of groups under consideration to the class of $\sigma$-compact groups. Then the notions of locally measur-

able sets and local null-sets coincide with the usual notions of Borel measurable sets and null-sets and the problem vanishes. We remark that [32, Proposition (2.22)] implies that an LCH group is $\sigma$-compact if and only if it is $\sigma$-finite with respect to the Haar measure.

For simplicity, we will use the second approach in this thesis, i.e., we will most of the time assume that the groups under consideration are $\sigma$-compact. We remark that this is the case for all Lie groups, since these are by definition second countable. ◀

One immediate question is whether $W_Q(L^\infty, Y)$ is a Quasi-Banach space, or at least a quasi-normed vector space. To answer this question, we first show $|f| \leq M_Q f$ almost everywhere. This will in particular imply that $\|\cdot\|_{W_Q(Y)}$ is positive definite.

**Lemma 2.3.3.** *Let $G$ be a $\sigma$-compact LCH group and assume that $Q \subset G$ is a measurable, precompact unit neighborhood. Then*

$$|f(x)| \leq M_Q f(x)$$

*holds for almost all $x \in G$ for each $f \in L^\infty_{\mathrm{loc}}(G)$.*

*Proof.* Since $Q$ is a unit neighborhood, there is an *open*, symmetric, precompact unit-neighborhood $P \subset G$ with $PP \subset Q$. Using the fact that $G$ is $\sigma$-compact, we can write $G = \bigcup_{n\in\mathbb{N}} K_n$ with compact sets $K_n \subset G$. By compactness, we can cover each $K_n$ by finitely many translates $x_1^{(n)}P, \ldots, x_{m_n}^{(n)}P$. Thus, there is a (countable) sequence $(x_i)_{i\in\mathbb{N}}$ with $G = \bigcup_{i\in\mathbb{N}} x_i P$.

Let $x \in P$ be arbitrary. For each $y \in xP$, we have $yP \subset xPP \subset xQ$ and hence

$$M_P f(y) = \|f \cdot \chi_{yP}\|_{L^\infty(G)} \leq \|f \cdot \chi_{xQ}\|_{L^\infty(G)} = M_Q f(x). \tag{2.3.1}$$

Now, for each $i \in \mathbb{N}$, there is some nullset $N_i \subset x_i P$ with

$$|f(x)| \leq \|f \cdot \chi_{x_i P}\|_{L^\infty} = M_P f(x_i)$$

for all $x \in x_i P \setminus N_i$. For such $x$, there is some $p \in P$ with $x = x_i p$. Hence, $x_i = xp^{-1} \in xP^{-1} = xP$. By equation (2.3.1), this yields

$$|f(x)| \leq M_P f(x_i) \leq M_Q f(x).$$

Since $x \in x_i P \setminus N_i$ was arbitrary, we have shown

$$|f(x)| \leq M_Q f(x) \qquad \forall i \in I \, \forall x \in x_i P \setminus N_i.$$

Now, note that $N := \bigcup_{i\in\mathbb{N}} N_i$ is a nullset. For $x \in G \setminus N$, we have $x \in x_i P$ for some $i \in \mathbb{N}$ and then $x \in x_i P \setminus N_i$, since $x \notin N$. As seen above, this implies $|f(x)| \leq M_Q f(x)$, so that this estimate is true for almost all $x \in G$. □

We now want to show that the Wiener amalgam space $W_Q(L^\infty, Y)$ is a quasi-normed vector space. For $f, g \in W_Q(L^\infty, Y) \subset L^\infty_{\mathrm{loc}}(G)$, it is easy to see $0 \leq M_Q(f+g) \leq M_Q f + M_Q g$, so that the solidity of $Y$ implies $M_Q(f+g) \in Y$ *if* $M_Q(f+g)$ is measurable. This measurability problem is usually neglected in treatments of Wiener amalgam spaces. The following lemma shows that it is safe to do so, since $M_Q f$ is always measurable for $f \in L^\infty_{\mathrm{loc}}(G)$.

**Lemma 2.3.4.** *Let $G$ be an LCH group, let $f \in L^\infty_{\mathrm{loc}}(G)$ and assume that $Q \subset G$ is measurable and precompact. Then $M_Q f : G \to [0, \infty)$ is lower semicontinuous and in particular Borel measurable.*

*If $G$ is $\sigma$-compact, and if $Q \subset G$ is a measurable, precompact unit neighborhood, then $W_Q(L^\infty, Y)$ is a quasi-normed space for each solid function space $(Y, \|\cdot\|_Y)$.*

*Each triangle constant $C > 0$ for $\|\cdot\|_Y$ is also a valid triangle constant for $\|\cdot\|_{W_Q(L^\infty,Y)}$. Furthermore, if $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0,1]$, then the same holds for $\|\cdot\|_{W_Q(L^\infty,Y)}$.*

*Finally, we have $W_Q(L^\infty, Y) \hookrightarrow Y$. More precisely, $\|f\|_Y \leq \|f\|_{W_Q(L^\infty,Y)}$ holds for each $f \in W_Q(L^\infty, Y)$.*

*We recall that $f : X \to (-\infty, \infty]$ is called* LOWER SEMICONTINUOUS *if $\{x \in X \mid f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$.*

*Proof.* We first show that $M_Q f$ is lower semicontinuous. Let $\alpha \in \mathbb{R}$ be arbitrary and assume $(M_Q f)(x_0) > \alpha$. Let $K \subset G$ be a compact neighborhood of $x_0$.

For each $x \in G$, the function $\chi_{xQ} \cdot f \in L^\infty(G)$ vanishes outside of a $\sigma$-finite set (because $Q$ is precompact). Using the characterization of the $L^\infty$-norm by duality (cf. [33, Theorem 6.14]), we see

$$(M_Q f)(x) = \operatorname*{ess\,sup}_{y \in xQ} |f(y)| = \|\chi_{xQ} \cdot f\|_{L^\infty} = \sup_{\substack{g \in L^1(G), \\ \|g\|_{L^1} \leq 1}} |\langle \chi_{xQ} \cdot f, g \rangle| .$$

Thus, there is $g_0 \in L^1(G)$ with $\|g_0\|_{L^1} \leq 1$ and $|\langle \chi_{x_0 Q} \cdot f, g_0 \rangle| > \alpha$.

For $x \in K$, we see

$$\begin{aligned} \langle \chi_{xQ} \cdot f, g_0 \rangle &= \left\langle \chi_{xQ} \cdot \chi_{K\overline{Q}} \cdot f, g_0 \right\rangle \\ &= \left\langle \chi_{K\overline{Q}} \cdot f, \chi_{xQ} \cdot g_0 \right\rangle \\ &= \left\langle \chi_{K\overline{Q}} \cdot f, L_x(\chi_Q \cdot L_{x^{-1}} g_0) \right\rangle . \end{aligned} \tag{2.3.2}$$

But the map

$$\Gamma : G \to L^1(G), x \mapsto L_x(\chi_Q \cdot L_{x^{-1}} g_0)$$

is continuous; indeed, for $y_0 \in G$ and $h := \chi_Q \cdot L_{y_0^{-1}} g_0 \in L^1(G)$, we have

$$\begin{aligned} &\|\Gamma(y) - \Gamma(y_0)\|_{L^1} \\ &\leq \left\| L_y \left( \chi_Q \cdot L_{y^{-1}} g_0 \right) - L_y h \right\|_{L^1} + \|L_y h - L_{y_0} h\|_{L^1} \end{aligned}$$

$$= \left\| \chi_Q \cdot \left( L_{y^{-1}} g_0 - L_{y_0^{-1}} g_0 \right) \right\|_{L^1} + \left\| L_y h - L_{y_0} h \right\|_{L^1}$$
$$\leq \left\| L_{y^{-1}} g_0 - L_{y_0^{-1}} g_0 \right\|_{L^1} + \left\| L_y h - L_{y_0} h \right\|_{L^1} \xrightarrow[y \to y_0]{} 0.$$

Since $f \in L^\infty_{\mathrm{loc}}(G)$, we have $\chi_{K\overline{Q}} \cdot f \in L^\infty(G)$. This implies that the map

$$\Theta : K \to \mathbb{R}_+, x \mapsto |\langle \chi_{xQ} \cdot f, g_0 \rangle| \overset{\text{Eq. (2.3.2)}}{=} \left| \left\langle \chi_{K\overline{Q}} \cdot f, \Gamma(x) \right\rangle \right|$$

is continuous. As seen above, $\Theta(x_0) > \alpha$, so that $V := \Theta^{-1}((\alpha, \infty))$ is a neighborhood of $x_0$, since $K$ is also a neighborhood of $x_0$.

But this implies

$$(M_Q f)(x) = \sup_{\substack{g \in L^1(G) \\ \|g\|_{L^1} \leq 1}} |\langle \chi_{xQ} \cdot f, g \rangle| \geq |\langle \chi_{xQ} \cdot f, g_0 \rangle| = \Theta(x) > \alpha$$

for all $x \in V$. Thus, $\{x \in G \mid M_Q f(x) > \alpha\}$ is a neighborhood of $x_0$, so that $M_Q f$ is lower semicontinuous.

Now, let us assume that $Q$ is a precompact, measurable unit neighborhood. If $f \in W_Q(L^\infty, Y)$, then $M_Q f \in Y$. But Lemma 2.3.3 shows $|f| \leq M_Q f$ almost everywhere. Since $f \in W_Q(L^\infty, Y) \subset L^\infty_{\mathrm{loc}}$ is a measurable function, solidity of $Y$ implies $f \in Y$ with

$$\|f\|_Y \leq \|M_Q f\|_Y = \|f\|_{W_Q(L^\infty, Y)}.$$

In particular, if $\|f\|_{W_Q(L^\infty,Y)} = 0$, then $\|f\|_Y = 0$ and hence $f \equiv 0$ almost everywhere. This shows that $\|\cdot\|_{W_Q(L^\infty,Y)}$ is positive definite. Once we know that $W_Q(L^\infty, Y)$ is a quasi-normed space, these considerations imply that the embedding $W_Q(L^\infty, Y) \hookrightarrow Y$ is valid.

It is clear that $M_Q(\alpha f) = |\alpha| \cdot M_Q f \in Y$ and thus

$$\begin{aligned} \|\alpha f\|_{W_Q(L^\infty,Y)} &= \|M_Q(\alpha f)\|_Y \\ &= |\alpha| \cdot \|M_Q f\|_Y \\ &= |\alpha| \cdot \|f\|_{W_Q(L^\infty,Y)} < \infty \end{aligned}$$

holds for each $\alpha \in \mathbb{C}$ and $f \in W_Q(L^\infty, Y)$. Hence, $W_Q(L^\infty, Y)$ is closed under scalar multiplications and $\|\cdot\|_{W_Q(L^\infty,Y)}$ is homogeneous.

Now, let $C > 0$ be a triangle constant for $\|\cdot\|_Y$. Pick $f, g \in W_Q(Y)$. Then $f + g \in L^\infty_{\mathrm{loc}}(G)$, so that $M_Q(f+g)$ is lower-semicontinuous and hence Borel measurable. Furthermore,

$$\begin{aligned} 0 &\leq M_Q[f+g](x) \\ &= \|\chi_{xQ} \cdot (f+g)\|_{L^\infty} \\ &\leq \|\chi_{xQ} \cdot f\|_{L^\infty} + \|\chi_{xQ} \cdot g\|_{L^\infty} \\ &= M_Q f(x) + M_Q g(x) \end{aligned}$$

holds for all $x \in G$. Using $M_Q f + M_Q g \in Y$, together with the solidity of $Y$, we conclude $M_Q [f+g] \in Y$ with

$$\begin{aligned}\|f+g\|_{W_Q(L^\infty,Y)} &= \|M_Q [f+g]\|_Y \leq \|M_Q f + M_Q g\|_Y \\ &\leq C \cdot \left[\|M_Q f\|_Y + \|M_Q g\|_Y\right] \\ &= C \cdot \left[\|f\|_{W_Q(L^\infty,Y)} + \|g\|_{W_Q(L^\infty,Y)}\right].\end{aligned}$$

This shows that $W_Q(L^\infty, Y)$ is closed under addition and that $\|\cdot\|_{W_Q(L^\infty,Y)}$ is a quasi-norm with triangle constant $C$. Similarly, if $\|\cdot\|$ is a $p$-norm, then

$$\begin{aligned}\|f+g\|^p_{W_Q(L^\infty,Y)} &= \|M_Q [f+g]\|^p_Y \leq \|M_Q f + M_Q g\|^p_Y \\ &\leq \|M_Q f\|^p_Y + \|M_Q g\|^p_Y \\ &= \|f\|^p_{W_Q(L^\infty,Y)} + \|g\|^p_{W_Q(L^\infty,Y)},\end{aligned}$$

so that $\|\cdot\|_{W_Q(L^\infty,Y)}$ is also a $p$-norm. □

Using the properties of solid function spaces from Section 2.2, together with the fact that $\|\cdot\|_{W_Q(L^\infty,Y)}$ is a quasi-norm, we can now show that the Wiener amalgam space $W_Q(L^\infty, Y)$ embeds into $L^\infty_{\mathrm{loc}}(G)$.

**Lemma 2.3.5.** *Let $G$ be a $\sigma$-compact, LCH group, let $(Y, \|\cdot\|_Y)$ be a solid function space on $G$ and assume that $Q \subset G$ is a precompact and measurable unit neighborhood.*

*Then $W_Q(L^\infty, Y)$ embeds continuously into $L^\infty_{\mathrm{loc}}(G)$. More precisely, for each compact $K \subset G$, there is a constant $C = C_{K,Q,Y} > 0$ with*

$$\|f \cdot \chi_K\|_{L^\infty(G)} \leq C_{K,Q,Y} \cdot \|f\|_{W_Q(L^\infty,Y)}$$

*for all $f \in W_Q(L^\infty, Y)$.*

*Proof.* Let $K \subset G$ be compact and assume towards a contradiction that the claim

$$\exists n \in \mathbb{N}\, \forall f \in W_Q(L^\infty, Y): \qquad \|f \cdot \chi_K\|_{L^\infty(G)} \leq n \cdot \|f\|_{W_Q(L^\infty,Y)}$$

is false. Thus, for every $n \in \mathbb{N}$, there is some $f_n \in W_Q(L^\infty, Y)$ with $\|f_n \cdot \chi_K\|_{L^\infty(G)} > n \cdot \|f_n\|_{W_Q(L^\infty,Y)}$.

In particular, $\|f_n \cdot \chi_K\|_{L^\infty(G)} > 0$ and thus $\|f_n\|_{W_Q(L^\infty,Y)} > 0$, since $\|\cdot\|_{W_Q(L^\infty,Y)}$ is a quasi-norm by Lemma 2.3.4. By normalizing, we can assume $\|f_n\|_{W_Q(L^\infty,Y)} = 1/n$ with

$$\|f_n \cdot \chi_K\|_{L^\infty(G)} > n \cdot \|f_n\|_{W_Q(L^\infty,Y)} = 1$$

for all $n \in \mathbb{N}$.

Because of $\|M_Q f_n\|_Y = \|f_n\|_{W_Q(L^\infty,Y)} \xrightarrow[n\to\infty]{} 0$ and since $G$ is $\sigma$-compact, Corollary 2.2.9 yields a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which satis-

fies $M_Q f_{n_k}(x) \xrightarrow[k\to\infty]{} 0$ for almost every $x \in G$, i.e. for every $x \in G \setminus N$, with $N \subset G$ of measure zero.

For arbitrary $y \in G$, the set $y\,(Q^\circ)^{-1}$ is nonempty and open. Since $N$ is of measure zero, $G \setminus N$ is dense in $G$. Hence, there is some $x \in G \setminus N$ with $x \in y\,(Q^\circ)^{-1}$. Thus, $y \in xQ^\circ$. Since $y \in G$ was arbitrary, we have shown

$$K \subset G \subset \bigcup_{x \in G \setminus N} xQ^\circ.$$

By compactness of $K$, there are $x_1, \dots, x_m \in G \setminus N$ with $K \subset \bigcup_{i=1}^m x_i Q^\circ$. Hence,

$$\begin{aligned} 1 < \|\chi_K \cdot f_{n_k}\|_{L^\infty(G)} &\leq \left\|\chi_{\bigcup_{i=1}^m x_i Q^\circ} \cdot f_{n_k}\right\|_{L^\infty(G)} \\ &\leq \sum_{i=1}^m \|\chi_{x_i Q} \cdot f_{n_k}\|_{L^\infty(G)} = \sum_{i=1}^m \left(M_Q f_{n_k}\right)(x_i) \xrightarrow[k\to\infty]{} 0, \end{aligned}$$

a contradiction. □

**Corollary 2.3.6.** *Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|)$ be a solid function space on $G$. Assume that $Q \subset G$ is a precompact, measurable unit neighborhood. Then $W_Q\left(C^0, Y\right)$ is a closed subspace of $W_Q\left(L^\infty, Y\right)$.*

*Proof.* Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $W_Q\left(C^0, Y\right)$ satisfying $f_n \to f$ for some $f \in W_Q\left(L^\infty, Y\right)$ with convergence in $W_Q\left(L^\infty, Y\right)$. We need to show that $f$ has a continuous representative with respect to equality almost everywhere.

Lemma 2.3.5 shows that for each compact set $K \subset G$, there is a constant $C_{K,Q} > 0$ with

$$\|g \cdot \chi_K\|_{L^\infty(G)} \leq C_{K,Q} \cdot \|g\|_{W_Q(L^\infty,Y)} \qquad \forall g \in W_Q\left(L^\infty, Y\right).$$

Now, let $\varnothing \neq U \subset G$ be an arbitrary open, precompact set. The above estimate implies

$$\begin{aligned} \|f_n - f_m\|_{\sup,U} &= \|(f_n - f_m) \cdot \chi_U\|_{L^\infty(G)} \\ &\leq \|(f_n - f_m) \cdot \chi_{\overline{U}}\|_{L^\infty(G)} \\ &\leq C_{\overline{U},Q} \cdot \|f_n - f_m\|_{W_Q(L^\infty,Y)} \xrightarrow[n,m\to\infty]{} 0. \end{aligned}$$

Here, the first step employed the fact that $f_n - f_m$ is continuous and that $U$ is open, so that the essential supremum coincides with the usual supremum.

By completeness of the space $C_b(U)$ of bounded continuous functions on $U$, equipped with the sup-norm, we see that there is a continuous, bounded function $g_U : U \to \mathbb{C}$ with $f_n \to g_U$ uniformly on $U$. It is easy to see $g_U = g_V$ on $U \cap V$ for open, precompact sets $U, V \subset G$.

By the gluing lemma, this proves existence of a continuous function $g : G \to \mathbb{C}$ with $f_n \to g$ uniformly on each open, precompact set $U \subset G$. In particular, $f_n \to g$ pointwise.

*The gluing lemma (cf. [60, Lemma 1.1']) states that if $f_i : U_i \to Y$ are continuous maps for $i \in I$ where $U_i \subset X$ is open with $X = \bigcup_{i \in I} U_i$ and if $f_i, f_j$ agree on the overlap $U_i \cap U_j$ for $i, j \in I$, then there is a unique continuous function $f : X \to Y$ extending each $f_i$.*

Since $W_Q(L^\infty, Y)$ is a solid function space by Lemma 2.3.4, we can employ Corollary 2.2.9 to obtain a subsequence $(f_{n_k})_{n \in \mathbb{N}}$ with $f_{n_k} \to f$ almost everywhere. This implies $f = g$ almost everywhere, so that $f$ indeed has a continuous representative. □

With these preparations, we can now show that if $Y$ is a solid QBF space, then so is the Wiener amalgam space $W_Q(L^\infty, Y)$. The completeness of $W_Q(L^\infty, Y)$ is also proved in [59, Theorem 2.6], but only under the additional assumption that the space $W_Q(L^\infty, Y)$ is invariant under right translations.

Showing the completeness *without* this assumption can be advantageous, since one can then use the closed graph theorem (which is also applicable to Quasi-Banach spaces, cf. the remark after Lemma 2.1.5 and [61, Theorem 2.15]) to show that each of the maps

$$R_x : W_Q(L^\infty, Y) \to W_Q(L^\infty, Y)$$

is bounded. Since $W_Q(L^\infty, Y)$ embeds into the Hausdorff space $L^\infty_{\mathrm{loc}}(G)$ (by Lemma 2.3.5) and because $R_x : L^\infty_{\mathrm{loc}}(G) \to L^\infty_{\mathrm{loc}}(G)$ is well-defined, it is thus sufficient to prove that $R_x : W_Q(L^\infty, Y) \to W_Q(L^\infty, Y)$ is well-defined; boundedness is automatic.

**Lemma 2.3.7.** *Let $G$ be a $\sigma$-compact LCH group and assume that $(Y, \|\cdot\|_Y)$ is a solid QBF space on $G$.*

*If $Q \subset G$ is a measurable, precompact unit neighborhood, then $W_Q(L^\infty, Y)$ is also a solid QBF space on $G$ and $W_Q(C^0, Y)$ is a Quasi-Banach space.*

*Proof.* By Lemma 2.3.4, we know that $\left(W_Q(L^\infty, Y), \|\cdot\|_{W_Q(L^\infty, Y)}\right)$ is a quasi-normed space. Furthermore, $W_Q(C^0, Y)$ is a closed subspace of $W_Q(L^\infty, Y)$ by Corollary 2.3.6 and hence complete if $W_Q(L^\infty, Y)$ is.

To see that $W_Q(L^\infty, Y)$ is a solid space, let $f, g : G \to \mathbb{C}$ be measurable with $|f| \leq |g|$ and $g \in W_Q(L^\infty, Y) \subset L^\infty_{\mathrm{loc}}(G)$. Because of $|f| \leq |g|$, we see $f \in L^\infty_{\mathrm{loc}}(G)$, so that Lemma 2.3.4 shows that $M_Q f : G \to [0, \infty)$ is measurable. Using the solidity of $Y$ together with

$$0 \leq M_Q f(x) = \|f \cdot \chi_{xQ}\|_{L^\infty(G)} \leq \|g \cdot \chi_{xQ}\|_{L^\infty(G)} = M_Q g(x)$$

for all $x \in G$, we conclude $M_Q f \in Y$ (i.e. $f \in W_Q(L^\infty, Y)$) with

$$\|f\|_{W_Q(L^\infty, Y)} = \|M_Q f\|_Y \leq \|M_Q g\|_Y = \|g\|_{W_Q(L^\infty, Y)}.$$

It remains to show completeness of $W_Q(L^\infty, Y)$. By Corollary 2.2.13, it is sufficient to show $f := \sum_{n=1}^\infty f_n \in W_Q(L^\infty, Y)$ for each sequence $(f_n)_{n \in \mathbb{N}}$ of nonnegative functions in $W_Q(L^\infty, Y)$, under the additional

assumption that $\|f_n\|_{W_Q(L^\infty,Y)} \leq C \cdot 2^{-n}$ for all $n \in \mathbb{N}$ and some $C > 0$. By Lemma 2.3.5, we get

$$\|f \cdot \chi_K\|_{L^\infty} \leq \sum_{n=1}^{\infty} \|f_n \cdot \chi_K\|_{L^\infty} \leq C_{K,Q} \cdot \sum_{n=1}^{\infty} \|f_n\|_{W_Q(L^\infty,Y)} < \infty$$

for each compact $K \subset G$ with a constant $C_{K,Q} = C(K, Q, Y) > 0$. Thus, $f \in L^\infty_{\mathrm{loc}}(G)$.

Now, Lemma 2.3.4 shows that $M_Q f$ is measurable. But we also have

$$M_Q f(x) = \|f \cdot \chi_{xQ}\|_{L^\infty} \leq \sum_{n=1}^{\infty} \|f_n \cdot \chi_{xQ}\|_{L^\infty} = \sum_{n=1}^{\infty} (M_Q f_n)(x).$$

Since $Y$ is a complete, solid function space with $M_Q f_n \in Y$ and with $\|M_Q f_n\|_Y = \|f_n\|_{W_Q(L^\infty,Y)} \leq C \cdot 2^{-n}$ for all $n \in \mathbb{N}$, Corollary 2.2.13 shows $\sum_{n=1}^{\infty} M_Q f_n \in Y$.

Using the solidity of $Y$, we get $M_Q f \in Y$ and hence $f \in W_Q(L^\infty, Y)$, which was to be shown. □

### 2.3.2 *A discrete characterization of Wiener amalgam spaces*

With the above definition of the Wiener amalgam space $W_Q(L^\infty, Y)$, a function $f \in L^\infty_{\mathrm{loc}}(G)$ is first transformed to the family $(M_Q f(x))_{x \in G}$ of sliding maxima, with "*continuous*" index set $G$. The "size" of this family is then measured using the quasi-norm $\|\cdot\|_Y$. Our next goal is to derive a description using *discrete* sequence spaces. More precisely, the continuously indexed family $(M_Q f(x))_{x \in G}$ will be replaced by the family $(\|\varphi_i \cdot f\|_{L^\infty})_{i \in I}$ for a suitable partition of unity $(\varphi_i)_{i \in I}$ of $G$ and the quasi-norm $\|\cdot\|_Y$ will be replaced by the quasi-norm $\|\cdot\|_{Y_d}$, where $Y_d$ is a certain sequence space derived from $Y$.

The advantage of this description is that each $f \in W_Q(L^\infty, Y)$ can be expressed as a (countable) sum $f = \sum_{i \in I} \varphi_i f$, with precise information on the size $\|\varphi_i f\|_{L^\infty}$ of each "piece" and on the support $\operatorname{supp}(\varphi_i f) \subset \operatorname{supp} \varphi_i$. This will be one of the main ingredients for the proofs of the convolution relations for Wiener amalgam spaces, cf. Theorems 2.3.24 and 2.3.31.

Before we state and prove this alternative characterization, we need to clarify our assumptions regarding the partition of unity $(\varphi_i)_{i \in I}$. To this end, we first introduce some notions pertaining to the distribution of discrete families in a locally compact group $G$.

**Definition 2.3.8.** (cf. [29, Definition 3.2])

Let $G$ be an LCH group, let $X = (x_i)_{i \in I}$ be a family in $G$ and let $V \subset G$ be relatively compact.

1. The family $X$ is called $V$-DENSE if $G = \bigcup_{i \in I} x_i V$.

2. The family $X$ is called $V$-SEPARATED if the family $(x_iV)_{i\in I}$ is pairwise disjoint.

3. The family $X$ is called RELATIVELY SEPARATED if for all compact sets $K \subset G$, the constant

$$C_{X,K} := \sup_{i\in I} \left|\left\{j \in I \,\middle|\, x_iK \cap x_jK \neq \varnothing\right\}\right|$$

is finite.

4. The family $X$ is called $V$-WELL-SPREAD if it is relatively separated as well as $V$-dense.

*Remark.* In [27, Lemma 2.9], it is stated that a family $X = (x_i)_{i\in I}$ is relatively separated if and only if for every compact set $K \subset G$, the family $X$ can be written as a finite union of $K$-separated families. This is a more intuitively accessible formulation of the definition, but we will not need it in the following, so we omit the proof.

Using these notions, we can now state our assumptions on the partition of unity $(\varphi_i)_{i\in I}$.

**Definition 2.3.9.** (cf. [29, Definition 3.6])
Let $G$ be an LCH group and let $U \subset G$ be a relatively compact unit neighborhood. A family $\Phi = (\varphi_i)_{i\in I}$ of continuous functions on $G$ is called a BOUNDED UNIFORM PARTITION OF UNITY OF SIZE $U$ (in short: $U$-BUPU), if the following are satisfied:

1. We have $0 \leq \varphi_i \leq 1$ for all $i \in I$,

2. There is a relatively separated family $(x_i)_{i\in I}$ with $\operatorname{supp}\varphi_i \subset x_iU$ for all $i \in I$, called a $U$-LOCALIZING FAMILY FOR $\Phi$,

3. We have $\sum_{i\in I} \varphi_i \equiv 1$ on $G$.

*We will see in Lemma 2.3.10 that the family $(x_iU)_{i\in I}$ is locally finite, so that the sum $\sum_{i\in I}\varphi_i$ is also locally finite. Hence, there are no convergence issues.*

*Remark.* Using $\sum_{i\in I}\varphi_i \equiv 1$ on $G$, together with $\operatorname{supp}\varphi_i \subset x_iU$, we conclude $G = \bigcup_{i\in I} x_iU$, so that the family $(x_i)_{i\in I}$ is $U$-well-spread in $G$.

The next lemma shows that relatively separated families yield "uniformly locally finite coverings". In particular, we see that relatively separated families are always countable if the group is $\sigma$-compact. This will be important later to remedy possible measurability issues.

**Lemma 2.3.10.** *Let $G$ be an LCH group. Assume that $X = (x_i)_{i\in I}$ is a relatively separated family in $G$ and that $K \subset G$ is compact. Then the family $(x_iK)_{i\in I}$ is "uniformly locally finite" in the following sense: For each compact $U \subset G$ and each $x \in G$, we have*

$$\left|\left\{j \in I \,\middle|\, x_jK \cap xU \neq \varnothing\right\}\right| \leq C_{X,KU^{-1}} < \infty$$

*with $C_{X,KU^{-1}}$ as in Definition 2.3.8.*

*If $G$ is $\sigma$-compact, the index set $I$ is countable.*

*Proof.* Let $U \subset G$ be compact and fix $x \in G$. If there is no $j \in I$ with $x_j K \cap xU \neq \varnothing$, the claim is trivial. Thus, fix some $i \in I$ with $x_i K \cap xU \neq \varnothing$.

Now, let $j \in I$ be arbitrary with $x_j K \cap xU \neq \varnothing$. Thus, there are $k, \kappa \in K$ and $u, v \in U$ with $x_i k = xu$ and $x_j \kappa = xv$. This implies

$$x_i k u^{-1} = x = x_j \kappa v^{-1}$$

and hence $x \in x_i K U^{-1} \cap x_j K U^{-1} \neq \varnothing$. Thus, we have shown

$$\left\{ j \in I \,\middle|\, x_j K \cap xU \neq \varnothing \right\} \subset \left\{ j \in I \,\middle|\, x_j K U^{-1} \cap x_i K U^{-1} \neq \varnothing \right\}.$$

By definition of $C_{X,KU^{-1}}$, this establishes the first part of the lemma.

For the second part, let $G = \bigcup_{n \in \mathbb{N}} K_n$ with compact sets $K_n \subset G$. Applying the first part with $K = \{1_G\}$, $x = 1_G$ and $U = K_n$, we see that

$$\left\{ j \in I \,\middle|\, x_j \in K_n \right\} = \left\{ j \in I \,\middle|\, x_j K \cap xU \neq \varnothing \right\}$$

is finite. Hence, $I = \bigcup_{n \in \mathbb{N}} \left\{ j \in I \,\middle|\, x_j \in K_n \right\}$ is countable. □

Using this lemma, we also obtain the following more convenient characterization of relatively separated families.

**Lemma 2.3.11.** *Let $G$ be an LCH group and let $(g_i)_{i \in I}$ be a family in $G$. Then $(g_i)_{i \in I}$ is relatively separated if and only if the function*

$$f_K : G \to [0, \infty], x \mapsto \sum_{i \in I} \chi_{g_i K}(x)$$

*is bounded for each compact set $K \subset G$.*

*For this, it is sufficient if $f_Q$ is bounded for* some *set $Q \subset G$ with nonempty interior.*

*Proof.* We first prove both implications of the claimed equivalence. "⇒": Let $(g_i)_{i \in I}$ be relatively separated and let $K \subset G$ be compact. Using Lemma 2.3.10, we get for arbitrary $x \in G$ the estimate

$$\begin{aligned} f_K(x) &= \left| \{ i \in I \mid x \in g_i K \} \right| \\ &= \left| \{ i \in I \mid g_i K \cap x \{1_G\} \neq \varnothing \} \right| \leq C_{(g_i)_{i \in I}, K\{1_G\}^{-1}} < \infty. \end{aligned}$$

Since the constant $C_{(g_i)_{i \in I}, K}$ does not depend on $x \in G$, we see that $f_K$ is bounded.

"⇐": Let $K \subset G$ be an arbitrary compact set and let $i_0 \in I$. We have

$$\begin{aligned} \left| \{ i \in I \mid x_{i_0} K \cap x_i K \neq \varnothing \} \right| &\leq \left| \left\{ i \in I \,\middle|\, x_{i_0} \in x_i K K^{-1} \right\} \right| \\ &= \sum_{i \in I} \chi_{x_i K K^{-1}}(x_{i_0}) = f_{KK^{-1}}(x_{i_0}) \leq \left\| f_{KK^{-1}} \right\|_{\sup}. \end{aligned}$$

Since $K \subset G$ was an arbitrary compact set, the condition in Definition 2.3.8 is satisfied. Here, we used that $KK^{-1} \subset G$ is compact, since $K$ is.

Now assume that $f_Q$ is bounded for *some* set $Q \subset G$ with nonempty interior. Let $K \subset G$ be an arbitrary compact set. By compactness, we get $K \subset \bigcup_{j=1}^{n} Q^\circ y_j$ for suitable $y_1, \dots, y_n \in G$, where $Q^\circ$ denotes the interior of $Q$. But this implies

$$f_K = \sum_{i \in I} \chi_{x_i K} \leq \sum_{i \in I} \sum_{j=1}^{n} \chi_{x_i Q y_j} = \sum_{j=1}^{n} \sum_{i \in I} R_{y_j^{-1}} \chi_{x_i Q} = \sum_{j=1}^{n} R_{y_j^{-1}} f_Q.$$

Because $f_Q$ is bounded, so is each $R_{y_j^{-1}} f_Q$. Since a finite sum of bounded functions is bounded, we see that $f_K$ is bounded. But $K \subset G$ was an arbitrary compact set. □

In [22, Theorem 2], Feichtinger establishes existence of BOUNDED PARTITIONS OF UNITY in a very general setting. By specializing to the case $B = C_0(G)$, we obtain the following existence result for $U$-BUPUs:

**Lemma 2.3.12.** *Let $G$ be an LCH group and let $U \subset G$ be a precompact unit neighborhood. Then there exists a $U$-BUPU $(\varphi_i)_{i \in I} \subset C_c(G)$.*

*Proof.* By specializing [22, Theorem 2] to the case of $B = C_0(G)$, we obtain a family $(\psi_i)_{i \in I}$ of functions in $C_0(G)$ and a relatively separated family $(x_i)_{i \in I}$ with $\operatorname{supp} \psi_i \subset x_i U$ for all $i \in I$ and $\sum_{i \in I} \psi_i \equiv 1$ on $G$, as well as $\sup_{i \in I} \|\psi_i\|_{\sup} < \infty$.

It remains to construct a modified family $(\varphi_i)_{i \in I}$ with the same properties, but which also satisfies $0 \leq \varphi_i \leq 1$ for all $i \in I$. Using Lemma 2.3.10 as well as $\operatorname{supp} \psi_i \subset x_i \overline{U}$ with $\overline{U}$ compact, we see that the series defining $\psi := \sum_{i \in I} |\psi_i|$ is locally finite, so that $\psi$ is continuous and well-defined. Furthermore,

$$1 = \left| \sum_{i \in I} \psi_i(x) \right| \leq \sum_{i \in I} |\psi_i(x)| = \psi(x)$$

holds for all $x \in G$. Thus, $\varphi_i := |\psi_i| / \psi$ is well-defined and continuous for each $i \in I$ and obviously satisfies $\operatorname{supp} \varphi_i \subset \operatorname{supp} \psi_i \subset x_i U$ as well as $0 \leq \varphi_i \leq 1$ and $\sum_{i \in I} \varphi_i \equiv 1$ on $G$. □

As discussed at the beginning of this subsection, our goal is to give an equivalent description of the Wiener amalgam space $W_Q(L^\infty, Y)$ using a discrete sequence space. More precisely, we want to show

$$\|f\|_{W_Q(L^\infty, Y)} \asymp \left\| \left( \|\varphi_i \cdot f\|_{L^\infty} \right)_{i \in I} \right\|_{Y_d} \text{ for all } f \in W_Q(L^\infty, Y),$$

where $Y_d$ is a suitable discrete sequence space obtained from $Y$. This space is introduced in the next definition:

**Definition 2.3.13.** (cf. [29, Definition 3.4] and [59, Equation (2.3)])

Let $G$ be an LCH group, let $X = (x_i)_{i\in I}$ be relatively separated in $G$ and let $Q \subset G$ be a relatively compact, measurable set of positive measure. Finally, assume that $(Y, \|\cdot\|_Y)$ is a solid function space on $G$. We call

$$Y_d(X,Q) := \left\{ (\lambda_i)_{i\in I} \in \mathbb{C}^I \,\middle|\, \sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} \in Y \right\}$$

the DISCRETE SEQUENCE SPACE ASSOCIATED TO $Y$. We also define

$$\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)} := \left\| \sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} \right\|_Y \text{ for } (\lambda_i)_{i\in I} \in Y_d(X,Q).$$

If $X$ and/or $Q$ are implied by the context, we omit them from the notation.

*Remark* 2.3.14. Let us make some brief remarks on the measurability of the function $\sum_{i\in I} |\lambda_i|\, \chi_{x_i Q}$, a point which is mostly neglected in other treatments. First of all, the series is locally uniformly convergent, since the family $(x_i Q)_{i\in I}$ is locally finite by Lemma 2.3.10. There are now two cases in which measurability is easy to ensure:

1. If $G$ is $\sigma$-compact, the index set $I$ is necessarily countable, see Lemma 2.3.10. Since each summand $|\lambda_i|\, \chi_{x_i Q}$ is measurable, the same is true of the countable(!) series $\sum_{i\in I} |\lambda_i|\, \chi_{x_i Q}$.

2. If the unit neighborhood $Q \subset G$ is open, then each summand $|\lambda_i|\, \chi_{x_i Q}$ is lower semicontinuous, see [33, Proposition 7.11]. By the same proposition,

$$f_{I_0} := \sum_{i\in I_0} |\lambda_i|\, \chi_{x_i Q}$$

is lower semicontinuous for each finite set $I_0 \subset I$. A final application of the same proposition shows that

$$\sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} = \sup \{ f_{I_0} \mid I_0 \subset I \}$$

is lower semicontinuous and hence also measurable.

By solidity of $Y$, as long as one of the two conditions above is satisfied, it is thus sufficient to find some $f \in Y$ with $\sum_{i\in I} |\lambda_i|\, \chi_{x_i Q} \leq f$ to conclude $(\lambda_i)_{i\in I} \in Y_d(X,Q)$ as well as $\left\|(\lambda_i)_{i\in I}\right\|_{Y_d} \leq \|f\|_Y$. ◀

Let us first treat the basic questions of solidity and completeness of the sequence space $Y_d(X,Q)$. Completeness as well as the embedding $Y_d(X,Q) \hookrightarrow \mathbb{C}^I$ are also shown in [59, Lemma 2.5], but the proof uses a somewhat unclear appeal to the solidity of $Y$ to justify the convergence of the sequences $(\lambda_i^{(n)})_{n\in\mathbb{N}}$ for each $i \in I$ as well as for the proof that $(\lambda_i^{(n)})_{i\in I}$ converges with respect to $\|\cdot\|_{Y_d}$ to the (pointwise)

limit $(\lambda_i)_{i\in I}$. Using our investigation of solid function spaces in Section 2.2, we clarify this argument.

**Lemma 2.3.15.** *Let $G$ be a $\sigma$-compact LCH group, let $X = (x_i)_{i\in I}$ be relatively separated in $G$ and let $Q \subset G$ be a relatively compact, measurable set of positive measure. Finally, assume that $(Y, \|\cdot\|_Y)$ is a solid function space on $G$.*

*If we consider $I$ as a measure space with the counting measure (and the power set as $\sigma$-algebra), then $Y_d(X,Q)$ is a solid function space on $I$.*

*If we define $c = (c_i)_{i\in I}$ by*

$$c_i := \begin{cases} \|\chi_{x_iQ}\|_Y , & \text{if } \chi_{x_iQ} \in Y, \\ 1, & \text{otherwise} \end{cases}$$

*We remark the the weight $c_i \in (0,\infty)$ can be chosen arbitrarily for the case $\chi_{x_iQ} \notin Y$. We just make the choice $c_i = 1$ for concreteness.*

*for $i \in I$, then $Y_d(X,Q)$ embeds continuously into $\ell^\infty_c(I)$ and thus also into $\mathbb{C}^I$, equipped with the product topology.*

*Furthermore,*

1. *Every triangle constant $C \geq 0$ for $\|\cdot\|_Y$ is also valid for $\|\cdot\|_{Y_d}$,*
2. *If $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0,1]$, the same is true for $\|\cdot\|_{Y_d}$,*
3. *$Y_d(X,Q)$ is complete if $Y$ is.*

*Proof.* Since $Q$ is of positive measure, $\chi_{x_iQ}$ does not vanish almost everywhere. This implies $c_i \in (0,\infty)$ for all $i \in I$. Furthermore, if $(\lambda_i)_{i\in I} \in Y_d(X,Q)$ and $j \in I$ are arbitrary, then the solidity of $Y$ yields

$$\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)} = \left\|\sum_{i\in I} |\lambda_i|\, \chi_{x_iQ}\right\|_Y \geq \left\||\lambda_j|\, \chi_{x_jQ}\right\|_Y = c_j \cdot |\lambda_j| .$$

Here, it is important to note that the identity $\left\||\lambda_j|\, \chi_{x_jQ}\right\|_Y = c_j \cdot |\lambda_j|$ also holds in the case $\chi_{x_jQ} \notin Y$, since this entails $\lambda_j = 0$. Indeed, if $\lambda_j \neq 0$, then measurability of $|\lambda_j|\, \chi_{x_jQ}$ together with solidity of $Y$ and with $|\lambda_j|\, \chi_{x_jQ} \leq \sum_{i\in I} |\lambda_i|\, \chi_{x_iQ} \in Y$ would imply $\chi_{x_jQ} \in Y$.

We have shown $(\lambda_i)_{i\in I} \in \ell^\infty_c(I)$ with $\left\|(\lambda_i)_{i\in I}\right\|_{\ell^\infty_c} \leq \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)}$, i.e. $Y_d(X,Q) \hookrightarrow \ell^\infty_c(I)$. In particular, this proves positive definiteness of $\|\cdot\|_{Y_d(X,Q)}$.

It is clear that $Y_d(X,Q)$ is closed under multiplication with complex scalars and that $\|\cdot\|_{Y_d(X,Q)}$ is homogeneous. To show that $\|\cdot\|_{Y_d(X,Q)}$ is a quasi-norm, let $C \geq 0$ be a triangle inequality for $\|\cdot\|_Y$. For sequences $\Lambda = (\lambda_i)_{i\in I}$ and $\Gamma = (\gamma_i)_{i\in I}$ in $Y_d(X,Q)$, we have

$$\sum_{i\in I} |(\Lambda+\Gamma)_i|\, \chi_{x_iQ} \leq \sum_{i\in I} |\lambda_i|\, \chi_{x_iQ} + \sum_{i\in I} |\gamma_i|\, \chi_{x_iQ}$$

and the right-hand side is contained in $Y$. Since the left-hand side is measurable (see Remark 2.3.14), solidity of $Y$ implies

$$\begin{aligned}\|\Lambda+\Gamma\|_{Y_d(X,Q)} &= \left\|\sum_{i\in I}|(\Lambda+\Gamma)_i|\,\chi_{x_iQ}\right\|_Y \\ &\le \left\|\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}+\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}\right\|_Y \\ &\le C\cdot\left[\left\|\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}\right\|_Y+\left\|\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}\right\|_Y\right] \\ &= C\cdot\left[\|\Lambda\|_{Y_d(X,Q)}+\|\Gamma\|_{Y_d(X,Q)}\right]<\infty.\end{aligned}$$

Thus, $Y_d(X,Q)$ is closed under addition and $\|\cdot\|_{Y_d(X,Q)}$ is a quasi-norm with triangle constant $C\ge 0$.

If $\|\cdot\|_Y$ is a $p$-norm, an analogous argument yields

$$\begin{aligned}\|\Lambda+\Gamma\|^p_{Y_d(X,Q)} &\le \left\|\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}+\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}\right\|^p_Y \\ &\le \left\|\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}\right\|^p_Y+\left\|\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}\right\|^p_Y \\ &= \|\Lambda\|^p_{Y_d(X,Q)}+\|\Gamma\|^p_{Y_d(X,Q)},\end{aligned}$$

so that $\|\cdot\|_{Y_d(X,Q)}$ is also a $p$-norm.

Solidity of $Y_d(X,Q)$ is straightforward: If $(\gamma_i)_{i\in I}\in Y_d(X,Q)$ and $(\lambda_i)_{i\in I}$ is a further sequence satisfying $|\lambda_i|\le|\gamma_i|$ for all $i\in I$ (which is the same as for *almost* all $i\in I$, since we are using the counting measure on $I$), then

$$0\le\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}\le\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}.$$

By solidity of $Y$, we get $\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}\in Y$, which means nothing but $(\lambda_i)_{i\in I}\in Y_d(X,Q)$. Furthermore,

$$\begin{aligned}\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)} &= \left\|\sum_{i\in I}|\lambda_i|\,\chi_{x_iQ}\right\|_Y \\ &\le \left\|\sum_{i\in I}|\gamma_i|\,\chi_{x_iQ}\right\|_Y=\left\|(\gamma_i)_{i\in I}\right\|_{Y_d(X,Q)}.\end{aligned}$$

It remains to show that $Y_d(X,Q)$ is complete if $Y$ is. By Corollary 2.2.13, we have to show $\lambda:=\sum_{n=1}^\infty\lambda^{(n)}\in Y_d(X,Q)$ for each collection of nonnegative sequences $\lambda^{(n)}=(\lambda_i^{(n)})_{i\in I}\in Y_d(X,Q)$ with $\|\lambda^{(n)}\|_{Y_d(X,Q)}\le C\cdot 2^{-n}$ for all $n\in\mathbb{N}$.

Since $Y$ is complete and each $f_n := \sum_{i\in I} \lambda_i^{(n)} \chi_{x_i Q} \in Y$ is a nonnegative function with $\|f_n\|_Y = \|\lambda^{(n)}\|_{Y_d(X,Q)} \leq C \cdot 2^{-n}$, the same corollary yields

$$\begin{aligned}\sum_{i\in I} \lambda_i \chi_{x_i Q} &= \sum_{i\in I} \left[ \left( \sum_{n=1}^{\infty} \lambda_i^{(n)} \right) \chi_{x_i Q} \right] \\ &= \sum_{n=1}^{\infty} f_n \in Y\end{aligned}$$

and hence $\lambda = (\lambda_i)_{i\in I} \in Y_d(X,Q)$. Here, we can interchange the sums since all summands are nonnegative (and because $I$ is countable by Lemma 2.3.10). □

The next lemma gives a sufficient criterion for independence of the space $Y_d(X,Q)$ from the choice of $Q$.

**Lemma 2.3.16.** *[59, Lemma 2.2] Let $G$ be a $\sigma$-compact LCH group and assume that the family $X = (x_i)_{i\in I}$ is relatively separated in $G$. Finally, let $(Y, \|\cdot\|_Y)$ be a solid function space on $G$.*

*If $Y$ is right invariant, then $Y_d(X,U) = Y_d(X,V)$ (with equivalent quasi-norms) holds for all measurable, precompact sets $U, V \subset G$ with nonempty interior.*

*The constant of equivalence for the quasi-norms can be chosen* independently *of the set $X$, only depending on $U, V, Y$.*

*We remark that the independence of the equivalence constant from the choice of $X$ is not explicitly stated in [59].*

*Proof.* Since $U$ has nonempty interior, we have $\overline{V} \subset G = \bigcup_{x\in G} U^\circ x$. By compactness of $\overline{V}$, we can thus choose $y_1, \dots, y_n \in G$ with

$$V \subset \overline{V} \subset \bigcup_{j=1}^{n} U^\circ y_j \subset \bigcup_{j=1}^{n} U y_j.$$

Let $C \geq 1$ be a triangle constant for $\|\cdot\|_Y$. By induction, we see $\left\| \sum_{j=1}^n f_j \right\|_Y \leq C^n \cdot \sum_{j=1}^n \|f_j\|_Y$ for all $f_1, \dots, f_n \in Y$.

Set $K := C^n \cdot \sum_{j=1}^n \| R_{y_j^{-1}} \|_{Y\to Y}$. Note that this choice of $K$ is independent of the family $X$.

Let $(\lambda_i)_{i\in I} \in Y_d(X,U)$. Using the identity $R_{y^{-1}} \chi_M = \chi_{My}$, we see

$$\begin{aligned}\sum_{i\in I} |\lambda_i| \chi_{x_i V} &\leq \sum_{i\in I} \left[ |\lambda_i| \sum_{j=1}^{n} \chi_{x_i U y_j} \right] \\ &= \sum_{j=1}^{n} R_{y_j^{-1}} \left( \sum_{i\in I} |\lambda_i| \chi_{x_i U} \right).\end{aligned}$$

By solidity of $Y$, this implies (since both sides of the inequality are measurable, cf. Remark 2.3.14)

$$\left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,V)} = \left\| \sum_{i\in I} |\lambda_i| \chi_{x_i V} \right\|_Y$$

$$\leq \left\| \sum_{j=1}^{n} R_{y_j^{-1}} \left( \sum_{i \in I} |\lambda_i| \, \chi_{x_i U} \right) \right\|_Y$$
$$\leq C^n \cdot \sum_{j=1}^{n} \left\| R_{y_j^{-1}} \left( \sum_{i \in I} |\lambda_i| \, \chi_{x_i U} \right) \right\|_Y$$
$$\leq K \cdot \left\| \sum_{i \in I} |\lambda_i| \, \chi_{x_i U} \right\|_Y = K \cdot \left\| (\lambda_i)_{i \in I} \right\|_{Y_d(X,U)}.$$

By symmetry, this completes the proof. □

We are now in a position to prove the promised characterization of $W(L^\infty, Y)$ using the discrete sequence space $Y_d$. Furthermore, we also establish independence of the space $W_Q(L^\infty, Y)$ from the choice of the (precompact) unit-neighborhood $Q \subset G$. The statement as well as the proof of the following theorem are essentially taken from [59, Theorem 2.3], with a few details added. The proof is given here to make the presentation more selfcontained.

**Theorem 2.3.17.** *[59, Theorem 2.3] Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid function space on $G$. Then the following statements are equivalent:*

1. *The space $W_Q(L^\infty, Y)$ is independent of the choice of the precompact, measurable unit neighborhood $Q \subset G$ (with equivalent quasi-norms for different choices).*
2. *For each relatively separated family $X = (x_i)_{i \in I}$ in $G$, the space $Y_d(X, Q)$ is independent of the choice of the precompact, measurable unit neighborhood $Q \subset G$ (with equivalent quasi-norms for different choices).*
3. *The Wiener amalgam space $W_Q(L^\infty, Y)$ is invariant under right translations (for each precompact, measurable unit neighborhood $Q$).*

*It is worth noting that Lemma 2.3.16 implies that condition 2 of the current theorem is satisfied if $Y$ is right invariant.*

*If these conditions are satisfied, an equivalent quasi-norm on $W_Q(L^\infty, Y)$ is given by*

$$\|f\|_{W(L^\infty, Y_d(X,Q))} := \left\| \left( \|\varphi_i \cdot f\|_{L^\infty} \right)_{i \in I} \right\|_{Y_d(X,Q)}, \tag{2.3.3}$$

*for each relatively separated family $X = (x_i)_{i \in I}$ in $G$ and each BUPU $(\varphi_i)_{i \in I}$ satisfying $\operatorname{supp} \varphi_i \subset x_i U$ for all $i \in I$ and some precompact set $U \subset G$.*

*The claimed equivalence of quasi-norms includes the assertion that for each $f \in L^\infty_{\mathrm{loc}}(G)$, finiteness of $\|f\|_{W(L^\infty, Y_d(X,Q))}$ is equivalent to $f \in W_Q(L^\infty, Y)$.*

*Furthermore, for each precompact, measurable unit neighborhood $Q \subset G$ and each compact set $K \subset G$, there is a constant $C_{K,Q} > 0$ with*

$$\left\| \left\| R_x \right\| \right\|_{W_Q(L^\infty, Y) \to W_Q(L^\infty, Y)} \leq C_{K,Q} \text{ for all } x \in K.$$

*Finally, under the above conditions, $(W_Q(L^\infty, Y))_d(X, U) = Y_d(X, U)$ holds for all measurable, precompact unit neighborhoods $U, Q \subset G$. The constant of equivalence for the quasi-norms of the two spaces can be chosen* independently *of $X$, only depending on $Q, U, Y$.*

*Remark.* The above result justifies the notations $W(L^\infty, Y)$ instead of $W_Q(L^\infty, Y)$ as well as $Y_d(X)$ instead of $Y_d(X, Q)$ which we will use in the following.

It is worth noting that the proof shows that the constant of norm equivalence between $\|\cdot\|_{Y_d(X,U)}$ and $\|\cdot\|_{Y_d(X,V)}$ in statement (2) can be chosen independently of the family $X$, only depending on $U, V, Y$.

*Proof.* We first show that statement (2) implies that the right-hand side of equation (2.3.3) yields an equivalent quasi-norm on $W_Q(L^\infty, Y)$ for each relatively separated family $X = (x_i)_{i\in I}$, each relatively compact, measurable unit neighorhood $Q \subset G$, each symmetric, measurable unit-neighborhood $U \subset G$ with $U \cdot U \subset Q$ and each $U$-BUPU $(\varphi_i)_{i\in I}$ satisfying $\operatorname{supp}\varphi_i \subset x_i U$ for all $i \in I$.

If $x \in G$ with $x_i U \subset xQ$, then

$$\begin{aligned}\|\varphi_i \cdot f\|_{L^\infty} &\leq \|f \cdot \chi_{x_i U}\|_{L^\infty} \\ &\leq \|f \cdot \chi_{xQ}\|_{L^\infty} \\ &= (M_Q f)(x).\end{aligned}$$

We now observe that $x \in x_i U$ implies $x_i U \subset xQ$, since $x \in x_i U$ means $x = x_i u$ for some $u \in U$, which yields

$$x_i v = x u^{-1} v \in x U^{-1} U = xUU \subset xQ$$

for all $v \in U$. All in all, we conclude

$$\begin{aligned}\sum_{i\in I} \|\varphi_i \cdot f\|_{L^\infty} \chi_{x_i U}(x) &\leq (M_Q f)(x) \cdot \sum_{i\in I} \chi_{x_i U}(x) \\ &\leq (M_Q f)(x) \cdot \left|\left\{i \in I \,\middle|\, x_i \overline{U} \cap x\{1_G\} \neq \varnothing\right\}\right| \\ &\leq C_{X,\overline{U}\{1_G\}^{-1}} \cdot (M_Q f)(x),\end{aligned}$$

where the last step used Lemma 2.3.10. Thus, if $f \in W_Q(L^\infty, Y)$, then the solidity of $Y$ (together with Remark 2.3.14 which is applicable since $G$ is $\sigma$-compact) yields $\sum_{i\in I} \|\varphi_i \cdot f\|_{L^\infty} \chi_{x_i U} \in Y$ with

$$\begin{aligned}\left\|(\|\varphi_i \cdot f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,U)} &= \left\|\sum_{i\in I} \|\varphi_i \cdot f\|_{L^\infty} \chi_{x_i U}\right\|_Y \\ &\leq C_{X,\overline{U}} \cdot \|M_Q f\|_Y \\ &= C_{X,\overline{U}} \cdot \|f\|_{W_Q(L^\infty,Y)}.\end{aligned}$$

By statement (2), we know that the quasi-norms $\|\cdot\|_{Y_d(X,U)}$ and $\|\cdot\|_{Y_d(X,Q)}$ are equivalent. Thus,

$$\begin{aligned}\|f\|_{W(L^\infty,Y_d(X,Q))} &= \left\|(\|\varphi_i \cdot f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,Q)} \\ &\lesssim \left\|(\|\varphi_i \cdot f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,U)} \\ &\leq C_{X,\overline{U}} \cdot \|f\|_{W_Q(L^\infty,Y)} < \infty.\end{aligned}$$

Conversely, assume $f \in L^\infty_{\mathrm{loc}}(G)$ with $(\|\varphi_i \cdot f\|_{L^\infty})_{i\in I} \in Y_d(X,Q)$ and recall from statement (2) that $Y_d(X,Q) = Y_d\left(X, UQ^{-1}\right)$. We note that each $x \in G$ with $xQ \cap x_i U \neq \varnothing$ already satisfies $x \in x_i UQ^{-1}$. Together with $\sum_{i\in I} \varphi_i \equiv 1$, we conclude

$$\begin{aligned} 0 \leq (M_Q f)(x) &= \|f \cdot \chi_{xQ}\|_{L^\infty} \\ &= \left\| \chi_{xQ} \cdot \sum_{i\in I \text{ with } xQ\cap x_i U \neq \varnothing} \varphi_i f \right\|_{L^\infty} \\ &\leq \sum_{i\in I \text{ with } xQ\cap x_i U\neq\varnothing} \|\varphi_i \cdot f\|_{L^\infty} \\ &\leq \sum_{i\in I} \|\varphi_i \cdot f\|_{L^\infty} \cdot \chi_{x_i UQ^{-1}}(x). \end{aligned}$$

By solidity of $Y$ (and since $M_Q f$ is measurable), this yields

$$\begin{aligned} \|f\|_{W_Q(L^\infty,Y)} &= \|M_Q f\|_Y \\ &\leq \left\| \sum_{i\in I} \|\varphi_i \cdot f\|_{L^\infty} \cdot \chi_{x_i UQ^{-1}} \right\|_Y \\ &= \left\| (\|\varphi_i \cdot f\|_{L^\infty})_{i\in I} \right\|_{Y_d(X,UQ^{-1})} \\ &\lesssim \left\| (\|\varphi_i \cdot f\|_{L^\infty})_{i\in I} \right\|_{Y_d(X,Q)} < \infty. \end{aligned}$$

Hence, we have established the property stated at the beginning of the proof and we have also shown that $\|f\|_{W(L^\infty,Y_d(X,Q))}$ is finite iff $f \in W_Q(L^\infty, Y)$, but only under more restrictive conditions on $U$ than in the statement of the theorem. We will remove these restrictions later in the proof.

As a consequence, we can now show the implication "(2)⇒(1)": Let $Q, Q' \subset G$ be two precompact, measurable unit neighborhoods. By choosing a symmetric, open unit neighborhood $U \subset G$ satisfying $U \cdot U \subset Q \cap Q'$ as well as a $U$-BUPU $\Phi = (\varphi_i)_{i\in I}$ with $\operatorname{supp} \varphi_i \subset x_i U$ for some relatively separated family $X = (x_i)_{i\in I}$, we conclude

$$\|f\|_{W_Q(L^\infty,Y)} \asymp \left\| (\|\varphi_i \cdot f\|_{L^\infty})_{i\in I} \right\|_{Y_d(X,U)} \asymp \|f\|_{W_{Q'}(L^\infty,Y)} \text{ for } f \in L^\infty_{\mathrm{loc}}(G),$$

where either side is finite iff $\left\| (\|\varphi_i \cdot f\|_{L^\infty})_{i\in I} \right\|_{Y_d(X,U)}$ is. We remark that a unit neighborhood $U$ as above exists by elementary properties of topological groups (cf. [32, Proposition (2.1)]) and that existence of the $U$-BUPU $\Phi$ is ensured by Lemma 2.3.12.

Next, we show that statement (2) implies that the right hand side of equation (2.3.3) yields equivalent quasi-norms for different BUPUs $(\varphi_i)_{i\in I}$ and $\left(\psi_j\right)_{j\in J}$ and different localizing families $X, Z$. To this end, let $X = (x_i)_{i\in I}$ and $Z = \left(z_j\right)_{j\in J}$ be relatively separated families in $G$ and let $V, W \subset G$ be precompact open unit neighborhoods with $\operatorname{supp} \varphi_i \subset x_i V$ and $\operatorname{supp} \psi_j \subset z_j W$ for all $i \in I$ and $j \in J$.

Set $U_0 := \overline{V} \cup \overline{W}$ and $U := U_0 \cup U_0^{-1}$ as well as $Q := U \cdot U$. Then $Q$ is a compact unit neighborhood and $U$ is a symmetric, compact (hence measurable) unit neighborhood with $U \cdot U \subset Q$. Using $\operatorname{supp} \varphi_i \subset x_i V \subset x_i U$ and $\operatorname{supp} \psi_j \subset z_j W \subset z_j U$, the statement at the beginning of the proof – together with statement (2) – implies

$$\begin{aligned}
\left\| \left( \| \varphi_i \cdot f \|_{L^\infty} \right)_{i \in I} \right\|_{Y_d(X,V)} &\asymp \left\| \left( \| \varphi_i \cdot f \|_{L^\infty} \right)_{i \in I} \right\|_{Y_d(X,Q)} \\
&\asymp \| f \|_{W_Q(L^\infty, Y)} \\
&\asymp \left\| \left( \| \psi_j \cdot f \|_{L^\infty} \right)_{j \in J} \right\|_{Y_d(Z,Q)} \\
&\asymp \left\| \left( \| \psi_j \cdot f \|_{L^\infty} \right)_{j \in J} \right\|_{Y_d(Z,W)},
\end{aligned}$$

which shows the independence of the right hand side of equation (2.3.3) from the choice of the BUPU and of the relatively separated family $X$.

For "(3)⇒(2)", let $U, V \subset G$ be relatively compact, measurable unit neighborhoods. Choose an open, symmetric unit neighborhood $Q$ with $Q \cdot Q \subset V$. Note that $Q$ only depends on $V$ and not on $X$. By assumption, $Z := W_Q(L^\infty, Y)$ is invariant under right translations. Thus, Lemma 2.3.16 yields some $C = C(Q, U, Z) = C(Q, U, Y) > 0$ with

$$\| \cdot \|_{Z_d(X,U)} \leq C \cdot \| \cdot \|_{Z_d(X,Q)}. \tag{2.3.4}$$

Now, let $(\lambda_i)_{i \in I} \in Y_d(X, V)$. We observe that for all $y \in G$ and $i \in I$, if $\chi_{xQ}(y) \neq 0 \neq \chi_{x_i Q}(y)$, then

$$x \in yQ^{-1} = yQ \subset x_i QQ \subset x_i V.$$

Hence,

$$\begin{aligned}
M_Q \left( \sum_{i \in I} |\lambda_i| \, \chi_{x_i Q} \right)(x) &= \left\| \chi_{xQ} \cdot \sum_{i \in I} |\lambda_i| \, \chi_{x_i Q} \right\|_{L^\infty} \\
&\leq \sum_{i \in I} |\lambda_i| \, \chi_{x_i V}(x). \qquad (2.3.5)
\end{aligned}$$

We recall from Lemma 2.3.3 that the estimate $|f| \leq M_Q f$ is true almost everywhere for any $f \in L^\infty_{\mathrm{loc}}(G)$. Using the solidity of $Y$, as well as the estimates (2.3.4) and (2.3.5), we conclude

$$\begin{aligned}
\left\| (\lambda_i)_{i \in I} \right\|_{Y_d(X,U)} &= \left\| \sum_{i \in I} |\lambda_i| \, \chi_{x_i U} \right\|_Y \\
&\leq \left\| M_Q \left[ \sum_{i \in I} |\lambda_i| \, \chi_{x_i U} \right] \right\|_Y
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{i\in I} |\lambda_i| \, \chi_{x_i U} \right\|_{W_Q(L^\infty,Y)} = \left\| (\lambda_i)_{i\in I} \right\|_{Z_d(X,U)} \\
&\leq C \cdot \left\| (\lambda_i)_{i\in I} \right\|_{Z_d(X,Q)} = C \cdot \left\| \sum_{i\in I} |\lambda_i| \, \chi_{x_i Q} \right\|_{W_Q(L^\infty,Y)} \\
&= C \cdot \left\| M_Q \left( \sum_{i\in I} |\lambda_i| \, \chi_{x_i Q} \right) \right\|_Y \\
&\leq C \cdot \left\| \sum_{i\in I} |\lambda_i| \, \chi_{x_i V} \right\|_Y = C \cdot \left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,V)} .
\end{aligned}
$$

By symmetry, this completes the proof of "(3)⇒(2)". Since this proves that $Y_d(X,\cdot)$ is independent of the chosen precompact unit neighborhood (with a constant of norm-equivalence that can be chosen independently of $X$), the above chain of inequalities also implies

$$
\begin{aligned}
\left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,W)} &\lesssim \left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,U)} \lesssim \left\| (\lambda_i)_{i\in I} \right\|_{Z_d(X,U)} \\
&\lesssim \left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,V)} \lesssim \left\| (\lambda_i)_{i\in I} \right\|_{Y_d(X,W)}
\end{aligned}
$$

for arbitrary precompact, measurable unit neighborhoods $U, W \subset G$. Here, the implied constants are independent of the family $X$, since the constant $C$ above is also independent of $X$ (see the discussion before equation (2.3.4)). By recalling $Z = W_Q(L^\infty, Y)$, this proves the last claim of the theorem.

For the proof of "(1)⇒(3)", fix some compact unit neighborhood $K_0 \subset G$. Let $K \subset G$ be an arbitrary compact set. Define $V := K_0 \cup \overline{Q}K$. Let $f \in W_Q(L^\infty, Y)$ be arbitrary. Using the translation invariance of the $L^\infty$-norm, we get for each $y \in K$ the estimate

$$
\begin{aligned}
0 \leq M_Q\left[R_y f\right](x) &= \left\| \chi_{xQ} \cdot R_y f \right\|_{L^\infty} \\
&= \left\| f \cdot R_{y^{-1}} \chi_{xQ} \right\|_{L^\infty} \\
&= \left\| f \cdot \chi_{xQy} \right\|_{L^\infty} \\
&\leq \left\| f \cdot \chi_{xV} \right\|_{L^\infty} = \left(M_V f\right)(x),
\end{aligned}
$$

since $y \in K$ implies $xQy \subset x\overline{Q}K \subset xV$. Using the solidity of $Y$ and the measurability of $M_Q\left[R_y f\right]$ and of $M_V f$, we conclude

$$
\begin{aligned}
\left\| R_y f \right\|_{W_Q(L^\infty,Y)} &= \left\| M_Q\left(R_y f\right) \right\|_Y \leq \left\| M_V f \right\|_Y \\
&= \left\| f \right\|_{W_V(L^\infty,Y)} \leq C_{V,Q,Y} \cdot \left\| f \right\|_{W_Q(L^\infty,Y)} < \infty,
\end{aligned}
$$

where the last step made use of the independence of $W_Q(L^\infty, Y)$ from the chosen unit neighborhood $Q$.

This yields $\left\| \left| R_y \right| \right\|_{W_Q(L^\infty,Y) \to W_Q(L^\infty,Y)} \leq C_{V,Q,Y}$ for all $y \in K$, as claimed. Here, we observe that $V = K_0 \cup \overline{Q}K$ only depends on $Q$ and $K$. □

The above theorem indicates that right invariance of $W_Q(L^\infty, Y)$ is an important property. Hence, it is worth noting that left/right translation invariance of $Y$ implies the corresponding invariance for the Wiener amalgam space $W_Q(L^\infty, Y)$. We remark that it is possible for $W_Q(L^\infty, Y)$ to be right invariant even though $Y$ is not, cf. [59, Section 6].

**Lemma 2.3.18.** *Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid function space on $G$. Finally, assume that $Q \subset G$ is a measurable, precompact unit neighborhood. Then the following are true:*

1. *For each $f \in L^\infty_{\mathrm{loc}}(G)$, we have*

$$M_Q[L_x f] = L_x[M_Q f] \qquad \text{and} \qquad M_Q[R_x f] = M_{Qx} f.$$

2. *(cf. [59, Lemma 3.3]) If $L_x : Y \to Y$ is bounded for some $x \in G$, then the same is true of $L_x : W_Q(L^\infty, Y) \to W_Q(L^\infty, Y)$. We even have*

$$|||L_x|||_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} \leq |||L_x|||_{Y\to Y}.$$

3. *If $\|\cdot\|_Y$ is a $p$-norm and if $R_x : Y \to Y$ is bounded for each $x \in G$, then so is $R_x : W_Q(L^\infty, Y) \to W_Q(L^\infty, Y)$. We have the estimate*

$$|||R_x|||_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} \leq \left(\sum_{i=1}^n |||R_{x_i}|||^p_{Y\to Y}\right)^{1/p}$$

*for each sequence $(x_i)_{i\in\underline{n}}$ in $G$ with $Qx \subset \bigcup_{i=1}^n x_i Q$.*

*Proof.* We first compute, using the translation invariance of $\|\cdot\|_{L^\infty}$,

$$\begin{aligned} M_Q[L_x f](y) &= \left\|\chi_{yQ} \cdot L_x f\right\|_{L^\infty} \\ &= \left\|L_x\left(\chi_{x^{-1}yQ} \cdot f\right)\right\|_{L^\infty} \\ &= \left\|\chi_{x^{-1}yQ} \cdot f\right\|_{L^\infty} \\ &= (M_Q f)(x^{-1}y) = L_x[M_Q f](y). \end{aligned}$$

Similarly,

$$\begin{aligned} M_Q[R_x f](y) &= \left\|\chi_{yQ} \cdot R_x f\right\|_{L^\infty} = \left\|R_x\left(\chi_{yQx} \cdot f\right)\right\|_{L^\infty} \\ &= \left\|\chi_{yQx} \cdot f\right\|_{L^\infty} = (M_{Qx} f)(y). \end{aligned}$$

Let us now assume that $L_x : Y \to Y$ is bounded. For $f \in W_Q(L^\infty, Y)$, this implies $M_Q[L_x f] = L_x[M_Q f] \in Y$ and thus $L_x f \in W_Q(L^\infty, Y)$ with

$$\begin{aligned} \|L_x f\|_{W_Q(L^\infty,Y)} &= \|M_Q[L_x f]\|_Y = \|L_x[M_Q f]\|_Y \\ &\leq |||L_x|||_{Y\to Y} \cdot \|M_Q f\|_Y = |||L_x|||_{Y\to Y} \cdot \|f\|_{W_Q(L^\infty,Y)}. \end{aligned}$$

For the right-invariance suppose $Qx \subset \bigcup_{i=1}^{n} x_i Q$. Since $\overline{Q}x$ is compact and because $Q$ is a unit neighborhood, one can always choose such a finite family. Using the identity from above together with $yQx \subset \bigcup_{i=1}^{n} yx_i Q$, we see

$$\begin{aligned} M_Q\left[R_x f\right](y) &= \left\|\chi_{yQx} \cdot f\right\|_{L^\infty} \\ &\leq \sum_{i=1}^{n} \left\|\chi_{yx_iQ} \cdot f\right\|_{L^\infty} \\ &= \sum_{i=1}^{n} M_Q f\left(yx_i\right) \\ &= \sum_{i=1}^{n} R_{x_i}\left[M_Q f\right](y). \end{aligned}$$

Using the solidity and the right-invariance of $Y$, we get $M_Q\left[R_x f\right] \in Y$, and thus $R_x f \in W_Q\left(L^\infty, Y\right)$ with

$$\begin{aligned} \left\|R_x f\right\|_{W_Q(L^\infty,Y)}^p = \left\|M_Q\left[R_x f\right]\right\|_Y^p &\leq \left\|\sum_{i=1}^{n} R_{x_i}\left[M_Q f\right]\right\|_Y^p \\ &\leq \sum_{i=1}^{n} \left\|R_{x_i}\left[M_Q f\right]\right\|_Y^p \\ &\leq \left(\sum_{i=1}^{n} \left|\left|\left|R_{x_i}\right|\right|\right|_{Y\to Y}^p\right) \cdot \left\|M_Q f\right\|_Y^p \\ &= \left(\sum_{i=1}^{n} \left|\left|\left|R_{x_i}\right|\right|\right|_{Y\to Y}^p\right) \cdot \left\|f\right\|_{W_Q(L^\infty,Y)}^p. \end{aligned}$$

We remark that a different proof can be obtained (as in [59, Corollary 4.2]) by noting that right invariance of $Y$ implies (by Lemma 2.3.16) that $Y_d\left(X, Q\right)$ is independent of $Q$, so that Theorem 2.3.17 proves that $W_Q\left(L^\infty, Y\right)$ is right-invariant. But this would not (directly) yield the estimate for $\left|\left|\left|R_x\right|\right|\right|_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)}$ obtained above. □

One consequence of Theorem 2.3.17 is the following "sampling" result for $W\left(C^0, Y\right)$ which will play a role for the proof of atomic decompositions for coorbit spaces in the next section, cf. in particular Lemma 2.4.14.

**Corollary 2.3.19.** *Let $G$ be a $\sigma$-compact LCH group and let $\left(Y, \|\cdot\|_Y\right)$ be a solid function space on $G$ such that $W_Q\left(L^\infty, Y\right)$ is right-invariant for each precompact, measurable unit neighborhood $Q$.*

*Then for each relatively separated family $X = (x_i)_{i\in I}$ which admits a BUPU $\Phi = (\varphi_i)_{i\in I}$ with $\operatorname{supp}\varphi_i \subset x_i U$ for all $i \in I$ and some relatively compact unit neighborhood $U$, the map*

$$\Gamma : W\left(C^0, Y\right) \to Y_d\left(X\right), f \mapsto \left(f\left(x_i\right)\right)_{i\in I}$$

*is well-defined and bounded.*

*Proof.* Let $f \in W\left(C^0, Y\right) \subset W\left(L^\infty, Y\right)$ be arbitrary. By Theorem 2.3.17, we get $\Lambda := \left(\|\varphi_i \cdot f\|_{L^\infty}\right)_{i\in I} \in Y_d\left(X\right)$ with

$$\|\Lambda\|_{Y_d(X,QQ)} \leq C_{Q,X,\Phi} \cdot \|f\|_{W(L^\infty,Y)} = C_{Q,X,\Phi} \cdot \|f\|_{W(C^0,Y)}$$

for some constant $C_{Q,X,\Phi} > 0$.

Furthermore, Lemma 2.3.10 yields a constant $C_{Q,X} > 0$ satisfying

$$\left|\left\{j \in I \,\middle|\, x_j Q \cap xQ \neq \varnothing\right\}\right| \leq C_{Q,X} \text{ for all } x \in G.$$

Now, we observe $f = \sum_{i\in I} \varphi_i f = \sum_{i\in I} L_{x_i} f_i$ with

$$f_i := L_{x_i^{-1}}\left(\varphi_i \cdot f\right) \in C\left(G\right) \cap L^\infty\left(G\right)$$

and $\operatorname{supp} f_i \subset Q$ for all $i \in I$. Note that $0 \neq \left(L_{x_i} f_i\right)\left(x_j\right) = f_i\left(x_i^{-1} x_j\right)$ implies on the one hand that $x_i^{-1} x_j \in Q$, i.e., $x_j \in x_i Q$ and hence $x_j Q \cap x_i Q \neq \varnothing$ and on the other hand that $x_j Q \subset x_i QQ$. Together, we arrive at

$$\begin{aligned}
\sum_{j\in I} \left|f\left(x_j\right)\right| \chi_{x_j Q}\left(x\right) &\leq \sum_{j\in I}\sum_{i\in I} \left|\left(L_{x_i} f_i\right)\left(x_j\right)\right| \cdot \chi_{x_j Q}\left(x\right) \\
&= \sum_{i\in I}\sum_{j\in I} \left|\left(L_{x_i} f_i\right)\left(x_j\right)\right| \cdot \chi_{x_j Q}\left(x\right) \\
&\leq \sum_{i\in I} \sum_{\substack{j\in I \\ x_j Q \cap x_i Q \neq \varnothing}} \|L_{x_i} f_i\|_{L^\infty} \chi_{x_i QQ}\left(x\right) \\
&\leq C_{Q,X} \cdot \sum_{i\in I} \|f_i\|_{L^\infty} \chi_{x_i QQ}\left(x\right).
\end{aligned}$$

Here, we used that $\left|f_i\left(x\right)\right| \leq \|f_i\|_{L^\infty}$ holds for all $x \in G$, since $f$ is continuous.

But $\left(\|f_i\|_{L^\infty}\right)_{i\in I} = \Lambda \in Y_d\left(X, QQ\right)$, so that the right-hand side of the above equation is contained in $Y$. By solidity of $Y$, we conclude $\sum_{j\in I} \left|f\left(x_j\right)\right| \chi_{x_j Q} \in Y$ and hence $\left(f\left(x_j\right)\right)_{j\in I} \in Y_d\left(X, Q\right)$ with

$$\begin{aligned}
\left\|\left(f\left(x_j\right)\right)_{j\in I}\right\|_{Y_d(X,Q)} &= \left\|\sum_{j\in I} \left|f\left(x_j\right)\right| \chi_{x_j Q}\right\|_Y \\
&\leq C_{Q,X} \cdot \left\|\sum_{i\in I} \|f_i\|_{L^\infty} \chi_{x_i QQ}\right\|_Y \\
&= C_{Q,X} \cdot \|\Lambda\|_{Y_d(X,QQ)} \\
&\leq C_{Q,X} \cdot C_{Q,X,\Phi} \cdot \|f\|_{W(C^0,Y)} < \infty. \qquad \square
\end{aligned}$$

As a further consequence of Theorem 2.3.17, we get the following result regarding embeddings between Wiener amalgam spaces:

**Corollary 2.3.20.** *Let $G$ be a $\sigma$-compact LCH group and let $\left(Y, \|\cdot\|_Y\right)$ and $\left(Z, \|\cdot\|_Z\right)$ be two right invariant solid function spaces on $G$.*

*If $Y_d(X,Q) \hookrightarrow Z_d(X,Q)$ holds for some precompact, measurable unit neighborhood $Q \subset G$ and every relatively separated family $X = (x_i)_{i\in I}$ in $G$, then*

$$W(L^\infty, Y) \hookrightarrow W(L^\infty, Z)$$

*is true.*

*Proof.* By Lemma 2.3.12, there is a $Q$-BUPU $(\varphi_i)_{i\in I}$ with some localizing family $X = (x_i)_{i\in I}$. Let $f \in W(L^\infty, Y)$. Using Theorem 2.3.17, we get the estimate

$$\begin{aligned}\|f\|_{W(L^\infty,Z)} &\lesssim \left\|(\|\varphi_i f\|_{L^\infty})_{i\in I}\right\|_{Z_d(X,Q)} \\ &\lesssim \left\|(\|\varphi_i f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,Q)} \lesssim \|f\|_{W(L^\infty,Y)} < \infty,\end{aligned}$$

where the implied constants are independent of $f$. □

### 2.3.3 *Convolution relations for Wiener amalgam spaces*

In this subsection, we want to establish two convolution relations for Wiener amalgam spaces. More precisely, we will show

$$W(L^\infty, Y) * W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(C^0, Y\right)$$

and

$$W^R_Q(L^\infty, W(L^\infty, Y)) * \left[W^R\left(L^\infty, W\left(L^\infty, L^p_w\right)\right)\right]^\vee \hookrightarrow W^R_Q(L^\infty, W(L^\infty, Y))$$

for a suitable $p \in (0,1]$ and certain weights $v, w$. The space $W^R_Q(L^\infty, Z)$ used here is a RIGHT-SIDED WIENER AMALGAM SPACE, which we will define in this subsection.

For the proof of these convolution relations, we will use the equivalent quasi-norm

$$\|f\|_{W(L^\infty, Y_d(X,Q))} := \left\|(\|\varphi_i \cdot f\|_{L^\infty})_{i\in I}\right\|_{Y_d(X,Q)},$$

for $W(L^\infty, Y)$ given in equation (2.3.3) of Theorem 2.3.17. To be precise, we will apply this for $Y = L^p_w$. Hence, we first have to verify the expected relation $Y_d(X,Q) = \ell^p_{w_X}(I)$ with $w_X(i) = w(x_i)$.

**Lemma 2.3.21.** *Let $G$ be a $\sigma$-compact LCH group and let $w : G \to (0,\infty)$ be a measurable, submultiplicative weight. Assume that $m : G \to (0,\infty)$ is measurable and right moderate with respect to $w$.*

*For each $p \in (0,\infty)$, each measurable precompact set $Q \subset G$ with positive measure and each relatively separated family $X = (x_i)_{i\in I}$ in $G$, we have*

$$\left(L^p_m(G)\right)_d(X,Q) = \ell^p_{m_X}(I)$$

*with equivalent quasi-norms, where $m_X$ is given by*

$$m_X : I \to (0,\infty), i \mapsto m(x_i).$$

*Proof.* Since $X$ is relatively separated, Lemma 2.3.10 yields a constant $N = N(X, Q) \in \mathbb{N}$ such that $|I_x| \leq N$ holds for all $x \in G$ with

$$I_x := \{i \in I \,|\, \chi_{x_i Q}(x) \neq 0\} \subset \left\{i \in I \,\middle|\, x_i \overline{Q} \cap x\{1_G\} \neq \varnothing\right\}.$$

This implies for each $x \in G$ and each sequence $(\lambda_i)_{i\in I} \in \mathbb{C}^I$ the estimate

$$\begin{aligned}\left(\sum_{i\in I} |\lambda_i| \chi_{x_i Q}(x)\right)^p &\leq (|I_x| \cdot \max\{|\lambda_i| \,|\, i \in I_x\})^p \\ &\leq N^p \cdot \max\left\{|\lambda_i|^p \,\middle|\, i \in I_x\right\} \\ &\leq N^p \cdot \sum_{i\in I_x} |\lambda_i|^p \\ &= N^p \cdot \sum_{i\in I} |\lambda_i|^p \chi_{x_i Q}(x).\end{aligned}$$

Conversely,

$$\begin{aligned}\left(\sum_{i\in I} |\lambda_i| \chi_{x_i Q}(x)\right)^p &\geq [\max\{|\lambda_i| \,|\, i \in I_x\}]^p \\ &= \max\left\{|\lambda_i|^p \,\middle|\, i \in I_x\right\} \\ &\geq \frac{1}{N} \cdot \sum_{i\in I_x} |\lambda_i|^p \\ &= \frac{1}{N} \cdot \sum_{i\in I} |\lambda_i|^p \chi_{x_i Q}(x).\end{aligned}$$

All in all, this implies

$$\begin{aligned}\left\|(\lambda_i)_{i\in I}\right\|^p_{\left(L^p_m\right)_d(X,Q)} &= \int_G \left(\sum_{i\in I} |\lambda_i| \chi_{x_i Q}(x)\right)^p \cdot (m(x))^p \, \mathrm{d}x \\ &\asymp \int_G \sum_{i\in I} |\lambda_i|^p (m(x))^p \chi_{x_i Q}(x) \, \mathrm{d}x \\ &= \sum_{i\in I} |\lambda_i|^p \int_Q (m(x_i q))^p \, \mathrm{d}q,\end{aligned}$$

where the implied constant only depends on $N$ and $p$. The interchange of summation and integration is justified by the monotone convergence theorem, since Lemma 2.3.10 shows that $I$ is countable.

Finally, since $m$ is right-moderate with respect to $w$, Corollary 2.2.23 yields a constant $C = C(Q, w) > 0$ with

$$\frac{1}{C} \cdot m(xq) \leq m(x) \leq C \cdot m(xq) \tag{2.3.6}$$

for all $x \in G$ and $q \in \overline{Q}$. Thus,

$$\int_Q (m(x_i q))^p \, \mathrm{d}q \asymp (m(x_i))^p = (m_X(i))^p ,$$

where the implied constant only depends on $C, Q$ and $p$. All in all, we have shown

$$\left\| (\lambda_i)_{i\in I} \right\|^p_{\left(L^p_m\right)_d(X,Q)} \asymp \sum_{i\in I} \left[ |\lambda_i|^p \cdot (m(x_i))^p \right] = \left\| (\lambda_i)_{i\in I} \right\|^p_{\ell^p_{m_X}(I)}$$

for each $p \in (0, \infty)$. □

For the proof of the convolution relations for Wiener amalgam spaces, we will need the fact that the application of the maximal function $M_Q$ can be "passed on" to the right factor of a convolution, i.e. $M_Q(f * g) \leq |f| * M_Q g$. To prove this, we first need to establish an "inversion" of the dependency of the variables for the maximal function: By definition of the maximal function, we have $|f(xq)| \leq (M_Q f)(x)$ for each $x \in G$ and all $q \in Q \setminus N_x$, where $N_x \subset Q$ is a null-set which potentially depends on $x$. But for many applications, including the estimate for $M_Q(f * g)$ mentioned above, one actually wants to choose $q \in Q$ freely, independent of $x \in G$. In most treatments of Wiener spaces, this is used without further justification. A formal proof – at least in the $\sigma$-compact case – is given in the following lemma:

**Lemma 2.3.22.** *Let $G$ be a $\sigma$-compact LCH group, let $f \in L^\infty_{\mathrm{loc}}(G)$ and assume that $Q \subset G$ is precompact and measurable. Then there is a null-set $N \subset G$ and for each $q \in Q \setminus N$, there is some null-set $N_q \subset G$ with*

$$|f(xq)| \leq (M_Q f)(x) \qquad \forall x \in G \setminus N_q.$$

*Proof.* Let

$$\Gamma := \{(x, q) \in G \times Q \mid |f(xq)| > (M_Q f)(x)\} .$$

Since $Q, f$ are Borel measurable and because $M_Q f$ is Borel measurable by Lemma 2.3.4, we see that $\Gamma$ is a Borel measurable subsets of $G \times G$.

Let $\mu$ be a Haar measure on $G$. Since $G$ is $\sigma$-compact, Fubini's theorem for Radon measures (cf. [33, Theorem 7.27]) implies

$$\int_G \mu(\Gamma^q) \, \mathrm{d}\mu(q) = \mu \mathbin{\widehat{\times}} \mu(\Gamma) = \int_G \mu(\Gamma_x) \, \mathrm{d}\mu(x) , \tag{2.3.7}$$

where $\Gamma_x := \{q \in G \mid (x, q) \in \Gamma\}$ and $\Gamma^q := \{x \in G \mid (x, q) \in \Gamma\}$ are the sections of $\Gamma$ and $\mu \mathbin{\widehat{\times}} \mu$ is the RADON PRODUCT of $\mu$ with itself, cf. [33, Theorem 7.26].

By definition of the maximal function $M_Q f$, there is for each $x \in G$ a null-set $N(x) \subset xQ$ with $|f(y)| \leq (M_Q f)(x)$ for all $y \in xQ \setminus N(x)$. For $q \in Q \setminus \left(x^{-1}N(x)\right)$, this yields $y := xq \in xQ \setminus N(x)$ and thus

$$|f(xq)| = |f(y)| \leq (M_Q f)(x),$$

which means $(x,q) \notin \Gamma$, i.e. $q \notin \Gamma_x$. Together with $\Gamma_x \subset Q$, we arrive at $\Gamma_x \subset x^{-1}N(x)$, which yields

$$0 \leq \mu(\Gamma_x) \leq \mu\left(x^{-1}N(x)\right) = \mu(N(x)) = 0.$$

Using equation (2.3.7), we derive

$$\int_G \mu(\Gamma^q)\, \mathrm{d}\mu(q) = \int_G \mu(\Gamma_x)\, \mathrm{d}\mu(x) = 0.$$

Since the integrand on the left-hand side is nonnegative, this yields a null-set $N \subset G$ with $\mu(\Gamma^q) = 0$ for all $q \in G \setminus N$. For $q \in Q \setminus N$, set $N_q := \Gamma^q$ and note that $x \in G \setminus N_q$ implies $x \notin \Gamma^q$, i.e. $(x,q) \notin \Gamma$ and thus $|f(xq)| \leq (M_Q f)(x)$, since $q \in Q$. This completes the proof. □

Now, we can prove the desired estimate for the maximal function of a convolution product, together with a continuity statement for the convolution product.

**Lemma 2.3.23.** *Let $G$ be a $\sigma$-compact LCH group and let $f \in L^1_{\mathrm{loc}}(G)$ with compact support and $g \in L^\infty_{\mathrm{loc}}(G)$. Then the convolution $f * g$ is pointwise well-defined with $f * g \in C(G) \subset L^\infty_{\mathrm{loc}}(G)$. Finally, the estimate*

$$M_Q[f * g](x) \leq M_Q[|f| * |g|](x) \leq (|f| * M_Q g)(x) \tag{2.3.8}$$

*is true for all $x \in G$ and each precompact, measurable set $Q \subset G$.*

*The estimate $M_Q[|f| * |g|](x) \leq (|f| * M_Q g)(x)$ even holds without the assumption that $f$ has compact support, with the understanding that if the right-hand side is finite, so is the left-hand side.*

*Proof.* Let $Q \subset G$ be precompact and measurable. Set $L := \operatorname{supp} f$ and $g^\vee(x) := g\left(x^{-1}\right)$ for each $x \in G$. For $x \in Q$, the substitution $z = x^{-1}y$ yields

$$\begin{aligned} f * g(x) &= \int_G f(y) \cdot g(y^{-1}x)\, \mathrm{d}y \\ &= \int_G f(y) \cdot \chi_L(y) \cdot g\left((x^{-1}y)^{-1}\right)\, \mathrm{d}y \\ &= \int_G f(xz) \cdot \chi_L(xz) \cdot g(z^{-1})\, \mathrm{d}z. \end{aligned}$$

But $\chi_L(xz) \neq 0$ implies $xz \in L$ and hence $z \in x^{-1}L \subset Q^{-1}L$, i.e. $z^{-1} \in L^{-1}\overline{Q}^{-1}$. Thus,

$$f * g(x) = \int_G f(xz) \cdot \left(g \cdot \chi_{L^{-1}\overline{Q}^{-1}}\right)^\vee(z)\, \mathrm{d}z = \left\langle L_{x^{-1}} f, \left[g \cdot \chi_{L^{-1}\overline{Q}^{-1}}\right]^\vee \right\rangle.$$

Using $g \in L^\infty_{\mathrm{loc}}(G)$, we see $\left[g \cdot \chi_{L^{-1}\overline{Q}^{-1}}\right]^\vee \in L^\infty(G)$. But $f \in L^1_{\mathrm{loc}}(G)$ with compact support, and hence $f \in L^1(G)$. Using the continuity of $G \to L^1(G), x \mapsto L_x f$, we see that the map

$$x \mapsto f * g(x) = \left\langle L_{x^{-1}} f, \left[g \cdot \chi_{L^{-1}\overline{Q}^{-1}}\right]^\vee \right\rangle$$

is continuous on $Q$. Because $Q \subset G$ was an arbitrary compact set and since $G$ is locally compact, we see that $f * g$ is a well-defined, continuous – and thus also locally bounded – function.

The estimate $|f * g| \leq |f| * |g|$ is trivial. Since $|f|, |g|$ satisfy the same assumptions as $f, g$, it suffices to show the second part of the target inequality (2.3.8). To this end, we drop the assumption that $f$ has compact support and let $x \in G$ be arbitrary. Lemma 2.3.22 yields a nullset $N \subset G$ and for each $q \in Q \setminus N$, a further null-set $N_q \subset G$ with

$$|g(zq)| \leq (M_Q g)(z) \text{ for all } z \in G \setminus N_q.$$

We now observe the equivalence

$$y^{-1}x \in N_q \iff x^{-1}y \in N_q^{-1} \iff y \in xN_q^{-1}.$$

Since the left- and right Haar measure on $G$ are mutually absolutely continuous (cf. [32, Proposition (2.31)]), $xN_q^{-1}$ is a null-set. Together, this proves

$$|g(y^{-1}xq)| \leq (M_Q g)(y^{-1}x)$$

for all $q \in Q \setminus N$ for almost all (depending on $q$) $y \in G$.

Hence,

$$\begin{aligned} |f| * |g|(xq) &= \int_G |f(y)| \cdot |g(y^{-1}xq)|\, \mathrm{d}y \\ &\leq \int_G |f(y)| \cdot (M_Q g)(y^{-1}x)\, \mathrm{d}y = (|f| * M_Q g)(x) \end{aligned}$$

holds for all $q \in Q \setminus N$. But $N$ is a null-set, which implies

$$M_Q(|f| * |g|)(x)) = \operatorname*{ess\,sup}_{y \in xQ} [|f| * |g|](y) \leq (|f| * M_Q g)(x). \qquad \square$$

With these preparations, we are ready to state and prove the first of the desired convolution relations. We remark that this convolution relation is a slight improvement of [59, Theorem 5.1(b)], since Rauhut's theorem does not include the continuity of $f * g$.

We finally note that we require $w(x) \geq \Delta(x^{-1}) \cdot |||R_{x^{-1}}|||_{W_Q(Y) \to W_Q(Y)}$ instead of $w(x) := \Delta(x^{-1}) \cdot |||R_{x^{-1}}|||_{W_Q(Y) \to W_Q(Y)}$ to account for the possibility that the map $x \mapsto |||R_x|||_{W_Q(Y) \to W_Q(Y)}$ might not measurable. For a further discussion of this issue (which seems to be missing in most other treatments of Wiener amalgam spaces), cf. Lemma 2.3.35 below.

**Theorem 2.3.24.** *(cf. [59, Theorem 5.1(b)]) Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid QBF space. Assume that $\|\cdot\|_Y$ is a $p$-norm for a suitable $p \in (0,1]$. Finally, assume that $W_Q(L^\infty, Y)$ is right invariant for each measurable, precompact unit neighborhood $Q \subset G$.*

*If $w : G \to (0,\infty)$ is submultiplicative and measurable with*

$$\Delta(x^{-1}) \cdot |||R_{x^{-1}}|||_{W_Q(L^\infty,Y) \to W_Q(L^\infty,Y)} \leq w(x) \tag{2.3.9}$$

*for all $x \in G$ and some measurable, precompact unit neighborhood $Q \subset G$, then we have*

$$W_Q(L^\infty, Y) * W_Q\left(L^\infty, L^p_w\right) \hookrightarrow W_Q\left(C^0, Y\right).$$

*In particular, the integral $\int_G |f(y)| \left|g\left(y^{-1}x\right)\right| \, \mathrm{d}y$ is absolutely convergent for* all *$x \in G$ if $f \in W(L^\infty, Y)$ and $g \in W\left(L^\infty, L^p_w\right)$.*

*Proof.* We start with more restrictive assumptions on the right factor of the convolution: Let $f \in W_Q(L^\infty, Y)$ and $g \in L^\infty(G)$ with $\operatorname{supp} g \subset xK$ for some fixed compact unit neighborhood $K \subset G$. Fix any compact set $U \subset G$. If $z \in U$ and $y \in G$ with $g\left(y^{-1}z\right) \neq 0$, the support assumption on $g$ implies $y^{-1}z \in xK$ and thus $z^{-1}y \in K^{-1}x^{-1}$, i.e. $y \in zK^{-1}x^{-1} \subset UK^{-1}x^{-1} =: L$.

For arbitrary $z \in U$, these considerations show

$$\begin{aligned}(f * g)(z) &= \int f(y) \cdot g\left(y^{-1}z\right) \, \mathrm{d}y \\ &= \int (f \cdot \chi_L)(y) \cdot g\left(y^{-1}z\right) \, \mathrm{d}y = [(f \cdot \chi_L) * g](z).\end{aligned}$$

But we have $f \in W(L^\infty, Y) \subset L^\infty_{\mathrm{loc}}(G)$ and thus $f\chi_L \in L^\infty(G)$, since $L \subset G$ is compact. By Lemma 2.3.23, this implies that $(f\chi_L) * g$ is continuous, so that $f * g$ is continuous on $U$. But $U \subset G$ was an arbitrary compact set, so that $f * g$ is continuous, because $G$ is locally compact.

Let $Q \subset G$ be as in the assumptions of the theorem. We estimate the maximal function of $|f| * |g|$ by

$$\begin{aligned}M_Q[|f| * |g|](z) &= \operatorname*{ess\,sup}_{y \in zQ} \int_G |f(\gamma)| \cdot |g(\gamma^{-1}y)| \, \mathrm{d}\gamma \\ &\leq \operatorname*{ess\,sup}_{y \in zQ} \|g\|_{L^\infty} \cdot \int_G |f(\gamma)| \cdot \chi_{xK}(\gamma^{-1}y) \, \mathrm{d}\gamma \\ &\leq \|g\|_{L^\infty} \cdot \int_G |f(\gamma)| \cdot \chi_{zQK^{-1}x^{-1}}(\gamma) \, \mathrm{d}\gamma,\end{aligned}$$

where the last step used the fact that $\chi_{xK}\left(\gamma^{-1}y\right) \neq 0$, together with $y \in zQ$ implies $\gamma^{-1}y \in xK$ and hence $\gamma \in yK^{-1}x^{-1} \subset zQK^{-1}x^{-1}$. We recall from [32, Proposition (2.24)] the formula

$$\int_G R_x h \, \mathrm{d}\mu = \Delta\left(x^{-1}\right) \cdot \int_G h \, \mathrm{d}\mu$$

for $h \in L^1(G)$ and $x \in G$, where $\Delta$ denotes the modular function of $G$ and $\mu$ denotes the Haar measure on $G$. We use this formula to continue the above estimate as follows:

$$\begin{aligned} M_Q\left[|f| * |g|\right](z) &\leq \|g\|_{L^\infty} \cdot \int_G R_x\left[\chi_{zQK^{-1}} \cdot R_{x^{-1}}|f|\right] \mathrm{d}\mu \\ &= \|g\|_{L^\infty} \cdot \Delta\left(x^{-1}\right) \cdot \int_G \chi_{zQK^{-1}} \cdot R_{x^{-1}}|f| \, \mathrm{d}\mu \\ &\leq \mu\left(zQK^{-1}\right) \cdot \Delta\left(x^{-1}\right) \cdot \|g\|_{L^\infty} \cdot M_{QK^{-1}}\left[R_{x^{-1}}|f|\right](z) \\ &= \mu\left(QK^{-1}\right) \cdot \Delta\left(x^{-1}\right) \cdot \|g\|_{L^\infty} \cdot M_{QK^{-1}}\left[R_{x^{-1}}|f|\right](z). \end{aligned}$$

But $f \in W\left(L^\infty, Y\right) = W_{QK^{-1}}\left(L^\infty, Y\right)$ and this space is right-invariant and solid. Thus, $R_{x^{-1}}|f| \in W_{QK^{-1}}\left(L^\infty, Y\right)$, which means nothing but $M_{QK^{-1}}\left[R_{x^{-1}}|f|\right] \in Y$. Using the solidity of $Y$, together with the measurability of $M_Q\left[|f| * |g|\right]$, we conclude $M_Q\left[|f| * |g|\right] \in Y$ and hence $|f| * |g| \in W_Q\left(L^\infty, Y\right)$ with

$$\begin{aligned} \||f| * |g|\|_{W_Q(L^\infty,Y)} &= \|M_Q\left(|f| * |g|\right)\|_Y \\ &\leq \mu\left(QK^{-1}\right) \cdot \Delta\left(x^{-1}\right) \cdot \|g\|_{L^\infty} \cdot \left\|M_{QK^{-1}}\left[R_{x^{-1}}|f|\right]\right\|_Y \\ &= \mu\left(QK^{-1}\right) \cdot \Delta\left(x^{-1}\right) \cdot \|g\|_{L^\infty} \cdot \|R_{x^{-1}}|f|\|_{W_{QK^{-1}}(L^\infty,Y)} \\ &\leq C_{Q,K} \cdot \Delta\left(x^{-1}\right) \cdot \|g\|_{L^\infty} \cdot \|R_{x^{-1}}|f|\|_{W_Q(L^\infty,Y)} \\ &\leq C_{Q,K} \cdot w(x) \cdot \|g\|_{L^\infty} \cdot \|f\|_{W_Q(L^\infty,Y)}. \end{aligned} \tag{2.3.10}$$

Here, we used the equivalence of norms for the space $W\left(L^\infty, Y\right)$ for different choices of $Q$, as well as the assumption in equation (2.3.9) on $w$.

Now, let $f \in W\left(L^\infty, Y\right)$ and $g \in W\left(L^\infty, L^p_w\right)$ be arbitrary. By Lemma 2.3.12, there is some $K$-BUPU $\Phi = (\varphi_i)_{i\in I}$ with associated localizing family $X = (x_i)_{i\in I}$. Using Lemma 2.3.21 and Theorem 2.3.17, we conclude that there are absolute constants $C = C(w, X, p) > 0$ and $C' = C'(w, X, Q, p) > 0$ with

$$\begin{aligned} \sum_{i\in I}\left[w(x_i) \cdot \|\varphi_i \cdot g\|_{L^\infty}\right]^p &\leq C \cdot \left\|\left(\|\varphi_i \cdot g\|_{L^\infty}\right)_{i\in I}\right\|^p_{\left(L^p_w\right)_d(X,Q)} \\ &\leq C' \cdot \|g\|^p_{W_Q\left(L^\infty, L^p_w\right)}. \end{aligned}$$

Using the estimate (2.3.10) for $\varphi_i \cdot g$ instead of $g$ and with $x = x_i$, noting that $\operatorname{supp}(\varphi_i g) \subset x_i K$, we conclude

$$\begin{aligned}\sum_{i\in I} \| |f| * |\varphi_i \cdot g| \|^p_{W_Q(L^\infty,Y)} &\leq C_{Q,K} \cdot \|f\|^p_{W_Q(L^\infty,Y)} \cdot \sum_{i\in I} [w(x_i) \cdot \|\varphi_i g\|_{L^\infty}]^p \\ &\leq C_{Q,K} C' \cdot \|f\|^p_{W_Q(L^\infty,Y)} \cdot \|g\|^p_{W_Q\left(L^\infty, L^p_w\right)}.\end{aligned}$$

But $\|\cdot\|_Y$ is a $p$-norm, so that the same is also true of $\|\cdot\|_{W_Q(L^\infty,Y)}$, by Lemma 2.3.4. Furthermore, $W_Q(L^\infty, Y)$ is complete by Lemma 2.3.7. Using Theorem 2.2.10, we conclude

$$H := \sum_{i\in I} [|f| * |\varphi_i g|] \in W_Q(L^\infty, Y)$$

with

$$\|H\|^p_{W_Q(Y)} \leq \sum_{i\in I} \| |f| * |\varphi_i g| \|^p_{W_Q(Y)} \leq C_{Q,K} C' \cdot \|f\|^p_{W_Q(Y)} \cdot \|g\|^p_{W_Q\left(L^p_w\right)}.$$

Finally, Lemma 2.3.5 yields for each open, precompact set $U \subset G$ some constant $C_U > 0$ with

$$\begin{aligned}\left\| \sum_{i\in I_0} f * (\varphi_i g) \right\|^p_{\sup,U} &\leq \left\| \sum_{i\in I_0} [|f| * |\varphi_i g|] \right\|^p_{\sup,U} \\ &= \left\| \chi_U \cdot \sum_{i\in I_0} [|f| * |\varphi_i g|] \right\|^p_{L^\infty} \\ &\leq C_U^p \cdot \left\| \sum_{i\in I_0} [|f| * |\varphi_i g|] \right\|^p_{W_Q(L^\infty,Y)} \\ &\leq C_U^p \cdot \sum_{i\in I_0} \| |f| * |\varphi_i g| \|^p_{W_Q(L^\infty,Y)} < \varepsilon\end{aligned}$$

for each finite set $I_0 \subset I \setminus \{i_1, \ldots, i_n\}$ for suitable $i_1, \ldots, i_n \in I$. Here, the step from the first to the second line is justified since $|f| * |\varphi_i g|$ is continuous for each $i \in I$ as observed at the beginning of the proof and because $U$ is open, so that the supremum coincides with the essential supremum.

This implies that the two series $\sum_{i\in I} [f * (\varphi_i g)]$ and $\sum_{i\in I} [|f| * |\varphi_i g|]$ converge locally uniformly on $G$. In particular, the limit of each of the two series is continuous. Using the monotone convergence theorem (which is applicable since $I$ is countable by Lemma 2.3.10), we also get

$$\begin{aligned}\infty > \sum_{i\in I} [|f| * |\varphi_i g|](x) &= \sum_{i\in I} \int_G |f(y)| \cdot |\varphi_i(y^{-1}x)\, g(y^{-1}x)| \, \mathrm{d}y \\ &= \int_G |f(y)| \cdot |g(y^{-1}x)| \cdot \sum_{i\in I} \varphi_i(y^{-1}x) \, \mathrm{d}y\end{aligned}$$

$$= \int_G |f(y)| \cdot |g(y^{-1}x)| \, \mathrm{d}y$$

for all $x \in G$. Thus, we can apply the dominated convergence theorem to conclude

$$\begin{aligned} H_0(x) := \sum_{i \in I} [f * (\varphi_i g)](x) &= \sum_{i \in I} \int_G f(y) \cdot \varphi_i(y^{-1}x) \cdot g(y^{-1}x) \, \mathrm{d}y \\ &= \int_G f(y) \cdot g(y^{-1}x) \, \mathrm{d}y = (f * g)(x). \end{aligned}$$

Using the locally uniform convergence of the series defining $H_0$, we see that $f * g$ is indeed continuous and hence measurable. Since $|H_0(x)| \leq H(x)$ holds for all $x \in G$, solidity of $W_Q(L^\infty, Y)$ finally implies $f * g = H_0 \in W_Q(L^\infty, Y)$ with

$$\|f * g\|^p_{W_Q(Y)} \leq \|H\|^p_{W_Q(Y)} \leq C_{Q,K} C' \cdot \|f\|^p_{W_Q(Y)} \cdot \|g\|^p_{W_Q\left(L^p_w\right)},$$

which completes the proof. □

The convolution relation given above is the same as in [59, Theorem 5.1(b)]. Following this theorem, Rauhut also proposes the convolution relation

$$W\left(L^\infty, L^p_v\right) * \left(W\left(L^\infty, Y^\vee\right)\right)^\vee \hookrightarrow W\left(L^\infty, Y\right) \tag{2.3.11}$$

for $v(x) := \left|\left|\left|L_{x^{-1}}\right|\right|\right|_{W(L^\infty,Y) \to W(L^\infty,Y)}$, see [59, Theorem 5.2]. This convolution relation is incorrect in two ways:

A. The first is the form of $v$. Indeed, the proof uses the weight $v(x) = \left|\left|\left|L_x\right|\right|\right|_{W(Y) \to W(Y)}$ instead of $v(x) = \left|\left|\left|L_{x^{-1}}\right|\right|\right|_{W(Y) \to W(Y)}$. Example 2.3.25 below shows that the convolution relation does not hold with $v$ as given in the statement of [59, Theorem 5.2].

B. Even with the modified form $v(x) = \left|\left|\left|L_x\right|\right|\right|_{W(L^\infty,Y) \to W(L^\infty,Y)}$, the proposed convolution relation is false. The mistake in the proof of [59, Theorem 5.2] occurs in the first displayed formula. More precisely, the estimate

$$\sup_{q \in Q} \int \chi_Q(y) \left|\left(R_q G\right)(y^{-1}x)\right| \mathrm{d}y \leq C \int \chi_{Q^2}(y) \left|G^\vee(x^{-1}y)\right| \mathrm{d}y \tag{2.3.12}$$

is *not* true in general.

In this case, the construction of a counterexample (to the convolution relation, not only to the estimate in equation (2.3.12)) is more involved, cf. Example 2.3.26. It is important to note that this example even shows that [59, Corollary 5.4] is false in general. This is a problem, since this corollary is used in a crucial way in the proof of [58, Theorem 4.3(b)] to show the independence of the coorbit space $\mathrm{Co}(Y)$ of the analyzing window.

After giving the counterexamples to the convolution relation proposed by Rauhut, we derive an alternative convolution relation. In Section 2.4, we will then modify the construction of the coorbit spaces proposed by Rauhut in [58] so that the incorrect convolution relation is not used any longer.

We first provide an example which shows that one can not use the weight $v(x) = \|\|L_{x^{-1}}\|\|_{W(L^\infty,Y)\to W(L^\infty,Y)}$ as in [59, Theorem 5.2].

**Example 2.3.25.** We consider the (locally compact, $\sigma$-compact) group $G = \mathbb{R}$ and the (sub)multiplicative weight $w : \mathbb{R} \to (0,\infty), x \mapsto e^x$. For $f \in Y := L^1_w(\mathbb{R})$ and each $x \in \mathbb{R}$, we have

$$\|L_x f\|_Y = \int_{\mathbb{R}} |f(y-x)| \cdot e^y \,\mathrm{d}y = \int_{\mathbb{R}} |f(z)| \cdot e^{z+x} \,\mathrm{d}z = e^x \cdot \|f\|_Y$$

and hence $\|\|L_x\|\|_{Y\to Y} = e^x$. Using Lemma 2.3.18, we conclude

$$v(x) := \|\|L_{x^{-1}}\|\|_{W(L^\infty,Y)\to W(L^\infty,Y)} \le \|\|L_{x^{-1}}\|\|_{Y\to Y} = e^{-x},$$

where we have written $G = \mathbb{R}$ multiplicatively, as well as additively, i.e. $x^{-1} = -x$. Finally, we observe

$$\left\|f^\vee\right\|_Y = \int_{\mathbb{R}} |f(-x)| \cdot e^x \,\mathrm{d}x = \int_{\mathbb{R}} |f(y)| \cdot e^{-y} \,\mathrm{d}y = \|f\|_{L^1_u}$$

for $u : \mathbb{R} \to (0,\infty), x \mapsto e^{-x}$ for each measurable function $f : \mathbb{R} \to \mathbb{C}$. Thus, $Y^\vee = L^1_u(\mathbb{R})$.

Now, let $M > 0$ be arbitrary and define $f := \chi_{(M,M+1)}$ as well as $g := \chi_{(-M-1,-M)}$. Using the (precompact, measurable) unit neighborhood $Q := (-1,1)$, we have

$$(M_Q \chi_A)(x) = \operatorname*{ess\,sup}_{y\in x+Q} \chi_A(y) \le \chi_{A+Q}(x)$$

for each measurable $A \subset \mathbb{R}$, since $\chi_A(y) \neq 0$ for some $y \in x+Q$ implies $x+q = y \in A$ for a suitable $q \in Q$, which in turn yields $x = y - q \in A - Q = A + Q$.

Since $Y = L^1_w(G)$ is a Banach space, we can choose $p = 1$. Thus,

$$\begin{aligned}\|f\|_{W_Q\left(L^\infty,L^p_v\right)} = \|M_Q f\|_{L^p_v} &\le \left\|\chi_{(M-1,M+2)}\right\|_{L^p_v} \\ &\le \int_{M-1}^{M+2} e^{-x} \,\mathrm{d}x \le 3 \cdot e^{-(M-1)} = 3e \cdot e^{-M}\end{aligned}$$

and

$$\begin{aligned}\|g\|_{\left(W_Q(L^\infty,Y^\vee)\right)^\vee} = \|M_Q[g^\vee]\|_{Y^\vee} &\le \left\|\chi_{(M-1,M+2)}\right\|_{L^1_u} \\ &= \int_{M-1}^{M+2} e^{-x} \,\mathrm{d}x \le 3e \cdot e^{-M}.\end{aligned}$$

But since $f, g \in L^2(\mathbb{R})$, it is easy to see that $f * g$ is continuous with

$$f * g(0) = \int_{\mathbb{R}} f(x) \cdot g(-x)\, \mathrm{d}x = \int_{\mathbb{R}} \chi_{(M,M+1)} \cdot \chi_{(M,M+1)}\, \mathrm{d}x = 1.$$

By continuity, the essential supremum coincides with the actual supremum. Hence, for arbitrary $x \in (-1,1)$, we have

$$M_Q[f * g](x) = \sup_{y \in x+(-1,1)} |(f * g)(y)| \geq f * g(0) = 1,$$

which yields

$$\|f * g\|_{W_Q(L^\infty,Y)} = \|M_Q[f * g]\|_{L^1_w} \geq \int_{(-1,1)} e^x\, \mathrm{d}x \geq 1.$$

But if the convolution relation was true, there would be an absolute constant $C = C(w, Q) > 0$ satisfying

$$\begin{aligned} 1 &\leq \|f * g\|_{W_Q(L^\infty,Y)} \leq C \cdot \|f\|_{W_Q\left(L^\infty,L^p_v\right)} \cdot \|g\|_{\left(W_Q(L^\infty,Y^\vee)\right)^\vee} \\ &\leq C \cdot 9e^2 \cdot e^{-2M} \xrightarrow[M\to\infty]{} 0, \end{aligned}$$

a contradiction. ◀

Thus, we have shown that the convolution relation in equation (2.3.11) can not hold with $v(x) = |||L_{x^{-1}}|||_{W(L^\infty,Y)\to W(L^\infty,Y)}$. The next example shows that it also fails for $v(x) = |||L_x|||_{W(L^\infty,Y)\to W(L^\infty,Y)}$.

**Example 2.3.26.** Let $G$ denote the "$ax+b$-group", i.e. $G := \mathbb{R} \times (0,\infty)$ with multiplication given by

$$(x,a) \cdot (y,b) = (x + ay, ab).$$

Neutral element and inverse in $G$ are given by $1_G = (0,1) \in G$ and $(x,a)^{-1} = \left(-\frac{x}{a}, a^{-1}\right)$, respectively. In [32], in the discussion after Proposition (2.21) it is shown that the Haar integral on $G$ is given by

$$\int_G f(g)\, \mathrm{d}g = \int_0^\infty \int_{\mathbb{R}} f(x,a)\, \mathrm{d}x\, \frac{\mathrm{d}a}{a^2}$$

with modular function $\Delta(x,a) = a^{-1}$.

Let us define

$$w : G \to (1,\infty), (x,a) \mapsto 1 + a = 1 + \Delta\left((x,a)^{-1}\right).$$

This weight is clearly continuous with

$$\begin{aligned} w((x,a)(y,b)) &= w((x + ay, ab)) \\ &= 1 + ab \\ &\leq (1+a)(1+b) \\ &= w(x,a) \cdot w(y,b), \end{aligned}$$

so that $w$ is a submultiplicative, continuous weight.

Let us set $Y := L^1_w(G)$. Since $L^1(G)$ is right invariant and isometrically left invariant, Lemma 2.2.19 and Lemma 2.3.18 show that $Y$ and $W(L^\infty, Y)$ are left and right invariant with

$$\begin{aligned} v(x,a) &:= \left|\!\left|\!\left| L_{(x,a)} \right|\!\right|\!\right|_{W(L^\infty,Y)\to W(L^\infty,Y)} \\ &\leq \left|\!\left|\!\left| L_{(x,a)} \right|\!\right|\!\right|_{Y\to Y} \leq w(x,a) = 1+a. \end{aligned} \tag{2.3.13}$$

Finally, we observe $Y = Y^\vee$, since [32, Proposition (2.31)] yields

$$\begin{aligned} \|f^\vee\|_{L^1_w} &= \int_G |f^\vee(g)| \cdot w(g)\, \mathrm{d}g \\ &= \int_G |f(g^{-1})|\, \mathrm{d}g + \int_G |f(g^{-1})| \cdot \Delta(g^{-1})\, \mathrm{d}g \\ &= \int_G |f(g)| \cdot \Delta(g^{-1})\, \mathrm{d}g + \int_G |f(g)|\, \mathrm{d}g \\ &= \int_G |f(g)| \cdot w(g)\, \mathrm{d}g = \|f\|_{L^1_w}. \end{aligned}$$

Now, let $\alpha \in (1,\infty)$ and $\beta \in (0,1)$ and set $\delta := \max\{\alpha,\beta\} = \alpha$ as well as

$$f : G \to (0,\infty), (x,a) \mapsto e^{-|x|} \cdot \min\left\{a^\alpha, a^{-\beta}\right\}.$$

This function is clearly continuous (and hence measurable) and is easily seen to satisfy $\|f\|_{\sup} \leq 1$. In the following, we use the (precompact, measurable) unit neighborhood $Q := (-1,1) \times \left(\frac{1}{2},2\right)$. With this choice, the substitutions $(z,c) = (y,b)^{-1}$ and $(\mu,\nu) = (z,c)\cdot(x,a)$ yield

$$\begin{aligned} \left(M_{Q^{-1}} f^\vee\right)(x,a) &\leq \sup_{(y,b)\in(x,a)Q^{-1}} \left|f^\vee(y,b)\right| = \sup_{(z,c)\in Q(x,a)^{-1}} |f(z,c)| \\ &= \sup_{(\mu,\nu)\in Q} \left|f\left((\mu,\nu)\cdot(x,a)^{-1}\right)\right| \\ &= \sup_{(\mu,\nu)\in Q} \left|f\left((\mu,\nu)\cdot\left(-\frac{x}{a},a^{-1}\right)\right)\right| \\ &= \sup_{(\mu,\nu)\in Q} \left|f\left(\mu-\frac{\nu}{a}x,\, \frac{\nu}{a}\right)\right| \\ &= \sup_{(\mu,\nu)\in Q} e^{-\left|\mu-\frac{\nu}{a}x\right|} \cdot \min\left\{(\nu/a)^\alpha, (\nu/a)^{-\beta}\right\}. \end{aligned}$$

But using $(\mu,\nu) \in Q = (-1,1)\times\left(\frac{1}{2},2\right)$, we see

$$\left|\mu-\frac{\nu}{a}x\right| \geq \frac{\nu}{a}|x| - |\mu| \geq \frac{\nu}{a}|x| - 1 \geq \frac{|x|}{2a} - 1$$

and thus

$$e^{-\left|\mu-\frac{\nu}{a}x\right|} \leq e^{-\left(\frac{|x|}{2a}-1\right)} = e\cdot e^{-|x|/2a}.$$

Furthermore, $\frac{1}{2a} \leq \frac{\nu}{a} \leq \frac{2}{a}$, which entails

$$\left(\frac{\nu}{a}\right)^{\alpha} \leq \left(\frac{2}{a}\right)^{\alpha} \leq 2^{\delta} \cdot a^{-\alpha}$$

and

$$\left(\frac{\nu}{a}\right)^{-\beta} = \left(\frac{a}{\nu}\right)^{\beta} \leq 2^{\beta} \cdot a^{\beta} \leq 2^{\delta} \cdot a^{\beta}.$$

All in all, we arrive at

$$\left(M_{Q^{-1}} f^{\vee}\right)(x,a) \leq 2^{\delta} e \cdot e^{-|x|/2a} \cdot \min\left\{a^{-\alpha}, a^{\beta}\right\} \tag{2.3.14}$$

for all $(x,a) \in G$.

Using the identity

$$C_1 := \int_{\mathbb{R}} e^{-|y|}\,\mathrm{d}y = \int_{\mathbb{R}} e^{-|x|/2a} \cdot \frac{\mathrm{d}x}{2a}$$

which holds for all $a \in (0,\infty)$, we conclude

$$\begin{aligned}
\left\|f^{\vee}\right\|_{W_{Q^{-1}}(Y)} &= \left\|M_{Q^{-1}} f^{\vee}\right\|_{Y} \\
&\leq 2^{\delta} e \int_{G} e^{-|x|/2a} \cdot \min\left\{a^{-\alpha}, a^{\beta}\right\} \cdot w(x,a)\ \mathrm{d}(x,a) \\
&= 2^{\delta} e \int_{0}^{\infty} \int_{\mathbb{R}} \frac{\min\left\{a^{-\alpha}, a^{\beta}\right\} \cdot (1+a)}{a^2} \cdot 2a \int_{\mathbb{R}} e^{-|x|/2a} \frac{\mathrm{d}x}{2a}\,\mathrm{d}a \\
&= 2^{\delta+1} e C_1 \int_{0}^{\infty} \frac{\min\left\{a^{-\alpha}, a^{\beta}\right\} \cdot (1+a)}{a}\,\mathrm{d}a \\
&\leq 2^{\delta+1} e C_1 \cdot \left[\int_{0}^{1} a^{\beta}(1+a)\frac{\mathrm{d}a}{a} + \int_{1}^{\infty} a^{-\alpha}(1+a)\frac{\mathrm{d}a}{a}\right].
\end{aligned}$$

Using $1 + a \leq 2$ for $a \in (0,1)$ and $1 + a \leq a + a = 2a$ for $a \in [1,\infty)$, we finally get

$$\left\|f^{\vee}\right\|_{W_{Q^{-1}}(L^{\infty},Y)} \leq 2^{\delta+2} e C_1 \cdot \left[\int_{0}^{1} a^{\beta}\frac{\mathrm{d}a}{a} + \int_{1}^{\infty} a^{1-\alpha}\frac{\mathrm{d}a}{a}\right] < \infty,$$

because of $\beta > 0$ and $\alpha > 1$.

This implies $f^{\vee} \in W(Y) = W(Y^{\vee})$ and thus $f \in \left(W(L^{\infty}, Y^{\vee})\right)^{\vee}$, as well as

$$f^{\vee} \in W(L^{\infty}, Y) = W(L^{\infty}, L^1_w) \hookrightarrow W(L^{\infty}, L^1_v),$$

where we used $v \leq w$ in the last step (cf. equation (2.3.13)). If the convolution relation in equation (2.3.11) was true, this would yield $f^{\vee} * f \in W(L^{\infty}, Y)$, since we can choose $p = 1$ because $Y = L^1_w(G)$ is a normed space. We will now show that $f^{\vee} * f \in W(L^{\infty}, Y)$ does *not* hold, thus disproving the convolution relation (2.3.11).

We first note $f^\vee \in W\left(L^\infty, L^1_w\right) \hookrightarrow L^1_w \hookrightarrow L^1$, because of $w \geq 1$. Furthermore, $f$ is bounded, so that $f^\vee * f : G \to \mathbb{C}$ is a well-defined, continuous, bounded function. We compute

$$\begin{aligned}
&\left(f^\vee * f\right)(x,a)\\
&= \int_G f^\vee(y,b) \cdot f\left((y,b)^{-1} \cdot (x,a)\right) \mathrm{d}(y,b)\\
&= \int_0^\infty \int_{\mathbb{R}} f\left(-\frac{y}{b}, \frac{1}{b}\right) \cdot f\left(\frac{x-y}{b}, \frac{a}{b}\right) \mathrm{d}y \frac{\mathrm{d}b}{b^2}\\
&= \int_0^\infty \min\left\{b^{-\alpha}, b^\beta\right\} \min\left\{\left(\frac{a}{b}\right)^\alpha, \left(\frac{a}{b}\right)^{-\beta}\right\} \cdot \frac{b}{b^2} \int_{\mathbb{R}} e^{-\left|-\frac{y}{b}\right|} e^{-\left|\frac{x-y}{b}\right|} \frac{\mathrm{d}y}{b} \mathrm{d}b\\
&= \int_0^\infty \min\left\{b^{-\alpha}, b^\beta\right\} \cdot \min\left\{\left(\frac{a}{b}\right)^\alpha, \left(\frac{b}{a}\right)^\beta\right\} \cdot \int_{\mathbb{R}} e^{-|z|} e^{-\left|z-\frac{x}{b}\right|} \mathrm{d}z \frac{\mathrm{d}b}{b}.
\end{aligned}$$

We now use this for $x = 0$ and $a \geq 1$: For each $b \in (0,1)$, we have $b^\beta \leq 1 \leq b^{-\alpha}$ as well as $b < 1 \leq a$ and thus $(b/a)^\beta \leq 1^\beta = 1 \leq (a/b)^\alpha$. Together with the abbreviation $C_2 := \int_{\mathbb{R}} \left(e^{-|z|}\right)^2 \mathrm{d}z \in (0,\infty)$, these considerations and the above calculation imply

$$\begin{aligned}
\left(f^\vee * f\right)(0,a) &\geq C_2 \cdot \int_0^1 b^\beta \cdot (b/a)^\beta \frac{\mathrm{d}b}{b}\\
&= C_2 \cdot a^{-\beta} \cdot \int_0^1 b^{2\beta-1} \mathrm{d}b\\
&= C_2 \cdot a^{-\beta} \cdot \frac{b^{2\beta}}{2\beta}\Big|_0^1\\
&= \frac{C_2}{2\beta} \cdot a^{-\beta} \text{ for } a \geq 1. \qquad (2.3.15)
\end{aligned}$$

We want to use this lower bound to obtain a lower bound on $M_Q\left(f^\vee * f\right)$. To this end, we first observe for $(y,b) \in G$ that

$$\begin{aligned}
(y,b)\,Q &= \{(y,b) \cdot (x,a) \mid x \in (-1,1) \text{ and } a \in (1/2, 2)\}\\
&= \{(y + bx, ab) \mid x \in (-1,1) \text{ and } a \in (1/2, 2)\}\\
&= [y + (-b, b)] \times \left(\frac{b}{2}, 2b\right)\\
&= B_b(y) \times \left(\frac{b}{2}, 2b\right).
\end{aligned}$$

For $b \in (1,\infty)$ and $y \in (-b,b)$, we thus have $(0,b) \in (y,b)\,Q$. Since $f^\vee * f$ is continuous and $(y,b)\,Q$ is open, the essential supremum coincides with the ordinary supremum. Thus,

$$\begin{aligned}
M_Q\left[f^\vee * f\right](y,b) &= \operatorname*{ess\,sup}_{(z,c) \in (y,b)Q} \left|\left(f^\vee * f\right)(z,c)\right|\\
&\geq \left(f^\vee * f\right)(0,b)\\
&\geq \frac{C_2}{2\beta} \cdot b^{-\beta} \text{ for } b \in (1,\infty) \text{ and } y \in (-b,b).
\end{aligned}$$

Here, we used estimate (2.3.15) in the last step. All in all, we conclude

$$\begin{aligned}\left\|f^{\vee} * f\right\|_{W_Q(L^\infty,Y)} &= \left\|M_Q\left(f^{\vee} * f\right)\right\|_{L^1_w} \\ &\geq \frac{C_2}{2\beta} \cdot \int_G \chi_{(1,\infty)}(b) \cdot \chi_{(-b,b)}(y) \cdot b^{-\beta} \cdot w(y,b)\, \mathrm{d}(y,b) \\ &= \frac{C_2}{2\beta} \cdot \int_1^\infty \frac{b^{-\beta}}{b^2} \cdot (1+b) \cdot \int_{-b}^{b} \mathrm{d}y\, \mathrm{d}b \\ &\geq \frac{C_2}{2\beta} \cdot \int_1^\infty b^{-\beta}\, \mathrm{d}b = \infty.\end{aligned}$$

because of $\beta \in (0,1)$. Thus, $f^{\vee} * f \notin W(L^\infty, Y)$. ◀

We remark that the above example also shows that the convolution relation

$$W\left(L^\infty, L^p_w\right) * \left(W\left(L^\infty, L^p_{w^*}\right)\right)^{\vee} \hookrightarrow W\left(L^\infty, L^p_w\right) \tag{2.3.16}$$

with $w^*(x) = \Delta\left(x^{-1}\right) \cdot w\left(x^{-1}\right)$ proposed in [59, Corollary 5.4] is also false in general (even for $p = 1$). To see this, note that the weight $w$ in the above example satisfies $w^* = w$ and that $f^{\vee} \in W\left(L^\infty, L^1_w\right)$ as well as $f \in \left(W\left(L^\infty, L^1_{w^*}\right)\right)^{\vee}$, but $f^{\vee} * f \notin W(L^\infty, Y) = W\left(L^\infty, L^1_w\right)$.

In the remainder of this section, we develop a convolution relation that will serve as a substitute for the one in equation (2.3.16) in the construction of coorbit spaces with respect to Quasi-Banach spaces in the next section. This convolution relation is somewhat similar to [29, Proposition 3.10].

To this end, it will be convenient to have a "right-sided version" of the Wiener amalgam spaces. These spaces have of course been considered before, see e.g. the discussion before Lemma 3.9 in [29].

**Definition 2.3.27.** Let $G$ be an LCH group, assume that $f : G \to \mathbb{C}$ or $f : G \to [0,\infty]$ is measurable and let $Q \subset G$ be measurable and precompact. We define the RIGHT-SIDED MAXIMAL FUNCTION OF $f$ with respect to $Q$ as

$$M^R_Q f : G \to [0,\infty], x \mapsto \operatorname*{ess\,sup}_{y \in Qx} |f(y)| = \left\|\chi_{Qx} \cdot f\right\|_{L^\infty}.$$

If $(Y, \|\cdot\|_Y)$ is a solid function space, we define the RIGHT-SIDED WIENER AMALGAM SPACE with window $Q$, local component $L^\infty$ and global component $Y$ as

$$W^R_Q(L^\infty, Y) := \left\{ f \in L^\infty_{\mathrm{loc}}(G) \,\middle|\, M^R_Q f \in Y \right\},$$

with

$$\|f\|_{W^R_Q(L^\infty,Y)} := \left\|M^R_Q f\right\|_Y \text{ for } f \in W^R_Q(L^\infty, Y).$$

The next lemmata show that the right-sided maximal function and the right-sided Wiener amalgam spaces can also be obtained directly

by "inversion" of the usual maximal function and of the usual Wiener amalgam spaces, respectively.

**Lemma 2.3.28.** *Let $G$ be an LCH group, let $f : G \to \mathbb{C}$ or $f : G \to [0,\infty]$ be measurable and assume that $Q \subset G$ is measurable and precompact. Then the following are true:*

1. *We have $M_Q^R f = \left[M_{Q^{-1}} f^\vee\right]^\vee$ and $M_Q f = \left[M_{Q^{-1}}^R f^\vee\right]^\vee$.*
2. *If $f \in L^\infty_{\mathrm{loc}}(G)$, then the maximal function $M_Q^R f$ is lower semicontinuous and hence measurable.*
3. *If $Q$ is a unit neighborhood, then $|f(x)| \leq M_Q^R f(x)$ holds for almost all $x \in G$.*
4. *For $f, g \in L^\infty_{\mathrm{loc}}(G)$, we have*

$$M_Q^R\left[|f| * |g|\right](x) \leq \left[(M_Q^R f) * |g|\right](x)$$

*and*

$$M_Q\left[|f| * |g|\right](x) \leq \left[|f| * M_Q g\right](x)$$

*for all $x \in G$.*

*Proof.* By [32, Proposition (2.31)], the left and right Haar measures are strongly equivalent and thus have the same nullsets. If $\mu$ is a left Haar measure on $G$, then $\varrho(A) := \mu\left(A^{-1}\right)$ defines a right Haar measure. Thus, $N \subset G$ is a nullset iff $N^{-1}$ is one. Hence, $\|g\|_{L^\infty} = \|g^\vee\|_{L^\infty}$ holds for all measurable $g : G \to \mathbb{C}$. This implies

$$\begin{aligned} M_Q^R f(x) &= \|\chi_{Qx} \cdot f\|_{L^\infty} = \left\|\chi_{x^{-1}Q^{-1}} \cdot f^\vee\right\|_{L^\infty} \\ &= \left[M_{Q^{-1}} f^\vee\right](x^{-1}) = \left[M_{Q^{-1}} f^\vee\right]^\vee(x) \end{aligned}$$

for all $x \in G$. The proof of the second identity is analogous.

Using the first part, we see for each $\alpha \in \mathbb{R}$ that

$$\left(M_Q^R f\right)^{-1}((\alpha,\infty)) = \left[\left(M_{Q^{-1}} f^\vee\right)^{-1}((\alpha,\infty))\right]^{-1}.$$

But since $M_{Q^{-1}} f^\vee$ is lower semicontinuous by Lemma 2.3.4, the set $\left(M_{Q^{-1}} f^\vee\right)^{-1}((\alpha,\infty))$ is open. Because $G \to G, x \mapsto x^{-1}$ is a homeomorphism, we conclude that $\left(M_Q^R f\right)^{-1}((\alpha,\infty))$ is also open, so that $M_Q^R f$ is lower semicontinuous.

If $Q$ is a precompact, measurable unit neighborhood, then so is $Q^{-1}$, so that an application of Lemma 2.3.3 shows $|f^\vee(x)| \leq \left[M_{Q^{-1}} f^\vee\right](x)$ for almost all $x \in G$, i.e. for all $x \in G \setminus N$ for some nullset $N \subset G$. As seen above, $N^{-1} \subset G$ is also a nullset and we have

$$|f(x)| = \left|f^\vee(x^{-1})\right| \leq \left[M_{Q^{-1}} f^\vee\right](x^{-1}) = \left(M_Q^R f\right)(x)$$

for all $x \in G \setminus N^{-1}$.

For the last claim, we recall that the second estimate was already established in Lemma 2.3.23. Regarding the first estimate, we invoke Lemma 2.3.22 to obtain a nullset $N \subset G$ and for each $q \in Q^{-1} \setminus N$, a further nullset $N_q \subset G$ with

$$|f(q^{-1}z^{-1})| = |f^\vee(zq)| \leq \left(M_{Q^{-1}} f^\vee\right)(z) = \left(M_Q^R f\right)(z^{-1})$$

for all $z \in G \setminus N_q$. As at the start of the proof, we see that $N^{-1} \subset G$ is also a nullset and that for each $q \in Q \setminus N^{-1}$, the set $N_{q^{-1}}^{-1} \subset G$ is also of measure zero and we have

$$|f(qz)| \leq \left(M_Q^R f\right)(z) \text{ for all } q \in Q \setminus N^{-1} \text{ and } z \in G \setminus N_{q^{-1}}^{-1},$$

and thus for almost every (depending on $q \in Q \setminus N^{-1}$) $z \in G$. For $q \in Q \setminus N^{-1}$, the substitution $z = q^{-1}y$ finally leads to

$$\begin{aligned}
(|f| * |g|)(qx) &= \int_G |f(y)| \cdot |g(y^{-1}qx)| \, \mathrm{d}y \\
&= \int_G |f(qz)| \cdot |g(z^{-1}x)| \, \mathrm{d}z \\
&\leq \int_G \left(M_Q^R f\right)(z) \cdot |g(z^{-1}x)| \, \mathrm{d}z \\
&= \left[\left(M_Q^R f\right) * |g|\right](x).
\end{aligned}$$

Since this holds for all $q \in Q \setminus N^{-1}$ and thus for almost all $q \in Q$, we have shown for all $x \in G$ that

$$M_Q^R\left[|f| * |g|\right](x) \leq \left[\left(M_Q^R f\right) * |g|\right](x) \qquad \square$$

We will now use the construction of the "inverse space" $Y^\vee$ (cf. Definition 2.2.20) to reduce the study of *right-sided* Wiener amalgam spaces to that of the usual Wiener amalgam spaces:

**Lemma 2.3.29.** *Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid function space on $G$. For each precompact, measurable unit neighborhood $Q \subset G$, we have*

$$W_Q^R(L^\infty, Y) = \left(W_{Q^{-1}}\left(L^\infty, Y^\vee\right)\right)^\vee.$$

*Furthermore, $W_Q^R(L^\infty, Y)$ has the following properties:*

1. *$W_Q^R(L^\infty, Y)$ is a solid function space satisfying $W_Q^R(L^\infty, Y) \hookrightarrow Y$.*
2. *$W_Q^R(L^\infty, Y)$ is complete if $Y$ is.*
3. *Every triangle constant for $\|\cdot\|_Y$ is also valid for $\|\cdot\|_{W_Q^R(L^\infty, Y)}$.*
4. *If $\|\cdot\|_Y$ is a $p$-norm for some $p \in (0, 1]$, then so is $\|\cdot\|_{W_Q^R(L^\infty, Y)}$.*

*Furthermore, if $W_Q^R(L^\infty, Y)$ is left invariant for every precompact, measurable unit neighborhood $Q \subset G$, then $W_Q^R(L^\infty, Y)$ is independent of the choice of the (precompact, measurable) unit neighborhood $Q \subset G$.*

*This is true if and only if $W_Q(L^\infty, Y^\vee)$ is right invariant for every precompact, measurable unit neighborhood $Q \subset G$. In particular, this is true if $Y$ is left invariant.*

*Proof.* By Lemma 2.3.28, we have for every $f \in L^\infty_{\mathrm{loc}}(G)$ the equivalence

$$\begin{aligned} f \in W_Q^R(L^\infty, Y) &\iff M_Q^R f \in Y \\ &\iff \left[M_{Q^{-1}} f^\vee\right]^\vee \in Y \\ &\iff M_{Q^{-1}} f^\vee \in Y^\vee \\ &\iff f^\vee \in W_{Q^{-1}}\left(L^\infty, Y^\vee\right) \\ &\iff f \in \left[W_{Q^{-1}}\left(L^\infty, Y^\vee\right)\right]^\vee \end{aligned}$$

as well as

$$\begin{aligned} \|f\|_{W_Q^R(L^\infty,Y)} &= \left\|M_Q^R f\right\|_Y = \left\|\left[M_{Q^{-1}} f^\vee\right]^\vee\right\|_Y \\ &= \left\|M_{Q^{-1}} f^\vee\right\|_{Y^\vee} = \left\|f^\vee\right\|_{W_{Q^{-1}}(L^\infty,Y^\vee)} \\ &= \|f\|_{\left[W_{Q^{-1}}(L^\infty,Y^\vee)\right]^\vee}, \end{aligned}$$

which proves the first claim.

A combination of Lemma 2.2.21, Lemma 2.3.4, Theorem 2.3.17 and Lemma 2.3.16 proves the remaining assertions. □

The central ingredient for the proof of our modified convolution relation is the following lemma:

**Lemma 2.3.30.** *Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid QBF space on $G$ which is invariant under right translations. Furthermore, assume that $p \in (0,1]$ with the property that $\|\cdot\|_Y$ is a $p$-norm.*

*Finally, assume that $w : G \to (0,\infty)$ is a measurable, submultiplicative weight with $w(x) \geq \|\!|R_x|\!\|_{Y\to Y}$ for all $x \in G$. Then, the convolution relation*

$$W(L^\infty, Y) * \left[W\left(L^\infty, L^p_w\right)\right]^\vee \hookrightarrow Y$$

*is valid.*

*More precisely, for arbitrary $F \in W(L^\infty, Y)$ and $H \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$, the convolution integral $\int |F(y)| \cdot |H(y^{-1}x)| \, \mathrm{d}y$ is finite for almost all $x \in G$ and we have*

$$\|F * H\|_Y \leq \|\!|F| * |H|\!\|_Y \leq C \cdot \|F\|_{W(L^\infty,Y)} \cdot \|H\|_{\left[W\left(L^\infty,L^p_w\right)\right]^\vee}$$

*for some absolute constant $C > 0$ which depends on the precise choice of the (precompact, measurable) unit neighborhood $Q \subset G$ used to calculate the quasi-norms $\|\cdot\|_{W_Q(L^\infty,Y)}$ and $\|\cdot\|_{W_Q\left(L^\infty,L^p_w\right)}$.*

*Remark.* We emphasize that we require the space $Y$ *itself* to be invariant under right translations, whereas [59, Theorem 5.2] only requires the Wiener amalgam space $W(L^\infty, Y)$ to have this property.

But this is no severe restriction, since in what follows, we will only use the above convolution relation for the choice $Y = L^p_v(G)$ with a submultiplicative weight $v$, cf. Corollary 2.3.33. Since this space is always right invariant by Lemma 2.2.19, the assumption that $Y$ has to be right invariant is no further restriction.

*Proof.* Let $Q \subset G$ be a precompact open unit neighborhood. By Lemma 2.3.12, there is a $Q$-BUPU $(\varphi_i)_{i\in I}$ with associated $Q$-localizing family $(x_i)_{i\in I}$. Since $w$ is submultiplicative, $Z := L^p_w(G)$ is right invariant by Lemma 2.2.19. Thus, Lemma 2.3.16 implies that Theorem 2.3.17 is applicable.

For $H \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$, i.e. $H^\vee \in W\left(L^\infty, L^p_w\right)$, this implies

$$\begin{aligned}\left\| \left( \left\| \varphi_i \cdot H^\vee \right\|_{L^\infty} \right)_{i\in I} \right\|_{\ell^p_{w_X}(I)} &\leq C \cdot \left\| \left( \left\| \varphi_i \cdot H^\vee \right\|_{L^\infty} \right)_{i\in I} \right\|_{Z_d(X,Q)} \\ &\leq C' \cdot \left\| H^\vee \right\|_{W\left(L^\infty, L^p_w\right)} \end{aligned} \tag{2.3.17}$$

for certain absolute constants $C, C' > 0$. Here, we also used the identity $\ell^p_{w_X}(I) = Z_d(X,Q)$ for $w_X(i) := w(x_i)$, cf. Lemma 2.3.21.

For each $i \in I$, set $H_i := L_{x_i^{-1}}(\varphi_i \cdot H^\vee)$ and

$$G_i := R_{x_i} H_i^\vee = R_{x_i} R_{x_i^{-1}} \left[ \left( \varphi_i \cdot H^\vee \right) \right]^\vee = \varphi_i^\vee \cdot H.$$

We note $\operatorname{supp} H_i \subset Q$ and $\operatorname{supp} G_i \subset Q^{-1} x_i^{-1}$. For $x, y \in G$ with $G_i\left(y^{-1}x\right) \neq 0$, this implies $y^{-1}x \in Q^{-1}x_i^{-1}$, which in turn yields $x^{-1}y \in x_i Q$ and $y \in x x_i Q$. Thus,

$$G_i\left(y^{-1}x\right) = G_i\left(y^{-1}x\right) \cdot \chi_{x x_i Q}(y).$$

Now, let $F \in W(L^\infty, Y)$ be arbitrary. We estimate the convolution of $|F|$ with the "individual parts" $G_i$ as follows:

$$\begin{aligned}\left[|F| * |G_i|\right](x) &= \int_G |F(y)| \cdot |G_i(y^{-1}x)| \, \mathrm{d}y \\ &= \int_G |F(y)| \, \chi_{x x_i Q}(y) \, \chi_{x x_i Q}(y) \, |G_i(y^{-1}x)| \, \mathrm{d}y \\ &\leq \left\| F \cdot \chi_{x x_i Q} \right\|_{L^\infty} \cdot \|G_i\|_{L^\infty} \cdot \mu\left(x x_i Q\right) \\ &\leq \mu(Q) \cdot \left(M_Q F\right)(x x_i) \cdot \|G_i\|_{L^\infty} \\ &= \mu(Q) \cdot \left(R_{x_i}\left[M_Q F\right]\right)(x) \cdot \|G_i\|_{L^\infty}\end{aligned}$$

where $\mu$ is the (left) Haar measure on $G$. In particular, the integral is finite for all $x \in G$ since $F \in W(L^\infty, Y) \subset L^\infty_{\mathrm{loc}}(G)$, so that $M_Q F(z)$ is finite for all $z \in G$.

Since $Y$ is solid and right-invariant, we get

$$\begin{aligned}\|F * G_i\|_Y \leq \||F| * |G_i|\|_Y &\leq \mu(Q) \cdot \|G_i\|_{L^\infty} \cdot \|R_{x_i}[M_Q F]\|_Y \\ &\leq \mu(Q) \cdot w(x_i) \cdot \|G_i\|_{L^\infty} \cdot \|M_Q F\|_Y\end{aligned}$$

and hence

$$\begin{aligned}\sum_{i\in I} \||F| * |G_i|\|_Y^p &\leq [\mu(Q)]^p \cdot \|M_Q F\|_Y^p \cdot \left\|\left(\|G_i\|_{L^\infty}\right)_{i\in I}\right\|_{\ell^p_{w_X}}^p \\ &\leq \left(C'\right)^p \cdot [\mu(Q)]^p \cdot \|F\|_{W_Q(Y)}^p \left\|H^\vee\right\|_{W\left(L^p_w\right)}^p < \infty.\end{aligned}$$

Here, we used $\|G_i\|_{L^\infty} = \|H_i^\vee\|_{L^\infty} = \|H_i\|_{L^\infty} = \|\varphi_i \cdot H^\vee\|_{L^\infty}$ and estimate (2.3.17).

But $Y$ is a complete solid function space and $\|\cdot\|_Y$ is a $p$-norm. By Theorem 2.2.10, this implies $\Gamma := \sum_{i\in I} |F| * |G_i| \in Y$ with

$$\begin{aligned}\|\Gamma\|_Y &\leq \left(\sum_{i\in I} \||F| * |G_i|\|_Y^p\right)^{\frac{1}{p}} \\ &\leq C'\mu(Q) \cdot \|F\|_{W_Q(Y)} \|H\|_{\left[W\left(L^p_w\right)\right]^\vee}.\end{aligned}$$

In particular, the series defining $\Gamma$ converges pointwise almost everywhere. By the monotone convergence theorem, this means

$$\begin{aligned}\infty > \Gamma(x) &= \sum_{i\in I} \int_G |F(y)| \cdot |G_i(y^{-1}x)| \, \mathrm{d}y \\ &= \int_G |F(y)| \cdot \sum_{i\in I} |G_i(y^{-1}x)| \, \mathrm{d}y \\ &= \int_G |F(y)| \cdot \sum_{i\in I} \varphi_i^\vee(y^{-1}x) \cdot |H(y^{-1}x)| \, \mathrm{d}y \\ &= \int_G |F(y)| \cdot |H(y^{-1}x)| \, \mathrm{d}y = (|F| * |H|)(x).\end{aligned}$$

Hence, the integral defining $|F| * |H|$ is finite for almost all $x \in G$. Finally, solidity of $Y$ implies $F * H \in Y$ and $|F| * |H| = \Gamma \in Y$ with

$$\begin{aligned}\|F * H\|_Y \leq \||F| * |H|\|_Y &= \|\Gamma\|_Y \\ &\leq C' \cdot \mu(Q) \cdot \|F\|_{W_Q(Y)} \cdot \|H\|_{\left[W\left(L^p_w\right)\right]^\vee}.\end{aligned}$$ □

Together with the fact that the maximal operators $M_Q$ and $M_Q^R$ can be "passed on" to the individual factors of a convolution product, we obtain the following theorem:

**Theorem 2.3.31.** *Let $G$ be a $\sigma$-compact LCH group and let $(Y, \|\cdot\|_Y)$ be a solid QBF space on $G$ which is invariant under right translations. Furthermore, assume that $p \in (0,1]$ with the property that $\|\cdot\|_Y$ is a $p$-norm.*

*Finally, assume that $w : G \to (0,\infty)$ is a measurable, submultiplicative weight with $w(x) \geq |||R_x|||_{Y\to Y}$ for all $x \in G$. Then, the convolution relations*

$$W_Q^R(W(L^\infty, Y)) * \left[W^R\left(W\left(L^\infty, L^p_w\right)\right)\right]^\vee \hookrightarrow W_Q^R(W(L^\infty, Y))$$

*and*

$$W_Q^R(W(L^\infty, Y)) * \left[W^R\left(W\left(L^\infty, L^p_w\right)\right)\right]^\vee \hookrightarrow W\left(W_Q^R(L^\infty, Y)\right)$$

*are valid for each precompact measurable unit neighborhood $Q \subset G$.*

*As usual, it is understood that the integral defining the convolution product in question converges absolutely for almost all $x \in G$.*

*Remark.* In the above theorem, the set $Q \subset G$ is mentioned explicitly in a few places since we do *not* require $Y$ or $W(L^\infty, Y)$ to be left invariant. Thus, the spaces $W_Q^R(W(L^\infty, Y))$ and $W_Q^R(L^\infty, Y)$ are possibly dependent on the choice of $Q$.

But in what follows, we will only use the theorem in the case $Y = L^p_v(G)$ with a submultiplicative weight $v$ on $G$. With this choice, $Y$ is left- and right-invariant by Lemma 2.2.19, so that the problem vanishes.

*Proof.* Let $f \in W_Q^R(W(L^\infty, Y))$ and $h \in \left[W^R\left(W\left(L^\infty, L^p_w\right)\right)\right]^\vee$. This implies in particular $f \in W(L^\infty, Y)$ and $h \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$, so that the convolution integral $(|f| * |h|)(x)$ is finite for almost every $x \in G$ by Lemma 2.3.30 above.

By definition of the Wiener amalgam spaces, we have

$$[M_Q h]^\vee = M^R_{Q^{-1}} h^\vee \in W\left(L^\infty, L^p_w\right)$$

and $F := M_Q^R f \in W(L^\infty, Y)$. Hence, $H := M_Q h \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$. Using Lemma 2.3.30, we conclude $F * H \in Y$ with

$$\begin{aligned}
\|F * H\|_Y &\leq C \cdot \|F\|_{W(L^\infty,Y)} \cdot \|H\|_{\left[W\left(L^\infty,L^p_w\right)\right]^\vee} \\
&= C \cdot \|f\|_{W_Q^R(W(L^\infty,Y))} \cdot \left\|(M_Q h)^\vee\right\|_{W\left(L^\infty,L^p_w\right)} \\
&= C \cdot \|f\|_{W_Q^R(W(L^\infty,Y))} \cdot \left\|M^R_{Q^{-1}} h^\vee\right\|_{W\left(L^\infty,L^p_w\right)} \\
&= C \cdot \|f\|_{W_Q^R(W(L^\infty,Y))} \cdot \|h\|_{\left[W^R\left(W\left(L^\infty,L^p_w\right)\right)\right]^\vee} < \infty.
\end{aligned}$$

In particular, $F * H$ is finite almost everywhere. But Lemma 2.3.28 implies

$$M_Q^R\left[M_Q\left[|f| * |h|\right]\right] \leq M_Q^R\left[|f| * M_Q h\right] \leq M_Q^R f * M_Q h = F * H$$

as well as

$$M_Q\left[M_Q^R\left[|f|*|h|\right]\right] \leq M_Q\left[M_Q^R f * |h|\right] \leq M_Q^R f * M_Q h = F * H.$$

Since $F * H$ is finite almost everywhere, and since Lemma 2.3.28 and Lemma 2.3.3 yield $|g| \leq M_Q^R g$ and $|g| \leq M_Q g$ almost everywhere, we get $M_Q^R\left[M_Q\left[|f|*|h|\right]\right] < \infty$ and $M_Q\left[M_Q^R\left[|f|*|h|\right]\right] < \infty$ on subsets of full measure, which are necessarily dense. By Lemma 2.3.32 below, this implies that $|f|*|h|$, $M_Q\left[|f|*|h|\right]$ and $M_Q^R\left[|f|*|h|\right]$ are all elements of $L^\infty_{\mathrm{loc}}(G)$.

Using the solidity of $Y$, we finally arrive at

$$\begin{aligned}\|f*h\|_{W_Q^R(W(L^\infty,Y))} &\leq \left\|M_Q\left[M_Q^R\left[|f|*|h|\right]\right]\right\|_Y \leq \|F*H\|_Y \\ &\leq C\,\|f\|_{W_Q^R(W(L^\infty,Y))}\,\|h\|_{\left[W^R\left(W\left(L^\infty,L^p_w\right)\right)\right]^\vee} < \infty.\end{aligned}$$

The estimate for $\|f*h\|_{W\left(L^\infty,W_Q^R(L^\infty,Y)\right)}$ is obtained analogously. □

**Lemma 2.3.32.** *Let $G$ be an LCH group and assume that $f : G \to \mathbb{C}$ or $f : G \to [0,\infty]$ is measurable. If $Q \subset G$ is a measurable unit neighborhood with $M_Q f(x) < \infty$ or $M_Q^R f(x) < \infty$ for all $x$ in a dense subset of $G$, then $f \in L^\infty_{\mathrm{loc}}(G)$.*

*Proof.* Using $M_Q^R f = \left(M_{Q^{-1}} f^\vee\right)^\vee$ and that an arbitrary subset $M \subset G$ is dense if and only if $M^{-1} \subset G$ is dense, and noting that $f \in L^\infty_{\mathrm{loc}}(G)$ is equivalent to $f^\vee \in L^\infty_{\mathrm{loc}}(G)$, we see that it suffices to prove the claim for $M_Q f$.

Since $Q$ is a unit neighborhood, there is a smaller open unit neighborhood $U \subset G$ with $U = U^{-1}$ and $U \cdot U \subset Q$.

Let $M \subset G$ be dense with $M_Q f(x) < \infty$ for all $x \in M$. For each $x \in G$, the set $xU \subset G$ is open. Hence, there is some $y \in M \cap xU$. This yields $x \in yU^{-1} = yU$ and hence $xU \subset yUU \subset yQ$. Thus,

$$\|\chi_{xU} \cdot f\|_{L^\infty} \leq \|\chi_{yQ} \cdot f\|_{L^\infty} = (M_Q f)(y) < \infty,$$

because of $y \in M$.

Now, for an arbitrary compact subset $K \subset G$, there are finitely many $x_1, \ldots, x_n \in G$ with $K \subset \bigcup_{i=1}^n x_i U$, since $(xU)_{x\in K}$ is an open cover of the compact set $K$. This yields

$$\|\chi_K \cdot f\|_{L^\infty} \leq \sum_{i=1}^n \|\chi_{x_i U} \cdot f\|_{L^\infty} < \infty$$

and thus $f \in L^\infty_{\mathrm{loc}}(G)$. □

In the next section, we will not need the convolution relation in Theorem 2.3.31 in its full generality; we will only invoke the following specialization to the case of weighted Lebesgue spaces:

**Corollary 2.3.33.** *Let $G$ be a $\sigma$-compact LCH group, let $p \in (0,1]$ and assume that $w : G \to (0,\infty)$ is a measurable, submultiplicative weight. Then*

$$W^R\left(W\left(L^\infty, L^p_w\right)\right) * \left[W^R\left(W\left(L^\infty, L^p_{w_{\vee,p}}\right)\right)\right]^\vee \hookrightarrow W^R\left(W\left(L^\infty, L^p_w\right)\right).$$

*For convenience of the reader, we recall the definition $w_{\vee,p}(x) = \left[\Delta\left(x^{-1}\right)\right]^{1/p} \cdot w\left(x^{-1}\right)$ which was introduced in Lemma 2.2.21.*

*Remark.* We will mainly use this corollary in the case $w = w_{\vee,p}$.

*Proof.* Let $Y := L^p_w(G)$. Then $Y$ is a solid QBF space which is invariant under left- and right translations. More precisely, since the definition of the modular function yields

$$\|R_x f\|^p_{L^p} = \int R_x |f|^p \, \mathrm{d}\mu = \Delta\left(x^{-1}\right) \cdot \int |f|^p \, \mathrm{d}\mu = \Delta\left(x^{-1}\right) \cdot \|f\|^p_{L^p},$$

we have $|||R_x|||_{L^p \to L^p} = \left[\Delta\left(x^{-1}\right)\right]^{1/p}$, so that Lemma 2.2.19 yields

$$|||R_x|||_{Y \to Y} \leq w\left(x^{-1}\right) \cdot \left[\Delta\left(x^{-1}\right)\right]^{1/p} = w_{\vee,p}(x).$$

But by Theorem 2.3.31, this entails

$$W^R_Q\left(W\left(L^\infty, Y\right)\right) * \left[W^R\left(W\left(L^\infty, L^p_{w_{\vee,p}}\right)\right)\right]^\vee \hookrightarrow W^R_Q\left(W\left(L^\infty, Y\right)\right). \quad \square$$

### 2.3.4 *Further properties of Wiener amalgam spaces*

In this subsection, we prove embeddings of Wiener amalgam spaces into certain weighted $L^\infty$ spaces and we investigate the measurability of the weights $x \mapsto |||R_x|||_{W(Y) \to W(Y)}$ and $x \mapsto |||L_x|||_{W(Y) \to W(Y)}$.

We start with the embedding of Wiener amalgam spaces into certain weighted $L^\infty$ spaces. This embedding will become important for the treatment of coorbit spaces in the next section (cf. in particular part (4) of Lemma 2.4.14). With slightly different proofs, analogous results are given by Rauhut in [59, Lemma 3.1 and Lemma 3.2], but under the assumption that $W\left(L^\infty, Y\right)$ is left invariant. Here, we only assume right invariance.

**Lemma 2.3.34.** *Let $G$ be a $\sigma$-compact LCH group and let $\left(Y, \|\cdot\|_Y\right)$ be a solid function space on $Y$. Finally, let $Q \subset G$ be a measurable, precompact unit neighborhood. Then the following are true:*

1. *If $W_Q\left(L^\infty, Y\right)$ is right-invariant and*

$$r(x) := |||R_x|||_{W_Q(L^\infty,Y) \to W_Q(L^\infty,Y)} \text{ for } x \in G,$$

*then for each compact unit neighborhood $K \subset G$, there is a constant $C_{K,Q,Y} > 0$ such that*

$$(r(x))^{-1} \cdot |f(x)| \leq (r(x))^{-1} \cdot \left(M^R_K f\right)(x) \leq C_{K,Q,Y} \cdot \|f\|_{W_Q(L^\infty,Y)}$$

*is true for each $f \in W_Q\left(L^\infty, Y\right)$ for almost all $x \in G$.*

2. *If $W_U(L^\infty, Y)$ is right invariant for every precompact, measurable unit neighborhood $U \subset G$, there is a constant $C_{Q,Y} > 0$ such that for each relatively separated family $X = (x_i)_{i\in I}$ in $G$ and each sequence $(\lambda_i)_{i\in I} \in Y_d(X, Q)$, the estimate*

$$\forall i \in I: \quad \frac{1}{r(x_i)} \cdot |\lambda_i| \leq C_{Q,Y} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)}$$

*is true. This means $Y_d(X, Q) \hookrightarrow \ell^\infty_{1/r_X}(I)$ with $r_X(i) := r(x_i)$.*

*Proof.* By Lemma 2.3.5, there is a constant $C_{K,Q,Y} > 0$ with

$$\|f \cdot \chi_K\|_{L^\infty} \leq C_{K,Q,Y} \cdot \|f\|_{W_Q(L^\infty,Y)}$$

for all $f \in W_Q(L^\infty, Y)$. For arbitrary $x \in G$, this yields

$$\begin{aligned} M^R_K f(x) = \|\chi_{Kx} \cdot f\|_{L^\infty} &= \|R_{x^{-1}}(\chi_K \cdot R_x f)\|_{L^\infty} \\ &= \|\chi_K \cdot R_x f\|_{L^\infty} \\ &\leq C_{K,Q,Y} \cdot \|R_x f\|_{W_Q(L^\infty,Y)} \\ &\leq C_{K,Q,Y} \cdot r(x) \cdot \|f\|_{W_Q(L^\infty,Y)}. \end{aligned}$$

Finally, the estimate $|f(x)| \leq M^R_K f(x)$ holds for almost all $x \in G$ by Lemma 2.3.28.

For the proof of the last part, set $Z := W_Q(L^\infty, Y)$. Since $W_U(L^\infty, Y)$ is right invariant for every $U$ by assumption, Theorem 2.3.17 yields some $M_{Q,Y} > 0$ with $\|\cdot\|_{Z_d(X,Q)} \leq M_{Q,Y} \cdot \|\cdot\|_{Y_d(X,Q)}$. The same theorem also shows that there is a constant $L_Q > 0$ with $r(q) \leq L_Q$ for all $q \in Q$.

Now, let $(\lambda_i)_{i\in I} \in Y_d(X, Q) = Z_d(X, Q)$ be arbitrary. This means $\sum_{i\in I} |\lambda_i| \chi_{x_i Q} \in Z = W_Q(L^\infty, Y)$. Part 1 implies

$$\begin{aligned} \frac{1}{r(x)} \cdot \sum_{i\in I} |\lambda_i| \chi_{x_i Q}(x) &\leq C_{Q,Y} \cdot \left\| \sum_{i\in I} |\lambda_i| \chi_{x_i Q} \right\|_{W_Q(L^\infty,Y)} \\ &= C_{Q,Y} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{Z_d(X,Q)} \\ &\leq M_{Q,Y} C_{Q,Y} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)} \end{aligned}$$

for almost every $x \in G$.

Now, let $j \in I$ be arbitrary. Since $Q$ is a measurable unit neighborhood, $x_j Q$ is of positive measure. Thus, there is some $x \in x_j Q$ for which the above estimate is true. In particular, this yields

$$\frac{1}{r(x)} \cdot |\lambda_j| \leq \frac{1}{r(x)} \cdot \sum_{i\in I} |\lambda_i| \chi_{x_i Q}(x) \leq M_{Q,Y} C_{Q,Y} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X,Q)}.$$

But $x = x_j q$ for some $q \in Q$, which implies

$$r(x) = r(x_j q) \leq r(x_j)\, r(q) \leq L_Q \cdot r(x_j).$$

All in all, we get

$$\frac{1}{r(x_j)} \cdot |\lambda_j| \leq \frac{L_Q}{r(x)} \cdot |\lambda_j| \leq L_Q M_{Q,Y} C_{Q,Y} \cdot \left\| (\lambda_i)_{i \in I} \right\|_{Y_d(X,Q)}.$$

Since $j \in I$ was arbitrary, the proof is complete. □

We close this section with a short measurability consideration: In [59] and [58], the function

$$w(x) := \left\| |L_x| \right\|_{W_Q(L^\infty,Y) \to W_Q(L^\infty,Y)},$$

as well as the analogous version for the right translation, are used as weights for other spaces. Implicitly, this assumes that the weight $w$ defined in this way is measurable. I do not know whether this is true in general, but the next lemma shows that a potential counterexample needs to be quite pathological.

**Lemma 2.3.35.** *Let $G$ be a $\sigma$-compact, first countable LCH group and assume that $(Y, \|\cdot\|_Y)$ is a solid function space on $G$ satisfying the Fatou property.*

1. *If $W_Q(L^\infty, Y)$ is left-invariant, then the function*

$$w : G \to (0, \infty), x \mapsto \left\| |L_x| \right\|_{W_Q(L^\infty,Y) \to W_Q(L^\infty,Y)}$$

   *is lower semicontinuous and thus measurable.*

2. *The same is true for $R_x$ instead of $L_x$ if $W_Q(L^\infty, Y)$ is right-invariant instead of left-invariant.*

*Proof.* First of all, we recall the fact that any lower semicontinuous function $F : G \to \mathbb{R}$ satisfies $F(y) \leq \liminf_{n\to\infty} F(y_n)$ if $y_n \to y$. To see this, choose a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ with

$$F(y_{n_k}) \to \liminf_{n\to\infty} F(y_n) \in [-\infty, \infty].$$

For $k \in \mathbb{N}$, let $M_k := \sup_{\ell \geq k} F(y_{n_\ell})$. Then $y_{n_\ell} \in \{x \in G \mid F(x) \leq M_k\}$ holds for all $\ell \geq k$. But the latter set is closed, since $F$ is lower semicontinuous. Hence, $F(y) \leq M_k$ as well. Letting $k \to \infty$, we finally conclude

$$\begin{aligned} F(y) &\leq \lim_{k\to\infty} M_k = \lim_{k\to\infty} \sup_{\ell \geq k} F(y_{n_\ell}) = \limsup_{k\to\infty} F(y_{n_k}) \\ &= \lim_{k\to\infty} F(y_{n_k}) = \liminf_{n\to\infty} F(y_n). \end{aligned}$$

We want to show that $U_\alpha := w^{-1}((\alpha, \infty))$ is open for each $\alpha \in \mathbb{R}$. For $\alpha \leq 0$, we have $U_\alpha = G$, which is trivially open. Hence, it suffices for $\alpha > 0$ to show that $A_\alpha := U_\alpha^c \subset G$ is closed.

Since $G$ is first countable, it is enough to show that for each sequence $(x_n)_{n\in\mathbb{N}}$ in $A_\alpha$ with $x_n \to x \in G$, we also have $x \in A_\alpha$. To this

end, let $f \in W_Q(L^\infty, Y)$ be arbitrary. Since $M_Q f$ is lower semicontinuous by Lemma 2.3.4, the claim from the start of the proof, together with the identity $M_Q[L_x f] = L_x[M_Q f]$ from Lemma 2.3.18, implies

$$\begin{aligned} 0 \leq M_Q[L_x f](y) &= M_Q f(x^{-1}y) \\ &\leq \liminf_{n\to\infty} M_Q f(x_n^{-1}y) = \liminf_{n\to\infty} M_Q[L_{x_n} f](y). \end{aligned}$$

Since $Y$ has the Fatou property and because of $w(x_n) \leq \alpha$ for all $n \in \mathbb{N}$, we conclude

$$\begin{aligned} \|L_x f\|_{W_Q(L^\infty,Y)} &= \|M_Q[L_x f]\|_Y \\ &\leq \liminf_{n\to\infty} \|M_Q[L_{x_n} f]\|_Y = \liminf_{n\to\infty} \|L_{x_n} f\|_{W_Q(L^\infty,Y)} \\ &\leq \liminf_{n\to\infty} w(x_n) \cdot \|f\|_{W_Q(L^\infty,Y)} \leq \alpha \cdot \|f\|_{W_Q(L^\infty,Y)}. \end{aligned}$$

Since this holds for all $f \in W_Q(L^\infty, Y)$, we get $w(x) \leq \alpha$ and thus $x \in A_\alpha$ as desired.

For the case of $R_x$ instead of $L_x$, let again $f \in W_Q(L^\infty, Y)$ and define $f^\vee : G \to \mathbb{C}, y \mapsto f(y^{-1})$. Since the left and right Haar measure are mutually absolutely continuous, $f \in L^\infty_{\mathrm{loc}}(G)$ implies $f^\vee \in L^\infty_{\mathrm{loc}}(G)$. Furthermore, the invariance of $\|\cdot\|_{L^\infty}$ under translations and inversions yields

$$\begin{aligned} M_Q[R_z f](y) &= \left\|\chi_{yQ} \cdot R_z f\right\|_{L^\infty} = \left\|R_z\left(\chi_{yQz} \cdot f\right)\right\|_{L^\infty} = \left\|\chi_{yQz} \cdot f\right\|_{L^\infty} \\ &= \left\|\chi_{z^{-1}Q^{-1}y^{-1}} \cdot f^\vee\right\|_{L^\infty} = M_{Q^{-1}y^{-1}}\left[f^\vee\right](z^{-1}) \end{aligned}$$

for all $y, z \in G$. By Lemma 2.3.4, we know that $M_{Q^{-1}y^{-1}}[f^\vee]$ is lower semicontinuous. As above, this implies

$$\begin{aligned} M_Q[R_x f](y) &= M_{Q^{-1}y^{-1}}\left[f^\vee\right](x^{-1}) \\ &\leq \liminf_{n\to\infty} M_{Q^{-1}y^{-1}}\left[f^\vee\right](x_n^{-1}) \\ &= \liminf_{n\to\infty} M_Q[R_{x_n} f](y) \end{aligned}$$

for all $y \in G$. Using the Fatou property of $Y$ in exactly the same way as above, we conclude

$$\|R_x f\|_{W_Q(L^\infty,Y)} \leq \liminf_{n\to\infty} \|R_{x_n} f\|_{W_Q(L^\infty,Y)} \leq \alpha \cdot \|f\|_{W_Q(L^\infty,Y)}$$

and thus $w(x) = \|\!|R_x|\!\|_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} \leq \alpha$. As in the case of the left translations, this establishes the lower semicontinuity of $w$. □

Our final result in this section is to show that $W(L^\infty, Y)$ inherits the Fatou property from $Y$.

**Lemma 2.3.36.** *Let $G$ be a $\sigma$-compact LCH group and assume that $Y$ is a function space on $G$. If $Y$ satisfies the Fatou property, then so does $W_Q(L^\infty, Y)$ for each precompact measurable unit neighborhood $Q \subset G$.*

*Proof.* Let us first verify that $L^\infty(G)$ has the Fatou property: Assume that $f : G \to \mathbb{C}$ is measurable with $|f(x)| \leq \liminf_{n\to\infty} |f_n(x)|$ for all $x \in G \setminus N_0$, for some null-set $N_0 \subset G$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(G)$. Then for each $n \in \mathbb{N}$, there is a null-set $N_n \subset G$ with $|f_n(x)| \leq \|f_n\|_{L^\infty}$ for all $x \in G \setminus N_n$. The set $N := \bigcup_{n\in\mathbb{N}_0} N_n \subset G$ is also of measure zero and we have

$$|f(x)| \leq \liminf_{n\to\infty} |f_n(x)| \leq \liminf_{n\to\infty} \|f_n\|_{L^\infty}$$

for all $x \in G \setminus N$, and hence $\|f\|_{L^\infty} \leq \liminf_{n\to\infty} \|f_n\|_{L^\infty}$. In particular, if $\liminf_{n\to\infty} \|f_n\|_{L^\infty}$ is finite, then $f \in L^\infty(G)$.

Now assume $f_n \in W_Q(L^\infty, Y)$ with $\liminf_{n\to\infty} \|f_n\|_{W_Q(L^\infty,Y)} < \infty$ and that $f : G \to \mathbb{C}$ is measurable with $|f| \leq \liminf_{n\to\infty} |f_n|$ almost everywhere. For each $x \in G$, this yields $|\chi_{xQ} f| \leq \liminf_{n\to\infty} |\chi_{xQ} f_n|$ almost everywhere and thus

$$(M_Q f)(x) = \|\chi_{xQ} \cdot f\|_{L^\infty} \leq \liminf_{n\to\infty} \|\chi_{xQ} \cdot f_n\|_{L^\infty} = \liminf_{n\to\infty} (M_Q f_n)(x).$$

Furthermore, $M_Q f_n \in Y$, since $f_n \in W_Q(L^\infty, Y)$. Finally, the identity $\|M_Q f_n\|_Y = \|f_n\|_{W_Q(L^\infty,Y)}$ yields

$$\liminf_{n\to\infty} \|M_Q f_n\|_Y = \liminf_{n\to\infty} \|f_n\|_{W_Q(L^\infty,Y)} < \infty.$$

Since $Y$ has the Fatou property by assumption, we conclude $M_Q f \in Y$ (and thus $f \in W_Q(L^\infty, Y)$) with

$$\begin{aligned} \|f\|_{W_Q(L^\infty,Y)} = \|M_Q f\|_Y &\leq \liminf_{n\to\infty} \|M_Q f_n\|_Y \\ &= \liminf_{n\to\infty} \|f_n\|_{W_Q(L^\infty,Y)} < \infty. \end{aligned}$$ □

## 2.4 COORBIT THEORY FOR QUASI-BANACH SPACES

In this section, we modify the coorbit theory for Quasi-Banach spaces as developed by Rauhut in [58], so that the convolution relation [58, Theorem 5.2] is not used anymore. This will necessitate some changes, especially for the space of analyzing vectors. We remark that the only property for which the modified convolution relation (as stated in Corollary 2.3.33) is used is well-definedness of the coorbit space $\mathrm{Co}(Y)$, i.e. independence from the chosen analyzing vector $g$, cf. Theorem 2.4.9.

We briefly recall from the introduction that the coorbit space $\mathrm{Co}(Y)$ will be defined as

$$\mathrm{Co}(Y) = \left\{ f \in \mathcal{R} \,\middle|\, W_g f \in W(L^\infty, Y) \right\}.$$

Here, $\mathcal{R}$ is a suitable RESERVOIR and $g$ is a suitable ANALYZING VECTOR and $W_g f$ is the WAVELET TRANSFORM of $f$ with respect

to $g$. This wavelet transform is defined using a representation $\pi$ of $G$. The exact form of the reservoir and the condition for analyzing vectors depend on $Y$.

This section is structured as follows: In the first subsection, we collect our standing assumptions on the group $G$, the representation $\pi$ and the space $Y$. We also introduce the space of ANALYZING VECTORS $\mathbb{A}_v^p$, whose importance will become clear later.

The next subsection is concerned with the REPRODUCING FORMULA $W_g f = \frac{1}{C_g} W_g f * W_g g$ for the wavelet transform. This formula will be used again and again, for example to show that the space $\mathrm{Co}(Y)$ is independent of the exact choice of the analzying vector $g$, but also to prove the ATOMIC DECOMPOSITION results for coorbit spaces. In a nutshell, the idea of coorbit theory is to reduce the study of the function space $\mathrm{Co}(Y)$ to the study of certain reproducing kernel Quasi-Banach spaces on $G$, cf. in particular Lemma 2.4.18.

In the third subsection, we construct the reservoir $\mathcal{R}$ which is then used in the next subsection to formally define the coorbit space $\mathrm{Co}(Y)$. Here, we also show the most important property of the coorbit spaces, i.e. completeness and independence of the analyzing vector $g$.

Finally, in the last subsection, we derive ATOMIC DECOMPOSITION results for the space $\mathrm{Co}(Y)$. As noted in the introduction, this means that one can express every $f \in \mathrm{Co}(Y)$ as

$$f = \sum_{i \in I} [\lambda_i(f) \cdot \pi(x_i) g]$$

for a suitable family $(x_i)_{i \in I}$ in $g$. One even has a quasi-norm equivalence $\left\| (\lambda_i(f))_{i \in I} \right\|_{Y_d} \asymp \|f\|_{\mathrm{Co}(Y)}$.

### 2.4.1 *Assumptions for coorbit theory*

To begin with, let us collect the standing assumptions for this section:

**Assumption 2.4.1.** *(cf. [28, §3.1] and [58, Sections 2 and 4])*

*In the following, we assume that $G$ is a $\sigma$-compact LCH group and that the following properties are satisfied:*

A. *$\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous, unitary, irreducible representation of $G$ for some (nontrivial) Hilbert space $\mathcal{H}$.*

B. *For $f, g \in \mathcal{H}$, we define the* (GENERALIZED) WAVELET TRANSFORM OF $f$ (WITH RESPECT TO $g$) *as*

$$W_g f : G \to \mathbb{C}, x \mapsto \langle f \mid \pi(x) g \rangle_{\mathcal{H}}.$$

*Thanks to the Cauchy-Schwarz inequality and to the (strong) continuity of $\pi$, $W_g f$ is always a bounded, continuous function.*

*The representation $\pi$ is called* SQUARE INTEGRABLE, *if there is some* ADMISSIBLE *$g \in \mathcal{H} \setminus \{0\}$. Here, $g \in \mathcal{H}$ is called admissible*

*We remark that the (generalized) wavelet transform is often called the* VOICE TRANSFORM *of $f$ and denoted by $V_g f$ instead of $W_g f$, see e.g. [28, §3.1].*

*if $W_g g \in L^2(G)$. The representation $\pi$ is called* INTEGRABLE *if $W_g g \in L^1(G)$ holds for some $g \in \mathcal{H} \setminus \{0\}$. In the following, we assume that $\pi$ is integrable (and hence also square integrable).*

*We remark that $W_g g \in L^1(G)$ already entails $W_g g \in L^1(G) \cap L^\infty(G) \subset L^2(G)$, so that every integrable representation is also square integrable.*

c. *$(Y, \|\cdot\|_Y)$ is a solid QBF space on $G$ with a $p$-norm $\|\cdot\|_Y$ for some fixed $p \in (0,1]$,*

d. *$W_Q(L^\infty, Y)$ is right invariant for each measurable, precompact unit neighborhood $Q \subset G$ and $w : G \to (0,\infty)$ is a fixed measurable, submultiplicative weight with*

$$w(x) \geq \left|\!\left|\!\left| R_x \right|\!\right|\!\right|_{W(L^\infty,Y)\to W(L^\infty,Y)}$$

*and*

$$w(x) \geq \Delta(x^{-1}) \left|\!\left|\!\left| R_{x^{-1}} \right|\!\right|\!\right|_{W(L^\infty,Y)\to W(L^\infty,Y)}$$

*for all $x \in G$ and with the additional property that $w$ is bounded from below, i.e. there is some $c > 0$ with $w(x) \geq c$ for all $x \in G$.*

e. *We also fix a submultiplicative, measurable weight $v : G \to (0,\infty)$ with the properties $v \geq w$ and $v \geq w_{\vee,p}$. We call each such weight $v$ a* CONTROL WEIGHT *for the space $Y$.*

*For convenience of the reader, we recall the definition $v_{\vee,p}(x) = v(x^{-1}) \cdot [\Delta(x^{-1})]^{1/p}$ from Lemma 2.2.21.*

*One possible choice (in fact the smallest choice) of $v$ is*

$$v_0(x) := \max\left\{ w(x), [\Delta(x^{-1})]^{1/p} \cdot w(x^{-1}) \right\}.$$

f. *(cf. [28, §4.1]) We define the class of* GOOD VECTORS *as*

$$\mathcal{G}_v := \left\{ g \in \mathcal{H} \;\middle|\; W_g g \in L^1_v(G) \right\}$$

*and we assume that $\mathcal{G}_v$ is nontrivial, i.e. $\mathcal{G}_v \supsetneq \{0\}$.*

g. *Finally, we define the class of* ANALYZING VECTORS *as*

$$\mathbb{A}^p_v := \left\{ g \in \mathcal{H} \;\middle|\; W_g g \in W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \right\}.$$

*As above, we assume that $\mathbb{A}^p_v$ is nontrivial, i.e. $\mathbb{A}^p_v \supsetneq \{0\}$.*

*Remark* 2.4.2. We remark that our terminology is slightly different from the usual one. In particular, the set $\mathcal{G}_v$ is usually denoted by $\mathcal{A}_v$ and called the class of ANALYZING VECTORS (see [28, §4.1]). But in our context, it makes more sense to call the set $\mathbb{A}^p_v$ the class of analyzing vectors, since with this definition we will be able to show (cf. Theorem 2.4.9) that the space

$$\mathrm{Co}(Y) := \left\{ f \in \mathcal{R} \;\middle|\; W_g f \in W(L^\infty, Y) \right\}$$

is independent of the choice of $g \in \mathbb{A}^p_v \setminus \{0\}$, so that each (nonzero) "analyzing vector" $g \in \mathbb{A}^p_v$ can indeed be used to "analyze" distributions $f \in \mathcal{R}$, i.e. to decide their membership in $\mathrm{Co}(Y)$.

In the classical setting of Banach spaces considered by Feichtinger and Gröchenig, one can even show this independence for $g \in \mathcal{A}_v$, so that the original terminology is suitable in their setting, but not in ours.

Furthermore, we remark that the sequence of embeddings

$$W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \hookrightarrow W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(L^\infty, L^1_v\right) \hookrightarrow L^1_v$$

implies $\mathbb{A}^p_v \subset \mathcal{G}_v$, so that $\mathcal{G}_v$ is automatically nontrivial if $\mathbb{A}^p_v$ is. Here, the embedding $W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(L^\infty, L^1_v\right)$ is a consequence of Corollary 2.3.20, together with Lemma 2.3.21 and with the embedding $\ell^p_v \hookrightarrow \ell^1_v$, which is valid since $p \in (0,1]$.

Finally, one could also relax the assumption that $\|\cdot\|_Y$ is a $p$-norm to the assumption that $(Y, \|\cdot\|_Y)$ admits an equivalent $p$-norm $|||\cdot|||_Y$ such that $(Y, |||\cdot|||_Y)$ is solid, since we can simply change to the equivalent quasi-norm. Here, we remark once more that Corollary 2.2.12 and Theorem 2.1.4 imply that *any* solid QBF space $(Y, \|\cdot\|_Y)$ admits, for some $p \in (0,1]$, an equivalent $p$-norm $|||\cdot|||_Y$ such that $(Y, |||\cdot|||_Y)$ is a solid QBF space. But one has to know (a lower bound for) $p$ to be able to determine the weight $v$ and the space $\mathbb{A}^p_v$ of analyzing vectors. Since each $p$-norm is also a $q$-norm for $q \in (0,p]$ by Remark 2.1.2, it is indeed sufficient to know only a lower bound for $p$.

But to avoid changing to an equivalent quasi-norm in each proof, we prefer to assume that $\|\cdot\|_Y$ is a $p$-norm to begin with. ◀

Coorbit theory is heavily based on the REPRODUCING FORMULA, which is discussed in the next subsection.

### 2.4.2 *The reproducing formula for square integrable representations*

The reproducing formula is a consequence of a general theorem of Duflo and Moore pertaining to square integrable, irreducible representations. In our notation, this theorem takes the following form:

**Theorem 2.4.3.** *[20, Theorem 3] There is a unique self-adjoint, positive operator $K : \operatorname{dom}(K) \to \mathcal{H}$ (with $\operatorname{dom}(K) \leq \mathcal{H}$) satisfying the following conditions:*

1. *For $g \in \mathcal{H}$, the following are equivalent:*
   a) *$g$ is* ADMISSIBLE, *i.e. $W_g g \in L^2(G)$,*
   b) *There is some $f \in \mathcal{H} \setminus \{0\}$ with $W_g f \in L^2(G)$,*
   c) *$W_g f \in L^2(G)$ holds for all $f \in \mathcal{H}$,*
   d) *$g \in \operatorname{dom}\left(K^{-1/2}\right)$.*
2. *For $\varphi, \psi \in \operatorname{dom}\left(K^{-1/2}\right)$ and arbitrary $f, g \in \mathcal{H}$, we have the* ORTHOGONALITY RELATION

$$\left\langle W_\psi f \,\middle|\, W_\varphi g \right\rangle_{L^2(G)} = \left\langle K^{-1/2}\varphi \,\middle|\, K^{-1/2}\psi \right\rangle_{\mathcal{H}} \cdot \langle f \,|\, g \rangle_{\mathcal{H}}\,. \tag{2.4.1}$$

*Remark* 2.4.4. It is an easy but important observation that each good vector is also admissible, i.e. that $\mathcal{G}_v \subset \operatorname{dom}\left(K^{-1/2}\right)$. To see this, note that $g \in \mathcal{G}_v \subset \mathcal{H}$ implies $W_g g \in L^1_v(G) \subset L^1(G)$, since $v$ is bounded below. Furthermore, the Cauchy Schwarz inequality easily yields $W_g g \in L^\infty(G)$. Together, we arrive at $W_g g \in L^1 \cap L^\infty \subset L^2$, which implies $g \in \operatorname{dom}\left(K^{-1/2}\right)$ by the theorem above.

For brevity, we will write $A := K^{-1/2}$ in the remainder of this section and we call $A$ the DUFLO-MOORE OPERATOR associated to $\pi$.

The orthogonality relations imply in particular that $A := K^{-1/2}$ is injective, since $Ag = 0$ would imply $\langle W_g g \mid W_g g \rangle_{L^2(G)} = 0$ and hence $W_g g \equiv 0$, since $W_g g$ is continuous. But this means

$$0 = W_g g(1_G) = \langle g \mid \pi(1_G)\, g \rangle_{\mathcal{H}} = \|g\|^2_{\mathcal{H}}$$

and hence $g = 0$.

For $g \in \mathcal{H} \setminus \{0\}$ with $W_g g \in L^2(G)$ and

$$C_g := \left\langle K^{-1/2} g \,\middle|\, K^{-1/2} g \right\rangle_{\mathcal{H}} = \frac{\|W_g g\|^2_{L^2(G)}}{\|g\|^2_{\mathcal{H}}} > 0, \tag{2.4.2}$$

the orthogonality relations entail the INVERSION FORMULA

$$f = \frac{1}{C_g} \cdot \int_G \left(W_g f\right)(x) \cdot \pi(x)\, g \, \mathrm{d}x. \tag{2.4.3}$$

This formula has to be interpreted *in the weak sense*, i.e. it is true when tested against each functional $\langle \cdot \mid h \rangle_{\mathcal{H}}$. Indeed, equation (2.4.1) implies

$$\begin{aligned}
\frac{1}{C_g} \int_G \left(W_g f\right)(x) \cdot \langle \pi(x)\, g \mid h \rangle_{\mathcal{H}} \, \mathrm{d}x &= \frac{1}{C_g} \int_G \left(W_g f\right)(x) \cdot \overline{\left(W_g h\right)(x)} \, \mathrm{d}x \\
&= \frac{1}{C_g} \langle W_g f \mid W_g h \rangle_{L^2(G)} \\
&= \frac{1}{C_g} \left\langle K^{-1/2} g \,\middle|\, K^{-1/2} g \right\rangle_{\mathcal{H}} \cdot \langle f \mid h \rangle_{\mathcal{H}} \\
&= \langle f \mid h \rangle_{\mathcal{H}}.
\end{aligned}$$

A further important consequence is the REPRODUCING FORMULA

$$W_\varphi f * W_g \psi = \langle A\psi \mid A\varphi \rangle_{\mathcal{H}} \, W_g f \;\forall \varphi, \psi \in \operatorname{dom}(A) \;\forall f, g \in \mathcal{H}, \tag{2.4.4}$$

which is essentially a consequence of the fact that the wavelet transform $W_\psi$ transforms applications of $\pi$ into left translations (i.e. the wavelet transform is an INTERTWINING OPERATOR). Indeed, we have

$$W_g\left[\pi(x) f\right](y) = \langle \pi(x) f \mid \pi(y) g \rangle_{\mathcal{H}}$$

$$\begin{aligned} &= \langle f \mid \pi(x^{-1}y)\, g \rangle_{\mathcal{H}} \\ &= W_g f\,(x^{-1}y) \\ &= L_x \left[ W_g f \right](y) \end{aligned} \tag{2.4.5}$$

as well as

$$\begin{aligned} \left( W_{\pi(x)g} f \right)(y) &= \langle f \mid \pi(y)\, \pi(x)\, g \rangle_{\mathcal{H}} \\ &= \langle f \mid \pi(yx)\, g \rangle_{\mathcal{H}} \\ &= W_g f\,(yx) \\ &= R_x \left[ W_g f \right](y) \end{aligned} \tag{2.4.6}$$

for arbitrary $x, y \in G$ and $f, g \in \mathcal{H}$. Furthermore,

$$\begin{aligned} \left( W_g f \right)(x) &= \langle f \mid \pi(x)\, g \rangle_{\mathcal{H}} \\ &= \overline{\langle \pi(x)\, g \mid f \rangle_{\mathcal{H}}} \\ &= \overline{\langle g \mid \pi(x^{-1})\, f \rangle_{\mathcal{H}}} \\ &= \overline{W_f g\,(x^{-1})} \end{aligned} \tag{2.4.7}$$

holds for all $x \in G$. Thus, $W_g f = \overline{W_f g}^{\vee}$.

Using the above relations, we get

$$W_g \psi\,(y^{-1}x) = \overline{W_\psi g\,(x^{-1}y)} = \overline{W_\psi \left[ \pi(x)\, g \right](y)}.$$

By invoking the orthogonality relation (2.4.1), this finally establishes

$$\begin{aligned} W_\varphi f * W_g \psi\,(x) &= \int_G W_\varphi f\,(y) \cdot W_g \psi\,(y^{-1}x)\ \mathrm{d}y \\ &= \int_G W_\varphi f\,(y) \cdot \overline{W_\psi \left[ \pi(x)\, g \right](y)}\, \mathrm{d}y \\ &= \left\langle W_\varphi f \mid W_\psi \left[ \pi(x)\, g \right] \right\rangle_{L^2(G)} \\ &= \langle A\psi \mid A\varphi \rangle_{\mathcal{H}} \cdot \langle f \mid \pi(x)\, g \rangle_{\mathcal{H}} \\ &= \langle A\psi \mid A\varphi \rangle_{\mathcal{H}} \cdot W_g f\,(x)\,, \end{aligned}$$

which is precisely the reproducing formula (2.4.4). ◀

In the remainder of this chapter, we will also need the following result, which gives sufficient conditions to ensure that a space defined by imposing a condition on the "diagonal" wavelet transform $W_g g$ forms a vector space. A form of this result is used (without further comment/proof) in the proofs of [28, Lemma 4.2] as well as [58, Theorem 4.3(b)]. But without knowing that the space of all $h \in \mathcal{H}$ with $W_h h \in L^1_w(G)$ forms a vector space, this is a nontrivial claim.

It is worth noting that irreducibility of the representation $\pi$ is crucially used in the proof of the lemma.

**Lemma 2.4.5.** *Assume that $(Z, \|\cdot\|_Z)$ is a solid function space on $G$ which satisfies $Z \hookrightarrow L^1(G)$ and which is left- and right invariant and also satisfies $Z * Z \hookrightarrow Z$ or $Z * Z^\vee \hookrightarrow Z$.*

*Then the space*

$$\mathcal{Q}_Z := \{g \in \mathcal{H} \mid W_g g \in Z\}$$

*is a $\pi$-invariant vector space with $W_h g \in Z$ for all $g, h \in \mathcal{Q}_Z$ and with $\mathcal{Q}_Z \subset \operatorname{dom}(A^2)$. If $\mathcal{Q}_Z$ is nontrivial, then $\mathcal{Q}_Z$ is dense in $\mathcal{H}$.*

*Finally, if $(Y, \|\cdot\|_Y)$ is another function space on $G$ which satisfies the convolution relation $Y * Z \hookrightarrow Y$, then we have the equivalence*

$$\forall f \in \mathcal{H}: \qquad W_g f \in Y \iff W_h f \in Y \tag{2.4.8}$$

*for each pair of* nonzero *$g, h \in \mathcal{Q}_Z$, together with a corresponding estimate*

$$C_{g,h}^{-1} \cdot \|W_h f\|_Y \leq \|W_g f\|_Y \leq C_{g,h} \cdot \|W_h f\|_Y .$$

*Proof.* For $g \in \mathcal{Q}_Z$, we have $W_g g \in Z \hookrightarrow L^1(G)$. The Cauchy Schwarz inequality also yields $W_g g \in L^\infty(G)$ and hence

$$W_g g \in L^1(G) \cap L^\infty(G) \hookrightarrow L^2(G) .$$

By Theorem 2.4.3, this means $g \in \operatorname{dom}(A)$, so that $Ag \in \mathcal{H}$ is well-defined.

We now show $Ag \in \operatorname{dom}(A^*)$. Since $A$ is self-adjoint, this will yield $Ag \in \operatorname{dom}(A)$ and hence $\mathcal{Q}_Z \subset \operatorname{dom}(A^2)$. For the proof of $Ag \in \operatorname{dom}(A^*)$, we may clearly assume $g \neq 0$. Then, we note that the orthogonality relation (2.4.1) yields, for any $f \in \operatorname{dom}(A)$, the identity

$$\begin{aligned} |\langle Af \mid Ag \rangle_{\mathcal{H}}| &= \frac{\left|\langle W_g g \mid W_f g \rangle_{L^2(G)}\right|}{\langle g \mid g \rangle_{\mathcal{H}}} \\ &\leq \frac{1}{\|g\|_{\mathcal{H}}^2} \cdot \|W_g g\|_{L^1(G)} \cdot \|W_f g\|_{L^\infty(G)} \\ &\lesssim \frac{1}{\|g\|_{\mathcal{H}}} \cdot \|f\|_{\mathcal{H}} \cdot \|W_g g\|_Z . \end{aligned} \tag{2.4.9}$$

Here, we used the embedding $Z \hookrightarrow L^1$, the Cauchy-Schwarz inequality and the definition of $W_f g$ in the last step. The estimate (2.4.9) shows that the linear functional

$$\operatorname{dom}(A) \to \mathbb{C}, f \mapsto \langle Af \mid Ag \rangle_{\mathcal{H}}$$

is bounded. By definition of the adjoint $A^*$, this is equivalent to $Ag \in \operatorname{dom}(A^*)$.

Now, let $g, h \in \mathcal{Q}_Z \setminus \{0\}$ be arbitrary. We want to show that there is some $\gamma \in \mathcal{Q}_Z$ with $\langle Ag \mid A\gamma \rangle_{\mathcal{H}} \neq 0 \neq \langle Ah \mid A\gamma \rangle_{\mathcal{H}}$. To see this, we distinguish two cases:

*Case* 1. $\langle Ag \mid Ah \rangle_{\mathcal{H}} \neq 0$. In this case, we can simply take $\gamma = g$, since equation (2.4.2) yields $\langle Ag \mid Ag \rangle_{\mathcal{H}} = C_g > 0$.

*Case* 2. $\langle Ag \mid Ah \rangle_{\mathcal{H}} = 0$. Since $h \neq 0$ and $A$ is injective, we have $Ah \neq 0$ and $A^2 h \neq 0$. But $\pi$ is irreducible, so that the family $(\pi(x)\, g)_{x \in G}$ spans a dense subset of $\mathcal{H}$. Thus, there is some $x \in G$ with

$$\begin{aligned} 0 &\neq \left\langle \pi(x)\, g \mid A^2 h \right\rangle_{\mathcal{H}} \\ &= \langle A\left[\pi(x)\, g\right] \mid Ah \rangle_{\mathcal{H}}. \end{aligned}$$

We now define $\gamma := g + \varepsilon \cdot \pi(x)\, g$, where the precise value of $\varepsilon > 0$ will be determined below.

For this choice of $\gamma$, we have

$$\begin{aligned} &\langle A\gamma \mid Ah \rangle_{\mathcal{H}} \\ &= \langle Ag \mid Ah \rangle_{\mathcal{H}} + \varepsilon \cdot \langle A\left[\pi(x)\, g\right] \mid Ah \rangle_{\mathcal{H}} \\ &= \varepsilon \cdot \langle A\left[\pi(x)\, g\right] \mid Ah \rangle_{\mathcal{H}} \\ &\neq 0 \end{aligned}$$

as well as

$$\begin{aligned} &\langle A\gamma \mid Ag \rangle_{\mathcal{H}} \\ &= \langle Ag \mid Ag \rangle_{\mathcal{H}} + \varepsilon \cdot \langle A\left[\pi(x)\, g\right] \mid Ag \rangle_{\mathcal{H}} \\ &= \|Ag\|_{\mathcal{H}}^2 + \varepsilon \cdot \langle A\left[\pi(x)\, g\right] \mid Ag \rangle_{\mathcal{H}} \\ &\neq 0 \end{aligned}$$

for $\varepsilon > 0$ small enough.

It remains to show $\gamma \in \mathcal{Q}_Z$. But the "intertwining properties" of the wavelet transform (equations (2.4.5) and (2.4.6)) yield

$$\begin{aligned} &W_\gamma \gamma \\ &= W_g g + \varepsilon\, W_{\pi(x) g} g + \varepsilon\, W_g\left[\pi(x)\, g\right] + \varepsilon^2\, W_{\pi(x) g}\left[\pi(x)\, g\right] \\ &= W_g g + \varepsilon\, R_x\left[W_g g\right] + \varepsilon\, L_x\left[W_g g\right] + \varepsilon^2\, L_x R_x\left[W_g g\right] \in Z, \end{aligned}$$

since $Z$ is left- and right-invariant.

Analogously to the calculation in the previous equation, we get

$$W_{\pi(x) f}\left[\pi(x)\, f\right] = L_x R_x\left[W_f f\right] \in Z$$

for $f \in \mathcal{Q}_Z$, so that $\mathcal{Q}_Z$ is indeed $\pi$-invariant.

Now, we show $W_h g \in Z$ by applying the reproducing formula (2.4.4) twice. Indeed,

$$\begin{aligned} W_h g &= \frac{1}{\langle Ah \mid A\gamma \rangle_{\mathcal{H}}} \cdot W_\gamma g * W_h h \\ &= \frac{1}{\langle Ah \mid A\gamma \rangle_{\mathcal{H}} \cdot \langle A\gamma \mid Ag \rangle_{\mathcal{H}}} \cdot W_g g * W_\gamma \gamma * W_h h. \end{aligned}$$

In the case of $Z * Z \hookrightarrow Z$, we see $W_h g \in Z$. If instead the convolution relation $Z * Z^\vee \hookrightarrow Z$ holds, we first note that equation (2.4.7) implies $\left|W_f f\right| = \left|W_f f\right|^\vee$ for each $f \in \mathcal{H}$. Together with solidity of $Z$, this implies $W_g g, W_\gamma \gamma, W_h h \in Z \cap Z^\vee$ and thus

$$F := W_g g * W_\gamma \gamma \in Z * Z^\vee \hookrightarrow Z$$

and finally $F * W_h h \in Z * Z^\vee \hookrightarrow Z$, which also yields $W_h g \in Z$.

We remark that the convolution is associative, since it is associative on $L^1$ and $Z \hookrightarrow L^1$. Finally, $W_h g \in Z$ is trivial for $g = 0$ or $h = 0$.

Since $\mathcal{Q}_Z$ is obviously closed under multiplication with complex scalars, it remains to prove $g + h \in \mathcal{Q}_Z$ for $g, h \in \mathcal{Q}_Z$. But we have

$$W_{g+h}(g+h) = W_g g + W_g h + W_h g + W_h h \in Z$$

since we just showed $W_g h, W_h g \in Z$. All in all, $\mathcal{Q}_Z \leq \mathcal{H}$ is a $\pi$-invariant subspace. If $\mathcal{Q}_Z$ is nontrivial, then $\mathcal{Q}_Z \subset \mathcal{H}$ is dense, since $\pi$ is irreducible.

It remains to prove the equivalence (2.4.8), as well as the corresponding quasi-norm estimate. By symmetry, it suffices to show one implication/estimate. Thus, let $g, h \in \mathcal{Q}_Z \setminus \{0\}$ be arbitrary and let $f \in \mathcal{H}$ with $W_g f \in Y$. Similar to the derivation above, we apply the reproducing formula (2.4.4) to get

$$W_h f = \frac{1}{\langle Ag \mid Ag \rangle_{\mathcal{H}}} \cdot W_g f * W_h g \in Y * Z \subset Y$$

with

$$\|W_h f\|_Y \leq \frac{C \cdot \|W_h g\|_Z}{\|Ag\|_{\mathcal{H}}^2} \cdot \|W_g f\|_Y,$$

where the constant $C > 0$ comes from the assumption $Y * Z \hookrightarrow Y$. □

In the sketch of coorbit theory in the introduction to this section, the coorbit space was defined as

$$\mathrm{Co}(Y) = \left\{ f \in \mathcal{R} \,\middle|\, W_g f \in W(L^\infty, Y) \right\},$$

where $\mathcal{R}$ was some unspecified reservoir. In the next subsection, we recall from [28, Section 4] and [29, Theorem 4.1] the construction of the reservoir $\mathcal{R} = \left(\mathcal{H}_v^1\right)^\neg$, as well as the extension of the wavelet transform to this space.

### 2.4.3 *Construction of the reservoir* $\mathcal{R} = \left(\mathcal{H}_v^1\right)^\neg$

As in [28, §4.1], we want to define the space of TEST VECTORS as

$$\mathcal{T}_v := \mathcal{H}_v^1 := \mathcal{H}_v^1(G, g) := \left\{ f \in \mathcal{H} \;\middle|\; W_g f \in L_v^1(G) \right\}$$

with norm

$$\|f\|_{\mathcal{H}_v^1} := \|f\|_{\mathcal{H}_v^1(G,g)} := \left\|W_g f\right\|_{L_v^1},$$

where $g \in \mathcal{G}_v \setminus \{0\}$ is fixed. For this definition to make sense, we first have to show that it does not depend on the choice of $g$.

*In [28, §4.1], the space $\mathcal{H}_v^1$ is* not *called the space of "test vectors". But this terminology seems suitable, since the space $\mathcal{H}_v^1$ will play the role that is usually played by the spaces $C_c^\infty(U)$ or $\mathcal{S}(\mathbb{R}^d)$ of test functions, i.e., the coorbit spaces will be subspaces of the space of (anti)linear functionals on $\mathcal{H}_v^1$.*

As a preparation for the proof of this independence, we need the strong continuity of the representation $L : G \to \mathrm{GL}\left(L_v^1(G)\right), x \mapsto L_x$.

**Lemma 2.4.6.** *Let $u : G \to (0,\infty)$ be a measurable weight which is* locally *bounded from above and from below and let $0 < p < \infty$. Then $C_c(G)$ is a dense subspace of $L_u^p(G)$.*

*Furthermore, if $u : G \to (0,\infty)$ is measurable and left moderate with respect to a locally bounded, measurable weight $u_0 : G \to (0,\infty)$, then the map $G \to L_u^p(G), x \mapsto L_x f$ is continuous for every $f \in L_u^p(G)$.*

*It is worth noting that the assumptions of the lemma are always fulfilled if $u$ is measurable and submultiplicative, thanks to Theorem 2.2.22.*

*Proof.* We first prove density of $C_c(G)$. Since $u$ is locally bounded, it is clear that $C_c(G)$ is contained in $L_u^p(G)$. By $\sigma$-compactness of $G$, we have $G = \bigcup_{n\in\mathbb{N}} K_n$ for suitable compact sets $K_n \subset G$ with $K_n \subset K_{n+1}$ for all $n \in \mathbb{N}$.

Let $f \in L_u^p(G)$ and $\varepsilon > 0$. We want to construct some $g \in C_c(G)$ with $\|f - g\|_{L_u^p} < \varepsilon$. Using the dominated convergence theorem (and $p < \infty$), we see

$$\|f - f \cdot \chi_{K_n}\|_{L_u^p} \xrightarrow[n\to\infty]{} 0.$$

Hence, it suffices to consider $\tilde{f} := f \cdot \chi_{K_n}$.

Let $U \subset G$ be a compact neighborhood of $K_n$. By assumption on the weight $u$, we have $c_U \leq u \leq C_U$ on $U$ for suitable constants $c_U, C_U > 0$. This implies that the (quasi)-norms $\|\cdot\|_{L_u^p}$ and $\|\cdot\|_{L^p}$ are equivalent for functions supported in $U$. In particular, $\tilde{f} \in L^p(G)$. But $C_c(G)$ is dense in $L^p(G)$, so that there is some $\psi \in C_c(G)$ with $\left\|\psi - \tilde{f}\right\|_{L^p} < \frac{1}{C_U} \cdot \varepsilon$.

By the Urysohn-Lemma (cf. [33, Lemma 4.32]), there is a function $\varphi \in C_c(G)$ with $0 \leq \varphi \leq 1$ and $\operatorname{supp}\varphi \subset U$ as well as $\varphi|_{K_n} \equiv 1$. In particular, $\tilde{f} = \varphi\tilde{f}$. All in all, we arrive at

$$\begin{aligned}
\left\|\varphi\psi - \tilde{f}\right\|_{L_u^p} &\leq C_U \cdot \left\|\varphi\psi - \tilde{f}\right\|_{L^p} \\
&= C_U \cdot \left\|\varphi\left[\psi - \tilde{f}\right]\right\|_{L^p} \\
&\leq C_U \cdot \left\|\psi - \tilde{f}\right\|_{L^p} < \varepsilon.
\end{aligned}$$

For the second part, assume that $u_0 : G \to (0,\infty)$ is locally bounded and measurable and that $u$ is left moderate with respect to $u_0$. By Corollary 2.2.23, we know that $u$ is bounded above and below on

compact sets, so that $C_c(G)$ is dense in $L^p_u(G)$ by the previous part. Let $f \in L^p_u(G)$. Lemma 2.2.19 shows that $L^p_u(G)$ is invariant under left-translations with $|||L_x|||_{L^p_u \to L^p_u} \leq u_0(x)$ for all $x \in G$. Thus,

$$F : G \to L^p_u(G), x \mapsto L_x f$$

is well-defined.

As usual, it suffices to show that $F$ is continuous at $1_G$. To this end, set $q := \min\{1, p\}$, let $K \subset G$ be an arbitrary compact unit neighborhood and let $\varepsilon > 0$. By assumption, $u_0$ is bounded on $K$ by some $C > 0$. By density of $C_c(G)$, there is some $g \in C_c(G)$ with $\|f - g\|_{L^p_u} \leq \left(\frac{\varepsilon}{1+C^q}\right)^{1/q}$.

Let $M := \operatorname{supp} g$. Since $u$ is locally bounded, it is bounded on $KM$ by some $C' > 0$. By [32, Proposition (2.6)], $g$ is (left) uniformly continuous. Hence, there is some unit neighborhood $U \subset K$ with

$$\|L_x g - g\|_{\sup} < \varepsilon^{1/q} \cdot \left(C' \cdot [\mu(KM)]^{1/p}\right)^{-1}$$

for all $x \in U$, where $\mu$ denotes the Haar measure on $G$.

But for $x \in U \subset K$, we have

$$\operatorname{supp}(L_x g) = x \operatorname{supp} g \subset xM \subset KM$$

and thus $|L_x g - g| \leq \|L_x g - g\|_{\sup} \cdot \chi_{KM}$. Hence,

$$\begin{aligned}\|L_x g - g\|^q_{L^p_u} &\leq \|L_x g - g\|^q_{\sup} \cdot \|\chi_{KM}\|^q_{L^p_u} \\ &\leq \|L_x g - g\|^q_{\sup} \cdot \left(C' \cdot [\mu(KM)]^{1/p}\right)^q < \varepsilon.\end{aligned}$$

Together with $|||L_x|||_{L^p_u \to L^p_u} \leq u_0(x) \leq C$ for $x \in U \subset K$, this implies

$$\begin{aligned}\|F(x) - F(1_G)\|^q_{L^p_u} &\leq \|L_x(f-g)\|^q_{L^p_u} + \|L_x g - g\|^q_{L^p_u} + \|g - f\|^q_{L^p_u} \\ &\leq |||L_x|||^q_{L^p_u \to L^p_u} \|f - g\|^q_{L^p_u} + \|L_x g - g\|^q_{L^p_u} + \|g - f\|^q_{L^p_u} \\ &\leq (C^q + 1) \cdot \frac{\varepsilon}{1 + C^q} + \|L_x g - g\|^q_{L^p_u} \\ &< 2\varepsilon\end{aligned}$$

for all $x \in U$. Hence, $F$ is continuous in $1_G$. □

With these preparations, we can now prove the following lemma which shows in particular that the space $\mathcal{H}^1_v(G, g)$ is independent of the choice of $g \in \mathcal{G}_v \setminus \{0\}$. The proof is essentially that of [28, Lemma 4.2] with a few additional details. In particular, the completeness is somewhat neglected in [28]. Without assuming that the weight $v$ is bounded below (which is explicitly done in [44], but not in [28]), it is not completely clear that $\mathcal{H}^1_v$ is necessarily complete. At least the proof below would break down in this case.

A further deviation from the proof in [28] is that density in $\mathcal{H}_v^1(G)$ of the space $\langle \pi(x)\, g \mid x \in G \rangle \leq \mathcal{H}_v^1(G)$ consisting of all finite linear combinations of "atoms" $\pi(x)\, g$ is derived in [28] as a consequence of an atomic decomposition theorem for the space $\mathcal{H}_v^1(G)$, see [28, Theorem 4.6]. The direct proof given here – using the theory of Bochner integration – seems to be new.

**Lemma 2.4.7.** *([28, Lemma 4.1])*

1. *$\mathcal{G}_v$ is a dense, $\pi$-invariant subspace of $\mathcal{H}$.*
2. *The space $\mathcal{H}_v^1(G, g)$ is independent of the choice of $g \in \mathcal{G}_v \setminus \{0\}$, with equivalent norms for different choices.*
3. *$(\mathcal{H}_v^1, \|\cdot\|_{\mathcal{H}_v^1})$ is a $\pi$-invariant Banach space which embeds continuously into $\mathcal{H}$. Furthermore, $\mathcal{G}_v \subset \mathcal{H}_v^1$, so that $\mathcal{H}_v^1$ is dense in $\mathcal{H}$. Finally, $\pi$ acts strongly continuous on $\mathcal{H}_v^1$ with*

$$\left|\!\left|\!\left| \pi(x) \right|\!\right|\!\right|_{\mathcal{H}_v^1 \to \mathcal{H}_v^1} \leq v(x) \text{ for every } x \in G. \tag{2.4.10}$$

4. *For each $f \in \mathcal{H}_v^1$ and $g \in \mathcal{G}_v \setminus \{0\}$, the map*

$$F_{f,g} : G \to \mathcal{H}_v^1, x \mapsto \frac{1}{C_g} \cdot W_g f(x) \cdot \pi(x)\, g$$

   *is continuous and Bochner integrable with $f = \int_G F_{f,g}(x)\, \mathrm{d}x$.*

   *The formula $f = \int_G F_{f,g}(x)\, \mathrm{d}x$ given here is a version of the wavelet inversion formula. The point is that this formula is valid not only in the weak sense, but as a Bochner integral.*

5. *([28, Corollary 4.8]) For each $g \in \mathcal{G}_v \setminus \{0\}$, the space*

$$\langle \pi(x)\, g \mid x \in G \rangle \leq \mathcal{H}_v^1$$

   *consisting of all finite linear combinations of "atoms" $\pi(x)\, g$ is norm-dense in $\mathcal{H}_v^1$.*

6. *For $g \in \mathcal{H}$ with $W_g g \in W\left(L^\infty, L_v^1\right) \subset L_v^1$, we have*

$$\mathcal{H}_v^1 = \left\{ f \in \mathcal{H} \mid W_g f \in W\left(L^\infty, L_v^1\right) \right\}$$

   *and $f \mapsto \left\| W_g f \right\|_{W(L^\infty, L_v^1)}$ defines an equivalent norm on $\mathcal{H}_v^1$.*

*Proof.* Let us first prove that $\mathcal{H}_v^1(G, g)$ is independent of the choice of $g \in \mathcal{G}_v \setminus \{0\}$. To this end, we first note that we have $\mathcal{G}_v = \mathcal{Q}_{L_v^1(G)}$ in the notation of Lemma 2.4.5. Observe that $v(x) \geq w(x) \geq c$ holds for all $x \in G$, which easily entails $L_v^1(G) \hookrightarrow L^1(G)$. Furthermore, $L_v^1(G)$ is left- and right-invariant by Lemma 2.2.19, since $v$ is submultiplicative.

To see that $L_v^1$ is closed under convolution products, we observe for $f, g \in L_v^1(G) \subset L^1(G)$ that

$$\begin{aligned} &\int_G |(f * g)(x)| \cdot v(x)\, \mathrm{d}x \\ &\leq \int_G \int_G v(y) \cdot v(y^{-1}x)\, |f(y) \cdot g(y^{-1}x)|\, \mathrm{d}y\, \mathrm{d}x \end{aligned} \tag{2.4.11}$$

$$\begin{aligned} &= \|[v \cdot |f|] * [v \cdot |g|]\|_{L^1} \\ &\leq \|v \cdot |f|\|_{L^1} \cdot \|v \cdot |g|\|_{L^1} \\ &= \|f\|_{L^1_v} \cdot \|g\|_{L^1_v} < \infty, \end{aligned} \tag{2.4.12}$$

so that $L^1_v(G)$ is a Banach algebra with the convolution product.

All in all, we see that Lemma 2.4.5 is applicable, so that $\mathcal{G}_v \leq \mathcal{H}$ is a dense, $\pi$-invariant subspace of $\mathcal{H}$ with $W_g f \in L^1_v(G)$ for all $f, g \in \mathcal{G}_v$. A final application of (the equivalence (2.4.8) in) Lemma 2.4.5 with $Y = Z = L^1_v(G)$ also shows that $\mathcal{H}^1_v(G, g)$ is indeed independent of the choice of $g \in \mathcal{G}_v \setminus \{0\}$ with equivalent norms for different choices. Thus, we can use $\mathcal{H}^1_v = \mathcal{H}^1_v(G, g)$ for some fixed $g \in \mathcal{G}_v \setminus \{0\}$ for the remainder of the proof.

We now show that $\mathcal{H}^1_v = \mathcal{H}^1_v(G, g)$ embeds continuously into $\mathcal{H}$. We first use $v \geq c$ to derive

$$\begin{aligned} \|W_g f\|_{L^1} &\leq \frac{1}{c} \cdot \|W_g f\|_{L^1_v} \\ &= \frac{1}{c} \cdot \|f\|_{\mathcal{H}^1_v(G,g)} \qquad \text{for } f \in \mathcal{H}^1_v(G, g). \end{aligned}$$

By the remark after Theorem 2.4.3, we know $g \in \mathcal{G}_v \subset \operatorname{dom}(A)$ and hence $W_g g \in L^2(G)$. Thus, the orthogonality relation (2.4.1), together with the reproducing formula (2.4.4) and with the convolution relation $L^1 * L^2 \hookrightarrow L^2$ yields

$$\begin{aligned} \|f\|^2_{\mathcal{H}} &= \frac{1}{C_g} \cdot \|W_g f\|^2_{L^2(G)} \\ &= \frac{1}{C_g} \cdot \left\| \frac{1}{C_g} \cdot W_g f * W_g g \right\|^2_{L^2(G)} \\ &\leq \frac{1}{C_g^3} \cdot \|W_g g\|^2_{L^2(G)} \cdot \|W_g f\|^2_{L^1(G)} \\ &\leq \frac{C'_g}{c^2} \cdot \|f\|^2_{\mathcal{H}^1_v(G,g)} < \infty \end{aligned}$$

and thus $\mathcal{H}^1_v \hookrightarrow \mathcal{H}$. In particular, we see that $\|\cdot\|_{\mathcal{H}^1_v}$ is positive definite and hence a norm, since the other properties of a norm easily carry over from $\|\cdot\|_{L^1_v}$ to $\|\cdot\|_{\mathcal{H}^1_v}$.

This also implies completeness of $\mathcal{H}^1_v$. Indeed, if $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}^1_v$, the above estimate implies that $(f_n)_{n \in \mathbb{N}}$ is also Cauchy in $\mathcal{H}$ and thus convergent to some $f \in \mathcal{H}$. Furthermore, that $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}^1_v$ implies that the sequence $(F_n)_{n \in \mathbb{N}}$ given by $F_n := W_g f_n$ is Cauchy in $L^1_v(G)$ and thus convergent to some $F \in L^1_v(G)$. But since

$$F_n(x) = W_g f_n(x) = \langle f_n \mid \pi(x)\, g \rangle_{\mathcal{H}} \xrightarrow[n \to \infty]{} \langle f \mid \pi(x)\, g \rangle_{\mathcal{H}} = W_g f(x)$$

holds for all $x \in G$, we conclude $W_g f = F \in L^1_v(G)$ and thus $f \in \mathcal{H}^1_v$ with

$$\|f - f_n\|_{\mathcal{H}^1_v} = \left\|W_g(f - f_n)\right\|_{L^1_v} = \|F_n - F\|_{L^1_v} \xrightarrow[n\to\infty]{} 0.$$

For $\pi$-invariance of $\mathcal{H}^1_v$, we note that equation (2.4.5) implies for $f \in \mathcal{H}^1_v$ that $W_g[\pi(x) f] = L_x[W_g f]$ and hence

$$\begin{aligned}\|\pi(x) f\|_{\mathcal{H}^1_v} &= \left\|W_g[\pi(x) f]\right\|_{L^1_v} = \left\|L_x[W_g f]\right\|_{L^1_v} \\ &\leq v(x) \cdot \left\|W_g f\right\|_{L^1_v} = v(x) \cdot \|f\|_{\mathcal{H}^1_v}.\end{aligned}$$

Here, we made use of the estimate $\left|\left|\left|L_x\right|\right|\right|_{L^1_v \to L^1_v} \leq v(x)$ which was established in Lemma 2.2.19. Similarly, we get

$$\|\pi(x) f - f\|_{\mathcal{H}^1_v} = \left\|L_x[W_g f] - W_g f\right\|_{L^1_v} \xrightarrow[x\to 1_G]{} 0,$$

since $\|L_x F - F\|_{L^1_v} \xrightarrow[x\to 1_G]{} 0$ holds for all $F \in L^1_v$ by Lemma 2.4.6. Hence, $\pi$ acts strongly continuous on $\mathcal{H}^1_v$.

Finally, for $g \in \mathcal{G}_v \setminus \{0\}$, we easily see $g \in \mathcal{H}^1_v(G, g)$. Since $\mathcal{H}^1_v(G, g)$ is independent of the choice of $g \in \mathcal{G}_v \setminus \{0\}$, we see $\mathcal{G}_v \subset \mathcal{H}^1_v$. In particular, $\mathcal{H}^1_v \leq \mathcal{H}$ is dense.

Next, we prove part 4. For $f \in \mathcal{H}^1_v \subset \mathcal{H}$, we know that $W_g f$ is a bounded, continuous function. Since $\pi$ acts strongly continuous on $\mathcal{H}^1_v$ with $g \in \mathcal{H}^1_v$, we also know that $G \to \mathcal{H}^1_v, x \mapsto \pi(x) g$ is well-defined and continuous. This implies that the map $F_{f,g}$ given in part 4 of the lemma is well-defined and continuous. Since $G$ is $\sigma$-compact (and hence $\sigma$-finite), the image $F_{f,g}(G) \subset \mathcal{H}^1_v$ is also $\sigma$-compact and in particular separable. Furthermore, $F_{f,g}$ vanishes outside of a $\sigma$-finite set. By [54, Chapter VI, §1, M11], this implies that $F_{f,g}$ is $\mu$-measurable, where $\mu$ is the Haar measure on $G$.

But the estimate for the operator norm of $\pi(x)$ in equation (2.4.10) also implies

$$\begin{aligned}\left\|F_{f,g}(x)\right\|_{\mathcal{H}^1_v} &= \frac{1}{C_g} \cdot \left|W_g f(x)\right| \cdot \|\pi(x) g\|_{\mathcal{H}^1_v} \\ &\leq \frac{\|g\|_{\mathcal{H}^1_v}}{C_g} \cdot v(x) \cdot \left|W_g f(x)\right| \in L^1(G),\end{aligned}$$

because of $f \in \mathcal{H}^1_v$, which entails $v \cdot W_g f \in L^1(G)$. By [54, Chapter VI, Corollary 5.9], this implies that $F_{f,g}$ is indeed Bochner integrable with respect to the Haar measure $\mu$, so that $h := \int_G F_{f,g} \, \mathrm{d}\mu \in \mathcal{H}^1_v$ is well-defined. If we let

$$\mathcal{V}_g := \overline{\langle \pi(x) g \mid x \in G \rangle} \leq \mathcal{H}^1_v,$$

we also have $F_{f,g}(x) \in \mathcal{V}_g$ for each $x \in G$ and thus also $h \in \mathcal{V}_g$, since $\mathcal{V}_g$ is a Banach space. We will now show $f = h$ which will in particular imply $f \in \mathcal{V}_g$. Since $f \in \mathcal{H}_v^1$ was arbitrary, this will imply that $\langle \pi(x)\, g \mid x \in G \rangle$ is indeed dense in $\mathcal{H}_v^1$ as claimed in part 5 of the lemma.

To see $h = f$, we note that $W_g : \mathcal{H}_v^1(G, g) \to L_v^1(G)$ is isometric and in particular injective, so that it suffices to show $W_g h = W_g f$ or even $\iota(W_g f) = \iota(W_g f)$, with $\iota : L_v^1(G) \hookrightarrow L^1(G)$. But since the Bochner integral commutes with bounded linear operators (cf. [54, Chapter VI, Theorem 4.1]), we have

$$\begin{aligned}
(\iota \circ W_g)\, h &= \int_G (\iota \circ W_g)\left[F_{f,g}\right] \,\mathrm{d}x \\
&= \frac{1}{C_g} \cdot \int_G W_g f(x) \cdot W_g\left[\pi(x)\, g\right] \,\mathrm{d}x \\
&= \frac{1}{C_g} \cdot \int_G \left(W_g f\right)(x) \cdot L_x\left[W_g g\right] \,\mathrm{d}x \\
&= \frac{1}{C_g} \cdot W_g f * W_g g \\
&= W_g f \\
&= (\iota \circ W_g)\, f.
\end{aligned}$$

To get to the last line, we used the reproducing formula (2.4.4) and before that, we employed the fact that the wavelet transform intertwines $\pi(x)$ and $L_x$, see equation (2.4.5). Finally, we used the Bochner integral interpretation of the convolution as in [32, Equation (2.38) and Appendix 3]. Note that if the Bochner integral exists, it coincides with the "weak" vector valued integral as defined in [32, Appendix 3].

It remains to establish part 6. The inclusion "$\supset$" (with a corresponding norm estimate) is a direct consequence of the embedding $W\left(L^\infty, L_v^1\right) \hookrightarrow L_v^1$. For the reverse inclusion, let $f \in \mathcal{H}_v^1$ be arbitrary and $g \in \mathcal{H} \setminus \{0\}$ with $W_g g \in W\left(L^\infty, L_v^1\right)$. In particular, $g \in \mathcal{G}_v \setminus \{0\}$, so that $\mathcal{H}_v^1 = \mathcal{H}_v^1(G, g)$. By equation (2.4.4), we have

$$W_g f = \frac{1}{C_g} \cdot W_g f * W_g g,$$

so that Lemma 2.3.23 implies

$$M_Q\left[W_g f\right] \leq \frac{1}{C_g} \cdot \left|W_g f\right| * M_Q\left[W_g g\right].$$

Thus, equation (2.4.12) yields

$$\begin{aligned}
\left\|W_g f\right\|_{W_Q(L_v^1)} &= \left\|M_Q\left[W_g f\right]\right\|_{L_v^1} \\
&\leq \frac{1}{C_g} \cdot \left\|W_g f\right\|_{L_v^1} \cdot \left\|M_Q\left[W_g g\right]\right\|_{L_v^1}
\end{aligned}$$

$$= C_g' \cdot \|f\|_{\mathcal{H}_v^1}. \qquad \square$$

The above lemma implies that the RESERVOIR

$$\begin{aligned}\mathcal{R}_v &:= \mathcal{T}_v^{\neg} = (\mathcal{H}_v^1)^{\neg} \\ &:= \{f : \mathcal{H}_v^1 \to \mathbb{C} \mid f \text{ continuous and antilinear}\}\end{aligned}$$

is a well-defined Banach space.

*The notation $\mathcal{R}_v$ is an abbreviation for "reservoir", but also for "ROUGH VECTORS", since the analogy between $\mathcal{T}_v$ and $\mathcal{S}(\mathbb{R}^d)$ implies that $\mathcal{R}_v$ corresponds to $\mathcal{S}'(\mathbb{R}^d)$, which contains more "rough" objects than $\mathcal{S}(\mathbb{R}^d)$.*

The next lemma summarizes the most important properties of this space. Especially, it shows how elements of $\mathcal{R}_v$ can be characterized in terms of their wavelet transform.

**Lemma 2.4.8.** *([29, Theorem 4.1]) For $f \in \mathcal{R}_v$ and $g \in \mathcal{T}_v$, we define $\langle f \mid g \rangle := f(g)$ as well as*

$$W_g f : G \to \mathbb{C}, x \mapsto \langle f \mid \pi(x)\, g \rangle.$$

*Then the following are true:*

1. *The map*
$$\iota : \mathcal{H} \hookrightarrow \mathcal{R}_v, f \mapsto (\mathcal{T}_v \ni g \mapsto \langle f \mid g \rangle_{\mathcal{H}})$$
*defines a continuous embedding of $\mathcal{H}$ into $\mathcal{R}_v$.*

2. *Under the above embedding, $\langle \cdot \mid \cdot \rangle : \mathcal{R}_v \times \mathcal{T}_v \to \mathbb{C}$ is an extension of $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$ and the* EXTENDED WAVELET TRANSFORM
$$W_g : \mathcal{R}_v \to C(G) \cap L^\infty_{1/v}(G)$$
*is a well-defined bounded map (into $L^\infty_{1/v}(G)$) which extends the usual wavelet transform*
$$W_g : \mathcal{H} \to C_b(G) \hookrightarrow C(G) \cap L^\infty_{1/v}(G).$$

*We remark that $\langle \cdot \mid \cdot \rangle$ is indeed linear in the first component and antilinear in the second component, similar to the usual inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$.*

3. *The representation $\pi$ can be extended to $\mathcal{R}_v$ by setting*
$$\pi(x) f : \mathcal{T}_v \to \mathbb{C}, g \mapsto f(\pi(x^{-1})\, g) \text{ for } f \in \mathcal{R}_v.$$
*Then $\pi : G \to \mathrm{GL}(\mathcal{R}_v)$ is a group homomorphism with*
$$|||\pi(x)|||_{\mathcal{R}_v \to \mathcal{R}_v} \leq v(x^{-1}) \text{ for all } x \in G.$$

4. *The extended wavelet transform $W_g : \mathcal{R}_v \to C(G) \cap L^\infty_{1/v}(G)$ is injective for each $g \in \mathcal{G}_v \setminus \{0\}$ and satisfies*
$$W_g[\pi(x) f] = L_x[W_g f] \text{ for } f \in \mathcal{R}_v \text{ and } g \in \mathcal{T}_v \tag{2.4.13}$$
*as well as*
$$W_h f = \frac{1}{C_g} \cdot W_g f * W_h g \tag{2.4.14}$$
*for $f \in \mathcal{R}_v$, $g \in \mathcal{G}_v \setminus \{0\}$ and $h \in \mathcal{T}_v$.*

5. *For every $g \in \mathcal{G}_v \setminus \{0\}$ and every function $F \in L^\infty_{1/v}(G)$ satisfying $F = \frac{1}{C_g} \cdot F * W_g g$, there is a unique $f \in \mathcal{R}_v$ with $F = W_g f$.*

   *More precisely, for every $g \in \mathcal{G}_v \setminus \{0\}$, there is a bounded linear operator*

   $$T_g : L^\infty_{1/v}(G) \to \mathcal{R}_v$$

   *satisfying $W_g\left[T_g F\right] = F$ for every function $F \in L^\infty_{1/v}(G)$ which satisfies $F = \frac{1}{C_g} \cdot F * W_g g$.*

6. *For each $g \in \mathcal{G}_v \setminus \{0\}$, the function $f \mapsto \left\|W_g f\right\|_{L^\infty_{1/v}}$ defines an equivalent norm on $\mathcal{R}_v$.*

7. *If $(f_\alpha)_{\alpha \in I}$ is a* bounded *net in $\mathcal{R}_v$ which satisfies $W_g f_\alpha \to F$ with pointwise convergence for some $F : G \to \mathbb{C}$ and some $g \in \mathcal{G}_v \setminus \{0\}$, then there is some $f \in \mathcal{R}_v$ with $F = W_g f$ and we have $f_\alpha \to f$ with respect to the weak-$*$-topology on $\mathcal{R}_v = \left(\mathcal{H}^1_v\right)^\urcorner$.*

   *The pointwise convergence $W_g f_\alpha \to W_g f$ is in fact uniform on every compact subset of $G$.*

*Proof.* 1. Using $\mathcal{T}_v \subset \mathcal{H}$, it is clear that $\imath f : \mathcal{T}_v \to \mathbb{C}$ is well-defined and antilinear for each $f \in \mathcal{H}$. Lemma 2.4.7 shows $\|g\|_{\mathcal{H}} \leq C \cdot \|g\|_{\mathcal{T}_v}$ for all $g \in \mathcal{T}_v$ and some absolute constant $C > 0$. This implies

$$|(\imath f)(g)| = |\langle f \,|\, g\rangle_{\mathcal{H}}| \leq \|f\|_{\mathcal{H}} \cdot \|g\|_{\mathcal{H}} \leq C \cdot \|f\|_{\mathcal{H}} \cdot \|g\|_{\mathcal{T}_v},$$

so that $\imath f : \mathcal{T}_v \to \mathbb{C}$ is a bounded antilinear functional with norm $\|\imath f\|_{\mathcal{R}_v} \leq C \cdot \|f\|_{\mathcal{H}}$. Since $\mathcal{T}_v$ is dense in $\mathcal{H}$ by Lemma 2.4.7, $\imath$ is an injective map. Because $\imath : \mathcal{H} \to \mathcal{R}_v$ is clearly linear, the proof of the first part is complete.

2. For $f \in \mathcal{H}$ and $g \in \mathcal{T}_v$, we have

$$\langle \imath f \,|\, g\rangle = (\imath f)(g) = \langle f \,|\, g\rangle_{\mathcal{H}},$$

so that $\langle \cdot \,|\, \cdot \rangle$ is an extension of $\langle \cdot \,|\, \cdot \rangle_{\mathcal{H}}$.

Lemma 2.4.7 implies that $\Gamma : G \to \mathcal{T}_v, x \mapsto \pi(x)\, g$ is continuous for every $g \in \mathcal{T}_v$. Hence, $W_g f = f \circ \Gamma$ is continuous for every $f \in \mathcal{R}_v$. Furthermore, the estimate (2.4.10) yields

$$\begin{aligned} \left|W_g f(x)\right| &= |\langle f \,|\, \pi(x)\, g\rangle| \\ &\leq \|f\|_{\mathcal{R}_v} \cdot \|\pi(x)\, g\|_{\mathcal{T}_v} \\ &\leq \|f\|_{\mathcal{R}_v} \cdot v(x) \cdot \|g\|_{\mathcal{T}_v} \end{aligned}$$

and hence $W_g f \in L^\infty_{1/v}(G)$ with $\left\|W_g f\right\|_{L^\infty_{1/v}} \leq \|f\|_{\mathcal{R}_v} \cdot \|g\|_{\mathcal{T}_v}$. Since $\langle \cdot \,|\, \cdot \rangle$ extends the inner product $\langle \cdot \,|\, \cdot \rangle_{\mathcal{H}}$ on $\mathcal{H}$, the extended wavelet transform is an extension of the "usual" wavelet transform.

3. Since $\mathcal{T}_v$ is $\pi$-invariant by Lemma 2.4.7, $\pi(x) f : \mathcal{T}_v \to \mathbb{C}$ is a well defined antilinear map with

$$\begin{aligned} |\langle \pi(x) f \,|\, g \rangle| &= |\langle f \,|\, \pi(x^{-1}) g \rangle| \\ &\leq \|f\|_{\mathcal{R}_v} \cdot \|\pi(x^{-1}) g\|_{\mathcal{T}_v} \\ &\leq \|f\|_{\mathcal{R}_v} \|g\|_{\mathcal{T}_v} \cdot v(x^{-1}). \end{aligned}$$

Hence, $\pi(x) f \in \mathcal{R}_v$ with

$$\|\pi(x) f\|_{\mathcal{R}_v} \leq v(x^{-1}) \cdot \|f\|_{\mathcal{R}_v}.$$

Furthermore, for $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle \pi(x) [\iota f] \,|\, g \rangle &= \langle \iota f \,|\, \pi(x^{-1}) g \rangle \\ &= \langle f \,|\, \pi(x^{-1}) g \rangle_{\mathcal{H}} \\ &= \langle \pi(x) f \,|\, g \rangle_{\mathcal{H}} \\ &= \langle \iota [\pi(x) f] \,|\, g \rangle \end{aligned}$$

for all $g \in \mathcal{T}_v$, so that $\pi(x) : \mathcal{R}_v \to \mathcal{R}_v$ is indeed an extension of $\pi(x) : \mathcal{H} \to \mathcal{H}$ for every $x \in G$. We finally note

$$\begin{aligned} \langle \pi(x) [\pi(y) f] \,|\, g \rangle &= \langle \pi(y) f \,|\, \pi(x^{-1}) g \rangle \\ &= \langle f \,|\, \pi(y^{-1}) \pi(x^{-1}) g \rangle \\ &= \langle f \,|\, \pi((xy)^{-1}) g \rangle \\ &= \langle \pi(xy) f \,|\, g \rangle \end{aligned}$$

for every $g \in \mathcal{T}_v$, so that $\pi : G \to \mathrm{GL}(\mathcal{R}_v)$ is indeed a homomorphism.

4. For $f \in \mathcal{R}_v$ and $g \in \mathcal{G}_v \setminus \{0\}$ with $W_g f \equiv 0$, we have

$$\langle f \,|\, \pi(x) g \rangle = W_g f(x) = 0$$

for all $x \in G$. But Lemma 2.4.7 shows that $\langle \pi(x) g \,|\, x \in G \rangle$ is a dense subspace of $\mathcal{T}_v = \mathcal{H}_v^1$, so that $f \equiv 0$ follows by continuity.

For the proof of the convolution relation, fix $g \in \mathcal{G}_v \setminus \{0\}$, $x \in G$ and $h \in \mathcal{T}_v = \mathcal{H}_v^1$. Let $f \in \mathcal{R}_v$ and define $\tilde{f} \in \mathcal{T}_v'$ by

$$\tilde{f}(\gamma) := \overline{f(\pi(x) \gamma)}$$

for each $\gamma \in \mathcal{T}_v$. Using the Bochner integrable map

$$F_{h,g} : G \to \mathcal{T}_v, y \mapsto \frac{1}{C_g} \cdot W_g h(y) \cdot \pi(y) g$$

defined in Lemma 2.4.7, we have $h = \int_G F_{h,g} \, \mathrm{d}\mu$, where the integral converges in $\mathcal{T}_v$.

Since $\tilde{f} \in \mathcal{T}_v'$ is a continuous linear map, this implies

$$\begin{aligned}
W_h f(x) &= \langle f \mid \pi(x) h \rangle \\
&= \overline{\tilde{f}(h)} \\
&= \overline{\int_G \tilde{f}\left(F_{h,g}(y)\right) \, \mathrm{d}y} \\
&= \overline{\int_G \frac{1}{C_g} W_g h(y) \cdot \tilde{f}(\pi(y) g) \, \mathrm{d}y} \\
&= \int_G \frac{1}{C_g} \overline{W_g h(y)} \cdot f(\pi(x) \pi(y) g) \, \mathrm{d}y \\
&= \int_G \frac{1}{C_g} W_h g(y^{-1}) \cdot f(\pi(xy) g) \, \mathrm{d}y \\
&= \int_G \frac{1}{C_g} W_h g(z^{-1}x) \cdot f(\pi(z) g) \, \mathrm{d}z \\
&= \frac{1}{C_g} \cdot \left(W_g f * W_h g\right)(x).
\end{aligned}$$

Finally, we have for each $g \in \mathcal{T}_v$, $f \in \mathcal{R}_v$ and $x \in G$ the identity

$$\begin{aligned}
W_g\left[\pi(x) f\right](y) &= \langle \pi(x) f \mid \pi(y) g \rangle = \langle f \mid \pi(x^{-1}) \pi(y) g \rangle \\
&= \langle f \mid \pi(x^{-1}y) g \rangle = W_g f(x^{-1}y) \\
&= L_x\left[W_g f\right](y),
\end{aligned}$$

which proves equation (2.4.13).

5. For $F \in L^\infty_{1/v}(G)$, we define

$$T_g F : \mathcal{T}_v \to \mathbb{C}, h \mapsto \frac{1}{C_g} \cdot \int_G F(y) \cdot \langle \pi(y) g \mid h \rangle_{\mathcal{H}} \, \mathrm{d}y.$$

This is well-defined, since the integrand is measurable with

$$\begin{aligned}
\int_G |F(y)| \cdot |\langle \pi(y) g \mid h \rangle_{\mathcal{H}}| \, \mathrm{d}y &= \int_G |F(y)| \, |\langle h \mid \pi(y) g \rangle_{\mathcal{H}}| \, \mathrm{d}y \\
&= \int_G \frac{1}{v(y)} \cdot |F(y)| \cdot v(y) \cdot \left|W_g h(y)\right| \, \mathrm{d}y \\
&\leq \|F\|_{L^\infty_{1/v}} \cdot \left\|W_g h\right\|_{L^1_v} \\
&= \|F\|_{L^\infty_{1/v}} \cdot \|h\|_{\mathcal{H}^1_v(G,g)} \\
&\lesssim \|F\|_{L^\infty_{1/v}} \cdot \|h\|_{\mathcal{T}_v}.
\end{aligned}$$

Since $T_g F$ is clearly antilinear, we see $T_g F \in \mathcal{T}_v^\neg = \mathcal{R}_v$. Furthermore, since $T_g$ is linear, the above estimate shows that $T_g : L^\infty_{1/v}(G) \to \mathcal{R}_v$ is a bounded linear map.

Finally, if $F$ satisfies $F = \frac{1}{C_g} F * W_g g$, we get for $x \in G$ that

$$W_g\left[T_g F\right](x) = T_g F(\pi(x) g) = \frac{1}{C_g} \int_G F(y) \cdot \langle \pi(y) g \mid \pi(x) g \rangle_{\mathcal{H}} \, \mathrm{d}y$$

$$= \frac{1}{C_g} \int_G F(y) \cdot W_g g(y^{-1}x) \, dy$$
$$= \frac{1}{C_g} \left(F * W_g g\right)(x) = F(x).$$

This shows existence of $f := T_g F \in \mathcal{R}_v$ with $W_g f = F$. Uniqueness follows by injectivity of $W_g$, cf. part 4.

6. Part 2 showed that $W_g : \mathcal{R}_v \to L^\infty_{1/v}(G)$ is a well-defined and bounded linear map. This proves the estimate $\|W_g f\|_{L^\infty_{1/v}(G)} \lesssim \|f\|_{\mathcal{R}_v}$ for all $f \in \mathcal{R}_v$.

For the reverse estimate, let $f \in \mathcal{R}_v$ and set $F := W_g f \in L^\infty_{1/v}(G)$. By part 4, $F$ satisfies the reproducing formula $F = \frac{1}{C_g} \cdot F * W_g g$. Part 5 shows $h := T_g F \in \mathcal{R}_v$ with $\|h\|_{\mathcal{R}_v} \lesssim \|F\|_{L^\infty_{1/v}}$ and

$$W_g h = W_g \left[T_g F\right] = F = W_g f.$$

Since $W_g : \mathcal{R}_v \to L^\infty_{1/v}$ is injective by part 4, we conclude $f = h$ and thus

$$\|f\|_{\mathcal{R}_v} = \|h\|_{\mathcal{R}_v} \lesssim \|F\|_{L^\infty_{1/v}} = \|W_g f\|_{L^\infty_{1/v}},$$

where the implied constant is independent of $f \in \mathcal{R}_v$.

7. By hypothesis, the net $\left(W_g f_\alpha(x)\right)_{\alpha \in I} = \left(f_\alpha(\pi(x) g)\right)_{\alpha \in I}$ is convergent (to $F(x)$) for every $x \in G$. By (anti)-linearity, this yields convergence of the net $\left(f_\alpha(h)\right)_{\alpha \in I}$ for every $h \in H := \langle \pi(x) g \mid x \in G \rangle$. We denote the limit by $f(h) \in \mathbb{C}$. It is clear that $f : H \to \mathbb{C}$ is an antilinear functional. Since the net $(f_\alpha)_{\alpha \in I}$ is assumed to be bounded in $\mathcal{R}_v = \left(\mathcal{H}^1_v\right)^\neg$, there is some $C > 0$ with $|f_\alpha(\gamma)| \leq C \cdot \|\gamma\|_{\mathcal{H}^1_v}$ for all $\gamma \in \mathcal{H}^1_v$. In particular, this yields

$$|f(h)| = \lim_\alpha |f_\alpha(h)| \leq C \cdot \|h\|_{\mathcal{H}^1_v} \text{ for each } h \in H,$$

so that $f \in H^\neg$ is a bounded, antilinear functional. But $H$ is dense in $\mathcal{H}^1_v$ by Lemma 2.4.7. Thus, $f$ extends (uniquely) to a continuous antilinear functional $\tilde{f}$ on $\mathcal{H}^1_v$ with $\|\tilde{f}\|_{\mathcal{R}_v} \leq C$. We denote this extension again by $f \in \mathcal{R}_v = \left(\mathcal{H}^1_v\right)^\neg$.

It is clear that $W_g f_\alpha(x) = f_\alpha(\pi(x) g) \to f(\pi(x) g) = W_g f(x)$ holds for every $x \in G$, and hence $F = W_g f$. It remains to show $f_\alpha \to f$ with respect to the weak-$*$-topology (i.e. pointwise). To this end, let $\gamma \in \mathcal{H}^1_v$ be arbitrary and let $\varepsilon > 0$. By density of $H$ in $\mathcal{H}^1_v$, there is some $h \in H$ with $\|\gamma - h\|_{\mathcal{H}^1_v} < \frac{\varepsilon/3}{1+C}$. By definition of $f$, we have $f_\alpha(h) \to f(h)$, so that there is some $\alpha_0 \in I$ with $|f_\alpha(h) - f(h)| < \frac{\varepsilon}{3}$ for all $\alpha \geq \alpha_0$. All in all, we get for each $\alpha \geq \alpha_0$ the estimate

$$\begin{aligned} |f_\alpha(\gamma) - f(\gamma)| &\leq |f_\alpha(\gamma) - f_\alpha(h)| + |f_\alpha(h) - f(h)| + |f(h) - f(\gamma)| \\ &\leq \|f_\alpha\|_{\mathcal{R}_v} \|\gamma - h\|_{\mathcal{H}^1_v} + \frac{\varepsilon}{3} + \|f\|_{\mathcal{R}_v} \cdot \|h - \gamma\|_{\mathcal{H}^1_v} \\ &\leq 2C \cdot \|\gamma - h\|_{\mathcal{H}^1_v} + \frac{\varepsilon}{3} < \varepsilon \end{aligned}$$

which proves $f_\alpha(\gamma) \to f(\gamma)$ for all $\gamma \in \mathcal{H}_v^1$.

Now, let $K \subset G$ be compact. Since $\Gamma : G \to \mathcal{H}_v^1, x \mapsto \pi(x) g$ is continuous by Lemma 2.4.7, we see that $\Gamma(K) \subset \mathcal{H}_v^1$ is compact. For $\varepsilon > 0$, we can thus cover $\Gamma(K)$ by finitely many balls $(B_\varepsilon(h_i))_{i \in \underline{n}}$ for suitable $h_1, \dots, h_n \in \Gamma(K)$. Since $f_\alpha(h_i) \to f(h_i)$ holds for each $i \in \underline{n}$, there is some $\alpha_0 \in I$ with $|f_\alpha(h_i) - f(h_i)| < \varepsilon$ for all $\alpha \geq \alpha_0$ and $i \in \underline{n}$.

Now, let $\alpha \geq \alpha_0$ and let $x \in K$ be arbitrary. Then there is some $i \in \underline{n}$ with $\pi(x) g = \Gamma(x) \in B_\varepsilon(h_i)$. Hence,

$$\begin{aligned}
&\left|W_g f_\alpha(x) - W_g f(x)\right| \\
&= |f_\alpha(\pi(x) g) - f(\pi(x) g)| \\
&\leq |f_\alpha(\pi(x) g) - f_\alpha(h_i)| + |f_\alpha(h_i) - f(h_i)| + |f(h_i) - f(\pi(x) g)| \\
&\leq 2C \cdot \|\pi(x) g - h_i\|_{\mathcal{H}_v^1} + \varepsilon \\
&\leq (1 + 2C) \cdot \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we see that the convergence $W_g f_\alpha \to W_g f$ is uniform on $K$. □

Now that we understand the basic properties of the reservoir $\mathcal{R}_v$ of "rough vectors", we are in a position to define the actual coorbit space $\mathrm{Co}(Y)$.

### 2.4.4 *Definition of coorbit spaces with respect to Quasi-Banach spaces*

For a fixed analyzing vector $g \in \mathbb{A}_v^p \setminus \{0\}$, we define the COORBIT SPACE WITH RESPECT TO $Y$ as

$$\mathrm{Co}(Y) := \mathrm{Co}_g(Y) := \left\{ f \in \mathcal{R}_v \,\middle|\, W_g f \in W(L^\infty, Y) \right\}$$

with quasi-norm

$$\|f\|_{\mathrm{Co}(Y)} := \|f\|_{\mathrm{Co}_g(Y)} := \left\|W_g f\right\|_{W(L^\infty, Y)} \text{ for } f \in \mathrm{Co}_g(Y).$$

One of the most important properties is that $\mathrm{Co}_g(Y)$ is independent of the choice of the analyzing vector $g \in \mathbb{A}_v^p \setminus \{0\}$. That this is indeed the case is shown in the following lemma, which is essentially the same as [58, Theorem 4.3], up to the fact that the proof of [58, Theorem 4.3] uses the (incorrect) convolution relation [59, Corollary 5.4]. As a replacement of this, we will use Corollary 2.3.33.

We remark that this change of the employed convolution relation makes it necessary to restrict to analyzing vectors $g \in \mathbb{A}_v^p \setminus \{0\}$, whereas in [58], the space $\mathbb{B}_w^p = \left\{ g \in \mathcal{H} \,\middle|\, W_g g \in W\left(L^\infty, L_w^p\right) \right\}$ is used.

Finally, we only show the independence of the reservoir $\mathcal{R}_v$ in the sense that using a "larger" weight will result in the same coorbit space, cf. [29, Theorem 4.2(iii)]. In contrast to this, [58, Theorem 4.3(c)] claims a much stronger independence of the reservoir. We will see below in Example 2.4.10 that this independence fails in general.

**Theorem 2.4.9.** *(cf. [58, Theorem 4.3])*

1. $\mathrm{Co}_g(Y)$ *is independent of the choice of* $g \in \mathbb{A}_v^p \setminus \{0\}$ *with equivalent quasi-norms for different choices.*
2. $\mathrm{Co}(Y)$ *is a Quasi-Banach space with* $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_v$ *and* $\|\cdot\|_{\mathrm{Co}(Y)}$ *is a p-norm.*
3. $\mathrm{Co}(Y)$ *is independent of the reservoir* $\mathcal{R}_v$ *in the following sense: If* $u : G \to (0,\infty)$ *is a different weight with* $v \leq C \cdot u$ *for some* $C > 0$ *and with* $\mathbb{A}_u^p \supsetneq \{0\}$*, then we have*

$$\mathrm{Co}(Y) = \left\{ f \in (\mathcal{H}_u^1)^\urcorner \,\middle|\, W_g f \in W(L^\infty, Y) \right\} \ \forall g \in \mathbb{A}_u^p \setminus \{0\}.$$

*This equality has to be understood as stating that the restriction map* $\varrho$ *is a bijection with*

$$\varrho : \mathrm{Co}(Y) \to \left\{ f \in (\mathcal{H}_u^1)^\urcorner \,\middle|\, W_g f \in W(L^\infty, Y) \right\}, f \mapsto f|_{\mathcal{H}_u^1}.$$

*Proof.* 1. Let $w$ and $v_0$ be defined as in Assumption 2.4.1, parts D and E and define

$$Z_0 := W^R\left(L^\infty, W\left(L^\infty, L_{v_0}^p\right)\right)$$

as well as $Z := W^R\left(L^\infty, W\left(L^\infty, L_v^p\right)\right)$. By assumption on $v$, we have $v \geq v_0$ and hence $Z \hookrightarrow Z_0$.

In the notation of Lemma 2.4.5, we have $\mathbb{A}_v^p = \mathcal{Q}_Z \subset \mathcal{Q}_{Z_0}$. Using $v_0(x) \geq w(x) \geq c$ for every $x \in G$, we derive

$$\left(L_{v_0}^p(G)\right)_d(X) = \ell^p_{(v_0)_X}(I) \hookrightarrow \ell^p(I) \hookrightarrow \ell^1(I) = \left(L^1(G)\right)_d(X)$$

for every relatively separated family $X = (x_i)_{i\in I}$ in $G$. Here, we made use of Lemma 2.3.21 which identifies the associated discrete sequence space for (weighted) Lebesgue spaces.

By Corollary 2.3.20 and Lemma 2.3.4, this yields

$$Z \hookrightarrow Z_0 \hookrightarrow W(L^\infty, L_{v_0}^p) \hookrightarrow W(L^\infty, L^1) \hookrightarrow L^1.$$

Furthermore, Corollary 2.3.33 shows $Z_0 * Z_0^\vee \hookrightarrow Z_0$, since $v_0$ is submultiplicative with $(v_0)_{\vee,p} = v_0$ by choice of $v_0$ (cf. Assumption 2.4.1). All in all, we have shown that Lemma 2.4.5 is applicable. Thus, $\mathbb{A}_{v_0}^p = \mathcal{Q}_{Z_0}$ is a vector space and $W_g h \in Z_0$ holds for arbitrary $g, h \in \mathbb{A}_{v_0}^p \supset \mathbb{A}_v^p$.

To show independence of $\mathrm{Co}_g(Y)$ from $g \in \mathbb{A}_v^p \setminus \{0\}$, choose arbitrary $g, h \in \mathbb{A}_v^p \setminus \{0\} \subset \mathbb{A}_{v_0}^p \setminus \{0\}$. As seen above, this yields $W_g h \in Z_0$. Now, let $f \in \mathrm{Co}_g(Y)$ be arbitrary. We use the reproduction formula on $\mathcal{H}_v^1$ (cf. Lemma 2.4.8, equation (2.4.14)) to derive

$$\begin{aligned} W_h f &= \frac{1}{C_g} W_g f * W_h g \\ &\in W(L^\infty, Y) * Z_0 \hookrightarrow W(L^\infty, Y) * W\left(L^\infty, L_{v_0}^p\right). \end{aligned}$$

But our choice of the weights $v_0, w$ implies

$$v_0(x) \geq w(x) \geq \Delta(x^{-1}) \cdot \left|\!\left|\!\left| R_{x^{-1}} \right|\!\right|\!\right|_{W(L^\infty,Y) \to W(L^\infty,Y)}$$

for all $x \in G$.

Thus, $W(L^\infty, Y) * W\left(L^\infty, L^p_{v_0}\right) \hookrightarrow W(L^\infty, Y)$ holds by Theorem 2.3.24. This implies $W_h f \in W(L^\infty, Y)$, i.e. $f \in \mathrm{Co}_h(Y)$ with

$$\begin{aligned} \|f\|_{\mathrm{Co}_h(Y)} = \|W_h f\|_{W(Y)} &\lesssim \frac{1}{C_g} \cdot \left\|W_g f\right\|_{W(Y)} \cdot \|W_h g\|_{W(L^p_{v_0})} \\ &\leq C_{g,h} \cdot \left\|W_g f\right\|_{W(Y)} = C_{g,h} \cdot \|f\|_{\mathrm{Co}_g(Y)}. \end{aligned}$$

By symmetry, this establishes the desired independence.

2. We first show $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_v$. To this end, fix $g \in \mathbb{A}^p_v \setminus \{0\}$. By Lemma 2.3.34, we get

$$\frac{1}{r(x)} \cdot \left|W_g f(x)\right| \leq C_{\overline{Q},Q,Y} \cdot \left\|W_g f\right\|_{W_Q(L^\infty,Y)} = C_{\overline{Q},Q,Y} \cdot \|f\|_{\mathrm{Co}_g(Y)}$$

with $r(x) := \left|\!\left|\!\left| R_x \right|\!\right|\!\right|_{W(Y) \to W(Y)}$ for each $f \in \mathrm{Co}(Y) = \mathrm{Co}_g(Y)$ and almost all $x \in G$. But our assumptions on the weights $v, w$ imply $v(x) \geq w(x) \geq \left|\!\left|\!\left| R_x \right|\!\right|\!\right|_{W(L^\infty,Y) \to W(L^\infty,Y)} = r(x)$ for all $x \in G$ and hence

$$\left\|W_g f\right\|_{L^\infty_{1/v}} \leq \operatorname*{ess\,sup}_{x \in G} \frac{1}{r(x)} \cdot \left|W_g f(x)\right| \leq C_{\overline{Q},Q,Y} \cdot \|f\|_{\mathrm{Co}_g(Y)}.$$

But Lemma 2.4.8 shows that $f \mapsto \left\|W_g f\right\|_{L^\infty_{1/v}}$ defines an equivalent norm on $\mathcal{R}_v$. Thus, $\|f\|_{\mathcal{R}_v} \lesssim \|f\|_{\mathrm{Co}_g(Y)}$, where the implied constant does not depend on $f \in \mathrm{Co}(Y)$.

With this embedding, we can prove completeness: If $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathrm{Co}_g(Y)$, the embedding $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_v$ implies that $(f_n)_{n \in \mathbb{N}}$ is Cauchy in the Banach space $\mathcal{R}_v = \left(\mathcal{H}^1_v\right)^\neg$ and hence convergent (in $\mathcal{R}_v$) to some $f \in \mathcal{R}_v$. Furthermore, $\left(W_g f_n\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W(L^\infty, Y)$ and hence convergent to some $F \in W(L^\infty, Y)$, since $W(L^\infty, Y)$ is complete, cf. Lemma 2.3.7. For a suitable subsequence $(f_{n_k})_{k \in \mathbb{N}}$, this implies $W_g f_{n_k}(x) \to F(x)$ for almost every $x \in G$, by Corollary 2.2.9. But the convergence $f_{n_k} \to f$ in $\mathcal{R}_v$ yields

$$W_g f_{n_k}(x) = f_{n_k}(\pi(x)\, g) \xrightarrow[k \to \infty]{} f(\pi(x)\, g) = W_g f(x)$$

and thus $F = W_g f$. Since $F \in W(L^\infty, Y)$, this means $f \in \mathrm{Co}(Y)$ with

$$\|f - f_n\|_{\mathrm{Co}_g(Y)} = \left\|W_g(f - f_n)\right\|_{W(Y)} = \left\|W_g f_n - F\right\|_{W(Y)} \xrightarrow[n \to \infty]{} 0.$$

Finally, $\|\cdot\|_{W(L^\infty,Y)}$ is a $p$-norm by Lemma 2.3.4, which easily entails that $\|\cdot\|_{\mathrm{Co}_g(Y)}$ is also a $p$-norm.

3. The inequality $v \leq C \cdot u$ yields $L^q_u(G) \hookrightarrow L^q_v(G)$ for arbitrary $q \in (0,\infty]$ and hence also $\mathbb{A}^q_u \subset \mathbb{A}^q_v$ as well as

$$\mathcal{H}^1_u = \mathcal{H}^1_u(G,g) \hookrightarrow \mathcal{H}^1_v(G,g) = \mathcal{H}^1_v$$

for each $g \in \mathbb{A}^p_u \setminus \{0\}$. This shows that the restriction map $\varrho$ is well-defined. Furthermore, $\mathcal{H}^1_u$ contains the space $\langle \pi(x)\, g \mid x \in G\rangle$ which is dense in $\mathcal{H}^1_v$ by Lemma 2.4.7. This implies injectivity of $\varrho$.

To see that $\varrho$ is surjective, let $f \in \left(\mathcal{H}^1_u\right)^\neg$ with $W_g f \in W(L^\infty, Y)$. Lemma 2.3.34 shows

$$\frac{1}{r(x)} \cdot \left|W_g f(x)\right| \leq C_{\overline{Q},Q,Y} \cdot \left\|W_g f\right\|_{W_Q(L^\infty,Y)} < \infty$$

for almost all $x \in G$, with $r(x) := \left\|\left|R_x\right|\right\|_{W_Q(Y)\to W_Q(Y)}$. But our assumptions on $v, w$ imply $v(x) \geq w(x) \geq r(x)$ for all $x \in G$ and hence $W_g f \in L^\infty_{1/v}(G)$.

By part 4 of Lemma 2.4.8 (applied to the weight $u$ instead of $v$), we know that $W_g f$ satisfies $W_g f = \frac{1}{C_g} \cdot W_g f * W_g g$. Together with $W_g f \in L^\infty_{1/v}(G)$, we can invoke part 5 of the same lemma (this time for the weight $v$), to obtain $\tilde{f} \in \left(\mathcal{H}^1_v\right)^\neg$ with $W_g f = W_g \tilde{f}$. This shows that the (antilinear) functionals $f, \varrho\tilde{f} \in \left(\mathcal{H}^1_u\right)^\neg$ agree on the set $\{\pi(x)\, g \mid x \in G\}$. But Lemma 2.4.7 shows that this set spans a dense subspace of $\mathcal{H}^1_u$. Hence, $f = \varrho\tilde{f}$, so that $\varrho$ is surjective. □

As mentioned above, it is claimed in [58, Theorem 4.3(c)] that

$$\mathrm{Co}(Y) = \left\{f \in \mathbb{S}^\neg \,\middle|\, W_g f \in W(L^\infty, Y)\right\} \tag{2.4.15}$$

holds for every $0 \neq g \in \mathbb{S} \cap \mathbb{B}(Y)$, where $\mathbb{S}$ is an arbitrary locally convex vector space that satisfies the following properties:

1. $\mathbb{S} \hookrightarrow \mathcal{H}^1_v$,
2. $\mathbb{S}$ is $\pi$-invariant,
3. $\mathbb{S} \cap \mathbb{B}(Y) \supsetneq \{0\}$ with

$$\mathbb{B}(Y) = \left\{g \in \mathcal{H} \,\middle|\, W_g g \in L^1_v(G) \cap W\left(L^\infty, L^p_w\right) \cap L^2(G)\right\},$$

for

$$w(x) = \max\left\{\left\|\left|R_x\right|\right\|_{W(Y)\to W(Y)}, \Delta\left(x^{-1}\right)\left\|\left|R_{x^{-1}}\right|\right\|_{W(Y)\to W(Y)}\right\},$$
$$v(x) = \max\left\{1, \left\|\left|L_{x^{-1}}\right|\right\|_{W(Y)\to W(Y)}\right\}. \tag{2.4.16}$$

Of course, the equality (2.4.15) has to be understood as stating

$$\left\{f|_{\mathbb{S}\cap\mathcal{H}^1_v} \,\middle|\, f \in \mathrm{Co}(Y)\right\} = \left\{f|_{\mathbb{S}\cap\mathcal{H}^1_v} \,\middle|\, f \in \mathbb{S}^\neg \text{ with } W_g f \in W(L^\infty, Y)\right\}.$$

The following example shows that this equality fails in general. The reason is that the reproducing formula $W_g f = \frac{1}{C_g} \cdot W_g f * W_g g$ can fail for suitable $f \in \mathbb{S}^\neg$, so that the argument in the proof of [58, Theorem 4.3(c)] is insufficient. The example shows that one can even choose a normed vector space $\mathbb{S}$ such that each $\pi(x) : \mathbb{S} \to \mathbb{S}$ is a bounded linear map and such that $\pi : G \to \mathrm{GL}(\mathbb{S})$ is strongly continuous, but the claimed identity is still invalid.

The idea of the proof is to construct, for a given function $F \in C_c(G)$, a normed vector space $\mathbb{S}$ fulfilling the properties from above and some $0 \neq f \in \mathbb{S} \cap \mathbb{B}(Y)$, as well as an antilinear functional $\varphi \in \mathbb{S}^\neg$ with wavelet transform $W_f \varphi = F$. In other words, an arbitrary wavelet transform in $C_c(G)$ can be realized. We will then show that the reproducing formula $W_f \varphi = C_f^{-1} \cdot W_f \varphi * W_f f$ does not hold, thus yielding a contradiction.

**Example 2.4.10.** We consider the "$ax + b$ group" $G = \mathbb{R} \times \mathbb{R}^*$ with multiplication given by $(x, a)(y, b) = (x + ay, ab)$. The Haar integral on $G$ is given by

$$\int_G f(g) \, \mathrm{d}g = \int_{\mathbb{R}^*} \int_{\mathbb{R}} f(x, a) \, \mathrm{d}x \, \frac{\mathrm{d}a}{a^2}$$

and the modular function is $\Delta(x, a) = a^{-1}$, cf. Example 2.3.26.

We will use the QUASI-REGULAR REPRESENTATION of $G$, given by

$$\pi : G \to \mathcal{U}\left(L^2(\mathbb{R})\right), (x, a) \mapsto L_x D_a,$$

where $D_a f(x) := |a|^{-1/2} \cdot f(x/a)$ is the $L^2$-isometric dilation. It is well-known (see e.g. [28, §7.2]) that $\pi$ is indeed an irreducible, square integrable representation of $G$.

In the following, we consider the solid BF space $Y = L^\infty(G)$, so that we can choose $p = 1$. We have $W(L^\infty, Y) = L^\infty$. Thus,

$$w(x, a) = \max\{1, \Delta((x, a)^{-1})\} = \max\{1, a\}$$

and

$$\begin{aligned} v(x, a) &= \max\{w(x, a), \Delta((x, a)^{-1}) \cdot w((x, a)^{-1})\} \\ &= \max\{1, a\} = w(x, a) \end{aligned}$$

are valid choices for the weights from Assumption 2.4.1, since the space $W(L^\infty, Y) = L^\infty$ is isometrically left- and right-invariant. Furthermore, these weights also dominate the weight functions $v, w$ as used in [58, Theorem 4.3(c)], cf. equation (2.4.16) above. In Theorem 4.2.11, we will see that each $f \in \mathcal{S}(\mathbb{R})$ with $\widehat{f} \in C_c^\infty(\mathbb{R}^*)$ satisfies

$$W_f f \in W^R\left(W\left(L^\infty, L^p_w\right)\right) \hookrightarrow L^1_v(G) \cap W\left(L^\infty, L^p_w\right) \cap L^2(G)$$

and hence $f \in \mathbb{B}(Y) \cap \mathbb{A}^p_v$. Here, we recall that we have $p = 1$.

Let us choose $\widehat{f} \in C_c^\infty((0,\infty))$ with $\operatorname{supp} \widehat{f} = [1,2]$ and with $\widehat{f} > 0$ on $(1,2)$. We claim that the family $(\pi(x,a) f)_{(x,a)\in G}$ is linearly independent. To see this, let $(x_1,a_1),\dots,(x_n,a_n) \in G$ be pairwise distinct for some $n \in \mathbb{N}$ and assume that there are $\alpha_1,\dots,\alpha_n \in \mathbb{C} \setminus \{0\}$ with

$$0 \equiv \sum_{i=1}^{n} \alpha_i \cdot \pi(x_i,a_i) f.$$

By taking the Fourier transform, this yields

$$0 \equiv \sum_{i=1}^{n} \alpha_i \cdot M_{-x_i} D_{a_i^{-1}} \widehat{f}. \tag{2.4.17}$$

We distinguish two cases:

CASE 1: There is some $i \in \underline{n}$ with $a_i > 0$. In this case, let

$$a := \max\{a_i \mid i \in \underline{n}\} > 0$$

and $I := \{i \in \underline{n} \mid a_i = a\} \neq \varnothing$ as well as $J := \{i \in \underline{n} \mid a_i > 0\} \supset I$. Finally, set

$$b := \begin{cases} 0, & \text{if } I = J, \\ \max\{a_i \mid i \in J \setminus I\} > 0, & \text{if } I \subsetneq J. \end{cases}$$

For $i \in \underline{n} \setminus J$, we have $\operatorname{supp}\left[M_{-x_i} D_{a_i^{-1}} \widehat{f}\right] \subset (-\infty, 0)$. Furthermore, for each $i \in J \setminus I$, it holds $a_i \leq b < a$ and hence

$$\operatorname{supp}\left[M_{-x_i} D_{a_i^{-1}} \widehat{f}\right] = a_i^{-1} \cdot \operatorname{supp} \widehat{f} = [a_i^{-1}, 2a_i^{-1}] \subset [b^{-1}, \infty).$$

Thus, by restricting the identity (2.4.17) to the set

$$\varnothing \neq (a^{-1}, \min\{b^{-1}, 2a^{-1}\}) \subset (0,\infty),$$

we arrive at

$$0 = \sum_{\ell \in I} \alpha_\ell \cdot M_{-x_\ell} D_{a_\ell^{-1}} \widehat{f}(\xi) = D_{a^{-1}} \widehat{f}(\xi) \cdot \sum_{\ell \in I} \alpha_\ell \cdot e^{-2\pi i x_\ell \xi}$$

for each $\xi \in \left(a^{-1}, \min\left\{b^{-1}, 2a^{-1}\right\}\right) \subset \left(a^{-1}, 2a^{-1}\right)$. But for this range of $\xi$, we have $D_{a^{-1}} \widehat{f}(\xi) > 0$. Hence,

$$\sum_{\ell \in I} \alpha_\ell \cdot e^{-2\pi i x_\ell \xi} = 0 \quad \text{for} \quad \xi \in (a^{-1}, \min\{b^{-1}, 2a^{-1}\}).$$

Since the left-hand side is a holomorphic function of $\xi \in \mathbb{C}$, this identity is valid for all $\xi \in \mathbb{C}$.

*Here, $\delta_x$ denotes the delta distribution at $x$, given by $\langle \delta_x, \varphi \rangle = \varphi(x)$.*

Recall $\mathcal{F}\left[e^{2\pi i x \cdot}\right] = \delta_x$ for $x \in \mathbb{R}$. Hence, by taking the Fourier transform, we arrive at

$$0 = \sum_{i \in I} \alpha_i \cdot \delta_{-x_i}.$$

But we have $a_i = a$ for all $i \in I$. Since the $((x_i, a_i))_{i \in \underline{n}}$ are pairwise distinct, this implies that $(x_i)_{i \in I}$ is a pairwise distinct family. It is now straightforward to see that $(\delta_{-x_i})_{i \in I}$ is linearly independent. Thus, $\alpha_i = 0$ for all $i \in I$, in contradiction to $\alpha_i \neq 0$ for all $i \in \underline{n}$.

CASE 2: Because of $a_i \in \mathbb{R}^*$ for all $i \in \underline{n}$, the remaining case is $a_i < 0$ for all $i \in \underline{n}$. By evaluating equation (2.4.17) at $-\xi \in \mathbb{R}$, we get

$$0 \equiv \sum_{i=1}^{n} \alpha_i \cdot M_{-(-x_i)} D_{(-a_i)^{-1}} \widehat{f}.$$

Since the family $((-x_i, -a_i))_{i \in \underline{n}}$ is pairwise distinct, with $-a_i > 0$ for all $i \in \underline{n}$, the first case yields a contradiction.

Together, these two cases show that $(\pi(x,a) f)_{(x,a) \in G}$ is indeed linearly independent. Now, define (as a set) $\mathrm{S} := \langle \pi(x) f \mid x \in G \rangle$ as the $\mathbb{C}$-linear span of the family $(\pi(x) f)_{x \in G}$. Fix an arbitrary $F \in C_c(G)$ with $F \not\equiv 0$. Since the family $(\pi(x) f)_{x \in G}$ is linearly independent, we can define

$$\Gamma : \mathrm{S} \to C_c(G), \sum_{x \in G} [\alpha_x \cdot \pi(x) f] \mapsto \sum_{x \in G} [\alpha_x \cdot L_x F],$$

where only finitely many of the $\alpha_x$ are nonzero. A direct computation yields

$$\begin{aligned}
\Gamma\left(\pi(x_0)\left[\sum_{x \in G} \alpha_x \cdot \pi(x) f\right]\right) &= \Gamma\left(\sum_{x \in G} \alpha_x \cdot \pi(x_0 x) f\right) \\
&= \Gamma\left(\sum_{y \in G} \alpha_{x_0^{-1} y} \cdot \pi(y) f\right) \\
&= \sum_{y \in G} \alpha_{x_0^{-1} y} \cdot L_y F \\
&= \sum_{x \in G} \alpha_x \cdot L_{x_0 x} F \\
&= L_{x_0}\left(\sum_{x \in G} \alpha_x \cdot L_x F\right) \\
&= L_{x_0}\left(\Gamma\left[\sum_{x \in G} \alpha_x \cdot \pi(x) f\right]\right),
\end{aligned}$$

which means $\Gamma(\pi(x_0) g) = L_{x_0}[\Gamma g]$ for all $g \in \mathrm{S}$ and $x_0 \in G$.

Because of $f \in \mathbb{A}_v^1$ and since $\mathbb{A}_v^1$ is a $\pi$-invariant vector space, we have $\mathbb{S} \subset \mathbb{A}_v^1 \subset \mathcal{H}_v^1$. Hence, we can define a norm on $\mathbb{S}$ by

$$\|g\|_{\mathbb{S}} := \|g\|_{\mathcal{H}_v^1} + \|\Gamma g\|_{\sup}.$$

With this choice, $\mathbb{S}$ becomes a normed vector space and in particular locally convex. Furthermore, $\mathbb{S} \hookrightarrow \mathcal{H}_v^1$ is trivial.

It is clear that $\mathbb{S}$ is $\pi$-invariant. Even more, each $\pi(x) : \mathbb{S} \to \mathbb{S}$ is bounded, since Lemma 2.4.7 yields $\|\!|\pi(x)|\!\|_{\mathcal{H}_v^1 \to \mathcal{H}_v^1} \leq v(x)$ for all $x \in G$, which entails

$$\begin{aligned}\|\pi(x) g\|_{\mathbb{S}} &= \|\pi(x) g\|_{\mathcal{H}_v^1} + \|\Gamma(\pi(x) g)\|_{\sup} \\ &\leq v(x) \cdot \|g\|_{\mathcal{H}_v^1} + \|L_x[\Gamma g]\|_{\sup} \\ &\leq \max\{1, v(x)\} \cdot \|g\|_{\mathbb{S}}\end{aligned}$$

for all $g \in \mathbb{S}$. Finally, $\pi : G \to \mathrm{GL}(\mathbb{S})$ is strongly continuous, because of

$$\begin{aligned}&\|\pi(x) g - \pi(x_0) g\|_{\mathbb{S}} \\ &= \|\pi(x) g - \pi(x_0) g\|_{\mathcal{H}_v^1} + \|\Gamma(\pi(x) g - \pi(x_0) g)\|_{\sup} \\ &= \|\pi(x) g - \pi(x_0) g\|_{\mathcal{H}_v^1} + \|L_x[\Gamma g] - L_{x_0}[\Gamma g]\|_{\sup} \xrightarrow[x \to x_0]{} 0.\end{aligned}$$

Here, we used the strong continuity of $\pi : G \to \mathrm{GL}\left(\mathcal{H}_v^1\right)$ which was shown in Lemma 2.4.7, and the uniform continuity of $\Gamma g \in C_c(G)$, cf. [32, Proposition (2.6)].

We now define an antilinear functional $\varphi \in \mathbb{S}^\neg$ by

$$\varphi : \mathbb{S} \to \mathbb{C}, g \mapsto \overline{(\Gamma g)(1_G)}.$$

Because of $|\varphi(g)| \leq \|\Gamma g\|_{\sup} \leq \|g\|_{\mathbb{S}}$, the functional $\varphi$ is indeed continuous. The wavelet transform of $\varphi$ is given by

$$\begin{aligned}H(x) := W_f\varphi(x) &= \varphi(\pi(x) f) = \overline{(\Gamma[\pi(x) f])(1_G)} \\ &= \overline{(L_x F)(1_G)} = \overline{F(x^{-1})}\end{aligned}$$

for all $x \in G$. Thus, $W_f\varphi \in C_c(G) \subset L^\infty(G) = W(L^\infty, Y)$.

If equality (2.4.15) was true, there would be some $g \in \left(\mathcal{H}_v^1\right)^\neg$ with $\varphi|_{\mathbb{S} \cap \mathcal{H}_v^1} = g|_{\mathbb{S} \cap \mathcal{H}_v^1}$. In particular, $H = W_f\varphi = W_f g$, so that the reproducing formula in Lemma 2.4.8, equation (2.4.14) yields

$$H = W_f g = \frac{1}{C_f} \cdot W_f g * W_f f = \frac{1}{C_f} \cdot H * W_f f.$$

But we have $W_f f \in L^2(G)$ as seen above. Furthermore, equation (2.4.7) yields $\left|\left(W_f f\right)^\vee\right| = \left|W_f f\right|$ and hence $\left(W_f f\right)^\vee \in L^2(G)$. Finally, the connected component $V$ of $1_G = (0,1)$ in $G = \mathbb{R} \times \mathbb{R}^*$ is given

by $V = \mathbb{R} \times (0, \infty)$, which is not compact. Hence, [37, Theorem 2.45] yields $H \equiv 0$ and thus $F \equiv 0$, a contradiction. Thus, the identity (2.4.15) fails in this case.

*We remark that a different proof (without using [37, Theorem 2.45]) can be given by showing that the reproducing formula $H = C_f^{-1} \cdot H * W_f f$, with $H \in C_c(G)$, forces $H$ to be smooth, by smoothness of $W_f f$. But since $H = \overline{F^\vee}$ can be chosen to be not differentiable, we get a contradiction.*

We remark that one can use a similar construction to design a space $\mathbb{S}$ with the property that $W_f \varphi \in W(L^\infty, Y)$ is *discontinuous* for some $\varphi \in \mathbb{S}^\neg$. Since $W_f g$ is continuous for each $g \in \mathcal{R}_v$ by Lemma 2.4.8, this provides an easier example for which the identity (2.4.15) fails. Of course, $\pi : G \to \mathrm{GL}(\mathbb{S})$ can not be strongly continuous in this case. ◀

Now that we have established well-definedness of the coorbit spaces, we will develop the announced atomic decomposition result for coorbit spaces in the next subsection.

### 2.4.5 *Atomic decomposition results*

The idea for the proof of the atomic decomposition result – as also explained in [58, Section 5] – is to discretize the reproducing formula

$$W_g f = \frac{1}{C_g} \cdot W_g f * W_g g$$

for $f \in \mathrm{Co}(Y) \subset \mathcal{R}_v$.

To this end, we consider – for suitable functions $H : G \to \mathbb{C}$ – the (CONTINUOUS) CONVOLUTION OPERATOR

$$T^H F := F * H$$

and for each $U$-BUPU $\Phi = (\varphi_i)_{i \in I}$ with associated localizing family $X = (x_i)_{i \in I}$ the DISCRETIZED CONVOLUTION OPERATOR

$$T^H_{U,\Phi,X} F := \sum_{i \in I} \langle F \mid \varphi_i \rangle \cdot L_{x_i} H. \tag{2.4.18}$$

The goal is to show

$$\left| \left| \left| T^H - T^H_{U,\Phi,X} \right| \right|_{W(L^\infty,Y) \to W(L^\infty,Y)} \xrightarrow[U \to \{1_G\}]{} 0$$

under suitable assumptions on $H$. Later on, we will apply this with $H = W_g g$ to derive the desired atomic decomposition result using a Neumann series argument.

In the current situation, this Neumann series argument has two subtleties that need to be accounted for:

A. If $Z$ is a Banach space, and $T : Z \to Z$ is linear and bounded, one needs to know $|||\mathrm{id} - T|||_{Z \to Z} < 1$ in order to be able to apply the Neumann series to conclude that $T$ is invertible. For a Quasi-Banach space $Z$, the same statement is true, but one should convince oneself that this is indeed the case.

B. As noted above, one needs $|||\mathrm{id} - T|||_{Z\to Z} < 1$, with

$$|||S|||_{Z\to Z} = \sup_{\|z\|_Z \le 1} \|Sz\|_Z .$$

Here it is essential to use the *same* (quasi)-norm $\|\cdot\|_Z$ to measure $\|Sz\|_Z$, as well as $\|z\|_Z$. In our situation, we will choose $Z = W(L^\infty, Y)$, whose norm is only well-defined up to a constant. To circumvent this, we fix some open, precompact unit neighborhood $Q \subset G$ for the remainder of this section and we will always use the (quasi)-norm $\|\cdot\|_{W_Q(L^\infty,Y)}$ for $W(L^\infty, Y)$.

Let us now clarify the applicability of a Neumann series argument in the case of Quasi-Banach spaces:

**Lemma 2.4.11.** *Let $(Z, \|\cdot\|_Z)$ be a quasi-normed space and let $(W, \|\cdot\|_W)$ be a Quasi-Banach space and assume that $\|\cdot\|_W$ is a $p$-norm for some $p \in (0, 1]$.*

*Then the space of bounded linear operators*

$$\mathcal{B}(Z, W) := \{T : Z \to W \mid T \text{ linear and } |||T|||_{Z\to W} < \infty\}$$

*equipped with the quasi-norm*

$$|||T|||_{Z\to W} := \sup_{\substack{z\in Z \\ \|z\|_Z \le 1}} \|Tz\|_W$$

*We remark that the quasi-norm $|||T|||_{Z\to W}$ was already defined in Lemma 2.1.6*

*is a Quasi-Banach space and $|||\cdot|||_{Z\to W}$ is a $p$-norm.*

*Finally, if $T \in \mathcal{B}(W) := \mathcal{B}(W, W)$ satisfies $|||T|||_{W\to W} = \alpha < 1$, then* $\mathrm{id}_W - T$ *is invertible with bounded inverse*

$$(\mathrm{id}_W - T)^{-1} = \sum_{n=0}^{\infty} T^n \in \mathcal{B}(W)$$

*and* $\left|\left|\left|(\mathrm{id}_W - T)^{-1}\right|\right|\right|_{W\to W} \le \frac{1}{(1-\alpha^p)^{1/p}}$.

*Proof.* For brevity, let us write $|||\cdot||| := |||\cdot|||_{Z\to W}$ for the rest of the proof. It is clear that $|||\cdot|||$ is homogeneous. Furthermore, $|||T||| = 0$ yields $Tz = 0$ for all $z \in B := \overline{B_1^{\|\cdot\|_Z}(0)}$. Since $T$ is linear and because of $Z = \bigcup_{n\in\mathbb{N}} nB$, we see $T \equiv 0$, so that $|||\cdot|||$ is definite.

Finally, for $T, S \in \mathcal{B}(Z, W)$, we have for each $z \in B$ the estimate

$$\|(T+S)z\|_W^p \le \|Tz\|_W^p + \|Sz\|_W^p \le |||T|||^p + |||S|||^p ,$$

which implies

$$|||T+S|||^p = \sup_{z\in B} \|(T+S)z\|_W^p \le |||T|||^p + |||S|||^p < \infty.$$

This shows that $\mathcal{B}(Z, W)$ is a vector space and that $|||\cdot|||$ is a $p$-norm (and thus a quasi-norm) on $\mathcal{B}(Z, W)$.

For completeness of $\mathcal{B}(Z,W)$ is suffices (by Lemma 2.1.8) to show that the sequence of partial sums $\sum_{n=1}^{N} T_n$ converges for $N \to \infty$ to some $T \in \mathcal{B}(Z,W)$, assuming $|||T_n||| \leq 2^{-n}$ for all $n \in \mathbb{N}$. Under this assumption, we have for each $z \in Z$, the estimate $\|T_n z\|_W \leq 2^{-n} \|z\|_Z$ with $\sum_{n=1}^{\infty} (2^{-n} \|z\|_Z)^p < \infty$. Since $W$ is complete, Lemma 2.1.8 implies that

$$Tz := \sum_{n=1}^{\infty} T_n z \in W$$

is well-defined. Clearly, $T : Z \to W$ is linear.

Finally, for $z \in B$ and $N \in \mathbb{N}_0$, continuity of $\|\cdot\|_W^p$ (cf. Lemma 2.1.5) implies

$$\begin{aligned}
\|(T - T_N) z\|_W^p &= \lim_{K\to\infty} \left\| \sum_{n=1}^{K} T_n z - \sum_{n=1}^{N} T_n z \right\|_W^p = \lim_{K\to\infty} \left\| \sum_{n=N+1}^{K} T_n z \right\|_W^p \\
&\leq \lim_{k\to\infty} \sum_{n=N+1}^{K} \|T_n z\|_W^p \leq \sum_{n=N+1}^{\infty} |||T_n|||^p \\
&\leq \sum_{n=N+1}^{\infty} 2^{-np} < \infty.
\end{aligned}$$

Choosing $N = 0$, we conclude $T \in \mathcal{B}(Z,W)$, since this holds for all $z \in B$. Finally, using the estimate for general $N \in \mathbb{N}$, we arrive at

$$|||T - T_N||| \leq \left( \sum_{n=N+1}^{\infty} 2^{-np} \right)^{1/p} \xrightarrow[N\to\infty]{} 0.$$

As seen above, this yields completeness of $\mathcal{B}(Z,W)$.

Now assume that $T \in \mathcal{B}(W)$ satisfies $\alpha := |||T|||_{W\to W} < 1$. It is easy to see that $\|Sz\|_W \leq |||S|||_{Z\to W} \cdot \|z\|_Z$ holds for all $z \in Z$ and $S \in \mathcal{B}(Z,W)$; in fact, this was already used above. For $S, T \in \mathcal{B}(W)$, this implies in particular

$$\begin{aligned}
|||S \circ T|||_{W\to W} &= \sup_{\|w\|_W \leq 1} \|S(Tw)\|_W \leq \sup_{\|w\|_W \leq 1} |||S|||_{W\to W} \|Tw\|_W \\
&= |||S|||_{W\to W} \cdot |||T|||_{W\to W}.
\end{aligned}$$

Inductively, we arrive at $|||T^n|||_{W\to W} \leq \alpha^n$ for all $n \in \mathbb{N}_0$. Because of $\sum_{n=0}^{\infty} \alpha^{np} < \infty$, Lemma 2.1.8 shows that $S := \sum_{n=0}^{\infty} T^n \in \mathcal{B}(W)$ is well-defined and satisfies

$$|||S|||_{W\to W}^p \leq \sum_{n=0}^{\infty} |||T^n|||_{W\to W}^p \leq \sum_{n=0}^{\infty} \alpha^{np} = \frac{1}{1-\alpha^p}.$$

Since composition $\mathcal{B}(W) \times \mathcal{B}(W) \to \mathcal{B}(W), (S,T) \mapsto S \circ T$ is a bounded (and hence continuous) bilinear map, we finally get

$$\begin{aligned} S \circ (\mathrm{id}_W - T) &= \lim_{N\to\infty} \sum_{n=0}^{N} T^n \circ (\mathrm{id}_W - T) = \lim_{N\to\infty} \left[ \sum_{n=0}^{N} T^n - \sum_{n=0}^{N} T^{n+1} \right] \\ &= \lim_{N\to\infty} \left[ \mathrm{id}_W - T^{N+1} \right] = \mathrm{id}_W \end{aligned}$$

because of $\left|\left|\left| T^{N+1} \right|\right|\right|_{W\to W} \leq \alpha^{N+1} \xrightarrow[N\to\infty]{} 0$.

Similarly, $(\mathrm{id}_W - T) \circ S = \mathrm{id}_W$, so that $\mathrm{id}_W - T$ is indeed invertible with the claimed inverse. □

Our first aim is to prove convergence of the series defining $T^H_{U,\Phi,X}$, as well as boundedness of the operator $T^H_{U,\Phi,X}$. This is done in the next three lemmata.

**Lemma 2.4.12.** *For each precompact unit neighborhood $U \subset G$ and each $U$-BUPU $\Phi = (\varphi_i)_{i\in I}$ with localizing family $X = (x_i)_{i\in I}$, the coefficient operator*

$$C_{U,\Phi,X} : W(L^\infty, Y) \to Y_d(X), F \mapsto (\langle F \mid \varphi_i \rangle)_{i\in I}$$

*is linear and bounded.*

*Remark.* It is important to note that it is *not* claimed that the operator norm $C_{U,\Phi,X}$ can be estimated independently of $U, \Phi, X$. This is the reason why we do not have to provide a distinguished neighborhood $V \subset G$ for the definition of $Y_d(X, V)$.

*Proof.* Using $\operatorname{supp} \varphi_i \subset x_i U$, we get

$$|\langle F \mid \varphi_i \rangle| \leq \int_G |\varphi_i \cdot F| \, \mathrm{d}x \leq \mu\left(x_i \overline{U}\right) \cdot \|\varphi_i \cdot F\|_{L^\infty} = \mu\left(\overline{U}\right) \cdot \|\varphi_i \cdot F\|_{L^\infty}.$$

But Theorem 2.3.17 shows that the sequence $\left(\|\varphi_i \cdot F\|_{L^\infty}\right)_{i\in I}$ is contained in $Y_d(X)$. Using the solidity of $Y_d$, we see $(\langle F \mid \varphi_i \rangle)_{i\in I} \in Y_d$ with

$$\|C_{U,\Phi,X} F\|_{Y_d(X)} \leq \mu\left(\overline{U}\right) \cdot \left\| \left(\|\varphi_i \cdot F\|_{L^\infty}\right)_{i\in I} \right\|_{Y_d} \lesssim \|F\|_{W(L^\infty,Y)}.$$

Here, the implied constant is provided by Theorem 2.3.17 and depends on $\Phi, X, Y, Q, U, w$, but not on $F$. □

The following lemma provides the second ingredient for the well-definedness and boundedness of $T^H_{U,\Phi,X}$. We remark that the lemma can be seen as a special case of a convolution relation between the space $W(M,Y)$ and $\left[W\left(L^\infty, L^p_w\right)\right]^\vee$, cf. [59, Theorem 5.1(a)] for a similar relation. But since a proper introduction of the Wiener Space $W(M,Y)$, where $M$ denotes the space of finite Borel measures, requires some technical overhead, we restrict ourselves to the following special case:

*In this lemma, it is understood that the weight $w$ is chosen as in Assumption 2.4.1. The proof shows that it would actually suffice if $w$ is submultiplicative with $w(x) \geq \|\|R_x\|\|_{W(Y)\to W(Y)}$, but we will not need this (stronger) statement in this thesis.*

**Lemma 2.4.13.** *Let $X = (x_i)_{i\in I}$ be a relatively separated family in $G$ for which there is a $U$-BUPU $(\varphi_i)_{i\in I}$ with $\operatorname{supp}\varphi_i \subset x_i U$ for all $i \in I$, where $U$ is a precompact unit neighborhood.*

*Then,*

$$\Gamma_X : Y_d(X) \times \left[W\left(L^\infty, L^p_w\right)\right]^\vee \to W(L^\infty, Y), \left((\lambda_i)_{i\in I}, F\right) \mapsto \sum_{i\in I} \lambda_i \cdot L_{x_i} F$$

*is a well-defined, bounded bilinear map. The series defining $\Gamma_X\left((\lambda_i)_{i\in I}, F\right)$ is almost everywhere absolutely convergent.*

*If the finitely supported sequences are dense in $Y_d(X)$, the series defining $\Gamma_X\left((\lambda_i)_{i\in I}, F\right)$ converges unconditionally in $W(L^\infty, Y)$.*

*Proof.* We first assume that $H \in L^\infty(G)$ satisfies $\operatorname{supp} H \subset U^{-1}$. In this case, we have

$$\begin{aligned} M_Q\left[L_{x_i}|H|\right](x) &= \|L_{x_i}|H| \cdot \chi_{xQ}\|_{L^\infty} \\ &= \left\|L_{x_i}\left[|H| \cdot \chi_{x_i^{-1}xQ}\right]\right\|_{L^\infty} \\ &= \left\||H| \cdot \chi_{\overline{U}^{-1}} \cdot \chi_{x_i^{-1}xQ}\right\|_{L^\infty} \\ &\leq \|H\|_{L^\infty} \cdot \chi_{x_i\overline{U}^{-1}\overline{Q}^{-1}}(x), \end{aligned}$$

because $\overline{U}^{-1} \cap x_i^{-1}xQ \neq \varnothing$ implies $x \in x_i\overline{U}^{-1}\overline{Q}^{-1}$.

Let $V := \overline{U}^{-1}\overline{Q}^{-1}$. Since $I$ is countable, the above estimate implies

$$\begin{aligned} M_Q\left(\sum_{i\in I} |\lambda_i| \cdot L_{x_i}|H|\right) &\leq \sum_{i\in I} |\lambda_i| \cdot M_Q\left[L_{x_i}|H|\right] \\ &\leq \|H\|_{L^\infty} \cdot \sum_{i\in I} |\lambda_i| \cdot \chi_{x_i V}. \end{aligned}$$

But since $(\lambda_i)_{i\in I} \in Y_d(X) = Y_d(X, V)$, we know that the right-hand side is contained in $Y$. Here, we used that the space $Y_d(X, V)$ is independent of $V$, by Theorem 2.3.17 and our assumptions on the space $W(L^\infty, Y)$, cf. Assumption 2.4.1.

By solidity, the above estimate implies

$$\begin{aligned} \left\|\sum_{i\in I} |\lambda_i| \cdot L_{x_i}|H|\right\|_{W(L^\infty,Y)} &= \left\|M_Q\left(\sum_{i\in I} |\lambda_i| \cdot L_{x_i}|H|\right)\right\|_Y \\ &\leq \|H\|_{L^\infty} \cdot \left\|\sum_{i\in I} |\lambda_i| \chi_{x_i V}\right\|_Y \\ &\lesssim \|H\|_{L^\infty} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)}. \end{aligned} \tag{2.4.19}$$

Here, the implied constant is independent of $H$ and of $(\lambda_i)_{i\in I}$, but (possibly) dependent on $X, Y, U, Q$.

Now, let $F \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$, i.e., $F^\vee \in W\left(L^\infty, L^p_w\right)$. For $i \in I$, define $H_i := L_{x_i^{-1}}\left(\varphi_i \cdot F^\vee\right) \in L^\infty$. Using Theorem 2.3.17 together with $\left(L^p_w\right)_d(X) = \ell^p_{w_X}(I)$ (cf. Lemma 2.3.21), we get

$$\begin{aligned}\sum_{i\in I}\left[w(x_i)\cdot\|H_i\|_{L^\infty}\right]^p &\lesssim \left\|\left(\|H_i\|_{L^\infty}\right)_{i\in I}\right\|^p_{\left(L^p_w\right)_d(X)}\\ &= \left\|\left(\left\|\varphi_i\cdot F^\vee\right\|_{L^\infty}\right)_{i\in I}\right\|^p_{\left(L^p_w\right)_d(X)}\\ &\lesssim \left\|F^\vee\right\|^p_{W\left(L^\infty,L^p_w\right)}\\ &= \|F\|^p_{\left[W\left(L^\infty,L^p_w\right)\right]^\vee} < \infty,\end{aligned}$$

where the implied constants depend on $X, Y, Q, \Phi$, but not on $F, (\lambda_i)_{i\in I}$.

Since $(\varphi_i)_{i\in I}$ is a partition of unity, we have $F^\vee = \sum_{i\in I} L_{x_i} H_i$ and $F = \sum_{i\in I} R_{x_i} H_i^\vee$ with $\operatorname{supp} H_i^\vee \subset U^{-1}$, because of $\operatorname{supp}\varphi_i \subset x_i U$. Since the operators $L_x$ and $R_y$ commute, we derive

$$\begin{aligned}\sum_{i\in I}|\lambda_i|\,L_{x_i}|F| &\le \sum_{i\in I}\left(|\lambda_i|\sum_{j\in I} L_{x_i}R_{x_j}\left|H_j^\vee\right|\right)\\ &= \sum_{j\in I}\left[R_{x_j}\left(\sum_{i\in I}|\lambda_i|\cdot L_{x_i}\left|H_j^\vee\right|\right)\right]\\ &=: \sum_{j\in I} F_j. \qquad (2.4.20)\end{aligned}$$

Using equation (2.4.19) together with the right invariance of $W(L^\infty, Y)$ and with our assumptions on $w$, we get $F_j \in W(L^\infty, Y)$ for all $j \in J$, with

$$\begin{aligned}&\left\|F_j\right\|_{W(L^\infty,Y)}\\ &\le \left\|R_{x_j}\right\|_{W(L^\infty,Y)\to W(L^\infty,Y)}\cdot\left\|\sum_{i\in I}|\lambda_i|\cdot L_{x_i}\left|H_j^\vee\right|\right\|_{W(L^\infty,Y)}\\ &\lesssim w(x_j)\cdot\left\|H_j^\vee\right\|_{L^\infty}\cdot\left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)}\\ &= \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)}\cdot w(x_j)\cdot\|H_j\|_{L^\infty}\cdot\end{aligned}$$

All in all, we arrive at

$$\begin{aligned}&\sum_{j\in I}\left\|F_j\right\|^p_{W(L^\infty,Y)}\\ &\lesssim \left\|(\lambda_i)_{i\in I}\right\|^p_{Y_d(X)}\cdot\sum_{j\in I}\left[w(x_j)\cdot\|H_j\|_{L^\infty}\right]^p\\ &\lesssim \left\|(\lambda_i)_{i\in I}\right\|^p_{Y_d(X)}\cdot\|F\|^p_{\left[W\left(L^\infty,L^p_w\right)\right]^\vee} < \infty.\end{aligned}$$

Since $W(L^\infty, Y)$ is a $p$-normed, solid Quasi-Banach space and because $I$ is countable, Theorem 2.2.10 yields absolute convergence almost everywhere of the series $\sum_{j\in I} F_j$ with

$$\left\|\sum_{j\in I} F_j\right\|_{W(Y)}^p \leq \sum_{j\in I} \|F_j\|_{W(Y)}^p \lesssim \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)}^p \|F\|_{\left[W\left(L^p_w\right)\right]^\vee}^p < \infty.$$

Finally, the solidity of $Y$, together with equation (2.4.20), implies absolute convergence almost everywhere of the series $\sum_{i\in I} \lambda_i \cdot L_{x_i} F$, with

$$\left\|\sum_{i\in I} \lambda_i \cdot L_{x_i} F\right\|_{W(Y)} \leq \left\|\sum_{j\in I} F_j\right\|_{W(Y)} \lesssim \left\|(\lambda_i)_{i\in I}\right\|_{Y_d(X)} \|F\|_{\left[W\left(L^p_w\right)\right]^\vee} < \infty.$$

This shows that $\Gamma_X$ is a well-defined, bounded bilinear map.

Finally, if the finitely supported sequences are dense in $Y_d$, then for $\varepsilon > 0$, there is, by Lemma 2.2.4, a finite subset $I_0 \subset I$ with $\|\Lambda\chi_J - \Lambda\|_{Y_d} < \varepsilon$ for each finite set $J \subset I$ with $J \supset I_0$, where we have set $\Lambda = (\lambda_i)_{i\in I}$. This implies

$$\begin{aligned}\left\|\Gamma(\Lambda, F) - \sum_{i\in J} \lambda_i \cdot L_{x_i} F\right\|_{W(L^\infty,Y)} &= \|\Gamma(\Lambda - \Lambda\chi_J,\, F)\|_{W(L^\infty,Y)} \\ &\lesssim \|\Lambda - \Lambda\chi_J\|_{Y_d} \cdot \|F\|_{\left[W\left(L^\infty, L^p_w\right)\right]^\vee} \\ &< \varepsilon \|F\|_{\left[W\left(L^\infty, L^p_w\right)\right]^\vee}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves the desired convergence. □

Finally, we apply our results to the concrete case of the discretized convolution operator $T^H_{U,\Phi,X}$, with $H = W_g g$. We remark that the usual treatments of coorbit theory do not give much reasoning for the fact that the discretized convolution operators indeed have image in $L^\infty_{1/v} * H$ or $W(L^\infty, Y) * H$, respectively, see e.g. [29, Proposition 5.3]. This question is not completely trivial, since the series defining $T^H_{U,\Phi,X} f$ does *not* necessarily converge in the norm of $L^\infty_{1/v}(G)$ or of $W(L^\infty, Y)$ if the finitely supported sequences are not dense in $Y_d(X)$. Nevertheless, it is always true that the ranges of these discretized convolution operators are contained in $L^\infty_{1/v} * H$ or $W(L^\infty, Y) * H$, respectively, as we will see.

**Lemma 2.4.14.** *Let $g \in \mathbb{A}^p_v \setminus \{0\}$. Then the following are true:*

1. *For each $\gamma \in \left(L^1_v\right)^\vee$, the convolution operator*

$$T^\gamma : L^\infty_{1/v}(G) \to L^\infty_{1/v}(G), f \mapsto f * \gamma$$

*is well-defined and bounded with $\left\|\left|T^\gamma\right|\right\|_{L^\infty_{1/v}\to L^\infty_{1/v}} \leq \|\gamma\|_{\left(L^1_v\right)^\vee}$.*

2. *Let $\Phi = (\varphi_i)_{i\in I}$ be a U-BUPU with localizing family $X = (x_i)_{i\in I}$. Then, for each $\gamma \in \left[W\left(C^0, L^1_v\right)\right]^\vee$, the operators*

$$C^\Phi : L^\infty_{1/v}(G) \to \ell^\infty_{1/v_X}(I), f \mapsto (\langle f \mid \varphi_i\rangle)_{i\in I}$$

*and*

$$S^{X,\gamma} : \ell^\infty_{1/v_X}(I) \to L^\infty_{1/v}(G), (\lambda_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i \cdot L_{x_i}\gamma$$

*are well-defined and bounded. The series defining $S^{X,\gamma}\left((\lambda_i)_{i\in I}\right)$ is pointwise absolutely convergent.*

3. *We have $H := W_g g \in \left[W\left(L^\infty, L^1_v\right)\right]^\vee$ as well as $H \in \left(L^1_v\right)^\vee$.*

4. *We have $W(L^\infty, Y) \hookrightarrow L^\infty_{1/v}(G)$ and $T^H$ restricts to a bounded operator*

$$T^H_Y := T^H|_{W(L^\infty,Y)} : W(L^\infty, Y) \to W(L^\infty, Y).$$

5. *We have $g \in \operatorname{dom}(A)$ and*

$$\ker\left(\mathrm{id}_{L^\infty_{1/v}} - \frac{1}{C_g} T^H\right) = L^\infty_{1/v} * H$$

$$\ker\left(\mathrm{id}_{W(L^\infty,Y)} - \frac{1}{C_g} T^H_Y\right) = W(L^\infty, Y) * H.$$

*In particular, these subspaces are closed in $L^\infty_{1/v}(G)$ and $W(L^\infty, Y)$, respectively.*

6. *Let $\Phi = (\varphi_i)_{i\in I}$ be a U-BUPU with localizing family $X = (x_i)_{i\in I}$. Then the discretized convolution operator*

$$T^H_{U,\Phi,X} : L^\infty_{1/v}(G) \to L^\infty_{1/v}(G) * H,\ f \mapsto \sum_{i\in I} \langle f \mid \varphi_i\rangle \cdot L_{x_i} H$$

*is well-defined and bounded with pointwise absolute convergence of the series.*

*Furthermore, the restriction*

$$T^H_{U,\Phi,X}|_{W(L^\infty,Y)} : W(L^\infty, Y) \to W(L^\infty, Y) * H$$

*is also well-defined and bounded.*

*Proof.* 1. Observe that $v(y) = v\left(xx^{-1}y\right) \le v(x)\, v\left(x^{-1}y\right)$ holds for all $x, y \in G$. Hence,

$$\begin{aligned}
\frac{1}{v(x)} \cdot (|f| * |\gamma|)(x) &= \int_G \frac{1}{v(x)} \cdot |f(y)| \cdot |\gamma(y^{-1}x)| \,\mathrm{d}y \\
&\le \int_G \frac{|f(y)|}{v(y)} \cdot \left|\gamma^\vee(x^{-1}y)\right| \cdot v(x^{-1}y) \,\mathrm{d}y \\
&\le \|f\|_{L^\infty_{1/v}} \cdot \left\|\gamma^\vee\right\|_{L^1_v} = \|f\|_{L^\infty_{1/v}} \cdot \|\gamma\|_{(L^1_v)^\vee} \qquad (2.4.21)
\end{aligned}$$

holds for all $x \in G$. This proves absolute convergence of the integral defining $f * \gamma$, as well as $f * \gamma \in L^\infty_{1/v}$ and $|f| * |\gamma| \in L^\infty_{1/v}$ with

$$\|f * \gamma\|_{L^\infty_{1/v}} \leq \|f\|_{L^\infty_{1/v}} \cdot \|\gamma\|_{(L^1_v)^\vee} < \infty.$$

Hence, $T^\gamma$ is well-defined and bounded with $\|\!|\!| T^\gamma |\!|\!|_{L^\infty_{1/v} \to L^\infty_{1/v}} \leq \|\gamma\|_{(L^1_v)^\vee}$.

2. For $f \in L^\infty_{1/v}(G)$, we have

$$\begin{aligned} |\langle f \mid \varphi_i \rangle| &\leq \int_{x_i \overline{U}} \left| \frac{1}{v(x)} f(x) \right| \cdot v(x) \, \mathrm{d}x \\ &\leq \|f\|_{L^\infty_{1/v}} \cdot \int_{\overline{U}} v(x_i y) \, \mathrm{d}y \\ &\leq v(x_i) \cdot \|f\|_{L^\infty_{1/v}} \cdot \mu(\overline{U}) \cdot C_{v,U}. \end{aligned}$$

Here, we used $v(x_i y) \leq v(x_i) v(y) \leq v(x_i) \cdot C_{v,U}$ for all $y \in \overline{U}$ in the last step. The existence of a finite constant $C_{v,U}$ for which this estimate is valid is ensured by the local boundedness of $v$, see Theorem 2.2.22. This implies $C^\Phi(f) \in \ell^\infty_{1/v_X}(I)$ with

$$\left\| (\langle f \mid \varphi \rangle)_{i \in I} \right\|_{\ell^\infty_{1/v_X}} \leq \mu(\overline{U}) \cdot C_{v,U} \cdot \|f\|_{L^\infty_{1/v}},$$

so that $C^\Phi$ is well-defined and bounded.

Next, we show boundedness of $S^{X,\gamma}$. Since $v$ is submultiplicative, the spaces $L^1_v(G)$ and $W\left(L^\infty, L^1_v\right)$ are left- and right-invariant with

$$\|\!|\!| L_x |\!|\!|_{W(L^1_v) \to W(L^1_v)} \leq \|\!|\!| L_x |\!|\!|_{L^1_v \to L^1_v} \leq v(x)$$

by Lemma 2.2.19 and Lemma 2.3.18. Thus, $L_x(\gamma^\vee) \in W\left(L^\infty, L^1_v\right)$ for each $x \in G$ and $\gamma \in \left[W\left(L^\infty, L^1_v\right)\right]^\vee$. But the SAMPLING MAP

$$\Gamma : W(C^0, L^1_v) \to (L^1_v)_d(X) = \ell^1_{v_X}(I), f \mapsto (f(x_i))_{i \in I}$$

is well-defined and bounded by a combination of Corollary 2.3.19 and Lemma 2.3.21. Hence,

$$\begin{aligned} ((L_{x_i} \gamma)(x))_{i \in I} &= (\gamma(x_i^{-1} x))_{i \in I} \\ &= \left(\gamma^\vee(x^{-1} x_i)\right)_{i \in i} \\ &= \Gamma\left[L_x\left(\gamma^\vee\right)\right] \in \ell^1_{v_X}(I) \end{aligned}$$

for each $x \in G$ with

$$\begin{aligned} \left\| ((L_{x_i} \gamma)(x))_{i \in I} \right\|_{\ell^1_{v_X}(I)} &\lesssim \left\| \Gamma\left(L_x\left(\gamma^\vee\right)\right) \right\|_{(L^1_v)_d(X)} \\ &\lesssim \left\| L_x\left(\gamma^\vee\right) \right\|_{W(C^0, L^1_v)} \\ &\leq \|\!|\!| L_x |\!|\!|_{W(L^\infty, L^1_v) \to W(L^\infty, L^1_v)} \cdot \left\| \gamma^\vee \right\|_{W(L^\infty, L^1_v)} \\ &\leq v(x) \cdot \|\gamma\|_{[W(L^\infty, L^1_v)]^\vee}. \end{aligned}$$

It is important to observe that the implied constants are independent of $x \in G$ and of $\gamma \in \left[W\left(L^\infty, L^1_v\right)\right]^\vee$.

Now, let $\Lambda = (\lambda_i)_{i\in I} \in \ell^\infty_{1/v_X}(I)$ be arbitrary. Then

$$\begin{aligned}
&\frac{1}{v(x)} \cdot \sum_{i\in I} |\lambda_i| \cdot |(L_{x_i}\gamma)(x)| \\
&\leq \frac{1}{v(x)} \cdot \left\|(\lambda_i)_{i\in I}\right\|_{\ell^\infty_{1/v_X}} \cdot \left\|((L_{x_i}\gamma)(x))_{i\in I}\right\|_{\ell^1_{v_X}} \\
&\lesssim \left\|(\lambda_i)_{i\in I}\right\|_{\ell^\infty_{1/v_X}} \cdot \|\gamma\|_{[W(L^\infty,L^1_v)]^\vee} < \infty, \qquad (2.4.22)
\end{aligned}$$

where all implied constants are independent of $\Lambda \in \ell^\infty_{1/v_X}(I)$ and of $x \in G$. This establishes absolute convergence of the series as well as boundedness of $S^{X,\gamma}$.

3. By definition of $\mathbb{A}^p_v$, we have

$$W_g g \in W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \hookrightarrow W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(L^\infty, L^1_v\right).$$

Here, the last embedding is a consequence of Corollary 2.3.20 and Lemma 2.3.21, together with the discrete embedding $\ell^p_{v_X}(I) \hookrightarrow \ell^1_{v_X}(I)$. Hence, $W_g g \in W\left(L^\infty, L^1_v\right)$. But we have $\left|W_g g\right| = \left|W_g g\right|^\vee$ and $W\left(L^1_v\right)$ is a solid space. Thus, $\left(W_g g\right)^\vee \in W\left(L^\infty, L^1_v\right) \hookrightarrow L^1_v$ as well.

4. Lemma 2.3.34, with $r(x) = \left|\left|\left|R_x\right|\right|\right|_{W(Y)\to W(Y)} \leq w(x) \leq v(x)$ yields the continuous embedding $W\left(L^\infty, Y\right) \hookrightarrow L^\infty_{1/v}(G)$. Boundedness of $T^H_Y$ is a direct consequence of the convolution relation in Theorem 2.3.24, since we have $H \in W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \hookrightarrow W\left(L^\infty, L^p_v\right)$ with

$$v(x) \geq w(x) \geq \Delta\left(x^{-1}\right) \cdot \left|\left|\left|R_{x^{-1}}\right|\right|\right|_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)}$$

for all $x \in G$.

5. As seen above, $W_g g \in W\left(L^\infty, L^1_v\right) \hookrightarrow L^1_v \hookrightarrow L^1$. By Lemma 2.4.5, this implies $g \in \operatorname{dom}\left(A^2\right) \subset \operatorname{dom}(A)$. Regarding the identity characterizing the spaces $L^\infty_{1/v} * H$ and $W\left(L^\infty, Y\right) * H$, we only prove the first identity, since the proof of the second one is analogous.

"$\subset$": For $f \in \ker\left(\mathrm{id}_{L^\infty_{1/v}} - \frac{1}{C_g} T^H\right) \subset L^\infty_{1/v}(G)$, we have

$$f = \frac{1}{C_g} \cdot T^H(f) = \left(\frac{1}{C_g} f\right) * H \in L^\infty_{1/v}(G) * H.$$

"$\supset$": Above, we saw $g \in \operatorname{dom}(A)$. Thus, the reproducing formula (2.4.4) yields

$$\begin{aligned}
H * H &= W_g g * W_g g \\
&= \langle Ag \mid Ag \rangle_{\mathcal{H}} \cdot W_g g \\
&= C_g \cdot W_g g \\
&= C_g \cdot H.
\end{aligned}$$

Now, if $f \in L^\infty_{1/v}(G) * H \subset L^\infty_{1/v}(G)$, then $f = h * H$ for some $h \in L^\infty_{1/v}(G)$. But this implies

$$\begin{aligned} f &= h * H \\ &= \frac{1}{C_g} \cdot h * (H * H) \\ &= \frac{1}{C_g} \cdot (h * H) * H \\ &= \frac{1}{C_g} \cdot f * H \\ &= \frac{1}{C_g} \cdot T^H f \end{aligned}$$

and hence $f \in \ker\left(\mathrm{id}_{L^\infty_{1/v}} - \frac{1}{C_g} T^H\right)$. Here, associativity of the convolution can be checked using the Fubini-Tonelli theorem by splitting $h, H$ into positive and negative parts (of the real/imaginary parts).

6. For the first part, simply note that $T^H_{U,\Phi,X} = S^{X,H} \circ C^\Phi$ is bounded by part 2. The same part also yields absolute convergence of the series.

Boundedness of $T^H_{U,\Phi,X}|_{W(L^\infty,Y)} : W(L^\infty, Y) \to W(L^\infty, Y)$ is a direct consequence of Lemmas 2.4.12 and 2.4.13, because of

$$H \in W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \hookrightarrow W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(L^\infty, L^p_w\right),$$

which implies $H \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$ by symmetry of $|H|$ and because $W\left(L^\infty, L^p_w\right)$ is solid.

It remains to show that the range of $T^H_{U,\Phi,X}$ is contained in the space $L^\infty_{1/v}(G) * H$ and analogously that the range of $T^H_{U,\Phi,X}$ restricted to $W(L^\infty, Y)$ is contained in $W(L^\infty, Y) * H$. As seen above, it suffices to show $\frac{1}{C_g} \cdot T^H\left(T^H_{U,\Phi,X} f\right) = T^H_{U,\Phi,X} f$ for every $f \in L^\infty_{1/v}(G)$. It is important to note that this also proves the claim regarding $W(L^\infty, Y)$.

Hence, let $f \in L^\infty_{1/v}(G)$ and define $\lambda_i := \langle f \mid \varphi_i \rangle$ for $i \in I$. The estimate in equation (2.4.22) shows that the series

$$\Theta := \sum_{i \in I} |\lambda_i| \cdot L_{x_i} |H|$$

is pointwise absolutely convergent and contained in $L^\infty_{1/v}(G)$. Together with equation (2.4.21), we see $\int_G \Theta(y) \cdot |H|\left(y^{-1}x\right) \mathrm{d}y < \infty$ for every $x \in G$. Since $I$ is countable, we can thus apply the dominated convergence theorem to interchange summation and integration in the calculation

$$\begin{aligned} T^H\left(T^H_{U,\Phi,X} f\right)(x) &= \int_G \sum_{i \in I} \lambda_i \cdot \left(L_{x_i} H\right)(y) \cdot H\left(y^{-1}x\right) \mathrm{d}y \\ &= \sum_{i \in I} \lambda_i \cdot \int_G \left(L_{x_i} H\right)(y) \cdot H\left(y^{-1}x\right) \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I} \lambda_i \cdot \left[(L_{x_i} H) * H\right](x) \\
&= \sum_{i\in I} \lambda_i \cdot L_{x_i} \left[H * H\right](x) \\
&= C_g \cdot \left(\sum_{i\in I} \lambda_i \cdot L_{x_i} H\right)(x) \\
&= C_g \cdot \left(T^H_{U,\Phi,X} f\right)(x).
\end{aligned}
$$

This completes the proof. □

We remark that the operator norm of the operators $T^H_{U,\Phi,X}$ in the above lemmata could still (heavily) depend on $U, X$ which provides an obstruction for passing to the limit $U \to \{1_G\}$. This is the problem we adress next, by introducing the concept of the (right-sided) $U$-oscillation of a (continuous) function.

**Definition 2.4.15.** (cf. [58, Discussion before Lemma 3.8]) Let $U \subset G$ be a relatively compact unit neighborhood and let $H : G \to \mathbb{C}$ be continuous.

We define the (RIGHT-SIDED) $U$-OSCILLATION of $H$ as

$$\operatorname{osc}_U H : G \to [0,\infty), x \mapsto \sup_{u\in U} |H(ux) - H(x)|.$$

The most important properties of the oscillation are discussed in the following lemma which is based in large parts on [58, Lemma 3.8].

**Lemma 2.4.16.** *Let $H : G \to \mathbb{C}$ be a continuous function. Then the following are true:*

1. *$\operatorname{osc}_U H$ is a continuous function for each relatively compact unit neighborhood $U \subset G$.*
2. *We have $\operatorname{osc}_U H \xrightarrow[U\to\{1_G\}]{} 0$ with locally uniform convergence. More precisely, for each $\varepsilon > 0$ and each compact set $K \subset G$, there is a precompact unit neighborhood $U_0 \subset G$ such that $|\operatorname{osc}_U H(x)| < \varepsilon$ holds for all $x \in K$ and all unit neighborhoods $U \subset U_0$.*
3. *For $y \in xU$, the estimate $\left|L_y H - L_x H\right| \leq L_y \left[\operatorname{osc}_U H\right]$ is true.*
4. *We have $\operatorname{osc}_U H \leq 2 \cdot M^R_U H$ for each open, precompact unit neighborhood $U \subset G$.*
5. *If $u : G \to (0,\infty)$ is a measurable, submultiplicative weight and $q \in (0,\infty)$ with $\operatorname{osc}_{U_0} H \in W\left(L^\infty, L^q_u\right)$ for some relatively compact unit neighborhood $U_0 \subset G$, then*

$$\|\operatorname{osc}_U H\|_{W_Q\left(L^\infty, L^q_u\right)} \xrightarrow[U\to\{1_G\}]{} 0.$$

*Furthermore, the condition on $H$ is satisfied for $H \in W^R\left(W\left(L^q_u\right)\right)$.*

6. *If $u : G \to (0,\infty)$ is a measurable, submultiplicative weight and $q \in (0,\infty)$ with $\operatorname{osc}_{U_0} H \in L^q_u$ for some relatively compact unit neighborhood $U_0 \subset G$, then*

$$\|\operatorname{osc}_U H\|_{L^q_u} \xrightarrow[U \to \{1_G\}]{} 0.$$

*Proof.* As a preparation, we recall the well-known fact that any continuous function $f : G \to \mathbb{C}$ is uniformly continuous on compact sets, i.e., for every compact $K \subset G$ and $\varepsilon > 0$, there is some unit neighborhood $W \subset G$ such that $|f(x) - f(y)| < \varepsilon$ holds for all $x, y \in K$ satisfying $x^{-1}y \in W$.

1. It suffices to show continuity of $\operatorname{osc}_U$ on $K$ for each compact set $K \subset G$. Since $L := \overline{U}K$ is compact, $H$ is uniformly continuous on this set, i.e., there is a unit neighborhood $W \subset G$ with $|H(x) - H(y)| < \varepsilon$ for all $x, y \in L$ with $x^{-1}y \in W$. For $x, y \in K$ with $x^{-1}y \in W$, this yields for each $u \in U$ the estimate

$$\begin{aligned} &|H(ux) - H(x)| \\ &\leq |H(ux) - H(uy)| + |H(uy) - H(y)| + |H(y) - H(x)| \\ &\leq 2\varepsilon + |H(uy) - H(y)| \\ &\leq 2\varepsilon + (\operatorname{osc}_U H)(y), \end{aligned}$$

because of $x, y, ux, uy \in L$ with $x^{-1}y \in W$ and $(ux)^{-1} uy = x^{-1}y \in W$. Thus, $(\operatorname{osc}_U H)(x) \leq (\operatorname{osc}_U H)(y) + 2\varepsilon$. By symmetry, we get

$$|\operatorname{osc}_U H(x) - \operatorname{osc}_U H(y)| \leq 2\varepsilon$$

for all $x, y \in K$ with $x^{-1}y \in W$. In particular, $\operatorname{osc}_U H$ is continuous on $K$.

2. Fix some compact unit neighborhood $V_0 \subset G$ and let $K \subset G$ be compact. Since $H^\vee$ is continuous, it is uniformly continuous on the compact set $L := K^{-1}V_0^{-1}$, so that there is some symmetric unit neighborhood $U_0 \subset V_0 \subset G$ with $|H^\vee(x) - H^\vee(y)| < \varepsilon$ for $x, y \in L$ with $x^{-1}y \in U_0$.

For $x \in K$ and $u \in U_0 \subset V_0$, this implies

$$|H(ux) - H(x)| = \left|H^\vee(x^{-1}u^{-1}) - H^\vee(x^{-1})\right| < \varepsilon,$$

because of $\left(x^{-1}u^{-1}\right)^{-1} x^{-1} = uxx^{-1} = u \in U_0$ and $x^{-1} \in K^{-1} \subset L$, as well as $x^{-1}u^{-1} \in K^{-1}V_0^{-1} = L$. Hence, $(\operatorname{osc}_{U_0} H)(x) < \varepsilon$ for all $x \in K$. Using the estimate $0 \leq \operatorname{osc}_U H \leq \operatorname{osc}_{U_0} H$ for $U \subset U_0$, we get the claim.

3. By assumption, $x^{-1}y \in U$. Thus,

$$\begin{aligned} \left(L_y[\operatorname{osc}_U H]\right)(z) &= (\operatorname{osc}_U H)(y^{-1}z) \\ &= \sup_{u \in U} |H(uy^{-1}z) - H(y^{-1}z)| \end{aligned}$$

$$\begin{aligned}&\geq |H(x^{-1}yy^{-1}z) - H(y^{-1}z)|\\ &= \left|(L_xH)(z) - \left(L_yH\right)(z)\right|\end{aligned}$$

holds for all $z \in G$.

4. Let $U \subset G$ be an open, precompact unit neighborhood. We have

$$\begin{aligned}(\operatorname{osc}_U H)(y) &\leq |H(y)| + \sup_{u\in U} |H(uy)|\\ &\leq 2 \cdot \sup_{u\in U} |H(uy)|\\ &= 2 \cdot \operatorname*{ess\,sup}_{z\in Uy} |H(z)|\\ &\leq 2 \cdot (M_U^R H)(y).\end{aligned}$$

This computation used that $U$ is open and that $H$ is continuous, so that the essential supremum and the ordinary supremum coincide.

5. Since $G$ is $\sigma$-compact and because of $M_Q[\operatorname{osc}_{U_0} H] \in L^q_u$, the dominated convergence theorem ensures existence of a compact set $K \subset G$ with

$$\int_{G\setminus K} (u \cdot M_Q[\operatorname{osc}_{U_0} H])^q \, \mathrm{d}x < \frac{\varepsilon^q}{2}.$$

Theorem 2.2.22 shows that $C_0 := \sup_{x\in K} u(x)$ is finite. Furthermore, part 2 yields a unit neighborhood $V_0 \subset U_0$ with

$$(\operatorname{osc}_U H)(x) < \frac{\varepsilon/2^{1/q}}{[\mu(K)]^{1/q} \cdot C_0} \text{ for all } x \in KQ \text{ and } U \subset V_0.$$

For $x \in K$, this implies

$$(M_Q[\operatorname{osc}_U H])(x) = \operatorname*{ess\,sup}_{y\in xQ} (\operatorname{osc}_U H)(y) \leq \frac{\varepsilon/2^{1/q}}{[\mu(K)]^{1/q} \cdot C_0}.$$

For $U \subset V_0 \subset U_0$, we arrive at

$$\begin{aligned}&\|M_Q[\operatorname{osc}_U H]\|^q_{L^q_u}\\ &\leq \int_{G\setminus K} (u \cdot M_Q[\operatorname{osc}_{U_0} H])^q \, \mathrm{d}\mu + \int_K (M_Q[\operatorname{osc}_U H] \cdot u)^q \, \mathrm{d}\mu\\ &< \frac{\varepsilon^q}{2} + \mu(K) \cdot \left(\frac{\varepsilon/2^{1/q}}{[\mu(K)]^{1/q} \cdot C_0} \cdot C_0\right)^q = \varepsilon^q.\end{aligned}$$

This proves the desired convergence.

Finally, if $H \in W^R\left(L^\infty, W\left(L^\infty, L^q_u\right)\right)$, then $M_U^R H \in W\left(L^\infty, L^q_u\right)$ for each open, precompact unit neighborhood $U \subset G$. Since $W\left(L^\infty, L^q_u\right)$ is solid and $\operatorname{osc}_U H$ is continuous (and hence measurable), the estimate in part 4 yields $\operatorname{osc}_U H \in W\left(L^\infty, L^q_u\right)$.

6. The proof of this part is similar to, but easier than that of part 5 and hence omitted. □

Using the above properties of the oscillation, we are in a position to derive a bound for $\left|\!\left|\!\left| T^H - T^H_{U,\Phi,X} \right|\!\right|\!\right|_{W(L^\infty,Y)\to W(L^\infty,Y)}$ which vanishes for $U \to \{1_G\}$. This estimate is identical to [58, Theorem 5.4].

**Lemma 2.4.17.** *[58, Theorem 5.4] Let $g \in \mathbb{A}^p_v \setminus \{0\}$ and set $H := W_g g$. Then $\operatorname{osc}_U H \in \left(L^1_v\right)^\vee$ for each precompact unit neighborhood $U \subset G$.*

*Here, it is crucial that the constant $C$ as well as the quantity $\|\operatorname{osc}_U H\|_Z$, for the different spaces $Z$, do* not *depend on the chosen U-BUPU $\Phi$ or on the family $X$, but only on $U$.*

*Furthermore, there is an absolute constant $C = C(Q, Y) > 0$ such that the estimates*

$$\left|\!\left|\!\left| T^H - T^H_{U,\Phi,X} \right|\!\right|\!\right|_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} \leq C \cdot \|\operatorname{osc}_U H\|_{W_Q\left(L^\infty,L^p_v\right)} \xrightarrow[U\to\{1_G\}]{} 0,$$
$$\left|\!\left|\!\left| T^H - T^H_{U,\Phi,X} \right|\!\right|\!\right|_{L^\infty_{1/v}\to L^\infty_{1/v}} \leq C \cdot \|\operatorname{osc}_U H\|_{\left(L^1_v\right)^\vee} \xrightarrow[U\to\{1_G\}]{} 0$$

*hold for each U-BUPU $\Phi = (\varphi_i)_{i\in I}$ with associated localizing family $X$.*

*Proof.* We first show $\operatorname{osc}_U H \in \left(L^1_v\right)^\vee$. The embedding

$$\left(L^p_v\right)_d = \ell^p_{v_X}(I) \hookrightarrow \ell^1_{v_X}(I) = \left(L^1_v\right)_d,$$

together with Corollary 2.3.20 and with the definition of $\mathbb{A}^p_v$ imply

$$H = W_g g \in W^R\left(L^\infty, W\left(L^\infty, L^p_v\right)\right) \hookrightarrow W^R\left(L^\infty, W\left(L^\infty, L^1_v\right)\right).$$

But $|H| = |H^\vee|$. Since $W^R\left(L^\infty, W\left(L^\infty, L^1_v\right)\right)$ is solid, we conclude $H^\vee \in W^R\left(L^\infty, W\left(L^\infty, L^1_v\right)\right)$ and thus (cf. Lemma 2.3.29)

$$\begin{aligned} H &\in \left[W^R\left(L^\infty, W\left(L^\infty, L^1_v\right)\right)\right]^\vee = W\left(L^\infty, \left[W\left(L^\infty, L^1_v\right)\right]^\vee\right) \\ &= W\left(L^\infty, W^R\left(L^\infty, \left(L^1_v\right)^\vee\right)\right) \hookrightarrow W^R\left(L^\infty, \left(L^1_v\right)^\vee\right). \end{aligned}$$

Using the continuity (and hence measurability) of $\operatorname{osc}_U H$, together with the solidity of $\left(L^1_v\right)^\vee$ and with the estimate

$$0 \leq \operatorname{osc}_V H \leq \operatorname{osc}_U H \leq 2 \cdot M^R_U H \in \left(L^1_v\right)^\vee$$

for $V \subset U$ and $U$ open (cf. Lemma 2.4.16), we get $\operatorname{osc}_V H \in \left(L^1_v\right)^\vee$ for every precompact unit neighborhood $V \subset G$.

The convergences $\|\operatorname{osc}_U H\|_{W_Q\left(L^\infty,L^p_v\right)} \to 0$ and $\|\operatorname{osc}_U H\|_{\left(L^1_v\right)^\vee} \to 0$ as $U \to \{1_G\}$ now follow from Lemma 2.4.16, since $\left(L^1_v\right)^\vee = L^1_{v_{\vee,1}}$, cf. Lemma 2.2.21.

Now, let $F \in L^\infty_{1/v}(G) \supset W(L^\infty, Y)$ be arbitrary. Since $H \in \left(L^1_v\right)^\vee$, Lemma 2.4.14 shows $\int_G |F(y)| \cdot \left|H\left(y^{-1}x\right)\right| \, \mathrm{d}y < \infty$, so that countability of $I$, together with dominated convergence implies

$$\left(T^H F\right)(x) = \sum_{i\in I} \int_G F(y) \cdot \varphi_i(y) \cdot \left(L_y H\right)(x) \, \mathrm{d}y.$$

Thus,

$$\begin{aligned}
&\left|\left(T^H F\right)(x) - \left(T^H_{U,\Phi,X} F\right)(x)\right| \\
&= \left|\sum_{i\in I} \int_G F(y)\cdot\varphi_i(y)\cdot\left[L_y H(x) - L_{x_i} H(x)\right]\,\mathrm{d}y\right| \\
&\leq \sum_{i\in I}\int_G |F(y)|\cdot\varphi_i(y)\cdot\left|\left(L_y H - L_{x_i} H\right)(x)\right|\,\mathrm{d}y \\
&\overset{(*)}{\leq} \sum_{i\in I}\int_G |F(y)|\cdot\varphi_i(y)\cdot\left(L_y\left[\mathrm{osc}_U H\right]\right)(x)\,\mathrm{d}y \\
&= \int_G |F(y)|\cdot\left(\mathrm{osc}_U H\right)\left(y^{-1}x\right)\,\mathrm{d}y \\
&= \left(|F| * \mathrm{osc}_U H\right)(x).
\end{aligned}$$

The step marked with $(*)$ used that $\varphi_i(y) \neq 0$ implies $y \in x_i U$, so that part 3 of Lemma 2.4.16 yields

$$\left|\left(L_y H - L_{x_i} H\right)(x)\right| \leq L_y\left[\mathrm{osc}_U H\right](x).$$

Using part 1 of Lemma 2.4.14, we derive

$$\left\|T^H F - T^H_{U,\Phi,X} F\right\|_{L^\infty_{1/v}} \leq \left\||F| * \mathrm{osc}_U H\right\|_{L^\infty_{1/v}} \leq \|F\|_{L^\infty_{1/v}} \cdot \left\|\mathrm{osc}_U H\right\|_{(L^1_v)^\vee}.$$

If even $F \in W\left(L^\infty, Y\right)$ holds, then the convolution relation in Theorem 2.3.24 implies

$$\begin{aligned}
\left\|T^H F - T^H_{U,\Phi,X} F\right\|_{W_Q(L^\infty,Y)} &\leq \left\||F| * \mathrm{osc}_U H\right\|_{W_Q(L^\infty,Y)} \\
&\leq C_{Q,Y}\,\|F\|_{W_Q(L^\infty,Y)}\,\left\|\mathrm{osc}_U H\right\|_{W_Q\left(L^\infty,L^p_v\right)}. \qquad \square
\end{aligned}$$

As a last preparation, we need to know that the wavelet transform $W_g$ yields an isomorphism between the coorbit space $\mathrm{Co}(Y)$ and the reproducing kernel space $W\left(L^\infty, Y\right) * W_g g \leq W\left(L^\infty, Y\right)$.

**Lemma 2.4.18.** *(cf. [58, Proposition 4.2])*
*Let $g \in \mathbb{A}^p_v \setminus \{0\}$ and set $H := W_g g$. Then*

$$W_g : \mathcal{R}_v \to L^\infty_{1/v}(G) * H, f \mapsto W_g f$$

*and*

$$W_g : \mathrm{Co}(Y) \to W\left(L^\infty, Y\right) * H, f \mapsto W_g f$$

*are isomorphisms of (Quasi)-Banach spaces.*

*Proof.* By Lemma 2.4.8, we know that each $f \in \mathcal{R}_v$ satisfies the reproducing formula $W_g f = \frac{1}{C_g} W_g f * W_g g \in L^\infty_{1/v} * H$ and that the map $f \mapsto \left\|W_g f\right\|_{L^\infty_{1/v}}$ defines an equivalent norm on $\mathcal{R}_v$.

For the first part, it thus suffices to show that $W_g : \mathcal{R}_v \to L^\infty_{1/v} * H$ is surjective. To this end, let $F \in L^\infty_{1/v} * H$ be arbitrary. By part 5 of

Lemma 2.4.14, this implies $F = \frac{1}{C_g} F * H$, so that part 5 of Lemma 2.4.8 yields some $f \in \mathcal{R}_v$ with $F = W_g f$.

For the second part, we note that $W_g : \mathrm{Co}(Y) \to W(L^\infty, Y) * H$ is well-defined, since each $f \in \mathrm{Co}(Y) \subset \mathcal{R}_v$ satisfies $W_g f \in W(L^\infty, Y)$ be definition of $\mathrm{Co}(Y) = \mathrm{Co}_g(Y)$ and thus also

$$W_g f = \frac{1}{C_g} W_g f * W_g g = \frac{1}{C_g} W_g f * H \in W(L^\infty, Y) * H.$$

Furthermore, we have $\|f\|_{\mathrm{Co}_g(Y)} = \|W_g f\|_{W(L^\infty, Y)}$, so that the map under consideration is injective and isometric.

For surjectivity, let $F \in W(L^\infty, Y) * H$ be arbitrary. By Lemma 2.4.14, we have $W(L^\infty, Y) \hookrightarrow L^\infty_{1/v}(G)$, so that the previous part of the proof yields some $f \in \mathcal{R}_v$ with $F = W_g f$. But part 4 of Lemma 2.4.14 yields

$$W_g f = F \in W(L^\infty, Y) * H \subset W(L^\infty, Y)$$

and hence $f \in \mathrm{Co}(Y)$. □

We are finally in a position to prove the desired atomic decomposition results.

**Theorem 2.4.19.** *(cf. [58, Theorem 5.5]) Let $g \in \mathbb{A}^p_v \setminus \{0\}$. Then the following are true:*

1. *Let $X = (x_i)_{i \in I}$ be a relatively separated family in $G$ for which there is a $U$-BUPU $(\varphi_i)_{i \in I}$ with $\operatorname{supp} \varphi_i \subset x_i U$ for all $i \in I$ for some precompact unit neighborhood $U \subset G$.*

   *Then, the* SYNTHESIS OPERATOR

   $$S_g^X : \ell^\infty_{1/v_x}(I) \to \mathcal{R}_v, (\lambda_i)_{i \in I} \mapsto \sum_{i \in I} [\lambda_i \cdot \pi(x_i) g]$$

   *and its restriction*

   $$S^X_{g,Y} := S_g^X|_{Y_d(X)} : Y_d(X) \to \mathrm{Co}(Y)$$

   *are well-defined and bounded.*

   *Convergence of the series defining $S_g^X(\Lambda)$ has to be understood as unconditional convergence in the weak-$*$-sense of $\mathcal{R}_v$. If the finite sequences are dense in $Y_d(X)$, then the series defining $S^X_{g,Y}(\Lambda)$ converges unconditionally in $\mathrm{Co}(Y)$ for each $\Lambda \in Y_d(X)$.*

2. *There is a unit neighborhood $U_0 = U_0(Y, Q, g, v) \subset G$ such that for each relatively separated family $X = (x_i)_{i \in I}$ which admits a BUPU $\Phi = (\varphi_i)_{i \in I}$ with $\operatorname{supp} \varphi_i \subset x_i U_0$ for all $i \in I$, there exists a bounded linear* ANALYSIS OPERATOR

   $$A_g^X : \mathcal{R}_v \to \ell^\infty_{1/v_X}(I)$$

*satisfying* $S_g^X \circ A_g^X = \mathrm{id}_{\mathcal{R}_v}$. *Furthermore, its restriction*

$$A_{g,Y}^X := A_g^X|_{\mathrm{Co}(Y)} : \mathrm{Co}(Y) \to Y_d(X)$$

*is well-defined and bounded with* $S_{g,Y}^X \circ A_{g,Y}^X = \mathrm{id}_{\mathrm{Co}(Y)}$.

*Proof.* 1. By Lemma 2.4.14, we get $H := W_g g \in \left[W\left(C^0, L_v^1\right)\right]^\vee$ as well as boundedness of the operator

$$S^{X,H} : \ell^\infty_{1/v_X}(I) \to L^\infty_{1/v}(G), (\lambda_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i \cdot L_{x_i} H$$

with pointwise absolute convergence of the series.

Let $(\lambda_i)_{i\in I} \in \ell^\infty_{1/v_X}$ be arbitrary. Define $M := \{J \subset I \mid J \text{ finite}\}$. Equipped with the relation "$\subset$", this becomes a directed set and unconditional convergence of $\sum_{i\in I} [\lambda_i \cdot \pi(x_i) g]$ in the weak-$*$-sense of $\mathcal{R}_v$ means precisely that the net $(S_J)_{J\in M}$ defined by

$$S_J := \sum_{i\in J} [\lambda_i \cdot \pi(x_i) g]$$

converges with respect to the weak-$*$-topology on $\mathcal{R}_v$, i.e. pointwise.

But on the one hand, since $W_g$ intertwines $\pi(x)$ and $L_x$, we get

$$W_g S_J = \sum_{i\in J} \left[\lambda_i \cdot L_{x_i} W_g g\right] = \sum_{i\in J} [\lambda_i \cdot L_{x_i} H] \longrightarrow S^{X,H}\left((\lambda_i)_{i\in I}\right)$$

with pointwise convergence. On the other hand, Lemma 2.4.8 shows that $f \mapsto \left\|W_g f\right\|_{L^\infty_{1/v}}$ defines an equivalent norm on $\mathcal{R}_v$. Hence,

$$\|S_J\|_{\mathcal{R}_v} \lesssim \left\|W_g S_J\right\|_{L^\infty_{1/v}} = \left\|\sum_{i\in J} \lambda_i \cdot L_{x_i} H\right\|_{L^\infty_{1/v}} \leq \left\|\sum_{i\in I} |\lambda_i| \cdot L_{x_i} |H|\right\|_{L^\infty_{1/v}} < \infty.$$

Here, the last norm is finite by Lemma 2.4.14, since $(|\lambda_i|)_{i\in I} \in \ell^\infty_{1/v_X}(I)$ and because of $|H| \in \left[W\left(C^0, L_v^1\right)\right]^\vee$.

All in all, we see that the net $(S_J)_{J\in M}$ is a bounded net in $\mathcal{R}_v$ with pointwise convergent wavelet transform. By Lemma 2.4.8, this means that the net $(S_J)_{J\in M}$ is convergent with respect to the weak-$*$-topology of $\mathcal{R}_v$ and the limit

$$h := \lim_J S_J = \sum_{i\in I} \lambda_i \cdot \pi(x_i) g = S_g^X\left((\lambda_i)_{i\in I}\right)$$

satisfies

$$W_g\left[S_g^X(\lambda_i)_{i\in I}\right] = \lim_J W_g S_J = S^{X,H}\left((\lambda_i)_{i\in I}\right) = \sum_{i\in I} \lambda_i \cdot L_{x_i} H \quad (2.4.23)$$

and thus

$$\left\|S_g^X\left((\lambda_i)_{i\in I}\right)\right\|_{\mathcal{R}_v} \lesssim \left\|W_g\left[S_g^X(\lambda_i)_{i\in I}\right]\right\|_{L^\infty_{1/v}} = \left\|S^{X,H}\left((\lambda_i)_{i\in I}\right)\right\|_{L^\infty_{1/v}}.$$

Since $S^{X,H}$ is a bounded operator, this establishes all desired properties of $S_g^X$.

It remains to consider $S^X_{g,Y}$. But $W_g : \mathrm{Co}(Y) \to W(L^\infty, Y)$ is isometric and for $f \in \mathcal{R}_v$, we have $f \in \mathrm{Co}(Y)$ iff $W_g f \in W(L^\infty, Y)$. Thus, it suffices to show that

$$W_g \circ S^X_{g,Y} : Y_d(X) \to W(L^\infty, Y)$$

is bounded (with unconditional convergence of the series in $W(L^\infty, Y)$ if the finitely supported sequences are dense in $Y_d(X)$). As seen in equation (2.4.23), we have

$$\left(W_g \circ S_g^X\right)\left((\lambda_i)_{i\in I}\right) = \sum_{i\in I} \lambda_i \cdot L_{x_i} H = \Gamma^X\left((\lambda_i)_{i\in I}, H\right)$$

with $\Gamma^X$ as defined in Lemma 2.4.13. But this lemma yields boundedness of $\Gamma^X(\cdot, H) : Y_d(X) \to W(L^\infty, Y)$ and unconditional convergence of the series if the finitely supported sequences are dense in $Y_d$. This completes the proof of the first part of the theorem. We note that Lemma 2.4.13 is indeed applicable, since $g \in \mathbb{A}_v^p$ implies

$$H = W_g g \in W\left(L^\infty, L^p_v\right) \hookrightarrow W\left(L^\infty, L^p_w\right)$$

and then also $H \in \left[W\left(L^\infty, L^p_w\right)\right]^\vee$ because of $|H| = |H^\vee|$.

2. As above, let $H := W_g g$. Lemma 2.4.17 implies that there is a precompact unit neighborhood $U_0 = U_0(Y, Q, g, v) \subset G$ such that the estimates

$$\left\|\left\|\frac{1}{C_g} T^H - \frac{1}{C_g} T^H_{U,\Phi,X}\right\|\right\|_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} < 1$$

and

$$\left\|\left\|\frac{1}{C_g} T^H - \frac{1}{C_g} T^H_{U,\Phi,X}\right\|\right\|_{L^\infty_{1/v}\to L^\infty_{1/v}} < 1$$

hold for all precompact unit neighborhoods $U \subset U_0$ and each $U$-BUPU $\Phi = (\varphi_i)_{i\in I}$ with associated localizing family $X = (x_i)_{i\in i}$. Let us fix $U, \Phi, X$ with these properties.

Furthermore, Lemma 2.4.14 implies that the operator $\frac{1}{C_g} T^H$ restricts to the identity operator on the Banach space $Z_1 := L^\infty_{1/v} * H$ as well as on the Quasi-Banach space $Z_2 := W(L^\infty, Y) * H$. The same lemma also shows that the operators

$$A_1 := \frac{1}{C_g} T^H_{U,\Phi,X}|_{Z_1} : Z_1 \to Z_1 \quad \text{and} \quad A_2 := \frac{1}{C_g} T^H_{U,\Phi,X}|_{Z_2} : Z_2 \to Z_2$$

are *well-defined* and bounded. Using the estimates above, we see $|||\mathrm{id}_{Z_1} - A_1|||_{Z_1 \to Z_1} < 1$ as well as $|||\mathrm{id}_{Z_2} - A_2|||_{Z_2 \to Z_2} < 1$.

Using Lemma 2.4.11 (about the Neumann series for Quasi-Banach spaces), we see that the operator $A_i = \mathrm{id}_{Z_i} - (\mathrm{id}_{Z_i} - A_i)$ is invertible on $Z_i$ for $i \in \{1, 2\}$ with inverse given by

$$A_i^{-1} = \sum_{n=0}^{\infty} A_i^n.$$

We note that $W(L^\infty, Y)$ is a subspace of $L^\infty_{1/v}$ by Lemma 2.4.14. Hence, $Z_2 \leq Z_1$ and $A_2$ is the restriction of $A_1$ to $Z_2$. Using the explicit formula $A_2^{-1} = \sum_{n=0}^{\infty} A_2^n = \sum_{n=0}^{\infty} (A_1|_{Z_2})^n$, we conclude $A_2^{-1} = A_1^{-1}|_{Z_2}$.

For $V_1 := \mathcal{R}_v$ and $V_2 := \mathrm{Co}(Y)$, Lemma 2.4.18 shows that the wavelet transform $W_g : V_i \to Z_i$ is an isomorphism. Let us define $W_1 := \ell^\infty_{1/v_X}(I)$ and $W_2 := Y_d(X)$. Lemma 2.4.12 and Lemma 2.4.14 show that the COEFFICIENT OPERATORS

$$\begin{aligned} C_1^{X,\Phi} &: L^\infty_{1/v}(G) \to \ell^\infty_{1/v_X}(I) = W_1, f \mapsto (\langle f \mid \varphi_i \rangle)_{i \in I}, \\ C_2^{X,\Phi} &: W(L^\infty, Y) \to Y_d(X) = W_2, f \mapsto (\langle f \mid \varphi_i \rangle)_{i \in I} \end{aligned}$$

are well-defined and bounded. Hence,

$$A_i^X := C_i^{X,\Phi} \circ A_i^{-1} \circ W_g : V_i \to W_i, f \mapsto \frac{1}{C_g} \cdot \left( \left\langle A_i^{-1} \left[ W_g f \right] \,\middle|\, \varphi_i \right\rangle \right)_{i \in I}$$

is a bounded operator for $i \in \{1, 2\}$ and $A_2$ is the restriction of $A_1$ to $V_2$.

Finally, the calculation in the first part (cf. equation (2.4.23)) implies

$$\begin{aligned} W_g \left[ \left( S_g^X \circ A_1^X \right) f \right] &= \frac{1}{C_g} \cdot \sum_{i \in I} \left\langle A_1^{-1} \left[ W_g f \right] \,\middle|\, \varphi_i \right\rangle \cdot L_{x_i} H \\ &= \frac{1}{C_g} \cdot T^H_{U,\Phi,X} \left( A_1^{-1} \left[ W_g f \right] \right) \\ &= A_1 A_1^{-1} \left[ W_g f \right] \\ &= W_g f \end{aligned}$$

and thus $(S_g^X \circ A_1^X) f = f$ for all $f \in \mathcal{R}_v$. Since $A_2^X$ is a restriction of $A_1^X$, and $S_Y^X$ is a restriction of $S^X$, we also get $(S^X_{g,Y} \circ A_2^X) f = f$ for all $f \in \mathrm{Co}(Y)$, which completes the proof. □

*Remark* 2.4.20. We remark that our formulation of Theorem 2.4.19 differs slightly from the usual formulation, which can be derived as follows:

Since the analysis operator $A_g^X : \mathcal{R}_v \to \ell^\infty_{1/v_X}(I)$ is bounded, so are the linear functionals

$$\lambda_i := \pi_i \circ A_g^X : \mathcal{R}_v \to \mathbb{C},$$

where $\pi_i : \ell^\infty_{1/v_X}(I) \to \mathbb{C}, (x_j)_{j\in I} \mapsto x_i$ denotes the projection to the $i$-th coordinate.

Theorem 2.4.19 now implies the following: If $f \in \mathcal{R}_v$ satisfies $(\lambda_i(f))_{i\in I} \in Y_d(X)$, then

$$f = \left(S_g^X \circ A_g^X\right)(f) = S_g^X\left((\lambda_i(f))_{i\in I}\right) = S_{g,Y}^X\left((\lambda_i(f))_{i\in I}\right) \in \mathrm{Co}(Y)$$

with

$$\|f\|_{\mathrm{Co}(Y)} = \left\|S_{g,Y}^X\left((\lambda_i(f))_{i\in I}\right)\right\|_{\mathrm{Co}(Y)} \lesssim \left\|(\lambda_i(f))_{i\in I}\right\|_{Y_d(X)}.$$

Conversely, for $f \in \mathrm{Co}(Y) \leq \mathcal{R}_v$, we have

$$(\lambda_i(f))_{i\in I} = A_g^X f = A_{g,Y}^X f \in Y_d(X)$$

with

$$\left\|(\lambda_i(f))_{i\in I}\right\|_{Y_d(X)} = \left\|A_{g,Y}^X f\right\|_{Y_d(X)} \lesssim \|f\|_{\mathrm{Co}(Y)}.$$

All in all, this implies that for $f \in \mathcal{R}_v$, we have the equivalence

$$f \in \mathrm{Co}(Y) \iff (\lambda_i(f))_{i\in I} \in Y_d(X),$$

together with an accompanying quasi-norm equivalence.

Finally, each $f \in \mathcal{R}_v \supset \mathrm{Co}(Y)$ can be written as

$$f = \left(S_g^X \circ A_g^X\right) f = S_g^X\left((\lambda_i(f))_{i\in I}\right) = \sum_{i\in I} \lambda_i(f) \cdot \pi(x_i)\, g,$$

with convergence of the series in the weak-$*$-topology of $\mathcal{R}_v$.

If the finitely supported sequences are dense in $Y_d(X)$, the series even converges in the quasi-norm of $\mathrm{Co}(Y)$ for each $f \in \mathrm{Co}(Y)$. ◀

# 3

# (FOURIER SIDE) DECOMPOSITION SPACES

In this chapter, we will formally define the Fourier-side decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, Y)$ and derive their most important properties, i.e. well-definedness and completeness. Here, well-definedness refers to the fact that the space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is independent of the chosen partition of unity $(\varphi_i)_{i\in I}$ subordinate to the covering $\mathcal{Q}$.

For the case $p \in [1, \infty]$, this follows from the fact that $\mathcal{F}L^p$ is a Banach module over $\mathcal{F}L^1$. This property is a consequence of the convolution relation $L^1 * L^p \hookrightarrow L^p$. But for the case of Quasi-Banach spaces, i.e. $p \in (0, 1)$, this convolution relation is no longer true. Thus, we will need suitable convolution relations for $L^p\left(\mathbb{R}^d\right)$ in the case of $0 < p < 1$.

We will see that the form of these convolution relations makes it necessary to impose certain additional assumptions on the covering $\mathcal{Q}$. To this end, we will introduce the new class of (TIGHT) SEMI-STRUCTURED ADMISSIBLE COVERINGS.

This chapter is structured as follows: In the first section, we establish the above-mentioned convolution relations for $L^p\left(\mathbb{R}^d\right)$ with $0 < p < 1$.

In the second section, we introduce certain classes of coverings, the (TIGHT) (SEMI)-STRUCTURED ADMISSIBLE COVERINGS. In connection with these coverings, we also introduce a special class of partitions of unity, the $L^p$-BAPUS and we prove their existence under mild additional assumptions on the covering.

The third section is concerned with several notions of subordinateness of different coverings and contains many auxiliary results which are needed in the subsequent section and also in the remainder of the thesis.

In the next section, we finally introduce the class of (FOURIER SIDE) DECOMPOSITION SPACES which we will consider in the remainder of this thesis. Using the previous results of this chapter, we prove that these decomposition spaces are indeed complete and independent of the chosen partition of unity.

Finally, in the last section of the chapter, we consider dilations of decomposition spaces. In particular, we will show

$$\|f \circ T\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} = \left|\det(T)\right|^{\frac{1}{p}-1} \cdot \|f\|_{\mathcal{D}\left(T\mathcal{Q}, L^p, \ell^q_u\right)},$$

i.e., dilating an element $f$ of a decomposition space with a matrix $T \in \mathrm{GL}\left(\mathbb{R}^d\right)$ amounts to dilating the covering $\mathcal{Q}$, at least as long as one is only interested in the decomposition space norm of the dilated version of $f$. Hence, dilation invariance of $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ is equiva-

lent to existence of an embedding $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \hookrightarrow \mathcal{D}\left(T\mathcal{Q}, L^p, \ell^q_u\right)$. We will use this property in Chapter 6 to asymptotically calculate the operator norm of the isotropic dilation on various decomposition spaces.

## 3.1 CONVOLUTION RELATIONS FOR $L^p\left(\mathbb{R}^d\right)$, $0 < p < 1$

In this section, which is mostly based on the book [66] of Triebel, we develop convolution relations for $L^p\left(\mathbb{R}^d\right)$ in the Quasi-Banach case, i.e., for $0 < p < 1$. These will be important later to show that the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, Y\right)$ is well-defined, i.e. independent of the chosen partition of unity, even for $p \in (0,1)$.

We repeat some of the arguments given in [66] in order to make the presentation self-contained and because we need to know the dependence of the constant $C$ in the convolution relation

$$\|\mathcal{F}^{-1}\left(f \cdot g\right)\|_{L^p} \leq C \cdot \|\mathcal{F}^{-1} f\|_{L^p} \cdot \|\mathcal{F}^{-1} g\|_{L^p} \tag{3.1.1}$$

with respect to the supports of $f$ and $g$. This dependence is not explicitly stated in [66].

We first note that the Young inequality $\|f * g\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p}$ fails completely for $p \in (0,1)$, even if both $f, g$ have compact Fourier support. This is shown in the following example:

**Example 3.1.1.** Let us define

$$g_1 : \mathbb{R} \to \mathbb{R}, \xi \mapsto \max\{0, 1 - |\xi|\} = \begin{cases} 0, & \text{if } \xi \leq -1, \\ 1 + \xi, & \text{if } -1 \leq \xi \leq 0, \\ 1 - \xi, & \text{if } 0 \leq \xi \leq 1, \\ 0, & \text{if } \xi \geq 1 \end{cases}$$

and

$$g_2 : \mathbb{R} \to \mathbb{R}, \xi \mapsto \begin{cases} 0, & \text{if } \xi \leq -1, \\ (\xi - 1)^2 \cdot (\xi + 1)^2, & \text{if } -1 \leq \xi \leq 1, \\ 0, & \text{if } \xi \geq 1. \end{cases}$$

Then $g_1, g_2$ are supported in $[-1,1]$. Hence, if we set $f_j := \mathcal{F}^{-1} g_j$ for $j \in \{1,2\}$, then $f_1, f_2$ are bandlimited.

A straightforward, but tedious calculation shows for $x \in \mathbb{R} \setminus \{0\}$ that

$$f_1(x) = -\frac{1}{2} \frac{\cos(2\pi x) - 1}{\pi^2 x^2},$$

and

$$f_2(x) = \frac{3}{2}\frac{\sin(2\pi x)}{\pi^5 x^5} - 3\frac{\cos(2\pi x)}{\pi^4 x^4} - 2\frac{\sin(2\pi x)}{\pi^3 x^3}.$$

Being (inverse) Fourier transforms of $L^1$-functions, $f_1, f_2$ are both bounded. Furthermore, $f_1$ decays (at least) like $x^{-2}$ and $f_2$ decays (at least) like $|x|^{-3}$ at $\pm\infty$. This implies $f_1 \in L^p(\mathbb{R})$ for all $p > \frac{1}{2}$ as well as $f_2 \in L^p(\mathbb{R})$ for $p > \frac{1}{3}$. In particular, $f_1, f_2 \in L^1(\mathbb{R})$.

Another calculation using the convolution theorem, i.e.

$$h := f_1 * f_2 = \mathcal{F}^{-1}\left(\widehat{f_1} \cdot \widehat{f_2}\right) = \mathcal{F}^{-1}(g_1 \cdot g_2)$$

leads to

$$h(x) = -\frac{15}{4}\frac{\cos(2\pi x) - 1}{\pi^6 x^6} - 6\frac{\sin(2\pi x)}{\pi^5 x^5} + 3\frac{\cos(2\pi x) + \frac{1}{2}}{\pi^4 x^4} + \frac{1/2}{\pi^2 x^2}$$

for $x \in \mathbb{R} \setminus \{0\}$. Since all terms except for the last one decay strictly faster than $x^{-2}$, we get $|(f_1 * f_2)(x)| = |h(x)| \asymp x^{-2}$ for large $|x|$.

Thus, $f_1 * f_2 \notin L^p(\mathbb{R})$ for $\frac{1}{3} < p \leq \frac{1}{2}$, although $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^p(\mathbb{R})$ for these values of $p$. This shows that the usual form of Young's inequality fails for $p < 1$, even if we assume the factors of the convolution to be bandlimited. ◀

In the example above, one should note that $f_1 * f_2 \in L^p(\mathbb{R})$ holds for $\frac{1}{2} < p < 1$, which is exactly the range of $p \in (0,1)$, for which $f_1, f_2 \in L^p(\mathbb{R})$ holds. This is indicative of the kind of convolution relation that we are after (cf. also equation (3.1.1)).

For technical reasons, we start our derivation of the convolution relations for bandlimited $L^p$-functions with $p \in (0,1)$ by showing that we can always approximate such functions by Schwartz functions in a suitable way. This will allow us to restrict to Schwartz functions for most of our proofs.

**Lemma 3.1.2.** *Let $\Omega \subset \mathbb{R}^d$ be compact and assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution with compact Fourier support* $\operatorname{supp} \widehat{f} \subset \Omega$.

*Then $f$ is given by (integration against) a smooth function $g \in C^\infty(\mathbb{R}^d)$ with polynomially bounded derivatives of all orders.*

*Furthermore, there is a sequence of Schwartz functions $(g_n)_{n\in\mathbb{N}}$ with the following properties:*

1. $\operatorname{supp} \widehat{g_n} \subset B_{1/n}(\Omega)$, *where*

$$B_{1/n}(\Omega) = \left\{\xi \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(\xi, \Omega) < n^{-1}\right\}.$$

   *is the $\frac{1}{n}$-neighborhood of $\Omega$.*

2. $|g_n(x)| \leq |g(x)|$ *for all $x \in \mathbb{R}^d$ and*

3. $g_n(x) \xrightarrow[n\to\infty]{} g(x)$ *for all $x \in \mathbb{R}^d$.*

*In particular,* $\|g_n - f\|_{L^p} \xrightarrow[n\to\infty]{} 0$ *holds for any* $p \in (0,\infty)$ *for which* $f \in L^p\left(\mathbb{R}^d\right)$ *is true.*

*Remark.* In the following, we will always identify $f$ with its "smooth version" $g$, i.e. we will write $f(x)$ instead of $g(x)$ for $x \in \mathbb{R}^d$.

*Proof.* (based upon ideas in the proof of [66, Theorem 1.4.1]) In case of $\Omega = \varnothing$, we have $f = 0$, so that all claims are trivial (with $g_n \equiv 0$). Hence, we can assume $\Omega \neq \varnothing$ in the following. That $f$ is given by (integration against) a smooth (even analytic) function $g \in C^\infty\left(\mathbb{R}^d\right)$ with polynomially bounded derivatives of all orders is a consequence of the PALEY-WIENER THEOREM (cf. [61, Theorem 7.23]). Hence, we only need to establish the existence of the sequence $(g_n)_{n\in\mathbb{N}}$.

To this end, choose $\psi \in C_c^\infty(B_1(0))$ with $0 \leq \psi \leq 1$ and $\psi(0) = 1$ and set $\widetilde{\varphi} := \mathcal{F}^{-1}\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$. Note that

$$\gamma := \widetilde{\varphi}(0) = \int_{\mathbb{R}^d} \psi(\xi)\, \mathrm{d}\xi > 0,$$

since $\psi$ is continuous and nonnegative with $\psi \not\equiv 0$. Finally, define $\varphi := \widetilde{\varphi}/\gamma$ and observe $\operatorname{supp}\widehat{\varphi} = \operatorname{supp}\left(\mathcal{F}\widetilde{\varphi}\right) = \operatorname{supp}\psi \subset B_1(0)$ as well as $\varphi(0) = 1$.

Furthermore, since the Fourier transform $\widehat{\varphi} = \psi/\gamma$ is nonnegative, $|\varphi|$ attains its global maximum at the origin, since

$$\begin{aligned}
|\varphi(x)| &= \left|\left(\mathcal{F}^{-1}\widehat{\varphi}\right)(x)\right| \\
&= \frac{1}{\gamma}\left|\left(\mathcal{F}^{-1}\psi\right)(x)\right| \\
&\leq \frac{1}{\gamma}\cdot\int_{\mathbb{R}^d}\left|\psi(\xi)\cdot e^{2\pi i\langle x,\xi\rangle}\right|\,\mathrm{d}\xi \\
&= \frac{1}{\gamma}\cdot\int_{\mathbb{R}^d}\psi(\xi)\,\mathrm{d}\xi = \frac{1}{\gamma}\cdot\gamma = 1
\end{aligned}$$

holds for all $x \in \mathbb{R}^d$.

Let us now set

$$g_n : \mathbb{R}^d \to \mathbb{C}, x \mapsto g(x)\cdot\varphi\left(\frac{x}{n}\right)$$

for $n \in \mathbb{N}$. Since $g$ is a smooth function with polynomially bounded derivatives, the Leibniz rule easily implies that all derivatives of $g_n$ decay rapidly, so that $g_n$ is indeed a Schwartz function. As seen above, $|\varphi(x)| \leq 1$ for all $x \in \mathbb{R}^d$, which immediately yields

$$|g_n(x)| = \left|g(x)\cdot\varphi\left(\frac{x}{n}\right)\right| \leq |g(x)|.$$

Furthermore, $\varphi(0) = 1$ together with continuity of $\varphi$ yields

$$g_n(x) = g(x)\cdot\varphi\left(\frac{x}{n}\right) \xrightarrow[n\to\infty]{} g(x)\cdot 1 = g(x).$$

Finally, the convolution theorem (cf. [61, Theorem 7.19(e)] and see [61, Definition 7.18] for the definition of the convolution of a tempered distribution and a Schwartz function) yields

$$\widehat{g_n} = \widehat{g} * \widehat{\varphi\left(\frac{\cdot}{n}\right)} = n^d \cdot \left[\widehat{g} * \left(\widehat{\varphi}\left(n\cdot\right)\right)\right].$$

Since $\widehat{\varphi}$ is supported in $B_r(0)$ for some $r \in (0,1)$, we see that $\widehat{\varphi}(n\cdot)$ has support in $B_{r/n}(0)$.

By definition of the convolution for tempered distributions, we have

$$\left[\widehat{g} * \left(\widehat{\varphi}\left(n\cdot\right)\right)\right](\xi) = \widehat{g}\left(L_\xi\left[\widehat{\varphi}\left(n\cdot\right)\right]^\vee\right),$$

with $\gamma^\vee(\xi) = \gamma(-\xi)$. But since $\left[\widehat{\varphi}\left(n\cdot\right)\right]^\vee$is supported in $B_{r/n}(0)$, we see that $L_\xi\left[\widehat{\varphi}\left(n\cdot\right)\right]^\vee$ is supported in $B_{r/n}(\xi)$. Thus, $\left[\widehat{g} * \widehat{\varphi}\left(n\cdot\right)\right](\xi) \neq 0$ can only happen if $\operatorname{supp}\widehat{g} \cap B_{r/n}(\xi) \neq \varnothing$, i.e. for $\xi \in B_{r/n}(\Omega)$, since $\operatorname{supp}\widehat{g} = \operatorname{supp}\widehat{f} \subset \Omega$. Hence,

$$\operatorname{supp}\widehat{g_n} \subset \overline{B_{r/n}(\Omega)} \subset B_{1/n}(\Omega).$$

The additional claim regarding the $L^p$-convergence is a direct consequence of the dominated convergence theorem, using the pointwise convergence, together with $|g_n| \leq |g|$ and with $f = g$ almost everywhere if $f \in L^p\left(\mathbb{R}^d\right)$. □

Using the preceding approximation result, we will now show that bandlimited functions which are integrable to a power $p$ are automatically integrable to every power $q \geq p$. This roughly reflects the fact that these functions are always (uniformly) continuous, so that the only obstruction to integrability is the decay at infinity.

This embedding into $L^q$-spaces with larger $q$ will be a central ingredient for the proof of the convolution relations in the Quasi-Banach regime $p \in (0,1)$. We remark that the corollary below is a special case of [66, 1.4.1(3)].

**Corollary 3.1.3.** *Let $\Omega \subset \mathbb{R}^d$ be compact and assume that $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is a tempered distribution with compact Fourier support* $\operatorname{supp}\widehat{f} \subset \Omega$.

*If $f \in L^p\left(\mathbb{R}^d\right)$ holds for some $p \in (0,2]$, then*

$$\|f\|_{L^q} \leq \left[\lambda(\Omega)\right]^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L^p} \tag{3.1.2}$$

*holds for all $q \in [p,\infty]$.*

*Furthermore, there is a constant $K = K(\Omega) > 0$ such that*

$$\|f\|_{L^q} \leq K \cdot \|f\|_{L^p} \tag{3.1.3}$$

*holds for all $p \in [1,\infty]$, $q \in [p,\infty]$ and $f \in L^p\left(\mathbb{R}^d\right)$ with* $\operatorname{supp}\widehat{f} \subset \Omega$.

*Proof.* We first observe that it suffices to establish the stated estimates for Schwartz functions, since the Fatou property for $L^q\left(\mathbb{R}^d\right)$ implies, using the sequence $(g_n)_{n\in\mathbb{N}}$ as given by Lemma 3.1.2, that

$$\begin{aligned}\|f\|_{L^q} &\leq \liminf_{n\to\infty} \|g_n\|_{L^q} \\ &\leq \liminf_{n\to\infty} \left[\lambda\left(B_{1/n}(\Omega)\right)\right]^{\frac{1}{p}-\frac{1}{q}} \cdot \|g_n\|_{L^p} \\ &\leq \left[\lambda(\Omega)\right]^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L^p}.\end{aligned}$$

This proves equation (3.1.2) in the general case. Observe that the last step used $|g_n| \leq |f|$ as well as $\lambda\left(B_{1/n}(\Omega)\right) \to \lambda(\Omega)$. This last convergence is a consequence of the continuity of the (Lebesgue) measure from above (cf. [33, Theorem 1.8(d)]) and of the identity $\Omega = \bigcap_{n\in\mathbb{N}} B_{1/n}(\Omega)$, which holds because $\Omega$ is compact.

Validity of equation (3.1.3) in the general case is derived similarly, with $K = K\left(\overline{B_1(\Omega)}\right)$ instead of $K = K(\Omega)$.

Hence, we can assume $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\operatorname{supp}\widehat{f} \subset \Omega$ in the following. Observe that by regularity of the Lebesgue measure, there is for each $\varepsilon > 0$ some open set $U_\varepsilon \supset \Omega$ with $\lambda(U_\varepsilon) < \lambda(\Omega) + \varepsilon$. Furthermore, there is some $\gamma_\varepsilon \in C_c^\infty(U_\varepsilon)$ with $0 \leq \gamma_\varepsilon \leq 1$ and $\gamma_\varepsilon \equiv 1$ on $\Omega$. Set $\psi_\varepsilon := \mathcal{F}^{-1}\gamma_\varepsilon \in \mathcal{S}\left(\mathbb{R}^d\right)$.

We first observe that the HAUSDORFF-YOUNG INEQUALITY (cf. [33, Theorem 8.30]) yields

$$\|\psi_\varepsilon\|_{L^{p'}} \leq \|\gamma_\varepsilon\|_{L^p} \leq \|\chi_{U_\varepsilon}\|_{L^p} = \left[\lambda(U_\varepsilon)\right]^{1/p} \leq \left[\lambda(\Omega)+\varepsilon\right]^{1/p} \quad (3.1.4)$$

for all $p \in [1,2]$, where $p' \in [2,\infty]$ is the conjugate exponent, i.e. $p' = \frac{p}{p-1}$ for $p \in (1,2]$ and $p' = \infty$ for $p = 1$.

Using the support assumption on $\widehat{f}$, we get $\widehat{f} = \widehat{f}\cdot\gamma_\varepsilon$ and hence

$$f = \mathcal{F}^{-1}\widehat{f} = \mathcal{F}^{-1}\left[\widehat{f}\cdot\gamma_\varepsilon\right] = f * \psi_\varepsilon.$$

For the proof of equation (3.1.2), we first note that it suffices to establish the case $q = \infty$. In the remaining case $p \leq q < \infty$, we get

$$\begin{aligned}\|f\|_{L^q}^q &= \int_{\mathbb{R}^d} |f|^p \cdot |f|^{q-p}\,\mathrm{d}x \\ &\leq \|f\|_{L^\infty}^{q-p} \cdot \int_{\mathbb{R}^d} |f|^p\,\mathrm{d}x \\ &\leq \left[\lambda(\Omega)\right]^{\frac{q-p}{p}} \cdot \|f\|_{L^p}^{q-p} \cdot \|f\|_{L^p}^p,\end{aligned}$$

where the last step made use of the case $q = \infty$. Taking $q$-th roots completes the proof of equation (3.1.2).

For the case $q = \infty$, we distinguish two cases:

CASE 1: If $p \in [1,2]$, then Hölder's inequality implies

$$\begin{aligned} |f(x)| &= |(f * \psi_\varepsilon)(x)| \\ &= \left| \int_{\mathbb{R}^d} f(y) \cdot \psi_\varepsilon(x-y) \, \mathrm{d}y \right| \\ &\leq \|f\|_{L^p} \cdot \|\psi_\varepsilon(x-\cdot)\|_{L^{p'}} \\ &\leq [\lambda(\Omega) + \varepsilon]^{1/p} \cdot \|f\|_{L^p} \end{aligned} \tag{3.1.5}$$

for all $x \in \mathbb{R}^d$. Here, equation (3.1.4) was used in the last step. Since $\varepsilon > 0$ was arbitrary, this is nothing but equation (3.1.2) for $q = \infty$ and $p \in [1,2]$.

CASE 2: $p \in (0,1)$. Here, we cannot use Hölder's inequality as in the last case. Instead, we apply a "flop"[62, Chapter 11]. In the present context, this means that we will derive an estimate of the form $Q \leq C \cdot Q^r$ for some exponent $r \in (0,1)$ with $Q := \sup_{y \in \mathbb{R}^d} |f(y)|$. By rearranging, this yields $Q \leq C^{\frac{1}{1-r}}$, which will imply the desired estimate. Here it is important to note that $Q$ is indeed a finite quantity because of $f \in \mathcal{S}(\mathbb{R}^d)$.

For the execution of this plan, note that

$$\begin{aligned} |f(x)| &= \left| \int_{\mathbb{R}^d} f(y) \cdot \psi_\varepsilon(x-y) \, \mathrm{d}y \right| \\ &\leq \int_{\mathbb{R}^d} |f(y)|^p \cdot |f(y)|^{1-p} \cdot |\psi_\varepsilon(x-y)| \, \mathrm{d}y \\ &\leq \|\psi_\varepsilon\|_{L^\infty} \cdot \|f\|_{L^p}^p \cdot \sup_{y \in \mathbb{R}^d} |f(y)|^{1-p} \\ &\leq (\lambda(\Omega) + \varepsilon) \cdot \|f\|_{L^p}^p \cdot Q^{1-p}. \end{aligned}$$

Here, we again used equation (3.1.4) (with $p = 1$) in the last step. Furthermore, it is important to observe that we can indeed interchange the supremum and the power $1-p$ because of $1 - p > 0$.

Taking the supremum over $x \in \mathbb{R}^d$ on the left-hand side yields

$$\sup_{x \in \mathbb{R}^d} |f(x)| \leq (\lambda(\Omega) + \varepsilon) \cdot \|f\|_{L^p}^p \cdot \left( \sup_{x \in \mathbb{R}^d} |f(x)| \right)^{1-p}.$$

Rearranging, taking $p$-th roots and letting $\varepsilon \downarrow 0$ completes the proof of equation (3.1.2) for $p \in (0,1)$ and $q = \infty$.

It remains to establish estimate (3.1.3). To this end, we simply note that Hölder's inequality implies as in equation (3.1.5) that

$$|f(x)| \leq \|f\|_{L^p} \cdot \|\psi_1(x-\cdot)\|_{L^{p'}} \leq K \cdot \|f\|_{L^p}$$

with $K := \max\{1, \|\psi_1\|_{L^1}, \|\psi_1\|_{L^\infty}\}$. The last estimate is due to the estimate $\|\psi\|_{L^r} \leq \max\{\|\psi\|_{L^1}, \|\psi\|_{L^\infty}\}$ for all $r \in [1,\infty]$, cf. [33, Proposition 6.10]. This establishes the claim for $q = \infty$.

In general, for $1 \leq p \leq q < \infty$, we have

$$\|f\|_{L^q}^q = \int_{\mathbb{R}^d} |f|^p \cdot |f|^{q-p} \, \mathrm{d}x \leq \|f\|_{L^\infty}^{q-p} \cdot \|f\|_{L^p}^p \leq K^{q-p} \cdot \|f\|_{L^p}^{p+q-p}.$$

Rearranging yields $\|f\|_{L^q} \leq K^{1-\frac{p}{q}} \cdot \|f\|_{L^p}$. Because of $K \geq 1$ and $0 \leq 1 - \frac{p}{q} \leq 1$, we have $K^{1-\frac{p}{q}} \leq K$, which completes the proof. □

With these preparations, we can finally establish the convolution relation for bandlimited functions in $L^p(\mathbb{R}^d)$ for $p \in (0,1)$. The result is essentially taken from Triebel [66, Proposition 1.5.1]; but Triebel does not state the form of the constant $[\lambda(Q-\Omega)]^{\frac{1}{p}-1}$ explicitly in the statement of the theorem. Furthermore, the estimate (3.1.6) which will be crucial for us later (see e.g. the proof of Lemma 4.5.6) is only used in the proof but not mentioned explicitly in the statement of [66, Proposition 1.5.1].

**Theorem 3.1.4.** *(cf. [66, Proposition 1.5.1]) Let $\Omega, Q \subset \mathbb{R}^d$ be compact and $p \in (0,1]$. Furthermore, let $\psi \in L^1(\mathbb{R}^d)$ with $\operatorname{supp}\psi \subset Q$ and such that $\mathcal{F}^{-1}\psi \in L^p(\mathbb{R}^d)$.*

*For each $f \in L^p(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ with Fourier support $\operatorname{supp}\widehat{f} \subset \Omega$, we have*

$$\mathcal{F}^{-1}(\psi \cdot \widehat{f}) = (\mathcal{F}^{-1}\psi) * f \in L^p(\mathbb{R}^d)$$

*with norm estimate*

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\psi \cdot \widehat{f})\right\|_{L^p} &= \|(\mathcal{F}^{-1}\psi) * f\|_{L^p} \\ &\leq \||\mathcal{F}^{-1}\psi| * |f|\|_{L^p} \\ &\leq [\lambda(Q-\Omega)]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\psi\|_{L^p} \cdot \|f\|_{L^p},\end{aligned}$$

*where*

$$Q - \Omega = \{q - \omega \mid q \in Q \text{ and } \omega \in \Omega\}$$

*is the* DIFFERENCE SET *of $Q$ and $\Omega$, which is compact and hence measurable.*

*Finally, we have the pointwise estimate*

$$\begin{aligned}&|[(\mathcal{F}^{-1}\psi) * f](x)| \\ &\leq [|\mathcal{F}^{-1}\psi| * |f|](x) \\ &\leq [\lambda(Q-\Omega)]^{\frac{1}{p}-1} \left[\int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y) \cdot f(x-y)|^p \, \mathrm{d}y\right]^{\frac{1}{p}} \qquad (3.1.6)\end{aligned}$$

*for all $x \in \mathbb{R}^d$.*

*Remark.* We observe that $\psi \cdot \widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$ is well-defined, even though $\psi$ might not be a smooth function. This is because Corollary 3.1.3 together with $p \leq 1$ yields $f \in L^1(\mathbb{R}^d)$ and thus $\widehat{f} \in C_0(\mathbb{R}^d)$. Hence, $\psi \cdot \widehat{f} \in L^1(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ because of $\psi \in L^1(\mathbb{R}^d)$.

*Proof.* The Riemann-Lebesgue lemma yields $\mathcal{F}^{-1}\psi \in C_0(\mathbb{R}^d)$ because of $\psi \in L^1(\mathbb{R}^d)$. Thus, $\mathcal{F}^{-1}\psi \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $q \in [p, \infty]$, cf. [33, Proposition 6.10]. For the same range of $q$, Corollary 3.1.3 yields $f \in L^q(\mathbb{R}^d)$.

Because of $p \in (0,1]$, we get $f, \mathcal{F}^{-1}\psi \in L^1(\mathbb{R}^d)$ and thus also $(\mathcal{F}^{-1}\psi) * f \in L^1(\mathbb{R}^d)$ with Fourier transform

$$\mathcal{F}[(\mathcal{F}^{-1}\psi) * f] = \psi \cdot \widehat{f}.$$

Further, $\psi \in L^1(\mathbb{R}^d)$ together with $\widehat{f} \in C_0(\mathbb{R}^d)$ yields $\psi \cdot \widehat{f} \in L^1(\mathbb{R}^d)$. By Fourier inversion, this implies

$$\mathcal{F}^{-1}(\psi \cdot \widehat{f}) = (\mathcal{F}^{-1}\psi) * f$$

as claimed.

We first observe that the pointwise estimate (3.1.6) implies the remaining claims. Indeed, equation (3.1.6), together with Fubini's theorem, yields

$$\begin{aligned}
&\|(\mathcal{F}^{-1}\psi) * f\|_{L^p}^p \\
&\leq \||\mathcal{F}^{-1}\psi| * |f|\|_{L^p}^p \\
&\leq [\lambda(Q-\Omega)]^{1-p} \cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y)|^p \cdot |f(x-y)|^p \, \mathrm{d}y \, \mathrm{d}x \\
&= [\lambda(Q-\Omega)]^{1-p} \cdot \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y)|^p \cdot \int_{\mathbb{R}^d} |f(x-y)|^p \, \mathrm{d}x \, \mathrm{d}y \\
&= [\lambda(Q-\Omega)]^{1-p} \cdot \|\mathcal{F}^{-1}\psi\|_{L^p}^p \cdot \|f\|_{L^p}^p,
\end{aligned}$$

so that taking $p$-th roots completes the proof.

In order to prove equation (3.1.6), fix $x \in \mathbb{R}^d$. Using $f \in L^\infty(\mathbb{R}^d)$ and $\mathcal{F}^{-1}\psi \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we see

$$F_x := \left(\mathcal{F}^{-1}\psi\right) \cdot L_x f^\vee \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d),$$

*Here, we recall the notations $f^\vee(y) = f(-y)$, $L_x f(y) = f(y-x)$ and $M_\xi f(x) = e^{2\pi i \langle x, \xi \rangle} f(x)$.*

so that the Fourier transform $\mathcal{F}F_x$ is well-defined with

$$\begin{aligned}
(\mathcal{F}F_x)(\xi) &= \int_{\mathbb{R}^d} \widehat{\psi}(-y) \cdot f(x-y) \cdot e^{-2\pi i \langle y, \xi \rangle} \, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} \widehat{\psi}(z) \cdot f(x+z) \cdot e^{2\pi i \langle z, \xi \rangle} \, \mathrm{d}z \\
&= \int_{\mathbb{R}^d} \psi(y) \cdot \mathcal{F}\left[z \mapsto f(x+z) \cdot e^{2\pi i \langle z, \xi \rangle}\right](y) \, \mathrm{d}y,
\end{aligned}$$

where the last step used $\psi, f \in L^1(\mathbb{R}^d)$ and that $\int \widehat{f} g = \int f \widehat{g}$ holds for $f, g \in L^1(\mathbb{R}^d)$, cf. [33, Lemma 8.25].

By elementary properties of the Fourier transform,

$$\mathcal{F}\left[z \mapsto f(x+z) \cdot e^{2\pi i \langle z,\xi\rangle}\right](y) = \mathcal{F}\left[M_\xi L_{-x} f\right](y) = \left(L_\xi M_x \widehat{f}\right)(y),$$

which yields

$$(\mathcal{F}F_x)(\xi) = \int_{\mathbb{R}^d} \psi(y) \cdot (M_x \widehat{f})(y-\xi)\, \mathrm{d}y = \left[\psi * (M_x \widehat{f})^\vee\right](\xi)$$

and hence

$$\operatorname{supp}(\mathcal{F}F_x) \subset \operatorname{supp}(\psi) + \operatorname{supp}\left([M_x \widehat{f}]^\vee\right) \subset Q + (-\Omega) = Q - \Omega.$$

Using Corollary 3.1.3 and $0 < p \leq 1$, we finally arrive at

$$\begin{aligned} |[(\mathcal{F}^{-1}\psi) * f](x)| &\leq (|\mathcal{F}^{-1}\psi| * |f|)(x) \\ &= \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}\psi)(y) \cdot f(x-y)|\, \mathrm{d}y \\ &= \|F_x\|_{L^1} \leq [\lambda(Q-\Omega)]^{\frac{1}{p}-\frac{1}{1}} \cdot \|F_x\|_{L^p}, \end{aligned}$$

which is nothing but the pointwise estimate (3.1.6). □

## 3.2 DIFFERENT TYPES OF COVERINGS AND THEIR PROPERTIES

In this section, we recall the definitions of different types of coverings and introduce a new class of coverings, the (TIGHT) SEMI-STRUCTURED ADMISSIBLE COVERINGS. This will be the main class of coverings that we will consider in the remainder of the thesis.

Before we introduce this new notion, we first recall the more basic concept of ADMISSIBILITY of a covering. This concept is based on the notion of neighbors of a set in a covering, which we introduce first.

**Definition 3.2.1.** (cf. [27, Definition 2.3])
Let $X \neq \varnothing$ be a set and assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is a family of subsets of $X$. For a subset $J \subset I$, we define the (INDEX)-CLUSTER OF $J$ (also called the SET OF NEIGHBORS OF $J$) as

$$J^* := \left\{ i \in I \,\middle|\, \exists j \in J : Q_i \cap Q_j \neq \varnothing \right\}.$$

Inductively, we set $J^{0*} := J$ and $J^{(n+1)*} := (J^{n*})^*$ for $n \in \mathbb{N}_0$. We also set $i^{k*} := \{i\}^{k*}$ for $i \in I$ and $k \in \mathbb{N}_0$.

Furthermore, for any subset $J \subset I$, we define

$$Q_J := \bigcup_{j\in J} Q_j$$

and we also introduce the shortcuts $Q_i^{k*} := Q_{i^{k*}}$ and $Q_i^* := Q_{i^*}$ for $i \in I$ and $k \in \mathbb{N}_0$.

If $(\varphi_i)_{i\in I}$ is a family of functions with values in some common vector space $Y$, we define

$$\varphi_J := \sum_{i\in J} \varphi_i$$

for every finite set $J \subset I$. For brevity, we also define $\varphi_i^{k*} := \varphi_{i^{k*}}$ and $\varphi_i^* := \varphi_{i^*}$ for $i \in I$ and $k \in \mathbb{N}_0$.

*Remark* 3.2.2. If we are considering two different families $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ with (possibly) nonempty intersection $I \cap J \neq \varnothing$, we will use notation similar to $i^{*\mathcal{Q}}$ and $i^{*\mathcal{P}}$ for $i \in I \cap J$ to indicate the family which is used to form the respective cluster.

By induction on $n \in \mathbb{N}_0$, it is easy to see for $i, j \in I$ that $i \in j^{n*}$ is equivalent to $j \in i^{n*}$ and that this holds if and only if there is a $\mathcal{Q}$-CHAIN (cf. [27, Definition 2.3]) of length $n$ from $j$ to $i$.

Here, a finite sequence $(i_\ell)_{\ell=0,\dots,n}$ in $I$ is called a $\mathcal{Q}$-chain of length $n$ from $i_0$ to $i_n$ if $Q_{i_{\ell-1}} \cap Q_{i_\ell} \neq \varnothing$ holds for all $\ell \in \underline{n}$.

For further properties of admissible coverings and of the cluster-sets $J^*$, we refer to [27, Lemma 2.1]. ◀

Now, we can define the notion of an admissible covering and of moderate weights.

**Definition 3.2.3.** (cf. [27, Definitions 2.1, 3.1 and 3.2] and [5, Definition 6])

Let $X \neq \varnothing$ be a set and assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is a family of *nonempty* subsets of $X$. We say that $\mathcal{Q}$ is an ADMISSIBLE COVERING OF $X$, if

1. $\mathcal{Q}$ is a covering of $X$, i.e. $X = \bigcup_{i\in I} Q_i$ and
2. The constant $N_{\mathcal{Q}} := \sup_{i\in I} |i^*|$ is finite.

Let $u : I \to (0,\infty), i \mapsto u_i := u(i)$ be a function . We say that $u$ is a $\mathcal{Q}$-MODERATE (DISCRETE) WEIGHT if the constant

$$C_{u,\mathcal{Q}} := \sup_{i\in I} \sup_{j\in i^*} \frac{u_i}{u_j}$$

is finite.

Similarly, a function $u : X \to (0,\infty)$ is called a $\mathcal{Q}$-MODERATE (CONTINUOUS) WEIGHT if the constant

$$C_{u,\mathcal{Q}} := \sup_{i\in I} \sup_{x,y\in Q_i} \frac{u(x)}{u(y)}$$

is finite.

*Remark.* It is important to observe that the admissibility condition above is dependent on the way the covering is indexed using the

set $I$. For example, if one set is "repeated infinitely often", i.e. if $\varnothing \neq Q = Q_i$ holds for all $i \in I_0 \subset I$, where $I_0$ is infinite, then this implies $i^* \supset I_0$ for all $i \in I_0$, so that $\mathcal{Q}$ is not admissible.

Furthermore, it is worth noting that every admissible covering is OF FINITE HEIGHT. This means that the cardinality $|\{i \in I \,|\, x \in Q_i\}|$ is uniformly bounded with respect to $x \in X$. More precisely, we have $|\{i \in I \,|\, x \in Q_i\}| \leq N_{\mathcal{Q}}$ for all $x \in X$. Equivalently, the function $\sum_{i \in I} \chi_{Q_i}$ is bounded by $N_{\mathcal{Q}}$.

To be able to express the connection between discrete and continuous $\mathcal{Q}$-moderate weights, we introduce the following definition.

**Definition 3.2.4.** Let $X \neq \varnothing$ be a set, assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is a family of *nonempty* subsets of $X$ and that $u : X \to (0, \infty)$ is a function.

A $\mathcal{Q}$-DISCRETIZATION OF $u$ is a (discrete) weight $u_d : I \to (0, \infty)$ such that for each $i \in I$, there is some $x_i \in Q_i$ with $u_d(i) = u(x_i)$.

Finally, if $u$ is $\mathcal{Q}$-moderate and $p \in (0, \infty]$, we set $\ell^p_u(I) := \ell^p_{u_d}(I)$ for an arbitrary $\mathcal{Q}$-discretization $u_d$ of $u$.

*Remark.* The next lemma will show that any two $\mathcal{Q}$-discretizations $u_d, \widetilde{u_d}$ of a $\mathcal{Q}$-moderate weight $u : X \to (0, \infty)$ are equivalent. In particular, $\ell^p_{u_d}(I) = \ell^p_{\widetilde{u_d}}(I)$ with equivalent (quasi)-norms. Thus, the notation $\ell^p_u(I)$ is well-defined, as long as the covering $\mathcal{Q}$ is clear from the context.

But for two different coverings $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J}$ which have the same index set $I = J$, the two spaces $\ell^p_{u_d}(I)$ and $\ell^p_{\widetilde{u_d}}(I)$ with a $\mathcal{Q}$-discretization $u_d$ of $u$ and a $\mathcal{P}$-discretization $\widetilde{u_d}$ of $u$ can be *very* different.

Thus, we will only use the notation $\ell^p_u(I)$ or $\ell^p_u$ when the covering $\mathcal{Q}$ is clear from the context. In particular, this will be the case when we form the decomposition space $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_u)$.

Finally, we mention that in *most* cases, we will use the notation $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_u)$ only when $u$ is a discrete weight on the index set $I$ of the covering $\mathcal{Q} = (Q_i)_{i \in I}$. The case that $u : X \to (0, \infty)$ is a continuous weight will be the exception.

As promised in the above remark, we now show that any two $\mathcal{Q}$-discretizations of a $\mathcal{Q}$-moderate weight are equivalent. Furthermore, we show that the $\mathcal{Q}$-discretization of a $\mathcal{Q}$-moderate weight is again $\mathcal{Q}$-moderate.

**Lemma 3.2.5.** *(cf. [27, Lemma 3.3]) Let $X \neq \varnothing$ be a set and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is a family of* nonempty *subsets of $X$. Finally, assume that $u : X \to (0, \infty)$ is $\mathcal{Q}$-moderate.*

*Then any $\mathcal{Q}$-discretization $u_d : I \to (0, \infty)$ of $u$ is $\mathcal{Q}$-moderate with $C_{u_d, \mathcal{Q}} \leq C^2_{u, \mathcal{Q}}$.*

*Furthermore, any two $\mathcal{Q}$-discretizations $u_d, \widetilde{u_d} : I \to (0, \infty)$ of $u$ are equivalent in the sense that*

$$C^{-1}_{u, \mathcal{Q}} \leq \frac{u_d}{\widetilde{u_d}} \leq C_{u, \mathcal{Q}}.$$

*Proof.* Let $u_d : I \to (0,\infty)$ be a $\mathcal{Q}$-discretization of $u$ and let $i \in I$ and $j \in i^*$. Hence, there is some $x \in Q_i \cap Q_j$. By definition of $u_d$, there are $x_i \in Q_i$ and $x_j \in Q_j$ with $u_d(i) = u(x_i)$ and $u_d(j) = u(x_j)$. Using the $\mathcal{Q}$-moderateness of $u$, we get

$$u_d(i) = u(x_i) \leq C_{u,\mathcal{Q}} \cdot u(x) \leq C_{u,\mathcal{Q}}^2 \cdot u(x_j) = C_{u,\mathcal{Q}}^2 \cdot u_d(j).$$

By symmetry, this shows that $u_d$ is $\mathcal{Q}$-moderate.

Now, let $u_d, \widetilde{u_d} : I \to (0,\infty)$ be two $\mathcal{Q}$-discretizations of $u$. For $i \in I$, this yields $x_i, y_i \in Q_i$ with $u_d(i) = u(x_i)$ and $\widetilde{u_d}(i) = u(y_i)$. Using the $\mathcal{Q}$-moderateness of $u$, we see

$$u_d(i) = u(x_i) \leq C_{u,\mathcal{Q}} \cdot u(y_i) = C_{u,\mathcal{Q}} \cdot \widetilde{u_d}(i).$$

By symmetry, this proves the desired equivalence. □

For the case $p \geq 1$, we can now define the class of partitions of unity which will turn out to be suitable for the definition of decomposition spaces. For the case $p < 1$, we will first have to place more restrictive assumptions on the covering $\mathcal{Q}$, cf. Definitions 3.2.7 and 3.2.8 below.

**Definition 3.2.6.** (cf. [27, Definition 2.2])
Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (Q_i)_{i \in i}$ be an admissible covering of $\mathcal{O}$. A family $\Phi = (\varphi_i)_{i \in I}$ of functions on $\mathcal{O}$ is called an $L^p$-BOUNDED ADMISSIBLE PARTITION OF UNITY ($L^p$-BAPU) for $\mathcal{Q}$ for all $1 \leq p \leq \infty$, if

1. $\varphi_i \in C_c^\infty(\mathcal{O})$ for all $i \in I$,
2. $\sum_{i \in I} \varphi_i(x) = 1$ for all $x \in \mathcal{O}$,
3. $\varphi_i(x) = 0$ for all $x \in \mathbb{R}^d \setminus Q_i$ for all $i \in I$, and
4. the constant $C_{\mathcal{Q},\Phi,p} := \sup_{i \in I} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^1}$ is finite.

*The series converges, since for each $x \in \mathcal{O}$, at most $N_{\mathcal{Q}}$ terms are nonzero.*

*Of course, the constant $C_{\mathcal{Q},\Phi,p}$ does not actually depend on $\mathcal{Q}$ and $p$; but below, we will introduce the concept of an $L^p$-BAPU also for $p \in (0,1)$ and in this case, the constant will depend on $\mathcal{Q}$ and $p$. For consistency, we write $C_{\mathcal{Q},\Phi,p}$ also for $p \in [1,\infty]$.*

We say that an admissible covering $\mathcal{Q}$ of $\mathcal{O}$ is an $L^p$-DECOMPOSITION COVERING of $\mathcal{O}$ for all $1 \leq p \leq \infty$, if there is an $L^p$-BAPU $\Phi$ for $\mathcal{Q}$.

*Remark.* The term "$L^p$-bounded" used above does *not* refer to the fact that the $L^p$-norm of the $\varphi_i$ is uniformly bounded, but to the fact that $(\varphi_i)_{i \in I}$ forms a uniformly bounded family of $L^p$ Fourier-multipliers, as a consequence of Young's inequality $L^1 * L^p \hookrightarrow L^p$.

We will now introduce a more restrictive class of coverings, the STRUCTURED ADMISSIBLE COVERINGS. These were first introduced (for the case $\mathcal{O} = \mathbb{R}^d$) by Borup and Nielsen in [5, Definition 7] and then slightly generalized by Führ and Voigtlaender in [41, Definition 13] to the form given here.

**Definition 3.2.7.** Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $I \neq \varnothing$ be an index set. Let $Q \subset \mathbb{R}^d$ be precompact and open. Finally, assume

that $(T_i)_{i\in I}$ and $(b_i)_{i\in I}$ are families of invertible linear transformations $T_i \in \mathrm{GL}(\mathbb{R}^d)$ and of translations $b_i \in \mathbb{R}^d$, respectively.

We say that the family $\mathcal{Q} = (Q_i)_{i\in I} := (T_iQ + b_i)_{i\in I}$ is a STRUCTURED (ADMISSIBLE) COVERING (of $\mathcal{O}$), if

1. There is an open, precompact set $P \subset \mathbb{R}^d$ with $\overline{P} \subset Q$ such that $\mathcal{Q}$ and $\mathcal{P} := (T_iP + b_i)_{i\in I}$ are both admissible coverings of $\mathcal{O}$ and

2. the constant
$$C_{\mathcal{Q}} := \sup_{i\in I} \sup_{j\in i^*} \left\| T_i^{-1} T_j \right\|$$
is finite.

We will see below (cf. Theorem 3.2.17) that structured admissible coverings always admit an $L^p$-BAPU for every $1 \leq p \leq \infty$ (and even for $p \in (0,1)$).

For technical reasons, it will frequently be useful to switch from a given covering $\mathcal{Q}$ to the "clustered" covering $\mathcal{Q}^{k*} := \left(Q_i^{k*}\right)_{i\in I}$. The disadvantage of the notion of a structured admissible covering is that $\mathcal{Q}^{k*}$ will in general *not* be structured admissible, even if $\mathcal{Q}$ is.

We thus propose the following more general class of coverings that will turn out to be closed under clustering (cf. Lemma 3.2.18). The difference is that we do not assume that all sets $Q_i$ are affine images of a *fixed* set $Q$, but we allow the set $Q = Q_i'$ to be different for each $i \in I$. Of course, we have to impose some form of control over the size of the $Q_i'$, so that there is a relation between the set $Q_i = T_iQ_i' + b_i$ and $T_i, b_i$. Hence, we require $Q_i' \subset B_R(0)$ for all $i \in I$. Note that we allow for the possibility that the $Q_i'$ get arbitrarily small; the notion which excludes this behaviour is TIGHTNESS of the covering $\mathcal{Q}$.

**Definition 3.2.8.** Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be an open set. We say that a family $\mathcal{Q} = (Q_i)_{i\in I}$ of subsets of $\mathcal{O}$ is a SEMI-STRUCTURED (ADMISSIBLE) COVERING of $\mathcal{O}$, if

1. $\mathcal{Q}$ is an admissible covering of $\mathcal{O}$ and

2. there are families $(T_i)_{i\in I}$, $(b_i)_{i\in I}$ and $(Q_i')_{i\in I}$ of linear invertible transformations $T_i \in \mathrm{GL}(\mathbb{R}^d)$, translations $b_i \in \mathbb{R}^d$ and of subsets $Q_i' \subset \mathbb{R}^d$ with the following properties:

   a) $Q_i = T_iQ_i' + b_i$ for all $i \in I$,

   b) the constant $R_{\mathcal{Q}} := \sup_{i\in I} \sup_{\xi\in Q_i'} |\xi|$ is finite,

   c) the constant $C_{\mathcal{Q}} := \sup_{i\in I} \sup_{j\in i^*} \|T_i^{-1}T_j\|$ is finite.

In this case, we call the collection of families $(T_i)_{i\in I}$, $(b_i)_{i\in I}$ and $(Q_i')_{i\in I}$ an ASSOCIATED COLLECTION OF STANDARDIZATIONS for $\mathcal{Q}$.

We say that such a collection of standardizations is TIGHT if there is some $\varepsilon > 0$ and for each $i \in I$ some $c_i \in \mathbb{R}^d$ satisfying $B_\varepsilon(c_i) \subset Q_i'$.

If this case, we say that $\mathcal{Q}$ is a TIGHT SEMI-STRUCTURED COVERING of $\mathcal{O}$ and we call the collection of families $(T_i)_{i\in I}, (b_i)_{i\in I}$ and $(Q'_i)_{i\in I}$ a TIGHT COLLECTION OF STANDARDIZATIONS for $\mathcal{Q}$.

Furthermore, we set

$$\varepsilon_{\mathcal{Q}} := \sup\left\{\varepsilon > 0 \,\middle|\, \forall i \in I \exists c_i \in \mathbb{R}^d : \, B_\varepsilon(c_i) \subset Q'_i\right\}.$$

*Remark* 3.2.9. We remark that strictly speaking the constants $R_{\mathcal{Q}}$, $C_{\mathcal{Q}}$ and $\varepsilon_{\mathcal{Q}}$ depend on the choice of the collection $(T_i)_{i\in I}, (b_i)_{i\in I}$ and $(Q'_i)_{i\in I}$ of standardizations for $\mathcal{Q}$. As above, we will usually supress this dependence. Furthermore, we emphasize that the phrase "let $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q'_i + b_i)_{i\in I}$ be a [tight] semi-structured covering of $\mathcal{O}$" will always implicitly entail that $(T_i)_{i\in I}, (b_i)_{i\in I}$ and $(Q'_i)_{i\in I}$ are a [tight] collection of standardizations for $\mathcal{Q}$.

Using Hadamard's inequality $|\det(A)| \leq \prod_{i=1}^d \|A_i\|$ for each matrix $A \in \mathbb{R}^{d\times d}$ with columns $A_1, \dots, A_d$, we see that the additional constant

$$D_{\mathcal{Q}} := \sup_{i\in I} \sup_{j\in i^*} \left|\det\left(T_i^{-1} T_j\right)\right| \leq C_{\mathcal{Q}}^d \tag{3.2.1}$$

is finite.

We also remark that in the remainder of the thesis, we will *not* use the fact that the constant $N_{\mathcal{Q}}$ is chosen *minimal* with the property $|i^*| \leq N_{\mathcal{Q}}$ for all $i \in I$, or that $\varepsilon_{\mathcal{Q}}$ is chosen maximal with the property that there are $(c_i)_{i\in I}$ in $\mathbb{R}^d$ with $B_\varepsilon(c_i) \subset Q'_i$. Analogous statements hold for $C_{\mathcal{Q}}, D_{\mathcal{Q}}$ and $R_{\mathcal{Q}}$. The main reason for this is the following: If a theorem yields – for a given covering $\mathcal{Q}$ – a constant $C = C(N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, \dots) > 0$ with a certain property (depending on $\mathcal{Q}$), then the constant $C_0 = C(N, \varepsilon, \dots)$ will statisfy the this particular property for any covering $\mathcal{Q}$ with $N_{\mathcal{Q}} \leq N$ and $\varepsilon_{\mathcal{Q}} \geq \varepsilon$ (and $R_{\mathcal{Q}} \leq R$, $\dots$).

Finally, we observe that the supremum in the definition of $\varepsilon_{\mathcal{Q}}$ is attained. To see this, we note that finiteness of $R_{\mathcal{Q}}$ implies that each set $Q'_i$ is bounded, so that $\overline{Q'_i} \subset \mathbb{R}^d$ is compact.

Now, for each $n \in \mathbb{N}$, $i \in I$ and $\varepsilon_n := \left(1 - \frac{1}{2n}\right)\varepsilon_{\mathcal{Q}}$, there is some $c_n \in \mathbb{R}^d$ with $c_n \in B_{\varepsilon_n}(c_n) \subset Q'_i \subset \overline{Q'_i}$. Considering a subsequence, we get $c_{n_k} \to c \in \overline{Q'_i}$ by compactness of $\overline{Q'_i}$.

Now, let $x \in B_{\varepsilon_{\mathcal{Q}}}(c)$ and let $\delta := \frac{\varepsilon_{\mathcal{Q}} - |x-c|}{2} > 0$. We have

$$|x - c_{n_k}| \xrightarrow[k\to\infty]{} |x - c| < \delta + |x - c| < \varepsilon_{\mathcal{Q}}$$

and thus $|x - c_{n_k}| < \delta + |x - c| < \varepsilon_{n_k}$ for sufficiently large $k \in \mathbb{N}$. This implies $x \in B_{\varepsilon_{n_k}}(c_{n_k}) \subset Q'_i$. We have thus shown $B_{\varepsilon_{\mathcal{Q}}}(c) \subset Q'_i$. ◀

Now that we have clarified the nature of the coverings we will consider for the case $p \in (0,1)$, we can introduce suitable conditions on our partitions of unity. We will show below, after formally introducing the notion of (Fourier side) decomposition spaces, that this

notion of BAPUs indeed leads to a well-defined notion of decomposition spaces.

*We will see in Corollary 5.1.4 that every $L^p$-BAPU for $\mathcal{Q}$ is automatically an $L^q$-BAPU for $\mathcal{Q}$ for all $q \in [p, \infty]$.*

**Definition 3.2.10.** (cf. [5, Definition 2])
Let $0 < p < 1$, let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ is a semi-structured covering of $\mathcal{O}$. We say that a family $\Phi = (\varphi_i)_{i\in I}$ is an $L^p$-BOUNDED ADMISSIBLE PARTITION OF UNITY ($L^p$-BAPU) for $\mathcal{Q}$, if

1. $\varphi_i \in C_c^\infty(\mathcal{O})$ for all $i \in I$,
2. $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in \mathcal{O}$,
3. $\varphi_i(x) = 0$ for all $x \in \mathbb{R}^d \setminus Q_i$ for all $i \in I$, and
4. the constant

$$C_{\mathcal{Q},\Phi,p} := \sup_{i\in I} |\det(T_i)|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\varphi_i\|_{L^p}$$

is finite.

*The series $\sum_{i\in I} \varphi_i(x)$ converges, since for each $x \in \mathcal{O}$, at most $N_{\mathcal{Q}}$ terms are nonzero.*

We say that $\mathcal{Q}$ is an $L^p$-DECOMPOSITION COVERING of $\mathcal{O}$ if there is an $L^p$-BAPU $\Phi$ for $\mathcal{Q}$.

*We emphasize that an $L^p$-decomposition covering for $p \in (0,1)$ is by definition semi-structured.*

An immediate consequence of the definition is that any $L^p$-BAPU is locally finite.

**Lemma 3.2.11.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$ with $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for some $p \in (0, \infty]$. Then $\Phi$ is locally finite, in the sense that the family of supports $(\operatorname{supp}\varphi_i)_{i\in I}$ is locally finite in $\mathcal{O}$. Furthermore, the family $\mathcal{Q}$ is also locally finite in $\mathcal{O}$.*

*While $\Phi$, $\mathcal{Q}$ are locally finite in $\mathcal{O}$, it is not true in general that $\Phi$ or $\mathcal{Q}$ are locally finite in $\mathbb{R}^d$.*

*Proof.* Let $x \in \mathcal{O}$ be arbitrary. Then there is some $i \in I$ with $\varphi_i(x) \neq 0$, since $\sum_{i\in I} \varphi_i \equiv 1$ on $\mathcal{O}$.

Hence, $U_x := \varphi_i^{-1}(\mathbb{C}^*) \subset Q_i$ is an open neighborhood of $x$. Let $I_x := i^*$. For any index $j \in I$ with $U_x \cap \operatorname{supp}\varphi_j \neq \varnothing$, we already have $U_x \cap \varphi_j^{-1}(\mathbb{C}^*) \neq \varnothing$, since $U_x$ is open. Indeed, if $U_x \cap \varphi_j^{-1}(\mathbb{C}^*) = \varnothing$ would hold, this would yield $\varphi_j^{-1}(\mathbb{C}^*) \subset U_x^c$ and thus

$$\operatorname{supp}\varphi_j = \overline{\varphi_j^{-1}(\mathbb{C}^*)} \subset U_x^c,$$

in contradiction to $U_x \cap \operatorname{supp}\varphi_j \neq \varnothing$.

Hence, any index $j \in I$ with $U_x \cap \operatorname{supp}\varphi_j \neq \varnothing$ already fulfills $\varnothing \neq U_x \cap \varphi_j^{-1}(\mathbb{C}^*) \subset Q_i \cap Q_j$ and thus $j \in i^* = I_x$. Since $I_x = i^*$ is finite by admissibility of $\mathcal{Q}$ and since $U_x$ is an open neighborhood of $x$, we see that $(\operatorname{supp}\varphi_i)_{i\in I}$ is a locally finite family in $\mathcal{O}$.

Finally, if $U_x \cap Q_j \neq \varnothing$ holds for some index $j \in I$, this implies $\varnothing \neq U_x \cap Q_j \subset Q_i \cap Q_j$ and hence $j \in i^* = I_x$. As above, we conclude that $\mathcal{Q}$ is locally finite in $\mathcal{O}$. □

The definition of a semi-structured admissible covering does not impose any restrictions on the topological properties of the family $(Q_i)_{i\in I}$, except that each $Q_i \subset \mathbb{R}^d$ is precompact. But if we assume that $\mathcal{Q}$ is an $L^p$-decomposition covering, then each set $\overline{Q_i}$ is automatically a compact subset of $\mathcal{O}$. Furthermore, one can slightly shrink the covering $\mathcal{Q}$ to obtain a covering consisting of open sets.

**Lemma 3.2.12.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ be an $L^p$-decomposition covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ for some $p \in (0,\infty]$. Then the following are true:*

1. *If $(\varphi_i)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{Q}$, then*

$$\varphi_M \equiv 1 \quad \text{on } Q_i \text{ for each } i \in I \text{ with } i^* \subset M.$$

*We remark that $\varphi_M = \sum_{i\in M} \varphi_i$ is well-defined even if $M$ is an infinite set, since at every point at most $N_{\mathcal{Q}}$ summands do not vanish.*

2. *Each set $\overline{Q_i}$ is a compact subset of $\mathcal{O}$,*

3. *If we set*

$$I' := \{i \in I \mid Q_i^\circ \neq \varnothing\},$$

*where $Q_i^\circ$ denotes the topological interior of $Q_i$, then*

$$\mathcal{Q}^\circ := (Q_i^\circ)_{i\in I'} = \left(T_i \left(Q_i'\right)^\circ + b_i\right)_{i\in I}$$

*is also an $L^p$-decomposition covering of $\mathcal{O}$ with*

$$L_{\mathcal{Q}^\circ} \leq L_{\mathcal{Q}} \text{ for all } L \in \{R, N, C, D\}.$$

4. *If $\mathcal{Q}$ is tight, then $I = I'$ and $\mathcal{Q}^\circ$ is also tight with $\varepsilon_{\mathcal{Q}^\circ} = \varepsilon_{\mathcal{Q}}$.*

5. *Every $L^p$-BAPU $(\varphi_i)_{i\in I}$ for $\mathcal{Q}$ yields an $L^p$-BAPU $(\varphi_i)_{i\in I'}$ for $\mathcal{Q}^\circ$.*

*Proof.* By assumption on $\mathcal{Q}$, there is an $L^p$-BAPU $(\varphi_i)_{i\in I}$ for $\mathcal{Q}$. In particular, this means $\varphi_i \in C_c^\infty(\mathcal{O})$ with $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$ and $\sum_{i\in I} \varphi_i \equiv 1$ on $\mathcal{O}$.

1. Note that for $x \in Q_i$ and $j \in I$ with $\varphi_j(x) \neq 0$, we get $x \in Q_i \cap Q_j$ and thus $j \in i^* \subset M$. This implies

$$1 = \sum_{j\in J} \varphi_j(x) = \sum_{j\in M} \varphi_j(x) = \varphi_M(x)$$

for all $x \in Q_i$.

2. Since $i^*$ is finite for every $i \in I$, we have $\varphi_i^* = \sum_{j\in i^*} \varphi_j \in C_c^\infty(\mathcal{O})$. Furthermore, the preceding part shows $\varphi_i^* \equiv 1$ on $Q_i$, which entails $\overline{Q_i} \subset \operatorname{supp} \varphi_i^*$. Since $\operatorname{supp} \varphi_i^* \subset \mathcal{O}$ is compact, we are done.

3. We have $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$ and thus $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i$. Because $\varphi_i^{-1}(\mathbb{C}^*) \subset \mathbb{R}^d$ is open, this implies $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i^\circ$. In particular, we have shown $\varphi_i \equiv 0$ for $i \in I \setminus I'$. Using this, we get

$$1 \equiv \sum_{i\in I} \varphi_i \equiv \sum_{i\in I'} \varphi_i \text{ on } \mathcal{O}, \tag{3.2.2}$$

which yields

$$\mathcal{O} \subset \bigcup_{i \in I'} \varphi_i^{-1}(\mathbb{C}^*) \subset \bigcup_{i \in I'} Q_i^\circ \subset \mathcal{O},$$

so that $\mathcal{Q}^\circ$ is a covering of $\mathcal{O}$.

Let $P_i' := (Q_i')^\circ$ for $i \in I'$. This implies $Q_i^\circ = T_i P_i' + b_i$ for all $i \in I'$ because of $T_i \in \mathrm{GL}(\mathbb{R}^d)$. Since $P_i' \subset Q_i'$, the estimate $R_{\mathcal{Q}^\circ} \leq R_{\mathcal{Q}}$ is trivial. Moreover, $Q_i^\circ \cap Q_j^\circ \neq \varnothing$ implies $Q_i \cap Q_j \neq \varnothing$ and thus $\left\|T_i^{-1} T_j\right\| \leq C_{\mathcal{Q}}$, which yields $C_{\mathcal{Q}^\circ} \leq C_{\mathcal{Q}}$. The estimate $D_{\mathcal{Q}^\circ} \leq D_{\mathcal{Q}}$ is proved analogously.

Using $Q_i^\circ \subset Q_i$, we easily see $i^{*\mathcal{Q}^\circ} \subset i^{*\mathcal{Q}}$ for all $i \in I'$ and hence

$$N_{\mathcal{Q}^\circ} = \sup_{i \in I'} |i^{*\mathcal{Q}^\circ}| \leq \sup_{i \in I} |i^{*\mathcal{Q}}| = N_{\mathcal{Q}} < \infty.$$

All in all, we have shown that $\mathcal{Q}^\circ = \left(T_i (Q_i')^\circ + b_i\right)_{i \in I}$ is a semi-structured admissible covering of $\mathcal{O}$.

4. As shown above, we have $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i^\circ$ and thus $\varphi_i \equiv 0$ on $\mathcal{O} \setminus Q_i^\circ$ for $i \in I \supset I'$. Together with equation (3.2.2), we see that $(\varphi_i)_{i \in I'}$ is an $L^p$-BAPU for $\mathcal{Q}^\circ$, since the other defining properties of an $L^p$-BAPU only depend on the family $(T_i)_{i \in I}$ which is the same for $\mathcal{Q}$ and $\mathcal{Q}^\circ$. All in all, $\mathcal{Q}^\circ$ is an $L^p$-decomposition covering and $(\varphi_i)_{i \in I'}$ is an $L^p$-BAPU for $\mathcal{Q}^\circ$.

5. Finally, if $\mathcal{Q}$ is tight, then $B_{\varepsilon_{\mathcal{Q}}}(c_i) \subset Q_i'$ holds for all $i \in I$ and suitable $c_i \in \mathbb{R}^d$. But this yields $B_{\varepsilon_{\mathcal{Q}}}(c_i) \subset (Q_i')^\circ = P_i'$. In particular, we get $Q_i^\circ \neq \varnothing$ for all $i \in I$ and thus $I = I'$ and we see that $\mathcal{Q}^\circ$ is tight with $\varepsilon_{\mathcal{Q}} \leq \varepsilon_{\mathcal{Q}^\circ}$. The estimate $\varepsilon_{\mathcal{Q}^\circ} \leq \varepsilon_{\mathcal{Q}}$ is a direct consequence of $(Q_i')^\circ \subset Q_i'$. □

An important property of *structured* admissible coverings is that they always possess $L^p$-BAPUs. This was observed in [5, Proposition 1] for the case $\mathcal{O} = \mathbb{R}^d$. In their proof, Borup and Nielsen first construct a family $(g_i)_{i \in I}$, where each $g_i$ is supported in $Q_i = T_i Q + b_i$ and with the property $g := \sum_{i \in I} g_i \geq 1$. They then use the "method of division" to construct a partition of unity by setting $\varphi_i := g_i / g$.

It was pointed out by Prof. Feichtinger in a private discussion that one can also construct a BAPU by "multiplying and subtracting" rather than by division. For the sake of completeness and to present this different method of construction, we give a proof of the existence of $L^p$-BAPUs for structured admissible coverings based on this approach below (cf. Theorem 3.2.17).

As a preparation, we need the following lemma which shows that the "normalization" $T_i^{-1}(Q_i - b_i) \subset B_R(0)$, which holds for $R = R_{\mathcal{Q}}$ for a semi-structured covering by definition, also works if the set $Q_i$ is replaced by a neighboring set $Q_j$, as long as one is willing to enlarge the ball $B_R(0)$.

This will allow us in particular to estimate the Lebesgue measure $\lambda\left(\overline{Q_i} - \overline{Q_j}\right)$, which appears as a constant in the convolution relation in Theorem 3.1.4, by a multiple of $|\det(T_i)|$, cf. Corollary 3.2.14.

We remark that the issues related to the constant $\left[\lambda\left(\overline{Q_i} - \overline{Q_j}\right)\right]^{\frac{1}{p}-1}$ in contrast to $|\det(T_i)|^{\frac{1}{p}-1}$ are somewhat neglected in the treatment of Borup and Nielsen[5].

**Lemma 3.2.13.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ be a semi-structured covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$.*

*Then the inclusion*

$$Q_j \subset T_j\left(\overline{B_R}(0)\right) + b_j \subset T_i\left[\overline{B_{(2C_{\mathcal{Q}}+1)^\ell R}}(0)\right] + b_i$$

*is valid for all $i \in I$ and $j \in i^{\ell *}$, as long as $R > 0$ is chosen such that $Q_i' \subset \overline{B_R}(0)$ holds for all $i \in I$.*

*Proof.* Let $K := C_{\mathcal{Q}}$. We first prove

$$T_j\left(\overline{B_R}(0)\right) + b_j \subset T_i\left(\overline{B_{(2K+1)R}}(0)\right) + b_i \quad \forall i \in I \text{ and } j \in i^* \tag{3.2.3}$$

as long as $\bigcup_{i\in I} Q_i' \subset \overline{B_R}(0)$ holds.

To see this, note that $i \in j^*$ implies $\|T_i^{-1}T_j\| \leq K$ as well as

$$\varnothing \neq Q_i \cap Q_j \subset \left[T_i\left(\overline{B_R}(0)\right) + b_i\right] \cap \left[T_j\left(\overline{B_R}(0)\right) + b_j\right],$$

which yields $c_1, c_2 \in \overline{B_R}(0)$ satisfying $T_i c_1 + b_i = T_j c_2 + b_j$.

Rearranging results in

$$b_j - b_i = T_i c_1 - T_j c_2 = T_i\left(c_1 - T_i^{-1}T_j c_2\right),$$

with

$$\left|c_1 - T_i^{-1}T_j c_2\right| \leq |c_1| + \left\|T_i^{-1}T_j\right\| |c_2| \leq R + KR = (K+1)R,$$

which implies $T_i^{-1}\left(b_j - b_i\right) = c_1 - T_i^{-1}T_j c_2 \in \overline{B_{(K+1)R}}(0)$.

Now, let $b \in \overline{B_R}(0)$ be arbitrary and set $x := T_i^{-1}\left(T_j b + \left(b_j - b_i\right)\right)$. On the one hand,

$$\begin{aligned} |x| &\leq \left|T_i^{-1}T_j b\right| + \left|T_i^{-1}\left(b_j - b_i\right)\right| \\ &\leq \left\|T_i^{-1}T_j\right\| |b| + (K+1)R \leq (2K+1)R. \end{aligned}$$

On the other hand, we have $T_i x + b_i = T_j b + \left(b_j - b_i\right) + b_i = T_j b + b_j$ and thus

$$T_j b + b_j = T_i x + b_i \in T_i\left(\overline{B_{(2K+1)R}}(0)\right) + b_i,$$

which proves the claimed inclusion (3.2.3).

We can now establish the general claim by induction on $\ell \in \mathbb{N}$. The base case $\ell = 1$ is a direct consequence of equation (3.2.3), since we have $Q_j = T_j Q'_j + b_j \subset T_j\left(\overline{B_R}(0)\right) + b_j$.

For the induction step, we note that $j \in i^{(\ell+1)*}$ yields some $k \in i^{\ell *}$ with $j \in k^*$. Let us set $R' := (2K+1)R \geq R$, so that

$$\bigcup_{i \in I} Q'_i \subset \overline{B_R}(0) \subset \overline{B_{R'}}(0)$$

holds. If we apply the induction hypothesis for $R'$ instead of $R$, we derive

$$T_k\left(\overline{B_{R'}}(0)\right) + b_k \subset T_i\left(\overline{B_{(2K+1)^\ell R'}}(0)\right) + b_i = T_i\left(\overline{B_{(2K+1)^{\ell+1} R}}(0)\right) + b_i.$$

But since $j \in k^*$, we can apply equation (3.2.3), which results in

$$\begin{aligned} Q_j &\subset T_j\left(\overline{B_R}(0)\right) + b_j \\ &\subset T_k\left(\overline{B_{(2K+1)R}}(0)\right) + b_k \\ &= T_k\left(\overline{B_{R'}}(0)\right) + b_k \\ &\subset T_i\left(\overline{B_{(2K+1)^{\ell+1} R}}(0)\right) + b_i. \end{aligned}$$

This completes the induction step. □

As a corollary, we can prove an estimate for the Lebesgue measure of the difference sets $\overline{Q_i} - \overline{Q_j}$. This estimate will be important when we want to apply the convolution relation for $L^p$ with $p \in (0,1)$ given in Theorem 3.1.4.

**Corollary 3.2.14.** *Let $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q'_i + b_i)_{i \in I}$ be a semi-structured covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$.*

*There is a constant $C_2 = C_2(R_\mathcal{Q}, d) > 0$, such that*

$$\lambda(Q_i^\circ) \leq \lambda\left(\overline{Q_i}\right) \leq C_2 \cdot |\det(T_i)|$$

*holds for all $i \in I$.*

*Further, for any $n \in \mathbb{N}$, there is a constant $C_1 = C_1(C_\mathcal{Q}, R_\mathcal{Q}, n, d) > 0$, such that*

$$\max_{j \in i^{n*}} \lambda\left(\overline{Q_i^{n*}} - \overline{Q_j^{n*}}\right) \leq C_1 \cdot |\det(T_i)|$$

*holds for all $i \in I$.*

*Finally, if $\mathcal{Q}$ is tight, there is also a constant $C_3 = C_3(\varepsilon_\mathcal{Q}, d) > 0$ satisfying*

$$C_3 \cdot |\det(T_i)| \leq \lambda(Q_i^\circ) \leq \lambda\left(\overline{Q_i}\right)$$

*for all $i \in I$.*

*In case of a tight covering, we thus have $\lambda(Q_i^\circ) \asymp \lambda(\overline{Q_i}) \asymp |\det(T_i)|$.*

*Proof.* Let $R := R_\mathcal{Q}$. We first observe

$$Q_i \subset \overline{Q_i} = T_i \overline{Q'_i} + b_i \subset T_i\left(\overline{B_R(0)}\right) + b_i$$

and hence

$$\lambda\left(Q_i^\circ\right) \leq \lambda\left(\overline{Q_i}\right) \leq |\det\left(T_i\right)| \cdot \lambda\left(\overline{B_R\left(0\right)}\right),$$

so that we can set $C_2 := \lambda\left(\overline{B_R\left(0\right)}\right) = v_d \cdot R^d$, where $v_d := \lambda\left(B_1\left(0\right)\right)$ denotes the measure of the $d$-dimensional Euclidean unit ball.

Conversely, if $\mathcal{Q}$ is tight, set $\varepsilon := \varepsilon_{\mathcal{Q}}$. By Remark 3.2.9, there is for each $i \in I$ some $c_i \in \mathbb{R}^d$ with $B_\varepsilon\left(c_i\right) \subset Q_i'$. Hence,

$$Q_i = T_i Q_i' + b_i \supset T_i\left(B_\varepsilon\left(c_i\right)\right) + b_i,$$

which yields

$$\lambda\left(\overline{Q_i}\right) \geq \lambda\left(Q_i^\circ\right) \geq |\det\left(T_i\right)| \cdot \lambda\left(B_\varepsilon\left(c_i\right)\right) = v_d \varepsilon^d \cdot |\det\left(T_i\right)|,$$

so that we can set $C_3 := v_d \varepsilon_{\mathcal{Q}}^d$.

It remains to establish the estimate regarding the difference set $\overline{Q_i^{n*}} - \overline{Q_j^{n*}}$ for $j \in i^{n*}$. Note that $j^{n*} \subset i^{2n*}$. Hence, $\overline{Q_i^{n*}} \subset \overline{Q_i^{2n*}}$ and $\overline{Q_j^{n*}} \subset \overline{Q_i^{2n*}}$.

Now, let $K := C_{\mathcal{Q}}$ and $R := R_{\mathcal{Q}}$. By Lemma 3.2.13, this yields for any $\ell \in i^{2n*}$ the inclusion

$$Q_\ell \subset T_i\left[\overline{B_{(2K+1)^{2n}R}\left(0\right)}\right] + b_i.$$

Since $\ell \in i^{2n*}$ was arbitrary, we get

$$\overline{Q_i^{2n*}} \subset T_i\left[\overline{B_{(2K+1)^{2n}R}\left(0\right)}\right] + b_i$$

and hence

$$\begin{aligned}
\overline{Q_i^{n*}} - \overline{Q_j^{n*}} &\subset \overline{Q_i^{2n*}} - \overline{Q_i^{2n*}} \\
&\subset \left(T_i\left[\overline{B_{(2K+1)^{2n}R}\left(0\right)}\right] + b_i\right) - \left(T_i\left[\overline{B_{(2K+1)^{2n}R}\left(0\right)}\right] + b_i\right) \\
&\subset T_i\left[\overline{B_{(2K+1)^{2n}R}\left(0\right)} - \overline{B_{(2K+1)^{2n}R}\left(0\right)}\right] \\
&\subset T_i\left[\overline{B_{(2K+1)^{2n}2R}\left(0\right)}\right].
\end{aligned}$$

But this implies

$$\lambda\left(\overline{Q_i^{n*}} - \overline{Q_j^{n*}}\right) \leq |\det\left(T_i\right)| \cdot \lambda\left(\overline{B_{(2K+1)^{2n}2R}\left(0\right)}\right),$$

so that the choice $C_1 = v_d \cdot \left[\left(2K+1\right)^{2n} 2R\right]^d$ is possible. □

As a further corollary, we obtain a more convenient version of the convolution relation in Theorem 3.1.4.

**Corollary 3.2.15.** *Let $p \in (0,1]$ and let $\mathcal{Q} = \left(Q_i\right)_{i\in I} = \left(T_i Q_i' + b_i\right)_{i\in I}$ be an $L^p$-decomposition covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$.*

*For each $n \in \mathbb{N}$, there is a constant $C = C(n, d, p, R_\mathcal{Q}, C_\mathcal{Q}) > 0$ with the following property: If $i \in I$ and*

A. *$f \in L^1(\mathbb{R}^d)$ with $\operatorname{supp} f \subset \overline{Q_i^{n*}}$ and $\mathcal{F}^{-1}f \in L^p(\mathbb{R}^d)$*

B. *and $g \in \mathcal{D}'(\mathcal{O})$ with $\operatorname{supp} g \subset \overline{Q_i^{n*}}$ and $\mathcal{F}^{-1}g \in L^p(\mathbb{R}^d)$,*

*then $\mathcal{F}^{-1}(fg) \in L^p(\mathbb{R}^d)$ with*

$$\|\mathcal{F}^{-1}(fg)\|_{L^p} \leq C\,|\det(T_i)|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p}.$$

*Proof.* Lemma 3.2.12 implies that $\overline{Q_i^{n*}} \subset \bigcup_{j \in i^{n*}} \overline{Q_j} \subset \mathcal{O}$ is compact, since $i^{n*}$ is finite. Hence, $g \in \mathcal{D}'(\mathcal{O})$ is a distribution with compact support in $\mathcal{O}$, which means that $g$ extends to a tempered distribution $g \in \mathcal{S}'(\mathbb{R}^d)$, cf. [61, Theorem 6.24(d) and Example 7.12(a)]. Hence, we have $\mathcal{F}^{-1}g \in \mathcal{S}'(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{\mathcal{F}^{-1}g} = \operatorname{supp} g \subset \overline{Q_i^{n*}}$. By Theorem 3.1.4, this implies

$$\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}\left(f \cdot \widehat{\mathcal{F}^{-1}g}\right) \in L^p(\mathbb{R}^d)$$

with

$$\begin{aligned}\left\|\mathcal{F}^{-1}(fg)\right\|_{L^p} &= \left\|\mathcal{F}^{-1}\left(f \cdot \widehat{\mathcal{F}^{-1}g}\right)\right\|_{L^p} \\ &\leq \left[\lambda\left(\overline{Q_i^{n*}} - \overline{Q_i^{n*}}\right)\right]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p}.\end{aligned}$$

But Corollary 3.2.14 yields a constant $C_1 = C_1(R_\mathcal{Q}, C_\mathcal{Q}, n, d) > 0$ with

$$\lambda\left(\overline{Q_i^{n*}} - \overline{Q_i^{n*}}\right) \leq C_1 \cdot |\det(T_i)| \text{ for all } i \in I.$$

Recall $p \in (0, 1]$ and hence $\frac{1}{p} - 1 \geq 0$. All in all, we conclude

$$\begin{aligned}\left\|\mathcal{F}^{-1}(fg)\right\|_{L^p} &\leq \left[\lambda\left(\overline{Q_i^{n*}} - \overline{Q_i^{n*}}\right)\right]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p} \\ &\leq C_1^{\frac{1}{p}-1} \cdot |\det(T_i)|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}f\|_{L^p} \cdot \|\mathcal{F}^{-1}g\|_{L^p},\end{aligned}$$

which completes the proof. □

As a final preparation for the construction of an $L^p$-BAPU for a structured admissible covering $\mathcal{Q}$, we show that every admissible covering by nonempty open sets is already countable.

**Lemma 3.2.16.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an admissible covering of $\mathcal{O}$ with $Q_i \subset \mathcal{O}$ open for each $i \in I$. Then $I$ is countable.*

*The word "countable" is supposed to include the case that $I$ is finite.*

*Proof.* As a subspace of $\mathbb{R}^d$, $\mathcal{O}$ is a second countable topological space and hence Lindelöff. By assumption, $\mathcal{Q}$ is an open cover of $\mathcal{O}$, so that there is a countable subcover $\mathcal{O} \subset \bigcup_{n \in \mathbb{N}} Q_{i_n}$ with suitable $i_n \in I$ for each $n \in \mathbb{N}$.

We now claim $I = \bigcup_{n\in\mathbb{N}} i_n^*$. By admissibility, each set $i_n^*$ is finite, so that this will indeed prove $I$ to be countable.

For the proof of $I = \bigcup_{n\in\mathbb{N}} i_n^*$, let $i \in I$ be arbitrary. By definition of an admissible covering, we have $Q_i \neq \varnothing$, so that there is some $x \in Q_i \subset \mathcal{O}$. Hence, there is some $n \in \mathbb{N}$ with $x \in Q_{i_n}$. But this shows $x \in Q_{i_n} \cap Q_i \neq \varnothing$ and thus $i \in i_n^*$. □

Using these preparatory results, we can now apply the "multiply and subtract" method to prove the existence of $L^p$-BAPUs for structured admissible coverings. Essentially the same method of construction is used in [26, Theorem 4.2].

**Theorem 3.2.17.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q + b_i)_{i\in I}$ be a* structured *admissible covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$.*

*Then $I$ is countably infinite and there is a family $\Phi = (\varphi_i)_{i\in I}$ in $C_c^\infty(\mathcal{O})$ such that $\Phi$ is an $L^p$-BAPU for $\mathcal{Q}$ for each $p \in (0,\infty]$.*

*Proof.* By definition of a structured admissible covering, there is some open set $P$ which is compactly contained in $Q$ with the property that $\mathcal{P} := (P_i)_{i\in I} := (T_i P + b_i)_{i\in I}$ still covers $\mathcal{O}$.

Lemma 3.2.16 shows that $I$ is countable. We have

$$\mathcal{O} = \bigcup_{i\in I} P_i \subset \bigcup_{i\in I} \overline{P_i} \subset \bigcup_{i\in I} Q_i = \mathcal{O}.$$

If $I$ was finite, this would imply that $\mathcal{O} = \bigcup_{i\in I} \overline{P_i}$ was compact, too. But $\mathcal{O}$ is a nonempty, open subset of $\mathbb{R}^d$. Since $\mathbb{R}^d$ is connected, this yields a contradiction. Hence, $I$ is countably infinite, so that we can write $I = \{i_n \mid n \in \mathbb{N}\}$ with pairwise distinct $(i_n)_{n\in\mathbb{N}}$. By reindexing, we can assume without loss of generality that $I = \mathbb{N}$.

Fix any $\gamma \in C_c^\infty(Q)$ with $\gamma \equiv 1$ on $P$ and define

$$\begin{aligned}\gamma_n(\xi) &:= \gamma\left(T_n^{-1}(\xi - b_n)\right)\\ &= \left[L_{b_n}\left(\gamma \circ T_n^{-1}\right)\right](\xi)\end{aligned}$$

for $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N} = I$. By construction, $\gamma_n \in C_c^\infty(Q_n)$ with $\gamma_n \equiv 1$ on $P_n = T_n P + b_n$. We define

$$\varphi_n := \gamma_n \cdot \prod_{j=1}^{n-1} \left(1 - \gamma_j\right) \text{ for } n \in \mathbb{N}.$$

This easily implies $\operatorname{supp}\varphi_n \subset \operatorname{supp}\gamma_n \subset Q_n \subset \mathcal{O}$.

Furthermore, induction yields for all $n \in \mathbb{N}$ the identity

$$\sum_{j=1}^{n} \varphi_j = 1 - \prod_{j=1}^{n} \left(1 - \gamma_j\right).$$

Indeed, for $n = 1$, this is clear because of $\varphi_1 = \gamma_1$. Furthermore, by induction

$$\begin{aligned}\sum_{j=1}^{n+1} \varphi_j &= \left[1 - \prod_{j=1}^{n} (1 - \gamma_j)\right] + \varphi_{n+1} \\ &= \left[1 - \prod_{j=1}^{n} (1 - \gamma_j)\right] + \gamma_{n+1} \prod_{j=1}^{n} (1 - \gamma_j) \\ &= 1 - (1 - \gamma_{n+1}) \prod_{j=1}^{n} (1 - \gamma_j) = 1 - \prod_{j=1}^{n+1} (1 - \gamma_j).\end{aligned}$$

Now, for arbitrary $x \in \mathcal{O}$, there is some $n \in \mathbb{N}$ with $x \in P_n$, so that $1 - \gamma_n(x) = 0$. But this implies

$$\sum_{j=1}^{\infty} \varphi_j(x) = 1 - \prod_{j=1}^{\infty} (1 - \gamma_j(x)) = 1 - 0 = 1.$$

It remains to verify that

$$C_p := C_{\mathcal{Q},\Phi,p} = \sup_{n \in \mathbb{N}} |\det(T_n)|^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\varphi_n\|_{L^p}$$

is finite for all $p \in (0,1]$. To this end, fix some $p \in (0,1]$ and choose the constant $C_0 = C(1, d, p, R_{\mathcal{Q}}, C_{\mathcal{Q}})$ as in Corollary 3.2.15. We can assume $C_0 \geq 1$ without loss of generality.

Now, let us first prove the following auxiliary result: For arbitrary $n \in \mathbb{N}$, $N \in \mathbb{N}_0$ and $f_0, \dots, f_N \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f_i \subset Q_n^*$ and $\|\mathcal{F}^{-1} f_i\|_{L^p} \leq C \cdot |\det(T_n)|^{1-\frac{1}{p}}$ for all $i \in \{0, \dots, N\}$ and some constant $C \in [1, \infty)$, we have

$$\left\| \mathcal{F}^{-1}\left[ f_0 \cdot \prod_{j=1}^{N} (1 - f_j) \right] \right\|_{L^p} \leq \left(2^{1/p} C_0 C\right)^N C \cdot |\det(T_n)|^{1-\frac{1}{p}}.$$

We prove this by induction on $N \in \mathbb{N}_0$. The case $N = 0$ is trivial.

For the induction step, let $g_N := f_0 \cdot \prod_{j=1}^{N} (1 - f_j)$ and note

$$\begin{aligned}g_{N+1} &:= f_0 \cdot \prod_{j=1}^{N+1} (1 - f_j) \\ &= g_N \cdot (1 - f_{N+1}) \\ &= g_N - g_N f_{N+1}.\end{aligned}$$

Because $\|\cdot\|_{L^p}$ is a $p$-norm, this implies

$$\|\mathcal{F}^{-1} g_{N+1}\|_{L^p}^p \leq \|\mathcal{F}^{-1} g_N\|_{L^p}^p + \|\mathcal{F}^{-1}(g_N f_{N+1})\|_{L^p}^p.$$

But Corollary 3.2.15, together with the support assumptions on the $f_i$, yields

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left(g_N f_{N+1}\right)\right\|_{L^p} \\
&\leq C_0 \cdot |\det(T_n)|^{\frac{1}{p}-1} \cdot \left\|\mathcal{F}^{-1} g_N\right\|_{L^p} \cdot \left\|\mathcal{F}^{-1} f_{N+1}\right\|_{L^p} \\
&\leq C_0 C \left(2^{1/p} C_0 C\right)^N C \cdot |\det(T_n)|^{\frac{1}{p}-1} \cdot |\det(T_n)|^{1-\frac{1}{p}} |\det(T_n)|^{1-\frac{1}{p}} \\
&= 2^{N/p} \left(C_0 C\right)^{N+1} C \cdot |\det(T_n)|^{1-\frac{1}{p}}.
\end{aligned}$$

Together, we obtain

$$\begin{aligned}
&\left\|\mathcal{F}^{-1} g_{N+1}\right\|_{L^p}^p \\
&\leq \left\|\mathcal{F}^{-1} g_N\right\|_{L^p}^p + \left\|\mathcal{F}^{-1}\left(g_N f_{N+1}\right)\right\|_{L^p}^p \\
&\leq \left[2^{N/p} \left(C_0 C\right)^N C \,|\det(T_n)|^{1-\frac{1}{p}}\right]^p + \left[2^{N/p} \left(C_0 C\right)^{N+1} C \,|\det(T_n)|^{1-\frac{1}{p}}\right]^p \\
&\leq 2 \cdot \left[2^{N/p} \left(C_0 C\right)^{N+1} C \,|\det(T_n)|^{1-\frac{1}{p}}\right]^p \\
&= \left[\left(2^{1/p} C_0 C\right)^{N+1} C \,|\det(T_n)|^{1-\frac{1}{p}}\right]^p
\end{aligned}$$

and hence the auxiliary claim for $N+1$ instead of $N$.

We now apply this as follows: For $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus n^*$, we have

$$Q_n \cap \operatorname{supp} \gamma_m \subset Q_n \cap Q_m = \varnothing$$

and hence $1 - \gamma_m \equiv 1$ on $Q_n$. Since $\gamma_n$ vanishes outside of $Q_n$, this implies

$$\begin{aligned}
\varphi_n &= \gamma_n \cdot \prod_{j=1}^{n-1} \left(1 - \gamma_j\right) \\
&= \gamma_n \prod_{j \in \underline{n-1} \cap n^*} \left(1 - \gamma_j\right).
\end{aligned}$$

But for $j \in n^*$ (in particular for $j = n$), standard properties of the Fourier transform (cf. [33, Theorem 8.22]) yield

$$\begin{aligned}
\left\|\mathcal{F}^{-1} \gamma_j\right\|_{L^p} &= \left|\det\left(T_j\right)\right| \cdot \left\|M_{b_j}\left[\left(\mathcal{F}^{-1}\gamma\right) \circ T_j^T\right]\right\|_{L^p} \\
&= \left|\det\left(T_j\right)\right|^{1-\frac{1}{p}} \cdot \left\|\mathcal{F}^{-1}\gamma\right\|_{L^p}.
\end{aligned}$$

But equation (3.2.1) yields $|\det(T_n)| \leq D_{\mathcal{Q}} \cdot \left|\det\left(T_j\right)\right|$ and hence (because of $1 - \frac{1}{p} \leq 0$)

$$\left\|\mathcal{F}^{-1} \gamma_j\right\|_{L^p} \leq D_{\mathcal{Q}}^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\gamma\right\|_{L^p} \cdot |\det(T_n)|^{1-\frac{1}{p}}.$$

By applying the auxiliary claim with $C := \max\left\{1, D_{\mathcal{Q}}^{-1+1/p} \left\|\mathcal{F}^{-1}\gamma\right\|_{L^p}\right\}$, we conclude

$$\begin{aligned}\left\|\mathcal{F}^{-1}\varphi_n\right\|_{L^p} &\leq \left(2^{1/p} C_0 C\right)^{|\underline{n-1} \cap n^*|} C \cdot |\det(T_n)|^{1-\frac{1}{p}} \\ &\leq \left(2^{1/p} C_0 C\right)^{N_{\mathcal{Q}}} C \cdot |\det(T_n)|^{1-\frac{1}{p}}.\end{aligned}$$

Since the constant $\left(2^{1/p} C_0 C\right)^{N_{\mathcal{Q}}} C$ is independent of $n \in \mathbb{N} = I$, this completes the proof. □

As the final result in this section, we show that the class of semi-structured coverings is indeed closed under forming clusters.

**Lemma 3.2.18.** *Assume that $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ is a [tight] semi-structured covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$. Then the following are true:*

1. *We have $\left|i^{k*}\right| \leq N_{\mathcal{Q}}^k$ for all $i \in I$ and $k \in \mathbb{N}_0$. This even holds if $\mathcal{Q}$ is any admissible family.*
2. *The family of $k$-clusters $\mathcal{Q}^{k*} = \left(Q_i^{k*}\right)_{i \in I}$ is a [tight] semi-structured covering of $\mathcal{O}$ with $N_{\mathcal{Q}^{k*}} \leq N_{\mathcal{Q}}^{2k+1}$ and every $L^p$-BAPU $(\varphi_i)_{i \in I}$ for $\mathcal{Q}$ is also an $L^p$-BAPU for $\mathcal{Q}^{k*}$.*
3. *There are suitable $P_i' \subset \mathbb{R}^d$ for $i \in I$ so that $(T_i)_{i \in I}, (P_i')_{i \in I}, (b_i)_{i \in I}$ is a [tight] collection of standardizations for $\mathcal{Q}^{k*} = (T_i P_i' + b_i)_{i \in I}$ with*

$$\begin{aligned}R_{\mathcal{Q}^{k*}} &\leq (2C_{\mathcal{Q}} + 1)^k R_{\mathcal{Q}}, \\ D_{\mathcal{Q}^{k*}} &\leq D_{\mathcal{Q}}^{2k+1}, \\ C_{\mathcal{Q}^{k*}} &\leq C_{\mathcal{Q}}^{2k+1}\end{aligned}$$

   *and (in the tight case) with $\varepsilon_{\mathcal{Q}^{k*}} \geq \varepsilon_{\mathcal{Q}}$.*
4. *If $u = (u_i)_{i \in I}$ or $v : \mathcal{O} \to (0, \infty)$ are $\mathcal{Q}$-moderate, they are also $\mathcal{Q}^{k*}$-moderate with*

$$C_{u, \mathcal{Q}^{k*}} \leq C_{u, \mathcal{Q}}^{2k+1} \quad \text{and} \quad C_{v, \mathcal{Q}^{k*}} \leq C_{v, \mathcal{Q}}^{2k+1}.$$

*Proof.* We first show by induction on $k \in \mathbb{N}_0$ the following auxiliary result: We have

$$\begin{aligned}&|i^{k*}| \leq N_{\mathcal{Q}}^k \text{ for all } i \in I, \\ \text{and } &\frac{u_i}{u_j} \leq C_{u, \mathcal{Q}}^k \text{ for } i \in I,\ j \in i^{k*}, \\ \text{and } &\frac{v(x)}{v(y)} \leq C_{v, \mathcal{Q}}^{2k+1} \text{ for } x, y \in Q_i^{k*}, \\ \text{and } &\left\|T_i^{-1} T_j\right\| \leq C_{\mathcal{Q}}^k \text{ for } i \in I,\ j \in i^{k*},\end{aligned}$$

where $u = (u_i)_{i \in I}$ and $v : \mathcal{O} \to (0, \infty)$ are $\mathcal{Q}$-moderate weights.

The case $k = 0$ is trivial. For the induction step, first note

$$|i^{(k+1)*}| = \left| \bigcup_{j \in i^{k*}} j^* \right| \leq \sum_{j \in i^{k*}} |j^*| \leq N_{\mathcal{Q}} \cdot |i^{k*}| \leq N_{\mathcal{Q}}^{k+1},$$

where we used the induction hypothesis in the last step. For the second and last estimates, let $j \in i^{(k+1)*}$ be arbitrary. Thus, there is some $\ell \in i^{k*}$ with $j \in \ell^*$. By induction hypothesis, this yields

$$\frac{u_i}{u_j} = \frac{u_i}{u_\ell} \cdot \frac{u_\ell}{u_j} \leq C_{u,\mathcal{Q}}^k \cdot C_{u,\mathcal{Q}} = C_{u,\mathcal{Q}}^{k+1}.$$

Similarly,

$$\left\| T_i^{-1} T_j \right\| = \left\| T_i^{-1} T_\ell T_\ell^{-1} T_j \right\| \leq \left\| T_i^{-1} T_\ell \right\| \cdot \left\| T_\ell^{-1} T_j \right\| \leq C_{\mathcal{Q}}^k \cdot C_{\mathcal{Q}} = C_{\mathcal{Q}}^{k+1}.$$

Finally, for $x, y \in Q_i^{(k+1)*}$, there are $j_1, j_2 \in i^{(k+1)*}$ with $x \in Q_{j_1}$ and $y \in Q_{j_2}$. Hence, there are $i_1, i_2 \in i^{k*}$ with $j_1 \in i_1^*$ and $j_2 \in i_2^*$. This yields suitable $z_1 \in Q_{j_1} \cap Q_{i_1}$ and $z_2 \in Q_{j_2} \cap Q_{i_2}$. Because of $i_1, i_2 \in i^{k*}$, this implies $z_1, z_2 \in Q_i^{k*}$ and hence (by induction hypothesis)

$$\frac{v(x)}{v(y)} = \frac{v(x)}{v(z_1)} \cdot \frac{v(z_1)}{v(z_2)} \cdot \frac{v(z_2)}{v(y)} \leq C_{v,\mathcal{Q}} \cdot C_{v,\mathcal{Q}}^{2k+1} \cdot C_{v,\mathcal{Q}} = C_{v,\mathcal{Q}}^{2(k+1)+1}.$$

This completes the proof of the auxiliary result.

Since the weight $(|\det(T_i)|)_{i \in I}$ is $\mathcal{Q}$-moderate with "moderateness constant" $D_{\mathcal{Q}}$, the auxiliary result implies in particular

$$\left|\det\left(T_i^{-1} T_j\right)\right| = \left|\det\left(T_j\right)\right| / \left|\det(T_i)\right| \leq D_{\mathcal{Q}}^k \text{ for } i \in i \text{ and } j \in i^{k*}.$$

Now, let $i \in I$ and $j \in i^{*_{\mathcal{Q}^{k*}}}$, i.e. with $Q_i^{k*} \cap Q_j^{k*} \neq \varnothing$. This implies $Q_m \cap Q_\ell \neq \varnothing$ for suitable $m \in i^{k*}, \ell \in j^{k*}$ and hence

$$j \in \ell^{k*} \subset m^{(k+1)*} \subset i^{(2k+1)*}.$$

Together, we conclude $i^{*_{\mathcal{Q}^{k*}}} \subset i^{(2k+1)*}$ and hence

$$|i^{*_{\mathcal{Q}^{k*}}}| \leq |i^{(2k+1)*_{\mathcal{Q}}}| \leq N_{\mathcal{Q}}^{2k+1},$$

so that $\mathcal{Q}^{k*}$ is admissible with $N_{\mathcal{Q}^{k*}} \leq N_{\mathcal{Q}}^{2k+1}$.

As shown above, we also have $u_i / u_j \leq C_{u,\mathcal{Q}}^{2k+1}$, because of $j \in i^{(2k+1)*}$ and the same argument also yields $\left|\det\left(T_i^{-1} T_j\right)\right| \leq D_{\mathcal{Q}}^{2k+1}$. Finally, for $x, y \in Q_i^{k*}$, we also have $v(x) / v(y) \leq C_{v,\mathcal{Q}}^{2k+1}$, as seen above.

Now, set $P_i' := T_i^{-1}\left(Q_i^{k*} - b_i\right)$ for each $i \in I$. Thus, $T_i P_i' + b_i = Q_i^{k*}$. We now show that $\mathcal{Q}^{k*}$ obeys all requirements of Definition 3.2.8.

1. As seen above, $\mathcal{Q}^{k*}$ is admissible. Now note

$$Q_i \subset Q_i^{k*} = \bigcup_{j \in i^{k*}} Q_j \subset \mathcal{O}$$

for all $i \in I$, so that $\mathcal{Q}^{k*}$ covers $\mathcal{O}$.

2. Lemma 3.2.13 yields

$$Q_i^{k*} = \bigcup_{j \in i^{k*}} Q_j \subset T_i \left( \overline{B_{(2C_\mathcal{Q}+1)^k R_\mathcal{Q}}}(0) \right) + b_i$$

and thus $P_i' = T_i^{-1} \left( Q_i^{k*} - b_i \right) \subset \overline{B_{(2C_\mathcal{Q}+1)^k R_\mathcal{Q}}}(0)$ for all $i \in I$, i.e.

$$R_{\mathcal{Q}^{k*}} \leq (2C_\mathcal{Q} + 1)^k R_\mathcal{Q}.$$

3. For $i \in I$ and $j \in i^{*_{\mathcal{Q}^{k*}}} \subset i^{(2k+1)*}$, the auxiliary statement at the beginning of the proof shows $\left\| T_i^{-1} T_j \right\| \leq C_\mathcal{Q}^{2k+1}$. Hence, $C_{\mathcal{Q}^{k*}} \leq C_\mathcal{Q}^{2k+1} < \infty$.

Finally, if $\mathcal{Q}$ is a tight covering, there is a family $(c_i)_{i \in I} \in \left(\mathbb{R}^d\right)^I$ with $B_{\varepsilon_\mathcal{Q}}(c_i) \subset Q_i'$ for all $i \in I$. This yields

$$\begin{aligned} B_{\varepsilon_\mathcal{Q}}(c_i) &\subset Q_i' \\ &= T_i^{-1}(Q_i - b_i) \\ &\subset T_i^{-1}(Q_i^{k*} - b_i) \\ &= P_i' \end{aligned}$$

for all $i \in I$, so that $\mathcal{Q}^{k*}$ is also a tight semi-structured covering with $\varepsilon_{\mathcal{Q}^{k*}} \geq \varepsilon_\mathcal{Q}$.

Finally, if $(\varphi_i)_{i \in I}$ is an $L^p$-BAPU for $\mathcal{Q}$, it is also an $L^p$-BAPU for $\mathcal{Q}^{k*}$, since the only two properties of an $L^p$-BAPU that depend on the covering are the following:

1. $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{k*}$. But because of $Q_i \subset Q_i^{k*}$ and since $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$, we trivially have $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{k*}$.

2. $\left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^p} \leq C \cdot \left| \det(T_i) \right|^{1 - \frac{1}{r}}$ for all $i \in I$ with $r = \min\{1, p\}$. Since the two coverings $\mathcal{Q}$ and $\mathcal{Q}^{k*}$ use the same family $(T_i)_{i \in I}$, this condition clearly holds for $\mathcal{Q}^{k*}$ if it holds for $\mathcal{Q}$. □

## 3.3 RELATIONS BETWEEN COVERINGS

In Chapter 5, we will study the existence of embeddings between two decomposition spaces that are defined using different coverings. For this, it will be convenient to have a clear terminology for describing the relations between two coverings, e.g. the fact that one is finer than the other. This terminology, as well as certain auxiliary results, are developed in the present section.

**Definition 3.3.1.** (cf. [27, Definition 3.3]) Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = \left(P_j\right)_{j \in J}$ are two families of subsets of $\mathbb{R}^d$. Then

1. For $i \in I$ we define the $\mathcal{P}$-INDEX-CLUSTER AROUND $i$ (or the SET OF $\mathcal{P}$-NEIGHBORS OF $i$) by

$$J_i := \left\{ j \in J \,\middle|\, Q_i \cap P_j \neq \varnothing \right\}.$$

   The $\mathcal{Q}$-INDEX-CLUSTER AROUND $j \in J$ is defined analogously and denoted by $I_j$.

*We remark that the notation $J_i$ is strictly speaking ambigous, at least in the case $I = J$. But it will always be clear from the context what is meant.*

2. We say that $\mathcal{Q}$ IS WEAKLY SUBORDINATE TO $\mathcal{P}$ if the constant

$$N(\mathcal{Q}, \mathcal{P}) := \sup_{i \in I} |J_i|$$

   is finite (for equivalent conditions, see [27, Definition 3.3 and Proposition 3.5]).

3. We say that $\mathcal{Q}$ IS ALMOST SUBORDINATE TO $\mathcal{P}$ if there is a constant $k = k(\mathcal{Q}, \mathcal{P}) \in \mathbb{N}_0$ such that each set $Q_i$ is contained in some $P_j^{k*}$, i.e. if

$$\forall i \in I \, \exists j_i \in J : \qquad Q_i \subset P_{j_i}^{k*}.$$

   If we can take $k(\mathcal{Q}, \mathcal{P}) = 0$, then we say that $\mathcal{Q}$ IS SUBORDINATE TO $\mathcal{P}$.

4. A weight $u = (u_i)_{i \in I}$ is called RELATIVELY $\mathcal{P}$-MODERATE if there is a constant $C_{u,\mathcal{Q},\mathcal{P}} > 0$ with

$$u_i \leq C_{u,\mathcal{Q},\mathcal{P}} \cdot u_\ell$$

   for all $j \in J$ and all $i, \ell \in I$ with $Q_i \cap P_j \neq \varnothing \neq Q_\ell \cap P_j$.

5. Now, let $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ be a semi-structured covering of some open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$. We say that $\mathcal{Q}$ IS MODERATE WITH RESPECT TO $\mathcal{P}$ or that $\mathcal{Q}$ IS RELATIVELY $\mathcal{P}$-MODERATE if there is a constant $C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}) > 0$ satisfying

$$\forall j \in J \, \forall i, \ell \in I_j : \qquad |\det(T_i^{-1} T_\ell)| \leq C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}).$$

   Since the determinant is multiplicative, this is equivalent to the fact that the weight $(|\det(T_i)|)_{i \in I}$ is relatively $\mathcal{P}$-moderate.

*Remark.* The condition of relative $\mathcal{P}$-moderateness of a weight $u$ means that if two sets $Q_i, Q_\ell$ of the covering $\mathcal{Q}$ are "close" to each other *measured with respect to* $\mathcal{P}$ (i.e., they intersect the same $P_j$), then $u_i, u_\ell$ are of similar size. Analogously, $\mathcal{Q}$ is moderate with respect to $\mathcal{P}$ if the determinants $|\det(T_i)|, |\det(T_\ell)|$ are of comparable size if $Q_i, Q_\ell$ are close to each other measured with respect to $\mathcal{P}$.

It seems that this property is a novel concept. Its significance will become clear in Chapter 5, in particular in Section 5.3.

The notions introduced in the above definition are of course not independent. In the next lemmata, we explore the connections between these concepts.

**Lemma 3.3.2.** *Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ are two families of nonempty subsets of $\mathbb{R}^d$. Then the following hold:*

1. *If $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and if $\mathcal{P}$ is admissible, then $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ with*

$$N(\mathcal{Q},\mathcal{P}) \leq N_{\mathcal{P}}^{2k(\mathcal{Q},\mathcal{P})+1}.$$

   *If $\mathcal{Q},\mathcal{P}$ are coverings of $\mathcal{O},\mathcal{O}' \subset \mathbb{R}^d$, respectively, then we also have $\mathcal{O} \subset \mathcal{O}'$.*

2. *If $Q_i \subset P_j^{k*}$ holds for some $i \in I$, $j \in J$ and $k \in \mathbb{N}_0$, then $J_i \subset \ell^{(2k+2)*}$ holds for all $\ell \in J_i$. In particular, $Q_i \subset P_\ell^{(2k+2)*}$ for all $\ell \in J_i$.*

*The inclusion $Q_i \subset P_\ell^{(2k+2)*}$ for all $\ell \in J_i$ will turn out to be very useful in Chapter 5: By only assuming $Q_i \subset P_j^{k*}$ for some $j \in J$, we can already conclude $Q_i \subset P_\ell^{m*}$ (for suitable $m > k$) for all $\ell \in J$ which intersect $Q_i$ nontrivially.*

3. *If $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$, then $Q_i \subset P_j^{(2k(\mathcal{Q},\mathcal{P})+2)*}$ holds for all $i \in I$ and all $j \in J_i$.*

*Proof.* 1. Choose $k := k(\mathcal{Q},\mathcal{P})$ so that for each $i \in I$ there is some $j_i \in J$ with $Q_i \subset P_{j_i}^{k*}$. Now let $j \in J_i$ for some $i \in I$. This implies

$$\varnothing \neq P_j \cap Q_i \subset P_j \cap P_{j_i}^{k*} \subset P_j^{k*} \cap P_{j_i}^{k*},$$

which means $J_i \subset j_i^{*_{\mathcal{P}^{k*}}}$, where the cluster is formed with respect to $\mathcal{P}^{k*}$.

But $\mathcal{P}^{k*}$ is an admissible covering by Lemma 3.2.18. This implies

$$\sup_{i\in I} |J_i| \leq \sup_{i\in I} |j_i^{*_{\mathcal{P}^{k*}}}| \leq \sup_{j\in J} |j^{*_{\mathcal{P}^{k*}}}| \leq N_{\mathcal{P}^{k*}} \leq N_{\mathcal{P}}^{2k+1} < \infty,$$

so that $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ with $N(\mathcal{Q},\mathcal{P}) \leq N_{\mathcal{P}}^{2k(\mathcal{Q},\mathcal{P})+1}$.

Finally, assume that $\mathcal{Q},\mathcal{P}$ are coverings of $\mathcal{O},\mathcal{O}'$, respectively. Then

$$\mathcal{O} = \bigcup_{i\in I} Q_i \subset \bigcup_{i\in I} P_{j_i}^{k*} \subset \bigcup_{j\in J} P_j \subset \mathcal{O}'.$$

2. Fix $\ell \in J_i$ and let $m \in J_i$ be arbitrary. This implies

$$\varnothing \neq Q_i \cap P_s \subset P_j^{k*} \cap P_s \text{ for } s \in \{m,\ell\}$$

and thus $j \in \ell^{(k+1)*}$ as well as $m \in j^{(k+1)*} \subset \ell^{(2k+2)*}$. We have thus shown $J_i \subset \ell^{(2k+2)*}$. Finally, note that $Q_i \subset P_j^{k*}$ and $J_i \subset \ell^{(2k+2)*}$ imply

$$Q_i \subset \bigcup_{m\in J} (P_m \cap Q_i) \subset \bigcup_{m\in J_i} P_m \subset P_\ell^{(2k+2)*}$$

for all $\ell \in J_i$.

3. This is a special case of part 2. □

We now prove that under certain *connectivity assumptions*, weak subordinateness already implies almost subordinateness. This fact will turn out to be very convenient for verifying almost subordinateness for concrete examples.

The proof of the lemma is very close to the proof of [27, Proposition 3.6], but a few unnecessary assumptions (like connectedness of the space $X$) have been removed and the statement of the lemma has been made more quantitative.

**Lemma 3.3.3.** *(cf. [27, Proposition 3.6]) Let $X$ be a topological space and let $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ be families of subsets of $X$.*

*Let $i \in I$ and assume that $Q_i$ is path-connected with $Q_i \subset \bigcup_{j\in J} P_j$ and so that*

$$J_i := \{j \in J \mid Q_i \cap P_j \neq \varnothing\}$$

*is finite. Furthermore, assume that $P_j$ is open for every $j \in J_i$.*

*Let $r := |J_i|$. We then have $J_i \subset j^{r*}$ and in particular $Q_i \subset P_j^{r*}$ for every $j \in J_i$.*

*Proof.* For $j \in J \setminus J_i$, we have $Q_i \cap P_j = \varnothing$. Together with the assumption $Q_i \subset \bigcup_{j\in J} P_j$, this yields $Q_i \subset \bigcup_{j\in J_i} P_j = P_{J_i}$. It is hence sufficient to prove $J_i \subset j^{r*}$ for every $j \in J_i$, since this will also entail $Q_i \subset P_{J_i} \subset P_{j^{r*}} = P_j^{r*}$ for every $j \in J_i$.

Fix $j_0 \in J_i$ and let $j \in J_i$ be arbitrary. We want to show $j \in j_0^{r*}$. Because of $j_0, j \in J_i$, there are $x \in Q_i \cap P_{j_0}$ and $y \in Q_i \cap P_j$. Since $Q_i$ is path-connected, there is a continuous map $\gamma : [0,1] \to Q_i$ with $\gamma(0) = x$ and $\gamma(1) = y$.

For $k \in J$ with $\varnothing \neq P_k \cap \gamma([0,1]) \subset P_k \cap Q_i$ (which in particular implies $k \in J_i$), define

$$t_k := \sup\{t \in [0,1] \mid \gamma(t) \in P_k\}. \tag{3.3.1}$$

We note $t_k > 0$, since $t_k = 0$ would imply $\gamma^{-1}(P_k) = \{0\}$, which is not open in $[0,1]$, in contradiction to the fact that $\gamma$ is continuous and that $P_k$ is open. In case of $t_k \neq 1$, we have $\gamma(t_k) \notin P_k$, because otherwise there would be (again since $\gamma$ is continuous and $P_k$ is open) some $0 < \varepsilon < 1 - t_k$ with $\gamma(t_k + \varepsilon) \in P_k$, in contradiction to the definition of $t_k$.

By induction, let us now construct a finite sequence $j_1, \ldots, j_{r+1} \in J_i$ with $\gamma([0,1]) \cap P_{j_\ell} \neq \varnothing$ for all $\ell \in \{1, \ldots, r+1\}$ and such that the following three properties hold for all $\ell \in \{2, \ldots, r+1\}$:

1. We have $j_\ell = j$ or $j_\ell \notin \{j_1, \ldots, j_{\ell-1}\}$.
2. We have $j_\ell = j$ if $t_{j_{\ell-1}} = 1$.
3. We have $\gamma(t_{j_{\ell-1}}) \in P_{j_\ell}$. In particular, this implies $t_{j_\ell} \geq t_{j_{\ell-1}}$.

For $\ell = 1$, let us set $j_1 := j_0$. Because of $x = \gamma(0) \in \gamma([0,1]) \cap P_{j_0}$, all desired properties are satisfied, since the three additional properties only have to hold for $\ell \geq 2$. Now let $\ell \in \underline{r}$ and assume that

$j_1, \dots, j_\ell$ have already been constructed such that the properties above are fulfilled.

In case of $j_\ell = j$, define $j_{\ell+1} = j$. With this choice, properties (1) and (2) are trivially satisfied. Furthermore, the identity $\gamma(1) = y \in P_j$ yields $t_{j_\ell} = t_j = 1$ and hence $\gamma(t_{j_{(\ell+1)-1}}) = \gamma(1) = y \in P_j = P_{j_{\ell+1}}$, so that condition (3) is also satisfied.

In case of $j_\ell \neq j$, we distinguish two subcases:

A. $\gamma(t_{j_\ell}) \in P_j$. In this case, set $j_{\ell+1} := j$. We have $j_{\ell+1} \notin \{j_1, \dots, j_\ell\}$, because otherwise there would be a *maximal* index $k \in \underline{\ell}$ with $j = j_{\ell+1} = j_k$. Because of $j_\ell \neq j$, this yields $k \in \underline{\ell-1}$. But $t_{j_k} = t_j = 1$, so that condition (2) implies $j_{k+1} = j$ with $k+1 \in \underline{\ell}$, in contradiction to maximality of $k$. Hence, condition (1) holds for $\ell+1$ instead of $\ell$.

   Condition (2) is trivial because of $j_{\ell+1} = j$ and condition (3) is true, since we have $\gamma(t_{j_{(\ell+1)-1}}) = \gamma(t_{j_\ell}) \in P_j = P_{j_{\ell+1}}$ by assumption of the current case.

B. $\gamma(t_{j_\ell}) \notin P_j$. We have $t_{j_{(\ell+1)-1}} = t_{j_\ell} \neq 1$, because $t_{j_\ell} = 1$ would imply $\gamma(t_{j_\ell}) = \gamma(1) = y \in P_j$. Hence, condition (2) is trivially satisfied for any choice of $j_{\ell+1} \in J_i$ with $\gamma([0,1]) \cap P_{j_{\ell+1}} \neq \varnothing$.

   We have $\gamma(t_{j_\ell}) \in \gamma([0,1]) \subset Q_i \subset \bigcup_{j \in J_i} P_j$, so that there is some $j_{\ell+1} \in J_i$ with
   $$\gamma(t_{j_{(\ell+1)-1}}) = \gamma(t_{j_\ell}) \in P_{j_{\ell+1}}.$$
   With this choice of $j_{\ell+1} \in J_i$, condition (3) is trivially satisfied, which also yields $t_{j_{\ell+1}} \geq t_{j_\ell}$.

   We even have $t_{j_\ell} < t_{j_{\ell+1}}$. Indeed, in case of $t_{j_{\ell+1}} = 1$, this is clear, since $t_{j_\ell} \neq 1$ was shown above. In case of $t_{j_{\ell+1}} \neq 1$, the considerations after equation (3.3.1) imply $\gamma(t_{j_{\ell+1}}) \notin P_{j_{\ell+1}}$. Because of $\gamma(t_{j_\ell}) \in P_{j_{\ell+1}}$, this yields $t_{j_{\ell+1}} \neq t_{j_\ell}$ and hence $t_{j_\ell} < t_{j_{\ell+1}}$.

   By condition (3) (applied to the previous indices), this yields $t_{j_k} \leq t_{j_\ell} < t_{j_{\ell+1}}$ for all $k \in \underline{\ell}$. In particular, $j_{\ell+1} \notin \{j_1, \dots, j_\ell\}$, so that condition (1) is also satisfied.

If $j_\ell \neq j$ was true for all $\ell \in \underline{r+1}$, condition (1) would imply that $j_1, \dots, j_{r+1} \in J_i$ are pairwise distinct, in contradiction to $r = |J_i|$. Thus, there is a *maximal* $k \in \underline{r+1}$ with $j_k = j$. This implies $t_{j_k} = t_j = 1$. In case of $k < r+1$, condition (2) would yield $j_{k+1} = j$, in contradiction to the maximality of $k$. Hence, $k = r+1$, which implies $j_{r+1} = j$.

Finally, let us show $P_{j_\ell} \cap P_{j_{\ell+1}} \neq \varnothing$ for all $\ell \in \underline{r}$. Condition (3) shows $\gamma(t_{j_\ell}) \in P_{j_{\ell+1}}$. By definition of $t_{j_\ell}$, there is a sequence $(s_n)_{n \in \mathbb{N}}$ with $\gamma(s_n) \in P_{j_\ell}$ and $s_n \uparrow t_{j_\ell} > 0$. Since $\gamma$ is continuous and $P_{j_{\ell+1}}$ is open with $\gamma(t_{j_\ell}) \in P_{j_{\ell+1}}$, we get $\gamma(s_n) \in P_{j_{\ell+1}}$ for $n \in \mathbb{N}$ large enough. Hence, $\gamma(s_n) \in P_{j_\ell} \cap P_{j_{\ell+1}} \neq \varnothing$ for those $n$.

All in all, we have shown that $(j_1, \dots, j_{r+1})$ is a $\mathcal{P}$-chain of length $r$ (cf. Remark 3.2.2) from $j_1 = j_0$ to $j_{r+1} = j$. Hence, $j \in j_0^{r*}$. As seen above, this completes the proof. □

The above lemma immediately yields the following corollary.

**Corollary 3.3.4.** *Let $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ be families of subsets of $\mathbb{R}^d$ such that each $Q_i$ is path-connected and such that each $P_j$ is open.*

*Assume that $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ with $\bigcup_{i\in I} Q_i \subset \bigcup_{j\in J} P_j$. Then $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ with $k(\mathcal{Q},\mathcal{P}) \leq N(\mathcal{Q},\mathcal{P})$.*

One of the fundamental tools that we will use again and again in the remainder of the thesis is the following DISJOINTIZATION LEMMA for admissible coverings that was developed by Feichtinger and Gröbner in [27, Lemma 2.9].

**Lemma 3.3.5.** *[27, Lemma 2.9] Let $\mathcal{O} \neq \varnothing$ be a set and assume that the family $\mathcal{Q} = (Q_i)_{i\in I}$ is an admissible covering of $\mathcal{O}$.*

*Let $m \in \mathbb{N}_0$ be arbitrary. Then there exists a finite partition $I = \biguplus_{r=1}^{r_0} I^{(r)}$ with $Q_i^{m*} \cap Q_j^{m*} = \varnothing$ for all $i, j \in I^{(r)}$ with $i \neq j$ and all $r \in \{1, \dots, r_0\}$.*

*We call the minimal number $r_0 = r_0(\mathcal{Q}, m)$ such that the above is true the* DISJOINTIZATION NUMBER OF ORDER $m$ FOR THE COVERING *$\mathcal{Q}$.*

The next lemma provides a negative result which shows that restricting attention solely to coverings for which one is moderate with respect to the other would prevent one from handling the case where the two covered sets are different.

The prerequisites of the lemma are always satisfied if $\mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ are of full measure (this is in particular the case if $\mathcal{O}, \mathcal{O}'$ are dual orbits of admissible dilation groups as in Chapter 4) with $\mathcal{O} \subsetneq \mathcal{O}'$.

**Lemma 3.3.6.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p \in (0, \infty]$ and assume that $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ and $\mathcal{P} = (S_j P_j' + c_j)_{j\in J}$ are $L^p$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively.*

*Assume $\mathcal{O}' \cap \partial\mathcal{O} \neq \varnothing$. Then*

1. *$\mathcal{P}$ is* not *weakly subordinate to $\mathcal{Q}$.*
2. *If $\mathcal{Q}$ is* tight *and almost subordinate to $\mathcal{P}$, then $\mathcal{Q}$ is* not *moderate with respect to $\mathcal{P}$.*

*Proof.* Choose $x \in \mathcal{O}' \cap \partial\mathcal{O}$. By Lemma 3.2.12, there is some $j \in J$ with $x \in P_j^\circ$. Choose a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{O}^{\mathbb{N}}$ with $x_n \xrightarrow[n\to\infty]{} x$ and note that we have $x_n \in P_j^\circ$ for $n$ sufficiently large, so that we can assume $x_n \in P_j^\circ$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, choose $i_n \in I$ with $x_n \in Q_{i_n}$. Assume towards a contradiction that there is some $i \in I$ with $i_n = i$ for infinitely many $n \in \mathbb{N}$. Since $\overline{Q_i} \subset \mathcal{O}$ holds by Lemma 3.2.12, this implies $x \in \overline{Q_i} \subset \mathcal{O} = \mathcal{O}^\circ$ in contradiction to $x \in \partial\mathcal{O}$. Thus, restricting to a subsequence, we can assume that the $(i_n)_{n\in\mathbb{N}}$ are pairwise distinct. Using $x_n \in Q_{i_n} \cap P_j^\circ$, we derive that $I_j \supset \{i_n \mid n \in \mathbb{N}\}$ is infinite, so that $\mathcal{P}$ is not weakly subordinate to $\mathcal{Q}$.

Now assume towards a contradiction that $\mathcal{Q}$ is tight, almost subordinate to $\mathcal{P}$ and moderate with respect to $\mathcal{P}$. Using Lemma 3.3.2 and the definition of a $\mathcal{P}$-moderate covering, we see that there is some $\ell \in \mathbb{N}$ with $Q_{i_n} \subset P_j^{\ell *}$ for all $n \in \mathbb{N}$ and some $C > 0$ with

$$|\det(T_{i_n}^{-1} T_{i_m})| \leq C$$

for all $n, m \in \mathbb{N}$ because of $Q_{i_n} \cap P_j \neq \varnothing$.

By tightness of $\mathcal{Q}$, there is some $\varepsilon > 0$ such that $B_\varepsilon(c_i) \subset Q_i'$ holds for all $i \in I$ and suitable $c_i \in \mathbb{R}^d$. This implies

$$\begin{aligned}
\lambda\left(Q_{i_n}^\circ\right) &= \lambda\left(T_{i_n}\left(Q_{i_n}'\right)^\circ + b_{i_n}\right) \\
&= |\det(T_{i_n})| \cdot \lambda\left(\left(Q_{i_n}'\right)^\circ\right) \\
&\geq |\det(T_{i_n})| \cdot c_\varepsilon \\
&\geq \frac{c_\varepsilon}{C} \cdot |\det(T_{i_1})| =: C'
\end{aligned}$$

for all $n \in \mathbb{N}$.

Since an admissible covering is of finite height, we get the estimate $\sum_{n\in\mathbb{N}} \chi_{Q_{i_n}^\circ} \leq \sum_{i\in I} \chi_{Q_i} \leq N_{\mathcal{Q}}$. But above, we showed $Q_{i_n} \subset P_j^{\ell *}$ for all $n \in \mathbb{N}$. Together, we conclude

$$\sum_{n\in\mathbb{N}} \chi_{Q_{i_n}^\circ} \leq N_{\mathcal{Q}} \cdot \chi_{\overline{P_j^{\ell *}}}.$$

Integration yields

$$\infty > N_{\mathcal{Q}} \cdot \lambda\left(\overline{P_j^{\ell *}}\right) \geq \sum_{n\in\mathbb{N}} \lambda\left(Q_{i_n}^\circ\right) \geq \sum_{n\in\mathbb{N}} C' = \infty,$$

contradiction. Here, the finite union $\overline{P_j^{\ell *}} = \bigcup_{m\in j^{\ell *}} \overline{P_m}$ is of finite measure, by compactness of each $\overline{P_m}$ (cf. Lemma 3.2.12). □

Using similar techniques as in the proof above, we now establish an easy way to calculate the cardinality $\left|I_j\right|$ of the set of $\mathcal{Q}$-neighbors of $P_j$. This method of estimating $\left|I_j\right|$ is implicitly used in [49] for the concrete setting of $\alpha$-modulation spaces, but not stated explicitly. In the present generality, it seems to be a new observation which will be of great use to us in Chapter 5, in particular in the second part of Section 5.3.

It is important to note that we assume $\mathcal{Q}$ to be relatively moderate with respect to $\mathcal{P}$. Without this assumption, it is easy to see that an estimate as in equation (3.3.2) below fails in general.

**Lemma 3.3.7.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and assume that the families $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ and $\mathcal{P} = \left(P_j\right)_{j\in J} = (S_j P_j' + c_j)_{j\in J}$ are* tight *semi-structured coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively.*

*Finally, let $I_0 \subset I$, $J_0 \subset J$ and assume that*

1. *$\mathcal{Q}_{I_0} := (Q_i)_{i\in I_0}$ is almost subordinate to $\mathcal{P}$,*

2. *$\mathcal{Q}_{I_0}$ is moderate with respect to $\mathcal{P}_{J_0} := \left(P_j\right)_{j\in J_0}$,*

3. *There is some $r \in \mathbb{N}_0$ such that (up to a set of measure zero)*

$$P_j \subset \bigcup_{i\in I_0\cap I_j} Q_i^{r*}$$

*holds for all $j \in J_0$ with $I_0 \cap I_j \neq \varnothing$.*

*The moderateness assumption means explicitly that there is a constant $C = C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}\right)$ with $|\det(T_i)| \leq C \cdot |\det(T_\ell)|$ for all $i, \ell \in I_0$ for which there is some $j \in J_0$ with $Q_i \cap P_j \neq \varnothing \neq Q_\ell \cap P_j$.*

*Then there are positive constants*

$$C_1 = C_1\left(d, k\left(\mathcal{Q}_{I_0}, \mathcal{P}\right), \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}\right), R_{\mathcal{P}}, C_{\mathcal{P}}\right)$$

*and*

$$C_2 = C_2\left(d, r, \varepsilon_{\mathcal{P}}, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}\right), R_{\mathcal{Q}}, C_{\mathcal{Q}}\right)$$

*with*

$$C_1^{-1} \cdot \left|I_0 \cap I_j\right| \leq \frac{\left|\det\left(S_j\right)\right|}{\left|\det\left(T_i\right)\right|} \leq C_2 \cdot \left|I_0 \cap I_j\right| \qquad \forall i \in I_0 \cap I_j \tag{3.3.2}$$

*for all $j \in J_0$. Finally,*

A. *assumption 3 is only needed for the right estimate in equation* (3.3.2),

B. *tightness of $\mathcal{Q}$ is only needed for the left estimate in equation* (3.3.2),

C. *tightness of $\mathcal{P}$ is only needed for the right estimate in equation* (3.3.2).

*Remark.* It is worth noting that the third requirement above is trivially satisfied (with $r = 0$ and arbitrary $J_0$) if we have $\mathcal{O}' \subset \mathcal{O}$ (up to a set of measure zero) and $I_0 = I$, because in that case we have (up to a set of measure zero)

$$P_j \subset \mathcal{O}' \subset \mathcal{O} = \bigcup_{i\in I} Q_i = \bigcup_{i\in I_0} Q_i^{0*}$$

which easily implies $P_j \subset \bigcup_{i\in I_0\cap I_j} Q_i^{0*}$ (up to a set of measure zero).

It should also be noted that the statement in equation (3.3.2) is void for $j \in J_0$ with $I_0 \cap I_j = \varnothing$.

*Proof.* Corollary 3.2.14 ensures existence of constants $c = c\left(\varepsilon_{\mathcal{Q}}, d\right) > 0$ and $C = C\left(R_{\mathcal{Q}}, d\right) > 0$ with

$$c^{-1} \cdot |\det(T_i)| \leq \lambda\left(Q_i^\circ\right) \leq \lambda\left(\overline{Q_i}\right) \leq C \cdot |\det(T_i)| \tag{3.3.3}$$

for all $i \in I$, as well as $c_1 = c_1\left(\varepsilon_{\mathcal{P}}, d\right) > 0$ and $C_1 = C_1\left(R_{\mathcal{P}}, d\right) > 0$ so that an analogous estimate holds for $\mathcal{P}$.

Define $K := C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}_{J_0}\right)$, so that $|\det\left(T_i^{-1} T_\ell\right)| \leq K$ holds for all $j \in J_0$ and $i, \ell \in I_j \cap I_0$. Lemma 3.2.18 shows that $\mathcal{Q}^{r*}$ is also a tight semi-structured covering with $\varepsilon_{\mathcal{Q}^{r*}} \geq \varepsilon_{\mathcal{Q}}$ and $R_{\mathcal{Q}^{r*}} \leq \left(2C_{\mathcal{Q}} + 1\right)^r R_{\mathcal{Q}}$, so that equation (3.3.3) is also valid for $\mathcal{Q}^{r*}$ with (possibly different) constants $c' = c'\left(d, \varepsilon_{\mathcal{Q}}\right) > 0$ and $C' = C'\left(d, r, C_{\mathcal{Q}}, R_{\mathcal{Q}}\right) > 0$.

Choose an arbitrary $j \in J_0$ with $I_0 \cap I_j \neq \varnothing$ (otherwise, the claim is trivial) and note $P_j \subset \bigcup_{i \in I_0 \cap I_j} Q_i^{r*}$ (up to a set of measure zero). Fix any $i_0 \in I_0 \cap I_j$. This yields $|\det(T_i)| \leq K \cdot |\det(T_{i_0})|$ for all $i \in I_j \cap I_0$. If $\left|I_0 \cap I_j\right|$ is infinite, the upper estimate in equation (3.3.2) is trivial. Hence, we can assume that $I_0 \cap I_j$ is finite. In this case, we get

$$\begin{aligned} c_1^{-1} \cdot \left|\det(S_j)\right| \leq \lambda(P_j^\circ) &\leq \lambda\Bigg(\bigcup_{i \in I_0 \cap I_j} \overline{Q_i^{r*}}\Bigg) \\ &\leq C' \cdot \sum_{i \in I_0 \cap I_j} |\det(T_i)| \\ &\leq C'K \cdot |\det(T_{i_0})| \cdot \left|I_0 \cap I_j\right|. \end{aligned}$$

Since $i_0 \in I_0 \cap I_j$ was arbitrary, the upper estimate in equation (3.3.2) is established.

For the lower estimate, note that Lemma 3.3.2 yields $Q_i \subset P_j^{\ell*}$ for all $i \in I_0$ and $j \in J_i$, where we defined $\ell := 2k(\mathcal{Q}_{I_0}, \mathcal{P}) + 2$. Furthermore, Lemma 3.2.18 shows that $\mathcal{P}^{\ell*}$ is a tight semi-structured covering, so that equation (3.3.3) also holds for $\mathcal{P}^{\ell*}$ with constants $c_1' > 0$ and $C_1' = C_1'(d, \ell, R_{\mathcal{P}}, C_{\mathcal{P}}) > 0$.

Finally, admissibility of $\mathcal{Q}$ easily entails $\sum_{i \in I} \chi_{Q_i}(x) \leq N_{\mathcal{Q}}$ for all $x \in \mathbb{R}^d$ and hence $\sum_{i \in I_0 \cap I_j} \chi_{Q_i}(x) \leq N_{\mathcal{Q}} \cdot \chi_{\overline{P_j^{\ell*}}}(x)$ for all $j \in J$. Now, let $j \in J_0$ be arbitrary, let $\Gamma \subset I_0 \cap I_j$ be a finite subset and let $i_0 \in I_0 \cap I_j$ be arbitrary. We have

$$\begin{aligned} \frac{1}{cK} \cdot |\det(T_{i_0})| \cdot |\Gamma| &\leq \sum_{i \in \Gamma} c^{-1} \cdot |\det(T_i)| \\ &\leq \sum_{i \in \Gamma} \lambda(Q_i^\circ) \\ &= \int_{\mathbb{R}^d} \sum_{i \in \Gamma} \chi_{Q_i^\circ}(x) \, \mathrm{d}x \\ &\leq N_{\mathcal{Q}} \cdot \int_{\mathbb{R}^d} \chi_{\overline{P_j^{\ell*}}}(x) \, \mathrm{d}x \\ &\leq N_{\mathcal{Q}} C_1' \cdot \left|\det(S_j)\right|. \end{aligned}$$

Since $\Gamma \subset I_0 \cap I_j$ was an arbitrary finite subset, we see that $I_0 \cap I_j$ is finite and that the lower bound in equation (3.3.2) is satisfied. □

## 3.4 (FOURIER SIDE) DECOMPOSITION SPACES

In this section, we will finally give a formal definition of the decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, Y)$. But before we do this, we first clarify our assumptions on the space $Y$.

**Definition 3.4.1.** (cf. [27, Definition 2.5])

Let $\mathcal{Q} = (Q_i)_{i \in I}$ be a covering of a set $X \neq \varnothing$. We say that a Quasi-Banach space $Y$ is $\mathcal{Q}$-REGULAR, if $Y$ is a solid sequence space over $I$

as in Lemma 2.2.4 and if $Y$ is INVARIANT UNDER $\mathcal{Q}$-CLUSTERING, i.e., if the $\mathcal{Q}$-CLUSTERING MAP

$$Y \to Y, (c_i)_{i\in I} \mapsto (c_i^*)_{i\in I} := \left(\sum_{j\in i^*} c_j\right)_{i\in I}$$

is well-defined and bounded.

*Remark.* If all of the sets $i^*$ are finite and if $Y$ is complete, then the closed graph theorem (cf. [61, Theorem 2.15]), together with the continuous embedding $Y \hookrightarrow \mathbb{C}^I$ (which follows from Lemma 3.4.12 below) and the fact that the clustering map is continuous w.r.t. the (Hausdorff!) product topology on $\mathbb{C}^I$, imply that the clustering map is bounded iff it is well-defined.

Here, we recall from the remark after Lemma 2.1.5 that each Quasi-Banach space is an $F$-space (in the sense of [61, Section 1.8]), so that the closed graph theorem is applicable.

The most important example of $\mathcal{Q}$-regular sequence spaces that we will consider are the weighted $\ell^q$-spaces. The following lemma can also be seen as a special case of [27, Lemma 3.2], but our proof gives an explicit constant for the norm of the $\mathcal{Q}$-clustering map. This constant is used frequently in Chapter 5, see especially Corollary 5.1.5 and the proof of Theorem 5.1.6.

**Lemma 3.4.2.** *Let $\mathcal{Q} = (Q_i)_{i\in I}$ be an admissible covering of a set $X \neq \varnothing$, let $u = (u_i)_{i\in I}$ be $\mathcal{Q}$-moderate and let $q \in (0,\infty]$. Then $\ell^q_u(I)$ is $\mathcal{Q}$-regular.*

*A bound for the norm of the $\mathcal{Q}$-clustering map is given by $N_{\mathcal{Q}}^{1+\frac{1}{q}} \cdot C_{u,\mathcal{Q}}$.*

*Proof.* Let us first assume $q = \infty$. In this case, we have

$$\begin{aligned}
|u_i \cdot c_i^*| &\leq \sum_{j\in i^*} u_i \left|c_j\right| \\
&\leq C_{u,\mathcal{Q}} \cdot \sum_{j\in i^*} u_j \left|c_j\right| \\
&\leq C_{u,\mathcal{Q}} \cdot |i^*| \cdot \|c\|_{\ell^\infty_u(I)} \\
&\leq C_{u,\mathcal{Q}} N_{\mathcal{Q}} \cdot \|c\|_{\ell^\infty_u(I)} < \infty
\end{aligned}$$

for all $c = (c_i)_{i\in I} \in \ell^\infty_u(I)$, which proves the claim in this case.

Now, let $q \in (0,\infty)$ and $c = (c_i)_{i\in I} \in \ell^q_u(I)$. For arbitrary $i \in I$, we first note

$$\begin{aligned}
\left(\sum_{j\in i^*} u_i \left|c_j\right|\right)^q &\leq \left[|i^*| \cdot \max\left\{u_i \left|c_j\right| \,\middle|\, j \in i^*\right\}\right]^q \\
&\leq N_{\mathcal{Q}}^q \cdot \max\left\{\left(u_i \left|c_j\right|\right)^q \,\middle|\, j \in i^*\right\} \\
&\leq N_{\mathcal{Q}}^q \cdot \sum_{j\in i^*} \left(u_i \left|c_j\right|\right)^q \\
&\leq (N_{\mathcal{Q}} C_{u,\mathcal{Q}})^q \cdot \sum_{j\in i^*} \left(u_j \left|c_j\right|\right)^q .
\end{aligned}$$

By using the equivalence $i \in j^* \iff j \in i^*$, this implies

$$\begin{aligned}
\left\|(c_i^*)_{i\in I}\right\|_{\ell_u^q}^q &= \sum_{i\in I} (u_i \cdot |c_i^*|)^q \\
&\leq \sum_{i\in I} \left( \sum_{j\in i^*} u_i \left|c_j\right| \right)^q \\
&\leq (N_{\mathcal{Q}} C_{u,\mathcal{Q}})^q \cdot \sum_{i\in I} \sum_{j\in i^*} \left(u_j \left|c_j\right|\right)^q \\
&= (N_{\mathcal{Q}} C_{u,\mathcal{Q}})^q \cdot \sum_{j\in I} \sum_{i\in j^*} \left(u_j \left|c_j\right|\right)^q \\
&= (N_{\mathcal{Q}} C_{u,\mathcal{Q}})^q \cdot \sum_{j\in I} |j^*| \left(u_j \left|c_j\right|\right)^q \\
&\leq N_{\mathcal{Q}} \left(N_{\mathcal{Q}} C_{u,\mathcal{Q}}\right)^q \cdot \|c\|_{\ell_u^q}^q < \infty.
\end{aligned}$$

This proves the claim for $q \in (0, \infty)$. □

For later use, we state the following result which connects iterated applications of the clustering map and summing over the clustered index sets $i^{(\ell+1)*}$.

**Lemma 3.4.3.** *Let $\mathcal{Q} = (Q_i)_{i\in I}$ be an admissible covering of a set $X \neq \varnothing$. Let*

$$\Gamma : \mathbb{C}^I \to \mathbb{C}^I, c \mapsto c^*$$

*denote the clustering map.*

*For each sequence $c = (c_i)_{i\in I}$ with nonnegative terms $c_i \geq 0$ and each $\ell \in \mathbb{N}$, we have*

$$\left(\Gamma^\ell c\right)_i \geq \sum_{j\in i^{\ell *}} c_j \quad \text{for all } i \in I. \tag{3.4.1}$$

*In particular, if $Y$ is $\mathcal{Q}$-regular, then the $\ell$-FOLD CLUSTERING MAP $\Theta_\ell : Y \to Y$ with*

$$\left(\Theta_\ell c\right)_i := \sum_{j\in i^{\ell *}} c_j$$

*is well-defined and bounded with $|||\Theta_\ell|||_{Y\to Y} \leq |||\Gamma|||_{Y\to Y}^\ell$.*

*Proof.* We first prove the estimate (3.4.1) by induction on $\ell \in \mathbb{N}$. For $\ell = 1$, we have equality by definition of $\Gamma$.

For the induction step, fix for each $j \in i^{(\ell+1)*}$ some $k_j \in i^{\ell *}$ with $j \in k_j^*$. Since each term $c_i$ is nonnegative, this yields

$$\begin{aligned}
\sum_{j\in i^{(\ell+1)*}} c_j &= \sum_{k\in i^{\ell *}} \sum_{\substack{j\in i^{(\ell+1)*} \\ \text{with } k_j = k}} c_j \\
&\leq \sum_{k\in i^{\ell *}} \sum_{j\in k^*} c_j = \sum_{k\in i^{\ell *}} (\Gamma c)_k \\
&\overset{(\dagger)}{\leq} \left(\Gamma^\ell \Gamma c\right)_i = \left(\Gamma^{\ell+1} c\right)_i,
\end{aligned}$$

where we employed the induction hypothesis (with the nonnegative(!) sequence $\Gamma c$ instead of $c$) at (†).

To prove boundedness of $\Theta_\ell$, let $(c_i)_{i\in I} \in Y$ be arbitrary and set $d := (|c_i|)_{i\in I}$. Observe $d \in Y$ with $\|d\|_Y = \|c\|_Y$, since $Y$ is solid. As seen above, we have

$$|(\Theta_\ell c)_i| \leq \sum_{j\in i^{\ell *}} |c_j| \leq \left(\Gamma^\ell d\right)_i$$

for all $i \in I$. Because of $\Gamma^\ell d \in Y$, solidity of $Y$ implies $\Theta_\ell c \in Y$ with

$$\|\Theta_\ell c\|_Y \leq \|\Gamma^\ell d\|_Y \leq |||\Gamma^\ell|||_{Y\to Y} \|d\|_Y \leq |||\Gamma|||^\ell_{Y\to Y} \cdot \|c\|_Y < \infty. \qquad \square$$

Now, we are in a position to define the Fourier-side decomposition spaces.

**Definition 3.4.4.** Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be an open set and let $p \in (0,\infty]$. Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$ with $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ and that $Y$ is a $\mathcal{Q}$-regular solid sequence space on $I$.

For $f \in \mathcal{D}'(\mathcal{O})$, define

$$\|f\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} := \|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} := \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i\in I} \right\|_Y \in [0,\infty],$$

with the convention that for a family $c = (c_i)_{i\in I}$ with $c_i \in [0,\infty]$, the expression $\|c\|_Y$ is to be read as $\infty$ if $c_i = \infty$ for some $i \in I$ or if $c \notin Y$.

Define the FOURIER-SIDE DECOMPOSITION SPACE $\mathcal{D}(\mathcal{Q}, L^p, Y)$ with respect to the covering $\mathcal{Q}$, integrability exponent $p$ and global component $Y$ as

$$\mathcal{D}_\mathcal{F}(\mathcal{Q}, L^p, Y) := \mathcal{D}(\mathcal{Q}, L^p, Y) := \left\{ f \in \mathcal{D}'(\mathcal{O}) \;\middle|\; \|f\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} < \infty \right\}.$$

*Remark.* We remark that $\varphi_i f$ is a distribution on $\mathcal{O}$ with compact support, which thus extends to a (tempered) distribution on $\mathbb{R}^d$. By Lemma 3.1.2, this implies that $\mathcal{F}^{-1}(\varphi_i f) \in \mathcal{S}'(\mathbb{R}^d)$ is given by (integration against) a smooth function. Thus, it makes sense to write $\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$, with the caveat that this expression could be infinite.

We finally remark that the notations $\|\cdot\|_{\mathcal{D}(\mathcal{Q},L^p,Y)}$ and $\mathcal{D}(\mathcal{Q}, L^p, Y)$ suppress the family $(\varphi_i)_{i\in I}$ used to define the (quasi)-norm above. We will see below (cf. Corollary 3.4.11) that the resulting space is independent of the chosen $L^p$-BAPU, with equivalent quasi-norms for different choices, so that this is justified.

For completeness, we also define a "space-side" version of these spaces. To this end, we first introduce the reservoir $Z'(\mathcal{O})$ which is used for these spaces.

**Definition 3.4.5.** For $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ open, we define

$$Z(\mathcal{O}) := \mathcal{F}(C_c^\infty(\mathcal{O})) := \{\widehat{f} \,|\, f \in C_c^\infty(\mathcal{O})\} \leq \mathcal{S}(\mathbb{R}^d)$$

and endow this space with the unique topology that makes the Fourier transform

$$\mathcal{F} : C_c^\infty(\mathcal{O}) \to Z(\mathcal{O})$$

a homeomorphism.

We equip the topological dual space $Z'(\mathcal{O}) := [Z(\mathcal{O})]'$ of $Z(\mathcal{O})$ with the weak-$*$-topology, i.e., with the topology of pointwise convergence on $Z(\mathcal{O})$.

Finally, as on the Schwartz space, we extend the Fourier transform by duality to $Z'(\mathcal{O})$, i.e. we define

$$\mathcal{F} : Z'(\mathcal{O}) \to \mathcal{D}'(\mathcal{O}), f \mapsto f \circ \mathcal{F}. \tag{3.4.2}$$

As usual, we write $\widehat{f} := \mathcal{F}f$ for $f \in Z'(\mathcal{O})$.

*Remark.* Since $\mathcal{F} : C_c^\infty(\mathcal{O}) \to Z(\mathcal{O})$ is a linear homeomorphism, the Fourier transform as defined in equation (3.4.2) is easily seen to be a homeomorphism.

With these definitions, we are finally in a position to define the space-side decomposition spaces.

**Definition 3.4.6.** Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be an open set and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$ with $L^p$-BAPU $(\varphi_i)_{i \in I}$ and that $Y$ is a $\mathcal{Q}$-regular solid sequence space on $I$.

For $f \in Z'(\mathcal{O})$, set

$$\|f\|_{\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y)} := \|\widehat{f}\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} = \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i \widehat{f}) \right\|_{L^p} \right)_{i \in I} \right\|_Y \in [0, \infty]$$

and define the SPACE-SIDE DECOMPOSITION SPACE $\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y)$ with respect to the covering $\mathcal{Q}$, integrability exponent $p$ and global component $Y$ by

$$\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y) := \left\{ f \in Z'(\mathcal{O}) \,\middle|\, \|f\|_{\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y)} < \infty \right\}.$$

*Remark* 3.4.7. Since the Fourier transform $\mathcal{F} : Z'(\mathcal{O}) \to \mathcal{D}'(\mathcal{O})$ is an isomorphism, it is clear that the Fourier transform restricts to an (isometric) isomorphism

$$\mathcal{F} : \mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y) \to \mathcal{D}(\mathcal{Q}, L^p, Y).$$

Hence, for all practical purposes, it does not matter whether one considers the "space-side" or the "Fourier-side" version of these spaces. But since the space $\mathcal{D}'(\mathcal{O})$ is a widely known standard space, whereas $Z'(\mathcal{O})$ is not, we prefer to work with the "Fourier-side" spaces.

The reasons for using the space of distributions $\mathcal{D}'(\mathcal{O})$ instead of the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions are the following:

A. We want to allow the case $\mathcal{O} \subsetneq \mathbb{R}^d$. As we will see in the next chapter, this is for example the case if $\mathcal{O}$ is the dual orbit of a dilation group $H \leq \mathrm{GL}(\mathbb{R}^d)$, which is one of our main applications. If we were to take the space $\mathcal{S}'(\mathbb{R}^d)$, the decomposition space (quasi)-norm would not be positive definite, or we would have to factor out a certain subspace of $\mathcal{S}'(\mathbb{R}^d)$. This is for example done in the usual definition of homogeneous Besov spaces, which are subspaces of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$, where $\mathcal{P}$ is the space of polynomials. Here, it seems more natural to use the space $\mathcal{D}'(\mathcal{O})$.

B. In case of $\mathcal{O} = \mathbb{R}^d$, one could use $\mathcal{S}'(\mathbb{R}^d)$ as the reservoir. For example, Borup and Nielsen[5] use the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions as their reservoir, i.e. they consider the space

$$\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, Y) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \;\middle|\; \|f\|_{\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y)} < \infty \right\}.$$

But as we will see below (cf. Example 3.4.14), this does in general *not* yield a complete space, even for the case $Y = \ell^1_u$ with a moderate weight $u$.

We finally remark that the notations $Z(\mathcal{O})$ and $Z'(\mathcal{O})$ are inspired by Triebel's book [65], cf. in particular [65, Section 2.2.1-2.2.4]. Triebel also defines spaces very similar to $\mathcal{D}_{\text{space}}(\mathcal{Q}, L^p, Y)$, but restricts to the case in which the covering $\mathcal{Q}$ consists of (closed) rectangles with sides parallel to the coordinate axes. This is due to the fact that he also considers spaces of Triebel-Lizorkin type as opposed to the spaces of Besov type that we consider here. ◀

In the remainder of this section, we show that the decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, Y)$ are indeed well-defined Quasi-Banach spaces, in particular independent of the chosen $L^p$-BAPU. Instead of just proving independence of the chosen $L^p$-BAPU, we will show – for later use – a slightly stronger claim for which we need the notion of an $L^p$-BOUNDED (CONTROL) SYSTEM. This concept is based on that of a BOUNDED CONTROL SYSTEM which was introduced in [27, Definition 2.6].

**Definition 3.4.8.** Let $p \in (0, \infty]$ and let $\mathcal{Q} = (Q_i)_{i \in I}$ be an admissible covering of an open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$. For $p \in (0, 1)$, assume additionally that $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$ is semi-structured.

A family $\Gamma = (\gamma_i)_{i \in I}$ of functions $\gamma_i \in C^\infty_c(\mathcal{O})$ is called an $L^p$-BOUNDED SYSTEM for $\mathcal{Q}$ if the following conditions are satisfied:

A. there is some $\ell = \ell_{\Gamma, \mathcal{Q}} \in \mathbb{N}_0$ with $\gamma_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{\ell *}$ for all $i \in I$,

*We remark once more that the terminology "$L^p$-bounded" does* not *refer to the fact that the $\mathcal{F}L^p$-norms $\left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p}$ are bounded, but to the fact that the $\gamma_i$ form a uniformly bounded family of $L^p$-Fourier multipliers, at least for $p \in [1,\infty]$. For $p \in (0,1)$, this is only true restricted to the space of $L^p$-functions with Fourier support near $Q_i$, cf. Corollary 3.2.15.*

b. the constant

$$C_{\mathcal{Q},\Gamma,p} := \begin{cases} \sup_{i\in I} \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^1}, & \text{for } p \in [1,\infty], \\ \sup_{i\in I} |\det(T_i)|^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p}, & \text{for } p \in (0,1) \end{cases}$$

is finite.

If furthermore $\gamma_i \equiv 1$ on $Q_i$ holds for all $i \in I$, we say that $\Gamma$ is an $L^p$-BOUNDED CONTROL SYSTEM for $\mathcal{Q}$.

*Remark* 3.4.9. Fix $\ell \in \mathbb{N}_0$. If for each $i \in I$, some set $M_i \subset i^{\ell *}$ is fixed, then the family $\Gamma = (\varphi_{M_i})_{i\in I}$ is an $L^p$-bounded system for $\mathcal{Q}$ if $(\varphi_i)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{Q}$. To see this, use the (quasi)-triangle inequality for $\|\cdot\|_{L^p}$, the uniform bound $|M_i| \leq \left|i^{\ell *}\right| \leq N_{\mathcal{Q}}^{\ell *}$ (cf. Lemma 3.2.18) and the moderateness of the weight $(|\det(T_i)|)_{i\in I}$, see equation (3.2.1).

This yields an estimate of the form $C_{\mathcal{Q},\Gamma,p} \leq C\left(p, \ell, N_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}\right)$. In case of $p \in [1,\infty]$, the dependence on $D_{\mathcal{Q}}$ can be dropped, since $C_{\mathcal{Q},\Gamma,p}$ does not involve $|\det(T_i)|$ in this case.

Finally, if $M_i \supset i^*$ holds for all $i \in I$, then Lemma 3.2.12 shows $\varphi_{M_i} \equiv 1$ on $Q_i$ for all $i \in I$, so that $\Gamma$ is an $L^p$-bounded *control* system for $\mathcal{Q}$. ◀

With these notions, we can now state the following result which essentially allows to use an $L^p$-bounded control system instead of an $L^p$-BAPU to determine the quasi-norm on $\mathcal{D}(\mathcal{Q}, L^p, Y)$. We remark that the statement of the theorem is essentially the same as [27, Corollary 2.5], but the case $p \in (0,1)$ is of course not covered by the setting considered in [27].

**Theorem 3.4.10.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i\in I}$ be an $L^p$-decomposition covering of $\mathcal{O}$ for some $p \in (0,\infty]$ and let $Y$ be a $\mathcal{Q}$-regular solid sequence space on $I$. Furthermore, let $r_0 \in \mathbb{N}$ and assume $I = \bigcup_{r=1}^{r_0} I^{(r)}$ for certain subsets $I^{(r)} \subset I$. Finally, assume that $\Gamma = (\gamma_i)_{i\in I}$ is an $L^p$-bounded system for $\mathcal{Q}$.*

*For $J \subset I$, we define $\|(c_i)_{i\in J}\|_{Y(J)}$ as $\|d\|_Y$, where $d = (d_i)_{i\in I}$ is the trivial extension of the sequence $(c_i)_{i\in J}$ onto $I$, i.e. $d_i = c_i$ for $i \in J$ and $d_i = 0$ for $i \in I \setminus J$.*

*For $f \in \mathcal{D}'(\mathcal{O})$, define*

$$\|f\|_{\Gamma,\left(I^{(r)}\right)_{r},L^p,Y} := \sum_{r=1}^{r_0} \left\| \left( \left\| \mathcal{F}^{-1}(\gamma_i f) \right\|_{L^p} \right)_{i\in I^{(r)}} \right\|_{Y\left(I^{(r)}\right)} \in [0,\infty].$$

*If $\Gamma : Y \to Y$ denotes the clustering map, then there is a positive constant*

$$C = \begin{cases} C\left(p, r_0, d, \ell_{\Gamma,\mathcal{Q}}, C_{\mathcal{Q},\Gamma,p}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, |||\Gamma|||_{Y\to Y}\right), & \text{if } p \in (0,1), \\ C\left(p, r_0, \ell_{\Gamma,\mathcal{Q}}, C_{\mathcal{Q},\Gamma,p}, N_{\mathcal{Q}}, |||\Gamma|||_{Y\to Y}\right), & \text{if } p \in [1,\infty] \end{cases}$$

*with*

$$\|f\|_{\Gamma,\left(I^{(r)}\right)_{r},L^p,Y} \leq C \cdot \|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)}$$

*for all $f \in \mathcal{D}'(\mathcal{O})$ and each $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{Q}$.*

*Conversely, if* $\Gamma$ *is an* $L^p$*-bounded* control *system for* $\mathcal{Q}$ *and if* $C_0 \geq 1$ *denotes a triangle constant for* $Y$*, then there is a constant*

$$C' = \begin{cases} C'\left(C_0, r_0, R_{\mathcal{Q}}, C_{\mathcal{Q}}, \ell_{\Gamma,\mathcal{Q}}, p, d\right), & \text{if } p \in (0,1), \\ C'\left(C_0, r_0\right), & \text{if } p \in [1,\infty] \end{cases}$$

*with*

$$\|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} \leq C' C_{\mathcal{Q},\Phi,p} \cdot \|f\|_{\Gamma,\left(I^{(r)}\right)_r,L^p,Y} \text{ for all } f \in \mathcal{D}'(\mathcal{O})$$

*for each* $L^p$*-BAPU* $\Phi = (\varphi_i)_{i\in I}$ *for* $\mathcal{Q}$*.*

*Proof.* Let $\ell := \ell_{\Gamma,\mathcal{Q}}$. Let $\Phi = (\varphi_i)_{i\in I}$ be an $L^p$-BAPU for $\mathcal{Q}$. Using Lemma 3.2.12, we get $\varphi_i^{(\ell+1)*} \equiv 1$ on $Q_i^{\ell*}$ for all $i \in I$. Because of $\gamma_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i^{\ell*}$, we conclude

$$\gamma_i = \varphi_i^{(\ell+1)*}\gamma_i = \sum_{j\in i^{(\ell+1)*}} \gamma_i\varphi_j.$$

We begin by proving the first estimate. To this end, let $f \in \mathcal{D}'(\mathcal{O})$ with $\|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} < \infty$ (otherwise, the estimate is trivial). Since $\|\cdot\|_{L^p}$ is a quasi-norm and since Lemma 3.2.18 yields the uniform bound $|i^{(\ell+1)*}| \leq N_{\mathcal{Q}}^{\ell+1}$, there is some $C_1 = C_1(N_{\mathcal{Q}}, \ell, p) > 0$ with

$$\begin{aligned} \left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^p} &= \left\|\sum_{j\in i^{(\ell+1)*}} \mathcal{F}^{-1}\left(\gamma_i\varphi_j f\right)\right\|_{L^p} \\ &\leq C_1 \sum_{j\in i^{(\ell+1)*}} \left\|\mathcal{F}^{-1}\left(\gamma_i\varphi_j f\right)\right\|_{L^p} \end{aligned}$$

for all $i \in I$. There are now two cases. For $p \in [1,\infty]$, Young's inequality $L^1 * L^p \hookrightarrow L^p$ yields

$$\begin{aligned} \left\|\mathcal{F}^{-1}\left(\gamma_i\varphi_j f\right)\right\|_{L^p} &\leq \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^1} \cdot \left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p} \\ &\leq C_{\mathcal{Q},\Gamma,p} \cdot \left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p}. \end{aligned}$$

For $p \in (0,1)$, we invoke Corollary 3.2.15, which yields a constant $C_2 = C_2(R_{\mathcal{Q}}, C_{\mathcal{Q}}, \ell, p, d) > 0$ with

$$\begin{aligned} \left\|\mathcal{F}^{-1}\left(\gamma_i\varphi_j f\right)\right\|_{L^p} &\leq C_2 \cdot |\det(T_i)|^{\frac{1}{p}-1} \cdot \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p} \left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p} \\ &\leq C_2 C_{\mathcal{Q},\Gamma,p} \cdot \left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p}, \end{aligned}$$

where $\mathcal{Q}\prime = (T_i Q_i' + b_i)_{i\in I}$ is semi-structured by definition of an $L^p$-decomposition covering for $p \in (0,1)$, cf. Definition 3.2.10.

Here, we used the inclusion $\operatorname{supp}\gamma_i \subset \overline{Q_i^{\ell *}} \subset \overline{Q_i^{(\ell+1)*}}$ as well as $\operatorname{supp}\varphi_j \subset \overline{Q_j} \subset \overline{Q_i^{(\ell+1)*}}$ for all $j \in i^{(\ell+1)*}$. Thus, if we set $C_2 := 1$ for $p \in [1,\infty]$, the estimate above is valid for all $i \in I$.

With the "higher order clustering map" $\Theta_{\ell+1}$ as in Lemma 3.4.3, we finally get, using the solidity of $Y$,

$$\begin{aligned}
&\left\| \left( \left\| \mathcal{F}^{-1}\left(\gamma_j f\right) \right\|_{L^p} \right)_{i \in I^{(r)}} \right\|_{Y\left(I^{(r)}\right)} \\
&\leq C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \left\| \left( \sum_{j \in i^{(\ell+1)*}} \left\| \mathcal{F}^{-1}\left(\varphi_j f\right) \right\|_{L^p} \right)_{i \in I} \right\|_Y \\
&= C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot \left\| \Theta_{\ell+1} \left[ \left( \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \right)_{i \in I} \right] \right\|_Y \\
&\leq C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot |||\Theta_{\ell+1}|||_{Y\to Y} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \right)_{i \in I} \right\|_Y \\
&= C_1 C_2 C_{\mathcal{Q},\Gamma,p} \cdot |||\Theta_{\ell+1}|||_{Y\to Y} \cdot \|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} .
\end{aligned}$$

Summing over $r \in \underline{r_0}$, this completes the proof of the first estimate, since Lemma 3.4.3 yields $|||\Theta_{\ell+1}|||_{Y\to Y} \leq |||\Gamma|||_{Y\to Y}^{\ell+1}$.

Now, let $f \in \mathcal{D}'(\mathcal{O})$ with $\|f\|_{\Gamma,\left(I^{(r)}\right)_r,L^p,Y} < \infty$ and assume that $\Gamma$ is an $L^p$-bounded control system for $\mathcal{Q}$, i.e. that $\gamma_i \equiv 1$ on $Q_i$ for all $i \in I$. This implies $\varphi_i = \varphi_i \gamma_i$. In case of $p \in [1,\infty]$, Young's inequality implies

$$\begin{aligned}
\left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} &= \left\| \mathcal{F}^{-1}\left(\varphi_i \gamma_i f\right) \right\|_{L^p} \\
&\leq \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^1} \cdot \left\| \mathcal{F}^{-1}\left(\gamma_i f\right) \right\|_{L^p} \\
&\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \left\| \mathcal{F}^{-1}\left(\gamma_i f\right) \right\|_{L^p}
\end{aligned}$$

with $C_2 := 1$ and in case of $p \in (0,1)$, we can use the same constant $C_2 = C_2(R_\mathcal{Q}, C_\mathcal{Q}, \ell, p, d) > 0$ provided by Corollary 3.2.15 as above to conclude

$$\begin{aligned}
\left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} &= \left\| \mathcal{F}^{-1}\left(\varphi_i \gamma_i f\right) \right\|_{L^p} \\
&\leq C_2 \left|\det\left(T_i\right)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\gamma_i f\right) \right\|_{L^p} \\
&\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \left\| \mathcal{F}^{-1}\left(\gamma_i f\right) \right\|_{L^p},
\end{aligned}$$

so that this estimate holds for all $i \in I$ and $p \in (0,\infty]$.

Let $c_i := \left\| \mathcal{F}^{-1}\left(\gamma_i f\right) \right\|_{L^p}$ for $i \in I$. Since $Y$ is a solid quasi-normed sequence space and since $I = \bigcup_{r=1}^{r_0} I^{(r)}$, together with $c_i \geq 0$ for all $i \in I$, yields

$$0 \leq c_i \leq \sum_{r=1}^{r_0} \left(c\chi_{I^{(r)}}\right)_i \text{ for all } i \in I,$$

we get a constant $C_3 = C_3(C_0, r_0) > 0$ with

$$\begin{aligned}
\|f\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} &= \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i\in I} \right\|_Y \\
&\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \|c\|_Y \\
&\leq C_2 C_{\mathcal{Q},\Phi,p} \cdot \left\| \sum_{r=1}^{r_0} c\chi_{I^{(r)}} \right\|_Y \\
&\leq C_2 C_3 C_{\mathcal{Q},\Phi,p} \cdot \sum_{r=1}^{r_0} \|c\chi_{I^{(r)}}\|_Y \\
&= C_2 C_3 C_{\mathcal{Q},\Phi,p} \cdot \|f\|_{\Gamma,\left(I^{(r)}\right)_r,L^p,Y} < \infty. \qquad \square
\end{aligned}$$

**Corollary 3.4.11.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $p \in (0,\infty]$. Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$ with $L^p$-BAPUs $(\varphi_i)_{i\in I}, (\psi_i)_{i\in I}$ and that $Y$ is a $\mathcal{Q}$-regular solid sequence space on $I$.*
*Then we have*

$$\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i\in I} \right\|_Y \asymp \left\| \left( \left\| \mathcal{F}^{-1}(\psi_i f) \right\|_{L^p} \right)_{i\in I} \right\|_Y$$

*for all $f \in \mathcal{D}'(\mathcal{O})$, where the implied constants are independent of $f$. Especially, the left-hand side is finite iff the right-hand side is.*

*Proof.* By Remark 3.4.9, we know that $\Gamma := (\varphi_i^*)_{i\in I}$ yields an $L^p$-bounded control system for $\mathcal{Q}$. By Theorem 3.4.10, this implies

$$\|\cdot\|_{\mathcal{D}_\Phi(\mathcal{Q},L^p,Y)} \asymp \|\cdot\|_{\Gamma,(I),L^p,Y} \asymp \|\cdot\|_{\mathcal{D}_\Psi(\mathcal{Q},L^p,Y)}$$

for $\Phi = (\varphi_i)_{i\in I}$ and $\Psi = (\psi_i)_{i\in I}$, where the implied constants only depend on $p, d, \mathcal{Q}, \Phi, \Psi$ and $Y$. $\square$

We will now show that the space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is indeed a Quasi-Banach space. As a last preparation, we need the simple observation that any solid sequence space embeds into a suitable weighted $\ell^\infty$ space.

**Lemma 3.4.12.** *Let $I \neq \varnothing$ be a set and let $(Y, \|\cdot\|_Y)$ be a solid sequence space on $I$. For $i \in I$, set*

$$u_i := \begin{cases} \|\delta_i\|_Y, & \text{if } \delta_i \in Y, \\ 1, & \text{if } \delta_i \notin Y. \end{cases}$$

*We remark that one could define $u_i$ arbitrarily in case of $\delta_i \notin Y$. For definiteness, we simply choose $u_i = 1$ in this case. Note that $u$ is a* positive *weight by definiteness of $\|\cdot\|_Y$.*

*Then $Y \hookrightarrow \ell^\infty_u(I)$. More precisely, we even have $\|x\|_{\ell^\infty_u} \leq \|x\|_Y$ for all $x = (x_i)_{i\in I} \in Y$.*

*Proof.* Let $x = (x_i)_{i\in I} \in Y$ be arbitrary and let $i \in I$. In case of $x_i \neq 0$, we have

$$\left|(\delta_i)_j\right| \leq \frac{|x_j|}{|x_i|} \text{ for all } j \in I.$$

By solidity of $Y$, this implies $\delta_i \in Y$ with

$$\|\delta_i\|_Y \leq \left\| \frac{x}{|x_i|} \right\|_Y = \frac{\|x\|_Y}{|x_i|},$$

whence

$$c_i \, |x_i| = \|\delta_i\|_Y \cdot |x_i| \leq \|x\|_Y .$$

In case of $x_i = 0$, we trivially have $u_i \cdot |x_i| = 0 \leq \|x\|_Y$.

Since $i \in I$ was arbitrary, we conclude $\|x\|_{\ell^\infty_u} \leq \|x\|_Y < \infty$, which completes the proof. □

Now, we can at last show that $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is a Quasi-Banach space which embeds into $\mathcal{D}'(\mathcal{O})$.

**Theorem 3.4.13.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $p \in (0, \infty]$. Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$ and that $Y$ is a $\mathcal{Q}$-regular, complete solid sequence space on $I$.*

*Then $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is a Quasi-Banach space which embeds continuously into $\mathcal{D}'(\mathcal{O})$.*

*If $Y$ is a normed vector space and $p \in [1, \infty]$, then $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is a Banach space, i.e. $\|\cdot\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)}$ is a norm.*

*Proof.* It is clear that the space $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is closed under multiplication with complex scalars and that $\|\cdot\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)}$ is homogeneous.

Let $C_1 \geq 1$ be a triangle constant for $\|\cdot\|_{L^p}$, let $C_2 \geq 1$ be a triangle constant for $\|\cdot\|_Y$ and let $(\varphi_i)_{i \in I}$ be an $L^p$-BAPU for $\mathcal{Q}$ which is used to define $\|\cdot\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)}$.

For $f, g \in \mathcal{D}(\mathcal{Q}, L^p, Y)$ and $i \in I$, we have

$$\mathcal{F}^{-1}[\varphi_i (f+g)] = \mathcal{F}^{-1}(\varphi_i f) + \mathcal{F}^{-1}(\varphi_i g) \in L^p(\mathbb{R}^d)$$

with

$$\left\| \mathcal{F}^{-1}[\varphi_i (f+g)] \right\|_{L^p} \leq C_1 \cdot \left[ \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} + \left\| \mathcal{F}^{-1}(\varphi_i g) \right\|_{L^p} \right].$$

By solidity of $Y$, this implies $\left( \left\| \mathcal{F}^{-1}[\varphi_i (f+g)] \right\|_{L^p} \right)_{i \in I} \in Y$ with

$$\begin{aligned}
&\|f+g\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} \\
&= \left\| \left( \left\| \mathcal{F}^{-1}[\varphi_i (f+g)] \right\|_{L^p} \right)_{i \in I} \right\|_Y \\
&\leq C_1 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} + \left\| \mathcal{F}^{-1}(\varphi_i g) \right\|_{L^p} \right)_{i \in I} \right\|_Y \\
&\leq C_1 C_2 \cdot \left[ \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I} \right\|_Y + \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i g) \right\|_{L^p} \right)_{i \in I} \right\|_Y \right] \\
&= C_1 C_2 \cdot \left[ \|f\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} + \|g\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)} \right] < \infty.
\end{aligned}$$

This shows that $\mathcal{D}(\mathcal{Q}, L^p, Y)$ is a vector space and that $\|\cdot\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)}$ is a quasi-norm (once we know that $\|\cdot\|_{\mathcal{D}(\mathcal{Q}, L^p, Y)}$ is positive definite,

which we show below). If $Y$ is a normed vector space and $p \in [1,\infty]$, we can take $C_1 = C_2 = 1$, so that $\|\cdot\|_{\mathcal{D}(\mathcal{Q},L^p,Y)}$ is a genuine norm.

Now, let us prove $\mathcal{D}(\mathcal{Q},L^p,Y) \hookrightarrow \mathcal{D}'(\mathcal{O})$. To this end, let $K \subset \mathcal{O}$ be an arbitrary compact set. Lemma 3.2.12 shows that $(Q_i^\circ)_{i\in I}$ covers $\mathcal{O}$. Since $K \subset \mathcal{O}$ is compact, there are finitely many $i_1,\dots,i_n \in I$ with $K \subset \bigcup_{\ell=1}^n Q_{i_\ell}^\circ \subset \bigcup_{\ell=1}^n Q_{i_\ell}$. The set $I_K := \{i_1,\dots,i_n\}^* \subset I$ is finite and it is easy to see $\varphi_{I_K} \equiv 1$ on $K$.

Now, choose $u = (u_i)_{i\in I}$ as in Lemma 3.4.12, so that $Y \hookrightarrow \ell^\infty_u(I)$. Let us set $C_K := \min_{i\in I_K} u_i > 0$. For arbitrary $\varphi \in C_c^\infty(\mathcal{O})$ with $\operatorname{supp}\varphi \subset K$ and $f \in \mathcal{D}(\mathcal{Q},L^p,Y)$, we now have

$$\begin{aligned}|\langle f,\varphi\rangle| &= \left|\langle f, \sum_{i\in I_K}\varphi_i\varphi\rangle\right| = \left|\sum_{i\in I_K}\langle \varphi_i f,\varphi\rangle\right| \\ &\leq \sum_{i\in I_K}\left|\left\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi}\right\rangle\right|.\end{aligned}$$

There are now two cases. For $p \in [1,\infty]$, we can apply Hölder's inequality to conclude

$$\left|\left\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi}\right\rangle\right| \leq \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \cdot \|\widehat{\varphi}\|_{L^{p'}}.$$

In case of $p \in (0,1)$, we observe $\operatorname{supp}\widehat{\mathcal{F}^{-1}(\varphi_i f)} = \operatorname{supp}(\varphi_i f) \subset \overline{Q_i}$ and apply Corollary 3.1.3 with $q = \infty$ to conclude

$$\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^\infty} \leq \left[\lambda\left(\overline{Q_i}\right)\right]^{\frac{1}{p}} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$$

and hence

$$\begin{aligned}\left|\left\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{\varphi}\right\rangle\right| &\leq \left[\lambda\left(\overline{Q_i}\right)\right]^{\frac{1}{p}} \cdot \|\widehat{\varphi}\|_{L^1} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \\ &\leq C_K' \cdot \|\widehat{\varphi}\|_{L^1} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\end{aligned}$$

for all $i \in I_K$, with $C_K' := \max_{i\in I_K}\left[\lambda\left(\overline{Q_i}\right)\right]^{1/p}$.

In any of the two cases, there is thus an exponent $r \in [1,\infty]$ and some constant $C_K' > 0$ satisfying

$$\begin{aligned}|\langle f,\varphi\rangle| &\leq C_K' \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{i\in I_K}\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \\ &\leq \frac{C_K'}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{i\in I_K}\left[u_i \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right] \\ &\leq \frac{C_K'}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right)_{i\in I}\right\|_{\ell^\infty_u} \cdot |I_K| \\ &\leq \frac{C_K' \cdot |I_K|}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \|f\|_{\mathcal{D}(\mathcal{Q},L^p,Y)}. \qquad (3.4.3)\end{aligned}$$

Here, the term $\|\widehat{\varphi}\|_{L^r}$ is finite because of $\varphi \in C_c^\infty(\mathcal{O}) \leq \mathcal{S}(\mathbb{R}^d)$. We recall from above the assumption $\operatorname{supp}\varphi \subset K$, where $K \subset \mathcal{O}$ was an *arbitrary* compact subset of $\mathcal{O}$. In particular, the above estimate proves $f \equiv 0$ as an element of $\mathcal{D}'(\mathcal{O})$ if $\|f\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} = 0$. Hence, $\|\cdot\|_{\mathcal{D}(\mathcal{Q},L^p,Y)}$ is positive definite.

Now, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\mathcal{Q},L^p,Y)$ which converges in $\mathcal{D}(\mathcal{Q},L^p,Y)$ to some $f \in \mathcal{D}(\mathcal{Q},L^p,Y)$, then the above estimate easily implies $|\langle f_n - f, \varphi\rangle| \xrightarrow[n\to\infty]{} 0$ for all $\varphi \in C_c^\infty(\mathcal{O})$ and hence $f_n \to f$ with convergence in $\mathcal{D}'(\mathcal{O})$. This proves the continuous embedding $\mathcal{D}(\mathcal{Q},L^p,Y) \hookrightarrow \mathcal{D}'(\mathcal{O})$. We note that it suffices to consider sequences to prove this continuity, since the topology of $\mathcal{D}(\mathcal{Q},L^p,Y)$ is first countable, cf. Lemma 2.1.5.

It remains to prove completeness of $\mathcal{D}(\mathcal{Q},L^p,Y)$. To this end, we first show for a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^d)$ with $\sum_{n=1}^\infty C_1^n \|f_n\|_{L^p} < \infty$ that $\sum_{n=1}^\infty f_n \in L^p(\mathbb{R}^d)$ with

$$\left\|\sum_{n=1}^\infty f_n\right\|_{L^p} \leq \sum_{n=1}^\infty C_1^n \|f_n\|_{L^p}, \tag{3.4.4}$$

where we recall from above that $C_1$ is a triangle constant for $L^p(\mathbb{R}^d)$. For $p \in [1,\infty]$, this is a simple consequence of the completeness of $L^p(\mathbb{R}^d)$, see [33, Theorem 5.1]. Hence, we can assume $p \in (0,1)$. By induction, we get $\left\|\sum_{n=1}^N |g_n|\right\|_{L^p} \leq \sum_{n=1}^N C_1^n \|g_n\|_{L^p}$ for arbitrary $g_1,\dots,g_N \in L^p(\mathbb{R}^d)$, because if we set $h_i := g_{i+1}$ for $i \in \underline{N}$, we have

$$\begin{aligned}
\left\|\sum_{n=1}^{N+1} |g_n|\right\|_{L^p} &\leq C_1 \left[\||g_1|\|_{L^p} + \left\|\sum_{n=2}^{N+1} |g_n|\right\|_{L^p}\right] \\
&= C_1 \left[\|g_1\|_{L^p} + \left\|\sum_{n=1}^{N} |h_n|\right\|_{L^p}\right] \\
&\leq C_1 \left[\|g_1\|_{L^p} + \sum_{n=1}^{N} C_1^n \|h_n\|_{L^p}\right] \\
&= \sum_{n=1}^{N+1} C_1^n \|g_n\|_{L^p}.
\end{aligned}$$

Since the sequence $\left(\left[\sum_{n=1}^N |f_n|\right]^p\right)_{N\in\mathbb{N}}$ is nondecreasing, the monotone convergence theorem implies

$$\begin{aligned}
\left\|\sum_{n=1}^\infty f_n\right\|_{L^p}^p \leq \left\|\sum_{n=1}^\infty |f_n|\right\|_{L^p}^p &= \lim_{N\to\infty} \left\|\sum_{n=1}^N |f_n|\right\|_{L^p}^p \\
&\leq \lim_{N\to\infty} \left(\sum_{n=1}^N C_1^n \|f_n\|_{L^p}\right)^p = \left(\sum_{n=1}^\infty C_1^n \|f_n\|_{L^p}\right)^p < \infty,
\end{aligned}$$

which proves estimate (3.4.4) also for the case $p \in (0,1)$.

By Corollary 2.2.12, there is an equivalent $q$-norm $|||\cdot|||_Y$ on $Y$, for some $q \in (0,1]$, such that $(Y, |||\cdot|||_Y)$ is still a solid, complete sequence space on $I$. Since $\|\cdot\|_Y$ and $|||\cdot|||_Y$ are equivalent, it is easy to see that the resulting quasi-norms on $\mathcal{D}(\mathcal{Q}, L^p, Y)$ will also be equivalent. Hence, we can assume without loss of generality that $\|\cdot\|_Y$ is a $q$-norm.

By Theorem 2.1.4, we know that $\mathcal{D}(\mathcal{Q}, L^p, Y)$ admits an equivalent $s$-norm of some $s \in (0,1]$. Because of $\sum_{k=1}^{\infty} (2C_1)^{-ks} < \infty$, Lemma 2.1.8 shows that it suffices to consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{Q}, L^p, Y)$ with $\|f_n\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} \leq (2C_1)^{-n}$ for all $n \in \mathbb{N}$ and to show that the sequence $g_N := \sum_{n=1}^{N} f_n$ converges to some $g \in \mathcal{D}(\mathcal{Q}, L^p, Y)$.

Let us set

$$\theta_i^{(n)} := \left\| \mathcal{F}^{-1}(\varphi_i f_n) \right\|_{L^p}$$

for $i \in I$ and $n \in \mathbb{N}$ and note $\theta^{(n)} := (\theta_i^{(n)})_{i\in I} \in Y$ with quasi-norm $\|\theta^{(n)}\|_Y = \|f_n\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} \leq (2C_1)^{-n}$. Since $Y$ is complete and $\|\cdot\|_Y$ is a $q$-norm and because of

$$\sum_{n=N}^{\infty} \left\| C_1^n \theta^{(n)} \right\|_Y^q \leq \sum_{n=N}^{\infty} 2^{-nq} < \infty,$$

Theorem 2.2.10 implies $\sum_{n=N}^{\infty} C_1^n \theta^{(n)} \in Y$ for all $N \in \mathbb{N}$ with

$$\left\| \sum_{n=N}^{\infty} C_1^n \theta^{(n)} \right\|_Y \leq \left( \sum_{n=N}^{\infty} 2^{-nq} \right)^{1/q} \xrightarrow[N\to\infty]{} 0. \tag{3.4.5}$$

In particular, we get $\sum_{n=1}^{\infty} C_1^n \theta_i^{(n)} < \infty$ for almost all $i \in I$ and hence for all $i \in I$, since we are considering the counting measure on $I$.

We first note that equation (3.4.3) implies for each compact $K \subset \mathcal{O}$ and each $\varphi \in C_c^\infty(\mathcal{O})$ with $\operatorname{supp}\varphi \subset K$ the estimate

$$\sum_{n=1}^{\infty} |\langle f_n, \varphi \rangle| \leq \frac{C_K' \cdot |I_K|}{C_K} \cdot \|\widehat{\varphi}\|_{L^r} \cdot \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} < \infty.$$

This shows that $g(\varphi) := \sum_{n=1}^{\infty} \langle f_n, \varphi \rangle$ converges for every $\varphi \in \mathcal{D}'(\mathcal{O})$. By [61, Theorem 6.17], this implies $g \in \mathcal{D}'(\mathcal{O})$. It remains to show $g \in \mathcal{D}(\mathcal{Q}, L^p, Y)$ and $\|g_n - g\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} \to 0$.

We recall from [61, Theorem 7.23] that the inverse Fourier transform $\mathcal{F}^{-1}(\varphi_i [g - g_N])$ is given by

$$\begin{aligned}
\left[\mathcal{F}^{-1}(\varphi_i [g - g_N])\right](x) &= \langle \varphi_i [g - g_N], e_{-x} \rangle = \langle g - g_N, \varphi_i e_{-x} \rangle \\
&= \sum_{n=N+1}^{\infty} \langle f_n, \varphi_i e_{-x} \rangle = \sum_{n=N+1}^{\infty} \langle \varphi_i f_n, e_{-x} \rangle \\
&= \sum_{n=1}^{\infty} \left[\mathcal{F}^{-1}(\varphi_i f_{n+N})\right](x)
\end{aligned}$$

for all $x \in \mathbb{R}^d$, with $e_z(y) = e^{2\pi i \langle z,y\rangle}$ for $y, z \in \mathbb{R}^d$. Above, we saw $\sum_{n=1}^{\infty} C_1^n \left\|\mathcal{F}^{-1}(\varphi_i f_n)\right\|_{L^p} = \sum_{n=1}^{\infty} C_1^n \theta_i^{(n)} < \infty$ for all $i \in I$. By equation (3.4.4) above, this yields $\mathcal{F}^{-1}(\varphi_i [g - g_N]) \in L^p(\mathbb{R}^d)$ with

$$\begin{aligned} 0 \leq \gamma_i := \left\|\mathcal{F}^{-1}(\varphi_i [g - g_N])\right\|_{L^p} &\leq \sum_{n=1}^{\infty} C_1^n \left\|\mathcal{F}^{-1}(\varphi_i f_{n+N})\right\|_{L^p} \\ &= \sum_{n=N+1}^{\infty} C_1^{n-N} \left\|\mathcal{F}^{-1}(\varphi_i f_n)\right\|_{L^p} \\ &\leq \sum_{n=N+1}^{\infty} C_1^n \left\|\mathcal{F}^{-1}(\varphi_i f_n)\right\|_{L^p} \\ &= \sum_{n=N+1}^{\infty} C_1^n \theta_i^{(n)}. \end{aligned}$$

But above, we also saw $\sum_{n=N+1}^{\infty} C_1^n \theta^{(n)} \in Y$. By solidity of $Y$ and using equation (3.4.5), we conclude $\gamma := (\gamma_i)_{i\in I} \in Y$ with

$$\|g - g_N\|_{\mathcal{D}(\mathcal{Q},L^p,Y)} = \|\gamma\|_Y \leq \left\|\sum_{n=N+1}^{\infty} C_1^n \theta^{(n)}\right\|_Y \xrightarrow[N\to\infty]{} 0.$$

In particular, we get $g = (g - g_N) + g_N \in \mathcal{D}(\mathcal{Q}, L^p, Y)$ and the above estimate proves $g = \lim_{N\to\infty} g_N$ with convergence in $\mathcal{D}(\mathcal{Q}, L^p, Y)$. As noted above, this completes the proof. □

We end this section with an example which shows that the completeness proved above is not completely trivial. More precisely, we show that completeness can fail if the reservoir $\mathcal{D}'(\mathcal{O})$ is replaced by $\mathcal{S}'(\mathbb{R}^d)$, even in the case $\mathcal{O} = \mathbb{R}^d$ and $Y = \ell^1_u$ with a suitably chosen weight $u$.

**Example 3.4.14.** In the following, we provide a specific example showing that the space $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^p, \ell^q_u)$ as defined in Remark 3.4.7 is in general *not* complete. Specifically, we will take $p = q = 1$.

*The covering considered here is usually used to define modulation spaces, but the weight u decays (for $x \to \infty$) much faster than the weights used for the ordinary modulation spaces.*

Let $I := \mathbb{Z}$, $T_i := \mathrm{id}_{\mathbb{R}}$ and $b_i := i$ for $i \in \mathbb{Z}$. Furthermore, define $Q := \left(-\frac{3}{4}, \frac{3}{4}\right)$ and $P := \left(-\frac{5}{8}, \frac{5}{8}\right)$, as well as

$$Q_i := T_i Q + b_i = \left(i - \frac{3}{4}, i + \frac{3}{4}\right).$$

It is then easy to see $\bigcup_{i\in I} (T_i P + b_i) = \mathbb{R}$ and that $x \in Q_i \cap Q_j \neq \varnothing$ implies

$$i - \frac{3}{4} < x < j + \frac{3}{4}$$

and hence $i - j < \frac{6}{4} < 2$. Because of $i - j \in \mathbb{Z}$ we conclude $i - j \leq 1$. By symmetry we get $|i - j| \leq 1$ and thus $i^* \subset \{i-1, i, i+1\}$. This shows that $\mathcal{Q} = (Q_i)_{i\in I}$ is a structured admissible covering of $\mathbb{R}$.

Now consider the weight $u_i := 10^{-i}$ for $i \in \mathbb{Z}$ and note that because of the estimate

$$\frac{u_i}{u_j} = 10^{j-i} \leq 10^{|j-i|} \leq 10$$

which is valid for all $i \in I$ and $j \in i^* \subset \{i-1, i, i+1\}$, the weight $u$ is $\mathcal{Q}$-moderate.

Theorem 3.2.17 guarantees the existence of an $L^p$-BAPU $(\varphi_i)_{i\in I}$ for $\mathcal{Q}$. Note that for $i \in I$ we have

$$\begin{aligned}\bigcup_{j\in I\setminus\{i\}} Q_j &\subset \left(-\infty, (i-1)+\frac{3}{4}\right) \cup \left((i+1)-\frac{3}{4}, \infty\right) \\ &= \left(-\infty, i-\frac{1}{4}\right) \cup \left(i+\frac{1}{4}, \infty\right).\end{aligned}$$

Together with $\varphi_j(x) = 0$ for $x \in \mathbb{R} \setminus Q_j$ and $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in \mathbb{R}$, this implies $\varphi_i(x) = 1$ for $x \in \left[i-\frac{1}{4}, i+\frac{1}{4}\right]$ for all $i \in I$.

Now choose a nonnegative function $\psi \in C_c^\infty\left(\left(-\frac{1}{4}, \frac{1}{4}\right)\right) \setminus \{0\}$ and define $f_n := \sum_{j=1}^n 4^j \cdot L_j\psi$ for $n \in \mathbb{N}$. Because of

$$\operatorname{supp}(L_n\psi) \subset \left(n-\frac{1}{4}, n+\frac{1}{4}\right) \subset \left(\bigcup_{j\in I\setminus\{n\}} Q_j\right)^c$$

it is easy to see that

$$\varphi_i \cdot L_n\psi = \begin{cases} 0, & \text{if } i \neq n, \\ L_i\psi, & \text{if } i = n \end{cases}$$

holds for $i, n \in \mathbb{Z}$. For $n \geq m \geq m_0$ we thus get

$$\begin{aligned}\|f_n - f_m\|_{\mathcal{D}(\mathcal{Q},L^1,\ell^1_u)} &= \sum_{i\in\mathbb{Z}} 10^{-i} \left\| \mathcal{F}^{-1}\left(\varphi_i \cdot \sum_{j=m+1}^n 4^j \cdot L_j\psi\right)\right\|_{L^1} \\ &= \sum_{i=m+1}^n 10^{-i}4^i \cdot \left\|\mathcal{F}^{-1}(L_i\psi)\right\|_{L^1} \\ &\leq \left\|\mathcal{F}^{-1}\psi\right\|_{L^1} \cdot \sum_{i=m_0+1}^\infty \left(\frac{4}{10}\right)^i \xrightarrow[m_0\to\infty]{} 0,\end{aligned}$$

so that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell^1_u)$.

Let us assume that there is some $f \in \mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell^1_u) \subset \mathcal{S}'(\mathbb{R})$ with $f_n \xrightarrow[n\to\infty]{\mathcal{D}_{\mathcal{S}'}(\mathcal{Q},L^1,\ell^1_u)} f$. Using the continuous embeddings

$$\mathcal{D}_{\mathcal{S}'}(\mathcal{Q}, L^1, \ell^1_u) \hookrightarrow \mathcal{D}(\mathcal{Q}, L^1, \ell^1_u) \hookrightarrow \mathcal{D}'(\mathbb{R}),$$

we conclude

$$\langle f, g\rangle_{\mathcal{D}',\mathcal{D}} = \lim_{n\to\infty} \langle f_n, g\rangle_{\mathcal{D}',\mathcal{D}} \qquad \text{for all } g \in C_c^\infty(\mathbb{R}).$$

Furthermore, by definition of the topology on $\mathcal{S}(\mathbb{R})$, by [33, Proposition 5.15] and because of $f \in \mathcal{S}'(\mathbb{R})$ there exists some $N \in \mathbb{N}$ and some $C > 0$ such that

$$\left|\langle f, g\rangle_{\mathcal{S}',\mathcal{S}}\right| \leq C \cdot \sup_{\substack{\alpha\in\mathbb{N}_0\\ \alpha\leq N}} \sup_{x\in\mathbb{R}} (1+|x|)^N \cdot |(\partial^\alpha g)(x)|$$

holds for all $g \in \mathcal{S}(\mathbb{R})$.

For $n \in \mathbb{N}$ and $g_n := L_n\psi$ we have

$$\operatorname{supp} g_n \subset \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \subset [0, n+1]$$

and thus

$$\begin{aligned}\sup_{\substack{\alpha\in\mathbb{N}_0\\ \alpha\leq N}} \sup_{x\in\mathbb{R}} (1+|x|)^N \cdot |(\partial^\alpha g_n)(x)| &\leq (n+2)^N \cdot \sup_{\substack{\alpha\in\mathbb{N}_0\\ \alpha\leq N}} \sup_{x\in\mathbb{R}} |(\partial^\alpha \psi)(x)| \\ &= (n+2)^N \cdot C_{\psi,N}\end{aligned}$$

for some constant $C_{\psi,N} \in (0,\infty)$.

But because of $\operatorname{supp}(L_n\psi) \cap \operatorname{supp}(L_i\psi) = \varnothing$ for $i, n \in \mathbb{Z}$ with $i \neq n$ we have, for $m \geq n$, the identity

$$\langle f_m, g_n\rangle_{\mathcal{S}',\mathcal{S}} = \sum_{j=1}^{m} 4^j \left\langle L_j\psi, L_n\psi\right\rangle_{\mathcal{S}',\mathcal{S}} = 4^n \cdot \langle \psi, \psi\rangle_{\mathcal{S}',\mathcal{S}}$$

and thus

$$\begin{aligned}4^n \cdot \|\psi\|_{L^2}^2 &= \lim_{m\to\infty} \left|\langle f_m, g_n\rangle_{\mathcal{S}',\mathcal{S}}\right| = \left|\langle f, g_n\rangle_{\mathcal{S}',\mathcal{S}}\right| \\ &\leq C \cdot \sup_{\substack{\alpha\in\mathbb{N}_0\\ \alpha\leq N}} \sup_{x\in\mathbb{R}} (1+|x|)^N \cdot |(\partial^\alpha g_n)(x)| \\ &\leq C C_{\psi,N} \cdot (n+2)^N\end{aligned}$$

for all $n \in \mathbb{N}$, a contradiction.

Thus, there is no $f \in \mathcal{D}_{\mathcal{S}'}\left(\mathcal{Q}, L^1, \ell^1_u\right)$ with $\|f_n - f\|_{\mathcal{D}(\mathcal{Q},L^1,\ell^1_u)} \xrightarrow[n\to\infty]{} 0$, so that $\mathcal{D}_{\mathcal{S}'}\left(\mathcal{Q}, L^1, \ell^1_u\right)$ is *not* complete. ◀

## 3.5 DILATIONS OF DECOMPOSITION SPACES

The aim of this section is to introduce a natural way in which a distribution $f \in \mathcal{D}'(\mathcal{O})$ can be dilated using a matrix $A \in \mathrm{GL}\left(\mathbb{R}^d\right)$ to obtain a new distribution

$$f_A = f \circ A \in \mathcal{D}'\left(A^{-1}\mathcal{O}\right)$$

and to investigate how the decomposition space norm $\|f_A\|_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_u)}$ of the dilated distribution relates to a (different) decomposition space norm of $f$.

We will see that a suitable choice for this different decomposition space norm is $\|f\|_{\mathcal{D}(A\mathcal{Q},L^p,\ell^q_u)}$. Before we state the relation between these two norms, we first formaly define the covering $A\mathcal{Q}$ and investigate its properties. The fact that the "form constants" $R_{A\mathcal{Q}}, C_{A\mathcal{Q}}$, etc., are the same for the dilated covering $A\mathcal{Q}$ as for $\mathcal{Q}$ will become important in Chapter 6, where we will asymptotically compute the operator norm of the isotropic dilation operators on certain decomposition spaces.

**Lemma 3.5.1.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_iQ'_i + b_i)_{i\in I}$ be a semi-structured admissible covering of an open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ and let $A \in \mathrm{GL}(\mathbb{R}^d)$.*

*Then the covering*

$$A\mathcal{Q} := (AQ_i)_{i\in I} = \left(AT_i \cdot Q'_i + Ab_i\right)_{i\in I}$$

*is a semi-structured admissible covering of the open set $\varnothing \neq A\mathcal{O} \subset \mathbb{R}^d$. In the following, we will always consider the covering $A\mathcal{Q}$ with this family of standardizations.*

*Then the following hold:*

1. *The index clusters $i^{k*}$ for $k \in \mathbb{N}_0$ are the same when considered as clusters for $\mathcal{Q}$ or for $A\mathcal{Q}$. In particular, we have $N_{\mathcal{Q}} = N_{A\mathcal{Q}}$.*
2. *For each discrete weight $u = (u_i)_{i\in I}$, $u$ is moderate with respect to $\mathcal{Q}$ if and only if it is moderate with respect to $A\mathcal{Q}$. In this case, we have $C_{u,\mathcal{Q}} = C_{u,A\mathcal{Q}}$.*
3. *We have*

$$\begin{aligned} R_{A\mathcal{Q}} &= R_{\mathcal{Q}}, \\ C_{A\mathcal{Q}} &= C_{\mathcal{Q}}, \\ D_{A\mathcal{Q}} &= D_{\mathcal{Q}}. \end{aligned}$$

4. *$\mathcal{Q}$ is tight if and only if $A\mathcal{Q}$ is. In this case, we have $\varepsilon_{A\mathcal{Q}} = \varepsilon_{\mathcal{Q}}$.*
5. *We have $r_0(m, A\mathcal{Q}) = r_0(m, \mathcal{Q})$ for all $m \in \mathbb{N}_0$.*
6. *If $\Phi = (\varphi_i)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{Q}$, then $\Phi_A = \left(\varphi_i \circ A^{-1}\right)_{i\in I}$ is an $L^p$-BAPU for $A\mathcal{Q}$ and we have*

$$C_{A\mathcal{Q},\Phi_A,p} = C_{\mathcal{Q},\Phi,p}.$$

7. *If $\mathcal{P} = \left(P_j\right)_{j\in J}$ is a covering of $\varnothing \neq \mathcal{O}' \subset \mathbb{R}^d$, then the following hold:*
   a) *If $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$, then $A\mathcal{Q}$ is almost subordinate to $A\mathcal{P} = \left(AP_j\right)_{j\in J}$ with $k(A\mathcal{Q}, A\mathcal{P}) = k(\mathcal{Q}, \mathcal{P})$.*

b) *If a weight $u = (u_i)_{i\in I}$, considered as a weight for $\mathcal{Q}$, is relatively moderate with respect to $\mathcal{P}$, then $u$, considered as a weight for $A\mathcal{Q}$, is relatively moderate with respect to $A\mathcal{P}$ and we have*

$$C_{u,A\mathcal{Q},A\mathcal{P}} = C_{u,\mathcal{Q},\mathcal{P}}.$$

c) *If $\mathcal{Q}$ is relatively moderate with respect to $\mathcal{P}$, then $A\mathcal{Q}$ is relatively moderate with respect to $A\mathcal{P}$ with*

$$C_{\mathrm{mod}}(A\mathcal{Q}, A\mathcal{P}) = C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}).$$

*Proof.* We first note that $A\mathcal{Q}$ covers $A\mathcal{O}$, since $\mathcal{Q}$ covers $\mathcal{O}$. Since $A \in \mathrm{GL}(\mathbb{R}^d)$ is a homeomorphism of $\mathbb{R}^d$, $A\mathcal{O} \subset \mathbb{R}^d$ is open.

1. Since $x \mapsto Ax$ is a bijective map, we have $Q_i \cap Q_j \neq \varnothing$ if and only if $AQ_i \cap AQ_j \neq \varnothing$ holds. This shows that $i^*$ is the same for $\mathcal{Q}$ as for $A\mathcal{Q}$. A straightforward induction shows that $i^{k*}$ is the same for $\mathcal{Q}$ as for $A\mathcal{Q}$. Furthermore, we get

$$N_{A\mathcal{Q}} = \sup_{i\in I} |i^*| = N_{\mathcal{Q}} < \infty.$$

In particular, $A\mathcal{Q}$ is an admissible covering of $\mathcal{O}$.

2. The weight $u$ is moderate with respect to $\mathcal{Q}$ if and only if

$$C_{u,\mathcal{Q}} = \sup_{i\in I} \sup_{j\in i^*} \frac{u_i}{u_j}$$

is finite. Since the cluster $i^*$ is the same with respect to $\mathcal{Q}$ and $A\mathcal{Q}$, we get $C_{u,\mathcal{Q}} = C_{u,A\mathcal{Q}}$. In particular, $u$ is $\mathcal{Q}$-moderate if and only if it is moderate with respect to $A\mathcal{Q}$.

3. We first note that $R_{\mathcal{Q}}$ only depends on the sets $Q_i'$, which are the same for $\mathcal{Q}$ and for $A\mathcal{Q}$. Hence, $R_{A\mathcal{Q}} = R_{\mathcal{Q}}$. Finally, we have

$$\begin{aligned} D_{A\mathcal{Q}} &= \sup_{i\in I} \sup_{j\in i^*} \left|\det\left((AT_i)^{-1} \cdot AT_j\right)\right| \\ &= \sup_{i\in I} \sup_{j\in i^*} \left|\det\left(T_i^{-1} T_j\right)\right| = D_{\mathcal{Q}} < \infty \end{aligned}$$

and

$$C_{A\mathcal{Q}} = \sup_{i\in I} \sup_{j\in i^*} \left\|(AT_i)^{-1} \cdot AT_j\right\| = \sup_{i\in I} \sup_{j\in i^*} \left\|T_i^{-1} T_j\right\| = C_{\mathcal{Q}} < \infty.$$

In particular, this implies that $A\mathcal{Q}$ is a semi-structured covering of $A\mathcal{O}$.

It is worth noting that the above calculations used that the sets $i^*$ are the same for $\mathcal{Q}$ and for $A\mathcal{Q}$, as shown above. Furthermore, they rely on the exact form of the chosen standardizations. For example, for $A = \lambda \cdot \mathrm{id}$, we could also have chosen the standardization

$A\mathcal{Q} = (T_i \cdot \lambda Q_i' + \lambda b_i)_{i \in I}$. With this choice, $A\mathcal{Q}$ would still be semi-structured, but we would not have $R_{\mathcal{Q}} = R_{A\mathcal{Q}}$ anymore.

4. Tightness, as well as the quantity $\varepsilon_{\mathcal{Q}}$, only depend on the family $(Q_i')_{i \in I}$ which is the same for $\mathcal{Q}$ and $A\mathcal{Q}$.

5. For $r_0 = r_0(\mathcal{Q}, m)$, there is a finite partition $I = \biguplus_{r=1}^{r_0} I^{(r)}$ of $I$, such that $Q_i^{m*} \cap Q_j^{m*} = \varnothing$ holds for all $i, j \in I^{(r)}$ with $r \in \underline{r_0}$ and $i \neq j$. By bijectivity of $x \mapsto Ax$, this implies

$$(AQ_i)^{m*} \cap (AQ_j)^{m*} = A(Q_i^{m*}) \cap A(Q_j^{m*}) = \varnothing$$

for all $i, j \in I^{(r)}$ with $r \in \underline{r_0}$ and $i \neq j$. Here, we again used that the set $i^{m*}$ is the same when the cluster is taken with respect to $\mathcal{Q}$ or with respect to $A\mathcal{Q}$. These considerations imply $r_0(A\mathcal{Q}, m) \leq r_0(\mathcal{Q}, m)$. The reverse inequality follows by symmetry, i.e. by considering $A^{-1}$ instead of $A$.

6. Let $\Phi = (\varphi_i)_{i \in I}$ be an $L^p$-BAPU for $\mathcal{Q}$. Since $A : \mathcal{O} \to A\mathcal{O}$ is a homeomorphism, we have $\varphi_i \circ A^{-1} \in C_c^\infty(A\mathcal{O})$ for all $i \in I$. Furthermore,

$$1 = \sum_{i \in I} \left(\varphi_i \circ A^{-1}\right)(x) = \sum_{i \in I} \varphi_i(A^{-1}x)$$

holds for all $x \in A\mathcal{O}$, since this yields $A^{-1}x \in \mathcal{O}$. Likewise, we have $\varphi_i \circ A^{-1} \equiv 0$ on $\mathbb{R}^d \setminus AQ_i$, since $\varphi_i$ vanishes outside of $Q_i$.

Finally, [33, Theorem 8.22] yields

$$\mathcal{F}^{-1}(\varphi_i \circ A^{-1}) = |\det(A)| \cdot (\mathcal{F}^{-1}\varphi_i) \circ A^T.$$

By virtue of the change-of-variables formula, we conclude

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\varphi_i \circ A^{-1}) \right\|_{L^q} &= |\det(A)| \cdot \|(\mathcal{F}^{-1}\varphi_i) \circ A^T\|_{L^q} \\ &= |\det(A)|^{1-\frac{1}{q}} \cdot \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^q} \end{aligned}$$

for all $q \in (0, \infty)$. For $p \in [1, \infty]$, this implies

$$C_{A\mathcal{Q}, \Phi_A, p} = \sup_{i \in I} \left\| \mathcal{F}^{-1}(\varphi_i \circ A^{-1}) \right\|_{L^1} = \sup_{i \in I} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^1} = C_{\mathcal{Q}, \Phi, p} < \infty,$$

so that $\Phi_A$ is an $L^p$-BAPU for $A\mathcal{Q}$. In case of $p \in (0, 1)$, we get (cf. Definition 3.2.10)

$$\begin{aligned} C_{A\mathcal{Q}, \Phi_A, p} &= \sup_{i \in I} |\det(AT_i)|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}(\varphi_i \circ A^{-1}) \right\|_{L^p} \\ &= \sup_{i \in I} |\det(AT_i)|^{\frac{1}{p}-1} |\det(A)|^{1-\frac{1}{p}} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \\ &= \sup_{i \in I} |\det(T_i)|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \\ &= C_{\mathcal{Q}, \Phi, p} < \infty, \end{aligned}$$

since the determinant is multiplicative. Hence, $\Phi_A$ is also an $L^p$-BAPU for $A\mathcal{Q}$ in this case.

7a. If $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$, let $k := k(\mathcal{Q}, \mathcal{P})$. Then, for each $i \in I$ there is some $j_i \in J$ with $Q_i \subset P_{j_i}^{k*}$. But this implies

$$AQ_i \subset A \cdot P_{j_i}^{k*} = [AP_{j_i}]^{k*},$$

where the last step used that the cluster $j_i^{k*}$ is the same when formed with respect to $\mathcal{P}$ or with respect to $A\mathcal{P}$.

Thus, $A\mathcal{Q}$ is almost subordinate to $A\mathcal{P}$ with $k(A\mathcal{Q}, A\mathcal{P}) \leq k(\mathcal{Q}, \mathcal{P})$. Considering $A^{-1}$ instead of $A$ and $A\mathcal{Q}, A\mathcal{P}$ instead of $\mathcal{Q}, \mathcal{P}$ yields the reverse estimate.

7b. If $u$, considered as a weight on $\mathcal{Q}$, is relatively $\mathcal{P}$-moderate, we get

$$u_i \leq C_{u,\mathcal{Q},\mathcal{P}} \cdot u_\ell$$

if $Q_i \cap P_j \neq \varnothing \neq Q_\ell \cap P_j$ holds for $i, \ell \in I$ and some $j \in J$.

But $AQ_i \cap AP_j \neq \varnothing \neq AQ_\ell \cap AP_j$ implies $Q_i \cap P_j \neq \varnothing \neq Q_\ell \cap P_j$ by bijectivity of $A$ and hence $u_i \leq C_{u,\mathcal{Q},\mathcal{P}} \cdot u_\ell$. This shows that $u$, considered as a weight for $A\mathcal{Q}$, is relatively $A\mathcal{P}$-moderate with

$$C_{u,A\mathcal{Q},A\mathcal{P}} \leq C_{u,\mathcal{Q},\mathcal{P}}.$$

By considering $A^{-1}$ instead of $A$ as above, we get the reverse estimate.

7c. Using the identity $(AT_i)^{-1} \cdot AT_\ell = T_i^{-1} T_\ell$, the proof is an easy modification of part 7b. □

Now that we have defined the dilated covering $A\mathcal{Q}$, we want to establish a notion of dilation for distributions. We first observe for $f \in L^1_{\mathrm{loc}}(\mathcal{O})$, $A \in \mathrm{GL}(\mathbb{R}^d)$, $f_A := f \circ A$ and $g \in C_c^\infty(A^{-1}\mathcal{O})$ the identity

$$\int_{A^{-1}\mathcal{O}} f_A(x) \cdot g(x)\, \mathrm{d}x = |\det(A)|^{-1} \cdot \int_{\mathcal{O}} f(y) \cdot g(A^{-1}y)\, \mathrm{d}y,$$

thanks to the change-of-variables formula. This motivates the following definition:

**Definition 3.5.2.** (cf. [33, page 285]) Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and let $A \in \mathrm{GL}(\mathbb{R}^d)$. Given a distribution $f \in \mathcal{D}'(\mathcal{O})$, define the dilated distribution $f \circ A \in \mathcal{D}'(A^{-1}\mathcal{O})$ by

$$f_A := f \circ A : C_c^\infty(A^{-1}\mathcal{O}) \to \mathbb{C}, g \mapsto |\det(A)|^{-1} \cdot f(g \circ A^{-1}).$$

Since the same definition is given in [33, page 285], we omit the straightforward verification that $f_A \in \mathcal{D}'(A^{-1}\mathcal{O})$ is indeed a distribution on $A^{-1}\mathcal{O}$. Note that the above calculation shows that for $f \in L^1_{\mathrm{loc}}(\mathcal{O})$, both definitions of $f \circ A$ coincide if $f$ is identified with the associated distribution.

Now, we derive the advertised relation between $\|f_A\|_{\mathcal{D}\left(\mathcal{Q},L^p,\ell^q_u\right)}$ and $\|f\|_{\mathcal{D}\left(A^{-1}\mathcal{Q},L^p,\ell^q_u\right)}$. To this end, we first investigate the (localized) inverse Fourier transform of $f_A$. Hence, let $\varphi \in C_c^\infty\left(A^{-1}\mathcal{O}\right)$ be arbitrary. Then $\varphi \cdot f_A \in \mathcal{D}'\left(A^{-1}\mathcal{O}\right)$ is a distribution with compact support. Hence, [61, Theorem 7.23] implies that the (inverse) Fourier transform $\mathcal{F}^{-1}\left(\varphi \cdot f_A\right)$ is given by (integration against) a smooth function which we again denote by $\mathcal{F}^{-1}\left(\varphi \cdot f_A\right)$. Furthermore, we have

$$\begin{aligned}
&\left(\mathcal{F}^{-1}\left[\varphi \cdot f_A\right]\right)(x) \\
&= \left(\varphi \cdot f_A\right)(e_x) = f_A\left(\varphi \cdot e_x\right) \\
&\overset{(\dagger)}{=} \left|\det(A)\right|^{-1} \cdot f\left(\left(\varphi \circ A^{-1}\right) \cdot e_{A^{-T}x}\right) \\
&= \left|\det(A)\right|^{-1} \cdot \left(\left(\varphi \circ A^{-1}\right) \cdot f\right)\left(e_{A^{-T}x}\right) \\
&= \left|\det(A)\right|^{-1} \cdot \left(\mathcal{F}^{-1}\left[\left(\varphi \circ A^{-1}\right) \cdot f\right]\right)\left(A^{-T}x\right), \qquad (3.5.1)
\end{aligned}$$

with $e_x : \mathbb{R}^d \to \mathbb{C}, y \mapsto e^{2\pi i\langle x,y\rangle}$. At $(\dagger)$, we employed the definition of $f_A$ and the identity

$$\left(e_x \circ A^{-1}\right)(y) = e^{2\pi i\left\langle x,A^{-1}y\right\rangle} = e^{2\pi i\left\langle A^{-T}x,y\right\rangle} = e_{A^{-T}x}(y).$$

Using equation (3.5.1), we can now state the appropriate transformation law for the decomposition space norm of a distribution under dilation.

**Lemma 3.5.3.** *Let $\mathcal{Q} = \left(T_i Q_i' + b_i\right)_{i\in I}$ be a semi-structured covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ and let $u = (u_i)_{i\in I}$ be $\mathcal{Q}$-moderate. Finally, let $A \in \mathrm{GL}\left(\mathbb{R}^d\right)$.*

*For $f \in \mathcal{D}'\left(A\mathcal{O}\right)$ and $p, q \in (0,\infty]$, we have*

$$\|f_A\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)} = \left|\det(A)\right|^{\frac{1}{p}-1} \cdot \|f\|_{\mathcal{D}_{\Phi_A}\left(A\mathcal{Q},L^p,\ell^q_u\right)}$$

*for every $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{Q}$.*

*Proof.* We first observe that $\Phi_A = \left(\varphi_i \circ A^{-1}\right)_{i\in I}$ is an $L^p$-BAPU for $A\mathcal{Q}$ by Lemma 3.5.1.

For $i \in I$, equation (3.5.1) and an application of the change-of-variables formula yield

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(\varphi_i \cdot f_A\right)\right\|_{L^p} &= \left|\det(A)\right|^{-1} \cdot \left\|\left(\mathcal{F}^{-1}\left[\left(\varphi_i \circ A^{-1}\right) \cdot f\right]\right) \circ A^{-T}\right\|_{L^p} \\
&= \left|\det(A)\right|^{\frac{1}{p}-1} \cdot \left\|\mathcal{F}^{-1}\left[\left(\varphi_i \circ A^{-1}\right) \cdot f\right]\right\|_{L^p}
\end{aligned}$$

in case of $0 < p < \infty$. For $p = \infty$, note that $A^{-T}$ and $A^T$ map null sets to null sets, so that the above identity remains valid.

By definition of the (quasi)-norms $\|\cdot\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)}$ and $\|\cdot\|_{\mathcal{D}_{\Phi_A}\left(A\mathcal{Q},L^p,\ell^q_u\right)}$, this easily implies the claim. □

# 4

# WAVELET COORBIT SPACES AS DECOMPOSITION SPACES

In this chapter, we will establish a close connection between the two types of spaces discussed in the previous chapters, namely between (wavelet) coorbit spaces and certain decomposition spaces.

More precisely, as described in the introduction (Chapter 1, especially Section 1.3), we will show that a natural form of the Fourier transform can be defined on (the reservoir of) the wavelet coorbit spaces and that this gives rise to an isomorphism

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right),$$

at least under certain assumptions on the weight $m$.

Put differently, there is a canonical identification

$$\Theta : \mathcal{R}_v \hookrightarrow Z'\left(\mathcal{O}\right) = \left[\mathcal{F}\left(C^\infty_c\left(\mathcal{O}\right)\right)\right]',$$

defined on the reservoir $\mathcal{R}_v$ used to define the coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$, which restricts to an isomorphism $\Theta : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}_{\mathrm{space}}\left(\mathcal{Q}, L^p, \ell^q_u\right)$.

In the above statements, $\mathcal{Q}$ is always an INDUCED COVERING of the DUAL ORBIT $\mathcal{O}$ (defined below and in the introduction) and the weight $u$ depends on $H, m$ and on the exponent $q$.

The structure of this chapter is as follows: In the first section, we lay the foundations for the study of WAVELET COORBIT SPACES. We recall from the introduction that these are usual coorbit spaces, but with a certain choice of the group $G$ and the representation $\pi$. More precisely, we assume the group to be of the form $G = \mathbb{R}^d \rtimes H$, for a closed subgroup $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$, called the DILATION GROUP for $G$. The group $G$ can be viewed as the smallest group of affine mappings containing all dilations $h \in H$ and arbitrary translations $x \in \mathbb{R}^d$.

With this view, $G$ acts naturally on $L^2\left(\mathbb{R}^d\right)$, namely by composition with the inverse mapping, and this action is called the QUASI-REGULAR REPRESENTATION $\pi : G \to \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right)$ of $G$. A wavelet coorbit space is any coorbit space defined using the group $G$ and the quasi-regular representation $\pi$. In this chapter, we will be even more restrictive and only consider the wavelet coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\right)$, where $L^{p,q}_m\left(G\right)$ is a certain mixed weighted Lebesgue space.

The main requirement for coorbit theory is that the representation $\pi$ be *irreducible* and *square integrable*. If this is the case, we call the dilation group $H$ ADMISSIBLE.

Thus, we begin the first section by recalling from [38] the characterization of admissible dilation groups. This characterization is in terms of the DUAL ACTION

$$p : H \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^{-T}\xi.$$

In particular, an admissible dilation group always has a single open orbit $\mathcal{O} = H^T\xi_0$ of full measure, called the DUAL ORBIT of $H$. This is the set on which the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ will "live".

In the remainder of the first section, we derive alternative, Fourier analytic expressions for the wavelet transform $W_\psi f$ and we prepare the next section by showing that, for an admissible dilation group $H$, the ORBIT MAP

$$p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$$

is a proper map for all $\xi \in \mathcal{O}$.

In the second section, we will use this property of the map $p_\xi$ to show that admissibility of the dilation group already implies that all assumptions for coorbit theory, with respect to the spaces $L^{p,q}_m(G)$, are satisfied, at least under certain mild growth restrictions on the weight $m$. Precisely, we will show that any $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with compact Fourier support $\operatorname{supp}\widehat{\psi} \subset \mathcal{O}$ is admissible as an analyzing vector $\psi \in \mathbb{A}^r_v$ for all $r \in (0, 1]$, with a suitable control weight $v$ for the space $L^{p,q}_m$. Thus, there is indeed a rich supply of WAVELET COORBIT SPACES.

The aim of the third section is the construction of the Fourier transform $\mathcal{F} : \mathcal{R}_v \to \mathcal{D}'(\mathcal{O})$ and of the canonical restriction map

$$\Theta : \mathcal{R}_v \to Z'(\mathcal{O}) = \left[\mathcal{F}\left(C^\infty_c(\mathcal{O})\right)\right]',$$

which are both defined on the (union of all) reservoir(s) $\mathcal{R}_v$ used to defined the wavelet coorbit spaces $\operatorname{Co}\left(L^{p,q}_m\right)$.

In this section, we also construct alternative coorbit spaces $\widetilde{\operatorname{Co}}\left(L^{p,q}_m\right)$ which are defined using the UNIFIED RESERVOIR $Z'(\mathcal{O})$ instead of $\mathcal{R}_v$. This has the advantage that the reservoir is *universal*, i.e. independent of the parameters $p, q$ and $m$, in contrast to the reservoir $\mathcal{R}_v$, where the control weight $v$ has to be chosen depending on $p, q, m$. Finally, we will see that the canonical restriction map $\Theta$ induces an isomorphism between $\widetilde{\operatorname{Co}}\left(L^{p,q}_m\right)$ and $\operatorname{Co}\left(L^{p,q}_m\right)$.

In Section 4.4, we will see that, by virtue of the dual action, the dilation group $H$ induces a certain covering of the frequency space $\mathbb{R}^d$, or more precisely of the dual orbit $\mathcal{O}$. To wit, if $(h_i)_{i\in I}$ is a well-spread family in $H$, we will show that there is always a set $Q \Subset \mathcal{O}$ such that $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ is a structured admissible covering of $\mathcal{O}$, as defined in the last chapter. We call each such covering $\mathcal{Q}$ A COVERING OF $\mathcal{O}$ INDUCED BY $H$.

The second important result of this section is construction of a special BAPU subordinate to the covering $\mathcal{Q}$.

In Section 4.5, we use this special BAPU to prove continuity of the Fourier transform $\mathcal{F} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$. Finally, Section 4.6 is devoted to proving that this map actually yields an isomorphism of (Quasi)-Banach spaces, i.e., we prove bijectivity of $\mathcal{F}$ and boundedness of the inverse map $\mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathrm{Co}\left(L^{p,q}_m\right)$.

The last section of this chapter is concerned with studying the property of *dilation invariance* of coorbit spaces. More precisely, we will show that invariance of a coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ under dilation with a matrix $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$ is equivalent to existence of an embedding

$$\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right) \hookrightarrow \mathrm{Co}\left(L^{p,q}_{g^{-1}mg}\left(\mathbb{R}^d \rtimes g^{-1}Hg\right)\right).$$

Together with the isomorphism of coorbit spaces and decomposition spaces, we conclude that the problem of dilation invariance of wavelet coorbit spaces reduces to deciding certain embeddings between decomposition spaces. As an immediate application of our results, we show that *all* coorbit spaces of the similitude group are invariant under dilation with *arbitrary* invertible matrices. In Section 6.5, we will see that this is a special property of the similitude group; the set of invertible matrices $g$ such that *all* coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ are invariant under dilation with $g$ will turn out to be a proper subset of $\mathrm{GL}\left(\mathbb{R}^2\right)$, cf. Theorem 6.5.9.

We finally mention that this chapter is based in parts on the paper [41] written by Hartmut Führ and the present author. In that paper, only the case of Banach spaces, i.e. $p, q \in [1, \infty]$, is considered. A discussion of the changes needed to account for the setting of Quasi-Banach spaces is given at the very end of Section 4.6.

## 4.1 ADMISSIBLE DILATION GROUPS AND THE DUAL ACTION

As mentioned in the introduction to this chapter, we restrict our attention to the following setting:

**Assumption 4.1.1.** *In the rest of this chapter, we assume the following:*

A. *$H$ is a closed subgroup of the matrix group $\mathrm{GL}\left(\mathbb{R}^d\right)$ for some fixed $d \in \mathbb{N}$. We call $H$ the* DILATION GROUP. *As a closed subgroup of the LCH group $\mathrm{GL}\left(\mathbb{R}^d\right)$, $H$ is an LCH group itself.*

B. *$G = \mathbb{R}^d \rtimes H$ is the affine group generated by arbitrary arbitrary translations and by all linear transformations $h \in H$. More precisely, as a set, $G$ is of the form $G = \mathbb{R}^d \times H$ with multiplication given by*

$$(x, h) \cdot (y, g) = (x + hy,\, hg).$$

*Then $G$ is an LCH group with* HAAR INTEGRAL *given by*

$$\int_G f(x,h)\ \mathrm{d}(x,h) = \int_H \int_{\mathbb{R}^d} f(x,h)\ \mathrm{d}x \frac{\mathrm{d}h}{|\det(h)|},$$

*where* $\mathrm{d}h$ *denotes the Haar measure on* $H$.

*The* MODULAR FUNCTION *of* $G$ *is*

$$\Delta_G(x,h) = \Delta_H(h) \cdot |\det(h)|^{-1}, \tag{4.1.1}$$

*where* $\Delta_H$ *is the modular function of* $H$.

c. *We assume that* $H$ *is* ADMISSIBLE. *By this, we mean that the* QUASI-REGULAR REPRESENTATION *of* $G$, *given by*

$$\pi : G \to \mathcal{U}\left(L^2(\mathbb{R}^d)\right), (x,h) \mapsto L_x D_h,$$

*is unitary, strongly continuous,* square-integrable *and* irreducible. *By [38, Corollary 21], this is the case if and only if the following two conditions hold:*

*We remark that [38, Corollary 21] does strictly speaking not contain the fact that the orbit $\mathcal{O}$ of full measure has to be open. But as a closed subgroup of $\mathrm{GL}(\mathbb{R}^d)$, $H$ is a Lie group. Since the orbit maps associated to (smooth) actions of Lie groups are maps of constant rank (cf. [55, Propostion 7.26]), it is a consequence of the global rank theorem (cf. [55, Theorem 4.14]), that an orbit of full measure is necessarily open.*

*a) There is some* $\xi_0 \in \mathbb{R}^d$ *such that the* DUAL ORBIT

$$\mathcal{O} := H^T\xi_0 = \left\{ h^T\xi_0 \,\middle|\, h \in H \right\} \subset \mathbb{R}^d$$

*is an open set of full measure, i.e.,* $\lambda(\mathcal{O}^c) = 0$, *where* $\lambda$ *denotes the Lebesgue measure on* $\mathbb{R}^d$.

*b) The isotropy group of* $\xi_0$ *(with respect to the dual action of* $H$*), given by*

$$H_{\xi_0} := \left\{ h \in H \,\middle|\, h^T\xi_0 = \xi_0 \right\}$$

*is compact. Using the identitiy* $H_{h^T\xi_0} = h^{-1}H_{\xi_0}h$, *we see that the isotropy group* $H_\xi \leq H$ *is compact for every* $\xi \in \mathcal{O}$.

The equivalent conditions for the admissibility of the dilation group $H$ indicate that the DUAL ACTION, i.e., the (left) action

$$p : H \times \mathbb{R}^d \to \mathbb{R}^d, (h,\xi) \mapsto h^{-T}\xi$$

plays an important role in understanding the quasi-regular representation. This is in part explained by computing the action of $G$ on $L^2(\mathbb{R}^d)$ on the Fourier side: We have

$$\begin{aligned}
\mathcal{F}[\pi(x,h)f] &= |\det(h)|^{-1/2} \cdot \mathcal{F}[L_x(f \circ h^{-1})] \\
&= |\det(h)|^{-1/2} \cdot M_{-x}[\mathcal{F}(f \circ h^{-1})] \\
&= |\det(h)|^{1/2} \cdot M_{-x}\left[\widehat{f} \circ h^T\right] \\
&= |\det(h)|^{1/2} \cdot M_{-x}\Delta_h \widehat{f}
\end{aligned} \tag{4.1.2}$$

with $\Delta_h g := g \circ h^T$. This identity is valid for all $f \in L^1\left(\mathbb{R}^d\right) + L^2\left(\mathbb{R}^d\right)$.

For us, the most important consequence of the above conditions will be that the ORBIT MAP

$$p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$$

is a PROPER MAP for each $\xi \in \mathcal{O}$, cf. Corollary 4.1.3. We will deduce this from the following more general lemma, the proof of which is similar to [32, Proposition (2.44)].

**Lemma 4.1.2.** *Let $G$ be a $\sigma$-compact LCH group, assume that $M$ is a* BAIRE SPACE *which is also Hausdorff. Finally, assume that $G$ acts on $M$ via the (left) action $\varrho : G \times M \to M$, where $\varrho$ is continuous in each component.*

*Let $m_0 \in M$ such that the orbit $Gm_0 \subset M$ is open and such that the isotropy group*

$$G_{m_0} := \{g \in G \mid \varrho(g, m_0) = m_0\}$$

*is compact. Then the orbit map*

$$\varrho_{m_0} : G \to Gm_0, g \mapsto \varrho(g, m_0)$$

*is a proper map, i.e., $\varrho_{m_0}^{-1}(K) \subset G$ is compact for each compact set $K \subset Gm_0$.*

*We recall from [56, §48] that $M$ is called a Baire space if for any countable sequence of closed subsets $A_n \subset M$ with empty interior, their union $\bigcup_{n\in\mathbb{N}} A_n$ also has empty interior.*

*Proof.* By assumption, $M$ is a Baire space and $Gm_0 \subset M$ is open. Invoking [56, Lemma 48.4], this implies that $Gm_0$ is also a Baire space.

By $\sigma$-compactness of $G$, we can write $G = \bigcup_{n\in\mathbb{N}} K_n$ for suitable compact sets $K_n \subset G$. This yields $Gm_0 = \bigcup_{n\in\mathbb{N}} \varrho_{m_0}(K_n)$, where each set $\varrho_{m_0}(K_n) \subset Gm_0$ is compact and hence closed, since $M$ is Hausdorff. Because $Gm_0$ is a Baire space, we conclude that the interior $U := (\varrho_{m_0}(K_n))^\circ$ is nonempty for some $n \in \mathbb{N}$. Let us fix $x_0 \in U$.

For each $g \in G$, the map $\varrho^g : M \to M, m \mapsto \varrho(g, m)$ is continuous by assumption. Because of $(\varrho^g)^{-1} = \varrho^{g^{-1}}$, we see that each $\varrho^g$ is a homeomorphism. In particular, $\varrho^g(U) \subset Gm_0$ is open for every $g \in G$.

Since $Gm_0$ is a single orbit with $x_0 \in U \subset Gm_0$, we get

$$K \subset Gm_0 = \bigcup_{g\in G} \{\varrho^g(x_0)\} \subset \bigcup_{g\in G} \varrho^g(U).$$

By compactness of $K$, this yields some $m \in \mathbb{N}$ and $g_1, \ldots, g_m \in G$ with $K \subset \bigcup_{i=1}^m \varrho^{g_i}(U)$.

Up to now, we did not use compactness of $G_{m_0}$. To see where this is used, let $g \in \varrho_{m_0}^{-1}(K)$. Because of $\varrho(g, m_0) \in K \subset \bigcup_{i=1}^m \varrho^{g_i}(U)$, there is some $i \in \underline{m}$ as well as $u \in U \subset \varrho_{m_0}(K_n)$ with

$$\begin{aligned}\varrho(g, m_0) &= \varrho^{g_i}(u) = \varrho(g_i, u)\\ &= \varrho(g_i, \varrho_{m_0}(k)) = \varrho(g_i, \varrho(k, m_0)) = \varrho(g_i k, m_0)\end{aligned}$$

for some $k \in K_n$. This yields

$$\begin{aligned}\varrho\left((g_i k)^{-1} g, m_0\right) &= \varrho\left((g_i k)^{-1}, \varrho(g, m_0)\right) \\ &= \varrho\left((g_i k)^{-1}, \varrho(g_i k, m_0)\right) = \varrho(1_G, m_0) = m_0\end{aligned}$$

and hence $(g_i k)^{-1} g \in G_{m_0}$, which finally implies

$$g \in g_i k G_{m_0} \subset \bigcup_{j=1}^{m} g_j K G_{m_0}.$$

Since $g \in \varrho_{m_0}^{-1}(K)$ was arbitrary, we see $\varrho_{m_0}^{-1}(K) \subset \bigcup_{j=1}^{m} g_j K G_{m_0}$, so that $\varrho_{m_0}^{-1}(K)$ is compact as a closed subset of a compact set. □

**Corollary 4.1.3.** *For each $\xi \in \mathcal{O}$, the* ORBIT MAP

$$p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$$

*is a proper map, i.e., $p_\xi^{-1}(K) \subset H$ is compact for every compact set $K \subset \mathcal{O}$. Furthermore, $p_\xi$ is also an open map.*

*Proof.* By Baire's category theorem, $\mathbb{R}^d$ is a Baire space, since it is a complete metric space. Furthermore, $H \subset \mathbb{R}^{d\times d}$ is $\sigma$-compact as a second countable LCH group.

We finally consider the left action

$$p : H \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^{-T}\xi$$

which is jointly continuous. By assumption on $H$, the isotropy group

$$\{h \in H \mid \varrho(h, \xi) = \xi\} = \{h \in H \mid h^{-T}\xi = \xi\} = \{h \in H \mid \xi = h^T\xi\} = H_\xi$$

is compact. Furthermore, $\varrho(H, \xi) = H^{-T}\xi = H^T\xi = \mathcal{O}$ is an open subset of $\mathbb{R}^d$.

By Lemma 4.1.2, this implies that the map $p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$ is proper.

Since $\mathcal{O}$ is a locally compact space, [32, Proposition (2.44)] shows that

$$H/H_\xi \to \mathcal{O}, hH_\xi \mapsto h^{-T}\xi$$

is a well-defined homeomorphism. But the proof actually shows that $H \to \mathcal{O}, h \mapsto h^{-T}\xi$ is an open map, so that $p_\xi$ is open. □

The properness of the orbit map $p_\xi$ will play a crucial role in the next section, in particular for the proof that every Schwartz function with compact Fourier support in $\mathcal{O}$ is an analyzing vector. Furthermore, it implies that dilations $h \in H$ that are mapped "close to each other" by the orbit map $p_\xi$ are already "close to each other" in $H$. This statement is made precise in the following lemma.

**Lemma 4.1.4.** *(cf. [41, Lemma 18]) Let $K_1, K_2 \subset \mathcal{O}$ be compact sets. Fix $\xi_0 \in \mathcal{O}$ and define*

$$L := L(K_1, K_2) := p_{\xi_0}^{-1}(K_1) \cdot H_{\xi_0} \cdot \left(p_{\xi_0}^{-1}(K_2)\right)^{-1}.$$

*Then $L \subset H$ is compact.*

*For arbitrary $g, h \in H$ satisfying $h^{-T}K_1 \cap g^{-T}K_2 \neq \varnothing$ we have $g \in hL$.*

*Proof.* Corollary 4.1.3 shows that the inverse image $p_{\xi_0}^{-1}(K_i) \subset H$ is compact for $i \in \{1, 2\}$. Thus, $L \subset H$ is also a compact set.

For the proof of the second claim, let $h, g \in H$ be arbitrary with $h^{-T}K_1 \cap g^{-T}K_2 \neq \varnothing$. This yields existence of suitable $h_1, h_2 \in H$ which satisfy $k_1 = h_1^T \xi_0 \in K_1 \subset \mathcal{O}$ and $k_2 = h_2^T \xi_0 \in K_2 \subset \mathcal{O}$, as well as $h^{-T}h_1^T \xi_0 = g^{-T}h_2^T \xi_0$. We conclude

$$\left(h_1 h^{-1} g h_2^{-1}\right)^T \xi_0 = \xi_0.$$

In other words, we have $h_1 h^{-1} g h_2^{-1} \in H_{\xi_0}$ and thus

$$g \in h h_1^{-1} H_{\xi_0} h_2 \subset h \cdot p_{\xi_0}^{-1}(K_1) \cdot H_{\xi_0} \cdot \left(p_{\xi_0}^{-1}(K_2)\right)^{-1} = h \cdot L.$$

Here, we used $(h_1^{-1})^{-T} \xi_0 = k_1 \in K_1$ and hence $h_1^{-1} \in p_{\xi_0}^{-1}(K_1)$ in the last step. Analogously, $h_2 \in \left(p_{\xi_0}^{-1}(K_2)\right)^{-1}$. □

Since we want to establish a connection between the wavelet coorbit spaces, which are defined in terms of the wavelet transform $W_\psi f$, and certain decomposition spaces, which are defined using the Fourier transform, the following collection of different expressions for the wavelet transform (most of which involve the Fourier transform) will turn out to be useful.

**Lemma 4.1.5.** *For $f, \psi \in L^2(\mathbb{R}^d)$ and $(x, h) \in G$, the following identities are true:*

$$\begin{aligned}(W_\psi f)(x, h) &= \left[f * \overline{D_h \psi}^\vee\right](x) \\ &= |\det(h)|^{1/2} \cdot \mathcal{F}\left[\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}}\right](-x) \\ &= |\det(h)|^{1/2} \cdot \mathcal{F}^{-1}\left[\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}}\right](x). \end{aligned} \tag{4.1.3}$$

*For convenience of the reader, we recall the notations $\Delta_h f = f \circ h^T$ and $D_h f = |\det(h)|^{-\frac{1}{2}} \cdot f \circ h^{-1}$, as well as $f^\vee(y) = f(-y)$.*

*For $f, \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have $W_\psi f(\cdot, h) \in L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ with*

$$\mathcal{F}\left[W_\psi f(\cdot, h)\right] = |\det(h)|^{1/2} \cdot \widehat{f} \cdot \overline{\Delta_h \widehat{\psi}},$$

*with equality in the $L^2$-sense.*

*If $W_\psi f(\cdot, h) \in L^1(\mathbb{R}^d)$, then the equality holds pointwise. This is in particular the case for $f, \psi \in \mathcal{S}(\mathbb{R}^d)$; this even yields $W_\psi f(\cdot, h) \in \mathcal{S}(\mathbb{R}^d)$.*

*In particular, if $\widehat{\psi}$ has compact support, then $\mathcal{F}\left[W_\psi f(\cdot, h)\right]$ vanishes (almost everywhere) outside of $h^{-T} \cdot \operatorname{supp} \widehat{\psi}$.*

*Proof.* Using the definition of the quasi-regular representation, we get

$$\begin{aligned} W_\psi f(x,h) &= \langle f \mid \pi(x,h)\,\psi \rangle \\ &= \int_{\mathbb{R}^d} f(y) \cdot \overline{L_x D_h \psi}(y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y) \cdot \overline{D_h \psi}^{\vee}(x-y)\, \mathrm{d}y \\ &= \left[ f * \overline{D_h \psi}^{\vee} \right](x). \end{aligned}$$

By using Plancherel's theorem and the action of $\pi$ on the Fourier side given in equation (4.1.2), we also get

$$\begin{aligned} W_\psi f(x,h) &= \left\langle \widehat{f} \,\middle|\, \mathcal{F}\left[\pi(x,h)\,\psi\right] \right\rangle \\ &= |\det(h)|^{1/2} \cdot \left\langle \widehat{f} \,\middle|\, M_{-x} \cdot \Delta_h \widehat{\psi} \right\rangle \\ &= |\det(h)|^{1/2} \cdot \int_{\mathbb{R}^d} \widehat{f}(\xi) \cdot \overline{e^{2\pi i \langle -x, \xi \rangle} \cdot \Delta_h \widehat{\psi}(\xi)}\, \mathrm{d}\xi \\ &= |\det(h)|^{1/2} \cdot \int_{\mathbb{R}^d} \widehat{f}(\xi) \cdot \Delta_h \overline{\widehat{\psi}}(\xi) \cdot e^{2\pi i \langle x, \xi \rangle}\, \mathrm{d}\xi \\ &= |\det(h)|^{1/2} \cdot \left( \mathcal{F}^{-1}\left[ \widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \right] \right)(x). \end{aligned}$$

In the last step, we used $\widehat{f}, \Delta_h \overline{\widehat{\psi}} \in L^2(\mathbb{R}^d)$, so that their product is in $L^1(\mathbb{R}^d)$. Because of $\mathcal{F}^{-1} h(x) = \mathcal{F} h(-x)$, this proves the first block of identities.

Now, $f, \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ yields $\widehat{f}, \Delta_h \overline{\widehat{\psi}} \in L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and thus $\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Using the Riemann-Lebesgue lemma and the Plancherel theorem, this implies

$$W_\psi f(\cdot, h) = |\det(h)|^{1/2} \cdot \mathcal{F}^{-1}\left[ \widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \right] \in L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$$

and

$$\mathcal{F}\left[W_\psi f(\cdot, h)\right] = |\det(h)|^{1/2} \cdot \widehat{f} \cdot \Delta_h \overline{\widehat{\psi}},$$

with equality in the $L^2$-sense. If $W_\psi f(\cdot, h) \in L^1(\mathbb{R}^d)$, then the left-hand side and the right-hand side of this identity are continuous functions (because of $f, \psi \in L^1(\mathbb{R}^d)$), so that the equality holds pointwise.

In case of $f, \psi \in \mathcal{S}(\mathbb{R}^d)$, we get $\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \in \mathcal{S}(\mathbb{R}^d)$ as a product of Schwartz functions (cf. [61, Theorem 7.4]). But this implies

$$W_\psi f(\cdot, h) = |\det(h)|^{\frac{1}{2}} \cdot \mathcal{F}^{-1}\left[ \widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \right] \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d). \qquad \square$$

## 4.2 APPLICABILITY OF COORBIT THEORY FOR THE SPACES $L^{p,q}_m$

In the previous section, we imposed certain admissibility assumptions on the dilation group $H$ which ensure that the quasi-regular representation is square-integrable and irreducible.

In order to apply coorbit theory (as developed in Chapter 2) for the space $Y = L^{p,q}_m(G)$, we need to construct a suitable CONTROL WEIGHT $v : G \to (0,\infty)$ and to verify that the space $\mathbb{A}^r_v$ of ANALYZING VECTORS is nontrivial. This is the purpose of the present section.

We begin with analyzing the (translation invariance) properties of the spaces $Y = L^{p,q}_m(G)$ and $W(L^\infty, Y)$.

In the following, we will often use the space of Schwartz functions which is defined using polynomial decay conditions. Thus, we restrict our attention to those weights that grow at most polynomially with respect to the first component:

**Definition 4.2.1.** A weight $v : G \to (0,\infty)$ is called OF MODERATE GROWTH if there exists a locally bounded weight $v_0 : H \to (0,\infty)$ and some $s \geq 0$ satisfying

$$v(x,h) \leq (1+|x|)^s \cdot v_0(h) \tag{4.2.1}$$

for all $x \in \mathbb{R}^d$. Each weight $v_0$ with these properties is called a DILATION WEIGHT for the weight $v$.

We denote the class of all weights of moderate growth by

$$WMG := \{v : G \to (0,\infty) \mid v \text{ of moderate growth}\}.$$

Finally,

$$WMG_{\text{sub}} = \{v \in WMG \mid v \text{ is submultiplicative}\}$$

denotes the class of all submultiplicative weights of moderate growth.

*Remark* 4.2.2. We first observe that any weight of moderate growth is locally bounded, since the right hand side of equation (4.2.1) is.

Finally, a *submultiplicative*, measurable weight $v : G \to (0,\infty)$ is of moderate growth if and only if

$$v(x,\mathrm{id}) \leq C \cdot (1+|x|)^s$$

holds for suitable $C > 0$ and $s \geq 0$ for all $x \in \mathbb{R}^d$. In this case, the dilation weight $v_0 : H \to (0,\infty)$ can be taken to be submultiplicative as well. ◀

*Proof.* "⇒": If $v : G \to (0,\infty)$ is submultiplicative and of moderate growth, then $v(x,\mathrm{id}) \leq (1+|x|)^s \cdot v_0(\mathrm{id})$ holds for some locally bounded weight $v_0 : H \to (0,\infty)$, some $s \geq 0$ and for all $x \in \mathbb{R}^d$, so that we can take $C := v_0(\mathrm{id})$.

"$\Leftarrow$": Let $v : G \to (0,\infty)$ be measurable and submultiplicative with

$$v(x, \mathrm{id}) \leq C \cdot (1 + |x|)^s$$

for all $x \in \mathbb{R}^d$. Theorem 2.2.22 shows that $v$ is locally bounded, so that the same is true of

$$v_0 : H \to (0,\infty), h \mapsto \max\{1, C\} \cdot v(0, h).$$

We observe

$$\begin{aligned} v_0(hg) &= \max\{1, C\} \cdot v(0, hg) = \max\{1, C\} \cdot v((0, h)(0, g)) \\ &\leq \max\{1, C\} \cdot v(0, h) \cdot v(0, g) \\ &\leq v_0(h) \cdot v_0(g), \end{aligned}$$

so that $v_0$ is submultiplicative. Finally,

$$\begin{aligned} v(x, h) &= v((x, \mathrm{id})(0, h)) \\ &\leq v(x, \mathrm{id}) \cdot v(0, h) \\ &\leq C \cdot (1 + |x|)^s \cdot v(0, h) \\ &\leq (1 + |x|)^s \cdot v_0(h) \end{aligned}$$

holds for all $(x, h) \in G$, so that $v$ is of moderate growth with submultiplicative dilation weight $v_0$. □

We recall from Assumption 2.4.1 that the control weights $v, w$ for the coorbit space $\mathrm{Co}(Y)$ depend on the norms $|||R_x|||_{W(L^\infty,Y)\to W(L^\infty,Y)}$. Our first aim is thus to show that $|||R_{(x,h)}|||_{W(L^\infty,Y)\to W(L^\infty,Y)}$, for the space $Y = L^{p,q}_m$, grows at most polynomially in $x$ if the weight $m$ does. To this end, we will employ the estimate

$$|||R_x|||_{W_Q(L^\infty,Y)\to W_Q(L^\infty,Y)} \leq \left(\sum_{i=1}^n |||R_{x_i}|||^p_{Y\to Y}\right)^{1/p} \tag{4.2.2}$$

from Lemma 2.3.18, which is valid if $(x_i)_{i\in\underline{n}}$ satisfies $Qx \subset \bigcup_{i=1}^n x_i Q$. The following elementary result about covering balls in $\mathbb{R}^d$ with balls of smaller radii will provide a way of choosing the $(x_i)_{i\in\underline{n}}$ while controlling $n$.

**Lemma 4.2.3.** *Let $d \in \mathbb{N}$ and $r, R > 0$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^d$.*
*Then there is a family $(x_i)_{i\in\underline{M}}$ satisfying*

$$M \leq \left(1 + 2 \cdot \frac{R}{r}\right)^d$$

*as well as*

$$B_R^{\|\cdot\|}(0) \subset \bigcup_{i=1}^M B_r^{\|\cdot\|}(x_i).$$

*Proof.* For brevity, let us write $B_s(x) := B_s^{\|\cdot\|}(x)$ for arbitrary $s > 0$ and $x \in \mathbb{R}^d$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^d$.

Let $N \in \mathbb{N}$ and let $(y_i)_{i \in \underline{N}}$ be a family in $B_R(0)$ such that the balls $(B_{r/2}(y_i))_{i \in \underline{N}}$ are pairwise disjoint. Since $y_i \in B_R(0)$, we have $B_{r/2}(y_i) \subset B_{R+\frac{r}{2}}(0)$. This implies

$$\begin{aligned} N \cdot \lambda(B_1(0)) \cdot \left(\frac{r}{2}\right)^d &= \sum_{i=1}^{N} \lambda(B_{r/2}(y_i)) \\ &= \lambda\left(\biguplus_{i=1}^{N} B_{r/2}(y_i)\right) \\ &\leq \lambda\left(B_{R+\frac{r}{2}}(0)\right) \\ &= \lambda(B_1(0)) \cdot \left(R + \frac{r}{2}\right)^d. \end{aligned}$$

Using $0 < \lambda(B_1(0)) < \infty$, we conclude

$$N \leq \left(\frac{2}{r}\right)^d \cdot \left(R + \frac{r}{2}\right)^d = \left(1 + 2 \cdot \frac{R}{r}\right)^d =: N_0.$$

We have thus shown that the set

$$\Gamma := \left\{ N \in \mathbb{N} \;\middle|\; \exists\, (y_i)_{i \in \underline{N}} \text{ in } B_R(0) \text{ with } (B_{r/2}(y_i))_{i \in \underline{N}} \text{ p.w. disjoint} \right\}$$

is bounded above by $N_0$. Thus, $M := \max \Gamma$ is well-defined, since $\Gamma$ is obviously nonempty. By definition of $M = \max \Gamma$, there is a family $(y_i)_{i \in \underline{M}}$ in $B_R(0)$ such that $(B_{r/2}(y_i))_{i \in \underline{M}}$ is pairwise disjoint. We claim that

$$B_R(0) \subset \bigcup_{i=1}^{M} B_r(y_i).$$

Otherwise, there would be some $y_{M+1} \in B_R(0) \setminus \bigcup_{i=1}^{M} B_r(y_i)$. But this implies $B_{r/2}(y_{M+1}) \cap B_{r/2}(y_i) = \varnothing$ for arbitrary $i \in \underline{M}$, since any $x \in B_{r/2}(y_{M+1}) \cap B_{r/2}(y_i)$ would yield

$$\|y_{M+1} - y_i\| \leq \|y_{M+1} - x\| + \|x - y_i\| < \frac{r}{2} + \frac{r}{2} = r$$

and hence $y_{M+1} \in B_r(y_i)$, contradicting the choice of $y_{M+1}$. But since the family $(B_{r/2}(y_i))_{i \in \underline{M}}$ is already disjoint, this means that $(B_{r/2}(y_i))_{i \in \underline{M+1}}$ is a pairwise disjoint family. Using $y_{M+1} \in B_R(0)$, we conclude $M + 1 \in \Gamma$, which contradicts $M = \max \Gamma$.

This contradiction proves $B_R(0) \subset \bigcup_{i=1}^{M} B_r(y_i)$. Because of $M \leq N_0$ as seen above, the proof is complete. □

This covering result and estimate (4.2.2) will allow us to prove the polynomial growth of the operator norms $\left|\!\left|\!\left| R_{(x,h)} \right|\!\right|\!\right|_{W\left(L^{p,q}_m\right) \to W\left(L^{p,q}_m\right)}$.

But first, let us formally define the spaces $L^{p,q}_m(G)$, which were up to now only used in the introduction.

**Definition 4.2.4.** Let $m : G \to (0,\infty)$ be a (measurable) weight and let $p,q \in (0,\infty]$ be arbitrary. We define the (weighted) MIXED LEBESGUE SPACE $L^{p,q}_m = L^{p,q}_m(G)$ as

$$L^{p,q}_m(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; f \text{ measurable and } \|f\|_{L^{p,q}_m} < \infty \right\},$$

with

$$\|f\|_{L^{p,q}_m} := \left[ \int_H \|m(\cdot,h) \cdot f(\cdot,h)\|^q_{L^p(\mathbb{R}^d)} \, \frac{\mathrm{d}h}{|\det(h)|} \right]^{1/q}$$

for $q \in (0,\infty)$ and with

$$\|f\|_{L^{p,q}_m} := \operatorname*{ess\,sup}_{h \in H} \|m(\cdot,h) \cdot f(\cdot,h)\|_{L^p(\mathbb{R}^d)}$$

for $q = \infty$.

*Remark* 4.2.5. We remark that for each measurable function $f : G \to \mathbb{C}$, the "inner norm"

$$h \mapsto \|f(\cdot,h)\|_{L^p(\mathbb{R}^d)}$$

is a measurable function.

Indeed, for $p \in (0,\infty)$, the function $G \to [0,\infty), (x,h) \mapsto |f(x,h)|^p$ is measurable. Since $G = \mathbb{R}^d \times H$, where $\mathbb{R}^d$ and $H$ are $\sigma$-compact and hence $\sigma$-finite with respect to their Haar measures, we can apply the Fubini-Tonelli Theorem [33, Theorem 2.37] (together with the fact that the Borel $\sigma$-algebra on $G$ is the product $\sigma$-algebra of the factors, cf. [33, Theorem 7.20]) to conclude that

$$H \to [0,\infty], h \mapsto \|f(\cdot,h)\|^p_{L^p} = \int_{\mathbb{R}^d} |f(x,h)|^p \, \mathrm{d}x$$

is a measurable map. Hence, the same is true of $h \mapsto \|f(\cdot,h)\|_{L^p}$.

Finally, if $p = \infty$, consider

$$M_n := \left\{ (x,h) \in [-n,n]^d \times H \;\middle|\; |f(x,h)| \leq n \right\}$$

and note that $f_n := f \cdot \chi_{M_n}$ is bounded and satisfies $f_n(\cdot,h) \equiv 0$ outside of $[-n,n]^d$. Hence, $f_n(\cdot,h) \in L^q(\mathbb{R}^d)$ for all $q \in (0,\infty]$ and $h \in H$. By [33, §6.1, Exercise 7], this implies

$$\|f_n(\cdot,h)\|_{L^\infty} = \lim_{k \to \infty} \|f_n(\cdot,h)\|_{L^k} \text{ for all } h \in H.$$

Using the result proved above for each $k \in (0,\infty)$, we conclude that $h \mapsto \|f_n(\cdot,h)\|_{L^\infty}$ is measurable. But using $|f_n| \nearrow |f|$ (with pointwise

convergence), it is easy to see $\|f_h(\cdot,h)\|_{L^\infty} \nearrow \|f(\cdot,h)\|_{L^\infty}$ as $n \to \infty$ for all $h \in H$, which implies measurability of

$$H \to [0,\infty], h \mapsto \|f(\cdot,h)\|_{L^\infty}. \qquad \blacktriangleleft$$

Now, let us estimate the operator norm $\left|\!\left|\!\left|R_{(x,h)}\right|\!\right|\!\right|_{W\left(L_m^{p,q}\right)\to W\left(L_m^{p,q}\right)}$.

**Lemma 4.2.6.** *Let $p,q \in (0,\infty]$ and let $m_0 : G \to (0,\infty)$ be a (measurable) weight. Finally, let $Q \subset G$ be a precompact measurable unit neighborhood. Then*

1. *$L_{m_0}^{p,q}(G)$ is a solid QBF space satisfying the Fatou property.*
2. *Set $r := \frac{\min\{1,p,q\}}{2}$. Then $L_{m_0}^{p,q}(G)$ admits an equivalent $r$-norm $|||\cdot|||$ for which $\left(L_{m_0}^{p,q}, |||\cdot|||\right)$ is a solid QBF space.*
3. *If $m_0$ is right moderate with respect to a (measurable) weight $m_1$ on $G$, then $L_{m_0}^{p,q}(G)$ is invariant under right translations with*

$$\left|\!\left|\!\left|R_{(x,h)}\right|\!\right|\!\right|_{L_{m_0}^{p,q}\to L_{m_0}^{p,q}} \leq \left(\frac{|\det(h)|}{\Delta_H(h)}\right)^{1/q} \cdot m_1\left((x,h)^{-1}\right) \quad \forall (x,h) \in G.$$

4. *If $m_0$ is left moderate with respect to a (measurable) weight $m_1$ on $G$, then $L_{m_0}^{p,q}(G)$ is invariant under left translations.*
5. *If the weight $m_1$ from the previous part of the lemma satisfies*

$$m_1(x,h) \leq (1+|x|)^s \cdot m(h) \qquad \forall (x,h) \in G$$

*for some $s \geq 0$ and some (measurable) weight $m : H \to (0,\infty)$, then the weights*

$$w_0 : H \to (0,\infty), h \mapsto \left|\!\left|\!\left|R_{(0,h)}\right|\!\right|\!\right|_{W_Q\left(L_{m_0}^{p,q}\right)\to W_Q\left(L_{m_0}^{p,q}\right)},$$
$$w_1 : G \to (0,\infty), (x,h) \mapsto \left|\!\left|\!\left|R_{(x,h)}\right|\!\right|\!\right|_{W_Q\left(L_{m_0}^{p,q}\right)\to W_Q\left(L_{m_0}^{p,q}\right)}$$

*are measurable and submultiplicative and in particular locally bounded. Furthermore, there is a constant $C = C(Q,p,q,r,d,s,m) > 0$ such that*

$$\left|\!\left|\!\left|R_{(x,h)}\right|\!\right|\!\right|_{W_Q\left(L_{m_0}^{p,q}\right)\to W_Q\left(L_{m_0}^{p,q}\right)} \leq C \cdot (1+|x|)^{s+\frac{d}{r}} \cdot w_0(h) \qquad (4.2.3)$$

*holds for all $(x,h) \in G$. Here, $r = \frac{\min\{1,p,q\}}{2}$ is as in part 2.*

*Proof.* Define the Borel measure $\nu$ on $H$ by $\mathrm{d}\nu(h) := |\det(h)|^{-1}\,\mathrm{d}h$, where $\mathrm{d}h$ denotes the Haar measure on $H$. Note that $\nu$ has the same nullsets as $\mathrm{d}h$ and that $\|f\|_{L^{p,q}} = \left\|h \mapsto \|f(\cdot,h)\|_{L^p(\mathbb{R}^d)}\right\|_{L^q(\nu)}$ holds for all measurable $f : G \to \mathbb{C}$.

We first show definiteness of $\|\cdot\|_{L^{p,q}}$. To this end, let $f : G \to \mathbb{C}$ be measurable with

$$0 = \|f\|_{L^{p,q}} = \|h \mapsto \|f(\cdot, h)\|_{L^p}\|_{L^q(\nu)}.$$

This implies that there is a $\nu$-nullset $N \subset H$ satisfying

$$\|f(\cdot, h)\|_{L^p} = 0 \text{ for all } h \in H \setminus N.$$

Hence, there is for each $h \in H \setminus N$ some nullset $N_h \subset \mathbb{R}^d$ satisfying $f(x, h) = 0$ for all $x \in \mathbb{R}^d \setminus N_h$.

But since $f$ is measurable, so is the set

$$M := \{(x, h) \in G \mid f(x, h) \neq 0\}.$$

Using the Fubini-Tonelli theorem and the form of the Haar measure $\mu$ on $G = \mathbb{R}^d \rtimes H$, we conclude

$$\begin{aligned}\mu(M) &= \int_H \int_{\mathbb{R}^d} \chi_M(x, h)\, \mathrm{d}x \frac{\mathrm{d}h}{|\det(h)|} \\ &= \int_{H \setminus N} \int_{\mathbb{R}^d \setminus N_h} \chi_M(x, h)\, \mathrm{d}x\, \mathrm{d}\nu(h) = 0.\end{aligned}$$

Here, we used $\chi_M(\cdot, h) \equiv 0$ on $\mathbb{R}^d \setminus N_h$ for each $h \in H \setminus N$ and the fact that $N, N_h$ are nullsets. All in all, we have shown $f = 0$ almost everywhere, so that $\|\cdot\|_{L^{p,q}}$ is positive definite.

Now, let us check that $\|\cdot\|_{L^{p,q}}$ satisfies the Fatou property. To this end, let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : G \to \mathbb{C}$ with $|f| \leq \liminf_{n \to \infty} |f_n|$ almost everywhere for some measurable $f : G \to \mathbb{C}$. This means that the set

$$M := \left\{(x, h) \in G \,\middle|\, |f(x, h)| > \liminf_{n \to \infty} |f_n(x, h)|\right\}$$

is a nullset. By the same argument as above, we conclude

$$0 = \mu(M) = \int_H \int_{\mathbb{R}^d} \chi_M(x, h)\, \mathrm{d}x\, \mathrm{d}\nu(h).$$

Since the integrand of the outer integral is nonnegative, this yields a nullset $N \subset H$ with $\mu(M_h) = \int_{\mathbb{R}^d} \chi_M(x, h)\, \mathrm{d}x = 0$ for all $h \in H \setminus N$ and

$$M_h := \left\{x \in \mathbb{R}^d \,\middle|\, (x, h) \in M\right\}.$$

By definition of $M$, we have $|f(x, h)| \leq \liminf_{n \to \infty} |f_n(x, h)|$ for all $x \in \mathbb{R}^d \setminus M_h$ and thus for almost all $x \in \mathbb{R}^d$, at least for $h \in H \setminus N$. Since $\|\cdot\|_{L^p(\mathbb{R}^d)}$ satisfies the Fatou property, we get

$$\|f(\cdot, h)\|_{L^p(\mathbb{R}^d)} \leq \liminf_{n \to \infty} \|f_n(\cdot, h)\|_{L^p(\mathbb{R}^d)} \text{ for all } h \in H \setminus N$$

and hence for $\nu$-almost all $h \in H$. But $\|\cdot\|_{L^q(\nu)}$ also satisfies the Fatou property. Thus,

$$\begin{aligned}\|f\|_{L^{p,q}} &= \left\| h \mapsto \|f(\cdot,h)\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(\nu)} \\ &\leq \liminf_{n\to\infty} \left\| h \mapsto \|f_n(\cdot,h)\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(\nu)} = \liminf_{n\to\infty} \|f_n\|_{L^{p,q}}.\end{aligned}$$

If the Limes inferior is finite, we get in particular $f \in L^{p,q}$. All in all, this shows that $\|\cdot\|_{L^{p,q}}$ satisfies the Fatou property. By Lemma 2.2.15, this implies that $L^{p,q}$ is a solid QBF space (in particular complete), at least once we have shown that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm.

To see that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm, let $s := \min\{1,p,q\}$ and $r := s/2$. We recall from Remark 2.1.2 that a $p$-norm is also a $q$-norm for all $q \in (0,p]$. Furthermore, Example 2.1.3 shows that $\|\cdot\|_{L^p(\mu)}$ is a $p$-norm for $p \in (0,1]$ and arbitrary measure spaces $(X,\mathcal{A},\mu)$. Since $\|\cdot\|_{L^p(\mu)}$ is a genuine norm (i.e., a 1-norm) for $p \in [1,\infty]$, we see that $\|\cdot\|_{L^p(\mu)}$ is a $\min\{1,p\}$-norm and hence also a $\min\{1,p,q\}$-norm for each $p \in (0,\infty]$. The same is true for $\|\cdot\|_{L^q(\mu)}$. Finally, Remark 2.1.2 shows that $2^{\frac{1}{s}-1}$ is a valid triangle inequality for each $s$-norm $\|\cdot\|$ and thus in particular for $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^q}$. For $f,g \in L^{p,q}(G)$, this implies

$$\begin{aligned}&\|f+g\|_{L^{p,q}} \\ &= \|h \mapsto \|f(\cdot,h)+g(\cdot,h)\|_{L^p}\|_{L^q(\nu)} \\ &\leq \left\| h \mapsto 2^{\frac{1}{s}-1}\left[\|f(\cdot,h)\|_{L^p} + \|g(\cdot,h)\|_{L^p}\right] \right\|_{L^q(\nu)} \\ &\leq 2^{\frac{1}{s}-1}2^{\frac{1}{s}-1} \cdot \left[ \|h \mapsto \|f(\cdot,h)\|_{L^p}\|_{L^q(\nu)} + \|h \mapsto \|g(\cdot,h)\|_{L^p}\|_{L^q(\nu)} \right] \\ &\leq 2^{\frac{2}{s}-1} \cdot \left[\|f\|_{L^{p,q}} + \|g\|_{L^{p,g}}\right] < \infty.\end{aligned}$$

Since $\|\cdot\|_{L^{p,q}}$ is obviously homogeneous, so that $L^{p,q}(G)$ is closed under multiplication with scalars, we see that $L^{p,q}(G)$ is a vector space and that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm with triangle constant $2^{\frac{1}{s/2}-1}$. The $\frac{s}{2}$-norm $|||\cdot|||$ constructed in Theorem 2.1.4 thus yields an equivalent quasi-norm on $L^{p,q}(G)$. Corollary 2.2.12 shows that $(L^{p,q}, |||\cdot|||)$ is a solid QBF space.

We have thus proved claims (1) and (2) in the unweighted case. But the weighted case is a direct consequence of Lemma 2.2.18 and with the fact that $f \mapsto |||f \cdot m_0|||$, with the quasi-norm $|||\cdot|||$ from the unweighted case, defines an equivalent $r$-norm on $L^{p,q}_{m_0}(G)$ for which $\left(L^{p,q}_{m_0}(G), |||\cdot|||\right)$ is a solid QBF space.

Let us now prove parts 3 and 4 for the unweighted case. The weighted case is then a direct consequence of Lemma 2.2.19. We first note

$$\begin{aligned}\left[R_{(y,T)}f\right](x,h) &= f((x,h)(y,T)) = f(x+hY, hT), \qquad (4.2.4)\\ \left[L_{(y,T)^{-1}}f\right](x,h) &= f((y,T)(x,h)) = f(y+Tx, Th).\end{aligned}$$

Let us first assume $q \in (0,\infty)$ and set $F(h) := \|f(\cdot,h)\|_{L^p}^q / |\det(h)|$. Using the translation invariance of $\|\cdot\|_{L^p(\mathbb{R}^d)}$, we get

$$\begin{aligned}\frac{\left\|\left[R_{(y,T)}f\right](\cdot,h)\right\|_{L^p}^q}{|\det(h)|} &= |\det(T)| \cdot \frac{\|f(\cdot,hT)\|_{L^p}^q}{|\det(hT)|} \\ &= |\det(T)| \cdot (R_T F)(h)\end{aligned}$$

for all $h \in H$. Likewise, we use the change-of-variables formula (for $p < \infty$), or the fact that $T$ and $T^{-1}$ map null sets to null sets to conclude

$$\begin{aligned}\frac{\left\|\left[L_{(y,T)^{-1}}f\right](\cdot,h)\right\|_{L^p}^q}{|\det(h)|} &= \frac{\left\|\left[L_{(y,T)^{-1}}f\right](y+T\cdot,Th)\right\|_{L^p}^q}{|\det(h)|} \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot \frac{\left\|\left[L_{(y,T)^{-1}}f\right](\cdot,Th)\right\|_{L^p}^q}{|\det(Th)|} \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot (L_{T^{-1}}F)(h). \qquad (4.2.5)\end{aligned}$$

By [32, Proposition (2.24)], we finally conclude

$$\begin{aligned}\left\|R_{(y,T)}f\right\|_{L^{p,q}}^q &= \int_H \frac{\left\|\left[R_{(y,T)}f\right](\cdot,h)\right\|_{L^p}^q}{|\det(h)|}\,\mathrm{d}h \\ &= |\det(T)| \cdot \int_H (R_T F)(h)\,\mathrm{d}h \\ &= \Delta_H(T^{-1}) \cdot |\det(T)| \cdot \int_H F\,\mathrm{d}h \\ &= \frac{|\det(T)|}{\Delta_H(T)} \cdot \int_H \|f(\cdot,h)\|_{L^p}^q \frac{\mathrm{d}h}{|\det(h)|} \\ &= \frac{|\det(T)|}{\Delta_H(T)} \cdot \|f\|_{L^{p,q}}^q < \infty.\end{aligned}$$

Similarly, left-invariance of the Haar measure on $H$ yields

$$\begin{aligned}\left\|L_{(y,T)^{-1}}f\right\|_{L^{p,q}}^q &= \int_H \frac{\left\|\left[L_{(y,T)^{-1}}f\right](\cdot,h)\right\|_{L^p}^q}{|\det(h)|}\,\mathrm{d}h \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot \int_H L_{T^{-1}}F\,\mathrm{d}h \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot \int_H F\,\mathrm{d}h \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot \int_H \frac{\|f(\cdot,h)\|_{L^p}^q}{|\det(h)|}\,\mathrm{d}h \\ &= |\det(T)|^{1-\frac{q}{p}} \cdot \|f\|_{L^{p,q}}^q < \infty.\end{aligned}$$

This proves parts 3 and 4 for the unweighted case for $q \in (0,\infty)$.

For $q = \infty$, we use equation (4.2.4), the translation invariance of $\|\cdot\|_{L^p(\mathbb{R}^d)}$ and the right invariance of $\|\cdot\|_{L^\infty(H)}$ to calculate

$$\begin{aligned}\left\|R_{(y,T)}f\right\|_{L^{p,\infty}} &= \left\|h \mapsto \left\|\left(R_{(y,T)}f\right)(\cdot,h)\right\|_{L^p}\right\|_{L^\infty(H)} \\ &= \|h \mapsto \|f(\cdot,hT)\|_{L^p}\|_{L^\infty(H)} \\ &= \|h \mapsto \|f(\cdot,h)\|_{L^p}\|_{L^\infty(H)} = \|f\|_{L^{p,\infty}},\end{aligned}$$

which proves part 3 for the unweighted case for $q = \infty$.

For part 4 in case of $q = \infty$, we calculate as in equation (4.2.5):

$$\begin{aligned}\left\|L_{(y,T)^{-1}}f\right\|_{L^{p,\infty}} &= \left\|h \mapsto \left\|\left(L_{(y,T)^{-1}}f\right)(\cdot,h)\right\|_{L^p}\right\|_{L^\infty(H)} \\ &= \|h \mapsto \|f(y + T\cdot, Th)\|_{L^p}\|_{L^\infty(H)} \\ &= |\det(T)|^{-\frac{1}{p}} \cdot \|h \mapsto \|f(\cdot,Th)\|_{L^p}\|_{L^\infty(H)} \\ &= |\det(T)|^{-\frac{1}{p}} \cdot \|g \mapsto \|f(\cdot,g)\|_{L^p}\|_{L^\infty(H)} \\ &= |\det(T)|^{-\frac{1}{p}} \cdot \|f\|_{L^{p,\infty}} < \infty.\end{aligned}$$

We finally prove part 5. As a topological space, $G = \mathbb{R}^d \times H$ is metrizable and hence first countable and $L^{p,q}_{m_0}(G)$ satisfies the Fatou property by part 2. Thus, Lemma 2.3.35 is applicable, so that $w_0, w_1$ are lower semicontinuous and thus also measurable. By Theorem 2.2.22, this implies that $w_0, w_1$ are locally bounded. Since the operator (quasi)-norm is sumbultiplicative, so are $w_0, w_1$.

It remains to establish estimate (4.2.3). Using $(x,h) = (x,\mathrm{id}) \cdot (0,h)$, we get

$$\begin{aligned}&|||R_{(x,h)}|||_{W_Q\left(L^{p,q}_{m_0}\right)\to W_Q\left(L^{p,q}_{m_0}\right)} \\ &\leq |||R_{(x,\mathrm{id})}|||_{W_Q\left(L^{p,q}_{m_0}\right)\to W_Q\left(L^{p,q}_{m_0}\right)} \cdot |||R_{(0,h)}|||_{W_Q\left(L^{p,q}_{m_0}\right)\to W_Q\left(L^{p,q}_{m_0}\right)} \\ &= |||R_{(x,\mathrm{id})}|||_{W_Q\left(L^{p,q}_{m_0}\right)\to W_Q\left(L^{p,q}_{m_0}\right)} \cdot w_0(h),\end{aligned}$$

so that it suffices to show

$$|||R_{(x,\mathrm{id})}|||_{W_Q\left(L^{p,q}_{m_0}\right)\to W_Q\left(L^{p,q}_{m_0}\right)} \leq C \cdot (1 + |x|)^{s+\frac{d}{r}}$$

for all $x \in \mathbb{R}^d$ and some absolute constant $C > 0$.

Since $Y = L^{p,q}_{m_0}(G)$ is right invariant by part 3, the Wiener amalgam space $W_Q(L^\infty, Y)$ is independent of the choice of the precompact, measurable unit neighborhood $Q \subset G$ by Lemma 2.3.16 and Theorem 2.3.17. Thus, we can make the choice

$$Q_0 := B_1(0) \times (H \cap B_{1/2}(\mathrm{id}))$$

instead of $Q$. Here, it is important to note that $\overline{B_{1/2}(\mathrm{id})} \subset \mathbb{R}^{d\times d}$ is closed and satisfies $\overline{B_{1/2}(\mathrm{id})} \subset \mathrm{GL}(\mathbb{R}^d)$ by [61, Theorem 10.7] (or by Lemma 2.4.11). Since $H \leq \mathrm{GL}(\mathbb{R}^d)$ is closed, we see that

$$\overline{Q_0} \subset \overline{B_1(0)} \times \left[H \cap \overline{B_{1/2}(\mathrm{id})}\right] \subset G$$

is closed, so that $Q_0 \subset G$ is an open, precompact unit neighborhood.

Now, let $x \in \mathbb{R}^d$ be arbitrary and set $R := 2(1+|x|)$ as well as $r := 1$. Lemma 4.2.3 yields a finite family $(x_i)_{i\in M}$ in $B_R(0)$ which satisfies $B_R(0) \subset \bigcup_{i=1}^M B_r(x_i)$ as well as

$$M \leq \left(1 + 2\cdot\frac{R}{r}\right)^d = (1+4(1+|x|))^d = (5+4|x|)^d \leq 5^d\cdot(1+|x|)^d.$$

Set $y := (x, \mathrm{id}) \in G$ and $y_i := (x_i, \mathrm{id}) \in G$ for $i \in \underline{M}$. We claim

$$Q_0 y \subset \bigcup_{i=1}^M y_i Q_0. \tag{4.2.6}$$

To see this, let $(z,T) \in Q_0$ be arbitrary. We have

$$\begin{aligned} |z+Tx| &\leq |z| + |Tx - \mathrm{id}x| + |x| \\ &< 1 + \|T-\mathrm{id}\|\,|x| + |x| \\ &\leq 1 + \frac{3}{2}|x| < 2(1+|x|) = R \end{aligned}$$

and hence $z + Tx \in B_R(0) \subset \bigcup_{i=1}^M B_r(x_i)$. Thus, there is $i \in \underline{M}$ with $z + Tx \in B_r(x_i)$, i.e. $z + Tx = x_i + v$ with $v \in B_r(0) = B_1(0)$. This implies $(v,T) \in Q_0$ and hence

$$\begin{aligned} (z,T)\,y &= (z,T)\,(x,\mathrm{id}) = (z+Tx, T) = (x_i+v, T) \\ &= (x_i,\mathrm{id})\,(v,T) = y_i\,(v,T) \in y_i Q_0. \end{aligned}$$

We now apply Lemma 2.3.18 to $\left(L^{p,q}_{m_0}, f \mapsto |||f\cdot m_0|||\right)$, where $|||\cdot|||$ is the equivalent $r$-norm on $L^{p,q}(G)$ constructed above. Since equivalent quasi-norms on $L^{p,q}_{m_0}(G)$ yield equivalent quasi-norms for the Wiener amalgam space $W_Q\left(L^\infty, L^{p,q}_{m_0}\right)$ and thus equivalent operator norms $\left|\left|\left|R_{(x,\mathrm{id})}\right|\right|\right|_{W\left(L^{p,q}_{m_0}\right)\to W\left(L^{p,q}_{m_0}\right)}$ with equivalence constants *independent* of the parameter $x \in \mathbb{R}^d$, this suffices. Part 3, together with the growth condition on $m_1$, yields for each $i \in \underline{M}$ the estimate

$$\begin{aligned} \left|\left|\left|R_{(x_i,\mathrm{id})}\right|\right|\right|_{L^{p,q}_{m_0}\to L^{p,q}_{m_0}} &\lesssim \left(\frac{|\det(\mathrm{id})|}{\Delta_H(\mathrm{id})}\right)^{1/q}\cdot m_1\left((x_i,\mathrm{id})^{-1}\right) \\ &= m_1(-x_i,\mathrm{id}) \leq (1+|-x_i|)^s\cdot m(\mathrm{id}) \\ &\leq m(\mathrm{id})\cdot(1+R)^s \leq m(\mathrm{id})\cdot(3+2|x|)^s \\ &\leq 3^s m(\mathrm{id})\cdot(1+|x|)^s. \end{aligned}$$

Finally, Lemma 2.3.18, together with the inclusion (4.2.6), implies

$$\begin{aligned}
\left|\!\left|\!\left| R_{(x,\mathrm{id})} \right|\!\right|\!\right|_{W_{Q_0}\left(L^{p,q}_{m_0}\right)\to W_{Q_0}\left(L^{p,q}_{m_0}\right)} &\leq \left( \sum_{i=1}^{M} \left|\!\left|\!\left| R_{(x_i,\mathrm{id})} \right|\!\right|\!\right|^r_{L^{p,q}_{m_0}\to L^{p,q}_{m_0}} \right)^{1/r} \\
&\lesssim \left( \sum_{i=1}^{M} \left[ 3^s m\left(\mathrm{id}\right) \left(1+|x|\right)^s \right]^r \right)^{1/r} \\
&= 3^s \cdot m\left(\mathrm{id}\right) \cdot \left(1+|x|\right)^s \cdot M^{1/r} \\
&\leq 5^{d/r} 3^s \cdot m\left(\mathrm{id}\right) \cdot \left(1+|x|\right)^{s+\frac{d}{r}} .
\end{aligned}$$

The implied constants only depend on $r > 0$ and on the constant of norm-equivalence between $\|\cdot\|_{L^{p,q}}$ and $|||\cdot|||$, but not on $x \in \mathbb{R}^d$. This completes the proof. □

We can now finally show that the space $L^{p,q}_m(G)$ admits a control weight $v : G \to (0,\infty)$ of moderate growth if the weight $m$ is itself of moderate growth. For the proof, we first observe the following closure properties of the class of (submultiplicative) weights of moderate growth.

**Lemma 4.2.7.** *Let $v, w : G \to (0,\infty)$ be submultiplicative. Then the following weights are also submultiplicative:*

1. *$\alpha v$ for each $\alpha \geq 1$,*
2. *$v + w$,*
3. *$\max\{v, w\}$,*
4. *$v^\vee$,*
5. *$v \cdot w$.*

*The same closure properties hold if "submultiplicative" is replaced by "of moderate growth".*

*Proof.* If $v, w$ are of moderate growth, let $v_0$ be a dilation weight for $v$ and $w_0$ be a dilation weight for $w$ with $v(x,h) \leq (1+|x|)^s \cdot v_0(h)$ and $w(x,h) \leq (1+|x|)^r \cdot w_0(h)$ for all $(x,h) \in G$ and suitable $r, s \geq 0$.

1. We have $\alpha v(x,h) \leq (1+|x|)^s \cdot (\alpha v_0)(h)$, so that $\alpha v$ is of moderate growth if $v$ is. Furthermore, if $v$ is submultiplicative, then

$$\alpha v(gh) \leq \alpha \cdot v(g) \cdot v(h) \leq (\alpha v)(g) \cdot (\alpha v)(h)$$

is true for all $g, h \in G$, since $\alpha \geq 1$.

2. Let $t := \max\{s, r\}$. Then

$$\begin{aligned}
(v+w)(x,h) &\leq (1+|x|)^s v_0(h) + (1+|x|)^r w_0(h) \\
&\leq (1+|x|)^t \cdot (v_0 + w_0)(h) ,
\end{aligned}$$

so that $v + w$ is of moderate growth if $v, w$ are. If $v, w$ are submultiplicative, then

$$\begin{aligned}(v+w)(gh) &\leq v(g)\, v(h) + w(g)\, w(h) \\ &\leq (v(g) + w(g)) \cdot (v(h) + w(h))\end{aligned}$$

holds for all $g, h \in G$, so that $v + w$ is submultiplicative.

3. As above, let $t := \max\{s, r\}$. This yields

$$\begin{aligned}\max\{v(x), w(x)\} &\leq \max\left\{(1+|x|)^s v_0(h), (1+|x|)^r w_0(h)\right\} \\ &\leq (1+|x|)^r \cdot \max\{v_0(h), w_0(h)\},\end{aligned}$$

so that $\max\{v, w\}$ is of moderate growth if $v, w$ are. If $v, w$ are submultiplicative and $\gamma \in \{v, w\}$, then

$$\gamma(gh) \leq \gamma(g)\,\gamma(h) \leq \max\{v(g), w(g)\} \cdot \max\{v(h), w(h)\}$$

holds for all $g, h \in G$, so that $\max\{v, w\}$ is submultiplicative.

4. We have

$$\begin{aligned}v^\vee(x,h) = v(-h^{-1}x,\, h^{-1}) &\leq (1+|h^{-1}x|)^s \cdot v_0(h^{-1}) \\ &\leq (1+\|h^{-1}\|\,|x|)^s \cdot v_0^\vee(h) \\ &\leq (1+|x|)^s \cdot (1+\|h^{-1}\|)^s \cdot v_0^\vee(h)\end{aligned}$$

for all $(x, h) \in G$, so that $v^\vee$ is of moderate growth (if $v$ is), with dilation weight $h \mapsto (1+\|h^{-1}\|)^s \cdot v_0^\vee(h)$. If $v$ is submultiplicative, then

$$v^\vee(gh) = v(h^{-1}g^{-1}) \leq v(h^{-1}) \cdot v(g^{-1}) = v^\vee(g) \cdot v^\vee(h)$$

holds for all $g, h \in G$, so that $v^\vee$ is submultiplicative.

5. We have

$$vw(x,h) \leq (1+|x|)^{s+r} \cdot (v_0 w_0)(h),$$

which proves moderate growth of $vw$, if $v, w$ are of moderate growth. If $v, w$ are submultiplicative, then

$$vw(gh) \leq v(g)\, v(h)\, w(g)\, w(h) = vw(g) \cdot vw(h),$$

so that $vw$ is submultiplicative. □

**Corollary 4.2.8.** *Let $p, q \in (0, \infty]$ and let $m, m_1 : G \to (0, \infty)$ be two (measurable) weights such that $m$ is right moderate with respect to $m_1$ and such that $m_1$ is of moderate growth.*

*Then there is a submultiplicative weight $v : G \to (0, \infty)$ of moderate growth which is a control weight for $Y = L^{p,q}_m(G)$ in the sense of Assumption 2.4.1.*

*Proof.* Let $Y := L_m^{p,q}(G)$. Lemma 4.2.6 shows that the weight

$$w_1 : G \to (0,\infty), (x,h) \mapsto \left|\left|\left|R_{(x,h)}\right|\right|\right|_{W(L^\infty,Y)\to W(L^\infty,Y)}$$

is measurable, submultiplicative and of moderate growth.

The modular function $\Delta_G$ is given by $\Delta_G(x,h) = \Delta_H(h) \cdot |\det(h)|^{-1}$ which is continuous and thus locally bounded. Furthermore, $\Delta_G$ is of moderate growth and (sub)multiplicative. Using Lemma 4.2.7, we conclude that $w := 1 + w_1 + \Delta_G^\vee \cdot w_1^\vee$ is also measurable, submultiplicative and of moderate growth and satisfies all properties required by part D of Assumption 2.4.1.

Finally, part E of Assumption 2.4.1 shows that

$$v := \max\left\{w, \left[\Delta_G^\vee\right]^{1/p} \cdot w^\vee\right\}$$

is a valid control weight for $Y = L_m^{p,q}(G)$. But the same arguments as in the last paragraph show that $v$ is measurable, submultiplicative and of moderate growth. For this, it is important to note that the modular function $\Delta_G$ is given by $\Delta_G(x,h) = \Delta_H(h) \cdot |\det(h)|^{-1}$, cf. equation (4.1.1), and is hence of moderate growth. □

Our main result in this section will be well-definedness of the map

$$\varrho : C_c^\infty(\mathcal{O}) \to W^R\left(L^\infty, W\left(L^\infty, L_v^p\right)\right), g \mapsto W_\psi\left[\mathcal{F}^{-1}g\right]$$

for every $p \in (0,\infty)$ and every $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with compact Fourier support $\operatorname{supp}\widehat{\psi} \subset \mathcal{O}$ for any weight $v : G \to (0,\infty)$ of moderate growth, cf. Theorem 4.2.11. We remark that this is an even stronger statement than [41, Theorem 9], where well-definedness of

$$C_c^\infty(\mathcal{O}) \to W^R\left(L^\infty, L_v^p\right), g \mapsto W_\psi\left[\mathcal{F}^{-1}g\right]$$

was shown. This stronger statement is needed here, since coorbit theory for *Quasi*-Banach places stronger restrictions on the analyzing vectors than coorbit theory for Banach spaces, cf. Chapter 2.

Before we can prove well-definedness of $\varrho$, we need to establish continuity of the maps $H \to \mathcal{S}\left(\mathbb{R}^d\right), h \mapsto D_h\psi$ and

$$H \to \mathcal{S}\left(\mathbb{R}^d\right), h \mapsto \left[W_\psi\left(\mathcal{F}^{-1}f\right)\right](\cdot, h).$$

This will be a consequence of the following more general lemma.

**Lemma 4.2.9.** *[41, Lemma 7] Let $\varnothing \neq U \subset \mathbb{R}^\ell$ be an open set and let $\gamma : U \times \mathbb{R}^d \to \mathbb{R}^d$ be smooth with the additional property that for all compact sets $L \subset U$ and $K \subset \mathbb{R}^d$ the set*

$$\bigcup_{p\in L} \left(\gamma(p,\cdot)\right)^{-1}(K) \subset \mathbb{R}^d$$

*is bounded. Furthermore, let $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ be arbitrary.*

*Then the map*

$$\Phi : U \to C_c^\infty\left(\mathbb{R}^d\right), p \mapsto \varphi\left(\gamma\left(p, \cdot\right)\right)$$

*is well-defined and continuous. In particular,* $\Phi : U \to \mathcal{S}\left(\mathbb{R}^d\right)$ *is continuous.*

*Remark.* The stated requirements for $\gamma$ are (under the identification $\mathbb{R}^{d\times d} \cong \mathbb{R}^{d^2}$) fulfilled for the choice

$$\gamma : \mathrm{GL}\left(\mathbb{R}^d\right) \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^T \xi.$$

*Proof of the remark.* Let $L \subset \mathrm{GL}\left(\mathbb{R}^d\right)$ and $K \subset \mathbb{R}^d$ be compact. Then $L^{-T} \subset \mathrm{GL}\left(\mathbb{R}^d\right)$ is also compact, which implies $\left\|h^{-T}\right\| \leq C_1$ for some $C_1 > 0$ and all $h \in L$. Furthermore, there is some $C_2 > 0$ such that $|\xi| \leq C_2$ holds for all $\xi \in K$.

For $y \in \bigcup_{h\in L} \left(\gamma\left(h, \cdot\right)\right)^{-1}(K)$ we then have $\xi := h^T y = \gamma\left(h, y\right) \in K$ for some $h \in L$. This implies

$$|y| = |h^{-T}\xi| \leq \|h^{-T}\| \cdot |\xi| \leq C_1 C_2. \qquad \square$$

*Proof of Lemma 4.2.9.* Since $\gamma$ (and hence also $\gamma\left(p, \cdot\right)$ for every $p \in U$) is smooth, we see that $\varphi\left(\gamma\left(p, \cdot\right)\right) \in C^\infty\left(\mathbb{R}^d\right)$ is also smooth. Now let $K := \operatorname{supp}\varphi$. Since $\gamma\left(p, \cdot\right)$ is continuous, $\left(\gamma\left(p, \cdot\right)\right)^{-1}(K) \subset \mathbb{R}^d$ is closed. It is thus easy to see that

$$\operatorname{supp}\left(\Phi\left(p\right)\right) = \operatorname{supp}\left(\varphi\left(\gamma\left(p, \cdot\right)\right)\right) \subset \left(\gamma\left(p, \cdot\right)\right)^{-1}(K) \tag{4.2.7}$$

holds for every $p \in U$. Now the assumption (with $L = \{p\}$) yields that the right-hand side is a bounded subset of $\mathbb{R}^d$.

Thus, $\varphi\left(\gamma\left(p, \cdot\right)\right) \in C_c^\infty\left(\mathbb{R}^d\right)$ is compactly supported, so that $\Phi$ is well-defined.

To prove the continuity of $\Phi$, let $\left(p_n\right)_{n\in\mathbb{N}} \in U^{\mathbb{N}}$ with $p_n \xrightarrow[n\to\infty]{} p_0$ for some $p_0 \in U$. Then $L := \{p_n \mid n \in \mathbb{N}\} \cup \{p_0\}$ is a compact subset of $U$. The assumption (and Heine-Borel) thus yield the compactness of

$$M := \overline{\bigcup_{p\in L} \left(\gamma\left(p, \cdot\right)\right)^{-1}(K)} \subset \mathbb{R}^d.$$

By equation (4.2.7) we see

$$\operatorname{supp}\Phi\left(p_0\right) \subset M \text{ and } \operatorname{supp}\Phi\left(p_n\right) \subset M \text{ for all } n \in \mathbb{N}. \tag{4.2.8}$$

Now for every multi-index $\beta \in \mathbb{N}_0^d$, the map

$$\Psi_\beta : U \times \mathbb{R}^d \to \mathbb{C},$$
$$(p, x) \mapsto \left(\partial^\beta\left(\Phi\left(p\right)\right)\right)(x) = \left.\frac{\partial^{|\beta|}\left(\left(\varphi \circ \gamma\right)\left(q, y\right)\right)}{\partial y_1^{\beta_1} \cdots \partial y_d^{\beta_d}}\right|_{(q,y)=(p,x)}$$

is smooth, hence continuous. In particular, $\Psi_\beta$ is uniformly continuous on the compact set $L \times M \subset U \times \mathbb{R}^d$. This yields, for arbitrary $\varepsilon > 0$, some $\delta > 0$ such that $\left|\Psi_\beta(p,x) - \Psi_\beta(q,y)\right| < \varepsilon$ holds for all $(p,x),(q,y) \in L \times M$ with $|(p,x) - (q,y)| < \delta$.

Let $n_0 \in \mathbb{N}$ with $|p_n - p_0| < \delta$ for $n \geq n_0$. For $n \geq n_0$ and $x \in \mathbb{R}^d$ there are two cases:

1. $x \notin M$. By equation (4.2.8), this means $x \notin \operatorname{supp} \Phi(p_n)$ and $x \notin \operatorname{supp} \Phi(p_0)$. Hence, we conclude $\Phi(p_0)|_U \equiv 0 \equiv \Phi(p_n)|_U$ for some neighborhood $U$ of $x$. In particular

$$\left|\left(\partial^\beta (\Phi(p_n) - \Phi(p_0))\right)(x)\right| = 0 < \varepsilon.$$

2. $x \in M$. Then $(p_n,x),(p_0,x) \in L \times M$ with

$$|(p_n,x) - (p_0,x)| = |p_n - p_0| < \delta.$$

By choice of $\delta$ this implies

$$\left|\left(\partial^\beta (\Phi(p_n) - \Phi(p_0))\right)(x)\right| = \left|\Psi_\beta(p_n,x) - \Psi_\beta(p,x)\right| < \varepsilon.$$

Now [61, Theorem 6.5] (and the associated remark) yield the convergence $\Phi(p_n) \xrightarrow[n\to\infty]{C_c^\infty(\mathbb{R}^d)} \Phi(p_0)$ (recall that the supports of $\Phi(p_n)$ are "uniformly compact" by equation (4.2.8)). Since the inclusion $C_c^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous by [61, Theorem 7.10], we are done. □

With these preparations, we can prove that the map

$$H \to \mathcal{S}(\mathbb{R}^d), h \mapsto \left(W_\psi f\right)(\cdot,h)$$

is well-defined and continuous with compact support, if $f,\psi$ have compact Fourier support in $\mathcal{O}$. For this, we first recall the identity

$$\left(W_\psi f\right)(x,h) = |\det(h)|^{1/2} \cdot \left(\mathcal{F}^{-1}\left[\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}}\right]\right)(x) \tag{4.2.9}$$

from Lemma 4.1.5, which is valid for $f,\psi \in L^2(\mathbb{R}^d)$ and all $(x,h) \in G$.

**Lemma 4.2.10.** *[41, Lemma 8] Let $f,\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{f},\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Then*

$$\Gamma : H \to \mathcal{S}(\mathbb{R}^d), h \mapsto \mathcal{F}^{-1}\left[\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}}\right]$$

*is well-defined and continuous with compact support*

$$\operatorname{supp} \Gamma \subset p_{\xi_0}^{-1}(\operatorname{supp} \widehat{f}) \cdot H_{\xi_0} \cdot \left(p_{\xi_0}^{-1}(\operatorname{supp} \widehat{\psi})\right)^{-1}, \tag{4.2.10}$$

*where we used $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^{-T}\xi_0$ for some fixed $\xi_0 \in \mathcal{O}$.*

*Proof.* We note that the multiplication map

$$\mu_{\widehat{f}} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right), g \mapsto \widehat{f} \cdot g$$

is (well-defined and) continuous by [61, Theorem 7.4(b)]. By [61, Theorem 7.7(b)], the same holds for the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$. Finally, Lemma 4.2.9 and the corresponding remark show that the map

$$\Phi : H \to \mathcal{S}\left(\mathbb{R}^d\right), h \mapsto \Delta_h \overline{\widehat{\psi}}$$

is well-defined and continuous. Here we used $\widehat{\psi} \in C_c^\infty\left(\mathbb{R}^d\right)$. In summary, this shows that $\Gamma = \mathcal{F}^{-1} \circ \mu_{\widehat{f}} \circ \Phi$ is well-defined and continuous.

Now let $h \in H$ with $0 \neq \Gamma(h) = \mathcal{F}^{-1}\left[\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}}\right]$. This implies $\widehat{f} \cdot \Delta_h \overline{\widehat{\psi}} \neq 0$ and thus there is some

$$\xi \in \operatorname{supp} \widehat{f} \cap \operatorname{supp}\left(\Delta_h \overline{\widehat{\psi}}\right) \subset \operatorname{supp} \widehat{f} \cap h^{-T}\left(\operatorname{supp} \widehat{\psi}\right).$$

The inclusions $\operatorname{supp} \widehat{f} \subset \mathcal{O} = H^T \xi_0$ and $\operatorname{supp} \widehat{\psi} \subset \mathcal{O} = H^T \xi_0$ yield $g_1, g_2 \in H$ satisfying $g_1^{-T} \xi_0 \in \operatorname{supp} \widehat{f}$ and $g_2^{-T} \xi_0 \in \operatorname{supp} \widehat{\psi}$ as well as $g_1^{-T} \xi_0 = \xi = h^{-T} g_2^{-T} \xi_0$.

This implies $\left(g_1^{-1} h g_2\right)^T \xi_0 = \xi_0$, which yields $g_1^{-1} h g_2 \in H_{\xi_0}$ and thus $h \in g_1 H_{\xi_0} g_2^{-1}$. Noting $g_1 \in p_{\xi_0}^{-1}\left(\operatorname{supp} \widehat{f}\right)$ and $g_2 \in p_{\xi_0}^{-1}\left(\operatorname{supp} \widehat{\psi}\right)$ completes the proof of the inclusion (4.2.10). Now Corollary 4.1.3 shows that $\Gamma$ indeed has compact support. □

Using the continuity proved in the above lemma, we are finally in a position to prove $W_\psi\left[\mathcal{F}^{-1} f\right] \in L_w^p(G)$ for any weight $w$ of moderate growth. In particular, the theorem below is applicable to a suitable control weight $v$ for the space $Y = L_m^{p,q}$, see Corollary 4.2.12.

**Theorem 4.2.11.** *Let $w : G \to (0, \infty)$ be a weight of moderate growth.*

*Fix some $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$ and let $\varnothing \neq Q_1, Q_2 \subset G$ be precompact and measurable and let $p \in (0, \infty)$. Then the map*

$$\varrho : C_c^\infty(\mathcal{O}) \to W_{Q_1}^R\left(L^\infty, W_{Q_2}\left(L^\infty, L_w^p\right)\right), g \mapsto W_\psi\left[\mathcal{F}^{-1} g\right]$$

*is well-defined.*

*If we have $p \geq 1$, then $\varrho$ is also continuous.*

*Remark.* We mention the sets $Q_1, Q_2$ explicitly, since our assumptions on $w$ do not necessarily imply that the space $L_w^p(G)$ is translation invariant. Thus, the definition of $W_Q\left(L^\infty, L_w^p\right)$ potentially depends on the choice of $Q$, as does the definition of the "outer" right sided Wiener amalgam space.

*Proof.* Since $w$ is of moderate growth, there is a locally bounded (measurable) weight $w_0 : H \to (0,\infty)$ and some $N \in \mathbb{N}_0$ with

$$w(x,h) \leq (1+|x|)^N \cdot w_0(h)$$

for all $(x,h) \in G$.

Let $K \subset \mathcal{O}$ be an arbitrary compact set and set

$$C^\infty_K(\mathcal{O}) := \{g \in C^\infty_c(\mathcal{O}) \mid \operatorname{supp} g \subset K\}.$$

Fix $\xi_0 \in \mathcal{O}$ and define $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^{-T}\xi_0$ as well as

$$L := p^{-1}_{\xi_0}(K) \cdot H_{\xi_0} \cdot \left(p^{-1}_{\xi_0}(\operatorname{supp} \widehat{\psi})\right)^{-1}.$$

Note that $L \subset H$ is compact by Corollary 4.1.3. We will show below that we have

$$f_h := W_\psi\left[\mathcal{F}^{-1}g\right](\cdot,h) \in \mathcal{S}(\mathbb{R}^d)$$

for all $h \in H$ with $f_h \equiv 0$ for $h \in H \setminus L$, for any $g \in C^\infty_K(\mathcal{O})$.

Before we do that, let us start by investigating how this implies the claim. To this end, we estimate the iterated maximal function of $f : G \to \mathbb{C}$. We have

$$\begin{aligned}
&M_{Q_2}\left[M^R_{Q_1} f\right](x,h)\\
&\leq \sup_{(z,S)\in Q_2} \left(M^R_{Q_1} f\right)((x,h)(z,S))\\
&\leq \sup_{(z,S)\in Q_2} \sup_{(y,T)\in Q_1} |f((y,T)(x,h)(z,S))|\\
&= \sup_{(z,S)\in Q_2} \sup_{(y,T)\in Q_1} |f(y+Tx+Thz, ThS)|. \qquad (4.2.11)
\end{aligned}$$

For $f(x,h) = f_h(x)$ with $f_h \in \mathcal{S}(\mathbb{R}^d)$ and $f_h \equiv 0$ for $h \in H \setminus L$ as above, we get

$$\begin{aligned}
|f_h(x)| &= (1+|x|)^{-N_0}(1+|x|)^{N_0} \cdot |f_h(x)|\\
&\leq (1+|x|)^{-N_0} \cdot |f_h|_{N_0} \qquad (4.2.12)
\end{aligned}$$

for each $N_0 \in \mathbb{N}_0$, where the norms $|\cdot|_\kappa$ are defined for $\kappa \in \mathbb{N}_0$ by

$$|f|_\kappa := \max_{\substack{\alpha\in\mathbb{N}^d_0\\ |\alpha|\leq\kappa}} \sup_{x\in\mathbb{R}^d} (1+|x|)^\kappa \cdot |(\partial^\alpha f)(x)|.$$

In view of estimate (4.2.11) above, we thus want to control the size of $|y+Tx+Thz|$ *from below*.

To this end, let $Q := \overline{Q_1} \cup \overline{Q_2} \subset G$. Since the projection

$$\pi_2 : G = \mathbb{R}^d \times H \to H, (x,h) \mapsto h$$

is continuous, the set

$$Q_3 := [\pi_2(Q)]^{-1} \cdot L \cdot [\pi_2(Q)]^{-1} \subset H$$

is compact and the constant

$$R := \max\left\{1, \max_{(x,h)\in Q} |x| + \|h\| + \|h^{-1}\|, \max_{(x,h)\in Q_3} \|h\|\right\}$$

is finite.

Using this constant, we have for arbitrary $(y,T) \in Q_1$, $(z,S) \in Q_2$ and $(x,h) \in G$ the estimate $|y| \leq R$, as well as

$$|x| = |T^{-1}Tx| \leq \|T^{-1}\| \cdot |Tx| \leq R \cdot |Tx|.$$

Furthermore, if $ThS \in L$, then

$$h = T^{-1} \cdot ThS \cdot S^{-1} \in [\pi_2(Q)]^{-1} \cdot L \cdot [\pi_2(Q)]^{-1} = Q_3.$$

This implies

$$\chi_L(ThS) \leq \chi_{Q_3}(h), \tag{4.2.13}$$

as well as $\|h\| \leq R$ and $|Thz| \leq \|T\| \|h\| |z| \leq R^3$. In particular,

$$\begin{aligned} |y + Tx + Thz| &\geq |Tx| - |y + Thz| \\ &\geq \frac{|x|}{R} - [|y| + |Thz|] \\ &\geq \frac{|x|}{R} - \left(R + R^3\right) \geq \frac{|x|}{R} - 2R^3. \end{aligned}$$

Rearranging, together with $R \geq 1$, finally yields

$$\begin{aligned} 1 + |x| &\leq 1 + R \cdot \left[2R^3 + |y + Tx + Thz|\right] \\ &\leq R \cdot \left(1 + 2R^3 + |y + Tx + Thz|\right) \\ &\leq R\left(1 + 2R^3\right) \cdot (1 + |y + Tx + Thz|) \end{aligned}$$

and thus

$$\begin{aligned} (1 + |y + Tx + Thz|)^{-N_0} &\leq \left[R \cdot \left(1 + 2R^3\right)\right]^{N_0} \cdot (1 + |x|)^{-N_0} \\ &=: C_1 \cdot (1 + |x|)^{-N_0}. \end{aligned}$$

Since $(y,T) \in Q_1$, $(z,S) \in Q_2$ and $(x,h) \in G$ with $ThS \in L$ were arbitrary, we can use equations (4.2.11), (4.2.12) and (4.2.13) to get

$$\begin{aligned} M_{Q_2}\left[M^R_{Q_1} f\right](x,h) &\leq \sup_{(z,S)\in Q_2} \sup_{(y,T)\in Q_1} |f(y + Tx + Thz, ThS)| \\ &\leq \sup_{(z,S)\in Q_2} \sup_{(y,T)\in Q_1} (1 + |y + Tx + Thz|)^{-N_0} \cdot |f_{ThS}|_{N_0} \\ &\leq C_1 \cdot (1 + |x|)^{-N_0} \cdot \chi_{Q_3}(h) \cdot \sup_{t\in H} |f_t|_{N_0} \end{aligned} \tag{4.2.14}$$

for each $N_0 \in \mathbb{N}_0$ and each family $(f_h)_{h\in H}$ with $f_h \in \mathcal{S}\left(\mathbb{R}^d\right)$ for each $h \in H$ and with $f_h \equiv 0$ for $h \in H \setminus L$. In the last step of the above estimate, we used equation (4.2.13) to introduce the indicator function $\chi_{Q_3}(h)$.

Let us now show that $W_\psi\left[\mathcal{F}^{-1}g\right]$, for $g \in C^\infty_K(\mathcal{O})$, can indeed be written in the above form. Fix $N_0 \in \mathbb{N}$ with $N_0 > N + \frac{d}{p}$. Since the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$ is continuous, there are $C_2 > 0$ and $N_1 \in \mathbb{N}$ such that

$$\left|\mathcal{F}^{-1}g\right|_{N_0} \leq C_2 \cdot |g|_{N_1} \tag{4.2.15}$$

holds for all $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. With $L$ as defined at the start of the proof, Lemma 4.2.10 shows that

$$\Gamma_g : H \to \mathcal{S}\left(\mathbb{R}^d\right), h \mapsto \mathcal{F}^{-1}\left[g \cdot \Delta_h\overline{\widehat{\psi}}\right] = \mathcal{F}^{-1}\left[\widehat{\mathcal{F}^{-1}g} \cdot \Delta_h\overline{\widehat{\psi}}\right]$$

is well-defined and continuous with $\operatorname{supp}\Gamma_g \subset L$ for each $g \in C^\infty_K(\mathcal{O})$.

Furthermore, Lemma 4.2.9, together with the ensuing remark, yields continuity of

$$\Phi : H \to \mathcal{S}\left(\mathbb{R}^d\right), h \mapsto \Delta_h\overline{\widehat{\psi}} = \overline{\Delta_h\widehat{\psi}}.$$

In turn, this yields continuity of the real valued function

$$H \to \mathbb{R}_+, h \mapsto \max_{|\alpha|\leq N_1} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \cdot \left|\Delta_h\overline{\widehat{\psi}}\right|_{N_1} = \max_{|\alpha|\leq N_1} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \cdot |\Phi(h)|_{N_1}.$$

On the compact set $L \subset H$, this function attains its maximum $C_3 \geq 0$. Using the Leibniz rule, we get – for arbitrary $h \in L$, $g \in \mathcal{S}\left(\mathbb{R}^d\right)$, $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq N_1$ – the estimate

$$\begin{aligned}
&(1+|x|)^{N_1} \cdot \left|\partial^\alpha\left[g \cdot \Delta_h\overline{\widehat{\psi}}\right](x)\right| \\
&\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \cdot \underbrace{\left|\partial^\beta g(x)\right|}_{\leq |g|_{N_1}} \cdot \underbrace{(1+|x|)^{N_1} \cdot \left|\left[\partial^{\alpha-\beta}\Phi(h)\right](x)\right|}_{\leq |\Phi(h)|_{N_1}} \\
&\leq C_3 \cdot |g|_{N_1}.
\end{aligned}$$

Together with equation (4.2.15), we derive for arbitrary $h \in L$ and $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ the estimate

$$\left|\Gamma_g(h)\right|_{N_0} = \left|\mathcal{F}^{-1}\left[g \cdot \Delta_h\overline{\widehat{\psi}}\right]\right|_{N_0} \leq C_2 \cdot \left|g \cdot \Delta_h\overline{\widehat{\psi}}\right|_{N_1} \leq C_2C_3 \cdot |g|_{N_1}.$$

Now assume in addition that $\operatorname{supp} g \subset K$ and recall from above that $\Gamma_g(h) = 0$ holds for all $h \in H \setminus L$. All in all, this proves

$$\left|\Gamma_g(h)\right|_{N_0} \leq C_2C_3 \cdot |g|_{N_1} \chi_L(h) \quad \forall h \in H \text{ and } g \in C^\infty_K(\mathcal{O}). \tag{4.2.16}$$

For $g \in C_K^\infty(\mathcal{O})$, Lemma 4.1.5, equation (4.1.3) implies

$$\begin{aligned} W_\psi\left[\mathcal{F}^{-1}g\right](x,h) &= |\det(h)|^{1/2} \cdot \mathcal{F}^{-1}\left[\widehat{\mathcal{F}^{-1}g} \cdot \Delta_h \widehat{\overline{\psi}}\right](x) \\ &= |\det(h)|^{1/2} \cdot \left[\Gamma_g(h)\right](x) \end{aligned}$$

for all $(x,h) \in G$. Setting $f := W_\psi\left[\mathcal{F}^{-1}g\right]$, i.e. $f_h := W_\psi\left[\mathcal{F}^{-1}g\right](\cdot,h)$, this identity, together with equation (4.2.16), implies $f_h \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $f_h \equiv 0$ for $h \in H \setminus L$ and

$$\begin{aligned} |f_h|_{N_0} &\leq C_2C_3 \cdot |g|_{N_1} \cdot |\det(h)|^{1/2} \cdot \chi_L(h) \\ &\leq C_2C_3C_4 \cdot |g|_{N_1} \end{aligned}$$

with $C_4 := \max_{h\in L} |\det(h)|^{1/2}$. Employing equation (4.2.14), we conclude

$$\begin{aligned} M_{Q_2}\left[M^R_{Q_1}f\right](x,h) &\leq C_1 \cdot (1+|x|)^{-N_0} \cdot \chi_{Q_3}(h) \cdot \sup_{t\in H} |f_t| \\ &\leq C_1C_2C_3C_4 \cdot |g|_{N_1} \cdot (1+|x|)^{-N_0} \chi_{Q_3}(h). \end{aligned}$$

Setting $C_5 := C_1C_2C_3C_4$, we finally arrive at

$$\begin{aligned} &\left\|W_\psi\left[\mathcal{F}^{-1}g\right]\right\|^p_{W^R_{Q_1}\left(W_{Q_2}\left(L^p_w\right)\right)} \\ &= \left\|M_{Q_2}\left[M^R_{Q_1}f\right]\right\|^p_{L^p_w} \\ &\leq C_5^p\,|g|^p_{N_1} \cdot \int_H \chi_{Q_3}(h) \int_{\mathbb{R}^d} \left[(1+|x|)^{-N_0} \cdot w(x,h)\right]^p \,\mathrm{d}x \frac{\mathrm{d}h}{|\det(h)|} \\ &\leq C_5^p\,|g|^p_{N_1} \cdot \int_{Q_3} \frac{[w_0(h)]^p}{|\det(h)|}\,\mathrm{d}h \cdot \int_{\mathbb{R}^d} (1+|x|)^{p(N-N_0)}\,\mathrm{d}x \\ &=: C_6 \cdot |g|^p_{N_1}. \end{aligned}$$

The constant $C_6$ thus defined is finite, since $Q_3$ is compact and the function $h \mapsto \frac{[w_0(h)]^p}{|\det(h)|}$ is bounded on compact sets. Furthermore, our choice of $N_0 > N + \frac{d}{p}$ implies $p(N-N_0) < -p \cdot \frac{d}{p} = -d$, so that the integral

$$\int_{\mathbb{R}^d} (1+|x|)^{p(N-N_0)}\,\mathrm{d}x$$

is also finite.

All in all, we have shown that the map $\varrho$ defined in the statement of the theorem is indeed well-defined. Furthermore, since the constant $C_6$ does not depend on $g \in C_K^\infty(\mathcal{O})$ and since $|\cdot|_{N_1}$ is a continuous semi-norm on $C_K^\infty(\mathcal{O})$, we have shown that

$$\varrho|_{C_K^\infty(\mathcal{O})} : C_K^\infty(\mathcal{O}) \to W^R_{Q_1}\left(W_{Q_2}\left(L^p_w\right)\right)$$

is continuous with respect to the usual topology on $C_K^\infty(\mathcal{O})$, cf. [61, Section 6.2]. But $L^p_w$ is a normed vector space for $p \geq 1$, so that

the same is true of $W_{Q_1}^R\left(W_{Q_2}\left(L_w^p\right)\right)$ by Lemma 2.3.4. In particular, $W_{Q_1}^R\left(W_{Q_2}\left(L_w^p\right)\right)$ is a *locally convex* topological vector space.

Since $K \subset \mathcal{O}$ was an arbitrary compact set, [61, Theorem 6.6] yields continuity of $\varrho$ for $p \geq 1$. □

**Corollary 4.2.12.** *Let $v : G \to (0,\infty)$ be a (measurable) submultiplicative weight of moderate growth which is bounded from below. Then the following properties hold:*

1. *We have $\mathcal{F}^{-1}\left[C_c^\infty(\mathcal{O})\right] \subset \mathbb{A}_v^r \subset \mathcal{G}_v$ for every $r \in (0,1]$.*
2. *The map*
$$\mathcal{F}^{-1} : C_c^\infty(\mathcal{O}) \to \mathcal{H}_v^1, g \mapsto \mathcal{F}^{-1}g$$
*is a well-defined continuous linear map.*

*Now, let $p,q \in (0,\infty]$ and let $m : G \to (0,\infty)$ be a (measurable) weight which is right-moderate with respect to a weight $m_1 : G \to (0,\infty)$ of moderate growth. Then there is a control weight $v : G \to (0,\infty)$ of moderate growth for the right-invariant solid QBF space $Y = L_m^{p,q}(G)$. By the above, $\mathcal{G}_v \supset \mathbb{A}_v^r \supsetneq \{0\}$, so that all assumptions for the applicability of coorbit theory for the space $Y = L_m^{p,q}(G)$ are fulfilled, cf. Assumption 2.4.1.*

*Proof.* The inclusion $\mathbb{A}_v^r \subset \mathcal{G}_v$ was already established in Chapter 2, see the remark after Assumption 2.4.1.

Further, if $g \in \mathcal{F}^{-1}\left[C_c^\infty(\mathcal{O})\right] \subset L^2\left(\mathbb{R}^d\right)$, then $\widehat{g} \in C_c^\infty(\mathcal{O})$, so that Theorem 4.2.11 shows

$$W_g g = W_g\left[\mathcal{F}^{-1}\widehat{g}\right] \in W^R\left(L^\infty, W\left(L^\infty, L_v^r\right)\right)$$

for all $r \in (0,1]$, since $v$ is of moderate growth. By definition, this means $g \in \mathbb{A}_v^r$.

Fix some $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\psi \neq 0$ and $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. By the preceding statement, we know $\psi \in \mathcal{G}_v$ and hence $\mathcal{H}_v^1 = \mathcal{H}_v^1(G,\psi)$, see Lemma 2.4.7. Theorem 4.2.11 shows that

$$\varrho : C_c^\infty(\mathcal{O}) \to W^R\left(L^\infty, W\left(L^\infty, L_v^1\right)\right), g \mapsto W_\psi\left[\mathcal{F}^{-1}g\right]$$

is a continuous linear map. Because of $W^R\left(L^\infty, W\left(L^\infty, L_v^1\right)\right) \hookrightarrow L_v^1(G)$, this implies $W_\psi\left[\mathcal{F}^{-1}g\right] \in L_v^1(G)$ and hence $\mathcal{F}^{-1}g \in \mathcal{H}_v^1(G,\psi)$ for all $g \in C_c^\infty(\mathcal{O})$.

The continuity of $\varrho$ finally implies

$$\begin{aligned}
\left\|\mathcal{F}^{-1}g - \mathcal{F}^{-1}g_0\right\|_{\mathcal{H}_v^1(G,\psi)} &= \left\|W_\psi\left[\mathcal{F}^{-1}(g-g_0)\right]\right\|_{L_v^1} \\
&\lesssim \left\|W_\psi\left[\mathcal{F}^{-1}(g-g_0)\right]\right\|_{W^R(W(L_v^1))} \\
&= \left\|\varrho(g-g_0)\right\|_{W^R(W(L_v^1))} \xrightarrow[g\to g_0]{} 0
\end{aligned}$$

for arbitrary $g_0 \in C_c^\infty(\mathcal{O})$, so that $\mathcal{F}^{-1} : C_c^\infty(\mathcal{O}) \to \mathcal{H}_v^1$ is indeed well-defined and continuous. □

We now add a further condition to our list of standing assumptions for this chapter.

**Assumption 4.2.13.** *For the rest of this chapter, fix $p, q \in (0, \infty]$ and assume that $m : G \to (0, \infty)$ is a fixed (measurable) weight which is right-moderate with respect to some weight $m_1 : G \to (0, \infty)$ of moderate growth.*

*Finally, we assume that $v : G \to (0, \infty)$ is of moderate growth and a control weight for the right-invariant solid QBF space $Y = L^{p,q}_m(G)$. In particular, $v$ is bounded from below, cf. Assumption 2.4.1.*

*Remark.* By Corollary 4.2.8, we know that there always is a control weight $v : G \to (0, \infty)$ with the above properties.

## 4.3 DEFINITION OF THE FOURIER TRANSFORM $\mathcal{F}_{\mathcal{R}} : \mathcal{R}_v \to \mathcal{D}'(\mathcal{O})$ AND AN ALTERNATIVE RESERVOIR

The first aim of this section is to define the Fourier transform on the union of all reservoirs $\mathcal{R}_v$, where $v : G \to (0, \infty)$ is of moderate growth. In particular, this will define the Fourier transform on all coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\right)$, where $m : G \to (0, \infty)$ is right-moderate with respect to some weight of moderate growth.

In this section, the codomain of the Fourier transform on $\mathcal{R}_v$ will be just the space $\mathcal{D}'(\mathcal{O})$ of distributions on $\mathcal{O}$. In Section 4.5 below, we will see that the Fourier transform actually maps each coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$ into a decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ for a suitable weight $u$ and the INDUCED COVERING $\mathcal{Q}$.

The second aim of this section is to show that one can use the space $Z'(\mathcal{O}) = [\mathcal{F}(C^\infty_c(\mathcal{O}))]'$ as an alternative reservoir for the definition of coorbit spaces, i.e., we will show that there is a canonical isomorphism between $\mathrm{Co}\left(L^{p,q}_m\right)$ and the space

*We note that for the space $\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right)$ to make sense, we still have to define the wavelet transform $W_\psi f$ for $f \in Z'(\mathcal{O})$. This is done in Definition 4.3.2 below.*

$$\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right) := \left\{ f \in Z'(\mathcal{O}) \,\middle|\, W_\psi f \in W\left(L^\infty, L^{p,q}_m\right) \right\},$$

where $\psi \in \mathcal{S}\left(\mathbb{R}^d\right) \setminus \{0\}$ with $\widehat{\psi} \in C^\infty_c(\mathcal{O})$ is arbitrary but fixed. This will be helpful in Section 4.6 where we show that the Fourier transform $\mathcal{F}_R : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ is actually an isomorphism.

We recall from Definition 3.4.5 that $\mathcal{F} : C^\infty_c(\mathcal{O}) \to Z(\mathcal{O})$ is a homeomorphism by choice of the topology on $Z(\mathcal{O}) = \mathcal{F}(C^\infty_c(\mathcal{O}))$. Furthermore, $Z'(\mathcal{O})$ is the topological dual space of $Z(\mathcal{O})$, equipped with the weak-$*$-topology.

Finally, we mention the following elementary identity which will be used frequently in this chapter: For $f \in L^1\left(\mathbb{R}^d\right)$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \left(\mathcal{F}^{-1}\overline{f}\right)(x) &= \int_{\mathbb{R}^d} \overline{f(\xi)} \cdot e^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}\xi \\ &= \overline{\int_{\mathbb{R}^d} f(\xi) \cdot e^{-2\pi i \langle x, \xi \rangle} \, \mathrm{d}\xi} = \overline{\widehat{f}(x)}. \end{aligned} \tag{4.3.1}$$

We are now in a position to state the first part of the main results in this section. The missing part is the isomorphism between $\mathrm{Co}\left(L^{p,q}_m\right)$ and $\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right)$. This will be established in Theorem 4.3.6 below.

We remark that part 3 of the next theorem is crucial, because it shows that the Fourier transform $\mathcal{F}_{\mathcal{R}}$ introduced here is indeed an extension of the usual Fourier transform. This justifies the terminology "Fourier transform" for the map $\mathcal{F}_{\mathcal{R}}$.

**Theorem 4.3.1.** *(cf. [41, Corollaries 10 and 11]) Let*

$$\mathcal{R} := \bigcup_{\substack{w \in WMG_{\mathrm{sub}} \\ w \text{ bounded below}}} \mathcal{R}_w.$$

*We recall from Definition 4.2.1 that $WMG_{\mathrm{sub}}$ is the class of all submultiplicative weights of moderate growth.*

*The Fourier transform $\mathcal{F}_{\mathcal{R}} : \mathcal{R} \to \mathcal{D}'(\mathcal{O})$ and the* CANONICAL RESTRICTION MAP *$\Theta : \mathcal{R} \to Z'(\mathcal{O}) = [\mathcal{F}(C^\infty_c(\mathcal{O}))]'$, defined by*

$$\mathcal{F}_{\mathcal{R}} f : C^\infty_c(\mathcal{O}) \to \mathbb{C}, g \mapsto f\left(\mathcal{F}^{-1}\overline{g}\right) \text{ for } f \in \mathcal{R}$$

*and*

$$\Theta f : \mathcal{F}(C^\infty_c(\mathcal{O})) \to \mathbb{C}, g \mapsto f(\overline{g}) \text{ for } f \in \mathcal{R}$$

*are well-defined maps. They satisfy the following properties:*

1. *We have*
$$\mathcal{F}_{\mathcal{R}} f = (\Theta f) \circ \mathcal{F} \text{ for each } f \in \mathcal{R}.$$

2. *Restricted to each reservoir $\mathcal{R}_w$, the maps $\mathcal{F}_{\mathcal{R}}$ and $\Theta$ are linear, injective and continuous with respect to the weak-$*$-topology on $\mathcal{R}_w$.*

3. *We can interpret each function $f \in L^2\left(\mathbb{R}^d\right)$ as a distribution on $\mathcal{O}$ by setting $\langle f, \varphi \rangle := \int f(x) \cdot \varphi(x)\, \mathrm{d}x$.*

   *With this identification, we have*

   $$\mathcal{F}_{\mathcal{R}} f = \mathcal{F}_{L^2} f \text{ for all } f \in L^2(\mathbb{R}^d) \hookrightarrow \mathcal{R}_w$$

   *for each sumbultiplicative weight $w : G \to (0, \infty)$ of moderate growth which is also bounded below.*

   *Here, $\mathcal{F}_{L^2} = \mathcal{F} : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$ denotes the ordinary Fourier transformation and $L^2\left(\mathbb{R}^d\right)$ is considered as a subspace of $\mathcal{R}_w$ by virtue of the embedding*

   $$\iota : L^2(\mathbb{R}^d) \hookrightarrow \mathcal{R}_w, f \mapsto \left(\mathcal{S}_w \ni g \mapsto \langle f \mid g \rangle_{L^2}\right),$$

   *see Lemma 2.4.8.*

*Proof.* 2. It is easy to see that the conjugation map

$$c : C^\infty_c(\mathcal{O}) \to C^\infty_c(\mathcal{O}), g \mapsto \overline{g}$$

is a well-defined, continuous antilinear map. Corollary 4.2.12 shows that the inverse Fourier transform $\mathcal{F}^{-1} : C_c^\infty(\mathcal{O}) \to \mathcal{H}_w^1$ is well-defined and continuous for each weight $w : G \to (0,\infty)$ of moderate growth which is also bounded from below.

For $w \in WMG_{\text{sub}}$ (with $w$ bounded below) and $f \in \mathcal{R}_w = \left(\mathcal{H}_w^1\right)^\neg$, we have $\mathcal{F}_\mathcal{R} f = f \circ \mathcal{F}^{-1} \circ c$, which is continuous as a composition of continuous maps. Furthermore, since $f$ and $c$ are antilinear, and since $\mathcal{F}^{-1}$ is linear, the composition $\mathcal{F}_\mathcal{R} f = f \circ \mathcal{F}^{-1} \circ c$ is linear. This proves $\mathcal{F}_\mathcal{R} f \in \mathcal{D}'(\mathcal{O})$, so that $\mathcal{F}_\mathcal{R} : \mathcal{R} \to \mathcal{D}'(\mathcal{O})$ is well-defined.

Finally, let $(f_\alpha)_{\alpha \in I}$ be a net in $\mathcal{R}_w$ with $f_\alpha \to f \in \mathcal{R}_w$ (with convergence in the weak-$*$-topology). Let $g \in C_c^\infty(\mathcal{O})$ be arbitrary. Corollary 4.2.12 shows $\mathcal{F}^{-1}\overline{g} \in \mathcal{H}_w^1$ and this implies

$$(\mathcal{F}_\mathcal{R} f_\alpha)(g) = f_\alpha(\mathcal{F}^{-1}\overline{g}) \longrightarrow f(\mathcal{F}^{-1}\overline{g}) = (\mathcal{F}_\mathcal{R} f)(g).$$

Since $g \in C_c^\infty(\mathcal{O})$ was arbitrary, we get $\mathcal{F}_\mathcal{R} f_\alpha \to \mathcal{F}_\mathcal{R} f$, so that $\mathcal{F}_\mathcal{R}$ is indeed continuous with respect to the weak-$*$-topology on $\mathcal{R}_w$.

For well-definedness of $\Theta f$ for $f \in \mathcal{R}_w$, let $g \in \mathcal{F}(C_c^\infty(\mathcal{O}))$ be arbitrary. Hence, $g = \widehat{h}$ for $h = \mathcal{F}^{-1} g \in C_c^\infty(\mathcal{O})$. By equation (4.3.1) and Corollary 4.2.12, this implies

$$\overline{g} = \overline{\widehat{h}} = \mathcal{F}^{-1}\overline{h} \in \mathcal{F}^{-1}(C_c^\infty(\mathcal{O})) \subset \mathcal{H}_w^1,$$

so that $(\Theta f)(g) = f(\overline{g}) \in \mathbb{C}$ is well-defined with

$$(\Theta f)(g) = f(\overline{g}) = f\left(\mathcal{F}^{-1}\overline{\mathcal{F}^{-1}\overline{g}}\right) = \mathcal{F}_\mathcal{R}\left(\mathcal{F}^{-1} g\right).$$

In particular, this proves $(\Theta f) \circ \mathcal{F} = \mathcal{F}_\mathcal{R} f$ and $\Theta f = (\mathcal{F}_\mathcal{R} f) \circ \mathcal{F}^{-1}$ and thus part 1 of the theorem.

Since the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{F}(C_c^\infty(\mathcal{O})) \to C_c^\infty(\mathcal{O})$ is a homeomorphism by definition of the topology on $\mathcal{F}(C_c^\infty(\mathcal{O}))$, we get $\Theta f \in [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$, so that $\Theta$ is well-defined. Linearity of $\Theta$ is obvious. The proof of continuity of $\Theta$ with respect to the weak-$*$-topology on $\mathcal{R}_w$ is essentially the same as for $\mathcal{F}_\mathcal{R}$ and hence omitted.

Since $\mathcal{F}^{-1} : \mathcal{F}(C_c^\infty(\mathcal{O})) \to C_c^\infty(\mathcal{O})$ is bijective, we see $\Theta f = 0$ if and only if $\mathcal{F}_\mathcal{R} f = 0$, so that it suffices to show that $\mathcal{F}_\mathcal{R}$ is injective on each reservoir $\mathcal{R}_w$. To this end, assume $f \in \mathcal{R}_w$ with $\mathcal{F}_\mathcal{R} f = 0$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be arbitrary with $\psi \neq 0$ and $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Equation (4.1.2) shows $\mathcal{F}[\pi(x,h)\psi] \in C_c^\infty(\mathcal{O})$ and hence $\overline{\mathcal{F}[\pi(x,h)\psi]} \in C_c^\infty(\mathcal{O})$ for every $(x,h) \in G$, But this implies

$$\begin{aligned} 0 &= (\mathcal{F}_\mathcal{R} f)\left(\overline{\mathcal{F}[\pi(x,h)\psi]}\right) = f\left(\mathcal{F}^{-1}\overline{\overline{\mathcal{F}[\pi(x,h)\psi]}}\right) \\ &= f(\pi(x,h)\psi) = \left(W_\psi f\right)(x,h). \end{aligned}$$

By Lemma 2.4.8, we know that $W_\psi : \mathcal{R}_w \to L^\infty_{1/w}(G)$ is injective, because of $\psi \in \mathcal{G}_w \setminus \{0\}$ (cf. Corollary 4.2.12). Hence, $f = 0$.

3. For $f \in L^2(\mathbb{R}^d)$ and $g \in C_c^\infty(\mathcal{O})$, Plancherel's theorem implies

$$\begin{aligned}[\mathcal{F}_{\mathcal{R}}(\iota f)](g) &= (\iota f)\left(\mathcal{F}^{-1}\overline{g}\right) = \left\langle f \,\middle|\, \mathcal{F}^{-1}\overline{g}\right\rangle_{L^2} \\ &= \left\langle \widehat{f} \,\middle|\, \overline{g}\right\rangle_{L^2} = \left\langle \widehat{f}, g\right\rangle = \langle \mathcal{F}_{L^2} f, g\rangle .\end{aligned}$$

Since this holds for all $g \in C_c^\infty(\mathcal{O})$, we conclude $\mathcal{F}_{\mathcal{R}}(\iota f) = \mathcal{F}_{L^2} f$ as distributions on $\mathcal{O}$. □

Our final goal for this section is to establish the canonical isomorphism between $\mathrm{Co}\left(L_m^{p,q}\right)$ and $\widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$. To this end, we first have to define the wavelet transform $W_\psi f$ for $f \in Z'(\mathcal{O}) = [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$.

**Definition 4.3.2.** (cf. [41, equation (8.1)]) Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ and assume $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. For $f \in Z'(\mathcal{O})$, we define the (GENERALIZED) WAVELET TRANSFORM $W_\psi f$ of $f$ with respect to $\psi$ by

$$W_\psi f : G \to \mathbb{C}, (x,h) \mapsto f\left(\overline{\pi(x,h)\psi}\right) = \left\langle f, \overline{\pi(x,h)\psi}\right\rangle .$$

*Remark.* Lemma 4.3.4 below will show that $W_\psi f$ is indeed a well-defined (continuous) function. Under this assumption, we show that the new wavelet transform is a consistent extension in the following sense: We have

$$W_\psi[\Theta f] = W_\psi f \tag{4.3.2}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$ and each $f \in \mathcal{R}$. To see this, simply note that the definition of $\Theta$ implies

$$\left(W_\psi[\Theta f]\right)(\alpha) = (\Theta f)\left(\overline{\pi(\alpha)\psi}\right) = f(\pi(\alpha)\psi) = \left(W_\psi f\right)(\alpha) .$$

Here, the last step is a direct consequence of the definition of $W_\psi f$ for $f \in \mathcal{R}_w$, cf. Lemma 2.4.8.

The main technical difficulty for the isomorphism between $\mathrm{Co}\left(L_m^{p,q}\right)$ and $\widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$ is to show that the wavelet transform $W_\psi f$ satisfies the REPRODUCING FORMULA $W_\psi f = \frac{1}{C_\psi} \cdot W_\psi f * W_\psi \psi$. We remark that it is important to not take this formula for granted, since failure of this property is exactly the reason why [58, Theorem 4.3(c)] is false in general, see also Example 2.4.10.

For the proof of the reproducing formula, we first need the following technical lemma.

**Lemma 4.3.3.** *Let $\varnothing \neq U \subset \mathbb{R}^d$ be an open set and let $K \subset U$ be compact and $k \in \mathbb{N}_0$. Then the map*

$$\Phi : C_K^k(U) \times \mathbb{R}^d \to C_K^k(U), (f,\xi) \mapsto \left(x \mapsto e^{i\langle x,\xi\rangle} \cdot f(x)\right)$$

*is well-defined and continuous.*

*Here, $C_K^k(U)$ is the space*

$$C_K^k(U) := \left\{ f \in C^k(U) \,\middle|\, \operatorname{supp} f \subset K \right\}$$

*with norm*

$$\|f\|_{C_K^k(U)} := \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq k}} \sup_{x \in U} |(\partial^\alpha f)(x)|.$$

*Proof.* Let $(f_n, \xi_n)_{n \in \mathbb{N}}$ be a sequence in $C_K^k(U) \times \mathbb{R}^d$ converging to $(f, \xi) \in C_K^k(U) \times \mathbb{R}^d$. Choose $R > 0$ so that $K \subset \overline{B_R}(0)$ holds. Because of $\left|\frac{d}{dt} e^{it}\right| = \left|ie^{it}\right| \leq 1$ for $t \in \mathbb{R}$, we see that $t \mapsto e^{it}$ obeys the estimate $\left|e^{it} - e^{is}\right| \leq |t - s|$ for all $t, s \in \mathbb{R}$. For $x \in K$, this implies

$$\left|e^{i\langle x, \xi_n\rangle} - e^{i\langle x, \xi\rangle}\right| \leq |\langle x, \xi_n\rangle - \langle x, \xi\rangle| \leq |x| \cdot |\xi_n - \xi| \leq R \cdot |\xi_n - \xi|.$$

For $x \in U$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, the Leibniz rule yields

$$\begin{aligned}
&|\partial_x^\alpha ((\Phi(f_n, \xi_n))(x) - (\Phi(f, \xi))(x))| \\
&= \left|\partial_x^\alpha \left(e^{i\langle x, \xi_n\rangle} (f_n(x) - f(x))\right) + \partial_x^\alpha \left(\left(e^{i\langle x, \xi_n\rangle} - e^{i\langle x, \xi\rangle}\right) \cdot f(x)\right)\right| \\
&= \left|\sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta} \left[\partial_x^\beta e^{i\langle x, \xi_n\rangle} (\partial^\gamma (f_n - f))(x) + (\partial^\gamma f)(x) \cdot \partial_x^\beta \left(e^{i\langle x, \xi_n\rangle} - e^{i\langle x, \xi\rangle}\right)\right]\right| \\
&\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left[\left|(i\xi_n)^\beta e^{i\langle x, \xi_n\rangle}\right| \cdot \|f_n - f\|_{C_K^k(U)} \right. \\
&\qquad \left. + \left|(i\xi_n)^\beta e^{i\langle x, \xi_n\rangle} - (i\xi)^\beta e^{i\langle x, \xi\rangle}\right| \cdot \|f\|_{C_K^k(U)}\right]
\end{aligned}$$

Note for $x \in U \setminus K$ that $\Phi(f_n, \xi_n) - \Phi(f, \xi) \equiv 0$ on the neighborhood $U \setminus K$ of $x$ and thus $|\partial_x^\alpha ((\Phi(f_n, \xi_n))(x) - (\Phi(f, \xi))(x))| = 0$.

For $x \in K$ we have

$$\begin{aligned}
&\left|(i\xi_n)^\beta e^{i\langle x, \xi_n\rangle} - (i\xi)^\beta \cdot e^{i\langle x, \xi\rangle}\right| \\
&\leq \left|(i\xi_n)^\beta e^{i\langle x, \xi_n\rangle} - (i\xi_n)^\beta e^{i\langle x, \xi\rangle}\right| + \left|(i\xi_n)^\beta e^{i\langle x, \xi\rangle} - (i\xi)^\beta e^{i\langle x, \xi\rangle}\right| \\
&\leq |\xi_n|^{|\beta|} \cdot \left|e^{i\langle x, \xi_n\rangle} - e^{i\langle x, \xi\rangle}\right| + \left|\xi_n^\beta - \xi^\beta\right| \\
&\leq |\xi_n|^{|\beta|} \cdot R \cdot |\xi_n - \xi| + \left|\xi_n^\beta - \xi^\beta\right|.
\end{aligned}$$

Using the above estimate, we deduce

$$\begin{aligned}
&\|\partial_x^\alpha [(\Phi(f_n, \xi_n))(x) - (\Phi(f, \xi))(x)]\|_{\sup} \\
&\leq \|f_n - f\|_{C_K^k(U)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\xi_n|^{|\beta|} \\
&\quad + \|f\|_{C_K^k(U)} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(R \cdot |\xi_n|^{|\beta|} |\xi_n - \xi| + \left|\xi_n^\beta - \xi^\beta\right|\right),
\end{aligned}$$

where the right-hand side vanishes for $n \to \infty$. This proves the claim, since $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ was arbitrary. □

Using this result on the continuity of modulation, we can now show that the extended wavelet transform as defined in Definition 4.3.2 indeed defines a continuous function that satisfies the expected reproducing formula.

**Lemma 4.3.4.** *(cf. [41, Lemma 40]) Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Then the following are true:*

1. *We have*

$$\mathcal{F}^{-1}\left[\overline{\pi(x,h)\,\psi}\right] = |\det(h)|^{1/2} \cdot M_x\left(\overline{\widehat{\psi}} \circ h^T\right)$$

*and*

$$\operatorname{supp}\left(\mathcal{F}^{-1}\left[\overline{\pi(x,h)\,\psi}\right]\right) \subset h^{-T} \operatorname{supp}\widehat{\psi} \subset \mathcal{O}, \tag{4.3.3}$$

*Thus, $\mathcal{F}^{-1}\left[\overline{\pi(x,h)\,\psi}\right] \in C_c^\infty(\mathcal{O})$ and $\overline{\pi(x,h)\,\psi} \in Z(\mathcal{O})$ for all $(x,h) \in G$.*

2. *In particular, the wavelet transform $W_\psi f$ as defined in Definition 4.3.2 is well-defined for every $f \in [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$.*

3. *We have $W_\psi\psi \in L^2(G)$. In case of $\psi \neq 0$, this implies that the constant $C_\psi > 0$ from equation (2.4.2) is well-defined.*

4. *For $\psi \neq 0$ and $C_\psi > 0$ as above, we have for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\varphi} \in C_c^\infty(\mathcal{O})$ the identity*

$$\mathcal{F}^{-1}\left[\overline{\pi(\alpha)\,\varphi}\right] = \frac{1}{C_\psi}\int_G \langle \pi(\beta)\,\psi \mid \pi(\alpha)\,\varphi\rangle \cdot \mathcal{F}^{-1}\left[\overline{\pi(\beta)\,\psi}\right] \mathrm{d}\beta, \tag{4.3.4}$$

*for all $\alpha \in G$, where the integral is to be understood in the weak sense in $C_c^\infty(\mathcal{O})$.*

5. *For $f \in [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$, $\psi \neq 0$ and $C_\psi > 0$ as above, the wavelet transformation $W_\psi f$ is a continuous function which satisfies the reproducing formula*

$$W_\psi f = \frac{1}{C_\psi} \cdot W_\psi f * W_\psi\psi. \tag{4.3.5}$$

*Proof.* Theorem 4.2.11 implies $W_\psi\psi = W_\psi\left[\mathcal{F}^{-1}\widehat{\psi}\right] \in L^2(G)$, so that the constant $C_\psi > 0$ from equation (2.4.2) is well-defined in case of $\psi \neq 0$.

Now, let us prove equation (4.3.3). Equation (4.3.1), together with the effect of $\pi(x,h)$ on the Fourier side (as given in formula (4.1.2)), yields

$$\begin{aligned}\mathcal{F}^{-1}\left[\overline{\pi(x,h)\psi}\right] &= \overline{\mathcal{F}(\pi(x,h)\psi)} \\ &= |\det(h)|^{1/2} \cdot \overline{M_{-x}\Delta_h\widehat{\psi}} \\ &= |\det(h)|^{1/2} \cdot \left(\overline{\widehat{\psi}} \circ h^T\right) \cdot e_x \qquad (4.3.6)\end{aligned}$$

with $e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i\langle x,\xi\rangle}$. We derive

$$\begin{aligned}\operatorname{supp}\left(\mathcal{F}^{-1}\left[\overline{\pi(x,h)\psi}\right]\right) &= \operatorname{supp}\left(\overline{\widehat{\psi}} \circ h^T\right) \\ &= h^{-T}\operatorname{supp}\widehat{\psi} \subset h^{-T}\mathcal{O} = \mathcal{O},\end{aligned}$$

as claimed. Since $(\overline{\widehat{\psi}} \circ h^T) \cdot e_x$ is a smooth function, we conclude $\mathcal{F}^{-1}\left[\overline{\pi(x,h)\psi}\right] \in C_c^\infty(\mathcal{O})$ and hence

$$\overline{\pi(x,h)\psi} \in Z(\mathcal{O}) = \mathcal{F}(C_c^\infty(\mathcal{O})).$$

Now, assume $\psi \neq 0$. Fix $\alpha = (x,h) \in G$. For $\beta = (y,g) \in G$ with $\langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle_{L^2} \neq 0$, the Plancherel theorem (together with equation (4.1.2)) yields

$$\begin{aligned}0 \neq \langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle &= \left\langle \widehat{\pi(\beta)\psi} \,\middle|\, \widehat{\pi(\alpha)\varphi}\right\rangle \\ &= |\det(gh)|^{1/2} \cdot \left\langle M_{-y}\Delta_g\widehat{\psi} \,\middle|\, M_{-x}\Delta_h\widehat{\varphi}\right\rangle,\end{aligned}$$

which implies

$$\varnothing \neq \operatorname{supp}\left(\Delta_g\widehat{\psi}\right) \cap \operatorname{supp}\left(\Delta_h\widehat{\varphi}\right) = h^{-T}\left[\operatorname{supp}\widehat{\varphi}\right] \cap g^{-T}\left[\operatorname{supp}\widehat{\psi}\right].$$

For $K_1 := \operatorname{supp}\widehat{\varphi}$, $K_2 := \operatorname{supp}\widehat{\psi} \subset \mathcal{O}$ and $L = L(K_1,K_2) \subset H$ compact as in Lemma 4.1.4, the same lemma yields $g \in hL$. Using the estimate on the support given in equation (4.3.3), we see

$$\begin{aligned}\operatorname{supp}\left(\mathcal{F}^{-1}\left[\overline{\pi(\beta)\psi}\right]\right) &\subset g^{-T}\operatorname{supp}\widehat{\psi} \\ &\subset h^{-T}L^{-T}K_2 =: K_3 \subset \mathcal{O}.\end{aligned}$$

Note that $K_3 = K_3(h)$ depends upon $h$ but that $\alpha = (x,h) \in G$ is fixed.

Together with equation (4.3.6), this shows for arbitrary $\ell \in \mathbb{N}_0$ that

$$\begin{aligned}\Phi_\ell : G &\longrightarrow C_{K_3}^\ell(\mathcal{O}), \\ \beta = (y,g) &\mapsto \langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle \cdot \mathcal{F}^{-1}\left[\overline{\pi(\beta)\psi}\right] \\ &= \langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle \cdot |\det(g)|^{\frac{1}{2}} \cdot e_y \cdot \left(\overline{\widehat{\psi}} \circ g^T\right)\end{aligned}$$

is well-defined with $\Phi_\ell(y,g) = 0$ for $g \notin hL$. The strong continuity of $\pi$ and Lemma 4.2.9 (with the ensuing remark), in tandem with Lemma 4.3.3, show that $\Phi_\ell$ is continuous. This implies that $\Phi_\ell$ is measurable and that $\Phi_\ell(G) \subset C^\ell_{K_3}(\mathcal{O})$ is $\sigma$-compact and hence separable, since $G$ is $\sigma$-compact. By [54, Chapter VI, §1, M11], this implies that $\Phi_\ell$ is $\mu$-measurable, where $\mu$ is the Haar measure on $G$.

The continuity of $\Gamma : H \to C^\infty_c\left(\mathbb{R}^d\right), g \mapsto \overline{\widehat{\psi}} \circ g^T$, which was shown in (the remark after) Lemma 4.2.9, implies finiteness of

$$C_\alpha := \max_{\gamma \le \alpha} \max_{g \in hL} \left\| \partial^\gamma \left[ \overline{\widehat{\psi}} \circ g^T \right] \right\|_{\sup}$$

for all $\alpha \in \mathbb{N}_0^d$. Using the Leibniz rule, we arrive at

$$\begin{aligned}
\left\| \partial^\alpha \left[ e_y \cdot \left( \overline{\widehat{\psi}} \circ g^T \right) \right] \right\|_{\sup} &= \left\| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \cdot \partial^\beta e_y \cdot \partial^{\alpha-\beta} \left[ \overline{\widehat{\psi}} \circ g^T \right] \right\|_{\sup} \\
&\le C_\alpha \cdot \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left\| (2\pi i y)^\beta e_y \right\|_{\sup} \\
&\le C_\alpha \cdot \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |2\pi y|^{|\beta|} \\
&\le C'_\alpha \cdot (1 + |y|)^{|\alpha|}
\end{aligned}$$

for $g \in hL$ and some constant $C'_\alpha > 0$. Define $C := \max_{g \in hL} |\det(g)|^{1/2}$ and $\zeta := \pi(\alpha)\varphi$ and note

$$|\langle \pi(\beta)\psi \mid \pi(\alpha)\varphi \rangle| = |\langle \zeta \mid \pi(\beta)\psi \rangle| = \left| \left( W_\psi \zeta \right)(\beta) \right|.$$

Together with $\Phi_\ell(y,g) = 0$ for $g \notin hL$, this proves the estimate

$$\begin{aligned}
\left\| \partial^\alpha \Phi_\ell(\beta) \right\|_{\sup} &= \left| \left( W_\psi \zeta \right)(\beta) \right| \cdot |\det(g)|^{1/2} \cdot \left\| \partial^\alpha \left[ e_y \cdot \left( \overline{\widehat{\psi}} \circ g^T \right) \right] \right\|_{\sup} \\
&\le C \cdot C'_\alpha \cdot (1 + |y|)^{|\alpha|} \cdot \left| \left( W_\psi \zeta \right)(\beta) \right|
\end{aligned}$$

for $\alpha \in \mathbb{N}_0^d$ and $\beta = (y,g) \in G$.

In summary, setting $C' := \max_{|\alpha| \le \ell} C'_\alpha$, we showed

$$\left\| \Phi_\ell(\beta) \right\|_{C^\ell_{K_3}} \le CC' \cdot (1 + |y|)^\ell \cdot \left| \left( W_\psi \zeta \right)(\beta) \right| \quad \forall \beta \in G.$$

But equation (4.1.2) yields $\widehat{\zeta} = |\det(h)|^{1/2} \cdot M_{-x} \Delta_h \widehat{\varphi}$, which implies that $\operatorname{supp} \widehat{\zeta} \subset h^{-T} \operatorname{supp} \widehat{\varphi} \subset \mathcal{O}$ is compact, so that Theorem 4.2.11 (with $w_0 \equiv 1$ and $N = \ell$) yields $W_\psi \zeta \in L^1_{(y,g) \mapsto (1+|y|)^\ell}(G)$. By [54, Chapter VI, Corollary 5.9], this means that $\Phi_\ell$ is Bochner integrable, so that the integral

$$\varphi_\ell := \frac{1}{C_\psi} \cdot \int_G \Phi_\ell(y,g) \, \mathrm{d}(y,g) \in C^\ell_{K_3}(\mathcal{O}) \hookrightarrow \bigcap_{p \in [1,\infty]} L^p\left(\mathbb{R}^d\right)$$

is well-defined.

We want to show $\varphi_\ell = \mathcal{F}^{-1}\left[\overline{\pi(\alpha)\varphi}\right]$. To this end, let $f \in L^2(\mathbb{R}^d)$ be arbitrary. The map $C^\ell_{K_3}(\mathcal{O}) \to \mathbb{C}, g \mapsto \langle g \mid f\rangle$ is clearly a bounded linear functional. Because of $W_\psi\psi \in L^2(G)$, the weak inversion formula (2.4.3) is applicable with $g = \psi$. Together with the left invariance of the Haar measure (applied to the substitution $\gamma = \alpha^{-1}\beta$) and Plancherel's theorem, we conclude

$$\begin{aligned}
\langle \varphi_\ell \mid f\rangle &= \frac{1}{C_\psi} \cdot \int_G \langle \Phi_\ell(\beta) \mid f\rangle \, \mathrm{d}\beta \\
&= \frac{1}{C_\psi} \cdot \int_G \left\langle \langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle \cdot \mathcal{F}^{-1}\left[\overline{\pi(\beta)\psi}\right] \,\middle|\, f\right\rangle \mathrm{d}\beta \\
&= \frac{1}{C_\psi} \cdot \int_G \langle \pi(\alpha^{-1}\beta)\psi \mid \varphi\rangle \cdot \left\langle \overline{\pi(\beta)\psi} \,\middle|\, \widehat{f}\right\rangle \mathrm{d}\beta \\
&= \frac{1}{C_\psi} \cdot \overline{\int_G \langle \varphi \mid \pi(\alpha^{-1}\beta)\psi\rangle \cdot \left\langle \pi(\alpha\alpha^{-1}\beta)\psi \,\middle|\, \overline{\widehat{f}}\right\rangle \mathrm{d}\beta} \\
&= \frac{1}{C_\psi} \cdot \overline{\int_G \left(W_\psi\varphi\right)(\gamma) \cdot \left\langle \pi(\alpha\gamma)\psi \,\middle|\, \overline{\widehat{f}}\right\rangle \mathrm{d}\gamma} \\
&= \frac{1}{C_\psi} \cdot \overline{\int_G \left(W_\psi\varphi\right)(\gamma) \cdot \left\langle \pi(\gamma)\psi \,\middle|\, \pi(\alpha^{-1})\overline{\widehat{f}}\right\rangle \mathrm{d}\gamma} \\
&= \overline{\left\langle \varphi \,\middle|\, \pi(\alpha^{-1})\overline{\widehat{f}}\right\rangle} \\
&= \left\langle \overline{\widehat{f}} \,\middle|\, \pi(\alpha)\varphi\right\rangle \\
&= \left\langle \overline{\pi(\alpha)\varphi} \,\middle|\, \widehat{f}\right\rangle \\
&= \left\langle \mathcal{F}^{-1}\left[\overline{\pi(\alpha)\varphi}\right] \,\middle|\, f\right\rangle.
\end{aligned}$$

Since this holds for every $f \in L^2(\mathbb{R}^d)$, we get $\mathcal{F}^{-1}\left[\overline{\pi(\alpha)\varphi}\right] = \varphi_\ell$ almost everywhere and then everywhere, because both sides of the identity are continuous functions.

Now, let $f \in \mathcal{D}'(\mathcal{O})$ be arbitrary. By [61, Theorem 6.8] and Hahn-Banach, we know that $f|_{C^\ell_{K_3}(\mathcal{O})\cap C^\infty_c(\mathcal{O})}$ admits a continuous extension $\widetilde{f}$ to $C^\ell_{K_3}(\mathcal{O})$ for a suitable $\ell \in \mathbb{N}_0$. Because the Bochner integral commutes with bounded linear operators (cf. [54, Chapter VI, Theorem 4.1]), we get

$$\begin{aligned}
f\left(\mathcal{F}^{-1}\left[\overline{\pi(\alpha)\varphi}\right]\right) &= f(\varphi_\ell) = \widetilde{f}(\varphi_\ell) \\
&= \frac{1}{C_\psi} \int_G \widetilde{f}(\Phi_\ell(\beta)) \, \mathrm{d}\beta \\
&= \frac{1}{C_\psi} \int_G \langle \pi(\beta)\psi \mid \pi(\alpha)\varphi\rangle \cdot f\left(\mathcal{F}^{-1}\left[\overline{\pi(\beta)\psi}\right]\right) \mathrm{d}\beta,
\end{aligned}$$

which proves equation (4.3.4), since $f \in \mathcal{D}'(\mathcal{O})$ was arbitrary.

Now, let $f \in [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$ be arbitrary. By definition of the topology on $\mathcal{F}(C_c^\infty(\mathcal{O}))$, this implies $g := f \circ \mathcal{F} \in \mathcal{D}'(\mathcal{O})$. By choosing $\varphi = \psi$ in equation (4.3.4), we get

$$\begin{aligned}
(W_\psi f)(\alpha) &\stackrel{\text{def}}{=} f\left(\overline{\pi(\alpha)\psi}\right) \\
&= g\left(\mathcal{F}^{-1}\left[\overline{\pi(\alpha)\psi}\right]\right) \\
&= \frac{1}{C_\psi}\int_G \langle \pi(\beta)\psi \mid \pi(\alpha)\psi\rangle \cdot g\left(\mathcal{F}^{-1}\left[\overline{\pi(\beta)\psi}\right]\right) \mathrm{d}\beta \\
&= \frac{1}{C_\psi}\int_G \langle \psi \mid \pi(\beta^{-1}\alpha)\psi\rangle \cdot f\left(\overline{\pi(\beta)\psi}\right) \mathrm{d}\beta \\
&= \frac{1}{C_\psi}\int_G W_\psi\psi(\beta^{-1}\alpha)\cdot (W_\psi f)(\beta)\,\mathrm{d}\beta \\
&= \frac{1}{C_\psi}\left(W_\psi f * W_\psi\psi\right)(\alpha),
\end{aligned}$$

which is nothing else than equation (4.3.5).

The only thing missing is continuity of $W_\psi f$. For this, let $h_0 \in H$ be arbitrary and choose a compact neighborhood $K_4 \subset H$ of $h_0$. For $\alpha = (x,h) \in \mathbb{R}^d \times K_4$ we then have

$$\operatorname{supp}\left(\mathcal{F}^{-1}\left[\overline{\pi(x,h)\psi}\right]\right) \subset h^{-T}K_2 \subset K_4^{-T}K_2 =: K_5,$$

cf. equation (4.3.3). For $f \in Z'(\mathcal{O})$, we have $g := f \circ \mathcal{F} \in \mathcal{D}'(\mathcal{O})$, so that (as above) there is some $\ell \in \mathbb{N}_0$ such that $g|_{C_c^\infty(\mathcal{O})\cap C_{K_5}^\ell(\mathcal{O})}$ admits a continuous extension $\widetilde{g}$ to all of $C_{K_5}^\ell(\mathcal{O})$. Using equation (4.3.6), this yields

$$\begin{aligned}
(W_\psi f)(x,h) &\stackrel{\text{def}}{=} f\left(\overline{\pi(x,h)\psi}\right) = g\left(\mathcal{F}^{-1}\left[\overline{\pi(x,h)\psi}\right]\right) \\
&= |\det(h)|^{1/2}\cdot g\left(\left[\overline{\widehat{\psi}}\circ h^T\right]\cdot e_x\right) \\
&= |\det(h)|^{1/2}\cdot \widetilde{g}\left(\left[\overline{\widehat{\psi}}\circ h^T\right]\cdot e_x\right)
\end{aligned}$$

for all $(x,h) \in \mathbb{R}^d \times K_4$. But Lemma 4.3.3 and (the remark after) Lemma 4.2.9 show that the right-hand side defines a continuous function in $(x,h)$, at least restricted to $\mathbb{R}^d \times K_4$. □

As an important consequence, we see that the wavelet transform $f \mapsto W_\psi f$ on $Z'(\mathcal{O}) = [\mathcal{F}(C_c^\infty(\mathcal{O}))]'$ is injective.

**Corollary 4.3.5.** *The wavelet transform*

$$W_\psi : Z'(\mathcal{O}) \to C(G), f \mapsto W_\psi f$$

*is an injective linear map for each* $\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ *with* $\widehat{\psi} \in C_c^\infty(\mathcal{O})$.

*Proof.* Lemma 4.3.4 shows continuity of $W_\psi f$ for each $f \in Z'(\mathcal{O})$. Linearity of $f \mapsto W_\psi f$ is clear. Thus, assume that $f \in Z'(\mathcal{O})$ satisfies

$W_\psi f \equiv 0$. We have $g := f \circ \mathcal{F} \in \mathcal{D}'(\mathcal{O})$ and we want to show $f \equiv 0$ on $\mathcal{F}(C_c^\infty(\mathcal{O}))$, which is equivalent to showing $g \equiv 0$ on $C_c^\infty(\mathcal{O})$.

To this end, let $h \in C_c^\infty(\mathcal{O})$ be arbitrary and define $\varphi := \overline{\widehat{h}} \in \mathcal{S}(\mathbb{R}^d)$. Using identity (4.3.1), we see $\varphi = \mathcal{F}^{-1}\overline{h}$ and hence $\widehat{\varphi} = \overline{h} \in C_c^\infty(\mathcal{O})$. The definition of $\varphi$ implies, for $\alpha = 1_G$,

$$\mathcal{F}^{-1}\overline{\pi(\alpha)\,\varphi} = \mathcal{F}^{-1}\overline{\varphi} = \mathcal{F}^{-1}\widehat{h} = h.$$

By invoking the identity (4.3.4) (with $\alpha = 1_G$), we conclude

$$\begin{aligned} g(h) &= g\left(\mathcal{F}^{-1}\overline{\pi(\alpha)\,\varphi}\right) \\ &= \frac{1}{C_\psi} \cdot \int_G \langle \pi(\beta)\,\psi \mid \pi(\alpha)\,\varphi\rangle \cdot g\left(\mathcal{F}^{-1}\left[\overline{\pi(\beta)\,\psi}\right]\right)\,\mathrm{d}\beta \\ &= \frac{1}{C_\psi} \cdot \int_G \langle \pi(\beta)\,\psi \mid \pi(\alpha)\,\varphi\rangle \cdot f\left(\overline{\pi(\beta)\,\psi}\right)\,\mathrm{d}\beta \\ &= \frac{1}{C_\psi} \cdot \int_G \langle \pi(\beta)\,\psi \mid \pi(\alpha)\,\varphi\rangle \cdot W_\psi f(\beta)\,\mathrm{d}\beta = 0, \end{aligned}$$

because of $W_\psi f \equiv 0$. Since $h \in C_c^\infty(\mathcal{O})$ was arbitrary, we have shown $g \equiv 0$ and thus $f \equiv 0$. □

Using the reproducing formula for the (generalized) wavelet transform, as well as the injectivity of $W_\psi$, we can now show that the canonical restriction map $\Theta$ from Theorem 4.3.1 indeed yields an isomorphism between $\mathrm{Co}\left(L_m^{p,q}\right)$ and $\widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$.

**Theorem 4.3.6.** *(cf. [41, Theorem 38])*

*For each weight $m : G \to (0,\infty)$ satisfying our standing assumptions (cf. Assumption 4.2.13) and each $\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$, the* CANONICAL RESTRICTION MAP *$\Theta : \mathcal{R} \to Z'(\mathcal{O})$ given in Theorem 4.3.1 restricts to an isometric isomorphism*

$$\Theta|_{\mathrm{Co}_\psi\left(L_m^{p,q}\right)} : \mathrm{Co}_\psi\left(L_m^{p,q}\right) \to \widetilde{\mathrm{Co}_\psi}\left(L_m^{p,q}\right),$$

*with*

$$\widetilde{\mathrm{Co}_\psi}\left(L_m^{p,q}\right) := \left\{ f \in Z'(\mathcal{O}) \,\middle|\, W_\psi f \in W\left(L^\infty, L_m^{p,q}\right)\right\}$$

*and*

$$\|f\|_{\widetilde{\mathrm{Co}_\psi}\left(L_m^{p,q}\right)} := \left\|W_\psi f\right\|_{W\left(L^\infty, L_m^{p,q}\right)}.$$

*Since $\mathrm{Co}_g\left(L_m^{p,q}\right)$ is independent of $g \in \mathbb{A}_v^r \setminus \{0\}$ (for a suitable $r \in (0,1]$) by Theorem 2.4.9 and hence (by Corollary 4.2.12) also independent of the choice of $\psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$, this shows in particular that*

$$\widetilde{\mathrm{Co}}\left(L_m^{p,q}\right) := \widetilde{\mathrm{Co}_\psi}\left(L_m^{p,q}\right)$$

*is independent of the choice of $\psi$ with equivalent quasi-norms $\|\cdot\|_{\widetilde{\mathrm{Co}_\psi}\left(L_m^{p,q}\right)}$ for different choices.*

*Proof.* We first note that $0 = \|f\|_{\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)} = \left\|W_\psi f\right\|_{W\left(L^\infty, L^{p,q}_m\right)}$ implies $W_\psi f \equiv 0$ almost everywhere and thus everywhere, since $W_\psi f$ is a continuous function by Corollary 4.3.5. By the same corollary, we get $f \equiv 0$, so that $\|\cdot\|_{\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)}$ is positive definite on $Z'(\mathcal{O})$. The remaining properties of a quasi-norm easily transfer from $\|\cdot\|_{W\left(L^\infty, L^{p,q}_m\right)}$ to $\|\cdot\|_{\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)}$, so that $\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)$ is a quasi-normed linear space.

Now, let $v : G \to (0, \infty)$ be the control weight of moderate growth that is used for the definition of the coorbit space

$$\mathrm{Co}_\psi\left(L^{p,q}_m\right) = \left\{f \in \mathcal{R}_v \,\middle|\, W_\psi f \in W\left(L^\infty, L^{p,q}_m\right)\right\},$$

cf. Assumption 4.2.13.

Equation (4.3.2) yields

$$W_\psi\left[\Theta f\right] = W_\psi f \in W\left(L^\infty, L^{p,q}_m\right)$$

for all $f \in \mathrm{Co}\left(L^{p,q}_m\right)$ and hence

$$\|\Theta f\|_{\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)} = \left\|W_\psi\left[\Theta f\right]\right\|_{L^{p,q}_m} = \left\|W_\psi f\right\|_{L^{p,q}_m} < \infty,$$

which shows that $\Theta|_{\mathrm{Co}_\psi\left(L^{p,q}_m\right)}$ is indeed a well-defined, isometric (and hence injective) map. Linearity of $\Theta$ on $\mathcal{R}_v \supset \mathrm{Co}\left(L^{p,q}_m\right)$ was already shown in Theorem 4.3.1.

It remains to prove that $\Theta$ maps $\mathrm{Co}_\psi\left(L^{p,q}_m\right)$ onto $\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)$. Thus, let $f \in \widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)$ be arbitrary. Lemma 4.3.4 implies that $F := W_\psi f$ satisfies the reproducing formula $F = \frac{1}{C_\psi} \cdot F * W_\psi \psi$. Furthermore, Lemma 2.3.34 shows

$$\frac{1}{r(\alpha)} \cdot |F(\alpha)| \leq C \cdot \|F\|_{W\left(L^\infty, L^{p,q}_m\right)}$$

for almost all $\alpha \in G$ with $r(\alpha) = \left\|\left|R_\alpha\right|\right\|_{W\left(L^{p,q}_m\right) \to W\left(L^{p,q}_m\right)} \leq v(\alpha)$ and some absolute constant $C > 0$. Hence, $F \in L^\infty_{1/v}(G)$. Finally, recall from Corollary 4.2.12 that $\psi \in \mathcal{G}_v \setminus \{0\}$. Hence, part 5 of Lemma 2.4.8 yields some $\tilde{f} \in \mathcal{R}_v$ with $W_\psi \tilde{f} = F$. In particular,

$$W_\psi \tilde{f} = F = W_\psi f \in W\left(L^\infty, L^{p,q}_m\right)$$

and hence $\tilde{f} \in \mathrm{Co}_\psi\left(L^{p,q}_m\right)$.

But now $g := \Theta\tilde{f}$ and $f$ are two elements of $Z'(\mathcal{O})$ with

$$W_\psi g = W_\psi\left[\Theta\tilde{f}\right] = W_\psi \tilde{f} = W_\psi f,$$

which entails $f = g = \Theta\tilde{f}$ by Corollary 4.3.5. Hence, $\Theta$ maps $\mathrm{Co}_\psi\left(L^{p,q}_m\right)$ onto $\widetilde{\mathrm{Co}_\psi}\left(L^{p,q}_m\right)$. □

## 4.4 THE INDUCED COVERING AND A SPECIAL BAPU

Our aim in this section is to define the INDUCED COVERING $\mathcal{Q}$ of the set $\mathcal{O}$. We will then show in the remaining two sections of this chapter that with this choice of $\mathcal{Q}$, the Fourier transform $\mathcal{F}_{\mathcal{R}} : \mathcal{R}_v \to \mathcal{D}'(\mathcal{O})$ indeed restricts to an isomorphism

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$$

for a suitable choice of the weight $u$.

An important ingredient of this proof is the choice of a special BAPU subordinate to the induced covering $\mathcal{Q}$. More precisely, we will show in this section that the definition

$$\varphi_U : \mathbb{R}^d \to \mathbb{R}_+, \xi \mapsto \int_U \left|\widehat{\psi}\left(h^T\xi\right)\right|^2 \mathrm{d}h$$

yields a test function $\varphi_U \in C^\infty_c(\mathcal{O})$ for each precompact set $U \subset H$, at least if $\widehat{\psi} \in C^\infty_c(\mathcal{O})$. Even more, we will prove that, for a suitable choice of the sequence $(V_i)_{i\in I}$, the family $(\varphi_{V_i})_{i\in I}$ yields (a multiple of) an $L^p$-BAPU.

The special structure of this BAPU will allow us to establish the desired connection between the coorbit norm $\left\|W_\psi f\right\|_{W\left(L^\infty, L^{p,q}_m\right)}$ and the decomposition space norm $\left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_{V_i}\cdot f\right)\right\|_{L^p}\right)_{i\in I}\right\|_{\ell^q_u}$ in the next section.

The first part of the current section is concerned with the construction of the induced covering $\mathcal{Q}$, whereas the second subsection is devoted to an analysis of the functions $\varphi_U$ as defined above.

### 4.4.1 *The induced covering $\mathcal{Q}$*

The idea for the construction of the induced covering is to take a relatively separated, $U$-dense family $(h_i)_{i\in I}$ in $H$ for some precompact open unit neighborhood $U \subset H$ and then to employ the orbit map $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^{-T}\xi_0$ to "transplant" the covering $(h_iU)_{i\in I}$ of $H$ to a covering of $\mathcal{O}$.

Our first aim is to show that this construction yields an admissible family of sets (essentially) if and only if the family $(h_i)_{i\in I}$ is relatively separated. The following lemma is the key step for showing that it is *sufficient* if $(h_i)_{i\in I}$ is relatively separated.

**Lemma 4.4.1.** *(cf. [41, Lemma 18]) Let $(h_i)_{i\in I}$ be a relatively separated family in $H$ and let $K_1, K_2 \subset \mathcal{O}$ be compact sets.*

*Given $h \in H$, let*

$$I_h := I_h(K_1, K_2) := \left\{ i \in I \,\middle|\, h^{-T}K_1 \cap h_i^{-T}K_2 \neq \varnothing \right\}.$$

*There are* $C_1 = C_1(K_1, K_2) > 0$ *and* $C_2 = C_2\left(K_1, K_2, (h_i)_{i\in I}\right) > 0$ *with*

$$|I_h| \leq C_2 \text{ for all } h \in H$$

*and with*

$$\left\|h^T h_i^{-T}\right\| \leq C_1 \text{ for all } i \in I_h.$$

*Proof.* Choose the compact set $L_1 := L(K_1, K_2) \subset H$ as in Lemma 4.1.4. For $h \in H$ and $i \in I_h$, this lemma implies $h_i \in hL_1$ and thus $h \in h_i L_1^{-1}$. In particular, we get

$$\left\|h^T h_i^{-T}\right\| = \left\|(h_i^{-1}h)^T\right\| = \|h_i^{-1}h\| \leq \max_{g\in L_1^{-1}} \|g\| =: C_1.$$

Note that $C_1$ only depends on $K_1, K_2$, but not on the family $(h_i)_{i\in I}$.

As seen above, $i \in I_h$ entails $h_i \in hL_1$. By Lemma 2.3.10, this implies

$$\begin{aligned}
|I_h| &\leq |\{i \in I \,|\, h_i \in hL_1\}| \\
&= |\{i \in I \,|\, h_i \{1_G\} \cap hL_1 \neq \varnothing\}| \\
&\leq C_{(h_i)_{i\in I}, \{1_G\}L_1^{-1}} \\
&= C_{(h_i)_{i\in I}, L_1^{-1}} < \infty,
\end{aligned}$$

where the constant $C_2 := C_{(h_i)_{i\in I}, L_1^{-1}}$ is defined as in Definition 2.3.8. Note that $C_2$ only depends on $K_1, K_2$ and $(h_i)_{i\in I}$, but not on $h \in H$. □

Using the above lemma, together with the caracterization of relatively separated families given in Lemma 2.3.11, we can now prove that $\left(h_i^{-T}Q\right)_{i\in I}$ is an admissible family if and only if $(h_i)_{i\in I}$ is relatively separated, at least if $Q$ has nonempty interior.

**Corollary 4.4.2.** *Let* $(h_i)_{i\in I}$ *be a family in* $H$*. Then* $(h_i)_{i\in I}$ *is relatively separated if and only if there is a set* $Q \subset \mathcal{O}$ *with nonempty interior such that the family* $\left(h_i^{-T}Q\right)_{i\in I}$ *is* admissible *in the sense of Definition 3.2.3. This means that*

$$C_Q := \sup_{i\in I} \left|\left\{j \in I \,\middle|\, h_i^{-T}Q \cap h_j^{-T}Q \neq \varnothing\right\}\right|$$

*is finite.*

*Proof.* "⇒": If $(h_i)_{i\in I}$ is relatively separated, then Lemma 4.4.1 shows that $\left(h_i^{-T}Q\right)_{i\in I}$ is admissible in the above sense for *every* compact set $Q \subset \mathcal{O}$. Since $\mathcal{O} \neq \varnothing$ is open, there is a compact subset $Q \subset \mathcal{O}$ with nonempty interior.

"⇐": Fix some $\xi \in \mathcal{O}$. The orbit map $p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$ is well-defined and continuous. Thus, $M := p_\xi^{-1}(Q^\circ) \subset H$ is an open set. Since $p_\xi$ is surjective, $M \neq \varnothing$. By Lemma 2.3.11 above, it suffices to show that the function $f_M := \sum_{i\in I} \chi_{h_i M}$ is bounded.

To this end, let $h \in H$. For $i, j \in I$ with $\chi_{h_i M}(h) = 1 = \chi_{h_j M}(h)$, we have

$$\begin{aligned} p_\xi(h) &\in p_\xi(h_i M) \cap p_\xi(h_j M) \\ &= \left[(h_i M)^{-T} \xi\right] \cap \left[(h_j M)^{-T} \xi\right] \\ &= h_i^{-T} M^{-T} \xi \cap h_j^{-T} M^{-T} \xi \\ &= \left[h_i^{-T} p_\xi(M)\right] \cap \left[h_j^{-T} p_\xi(M)\right] \\ &\subset h_i^{-T} Q^\circ \cap h_j^{-T} Q^\circ \\ &\subset h_i^{-T} Q \cap h_j^{-T} Q. \end{aligned}$$

For some fixed $i = i_h \in I$ with $\chi_{h_i M}(h) = 1$, this implies

$$\begin{aligned} f_M(h) &= \left| \left\{ j \in I \,\middle|\, \chi_{h_j M}(h) = 1 \right\} \right| \\ &\leq \left| \left\{ j \in I \,\middle|\, h_i^{-T} Q \cap h_j^{-T} Q \neq \varnothing \right\} \right| \leq C_Q < \infty. \end{aligned}$$

If there is no such $i \in I$, we trivially have $f_M(h) = 0 \leq C_Q$. All in all, we have shown $0 \leq f_M \leq C_Q$, so that $f_M$ is bounded. □

This characterization motivates the following definition:

**Definition 4.4.3.** We say that a family $\mathcal{Q} = (Q_i)_{i \in I}$ of subsets of $\mathcal{O}$ is A COVERING OF $\mathcal{O}$ INDUCED BY $H$, if the following conditions are satisfied:

A. $\mathcal{Q}$ is a covering of $\mathcal{O}$,

B. there is a relatively separated family $(h_i)_{i \in I}$ in $H$ and some set $Q \Subset \mathcal{O}$ such that $Q_i = h_i^{-T} Q$ holds for all $i \in I$.

*We recall that the notation $M \Subset U$ means that $U \subset \mathbb{R}^d$ is open and that $M \subset \mathbb{R}^d$ is precompact with $\overline{M} \subset U$.*

If the group $H$ is clear from the context, we simply say that $\mathcal{Q}$ is AN INDUCED COVERING OF $\mathcal{O}$. In the following, if we write "let $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i \in I}$ be an induced covering of $\mathcal{O}$", this always implicitly entails $Q \Subset \mathcal{O}$ and that $(h_i)_{i \in I}$ is relatively separated in $H$.

We now show that each induced covering is a semi-structured admissible covering of $\mathcal{O}$ in the sense of Definition 3.2.8. We will see in the next theorem below that one can enlarge the set $Q$ to obtain a *structured* admissible covering.

**Lemma 4.4.4.** *Let $\mathcal{Q} = (Q_i)_{i \in I} = \left(h_i^{-T} Q\right)_{i \in I}$ be an induced covering of $\mathcal{O}$. Then there is some compact unit neighborhood $U \subset H$ such that $(h_i)_{i \in I}$ is $U$-dense. Furthermore, $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i \in I}$ is a semi-structured admissible covering of $\mathcal{O}$.*

*Proof.* Fix $\xi_0 \in \mathcal{O}$ and choose the compact set $L = L\left(\overline{Q}, \{\xi_0\}\right) \subset H$ as in Lemma 4.1.4. For arbitrary $h \in H$, we have

$$h^{-T} \xi_0 \in \mathcal{O} = \bigcup_{i \in I} h_i^{-T} Q$$

and thus $h_i^{-T}\overline{Q} \cap h^{-T}\{\xi_0\} \neq \varnothing$ for some $i \in I$. By choice of $L$, this yields $h \in h_i L$. Since $h \in H$ was arbitrary, we conclude

$$H \subset \bigcup_{i\in I} h_i L \subset \bigcup_{i\in I} h_i U,$$

where $U \subset H$ is a compact unit neighborhood with $U \supset L$. Since $L$ is compact, we can choose such a set $U$. Thus, $(h_i)_{i\in I}$ is $U$-dense in $H$.

Let us now verify the conditions given in Definition 3.2.8. Since $\mathcal{Q}$ covers $\mathcal{O}$ by assumption, Corollary 4.4.2 shows that $\mathcal{Q}$ is an *admissible covering* of $\mathcal{O}$.

Regarding the second part of Definition 3.2.8, let $Q_i' := Q$, $T_i := h_i^{-T}$ and $b_i := 0$ for $i \in I$. This yields $Q_i = T_i Q_i' + b_i$ as well as

$$R_{\mathcal{Q}} = \sup_{i\in I} \sup_{\xi\in Q_i'} |\xi| = \sup_{\xi\in Q} |\xi| \leq \sup_{\xi\in\overline{Q}} |\xi| < \infty,$$

since $\overline{Q} \subset \mathcal{O}$ is compact. Furthermore, Lemma 4.4.1 implies

$$C_{\mathcal{Q}} = \sup_{i\in I} \sup_{j\in i^*} \left\| T_i^{-1} T_j \right\| = \sup_{i\in I} \sup_{j\in I_{h_i}} \|h_i^T h_j^{-T}\| \leq C' < \infty$$

with an absolute constant $C' = C'(Q) > 0$. □

Depending on the nature of the set $Q$, it is of course possible that there is no partitition of unity subordinate to a given induced covering $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$. For brevity, it makes sense to give a special name to coverings for which such a partition of unity exists.

**Definition 4.4.5.** Let $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$ be an induced covering of $\mathcal{O}$ and let $p \in (0,\infty]$. We say that $\mathcal{Q}$ is an INDUCED $L^p$-DECOMPOSITION COVERING OF $\mathcal{O}$, if $\mathcal{Q}$ is an $L^p$-decomposition covering of $\mathcal{O}$. If this is true for every $p \in (0,\infty]$, then $\mathcal{Q}$ is called an INDUCED DECOMPOSITION COVERING OF $\mathcal{O}$.

*We remark that if $\mathcal{Q}$ is an induced decomposition covering of $\mathcal{O}$, it might be necessary to choose different $L^p$-BAPUs $(\varphi_i)_{i\in I}$ for different values of $p \in (0,\infty]$.*

Finally, we show that there are induced coverings which are *structured* admissible coverings of $\mathcal{O}$. In fact, one can enlarge any induced covering to obtain a structured admissible covering.

**Theorem 4.4.6.** *Let $(h_i)_{i\in I}$ be a family in $H$. Then the following are equivalent:*

A. *The family $(h_i)_{i\in I}$ is well-spread in $H$,*

B. *There is $Q \Subset \mathcal{O}$ with nonempty interior such that $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$ is an admissible covering of $\mathcal{O}$.*

*In this case, one can choose for any compact set $Q_0 \subset \mathcal{O}$ an open set $Q \Subset \mathcal{O}$ with $Q_0 \subset Q$ and such that $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$ is a* structured *admissible covering of $\mathcal{O}$. In particular, $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$ is an induced decomposition covering of $\mathcal{O}$.*

*Proof.* The implication "(B)⇒(A)" is a consequence of Lemma 4.4.4 and Corollary 4.4.2.

For the proof of "(A)⇒(B)", let $Q_0 \subset \mathcal{O}$ be compact and assume that $(h_i)_{i\in I}$ is well-spread in $H$. In particular, $(h_i)_{i\in I}$ is $V$-dense in $H$ for some compact unit neighborhood $V \subset H$. Fix some $\xi_0 \in \mathcal{O}$. The set $K := Q_0 \cup V^{-T}\xi_0 \subset \mathcal{O}$ is compact. Since $\mathcal{O} \subset \mathbb{R}^d$ is open, there are open subsets $P \Subset Q \Subset \mathcal{O}$ with $K \subset P$.

Since $\mathcal{O}$ is a single orbit, we have

$$\begin{aligned}\mathcal{O} = H^T\xi_0 = H^{-T}\xi_0 &= \bigcup_{i\in I} (h_i V)^{-T}\,\xi_0 \\ &\subset \bigcup_{i\in I} h_i^{-T} K \subset \bigcup_{i\in I} h_i^{-T} P \subset \bigcup_{i\in I} h_i^{-T} Q.\end{aligned}$$

Thus, $\mathcal{P} = \left(h_i^{-T}P\right)_{i\in I}$ and $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ are coverings of $\mathcal{O}$. Since $(h_i)_{i\in I}$ is relatively separated in $H$, we see that $\mathcal{P}, \mathcal{Q}$ are induced coverings of $\mathcal{O}$. By Lemma 4.4.4, this implies that $\mathcal{P}, \mathcal{Q}$ are semi-structured coverings and in particular admissible coverings.

Finally, Lemma 4.4.1 implies

$$C_{\mathcal{Q}} = \sup_{i\in I} \sup_{j\in i^*} \left\|T_i^{-1}T_j\right\| = \sup_{i\in I} \sup_{j\in I_{h_i}} \left\|h_i^T h_j^{-T}\right\| \leq C' < \infty$$

with an absolute constant $C' = C'(Q) > 0$. Hence, all conditions of Definition 3.2.7 are satisfied, so that $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ is a structured admissible covering of $\mathcal{O}$.

By Theorem 3.2.17, there is a partition of unity $\Phi = (\varphi_i)_{i\in I}$ such that $\Phi$ is an $L^p$-BAPU subordinate to $\mathcal{Q}$ for every $p \in (0,\infty]$. Hence, $\mathcal{Q}$ is an induced decomposition covering of $\mathcal{O}$. □

We conclude this subsection by describing a way in which weights on the dilation group $H$ can be TRANSPLANTED to the dual orbit $\mathcal{O}$. Beginning in the next section, we will always assume that the weight $m$ is of the form $m(x,h) = m(h)$ for some weight $m : H \to (0,\infty)$. Thus, we will be able to "transplant" (a modified version of) this weight onto the dual orbit $\mathcal{O}$ to obtain a new weight $u : \mathcal{O} \to (0,\infty)$. This will be the weight for which the isomorphism between $\mathrm{Co}\left(L^{p,q}_m\right)$ and $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ will hold.

**Definition 4.4.7.** Let $m : H \to (0,\infty)$ and fix some $\xi_0 \in \mathcal{O}$. We say that a function $u : \mathcal{O} \to (0,\infty)$ is a $\xi_0$-TRANSPLANT OF $m$ FROM $H$ ONTO $\mathcal{O}$ if for each $\xi \in \mathcal{O}$, there is some $h_\xi \in H$ with $\xi = h_\xi^{-T}\xi_0$ and with $u(\xi) = m\left(h_\xi\right)$.

We say that $u$ is a TRANSPLANT OF $m$ FROM $H$ ONTO $\mathcal{O}$ if $u$ is a $\xi_0$-transplant of $m$ from $H$ onto $\mathcal{O}$ for some $\xi_0 \in \mathcal{O}$.

*We remark that we do* not *assume $u : \mathcal{O} \to (0,\infty)$ to be measurable. Since we will use $u$ only as a weight for $\ell^q_u(I)$, only a discretized version of $u$ will be used, so that measurability is irrelevant.*

An immediate question is whether two transplants of the same weight $m : H \to (0,\infty)$ are equivalent. The next lemma shows that this is the case if $m$ is a right-moderate weight. Furthermore, it turns

out that transplanted weights are $\mathcal{Q}$-moderate with respect to *any* induced covering $\mathcal{Q}$.

**Lemma 4.4.8.** *Let $m : H \to (0,\infty)$ be right-moderate with respect to a measurable, locally bounded weight $m_0 : H \to (0,\infty)$.*

*If $u : \mathcal{O} \to (0,\infty)$ is a transplant of $m$ from $H$ onto $\mathcal{O}$, and if $Q \subset \mathcal{O}$ is compact, then there is a constant $C = C(u,Q) > 0$ with*

$$C^{-1} \cdot m(h) \leq u(\xi) \leq C \cdot m(h)$$

*for all $\xi \in \mathcal{O}$ and $h \in H$ with $\xi \in h^{-T}Q$.*

*In particular, $u$ is $\mathcal{Q}$-moderate for any induced covering $\mathcal{Q}$ of $\mathcal{O}$, and any two transplants $u, v : \mathcal{O} \to (0,\infty)$ of $m$ are equivalent.*

*We remark that the lemma applies in particular if $m_0$ is of moderate growth.*

*Proof.* Fix some $\xi_0 \in \mathcal{O}$ such that $u$ is a $\xi_0$-transplant of $m$ from $H$ onto $\mathcal{O}$. Choose $L := L(\{\xi_0\}, Q)$ as in Lemma 4.1.4. Note that $L \subset H$ is compact. By Corollary 2.2.23, there is a constant $C = C(L, m_0) > 0$ with

$$\frac{1}{C} \cdot m(hk) \leq m(h) \leq C \cdot m(hk)$$

for all $h \in H$ and $k \in L$.

Let $h \in H$ and $\xi \in \mathcal{O}$ with $\xi \in h^{-T}Q$. By definition of a $\xi_0$-transplant, there is some $h_\xi \in H$ with $u(\xi) = m(h_\xi)$ and with $h_\xi^{-T}\xi_0 = \xi \in h^{-T}Q$. In particular,

$$h_\xi^{-T}\{\xi_0\} \cap h^{-T}Q \neq \varnothing.$$

By Lemma 4.1.4, this implies $h \in h_\xi L$, i.e., $h = h_\xi k$ for some $k \in L$. As seen above, we get

$$\frac{1}{C} \cdot m(h) = \frac{1}{C} \cdot m(h_\xi k) \leq m(h_\xi) \leq C \cdot m(h_\xi k) = C \cdot m(h).$$

Because of $m(h_\xi) = u(\xi)$, this completes the first part of the proof.

If $\mathcal{Q}$ is an induced covering, then $\mathcal{Q} = (h_i^{-T}Q)_{i\in I}$ with some relatively separated family $(h_i)_{i\in I}$ in $H$ and some set $Q \Subset \mathcal{O}$. For $\xi, \eta \in h_i^{-T}Q$ for some $i \in I$, the estimate above implies

$$u(\xi) \leq C \cdot m(h_i) \leq C^2 \cdot u(\eta)$$

with some constant $C = C(u,Q) > 0$ independent of $i \in I$. By symmetry, this shows that $u$ is $\mathcal{Q}$-moderate.

Finally, assume that $u, v$ are two transplants of $m$ onto $\mathcal{O}$. Fix some $\eta \in \mathcal{O}$ and let $Q := \{\eta\}$. Choose the constants $C_1 = C_1(u,Q)$ and $C_2 = C_2(v,Q)$ as above. For arbitrary $\xi \in \mathcal{O} = H^T\eta$, choose some $h \in H$ with $\xi = h^{-T}\eta \in h^{-T}Q$. By choice of $C_1, C_2$, this implies

$$C_1^{-1}C_2^{-1} \cdot v(\xi) \leq C_1^{-1} \cdot m(h) \leq u(\xi) \leq C_1 \cdot m(h) \leq C_1C_2 \cdot v(\xi).$$

Since $C_1, C_2$ are independent of $\xi \in \mathcal{O}$, this shows that $u, v$ are equivalent. □

In the following sections, the ensuing lemma will be useful, because it will allow us to switch to a "larger" induced covering, without changing the associated decomposition spaces:

**Lemma 4.4.9.** *Let $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i \in I}$ be an induced $L^p$-decomposition covering of $\mathcal{O}$ for some $p \in (0, \infty]$ and let $Q_0 \Subset \mathcal{O}$ be arbitrary with $Q \subset Q_0$.*

*Then $\mathcal{P} = \left(h_i^{-T} Q_0\right)_{i \in I}$ is also an induced $L^p$-decomposition covering of $\mathcal{O}$. If $u : \mathcal{O} \to (0, \infty)$ is $\mathcal{P}$-moderate, then it is also $\mathcal{Q}$-moderate and we have*

$$\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) = \mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$$

*with equivalent (quasi)-norms.*

*We remark that the moderateness assumption on $u$ will always be satisfied if $u$ is a transplant of a right-moderate weight $m : H \to (0, \infty)$, cf. Lemma 4.4.8, at least if the "moderating weight" is locally bounded.*

*Proof.* Since $\mathcal{Q}$ covers $\mathcal{O}$ and because of $Q_0 \supset Q$, we see that $\mathcal{P}$ also covers $\mathcal{O}$. Hence, $\mathcal{P}$ is an induced covering of $\mathcal{O}$. By Lemma 4.4.4, this implies that $\mathcal{P}$ is a semi-structured covering of $\mathcal{O}$.

Since $\mathcal{Q}$ is an $L^p$-decomposition covering of $\mathcal{O}$, there is an $L^p$-BAPU $(\varphi_i)_{i \in I}$ for $\mathcal{Q}$. Clearly, $(\varphi_i)_{i \in I}$ is also subordinate to $\mathcal{P}$. Since $\mathcal{P}, \mathcal{Q}$ use the same family $\left(h_i^{-T}\right)_{i \in I}$ of normalizations, we see that $(\varphi_i)_{i \in I}$ is also an $L^p$-BAPU for $\mathcal{P}$, cf. Definition 3.2.10.

Since decomposition spaces are well-defined, i.e. independent of the choice of a specific $L^p$-BAPU (cf. Corollary 3.4.11), we can use the same $L^p$-BAPU $(\varphi_i)_{i \in I}$ to calculate the quasi-norms for $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ as well as $\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$.

Furthermore, since $u$ is $\mathcal{P}$-moderate, it is clearly also $\mathcal{Q}$-moderate. If we fix $\xi \in Q \subset Q_0$, then $u_d(i) := u\left(h_i^{-T} \xi\right)$ yields a $\mathcal{Q}$-discretization $u_d$ of $u$ which is also a $\mathcal{P}$-discretization. Since the space $\ell^q_u(I)$ is independent of the chosen discretization (with equivalent quasi-norms for different choices) by Lemma 3.2.5, we see that

$$f \mapsto \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I} \right\|_{\ell^q_{u_d}(I)}$$

is a valid choice for the quasi-norm on the space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ as well as on $\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$, which completes the proof. □

### 4.4.2 *A special BAPU*

As noted in the introduction to this section, we want to show that the family $(\varphi_{V_i})_{i \in I}$, with

$$\varphi_U : \mathbb{R}^d \to \mathbb{R}_+, \xi \mapsto \int_U \left|\widehat{\psi}\left(h^T \xi\right)\right|^2 \mathrm{d}h$$

for each measurable subset $U \subset H$, yields (a multiple of) a BAPU.

The following lemma is essential for proving that the family $(\varphi_{V_i})_{i\in I}$ indeed yields a partition of unity on $\mathcal{O}$ if $(V_i)_{i\in I}$ forms a partition of the dilation group $H$. The proof is very similar to that of [36, Lemma 9] and is only provided to make the presentation more self-contained.

**Lemma 4.4.10.** *(cf. [41, Lemma 31]) Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$ and $\psi \neq 0$. By Lemma 4.3.4, this implies that the constant $C_\psi > 0$ from equation (2.4.2) is well-defined.*

*With this choice, we have*

$$\frac{1}{C_\psi} \cdot \int_H \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}h = 1 \quad \forall \xi \in \mathcal{O}. \tag{4.4.1}$$

*Proof.* Let $K := \operatorname{supp}\widehat{\psi} \subset \mathcal{O}$. For arbitrary $\xi \in \mathcal{O}$, the map

$$F_\xi : H \to [0,\infty), h \mapsto \left|\widehat{\psi}(h^T\xi)\right|^2$$

is continuous with $\operatorname{supp} F_\xi \subset [p_\xi^{-1}(K)]^{-1}$, where the orbit map

$$p_\xi : H \to \mathcal{O}, h \mapsto h^{-T}\xi$$

is defined as in Corollary 4.1.3. That corollary shows that $p_\xi$ is proper, so that $[p_\xi^{-1}(K)]^{-1} \subset H$ is compact.

We conclude $F_\xi \in C_c(H) \subset L^1(H)$, so that the integral in equation (4.4.1) is well-defined for every $\xi \in \mathcal{O}$.

Now, let $\xi, \eta \in \mathcal{O}$ be arbitrary. Since $\mathcal{O} = H^T\xi_0$ is a single orbit, there is some $h_0 \in H$ with $\eta = h_0^T\xi$. By left invariance of the Haar measure on $H$, we conclude

$$\int_H \left|\widehat{\psi}(h^T\eta)\right|^2 \mathrm{d}h = \int_H \left|\widehat{\psi}((h_0h)^T\xi)\right|^2 \mathrm{d}h = \int_H \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}h, \tag{4.4.2}$$

so that the integral in question is independent of the coice of $\xi \in \mathcal{O}$.

Recall from equation (4.2.9) and Lemma 4.2.10 that for every $h \in H$,

$$\left(W_\psi\psi\right)(\cdot,h) = |\det(h)|^{1/2} \cdot \mathcal{F}^{-1}\left[\widehat{\psi} \cdot \Delta_h\overline{\widehat{\psi}}\right].$$

Using Plancherel's and Fubini's theorem, this yields for $\xi_0 \in \mathcal{O}$

$$\begin{aligned}
\left\|W_\psi\psi\right\|_{L^2(G)}^2 &= \int_H \int_{\mathbb{R}^d} \left|\left(W_\psi\psi\right)(x,h)\right|^2 \mathrm{d}x \frac{\mathrm{d}h}{|\det(h)|} \\
&= \int_H \left\|\left(W_\psi\psi\right)(\cdot,h)\right\|_{L^2(\mathbb{R}^d)}^2 \frac{\mathrm{d}h}{|\det(h)|} \\
&= \int_H \left\|\widehat{\psi} \cdot \Delta_h\overline{\widehat{\psi}}\right\|_{L^2(\mathbb{R}^d)}^2 \mathrm{d}h \\
&= \int_H \int_{\mathbb{R}^d} \left|\widehat{\psi}(\xi)\right|^2 \cdot \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}\xi \,\mathrm{d}h \\
&\overset{(\dagger)}{=} \|\psi\|_{L^2(\mathbb{R}^d)}^2 \cdot \int_H \left|\widehat{\psi}(h^T\xi_0)\right|^2 \mathrm{d}h.
\end{aligned}$$

Here, equation (4.4.2) and $\|\psi\|^2_{L^2(\mathbb{R}^d)} = \|\widehat{\psi}\|^2_{L^2(\mathbb{R}^d)}$ were used at (†).

Recalling the definition $C_\psi = \|W_\psi \psi\|^2_{L^2(G)} / \|\psi\|^2_{L^2(\mathbb{R}^d)}$ from equation (2.4.2) finishes the proof. □

Our next aim is to show that $\varphi_U$ indeed defines a test function $\varphi_U \in C_c^\infty(\mathcal{O})$ and to derive a bound for the support of $\varphi_U$.

**Lemma 4.4.11.** *(cf. [41, Lemma 27]) Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ and assume that $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Let $U \subset H$ be precompact and measurable. Then*

$$\varphi_U : \mathbb{R}^d \to [0,\infty), \xi \mapsto \int_U \left|\widehat{\psi}(h^T \xi)\right|^2 \mathrm{d}h$$

*is a well-defined function with $\varphi_U \in C_c^\infty(\mathcal{O})$.*

*Furthermore, $\varphi_U$ vanishes on $\mathbb{R}^d \setminus \left(U^{-T} \cdot \widehat{\psi}^{-1}(\mathbb{C}^*)\right)$ and thus*

$$\operatorname{supp} \varphi_U \subset \overline{U}^{-T} \cdot \operatorname{supp} \widehat{\psi} \subset H^T \mathcal{O} = \mathcal{O}.$$

*Proof.* Let $\varphi := |\widehat{\psi}|^2 \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathbb{R}^d)$. Lemma 4.2.9 shows that

$$\Phi : \mathrm{GL}(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d), h \mapsto \varphi(h^T \cdot) = \left|\widehat{\psi}(h^T \cdot)\right|^2$$

is well-defined and continuous, so that $\Phi$ is in particular continuous on the compact set $\overline{U} \subset H$.

For $k \in \mathbb{N}_0$ the inclusion $\iota_k : C_c^\infty(\mathbb{R}^d) \hookrightarrow C_b^k(\mathbb{R}^d)$ into the Banach space

$$C_b^k(\mathbb{R}^d) := \left\{ f \in C^k(\mathbb{R}^d) \;\middle|\; \|f\|_{C_b^k} := \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{\sup} < \infty \right\}$$

is continuous. The function $\iota_k \circ \Phi : \overline{U} \to C_b^k(\mathbb{R}^d)$ is continuous and hence measurable. Furthermore, the image $\iota_k \circ \Phi(\overline{U}) \subset C_b^k(\mathbb{R}^d)$ is a compact subset of a normed vector space and thus separable. By [54, Chapter VI, §1, M11], this shows that $\iota_k \circ \Phi : \overline{U} \to C_b^k(\mathbb{R}^d)$ is $\mu$-measurable, where $\mu$ is the Haar measure on $H$. Since $\iota_k \circ \Phi$ is bounded on $\overline{U}$ with $\mu(\overline{U}) < \infty$ and since $U$ is measurable, [54, Chapter VI, Corollary 5.9] finally implies that the Bochner integral

$$\psi_k := \int_U (\iota_k \circ \Phi)(h) \, \mathrm{d}h \in C_b^k(\mathbb{R}^d)$$

is well-defined.

Since the evaluation mapping $\alpha_\xi : C_b^k(\mathbb{R}^d) \to \mathbb{C}, f \mapsto f(\xi)$ is continuous for every $\xi \in \mathbb{R}^d$, we easily see $\psi_k(\xi) = \varphi_U(\xi)$ for each $\xi \in \mathbb{R}^d$, so that we conclude $\varphi_U = \psi_k \in C_b^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$. Hence, $\varphi_U \in C^\infty(\mathbb{R}^d)$ is smooth.

Finally, if we have $0 \neq \varphi_U(\xi) = \int_U \left|\widehat{\psi}\left(h^T\xi\right)\right|^2 \mathrm{d}h$ for some $\xi \in \mathbb{R}^d$, then there is some $h \in U$ with $h^T\xi \in \widehat{\psi}^{-1}(\mathbb{C}^*)$, i.e.

$$\xi \in h^{-T} \cdot \widehat{\psi}^{-1}(\mathbb{C}^*) \subset U^{-T} \cdot \widehat{\psi}^{-1}(\mathbb{C}^*). \qquad \square$$

In the next lemma, we give an explicit formula for the (inverse) Fourier transform of $\varphi_U$. This has two applications: First, we will use this formula to show that $(\varphi_{V_i})_{i\in I}$ yields an $L^p$-BAPU (cf. Definitions 3.2.6 and 3.2.10) for all $0 < p \leq \infty$. This is a slight improvement on [41, Lemma 28], where only the case $1 \leq p \leq \infty$ was considered.

Second, this formula will play a crucial role in the next section for establishing a connection between the frequency-localized "pieces" $\mathcal{F}^{-1}\left(\varphi_U \widehat{f}\right)$ and the wavelet transform $W_\psi f$, cf. Lemma 4.5.6.

**Lemma 4.4.12.** *(cf. [41, Lemma 28]) Let $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Set $\gamma := \mathcal{F}^{-1}\left|\widehat{\psi}\right|^2 \in \mathcal{S}\left(\mathbb{R}^d\right)$ and let $U \subset H$ be an arbitrary precompact, measurable set.*

*Then the inverse Fourier transform of $\varphi_U$ (defined as in Lemma 4.4.11) is given by*

$$\left(\mathcal{F}^{-1}\varphi_U\right)(x) = \int_U \frac{\gamma\left(h^{-1}x\right)}{|\det(h)|}\,\mathrm{d}h \qquad \text{for all } x \in \mathbb{R}^d. \tag{4.4.3}$$

*Finally, for each $p \in (0,\infty)$ there is a constant $C = C(U,\psi,d,p) > 0$ such that*

$$\left\|\mathcal{F}^{-1}\varphi_V\right\|_{L^p\left(\mathbb{R}^d\right)} \leq C \cdot \left|\det\left(h_0^{-T}\right)\right|^{1-\frac{1}{p}}$$

*holds for all measurable sets $V \subset H$ and all $h_0 \in H$ with $V \subset h_0 U$.*

*Proof.* Let $\varphi := \left|\widehat{\psi}\right|^2 \in C_c^\infty(\mathcal{O})$. Lemma 4.2.9 (and the ensuing remark) shows that

$$\begin{aligned}\Phi : \mathrm{GL}\left(\mathbb{R}^d\right) &\to C_c^\infty\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{S}\left(\mathbb{R}^d\right) \hookrightarrow L^1\left(\mathbb{R}^d\right),\\ h &\mapsto \left(\xi \mapsto \varphi\left(h^T\xi\right) = \left|\widehat{\psi}\left(h^T\xi\right)\right|^2\right)\end{aligned}$$

is well-defined and continuous. Hence

$$\begin{aligned}\int_U \int_{\mathbb{R}^d} \left|\left|\widehat{\psi}\left(h^T\xi\right)\right|^2 \cdot e^{2\pi i\langle x,\xi\rangle}\right| \mathrm{d}\xi\,\mathrm{d}h &= \int_U \left\|\Phi(h)\right\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}h\\ &\leq \mu_H\left(\overline{U}\right) \cdot \sup_{h\in\overline{U}} \left\|\Phi(h)\right\|_{L^1(\mathbb{R}^d)} < \infty,\end{aligned}$$

where $\mu_H$ denotes the Haar measure on $H$. Here, we used that $\overline{U}$ is compact, which implies that the right-hand side of the estimate is indeed finite.

Since $\varphi_U \in C_c^\infty(\mathcal{O}) \subset L^1\left(\mathbb{R}^d\right)$, the inverse Fourier transform $\mathcal{F}^{-1}\varphi_U$ can be computed by the usual integral formula. Fubini's theorem,

the change of variables formula (with $\varrho = h^T\xi$) and Fourier inversion (which is applicable since $\gamma \in \mathcal{S}(\mathbb{R}^d)$) now yield

$$\begin{aligned}
&\left[\mathcal{F}^{-1}\varphi_U\right](x) \\
&= \int_{\mathbb{R}^d}\int_U \left|\widehat{\psi}(h^T\xi)\right|^2 \,\mathrm{d}h\, e^{2\pi i\langle x,\xi\rangle}\,\mathrm{d}\xi \\
&= \int_U \frac{1}{|\det(h)|}\cdot\int_{\mathbb{R}^d}\left|\widehat{\psi}(h^T\xi)\right|^2\cdot e^{2\pi i\left\langle h^{-1}x,h^T\xi\right\rangle}\cdot|\det(h^T)|\,\mathrm{d}\xi\,\mathrm{d}h \\
&= \int_U \frac{1}{|\det(h)|}\cdot\int_{\mathbb{R}^d}\widehat{\gamma}(\varrho)\cdot e^{2\pi i\left\langle h^{-1}x,\varrho\right\rangle}\,\mathrm{d}\varrho\,\mathrm{d}h \\
&= \int_U \frac{\gamma\left(h^{-1}x\right)}{|\det(h)|}\,\mathrm{d}h.
\end{aligned}$$

It remains to prove the estimate for the $L^p$-norm of $\mathcal{F}^{-1}\varphi_V$. To this end, choose $N = N(d,p) \in \mathbb{N}$ with $N > \frac{d}{p}$, define $C_1 := \max_{h\in\overline{U}}\|h\|$ and

$$C_2 := \sup_{x\in\mathbb{R}^d}(1+|x|)^N\cdot|\gamma(x)|,$$

as well as

$$C_3 := \int_U |\det(g^{-1})|\,\mathrm{d}g$$

and finally

$$C_4 := \left[\int_{\mathbb{R}^d}\left(1+C_1^{-1}\cdot|z|\right)^{-Np}\,\mathrm{d}z\right]^{1/p}.$$

Note that $C_3$ is finite, since $g \mapsto \left|\det\left(g^{-1}\right)\right|$ is continuous and $\overline{U}$ is compact. Finiteness of $C_4$ follows because of $-Np < -d$ and $C_2$ is finite because of $\gamma \in \mathcal{S}(\mathbb{R}^d)$.

Choose $h_0 \in H$ and let $V \subset h_0U$ be measurable. For $h \in V \subset h_0U$, we have $h_0^{-1}h \in U$ and thus also $\|h_0^{-1}h\| \leq C_1$. This implies for arbitrary $y \in \mathbb{R}^d$ the estimate

$$\begin{aligned}
|h_0^{-1}y| &= |h_0^{-1}hh^{-1}y| \\
&\leq \|h_0^{-1}h\|\cdot|h^{-1}y| \\
&\leq C_1\cdot|h^{-1}y|
\end{aligned}$$

and hence $|h^{-1}y| \geq C_1^{-1}\cdot|h_0^{-1}y|$.

We conclude

$$(1+|h^{-1}y|)^{-N} \leq \left(1+C_1^{-1}|h_0^{-1}y|\right)^{-N},$$

which finally yields

$$\begin{aligned}
|\gamma(h^{-1}y)| &= (1+|h^{-1}y|)^{-N}\cdot(1+|h^{-1}y|)^N|\gamma(h^{-1}y)| \\
&\leq C_2\cdot\left(1+C_1^{-1}|h_0^{-1}y|\right)^{-N}.
\end{aligned}$$

Since $U$ is precompact, so is $V$. Hence, we can apply the formula for $\mathcal{F}^{-1}\varphi_V$ obtained above. Hence,

$$\begin{aligned}
\left|\left[\mathcal{F}^{-1}\varphi_V\right](y)\right| &= \left|\int_V \frac{\gamma(h^{-1}y)}{|\det(h)|}\,\mathrm{d}h\right| \\
&\leq \int_V |\det(h^{-1})| \cdot |\gamma(h^{-1}y)|\,\mathrm{d}h \\
&\leq C_2 \cdot \left(1 + C_1^{-1}\,|h_0^{-1}y|\right)^{-N} \cdot \int_{h_0 U} |\det(h^{-1})|\,\mathrm{d}h \\
&= C_2 \cdot \left(1 + C_1^{-1}\,|h_0^{-1}y|\right)^{-N} \cdot \int_U |\det((h_0 g)^{-1})|\,\mathrm{d}g \\
&= C_2 C_3 \cdot \left(1 + C_1^{-1}\left|h_0^{-1}y\right|\right)^{-N} \cdot |\det(h_0)|^{-1}.
\end{aligned}$$

All in all, we arrive at

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\varphi_V\right\|_{L^p(\mathbb{R}^d)}^p \\
&= \int_{\mathbb{R}^d} \left|\left[\mathcal{F}^{-1}\varphi_V\right](y)\right|^p \mathrm{d}y \\
&\leq C_2^p C_3^p \cdot |\det(h_0)|^{1-p} \cdot \int_{\mathbb{R}^d} \left(1 + C_1^{-1} \cdot |h_0^{-1}y|\right)^{-Np} |\det(h_0^{-1})|\,\mathrm{d}y \\
&= C_2^p C_3^p \cdot |\det(h_0)|^{1-p} \cdot \int_{\mathbb{R}^d} \left(1 + C_1^{-1} \cdot |z|\right)^{-Np} \mathrm{d}z \\
&= \left(C_2 C_3 C_4 \cdot |\det(h_0^{-T})|^{1-\frac{1}{p}}\right)^p.
\end{aligned}$$

This completes the proof. □

We are finally in a position to show how to construct the special BAPU $(\varphi_{V_i})_{i\in I}$.

**Theorem 4.4.13.** *(cf. [41, Theorem 32]) Let $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ be a covering of $\mathcal{O}$ induced by $H$ and let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$ and $\psi \neq 0$. Then*

1. *$I$ is countably infinite and there is a compact unit neighborhood $U$ in $H$ with $H = \bigcup_{i\in I} h_i U$.*
2. *There are measurable sets $V_i \subset h_i U$ with $H = \biguplus_{i\in I} V_i$.*
3. *The set $Q_0 := Q \cup U^{-T}\left[\widehat{\psi}^{-1}(\mathbb{C}^*)\right]$ satisfies $Q_0 \Subset \mathcal{O}$, the family $\mathcal{P} = \left(h_i^{-T}Q_0\right)_{i\in I}$ is a semi-structured covering of $\mathcal{O}$ induced by $H$ and the family $\left(\frac{1}{C_\psi}\varphi_{V_i}\right)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{P}$ for every $p \in (0,\infty]$.*

*Proof.* 1. By definition of an induced covering, the family $(h_i)_{i\in I}$ is relatively separated. By Lemma 2.3.10, this implies that $I$ is countable. Furthermore, Theorem 4.4.6 shows that $(h_i)_{i\in I}$ is well-spread in $H$, which implies existence of a compact unit neighborhood $U \subset H$ with $H = \bigcup_{i\in I} h_i U$. If $I$ was finite, this would imply that $H$ is compact, so that $\mathcal{O} = H^T\xi_0$ would also be compact. But $\mathcal{O} \subset \mathbb{R}^d$ is open. By connectedness of $\mathbb{R}^d$, this would yield $\mathcal{O} \in \{\varnothing, \mathbb{R}^d\}$ and hence

$\mathcal{O} = \varnothing$ by compactness. This contradiction shows that $I$ is countably infinite.

2. Since $I$ is countably infinite, there is an enumeration $(i_n)_{n\in\mathbb{N}}$ of $I$. Now, define

$$V_{i_n} := h_{i_n} U \setminus \bigcup_{m=1}^{n-1} h_{i_m} U \quad \text{for } n \in \mathbb{N}.$$

Since all $h_{i_n} U$ are compact and hence measurable, $V_{i_n}$ is measurable for each $n \in \mathbb{N}$. Now, for arbitrary $h \in H$, there is a minimal $n \in \mathbb{N}$ with $h \in h_{i_n} U$, which yields $h \in V_{i_n}$ and hence $H = \bigcup_{i\in I} V_i$.

Finally, for $n < m$, we have $V_{i_n} \subset h_{i_n} U$ and

$$V_{i_m} \subset H \setminus \bigcup_{j=1}^{m-1} h_{i_j} U \subset H \setminus h_{i_n} U,$$

so that the family $(V_i)_{i\in I}$ is pairwise disjoint.

3. We have $\widehat{\psi}^{-1}(\mathbb{C}^*) \subset \operatorname{supp} \widehat{\psi} =: K$, where $K \subset \mathcal{O}$ is a compact set. Since $U \subset H$ is also compact, we see that $U^{-T}K \subset \mathcal{O}$ is compact, whence $U^{-T}\left[\widehat{\psi}^{-1}(\mathbb{C}^*)\right] \subset U^{-T}K \subset \mathcal{O}$ is compactly contained in $\mathcal{O}$. Thus, $Q_0 \Subset \mathcal{O}$. Because of $Q_0 \supset Q$, we see that $\mathcal{P}$ covers $\mathcal{O}$. By definition, this means that $\mathcal{P}$ is an induced covering of $\mathcal{O}$, since $(h_i)_{i\in I}$ is relatively separated. By Lemma 4.4.4, this implies that $\mathcal{P}$ is a semi-structured covering of $\mathcal{O}$.

Finally, since $V_i \subset h_i U$ is precompact and measurable for each $i \in I$, Lemma 4.4.11 yields $\varphi_{V_i} \in C_c^\infty(\mathcal{O})$, where $\varphi_{V_i}$ vanishes on the set $\mathbb{R}^d \setminus \left(V_i^{-T} \cdot \widehat{\psi}^{-1}(\mathbb{C}^*)\right)$. Hence, $\varphi_{V_i} \equiv 0$ outside of

$$V_i^{-T}\widehat{\psi}^{-1}(\mathbb{C}^*) \subset h_i^{-T} U^{-T}\left[\widehat{\psi}^{-1}(\mathbb{C}^*)\right] \subset h_i^{-T} Q_0.$$

By definition of $\varphi_{V_i}$ and by Lemma 4.4.10, we get

$$\begin{aligned}\sum_{i\in I} \varphi_{V_i}(\xi) &= \sum_{i\in I} \int_{V_i} \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}h \\ &= \int_H \left(\sum_{i\in I} \chi_{V_i}(h)\right) \cdot \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}h \\ &= \int_H \left|\widehat{\psi}(h^T\xi)\right|^2 \mathrm{d}h = C_\psi\end{aligned}$$

for all $\xi \in \mathcal{O}$. Here, we used that $H = \biguplus_{i\in I} V_i$. The interchange of integration and summation is justified by the monotone convergence theorem, since $I$ is countable. Together, we have shown that $\left(\frac{1}{C_\psi}\varphi_{V_i}\right)_{i\in I}$ is indeed a partition of unity on $\mathcal{O}$, subordinate to $\mathcal{P}$.

Finally, Lemma 4.4.12 yields for each $p \in (0,1]$ a suitable constant $C = C(U, \psi, d, p) > 0$ with

$$\left\|\mathcal{F}^{-1}\varphi_{V_i}\right\|_{L^p} \leq C \cdot \left|\det\left(h_i^{-T}\right)\right|^{1-\frac{1}{p}}$$

for all $i \in I$, since $V_i \subset h_i U$. For $p \in (0,1)$, this implies that $\left(\frac{1}{C_\psi}\varphi_{V_i}\right)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{P} = \left(h_i^{-T} Q_0\right)_{i\in I}$, cf. Definition 3.2.10. For $p = 1$, we see that $\sup_{i\in I}\left\|\mathcal{F}^{-1}\varphi_{V_i}\right\|_{L^1}$ is finite, so that $\left(\frac{1}{C_\psi}\varphi_{V_i}\right)_{i\in I}$ is an $L^p$-BAPU for $\mathcal{P}$ for all $p \in [1,\infty]$, cf. Definition 3.2.6. □

## 4.5 CONTINUITY OF $\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$

In this section, we use the special BAPU $\left(\frac{1}{C_\psi}\varphi_{V_i}\right)_{i\in I}$ constructed in the previous section to show that the Fourier transform $\mathcal{F}_{\mathcal{R}} : \mathcal{R} \to \mathcal{D}'(\mathcal{O})$ defined in Theorem 4.3.1 restricts to a continuous linear map

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right).$$

To this end, we first need to specify the space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ completely. The covering $\mathcal{Q}$ will be any induced $L^p$-decomposition covering as defined in Definition 4.4.5 and the exponents $p, q$ are the same as for the space $L^{p,q}_m(G)$. But we have not yet indicated how the weight $u$ has to be chosen.

In order to do this, we first have to introduce an additional assumption on the weight $m : G \to (0,\infty)$ which is used to define the space $L^{p,q}_m(G)$. Thus, for the remainder of the thesis, we will always assume the following:

**Assumption 4.5.1.** *We assume that the weight $m : G \to (0,\infty)$ only depends on the second factor, i.e. we assume that $m(x,h) = m(y,h)$ holds for all $x, y \in \mathbb{R}^d$ and $h \in H$. By abuse of notation, we will then write $m(h) = m(0,h)$ and consider $m$ as a weight on $G$ as well as a weight on $H$, depending on the context.*

*Remark* 4.5.2. We recall from Assumption 4.2.13 that we assume that the weight $m : G \to (0,\infty)$ is right-moderate with respect to some weight $m_1 : G \to (0,\infty)$ of moderate growth. Under the above assumptions, we can also assume that $m_1$ only "depends on the second factor," because if we set $m_1(h) := m_1(0,h)$ for each $h \in H$, we have

$$\begin{aligned} m(hg) = m(0,hg) &= m\left((0,h)(0,g)\right) \\ &\leq m(0,h)\, m_1(0,g) = m(h) \cdot m_1(g) \end{aligned}$$

for all $h, g \in H$. Furthermore, $m_1 : H \to (0,\infty)$ is clearly measurable and locally bounded. In what follows, we will thus always assume $m_1$ to only depend on the second factor.

Further, we remark that *any* measurable, locally bounded weight $n : H \to (0,\infty)$, considered as a weight on $G$ by $n(x,h) := n(h)$, is *always* of moderate growth.

Finally, it is worth noting that the control weight $v : G \to (0,\infty)$ can in general *not* be assumed to depend only on the second factor, since the operator norm $\left|\left|\left|R_{(x,h)}\right|\right|\right|_{W\left(L^{p,q}_m\right)\to W\left(L^{p,q}_m\right)}$ could depend also on the first factor, cf. the proof of Lemma 4.2.6. ◀

Now, we can use the construction of a "transplanted weight" (cf. Definition 4.4.7) to define the weight $u$.

**Definition 4.5.3.** Any transplant onto $\mathcal{O}$ of the weight

$$m^{(q)} : H \to (0,\infty), h \mapsto |\det(h)|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h)$$

is called a DECOMPOSITION WEIGHT for the coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$.

The following lemma provides a more concrete description of decomposition weights.

**Lemma 4.5.4.** *If $\mathcal{Q} = (Q_i)_{i\in I} = \left(h_i^{-T}Q\right)_{i\in I}$ is an induced covering of $\mathcal{O}$ for a well-spread family $(h_i)_{i\in I}$, and if $u : \mathcal{O} \to (0,\infty)$ is a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\right)$, then $u$ is $\mathcal{Q}$-moderate and satisfies*

$$u(\xi) \asymp |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h_i) \text{ for all } i \in I \text{ and } \xi \in Q_i, \tag{4.5.1}$$

*where the implied constant are independent of $\xi \in Q_i$ and of $i \in I$.*

*Proof.* Fix some $\xi_0 \in Q$. Since $m : H \to (0,\infty)$ is right moderate with respect to a locally bounded, measurable weight $m_1 : H \to (0,\infty)$, it is easy to see that the weight $m^{(q)}$ is right-moderate with respect to the measurable, locally bounded weight $h \mapsto |\det(h)|^{\frac{1}{2}-\frac{1}{q}} \cdot m_1(h)$. Hence, Lemma 4.4.8 shows that any transplant $u$ of $m^{(q)}$ onto $\mathcal{O}$ is $\mathcal{Q}$-moderate and (thus) satisfies

*We have $(uv)(gh) \leq (uv)(g)(u_1v_1)(h)$ if $u, v$ are right-moderate with respect to $u_1, v_1$, respectively. This shows that $m^{(q)}$ is indeed right-moderate with respect to the weight given in the proof.*

$$u(\xi) \asymp u\left(h_i^{-T}\xi_0\right) \asymp m^{(q)}(h_i) = |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h_i)$$

for all $i \in I$ and $\xi \in h_i^{-T}Q$, with implied constants depending only on $u$ and $Q$. □

In the following, we will repeatedly need to know how convolutions and $L^p$ (quasi)-norms are affected by dilations. For future reference, we collect these straightforward identities in the following lemma.

*We recall once more the notation $\Delta_h f = f \circ h^T$.*

**Lemma 4.5.5.** *Let $f, g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $h \in \mathrm{GL}\left(\mathbb{R}^d\right)$. Then*

$$\Delta_h(f * g) = |\det(h)| \cdot (\Delta_h f) * (\Delta_h g).$$

*For $p \in (0,\infty]$, $f \in L^p\left(\mathbb{R}^d\right)$ and $h \in \mathrm{GL}\left(\mathbb{R}^d\right)$, we have*

$$\|\Delta_h f\|_{L^p} = |\det(h)|^{-\frac{1}{p}} \cdot \|f\|_{L^p}.$$

*Proof.* For the first identity, we calculate, using the change-of-variables formula:

$$\begin{aligned}
&\left[\Delta_h \left(f * g\right)\right](x) \\
&= f * g\left(h^T x\right) = \int_{\mathbb{R}^d} f(y) \cdot g\left(h^T x - y\right) \, \mathrm{d}y \\
&= \left|\det(h)\right| \cdot \int_{\mathbb{R}^d} f\left(h^T h^{-T} y\right) \cdot g\left(h^T\left(x - h^{-T} y\right)\right) \cdot \left|\det\left(h^{-T}\right)\right| \, \mathrm{d}y \\
&= \left|\det(h)\right| \cdot \int_{\mathbb{R}^d} f\left(h^T z\right) \cdot g\left(h^T(x - z)\right) \, \mathrm{d}z \\
&= \left|\det(h)\right| \cdot \left[\left(\Delta_h f\right) * \left(\Delta_h g\right)\right](x).
\end{aligned}$$

For the second identity, let us first assume $p \in (0, \infty)$. We again use the change-of-variables formula to compute

$$\begin{aligned}
\left\|\Delta_h f\right\|_{L^p}^p &= \left|\det(h)\right|^{-1} \cdot \int_{\mathbb{R}^d} \left|f\left(h^T x\right)\right|^p \cdot \left|\det\left(h^T\right)\right| \, \mathrm{d}x \\
&= \left|\det(h)\right|^{-1} \cdot \int_{\mathbb{R}^d} |f(y)|^p \, \mathrm{d}y \\
&= \left|\det(h)\right|^{-1} \cdot \|f\|_{L^p}^p,
\end{aligned}$$

which implies the claim. For the case $p = \infty$, we note $N \subset \mathbb{R}^d$ is a nullset if and only if $h^T N \subset \mathbb{R}^d$ is. This implies $|f| \leq C$ almost everywhere if and only if $\left|f \circ h^T\right| \leq C$ almost everywhere and thus

$$\left\|\Delta_h f\right\|_{L^\infty} = \|f\|_{L^\infty} = \left|\det(h)\right|^{-\frac{1}{p}} \cdot \|f\|_{L^\infty}. \qquad \square$$

The following lemma contains the core argument for the continuity of the Fourier transform $\mathcal{F} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$. The main ingredient is the explicit formula

$$\left(\mathcal{F}^{-1} \varphi_V\right)(x) = \int_V \frac{\gamma\left(h^{-1} x\right)}{\left|\det(h)\right|} \, \mathrm{d}h \text{ with } \gamma = \mathcal{F}^{-1}\left|\widehat{\psi}\right|^2,$$

which was given in equation (4.4.3). This yields (at least formally) the identity

$$\begin{aligned}
\mathcal{F}^{-1}\left[\varphi_V \widehat{f}\right] &= \left[\mathcal{F}^{-1} \varphi_V\right] * \left[\mathcal{F}^{-1} \widehat{f}\right] \\
&= \int_V f * \Delta_{h^{-T}} \gamma \frac{\mathrm{d}h}{\left|\det(h)\right|}
\end{aligned}$$

for the frequency-localized "pieces" of $f$.

By employing equation (4.5.2) below, this establishes a link between the frequency-localized "pieces" of $f$ and the wavelet transform $W_\psi f$. Precisely this will enable us to prove the desired continuity, cf. the proof of the lemma below and the proof of Lemma 4.5.7.

**Lemma 4.5.6.** *Let $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Define $\gamma = \mathcal{F}^{-1}\left|\widehat{\psi}\right|^2$ as in Lemma 4.4.12.*

*Then $\gamma = \psi * \overline{\psi}^{\vee} \in \mathcal{S}\left(\mathbb{R}^d\right)$ and we have*

$$f * \Delta_{h^{-T}} \gamma = |\det(h)|^{-1/2} \cdot \left[\left(W_\psi f\right)(\cdot, h)\right] * \Delta_{h^{-T}} \psi \tag{4.5.2}$$

*for all $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $h \in H$.*

*Finally, let $U \subset H$ be a precompact open set and let $p \in (0, \infty]$ and $q \in (0, \infty)$ be arbitrary. Then, there is a constant $C = C(U, \psi, p, q, d) > 0$ such that*

*We recall that $M_Q f(x) = \|\chi_{xQ} \cdot f\|_{L^\infty}$ denotes the* MAXIMAL FUNCTION *of $f$, cf. Definition 2.3.1.*

$$\left\| \mathcal{F}^{-1}\left[\varphi_V \widehat{f}\right] \right\|_{L^p}^q \leq C \int_{h_0 U} \left[ |\det(h)|^{-\frac{1}{2}} \left\| \left(M_{W \times U^{-1} U}\left[W_\psi f\right]\right)(\cdot, h) \right\|_{L^p} \right]^q \mathrm{d}h$$

*holds for all $f \in \mathcal{S}\left(\mathbb{R}^d\right)$, each open unit neighborhood $W \subset \mathbb{R}^d$, each $h_0 \in H$ and each measurable set $V \subset h_0 U$.*

*Here, the function $\varphi_V$ is defined as in Lemma 4.4.11.*

*Proof.* Lemma 4.5.5 implies $(f * g)^{\vee} = f^{\vee} * g^{\vee}$ for $f, g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Using the "inverse" convolution theorem (cf. [61, Theorem 7.8]), we get

$$\begin{aligned} \mathcal{F}^{-1}[f \cdot g] &= \left(\mathcal{F}[f \cdot g]\right)^{\vee} = \left(\widehat{f} * \widehat{g}\right)^{\vee} \\ &= \widehat{f}^{\vee} * \widehat{g}^{\vee} = \left(\mathcal{F}^{-1} f\right) * \left(\mathcal{F}^{-1} g\right). \end{aligned} \tag{4.5.3}$$

*The identity $\overline{\widehat{\psi}} = \mathcal{F}^{-1}\overline{\psi} = \left(\mathcal{F}\overline{\psi}\right)^{\vee} = \mathcal{F}\overline{\psi}^{\vee}$ is a consequence of equation (4.3.1).*

Applied to $f = \widehat{\psi}$ and $g = \overline{\widehat{\psi}} = \mathcal{F}\left(\overline{\psi}^{\vee}\right)$, this yields

$$\begin{aligned} \gamma &= \mathcal{F}^{-1}\left|\widehat{\psi}\right|^2 = \mathcal{F}^{-1}\left[\widehat{\psi} \cdot \overline{\widehat{\psi}}\right] = \mathcal{F}^{-1}\left[\widehat{\psi} \cdot \mathcal{F}\left(\overline{\psi}^{\vee}\right)\right] \\ &= \left(\mathcal{F}^{-1}\widehat{\psi}\right) * \mathcal{F}^{-1}\left[\mathcal{F}\overline{\psi}^{\vee}\right] = \psi * \overline{\psi}^{\vee}. \end{aligned}$$

Using Lemma 4.5.5, the commutativity and associativity of convolution on $\mathbb{R}^d$ and the convolutional expression for $W_\psi f$ given in Lemma 4.1.5, we conclude

$$\begin{aligned} f * \Delta_{h^{-T}} \gamma &= f * \Delta_{h^{-T}}\left(\psi * \overline{\psi}^{\vee}\right) \\ &= |\det(h^{-T})| \cdot f * \left[\Delta_{h^{-T}} \psi * \Delta_{h^{-T}} \overline{\psi}^{\vee}\right] \\ &= |\det(h)|^{-1} \cdot f * \left[\Delta_{h^{-T}} \overline{\psi}^{\vee} * \Delta_{h^{-T}} \psi\right] \\ &= |\det(h)|^{-1} \cdot \left(f * \Delta_{h^{-T}} \overline{\psi}^{\vee}\right) * \Delta_{h^{-T}} \psi \\ &= |\det(h)|^{-1/2} \cdot \left(f * D_h \overline{\psi}^{\vee}\right) * \Delta_{h^{-T}} \psi \\ &= |\det(h)|^{-1/2} \cdot \left[\left(W_\psi f\right)(\cdot, h)\right] * \Delta_{h^{-T}} \psi \end{aligned}$$

for every $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. This proves equation (4.5.2).

Now, let $p, q, U, h_0, V, W$ be as in the assumptions, let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ and define $F := \mathcal{F}^{-1}\left[\varphi_V \cdot \widehat{f}\right]$. We have $\varphi_V \in C_c^\infty(\mathcal{O}) \subset \mathcal{S}\left(\mathbb{R}^d\right)$ by Lemma 4.4.11. Using equation (4.5.3), Fubini's theorem and the

explicit expression for $\mathcal{F}^{-1}\varphi_V$ obtained in Lemma 4.4.12 (equation (4.4.3)), we conclude

$$\begin{aligned} F(x) &= \left(\mathcal{F}^{-1}\varphi_V * \mathcal{F}^{-1}\widehat{f}\right)(x) = \left(f * \mathcal{F}^{-1}\varphi_V\right)(x) \\ &= \int_{\mathbb{R}^d} f(y) \cdot \mathcal{F}^{-1}\varphi_V(x-y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y) \cdot \int_V \gamma\left(h^{-1}(x-y)\right) \frac{\mathrm{d}h}{|\det(h)|}\, \mathrm{d}y \\ &= \int_V \int_{\mathbb{R}^d} f(y)\, \gamma\left(h^{-1}(x-y)\right)\, \mathrm{d}y \frac{\mathrm{d}h}{|\det(h)|} \\ &= \int_V \left(f * \Delta_{h^{-T}}\gamma\right)(x) \frac{\mathrm{d}h}{|\det(h)|} \\ &= \int_V \left(\left[\left(W_\psi f\right)(\cdot, h)\right] * \Delta_{h^{-T}}\psi\right)(x) \frac{\mathrm{d}h}{|\det(h)|^{3/2}}. \end{aligned} \tag{4.5.4}$$

Here, the last step is a consequence of equation (4.5.2). We remark that the appeal to Fubini's theorem is justified, since

$$\begin{aligned} &\int_V \int_{\mathbb{R}^d} \left|f(y)\, \gamma\left(h^{-1}(x-y)\right)\right|\, \mathrm{d}y \frac{\mathrm{d}h}{|\det(h)|} \\ \leq\ &\|\gamma\|_{\sup} \|f\|_{L^1} \int_{\overline{V}} |\det(h)|^{-1}\, \mathrm{d}h \end{aligned}$$

is finite, because $\overline{V} \subset h_0\overline{U}$ is a compact subset of $H$.

Before we estimate $F(x)$ further using equation (4.5.4), we use the compactness of $\overline{U}$ to derive estimates for the integrand of the right-hand side of equation (4.5.4) which are *independent* of $h \in V$.

To this end, let us first make a few definitions. Set

$$|\psi|_N := \sup_{x \in \mathbb{R}^d} (1+|x|)^N \cdot |\psi(x)|$$

for each $N \in \mathbb{N}$. Note that $|\psi|_N$ is finite because of $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$. Furthermore, write $\Gamma_N(w) := (1+|w|)^{-N}$ for $w \in \mathbb{R}^d$. Finally, set

$$C_U := \max\left\{1, \max_{h \in \overline{U}^{-1}\overline{U}} \|h\|\right\}$$

and note that this is a finite constant, since $\overline{U} \subset H$ is compact.

Now, let $h, g \in h_0 U$ be arbitrary, i.e. $h = h_0 u$ and $g = h_0 v$ for suitable $u, v \in U$. This implies $g = h_0 v = h u^{-1} v$ and $h = h_0 u = g v^{-1} u$. On the one hand, this yields

$$\begin{aligned} |\det(h)|^{-\frac{1}{2}-\frac{1}{p}} &= |\det(g)|^{-\frac{1}{2}-\frac{1}{p}} \cdot |\det(v^{-1}u)|^{-\frac{1}{2}-\frac{1}{p}} \\ &\leq C_{U,p} \cdot |\det(g)|^{-\frac{1}{2}-\frac{1}{p}} \end{aligned} \tag{4.5.5}$$

with $C_{U,p} := \max_{v,u\in\overline{U}} \left|\det\left(v^{-1}u\right)\right|^{-\frac{1}{2}-\frac{1}{p}}$ and

$$\begin{aligned}|\det(h)|^{-1/2} &= |\det(g)|^{-1/2}\cdot|\det(v^{-1}u)|^{-1/2}\\ &\leq C_U^{(1)}\cdot|\det(g)|^{-1/2}\end{aligned} \tag{4.5.6}$$

with $C_U^{(1)} := \max_{v,u\in\overline{U}} \left|\det\left(v^{-1}u\right)\right|^{-1/2}$.

On the other hand, we get

$$|g^{-1}w| = |(hu^{-1}v)^{-1}w| = |v^{-1}uh^{-1}w| \leq C_U\cdot|h^{-1}w|$$

for all $w\in\mathbb{R}^d$. This implies

$$(1+|h^{-1}w|) \geq \left(1+C_U^{-1}\,|g^{-1}w|\right) \geq C_U^{-1}\cdot(1+|g^{-1}w|)$$

and thus

$$\begin{aligned}|(\Delta_{h^{-T}}\psi)(x-z)|^p &= |\psi(h^{-1}(x-z))|^p\\ &\leq (1+|h^{-1}(x-z)|)^{-Np}\cdot|\psi|_N^p\\ &\leq C_U^{Np}\cdot(1+|g^{-1}(x-z)|)^{-Np}\cdot|\psi|_N^p\\ &= C_U^{Np}\,|\psi|_N^p\cdot\left[\left(\Delta_{g^{-T}}\Gamma_N\right)(x-z)\right]^p.\end{aligned} \tag{4.5.7}$$

Finally, $g^{-1}h = v^{-1}h_0^{-1}h_0u = v^{-1}u \in U^{-1}U$, which leads to

$$(z,h) = (z,g)\cdot(0,g^{-1}h) \in (z,g)\,Q$$

for the open unit neighborhood $Q := W\times U^{-1}U$. Since $W_\psi f$ is continuous and $Q$ is open, we get

$$\left|W_\psi f(z,h)\right| \leq \left(M_Q\left[W_\psi f\right]\right)(z,g) = \Lambda(z,g) \tag{4.5.8}$$

for $\Lambda := M_Q\left[W_\psi f\right]$, because the essential supremum coincides with the usual supremum.

We now continue estimate (4.5.4). To this end, we distinguish the cases $p\in[1,\infty]$ and $p\in(0,1)$. In the first case, we employ Young's inequality (i.e. the convolution relation $L^p * L^1 \hookrightarrow L^p$) and Minkowski's inequality for integrals (cf. [33, Theorem 6.19]) to conclude

$$\begin{aligned}\|F\|_{L^p} &= \left\|\int_V \left(W_\psi f\right)(\cdot,h) * \Delta_{h^{-T}}\psi\,\frac{\mathrm{d}h}{|\det(h)|^{3/2}}\right\|_{L^p}\\ &\leq \int_V \left\|\left(W_\psi f\right)(\cdot,h) * \Delta_{h^{-T}}\psi\right\|_{L^p}\frac{\mathrm{d}h}{|\det(h)|^{3/2}}\\ &\leq \int_V \left\|W_\psi f(\cdot,h)\right\|_{L^p}\cdot\left\|\Delta_{h^{-T}}\psi\right\|_{L^1}\frac{\mathrm{d}h}{|\det(h)|^{3/2}}\\ &= \|\psi\|_{L^1}\cdot\int_V \left\|W_\psi f(\cdot,h)\right\|_{L^p}\frac{\mathrm{d}h}{|\det(h)|^{1/2}}.\end{aligned} \tag{4.5.9}$$

Here, the last step made use of Lemma 4.5.5.

As seen in equation (4.5.8), we have $\left|W_\psi f\left(\cdot, h\right)\right| \leq \Lambda\left(\cdot, g\right)$ for arbitrary $h \in V \subset h_0 U$ and $g \in h_0 U$. For the same $h, g$, equation (4.5.6) shows $\left|\det\left(h\right)\right|^{-1/2} \leq C^{(1)}_U \cdot \left|\det\left(g\right)\right|^{-1/2}$. Thus, we get

$$\begin{aligned}\|F\|^q_{L^p} &\leq \left(\|\psi\|_{L^1} \cdot \mu_H\left(V\right)\right)^q \cdot \left[C^{(1)}_U \cdot \|\Lambda\left(\cdot, g\right)\|_{L^p} \left|\det\left(g\right)\right|^{-1/2}\right]^q \\ &\leq \left(\|\psi\|_{L^1} \cdot C^{(1)}_U \cdot \mu_H\left(U\right)\right)^q \cdot \left[\|\Lambda\left(\cdot, g\right)\|_{L^p} \left|\det\left(g\right)\right|^{-1/2}\right]^q,\end{aligned}$$

because of $\mu_H\left(V\right) \leq \mu_H\left(h_0 U\right) = \mu_H\left(U\right)$. But this estimate for $\|F\|^q_{L^p}$ is true for arbitrary $g \in h_0 U$, so that we can take the average over $g \in h_0 U$. This finally implies

$$\|F\|^q_{L^p} \leq \left(C^{(1)}_U \|\psi\|_{L^1}\right)^q \cdot \left(\mu_H\left(U\right)\right)^{q-1} \int_{h_0 U} \left[\left|\det\left(g\right)\right|^{-\frac{1}{2}} \|\Lambda\left(\cdot, g\right)\|_{L^p}\right]^q \mathrm{d}g.$$

Because of $F = \mathcal{F}^{-1}\left[\varphi_V \widehat{f}\right]$ and $\Lambda = M_{W \times U^{-1} U}\left[W_\psi f\right]$, this completes the proof in the case $p \in [1, \infty]$.

In the case $p \in (0, 1]$, we first note that the Fourier transform of $\Delta_{h^{-T}} \psi$ satisfies

$$\begin{aligned}\mathcal{F}\left[\Delta_{h^{-T}} \psi\right] &= \mathcal{F}\left[\psi \circ h^{-1}\right] \\ &= \left|\det\left(h\right)\right| \cdot \left(\widehat{\psi} \circ h^T\right) \\ &= \left|\det\left(h\right)\right| \cdot \Delta_h \widehat{\psi}.\end{aligned}$$

In particular, we have $\mathrm{supp}\left(\mathcal{F}\left[\Delta_{h^{-T}} \psi\right]\right) \subset h^{-T} \,\mathrm{supp}\, \widehat{\psi}$. Likewise, Lemma 4.1.5 implies

$$\mathrm{supp}\left(\mathcal{F}\left[W_\psi f\left(\cdot, h\right)\right]\right) \subset h^{-T} \,\mathrm{supp}\, \widehat{\psi}.$$

Let us now set $K := \left[\mathrm{supp}\, \widehat{\psi}\right] - \left[\mathrm{supp}\, \widehat{\psi}\right]$ and observe

$$\begin{aligned}\lambda\left(\left[h^{-T} \,\mathrm{supp}\, \widehat{\psi}\right] - \left[h^{-T} \,\mathrm{supp}\, \widehat{\psi}\right]\right) &= \lambda\left(h^{-T} K\right) \\ &= \left|\det\left(h^{-T}\right)\right| \cdot \lambda\left(K\right) \\ &= \left|\det\left(h\right)\right|^{-1} \cdot \lambda\left(K\right),\end{aligned}$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$. Finally, Lemma 4.1.5 shows $F_h := W_\psi f\left(\cdot, h\right) \in \mathcal{S}\left(\mathbb{R}^d\right)$. Since convolution is commutative on $L^1\left(\mathbb{R}^d\right)$, we conclude

$$\begin{aligned}\left|\left[\left(W_\psi f\right)\left(\cdot, h\right) * \Delta_{h^{-T}} \psi\right]\left(x\right)\right| &= \left|\left[\left(\widehat{\mathcal{F}^{-1} \Delta_{h^{-T}} \psi}\right) * W_\psi f\left(\cdot, h\right)\right]\left(x\right)\right| \\ &= \left|\left[\left(\widehat{\mathcal{F}^{-1} \Delta_{h^{-T}} \psi}\right) * F_h\right]\left(x\right)\right|\end{aligned}$$

for all $x \in \mathbb{R}^d$.

By Theorem 3.1.4, equation (3.1.6), the considerations in the preceding paragraph imply

$$\begin{aligned}
&\left|\left[\left(W_\psi f\right)(\cdot,h) * \Delta_{h^{-T}}\psi\right](x)\right| \\
&= \left|\left[\left(\mathcal{F}^{-1}\widehat{\Delta_{h^{-T}}\psi}\right) * F_h\right](x)\right| \\
&\leq \left[|\det(h)|^{-1}\lambda(K)\right]^{\frac{1}{p}-1}\left[\int_{\mathbb{R}^d}\left|\left(\mathcal{F}^{-1}\widehat{\Delta_{h^{-T}}\psi}\right)(y)\right|^p |F_h(x-y)|^p \,\mathrm{d}y\right]^{1/p} \\
&= \left[|\det(h)|^{-1}\lambda(K)\right]^{\frac{1}{p}-1}\left[\int_{\mathbb{R}^d}|(\Delta_{h^{-T}}\psi)(x-z)|^p |F_h(z)|^p \,\mathrm{d}z\right]^{1/p},
\end{aligned}$$

where the last step is due to the change of variables $z = x - y$. Plugging this into equation (4.5.4), we arrive at

$$\begin{aligned}
&|F(x)| \qquad (4.5.10)\\
&\leq (\lambda(K))^{\frac{1}{p}-1}\int_V |\det(h)|^{-\frac{1}{2}-\frac{1}{p}}\left[\int_{\mathbb{R}^d}|(\Delta_{h^{-T}}\psi)(x-z)|^p |F_h(z)|^p \,\mathrm{d}z\right]^{\frac{1}{p}}\mathrm{d}h.
\end{aligned}$$

By combining the estimates (4.5.7) and (4.5.8), we obtain for each $h \in V \subset h_0 U$ that

$$\begin{aligned}
&\int_{\mathbb{R}^d}|(\Delta_{h^{-T}}\psi)(x-z)|^p |F_h(z)|^p \,\mathrm{d}z \\
&\leq C_U^{Np}|\psi|_N^p \cdot \left[\int_{\mathbb{R}^d}\left[\left(\Delta_{g^{-T}}\Gamma_N\right)(x-z)\right]^p [\Lambda(z,g)]^p \,\mathrm{d}z\right]^{1/p} \\
&= C_U^{Np}|\psi|_N^p \cdot \left[\left(\left[\Delta_{g^{-T}}\Gamma_N\right]^p * [\Lambda(\cdot,g)]^p\right)(x)\right]^{1/p}.
\end{aligned}$$

Note that the right-hand side is *independent* of $h \in V$.

Thus, we set $C_{U,\psi,p,N} := C_U^{Np}|\psi|_N^p (\lambda(K))^{\frac{1}{p}-1}$ and insert the above inequality into equation (4.5.10). We conclude

$$\begin{aligned}
&|F(x)| \\
&\leq C_{U,\psi,p,N}\cdot\int_V |\det(h)|^{-\frac{1}{2}-\frac{1}{p}}\,\mathrm{d}h \cdot \left[\left(\left[\Delta_{g^{-T}}\Gamma_N\right]^p * [\Lambda(\cdot,g)]^p\right)(x)\right]^{1/p} \\
&\leq \mu_H(U)\, C_{U,\psi,p,N} C_{U,p}\cdot|\det(g)|^{-\frac{1}{2}-\frac{1}{p}}\left[\left(\left[\Delta_{g^{-T}}\Gamma_N\right]^p * [\Lambda(\cdot,g)]^p\right)(x)\right]^{\frac{1}{p}}
\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $g \in h_0 U$. In the last step, we used the estimate $\mu_H(V) \leq \mu_H(h_0 U) = \mu_H(U)$. For brevity, we introduce the constant $C^{(2)}_{U,\psi,p,N} := C_{U,\psi,p,N} C_{U,p}\cdot\mu_H(U)$.

We now take the $L^p$ (quasi)-norm in the last estimate. Using Young's inequality (i.e. $L^1 * L^1 \hookrightarrow L^1$), this yields

$$\begin{aligned}
\|G\|_{L^p} &\leq C^{(2)}_{U,\psi,p,N}\cdot|\det(g)|^{-\frac{1}{2}-\frac{1}{p}}\cdot\left\|\left(\left[\Delta_{g^{-T}}\Gamma_N\right]^p * [\Lambda(\cdot,g)]^p\right)^{1/p}\right\|_{L^p} \\
&= C^{(2)}_{U,\psi,p,N}\cdot|\det(g)|^{-\frac{1}{2}-\frac{1}{p}}\cdot\left\|\left(\left[\Delta_{g^{-T}}\Gamma_N\right]^p * [\Lambda(\cdot,g)]^p\right)(x)\right\|_{L^1}^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq C^{(2)}_{U,\psi,p,N} \cdot |\det(g)|^{-\frac{1}{2}-\frac{1}{p}} \cdot \left\| \left[\Delta_{g^{-T}} \Gamma_N\right]^p \right\|^{1/p}_{L^1} \cdot \left\| [\Lambda(\cdot, g)]^p \right\|^{1/p}_{L^1} \\
&= C^{(2)}_{U,\psi,p,N} \cdot |\det(g)|^{-\frac{1}{2}-\frac{1}{p}} \cdot \left\| \Delta_{g^{-T}} \Gamma_N \right\|_{L^p} \cdot \|\Lambda(\cdot, g)\|_{L^p} \\
&\leq C^{(2)}_{U,\psi,p,N} \cdot \|\Gamma_N\|_{L^p} \cdot |\det(g)|^{-\frac{1}{2}} \cdot \|\Lambda(\cdot, g)\|_{L^p},
\end{aligned}$$

where Lemma 4.5.5 was used in the last step. Here, the term $\|\Gamma_N\|_{L^p}$ is finite as soon as $N > d/p$, which we assume in what follows.

We note that $g \in h_0 U$ can be chosen arbitrarily in this estimate. Taking the estimate to the power $q$ and averaging over $g \in h_0 U$, we conclude

$$\|F\|^q_{L^p} \leq \frac{\left(C^{(2)}_{U,\psi,p,N} \|\Gamma_N\|_{L^p}\right)^q}{\mu_H(U)} \cdot \int_{h_0 U} \left[ |\det(g)|^{-\frac{1}{2}} \|\Lambda(\cdot, g)\|_{L^p} \right]^q \mathrm{d}g.$$

Because of $\Lambda = M_{W \times U^{-1}U}\left[W_\psi f\right]$ and $F = \mathcal{F}^{-1}\left[\varphi_V \widehat{f}\right]$, this proves the desired estimate in the case $p \in (0, 1]$. □

We can now prove the desired continuity statement for the Fourier transform restricted to $\mathrm{Co}\left(L^{p,q}_m\right) \cap \mathcal{S}\left(\mathbb{R}^d\right)$. We will then use a density argument (using the atomic decomposition result for coorbit spaces) for the general case.

**Lemma 4.5.7.** *(cf. [41, Lemma 35]) Let $\mathcal{Q}$ be an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and let $u : \mathcal{O} \to (0, \infty)$ be a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\right)$.*

*Then there is a constant $C > 0$ such that any $f \in \mathcal{S}\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p,q}_m\right)$ satisfies $\widehat{f} \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ with*

$$\|\widehat{f}\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} \leq C \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)}.$$

*Proof.* Choose some $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\psi \neq 0$ and $\widehat{\psi} \in C^\infty_c(\mathcal{O})$. Let $(h_i)_{i \in I}$ be a relatively separated family in $\mathcal{O}$ and let $Q \Subset \mathcal{O}$ with $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i \in I}$.

Using Theorem 4.4.13, choose a set $Q_0 \Subset \mathcal{O}$ with $Q \subset Q_0$ such that the family $\mathcal{P} = \left(h_i^{-T} Q_0\right)_{i \in I}$ is a semi-structured covering of $\mathcal{O}$ induced by $H$ and choose a measurable partition $H = \biguplus_{i \in I} V_i$ such that $(\gamma_i)_{i \in I} := \left(\frac{1}{C_\psi} \varphi_{V_i}\right)_{i \in I}$ is an $L^p$-BAPU for $\mathcal{P}$ for every $p \in (0, \infty]$ and such that $V_i \subset h_i U'$ holds for all $i \in I$ for a suitable compact unit neighorhood $U' \subset H$.

By Lemma 4.4.8, we know that $u$ is $\mathcal{P}$-moderate. Thus, Lemma 4.4.9 yields $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) = \mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$ with equivalent quasi-norms, so that we can without loss of generality assume $\mathcal{Q} = \mathcal{P}$.

Let $U \subset H$ be an *open*, precompact unit neighborhood with $U' \subset U$, fix an open neighborhood $W \subset \mathbb{R}^d$ of the origin and let us define

$$Q := W \times U^{-1} U.$$

Let $(u_i)_{i\in I}$ be a $\mathcal{P}$-discretization of $u$. By Lemma 4.5.4, this implies

$$u_i \leq C_0 \cdot |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h_i) \text{ for all } i \in I \tag{4.5.11}$$

for some constant $C_0 = C_0\left((u_i)_{i\in I}\right)$. Furthermore, since $m$ is right-moderate with respect to some measurable, locally bounded weight $m_1 : H \to (0,\infty)$, Corollary 2.2.23 yields some $C_1 = C_1\left(m_1, \overline{U}\right) > 0$ with

$$m(h_i) \leq C_1 \cdot m(h) \text{ for all } h \in h_i U. \tag{4.5.12}$$

Finally, setting $C_{U,q} := \sup_{v\in\overline{U}} |\det(v)|^{\frac{1}{q}-\frac{1}{2}}$, we also have for each $h = h_i v \in h_i U$ the estimate

$$\begin{aligned} |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} &= |\det(h)|^{\frac{1}{2}-\frac{1}{q}} \cdot |\det(v^{-1})|^{\frac{1}{2}-\frac{1}{q}} \\ &\leq C_{U,q} \cdot |\det(h)|^{\frac{1}{2}-\frac{1}{q}}. \end{aligned} \tag{4.5.13}$$

For brevity, let us set $F := M_Q\left[W_\psi f\right]$. By Lemma 4.5.6, there is for each $r \in (0,\infty)$ a suitable constant $C^{(r)} = C(U,\psi,p,r,d) > 0$ with

$$\left\|\mathcal{F}^{-1}[\varphi_V \widehat{f}]\right\|_{L^p}^r \leq C^{(r)} \int_{h_0 U} \left[|\det(h)|^{-\frac{1}{2}} \|F(\cdot,h)\|_{L^p}\right]^r \mathrm{d}h \tag{4.5.14}$$

for each $h_0 \in H$ and each measurable set $V \subset h_0 U$.

Let us first assume $q \in (0,\infty)$. We now apply estimate (4.5.14) (with $r = q$) to $h_0 = h_i$ and $V = V_i \subset h_i U' \subset h_i U$. Hence,

$$\left\|\mathcal{F}^{-1}\left[\gamma_i \widehat{f}\right]\right\|_{L^p}^q \leq \frac{C^{(q)}}{C_\psi^q} \int_{h_i U} \left[|\det(h)|^{-\frac{1}{2}} \|F(\cdot,h)\|_{L^p}\right]^q \mathrm{d}h$$

holds for all $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $i \in I$. Together with the estimates (4.5.11), (4.5.12) and (4.5.13), this yields

$$\begin{aligned} &\sum_{i\in I} \left(u_i \cdot \left\|\mathcal{F}^{-1}\left[\gamma_i \widehat{f}\right]\right\|_{L^p}\right)^q \\ &\leq \frac{C_0^q C^{(q)}}{C_\psi^q} \sum_{i\in I} \int_{h_i U} \left(|\det(h)|^{-\frac{1}{2}} |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h_i) \cdot \|F(\cdot,h)\|_{L^p}\right)^q \mathrm{d}h \\ &\leq \frac{C_0^q C_1^q C_{U,q}^q C^{(q)}}{C_\psi^q} \sum_{i\in I} \int_{h_i U} \left(|\det(h)|^{-\frac{1}{q}} m(h) \cdot \|F(\cdot,h)\|_{L^p}\right)^q \mathrm{d}h \\ &= \frac{C_0^q C_1^q C_{U,q}^q C^{(q)}}{C_\psi^q} \int_H \left(\sum_{i\in I} \chi_{h_i\overline{U}}(h)\right) \cdot \left(m(h) \cdot \|F(\cdot,h)\|_{L^p}\right)^q \frac{\mathrm{d}h}{|\det(h)|}. \end{aligned}$$

But Lemma 2.3.11 implies that $f_{\overline{U}} : H \to [0,\infty], h \mapsto \sum_{i\in I} \chi_{h_i\overline{U}}(h)$ is bounded by some constant $C_2 = C_2\left((h_i)_{i\in I}, U\right) < \infty$.

Setting $C_3 := \frac{C_0 C_1 C_{U,Q}}{C_\psi}$, we conclude

$$\begin{aligned}\|\widehat{f}\|_{\mathcal{D}\left(\mathcal{P},L^p,\ell^q_u\right)} &\leq C_3 \left(C^{(q)} C_2\right)^{\frac{1}{q}} \cdot \left[\int_H \left(m\left(h\right) \cdot \|F\left(\cdot,h\right)\|_{L^p}\right)^q \frac{\mathrm{d}h}{|\det\left(h\right)|}\right]^{1/q} \\ &= C_3 \left(C^{(q)} C_2\right)^{\frac{1}{q}} \cdot \left\|M_Q\left[W_\psi f\right]\right\|_{L^{p,q}_m} \\ &= C_3 \left(C^{(q)} C_2\right)^{\frac{1}{q}} \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)} < \infty,\end{aligned}$$

since $\psi \in \mathbb{A}^r_v \setminus \{0\}$ is admissible as an analyzing vector by Corollary 4.2.12, where $r \in (0,1]$ is chosen so that $L^{p,q}_m$ admits an equivalent $r$-norm, cf. Lemma 4.2.6. Here, it is worth recalling that the quasi-norm on the coorbit space is given by $\|f\|_{\mathrm{Co}(Y)} = \left\|W_\psi f\right\|_{W(L^\infty,Y)}$ and not by $\|f\|_{\mathrm{Co}(Y)} = \left\|W_\psi f\right\|_Y$.

It remains to consider the case $q = \infty$. Here, we employ the inequality (4.5.14) for $r = 1$, $h_0 = h_i$ and $V = V_i$ and the estimates (4.5.11), (4.5.13) and (4.5.12) (with $q = \infty$), to obtain

$$\begin{aligned}u_i \cdot \left\|\mathcal{F}^{-1}\left[\gamma_i \widehat{f}\right]\right\|_{L^p} &\leq \frac{C^{(1)}}{C_\psi} \cdot u_i \int_{h_i U} |\det\left(h\right)|^{-\frac{1}{2}} \|F\left(\cdot,h\right)\|_{L^p} \,\mathrm{d}h \\ &\leq \frac{C_{U,\infty} C_0 C_1 C^{(1)}}{C_\psi} \int_{h_i U} m\left(h\right) \cdot \|F\left(\cdot,h\right)\|_{L^p} \,\mathrm{d}h \\ &\leq \frac{C_{U,\infty} C_0 C_1 C^{(1)}}{C_\psi} \mu\left(h_i U\right) \cdot \left\|M_Q\left[W_\psi f\right]\right\|_{L^{p,\infty}_m} \\ &\leq \frac{C_{U,\infty} C_0 C_1 C^{(1)}}{C_\psi} \mu\left(\overline{U}\right) \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)} < \infty\end{aligned}$$

for all $i \in I$. By definition of $\|\widehat{f}\|_{\mathcal{D}\left(\mathcal{P},L^p,\ell^q_u\right)}$ and because of $q = \infty$, this completes the proof. □

In order to employ the density argument using the atomic decomposition result for coorbit spaces, we need the following variant of the Fatou property for decomposition spaces.

**Lemma 4.5.8.** *(cf. [41, Lemma 36]) Let $\varnothing \neq U \subset \mathbb{R}^d$ be an open set, let $p, q \in (0,\infty]$, let $\mathcal{P} = \left(P_j\right)_{j \in J}$ be an $L^p$-decomposition covering of $U$ and let $u : J \to (0,\infty)$ be $\mathcal{P}$-moderate.*

*Let $\left(f_n\right)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$ and assume that $f \in \mathcal{D}'\left(U\right)$ is a distribution with $f_n \to f$ in the sense of distributions. Finally, assume that $\liminf_{n\to\infty} \|f_n\|_{\mathcal{D}\left(\mathcal{P},L^p,\ell^q_u\right)}$ is finite. Then $f \in \mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$ with*

$$\|f\|_{\mathcal{D}\left(\mathcal{P},L^p,\ell^q_u\right)} \leq \liminf_{n\to\infty} \|f_n\|_{\mathcal{D}\left(\mathcal{P},L^p,\ell^q_u\right)}.$$

*Proof.* Let $(\varphi_j)_{j\in J}$ be an $L^p$-BAPU for $\mathcal{P}$. For $j \in J$ and $x \in \mathbb{R}^d$, an analog of [61, Theorem 7.23] (applied to $\mathcal{F}^{-1}$ instead of $\mathcal{F}$) yields, with $e_x : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto e^{2\pi i \langle x, \xi\rangle}$,

$$\begin{aligned}\left[\mathcal{F}^{-1}\left(\varphi_j f\right)\right](x) &= \left(\varphi_j f\right)(e_x)\\ &= f\left(\varphi_j e_x\right)\\ &= \lim_{n\to\infty} f_n\left(\varphi_j e_x\right)\\ &= \lim_{n\to\infty} \left(\varphi_j f_n\right)(e_x)\\ &= \lim_{n\to\infty} \left[\mathcal{F}^{-1}\left(\varphi_j f_n\right)\right](x),\end{aligned}$$

where we used the fact that $\varphi_j f$ and $\varphi_j f_n$ are distributions with compact support so that [61, Theorem 7.23] is applicable. The step from the first to the second line follows because of $f_n \to f$ in the sense of distributions and because $\varphi_j e_x \in C_c^\infty(U)$ is a (fixed) test function.

We get $\mathcal{F}^{-1}\left(\varphi_j f\right) \in L^p\left(\mathbb{R}^d\right)$ for all $j \in J$, since $L^p\left(\mathbb{R}^d\right)$ satisfies the Fatou property. This also yields the estimate

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p} &\le \liminf_{n\to\infty} \left\|\mathcal{F}^{-1}\left(\varphi_j f_n\right)\right\|_{L^p}\\ &\le \liminf_{n\to\infty} \frac{1}{u_j} \|f_n\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)} < \infty.\end{aligned}$$

But because $\ell^q(J)$ also enjoys the Fatou property and is solid, we finally get

$$\begin{aligned}\|f\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)} &= \left\|\left(u_j \cdot \left\|\mathcal{F}^{-1}\left(\varphi_j f\right)\right\|_{L^p}\right)\right\|_{\ell^q(J)}\\ &\le \left\|\left(u_j \cdot \liminf_{n\to\infty} \left\|\mathcal{F}^{-1}\left(\varphi_j f_n\right)\right\|_{L^p}\right)\right\|_{\ell^q(J)}\\ &\le \liminf_{n\to\infty} \left\|\left(u_j \cdot \left\|\mathcal{F}^{-1}\left(\varphi_j f_n\right)\right\|_{L^p}\right)\right\|_{\ell^q(J)}\\ &= \liminf_{n\to\infty} \|f_n\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)} < \infty.\end{aligned}$$

In particular, we have $f \in \mathcal{D}\left(\mathcal{P}, L^p, \ell^q_u\right)$. □

Using this Fatou property, we can finally complete the proof of the continuity of the Fourier transform.

**Theorem 4.5.9.** *Let $\mathcal{Q}$ be an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and let $u : \mathcal{O} \to (0,\infty)$ be a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\right)$.*

*Then the Fourier transform $\mathcal{F}_\mathcal{R} : \mathcal{R} \to \mathcal{D}'(\mathcal{O})$ constructed in Theorem 4.3.1 restricts to a continuous linear map*

$$\mathcal{F}_\mathcal{R} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right).$$

*Proof.* Lemma 4.5.7, yields some $C > 0$ with

$$\|\widehat{f}\|_{\mathcal{D}\left(\mathcal{Q},L^p,\ell^q_u\right)} \leq C \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)}$$

for all $f \in \mathcal{S}\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p,q}_m\right)$.

Fix some function $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\psi \neq 0$ and $\widehat{\psi} \in C^\infty_c\left(\mathcal{O}\right)$ and let $v : G \to (0,\infty)$ be a control weight for $L^{p,q}_m$ of moderate growth. By Corollary 4.2.12, this implies $\psi \in \mathbb{A}^r_v$ for all $r \in (0,1]$. Now, choose a unit neighborhood

$$U_0 = U_0\left(L^{p,q}_m, \psi, v\right) \subset G = \mathbb{R}^d \rtimes H$$

as in the "analysis" part of the atomic decomposition theorem (part 2 of Theorem 2.4.19).

By Lemma 2.3.12, there is a $U_0$-BUPU $\Phi = (\varphi_i)_{i\in I}$ with associated localizing family $X = (x_i)_{i\in I}$. Let $S^X_\psi, A^X_\psi$ be the associated synthesis and analysis operators as in Theorem 2.4.19.

Now, let $f \in \mathrm{Co}\left(L^{p,q}_m\right) \subset \mathcal{R}_v$ be arbitrary and let

$$\Lambda := (\lambda_i)_{i\in I} := A^X_\psi f \in Y_d\left(X\right).$$

Since $G$ is $\sigma$-compact, we see that $I$ is a countable set, thanks to Lemma 2.3.10. Hence, we can choose a non-decreasing family $(I_n)_{n\in\mathbb{N}}$ of *finite* subsets of $I$ with $I = \bigcup_{n\in\mathbb{N}} I_n$. Let us define $\Lambda^{(n)} := \Lambda \cdot \chi_{I_n}$. Since $Y_d\left(X\right)$ is solid, we get $\Lambda^{(n)} \in Y_d\left(X\right)$ with

$$\|\Lambda^{(n)}\|_{Y_d} \leq \|\Lambda\|_{Y_d} \leq \|A^X_\psi\| \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)} \tag{4.5.15}$$

for all $n \in \mathbb{N}$.

By the atomic decomposition theorem (Theorem 2.4.19), we know that the series

$$f = S^X_\psi A^X_\psi f = S^X_\psi \Lambda = \sum_{i\in I} \lambda_i \cdot \pi\left(x_i\right)\psi$$

converges unconditionally with respect to the weak-$*$-topology of $\mathcal{R}_v$. But this means that $f^{(n)} := S^X_\psi \Lambda^{(n)} \in \mathrm{Co}\left(L^{p,q}_m\right)$ converges for $n \to \infty$ to $f$ in the weak-$*$-topology of $\mathcal{R}_v$.

By Theorem 4.3.1, this implies $\mathcal{F}_{\mathcal{R}} f^{(n)} \to \mathcal{F}_{\mathcal{R}} f$ with convergence in $\mathcal{D}'\left(\mathcal{O}\right)$. But

$$f^{(n)} = \sum_{i\in I_n} \lambda_i \cdot \pi\left(x_i\right)\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$$

as a (finite) linearcombination of Schwartz functions. Consequently, $f^{(n)} \in \mathcal{S}\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p,q}_m\right)$. Finally, $\mathcal{F}_{\mathcal{R}} f^{(n)} = \widehat{f^{(n)}}$ where $\widehat{f^{(n)}} \in \mathcal{S}\left(\mathbb{R}^d\right)$ is the usual Fourier transform of $f^{(n)}$, but interpreted as a distribution on $\mathcal{O}$, cf. Theorem 4.3.1.

We now use the Fatou property established in Lemma 4.5.8 to conclude $\mathcal{F}_{\mathcal{R}} f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ with

$$\begin{aligned}
\|\mathcal{F}_{\mathcal{R}} f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} &\leq \liminf_{n\to\infty} \left\|\mathcal{F}_{\mathcal{R}} f^{(n)}\right\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} \\
&= \liminf_{n\to\infty} \left\|\widehat{f^{(n)}}\right\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)} \\
&\leq C \cdot \liminf_{n\to\infty} \|f_n\|_{\mathrm{Co}\left(L^{p,q}_m\right)} \\
&= C \cdot \liminf_{n\to\infty} \left\|S^X_\psi \Lambda^{(n)}\right\|_{\mathrm{Co}\left(L^{p,q}_m\right)} \\
&\leq C \cdot \|S^X_\psi\| \cdot \liminf_{n\to\infty} \|\Lambda^{(n)}\|_{Y_d(X)} \\
&\leq C \cdot \|S^X_\psi\| \cdot \|A^X_\psi\| \cdot \|f\|_{\mathrm{Co}\left(L^{p,q}_m\right)} < \infty.
\end{aligned}$$

Here, we used equation (4.5.15) in the last step. Since $f \in \mathrm{Co}\left(L^{p,q}_m\right)$ was arbitrary, the proof is complete. □

## 4.6 CONTINUITY OF $\mathcal{F}^{-1}_{\mathcal{R}} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathrm{Co}\left(L^{p,q}_m\right)$

In this section, we will show that the Fourier transform

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right),$$

which was proven to be continuous in the preceding section, is an isomorphism of Quasi-Banach spaces.

To this end, we first show that $\mathcal{F}^{-1} : \mathcal{D}'(\mathcal{O}) \to Z'(\mathcal{O})$ restricts to a bounded linear map

$$\mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \widetilde{\mathrm{Co}}\left(L^{p,q}_m\right),$$

where $\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right)$ is the coorbit space using the "universal reservoir" $Z'(\mathcal{O})$, cf. Theorem 4.3.6.

The main estimate for showing this continuity of $\mathcal{F}^{-1}$ is given in the following lemma. We remark that the proof for $p \in (0,1)$ is rather technical. A high-level description of the argument (ignoring many details) is given at the very end of this section.

**Lemma 4.6.1.** *Let $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ and let $U \subset \mathbb{R}^d$ and $V \subset H$ be open, precompact sets. Then the following are true:*

1. *For $p \in [1,\infty]$, there is a constant $C = C(U, V, \psi, p, d) > 0$ such that any $F \in L^p\left(\mathbb{R}^d\right)$ satisfies $\left[M_{U\times V}\left(W_\psi F\right)\right](\cdot, h) \in L^p\left(\mathbb{R}^d\right)$ with*

$$\left\|\left[M_{U\times V}\left(W_\psi F\right)\right](\cdot, h)\right\|_{L^p} \leq C \cdot |\det(h)|^{1/2} \cdot \|F\|_{L^p}$$

*for all $h \in H$.*

2. *Assume that* $\operatorname{supp} \widehat{\psi} \subset \mathbb{R}^d$ *is compact. Then for each* $p \in (0,1]$*, there is a constant* $C = C(U, V, \psi, p, d) > 0$ *such that every function* $F \in L^p(\mathbb{R}^d) \cap L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ *with* $\operatorname{supp} \widehat{F} \subset \Omega$ *for some compact set* $\Omega \subset \mathbb{R}^d$ *satisfies* $\left[M_{U\times V}(W_\psi F)\right](\cdot, h) \in L^p(\mathbb{R}^d)$ *with*

$$\begin{aligned}&\left\|\left[M_{U\times V}(W_\psi F)\right](\cdot,h)\right\|_{L^p}\\ &\leq C \cdot |\det(h)|^{\frac{1}{p}-\frac{1}{2}} \cdot \lambda\left(\left(h^{-T}V^{-T}\operatorname{supp}\widehat{\psi}\right) - \Omega\right)^{\frac{1}{p}-1} \cdot \|F\|_{L^p}\end{aligned}$$

*for all* $h \in H$*, where* $\lambda$ *denotes the Lebesgue measure on* $\mathbb{R}^d$*.*

*In both cases, the wavelet transform* $W_\psi F$ *is understood as the integral*

$$W_\psi F(x,h) = \int_{\mathbb{R}^d} F(y) \cdot \overline{[\pi(x,h)\,\psi](y)}\,\mathrm{d}y$$

*and this integral converges absolutely for all values of* $(x,h) \in G$ *and defines a measurable function.*

*Proof.* In case of $p \in (0,1]$, Corollary 3.1.3 yields $F \in L^1(\mathbb{R}^d)$. Because of $\pi(x,h)\,\psi \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, this implies that the integral defining $W_\psi F(x,h)$ converges absolutely for all $(x,h) \in G$. Likewise, in case of $p \in [1,\infty]$, we have $\pi(x,h)\,\psi \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for the conjugate exponent $q \in [1,\infty]$, i.e. with $\frac{1}{p} + \frac{1}{q} = 1$. Again, this implies absolute convergence of the integral defining $W_\psi F(x,h)$.

Finally, the integrand

$$\begin{aligned}\Theta : \mathbb{R}^d \times H \times \mathbb{R}^d \to \mathbb{C}, (x,h,y) &\mapsto F(y) \cdot \overline{[\pi(x,h)\,\psi](y)}\\ &= F(y) \cdot |\det(h)|^{-\frac{1}{2}}\,\psi\left(h^{-1}(y-x)\right)\end{aligned}$$

is a measurable function. It is now a consequence of the Fubini-Tonelli theorem ([33, Theorem 2.37]) that $(x,h) \mapsto W_\psi F(x,h)$ defines a measurable function. One way of seeing this is to decompose $\Theta$ into the positive/negative parts of its real/imaginary parts and then to apply Tonelli's theorem to each of these functions.

*The wavelet transform* $W_\psi F$ *is of course not just measurable, but even continuous, but we will not need this here.*

Furthermore, a direct computation using the change of variables $z = x - y$ yields

$$\begin{aligned}W_\psi F(x,h) &= |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} F(y)\,\overline{\psi\left(h^{-1}(y-x)\right)}\,\mathrm{d}y\\ &= |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} F(y) \cdot \overline{\psi}^\vee\left(h^{-1}(x-y)\right)\,\mathrm{d}y \qquad (4.6.1)\\ &= |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} \left(\Delta_{h^{-T}}\overline{\psi}^\vee\right)(z) \cdot F(x-z)\,\mathrm{d}z\\ &= |\det(h)|^{-1/2} \cdot \left[\left(\Delta_{h^{-T}}\overline{\psi}^\vee\right) * F\right](x). \qquad (4.6.2)\end{aligned}$$

For $N \in \mathbb{N}$, let us write

$$|\psi|_N := \sup_{x\in\mathbb{R}^d} (1+|x|)^N\,|\psi(x)|$$

and $\Gamma_N(w) := (1+|w|)^{-N}$. Since $U, V$ are precompact sets, there is a constant $R = R(U,V) \geq 1$ with $|u| \leq R$ and $\|v\| \leq R$ for all $(u,v) \in U \times V$. Now, for $(x,h) \in G$ and $(y,g) \in (x,h) \cdot (U \times V)$, there are $(u,v) \in U \times V$ with

$$(y,g) = (x,h)(u,v) = (x+hu, hv).$$

Hence, $x = y - hu$ and $h = gv^{-1}$, which entails $h^{-1} = vg^{-1}$. For arbitrary $z \in \mathbb{R}^d$, this implies

$$\begin{aligned}
1 + |h^{-1}(x-z)| &= 1 + |h^{-1}(y - hu - z)| \\
&= 1 + |h^{-1}(y-z) - u| \\
&\leq 1 + |vg^{-1}(y-z)| + |u| \\
&\leq \|v\| \cdot |g^{-1}(y-z)| + R + 1 \\
&\leq R(1 + |g^{-1}(y-z)|) + 1 \\
&\leq 2R \cdot (1 + |g^{-1}(z-y)|).
\end{aligned}$$

Hence, the estimate

$$\begin{aligned}
\left| \left( \Delta_{g^{-T}} \overline{\psi}^\vee \right)(y-z) \right| &= \left| \overline{\psi}^\vee \left( g^{-1}(y-z) \right) \right| \\
&= \left| \psi \left( g^{-1}(z-y) \right) \right| \\
&\leq |\psi|_N \cdot \left( 1 + \left| g^{-1}(z-y) \right| \right)^{-N} \\
&\leq (2R)^N |\psi|_N \cdot \left( 1 + \left| h^{-1}(x-z) \right| \right)^{-N} \\
&= (2R)^N |\psi|_N \cdot (\Delta_{h^{-T}} \Gamma_N)(x-z) \qquad (4.6.3)
\end{aligned}$$

holds for all $z \in \mathbb{R}^d$, $(x,h) \in G$ and $(y,g) \in (x,h) \cdot (U \times V)$.

Furthermore, we have

$$\begin{aligned}
|\det(g)|^{-1/2} = |\det(hv)|^{-1/2} &= |\det(h)|^{-1/2} \cdot |\det(v)|^{-1/2} \\
&\leq C_V \cdot |\det(h)|^{-1/2} \qquad (4.6.4)
\end{aligned}$$

with $C_V := \max_{v \in \overline{V}} |\det(v)|^{-1/2}$ for all $(x,h) \in G$ and arbitrary $(y,g) \in (x,h) \cdot (U \times V)$.

In the case $p \in (0,1)$, we also need an estimate for the support of $\mathcal{F}\left[ \Delta_{g^{-T}} \overline{\psi}^\vee \right]$. To this end, we calculate, using well-known properties of the Fourier transform (cf. [33, Theorem 8.22]) and equation (4.3.1)

$$\begin{aligned}
\mathcal{F}\left[ \Delta_{g^{-T}} \overline{\psi}^\vee \right] &= |\det(g)| \cdot \left( \mathcal{F} \overline{\psi}^\vee \right) \circ g^T \\
&= |\det(g)| \cdot \left( \mathcal{F}^{-1} \overline{\psi} \right) \circ g^T \\
&= |\det(g)| \cdot \overline{\widehat{\psi}} \circ g^T.
\end{aligned}$$

For $K := \operatorname{supp} \widehat{\psi} \subset \mathbb{R}^d$, this implies

$$\begin{aligned} \operatorname{supp} \mathcal{F}\left[\Delta_{g^{-T}} \overline{\psi}^{\vee}\right] &\subset g^{-T} K = (hv)^{-T} K \\ &= h^{-T} v^{-T} K \subset h^{-T} V^{-T} K. \end{aligned} \tag{4.6.5}$$

We now distinguish the two cases $p \in [1, \infty]$ and $p \in (0, 1]$. In the first case, equations (4.6.1), (4.6.3) and (4.6.4) yield

$$\begin{aligned} &\left|\left(W_\psi F\right)(y, g)\right| \\ &\leq |\det(g)|^{-1/2} \cdot \int_{\mathbb{R}^d} |F(z)| \cdot \left|\overline{\psi}^{\vee}\left(g^{-1}(y - z)\right)\right| \, \mathrm{d}z \\ &\leq (2R)^N |\psi|_N C_V \cdot |\det(h)|^{-1/2} \cdot \int_{\mathbb{R}^d} |F(z)| \cdot \left(\Delta_{h^{-T}} \Gamma_N\right)(x - z) \, \mathrm{d}z \\ &= (2R)^N |\psi|_N C_V \cdot |\det(h)|^{-1/2} \cdot \left(|F| * \Delta_{h^{-T}} \Gamma_N\right)(x) \end{aligned}$$

for all $(y, g) \in (x, h) \cdot (U \times V)$. Since the right-hand side is *independent* of $(y, g) \in (x, h) \cdot (U \times V)$, we conclude for $(x, h) \in G$ that

$$\left(M_{U \times V}\left[W_\psi F\right]\right)(x, h) \leq (2R)^N |\psi|_N C_V \cdot |\det(h)|^{-\frac{1}{2}} \left(|F| * \Delta_{h^{-T}} \Gamma_N\right)(x).$$

Using Young's inequality [33, Theorem 8.7], i.e., the convolution relation $L^p(\mathbb{R}^d) * L^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, this implies

$$\begin{aligned} &\left\|\left(M_{U \times V}\left[W_\psi F\right]\right)(\cdot, h)\right\|_{L^p} \\ &\leq (2R)^N |\psi|_N C_V \cdot |\det(h)|^{-1/2} \cdot \left\||F|\right\|_{L^p} \left\|\Delta_{h^{-T}} \Gamma_N\right\|_{L^1} \\ &= (2R)^N |\psi|_N C_V \cdot |\det(h)|^{1/2} \cdot \|F\|_{L^p} \cdot \|\Gamma_N\|_{L^1}, \end{aligned}$$

where the last step is due to Lemma 4.5.5. For $N > d$, the term $\|\Gamma_N\|_{L^1}$ is finite. This proves the claim in the case $p \in [1, \infty]$.

Now, assume $p \in (0, 1]$ and let $(x, h) \in G$. For brevity, let us set $C_h := \lambda\left([h^{-T} V^{-T} K] - \Omega\right)$. Using the estimate on the Fourier support of $\Delta_{g^{-T}} \overline{\psi}^{\vee}$ in equation (4.6.5) together with $\operatorname{supp} \widehat{F} \subset \Omega$, equation (3.1.6) of Theorem 3.1.4 yields the pointwise estimate

$$\begin{aligned} &\left|\left[\left(\Delta_{g^{-T}} \overline{\psi}^{\vee}\right) * F\right](y)\right| \\ &= \left|\left(\mathcal{F}^{-1}\left[\widehat{\Delta_{g^{-T}} \overline{\psi}^{\vee}}\right] * F\right)(y)\right| \\ &\leq \left[\lambda\left(h^{-T} V^{-T} K - \Omega\right)\right]^{\frac{1}{p}-1} \left[\int_{\mathbb{R}^d} \left|\left(\mathcal{F}^{-1}\left[\widehat{\Delta_{g^{-T}} \overline{\psi}^{\vee}}\right]\right)(w) \cdot F(y - w)\right|^p \mathrm{d}w\right]^{\frac{1}{p}} \\ &= C_h^{\frac{1}{p}-1} \cdot \left[\int_{\mathbb{R}^d} \left|\left(\Delta_{g^{-T}} \overline{\psi}^{\vee}\right)(y - z) \cdot F(z)\right|^p \, \mathrm{d}z\right]^{\frac{1}{p}} \\ &\overset{(*)}{\leq} (2R)^N |\psi|_N \cdot C_h^{\frac{1}{p}-1} \cdot \left[\int_{\mathbb{R}^d} \left|\left(\Delta_{h^{-T}} \Gamma_N\right)(x - z)\right|^p \cdot |F(z)|^p \, \mathrm{d}z\right]^{\frac{1}{p}} \\ &= (2R)^N |\psi|_N \cdot C_h^{\frac{1}{p}-1} \cdot \left[\left(\left|\Delta_{h^{-T}} \Gamma_N\right|^p * |F|^p\right)(x)\right]^{1/p} \end{aligned}$$

for all $(y,g) \in (x,h)\cdot(U\times V)$. Here, we made use of equation (4.6.3) in the step marked with $(*)$.

Together with equations (4.6.2) and (4.6.4), we conclude

$$\begin{aligned}
&\left|\left(W_\psi F\right)(y,g)\right| \\
&= \left|\det(g)\right|^{-1/2} \cdot \left|\left[\left(\Delta_{g^{-T}}\overline{\psi}^\vee\right) * F\right](y)\right| \\
&\leq C_V \left|\det(h)\right|^{-1/2} \cdot \left|\left[\left(\Delta_{g^{-T}}\overline{\psi}^\vee\right) * F\right](y)\right| \\
&\leq C_V (2R)^N |\psi|_N \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left[\left(\left|\Delta_{h^{-T}}\Gamma_N\right|^p * |F|^p\right)(x)\right]^{1/p}.
\end{aligned}$$

As in the case $p \in [1,\infty]$, we observe that the right-hand side of this inequality is independent of $(y,g) \in (x,h)\cdot(U\times V)$. This implies

$$\begin{aligned}
&\left(M_{U\times V}\left[W_\psi F\right]\right)(x,h) \\
&\leq C_{U,V,N,\psi} \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left[\left(\left|\Delta_{h^{-T}}\Gamma_N\right|^p * |F|^p\right)(x)\right]^{1/p}
\end{aligned}$$

for all $(x,h) \in G$, with $C_{U,V,N,\psi} := C_V (2R)^N |\psi|_N$.

We now take the $L^p$ quasi-norm in this estimate and use the convolution relation $L^1\left(\mathbb{R}^d\right) * L^1\left(\mathbb{R}^d\right) \hookrightarrow L^1\left(\mathbb{R}^d\right)$ to conclude

$$\begin{aligned}
&\left\|\left(M_{U\times V}\left[W_\psi F\right]\right)(\cdot,h)\right\|_{L^p} \\
&\leq C_{U,V,N,\psi} \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left\|\left[\left|\Delta_{h^{-T}}\Gamma_N\right|^p * |F|^p\right]^{1/p}\right\|_{L^p} \\
&= C_{U,V,N,\psi} \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left\|\left|\Delta_{h^{-T}}\Gamma_N\right|^p * |F|^p\right\|_{L^1}^{1/p} \\
&\leq C_{U,V,N,\psi} \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left\|\left|\Delta_{h^{-T}}\Gamma_N\right|^p\right\|_{L^1}^{1/p} \cdot \left\||F|^p\right\|_{L^1}^{1/p} \\
&= C_{U,V,N,\psi} \cdot \left|\det(h)\right|^{-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \left\|\Delta_{h^{-T}}\Gamma_N\right\|_{L^p} \cdot \|F\|_{L^p} \\
&= C_{U,V,N,\psi} \|\Gamma_N\|_{L^p} \cdot \left|\det(h)\right|^{\frac{1}{p}-\frac{1}{2}} C_h^{\frac{1}{p}-1} \cdot \|F\|_{L^p}.
\end{aligned}$$

Here, we used Lemma 4.5.5 in the last step. The term $\|\Gamma_N\|_{L^p}$ is finite as soon as $N > d/p$. Recalling the definition $C_h := \lambda\left(\left[h^{-T}V^{-T}K\right] - \Omega\right)$ from above, we see that the proof is complete. □

The following lemma essentially proves the claimed isomorphism between the coorbit space $\mathrm{Co}\left(L^{p,q}_m\right)$ and the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$. All that is missing is to employ the isomorphism between $\mathrm{Co}\left(L^{p,q}_m\right)$ and $\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right)$ given in Theorem 4.3.6 and to check that the map thus constructed is indeed inverse to the Fourier transform $\mathcal{F}_\mathcal{R} : \mathrm{Co}\left(L^{p,q}_m\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ from Theorem 4.5.9. This is done in Theorem 4.6.3 below.

**Lemma 4.6.2.** *(cf. [41, Lemma 41]) Let $\mathcal{Q}$ be an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and let $u : \mathcal{O} \to (0,\infty)$ be a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\right)$.*

*Then the inverse Fourier transform*

$$\mathcal{F}^{-1} : \mathcal{D}'(\mathcal{O}) \to Z'(\mathcal{O})', f \mapsto f \circ \mathcal{F}^{-1}$$

*restricts to a bounded linear map*

$$\mathcal{F}^{-1} : \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q) \to \widetilde{\mathrm{Co}}(L_m^{p,q}),$$

*where* $\widetilde{\mathrm{Co}}(L_m^{p,q})$ *is defined as in Theorem 4.3.6.*

*Proof.* First of all, we note that $\mathcal{F}^{-1} : Z(\mathcal{O}) \to C_c^\infty(\mathcal{O})$ is a continuous linear map by definition of the topology on $Z(\mathcal{O}) = \mathcal{F}(C_c^\infty(\mathcal{O}))$. By duality, this shows that $\mathcal{F}^{-1} : \mathcal{D}'(\mathcal{O}) \to Z'(\mathcal{O})$ is well-defined.

It remains to prove boundedness (and well-definedness) of the restricted version of $\mathcal{F}^{-1}$. We divide the rest of the proof into 5 steps.

STEP 1 (Setting up constants): By definition of an induced covering, there is a relatively separated family $(h_i)_{i\in I}$ in $H$ and some set $Q \Subset \mathcal{O}$ with $\mathcal{Q} = (Q_i)_{i\in I} = \left(h_i^{-T} Q\right)_{i\in I}$. Furthermore, there is some $L^p$-BAPU $(\varphi_i)_{i\in I}$ for $\mathcal{Q}$. Let us fix some $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\psi \neq 0$ and $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. Set $K := \operatorname{supp} \widehat{\psi} \subset \mathcal{O}$ and fix open, precompact unit neighborhoods $U \subset \mathbb{R}^d$ and $V \subset H$.

For $h \in H$, we define

$$I_h := \left\{ i \in I \,\middle|\, h^{-T} K \cap h_i^{-T} \overline{Q} \neq \varnothing \right\}.$$

Note that $\overline{Q} \subset \mathcal{O}$ and $K \subset \mathcal{O}$ are compact. Therefore, Lemma 4.4.1 yields a constant $C_1 = C_1\left((h_i)_{i\in I}, \overline{Q}, K\right) > 0$ with $|I_h| \leq C_1$ for all $h \in H$. Likewise, Lemma 4.1.4 yields a compact set $L = L(Q, K) \subset H$ such that $h \in h_i L$ holds for all $h \in H$ and all $i \in I_h$.

Now, Theorem 4.4.6 shows that the family $(h_i)_{i\in I}$ is well-spread in $H$, so that there is a compact unit neighborhood $W \subset H$ with $H = \bigcup_{i\in I} h_i W$. Note that

$$P := \overline{Q} \cup \left(L W^{-1}\right)^{-T} \overline{Q} \subset \mathcal{O}$$

is a compact set and that $\mathcal{P} := (P_i)_{i\in I} := \left(h_i^{-T} P\right)_{i\in I}$ covers $\mathcal{O}$ because of $Q \subset P$. Hence, $\mathcal{P}$ is an induced covering of $\mathcal{O}$ and in particular an admissible, semi-structured covering of $\mathcal{O}$, cf. Lemma 4.4.4.

Let us set

$$m_{(q)} : H \to (0, \infty), h \mapsto |\det(h)|^{-\frac{1}{q}} \cdot m(h).$$

It is easy to see that this weight is right-moderate with respect to some measurable, locally bounded weight on $H$, cf. the proof of Lemma 4.5.4.

With this definition, there is a constant $C_2 > 0$ such that

$$i \in j^{*\mathcal{P}} \text{ and } m_{(q)}(h) \leq C_2 \cdot m_{(q)}(h_i) \tag{4.6.6}$$

holds for all $i, j \in I$ for which there is some $h \in h_j W$ with $i \in I_h$. Indeed, as above, we get $h \in h_i L$ and hence $h_j \in hW^{-1} \subset h_i L W^{-1}$. This implies

$$\varnothing \neq h_j^{-T} Q \subset \left[h_j^{-T} Q\right] \cap \left[h_i^{-T} \left(LW^{-1}\right)^{-T} Q\right] \subset P_j \cap P_i$$

and hence $i \in j^{*_{\mathcal{P}}}$. Furthermore, because $m^{(q)}$ is right-moderate and $h \in h_i L$, Corollary 2.2.23 yields a constant $C_2 = C_2\left(L, m_{(q)}\right) > 0$ with $m_{(q)}(h) \leq C_2 \cdot m_{(q)}(h_i)$ as desired.

STEP 2 (The localization identity): Now, we observe

$$\sum_{i \in I_h} \varphi_i(x) = 1 \qquad \text{for all } x \in h^{-T} K, \tag{4.6.7}$$

because for $x \in h^{-T} K \subset \mathcal{O}$ and $i \in I$ with $\varphi_i(x) \neq 0$ we have $x \in \varphi_i^{-1}(\mathbb{C}^*) \subset h_i^{-T} Q$ and thus $x \in h^{-T} K \cap h_i^{-T} \overline{Q}$, i.e. $i \in I_h$. This shows $1 = \sum_{i \in I} \varphi_i(x) = \sum_{i \in I_h} \varphi_i(x)$.

Let $(x, h) \in G$ be arbitrary. By equations (4.3.1) and (4.1.2), we know

$$F_{x,h} := \mathcal{F}^{-1} \overline{\pi(x,h)\psi} = \overline{\mathcal{F}\left[\pi(x,h)\psi\right]} \in C_c^\infty(\mathcal{O})$$

with

$$\operatorname{supp}(F_{x,h}) \subset h^{-T} \operatorname{supp} \widehat{\psi} = h^{-T} K.$$

Together with equation (4.6.7), this yields

$$F_{x,h} = \sum_{i \in I_h} \varphi_i F_{x,h}.$$

Now, let $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)$ be arbitrary. For brevity, let us set $F := W_\psi\left(\mathcal{F}^{-1} f\right)$. Recalling the definition of the wavelet transform on $Z'(\mathcal{O})$ (see Definition 4.3.2), together with the considerations above, yields the fundamental LOCALIZATION IDENTITY

$$\begin{aligned}
F(x,h) &= \left(W_\psi\left(\mathcal{F}^{-1} f\right)\right)(x,h) \\
&= \left(\mathcal{F}^{-1} f\right)\left(\overline{\pi(x,h)\psi}\right) \\
&= f\left(\mathcal{F}^{-1} \overline{\pi(x,h)\psi}\right) \\
&= f(F_{x,h}) \\
&= \sum_{i \in I_h} f(\varphi_i F_{x,h}) \\
&= \sum_{i \in I_h} (\varphi_i f)\left(\mathcal{F}^{-1} \overline{\pi(x,h)\psi}\right) \\
&= \sum_{i \in I_h} \left[\mathcal{F}^{-1}(\varphi_i f)\right]\left(\overline{\pi(x,h)\psi}\right) \\
&= \sum_{i \in I_h} \left(W_\psi\left[\mathcal{F}^{-1}(\varphi_i f)\right]\right)(x,h),
\end{aligned} \tag{4.6.8}$$

which is valid for all $(x,h) \in G$. Because of $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)$, we have $\mathcal{F}^{-1}(\varphi_i f) \in L^p\left(\mathbb{R}^d\right)$ for all $i \in I$.

STEP 3 (Estimating the wavelet transform of the pieces $\varphi_i f$ by $\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$): There are two cases. For $p \in [1,\infty]$, Lemma 4.6.1 yields a constant $C_3 = C_3(U,V,\psi,p,d) > 0$ with

$$\left\|M_{U\times V}\left(W_\psi\left[\mathcal{F}^{-1}(\varphi_i f)\right]\right)(\cdot,h)\right\|_{L^p} \leq C_3 \cdot |\det(h)|^{1/2} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$$

for all $i \in I$. For $i \in I_h$, we have $h = h_i k \in h_i L$ for some $k \in L$ as seen in step 1. Hence,

$$|\det(h)|^{1/2} = |\det(h_i)|^{1/2} |\det(k)|^{1/2} \leq C_4 \cdot |\det(h_i)|^{1/2}$$

with $C_4 := \max_{\ell\in L} |\det(\ell)|^{1/2}$. This implies

$$\begin{aligned} &\left\|M_{U\times V}\left(W_\psi\left[\mathcal{F}^{-1}(\varphi_i f)\right]\right)(\cdot,h)\right\|_{L^p} \\ &\leq C_5 \cdot |\det(h_i)|^{\frac{1}{2}} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \end{aligned} \tag{4.6.9}$$

for all $h \in H$ and $i \in I_h$ and some constant $C_5 = C_5(U,V,\psi,d,p,L)$.

In case of $p \in (0,1)$, we observe for $i \in I_h$ that the tempered distribution $F_i := \mathcal{F}^{-1}(\varphi_i f)$ satisfies

$$\operatorname{supp} \widehat{F_i} \subset \operatorname{supp} \varphi_i \subset h_i^{-T}\overline{Q} =: \Omega_i.$$

As seen above, we also have $h \in h_i L$, i.e. $h = h_i k$ for some $k \in L$. For the compact set $Z := L^{-T}V^{-T}K - \overline{Q} \subset \mathbb{R}^d$, this implies

$$\begin{aligned} \left[h^{-T}V^{-T}\operatorname{supp}\widehat{\psi}\right] - \Omega_i &\subset h_i^{-T}L^{-T}V^{-T}K - h_i^{-T}\overline{Q} \\ &= h_i^{-T}\left(L^{-T}V^{-T} - \overline{Q}\right) \\ &= h_i^{-T}Z \end{aligned}$$

and hence

$$\begin{aligned} \left[\lambda\left(h^{-T}V^{-T}\operatorname{supp}\widehat{\psi} - \Omega_i\right)\right]^{\frac{1}{p}-1} &\leq \left[\lambda\left(h_i^{-T}Z\right)\right]^{\frac{1}{p}-1} \\ &= |\det(h_i)|^{1-\frac{1}{p}} \cdot [\lambda(Z)]^{\frac{1}{p}-1}. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} |\det(h)|^{\frac{1}{p}-\frac{1}{2}} &= |\det(h_i)|^{\frac{1}{p}-\frac{1}{2}} \cdot |\det(k)|^{\frac{1}{p}-\frac{1}{2}} \\ &\leq C_{L,p} \cdot |\det(h_i)|^{\frac{1}{p}-\frac{1}{2}} \end{aligned}$$

with $C_{L,p} := \max_{k\in L} |\det(k)|^{\frac{1}{p}-\frac{1}{2}}$.

But $F_i := \mathcal{F}^{-1}(\varphi_i f)$ is an element of $L^p\left(\mathbb{R}^d\right)$, a tempered distribution, as well as (by Lemma 3.1.2) a smooth function and hence an

element of $L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$. Thus, Lemma 4.6.1 yields an absolute constant $C_6 = C_6\left(U, V, \psi, p, d\right) > 0$ with

$$\begin{aligned}
&\left\| M_{U\times V}\left(W_\psi\left[\mathcal{F}^{-1}\left(\varphi_i f\right)\right]\right)(\cdot, h)\right\|_{L^p} \\
&\leq C_6 \left|\det(h)\right|^{\frac{1}{p}-\frac{1}{2}} \left[\lambda\left(\left(h^{-T}V^{-T}\operatorname{supp}\widehat{\psi}\right) - \Omega_i\right)\right]^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p} \\
&\leq C_6 C_{L,p}\left[\lambda(Z)\right]^{\frac{1}{p}-1} \cdot \left|\det(h_i)\right|^{\frac{1}{p}-\frac{1}{2}} \left|\det(h_i)\right|^{1-\frac{1}{p}} \cdot \left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p} \\
&=: C_7 \cdot \left|\det(h_i)\right|^{1/2} \cdot \left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p}.
\end{aligned}$$

All in all, we have shown that the estimate in equation (4.6.9) holds (potentially with a different constant $C_5'$ instead of $C_5$) for arbitrary $p \in (0, \infty]$, all $h \in H$ and all $i \in I_h$.

STEP 4 (Proof of the claim in the case $q < \infty$): Since $\|\cdot\|_{L^p\left(\mathbb{R}^d\right)}$ is a solid quasi-norm, since taking the maximal function $M_{U\times V}$ is subadditive, and since the cardinality $|I_h| \leq C_1$ is uniformly bounded, we can employ the localization identity (equation (4.6.8)) and equation (4.6.9) to get some constant $C_8 = C_8\left(C_1, p\right) > 0$ with

$$\begin{aligned}
\left\|\left(M_{U\times V}F\right)(\cdot, h)\right\|_{L^p} &\leq C_8 \cdot \sum_{i\in I_h} \left\| M_{U\times V}\left(W_\psi\left[\mathcal{F}^{-1}\left(\varphi_i f\right)\right]\right)(\cdot, h)\right\|_{L^p} \\
&\leq C_5' C_8 \cdot \sum_{i\in I_h} \left|\det(h_i)\right|^{\frac{1}{2}} \left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p}. \quad (4.6.10)
\end{aligned}$$

Again, since $|I_h| \leq C_1$ is uniformly bounded, there is an absolute constant $C_9 = C_9\left(C_1, q\right) > 0$ with

$$\left\|\left(M_{U\times V}F\right)(\cdot, h)\right\|_{L^p}^q \leq C_5' C_8 C_9 \sum_{i\in I_h} \left[\left|\det(h_i)\right|^{\frac{1}{2}} \left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p}\right]^q. \quad (4.6.11)$$

For brevity, set $C_{10} := C_5' C_8 C_9$ and let us agree that the cluster $i^*$ is understood with respect to the covering $\mathcal{P}$ for the rest of the proof. Finally, set $N := \max_{i\in I} |i^*|$.

Because of $H = \bigcup_{j\in I} h_j W$ and using equations (4.6.6) and (4.6.11), we can now estimate

$$\left\|\mathcal{F}^{-1} f\right\|_{\widetilde{\mathrm{Co}}\left(L^{p,q}_m\right)} \asymp \left\|W_\psi\left[\mathcal{F}^{-1} f\right]\right\|_{W_{U\times V}\left(L^\infty, L^{p,q}_m\right)} = \|F\|_{W_{U\times V}\left(L^\infty, L^{p,q}_m\right)}$$

by

$$\begin{aligned}
&\|F\|^q_{W_{U\times V}\left(L^\infty, L^{p,q}_m\right)} \\
&= \int_H \left(\frac{m(h)}{\left|\det(h)\right|^{1/q}}\right)^q \cdot \left\|\left(M_{U\times V}F\right)(\cdot, h)\right\|^q_{L^p} \, \mathrm{d}h \\
&\leq \sum_{j\in I} \int_{h_j W} \left(m_{(q)}(h)\right)^q \cdot \left\|\left(M_{U\times V}F\right)(\cdot, h)\right\|^q_{L^p} \, \mathrm{d}h
\end{aligned}$$

$$
\begin{aligned}
&\leq C_2^q C_{10} \cdot \sum_{j\in I} \int_{h_j W} \sum_{i\in I_h} \left(m_{(q)}(h_i)\right)^q \left[|\det(h_i)|^{1/2} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right]^q \mathrm{d}h \\
&\leq C_2^q C_{10} \cdot \sum_{j\in I} \int_{h_j W} \sum_{i\in j^{*\mathcal{P}}} \left[|\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} m(h_i) \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right]^q \mathrm{d}h \\
&\overset{(*)}{=} C_2^q C_{10} \mu_H(W) \cdot \sum_{i\in I} \sum_{j\in i^{*\mathcal{P}}} \left[|\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} m(h_i) \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right]^q \\
&\leq C_2^q C_{10} \mu_H(W) N \cdot \sum_{i\in I} \left[|\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} m(h_i) \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right]^q \\
&\overset{(\dagger)}{\lesssim} C_2^q C_{10} \mu_H(W) N \cdot \left\| \left( \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \right)_{i\in I} \right\|_{\ell_u^q(I)}^q \\
&= C_2^q C_{10} \mu_H(W) N \cdot \|f\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_u^q)}^q < \infty
\end{aligned}
$$

Here, we used the equivalence $i \in j^* \Leftrightarrow j \in i^*$ at $(*)$. The step marked with $(\dagger)$ is justified by Lemma 4.5.4, which shows that any discretization $u_d$ of $u$ satisfies $(u_d)_i \asymp |\det(h)|^{\frac{1}{2}-\frac{1}{q}} m(h_i)$ for all $i \in I$.

Since all constant are independent of the choice of $f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$, this completes the proof in the case $q < \infty$.

STEP 5 (Proof of the claim for $q = \infty$): Let $h \in H$ be arbitrary. Because of $H = \bigcup_{j\in I} h_j W$, there is some $j \in I$ with $h \in h_j W$. Note that $q = \infty$ entails $m_{(q)} = m$. Hence, we can use equations (4.6.6) and (4.6.10) to derive

$$
\begin{aligned}
&m(h) \cdot \|(M_{U\times V} F)(\cdot, h)\|_{L^p} \\
&\leq C_5' C_8 \cdot \sum_{i\in I_h} \left[m(h) \cdot |\det(h_i)|^{1/2} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right] \\
&\leq C_2 C_5' C_8 \cdot \sum_{i\in I_h} \left[m(h_i) \cdot |\det(h_i)|^{1/2} \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right] \\
&\overset{(\dagger)}{\lesssim} C_2 C_5' C_8 \cdot |I_h| \cdot \left\| \left(\|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p}\right)_{i\in I} \right\|_{\ell_u^q(I)} \\
&\leq C_1 C_2 C_5' C_8 \cdot \|f\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_u^q)} < \infty.
\end{aligned}
$$

In the step marked with $(\dagger)$, we invoked Lemma 4.5.4 as in the previous step. This completes the proof. □

It is now easy to show that the Fourier transform from Theorem 4.5.9 yields an isomorphism of (Quasi)-Banach spaces.

**Theorem 4.6.3.** *Let $\mathcal{Q}$ be an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and let $u : \mathcal{O} \to (0, \infty)$ be a decomposition weight for $\mathrm{Co}(L_m^{p,q})$.*

*Then the Fourier transform*

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}(L_m^{p,q}) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$$

*as defined in Theorem 4.5.9 and Theorem 4.3.1 is an isomorphism of (Quasi)-Banach spaces.*

*The inverse map is given by*

$$\Theta^{-1} \circ \mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) \to \mathrm{Co}\left(L_m^{p,q}\right),$$

*where the canonical restriction map* $\Theta : \mathrm{Co}\left(L_m^{p,q}\right) \to \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$ *is defined as in Theorem 4.3.6 and* $\mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) \to \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$ *as in Lemma 4.6.2.*

*Furthermore,* $\mathcal{F}_{\mathcal{R}}^{-1} f = \mathcal{F}^{-1} f$ *holds for all* $f \in L^2\left(\mathbb{R}^d\right) \cap \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)$*, where we consider* $\mathcal{F}^{-1} f \in L^2\left(\mathbb{R}^d\right)$ *as an element of* $\mathcal{R}$ *using the embedding* $\iota$ *from part 3 of Theorem 4.3.1.*

*Proof.* From the previous results, we know that $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}^{-1}$ are indeed well-defined and bounded linear maps with domains and codomains as in the statement of the theorem. Furthermore, Theorem 4.3.6 shows that $\Theta : \mathrm{Co}\left(L_m^{p,q}\right) \to \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$ yields an isomorphism of (Quasi)-Banach spaces.

Finally, $\mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) \to \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$ is a restriction of the map

$$\mathcal{F}^{-1} : \mathcal{D}'(\mathcal{O}) \to Z'(\mathcal{O}), f \mapsto f \circ \mathcal{F}^{-1},$$

cf. Lemma 4.6.2. Since $\mathcal{F}^{-1} : \mathcal{F}\left(C_c^\infty(\mathcal{O})\right) \to C_c^\infty(\mathcal{O})$ is bijective, it follows that $\mathcal{F}^{-1} : \mathcal{D}'(\mathcal{O}) \to Z'(\mathcal{O})$ is injective and so is its restriction $\mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) \to \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right)$. Together with bijectivity of $\Theta$, we conclude that $\Theta^{-1} \circ \mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) \to \mathrm{Co}\left(L_m^{p,q}\right)$ is injective.

It is thus sufficient to show that this map is also surjective (and hence bijective) with right inverse $\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L_m^{p,q}\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)$. To this end, let $f \in \mathrm{Co}\left(L_m^{p,q}\right)$ be arbitrary and define

$$g := \left(\Theta^{-1} \circ \mathcal{F}^{-1}\right)\left[\mathcal{F}_{\mathcal{R}} f\right] \in \mathrm{Co}\left(L_m^{p,q}\right).$$

We will show $\Theta f = \Theta g$. By injectivity of $\Theta$, this will imply $f = g$ and hence $\left(\Theta^{-1} \circ \mathcal{F}^{-1}\right) \circ \mathcal{F}_{\mathcal{R}} = \mathrm{id}_{\mathrm{Co}\left(L_m^{p,q}\right)}$, thus completing the proof.

Now, let $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\psi \neq 0$ and $\widehat{\psi} \in C_c^\infty(\mathcal{O})$. We have

$$\Theta g = \left(\mathcal{F}^{-1} \circ \mathcal{F}_{\mathcal{R}}\right) f \in \widetilde{\mathrm{Co}}\left(L_m^{p,q}\right) \leq Z'(\mathcal{O}),$$

but it is not clear whether we can simply "cancel" $\mathcal{F}^{-1} \circ \mathcal{F}_{\mathcal{R}}$. But using the definitions (cf. Definition 4.3.2 and Theorem 4.3.1), we see

$$\begin{aligned}
\left(W_\psi\left[\Theta g\right]\right)(x,h) &= (\Theta g)\left(\overline{\pi(x,h)\psi}\right) \\
&= \left[\left(\mathcal{F}^{-1} \circ \mathcal{F}_{\mathcal{R}}\right) f\right]\left(\overline{\pi(x,h)\psi}\right) \\
&= \left(\mathcal{F}_{\mathcal{R}} f\right)\left(\mathcal{F}^{-1}\overline{\pi(x,h)\psi}\right) \\
&= f\left(\mathcal{F}^{-1}\overline{\mathcal{F}^{-1}\overline{\pi(x,h)\psi}}\right) \\
&\overset{(*)}{=} f\left(\overline{\mathcal{F}\mathcal{F}^{-1}\overline{\pi(x,h)\psi}}\right) \\
&= f\left(\overline{\overline{\pi(x,h)\psi}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\Theta f)\left(\overline{\pi(x,h)\,\psi}\right) \\
&= \left(W_\psi[\Theta f]\right)(x,h)
\end{aligned}
$$

for all $(x,h) \in G$. Here, we made use of equation (4.3.1) in the step marked with $(*)$.

But Corollary 4.3.5 shows that $W_\psi : Z'(\mathcal{O}) \to C(G)$ is injective. Hence $\Theta f = \Theta g$. As explained above, this shows that $\mathcal{F}_{\mathcal{R}}$ is indeed an isomorphism of Quasi-Banach spaces with the claimed inverse map.

For the final claim, let $f \in L^2(\mathbb{R}^d) \cap \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$ be arbitrary, let $g_1 := \mathcal{F}_{\mathcal{R}}^{-1} f \in \mathrm{Co}(L_m^{p,q}) \subset \mathcal{R}_v$ and $g_2 := \iota(\mathcal{F}^{-1} f) \in \mathcal{R}_v$, where $v : G \to (0,\infty)$ is a control weight for $L_m^{p,g}(G)$ of moderate growth, cf. Assumption 4.2.13. Here, the map $\iota$ is defined as in part 3 of Theorem 4.3.1. Our goal is to show $g_1 = g_2$. But for $\psi$ as above, Plancherel's theorem implies

$$
\begin{aligned}
\left(W_\psi g_2\right)(x,h) &= \langle g_2 \mid \pi(x,h)\,\psi\rangle_{\mathcal{R}_v, \mathcal{S}_v} \\
&= \left\langle \mathcal{F}^{-1} f \,\middle|\, \pi(x,h)\,\psi \right\rangle_{L^2} \\
&= \langle f \mid \mathcal{F}[\pi(x,h)\,\psi]\rangle_{L^2} \\
&= \left\langle f, \overline{\mathcal{F}[\pi(x,h)\,\psi]} \right\rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\
&\overset{(\dagger)}{=} \left\langle \mathcal{F}_{\mathcal{R}} g_1, \overline{\mathcal{F}[\pi(x,h)\,\psi]} \right\rangle_{\mathcal{D}'(\mathcal{O}), C_c^\infty(\mathcal{O})} \\
&\overset{(\ddagger)}{=} g_1\left(\mathcal{F}^{-1}\overline{\mathcal{F}[\pi(x,h)\,\psi]}\right) \\
&= g_1(\pi(x,h)\,\psi) \\
&= \left(W_\psi g_1\right)(x,h)
\end{aligned}
$$

for arbitrary $(x,h) \in G$. Here, we used $\mathcal{F}_{\mathcal{R}} g_1 = f$ at $(\dagger)$ and the definition of $\mathcal{F}_{\mathcal{R}}$ at $(\ddagger)$.

This shows $W_\psi g_1 = W_\psi g_2$ with $g_1, g_2 \in \mathcal{R}_v$. But Lemma 2.4.8 shows that $W_\psi$ is injective on $\mathcal{R}_v$, since $\psi \in \mathbb{A}_v^r \setminus \{0\} \subset \mathcal{G}_v \setminus \{0\}$ holds for all $r \in (0,1]$ by Corollary 4.2.12. We conclude $g_1 = g_2$ as desired. □

Using the "space-sided" version of the decomposition spaces, we can now easily show $\mathrm{Co}(L_m^{p,q}) = \mathcal{D}_{\mathrm{space}}(\mathcal{Q}, L^p, \ell_u^q)$, at least up to (canonical) identifications. In this sense, wavelet coorbit spaces *are* (special) decomposition spaces.

**Theorem 4.6.4.** *Let $\mathcal{Q}$ be an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and let $u : \mathcal{O} \to (0,\infty)$ be a decomposition weight for $\mathrm{Co}(L_m^{p,q})$.*

*The canonical restriction map $\Theta : \mathcal{R} \to Z'(\mathcal{O})$ defined in Theorem 4.3.1 restricts to an isomorphism of Quasi-Banach spaces*

$$
\Theta : \mathrm{Co}(L_m^{p,q}) \to \mathcal{D}_{\mathrm{space}}(\mathcal{Q}, L^p, \ell_u^q).
$$

*Proof.* Theorem 4.6.3 shows that

$$\Theta^{-1} \circ \mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathrm{Co}\left(L^{p,q}_m\right)$$

is an isomorphism of Quasi-Banach spaces.

But Remark 3.4.7 shows that

$$\mathcal{F} : \mathcal{D}_{\mathrm{space}}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$$

is an (isometric) isomorphism. Hence,

$$\Theta^{-1} = \left(\Theta^{-1} \circ \mathcal{F}^{-1}\right) \circ \mathcal{F} : \mathcal{D}_{\mathrm{space}}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathrm{Co}\left(L^{p,q}_m\right)$$

is an isomorphism of Quasi-Banach spaces. □

We close this section by comparing the proof techniques of this chapter with those in [41], which was written by Hartmut Führ and the present author. In [41], only the case $p, q \in [1, \infty]$ was considered. To extend the results to the whole range $p, q \in (0, \infty]$, three major changes had to be implemented:

A. In the Quasi-Banach case, coorbit theory imposes stronger restrictions on the set of analyzing vectors. Thus, [41, Theorem 9] had to be generalized to Theorem 4.2.11, i.e. we showed

$$W_\psi \psi \in W^R\left(L^\infty, W\left(L^\infty, L^{p,q}_w\right)\right)$$

for all $\psi \in \mathcal{S}\left(\mathbb{R}^d\right)$ with compact Fourier support in $\mathcal{O}$, every weight $w$ of moderate growth and all $p, q \in (0, \infty]$. In contrast, [41, Theorem 9] only established $W_\psi \psi \in W^R\left(L^\infty, L^{p,q}_w\right)$ under the same assumptions.

B. In [41, Lemma 41], it sufficed to establish the estimate

$$\left\| W_\psi\left(f \circ \mathcal{F}^{-1}\right)\right\|_{L^{p,q}_m} \lesssim \|f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)}$$

for $f \in \mathcal{S}\left(\mathbb{R}^d\right) \cap \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$. But since coorbit spaces for Quasi-Banach spaces are defined using the space $W\left(L^\infty, L^{p,q}_m\right)$ instead of $L^{p,q}_m$, the stronger estimate

$$\left\| W_\psi\left(f \circ \mathcal{F}^{-1}\right)\right\|_{W\left(L^\infty, L^{p,q}_m\right)} \lesssim \|f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)}$$

had to be established, cf. Lemmas 4.6.1 and 4.6.2.

C. In most of the proofs in [41], Young's convolution inequality $L^1 * L^p \hookrightarrow L^p$ is used repeatedly. As we observed in Section 3.1, this inequality fails in general for $p \in (0, 1)$. Thus, all of these estimates had to be modified to use the convolution relation given in Theorem 3.1.4 instead.

An important technical detail is that in most cases the simple estimate

$$\left\| \mathcal{F}^{-1}(\psi \cdot \widehat{f}) \right\|_{L^p} \leq [\lambda(Q - \Omega)]^{\frac{1}{p}-1} \cdot \|\mathcal{F}^{-1}\psi\|_{L^p} \cdot \|f\|_{L^p}$$

for $\operatorname{supp}\psi \subset Q$ and $\operatorname{supp}\widehat{f} \subset \Omega$ is not strong enough to derive at the desired conclusion. Instead, one needs to use the pointwise estimate

$$\begin{aligned} &[|\mathcal{F}^{-1}\psi| * |f|](x) \\ &\leq [\lambda(Q - \Omega)]^{\frac{1}{p}-1} \left[ \int_{\mathbb{R}^d} |\mathcal{F}^{-1}\psi(y) \cdot f(x-y)|^p \, \mathrm{d}y \right]^{\frac{1}{p}} \end{aligned}$$

given in equation (3.1.6).

The proofs (see e.g. the proof of Lemma 4.6.1) then typically run as follows: We want to estimate (see equation 4.6.2)

$$\begin{aligned} &\|(M_Q[W_\gamma f])(\cdot, g)\|_{L^p(\mathbb{R}^d)} \\ &= \left\| \left[ M_Q \left( (x,h) \mapsto |\det(h)|^{-\frac{1}{2}} \left[ (\Delta_{h^{-T}} \overline{\gamma}^\vee) * f \right](x) \right) \right](\cdot, g) \right\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Essentially, this means (with notation as in the convolution relation above) that we have $\mathcal{F}^{-1}\psi = \gamma \circ h^{-1}$ for $h$ near $g$ and some (fixed) $\gamma \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\gamma} \in C_c^\infty(\mathbb{R}^d)$. Note that we are considering the maximal function, so that the supremum over all $h$ near $g$ has to be calculated *before* taking the $L^p$-norm.

Thus, we first have to use the (Fourier) support conditions on the factors $\psi, f$ to obtain the *pointwise* estimate for the convolution product as above. This yields

$$\left| \left[ (\Delta_{h^{-T}} \overline{\gamma}^\vee) * f \right](x) \right| \lesssim \left[ \int_{\mathbb{R}^d} |\gamma(h^{-1}y)|^p \cdot |f(x-y)|^p \, \mathrm{d}y \right]^{1/p},$$

where the implied constant depends (in a controlled way) on $g$.

Afterwards, one can estimate $\gamma$ by a less regular, but simpler function, like $(1+|x|)^{-N}$, i.e.

$$|\gamma(h^{-1}y)| \lesssim (1 + |h^{-1}y|)^{-N} \lesssim (1 + |g^{-1}y|)^{-N},$$

where the last estimate used that $h$ is near to $g$. This estimate could *not* have been applied before, since this would have destroyed the Fourier support property of $\psi$.

Using $|f|^p \in L^1(\mathbb{R}^d)$, one can finally apply the convolution relation $L^1 * L^1 \hookrightarrow L^1$ to complete the estimate.

## 4.7 DILATION INVARIANCE OF COORBIT SPACES

One of the motivations for establishing the isomorphism between wavelet coorbit spaces and decomposition spaces was the desire to compare coorbit spaces "living" on different groups. In this section, we will see that this question of comparing coorbit spaces on different groups arises quite naturally, even if one is only interested in properties of coorbit spaces living on a fixed group.

More precisely, we will show that invariance of a coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ under dilation with a matrix $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$ is equivalent to existence of an embedding

$$\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right) \hookrightarrow \mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^d \rtimes g^{-1}Hg\right)\right).$$

Here, it is worth noting that the coorbit space on the left-hand side is a subspace of $\mathcal{R}_v = \left(\mathcal{H}^1_v\right)^{\neg}$, where the space $\mathcal{H}^1_v = \mathcal{H}^1_v\left(\mathbb{R}^d \rtimes H\right)$ depends on the group $\mathbb{R}^d \rtimes H$ and on the control weight $v$, which in turn depends on $p, q, m$. Analogously, the coorbit space on the right-hand side is a subspace of $\left(\mathcal{H}^1_v\left(\mathbb{R}^d \rtimes g^{-1}Hg\right)\right)^{\neg}$, which is (possibly) a completely different object. Hence, it is not clear what we mean by an embedding between these two coorbit spaces.

Luckily, both coorbit spaces are defined using the quasi-regular representation of the respective group, which in both cases acts on $L^2\left(\mathbb{R}^d\right)$. Thus, we can make sense of the notion of an embedding between the two spaces in the following way:

**Definition 4.7.1.** Let $p_1, p_2, q_1, q_2 \in (0, \infty]$.

A. Let $H_1, H_2 \leq \mathrm{GL}\left(\mathbb{R}^d\right)$ be two closed subgroups fulfilling our standing assumptions and assume furthermore that the weights $m_i : H_i \to (0, \infty)$ also fulfill our standing assumptions for $i \in \underline{2}$. Define $G_i := \mathbb{R}^d \rtimes H_i$ for $i \in \underline{2}$. We then say that a bounded linear map

$$\iota : \mathrm{Co}\left(L^{p_1,q_1}_{m_1}\left(G_1\right)\right) \to \mathrm{Co}\left(L^{p_2,q_2}_{m_2}\left(G_2\right)\right)$$

is an EMBEDDING OF COORBIT SPACES if $\iota f = f$ holds for all $f \in L^2\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p_1,q_1}_{m_1}\left(\mathbb{R}^d \rtimes H_1\right)\right)$.

B. Let $\varnothing \neq \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q}^{(i)}$ is an $L^{p_i}$-decomposition covering of $\mathcal{O}_i$ for $i \in \underline{2}$. Finally, assume that $u^{(i)}$ is a $\mathcal{Q}^{(i)}$-moderate weight for $i \in \underline{2}$. We then say that a bounded linear map

$$\theta : \mathcal{D}\left(\mathcal{Q}^{(1)}, L^{p_1}, \ell^{q_1}_{u^{(1)}}\right) \to \mathcal{D}\left(\mathcal{Q}^{(2)}, L^{p_2}, \ell^{q_2}_{u^{(2)}}\right)$$

is an EMBEDDING OF DECOMPOSITION SPACES if $\theta f = f$ holds for all $f \in L^2\left(\mathbb{R}^d\right) \cap \mathcal{D}\left(\mathcal{Q}^{(1)}, L^{p_1}, \ell^{q_1}_{u^{(1)}}\right)$.

*Remark.* It is worth noting that an embedding in the above sense is *not* required to be an injective map.

Using the results from this chapter, the existence of an embedding of coorbit spaces can be characterized by the existence of an embedding between the associated decomposition spaces as follows:

**Lemma 4.7.2.** *Let $p_1, p_2, q_1, q_2 \in (0, \infty]$ and let $H_1, H_2, G_1, G_2$ and $m_1, m_2$ be as in Definition 4.7.1. Finally, let $\mathcal{Q}^{(j)} = (\mathcal{Q}_i^{(j)})_{i \in I^{(j)}}$ be an induced $L^{p_j}$-decomposition covering of the dual orbit $\mathcal{O}_j \subset \mathbb{R}^d$ of the group $H_j$ and let $u^{(j)} : \mathcal{O}_j \to (0, \infty)$ be a decomposition weight for $\mathrm{Co}\left(L^{p_j,q_j}_{m_j}(G_j)\right)$ for $j \in \underline{2}$.*

*Then*

$$\iota : \mathrm{Co}\left(L^{p_1,q_1}_{m_1}(G_1)\right) \to \mathrm{Co}\left(L^{p_2,q_2}_{m_2}(G_2)\right)$$

*is an embedding of coorbit spaces if and only if*

$$\mathcal{F} \circ \iota \circ \mathcal{F}^{-1} : \mathcal{D}\left(\mathcal{Q}^{(1)}, L^{p_1}, \ell^{q_1}_{u^{(1)}}\right) \to \mathcal{D}\left(\mathcal{Q}^{(2)}, L^{p_2}, \ell^{q_2}_{u^{(2)}}\right)$$

*is an embedding of decomposition spaces. Here, $\mathcal{F} = \mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}^{-1} = \mathcal{F}_{\mathcal{R}}^{-1}$ are defined as in Theorems 4.5.9 and 4.6.3, respectively.*

*Proof.* Since $\mathcal{F}, \mathcal{F}^{-1}$ are isomorphisms of Quasi-Banach spaces by Theorem 4.6.3, we see that $\iota$ is bounded if and only if $\mathcal{F} \circ \iota \circ \mathcal{F}^{-1}$ is bounded.

Now, assume that $\iota$ is an embedding of coorbit spaces and choose any $f \in L^2(\mathbb{R}^d) \cap \mathcal{D}(\mathcal{Q}^{(1)}, L^{p_1}, \ell^{q_1}_{u^{(1)}})$. Theorem 4.6.3 shows that

$$\mathcal{F}^{-1} f \in \mathrm{Co}\left(L^{p_1,q_1}_{m_1}(G_1)\right) \cap L^2(\mathbb{R}^d)$$

is just the "usual" inverse Fourier transform of $f \in L^2(\mathbb{R}^d)$. By definition of an embedding of coorbit spaces, this yields $\iota\left(\mathcal{F}^{-1} f\right) = \mathcal{F}^{-1} f$.

But Theorem 4.3.1 shows that the Fourier transform $\mathcal{F}_{\mathcal{R}}$ coincides with the usual Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}^d)$. Hence,

$$\left(\mathcal{F} \circ \iota \circ \mathcal{F}^{-1}\right) f = \mathcal{F}\mathcal{F}^{-1} f = f.$$

We have thus shown that $\mathcal{F} \circ \iota \circ \mathcal{F}^{-1}$ is an embedding of decomposition spaces.

The proof of the reverse direction is completely analogous and hence omitted. □

Finally, we show that dilation invariance of $\mathrm{Co}\left(L^{p,q}_m(\mathbb{R}^d \rtimes H)\right)$ indeed amounts to existence of an embedding between two coorbit spaces which "live" on different groups. To this end, we introduce the FULL QUASI-REGULAR REPRESENTATION

$$\sigma : \mathbb{R}^d \rtimes \mathrm{GL}(\mathbb{R}^d) \to \mathcal{U}\left(L^2(\mathbb{R}^d)\right), (x, h) \mapsto L_x D_h,$$

which is an extension of all quasi-regular representations $\pi$ on $\mathbb{R}^d \rtimes H$ for arbitrary dilation groups $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$.

With this notation, we have the following:

**Lemma 4.7.3.** *Let $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$ be a closed subgroup fulfilling our standing assumptions. For arbitrary $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$, define $H^g := g^{-1}Hg$. Then the following are true:*

1. *$H^g$ is admissible. More precisely, if $\mathcal{O} = H^T\xi_0$ denotes the open dual orbit of $H$, then the open dual orbit of $H^g$ is given by*

$$\mathcal{O}^g := g^T\mathcal{O} = \left(H^g\right)^T g^T\xi_0$$

   *and the stabilizier of $g^T\xi_0$ is given by $H^g_{g^T\xi_0} = g^{-1}H_{\xi_0}g$, which is compact.*

2. *If $\psi_1 \in \mathcal{S}\left(\mathbb{R}^d\right)$ satisfies $\widehat{\psi_1} \in C_c^\infty\left(\mathcal{O}\right)$, then $\psi_2 := \sigma\left(0, g^{-1}\right)\psi_1$ satisfies $\widehat{\psi_2} \in C_c^\infty\left(\mathcal{O}^g\right)$.*

3. *For $\psi_1, \psi_2$ as in the previous parts and $f \in L^2\left(\mathbb{R}^d\right)$, we have*

$$W^1_{\psi_1}\left[\sigma\left(0,g\right)f\right]\left(x,h\right) = \left(W^2_{\psi_2}f\right)\left(g^{-1}x, g^{-1}hg\right)$$

   *for all $(x,h) \in \mathbb{R}^d \rtimes H$, where $W^1_{\psi_1}$ denotes the wavelet transform on $\mathbb{R}^d \rtimes H$ and $W^2_{\psi_2}$ denotes the wavelet transform on $\mathbb{R}^d \rtimes H^g$.*

4. *If $m : H \to (0,\infty)$ is a weight, define*

$$m^g : H^g \to (0,\infty), h \mapsto m\left(ghg^{-1}\right).$$

   *If $m$ is right-moderate with respect to some weight $m_0 : H \to (0,\infty)$ of moderate growth, then $m^g$ is right-moderate with respect to $m_0^g$, which is again of moderate growth.*

5. *For $f \in L^2\left(\mathbb{R}^d\right)$, we have*

$$\left\|\sigma\left(0,g\right)f\right\|_{\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)} \asymp \left\|f\right\|_{\mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^d \rtimes H^g\right)\right)},$$

   *where the implied constants only depend on $g, p, q$.*

6. *If $\mathcal{Q} = \left(h_i^{-T}Q\right)_{i\in I}$ is a covering of $\mathcal{O}$ induced by $H$, then the family $g\mathcal{Q} = \left(gh_i^{-T}Q\right)_{i\in I}$ is a covering of $\mathcal{O}^g$ induced by $H^g$.*

   *Furthermore, if $u = \left(u_i\right)_{i\in I}$ is a $\mathcal{Q}$-discretization of a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$, then $u$ is also a $g\mathcal{Q}$-discretization of a decomposition weight for $\mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^d \rtimes H^g\right)\right)$.*

*Remark* 4.7.4. In the following, we will say that $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is INVARIANT UNDER DILATION BY $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$, if the map

$$L^2\left(\mathbb{R}^d\right) \cap \mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right) \to \mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right), f \mapsto \sigma\left(0,g\right)f$$

is well-defined and bounded.

By the above lemma, this is equivalent to well-definedness and boundedness of

$$L^2(\mathbb{R}^d) \cap \mathrm{Co}\left(L^{p,q}_m(\mathbb{R}^d \rtimes H)\right) \to \mathrm{Co}\left(L^{p,q}_{m^g}(\mathbb{R}^d \rtimes H^g)\right), f \mapsto f.$$

In particular, if there is an embedding of coorbit spaces

$$T : \mathrm{Co}\left(L^{p,q}_m(\mathbb{R}^d \rtimes H)\right) \to \mathrm{Co}\left(L^{p,q}_{m^g}(\mathbb{R}^d \rtimes H^g)\right),$$

then $\mathrm{Co}\left(L^{p,q}_m(\mathbb{R}^d \rtimes H)\right)$ is invariant under dilation by $g$. Using the above lemma and Lemma 4.7.2, it is thus sufficient if there is an embedding of decomposition spaces

$$S : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(g^T\mathcal{Q}, L^p, \ell^q_u\right),$$

where $\mathcal{Q}$ is an $L^p$-decomposition covering of $\mathcal{O}$ induced by $H$ and where $u$ is a $\mathcal{Q}$-discretization of a decomposition weight for the coorbit space $\mathrm{Co}\left(L^{p,q}_m(\mathbb{R}^d \rtimes H)\right)$. ◀

*Proof.* 1. Since the conjugation map $\mathrm{GL}(\mathbb{R}^d) \to \mathrm{GL}(\mathbb{R}^d), h \mapsto g^{-1}hg$ is a homeomorphism, and also an automorphism, $H^g \leq \mathrm{GL}(\mathbb{R}^d)$ is a closed subgroup. Furthermore, we get

$$(H^g)^T g^T\xi_0 = g^T H^T (g^{-1})^T g^T\xi_0 = g^T H^T\xi_0 = g^T\mathcal{O},$$

which is open. Finally,

$$\left[(H^g)^T g^T\xi_0\right]^c = g^T\mathcal{O}^c \subset \mathbb{R}^d$$

is a nullset, so that the open dual orbit of $H^g$ is of full measure and thus the unique open dual orbit.

Finally, for $h \in g^{-1}Hg$ we have

$$\begin{aligned} h \in H^g_{g^T\xi_0} &\iff h^T g^T\xi_0 = g^T\xi_0 \\ &\iff (ghg^{-1})^T\xi_0 = \xi_0 \\ &\iff ghg^{-1} \in H_{\xi_0} \\ &\iff h \in g^{-1}H_{\xi_0}g \leq H^g, \end{aligned}$$

where $g^{-1}H_{\xi_0}g$ is compact as the image of a compact set under the conitnuous conjugation map.

2. We have

$$\mathcal{F}\psi_2 = \mathcal{F}\left[\sigma(0, g^{-1})\,\psi_1\right] = |\det(g)|^{-1/2} \cdot \widehat{\psi_1} \circ g^{-T},$$

so that

$$\mathrm{supp}\left[\mathcal{F}\psi_2\right] = g^T \,\mathrm{supp}\,\widehat{\psi_1} \subset g^T\mathcal{O} = \mathcal{O}^g$$

is compact.

3. We calculate

$$\begin{aligned}\left(W^2_{\psi_2} f\right)(g^{-1}x, g^{-1}hg) &= \langle f \mid \pi(g^{-1}x, g^{-1}hg)\,\psi_2\rangle \\ &= \langle f \mid \sigma((0,g^{-1})(x,h)(0,g))\,\psi_2\rangle \\ &= \langle \sigma(0,g) f \mid \sigma(x,h)\,\sigma(0,g)\,\psi_2\rangle \\ &= \langle \sigma(0,g) f \mid \sigma(x,h)\,\psi_1\rangle \\ &= \left(W^1_{\psi_1}[\sigma(0,g) f]\right)(x,h).\end{aligned}$$

Here, we used that $\sigma$ is indeed a representation of $\mathbb{R}^d \rtimes \mathrm{GL}\left(\mathbb{R}^d\right)$.

4. We have

$$\begin{aligned}m^g(xy) &= m(gxyg^{-1}) = m(gxg^{-1}gyg^{-1}) \\ &\leq m(gxg^{-1}) \cdot m_0(gyg^{-1}) \\ &= m^g(x) \cdot m_0^g(y)\end{aligned}$$

for all $x, y \in H^g$. Hence, $m^g$ is right-moderate with respect to $m_0^g$. Finally, a weight which only depends on the second factor is of moderate growth if and only if it is locally bounded. Since this holds for $m_0$, it clearly holds for $m_0^g$.

5. It is not hard to see that a left Haar-measure on $H^g = g^{-1}Hg$ is given by

$$\nu(M) = \mu(gMg^{-1}),$$

where $\mu$ is the left Haar measure on $H$. Since this is the image measure of the map $H \to H^g, h \mapsto g^{-1}hg$, we conclude (see e.g. [33, Proposition 10.1])

$$\int_{H^g} f \,\mathrm{d}\nu = \int_H f(g^{-1}hg)\,\mathrm{d}h$$

for each measurable function $f : H^g \to \mathbb{C}$, whenever either side of the equation is defined.

Now, pick any nonzero $\psi_1 \in \mathcal{S}\left(\mathbb{R}^d\right)$ with $\widehat{\psi_1} \in C_c^\infty(\mathcal{O})$. As seen above, this yields $\widehat{\psi_2} \in C_c^\infty(\mathcal{O}^g)$ for $\psi_2 := \sigma\left(0, g^{-1}\right)\psi_1$. By Corollary 4.2.12, this shows that $\psi_1, \psi_2$ are admissible as analyzing vectors for the respective coorbit spaces. Now, let $U_1 \subset \mathbb{R}^d$ and $U_2 \subset H$ be precompact open unit neighborhoods. Clearly, $U_2^g = g^{-1}U_2 g \subset H^g$ is also a precompact open unit neighborhood, as is $g^{-1}U_1 \subset \mathbb{R}^d$. For brevity, let us write $V := U_1 \times U_2$ and $V^g := g^{-1}U_1 \times U_2^g$.

Since $W^1_{\psi_1}[\sigma(0,g) f]$ is continuous and since $V, V^g$ are open, we get

$$\begin{aligned}&\left[K_V\left(W^1_{\psi_1}[\sigma(0,g) f]\right)\right](x,h) \\ &= \sup_{(y,k)\in U_1\times U_2} \left|\left(W^1_{\psi_1}[\sigma(0,g) f]\right)((x,h)(y,k))\right| \\ &= \sup_{(y,k)\in U_1\times U_2} \left|\left(W^2_{\psi_2} f\right)[(g^{-1}(x+hy),\, g^{-1}hkg)]\right|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{(y,k)\in U_1\times U_2} \left|\left(W^2_{\psi_2} f\right)\left[(g^{-1}x + g^{-1}hy,\, g^{-1}hg\cdot g^{-1}kg)\right]\right| \\
&\overset{(\dagger)}{=} \sup_{(z,\ell)\in g^{-1}U_1\times U_2^g} \left|\left(W^2_{\psi_2} f\right)\left[(g^{-1}x + g^{-1}hg\cdot z,\, g^{-1}hg\cdot \ell)\right]\right| \\
&= \sup_{(z,\ell)\in g^{-1}U_1\times U_2^g} \left|\left(W^2_{\psi_2} f\right)\left[((g^{-1}x, g^{-1}hg)\,(z,\ell))\right]\right| \\
&= \left[K_{V^g}\left(W^2_{\psi_2} f\right)\right](g^{-1}x,\, g^{-1}hg)\,,
\end{aligned}
$$

where the change of variables $z = g^{-1}y$ and $\ell = g^{-1}kg$ was used at $(\dagger)$.

Using the change-of-variables formula, we arrive at

$$
\begin{aligned}
&\left\|\left[K_V\left(W^1_{\psi_1}\left[\sigma\,(0,g)\,f\right]\right)\right](\cdot,h)\right\|_{L^p\left(\mathbb{R}^d\right)} \\
&= \left\|\left[K_{V^g}\left(W^2_{\psi_2} f\right)\right](g^{-1}\cdot, g^{-1}hg)\right\|_{L^p\left(\mathbb{R}^d\right)} \\
&= \left|\det(g)\right|^{1/p}\cdot\left\|\left[K_{V^g}\left(W^2_{\psi_2} f\right)\right](\cdot,\, g^{-1}hg)\right\|_{L^p\left(\mathbb{R}^d\right)}
\end{aligned}
$$

for all $h \in H$.

Let us assume $q \in (0,\infty)$. Since $\left|\det(g)\right|^{1/p}$ is an absolute constant, independent of $f$, we get

$$
\begin{aligned}
&\left\|\sigma\,(0,g)\,f\right\|^q_{\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d\rtimes H\right)\right)} \\
&= \left\|W^1_{\psi_1}\left[\sigma\,(0,g)\,f\right]\right\|^q_{W\left(L^\infty, L^{p,q}_m\left(\mathbb{R}^d\rtimes H\right)\right)} \\
&= \int_H \left[\left\|\left[K_V\left(W^1_{\psi_1}\left[\sigma\,(0,g)\,f\right]\right)\right](\cdot,h)\right\|_{L^p}\cdot m\,(h)\right]^q \frac{\mathrm{d}h}{\left|\det(h)\right|} \\
&\asymp \int_H \left[\left\|\left[K_{V^g}\left(W^2_{\psi_2} f\right)\right](\cdot,\, g^{-1}hg)\right\|_{L^p}\cdot m^g\,(g^{-1}hg)\right]^q \frac{\mathrm{d}h}{\left|\det(g^{-1}hg)\right|} \\
&= \int_{H^g} \left[\left\|\left[K_{V^g}\left(W^2_{\psi_2} f\right)\right](\cdot,h)\right\|_{L^p}\cdot m^g\,(h)\right]^q \frac{\mathrm{d}h}{\left|\det(h)\right|} \\
&= \left\|W^2_{\psi_2} f\right\|^q_{W\left(L^\infty, L^{p,q}_{m^g}\left(\mathbb{R}^d\rtimes H^g\right)\right)} \\
&= \left\|f\right\|^q_{\mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^d\rtimes H^g\right)\right)}.
\end{aligned}
$$

Here, we used that $V^g \subset \mathbb{R}^d \rtimes H^g$ is an open, precompact unit neighborhood and that we are free in our choice of the unit neighborhood for defining the quasi-norm $\|\cdot\|_{W\left(L^\infty, L^{p,q}_{m^g}\right)}$.

The argument for $q = \infty$ is analogous.

6. By definition of an induced covering (cf. Definition 4.4.3), we know that $\overline{Q} \subset \mathcal{O}$ is compact and that $(h_i)_{i\in I}$ is a relatively separated

family in $H$ with $\mathcal{O} = \bigcup_{i\in I} h_i^{-T} Q$. It is easy to see that $(g^{-1}h_i g)_{i\in I}$ is relatively separated in $H^g$. Furthermore, $\overline{g^T Q} \subset \mathcal{O}^g$ is compact with

$$\mathcal{O}^g = g^T \mathcal{O} = \bigcup_{i\in I} g^T h_i^{-T} Q = \bigcup_{i\in I} \left[ \left( g^{-1} h_i g \right)^{-T} \cdot g^T Q \right].$$

Again by definition, this means that

$$\left( g^T h_i^{-T} Q \right)_{i\in I} = \left( \left[ g^{-1} h_i g \right]^{-T} \cdot g^T Q \right)_{i\in I}$$

is a covering of $\mathcal{O}^g$ induced by $H^g$.

Finally, by definition of a decomposition weight (cf. Definition 4.5.3) for $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ and of a $\mathcal{Q}$-discretization, there is some $\xi_0 \in \mathcal{O}$ and for each $i \in I$ some $k_i \in H$ with $k_i^{-T}\xi_0 \in h_i^{-T} Q$ and with $u_i = m^{(q)}(k_i)$, where

$$m^{(q)} : H \to (0,\infty), h \mapsto \left|\det(h)\right|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h).$$

Note that conjugation invariance of the determinant yields

$$\begin{aligned} \left(m^g\right)^{(q)}(h) &= \left|\det(h)\right|^{\frac{1}{2}-\frac{1}{q}} \cdot m^g(h) \\ &= \left|\det(ghg^{-1})\right|^{\frac{1}{2}-\frac{1}{q}} \cdot m(ghg^{-1}) \\ &= \left(m^{(q)}\right)^g(h) \end{aligned}$$

for all $h \in H^g$. This implies

$$u_i = m^{(q)}(k_i) = \left(m^{(q)}\right)^g (g^{-1}k_i g) = \left(m^g\right)^{(q)} (g^{-1}k_i g),$$

where we have

$$\left(g^{-1}k_i g\right)^{-T} g^T \xi_0 = g^T k_i^{-T} g^{-T} g^T \xi_0 = g^T k_i^{-T} \xi_0 \in g^T h_i^{-T} Q.$$

Hence, $u$ is a $g\mathcal{Q}$-discretization of a $g^T\xi_0$-transplant of $\left(m^g\right)^{(q)}$. By definition, this means that $u$ is a $g\mathcal{Q}$-discretization of a decomposition weight for $\mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^d \rtimes H^g\right)\right)$. □

As a quick application of our results in this section, we now show that *all* coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ are invariant under dilation by *arbitrary* $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$, if $H$ is the SIMILITUDE GROUP. This is probably well-known, but I do not know any convenient source for this results, except for [41, Section 9], which is written by Hartmut Führ and the present author.

We will see in Section 6.5.2 of Chapter 6 that this invariance property is indeed special to the similitude group. In fact, we will completely characterize the set of matrices under which all coorbit spaces with respect to the shearlet-type groups $H_c$ are dilation invariant. For the precise definition of these groups, we refer to Chapter 6.

**Lemma 4.7.5.** *Assume $d \geq 2$ and let*

$$H = \mathbb{R}_+^* \cdot \mathrm{SO}\left(\mathbb{R}^d\right) = \left\{r \cdot A \,|\, r \in (0,\infty) \text{ and } A \in \mathrm{SO}\left(\mathbb{R}^d\right)\right\}.$$

*Then $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$ is admissible with open dual orbit*

$$\mathcal{O} = H^T e_1 = \mathbb{R}^d \setminus \{0\},$$

*with $e_1 = (1, 0, \ldots, 0)$.*

*A structured admissible covering of $\mathcal{O}$ induced by $H$ is given by*

$$\mathcal{Q} = (Q_k)_{k\in\mathbb{Z}} = \left(2^k \mathrm{id} \cdot B\right)_{k\in\mathbb{Z}},$$

*with $B := B_4(0) \setminus \overline{B_{\frac{1}{4}}(0)}$.*

*If $m : H \to (0,\infty)$ is right-moderate with respect to a locally bounded weight $m_0 : H \to (0,\infty)$, then $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is invariant under dilation by $g$ for arbitrary $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$.*

*Remark.* It is worth noting that the same claim (even with the same proof) holds for any admissible dilation group $H$ which admits a well-spread family $(h_i)_{i\in I}$ commuting with $g$, i.e. which satisfies $gh_i = h_i g$ for all $i \in I$.

*Proof.* Let $x \in S^{d-1}$ be arbitrary. By extending $x$ to an orthonormal basis $A := \left(x, x^{(2)}, \ldots, x^{(d)}\right)$ of $\mathbb{R}^d$, we obtain an orthogonal matrix $A \in O\left(\mathbb{R}^d\right)$ with $Ae_1 = x$. If necessary, we can multiply $x^{(d)}$ by $\pm 1$ to achieve $\det A = 1$, i.e. $A \in \mathrm{SO}\left(\mathbb{R}^d\right)$. Note that this is only possible because of $d \geq 2$.

This shows $x = \left(A^T\right)^T e_1 \in \mathcal{O}$ for all $x \in S^{d-1}$. Since $H$ contains all positive scalars, we get $\mathcal{O} = H^T e_1 = \mathbb{R}^d \setminus \{0\}$, which is open and of full measure.

Finally, if $(r \cdot A)^T e_1 = e_1$ holds, we get $r = \left|(r \cdot A)^T e_1\right| = |e_1| = 1$ and hence $r \cdot A = A \in \mathrm{SO}\left(\mathbb{R}^d\right)$, which is a compact subgroup of $H$. Hence, we see that the isotropy group

$$H_{e_1} = \left\{h \in H \,\middle|\, h^T e_1 = e_1\right\} \leq \mathrm{SO}\left(\mathbb{R}^d\right)$$

is compact as a closed subgroup of a compact group. In conclusion, $H$ is admissible.

Note that

$$U := \left\{h \in H \,\middle|\, \frac{1}{\sqrt{2}} < \|h\| < \sqrt{2}\right\} \subset H$$

is a precompact open unit neighorhood, because of

$$\overline{U} \subset \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right] \cdot \mathrm{SO}\left(\mathbb{R}^d\right),$$

where the right-hand side is compact. Furthermore, note that we have $\|r \cdot A\| = r$ for all $A \in \mathrm{SO}\left(\mathbb{R}^d\right)$. This yields $\|hg\| = \|h\| \, \|g\|$ and $\left\|h^{-1}\right\| = \|h\|^{-1}$ for all $h, g \in H$.

For $k \in \mathbb{Z}$, define $h_k := 2^k \cdot \mathrm{id} \in H$. Let $h \in H$ be arbitrary. For $k, \ell \in \mathbb{Z}$ with $h \in h_k U \cap h_\ell U$, we have $h_k^{-1} h, h_\ell^{-1} h \in U$ and hence

$$2^{\ell - k} = \left\|h_k^{-1} h_\ell\right\| = \left\|h_k^{-1} h\right\| \cdot \left\|h^{-1} h_\ell\right\| = \frac{\left\|h_k^{-1} h\right\|}{\left\|h_\ell^{-1} h\right\|} < \frac{\sqrt{2}}{1/\sqrt{2}} = 2.$$

By symmetry, we also get $2^{k-\ell} < 2$. But this implies $\ell - k \leq 0$ and $k - \ell \leq 0$, i.e. $k = \ell$. All in all, we have shown that the function

$$f_U = \sum_{k \in \mathbb{Z}} \chi_{h_k U}$$

is bounded by 1. Using Lemma 2.3.11, we conclude that $(h_k)_{k \in \mathbb{Z}}$ is indeed relatively separated in $H$.

It is clear that $B$ is precompact with $\overline{B} \subset \mathbb{R}^d \setminus \{0\} = \mathcal{O}$. Furthermore, any $x \in \mathbb{R}^d \setminus \{0\}$ satisfies

$$|x| \in (0, \infty) = \bigcup_{k \in \mathbb{Z}} \left(2^{k-1}, 2^{k+1}\right)$$

and hence

$$x \in B_{2^{k+1}}(0) \setminus \overline{B_{2^{k-1}}}(0) = h_k P \quad \text{for some } k \in \mathbb{Z},$$

with $P := B_2(0) \setminus \overline{B_{1/2}}(0)$. Hence, $\mathcal{P} = (h_k \cdot P)_{k \in \mathbb{Z}}$ is a covering of $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$. In particular, $\mathcal{Q}$ also covers $\mathcal{O}$, so that $\mathcal{P}, \mathcal{Q}$ are coverings of $\mathcal{O}$ induced by $H$, cf. Definition 4.4.3.

Using Lemma 4.4.4, we conclude that $\mathcal{P}, \mathcal{Q}$ are semi-structured coverings. By comparing the definitions and by noting $\overline{P} \subset B$, this easily shows that $\mathcal{Q}$ is a structured admissible covering of $\mathcal{O}$ induced by $H$.

*The identity $h_k = g^{-1} h_k g$ holds because of $h_k = 2^k \mathrm{id}$.*

Now, let $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$ be arbitrary. It is easy to see that the family $(h_k)_{k \in \mathbb{Z}} = \left(g^{-1} h_k g\right)_{k \in \mathbb{Z}}$ is relatively separated in $H^g = g^{-1} H g$. By Lemma 4.7.3, we know that $H^g$ is admissible with open dual orbit

$$\mathcal{O}^g = g^T \mathcal{O} = \mathbb{R}^d \setminus \{0\} = \mathcal{O}.$$

The definitions show that $\mathcal{Q}$ is thus also a covering of $\mathcal{O}^g$ induced by $H^g$.

Now, let

$$m^{(q)} : H \to (0, \infty), h \mapsto |\det(h)|^{\frac{1}{2} - \frac{1}{q}} \cdot m(h)$$

as in Definition 4.5.3. A $\mathcal{Q}$-discretization of an $e_1$-transplant of $m^{(q)}$ is given by

$$u_k = m^{(q)}\left(h_k^{-1}\right),$$

since we have $(h_k^{-1})^{-T} e_1 = 2^k e_1 \in Q_k$ for all $k \in \mathbb{Z}$. Likewise, a $\mathcal{Q}$-discretization of an $e_1$-transplant of $(m^g)^{(q)}$ (which is a weight on $H^g$) is given by

$$\begin{aligned} v_k &= (m^g)^{(q)}(h_k^{-1}) = |\det(h_k^{-1})|^{\frac{1}{2}-\frac{1}{q}} \cdot m^g(h_k^{-1}) \\ &= |\det(h_k^{-1})|^{\frac{1}{2}-\frac{1}{q}} \cdot m(g h_k^{-1} g^{-1}) \\ &= |\det(h_k^{-1})|^{\frac{1}{2}-\frac{1}{q}} \cdot m(h_k^{-1}) \\ &= m^{(q)}(h_k^{-1}) = u_k. \end{aligned}$$

Here, we crucially used the identity $g h_k^{-1} g^{-1} = h_k^{-1}$, since $h_k = 2^k \mathrm{id}$.

All in all, we conclude that the two associated decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$ and $\mathcal{D}(\mathcal{Q}, L^p, \ell_v^q)$ to the coorbit spaces $\mathrm{Co}\left(L_m^{p,q}(\mathbb{R}^d \rtimes H)\right)$ and $\mathrm{Co}\left(L_{m^g}^{p,q}(\mathbb{R}^d \rtimes H^g)\right)$ are *literally the same*. By Lemma 4.7.2, we conclude that the map

$$\mathcal{F}^{-1} \circ \mathrm{id} \circ \mathcal{F} : \mathrm{Co}\left(L_m^{p,q}(\mathbb{R}^d \rtimes H)\right) \to \mathrm{Co}\left(L_{m^g}^{p,q}(\mathbb{R}^d \rtimes H^g)\right)$$

is an embedding of coorbit spaces, with

$$\mathrm{id} : \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q) \to \mathcal{D}(\mathcal{Q}, L^p, \ell_v^q), f \mapsto f$$

and where $\mathcal{F} = \mathcal{F}_{\mathcal{R}}, \mathcal{F}^{-1} = \mathcal{F}_{\mathcal{R}}^{-1}$ are defined as in Theorem 4.5.9 and Theorem 4.6.3.

Recalling the remark after Lemma 4.7.3, we see that this suffices for the claimed invariance of $\mathrm{Co}\left(L_m^{p,q}(\mathbb{R}^d \rtimes H)\right)$ under dilation by $g$. □

# 5

# EMBEDDINGS BETWEEN DECOMPOSITION SPACES

The aim of this chapter is to derive sufficient conditions and necessary restrictions for the existence of an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$

between two (Fourier-side) decomposition spaces. The coverings $\mathcal{Q}, \mathcal{P}$ used above are allowed to behave very differently. Most of the time, we will (only) assume $\mathcal{Q}$ to be almost subordinate to $\mathcal{P}$. In this case, we also study existence of the reverse embedding

$$\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right).$$

The most general setting which we will consider is that of two coverings $\mathcal{R}, \mathcal{S}$ where $\mathcal{R}$ is almost subordinate to $\mathcal{S}$ near a certain set $A$ and vice versa near a certain set $B$.

The structure of this chapter is as follows: In the first section, we study sufficient criteria for the existence of embeddings between decomposition spaces. As mentioned in the introduction, our main goal is to show that the existence of certain embeddings between nested sequence spaces suffices for the existence of embeddings between decomposition spaces. These sequence spaces are defined using the INTERSECTION SETS $I_j$ which encode the geometric relations between the coverings $\mathcal{Q}, \mathcal{P}$.

The formulation of our sufficient criteria using embeddings between nested sequence spaces is well-adapted to our proof technique, but cumbersome for applications. Thus, the second section is devoted to a considerable simplification of these assumptions. We will show that it suffices to check finiteness of the nested sequence-space norm of a *single* sequence in order to guarantee the desired embedding between the nested sequence spaces. We remark that this simplification is only possible, since the family of intersection sets $\left(I_j\right)_{j \in J}$ is "almost disjoint", at least if $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

The third section is devoted to studying sharpness of the sufficient criteria developed before. In the first half of this section, we consider the general case, in which we do *not* assume that $\mathcal{Q}$ is relatively moderate with respect to $\mathcal{P}$. In this case, we show that our sufficient conditions for the existence of embeddings are also necessary, but only for a certain range of exponents $p_1, p_2$. For the remaining range of exponents, which is considered in the second half of this section, we assume $\mathcal{Q}$ to be relatively moderate with respect to $\mathcal{P}$. In this case,

our sufficient criteria can be simplified even more and turn out to be sharp for *all* values of $p, q$.

After that, the fourth section summarizes those results which are most suited for applications to concrete examples. Readers who want to get a quick overview of our results should skim this section first.

Finally, we close this chapter by giving sufficient criteria for a continuous embedding $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \hookrightarrow \mathcal{S}'\left(\mathbb{R}^d\right)$. This is nontrivial, since we use $\mathcal{D}'\left(\mathcal{O}\right)$ as our reservoir.

The reader will notice that we pay closer attention than usual to the constant factors whose existence is ensured by various theorems. That is, we do not simply state that there is a constant $C = C\left(\mathcal{Q}, \mathcal{P}\right)$ with certain properties; instead, we indicate on which parameters of the coverings $\mathcal{Q}, \mathcal{P}$ the constant depends. The reason for this will become clear in Chapter 6: One application of our embedding results is that we can study the dilation invariance of decomposition spaces. In fact, we saw in Section 3.5 that invariance of the space $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$ under dilation with $h \in \mathrm{GL}\left(\mathbb{R}^d\right)$ is equivalent to existence of an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) \hookrightarrow \mathcal{D}\left(h^{-1}\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right).$$

*We recall that $u = (u_i)_{i\in I}$ is a discrete weight on the index set of the covering $\mathcal{Q} = (Q_i)_{i\in I}$. Since $h^{-1}\mathcal{Q} = \left(h^{-1}Q_i\right)_{i\in I}$ uses the same index-set, the same weight $u$ can be used on both sides of the embedding.*

Furthermore, the operator norm of the dilation operator is equivalent to the operator norm of the embedding. Thus, knowing the dependence of various constants on the covering $h^{-1}\mathcal{Q}$ will be crucial.

## 5.1 SUFFICIENT CONDITIONS

As indicated above, the goal of this section is to show that existence of embeddings for certain nested sequence spaces implies existence of embeddings for certain decomposition spaces. For this, we first consider the "local" setting, i.e. we fix an index $j_0 \in J$ and consider a set of indices $I_0$ with the property $Q_i \subset P^{k*}_{j_0}$ for all $i \in I_0$ and a certain fixed $k \in \mathbb{N}_0$. In this setting, we are interested in estimating $\left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p}$ – or actually the whole family $\left(\left\|\mathcal{F}^{-1}\left(\varphi_i f\right)\right\|_{L^p}\right)_{i\in I_0}$, measured in a certain sequence space norm – by $\left\|\mathcal{F}^{-1}\left(\psi_{j_0} f\right)\right\|_{L^p}$ and vice versa. Determining the correct sequence space norm is part of the problem.

We will derive this local estimate from Plancherel's theorem by interpolation. It is worth noting that the usual approach for establishing embeddings seems to be to handle some special cases explicitly and to obtain the remaining cases by interpolation of the *whole* decomposition space (cf. e.g. [43, 49]). In contrast to this, we only apply interpolation once in the local setting and use suitable embedding assumptions for nested sequence spaces to derive the global result.

In the statement of the lemma below, we use the notion of the CONJUGATE EXPONENT $p'$ for $p \in (0,\infty]$. We define this exponent by

$$p' := \begin{cases} \infty, & \text{for } p \in (0,1], \\ \frac{p}{p-1}, & \text{for } p \in (1,\infty), \\ 1, & \text{for } p = \infty. \end{cases}$$

We remark that this coincides with the usual definition for $p \in [1,\infty]$, which means $\frac{1}{p} + \frac{1}{p'} = 1$ for $p \in [1,\infty]$. But for $p \in (0,1)$, this is false, since $\frac{1}{p} + \frac{1}{p'} = \frac{1}{p} > 1$. Thus, we define the DUALITY DEFECT $p_d$ by

*We remark that the "d" in $p_d$ stands for "defect" and has nothing to do with the dimension of the space $\mathbb{R}^d$.*

$$\begin{aligned} p_d &:= \left(1 - \frac{1}{p}\right)_- = \min\left\{0, 1 - \frac{1}{p}\right\} \\ &= \begin{cases} 0, & \text{if } p \in [1,\infty], \\ 1 - \frac{1}{p}, & \text{if } p \in (0,1). \end{cases} \end{aligned} \tag{5.1.1}$$

For an $L^p$-decomposition covering $\mathcal{Q}$, we define the DUALITY DEFECT WEIGHT $w^{\mathcal{Q},p}$ by

$$w_i^{\mathcal{Q},p} := \begin{cases} 1, & \text{if } p \in [1,\infty], \\ |\det(T_i)|^{-p_d} = |\det(T_i)|^{\frac{1}{p}-1}, & \text{if } p \in (0,1), \end{cases} \tag{5.1.2}$$

where for $p \in (0,1)$, the $L^p$-decomposition covering $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ is semi-structured, cf. Definition 3.2.10. We observe that equation (3.2.1) easily implies that $w^{\mathcal{Q},p}$ is a $\mathcal{Q}$-moderate weight.

Finally, we define the UPPER CONJUGATE EXPONENT $p^\triangle$ OF $p$ by

*The idea of the notation is that the upward pointing triangle indicates the* upper *conjugate exponent. Further, the terminology "*upper *conjugate exponent" is chosen, since we are taking the* maximum *(as opposed to the minimum) of $p$ and $p'$.*

$$p^\triangle := \max\{p, p'\} = \begin{cases} p, & \text{for } p \in [2,\infty], \\ p', & \text{for } p \in (0,2]. \end{cases} \tag{5.1.3}$$

With these notions, we can finally state and prove our first local embedding result.

**Lemma 5.1.1.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let $p \in (0,\infty]$. Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ are $L^p$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively, with corresponding $L^p$-BAPUs $\Phi = (\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in J}$.*

*Let $j_0 \in J$ and $I_0 \subset I_{j_0}$ be arbitrary. Assume that there is some $k \in \mathbb{N}_0$, such that for each $i \in I_0$, the inclusion $Q_i \subset P_{j_i}^{k*}$ holds for some $j_i \in J$.*

*Then for $p \in [1,\infty]$ and $f \in \mathcal{D}'(\mathcal{O}')$, we have*

$$\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i\in I_0} \right\|_{\ell^{p^\triangle}} \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p} \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p}. \tag{5.1.4}$$

*We recall the notion of the $\mathcal{Q}$-*INDEX CLUSTER *$I_j$ around $j$ from Definition 3.3.1. We have $I_j = \{i \in I \mid Q_i \cap P_j \neq \varnothing\}$. Occasionally, we will also call $I_j$ an* INTERSECTION SET.

*For $p \in (0,1)$, there is a constant $C = C(R_{\mathcal{P}}, C_{\mathcal{P}}, d, p, k) > 0$ with*

$$\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^{p^\triangle}_{w^{\mathcal{Q},p}}} \leq C C_{\mathcal{Q},\Phi,p} \cdot w^{\mathcal{P},p}_{j_0} \cdot \left\| \mathcal{F}^{-1}\left( \psi^{(2k+3)*}_{j_0} f \right) \right\|_{L^p} \tag{5.1.5}$$

*for all $f \in \mathcal{D}'(\mathcal{O}')$.*

*Remark.* It is worth noting that we have $\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset \overline{P^{k*}_{j_i}} \subset \mathcal{O}'$ for each $i \in I_0$ by Lemma 3.2.12, so that $\varphi_i f \in \mathcal{D}'(\mathcal{O}')$ is a well-defined distribution with compact support and thus a tempered distribution, so that $\mathcal{F}^{-1}(\varphi_i f) \in \mathcal{S}'(\mathbb{R}^d)$ is well-defined (even a smooth function, cf. [61, Theorem 7.23]).

Finally, we recall from above that we have $w^{\mathcal{P},p} \equiv 1 \equiv w^{\mathcal{Q},p}$ for $p \in [1,\infty]$, so that the above estimates can be uniformly written as

$$\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^{p^\triangle}_{w^{\mathcal{Q},p}}} \lesssim w^{\mathcal{P},p}_{j_0} \cdot \left\| \mathcal{F}^{-1}\left( \psi^{(2k+3)*}_{j_0} \cdot f \right) \right\|_{L^p}, \tag{5.1.6}$$

where the implied constant only depends on $N_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}, R_{\mathcal{P}}, C_{\mathcal{P}}, d, p, k$.

*Proof.* In the whole proof, we can assume that $\left\| \mathcal{F}^{-1}(\psi^{(2k+3)*}_{j_0} f) \right\|_{L^p}$ is finite, since otherwise there is nothing to show.

We start by observing

$$\varphi_i = \varphi_i \psi^{(2k+3)*}_{j_0} \qquad \forall i \in I_0. \tag{5.1.7}$$

To see this, note that for each $i \in I_0$, we have $Q_i \subset P^{k*}_{j_i}$ for some $j_i \in J$ by assumption. Using this, together with $i \in I_0 \subset I_{j_0}$, which implies $j_0 \in J_i$, Lemma 3.3.2 shows that even $Q_i \subset P^{(2k+2)*}_{j_0}$ is true. But Lemma 3.2.12 implies $\psi^{(2k+3)*}_{j_0} \equiv 1$ on $P^{(2k+2)*}_{j_0} \supset Q_i$ and thus $\varphi_i = \varphi_i \psi^{(2k+3)*}_{j_0}$.

Let us first consider the case $p \in (0,1)$. For this range of $p$, every $L^p$-decomposition covering is semi-structured, by definition. Thus, we have $\mathcal{P} = (S_j P'_j + c_j)_{j \in J}$ and $\mathcal{Q} = (T_i Q'_i + b_i)_{i \in I}$. Because of $p \in (0,1)$, we have $p' = \infty$ and hence $p^\triangle = \max\{p, p'\} = \infty$. Furthermore, the considerations in the preceding paragraph imply

$$\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset \overline{P^{(2k+2)*}_{j_0}} \subset \overline{P^{(2k+3)*}_{j_0}}$$

and we trivially have

$$\operatorname{supp} \psi^{(2k+3)*}_{j_0} \subset \overline{P^{(2k+3)*}_{j_0}}.$$

Hence, Corollary 3.2.15 yields a constant $C = C(R_{\mathcal{P}}, C_{\mathcal{P}}, d, p, k) > 0$ with

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \\
&= \left\| \mathcal{F}^{-1}\left( \varphi_i \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p} \\
&\leq C \cdot \left| \det\left(S_{j_0}\right) \right|^{\frac{1}{p}-1} \cdot \left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^p} \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p} \\
&\leq C C_{\mathcal{Q},\Phi,p} \cdot \left| \det\left(S_{j_0}\right) \right|^{\frac{1}{p}-1} \left| \det\left(T_i\right) \right|^{1-\frac{1}{p}} \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p} \\
&= C C_{\mathcal{Q},\Phi,p} \cdot w_{j_0}^{\mathcal{P},p} \cdot \left( w_i^{\mathcal{Q},p} \right)^{-1} \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p}.
\end{aligned}$$

Rearranging yields

$$w_i^{\mathcal{Q},p} \cdot \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \leq C C_{\mathcal{Q},\Phi,p} \cdot w_{j_0}^{\mathcal{P},p} \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} f \right) \right\|_{L^p}.$$

Since this holds for all $i \in I_0$ and because of $p^{\triangle} = \infty$, the proof in the case $p \in (0,1)$ is complete.

Now, let us establish the case $p \in [2,\infty]$. To this end, consider the family of linear maps

$$\Gamma_p : L^p(\mathbb{R}^d) \to \ell^p\left(I; L^p(\mathbb{R}^d)\right), g \mapsto \left( \mathcal{F}^{-1}(\varphi_i \widehat{g}) \right)_{i \in I}$$

for $p \in [2,\infty]$. We will show that $\Gamma_2$ and $\Gamma_\infty$ are well-defined and bounded with operator norm

$$\left|\left|\left| \Gamma_2 \right|\right|\right|, \left|\left|\left| \Gamma_\infty \right|\right|\right| \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p}.$$

By a standard complex interpolation argument (for vector-valued $L^p$-spaces, cf. [2, Theorems 5.1.2 and 5.1.1]), it then follows that each of the maps $\Gamma_p$ for $p \in [2,\infty]$ is well-defined and bounded with norm $\left|\left|\left| \Gamma_p \right|\right|\right| \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p}$.

*We remark that for $p \in (2,\infty]$, the Fourier transform $\widehat{g}$ is not necessarily a function, but rather a tempered distribution. Nevertheless, $\varphi_i \widehat{g}$ is a tempered distribution with compact support, so that $\mathcal{F}^{-1}(\varphi_i \widehat{g})$ is a (smooth) function.*

Assuming this, we can complete the proof of equation (5.1.4) in the case $p \in [2,\infty]$ as follows: Let $f \in \mathcal{D}'(\mathcal{O}')$ so that the right-hand side of equation (5.1.4) is finite (otherwise there is nothing to show), i.e. so that $g := \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} \cdot f \right) \in L^p(\mathbb{R}^d)$. Using equation (5.1.7), we see

$$\left(\Gamma_p(g)\right)_i = \mathcal{F}^{-1}(\varphi_i \widehat{g}) = \mathcal{F}^{-1}\left( \varphi_i \psi_{j_0}^{(2k+3)*} \cdot f \right) = \mathcal{F}^{-1}(\varphi_i f) \quad (5.1.8)$$

for all $i \in I_0$ and thus

$$\begin{aligned}
\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^p(I_0)} &= \left\| \Gamma_p(g) \right\|_{\ell^p\left(I_0; L^p\left(\mathbb{R}^d\right)\right)} \\
&\leq \left\| \Gamma_p(g) \right\|_{\ell^p\left(I; L^p\left(\mathbb{R}^d\right)\right)} \leq C \cdot \|g\|_{L^p} \\
&= C \cdot \left\| \mathcal{F}^{-1}\left( \psi_{j_0}^{(2k+3)*} \cdot f \right) \right\|_{L^p} \quad (5.1.9)
\end{aligned}$$

with $C = \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p}$. Because of $p^{\triangle} = p$ for $p \in [2,\infty]$, this proves the claim in the case $p \in [2,\infty]$.

For the proof of equation (5.1.4) in the case $p \in [2,\infty]$, it thus remains to establish boundedness of $\Gamma_\infty$ and $\Gamma_2$. For $\Gamma_\infty$, we simply note that for $g \in L^\infty\left(\mathbb{R}^d\right)$, we have

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^\infty} &= \left\|\left(\mathcal{F}^{-1}\varphi_i\right) * \left(\mathcal{F}^{-1}\widehat{g}\right)\right\|_{L^\infty} \\ &\leq \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^1} \cdot \|g\|_{L^\infty} \leq C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^\infty}\end{aligned}$$

by Young's inequality and by definition of $C_{\mathcal{Q},\Phi,p}$ for $p \geq 1$. But this means

$$\|\Gamma_\infty(g)\|_{\ell^\infty\left(I;L^\infty\left(\mathbb{R}^d\right)\right)} = \left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^\infty}\right)_{i\in I}\right\|_{\ell^\infty(I)} \leq C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^\infty}.$$

For the boundedness of $\Gamma_2$, we use the Plancherel theorem. For $g \in L^2\left(\mathbb{R}^d\right)$ and an arbitrary finite subset $L \subset I$, we have

$$\begin{aligned}\left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^2}\right)_{i\in L}\right\|_{\ell^2(L)}^2 &= \sum_{i\in L}\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^2}^2 \\ &= \sum_{i\in L}\left\|\varphi_i \widehat{g}\right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2 \sum_{i\in L} |\varphi_i(\xi)|^2 \, \mathrm{d}\xi. \qquad (5.1.10)\end{aligned}$$

By Fourier inversion (which holds since $\varphi_i \in C_c^\infty(\mathcal{O}) \subset \mathcal{S}\left(\mathbb{R}^d\right)$), we get

$$|\varphi_i(\xi)| = \left|\widehat{\mathcal{F}^{-1}\varphi_i}(\xi)\right| \leq \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^1} \leq C_{\mathcal{Q},\Phi,p}$$

for all $\xi \in \mathbb{R}^d$.

But for $\xi \in \mathbb{R}^d \setminus \mathcal{O}$, we have $\varphi_i(\xi) = 0$ for all $i \in L$ and thus

$$\sum_{i\in L} |\varphi_i(\xi)|^2 = 0 \leq N_{\mathcal{Q}} \cdot C_{\mathcal{Q},\Phi,p}^2.$$

In case of $\xi \in \mathcal{O}$, we have $\xi \in Q_{i_\xi}$ for some $i_\xi \in I$. Hence, if we have $\varphi_i(\xi) \neq 0$ for some $i \in L$, this implies $\xi \in Q_{i_\xi} \cap Q_i \neq \varnothing$, which means $i \in i_\xi^*$. Thus,

$$\begin{aligned}\sum_{i\in L} |\varphi_i(\xi)|^2 &= \sum_{i\in L\cap i_\xi^*} |\varphi_i(\xi)|^2 \\ &\leq \left|L \cap i_\xi^*\right| \cdot C_{\mathcal{Q},\Phi,p}^2 \\ &\leq N_{\mathcal{Q}} \cdot C_{\mathcal{Q},\Phi,p}^2,\end{aligned}$$

where we used $|i_\xi^*| \leq N_{\mathcal{Q}}$. Thus, this estimate is true for all $\xi \in \mathbb{R}^d$.

Continuing estimate (5.1.10), and applying Plancherel's theorem a second time, we conclude

$$\begin{aligned}\left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^2}\right)_{i\in L}\right\|_{\ell^2(L)}^2 &= \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2 \sum_{i\in L} |\varphi_i(\xi)|^2 \,\mathrm{d}\xi \\ &\leq N_{\mathcal{Q}} \cdot C_{\mathcal{Q},\Phi,p}^2 \cdot \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2 \,\mathrm{d}\xi \\ &= N_{\mathcal{Q}} \cdot C_{\mathcal{Q},\Phi,p}^2 \cdot \|g\|_{L^2}^2 .\end{aligned}$$

Since $L \subset I$ was an arbitrary finite set, we have shown

$$\|\Gamma_2(g)\|_{\ell^2\left(I;L^2\left(\mathbb{R}^d\right)\right)} \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^2},$$

which completes the proof of equation (5.1.4) in the case $p \in [2,\infty]$.

The proof in the case $p \in [1,2]$ is similar. Here, we have $p^{\triangle} = p'$. Thus, we consider the family of maps

$$\Theta_p : L^p\left(\mathbb{R}^d\right) \to \ell^{p'}\left(I; L^p\left(\mathbb{R}^d\right)\right), g \mapsto \left(\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right)_{i\in I}$$

for $p \in [1,2]$. Once we have shown that each of the maps $\Theta_p$ is well-defined and bounded with operator norm $|||\Theta_p||| \leq \sqrt{N_{\mathcal{Q}}} C_{\mathcal{Q},\Phi,p}$, the proof proceeds along the same lines as in the case $p \in [2,\infty]$ above: Simply replace $\Gamma_p$ by $\Theta_p$ everywhere and $p$ by $p' = p^{\triangle}$ in the appropriate places in equations (5.1.8) and (5.1.9).

We note that we have $\Theta_2 = \Gamma_2$ and thus $|||\Theta_2||| \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p}$, because the conjugate exponent to $p = 2$ is $p' = 2$. For the boundedness of $\Theta_1$, note that the conjugate exponent to $p = 1$ is $p' = \infty$. For $g \in L^1\left(\mathbb{R}^d\right)$, Young's inequality and the definition of $C_{\mathcal{Q},\Phi,p}$ for $p \in [1,\infty]$ yield

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^1} &= \left\|\left(\mathcal{F}^{-1}\varphi_i\right) * \left(\mathcal{F}^{-1}\widehat{g}\right)\right\|_{L^1} \\ &\leq \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^1} \cdot \|g\|_{L^1} \leq C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^1}\end{aligned}$$

and thus

$$\begin{aligned}\|\Theta_1(g)\|_{\ell^\infty\left(I;L^1\left(\mathbb{R}^d\right)\right)} &= \sup_{i\in I} \left\|\mathcal{F}^{-1}\left(\varphi_i \widehat{g}\right)\right\|_{L^1} \\ &\leq C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^1} \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p} \cdot \|g\|_{L^1} .\end{aligned}$$

It remains to note that we have $\frac{1}{p} \in \left(\frac{1}{2},1\right)$ for each $p \in (1,2)$ and thus $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{2} + \frac{\theta}{2}$ for some $\theta \in (0,1)$. But this implies

$$\frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1}{2} - \frac{\theta}{2} = 1 - \frac{1}{p} = \frac{1}{p'}$$

and hence (complex interpolation) that

$$\ell^{p'}\left(I; L^p\left(\mathbb{R}^d\right)\right) = \left(\ell^\infty\left(I; L^1\left(\mathbb{R}^d\right)\right), \ell^2\left(I; L^2\left(\mathbb{R}^d\right)\right)\right)_{[\theta]}$$

by [2, Theorems 5.1.2 and 5.1.1]. The boundedness of $\Theta_p$ as well as $\left|\left|\left|\Theta_p\right|\right|\right| \leq \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p}$ then follow by complex interpolation. □

Using the above lemma, we will now obtain – by duality – a "reverse" estimate. This uses the notion of the LOWER CONJUGATE EXPONENT $p^\triangledown$ OF $p$, which is defined by

$$p^\triangledown := \min\left\{p, p'\right\} = \begin{cases} p, & \text{for } p \in (0,2], \\ p', & \text{for } p \in [2,\infty]. \end{cases} \tag{5.1.11}$$

For use in the ensuing proof, we note for $p \in (0,\infty]$ the equivalences $p \in (0,2] \Leftrightarrow p' \in [2,\infty]$ and $p \in [2,\infty] \Leftrightarrow p' \in (0,2] \Leftrightarrow p' \in [1,2]$. These equivalences imply

$$\begin{aligned} \left(p^\triangledown\right)' &= \begin{cases} p', & \text{for } p \in (0,2], \\ \left(p'\right)', & \text{for } p \in [2,\infty] \end{cases} \\ &= \begin{cases} p', & \text{for } p' \in [2,\infty], \\ \left(p'\right)', & \text{for } p' \in (0,2] \end{cases} \\ &= \left(p'\right)^\triangle. \end{aligned}$$

With these preparations, we can now state and prove the dual result to the above lemma.

**Lemma 5.1.2.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and $p \in (0,\infty]$. Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = \left(P_j\right)_{j\in J}$ are $L^p$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively, and let $\Phi = (\varphi_i)_{i\in I}$ be an $L^p$-BAPU for $\mathcal{Q}$.*

*Let $j_0 \in J$ and $I_0 \subset I_{j_0}$ be arbitrary. Assume that there is some $k \in \mathbb{N}_0$, such that for each $i \in I_0$, the inclusion $Q_i \subset P_{j_i}^{k*}$ holds for some $j_i \in J$.*

*We emphasize that the constant $C$ does neither depend on $I_0 \subset I_{j_0}$, nor on $j_0 \in J$ or on the distribution $f \in \mathcal{D}'(\mathcal{O})$.*

*There is a constant $C = C\left(R_{\mathcal{Q}}, C_{\mathcal{Q}}, p, d, k, C_{\mathcal{Q},\Phi,p}, C_{\mathcal{P},\Psi,p}, N_{\mathcal{Q}}, N_{\mathcal{P}}\right) > 0$ such that for every distribution $f \in \mathcal{D}'(\mathcal{O})$ with $\mathcal{F}^{-1}(\varphi_i^* f) \in L^p\left(\mathbb{R}^d\right)$ for all $i \in I_0$ and with $\left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}\right)_{i\in I_0}\right\|_{\ell^{p^\triangledown}} < \infty$ the following holds:*

*The map*

$$f_{I_0} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathbb{C}, g \mapsto \sum_{i\in I_0} (\varphi_i f)(g)$$

*is well-defined with absolute convergence of the series for every $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Furthermore, $f_{I_0} \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is a tempered distribution with compact support* $\operatorname{supp} f_{I_0} \subset \overline{P_{j_0}^{(2k+2)*}}$ *and with*

$$\left\|\mathcal{F}^{-1} f_{I_0}\right\|_{L^p} \leq C \cdot \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}\right)_{i\in I_0}\right\|_{\ell^{p^\triangledown}}. \tag{5.1.12}$$

*More precisely, we can choose* $C = \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p} \cdot C_{\mathcal{P},\Psi,p} \cdot N_{\mathcal{P}}^{2k+3}$ *for* $p \in [1,\infty]$, *where* $(\psi_j)_{j\in J}$ *is a fixed* $L^p$*-BAPU for* $\mathcal{P}$. *For* $p \in (0,1]$, *we can take* $C = C_1 \cdot C_{\mathcal{Q},\Phi,p}$ *for some constant* $C_1 = C_1(R_{\mathcal{Q}}, C_{\mathcal{Q}}, p, d) > 0$.

*Remark.* We have $\varphi_i g \in C_c^\infty(\mathcal{O})$ for all $g \in \mathcal{S}(\mathbb{R}^d)$, so that the expression $(\varphi_i f)(g) = f(\varphi_i g) \in \mathbb{C}$ is well-defined for each $i \in I$. Thus, the definition of $f_{I_0}$ makes sense, at least once the convergence issues have been settled. This is done in the proof below.

*Proof.* Let us first consider the (easier) case $p \in (0,1]$. Corollary 3.2.15 yields a constant $C_1 = C_1(R_{\mathcal{Q}}, C_{\mathcal{Q}}, p, d) > 0$ with

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} &= \left\|\mathcal{F}^{-1}(\varphi_i \varphi_i^* f)\right\|_{L^p} \\ &\leq C_1 \cdot |\det(T_i)|^{\frac{1}{p}-1} \cdot \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^p} \cdot \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p} \\ &\leq C_1 C_{\mathcal{Q},\Phi,p} \cdot \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p} < \infty\end{aligned}$$

for all $i \in I_0$. Here, we used $\operatorname{supp}\varphi_i \subset \overline{Q_i} \subset \overline{Q_i^*}$ and $\operatorname{supp}\varphi_i^* \subset \overline{Q_i^*}$, together with $\varphi_i = \varphi_i \varphi_i^*$, which follows from $\varphi_i^* \equiv 1$ on $Q_i$, which is in turn a consequence of Lemma 3.2.12.

Since $L^p(\mathbb{R}^d)$ is a $p$-normed Quasi-Banach space by Example 2.1.3, Theorem 2.2.10 implies $F := \sum_{i\in I_0} \left|\mathcal{F}^{-1}(\varphi_i f)\right| \in L^p(\mathbb{R}^d)$ with

$$\begin{aligned}\|F\|_{L^p}^p &\leq \sum_{i\in I_0} \left\|\left|\mathcal{F}^{-1}(\varphi_i f)\right|\right\|_{L^p}^p \\ &\leq \left(C_1 C_{\mathcal{Q},\Phi,p}\right)^p \cdot \sum_{i\in I_0} \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}^p \\ &= \left(C_1 C_{\mathcal{Q},\Phi,p}\right)^p \cdot \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}\right)_{i\in I_0}\right\|_{\ell^{p^\nabla}}^p =: C_2 < \infty.\end{aligned} \tag{5.1.13}$$

Here, we used $p^\nabla = \min\{p, p'\} = p$, since $p \in (0,1]$. We also used that $\mathcal{F}^{-1}(\varphi_i^* f) \not\equiv 0$ can only hold for countably many $i \in I_0$, since $\sum_{i\in I_0} \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}^p$ is finite and since each function $\mathcal{F}^{-1}(\varphi_i^* f)$ is smooth.

By assumption, $Q_i \subset P_{j_i}^{k*}$ holds for each $i \in I_0 \subset I_{j_0}$ for some $j_i \in J$. By Lemma 3.3.2, this entails $Q_i \subset P_{j_0}^{(2k+2)*}$, because of $i \in I_{j_0}$. But this implies

$$\operatorname{supp}(\varphi_i f) \subset \overline{Q_i} \subset \overline{P_{j_0}^{(2k+2)*}} =: K$$

for all $i \in I_0$. Using Corollary 3.1.3, we get a constant $C_3 = C_3(K) > 0$ with

$$\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^1} \leq C_3 \cdot \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} < \infty \text{ for all } i \in I_0.$$

Together with the norm-decreasing embedding $\ell^p(I_0) \hookrightarrow \ell^1(I_0)$, we conclude

$$\begin{aligned}\sum_{i\in I_0} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^1} &\leq C_3 \cdot \sum_{i\in I_0} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \\ &\leq C_3 \cdot \left( \sum_{i\in I_0} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p}^p \right)^{1/p} \\ &\leq C_2^{1/p} C_3 < \infty.\end{aligned}$$

*In fact, we have even shown that $f_{I_0}$ is given by integration against an $L^\infty$ function. A slight twist of the argument would even yield that $f_{I_0}$ is given by integration against a function $f_{I_0} \in C_0(\mathbb{R}^d)$, but we do not need this here.*

For arbitrary $g \in \mathcal{S}(\mathbb{R}^d)$ and by definition of the Fourier transform of tempered distributions, this implies

$$\begin{aligned}|f_{I_0}(g)| &\leq \sum_{i\in I_0} |(\varphi_i f)(g)| \\ &= \sum_{i\in I_0} \left| \left\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{g} \right\rangle \right| \\ &\leq \sum_{i\in I_0} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^1} \|\widehat{g}\|_{L^\infty} \\ &\leq C_2^{1/p} C_3 \cdot \|g\|_{L^1} < \infty.\end{aligned}$$

This estimate proves absolute convergence of the series defining $f_{I_0}$ as well as $f_{I_0} \in \mathcal{S}'(\mathbb{R}^d)$.

As seen above, we have $\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset \overline{P_{j_0}^{(2k+2)*}}$ for all $i \in I_0$. For $g \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} g \subset \overline{P_{j_0}^{(2k+2)*}}^c$, this implies $\varphi_i g \equiv 0$ and hence $(\varphi_i f)(g) = f(\varphi_i g) = 0$ for all $i \in I_0$, which yields $f_{I_0}(g) = 0$. But by definition of the support of a (tempered) distribution (see e.g. [61, Definition 6.22]), this means $\operatorname{supp} f_{I_0} \subset \overline{P_{j_0}^{(2k+2)*}}$.

By Lemma 3.2.12, we know that $\overline{P_{j_0}^{(2k+2)*}} \subset \mathbb{R}^d$ is compact, so that the explicit formula for the (inverse) Fourier transform of compactly supported distributions (cf. [61, Theorem 7.23]) yields

$$\begin{aligned}\left(\mathcal{F}^{-1} f_{I_0}\right)(x) = f_{I_0}(\gamma e_x) &= \sum_{i\in I_0} (\varphi_i f)(\gamma e_x) \\ &= \sum_{i\in I_0} f(\varphi_i \gamma e_x) = \sum_{i\in I_0} f(\varphi_i e_x) \\ &= \sum_{i\in I_0} (\varphi_i f)(e_x) = \sum_{i\in I_0} \left[\mathcal{F}^{-1}(\varphi_i f)\right](x)\end{aligned}$$

for any $\gamma \in C_c^\infty(\mathbb{R}^d)$ with $\gamma \equiv 1$ on a neighborhood of $\overline{P_{j_0}^{(2k+2)*}}$ and $e_x : \mathbb{R}^d \to \mathbb{C}, y \mapsto e^{2\pi i \langle x,y\rangle}$. Here, we used $\varphi_i \gamma = \varphi_i$, since $\gamma \equiv 1$ on the support of $\varphi_i$. This shows $\left|\mathcal{F}^{-1} f_{I_0}\right| \leq F$, which, together with equation (5.1.13), implies validity of equation (5.1.12) by solidity of $L^p(\mathbb{R}^d)$. This completes the proof for $p \in (0,1]$.

Now, let us consider the case $p \in [1,\infty]$. To this end, fix an $L^p$-BAPU $\Psi = (\psi_j)_{j\in J}$ for $\mathcal{P}$. As above, we have $\varphi_i^* \varphi_i = \varphi_i$ for all $i \in I$. For arbitrary $h \in \mathcal{S}(\mathbb{R}^d) \subset L^{p'}(\mathbb{R}^d)$, this implies

$$\begin{aligned}
&\sum_{i\in I_0} \left|\left\langle \mathcal{F}^{-1}(\varphi_i f), h\right\rangle\right| \\
&= \sum_{i\in I_0} \left|\left\langle \varphi_i f, \mathcal{F}^{-1} h\right\rangle\right| \\
&= \sum_{i\in I_0} \left|\left\langle \varphi_i \varphi_i^* f, \mathcal{F}^{-1} h\right\rangle\right| \\
&= \sum_{i\in I_0} \left|\left\langle \varphi_i^* f,\ \varphi_i \cdot \mathcal{F}^{-1} h\right\rangle\right| \\
&= \sum_{i\in I_0} \left|\left\langle \mathcal{F}^{-1}(\varphi_i^* f),\ \mathcal{F}\left(\varphi_i \cdot \mathcal{F}^{-1} h\right)\right\rangle\right| \\
&\leq \sum_{i\in I_0} \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p} \cdot \left\|\mathcal{F}\left(\varphi_i \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}} \\
&\leq \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}\right)_{i\in I_0}\right\|_{\ell^{p^\vee}} \cdot \left\|\left(\left\|\mathcal{F}\left(\varphi_i \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}}\right)_{i\in I_0}\right\|_{\ell^{(p^\vee)'}}.
\end{aligned}$$

Here, we used Hölder's inequality in the last two steps.

As observed before the lemma, we have $(p^\vee)' = (p')^\wedge$, where $(p')^\wedge$ is defined as in equation (5.1.3). Because of $p' \in [1,\infty]$, Lemma 5.1.1 (marked with (‡)) and Young's inequality $L^1 * L^{p'} \hookrightarrow L^{p'}$ imply

$$\begin{aligned}
\left\|\left(\left\|\mathcal{F}\left(\varphi_i \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}}\right)_{i\in I_0}\right\|_{\ell^{(p^\vee)'}} &\overset{(\dagger)}{=} \left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}}\right)_{i\in I_0}\right\|_{\ell^{(p')^\wedge}} \\
&\overset{(\ddagger)}{\leq} C_4 \cdot \left\|\mathcal{F}^{-1}\left(\psi_{j_0}^{(2k+3)*} \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}} \\
&\overset{(\dagger)}{=} C_4 \cdot \left\|\mathcal{F}\left(\psi_{j_0}^{(2k+3)*} \cdot \mathcal{F}^{-1} h\right)\right\|_{L^{p'}} \\
&= C_4 \cdot \left\|\left(\mathcal{F}\psi_{j_0}^{(2k+3)*}\right) * h\right\|_{L^{p'}} \\
&\leq C_4 \cdot \left\|\mathcal{F}\psi_{j_0}^{(2k+3)*}\right\|_{L^1} \cdot \|h\|_{L^{p'}} \\
&\overset{(\dagger)}{\leq} C_4 \cdot C_{\mathcal{P},\Psi,p} \cdot N_{\mathcal{P}}^{2k+3} \cdot \|h\|_{L^{p'}}
\end{aligned}$$

with $C_4 := \sqrt{N_{\mathcal{Q}}} \cdot C_{\mathcal{Q},\Phi,p'} = \sqrt{N_{\mathcal{Q}}} C_{\mathcal{Q},\Phi,p}$. Here, we used the identity $\left\|\mathcal{F}^{-1} g\right\|_{L^p} = \|\mathcal{F} g\|_{L^p}$ at (†). In the last step, we employed the triangle inequality for $L^1(\mathbb{R}^d)$, as well as the estimate $|j_0^{(2k+3)*}| \leq N_{\mathcal{P}}^{2k+3}$ (cf. Lemma 3.2.18).

Using the abbreviation $C_5 := C_4 \cdot C_{\mathcal{P},\Psi,p} \cdot N_{\mathcal{P}}^{2k+3}$, we arrive at

$$\sum_{i\in I_0} \left|\left\langle \mathcal{F}^{-1}(\varphi_i f), h\right\rangle\right| \leq C_5 \cdot \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}\right)_{i\in I_0}\right\|_{\ell^{p^\vee}} \cdot \|h\|_{L^{p'}} < \infty \tag{5.1.14}$$

for all $h \in \mathcal{S}(\mathbb{R}^d) \subset L^{p'}(\mathbb{R}^d)$.

For arbitrary $g \in \mathcal{S}\left(\mathbb{R}^d\right)$, we have $\widehat{g} \in \mathcal{S}\left(\mathbb{R}^d\right) \hookrightarrow L^{p'}\left(\mathbb{R}^d\right)$ and thus

$$\begin{aligned}
|f_{I_0}(g)| &\leq \sum_{i \in I_0} |(\varphi_i f)(g)| \\
&= \sum_{i \in I_0} \left| \left\langle \mathcal{F}^{-1}(\varphi_i f), \widehat{g} \right\rangle \right| \\
&\leq C_5 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^{p\triangledown}} \cdot \|\widehat{g}\|_{L^{p'}} < \infty.
\end{aligned}$$

This shows that $f_{I_0}$ is well-defined with absolute convergence of the series for each $g \in \mathcal{S}\left(\mathbb{R}^d\right)$.

By continuity of $\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right) \hookrightarrow L^{p'}\left(\mathbb{R}^d\right)$, we also see $f_{I_0}(g) \to 0$ for $g \to 0$ in $\mathcal{S}\left(\mathbb{R}^d\right)$, so that $f_{I_0} \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is a tempered distribution. Exactly as in the case $p \in (0,1]$, we see that $f_{I_0}$ is supported in $\overline{P_{j_0}^{(2k+2)*}}$, which is compact by Lemma 3.2.12.

Finally, we establish equation (5.1.12). To this end, note that [61, Theorem 7.23] implies that the compactly supported, tempered distribution $\mathcal{F}^{-1} f_{I_0} \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is actually given by (integration against) a smooth, polynomially bounded function $\theta \in C^\infty\left(\mathbb{R}^d\right)$. For arbitrary $g \in C_c^\infty\left(\mathbb{R}^d\right)$, this implies

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} \theta(x) \cdot g(x) \, \mathrm{d}x \right| &= \left| \left\langle \mathcal{F}^{-1} f_{I_0}, g \right\rangle \right| \\
&= \left| \left\langle f_{I_0}, \mathcal{F}^{-1} g \right\rangle \right| \\
&\leq \sum_{i \in I_0} \left| \left\langle \varphi_i f, \mathcal{F}^{-1} g \right\rangle \right| \\
&= \sum_{i \in I_0} \left| \left\langle \mathcal{F}^{-1}(\varphi_i f), g \right\rangle \right| \\
&\leq C_5 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^{p\triangledown}} \cdot \|g\|_{L^{p'}},
\end{aligned}$$

where we applied equation (5.1.14) in the last step.

Using standard duality arguments, this yields $\mathcal{F}^{-1} f_{I_0} = \theta \in L^p\left(\mathbb{R}^d\right)$ with

$$\left\| \mathcal{F}^{-1} f_{I_0} \right\|_{L^p} \leq C_5 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell^{p\triangledown}(I_0)}.$$

Note that it is (at least in the case $p = 1$) crucial to know *in advance* that $\mathcal{F}^{-1} f_{I_0}$ is given by integration against *some* function, since there are bounded linear functionals on $L^\infty\left(\mathbb{R}^d\right)$ that are not given by integration against an $L^1$-function. □

In the remainder of this section, we will use the local embedding results given in the two lemmata above to derive global embedding results for decomposition spaces. In these results, we assume the existence of embeddings between certain nested sequence spaces to guarantee the existence of the embedding for decomposition spaces.

See for example Theorem 5.1.8, in particular equation (5.1.17), for such an assumption.

These assumptions have two (apparent) drawbacks:

A. The existence of embeddings between nested sequence spaces that we require seems to be hard to check.

B. It is not clear whether these conditions are sharp, i.e. whether the existence of the embedding for discrete sequence spaces is a necessary condition for the existence of an embedding between the decomposition spaces.

These two flaws will be adressed in the next two sections, where we show that they are not as serious as they at first appear. In fact

A. We will reduce the existence of embeddings between nested sequence spaces to an equivalent condition which just requires finiteness of a certain nested sequence space norm of just *one* single fixed sequence.

B. We will show that under natural conditions our results are sharp, i.e. our sufficient conditions are in fact necessary for the existence of an embedding.

But first of all, we have to formulate and prove our sufficient conditions for the existence of embeddings between decomposition spaces.

To this end, we remark that up to now we only estimated certain $\mathcal{F}L^p$ (quasi)-norms by other $\mathcal{F}L^p$ (quasi)-norms for the *same* exponent $p \in (0, \infty]$. But we want to establish results that guarantee an embedding

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$

for possibly distinct exponents $p_1 \neq p_2$.

Thus, we need to establish a "local" embedding between $L^p$-spaces of suitably bandlimited functions, which we do now. We mention that the ensuing result is very similar to Corollary 3.1.3, but more specialized to the setting of semi-structured coverings.

**Lemma 5.1.3.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ be a semi-structured admissible covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$.*

*Let $p_1, p_2 \in (0, \infty]$ with $p_1 \leq p_2$. Then there is a suitable constant $C = C(d, p_1, p_2, R_{\mathcal{Q}})$, so that*

$$\left\| \mathcal{F}^{-1}(\gamma_i f) \right\|_{L^{p_2}} \leq C \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot \left\| \mathcal{F}^{-1}(\gamma_i f) \right\|_{L^{p_1}}$$

*holds for each distribution $f \in \mathcal{D}'(\mathcal{O})$, all $i \in I$ and each $\gamma_i \in C_c^\infty(\mathcal{O})$ with $\gamma_i \equiv 0$ on $\mathcal{O} \setminus Q_i$.*

*Proof.* Let $R := R_{\mathcal{Q}}$ and fix a function $\eta \in C_c^\infty\left(\mathbb{R}^d\right)$ with $\eta|_{\overline{B_R}(0)} \equiv 1$. Let $f \in \mathcal{D}'(\mathcal{O})$ and $i \in I$ with $\mathcal{F}^{-1}(\gamma_i f) \in L^{p_1}\left(\mathbb{R}^d\right)$ (otherwise, there is nothing to show).

Let us first consider the case $p_1 \in (0,1)$. Here, Corollary 3.1.3 implies $\mathcal{F}^{-1}(\gamma_i f) \in L^{p_2}(\mathbb{R}^d)$ with

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_2}} &\leq \left[\lambda\left(\overline{Q_i}\right)\right]^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_1}} \\ &\leq \left[\lambda\left(T_i\left(\overline{B_R}(0)\right)+b_i\right)\right]^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_1}} \\ &= \lambda(B_1(0)) \cdot R^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_1}}.\end{aligned}$$

Here, we used

$$\operatorname{supp}\left(\widehat{\mathcal{F}^{-1}(\gamma_i f)}\right) \subset \operatorname{supp}\gamma_i \subset \overline{Q_i} \subset T_i\left(\overline{B_R}(0)\right)+b_i$$

as well as $\frac{1}{p_1}-\frac{1}{p_2} \geq 0$. This completes the proof in the case $p_1 \in (0,1)$.

Now, let us assume $p_1 \in [1,\infty]$, which also entails $p_2 \in [1,\infty]$, because of $p_2 \geq p_1$. For $i \in I$, define

$$\eta_i : \mathbb{R}^d \to \mathbb{C}, \xi \mapsto \eta\left(T_i^{-1}(\xi - b_i)\right).$$

With this definition, we have $\eta_i \equiv 1$ on $Q_i$. To see this, note that $\xi \in Q_i = T_i Q_i' + b_i \subset T_i\left(\overline{B_R}(0)\right)+b_i$ entails $T_i^{-1}(\xi - b_i) \in \overline{B_R}(0)$ and thus $\eta_i(\xi) = 1$. Hence, $\eta_i \gamma_i = \gamma_i$ and furthermore

$$\mathcal{F}^{-1}(\gamma_i f) = \mathcal{F}^{-1}(\eta_i \cdot \gamma_i f) = (\mathcal{F}^{-1}\eta_i) * \mathcal{F}^{-1}(\gamma_i f).$$

Because of $1 \leq p_1 \leq p_2$, we see $-1 \leq -\frac{1}{p_1} \leq \frac{1}{p_2}-\frac{1}{p_1} \leq 0$ and hence $1+\frac{1}{p_2}-\frac{1}{p_1} \in [0,1]$. We can thus define $q \in [1,\infty]$ by $\frac{1}{q} = 1+\frac{1}{p_2}-\frac{1}{p_1}$ with the understanding of $q = \infty$ in the case $1+\frac{1}{p_2}-\frac{1}{p_1} = 0$. By the general form of Young's inequality (cf. [33, Proposition 8.9(a)]), we derive

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_2}} &= \left\|\left(\mathcal{F}^{-1}\eta_i\right) * \mathcal{F}^{-1}(\gamma_i f)\right\|_{L^{p_2}} \\ &\leq \|\mathcal{F}^{-1}\eta_i\|_{L^q} \cdot \|\mathcal{F}^{-1}(\gamma_i f)\|_{L^{p_1}} < \infty,\end{aligned}$$

where we used $\eta_i \in C_c^\infty(\mathbb{R}^d)$ and thus $\mathcal{F}^{-1}\eta_i \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$.

Since $\eta_i = L_{b_i}(\eta \circ T_i^{-1})$, a straightforward calculation using elementary properties of the Fourier transform (cf. [33, Theorem 8.22]) yields

*For convenience of the reader, we recall $\Delta_h f = f \circ h^T$.*

$$\mathcal{F}^{-1}\eta_i = |\det(T_i)| \cdot M_{b_i}\left(\Delta_{T_i}\left[\mathcal{F}^{-1}\eta\right]\right),$$

which – together with Lemma 4.5.5 – implies

$$\begin{aligned}\|\mathcal{F}^{-1}\eta_i\|_{L^q} &= |\det(T_i)| \cdot \|\Delta_{T_i}\left[\mathcal{F}^{-1}\eta\right]\|_{L^q} \\ &= |\det(T_i)|^{1-\frac{1}{q}} \cdot \|\mathcal{F}^{-1}\eta\|_{L^q} \\ &= |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \|\mathcal{F}^{-1}\eta\|_{L^q}.\end{aligned}$$

Finally, note that $q$ depends only on $p_1, p_2$ and that $\eta$ depends only on $d$ and $R_{\mathcal{Q}}$, so that the quantity $C(d, p_1, p_2, R_{\mathcal{Q}}) := \left\|\mathcal{F}^{-1}\eta\right\|_{L^q}$ only depends on $d, p_1, p_2, R_{\mathcal{Q}}$. □

As a corollary, we see that every $L^p$-BAPU $(\varphi_i)_{i\in I}$ is also an $L^q$-BAPU for $q \geq p$.

**Corollary 5.1.4.** *Assume that $\mathcal{Q} = (Q_i)_{i\in I}$ is an $L^p$-decomposition covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ for some $p \in (0,\infty]$ and let $\Phi = (\varphi_i)_{i\in I}$ be an $L^p$-BAPU for $\mathcal{Q}$.*

*Then $\Phi$ is an $L^q$-BAPU for $\mathcal{Q}$ for any $q \in [p,\infty]$ and we have an estimate*

$$C_{\mathcal{Q},\Phi,q} \leq C \cdot C_{\mathcal{Q},\Phi,p}$$

*with $C = 1$ for $p \in [1,\infty]$ and $C = C(d, p, q, R_{\mathcal{Q}})$ for $p \in (0,1)$.*

*Proof.* Since the definition of an $L^p$-BAPU is independent of $p$ for $p \in [1,\infty]$, the claim is clear (with $C = 1$) for $p \in [1,\infty]$.

Hence, let us assume $p \in (0,1)$. In particular, this implies that $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ is a semi-structured admissible covering. For any $r \geq p$, Lemma 5.1.3 (with $f \equiv 1$) yields a constant $C_r = C(d, p, r, R_{\mathcal{Q}})$ with

$$\begin{aligned}
\|\mathcal{F}^{-1}\varphi_i\|_{L^r} &\leq C_r \cdot |\det(T_i)|^{\frac{1}{p}-\frac{1}{r}} \|\mathcal{F}^{-1}\varphi_i\|_{L^p} \\
&\leq C_r \cdot |\det(T_i)|^{\frac{1}{p}-\frac{1}{r}} |\det(T_i)|^{1-\frac{1}{p}} C_{\mathcal{Q},\Phi,p} \\
&= C_r \cdot |\det(T_i)|^{1-\frac{1}{r}} C_{\mathcal{Q},\Phi,p}
\end{aligned}$$

for all $i \in I$.

In case of $q \in (0,1)$, we choose $r = q$ and get $C_{\mathcal{Q},\Phi,q} \leq C_q C_{\mathcal{Q},\Phi,p}$. In case of $q \in [1,\infty]$, we apply the above with $r = 1$ to conclude $C_{\mathcal{Q},\Phi,q} \leq C_1 C_{\mathcal{Q},\Phi,p}$. □

As a further preparation for the proof of our embedding results, we remark that the lemmata 5.1.1 and 5.1.2 involve quantities like $\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^p}$ and $\left\|\mathcal{F}^{-1}\left(\psi_{j_0}^{(2k+3)*} f\right)\right\|_{L^p}$, whereas the quasi-norms used to define the decomposition spaces only contain the quantities $\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}$ and $\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}$. By Theorem 3.4.10, this is no problem, since the families $(\varphi_i^*)_{i\in I}$ and $\left(\psi_j^{(2k+3)*}\right)_{j\in J}$ are bounded control systems for $\mathcal{Q}$ and $\mathcal{P}$, respectively.

For concreteness, we state the following variant of Theorem 3.4.10. Given that theorem, the corollary below is an immediate consequence, once we recall that Lemma 3.4.2 allows to estimate the operator norm of the clustering map $c \mapsto c^*$ on the space $\ell_u^q(I)$ only in terms of $q, N_{\mathcal{Q}}$ and $C_{u,\mathcal{Q}}$ and that the triangle constant for $\ell_u^q$ only depends on $q \in (0,\infty]$.

**Corollary 5.1.5.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (Q_i)_{i\in I}$ be an $L^p$-decomposition covering of $\mathcal{O}$ for some $p \in (0,\infty]$ and let $u = (u_i)_{i\in I}$ be*

*$\mathcal{Q}$-moderate. Furthermore, let $r_0 \in \mathbb{N}$ and assume $I = \bigcup_{r=1}^{r_0} I^{(r)}$ for certain subsets $I^{(r)} \subset I$. Finally, assume that $\Gamma = (\gamma_i)_{i\in I}$ is an $L^p$-bounded system for $\mathcal{Q}$.*

*For $f \in \mathcal{D}'(\mathcal{O})$, define (as in Theorem 3.4.10)*

$$\|f\|_{\Gamma,\left(I^{(r)}\right)_{r'},L^p,\ell^q_u} := \sum_{r=1}^{r_0} \left\| \left( \left\| \mathcal{F}^{-1}(\gamma_i f) \right\|_{L^p} \right)_{i\in I^{(r)}} \right\|_{\ell^q_u\left(I^{(r)}\right)} \in [0,\infty].$$

*Then there is a constant*

$$C = \begin{cases} C\left(p,q,r_0,d,\ell_{\Gamma,\mathcal{Q}},C_{\mathcal{Q},\Gamma,p},N_{\mathcal{Q}},R_{\mathcal{Q}},C_{\mathcal{Q}},C_{u,\mathcal{Q}}\right), & \text{if } p\in(0,1), \\ C\left(p,q,r_0,\ell_{\Gamma,\mathcal{Q}},C_{\mathcal{Q},\Gamma,p},N_{\mathcal{Q}},C_{u,\mathcal{Q}}\right), & \text{if } p\in[1,\infty] \end{cases}$$

*with*

$$\|f\|_{\Gamma,\left(I^{(r)}\right)_{r'},L^p,\ell^q_u} \le C\cdot\|f\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)}$$

*for all $f \in \mathcal{D}'(\mathcal{O})$ and each $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{Q}$.*

*The $L^p$-BAPU $\Phi$ is mentioned on the right-hand side of the estimate mainly to emphasize that the same estimate holds for every choice of $\Phi$, although none of the constants depends on $\Phi$.*

*Conversely, if $\Gamma$ is an $L^p$-bounded* control *system for $\mathcal{Q}$ then there is a constant*

$$C' = \begin{cases} C'\left(r_0,R_{\mathcal{Q}},C_{\mathcal{Q}},\ell_{\Gamma,\mathcal{Q}},p,q,d\right), & \text{if } p\in(0,1), \\ C'\left(r_0,q\right), & \text{if } p\in[1,\infty] \end{cases}$$

*with*

$$\|f\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)} \le C' C_{\mathcal{Q},\Phi,p}\cdot\|f\|_{\Gamma,\left(I^{(r)}\right)_{r'},L^p,\ell^q_u}$$

*for all $f \in \mathcal{D}'(\mathcal{O})$ and each $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{Q}$.*

With these preparations, we can finally state our first sufficient condition for the existence of an embedding between decomposition spaces. We start with the embedding of a decomposition space with respect to a "coarse" covering into a decomposition space with respect to a "fine" covering.

We remark that the formulation of the assumptions, in particular the required embedding for nested sequence spaces (cf. equation (5.1.15)) is chosen as to make the proof simple and transparent, not with the aim of making the theorem easily applicable. In the next section, we will simplify the assumptions to obtain embedding results whose assumptions are easier to verify, cf. in particular Corollaries 5.2.8 and 5.2.9.

The purpose of the sets $J_0, I_0$ in the ensuing theorem is to allow for the case in which only *a part* of the covering $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. Apart from the assumption that $\mathcal{Q}_{I_0}$ (as opposed to $\mathcal{Q}$) is almost subordinate to $\mathcal{P}$, the sets $I_0, J_0$ are only used in equation (5.1.15) and in the definition of the map $\iota$. The role of $\iota$ is to "cut down" the distribution $f \in \mathcal{D}\left(\mathcal{P},L^{p_2},\ell^{q_2}_v\right)$ to $\sum_{i\in I_0}\varphi_i f$. This ensures that only

those parts of $f$ which belong to the part of $\mathcal{O}$ near which $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ are left intact.

Ideally, we would like to be able to select the set $I_0$ arbitrarily. But (at least with the method of proof as below), we have to start from the set $J_0 \subset J$, which then determines $I_0$.

The most important special case is given by $\mathcal{O} = \mathcal{O}'$ and $J_0 = J$. Here, we get $I_0 = I$ and thus $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$, so that $\iota$ is a genuine embedding.

**Theorem 5.1.6.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2, q_1, q_2 \in (0, \infty]$, set $p := \min\{p_1, p_2\}$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ and $\mathcal{P} = \left(P_j\right)_{j \in J}$ are $L^p$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = (u_i)_{i \in I}$ and $v = \left(v_j\right)_{j \in J}$ be $\mathcal{Q}$- and $\mathcal{P}$-moderate, respectively. Finally, assume that $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ is semi-structured.*

*Let $J_0 \subset J$ and define*

$$I_0 := \left\{ i \in I \,\middle|\, Q_i \cap P_j \neq \varnothing \text{ for some } j \in J_0 \right\} = \bigcup_{j \in J_0} I_j.$$

*We note that $\mathcal{Q}$ is required to be an $L^p$-decomposition covering with $p = p_2$, although only the space $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ is considered. The reason for this is that the $L^{p_2}$-BAPU $\Phi$ is used for the "localization" of $f$, cf. the definition of $\iota$.*

*Assume that*

1. *$\mathcal{Q}_{I_0} = (Q_i)_{i \in I_0}$ is almost subordinate to $\mathcal{P}$,*
2. *$p_2 \leq p_1$,*
3. *there is a finite constant $C > 0$ such that*

$$\left\| \left( |\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \leq C \left\| \left( \left\| (c_i)_{i \in I_j} \right\|_{\ell^{p_2}_{w^{\mathcal{Q}, p_2}}} \Big/ w_j^{\mathcal{P}, p_2} \right)_{j \in J_0} \right\|_{\ell_v^{q_2}} \tag{5.1.15}$$

*holds for all sequences $(c_i)_{i \in I_0}$.*

*Note that we have $I_j \subset I_0$ for all $j \in J_0$, by definition of $I_0$. Thus, the "inner" norm on the right-hand side is well-defined.*

*Let $\Phi = (\varphi_i)_{i \in I}$ be an $L^p$-BAPU for $\mathcal{Q}$. For each $f \in \mathcal{D}'(\mathcal{O}')$, the map*

$$\iota f : C_c^\infty(\mathcal{O}) \to \mathbb{C}, g \mapsto \sum_{i \in I_0} (\varphi_i f)(g)$$

*In case of $J_0 = \varnothing$, we get $I_0 = \varnothing$ and hence $\iota \equiv 0$, so that the claim of the theorem is trivial.*

*is a well-defined distribution on $\mathcal{O}$. In fact, the series has only finitely many nonvanishing terms for each $g \in C_c^\infty(\mathcal{O})$.*

*The linear map $\iota : \mathcal{D}'(\mathcal{O}') \to \mathcal{D}'(\mathcal{O})$ thus defined restricts to a bounded linear map*

$$\iota : \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$$

*with $|||\iota||| \leq C \cdot C_1$ for a constant $C_1 > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on*

$$k(\mathcal{Q}_{I_0}, \mathcal{P}), N_{\mathcal{Q}}, C_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},$$

*where $\Phi$ is used to calculate the quasi-norm on $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ and the $L^{p_2}$-BAPU $\Psi$ is used to calculate the quasi-norm on $\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$.*

*Here, the dependence of $C_1$ on $R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}$ can be dropped for $p_2 \in [1, \infty]$.*

*For simplicity, we use the same BAPU $\Phi$ for the "localization" in the definition of $\iota$ and for the calculation of the quasi-norm on $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$. For the calculation of the quasi-norm, one could of course use any $L^{p_1}$-BAPU $\Phi'$ instead of $\Phi$, but then the constant $C_1$ will also depend on $C_{\mathcal{Q},\Phi',p_1}$.*

*Remark* 5.1.7. Several remarks are in order:

1. Considering $\iota$ as an *embedding* is not always justified. For example in the case that $\mathcal{O}' \setminus \bigcup_{i\in I_0} Q_i$ has nonempty interior, $\iota$ will not be injective, since $\iota f = 0$ holds for all $f \in C_c^\infty(\mathcal{O}')$ which are supported in the aforementioned interior.

2. We remark that if $\mathcal{Q}$ admits a family $\Phi = (\varphi_i)_{i\in I}$ which is an $L^p$-BAPU for $\mathcal{Q}$ for all $p \in (0,\infty]$, then $\iota$ can be chosen *independently* of $p_1, p_2, q_1, q_2$, in fact depending only on $I_0, \Phi, \mathcal{O}, \mathcal{O}'$. By Theorem 3.2.17, this is in particular the case if $\mathcal{Q}$ is a structured admissible covering of $\mathcal{O}$.

3. The assumption in equation (5.1.15) can be simplified by distinguishing the cases $p_2 \in (0,1)$ and $p_2 \in [1,\infty]$. For $p_2 \in [1,\infty]$, we have $w_j^{\mathcal{P},p_2} \equiv 1 \equiv w_j^{\mathcal{Q},p_2}$, so that equation (5.1.15) reduces to

$$\left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} c_i \right)_{i\in I_0} \right\|_{\ell_u^{q_1}} \lesssim \left\| \left( \left\| (c_i)_{i\in I_j} \right\|_{\ell^{p_2}} \right)_{j\in J_0} \right\|_{\ell_v^{q_2}}.$$

   For $p_2 \in (0,1)$, we have $w_i^{\mathcal{Q},p_2} = |\det(T_i)|^{\frac{1}{p_2}-1}$ (cf. equation (5.1.2)). Replacing $c_i$ by $c_i |\det(T_i)|^{1-\frac{1}{p_2}}$ overall (which yields a bijection of the set of all sequences to itself), we see that equation (5.1.15) reduces to

$$\left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} c_i \right)_{i\in I_0} \right\|_{\ell_u^{q_1}} \lesssim \left\| \left( |\det(S_j)|^{1-\frac{1}{p_2}} \left\| (c_i)_{i\in I_j} \right\|_{\ell^{p_2}} \right)_{j\in J_0} \right\|_{\ell_v^{q_2}},$$

   where $\mathcal{P} = (S_j P_j' + c_j)_{j\in J}$ is a semi-structured admissible covering, since it is an $L^{p_2}$-decomposition covering with $p_2 \in (0,1)$.

4. Since $\mathcal{Q}_{I_0}$ is almost subordinate to $\mathcal{P}$, there is some $k \in \mathbb{N}_0$ such that for each $i \in I_0$, the inclusion $\overline{Q_i} \subset \overline{P_{j_i}^{k*}} \subset \mathcal{O}'$ holds for some $j_i \in J$. This implies $\varphi_i \in C_c^\infty(\mathcal{O}')$, so that $\varphi_i f \in \mathcal{D}'(\mathbb{R}^d)$ is well-defined. Furthermore, this proves that each summand of the series defining $(\iota f)(g)$ makes sense.

   *The inclusion $\overline{P_{j_i}^{k*}} \subset \mathcal{O}'$ is a consequence of Lemma 3.2.12.*

5. For $f \in \mathcal{D}'(\mathcal{O}')$ and $g \in C_c^\infty(\mathcal{O})$ with $\operatorname{supp} g \subset P_{J_0} \subset \mathcal{O}'$, we have

$$\langle \iota f, g\rangle = \langle f, g\rangle,$$

   because $(\varphi_i)_{i\in I}$ is a (locally finite) partition of unity on $\mathcal{O}$, which implies $g = \sum_{i\in I} \varphi_i g$, where only finitely many terms of the sum do not vanish, since $g \in C_c^\infty(\mathcal{O})$.

   *Local finiteness of $(\varphi_i)_{i\in I}$ is a consequence of Lemma 3.2.11.*

   In case of $\varphi_i g \not\equiv 0$, we have

$$\varnothing \neq Q_i \cap \operatorname{supp} g \subset Q_i \cap P_{J_0} = \bigcup_{j\in J_0} Q_i \cap P_j$$

and thus $i \in I_0$, which implies

$$\begin{aligned}\langle \iota f, g\rangle &= \sum_{i\in I_0} (\varphi_i f)(g) = \sum_{i\in I_0} f(\varphi_i g) = f\left(\sum_{i\in I_0} \varphi_i g\right)\\ &= f\left(\sum_{i\in I} \varphi_i g\right) = f(g) = \langle f, g\rangle .\end{aligned}$$

6. If $P_{J_0} \subset \mathbb{R}^d$ is open and $f \in \mathcal{D}'(\mathcal{O}')$ with $\operatorname{supp} f \subset P_{J_0}$, then we even have $\iota f = f$ as elements of $\mathcal{D}'(\mathcal{O} \cap \mathcal{O}')$, because as seen in part 5 of this remark, we have $\iota f = f$ on the open set $\mathcal{O} \cap P_{J_0}$, i.e. $\operatorname{supp}(f - \iota f) \subset P_{J_0}^c$.

   But we also have $\iota f = 0 = f$ on the open set $\mathcal{O} \cap [\mathcal{O}' \setminus \operatorname{supp} f]$ and thus $\operatorname{supp}(f - \iota f) \subset \mathcal{O} \cap \operatorname{supp} f \subset P_{J_0}$, which leads to $f = \iota f$ as elements of $\mathcal{D}'(\mathcal{O} \cap \mathcal{O}')$.

7. In the case that $\mathcal{Q}$ (as opposed to $\mathcal{Q}_{I_0}$) is almost subordinate to $\mathcal{P}$, we can take $J_0 := J$, which easily implies $I_0 = I$, since $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. Furthermore, Lemma 3.3.2 yields $\mathcal{O} \subset \mathcal{O}'$. Since $P_{J_0} = \mathcal{O}'$ is open, part 6 of this remark shows $\iota f = f|_{C_c^\infty(\mathcal{O})}$ for all $f \in \mathcal{D}'(\mathcal{O}')$. Hence, $\iota f$ is – in this case – just the restriction of $f$ to the subspace $C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathcal{O}')$. In case of $\mathcal{O} = \mathcal{O}'$, this means $\iota f = f$. ◀

*Proof of Theorem 5.1.6.* Let $f \in \mathcal{D}'(\mathcal{O}')$ and let $K \subset \mathcal{O}$ be compact. Lemma 3.2.12 implies that $(Q_i^\circ)_{i\in I}$ is an open cover of $\mathcal{O} \supset K$, so that there are finitely many $i_1, \dots, i_n \in I$ with $K \subset \bigcup_{\ell=1}^n Q_{i_\ell}$. For $i \in I_0$ with $\varphi_i \not\equiv 0$ on $K$, this yields

$$\varnothing \neq Q_i \cap K \subset \bigcup_{\ell=1}^n Q_{i_\ell} \cap Q_i$$

and thus $i \in I_K := I_0 \cap \bigcup_{\ell=1}^n i_\ell^*$, where $I_K \subset I_0$ is finite.

For

$$g \in \mathcal{D}_K(\mathcal{O}) := \{h \in C_c^\infty(\mathcal{O}) \mid \operatorname{supp} h \subset K\},$$

we thus have $\varphi_i g \equiv 0$ for all $i \in I_0 \setminus I_K$ and hence

$$(\iota f)(g) = \sum_{i\in I_0} (\varphi_i f)(g) = \sum_{i\in I_K} f(\varphi_i g).$$

Let $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$. With this choice and Lemma 3.2.12, we see that $\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset \overline{P_{j_i}^{k*}} \subset \mathcal{O}'$ holds for a suitable $j_i \in J$ for each $i \in I_0$, so that $\operatorname{supp} \varphi_i$ is compactly contained in $\mathcal{O}'$, which shows that the map $C_c^\infty(\mathcal{O}) \to C_c^\infty(\mathcal{O}'), g \mapsto \varphi_i g$ is continuous. This also implies continuity of $C_c^\infty(\mathcal{O}) \to \mathbb{C}, g \mapsto f(\varphi_i g)$, so that $\iota f$ restricted to $\mathcal{D}_K(\mathcal{O})$ is continuous as a finite sum of continuous functionals. Since $K \subset \mathcal{O}$ was an arbitrary compact set, we conclude that $\iota f \in \mathcal{D}'(\mathcal{O})$ is a well-defined distribution on $\mathcal{O}$ (cf. [61, Theorem 6.6(d)]).

Now, let $f \in \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$ and let $i \in I$ be arbitrary. We then have

$$\begin{aligned}(\varphi_i \cdot \iota f)(g) &= \langle \iota f, \varphi_i g\rangle \\ &= \sum_{\ell \in I_0} (\varphi_\ell f)(\varphi_i g) \\ &= \sum_{\ell \in I_0 \cap i^*} (\varphi_i \varphi_\ell f)(g) \\ &= \left(\varphi_i \cdot \sum_{\ell \in I_0 \cap i^*} \varphi_\ell f\right)(g), \end{aligned} \tag{5.1.16}$$

for all $g \in C_c^\infty(\mathcal{O})$ because $\varphi_i \varphi_\ell \not\equiv 0$ implies $Q_i \cap Q_\ell \neq \varnothing$ and thus $\ell \in i^*$.

We now distinguish two cases: For $p_1 \in [1, \infty]$, we can apply the triangle inequality for $L^{p_1}(\mathbb{R}^d)$ and Young's inequality $L^1 * L^{p_1} \hookrightarrow L^{p_1}$, which implies

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\varphi_i \cdot \iota f)\right\|_{L^{p_1}} &= \left\|\mathcal{F}^{-1}\left(\varphi_i \cdot \sum_{\ell \in I_0 \cap i^*} \varphi_\ell f\right)\right\|_{L^{p_1}} \\ &\leq \sum_{\ell \in I_0 \cap i^*} \left\|\mathcal{F}^{-1}(\varphi_i \cdot \varphi_\ell f)\right\|_{L^{p_1}} \\ &\leq \sum_{\ell \in I_0 \cap i^*} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^1} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \\ &\leq C_{\mathcal{Q},\Phi,p_1} \cdot \left(\left(\left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \chi_{I_0}(\ell)\right)_{\ell \in I}\right)_i^*, \end{aligned}$$

where as usual $c_i^* = \sum_{j \in i^*} c_j$.

In case of $p_1 \in (0,1)$, we can use the fact that $\|\cdot\|_{L^{p_1}}$ is a quasi-norm (with triangle constant only depending on $p_1$), together with the uniform bound $|I_0 \cap i^*| \leq |i^*| \leq N_{\mathcal{Q}}$ and with Corollary 3.2.15 to obtain constants $C_1 = C_1(N_{\mathcal{Q}}, p_1)$ and $C_2 = C_2(R_{\mathcal{Q}}, C_{\mathcal{Q}}, p_1, d) > 0$ with

*We remark that Corollary 5.1.4 shows $C_{\mathcal{Q},\Phi,p_1} \lesssim C_{\mathcal{Q},\Phi,p}$, where the implied constant only depends on $d, p_1, p_2, R_{\mathcal{Q}}$.*

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\varphi_i \cdot \iota f)\right\|_{L^{p_1}} &= \left\|\mathcal{F}^{-1}\left(\varphi_i \cdot \sum_{\ell \in I_0 \cap i^*} \varphi_\ell f\right)\right\|_{L^{p_1}} \\ &\leq C_1 \sum_{\ell \in I_0 \cap i^*} \left\|\mathcal{F}^{-1}(\varphi_i \cdot \varphi_\ell f)\right\|_{L^{p_1}} \\ &\leq C_1 C_2 \sum_{\ell \in I_0 \cap i^*} |\det(T_i)|^{\frac{1}{p_1}-1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \\ &\leq C_3 C_{\mathcal{Q},\Phi,p_1} \cdot \left(\left(\left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \chi_{I_0}(\ell)\right)_{\ell \in I}\right)_i^* \end{aligned}$$

for $C_3 := C_1 C_2$. Here, we used $\operatorname{supp}\varphi_i \subset \overline{Q_i} \subset \overline{Q_i^*}$ as well as $\operatorname{supp}\varphi_\ell \subset \overline{Q_\ell} \subset \overline{Q_i^*}$ for $\ell \in i^*$. Finally, the first step is justified by equation (5.1.16). Thus, the above estimate is valid (with $C_3 := 1$ for $p_1 \in [1, \infty]$) for arbitrary $p_1 \in (0, \infty]$.

Because of $p_2 \leq p_1$, we can use Lemma 5.1.3 to obtain a constant $C_4 = C_4(d, p_1, p_2, R_{\mathcal{Q}})$ with

$$\left\| \mathcal{F}^{-1}(\varphi_\ell f) \right\|_{L^{p_1}} \leq C_4 \cdot |\det(T_\ell)|^{\frac{1}{p_2}-\frac{1}{p_1}} \cdot \left\| \mathcal{F}^{-1}(\varphi_\ell f) \right\|_{L^{p_2}}$$

for all $\ell \in I$. All in all, this yields

$$\left\| \mathcal{F}^{-1}(\varphi_i \cdot \iota f) \right\|_{L^{p_1}} \leq C_5 \left( \left( |\det(T_\ell)|^{\frac{1}{p_2}-\frac{1}{p_1}} \left\| \mathcal{F}^{-1}(\varphi_\ell f) \right\|_{L^{p_2}} \chi_{I_0}(\ell) \right)_{\ell \in I} \right)_i^*$$

with $C_5 := C_3 C_4 C_{\mathcal{Q},\Phi,p_1}$ and for all $i \in I$ .

Let $\Gamma : \ell_u^{q_1}(I) \to \ell_u^{q_1}(I), c \mapsto c^*$ denote the clustering map. Using the preceding estimates, we conclude

$$\begin{aligned}
&\|\iota f\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} \\
&= \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i \cdot \iota f) \right\|_{L^{p_1}} \right)_{i \in I} \right\|_{\ell_u^{q_1}} \\
&\leq C_5 \cdot \left\| \Gamma \left( |\det(T_\ell)|^{\frac{1}{p_2}-\frac{1}{p_1}} \left\| \mathcal{F}^{-1}(\varphi_\ell f) \right\|_{L^{p_2}} \chi_{I_0}(\ell) \right)_{\ell \in I} \right\|_{\ell_u^{q_1}} \\
&\leq C_5 \left\| \left| \Gamma \right| \right\| \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^{p_2}} \chi_{I_0}(i) \right)_{i \in I} \right\|_{\ell_u^{q_1}} \\
&= C_5 \left\| \left| \Gamma \right| \right\| \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^{p_2}} \right)_{i \in I_0} \right\|_{\ell_u^{q_1}(I_0)} \\
&\overset{(*)}{\leq} C C_5 \left\| \left| \Gamma \right| \right\| \cdot \left\| \left[ \left( w_j^{\mathcal{P},p_2} \right)^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^{p_2}} \right)_{i \in I_j} \right\|_{\ell^{p_2}_{w^{\mathcal{Q},p_2}}} \right]_{j \in J_0} \right\|_{\ell_v^{q_2}} \\
&\overset{(\dagger)}{\leq} C C_5 C_6 \left\| \left| \Gamma \right| \right\| \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left( \psi_j^{(2k+3)*} \cdot f \right) \right\|_{L^{p_2}} \right)_{j \in J_0} \right\|_{\ell_v^{q_2}} \\
&\overset{(\ddagger)}{\leq} C C_5 C_6 C_7 \left\| \left| \Gamma \right| \right\| \cdot \|f\|_{\mathcal{D}_\Psi\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)} < \infty.
\end{aligned}$$

Here, we used the assumed embedding from equation (5.1.15) at $(*)$. Furthermore, the constant $C_6 > 0$ (with $C_6 = \sqrt{N_{\mathcal{Q}}} C_{\mathcal{Q},\Phi,p_2}$ for $p_2 \in [1,\infty]$ and $C_6 = C_6\left(R_{\mathcal{P}}, C_{\mathcal{P}}, d, p_2, k, C_{\mathcal{Q},\Phi,p_2}\right)$ for $p_2 \in (0,1)$) in the step marked with $(\dagger)$ is provided by Lemma 5.1.1, see especially equation (5.1.6). Finally, the step marked with $(\ddagger)$ is justified by applying Corollary 5.1.5 to the family $\Theta = (\psi_j^{(2k+3)*})_{j \in J}$, which yields the constant

*Here, we apply the "local" result from Lemma 5.1.1 for each $j \in J_0$ separately, with $j_0 = j$ and $I_0 = I_j$ (in the notation of the local lemma).*

$$C_7 = \begin{cases} C_7\left(p_2, q_2, d, \ell_{\Theta,\mathcal{P}}, C_{\mathcal{P},\Theta,p_2}, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}\right), & \text{if } p_2 \in (0,1), \\ C_7\left(p_2, q_2, \ell_{\Theta,\mathcal{P}}, C_{\mathcal{P},\Theta,p_2}, N_{\mathcal{P}}, C_{v,\mathcal{P}}\right), & \text{if } p_2 \in [1,\infty] \end{cases}$$

Here, we have $\ell_{\Theta,\mathcal{P}} \leq 2k+3$ and in view of Remark 3.4.9, we can estimate $C_{\mathcal{P},\Theta,p_2}$ by a constant only depending on $p_2, k, N_{\mathcal{P}}, D_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}$. For $p_2 \in [1,\infty]$, the dependence on $D_{\mathcal{P}}$ can be omitted.

For the applicability of Lemma 5.1.1, we observe that $I_j \subset I_0$ holds for all $j \in J_0$, so that we get $Q_i \subset P_{j_i}^{k*}$ for some $j_i \in J$ for each $i \in I_j \subset I_0$ by choice of $k = k(\mathcal{Q}_{I_0}, \mathcal{P})$. Furthermore, we note that $\Phi$ is assumed to be an $L^{p_2}$-BAPU for $\mathcal{Q}$, because of $p = \min\{p_1, p_2\} = p_2$.

We finally note that Lemma 3.4.2 yields $|||\Gamma||| \leq N_{\mathcal{Q}}^{1+\frac{1}{q_1}} C_{u,\mathcal{Q}}$. □

The next theorem establishes the "inverse" direction of the above embedding, that is, it establishes embeddings of a decomposition space with respect to a "fine" covering into the decomposition space with respect to a "coarse" covering.

As above, the purpose of the set $I_0$ is to allow for the possibility that only *a part* of the covering $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

**Theorem 5.1.8.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2, q_1, q_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = (P_j)_{j \in J}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}'$. Let $u = (u_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be $\mathcal{Q}$- and $\mathcal{P}$-moderate, respectively. Finally, assume that $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ is semi-structured and that $\Phi = (\varphi_i)_{i \in I}$ is an $L^{p_1}$-BAPU for $\mathcal{Q}$.*

*Let $I_0 \subset I$ be arbitrary and assume that*

1. *$\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to $\mathcal{P}$,*
2. *$p_1 \leq p_2$,*
3. *there is a finite constant $C > 0$ such that*

$$\left\| \left( \left\| (c_i)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2}}^{\vee} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq C \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}} \cdot c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \tag{5.1.17}$$

*holds for all sequences $(c_i)_{i \in I_0}$.*

*Then, the map*

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto \iota f$$

*is well-defined and bounded with $|||\iota||| \leq C_1 \cdot C$ for some constant $C_1 > 0$ depending only on*

$$\begin{cases} d, p_1, p_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), N_{\mathcal{Q}}, R_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, N_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}, & \text{if } p_2 \in [1, \infty], \\ d, p_1, p_2, k(\mathcal{Q}_{I_0}, \mathcal{P}), N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}, & \text{if } p_2 \in (0, 1), \end{cases}$$

*where the $L^{p_2}$-BAPU $\Psi$ is used to calculate the quasi-norm on $\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$.*

*Here, $\iota f$ is defined by*

$$\iota f : C_c^\infty(\mathcal{O}') \to \mathbb{C}, g \mapsto \sum_{i \in I_0} (\varphi_i f)(g)$$

*for each $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$. More precisely, the series defining $(\iota f)(g)$ is absolutely convergent for every $g \in C_c^\infty(\mathcal{O}')$ and we have $\iota f \in \mathcal{D}'(\mathcal{O}')$.*

*Remark.* Similar to the previous theorem, we remark that if $\mathcal{Q}$ admits a family $\Phi = (\varphi_i)_{i\in I}$ which is an $L^p$-BAPU for $\mathcal{Q}$ for all $p \in (0,\infty]$, then $\iota$ can be chosen independently of $p_1, p_2, q_1, q_2$, in fact depending only on $I_0, \Phi, \mathcal{O}, \mathcal{O}'$. By Theorem 3.2.17, this is in particular the case if $\mathcal{Q}$ is a structured admissible covering of $\mathcal{O}$.

But it is important to observe in this case that (in contrast to the previous theorem) the convergence of the series defining $\iota f$ is only guaranteed for $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ (and only if equation (5.1.17) holds) and *not* for arbitrary $f \in \mathcal{D}'(\mathcal{O})$.

*Proof.* Set $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$ and fix some $L^{p_2}$-BAPU $\Psi = \left(\psi_j\right)_{j\in J}$ for $\mathcal{P}$.

Let $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ be arbitrary. Since $\|\cdot\|_{L^{p_2}}$ is a (quasi)-norm and because of the uniform bound $|i^*| \leq N_{\mathcal{Q}}$ for all $i \in I$, there is a constant $C_1 = C_1(N_{\mathcal{Q}}, p_2) > 0$ with

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}} &= \left\|\sum_{\ell\in i^*} \mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_2}} \\ &\leq C_1 \sum_{\ell\in i^*} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_2}}\end{aligned}$$

for all $i \in I$. Because of $p_1 \leq p_2$, Lemma 5.1.3 yields a constant $C_2 = C_2(d, p_1, p_2, R_{\mathcal{Q}})$, such that

$$\begin{aligned}&\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}} \\ &\leq C_1 \sum_{\ell\in i^*} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_2}} \\ &\leq C_1 C_2 \sum_{\ell\in i^*} |\det(T_\ell)|^{\frac{1}{p_1}-\frac{1}{p_2}} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \\ &\leq C_1 C_2 D_{\mathcal{Q}}^{\frac{1}{p_1}-\frac{1}{p_2}} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \sum_{\ell\in i^*} \left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}} \\ &= C_1 C_2 D_{\mathcal{Q}}^{\frac{1}{p_1}-\frac{1}{p_2}} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left(\left(\left\|\mathcal{F}^{-1}(\varphi_\ell f)\right\|_{L^{p_1}}\right)_{\ell\in I}\right)_i^* \qquad (5.1.18)\end{aligned}$$

holds for all $i \in I$, where as usual $c_i^* = \sum_{\ell\in i^*} c_\ell$ for $c = (c_i)_{i\in I}$. In the above estimate, we used the definition of $D_{\mathcal{Q}}$ (cf. equation (3.2.1)), as well as $\frac{1}{p_1} - \frac{1}{p_2} \geq 0$.

Let us denote the clustering map by $\Gamma : \ell_u^{q_1}(I) \to \ell_u^{q_1}(I), c \mapsto c^*$ and define $C_3 := C_1 C_2 D_{\mathcal{Q}}^{\frac{1}{p_1}-\frac{1}{p_2}}$. Using the above estimate (at $(*)$), we conclude

$$\begin{aligned}&\left\|\left(\left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}}\right)_{i\in I_0\cap I_j}\right\|_{\ell^{p_2}_\nabla}\right)_{j\in J}\right\|_{\ell_v^{q_2}} \\ &\overset{(\ddagger)}{\leq} C \cdot \left\|\left(|\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \cdot \left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}}\right)_{i\in I_0}\right\|_{\ell_u^{q_1}}\end{aligned}$$

$$\begin{aligned}
&\overset{(*)}{\leq} C C_3 \cdot \left\| \left( \left[ \left( \left\| \mathcal{F}^{-1}(\varphi_\ell f) \right\|_{L^{p_1}} \right)_{\ell \in I} \right]_i^* \right)_{i \in I} \right\|_{\ell_u^{q_1}} \\
&= C C_3 \cdot \left\| \Gamma \left[ \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^{p_1}} \right)_{i \in I} \right] \right\|_{\ell_u^{q_1}} \\
&\leq C C_3 \, |||\Gamma||| \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^{p_1}} \right)_{i \in I} \right\|_{\ell_u^{q_1}} \\
&= C C_3 \, |||\Gamma||| \cdot \|f\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} < \infty. \qquad (5.1.19)
\end{aligned}$$

Here, the step marked with (‡) is due to the assumed embedding for nested sequence spaces, cf. equation (5.1.17).

The above estimate shows that $\left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f) \right\|_{L^{p_2}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2^\triangledown}}$ is finite for all $j \in J$. Furthermore, $Q_i \subset P_{j_i}^{k*}$ holds for every $i \in I_j \cap I_0 \subset I_0$ for a suitable $j_i \in J$ by assumption 3 and by choice of $k = k(\mathcal{Q}_{I_0}, \mathcal{P})$. Together, we see that Lemma 5.1.2 is applicable, so that we can define $f_{I_j \cap I_0}$ for each $j \in J$ as in that lemma. Equation (5.1.12) of that lemma yields a constant

$$C_4 = \begin{cases} C_4\left(k, N_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_2}, N_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right), & \text{if } p_2 \in [1,\infty], \\ C_4\left(d, p_2, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_2}\right), & \text{if } p_2 \in (0,1), \end{cases}$$

such that

$$\left\| \mathcal{F}^{-1} f_{I_0 \cap I_j} \right\|_{L^{p_2}} \leq C_4 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f) \right\|_{L^{p_2}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2^\triangledown}} \qquad (5.1.20)$$

holds for all $j \in J$. Here, we recall from Corollary 5.1.4 the estimate $C_{\mathcal{Q},\Phi,p_2} \leq C_5 C_{\mathcal{Q},\Phi,p_1}$ with a suitable constant $C_5 = C_5(d, p_1, p_2, R_{\mathcal{Q}})$.

Now, let $j \in J$ be arbitrary and let $g \in C_c^\infty(\mathcal{O}')$. For $i \in I_0$ with $\varphi_i \psi_j g \not\equiv 0$, we have $Q_i \cap P_j \neq \varnothing$ and thus $i \in I_0 \cap I_j$. This yields

$$\begin{aligned}
(\iota f)(\psi_j g) &= \sum_{i \in I_0} (\varphi_i f)(\psi_j g) \\
&= \sum_{i \in I_0} f(\varphi_i \psi_j g) \\
&= \sum_{i \in I_0 \cap I_j} (\varphi_i f)(\psi_j g) \\
&= f_{I_0 \cap I_j}(\psi_j g) \\
&= \left(\psi_j \cdot f_{I_0 \cap I_j}\right)(g). \qquad (5.1.21)
\end{aligned}$$

By Lemma 5.1.2, we know that the sum $\sum_{i \in I_0 \cap I_j} (\varphi_i f)(\psi_j g)$ converges absolutely, which shows that the series defining $(\iota f)(\psi_j g)$ also converges absolutely, because all terms for $i \in I_0 \setminus I_j$ vanish, as seen above.

Now, let $K \subset \mathcal{O}'$ be an arbitrary compact set. Lemma 3.2.12 establishes $K \subset \bigcup_{j \in J} P_j^\circ$, so that there are finitely many $j_1, \ldots, j_k \in J$ with

$K \subset \bigcup_{\ell=1}^{k} P_{j_\ell}$. Set $J_K := \{j_1, \dots, j_k\}^*$. By Lemma 3.2.12, this yields $\psi_{J_K} \equiv 1$ on $\bigcup_{\ell=1}^{k} P_{j_\ell} \supset K$.

For $g \in C_c^\infty(\mathcal{O}')$ with $\operatorname{supp} g \subset K$, this implies $g = \psi_{J_K} g$. For $i \in I_0$, we thus have

$$|(\varphi_i f)(g)| = |(\varphi_i f)(\psi_{J_K} g)| \leq \sum_{j \in J_K} \left|(\varphi_i f)(\psi_j g)\right|,$$

which implies

$$\begin{aligned}\sum_{i \in I_0} |(\varphi_i f)(g)| &\leq \sum_{i \in I_0} \sum_{j \in J_K} \left|(\varphi_i f)(\psi_j g)\right| \\ &= \sum_{j \in J_K} \sum_{i \in I_0} \left|(\varphi_i f)(\psi_j g)\right| < \infty,\end{aligned}$$

since $\sum_{j \in J_K}$ is a finite sum and since each of the sums $\sum_{i \in I_0} (\varphi_i f)(\psi_j g)$ converges absolutely, as shown above.

Finally, an application of equation (5.1.21) yields

$$\begin{aligned}(\iota f)(g) &= (\iota f)(\psi_{J_K} g) \\ &= \sum_{j \in J_K} (\iota f)(\psi_j g) \\ &= \sum_{j \in J_K} \left(\psi_j \cdot f_{I_0 \cap I_j}\right)(g).\end{aligned}$$

But Lemma 5.1.2 shows that each of the functionals $f_{I_0 \cap I_j}$ and thus also $\psi_j \cdot f_{I_0 \cap I_j}$ is continuous on $\mathcal{S}'(\mathbb{R}^d)$ and hence also on $C_c^\infty(\mathcal{O}')$. This shows that $\iota f$, restricted to

$$\mathcal{D}_K(\mathcal{O}') = \left\{g \in C_c^\infty(\mathcal{O}') \,\middle|\, \operatorname{supp} g \subset K\right\}$$

is continuous as a finite sum of continuous functionals. Using [61, Theorem 6.6], we conclude $\iota f \in \mathcal{D}'(\mathcal{O}')$.

Now, we distinguish the cases $p_2 \in [1, \infty]$ and $p_2 \in (0, 1)$. In the first case, Young's inequality $L^1 * L^{p_2} \hookrightarrow L^{p_2}$ implies

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\psi_j \cdot f_{I_0 \cap I_j}\right)\right\|_{L^{p_2}} &\leq \left\|\mathcal{F}^{-1}\psi_j\right\|_{L^1} \cdot \left\|\mathcal{F}^{-1} f_{I_0 \cap I_j}\right\|_{L^{p_2}} \\ &\leq C_{\mathcal{P},\Psi,p_2} \cdot \left\|\mathcal{F}^{-1} f_{I_0 \cap I_j}\right\|_{L^{p_2}} \\ &\leq C_4 C_{\mathcal{P},\Psi,p_2} \cdot \left\|\left(\left\|\mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}}\right)_{i \in I_0 \cap I_j}\right\|_{\ell^{p_2}_\nabla}.\end{aligned}$$

Here, equation (5.1.20) was used in the last step.

In case of $p_2 \in (0, 1)$, Definition 3.2.10 implies that $\mathcal{P} = (S_j P_j' + c_j)_{j \in J}$ is semi-structured. Furthermore, Lemma 5.1.2 yields

$$\operatorname{supp} f_{I_0 \cap I_j} \subset \overline{P_j^{(2k+2)*}}$$

and we trivially have $\operatorname{supp} \psi_j \subset \overline{P_j} \subset \overline{P_j^{(2k+2)*}}$. In view of these inclusions, Corollary 3.2.15 yields a constant $C_5 = C_5(R_{\mathcal{P}}, C_{\mathcal{P}}, p_2, d, k) > 0$, such that $C_6 := C_4 C_5 C_{\mathcal{P},\Psi,p_2}$ satisfies

$$\begin{aligned}\left\| \mathcal{F}^{-1}\left(\psi_j \cdot f_{I_0 \cap I_j}\right)\right\|_{L^{p_2}} &\leq C_5 \left|\det(S_j)\right|^{\frac{1}{p_2}-1} \left\|\mathcal{F}^{-1}\psi_j\right\|_{L^{p_2}} \left\|\mathcal{F}^{-1} f_{I_0\cap I_j}\right\|_{L^{p_2}} \\ &\leq C_6 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2^\nabla}},\end{aligned}$$

where equation (5.1.20) was used in the last step. All in all, if we define $C_6 := C_4 C_{\mathcal{P},\Psi,p_2}$ for $p_2 \in [1,\infty]$, the above estimate holds for all $p_2 \in (0,\infty]$.

Equation (5.1.21) shows $\psi_j \cdot \iota f = \psi_j \cdot f_{I_0 \cap I_j}$ for all $j \in J$. This implies

$$\begin{aligned}\|\iota f\|_{\mathcal{D}_\Psi(\mathcal{P},L^{p_2},\ell_v^{q_2})} &= \left\| \left( \left\| \mathcal{F}^{-1}(\psi_j \cdot \iota f)\right\|_{L^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ &= \left\| \left( \left\| \mathcal{F}^{-1}\left(\psi_j \cdot f_{I_0\cap I_j}\right)\right\|_{L^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ &\leq C_6 \cdot \left\| \left( \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i^* f)\right\|_{L^{p_2}} \right)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2^\nabla}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ &\leq C C_3 C_6 \left|\!\left|\!\left|\Gamma\right|\!\right|\!\right| \cdot \|f\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} < \infty,\end{aligned}$$

where equation (5.1.19) was used in the last step. Since Lemma 3.4.2 shows $\left|\!\left|\!\left|\Gamma\right|\!\right|\!\right| \leq N_{\mathcal{Q}}^{1+\frac{1}{q_1}} C_{u,\mathcal{Q}}$, the proof is complete. □

A very special case of embeddings between decomposition spaces is the case in which two decomposition spaces turn out to be equal, even though the two coverings are different. Our next result shows that this is the case if the two coverings $\mathcal{Q}, \mathcal{P}$ are *weakly* equivalent. Since this is a strictly weaker assumption than almost subordinateness of $\mathcal{Q}$ to $\mathcal{P}$ and vice versa, this result is *not* a special case of our embedding results.

**Lemma 5.1.9.** *Let $p, q \in (0,\infty]$, let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open and assume that $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ are two $L^p$-decomposition coverings of $\mathcal{O}$. Let $u = (u_i)_{i\in I}$ be $\mathcal{Q}$-moderate and let $v = (v_j)_{j\in J}$ be $\mathcal{P}$-moderate.*

*Furthermore, assume that there is a constant $C_0 > 0$ with*

$$C_0^{-1} \cdot u_i \leq v_j \leq C_0 \cdot u_i$$

*for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$.*

*Finally, assume that $\mathcal{Q}, \mathcal{P}$ are* WEAKLY EQUIVALENT, *i.e. $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ and vice versa.*

*For $p \in (0,1)$, the coverings $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ and $\mathcal{P} = (S_j P_j' + c_j)_{j\in J}$ are semi-structured. In this case, assume furthermore that there is a constant $C_1 > 0$ such that one of the following two conditions holds:*

*For convenience of the reader, we recall from Definition 3.3.1 that $\mathcal{Q}$ is weakly subordinate to $\mathcal{P}$ if $N(\mathcal{Q},\mathcal{P}) = \sup_{i\in I} |J_i|$ is finite.*

1. *For arbitrary $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$, we have*

$$\left\|T_i^{-1} S_j\right\| \leq C_1 \quad \text{and} \quad \left\|S_j^{-1} T_i\right\| \leq C_1.$$

2. *$\mathcal{P}$ is almost subordinate to $\mathcal{Q}$ and we have*

$$\left|\det\left(S_j^{-1} T_i\right)\right| \leq C_1$$

*for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$.*

*Then we have $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right) = \mathcal{D}\left(\mathcal{P}, L^p, \ell_v^q\right)$ with equivalent quasi-norms.*

*Proof.* Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = \left(\psi_j\right)_{j \in J}$ be $L^p$-BAPUs for $\mathcal{Q}$ and $\mathcal{P}$, respectively. We first show $\varphi_i \psi_{J_i} = \varphi_i$. For $x \in \mathbb{R}^d$ with $\varphi_i(x) = 0$, this is clear. If otherwise $\varphi_i(x) \neq 0$, then $x \in Q_i \subset \mathcal{O}$ and each $j \in J$ with $\psi_j(x) \neq 0$ hence satisfies $x \in Q_i \cap P_j \neq \varnothing$, i.e. $j \in J_i$. Thus,

*For the identity $\sum_{j \in J} \psi_j(x) = 1$, we crucially use that $\mathcal{Q}, \mathcal{P}$ both cover the same set $\mathcal{O}$.*

$$1 = \sum_{j \in J} \psi_j(x) = \sum_{j \in J_i} \psi_j(x) = \psi_{J_i}(x),$$

which implies $\varphi_i(x)\, \psi_{J_i}(x) = \varphi_i(x)$ also in this case. Completely analogous, one can show $\psi_j \varphi_{I_j} = \psi_j$.

We first prove the claim for $p \in [1, \infty]$. Here, it suffices to show $\|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)} \lesssim \|\cdot\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell_u^q\right)}$, since the assumptions are symmetric in $\mathcal{Q}, \mathcal{P}$. Hence, let $f \in \mathcal{D}\left(\mathcal{P}, L^p, \ell_u^q\right)$ be arbitrary. As seen above, we have $\varphi_i = \varphi_i \psi_{J_i}$. Together with Young's inequality $L^1 * L^p \hookrightarrow L^p$ and with the triangle inequality for $L^p$, this yields

$$\begin{aligned}
u_i \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} &= u_i \left\|\mathcal{F}^{-1}\left(\varphi_i \psi_{J_i} f\right)\right\|_{L^p} \\
&\leq \sum_{j \in J_i} u_i \left\|\mathcal{F}^{-1}\left(\varphi_i \psi_j f\right)\right\|_{L^p} \\
&\leq C_0 \sum_{j \in J_i} \left[v_j \left\|\mathcal{F}^{-1} \varphi_i\right\|_{L^1} \left\|\mathcal{F}^{-1}\left(\psi_j f\right)\right\|_{L^p}\right] \\
&\leq C_0 C_{\mathcal{Q}, \Phi, p} \cdot \sum_{j \in J_i} \left[v_j \left\|\mathcal{F}^{-1}\left(\psi_j f\right)\right\|_{L^p}\right] \qquad (5.1.22)
\end{aligned}$$

for all $i \in I$.

In case of $q = \infty$, this implies

$$\begin{aligned}
\|f\|_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)} &= \sup_{i \in I} u_i \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} \\
&\leq \sup_{i \in I} C_0 C_{\mathcal{Q}, \Phi, p} \cdot |J_i| \cdot \|f\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell_v^q\right)} \\
&\leq C_0 C_{\mathcal{Q}, \Phi, p} \cdot N(\mathcal{Q}, \mathcal{P}) \cdot \|f\|_{\mathcal{D}\left(\mathcal{P}, L^p, \ell_v^q\right)} < \infty.
\end{aligned}$$

In case of $q \in (0,\infty)$, we use the uniform bound $|J_i| \leq N(\mathcal{Q},\mathcal{P}) =: N$ to obtain a constant $C_2 = C_2(N,q) > 0$ with

$$\left(\sum_{j\in J_i}\left[v_j\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}\right]\right)^q \leq C_2 \cdot \sum_{j\in J_i}\left[v_j\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}\right]^q$$

for all $i \in I$. This implies

$$\begin{aligned}
\|f\|^q_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_u)} &= \sum_{i\in I}\left[u_i\left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p}\right]^q \\
&\leq \left(C_0 C_{\mathcal{Q},\Phi,p}\right)^q C_2 \cdot \sum_{i\in I}\sum_{j\in J_i}\left[v_j\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}\right]^q \\
&= \left(C_0 C_{\mathcal{Q},\Phi,p}\right)^q C_2 \cdot \sum_{j\in J}\sum_{i\in I_j}\left[v_j\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}\right]^q \\
&\leq \left(C_0 C_{\mathcal{Q},\Phi,p}\right)^q C_2 \cdot N(\mathcal{P},\mathcal{Q}) \cdot \sum_{j\in J}\left[v_j\left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p}\right]^q \\
&= \left(C_0 C_{\mathcal{Q},\Phi,p}\right)^q C_2 \cdot N(\mathcal{P},\mathcal{Q}) \cdot \|f\|^q_{\mathcal{D}(\mathcal{P},L^p,\ell^q_v)} < \infty.
\end{aligned}$$

Here, we used the equivalence $i \in I_j \Leftrightarrow j \in J_i$, as well as the estimate $|I_j| \leq N(\mathcal{P},\mathcal{Q})$ for all $j \in J$.

Now, we consider the case $p \in (0,1)$. Here, we first show that both of the additional assumptions imply that there are constants $C_3, R > 0$ such that

$$|\det(S_j^{-1}T_i)| \leq C_3 \tag{5.1.23}$$

and

$$P_j \subset T_i\left(\overline{B_R}(0)\right) + b_i$$

hold for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$.

The inequality concerning the determinant is clear in case of the second assumption. In case of the first assumption, we use Hadamard's inequality $|\det(A)| \leq \|A\|^d$ for arbitrary matrices $A \in \mathbb{R}^{d\times d}$ to deduce

$$|\det(S_j^{-1}T_i)| \leq \|S_j^{-1}T_i\|^d \leq C_1^d$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$.

For the inclusion $P_j \subset T_i\left(\overline{B_R}(0)\right) + b_i$, let us first consider the case of the second assumption. Since $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$, the constant $k := k(\mathcal{P},\mathcal{Q})$ is well-defined and we have $P_j \subset Q_{i_j}^{k*}$ for all $j \in J$ and suitable $i_j \in I$. In case of $Q_i \cap P_j \neq \varnothing$, this implies $P_j \subset Q_i^{(2k+2)*}$ by Lemma 3.3.2. But Lemma 3.2.13 yields

$$P_j \subset Q_i^{(2k+2)*} \subset T_i\left[\overline{B_{(2C_{\mathcal{Q}}+1)^{2k+2}R_{\mathcal{Q}}}}(0)\right] + b_i$$

for all $i \in I$.

In case of the first assumption, define $R_0 := \max\{R_{\mathcal{Q}}, R_{\mathcal{P}}\}$ and let $x \in Q_i \cap P_j \neq \varnothing$. This yields $y \in Q_i' \subset \overline{B_{R_0}}(0)$ and $z \in P_j' \subset \overline{B_{R_0}}(0)$ with

$$x = T_i y + b_i = S_j z + c_j.$$

Hence, $b_i = S_j z + c_j - T_i y$. Now, let $w \in \overline{B_{R_0}}(0)$ be arbitrary. We have

$$\begin{aligned}\left|T_i^{-1}\left(S_j w + c_j - b_i\right)\right| &= \left|T_i^{-1}\left(S_j w + c_j - \left(S_j z + c_j - T_i y\right)\right)\right| \\ &\leq \left|T_i^{-1} S_j w\right| + \left|T_i^{-1} S_j z\right| + |y| \\ &\leq C_1 |w| + C_1 |z| + |y| \\ &\leq R_0 (1 + 2C_1) =: R\end{aligned}$$

and hence

$$S_j w + c_j \in T_i\left(\overline{B_R}(0)\right) + b_i.$$

Since $w \in \overline{B_{R_0}}(0)$ was arbitrary, this implies

$$P_j = S_j P_j' + c_j \subset S_j\left[\overline{B_{R_0}}(0)\right] + c_j \subset T_i\left(\overline{B_R}(0)\right) + b_i$$

for all $i \in I$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$.

With this preparation, we can prove the actual claim: The argument is almost the same as for $p \in [1, \infty]$; only the application of Young's inequality and of the triangle inequality for $L^p$ in equation (5.1.22) need to be replaced. In short, we only need to establish equation (5.1.22) also for $p \in (0, 1)$, the rest of the proof then proceeds as for $p \in [1, \infty]$. The only caveat is that since the assumptions are not symmetric in $\mathcal{Q}, \mathcal{P}$ anymore, we also need to show the "reverse" version of equation (5.1.22).

Since $\|\cdot\|_{L^p}$ is a quasi-norm and because of the uniform bound $|J_i| \leq N(\mathcal{Q}, \mathcal{P}) = N$, there is a constant $C_4 = C_4(N, p) > 0$ with

$$\begin{aligned}u_i \left\|\mathcal{F}^{-1}(\varphi_i f)\right\|_{L^p} &= u_i \left\|\mathcal{F}^{-1}(\varphi_i \psi_{J_i} f)\right\|_{L^p} \\ &\leq C_4 \cdot u_i \sum_{j \in J_i} \left\|\mathcal{F}^{-1}(\varphi_i \psi_j f)\right\|_{L^p} \\ &\overset{(\dagger)}{\leq} C_4 C_5 \cdot u_i \sum_{j \in J_i} |\det(T_i)|^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^p} \left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p} \\ &\leq C_0 C_4 C_5 C_{\mathcal{Q},\Phi,p} \cdot \sum_{j \in J_i} v_j \left\|\mathcal{F}^{-1}(\psi_j f)\right\|_{L^p} \qquad (5.1.24)\end{aligned}$$

for all $i \in I$. Using this estimate instead of equation (5.1.22), the proof for $p \in (0, 1)$ can be completed just as for $p \in [1, \infty]$.

In equation (5.1.24), the step marked with $(\dagger)$ is justified as follows: By possibly enlarging $R$, we can assume $R \geq R_0$. This yields $\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset T_i\left(\overline{B_R}(0)\right) + b_i =: M_i$ and

$$\operatorname{supp} \psi_j \subset \overline{P_j} \subset T_i\left(\overline{B_R}(0)\right) + b_i = M_i$$

for all $j \in J_i$. Thus, Theorem 3.1.4 shows

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left(\varphi_i \psi_j f\right) \right\|_{L^p} \\
&\leq \left[\lambda\left(M_i - M_i\right)\right]^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\psi_j f\right) \right\|_{L^p} \\
&= \left[\lambda\left(T_i\left[\overline{B_R}\,(0) - \overline{B_R}\,(0)\right]\right)\right]^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\psi_j f\right) \right\|_{L^p} \\
&\leq \left[\lambda\left(\overline{B_{2R}}\,(0)\right)\right]^{\frac{1}{p}-1} \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\psi_j f\right) \right\|_{L^p} \\
&=: C_5 \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\psi_j f\right) \right\|_{L^p}, \qquad (5.1.25)
\end{aligned}$$

where the constant $C_5$ only depends on $R > 0$ and on $p \in (0,1)$.

For the "reverse" version of equation (5.1.22), a similar argument applies: Because of the uniform bound $\left|I_j\right| \leq N\left(\mathcal{P}, Q\right) =: N_2$, there is a constant $C_6 = C_6\left(N_2, p\right) > 0$ with

$$\begin{aligned}
&v_j \left\| \mathcal{F}^{-1}\left(\psi_j f\right) \right\|_{L^p} \\
&= v_j \left\| \mathcal{F}^{-1}\left(\psi_j \varphi_{I_j} f\right) \right\|_{L^p} \\
&\leq C_6 \cdot v_j \sum_{i \in I_j} \left\| \mathcal{F}^{-1}\left(\psi_j \varphi_i f\right) \right\|_{L^p} \\
&\overset{(\dagger)}{\leq} C_5 C_6 \cdot v_j \sum_{i \in I_j} \left|\det\left(T_i\right)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\psi_j \right\|_{L^p} \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \\
&\overset{(\ddagger)}{\leq} C_0 C_3^{\frac{1}{p}-1} C_5 C_6 \cdot \sum_{i \in I_j} \left[ \left|\det\left(S_j\right)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1}\psi_j \right\|_{L^p} \cdot u_i \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \right] \\
&\leq C_0 C_3^{\frac{1}{p}-1} C_5 C_6 C_{\mathcal{P},\Psi,p} \cdot \sum_{i \in I_j} \left[ u_i \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \right].
\end{aligned}$$

Here, we used $\frac{1}{p} - 1 > 0$, together with estimate (5.1.23) at $(\ddagger)$. The justification for $(\dagger)$ is exactly as in equation (5.1.25) above. □

Our final goal in this section is to combine our results to establish an embedding in the case where the intersection $\mathcal{O} \cap \mathcal{O}'$ can be expressed as a union of two sets, one near which the covering $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and one where the covering $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$.

For the proof of this result, we need the following lemma, which shows that arbitrary sums over subfamilies of a BAPU yield uniformly bounded multipliers for decomposition spaces. We remark that this is essentially the same as [27, Theorem 2.2(C)], but adapted to the slightly different class of decomposition spaces considered here.

**Lemma 5.1.10.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0,\infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an $L^p$-decomposition covering of $\mathcal{O}$. Let $u = (u_i)_{i \in I}$ be $\mathcal{Q}$-moderate and let $\Phi = (\varphi_i)_{i \in I}$ be an $L^p$-BAPU for $\mathcal{Q}$.*

*Then there is a constant*

$$C = \begin{cases} N_{\mathcal{Q}} C_{\mathcal{Q},\Phi,p}, & \text{if } p \in [1,\infty], \\ C\left(d, p, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}\right), & \text{if } p \in (0,1) \end{cases}$$

*with the following property: For any subset $L \subset I$, the map*

$$m_L : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right), f \mapsto \sum_{i \in L} \varphi_i f$$

*is well-defined and bounded with operator norm $|||m_L||| \leq C$.*

*Proof.* By Lemma 3.2.11, we know that $\Phi = (\varphi_i)_{i \in I}$ is a locally finite partition of unity, so that $\sum_{i \in L} \varphi_i f \in \mathcal{D}'(\mathcal{O})$ is a well-defined distribution on $\mathcal{O}$ for each $f \in \mathcal{D}'(\mathcal{O})$.

Let $\Gamma = (\gamma_i)_{i \in I}$ be an arbitrary $L^p$-BAPU for $\mathcal{Q}$. For $\ell \in I$, we have

$$\gamma_\ell \cdot m_L(f) = \sum_{i \in L} \varphi_i \gamma_\ell f = \sum_{i \in L \cap \ell^*} \varphi_i \gamma_\ell f,$$

because of $\varphi_i \gamma_\ell \equiv 0$ for $i \notin \ell^*$.

Since $\|\cdot\|_{L^p}$ is a (quasi)-norm and because of the uniform bound $|L \cap \ell^*| \leq |\ell^*| \leq N_{\mathcal{Q}}$, there is a constant $C_1 = C_1(p, N_{\mathcal{Q}}) > 0$ (with $C_1 = 1$ for $p \in [1,\infty]$) with

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\gamma_\ell \cdot m_L(f)) \right\|_{L^p} &= \left\| \sum_{i \in L \cap \ell^*} \mathcal{F}^{-1}(\varphi_i \gamma_\ell f) \right\|_{L^p} \\ &\leq C_1 \sum_{i \in L \cap \ell^*} \left\| \mathcal{F}^{-1}(\varphi_i \gamma_\ell f) \right\|_{L^p}. \end{aligned}$$

As usual, we distinguish the cases $p \in [1,\infty]$ and $p \in (0,1)$. In the first case, Young's inequality $L^1 * L^p \hookrightarrow L^p$ yields

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\varphi_i \gamma_\ell f) \right\|_{L^p} &\leq \left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^1} \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \\ &\leq C_{\mathcal{Q},\Phi,p} \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p}. \end{aligned}$$

In case of $p \in (0,1)$, the covering $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ is semi-structured and we have $\operatorname{supp} \varphi_i \subset \overline{Q_i} \subset \overline{Q_i^*}$ and

$$\operatorname{supp}(\gamma_\ell f) \subset \operatorname{supp} \gamma_\ell \subset \overline{Q_\ell} \subset \overline{Q_i^*}$$

for all $i \in \ell^*$, so that Corollary 3.2.15 yields

$$\begin{aligned} \left\| \mathcal{F}^{-1}(\varphi_i \gamma_\ell f) \right\|_{L^p} &\leq C_2 \left|\det(T_i)\right|^{\frac{1}{p}-1} \left\| \mathcal{F}^{-1} \varphi_i \right\|_{L^p} \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \\ &\leq C_2 C_{\mathcal{Q},\Phi,p} \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \end{aligned}$$

for some constant $C_2 = C_2(d, p, R_{\mathcal{Q}}, C_{\mathcal{Q}}) > 0$. Setting $C_2 := 1$ for $p \in [1, \infty]$ and $C_3 := C_1 C_2 C_{\mathcal{Q},\Phi,p}$, we conclude

$$\begin{aligned}
\left\| \mathcal{F}^{-1}(\gamma_\ell \cdot m_L(f)) \right\|_{L^p} &\leq C_1 \sum_{i \in L \cap \ell^*} \left\| \mathcal{F}^{-1}(\varphi_i \gamma_\ell f) \right\|_{L^p} \\
&\leq C_3 \cdot \sum_{i \in L \cap \ell^*} \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \\
&= C_3 \cdot |L \cap \ell^*| \cdot \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \\
&\leq C_3 N_{\mathcal{Q}} \cdot \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p}
\end{aligned}$$

for all $i \in I$. By solidity of $\ell^q_u(I)$, this yields

$$\begin{aligned}
\|m_L(f)\|_{\mathcal{D}_\Gamma(\mathcal{Q}, L^p, \ell^q_u)} &= \left\| \left( \left\| \mathcal{F}^{-1}(\gamma_\ell \cdot m_L(f)) \right\|_{L^p} \right)_{\ell \in I} \right\|_{\ell^q_u} \\
&\leq C_3 N_{\mathcal{Q}} \cdot \left\| \left( \left\| \mathcal{F}^{-1}(\gamma_\ell f) \right\|_{L^p} \right)_{\ell \in I} \right\|_{\ell^q_u} \\
&= C_3 N_{\mathcal{Q}} \cdot \|f\|_{\mathcal{D}_\Gamma(\mathcal{Q}, L^p, \ell^q_u)} < \infty,
\end{aligned}$$

which completes the proof. □

With this preparation, we can now state our most general sufficient embedding result. We will deviate from our earlier nomenclature in using $\mathcal{R}, \mathcal{S}$ as the names for our coverings. We do this because the names $\mathcal{P}, \mathcal{Q}$ are burdened with the implicit assumption that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

Our (somewhat technical) result is the following:

**Corollary 5.1.11.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$, define $p := \min\{p_1, p_2\}$ and assume that $\mathcal{R} = (R_\lambda)_{\lambda \in \Lambda} = (T_\lambda R'_\lambda + b_\lambda)_{\lambda \in \Lambda}$ and $\mathcal{S} = (S_\omega)_{\omega \in \Omega} = (M_\omega S'_\omega + c_\omega)_{\omega \in \Omega}$ are semi-structured $L^p$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = (u_\lambda)_{\lambda \in \Lambda}$ and $v = (v_\omega)_{\omega \in \Omega}$ be $\mathcal{R}$- and $\mathcal{S}$-moderate, respectively.*

*Let $q_1, q_2 \in (0, \infty]$ and assume that*

1. *there are $k, \ell \in \mathbb{N}_0$ and sets $A, B \subset \mathcal{O} \cap \mathcal{O}'$ with $\mathcal{O} \cap \mathcal{O}' = A \cup B$ and with the following two properties:*

   *a) for each $\lambda \in \Lambda_A$ with*

   $$\Lambda_A := \{\lambda \in \Lambda \mid R_\lambda \cap A \neq \varnothing\},$$

   *there is some $\omega_\lambda \in \Omega$ with $R_\lambda \subset S^{k*}_{\omega_\lambda}$,*

   *b) for each $\omega \in \Omega_B$ with*

   $$\Omega_B := \{\omega \in \Omega \mid S_\omega \cap B \neq \varnothing\},$$

   *there is some $\lambda_\omega \in \Lambda$ with $S_\omega \subset R^{\ell*}_{\lambda_\omega}$,*

*Put differently, we assume that $\mathcal{R}_{\Lambda_A}$ is almost subordinate to $\mathcal{S}$ and that $\mathcal{S}_{\Omega_B}$ is almost subordinate to $\mathcal{R}$.*

2. *we have* $p_1 \leq p_2$,

3. *with the definition*

$$\Omega_0 := \{\omega \in \Omega \mid S_\omega \cap R_\lambda \neq \varnothing \text{ for some } \lambda \in \Lambda \setminus \Lambda_A\} = \bigcup_{\lambda \in \Lambda \setminus \Lambda_A} \Omega_\lambda,$$

*there are finite constants* $C_1, C_2 > 0$, *such that the estimates*

$$\left\| \left( \left\| \left( |\det(T_\lambda)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_\lambda \right)_{\lambda \in \Lambda_\omega \cap \Lambda_A} \right\|_{\ell^{p_2}_{\square}} \right)_{\omega \in \Omega} \right\|_{\ell^{q_2}_v} \leq C_1 \left\| (c_\lambda)_{\lambda \in \Lambda_A} \right\|_{\ell^{q_1}_u} \tag{5.1.26}$$

*and*

$$\begin{aligned} &\left\| \left( |\det(M_\omega)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_\omega \right)_{\omega \in \Omega_0} \right\|_{\ell^{q_2}_v} \\ &\leq C_2 \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda} \right\|_{\ell^{p_1}_{w^{\mathcal{S},p_1}}} / w_\lambda^{\mathcal{R},p_1} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell^{q_1}_u} \end{aligned} \tag{5.1.27}$$

*hold for all sequences* $(c_\lambda)_{\lambda \in \Lambda_A}$ *and* $(c_\omega)_{\omega \in \Omega_0}$.

*Then, there is a bounded linear map*

$$\iota : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell^{q_2}_v\right)$$

*with* $|||\iota||| \leq C \cdot (C_1 + C_2)$ *for some constant* $C > 0$ *depending only on the quantities* $d, p_1, p_2, q_1, q_2$ *and on*

$$k, \ell, N_\mathcal{R}, C_\mathcal{R}, R_\mathcal{R}, D_\mathcal{R}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p}, N_\mathcal{S}, R_\mathcal{S}, C_\mathcal{S}, C_{v,\mathcal{S}}, C_{\mathcal{S},\Psi,p}$$

*where the* $L^p$*-BAPU* $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ *is used to calculate the quasi-norm on* $\mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right)$ *and the* $L^p$*-BAPU* $\Psi = (\psi_\omega)_{\omega \in \Omega}$ *is used to calculate the quasi-norm on* $\mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell^{q_2}_v\right)$.

*Because of* $p = p_1 \leq p_2$, *Corollary 5.1.4 shows that* $\Psi$ *is also an* $L^{p_2}$*-BAPU.*

*The map* $\iota$ *has the following additional properties:*

1. *For* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right)$ *and* $\gamma \in C^\infty_c(\mathcal{O} \cap \mathcal{O}')$, *we have*

$$\langle f, \gamma \rangle = \langle \iota f, \gamma \rangle .$$

2. *For* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right)$, *the inclusion* $\operatorname{supp} \iota f \subset \mathcal{O}' \cap \overline{\operatorname{supp} f}$ *is valid.*

3. *If* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right)$ *is given by integration against a measurable function* $f : \mathbb{R}^d \to \mathbb{C}$ *with* $f \in L^1_{\mathrm{loc}}(\mathcal{O}) \cap L^1_{\mathrm{loc}}(\mathcal{O}')$ *and with* $f(x) = 0$ *a.e. on* $\mathcal{O}' \setminus \mathcal{O}$, *then* $\iota f = f$.

4. *In particular, we have* $\iota f = f$ *for all* $f \in C^\infty_c(\mathcal{O} \cap \mathcal{O}')$.

*Note that* $f = 0$ *a.e. on* $\mathcal{O}' \setminus \mathcal{O}$ *is always fulfilled if* $\mathcal{O} \subset \mathbb{R}^d$ *is of full measure. In this case,* $\iota f = f$ *holds for every* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right)$ *which is given by (integration against) a function* $f \in L^1_{\mathrm{loc}}(\mathcal{O}) \cap L^1_{\mathrm{loc}}(\mathcal{O}')$.

*Proof.* We apply Theorem 5.1.8 (with $\mathcal{Q} = \mathcal{R}$, $\mathcal{P} = \mathcal{S}$ and $I_0 = \Lambda_A$) to derive boundedness of the map

$$\iota_1 : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto \left(g \mapsto \sum_{\lambda \in \Lambda_A} \langle \varphi_\lambda f, g \rangle\right)$$

with $|||\iota_1||| \leq C_3 C_1$ for some constant $C_3 > 0$ depending only on

$$d, p_1, p_2, k, N_{\mathcal{R}}, C_{\mathcal{R}}, R_{\mathcal{R}}, D_{\mathcal{R}}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}, N_{\mathcal{S}}, R_{\mathcal{S}}, C_{\mathcal{S}}, C_{\mathcal{S},\Psi,p_2}.$$

Here, the required almost subordinateness is ensured by assumption 1a. The required embedding of the discrete sequence spaces (cf. equation (5.1.17)) is guaranteed by equation (5.1.26). Here it is worth noting that we moved the factor $|\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}}$ to the left-hand side, which amounts to replacing $c_i$ by $c_i |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}}$ on both sides of equation (5.1.17). Since this is a bijection, the two conditions are equivalent. Finally, Corollary 5.1.4 shows $C_{\mathcal{S},\Psi,p_2} \lesssim C_{\mathcal{S},\Psi,p}$, where the implied constant only depends on $d, p_1, p_2, \mathcal{R}_{\mathcal{S}}$.

Furthermore, Lemma 5.1.10 yields boundedness of the multiplication operator

$$m_{\Lambda \setminus \Lambda_A} : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto \sum_{\lambda \in \Lambda \setminus \Lambda_A} \varphi_\lambda f$$

with $\left|\left|\left|m_{\Lambda \setminus \Lambda_A}\right|\right|\right| \leq C_4$ for a suitable constant

$$C_4 = C_4\left(d, p_1, N_{\mathcal{R}}, R_{\mathcal{R}}, C_{\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}\right).$$

Finally, Theorem 5.1.6 with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$, $J_0 := \Lambda \setminus \Lambda_A$ and

$$I_0 = \{\omega \in \Omega \mid S_\omega \cap R_\lambda \neq \varnothing \text{ for some } \lambda \in J_0\} = \Omega_0,$$

and with interchanged roles of $p_1, p_2$ and $q_1, q_2$ as well as $u, v$, implies boundedness of the map

$$\iota_2 : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto \left(g \mapsto \sum_{\omega \in \Omega_0} (\psi_\omega f)(g)\right)$$

with $|||\iota_2||| \leq C_5 C_2$ for a constant $C_5 > 0$ depending only on

$$p_1, p_2, q_1, q_2, d, \ell, N_{\mathcal{S}}, C_{\mathcal{S}}, R_{\mathcal{S}}, C_{v,\mathcal{S}}, N_{\mathcal{R}}, R_{\mathcal{R}}, C_{\mathcal{R}}, D_{\mathcal{R}}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}, C_{\mathcal{S},\Psi,p}.$$

Here, we note that $\mathcal{Q}_{I_0} = \mathcal{S}_{\Omega_0} = (S_\omega)_{\omega \in \Omega_0}$ is indeed almost subordinate to $\mathcal{P} = \mathcal{R}$, because for $\omega \in \Omega_0$, there exists some

$$x \in S_\omega \cap R_\lambda \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$$

for a suitable $\lambda \in J_0 = \Lambda \setminus \Lambda_A$. In case of $x \in A$, we would have $x \in R_\lambda \cap A \neq \varnothing$ which contradicts $\lambda \in \Lambda \setminus \Lambda_A$. Therefore, we must

have $x \in B$, so that we get $\omega \in \Omega_B$ because of $x \in B \cap S_\omega$. By assumption 1b, this implies $S_\omega \subset R^{\ell *}_{\lambda_\omega}$ for a suitable $\lambda_\omega \in \Lambda$.

We finally observe that the required embedding of nested sequence spaces (cf. equation (5.1.15)) is ensured by equation (5.1.27).

The above implies that

$$\iota : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell^{q_1}_u\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell^{q_2}_v\right), f \mapsto \iota_1 f + \iota_2\left(m_{\Lambda\setminus\Lambda_A} f\right)$$

is well-defined and bounded with

$$\begin{aligned}|||\iota||| &\leq |||\iota_1||| + |||\iota_2||| \cdot \left|||m_{\Lambda\setminus\Lambda_A}|||\right| \leq C_3 C_1 + C_4 C_5 C_2 \\ &\leq \max\left\{C_3, C_4 C_5\right\} \cdot (C_1 + C_2).\end{aligned}$$

It remains to check the claimed properties of $\iota$.

1. We first establish the identity

$$\varphi_\lambda(x) = \sum_{\omega\in\Omega_0} \varphi_\lambda(x)\,\psi_\omega(x) \qquad \forall x \in \mathcal{O}\cap\mathcal{O}'\,\forall\lambda\in\Lambda\setminus\Lambda_A. \quad (5.1.28)$$

This is trivial in the case $\varphi_\lambda(x) = 0$, so that we can assume $\varphi_\lambda(x) \neq 0$, and hence $x \in R_\lambda$. Now, $\sum_{\omega\in\Omega}\psi_\omega(x) = 1$ because of $x \in \mathcal{O}\cap\mathcal{O}'$. For each $\omega \in \Omega$ with $\psi_\omega(x) \neq 0$, we derive $x \in S_\omega \cap R_\lambda$ and thus $\omega \in \Omega_0$ because of $\lambda \in \Lambda\setminus\Lambda_A$. This shows $1 = \sum_{\omega\in\Omega}\psi_\omega(x) = \sum_{\omega\in\Omega_0}\psi_\omega(x)$ in the case $\varphi_\lambda(x) \neq 0$. Thus, the above identity is also true in this case.

Now, let $\gamma \in C^\infty_c(\mathcal{O}\cap\mathcal{O}')$ be arbitrary. Since $(\varphi_\lambda)_{\lambda\in\Lambda}$ is a locally finite partition of unity on $\mathcal{O}$ (cf. Lemma 3.2.11), an application of equation (5.1.28) yields

$$\begin{aligned}\gamma &= \sum_{\lambda\in\Lambda_A}\varphi_\lambda\gamma + \sum_{\lambda\in\Lambda\setminus\Lambda_A}\varphi_\lambda\gamma \\ &= \sum_{\lambda\in\Lambda_A}\varphi_\lambda\gamma + \sum_{\lambda\in\Lambda\setminus\Lambda_A}\sum_{\omega\in\Omega_0}\psi_\omega\varphi_\lambda\gamma \\ &= \sum_{\lambda\in\Lambda_A}\varphi_\lambda\gamma + \sum_{\omega\in\Omega_0}\sum_{\lambda\in\Lambda\setminus\Lambda_A}\psi_\omega\varphi_\lambda\gamma,\end{aligned}$$

where the interchange in summation is allowed because the local finiteness of the partitions of unity $(\varphi_\lambda)_{\lambda\in\Lambda}$ on $\mathcal{O}$ and $(\psi_\omega)_{\omega\in\Omega}$ on $\mathcal{O}'$, together with compactness of $\operatorname{supp}\gamma \subset \mathcal{O}\cap\mathcal{O}'$, implies that only finitely many summands do not vanish. Thus, there is no convergence issue, so that application of $f$ yields

$$\begin{aligned}\langle f,\gamma\rangle &= \sum_{\lambda\in\Lambda_A}\langle f,\varphi_\lambda\gamma\rangle + \sum_{\omega\in\Omega_0}\sum_{\lambda\in\Lambda\setminus\Lambda_A}\langle f,\psi_\omega\varphi_\lambda\gamma\rangle \\ &= \langle\iota_1 f,\gamma\rangle + \sum_{\omega\in\Omega_0}\left\langle m_{\Lambda\setminus\Lambda_A} f,\psi_\omega\gamma\right\rangle \\ &= \langle\iota_1 f,\gamma\rangle + \left\langle\iota_2\left(m_{\Lambda\setminus\Lambda_A} f\right),\gamma\right\rangle \\ &= \langle\iota f,\gamma\rangle.\end{aligned}$$

2. Let $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \subset \mathcal{D}'(\mathcal{O})$ and set $A := \overline{\operatorname{supp} f}$. Note that $A \subset \mathbb{R}^d$ is closed, so that $U := \mathcal{O}' \setminus A$ is open. Now, let $\gamma \in C_c^\infty(\mathcal{O}')$ with $\operatorname{supp}\gamma \subset U = \mathcal{O}' \setminus A$. Note that we have $\operatorname{supp}\varphi_\lambda \subset \mathcal{O}$ and hence

$$\begin{aligned}\operatorname{supp}(\varphi_\lambda \gamma) &\subset \mathcal{O} \cap U \\ &= \mathcal{O} \cap (\mathcal{O}' \setminus A) \\ &\subset \mathcal{O} \setminus A \\ &= \mathcal{O} \setminus \overline{\operatorname{supp} f} \\ &\subset \mathcal{O} \setminus \operatorname{supp} f\end{aligned}$$

for each $\lambda \in \Lambda$. Likewise, $\operatorname{supp}(\varphi_\lambda \psi_\omega \gamma) \subset \mathcal{O} \setminus \operatorname{supp} f$.

Since $f$ vanishes on $\mathcal{O} \setminus \operatorname{supp} f$, this yields

$$\langle f, \varphi_\lambda \gamma \rangle = 0 = \langle f, \varphi_\lambda \psi_\omega \gamma \rangle$$

for all $\lambda \in \Lambda$ and $\omega \in \Omega$. Hence,

$$\begin{aligned}\langle \iota f, \gamma \rangle &= \sum_{\Lambda \in \Lambda_A} \langle \varphi_\lambda f, \gamma \rangle + \sum_{\omega \in \Omega_0} \langle \psi_\omega \cdot m_{\Lambda \setminus \Lambda_A} f, \gamma \rangle \\ &= \sum_{\Lambda \in \Lambda_A} \langle f, \varphi_\lambda \gamma \rangle + \sum_{\omega \in \Omega_0} \sum_{\lambda \in \Lambda \setminus \Lambda_A} \langle \varphi_\lambda f, \psi_\omega \gamma \rangle \\ &= \sum_{\Lambda \in \Lambda_A} \langle f, \varphi_\lambda \gamma \rangle + \sum_{\omega \in \Omega_0} \sum_{\lambda \in \Lambda \setminus \Lambda_A} \langle f, \varphi_\lambda \psi_\omega \gamma \rangle \\ &= 0.\end{aligned}$$

All in all, we have shown that $\iota f$ vanishes on the open(!) set $U \subset \mathcal{O}'$. By definition of the support of a distribution (cf. [61, Definition 6.22]), this means

$$\operatorname{supp}(\iota f) \subset \mathcal{O}' \setminus U \subset \mathcal{O}' \cap A = \mathcal{O}' \cap \overline{\operatorname{supp} f}$$

as claimed.

3. For each $x \in \mathcal{O} \cap \mathcal{O}'$, Lemma 3.2.12 implies $x \in R_{\lambda_x}^\circ \cap S_{\omega_x}^\circ$ for suitable $\lambda_x \in \Lambda$ and $\omega_x \in \Omega$. Since $\mathbb{R}^d$ and hence also $\mathcal{O} \cap \mathcal{O}'$ are second-countable, this yields a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{O} \cap \mathcal{O}'$ with

$$\mathcal{O} \cap \mathcal{O}' \subset \bigcup_{n \in \mathbb{N}} \left[ R_{\lambda_{x_n}}^\circ \cap S_{\omega_{x_n}}^\circ \right].$$

Let

$$\Lambda' := \bigcup_{n \in \mathbb{N}} \lambda_{x_n}^* \qquad \text{and} \qquad \Omega' := \bigcup_{n \in \mathbb{N}} \omega_{x_n}^*. \tag{5.1.29}$$

For an arbitrary $x \in \mathcal{O} \cap \mathcal{O}'$ we then have $x \in R_{\lambda_{x_n}}^\circ \cap S_{\omega_{x_n}}^\circ$ for a suitable $n \in \mathbb{N}$. For $\lambda \in \Lambda$ with $\varphi_\lambda(x) \neq 0$, this implies $x \in R_\lambda \cap R_{\lambda_{x_n}} \neq \varnothing$ and thus $\lambda \in \lambda_{x_n}^* \subset \Lambda'$. Analogously, $\psi_\omega(x) \neq 0$ for some $\omega \in \Omega$ is only possible for $\omega \in \omega_{x_n}^* \subset \Omega'$ for a suitable $n \in \mathbb{N}$.

We recall that Fourier inversion implies

$$|\varphi_\lambda(x)| = \left|\widehat{\mathcal{F}^{-1}\varphi_\lambda}(x)\right| \leq \left\|\mathcal{F}^{-1}\varphi_\lambda\right\|_{L^1} \leq C_{\mathcal{R},\Phi,\infty} =: K_1$$

for all $\lambda \in \Lambda$ and $x \in \mathbb{R}^d$. Analogously, $|\psi_\omega(x)| \leq C_{\mathcal{S},\Psi,\infty} =: K_2$ holds for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$. Here, we implicitly used Corollary 5.1.4, which shows that any $L^{p_i}$-BAPU is also an $L^\infty$-BAPU.

Now, let $f$ be as in part 3 of the corollary and let $\varphi \in C_c^\infty(\mathcal{O}')$ be arbitrary. Set $K := \operatorname{supp}\varphi$. Let $x \in \mathcal{O} \cap \mathcal{O}'$ be arbitrary and choose $n \in \mathbb{N}$ with $x \in R_{\lambda_{x_n}} \cap S_{\omega_{x_n}}$. We get

$$\begin{aligned}
&\sum_{\lambda\in\Lambda_A} |\varphi_\lambda(x) f(x) \varphi(x)| + \sum_{\omega\in\Omega_0} \sum_{\lambda\in\Lambda\setminus\Lambda_A} |\varphi_\lambda(x) f(x) \psi_\omega(x) \varphi(x)| \\
&\leq \|\varphi\|_{\sup} \chi_K(x) |f(x)| \cdot \left[ \sum_{\lambda\in\lambda^*_{x_n}} |\varphi_\lambda(x)| + \sum_{\omega\in\omega^*_{x_n}} |\psi_\omega(x)| \cdot \sum_{\lambda\in\lambda^*_{x_n}} |\varphi_\lambda(x)| \right] \\
&\leq \|\varphi\|_{\sup} \cdot [K_1 \cdot N_{\mathcal{R}} + K_1 K_2 \cdot N_{\mathcal{R}} N_{\mathcal{S}}] \cdot \chi_K(x) \cdot |f(x)| \in L^1(\mathbb{R}^d).
\end{aligned} \tag{5.1.30}$$

In the last line, we used $|\lambda^*_{x_n}| \leq N_{\mathcal{R}}$ and $|\omega^*_{x_n}| \leq N_{\mathcal{S}}$, as well as $f \in L^1_{\mathrm{loc}}(\mathcal{O}')$ and compactness of $K \subset \mathcal{O}'$ to conclude $\chi_K f \in L^1(\mathbb{R}^d)$.

It is important to note that estimate (5.1.30) is in fact true for almost all $x \in \mathbb{R}^d$ and not only for $x \in \mathcal{O} \cap \mathcal{O}'$, because $x \in \mathbb{R}^d \setminus \mathcal{O}'$ implies $\varphi(x) = 0$, which makes the estimate trivial and $x \in \mathcal{O}' \setminus \mathcal{O}$ implies $f(x) = 0$ except on a set of measure zero, so that the estimate is also trivial.

As shown above, $\lambda \in \Lambda \setminus \Lambda'$ implies $\varphi_\lambda(x) \cdot f(x) \cdot \varphi(x) = 0$ for all $x \in \mathcal{O} \cap \mathcal{O}'$ and thus for almost all $x \in \mathbb{R}^d$, since we have $f(x) = 0$ for almost all $x \in \mathcal{O}' \setminus \mathcal{O}$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^d \setminus \mathcal{O}'$. This entails $\langle \varphi_\lambda f, \varphi\rangle = 0$ as well as $\langle \varphi_\lambda f, \psi_\omega\varphi\rangle = 0$ for $\lambda \in \Lambda \setminus \Lambda'$. Analogously, we have $\langle \varphi_\lambda f, \psi_\omega\varphi\rangle = 0$ for arbitrary $\lambda \in \Lambda$ and $\omega \in \Omega \setminus \Omega'$. Unwinding the definition of $\iota$, this implies

$$\begin{aligned}
\langle \iota f, \varphi\rangle &= \sum_{\lambda\in\Lambda_A} \langle \varphi_\lambda f, \varphi\rangle + \sum_{\omega\in\Omega_0} \langle \psi_\omega \cdot m_{\Lambda\setminus\Lambda_A} f, \varphi\rangle \\
&= \sum_{\lambda\in\Lambda_A} \langle \varphi_\lambda f, \varphi\rangle + \sum_{\omega\in\Omega_0} \sum_{\lambda\in\Lambda\setminus\Lambda_A} \langle \varphi_\lambda f, \psi_\omega\varphi\rangle \\
&= \sum_{\lambda\in\Lambda_A\cap\Lambda'} \int_{\mathbb{R}^d} \varphi_\lambda f \varphi \,\mathrm{d}x + \sum_{\omega\in\Omega_0\cap\Omega'} \sum_{\lambda\in\Lambda'\setminus\Lambda_A} \int_{\mathbb{R}^d} \varphi_\lambda \psi_\omega f \varphi \,\mathrm{d}x \\
&\overset{(\dagger)}{=} \int_{\mathbb{R}^d} \left[ \sum_{\lambda\in\Lambda_A\cap\Lambda'} \varphi_\lambda + \sum_{\omega\in\Omega_0\cap\Omega'} \sum_{\lambda\in\Lambda'\setminus\Lambda_A} \psi_\omega \varphi_\lambda \right] \cdot f \cdot \varphi \,\mathrm{d}x \\
&= \int_{\mathcal{O}\cap\mathcal{O}'} \left[ \sum_{\lambda\in\Lambda_A\cap\Lambda'} \varphi_\lambda + \sum_{\omega\in\Omega_0\cap\Omega'} \sum_{\lambda\in\Lambda'\setminus\Lambda_A} \psi_\omega \varphi_\lambda \right] \cdot f \cdot \varphi \,\mathrm{d}x.
\end{aligned} \tag{5.1.31}$$

The last step is justified because of $f\varphi = 0$ almost everywhere on $\mathbb{R}^d \setminus (\mathcal{O} \cap \mathcal{O}')$. At (†), we used the dominated convergence theorem together with estimate (5.1.30) and with the fact that all sums have only countably many terms.

For $x \in \mathcal{O} \cap \mathcal{O}'$, there are two cases:

*Case* 1. There is some $\lambda_0 \in \Lambda \setminus \Lambda_A$ with $\varphi_{\lambda_0}(x) \neq 0$. For $\omega \in \Omega$ with $\psi_\omega(x) \neq 0$, this implies $x \in R_{\lambda_0} \cap S_\omega \neq \varnothing$ and thus $\omega \in \Omega_0$. As seen in the discussion after equation (5.1.29), we also have $\omega \in \Omega'$. These considerations show

$$1 = \sum_{\omega \in \Omega} \psi_\omega(x) = \sum_{\omega \in \Omega_0 \cap \Omega'} \psi_\omega(x)$$

and thus

$$\begin{aligned} 1 = \sum_{\lambda \in \Lambda'} \varphi_\lambda(x) &= \sum_{\lambda \in \Lambda_A \cap \Lambda'} \varphi_\lambda(x) + \sum_{\lambda \in \Lambda' \setminus \Lambda_A} \varphi_\lambda(x) \\ &= \sum_{\lambda \in \Lambda_A \cap \Lambda'} \varphi_\lambda(x) + \sum_{\omega \in \Omega_0 \cap \Omega'} \sum_{\lambda \in \Lambda' \setminus \Lambda_A} (\psi_\omega \varphi_\lambda)(x). \end{aligned}$$

*Case* 2. We have $\varphi_\lambda(x) = 0$ for all $\lambda \in \Lambda \setminus \Lambda_A$. In this case, we derive

$$\begin{aligned} 1 = \sum_{\lambda \in \Lambda} \varphi_\lambda(x) &= \sum_{\lambda \in \Lambda' \cap \Lambda_A} \varphi_\lambda(x) \\ &= \sum_{\lambda \in \Lambda' \cap \Lambda_A} \varphi_\lambda(x) + \sum_{\omega \in \Omega_0 \cap \Omega'} \sum_{\lambda \in \Lambda' \setminus \Lambda_A} (\psi_\omega \varphi_\lambda)(x). \end{aligned} \tag{5.1.32}$$

Hence, equation (5.1.32) is true for all $x \in \mathcal{O} \cap \mathcal{O}'$. Substituting this identity into equation (5.1.31) yields

$$\langle \iota f, \varphi \rangle = \int_{\mathcal{O} \cap \mathcal{O}'} f(x)\, \varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} f(x)\, \varphi(x)\, \mathrm{d}x = \langle f, \varphi \rangle,$$

where we used once again that $f(x)\,\varphi(x) = 0$ holds for almost all $x \in \mathbb{R}^d \setminus (\mathcal{O} \cap \mathcal{O}')$, since $\varphi$ vanishes outside of $\mathcal{O}'$ and $f$ vanishes (almost everywhere) on $\mathcal{O}' \setminus \mathcal{O}$.

4. This is an immediate consequence of the prior property. □

The above embedding results seem (at least on first sight) quite technical and involved. In particular, checking the embeddings for nested sequence spaces required by equations (5.1.26) and (5.1.27) is not straightforward.

Hence, the next section is devoted to deriving significant simplifications of these conditions.

## 5.2 SIMPLIFICATION OF THE SUFFICIENT CONDITIONS

In this section, we will show that the validity of the embeddings for nested sequence spaces which appear as prerequisites in Theorems 5.1.6, 5.1.8 and in Corollary 5.1.11 can be decided by calculating the nested $\ell^p$-$\ell^q$-norm of a *single* sequence. This will make application of these results much more painless. Nevertheless, the calculation of these nested $\ell^p$-norms can still present a serious obstacle in concrete applications.

We begin with Lemma 5.2.2, which simplifies the problem of deciding whether there exists an embedding of a weighted $\ell^p$ sequence space into a nested weighted $\ell^q$-space , as long as the "subdivision" of the nested $\ell^q$-space is in fact a *partition*. After that, we establish the reverse result.

Finally, Lemma 5.2.5 will show that we can always reduce to the case of a disjoint subdivision in the settings of Theorems 5.1.6, 5.1.8 and Corollary 5.1.11.

The rest of this section is concerned with applying these results to obtain simplified versions of Theorems 5.1.6, 5.1.8 and of Corollary 5.1.11.

Before we state our results, we recall from the previous section the notation

$$p' := \begin{cases} \infty, & \text{for } p \in (0,1], \\ \frac{p}{p-1}, & \text{for } p \in (1,\infty), \\ 1, & \text{for } p = \infty, \end{cases}$$

where for $p \in [1,\infty]$, $p'$ is just the usual conjugate exponent, i.e. such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Using this notation, we have the following result about the characterization of certain sequence norms by duality, which is well-known for the range $p \in [1,\infty]$. Even for $p \in (0,1)$, no originality is claimed.

**Lemma 5.2.1.** *Let $p \in (0,\infty]$ be arbitrary and let $I \neq \varnothing$. We have*

$$\sum_{i\in I} |x_i y_i| \leq \left\|(x_i)_{i\in I}\right\|_{\ell^p} \cdot \left\|(y_i)_{i\in I}\right\|_{\ell^{p'}}$$

*for arbitrary sequences $(x_i)_{i\in I}, (y_i)_{i\in I} \in \mathbb{C}^I$.*

*Conversely, if $(y_i)_{i\in I}$ is a fixed sequence which satisfies*

$$\left|\sum_{i\in I} x_i y_i\right| \leq C \cdot \left\|(x_i)_{i\in I}\right\|_{\ell^p} \qquad \forall\, (x_i)_{i\in I} \in \ell_0(I)$$

*for some $C \in [0,\infty)$, then we have $\left\|(y_i)_{i\in I}\right\|_{\ell^{p'}} \leq C < \infty$.*

*Here, $\ell_0(I) := \langle \delta_i \,|\, i \in I\rangle$ is the* SPACE OF FINITELY SUPPORTED SEQUENCES.

*Proof.* For $p \in [1,\infty]$ the first statement is just Hölder's inequality and the second is the characterization of the $\ell^p$-norm by duality (cf. [33, Theorem 6.14], noting that the counting measure is semi-finite).

It remains to consider the case $p < 1$. By [33, Proposition 6.11], we have $\ell^p \hookrightarrow \ell^1$, where the embedding is norm-decreasing. Hence,

$$\sum_{i\in I} |x_i y_i| \leq \left\|(x_i)_{i\in I}\right\|_{\ell^1} \cdot \left\|(y_i)_{i\in I}\right\|_{\ell^\infty} \leq \left\|(x_i)_{i\in I}\right\|_{\ell^p} \cdot \left\|(y_i)_{i\in I}\right\|_{\ell^{p'}},$$

where we used $p' = \infty$ for $p < 1$.

For the converse inequality, let $(y_i)_{i\in I}$ be arbitrary. For $(y_i)_{i\in I} = 0$, the claim is trivial, so that we can assume $(y_i)_{i\in I} \not\equiv 0$.

Let $0 \leq \gamma < \left\|(y_i)_{i\in I}\right\|_{\ell^{p'}} = \left\|(y_i)_{i\in I}\right\|_{\ell^\infty}$ be arbitrary. Then there is some $i_0 \in I$ satisfying $\gamma < |y_{i_0}|$. Let $x_i := \delta_{i,i_0}$, so that $(x_i)_{i\in I} \in \ell_0(I)$. We then get

$$\gamma < |y_{i_0}| = \left|\sum_{i\in I} x_i y_i\right| \leq C \cdot \left\|(x_i)_{i\in I}\right\|_{\ell^p} = C.$$

Since $0 \leq \gamma < \left\|(y_i)_{i\in I}\right\|_{\ell^{p'}}$ was arbitrary, we are done. □

The next lemma is our first step for the simplification of the sufficient conditions for embeddings of decomposition spaces, which we developed in the previous section. It is important to note that the sets $(I^{(j)})_{j\in J}$ are required to partition the set $I$ in the present lemma. This is not (necessarily) the case for the families $(I_j)_{j\in J}$ and $(J_i)_{i\in I}$ used in the context of embeddings for decomposition spaces. Nevertheless, we will see below (cf. Lemma 5.2.5), that one can reduce to the case of a partition.

**Lemma 5.2.2.** *Let $I, J \neq \varnothing$ be arbitrary index sets and let $(u_i)_{i\in I}$, $(v_j)_{j\in J}$ and $(\alpha_i)_{i\in I}$ be families of positive numbers. Furthermore, assume that for each $j \in J$, we are given some subset $I^{(j)} \subset I$, such that the family $(I^{(j)})_{j\in J}$ satisfies*

$$I = \biguplus_{j\in J} I^{(j)}.$$

*Let $p, q, r \in (0,\infty]$ and $C \in [0,\infty)$.*

*Then, the estimate*

$$\left\|\left(\left\|(\alpha_i \cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell^q_v} \leq C \cdot \left\|(x_i)_{i\in I}\right\|_{\ell^p_u} \tag{5.2.1}$$

*is valid for all $(x_i)_{i\in I} \in \ell_0(I)$ if and only if we have*

$$\left\|\left(\left\|(u_i^{-1}\alpha_i)_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}\right)_{j\in J}\right\|_{\ell^{q\cdot(p/q)'}_v} \leq C, \tag{5.2.2}$$

*where $r\cdot(p/r)'$ has to be interpreted as $\infty$ for $r = \infty$ and $q\cdot(p/q)'$ also has to be interpreted as $\infty$ in case of $q = \infty$.*

*If estimate (5.2.2) holds, then equation (5.2.1) is valid for arbitrary sequences $(x_i)_{i\in I}$, not just for those in $\ell_0(I)$.*

*Remark* 5.2.3. As indicated by the above lemma, exponents of the form $q \cdot (p/q)'$ (which are to be interpreted as $\infty$ for $q = \infty$) will play an important role in checking the existence of embeddings as in equation (5.2.1).

For this reason, we note that $q \cdot (p/q)' < \infty$ is equivalent to $q < p$ and we give a simpler form for the reciprocal $\frac{1}{q\cdot(p/q)'}$, namely

$$\frac{1}{q \cdot (p/q)'} = \left(\frac{1}{q} - \frac{1}{p}\right)_+, \tag{5.2.3}$$

where we set $x_+ := \max\{0, x\}$ for $x \in \mathbb{R}$.

To see this, observe for $q < \infty$ the equivalence

$$\begin{aligned} q \cdot (p/q)' < \infty &\iff (p/q)' < \infty \iff p/q > 1 \\ &\iff p > q \iff \frac{1}{q} - \frac{1}{p} > 0. \end{aligned}$$

This entails

$$\frac{1}{q \cdot (p/q)'} = \begin{cases} \frac{1}{\infty} = 0 = \left(\frac{1}{q} - \frac{1}{p}\right)_+, & \text{if } q = \infty, \\ \frac{1}{\infty} = 0 = \left(\frac{1}{q} - \frac{1}{p}\right)_+, & \text{if } p \leq q < \infty, \\ \frac{1}{q}\left(1 - \frac{1}{p/q}\right) = \frac{1}{q} - \frac{1}{p} = \left(\frac{1}{q} - \frac{1}{p}\right)_+, & \text{if } p > q. \end{cases}$$

◀

Before we give the proof of Lemma 5.2.2, we remark that we will frequently use the identity

$$\left\|(x_i)_{i\in I}\right\|_{\ell^{p\cdot q}}^p = \left\|\left(|x_i|^p\right)_{i\in I}\right\|_{\ell^q}$$

which is valid for arbitrary sequences $(x_i)_{i\in I}$ and $p, q \in (0, \infty)$. A moment's thought shows that this identity also holds for $q = \infty$.

*Proof of Lemma 5.2.2.* Because of $\|\cdot\|_{\ell^s\left(I^{(j)}\right)} \equiv 0$ for $I^{(j)} = \varnothing$, we can assume $I^{(j)} \neq \varnothing$ for all $j \in J$ by discarding the empty sets.

It is easy to see that the validity of the estimate (5.2.1) for all $(x_i)_{i\in I} \in \ell_0(I)$ (or for all $(x_i)_{i\in I} \in \mathbb{C}^I$) is equivalent to

$$\left\|\left(\left\|(\alpha_i u_i^{-1} \cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q} \leq C \cdot \left\|(x_i)_{i\in I}\right\|_{\ell^p} \tag{5.2.4}$$

for all $(x_i)_{i\in I} \in \ell_0(I)$ (or for all $(x_i)_{i\in I} \in \mathbb{C}^I$).

Let us first assume $r < \infty$ and that estimate (5.2.1) (and hence also (5.2.4)) holds for all $(x_i)_{i\in I} \in \ell_0(I)$. Let $j \in J$ be arbitrary. Then inequality (5.2.4) yields (setting $x_i \equiv 0$ on $I \setminus I^{(j)}$):

$$v_j \cdot \left\|(\alpha_i u_i^{-1} \cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r} \leq C \cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p} \qquad \forall\, (x_i)_{i\in I^{(j)}} \in \ell_0\left(I^{(j)}\right).$$

Let $(y_i)_{i\in I^{(j)}} \in \ell_0(I^{(j)})$ be arbitrary and choose $(x_i)_{i\in I^{(j)}} := (|y_i|^{1/r})_{i\in I^{(j)}}$. This implies

$$\begin{aligned}\left|\sum_{i\in I^{(j)}} (\alpha_i u_i^{-1})^r \cdot y_i\right| &\leq \sum_{i\in I^{(j)}} (\alpha_i u_i^{-1} x_i)^r \\ &\leq \left[\frac{C}{v_j}\cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p}\right]^r \\ &= \left(\frac{C}{v_j}\right)^r \cdot \left\|(y_i)_{i\in I^{(j)}}\right\|_{\ell^{p/r}}.\end{aligned}$$

Using Lemma 5.2.1 (together with $p/r \in (0,\infty]$ because of $r<\infty$), we conclude

$$\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}^r = \left\|\left((\alpha_i u_i^{-1})^r\right)_{i\in I^{(j)}}\right\|_{\ell^{(p/r)'}} \leq \left(\frac{C}{v_j}\right)^r < \infty$$

for all $j\in J$. In case of $q=\infty$, we have thus established estimate (5.2.2), so that we can assume $q<\infty$.

Let $\varepsilon\in(0,1)$. Note that $I^{(j)}\neq\varnothing$ implies $\left\|\left((\alpha_i u_i^{-1})^r\right)_{i\in I^{(j)}}\right\|_{\ell^{(p/r)'}} > 0$. Hence, Lemma 5.2.1 shows that for each $j\in J$, there is a finitely supported sequence $(y_i)_{i\in I^{(j)}}\in\ell_0(I^{(j)})$ satisfying

$$\begin{aligned}\sum_{i\in I^{(j)}} (\alpha_i u_i^{-1})^r |y_i| &\geq \left|\sum_{i\in I^{(j)}} (\alpha_i u_i^{-1})^r y_i\right| \\ &> (1-\varepsilon)^r \left\|(y_i)_{i\in I^{(j)}}\right\|_{\ell^{p/r}} \left\|\left((\alpha_i u_i^{-1})^r\right)_{i\in I^{(j)}}\right\|_{\ell^{(p/r)'}}.\end{aligned} \tag{5.2.5}$$

This implies in particular $(y_i)_{i\in I^{(j)}}\not\equiv 0$, so that (by rescaling), we can assume

$$\left\|(y_i)_{i\in I^{(j)}}\right\|_{\ell^{p/r}} = 1. \tag{5.2.6}$$

Note that we can *unambiguously* write $y_i$ for $i\in I$, since there is a *unique* $j_i\in J$ satisfying $i\in I^{(j_i)}$, so that $y_i$ denotes the $i$-component of $(y_\ell)_{\ell\in I^{(j_i)}}$. Here, we used the assumption $I=\biguplus_{j\in J} I^{(j)}$.

Now, let $\varrho = (\varrho_j)_{j\in J}\in\ell_0(J)$ be arbitrary and define $z=(z_i)_{i\in I}$ by

$$z_i := |y_i|^{1/r}\cdot|\varrho_{j_i}|^{1/q}$$

for $i\in I$. We note that

$$\operatorname{supp} z \subset \bigcup_{j\in\operatorname{supp}\varrho} \operatorname{supp}(y_i)_{i\in I^{(j)}}$$

is finite, which implies $z\in\ell_0(I)$.

Note that we have $j_i = j$ for all $i \in I^{(j)}$. We now apply this observation, together with equations (5.2.5) and (5.2.6) (at $(\dagger)$), as well as (5.2.4) (at $(*)$), to derive

$$\begin{aligned}
&(1-\varepsilon)\cdot\left(\sum_{j\in J}\left|\varrho_j\right|\cdot\left\|\left(\alpha_i u_i^{-1}\right)_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}^{q}\cdot v_j^q\right)^{1/q}\\
&=\left\|\left(\left|\varrho_j\right|^{1/q}\cdot\left[(1-\varepsilon)^r\left\|\left(\left(\alpha_i u_i^{-1}\right)^r\right)_{i\in I^{(j)}}\right\|_{\ell^{(p/r)'}}\right]^{1/r}\right)_{j\in J}\right\|_{\ell_v^q}\\
&\overset{(\dagger)}{\leq}\left\|\left(\left|\varrho_j\right|^{1/q}\cdot\left[\sum_{i\in I^{(j)}}\left(\alpha_i u_i^{-1}\right)^r\left|y_i\right|\right]^{1/r}\right)_{j\in J}\right\|_{\ell_v^q}\\
&=\left\|\left(\left[\sum_{i\in I^{(j)}}\left(\alpha_i u_i^{-1}\cdot z_i\right)^r\right]^{1/r}\right)_{j\in J}\right\|_{\ell_v^q}\\
&=\left\|\left(\left\|\left(\alpha_i u_i^{-1}\cdot z_i\right)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q}\\
&\overset{(*)}{\leq} C\cdot\left\|\left(z_i\right)_{i\in I}\right\|_{\ell^p}. \qquad (5.2.7)
\end{aligned}$$

Next, we note that for $p<\infty$, the requirement $I=\biguplus_{j\in J} I^{(j)}$, together with the normalization $\left\|\left(y_i\right)_{i\in I^{(j)}}\right\|_{\ell^{p/r}}=1$, yields

$$\begin{aligned}
\left\|\left(z_i\right)_{i\in I}\right\|_{\ell^p}^p &= \sum_{j\in J}\sum_{i\in I^{(j)}}\left|z_i\right|^p\\
&= \sum_{j\in J}\sum_{i\in I^{(j)}}\left[\left|y_i\right|^{p/r}\cdot\left|\varrho_j\right|^{p/q}\right]\\
&= \sum_{j\in J}\left[\left|\varrho_j\right|^{p/q}\cdot\left\|\left(y_i\right)_{i\in I^{(j)}}\right\|_{\ell^{p/r}}^{p/r}\right]\\
&= \sum_{j\in J}\left[\left|\varrho_j\right|^{p/q}\right]=\left\|\left(\varrho_j\right)_{j\in J}\right\|_{\ell^{p/q}}^{p/q}.
\end{aligned}$$

In case of $p=\infty$, we recall our present assumptions $r<\infty$ and $q<\infty$. This yields $p/r=\infty=p/q$ and hence

$$\begin{aligned}
\left\|\left(z_i\right)_{i\in I}\right\|_{\ell^p}=\sup_{i\in I}\left|z_i\right| &= \sup_{j\in J}\sup_{i\in I^{(j)}}\left|y_i\right|^{1/r}\cdot\left|\varrho_j\right|^{1/q}\\
&= \sup_{j\in J}\left[\left|\varrho_j\right|^{1/q}\cdot\left\|\left(y_i\right)_{i\in I^{(j)}}\right\|_{\ell^{p/r}}^{1/r}\right]\\
&= \sup_{j\in J}\left|\varrho_j\right|^{1/q}\\
&= \left\|\left(\varrho_j\right)_{j\in J}\right\|_{\ell^{p/q}}^{1/q},
\end{aligned}$$

so that we have $\left\|\left(z_i\right)_{i\in I}\right\|_{\ell^p}=\left\|\left(\varrho_j\right)_{j\in J}\right\|_{\ell^{p/q}}^{1/q}$ for all $p\in(0,\infty]$.

Substituting this into estimate (5.2.7) entails (for $p < \infty$ as well as for $p = \infty$):

$$\sum_{j\in J}\left[|\varrho_j|\cdot\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}^q\cdot v_j^q\right]\leq\left(\frac{C}{1-\varepsilon}\right)^q\cdot\left\|(\varrho_j)_{j\in J}\right\|_{\ell^{p/q}}.$$

Since $\varrho\in\ell_0(J)$ was arbitrary, Lemma 5.2.1 yields

$$\begin{aligned}&\left\|\left(\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}\right)_{j\in J}\right\|_{\ell_v^{q\cdot(p/q)'}}^q\\&=\left\|\left(\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}}^q\cdot v_j^q\right)_{j\in J}\right\|_{\ell^{(p/q)'}}\\&\leq\left(\frac{C}{1-\varepsilon}\right)^q.\end{aligned}$$

Sending $\varepsilon\downarrow 0$ establishes equation (5.2.2) in case of $r<\infty$.

Let us now consider the case $r=\infty$. Recall (cf. the statement of the lemma) the convention $r\cdot(p/r)'=\infty$ in this case. Let $j\in J$ and choose $i_0\in I^{(j)}$ arbitrary. Define $x_i:=\delta_{i,i_0}$ for $i\in I$ and apply estimate (5.2.4). This yields

$$\begin{aligned}v_j\cdot\alpha_{i_0}u_{i_0}^{-1}&\leq v_j\cdot\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\\&\leq C\cdot\left\|(x_i)_{i\in I}\right\|_{\ell^p}\\&=C.\end{aligned}$$

Since $i_0\in I^{(j)}$ was arbitrary, we derive $v_j\cdot\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty}\leq C$. In case of $q=\infty$, this is just the target inequality (5.2.2) (since $j\in J$ was arbitrary), so that we can assume $q<\infty$.

*We note that $i_j\neq i_\ell$ holds for $j\neq\ell$, since the family $(I^{(j)})_{j\in J}$ is pairwise disjoint.*

Let $\varepsilon\in(0,1)$. For each $j\in J$, there exists some $i_j\in I^{(j)}$ with

$$\alpha_{i_j}u_{i_j}^{-1}\geq(1-\varepsilon)\cdot\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty}.$$

Define $M:=\left\{i_j\,\middle|\,j\in J\right\}$ and let $\varrho=(\varrho_j)_{j\in J}\in\ell_0(J)$ be arbitrary. For $i\in I$, let $j_i\in J$ be the *unique* index such that $i\in I^{(j_i)}$. Define

$$x_i:=|\varrho_{j_i}|^{1/q}\cdot\chi_M(i)$$

for $i\in I$. Using the identities $M\cap I^{(\ell)}=\{i_\ell\}$ for all $\ell\in J$ and $j_\ell=j$ for $\ell\in I^{(j)}$, we see

$$\operatorname{supp}(x_i)_{i\in I}\subset\bigcup_{\ell\in\operatorname{supp}\varrho}\{i_\ell\},$$

which yields $(x_i)_{i\in I}\in\ell_0(I)$. Furthermore, the same identities imply

$$\begin{aligned}\left((1-\varepsilon)\cdot\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty}\cdot v_j\right)^q\cdot|\varrho_j|&\leq\left(\alpha_{i_j}u_{i_j}^{-1}\cdot v_j\right)^q\cdot|\varrho_j|\\&=\left(\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^\infty}\cdot v_j\right)^q.\end{aligned}$$

By recalling $I = \biguplus_{j\in J} I^{(j)}$ and $M \cap I^{(j)} = \{i_j\}$ for all $j \in J$ and using equation (5.2.4) (at $(\dagger)$), this finally leads to

$$\begin{aligned}\left|\sum_{j\in J} \left((1-\varepsilon)\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty} v_j\right)^q \cdot \varrho_j\right| &\leq \sum_{j\in J} \left(\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^\infty} v_j\right)^q \\ &\overset{(\dagger)}{\leq} C^q \cdot \left\|(x_i)_{i\in I}\right\|_{\ell^p}^q \\ &= C^q \cdot \left\|\left(\left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p}\right)_{j\in J}\right\|_{\ell^p}^q \\ &= C^q \cdot \left\|\left(|\varrho_{j_{i_j}}|^{1/q}\right)_{j\in J}\right\|_{\ell^p}^q \\ &= C^q \cdot \left\|\left(\varrho_j\right)_{j\in J}\right\|_{\ell^{p/q}}.\end{aligned}$$

Since $\varrho \in \ell_0(J)$ was arbitrary, Lemma 5.2.1 yields

$$\begin{aligned}&\left\|\left(\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty}\right)_{j\in J}\right\|_{\ell_v^{q(p/q)'}}^q \\ &= \left\|\left(\left(\left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty}\cdot v_j\right)^q\right)_{j\in J}\right\|_{\ell^{(p/q)'}} \\ &\leq \left(\frac{C}{1-\varepsilon}\right)^q.\end{aligned}$$

By sending $\varepsilon \downarrow 0$, this establishes inequality (5.2.2) also for $r = \infty$.

It remains to show that estimate (5.2.2) implies that inequality (5.2.4) (and hence also inequality (5.2.1)) is valid, even for arbitrary sequences $x = (x_i)_{i\in I}$, not just for $x \in \ell_0(I)$. To this end, we first consider the case $r < \infty$, in which Lemma 5.2.1 yields

$$\begin{aligned}\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r} &= \left(\sum_{i\in I^{(j)}} (\alpha_i u_i^{-1})^r \cdot |x_i|^r\right)^{1/r} \\ &\leq \left[\left\|\left((\alpha_i u_i^{-1})^r\right)_{i\in I^{(j)}}\right\|_{\ell^{(p/r)'}} \cdot \left\|\left(|x_i|^r\right)_{i\in I^{(j)}}\right\|_{\ell^{p/r}}\right]^{1/r} \\ &= \left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}} \cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p}.\end{aligned}$$

In case of $r = \infty$, we have

$$\begin{aligned}\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r} &\leq \left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^\infty} \cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^\infty} \\ &\leq \left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}} \cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p}. \qquad (5.2.8)\end{aligned}$$

We have thus shown that equation (5.2.8) holds for all $r \in (0,\infty]$.

For $q < \infty$, this estimate, together with Lemma 5.2.1 (used at $(*)$) and with equation (5.2.2) (used at $(\dagger)$), implies

$$\begin{aligned}&\left\|\left(\left\|(\alpha_i u_i^{-1}\cdot x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q}^q \\ &\leq \sum_{j\in J}\left[v_j \cdot \left\|(\alpha_i u_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{r\cdot(p/r)'}} \cdot \left\|(x_i)_{i\in I^{(j)}}\right\|_{\ell^p}\right]^q\end{aligned}$$

$$\begin{aligned}
&\overset{(*)}{\leq} \left\| \left( \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^p}^q \right)_{j\in J} \right\|_{\ell^{p/q}} \cdot \left\| \left( v_j^q \cdot \left\| (\alpha_i u_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{r\cdot(p/r)'}}^q \right)_{j\in J} \right\|_{\ell^{(p/q)'}} \\
&= \left\| (x_i)_{i\in I} \right\|_{\ell^p}^q \cdot \left\| \left( \left\| (\alpha_i u_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{r\cdot(p/r)'}} \right)_{j\in J} \right\|_{\ell_v^{q\cdot(p/q)'}}^q \\
&\overset{(\dagger)}{\leq} C^q \cdot \left\| (x_i)_{i\in I} \right\|_{\ell^p}^q ,
\end{aligned}$$

which is nothing but equation (5.2.4) for arbitrary sequences $(x_i)_{i\in I}$.

Finally, for $q = \infty$, let $j \in J$ be arbitrary and use estimates (5.2.8) and (5.2.2) to derive

$$\begin{aligned}
v_j \cdot \left\| (\alpha_i u_i^{-1} \cdot x_i)_{i\in I^{(j)}} \right\|_{\ell^r} &\leq v_j \cdot \left\| (\alpha_i u_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{r\cdot(p/r)'}} \cdot \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^p} \\
&\leq \left\| \left( \left\| (\alpha_i u_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{r\cdot(p/r)'}} \right)_{j\in J} \right\|_{\ell_v^{q\cdot(p/q)'}} \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^p} \\
&\overset{(\dagger)}{\leq} C \cdot \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^p} \leq C \cdot \left\| (x_i)_{i\in I} \right\|_{\ell^p} ,
\end{aligned}$$

where equation (5.2.2) was used at $(\dagger)$. Because of $q = \infty$, this proves equation (5.2.4) for arbitrary sequences $(x_i)_{i\in I}$. □

Our next result can be seen as a reverse version of the above lemma.

**Lemma 5.2.4.** *Let $I, J \neq \varnothing$ be arbitrary index sets and let $(u_i)_{i\in I}$, $(v_j)_{j\in J}$ and $(\alpha_i)_{i\in I}$ be families of positive numbers. Furthermore, assume that for each $j \in J$, we are given some subset $I^{(j)} \subset I$, such that the family $(I^{(j)})_{j\in J}$ satisfies*

$$I = \biguplus_{j\in J} I^{(j)}.$$

*Let $p, q, r \in (0, \infty]$ and $C \in [0, \infty)$.*

*Then, the estimate*

$$\left\| (x_i)_{i\in I} \right\|_{\ell_u^p} \leq C \cdot \left\| \left( \left\| (\alpha_i \cdot x_i)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^q} \tag{5.2.9}$$

*is valid for all $(x_i)_{i\in I} \in \ell_0(I)$ if and only if we have*

$$\left\| \left( v_j^{-1} \cdot \left\| (u_i \alpha_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{p\cdot(r/p)'}} \right)_{j\in J} \right\|_{\ell^{p\cdot(q/p)'}} \leq C, \tag{5.2.10}$$

*where the exponents $p \cdot (r/p)'$ and $p \cdot (q/p)'$ have to be interpreted as $\infty$ in case of $p = \infty$.*

*If estimate* (5.2.10) *holds, then equation* (5.2.9) *is valid for arbitrary sequences $(x_i)_{i\in I}$, not just for those in $\ell_0(I)$.*

*Proof.* As in the previous lemma, we can assume $I^{(j)} \neq \varnothing$ for all $j \in J$ by discarding all indices $j \in J$ with $I^{(j)} = \varnothing$.

Let us first consider the case $p \in (0,\infty)$. We begin by showing that condition (5.2.9) implies estimate (5.2.10). Let $j_0 \in J$ be arbitrary. For any $(y_i)_{i\in I^{(j_0)}} \in \ell_0\left(I^{(j_0)}\right)$, let

$$x_i := \begin{cases} \alpha_i^{-1} \cdot |y_i|^{1/p}, & \text{if } i \in I^{(j_0)}, \\ 0, & \text{if } i \in I \setminus I^{(j_0)} \end{cases}$$

and note $(x_i)_{i\in I} \in \ell_0(I)$. Using equation (5.2.9) (at $(*)$), we conclude

$$\begin{aligned} \left| \sum_{i\in I^{(j_0)}} (u_i\alpha_i^{-1})^p \cdot y_i \right|^{1/p} &\leq \left[ \sum_{i\in I^{(j_0)}} (u_i x_i)^p \right]^{1/p} \\ &= \left\| (x_i)_{i\in I} \right\|_{\ell^p_u} \\ &\overset{(*)}{\leq} C \cdot \left\| \left( \left\| (\alpha_i \cdot x_i)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell^q_v} \\ &\overset{(\dagger)}{=} C \cdot v_{j_0} \cdot \left\| (\alpha_i \cdot x_i)_{i\in I^{(j_0)}} \right\|_{\ell^r} \\ &= C \cdot v_{j_0} \cdot \left\| \left( |y_i|^{1/p} \right)_{i\in I^{(j_0)}} \right\|_{\ell^r} \\ &= C \cdot v_{j_0} \cdot \left\| (y_i)_{i\in I^{(j_0)}} \right\|^{1/p}_{\ell^{r/p}}, \end{aligned}$$

where a moment's thought shows that the last step is indeed also valid for $r = \infty$. In the step marked with $(\dagger)$, we used disjointness of the family $\left(I^{(j)}\right)_{j\in J}$.

The above estimate is equivalent to

$$\left| \sum_{i\in I^{(j_0)}} (u_i\alpha_i^{-1})^p \cdot y_i \right| \leq C^p \cdot v_{j_0}^p \cdot \left\| (y_i)_{i\in I^{(j_0)}} \right\|_{\ell^{r/p}} < \infty.$$

Since this holds for arbitrary $(y_i)_{i\in I^{(j_0)}} \in \ell_0\left(I^{(j_0)}\right)$, Lemma 5.2.1 implies

$$\left\| (u_i\alpha_i^{-1})_{i\in I^{(j_0)}} \right\|_{\ell^{p\cdot(r/p)'}} = \left\| \left((u_i\alpha_i^{-1})^p\right)_{i\in I^{(j_0)}} \right\|^{1/p}_{\ell^{(r/p)'}} \leq C \cdot v_{j_0} < \infty$$

for all $j_0 \in J$. Here, we also used $r/p \in (0,\infty]$, which is true because of $p < \infty$.

Let $\varepsilon > 0$. Because of $I^{(j)} \neq \varnothing$, we have $\left\| \left((u_i\alpha_i^{-1})^p\right)_{i\in I^{(j)}} \right\|_{\ell^{(r/p)'}} > 0$. Thus, Lemma 5.2.1 yields for each $j \in J$ a finitely supported sequence $(y_i)_{i\in I^{(j)}} \in \ell_0\left(I^{(j)}\right)$ satisfying

$$\left| \sum_{i\in I^{(j)}} (u_i\alpha_i^{-1})^p \cdot y_i \right| > (1-\varepsilon) \cdot \left\| \left((u_i\alpha_i^{-1})^p\right)_{i\in I^{(j)}} \right\|_{\ell^{(r/p)'}} \cdot \left\| (y_i)_{i\in I^{(j)}} \right\|_{\ell^{r/p}}. \tag{5.2.11}$$

This implies in particular $(y_i)_{i\in I^{(j)}} \not\equiv 0$, so that we can assume

$$\left\|(y_i)_{i\in I^{(j)}}\right\|_{\ell^{r/p}} = 1 \tag{5.2.12}$$

by rescaling.

Observe that we can *unambiguously* write $y_i$ for $i \in I$, since there is a *unique* $j_i \in J$ satisfying $i \in I^{(j_i)}$, so that $y_i$ denotes the $i$-component of $(y_\ell)_{\ell\in I^{(j_i)}}$. Now, let $\varrho = (\varrho_j)_{j\in J} \in \ell_0(J)$ be arbitrary. Define $z = (z_i)_{i\in I}$ by

$$z_i := \alpha_i^{-1} |y_i|^{1/p} \cdot v_{j_i}^{-1} \cdot |\varrho_{j_i}|^{1/p} \text{ for } i \in I.$$

We have

$$\operatorname{supp} z \subset \bigcup_{j\in\operatorname{supp}\varrho} \operatorname{supp}(y_i)_{i\in I^{(j)}},$$

and thus $z \in \ell_0(I)$.

Note that we have $j_i = j$ for all $i \in I^{(j)}$. This observation, together with equation (5.2.11) and with the normalization (5.2.12), implies for arbitrary $j \in J$ the estimate

$$\begin{aligned}
(1-\varepsilon)\cdot \left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|^p_{\ell^{p\cdot(r/p)'}} |\varrho_j| &= (1-\varepsilon)\cdot|\varrho_j| \left\|\left((u_i\alpha_i^{-1})^p\right)_{i\in I^{(j)}}\right\|_{\ell^{(r/p)'}} \\
&\le |\varrho_j| \left|\sum_{i\in I^{(j)}} (u_i\alpha_i^{-1})^p \cdot y_i\right| \\
&\le |\varrho_j| \sum_{i\in I^{(j)}} (u_i\alpha_i^{-1})^p \cdot |y_i| \\
&= v_j^p \cdot \sum_{i\in I^{(j)}} (u_i z_i)^p .
\end{aligned}$$

Employing assumption (5.2.9) (which is used at $(\dagger)$), as well as the normalization in equation (5.2.12) (used at $(*)$), we arrive at

$$\begin{aligned}
&\left[\sum_{j\in J} (1-\varepsilon)\cdot \left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|^p_{\ell^{p\cdot(r/p)'}} \cdot v_j^{-p} \cdot |\varrho_j|\right]^{1/p} \\
&\le \left(\sum_{j\in J}\sum_{i\in I^{(j)}} (u_i z_i)^p\right)^{1/p} = \left(\sum_{i\in I} (u_i z_i)^p\right)^{1/p} = \left\|(z_i)_{i\in I}\right\|_{\ell^p_u} \\
&\overset{(\dagger)}{\le} C\cdot \left\|\left(\left\|(\alpha_i\cdot z_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell^q_v} \\
&= C\cdot \left\|\left(|\varrho_j|^{1/p}\cdot v_j^{-1}\cdot \left\|\left(|y_i|^{1/p}\right)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell^q_v} \\
&= C\cdot \left\|\left(|\varrho_j|^{1/p}\cdot v_j^{-1}\cdot \left\|(y_i)_{i\in I^{(j)}}\right\|^{1/p}_{\ell^{r/p}}\right)_{j\in J}\right\|_{\ell^q_v} \\
&\overset{(*)}{=} C\cdot \left\|\left(|\varrho_j|^{1/p}\right)_{j\in J}\right\|_{\ell^q} = C\cdot \left\|(\varrho_j)_{j\in J}\right\|^{1/p}_{\ell^{q/p}}.
\end{aligned}$$

Rearranging yields

$$\begin{aligned}\left|\sum_{j\in J}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p\cdot(r/p)'}}^{p}\cdot v_j^{-p}\cdot\varrho_j\right| &\leq \sum_{j\in J}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p\cdot(r/p)'}}^{p}\cdot v_j^{-p}\cdot|\varrho_j|\\ &\leq \frac{C^p}{1-\varepsilon}\cdot\left\|(\varrho_j)_{j\in J}\right\|_{\ell^{q/p}}.\end{aligned}$$

Since $\varrho\in\ell_0(J)$ was arbitrary, Lemma 5.2.1 shows

$$\begin{aligned}&\left\|\left(v_j^{-1}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p\cdot(r/p)'}}\right)_{j\in J}\right\|_{\ell^{p\cdot(q/p)'}}\\ &=\left\|\left(v_j^{-p}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p\cdot(r/p)'}}^{p}\right)_{j\in J}\right\|_{\ell^{(q/p)'}}^{1/p}\leq\frac{C}{(1-\varepsilon)^{\frac{1}{p}}}.\end{aligned}$$

Taking the limit $\varepsilon\downarrow 0$ proves estimate (5.2.10) for $p<\infty$.

For the converse direction, assume that estimate (5.2.10) holds. Using Lemma 5.2.1 (at (†)), we conclude

$$\begin{aligned}&\left\|(x_i)_{i\in I}\right\|_{\ell_u^p}^p\\ &=\sum_{j\in J}\sum_{i\in I^{(j)}}(u_i\alpha_i^{-1})^p\cdot(\alpha_i\,|x_i|)^p\\ &\overset{(\dagger)}{\leq}\sum_{j\in J}\left[\left\|\left((u_i\alpha_i^{-1})^p\right)_{i\in I^{(j)}}\right\|_{\ell^{(r/p)'}}\cdot\left\|\left((\alpha_i\,|x_i|)^p\right)_{i\in I^{(j)}}\right\|_{\ell^{r/p}}\right]\\ &=\sum_{j\in J}\left[\left(v_j^{-1}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p(r/p)'}}\right)^p\cdot\left(v_j\cdot\left\|(\alpha_i\,|x_i|)_{i\in I^{(j)}}\right\|_{\ell^r}\right)^p\right]\\ &\overset{(\dagger)}{\leq}\left\|\left(\left(v_j^{-1}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p(r/p)'}}\right)^p\right)_{j\in J}\right\|_{\ell^{(q/p)'}}\left\|\left(\left(v_j\left\|(\alpha_i\,|x_i|)_{i\in I^{(j)}}\right\|_{\ell^r}\right)^p\right)_{j\in J}\right\|_{\ell^{q/p}}\\ &=\left\|\left(v_j^{-1}\left\|(u_i\alpha_i^{-1})_{i\in I^{(j)}}\right\|_{\ell^{p(r/p)'}}\right)_{j\in J}\right\|_{\ell^{p\cdot(q/p)'}}^{p}\cdot\left\|\left(\left\|(\alpha_i\,|x_i|)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q}^p\\ &\leq\left[C\cdot\left\|\left(\left\|(\alpha_i x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q}\right]^p,\end{aligned}$$

where we used estimate (5.2.10) in the last step. This easily implies validity of estimate (5.2.9) for arbitrary sequences $(x_i)_{i\in I}\in\mathbb{C}^I$.

We finally handle the case $p=\infty$. Note that we have $p\cdot(r/p)'=\infty$ and $p\cdot(q/p)'=\infty$. First, assume that inequality (5.2.9) is satisfied. Let $j\in J$ be arbitrary. For each $i_0\in I^{(j)}$, application of equation (5.2.9) to $(x_i)_{i\in I}=(\delta_{i,i_0})_{i\in I}$ yields

$$u_{i_0}=\left\|(x_i)_{i\in I}\right\|_{\ell_u^p}\leq C\cdot\left\|\left(\left\|(\alpha_i x_i)_{i\in I^{(j)}}\right\|_{\ell^r}\right)_{j\in J}\right\|_{\ell_v^q}=C\cdot\alpha_{i_0}v_j,$$

where we used disjointness of the family $(I^{(j)})_{j\in J}$ in the last step. Since $i_0\in I^{(j)}$ was arbitrary, this yields

$$v_j^{-1}\cdot\left\|(\alpha_i^{-1}u_i)_{i\in I^{(j)}}\right\|_{\ell^\infty}\leq C.$$

Since $j \in J$ was arbitrary, and because of $p = \infty$, we easily see that the estimate in equation (5.2.10) holds.

Now, assume that equation (5.2.10) holds and let $(x_i)_{i\in I}$ be an arbitrary sequence. Choose $i_0 \in I$ and let $j_0 \in J$ with $i_0 \in I^{(j_0)}$. We have

$$\begin{aligned}
u_{i_0} |x_{i_0}| &= v_{j_0}\alpha_{i_0} |x_{i_0}| \cdot v_{j_0}^{-1} \cdot u_{i_0}\alpha_{i_0}^{-1} \\
&\leq v_{j_0}\alpha_{i_0} |x_{i_0}| \cdot v_{j_0}^{-1} \cdot \left\| (u_i\alpha_i^{-1})_{i\in I^{(j_0)}} \right\|_{\ell^{p\cdot(r/p)'}} \\
&\leq v_{j_0}\alpha_{i_0} |x_{i_0}| \cdot \left\| \left( v_j^{-1} \cdot \left\| (u_i\alpha_i^{-1})_{i\in I^{(j)}} \right\|_{\ell^{p\cdot(r/p)'}} \right)_{j\in J} \right\|_{\ell^{p\cdot(q/p)'}} \\
&\overset{(\dagger)}{\leq} C \cdot v_{j_0}\alpha_{i_0} |x_{i_0}| \\
&\leq C \cdot v_{j_0} \cdot \left\| (\alpha_i x_i)_{i\in I^{(j_0)}} \right\|_{\ell^r} \\
&\leq C \cdot \left\| \left( \left\| (\alpha_i x_i)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^q},
\end{aligned}$$

where equation (5.2.10) was used at $(\dagger)$. Since $i_0 \in I$ was arbitrary, this establishes equation (5.2.9) for $p = \infty$ and arbitrary sequences $(x_i)_{i\in I} \in \mathbb{C}^I$. □

To apply the above results in the setting of Section 5.1, we still have to show that we can reduce to the case of disjoint sets $(I^{(j)})_{j\in J}$ instead of the sets $(I_j)_{j\in J}$. The following lemma shows that this is indeed the case.

**Lemma 5.2.5.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ be admissible coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively.*

*Let $I_0 \subset I$ so that $\mathcal{Q}_{I_0}$ is almost subordinate to $\mathcal{P}$, i.e. there is some $k \in \mathbb{N}_0$ such that for each $i \in I_0$, we have $Q_i \subset P_{j_i}^{k*}$ for a suitable $j_i \in J$.*

*Then, for each $i \in I_0$, there is some $j \in J$ with $Q_i \cap P_j \neq \varnothing$.*

*We remark that $I^{(j)} = \varnothing$ can hold for some $j \in J$, but not all $j \in J$, unless $I_0 = \varnothing$.*

*Now, for each $i \in I_0$, select some $j^{(i)} \in J$ with $Q_i \cap P_{j^{(i)}} \neq \varnothing$ and set*

$$I^{(j)} := \left\{ i \in I_0 \,\middle|\, j^{(i)} = j \right\} \subset I_j \cap I_0$$

*for each $j \in J$.*

*We then have $I_0 = \biguplus_{j\in J} I^{(j)}$ and for each $\mathcal{P}$-moderate weight $v = (v_j)_{j\in J}$ and all $p, r \in (0, \infty]$, there is a constant $C = C(r, p, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}) > 0$ such that*

$$\begin{aligned}
\left\| \left( \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^p} &\leq \left\| \left( \left\| (x_i)_{i\in I_j\cap I_0} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^p} \\
&\leq C \cdot \left\| \left( \left\| (x_i)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^p} \qquad (5.2.13)
\end{aligned}$$

*holds for all sequences $(x_i)_{i\in I_0}$.*

*Proof.* For $i \in I_0$, we have $Q_i \neq \varnothing$ by definition of an admissible covering (cf. Definition 3.2.3). Let $x \in Q_i$ be arbitrary. Since $Q_i \subset P_{j_i}^{k*}$

holds by assumption, there is some $j \in j_i^{k*}$ with $x \in P_j$. This implies $x \in Q_i \cap P_j \neq \varnothing$, which proves the first claim.

Let us now prove estimate (5.2.13). The lower part of the estimate is trivial because of $I^{(j)} \subset I_j \cap I_0$ for all $j \in J$. It remains to establish the upper bound.

To this end, let $j \in J$. For $\ell \in J$ with $\varnothing \neq I_j \cap I^{(\ell)} \subset I_0 \cap I_j \cap I_\ell$, there is some $i \in I_0$ satisfying

$$\varnothing \neq Q_i \cap P_j \subset P_{j_i}^{k*} \cap P_j \qquad \text{and} \qquad \varnothing \neq Q_i \cap P_\ell \subset P_{j_i}^{k*} \cap P_\ell.$$

This implies $j_i \in j^{(k+1)*}$ and thus $\ell \in j_i^{(k+1)*} \subset j^{(2k+2)*}$. We have hence shown $I_j \cap I^{(\ell)} = \varnothing$ for $\ell \in J \setminus j^{(2k+2)*}$.

Since the (nonempty) inverse images of the map $I_0 \to J, i \mapsto j^{(i)}$ form a partition of $I_0$, we get $I_0 = \biguplus_{j \in J} I^{(j)}$. Together with the conclusion of the preceding paragraph, we arrive at

$$I_j \cap I_0 = \biguplus_{\ell \in J} I_j \cap I^{(\ell)} = \biguplus_{\ell \in j^{(2k+2)*}} I_j \cap I^{(\ell)}. \tag{5.2.14}$$

For $m \in \mathbb{N}$, let $\Theta_m : \ell_v^p(J) \to \ell_v^p(J)$ denote the "higher order" clustering map as defined in Lemma 3.4.3, i.e.

$$(\Theta_m c)_j = \sum_{\ell \in j^{m*}} c_\ell \text{ for } j \in J.$$

Now, let $(x_i)_{i \in I_0}$ be an arbitrary sequence. Since $\ell^r$ is a quasi-norm (more precisely, a $\min\{1, r\}$-norm, see Example 2.1.3) and because of the uniform bound $|j^{(2k+2)*}| \leq N_{\mathcal{P}}^{2k+2}$ (cf. Lemma 3.2.18), there is some constant $C_1 = C_1(r, N_{\mathcal{P}}, k) > 0$ with

$$\begin{aligned}
\left\| (x_i)_{i \in I_j \cap I_0} \right\|_{\ell^r} &= \left\| \sum_{\ell \in j^{(2k+2)*}} (x_i \cdot \chi_{I^{(\ell)}}(i))_{i \in I_j \cap I_0} \right\|_{\ell^r} \\
&\leq C_1 \sum_{\ell \in j^{(2k+2)*}} \left\| (x_i \cdot \chi_{I^{(\ell)}}(i))_{i \in I_j \cap I_0} \right\|_{\ell^r} \\
&\leq C_1 \cdot \left[ \Theta_{2k+2} \left( \left\| (x_i)_{i \in I^{(\ell)}} \right\|_{\ell^r} \right)_{\ell \in J} \right]_j
\end{aligned}$$

for all $j \in J$. Here, identity (5.2.14) was used in the first step.

By solidity of $\ell_v^p(J)$, we conclude

$$\begin{aligned}
\left\| \left( \left\| (x_i)_{i \in I_j \cap I_0} \right\|_{\ell^r} \right)_{j \in J} \right\|_{\ell_v^p} &\leq C_1 \cdot \left\| \Theta_{2k+2} \left( \left\| (x_i)_{i \in I^{(\ell)}} \right\|_{\ell^r} \right)_{\ell \in J} \right\|_{\ell_v^p} \\
&\leq C_1 \left\| \Theta_{2k+2} \right\|_{\ell_v^p \to \ell_v^p} \cdot \left\| \left( \left\| (x_i)_{i \in I^{(\ell)}} \right\|_{\ell^r} \right)_{\ell \in J} \right\|_{\ell_v^p} \\
&\leq C_1 C_2 \cdot \left\| \left( \left\| (x_i)_{i \in I^{(j)}} \right\|_{\ell^r} \right)_{j \in J} \right\|_{\ell_v^p},
\end{aligned}$$

where the constant $C_2 = C_2(k, p, N_{\mathcal{P}}, C_{v,\mathcal{P}})$ is obtained by combining Lemma 3.4.2 and Lemma 3.4.3. This completes the proof. □

Using these results, we can rephrase the sufficient conditions for the existence of embeddings between decomposition spaces obtained in Section 5.1 in a more compact form. The following corollary is a preparation for this.

**Corollary 5.2.6.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let*

$$\mathcal{R} = (R_\lambda)_{\lambda\in\Lambda} = \left(T_\lambda R'_\lambda + b_\lambda\right)_{\lambda\in\Lambda}$$

*and*

$$\mathcal{S} = (S_\omega)_{\omega\in\Omega} = \left(M_\omega S'_\omega + c_\omega\right)_{\omega\in\Omega}$$

*be semi-structured coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = (u_\lambda)_{\lambda\in\Lambda}$ and $v = (v_\omega)_{\omega\in\Omega}$ be $\mathcal{R}$- and $\mathcal{S}$-moderate, respectively.*

*Finally, assume that there are $k, \ell \in \mathbb{N}_0$ and sets $A, B \subset \mathcal{O} \cap \mathcal{O}'$ with $\mathcal{O} \cap \mathcal{O}' = A \cup B$ and with the following two properties:*

1. *for each $\lambda \in \Lambda_A := \{\lambda \in \Lambda \mid R_\lambda \cap A \neq \varnothing\}$, there is some $\omega_\lambda \in \Omega$ with $R_\lambda \subset S^{k*}_{\omega_\lambda}$,*
2. *for each $\omega \in \Omega_B := \{\omega \in \Omega \mid S_\omega \cap B \neq \varnothing\}$, there is some $\lambda_\omega \in \Lambda$ with $S_\omega \subset R^{\ell*}_{\lambda_\omega}$.*

*Define*

$$\Omega_0 := \{\omega \in \Omega \mid S_\omega \cap R_\lambda \neq \varnothing \text{ for some } \lambda \in \Lambda \setminus \Lambda_A\} = \bigcup_{\lambda\in\Lambda\setminus\Lambda_A} \Omega_\lambda.$$

*For each $\lambda \in \Lambda_A$, fix some $\omega^{(\lambda)} \in \Omega$ with $R_\lambda \cap S_{\omega^{(\lambda)}} \neq \varnothing$ and for each $\omega \in \Omega_0$, fix some $\lambda^{(\omega)} \in \Lambda \setminus \Lambda_A$ with $S_\omega \cap R_{\lambda^{(\omega)}} \neq \varnothing$. Define*

$$\Lambda^{(\omega)} := \left\{\lambda \in \Lambda_A \,\middle|\, \omega^{(\lambda)} = \omega\right\} \subset \Lambda_A \cap \Lambda_\omega,$$
$$\Omega_0^{(\lambda)} := \left\{\omega \in \Omega_0 \,\middle|\, \lambda^{(\omega)} = \lambda\right\} \subset \Omega_0 \cap \Omega_\lambda.$$

*Let $r \in \mathbb{R}$ and $t_1, t_2, t_3, t_4 \in (0, \infty]$ be arbitrary and set*

$$K_1 := \left\| \left( \left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^r \right)_{\lambda\in\Lambda^{(\omega)}} \right\|_{\ell^{t_3\cdot(t_1/t_3)'}} \right)_{\omega\in\Omega} \right\|_{\ell_v^{t_2\cdot(t_1/t_2)'}}$$

$$K'_1 := \left\| \left( \left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^r \right)_{\lambda\in\Lambda_A\cap\Lambda_\omega} \right\|_{\ell^{t_3\cdot(t_1/t_3)'}} \right)_{\omega\in\Omega} \right\|_{\ell_v^{t_2\cdot(t_1/t_2)'}}$$

$$K_2 := \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \cdot |\det(M_\omega)|^r \right)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{t_2\cdot(t_4/t_2)'}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell^{t_2\cdot(t_1/t_2)'}}$$

$$K'_2 := \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \cdot |\det(M_\omega)|^r \right)_{\omega\in\Omega_0\cap\Omega_\lambda} \right\|_{\ell^{t_2\cdot(t_4/t_2)'}} \right)_{\lambda\in\Lambda} \right\|_{\ell^{t_2\cdot(t_1/t_2)'}}$$

$$K''_2 := \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \cdot |\det(M_\omega)|^r \right)_{\omega\in\Omega_B\cap\Omega_\lambda} \right\|_{\ell^{t_2\cdot(t_4/t_2)'}} \right)_{\lambda\in\Lambda} \right\|_{\ell^{t_2\cdot(t_1/t_2)'}}$$

*Finally, let* $C_1, C_2, C_2'' \in [0,\infty]$ *be minimal with*

$$\left\| \left( \left\| \left( |\det(T_\lambda)|^r \cdot c_\lambda \right)_{\lambda \in \Lambda_\omega \cap \Lambda_A} \right\|_{\ell^{t_3}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}} \leq C_1 \cdot \left\| (c_\lambda)_{\lambda \in \Lambda_A} \right\|_{\ell_u^{t_1}}$$

*for all sequences* $(c_\lambda)_{\lambda \in \Lambda_A}$ *and with*

$$\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega \in \Omega_0} \right\|_{\ell_v^{t_2}} \leq C_2 \cdot \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell_u^{t_1}},$$

$$\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega \in \Omega_B} \right\|_{\ell_v^{t_2}} \leq C_2'' \cdot \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda \cap \Omega_B} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}}$$

*for all sequences* $(c_\omega)_{\omega \in \Omega}$.

*We have* $\Omega_0 \subset \Omega_B$ *and there are constants* $L_1, \ldots, L_4$ *of the form*

$$L_i = L_i(t_1, t_2, t_3, k, N_{\mathcal{S}}, C_{v,\mathcal{S}}) \text{ for } i \in \{1,2\},$$
$$L_i = L_i(t_1, t_2, t_4, \ell, N_{\mathcal{R}}, C_{u,\mathcal{R}}) \text{ for } i \in \{3,4\}$$

*such that we have*

$$L_1^{-1} \cdot C_1 \leq K_1 \leq K_1' \leq L_2 \cdot C_1, \tag{5.2.15}$$

*and*

$$C_2 \leq K_2 \leq K_2' \leq L_3 \cdot C_2 \tag{5.2.16}$$

*as well as* $K_2' \leq K_2''$ *and*

$$C_2'' \leq K_2'' \leq L_4 \cdot C_2''. \tag{5.2.17}$$

*Remark.* In applications of this corollary, we denote the constant $K_1$ by $K_1(t_1, t_2, t_3, r, u, v)$ to indicate the precise choices of $t_1, t_2, t_3, r, u, v$. The same notation applies to $K_1'$ and $C_1$.

Likewise, we will write $K_2 = K_2(t_1, t_2, t_4, r, u, v)$ and the same notation applies to $K_2', K_2''$ and $C_2, C_2''$.

*Proof.* Since $\mathcal{R}, \mathcal{S}$ are admissible coverings, we have $R_\lambda \neq \varnothing \neq S_\omega$ for all $(\lambda, \omega) \in \Lambda \times \Omega$. By hypothesis, $\mathcal{R}_{\Lambda_A}$ is almost subordinate to $\mathcal{S}$.

Furthermore, for each $\omega \in \Omega_0$ there is some $\lambda \in \Lambda \setminus \Lambda_A$ and some $x \in S_\omega \cap R_\lambda \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$. In case of $x \in A$, we would have $x \in R_\lambda \cap A \neq \varnothing$ in contradiction to $\lambda \in \Lambda \setminus \Lambda_A$. This implies $x \in B$ and thus $x \in B \cap S_\omega \neq \varnothing$, which means $\omega \in \Omega_B$ and hence $\Omega_0 \subset \Omega_B$. By assumption, $\mathcal{S}_{\Omega_B}$ is almost subordinate to $\mathcal{R}$, so that the same is true of $\mathcal{S}_{\Omega_0}$ with $k(\mathcal{S}_{\Omega_0}, \mathcal{R}) \leq k(\mathcal{S}_{\Omega_B}, \mathcal{R}) \leq \ell$.

We observe the obvious inclusions $\Lambda^{(\omega)} \subset \Lambda_A \cap \Lambda_\omega$ as well as $\Omega_0^{(\lambda)} \subset \Omega_0 \cap \Omega_\lambda$. Together with $\Omega_0 \subset \Omega_B$, these inclusions easily entail $K_1 \leq K_1'$ and $K_2 \leq K_2' \leq K_2''$, so that it remains to establish inequalities (5.2.15) – (5.2.17).

But Lemma 5.2.5 (with $\mathcal{Q} = \mathcal{R}$, $I_0 = \Lambda_A$ and $\mathcal{P} = \mathcal{S}$ and with the sequence $x_\lambda = |\det(T_\lambda)|^r c_\lambda$) yields some $L_1 = L_1(t_2, t_3, k, N_\mathcal{S}, C_{v,\mathcal{S}}) \geq 1$ such that

$$\begin{aligned}
&\left\| \left( \left\| \left( |\det(T_\lambda)|^r c_\lambda \right)_{\lambda \in \Lambda^{(\omega)}} \right\|_{\ell^{t_3}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}} \\
&\leq \left\| \left( \left\| \left( |\det(T_\lambda)|^r c_\lambda \right)_{\lambda \in \Lambda_\omega \cap \Lambda_A} \right\|_{\ell^{t_3}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}} \\
&\leq L_1 \cdot \left\| \left( \left\| \left( |\det(T_\lambda)|^r c_\lambda \right)_{\lambda \in \Lambda^{(\omega)}} \right\|_{\ell^{t_3}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}}
\end{aligned}$$

holds for all sequences $(c_\lambda)_{\lambda \in \Lambda_A}$. Now, choose $C_1' \in [0, \infty]$ minimal with

$$\left\| \left( \left\| \left( |\det(T_\lambda)|^r \cdot c_\lambda \right)_{\lambda \in \Lambda^{(\omega)}} \right\|_{\ell^{t_3}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}} \leq C_1' \cdot \left\| (c_\lambda)_{\lambda \in \Lambda_A} \right\|_{\ell_u^{t_1}}$$

for all sequences $(c_\lambda)_{\lambda \in \Lambda_A}$. The above implies $C_1' \leq C_1 \leq L_1 C_1'$.

Furthermore, by definition of $K_1$ and by Lemma 5.2.2 (with $J = \Omega$, $I = \Lambda_A$ and $I^{(j)} = \Lambda^{(j)}$ for $j \in \Omega$ as well as $\alpha_i = |\det(T_i)|^r$ and $r = t_3$, $p = t_1$, $q = t_2$), we get $C_1' = K_1$. Thus, we have shown $K_1 = C_1' \leq C_1 \leq L_1 C_1' = L_1 K_1$.

But Lemma 5.2.5 (with $\mathcal{Q} = \mathcal{R}$, $I_0 = \Lambda_A$ and $\mathcal{P} = \mathcal{S}$ as well as $x_i = u_i^{-1} |\det(T_i)|^r$) yields a constant

$$L_2 = L_2(t_1, t_2, t_3, k, N_\mathcal{S}, C_{v,\mathcal{S}}) \geq 1$$

with $K_1 \leq K_1' \leq L_2 \cdot K_1$. Summing up, we have shown

$$L_1^{-1} C_1 \leq K_1 \leq K_1' \leq L_2 \cdot K_1 \leq L_2 \cdot C_1,$$

which proves equation (5.2.15).

For the proof of equation (5.2.16), we first apply Lemma 5.2.5 with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$ and $I_0 = \Omega_0$, as well as $(x_i)_{i \in I_0} = (c_\omega)_{\omega \in \Omega_0}$ to obtain

$$\begin{aligned}
\left\| \left( \left\| (c_\omega)_{\omega \in \Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell_u^{t_1}} &\leq \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell_u^{t_1}} \\
&\overset{(\dagger)}{=} \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda \cap \Omega_0} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell_u^{t_1}} \\
&\leq \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda \cap \Omega_0} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}} \\
&\leq L_3 \cdot \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}} \\
&\overset{(\ddagger)}{=} L_3 \cdot \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell_u^{t_1}}
\end{aligned}$$

for some constant

$$L_3 = L_3\left(t_1, t_4, \ell, N_{\mathcal{R}}, C_{u,\mathcal{R}}\right)$$

and arbitrary sequences $(c_\omega)_{\omega\in\Omega_0}$. At (†), we used $\Omega_\lambda \subset \Omega_0$ for $\lambda \in \Lambda \setminus \Lambda_A$. For the justification of (‡), observe that $\Omega_0^{(\lambda)} = \varnothing$ holds for all $\lambda \in \Lambda_A$, because the $\lambda^{(\omega)}$ are all chosen so that $\lambda^{(\omega)} \in \Lambda \setminus \Lambda_A$.

Now, let $M \in [0,\infty]$ be minimal with

$$\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_0} \right\|_{\ell_v^{t_2}} \le M \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}}$$

for all sequences $(c_\omega)_{\omega\in\Omega_0}$. Using the above estimate and the definition of $C_2$, we get

$$\begin{aligned}\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_0} \right\|_{\ell_v^{t_2}} &\le M \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}} \\ &\le M \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_\lambda} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}}\end{aligned}$$

and

$$\begin{aligned}\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_0} \right\|_{\ell_v^{t_2}} &\le C_2 \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_\lambda} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}} \\ &\le L_3 C_2 \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}}\end{aligned}$$

for all sequences $(c_\omega)_{\omega\in\Omega_0}$. By minimality, this yields

$$C_2 \le M \le L_3 C_2.$$

Since the map

$$(d_\omega)_{\omega\in\Omega_0} \mapsto \left( |\det(M_\omega)|^{-r} \cdot d_\omega \right)_{\omega\in\Omega_0}$$

yields a bijection of $\mathbb{C}^{\Omega_0}$ with itself, we see that $M \in [0,\infty]$ is minimal with

$$\left\| (d_\omega)_{\omega\in\Omega_0} \right\|_{\ell_v^{t_2}} \le M \cdot \left\| \left( \left\| \left( |\det(M_\omega)|^{-r} d_\omega \right)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{t_1}}$$

for all sequences $(d_\omega)_{\omega\in\Omega_0}$. Now, an application of Lemma 5.2.4 (with $I = \Omega_0$ and $I^{(j)} = \Omega_0^{(j)}$ for $j \in J := \Lambda \setminus \Lambda_A$ and with interchanged roles of $u, v$) yields $M = K_2$. Here, we note that $\Omega_0 = \biguplus_{\lambda\in\Lambda\setminus\Lambda_A} \Omega_0^{(\lambda)}$

holds, because $\Omega_0$ is partitioned by the family of inverse images $\Omega_0^{(\lambda)} = (\omega \mapsto \lambda^{(\omega)})^{-1}(\{\lambda\})$ for $\lambda \in \Lambda \setminus \Lambda_A$. Hence, we have shown

$$C_2 \leq M = K_2 = M \leq L_3 \cdot C_2.$$

We finally use Lemma 5.2.5 (with $\mathcal{Q} = \mathcal{S}$, $I = \Omega$, $I_0 = \Omega_0$, $J = \Lambda$, $I^{(j)} = \Omega_0^{(j)}$ and $\mathcal{P} = \mathcal{R}$ as well as $x_i = v_i \left|\det(M_i)\right|^r$) to furnish a constant $L_4 = L_4(t_1, t_2, t_4, \ell, N_{\mathcal{R}}, C_{u,\mathcal{R}}) \geq 1$ with

*To obtain the exact dependencies of $L_4$ as stated, we implicitly used the identity $C_{u,\mathcal{R}} = C_{u^{-1},\mathcal{R}}$, which is a direct consequence of the definitions.*

$$\begin{aligned}
K_2 &\overset{(\dagger)}{=} \left\| \left( \left\| \left( v_i \cdot \left|\det(M_i)\right|^r \right)_{i \in I^{(j)}} \right\|_{\ell^{t_2 \cdot (t_4/t_2)'}} \right)_{j \in J} \right\|_{\ell_{u^{-1}}^{t_2 \cdot (t_1/t_2)'}} \\
&\leq \left\| \left( \left\| \left( v_i \cdot \left|\det(M_i)\right|^r \right)_{i \in I_j \cap I_0} \right\|_{\ell^{t_2 \cdot (t_4/t_2)'}} \right)_{j \in J} \right\|_{\ell_{u^{-1}}^{t_2 \cdot (t_1/t_2)'}} \\
&= \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \cdot \left|\det(M_\omega)\right|^r \right)_{\omega \in \Omega_\lambda \cap \Omega_0} \right\|_{\ell^{t_2 \cdot (t_4/t_2)'}} \right)_{\lambda \in \Lambda} \right\|_{\ell^{t_2 \cdot (t_1/t_2)'}} \\
&= K_2' \\
&\overset{(\ddagger)}{\leq} L_4 \cdot K_2.
\end{aligned}$$

The step marked with (†) used that $\Omega_0^{(j)} = \varnothing$ holds for $j \in \Lambda_A$, because of $\lambda^{(\omega)} \in \Lambda \setminus \Lambda_A$ for all $\omega \in \Omega_0$. We also adopted the notation of Lemma 5.2.5 (i.e. $J$ instead of $\Lambda$, etc.) at (†). The step marked with (‡) is a consequence of Lemma 5.2.5 and (†).

In summary, we have established

$$C_2 \leq K_2 \leq L_3 \cdot C_2 \qquad \text{and} \qquad K_2 \leq K_2' \leq L_4 \cdot K_2 \leq L_3 L_4 \cdot C_2.$$

Together, this yields the statement of equation (5.2.16).

Let us finally consider equation (5.2.17). To this end, choose for each (remaining) $\omega \in \Omega_B \setminus \Omega_0$ some $\lambda^{(\omega)} \in \Lambda$ with $S_\omega \cap R_{\lambda^{(\omega)}} \neq \varnothing$, which is possible because of $\varnothing \neq S_\omega \cap B \subset S_\omega \cap \mathcal{O}$ and since $(R_\lambda)_{\lambda \in \Lambda}$ covers $\mathcal{O}$. Now, define

$$\Omega^{(\lambda)} := \left\{ \omega \in \Omega_B \,\middle|\, \lambda^{(\omega)} = \lambda \right\} \subset \Omega_B \cap \Omega_\lambda$$

for each $\lambda \in \Lambda$. We can now apply Lemma 5.2.5 with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$ and $I_0 = \Omega_B$, and with the choice $(x_i)_{i \in I_0} = (c_\omega)_{\omega \in \Omega_B}$ to obtain

$$\begin{aligned}
\left\| \left( \left\| (c_\omega)_{\omega \in \Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}} &\leq \left\| \left( \left\| (c_\omega)_{\omega \in \Omega_\lambda \cap \Omega_B} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}} \\
&\leq L_5 \cdot \left\| \left( \left\| (c_\omega)_{\omega \in \Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda \in \Lambda} \right\|_{\ell_u^{t_1}}
\end{aligned}$$

for each sequence $(c_\omega)_{\omega \in \Omega_B}$ and a constant $L_5 = L_5(t_1, t_4, \ell, N_{\mathcal{R}}, C_{u,\mathcal{R}})$. Here, it is worth noting that for each $\omega \in \Omega_B$, our assumptions yield some $\lambda_\omega$ with $S_\omega \subset R_{\lambda_\omega}^{\ell *}$, so that Lemma 5.2.5 is indeed applicable.

Let us now choose a minimal $N \in [0,\infty]$ with

$$\left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_B} \right\|_{\ell_v^{t_2}} \le N \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}} \tag{5.2.18}$$

for all sequences $(c_\omega)_{\omega\in\Omega_B}$. The above estimate and the definition of $C_2''$ imply

$$\begin{aligned} \left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_B} \right\|_{\ell_v^{t_2}} &\le N \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}} \\ &\le N \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_\lambda\cap\Omega_B} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}} \end{aligned}$$

and

$$\begin{aligned} \left\| \left( |\det(M_\omega)|^r \cdot c_\omega \right)_{\omega\in\Omega_B} \right\|_{\ell_v^{t_2}} &\le C_2'' \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_\lambda\cap\Omega_B} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}} \\ &\le L_5 C_2'' \cdot \left\| \left( \left\| (c_\omega)_{\omega\in\Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}} \end{aligned}$$

and hence $C_2'' \le N \le L_5 C_2''$.

But moving the factor $|\det(M_\omega)|^r$ in equation (5.2.18) to the right-hand side as usual, we see that $N \in [0,\infty]$ is minimal with

$$\left\| (d_\omega)_{\omega\in\Omega_B} \right\|_{\ell_v^{t_2}} \le N \cdot \left\| \left( \left\| \left( |\det(M_\omega)|^{-r} d_\omega \right)_{\omega\in\Omega^{(\lambda)}} \right\|_{\ell^{t_4}} \right)_{\lambda\in\Lambda} \right\|_{\ell_u^{t_1}}$$

for all sequence $(d_\omega)_{\omega\in\Omega_B}$. Applying Lemma 5.2.4 (with $I = \Omega_B$ and $I^{(j)} = \Omega^{(j)}$ for $j \in \Lambda := J$ as well as $\alpha_\omega = |\det(M_\omega)|^{-r}$ for $\omega \in I = \Omega_B$ and with interchanged roles of $u, v$), this implies

$$N = \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \, |\det(M_\omega)|^r \right)_{\omega\in\Omega^{(\lambda)}} \right\|_{\ell^{t_2\cdot(t_4/t_2)'}} \right)_{\lambda\in\Lambda} \right\|_{\ell^{t_2\cdot(t_1/t_2)'}}$$

In view of this expression for $N$, Lemma 5.2.5 (with $\mathcal{Q} = \mathcal{S}$, $I_0 = \Omega_B$, $\mathcal{P} = \mathcal{R}$, $v = u^{-1}$ and $x_i = v_i \, |\det(M_i)|^r$ for $i \in \Omega_B$) yields a constant $L_6 = L_6(t_1, t_2, t_4, \ell, N_\mathcal{R}, C_{u,\mathcal{R}})$ with $N \le K_2'' \le L_6 \cdot N$. In summary, we have shown

*Here, we again use the identity $C_{u,\mathcal{R}} = C_{u^{-1},\mathcal{R}}$.*

$$C_2'' \le N \le K_2'' \le L_6 \cdot N \le L_5 L_6 \cdot C_2'',$$

which completes the proof of equation (5.2.17). □

Now, we can put Corollary 5.1.11 in a more user friendly form. The simplification lies in the fact that instead of assuming an embedding of nested sequence spaces, i.e. an infinite number of estimates, for arbitrary sequences $(c_i)_{i\in I}$, only finiteness of a nested sequence norm of a *single* sequence is needed.

Even though the present reformulation of Corollary 5.1.11 might still look rather technical, we will see in the remainder of this section that the resulting criteria are readily applicable in the special case where the covering $\mathcal{Q}$ is almost subordinate to the covering $\mathcal{P}$.

**Corollary 5.2.7.** *Assume that the prerequisites of Corollary 5.2.6 regarding the coverings $\mathcal{R}, \mathcal{S}$ and the weights $u, v$ are met.*

*Let $p_1, p_2, q_1, q_2 \in (0, \infty]$. If we have $p_1 \leq p_2$ as well as*

$$K_1 := K_1\left(q_1, q_2, p_2^{\triangledown}, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right) < \infty$$

*The constants $K_1(\dots)$ and $K_2(\dots)$ used here are defined as in Corollary 5.2.6.*

*and*

$$K_2 := K_2\left(q_1, q_2, p_1^{\triangle}, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, \left(u_\lambda \left|\det\left(T_\lambda\right)\right|^{(p_1)_d}\right)_\lambda, v\right) < \infty,$$

*and if $\mathcal{R}, \mathcal{S}$ are $L^{p_1}$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively, then there is a bounded linear map*

$$\iota : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell_v^{q_2}\right)$$

*with the properties stated in Corollary 5.1.11 and with*

$$|||\iota||| \leq C\left[K_1 + K_2\right]$$

*for some constant $C > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on*

$$k, \ell, N_{\mathcal{R}}, C_{\mathcal{R}}, R_{\mathcal{R}}, D_{\mathcal{R}}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}, N_{\mathcal{S}}, C_{\mathcal{S}}, R_{\mathcal{S}}, C_{v,\mathcal{S}}, C_{\mathcal{S},\Psi,p_1},$$

*where the $L^{p_1}$-BAPUs $\Phi$ for $\mathcal{R}$ and $\Psi$ for $\mathcal{S}$ are used to calculate the operator norm $|||\iota|||$.*

*Remark.* For the sake of concreteness, let us spell out the constants explicitly: We have

$$K_1 = \left\| \left( \left\| \left( u_\lambda^{-1} \cdot \left|\det\left(T_\lambda\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{\lambda \in \Lambda^{(\omega)}} \right\|_{\ell^{p_2^{\triangledown} \cdot (q_1/p_2^{\triangledown})'}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}}, \tag{5.2.19}$$

where for each $\lambda \in \Lambda_A$, some $\omega^{(\lambda)} \in \Omega$ with $R_\lambda \cap S_{\omega^{(\lambda)}} \neq \varnothing$ was chosen and where

$$\Lambda^{(\omega)} = \left\{ \lambda \in \Lambda_A \,\middle|\, \omega^{(\lambda)} = \omega \right\}.$$

For the second constant, recall from equation (5.1.1) that $(p_1)_d = 0$ for $p_1 \in [1, \infty]$ and $(p_1)_d = 1 - \frac{1}{p_1}$ for $p_1 \in (0, 1)$. For $p_1 \in [1, \infty]$, this yields

$$K_2 = \left\| \left( u_\lambda^{-1} \left\| \left( \left|\det\left(M_\omega\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2}} v_\omega \right)_{\omega \in \Omega_0^{(\lambda)}} \right\|_{\ell^{q_2 \cdot \left(p_1^{\triangle}/q_2\right)'}} \right)_{\lambda \in \Lambda \setminus \Lambda_A} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}. \tag{5.2.20}$$

Similarly, for $p_1 \in (0, 1)$, we have

$$\frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d = \frac{1}{p_1} - \frac{1}{p_2} + \left(1 - \frac{1}{p_1}\right) = 1 - \frac{1}{p_2}$$

and $p_1^{\triangle} = \infty$, which entails $q_2 \cdot \left(p_1^{\triangle}/q_2\right)' = q_2 \cdot \infty' = q_2$ in case of $q_2 \in (0,\infty)$ and $q_2 \cdot \left(p_1^{\triangle}/q_2\right)' = \infty = q_2$ for $q_2 = \infty$. Thus,

$$K_2 = \left\| \left( \frac{|\det(T_\lambda)|^{\frac{1}{p_1}-1}}{u_\lambda} \left\| \left( v_\omega \, |\det(M_\omega)|^{1-\frac{1}{p_2}} \right)_{\omega\in\Omega_0^{(\lambda)}} \right\|_{\ell^{q_2}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell^{q_2(q_1/q_2)'}} \tag{5.2.21}$$

for $p_1 \in (0,1)$.

Finally, we observe $\Lambda^{(\omega)} \subset \Lambda_A \cap \Lambda_\omega$ and $\Omega_0^{(\lambda)} \subset \Omega_0 \cap \Omega_\lambda$ for all $\omega \in \Omega$ and $\lambda \in \Lambda$, so that one can also replace the set $\Lambda^{(\omega)}$ by $\Lambda_A \cap \Lambda_\omega$ and the set $\Omega_0^{(\lambda)}$ by $\Omega_0 \cap \Omega_\lambda$ in the definition of $K_1, K_2$ above. The statement of the corollary will then still be valid.

*Proof.* Comparing the definitions, we see that the constant

$$C_1 := C_1\left(q_1, q_2, p_2^{\triangledown}, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right)$$

from Corollary 5.2.6 is a valid choice for the constant $C_1$ required in Corollary 5.1.11.

Furthermore, the constant $C_2$ required in Corollary 5.1.11 has to satisfy

$$\left\| \left( |\det(M_\omega)|^{\frac{1}{p_1}-\frac{1}{p_2}} \, c_\omega \right)_{\omega\in\Omega_0} \right\|_{\ell_v^{q_2}} \le C_2 \left\| \left( \left\| (c_\omega)_{\omega\in\Omega_\lambda} \right\|_{\ell^{p_1^{\triangle}}_{w^{\mathcal{S},p_1}}} / w_\lambda^{\mathcal{R},p_1} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell_u^{q_1}}$$

for all sequences $(c_\omega)_{\omega\in\Omega_0}$. Here, we recall $w_\lambda^{\mathcal{R},p_1} = |\det(T_\lambda)|^{-(p_1)_d}$ and $w_\omega^{\mathcal{S},p_1} = |\det(M_\omega)|^{-(p_1)_d}$ from equation (5.1.2). By considering the sequence

$$(c_\omega)_{\omega\in\Omega_0} = \left( |\det(M_\omega)|^{(p_1)_d} \, d_\omega \right)_{\omega\in\Omega_0}$$

for an arbitrary sequence $(d_\omega)_{\omega\in\Omega_0}$, these considerations show that the constant $C_2$ equivalently has to satisfy

$$\begin{aligned} &\left\| \left( |\det(M_\omega)|^{\frac{1}{p_1}-\frac{1}{p_2}+(p_1)_d} \cdot d_\omega \right)_{\omega\in\Omega_0} \right\|_{\ell_v^{q_2}} \\ &\le C_2 \cdot \left\| \left( \left\| (d_\omega)_{\omega\in\Omega_\lambda} \right\|_{\ell^{p_1^{\triangle}}} \right)_{\lambda\in\Lambda\setminus\Lambda_A} \right\|_{\ell^{q_1}_{\left(u_\lambda |\det(T_\lambda)|^{(p_1)_d}\right)_\lambda}} \end{aligned}$$

for arbitrary sequences $(d_\omega)_{\omega\in\Omega_0}$. We conclude that the constant

$$C_2 := C_2\left(q_1, q_2, p_1^{\triangle}, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, \left(u_\lambda \, |\det(T_\lambda)|^{(p_1)_d}\right)_\lambda, v\right)$$

from Corollary 5.2.6 is a valid choice for the constant $C_2$ required in Corollary 5.1.11.

But Corollary 5.2.6 yields $C_1 \leq L_1 K_1 < \infty$ for some constant

$$L_1 = L_1\left(q_1, q_2, p_2^{\triangledown}, k, N_{\mathcal{S}}, C_{v,\mathcal{S}}\right),$$

as well as $C_2 \leq K_2 < \infty$.

Finally, an application of Corollary 5.1.11 yields existence of a map $\iota$ with the claimed properties and with

$$\begin{aligned} |||\iota||| &\leq C \cdot (C_1 + C_2) \\ &\leq \max\{1, L_1\}\, C \cdot (K_1 + K_2) \end{aligned}$$

for some constant $C > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on

$$k, \ell, N_{\mathcal{R}}, C_{\mathcal{R}}, R_{\mathcal{R}}, D_{\mathcal{R}}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}, N_{\mathcal{S}}, R_{\mathcal{S}}, C_{\mathcal{S}}, C_{v,\mathcal{S}}, C_{\mathcal{S},\Psi,p_1}.$$

This completes the proof. □

We now consider the special case in which the covering $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. One could of course deduce the embedding statements for this case from Corollary 5.2.7, but we will instead use Theorems 5.1.6 and 5.1.8, since these theorems impose less strict requirements on $\mathcal{P}$, i.e., $\mathcal{P}$ does not have to be semi-structured.

We begin with the embedding $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$.

**Corollary 5.2.8.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ be a semi-structured admissible covering of $\mathcal{O}$ and let $\mathcal{P} = \left(P_j\right)_{j \in J}$ be an admissible covering of $\mathcal{O}'$. Assume that $u = (u_i)_{i \in I}$ is $\mathcal{Q}$-moderate and that $v = \left(v_j\right)_{j \in J}$ is $\mathcal{P}$-moderate.*

*Further, let $p_1, p_2, q_1, q_2 \in (0, \infty]$ such that $\mathcal{Q}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ and $\mathcal{P}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}'$ and assume that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.*

*Finally, assume that we have $p_1 \leq p_2$ and that the constant*

$$K := \left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I^{(j)}} \right\|_{\ell^{p_2^{\triangledown} \cdot (q_1/p_2^{\triangledown})'}} \right)_{j \in J} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \tag{5.2.22}$$

*is finite, where for each $i \in I$, some $j^{(i)} \in J$ is chosen with $Q_i \cap P_{j^{(i)}} \neq \varnothing$ and the set $I^{(j)}$ is given by*

$$I^{(j)} := \left\{ i \in I \,\middle|\, j^{(i)} = j \right\}.$$

*Then $\mathcal{O} \subset \mathcal{O}'$ and there is a bounded linear map*

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$

*given by*

$$\langle \iota f, \varphi \rangle = \sum_{i \in I} \langle f, \varphi_i \varphi \rangle \quad \text{for all } \varphi \in C_c^\infty(\mathcal{O}') \text{ and } f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right),$$

*where $\Phi = (\varphi_i)_{i \in I}$ is an $L^{p_1}$-BAPU for $\mathcal{Q}$. This map $\iota$ has the following additional properties:*

1. *For each $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \leq \mathcal{D}'(\mathcal{O})$, the distribution $\iota f \in \mathcal{D}'(\mathcal{O}')$ is an extension of $f$.*
2. *If $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ is given by integration against a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ with $f \in L^1_{\mathrm{loc}}(\mathcal{O}')$ and with $f(x) = 0$ a.e. on $\mathcal{O}' \setminus \mathcal{O}$, then $\iota f = f$.*
3. *We have $\operatorname{supp}(\iota f) \subset \mathcal{O}' \cap \overline{\operatorname{supp} f}$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$.*

*If the $L^{p_1}$-BAPU $\Phi$ for $\mathcal{Q}$ and the $L^{p_2}$-BAPU $\Psi$ are used to calculate the quasi-norms on the two decomposition spaces, then $|||\iota||| \leq C \cdot K$ holds for some constant $C > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on*

$$k(\mathcal{Q}, \mathcal{P}), N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},$$

*where the dependence on $R_{\mathcal{P}}, C_{\mathcal{P}}$ can be dropped for $p_2 \in [1, \infty]$.*

*If $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate, if $\mathcal{P} = (S_j P_j' + c_j)_{j \in J}$ is semi-structured and if $\mathcal{Q}$ is tight, then finiteness of $K$ can be replaced by finiteness of*

$$K' := \left\| \left( \frac{\left|\det(S_j)\right|^{\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+}}{u_{i_j}} \left|\det(T_{i_j})\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}},$$

*where for each $j \in J_{\mathcal{O}}$ some $i_j \in I$ with $P_j \cap Q_{i_j} \neq \varnothing$ is fixed. Here,*

$$J_{\mathcal{O}} := \left\{ j \in J \,\middle|\, P_j \cap \mathcal{O} \right\} \neq \varnothing.$$

*In this case, the constant $C$ from above also depends on $C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P})$, $\varepsilon_{\mathcal{Q}}$ and on $C_{u,\mathcal{Q},\mathcal{P}}$. Furthermore, the dependence on $\mathcal{R}_{\mathcal{P}}, C_{\mathcal{P}}$ can not be dropped in this case.*

*Proof.* For brevity, set $k := k(\mathcal{Q}, \mathcal{P})$. We apply Theorem 5.1.8. To this end, we note that equation (5.2.22), together with Lemma 5.2.2, yields

$$\left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} x_i \right)_{i \in I^{(j)}} \right\|_{\ell^{p_2^\nabla}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq K \cdot \left\| (x_i)_{i \in I} \right\|_{\ell_u^{q_1}}$$

for all sequences $(x_i)_{i \in I}$. Together with Lemma 5.2.5, this yields

$$\left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} x_i \right)_{i \in I_j} \right\|_{\ell^{p_2^\nabla}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq CK \cdot \left\| (x_i)_{i \in I} \right\|_{\ell_u^{q_1}}$$

for all sequences $(x_i)_{i\in I}$ and some constant $C = C(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}})$. Thus, all assumptions of Theorem 5.1.8 (with $I_0 = I$) are satisfied, so that the map $\iota$ as given in the statement of the current theorem is indeed bounded with $|||\iota||| \lesssim K$, where the implied constants only depend on quantities mentioned in the theorem.

It remains to prove the additional properties of $\iota$.

1. Let $\varphi \in C_c^\infty(\mathcal{O})$ be arbitrary. Lemma 3.2.11 shows that $(\varphi_i)_{i\in I}$ is a locally finite partition of unity on $\mathcal{O}$, so that we get $\varphi = \sum_{i\in I} \varphi_i \varphi$, where only finitely many terms do not vanish identically. Hence,

$$\begin{aligned}\langle f, \varphi\rangle &= \Big\langle f, \sum_{i\in I} \varphi_i \varphi\Big\rangle \\ &= \sum_{i\in I} \langle f, \varphi_i\varphi\rangle \\ &= \langle \iota f, \varphi\rangle .\end{aligned}$$

We have thus shown that $\iota f$ is indeed an extension of $f$.

2. Let

$$I' := \{i \in I \mid Q_i^\circ \neq \varnothing\} .$$

Lemma 3.2.12 shows that $\mathcal{Q}^\circ = (Q_i^\circ)_{i\in I'}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ with $L^{p_1}$-BAPU $(\varphi_i)_{i\in I'}$. Furthermore, Lemma 3.2.16 shows that $I'$ is countable. Finally, we have $\varphi_i \equiv 0$ for $i \in I \setminus I'$, because of $\varphi_i^{-1}(\mathbb{C}^*) \subset Q_i^\circ = \varnothing$ for $i \in I \setminus I'$.

By Fourier inversion (which holds, because of $\varphi_i \in C_c^\infty(\mathbb{R}^d)$),

$$\begin{aligned}|\varphi_i(\xi)| &= |\mathcal{F}[\mathcal{F}^{-1}\varphi_i](\xi)| \\ &\leq \|\mathcal{F}^{-1}\varphi_i\|_{L^1} \\ &\leq C_{\mathcal{Q},\Phi,\infty} < \infty\end{aligned}$$

holds for all $\xi \in \mathbb{R}^d$. Here, the constant $C_{\mathcal{Q},\Phi,\infty}$ is finite by Corollary 5.1.4. Because of $\varphi_i \equiv 0$ on $\mathbb{R}^d \setminus Q_i$, we conclude

$$\begin{aligned}\sum_{i\in I} |\varphi_i(\xi)| &\leq C_{\mathcal{Q},\Phi,\infty} \sum_{i\in I} \chi_{Q_i}(x) \\ &\leq C_{\mathcal{Q},\Phi,\infty} N_{\mathcal{Q}} \\ &=: C_0\end{aligned}$$

for all $\xi \in \mathbb{R}^d$.

Finally, assume $f \in L^1_{\mathrm{loc}}(\mathcal{O}') \cap \mathcal{D}(\mathcal{Q}, L^{p_1}, \ell_u^{q_1})$ with $f \equiv 0$ almost everywhere on $\mathcal{O}' \setminus \mathcal{O}$ and let $\varphi \in C_c^\infty(\mathcal{O}')$ be arbitrary. Denote the support of $\varphi$ by $M := \operatorname{supp}\varphi$. Recalling the estimate from above, we conclude

$$\sum_{i\in I'} |f(\xi)\,\varphi_i(\xi)\,\varphi(\xi)| \leq C_0 \|\varphi\|_{\sup} \cdot |f(\xi)|\,\chi_M(\xi) \in L^1(\mathbb{R}^d),$$

because of $f \in L^1_{\mathrm{loc}}(\mathcal{O}')$ and since $M \subset \mathcal{O}'$ is compact. Furthermore, the sum only has countably many terms. Thus, the dominated convergence theorem yields

$$\begin{aligned}
\langle \iota f,\, \varphi \rangle &= \sum_{i \in I} \langle f,\, \varphi_i \varphi \rangle = \sum_{i \in I'} \langle f,\, \varphi_i \varphi \rangle \\
&= \sum_{i \in I'} \int_{\mathcal{O}} f(\xi)\, \varphi_i(\xi)\, \varphi(\xi)\, \mathrm{d}\xi \\
&= \int_{\mathcal{O}} f(\xi)\, \varphi(\xi) \sum_{i \in I'} \varphi_i(\xi)\, \mathrm{d}\xi \\
&= \int_{\mathcal{O}} f(\xi)\, \varphi(\xi)\, \mathrm{d}\xi \\
&\overset{(\dagger)}{=} \int_{\mathcal{O}'} f(\xi)\, \varphi(\xi)\, \mathrm{d}\xi = \langle f, \varphi \rangle,
\end{aligned}$$

where the step marked with $(\dagger)$ is justified by $f \equiv 0$ almost everywhere on $\mathcal{O}' \setminus \mathcal{O}$.

3. Let $\varphi \in C_c^\infty(\mathcal{O}')$ with $\operatorname{supp} \varphi \subset \mathcal{O}' \setminus \overline{\operatorname{supp} f}$. We have to show $\langle \iota f, \varphi \rangle = 0$. But for $i \in I$, we have $\varphi_i \varphi \in C_c^\infty(\mathcal{O})$ with

$$\operatorname{supp}(\varphi_i \varphi) \subset \mathcal{O} \cap \operatorname{supp} \varphi \subset \mathcal{O} \setminus \overline{\operatorname{supp} f} \subset \mathcal{O} \setminus [\operatorname{supp} f],$$

which implies $\langle f,\, \varphi_i \varphi \rangle = 0$. Since $i \in I$ was arbitrary, we conclude $\langle \iota f,\, \varphi \rangle = 0$ as desired.

Finally, let us assume that $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate, that $\mathcal{P} = (S_j P_j' + c_j)_{j \in J}$ is semi-structured and that $\mathcal{Q}$ is tight. For $j \in J$ with $I^{(j)} \neq \varnothing$, there is some $i \in I^{(j)}$, i.e. with $j^{(i)} = j$. This implies

$$\varnothing \neq Q_i \cap P_{j^{(i)}} = Q_i \cap P_j \subset \mathcal{O} \cap P_j$$

and hence $j \in J_{\mathcal{O}}$. Put differently, we have $I^{(j)} = \varnothing$ for all $j \in J \setminus J_{\mathcal{O}}$.

For $j \in J_{\mathcal{O}}$, we have $\varnothing \neq \mathcal{O} \cap P_j$ and hence also $Q_{i_j} \cap P_j \neq \varnothing$ for some $i_j \in I$, so that a choice as in the statement of the corollary is possible. With this choice, Lemma 3.3.7, together with the inclusion $I^{(j)} \subset I_j$, implies

$$|I^{(j)}| \leq |I_j| \leq C_1 \cdot \frac{|\det(S_j)|}{|\det(T_{i_j})|}$$

for all $j \in J_{\mathcal{O}}$ and $C_1 = C_1(d, k, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, N_{\mathcal{Q}}, C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), R_{\mathcal{P}}, C_{\mathcal{P}})$.

But since $u$ and $(|\det(T_i)|)_{i \in I}$ are relatively $\mathcal{P}$-moderate, there is a constant

$$C_2 = C_2(C_{u,\mathcal{Q},\mathcal{P}}, C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), p_1, p_2) > 0$$

with

$$u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \leq C_2 \cdot u_{i_j}^{-1} \cdot |\det(T_{i_j})|^{\frac{1}{p_1} - \frac{1}{p_2}}$$

for all $i \in I^{(j)} \subset I_j$.

All in all, we conclude

$$\begin{aligned}
&\left\| \left( u_i^{-1} \left|\det\left(T_i\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2^\triangledown\cdot(q_1/p_2^\triangledown)'}} \\
&\leq C_2 \cdot u_{i_j}^{-1} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left| I^{(j)} \right|^{\frac{1}{p_2^\triangledown\cdot(q_1/p_2^\triangledown)'}} \\
&\overset{(\dagger)}{=} C_2 \cdot u_{i_j}^{-1} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left| I^{(j)} \right|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \\
&\leq C_1^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \cdot C_2 \cdot u_{i_j}^{-1} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \cdot \left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+}
\end{aligned}$$

for all $j \in J_{\mathcal{O}}$. At $(\dagger)$, we used Remark 5.2.3, which gives the simplified expression

$$\frac{1}{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'} = \left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+ .$$

Using the estimate above, it is easy to see

$$K \leq C_1^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \cdot C_2 \cdot K' < \infty,$$

which completes the proof. □

As the last result of this section, we consider the "reverse" embedding, i.e. $\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$.

**Corollary 5.2.9.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ be a semi-structured admissible covering of $\mathcal{O}$ and let $\mathcal{P} = \left(P_j\right)_{j\in J}$ be an admissible covering of $\mathcal{O}'$. Assume that $u = (u_i)_{i\in I}$ is $\mathcal{Q}$-moderate and that $v = \left(v_j\right)_{j\in J}$ is $\mathcal{P}$-moderate.*

*Furthermore, let $p_1, p_2, q_1, q_2 \in (0,\infty]$ with $p_2 \leq p_1$. Assume that $\mathcal{Q}$ and $\mathcal{P}$ are $L^{p_2}$-decomposition coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively, and that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.*

*Finally, set $r := \frac{1}{p_2} - \frac{1}{p_1} + (p_2)_d$ and assume that the constant*

$$K := \left\| \left( \left( v_j / w_j^{\mathcal{P},p_2} \right)^{-1} \left\| \left( \left|\det\left(T_i\right)\right|^r u_i \right)_{i\in I^{(j)}} \right\|_{\ell^{q_1}\left(p_2^\triangle/q_1\right)'} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'}$$

*is finite, where for each $i \in I$, some $j^{(i)} \in J$ is chosen with $Q_i \cap P_{j^{(i)}} \neq \varnothing$ and the set $I^{(j)}$ is given by*

$$I^{(j)} := \left\{ i \in I \,\middle|\, j^{(i)} = j \right\}.$$

*Then, we have $\mathcal{O} \subset \mathcal{O}'$ and the map*

$$\iota : \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f|_{C_c^\infty(\mathcal{O})}$$

*For convenience of the reader, we recall the definition $w_j^{\mathcal{P},p_2} = 1$ for $p_2 \in [1,\infty]$ and $w_j^{\mathcal{P},p_2} = \left|\det\left(S_j\right)\right|^{-(p_2)_d}$ for $p_2 \in (0,1)$. Here, the covering $\mathcal{P} = (S_j P_j' + c_j)_{j\in J}$ is semi-structured for $p_2 \in (0,1)$, since it is an $L^{p_2}$-decomposition covering.*

*is well-defined and bounded. If the $L^{p_2}$-BAPUs $\Phi$ for $\mathcal{Q}$ and $\Psi$ for $\mathcal{P}$ are used to calculate the quasi-norms on the respective decomposition spaces, then $|||\iota||| \leq C \cdot K$ holds for some constant $C > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on*

$$k(\mathcal{Q}, \mathcal{P}), N_{\mathcal{P}}, C_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_2},$$

*where the dependence on $C_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}$ can be dropped for $p_2 \in [1, \infty]$.*

*If $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate, if $\mathcal{P} = (S_j P_j' + c_j)_{j \in J}$ is semi-structured and if $\mathcal{Q}$ is tight, then finiteness of $K$ can be replaced by finiteness of*

$$K' := \left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+ - (p_2)_d} \left|\det\left(T_{i_j}\right)\right|^{r - \left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_1}(q_2/q_1)'},$$

*where for each $j \in J_{\mathcal{O}}$ some $i_j \in I$ with $P_j \cap Q_{i_j} \neq \varnothing$ is fixed. Here,*

$$J_{\mathcal{O}} := \left\{ j \in J \,\middle|\, P_j \cap \mathcal{O} \right\} \neq \varnothing.$$

*In this case, the constant $C$ from above also depends on $C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P})$, $\varepsilon_{\mathcal{Q}}$ and on $C_{u,\mathcal{Q},\mathcal{P}}$. Furthermore, the dependence on $C_{\mathcal{P}}, R_{\mathcal{P}}$ can not be dropped in this case.*

*Remark* 5.2.10. In the last part of the above corollary, the exponent

$$\left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+ - (p_2)_d$$

is used. By introducing the concept of the SIGNED UPPER CONJUGATE EXPONENT $p^{\pm\triangle}$ of $p \in (0, \infty]$, this exponent can be simplified: We define $p^{\pm\triangle}$ by

$$\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}, \tag{5.2.23}$$

i.e.

$$p^{\pm\triangle} = \begin{cases} p, & \text{if } p \in [2, \infty], \\ \frac{p}{p-1}, & \text{if } p \in (0,2) \setminus \{1\}, \\ \infty, & \text{if } p = 1. \end{cases}$$

With this notation, we will see below that

$$\left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+ - (p_2)_d = \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+, \tag{5.2.24}$$

*Our terminology is motivated as follows: We have $p^{\triangle} = \max\{p, p'\}$ with $p, p' \geq 0$. Hence, $\frac{1}{p^{\triangle}} = \min\left\{\frac{1}{p}, \frac{1}{p'}\right\}$, with $p' = \infty$ for $p \in (0,1)$, to prevent it from becoming negative. But if we were to define $p'$ as $\frac{1}{p'} = 1 - \frac{1}{p}$ even for $p \in (0,1)$, we would arrive at $\frac{1}{p^{\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$. Since this can attain negative values, we call (the reciprocal of) this quantity the* signed *upper conjugate exponent.*

so that the constant $K'$ from above takes the simplified form

$$K' = \left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_2} - \frac{1}{p_1} - \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_1}(q_2/q_1)'}.$$

For the proof of equation (5.2.24), we distinguish two cases:

*Case* 1. $p_2 \in [1, \infty]$. This implies $(p_2)_d = 0$ and $p_2^{\pm\triangle} = p_2^{\triangle}$. Hence, equation (5.2.24) clearly holds.

*Case* 2. $p_2 \in (0,1)$. Hence, $p_2^{\triangle} = p_2' = \infty$ and $(p_2)_d = 1 - \frac{1}{p_2}$. This implies

$$\begin{aligned}\left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+ - (p_2)_d &= \frac{1}{q_1} - \left(1 - \frac{1}{p_2}\right) \\ &= \frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}} = \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+,\end{aligned}$$

because of

$$\frac{1}{p_2^{\pm\triangle}} = \min\left\{\frac{1}{p_2}, 1 - \frac{1}{p_2}\right\} = 1 - \frac{1}{p_2} < 0,$$

which entails $\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}} > \frac{1}{q_1} \geq 0$. ◀

*Proof.* For brevity, set $k := k(\mathcal{Q}, \mathcal{P})$. We first note that Lemma 3.3.2 implies $\mathcal{O} \subset \mathcal{O}'$, since $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

We apply Theorem 5.1.6. To this end, we note that finiteness of $K$, together with Lemma 5.2.4 (with $\alpha_i = |\det(T_i)|^{-r}$ and with the modified weight $w_j := v_j / w_j^{\mathcal{P},p_2}$ instead of $v_j$), yields

$$\begin{aligned}\left\|(x_i)_{i\in I}\right\|_{\ell_u^{q_1}} &\leq K \cdot \left\| \left( \left\| \left( |\det(T_i)|^{-r} x_i \right)_{i \in I^{(j)}} \right\|_{\ell^{p_2^{\triangle}}} \right)_{j \in J} \right\|_{\ell_w^{q_2}} \\ &\leq K \cdot \left\| \left( \left\| \left( |\det(T_i)|^{-r} x_i \right)_{i \in I_j} \right\|_{\ell^{p_2^{\triangle}}} / w_j^{\mathcal{P},p_2} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \\ &= K \cdot \left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} x_i \right)_{i \in I_j} \right\|_{\ell^{p_2^{\triangle}}_{w^{\mathcal{Q},p_2}}} / w_j^{\mathcal{P},p_2} \right)_{j \in J} \right\|_{\ell_v^{q_2}}\end{aligned}$$

for all sequences $(x_i)_{i\in I}$.

Thus, all assumptions of Theorem 5.1.6 (with $J_0 = J$, which yields $I_0 = I$, since we have $\varnothing \neq Q_i \subset \mathcal{O} \subset \mathcal{O}'$ for all $i \in I$, so that $Q_i$ intersects some $P_j$ nontrivially) are satisfied, so that the map $\iota$ as given in that theorem is indeed bounded with $|||\iota||| \lesssim K$, where the implied constants only depend on quantities mentioned in the

current theorem. Finally, if $\Phi = (\varphi_i)_{i\in I}$ is the $L^{p_2}$-BAPU used for the definition of $\iota$, we have (cf. Theorem 5.1.6)

$$\langle \iota f, g\rangle = \sum_{i\in I} \langle \varphi_i f, g\rangle = \sum_{i\in I} \langle f, \varphi_i g\rangle = \left\langle f, \sum_{i\in I} \varphi_i g \right\rangle = \langle f, g\rangle$$

for all $g \in C_c^\infty(\mathcal{O}) \subset C_c^\infty(\mathcal{O}')$, since $(\varphi_i)_{i\in I}$ is a locally finite partition of unity for $\mathcal{O}$ by Lemma 3.2.11. Thus, $\iota f = f|_{C_c^\infty(\mathcal{O})}$ holds for all $f \in \mathcal{D}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})$.

Now, assume in addition that $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate and that $\mathcal{Q}$ is tight. For $j \in J$ with $I^{(j)} \neq \varnothing$, there is some $i \in I^{(j)}$, i.e. with $j^{(i)} = j$. This implies

$$\varnothing \neq Q_i \cap P_{j^{(i)}} = Q_i \cap P_j \subset \mathcal{O} \cap P_j$$

and hence $j \in J_{\mathcal{O}}$. Put differently, we have $I^{(j)} = \varnothing$ for all $j \in J \setminus J_{\mathcal{O}}$.

For $j \in J_{\mathcal{O}}$, we have $\varnothing \neq \mathcal{O} \cap P_j$ and hence also $Q_{i_j} \cap P_j \neq \varnothing$ for some $i_j \in I$, so that a choice as in the statement of the corollary is possible. With this choice, Lemma 3.3.7, together with the inclusion $I^{(j)} \subset I_j$, implies

$$|I^{(j)}| \leq \left|I_j\right| \leq C_1 \cdot \frac{\left|\det\left(S_j\right)\right|}{|\det(T_{i_j})|}$$

for all $j \in J_{\mathcal{O}}$ and $C_1 = C_1(d, k, \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), R_{\mathcal{P}}, C_{\mathcal{P}})$.

But since $u$ and $(|\det(T_i)|)_{i\in I}$ are relatively $\mathcal{P}$-moderate, there is a constant

$$C_2 = C_2(C_{u,\mathcal{Q},\mathcal{P}}, C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), r) > 0$$

with

$$u_i \cdot |\det(T_i)|^r \leq C_2 \cdot u_{i_j} \cdot |\det(T_{i_j})|^r$$

for all $i \in I^{(j)} \subset I_j$. All in all, we conclude

$$\begin{aligned}
&\left\| \left(u_i \cdot |\det(T_i)|^r\right)_{i\in I^{(j)}} \right\|_{\ell^{q_1\cdot\left(p_2^\triangle/q_1\right)'}} \\
&\leq C_2 \cdot u_{i_j} |\det(T_{i_j})|^r \cdot \left|I^{(j)}\right|^{\frac{1}{q_1\cdot\left(p_2^\triangle/q_1\right)'}} \\
&\overset{(\dagger)}{=} C_2 \cdot u_{i_j} |\det(T_{i_j})|^r \cdot \left|I^{(j)}\right|^{\left(\frac{1}{q_1}-\frac{1}{p_2^\triangle}\right)_+} \\
&\leq C_1^{\left(\frac{1}{q_1}-\frac{1}{p_2^\triangle}\right)_+} C_2 \cdot u_{i_j} |\det(T_{i_j})|^{r-\left(\frac{1}{q_1}-\frac{1}{p_2^\triangle}\right)_+} \cdot \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q_1}-\frac{1}{p_2^\triangle}\right)_+}
\end{aligned}$$

for all $j \in J_{\mathcal{O}}$. At $(\dagger)$, we used Remark 5.2.3 in form of the identity

$$\left(q_1 \cdot \left(p_2^\triangle/q_1\right)'\right)^{-1} = \left(q_1^{-1} - \left(p_2^\triangle\right)^{-1}\right)_+ .$$

Using the estimate above and the identity $w_j^{\mathcal{P},p_2} = \left|\det\left(S_j\right)\right|^{-(p_2)_d}$, it is easy to see

$$K \leq C_1^{\left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+} C_2 \cdot K' < \infty,$$

which completes the proof. □

In Chapter 6, we will see many applications of the above results. But before this, we want to study the extent to which our sufficient criteria are sharp, or even necessary for the existence of an embedding between decomposition spaces. This is the main topic of the next section.

## 5.3 NECESSARY CONDITIONS

Roughly speaking, the setting considered in this section is as follows: We assume that an embedding of the form

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \tag{5.3.1}$$

is true, where the two decomposition spaces are defined with respect to the ((tight) semi-structured) coverings $\mathcal{Q} = (T_i Q_i' + b_i)_{i\in I}$ of $\mathcal{O}$ and $\mathcal{P} = (S_j P_j' + c_j)_{j\in J}$ of $\mathcal{O}'$. We will then show that the existence of such an embedding necessarily implies the existence of certain embeddings between discrete sequence spaces, similar (and often identical) to the sufficient conditions which we derived in the previous two sections.

In the first subsection, we derive criteria under the mild assumptions that $\mathcal{Q}$ is a tight semi-structured covering, such that (a subfamily of) $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

The second subsection is concerned with a much more restrictive setting: Here, we assume that the covering $\mathcal{Q}$ and the weight $u$ are relatively $\mathcal{P}$-moderate.

### 5.3.1 *The general case*

In the following, we will actually consider a slightly more general setting than in equation (5.3.1). Precisely, we will assume that there is an embedding

$$\left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), \tag{5.3.2}$$

for some subset $K \subset \mathcal{O} \cap \mathcal{O}'$, where

$$\mathcal{D}_K := \{f \in C_c^\infty\left(\mathbb{R}^d\right) \mid \operatorname{supp} f \subset K\}$$

is the set of test functions with support in $K$.

Our first necessary condition is rather simple: We observe that in Corollaries 5.1.11 and 5.2.7 to 5.2.9, the inequality $p_1 \leq p_2$ was always part of the assumptions.

In the next lemma, we use the scaling behaviour of the $L^p$-norm to show that this is inevitable. We also show that the condition $v \lesssim u$ (in the sense $v_j \leq C \cdot u_i$ for $Q_i^\circ \cap P_j^\circ \neq \varnothing$) is necessary for the existence of an embedding of the form

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$

in case of $p_1 = p_2$.

It is worth noting that the lemma below imposes virtually no assumptions on the coverings $\mathcal{Q}, \mathcal{P}$ and their relations.

**Lemma 5.3.1.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = \left(P_j\right)_{j \in J}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}'$.*

*Let $K \subset \mathbb{R}^d$ with $K^\circ \cap \mathcal{O} \cap \mathcal{O}' \neq \varnothing$. Let $q_1, q_2 \in (0, \infty]$ and assume that $u = (u_i)_{i \in I}$ and $v = \left(v_j\right)_{j \in J}$ are moderate with respect to $\mathcal{Q}$ or $\mathcal{P}$, respectively.*

*If the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is well-defined and continuous, then we have $p_1 \leq p_2$.*

*In case of $p_1 = p_2$, there exists a constant $C > 0$ depending only on*

$$d, p_1, p_2, q_1, q_2, N_{\mathcal{P}}, C_{v,\mathcal{P}}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, C_{u,\mathcal{Q}}$$

*such that $v_j \leq C \, |||\iota||| \cdot u_i$ holds for all $(i,j) \in I \times J$ with $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \varnothing$.*

*Here, the norm $|||\iota|||$ is calculated using the $L^{p_1}$-BAPU $\Phi = (\varphi_i)_i$ for $\mathcal{Q}$ and the $L^{p_2}$-BAPU $\Psi = \left(\psi_j\right)_{j \in J}$ for $\mathcal{P}$. The dependence of $C$ on $R_{\mathcal{Q}}, C_{\mathcal{Q}}$ can be dropped for $p_1 \in [1, \infty]$.*

*Proof.* Let us fix a non-trivial test function $\gamma \in C_c^\infty\left(B_1(0)\right)$ and let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = \left(\psi_j\right)_{j \in J}$ be $L^{p_1}/L^{p_2}$-BAPUs for $\mathcal{Q}$ and $\mathcal{P}$, respectively.

For the proof of $p_1 \leq p_2$, choose an arbitrary $x_0 \in K^\circ \cap \mathcal{O} \cap \mathcal{O}'$. Lemma 3.2.12 shows that $(Q_i^\circ)_{i \in I}$ and $(P_j^\circ)_{j \in J}$ cover $\mathcal{O}$ and $\mathcal{O}'$, respectively. Thus, there are $i_0 \in I$ and $j_0 \in J$ as well as $\varepsilon > 0$ with $B_\varepsilon(x_0) \subset K^\circ \cap Q_{i_0}^\circ \cap P_{j_0}^\circ$.

For the proof of the second claim, we can take $x_0 \in K^\circ \cap Q_i^\circ \cap P_j^\circ$ and choose $i_0 = i$ and $j_0 = j$, allowing us to handle both claims simultaneously.

For $n \in \mathbb{N}$, define

$$\gamma_n : \mathbb{R}^d \to \mathbb{C}, x \mapsto \gamma\left(\frac{n}{\varepsilon}(x - x_0)\right).$$

We then have

$$\gamma_n \in C_c^\infty\left(B_{\varepsilon/n}(x_0)\right) \subset C_c^\infty\left(B_\varepsilon(x_0)\right) \subset C_c^\infty\left(K^\circ \cap P_{j_0}^\circ \cap Q_{i_0}^\circ\right).$$

But Lemma 3.2.12 implies $\psi_{j_0}^* \equiv 1$ on $P_{j_0} \supset \operatorname{supp}\gamma_n$, and hence $\psi_{j_0}^*\gamma_n = \gamma_n$. Furthermore, $\gamma_n \in \mathcal{D}_K$.

Since $\|\cdot\|_{L^{p_2}}$ is a quasi-norm and because of the uniform bound $|j_0^*| \leq N_\mathcal{P}$, there is a constant $C_1 = C_1(N_\mathcal{P}, p_2) > 0$ with

$$\begin{aligned}
v_{j_0} \cdot \left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_2}} &= v_{j_0} \cdot \left\|\mathcal{F}^{-1}\left(\psi_{j_0}^*\gamma_n\right)\right\|_{L^{p_2}} \\
&\leq C_1 \cdot \sum_{\ell \in j_0^*} v_{j_0} \left\|\mathcal{F}^{-1}\left(\psi_\ell\gamma_n\right)\right\|_{L^{p_2}} \\
&\leq C_1 C_{v,\mathcal{P}} \cdot \sum_{\ell \in j_0^*} v_\ell \left\|\mathcal{F}^{-1}\left(\psi_\ell\gamma_n\right)\right\|_{L^{p_2}} \\
&\leq C_1 C_{v,\mathcal{P}} \, |j_0^*| \cdot \|\gamma_n\|_{\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)} \\
&\leq C_2 \, |||\iota||| \cdot \|\gamma_n\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} \qquad (5.3.3)
\end{aligned}$$

with $C_2 := C_1 C_{v,\mathcal{P}} N_\mathcal{P}$.

Now note that $\varphi_i\gamma_n \not\equiv 0$ implies existence of $x \in \operatorname{supp}\gamma_n \subset Q_{i_0}$ with $\varphi_i(x) \neq 0$. Hence, $x \in Q_i \cap Q_{i_0} \neq \varnothing$, i.e. $i \in i_0^*$. Since $\ell^{q_1}$ is a quasi-norm and because of the uniform bound $|i^*| \leq N_\mathcal{Q}$ for all $i \in I$, there is some constant $C_3 = C_3(N_\mathcal{Q}, q_1) > 0$ with

$$\begin{aligned}
\|\gamma_n\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} &= \left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}}\right)_{i \in I}\right\|_{\ell_u^{q_1}} \\
&= \left\|\left(\left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}}\right)_{i \in i_0^*}\right\|_{\ell_u^{q_1}} \\
&\leq C_3 \sum_{i \in i_0^*} \left[u_i \cdot \left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}}\right]. \qquad (5.3.4)
\end{aligned}$$

In case of $p_1 \in [1, \infty]$, Young's inequality ($L^1 * L^{p_1} \hookrightarrow L^{p_1}$) yields

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}} &= \left\|\left(\mathcal{F}^{-1}\varphi_i\right) * \left(\mathcal{F}^{-1}\gamma_n\right)\right\|_{L^{p_1}} \\
&\leq \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^1} \cdot \left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_1}} \\
&\leq C_{\mathcal{Q},\Phi,p_1} \cdot \|\mathcal{F}^{-1}\gamma_n\|_{L^{p_1}}.
\end{aligned}$$

As usual, in case of $p_1 \in (0,1)$, we note $\operatorname{supp}\varphi_i \subset \overline{Q_i} \subset \overline{Q_i^*}$, as well as $\operatorname{supp}\gamma_n \subset Q_{i_0}^\circ \subset \overline{Q_i^*}$ for all $i \in i_0^*$, so that Corollary 3.2.15 yields a constant $C_4 = C_4(d, p_1, R_\mathcal{Q}, C_\mathcal{Q}) > 0$ with

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}} &\leq C_4 \left|\det\left(T_i\right)\right|^{\frac{1}{p_1}-1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}} \left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_1}} \\
&\leq C_4 C_{\mathcal{Q},\Phi,p_1} \cdot \|\mathcal{F}^{-1}\gamma_n\|_{L^{p_1}}
\end{aligned}$$

for all $i \in i_0^*$.

If we set $C_4 := 1$ for $p \in [1,\infty]$, then a combination of the above estimates and equation (5.3.4) shows

$$\begin{aligned}\|\gamma_n\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} &\leq C_3 \sum_{i\in i_0^*}\left[u_i \cdot \left\|\mathcal{F}^{-1}\left(\varphi_i\gamma_n\right)\right\|_{L^{p_1}}\right] \\ &\leq C_3C_4C_{\mathcal{Q},\Phi,p_1}\cdot\Big(\sum_{i\in i_0^*}u_i\Big)\cdot\left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_1}} \qquad (5.3.5)\end{aligned}$$

for all $n \in \mathbb{N}$.

We have $\mathcal{F}^{-1}\gamma_n = \left(\frac{\varepsilon}{n}\right)^d \cdot e^{2\pi i\langle\cdot,x_0\rangle}\cdot\left(\mathcal{F}^{-1}\gamma\right)\left(\frac{\varepsilon}{n}\cdot\right)$, so that Lemma 4.5.5 yields

$$\begin{aligned}\left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^p} &= \left(\frac{\varepsilon}{n}\right)^d\cdot\left\|\Delta_{\frac{\varepsilon}{n}\mathrm{id}}\left(\mathcal{F}^{-1}\gamma\right)\right\|_{L^p} \\ &= \left(\frac{\varepsilon}{n}\right)^{d\left(1-\frac{1}{p}\right)}\cdot\left\|\mathcal{F}^{-1}\gamma\right\|_{L^p}\end{aligned}$$

for all $p \in (0,\infty]$.

For brevity, set $C_5 := C_2C_3C_4C_{\mathcal{Q},\Phi,p_1}$. With this notation, a combination of estimates (5.3.3) and (5.3.5) yields

$$\begin{aligned}&v_{j_0}\cdot\left(\frac{\varepsilon}{n}\right)^{d\left(1-\frac{1}{p_2}\right)}\cdot\left\|\mathcal{F}^{-1}\gamma\right\|_{L^{p_2}} \\ &= v_{j_0}\cdot\left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_2}} \\ &\leq C_2\cdot|||\iota|||\cdot\|\gamma_n\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} \\ &\leq C_5\cdot|||\iota|||\cdot\Big(\sum_{i\in i_0^*}u_i\Big)\cdot\left\|\mathcal{F}^{-1}\gamma_n\right\|_{L^{p_1}} \\ &= C_5\cdot|||\iota|||\cdot\Big(\sum_{i\in i_0^*}u_i\Big)\cdot\left(\frac{\varepsilon}{n}\right)^{d\left(1-\frac{1}{p_1}\right)}\cdot\left\|\mathcal{F}^{-1}\gamma\right\|_{L^{p_1}}. \qquad (5.3.6)\end{aligned}$$

But this implies

$$n^{d\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} \leq \varepsilon^{d\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}\cdot C_5\cdot|||\iota|||\cdot\frac{\left(\sum_{i\in i_0^*}u_i\right)}{v_{j_0}}\cdot\frac{\left\|\mathcal{F}^{-1}\gamma\right\|_{L^{p_1}}}{\left\|\mathcal{F}^{-1}\gamma\right\|_{L^{p_2}}}$$

for all $n \in \mathbb{N}$. This is only possible in the case $\frac{1}{p_2}-\frac{1}{p_1} \leq 0$, that is, $p_1 \leq p_2$.

Finally, note that for $p_1 = p_2$, estimate (5.3.6) yields the inequality

$$\begin{aligned}v_{j_0} &\leq C_5\cdot|||\iota|||\cdot\left(\sum_{i\in i_0^*}u_i\right) \\ &\leq C_5\cdot|||\iota|||\cdot N_{\mathcal{Q}}\cdot C_{u,\mathcal{Q}}\cdot u_{i_0},\end{aligned}$$

where we used $u_i \leq C_{u,\mathcal{Q}}\cdot u_{i_0}$ for $i \in i_0^*$ as well as $|i_0^*| \leq N_{\mathcal{Q}}$. Because we chose $i_0 = i$ and $j_0 = j$ at the beginning of the proof, this establishes the second claim. □

As a next step towards proving stronger necessary conditions for embeddings between decomposition spaces, we construct a family $(\gamma_i)_{i\in I}$ of smooth functions which are supported in $\operatorname{supp}\gamma_i \subset Q_i$ and with the property $\left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p} \asymp |\det(T_i)|^{1-\frac{1}{p}}$. We will then use these building blocks to construct suitable functions for "testing" the embedding.

**Lemma 5.3.2.** *Let $\mathcal{Q} = (Q_i)_{i\in I} = (T_iQ_i' + b_i)_{i\in I}$ be a semi-structured covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ with $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for some $p \in (0,\infty]$.*

*Let $q \in [\min\{1,p\},\infty]$ and $\ell \in \mathbb{N}_0$. Then there is a constant*

$$C_1 = C_1\left(d,p,q,\ell,N_{\mathcal{Q}},R_{\mathcal{Q}},D_{\mathcal{Q}},C_{\mathcal{Q},\Phi,p}\right) > 0$$

*such that*

$$\|\mathcal{F}^{-1}\varphi_{M_i}\|_{L^q} \leq C_1 \cdot |\det(T_i)|^{1-\frac{1}{q}} \tag{5.3.7}$$

*holds for all $i \in I$ and all sets $M_i \subset i^{\ell *}$.*

*If $\mathcal{Q}$ is tight, there is a family of functions $(\gamma_i)_{i\in I}$ with $\gamma_i \in C_c^\infty(\mathcal{O})$ and $\operatorname{supp}\gamma_i \subset Q_i$ for all $i \in I$ and for each $p \in (0,\infty]$, there is a constant $C_2 = C_2(d,p) > 0$ with*

$$\|\mathcal{F}^{-1}\gamma_i\|_{L^p} = C_2 \cdot \varepsilon_{\mathcal{Q}}^{d\left(1-\frac{1}{p}\right)} \cdot |\det(T_i)|^{1-\frac{1}{p}}$$

*for all $i \in I$.*

*Proof.* Let $r := \min\{1,p\}$. By definition of an $L^p$-BAPU, we know that

$$C_{\mathcal{Q},\Phi,p} = C_{\mathcal{Q},\Phi,r} = \sup_{i\in I} |\det(T_i)|^{\frac{1}{r}-1} \|\mathcal{F}^{-1}\varphi_i\|_{L^r}$$

is finite. By applying Lemma 5.1.3 (with $p_1 = r$ and $p_2 = q \geq r = p_1$, as well as $f \equiv 1$), we get a constant $C_0 = C_0(d,r,q,R_{\mathcal{Q}}) > 0$ with

$$\begin{aligned}
\|\mathcal{F}^{-1}\varphi_i\|_{L^q} &\leq C_0 \cdot |\det(T_i)|^{\frac{1}{r}-\frac{1}{q}} \|\mathcal{F}^{-1}\varphi_i\|_{L^r} \\
&\leq C_0 C_{\mathcal{Q},\Phi,p} |\det(T_i)|^{\frac{1}{r}-\frac{1}{q}} |\det(T_i)|^{1-\frac{1}{r}} \\
&= C_0 C_{\mathcal{Q},\Phi,p} |\det(T_i)|^{1-\frac{1}{q}}
\end{aligned}$$

for all $i \in I$, because of $\varphi_i \equiv 0$ on $\mathcal{O} \setminus Q_i$.

Since $\|\cdot\|_{L^q}$ is a quasi-norm and because of the uniform bound $|M_i| \leq \left|i^{\ell *}\right| \leq N_{\mathcal{Q}}^\ell$ (which was shown in Lemma 3.2.18), we obtain a constant $C_1 = C_1(N_{\mathcal{Q}},\ell,q) > 0$ with

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\varphi_{M_i}\right\|_{L^q} &\leq C_1 \cdot \sum_{j\in M_i} \left\|\mathcal{F}^{-1}\varphi_j\right\|_{L^q} \\
&\leq C_0 C_1 C_{\mathcal{Q},\Phi,p} \cdot \sum_{j\in M_i} \left|\det(T_j)\right|^{1-\frac{1}{q}}
\end{aligned}$$

for all $i \in I$ and each subset $M_i \subset i^{\ell *}$. By $\mathcal{Q}$-moderateness of the weight $(|\det(T_i)|)_{i\in I}$ (with "moderateness constant" $D_{\mathcal{Q}}$), it is easy to see that there is a constant $C_2 = C_2(D_{\mathcal{Q}}, \ell, q) > 0$ with

$$C_2^{-1} |\det(T_i)|^{1-\frac{1}{q}} \leq \left|\det\left(T_j\right)\right|^{1-\frac{1}{q}} \leq C_2 |\det(T_i)|^{1-\frac{1}{q}}$$

for all $i \in I$ and $j \in i^{\ell *} \supset M_i$.

All in all, we conclude

$$\begin{aligned}\left\|\mathcal{F}^{-1}\varphi_{M_i}\right\|_{L^q} &\leq C_0 C_1 C_2 C_{\mathcal{Q},\Phi,p} \cdot \sum_{j\in M_i} |\det(T_i)|^{1-\frac{1}{q}} \\ &\leq C_0 C_1 C_2 N_{\mathcal{Q}}^{\ell} C_{\mathcal{Q},\Phi,p} \cdot |\det(T_i)|^{1-\frac{1}{q}}\end{aligned}$$

for all $i \in I$ and each subset $M_i \subset i^{\ell *}$. Here, we again used the bound $|M_i| \leq \left|i^{\ell *}\right| \leq N_{\mathcal{Q}}^{\ell}$ in the last step. We have thus established equation (5.3.7).

For the construction of $(\gamma_i)_{i\in I}$, fix a non-trivial $\gamma \in C_c^\infty(B_1(0))$ and let $\varepsilon := \varepsilon_{\mathcal{Q}}$. For each $i \in I$, choose $c_i \in \mathbb{R}^d$ with $B_\varepsilon(c_i) \subset Q_i'$ (cf. Remark 3.2.9 for the existence of $c_i$) and define

$$\gamma_i := L_{b_i}\left[\left(L_{c_i}\left[\gamma \circ \varepsilon^{-1}\mathrm{id}\right]\right) \circ T_i^{-1}\right].$$

We then have

$$\begin{aligned}\operatorname{supp}\gamma_i &= \operatorname{supp}\left(\left(L_{c_i}\left[\gamma \circ \varepsilon^{-1}\mathrm{id}\right]\right) \circ T_i^{-1}\right) + b_i \\ &= T_i\left(\operatorname{supp}\left(L_{c_i}\left[\gamma \circ \varepsilon^{-1}\mathrm{id}\right]\right)\right) + b_i \\ &= T_i\left(\operatorname{supp}\left(\gamma \circ \varepsilon^{-1}\mathrm{id}\right) + c_i\right) + b_i \\ &\subset T_i\left(B_\varepsilon(c_i)\right) + b_i \\ &\subset T_i Q_i' + b_i = Q_i \subset \mathcal{O},\end{aligned}$$

and thus $\gamma_i \in C_c^\infty(\mathcal{O})$ as well as $\varphi_M \gamma_i = \gamma_i$ for $M \supset i^*$, because Lemma 3.2.12 implies $\varphi_M \equiv 1$ on $Q_i$.

A direct calculation using standard properties of the Fourier transform (cf. [33, Theorem 8.22]) shows

$$\left(\mathcal{F}^{-1}\gamma_i\right)(x) = \varepsilon^d \cdot |\det(T_i)| \cdot M_{b_i}\left[\left(M_{c_i}\left[\left(\mathcal{F}^{-1}\gamma\right) \circ \varepsilon\,\mathrm{id}\right]\right) \circ T_i^T\right]$$

and thus (cf. Lemma 4.5.5)

$$\begin{aligned}\left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^r} &= \varepsilon^d \cdot |\det(T_i)| \cdot |\det(T_i)|^{-\frac{1}{r}} \cdot \left\|\left(\mathcal{F}^{-1}\gamma\right) \circ \varepsilon\,\mathrm{id}\right\|_{L^r} \\ &= \left\|\mathcal{F}^{-1}\gamma\right\|_{L^r} \cdot |\det(T_i)|^{1-\frac{1}{r}} \cdot \varepsilon^{d\left(1-\frac{1}{r}\right)}\end{aligned}$$

for all $r \in (0, \infty]$. This proves that the family $(\gamma_i)_{i\in I}$ indeed has the required properties with $C_2 = C_2(d, p) = \left\|\mathcal{F}^{-1}\gamma\right\|_{L^p}$, where $\gamma$ depends only on $d \in \mathbb{N}$. □

The main ingredient for our general sharpness results is the following theorem which describes the asymptotic behaviour of the $L^p$-(quasi)-norm of

$$\sum_{i=1}^{n} L_{x_i} f_i$$

on a locally compact group $G$, as the $(x_i)_{i=1,\dots,n}$ spread more and more. It is probably well-known, but since it is central to our argument, we provide a proof.

**Theorem 5.3.3.** *Let $G$ be an LCH group with (left) Haar-measure $\mu$ and let $p \in (0,\infty]$ and $f_1,\dots,f_n \in L^p(G)$. In case of $p=\infty$, assume additionally that we have*

$$f_1,\dots,f_n \in \overline{\{f \in L^\infty(G) \mid f \text{ compactly supported}\}},$$

*where the closure is taken in $L^\infty(G)$.*

*We remark that the additional assumption for the case $p=\infty$ is satisfied for $f_1,\dots,f_n \in C_0(G)$, cf. [33, Proposition 4.35].*

*For each $\varepsilon > 0$, there is a compact set $K = K(\varepsilon, f_1,\dots,f_n,p) \subset G$ with*

$$\left| \left\| \sum_{i=1}^{n} L_{x_i} f_i \right\|_{L^p} - \left\| \left( \|f_i\|_{L^p} \right)_{i\in\underline{n}} \right\|_{\ell^p} \right| < \varepsilon$$

*for all $x_1,\dots,x_n \in G$ with $x_iK \cap x_jK = \varnothing$ for distinct $i,j \in \underline{n}$.*

*Proof.* We divide the proof into two steps:

STEP 1: First assume that $f_1,\dots,f_n$ are compactly supported. Let $K := \bigcup_{i=1}^n \operatorname{supp} f_i$ and choose $x_1,\dots,x_n \in G$ as in the assumption. This implies

$$\begin{aligned} \operatorname{supp}(L_{x_i} f_i) \cap \operatorname{supp}(L_{x_j} f_j) &= x_i \operatorname{supp}(f_i) \cap x_j \operatorname{supp}(f_j) \\ &\subset x_iK \cap x_jK = \varnothing \end{aligned}$$

for $i \neq j$. Thus, for each $x \in G$, *at most* one summand of $\sum_{j=1}^n (L_{x_j} f_j)(x)$ does *not* vanish. For $0 < p < \infty$, this implies

$$\left| \sum_{j=1}^{n} \left( L_{x_j} f_j \right)(x) \right|^p = \sum_{j=1}^{n} \left| \left( L_{x_j} f_j \right)(x) \right|^p$$

for all $x \in G$ and hence

$$\begin{aligned} \left\| \sum_{j=1}^{n} L_{x_j} f_j \right\|_{L^p}^p &= \int_G \left| \sum_{j=1}^{n} \left( L_{x_j} f_j \right)(x) \right|^p \mathrm{d}\mu(x) \\ &= \int_G \sum_{j=1}^{n} \left| \left( L_{x_j} f_j \right)(x) \right|^p \mathrm{d}\mu(x) \\ &= \sum_{j=1}^{n} \int_G |f_j(x)|^p \,\mathrm{d}\mu(x) = \left\| \left( \|f_i\|_{L^p} \right)_{i\in\underline{n}} \right\|_{\ell^p}^p, \end{aligned}$$

where we used the left-invariance of $\mu$.

For $p = \infty$, and (almost all) $x \in G$, we either have

$$\left|\sum_{j=1}^{n} \left(L_{x_j} f_j\right)(x)\right| = 0 \leq \left\| \left( \|f_i\|_{L^p} \right)_{i \in \underline{n}} \right\|_{\ell^p} =: C,$$

or we have $(L_{x_k} f_k)(x) \neq 0$ for exactly one $k \in \underline{n}$ and thus

$$\left|\sum_{j=1}^{n} \left(L_{x_j} f_j\right)(x)\right| = |(L_{x_k} f_k)(x)| \leq \|f_k\|_{L^\infty} \leq C,$$

which implies $\left\|\sum_{j=1}^{n} L_{x_j} f_j\right\|_{L^\infty} \leq C$.

For the converse direction, choose $k \in \underline{n}$ with $C = \|f_k\|_{L^\infty}$ and note

$$\begin{aligned} C = \|f_k\|_{L^\infty} &= \operatorname*{ess\,sup}_{x \in \operatorname{supp} f_k} |f_k(x)| \\ &= \operatorname*{ess\,sup}_{x \in x_k \operatorname{supp} f_k} |(L_{x_k} f_k)(x)| \\ &\leq \operatorname*{ess\,sup}_{x \in x_k K} \left|\sum_{j=1}^{n} \left(L_{x_j} f_j\right)(x)\right| \\ &\leq \left\|\sum_{j=1}^{n} L_{x_j} f_j\right\|_{L^\infty}. \end{aligned}$$

All in all we have shown (for arbitrary $0 < p \leq \infty$) that for the above choice of $K$ and $x_1, \ldots, x_n \in G$, it holds

$$\left\|\sum_{j=1}^{n} L_{x_j} f_j\right\|_{L^p} = \left\| \left( \|f_i\|_{L^p} \right)_{i \in \underline{n}} \right\|_{\ell^p}.$$

STEP 2: For the general case, let $r := \min\{1, p\} \in (0, 1]$ and let $\varepsilon > 0$. Since the map $[0, \infty) \to \mathbb{R}, x \mapsto x^{1/r}$ is continuous at $x_0 := \left\| \left( \|f_i\|_{L^p} \right)_{i \in \underline{n}} \right\|_{\ell^p}^r$, there is some $\delta > 0$ such that

$$|x^{1/r} - x_0^{1/r}| < \varepsilon \tag{5.3.8}$$

holds for all $x \in [0, \infty)$ with $|x - x_0| < \delta$.

Note that $\|\cdot\|_{L^p}$ is an $r$-norm; for $p \in (0, 1)$, this is a consequence of Example 2.1.3 and for $p \in [1, \infty]$, we have $r = 1$ and $\|\cdot\|_{L^p}$ is a genuine norm. This $r$-norm property of $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\ell^p}$ especially implies that the INVERSE $r$-TRIANGLE INEQUALITY

$$\left| \|f\|_{L^p}^r - \|g\|_{L^p}^r \right| \leq \|f - g\|_{L^p}^r \quad \text{and} \quad \left| \|c\|_{\ell^p}^r - \|d\|_{\ell^p}^r \right| \leq \|c - d\|_{\ell^p}^r$$

is fulfilled. In particular, $\|\cdot\|_{L^p}$ is continuous. Thus, we can choose $g_j \in L^p(G)$ with compact support and with $\|f_j - g_j\|_{L^p} < \left(\frac{\delta}{2n}\right)^{1/r}$ as well as $\left| \|f_j\|_{L^p} - \|g_j\|_{L^p} \right| < \left(\frac{\delta}{2n}\right)^{1/r}$ for each $j \in \underline{n}$. Such $g_j$ exist by density of $C_c(G) \subset L^p(G)$ for $0 < p < \infty$ and by assumption on $f_1, \ldots, f_n$ in case of $p = \infty$.

Let $K := \bigcup_{i=1}^{n} \operatorname{supp} g_i$ as in the first step and let $x_1, \dots, x_n \in G$ with $x_i K \cap x_j K = \varnothing$ for all $i, j \in \underline{n}$ with $i \neq j$. Using the identity

$$\left\| \sum_{j=1}^{n} L_{x_j} g_j \right\|_{L^p} = \left\| \left( \|g_j\|_{L^p} \right)_{j \in \underline{n}} \right\|_{\ell^p}$$

from step one, the inverse $r$-triangle inequality for $\ell^p$ and $L^p$ (marked with $(\dagger)$) and the isometry of $L_x : L^p(G) \to L^p(G)$, we derive

$$\begin{aligned}
&\left| \left\| \sum_{j=1}^{n} L_{x_j} f_j \right\|_{L^p}^r - \left\| \left( \|f_j\|_{L^p} \right)_{j \in \underline{n}} \right\|_{\ell^p}^r \right| \\
\leq &\left| \left\| \sum_{j=1}^{n} L_{x_j} f_j \right\|_{L^p}^r - \left\| \sum_{j=1}^{n} L_{x_j} g_j \right\|_{L^p}^r \right| + \left| \left\| \left( \|g_j\|_{L^p} \right)_{j \in \underline{n}} \right\|_{\ell^p}^r - \left\| \left( \|f_j\|_{L^p} \right)_{j \in \underline{n}} \right\|_{\ell^p}^r \right| \\
\overset{(\dagger)}{\leq} &\left\| \sum_{j=1}^{n} L_{x_j} f_j - \sum_{j=1}^{n} L_{x_j} g_j \right\|_{L^p}^r + \left\| \left( \|g_j\|_{L^p} - \|f_j\|_{L^p} \right)_{j \in \underline{n}} \right\|_{\ell^p}^r \\
\overset{(*)}{\leq} &\sum_{j=1}^{n} \left\| L_{x_j} \left( f_j - g_j \right) \right\|_{L^p}^r + \left\| \left( (\delta / 2n)^{1/r} \right)_{j \in \underline{n}} \right\|_{\ell^p}^r \\
< &\left[ \sum_{j=1}^{n} \frac{\delta}{2n} \right] + \frac{\delta}{2n} \cdot n^{r/p} \leq \frac{\delta}{2} + \frac{\delta}{2n} \cdot n = \delta.
\end{aligned}$$

At $(*)$, we used the $r$-triangle inequality for $\|\cdot\|_{L^p}$, the solidity of $\ell^p$ and the estimate $\left| \|g_j\|_{L^p} - \|f_j\|_{L^p} \right| < \left( \frac{\delta}{2n} \right)^{1/r}$. Furthermore, the last line employed the estimate $\frac{r}{p} = \min\{\frac{1}{p}, 1\} \leq 1$, which implies $n^{r/p} \leq n$.

By choice of $x_0$ and $\delta > 0$ above (cf. equation (5.3.8)), we conclude

$$\left| \left\| \sum_{j=1}^{n} L_{x_j} f_j \right\|_{L^p} - \left\| \left( \|f_i\|_{L^p} \right)_{i \in \underline{n}} \right\|_{\ell^p} \right| < \varepsilon$$

for each choice of $x_1, \dots, x_n \in G$ with $x_i K \cap x_j K = \varnothing$ for $i, j \in \underline{n}$ with $i \neq j$. This completes the proof. □

The following corollary specializes the above theorem to the case $G = \mathbb{R}^d$.

**Corollary 5.3.4.** *Let $0 < p \leq \infty$ and $f_1, \dots, f_n \in L^p(\mathbb{R}^d)$. In case of $p = \infty$, assume additionally that $f_1, \dots, f_n \in C_0(\mathbb{R}^d)$. Let $\varepsilon > 0$. Then there is a constant $R = R(\varepsilon, f_1, \dots, f_n, p) > 0$ such that*

$$\left| \left\| \sum_{j=1}^{n} L_{x_j} f_j \right\|_{L^p} - \left\| \left( \|f_i\|_{L^p} \right)_{i \in \underline{n}} \right\|_{\ell^p} \right| < \varepsilon$$

*holds for all $x_1, \dots, x_n \in \mathbb{R}^d$ that satisfy $|x_i - x_j| > R$ for all $i, j \in \underline{n}$ with $i \neq j$.*

*Furthermore, such a family $(x_i)_{i \in \underline{n}}$ always exists.*

*Proof.* Choose a compact set $K = K(\varepsilon, f_1, \dots, f_n, p) > 0$ as in Theorem 5.3.3. By enlarging $K$ if necessary, we can assume $K \neq \varnothing$. Let $R := 2 \cdot \max_{x \in K} |x|$.

To show the claim, it suffices to show $(x_i + K) \cap (x_j + K) = \varnothing$ for $i \neq j$, since Theorem 5.3.3 implies the desired estimate in this case. But if $(x_i + K) \cap (x_j + K) \neq \varnothing$ would hold for some $i \neq j$, this would imply

$$x_i - x_j \in K - K \subset \overline{B_{R/2}}(0) - \overline{B_{R/2}}(0) \subset \overline{B_R}(0)$$

and hence $|x_i - x_j| \leq R$, a contradition.

Finally, note that the definition $x_i := (i(R+1), 0, \dots, 0) \in \mathbb{R}^d$ for $i \in \underline{n}$ leads to $|x_i - x_j| > R$ for $i \neq j$. This establishes existence of a family with the desired properties. □

The next corollary indicates the way in which we will use this asymptotic behaviour of sums of translations. The idea is that a modulation on the Fourier side amounts to a translation on the space side. This will allow us to construct functions that are located close to each other in the Fourier domain, but are strongly separated in the space domain. The norm of the sum of these function then behaves very differently (at least for $p \neq 2$), when one localizes the individual pieces (on the Fourier side) before one takes the $L^p$-norm (on the space side).

We remark that the formulation that equation (5.3.9) holds for *all* subsets $S \subset M$ is a direct consequence of showing the estimate only for the (arbitrary) set $M$, as the proof shows. But in applications, the set $M \subset I$ will be the support of a certain finitely supported sequence. Localization of $f = \sum_{i \in M} M_{z_i} f_i$ using the BAPU $(\psi_j)_{j \in J}$ will then yield (essentially) the function $f_S^{(z)}$ for $S = M \cap I_j$, so that it is useful to have control over $f_S^{(z)}$ for *all* subsets $S \subset M$ simultaneously.

**Corollary 5.3.5.** *Let $p \in (0, \infty]$ and let $M \neq \varnothing$ be a* finite *index set such that for each $i \in M$ a Schwartz-function $f_i \in \mathcal{S}(\mathbb{R}^d)$ is given.*

*For $S \subset M$ and any family $z = (z_i)_{i \in M}$ in $\mathbb{R}^d$, set*

$$f_S^{(z)} := \sum_{i \in S} M_{z_i} f_i \in \mathcal{S}(\mathbb{R}^d).$$

*Then there is a constant $R = R((f_i)_{i \in M}, M, p) > 0$ such that for every family $(z_i)_{i \in M}$ in $\mathbb{R}^d$ which satisfies $|z_i - z_j| \geq R$ for all distinct $i, j \in M$, the estimate*

$$\begin{aligned} \frac{1}{2} \cdot \left\| (\|\mathcal{F}^{-1} f_i\|_{L^p})_{i \in S} \right\|_{\ell^p} &\leq \left\| \mathcal{F}^{-1} f_S^{(z)} \right\|_{L^p} \\ &\leq 2 \cdot \left\| (\|\mathcal{F}^{-1} f_i\|_{L^p})_{i \in S} \right\|_{\ell^p} \end{aligned} \tag{5.3.9}$$

*is true for every $S \subset M$. Furthermore, such a family $(z_i)_{i \in M}$ always exists.*

*Proof.* We first show the claim for the case $S = M$. In case of $f_i \equiv 0$ for all $i \in M$, choose $R = 1$ and note that equation (5.3.9) holds trivially.

Otherwise, there is some $i \in M$ with $f_i \not\equiv 0$. In particular, $M \neq \varnothing$, so that we can assume $M = \{i_1, \dots, i_n\}$ with $n := |M|$ and (necessarily) pairwise distinct $i_1, \dots, i_n \in M$.

Because of $\mathcal{F}^{-1} f_i \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^p\left(\mathbb{R}^d\right)$, the number

$$\varepsilon := \frac{1}{2} \cdot \left\| \left( \|\mathcal{F}^{-1} f_i\|_{L^p} \right)_{i \in M} \right\|_{\ell^p} > 0$$

is finite.

For $\ell \in \underline{n}$, let $g_\ell := \mathcal{F}^{-1} f_{i_\ell} \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^p\left(\mathbb{R}^d\right)$ and note that we even have $g_\ell \in \mathcal{S}\left(\mathbb{R}^d\right) \subset C_0\left(\mathbb{R}^d\right)$, so that Corollary 5.3.4 yields some $R > 0$ such that

$$\begin{aligned} & \left| \left\| \mathcal{F}^{-1} \left( \sum_{\ell=1}^n M_{-y_\ell} f_{i_\ell} \right) \right\|_{L^p} - 2\varepsilon \right| \\ = & \left| \left\| \sum_{\ell=1}^n L_{y_\ell} \left( \mathcal{F}^{-1} f_{i_\ell} \right) \right\|_{L^p} - \left\| \left( \|\mathcal{F}^{-1} f_i\|_{L^p} \right)_{i \in M} \right\|_{\ell^p} \right| \\ = & \left| \left\| \sum_{\ell=1}^n L_{y_\ell} g_\ell \right\|_{L^p} - \left\| \left( \|g_\ell\|_{L^p} \right)_{\ell \in \underline{n}} \right\|_{\ell^p} \right| < \varepsilon \end{aligned}$$

holds for every family $(y_\ell)_{\ell=1,\dots,n}$ which satisfies $|y_\ell - y_k| \geq R$ for all $\ell, k \in \underline{n}$ with $\ell \neq k$. Here, we used the identity

$$\mathcal{F}^{-1} \left( M_{-z} f \right) = L_z \left( \mathcal{F}^{-1} f \right).$$

For $y_\ell := -z_{i_\ell}$, this implies

$$\left\| \mathcal{F}^{-1} f_M^{(z)} \right\|_{L^p} = \left\| \mathcal{F}^{-1} \left( \sum_{\ell=1}^n M_{-y_\ell} f_{i_\ell} \right) \right\|_{L^p} \in (2\varepsilon - \varepsilon, 2\varepsilon + \varepsilon)$$

as soon as we have $|y_k - y_\ell| = \left| z_{i_\ell} - z_{i_k} \right| \geq R$ for all $\ell \neq k$. Using the definition of $\varepsilon$, this easily implies the claim for $S = M$.

Now the above (applied to $(f_i)_{i \in S}$ for $S \subset M$) yields some $R_S > 0$ for each $S \subset M$ so that equation (5.3.9) is satisfied for every family $(z_i)_{i \in S}$ satisfying $\left| z_i - z_j \right| \geq R_S$ for all $i, j \in S$ with $i \neq j$.

Since $M$ is finite, the same is true of the power set $\mathcal{P}(M)$, so that $R := \max_{S \subset M} R_S$ is finite. It is now easy to see that the claim holds for this choice of $R$. □

We will now impose slightly stronger assumptions on the coverings $\mathcal{Q}, \mathcal{P}$ as well as on the relation between these coverings. The main restriction in the theorem below is that we assume $\mathcal{Q}_{I_0}$ to be almost subordinate to $\mathcal{P}$. This will allow us to prove that the embedding between decomposition spaces given in equation (5.3.2) indeed entails an accompanying embedding of discrete sequence spaces similar to equation (5.1.17) of Theorem 5.1.8. The only real difference is that we get the $\ell^{p_2}$-norm instead of the $\ell^{p_2^\nabla}$-norm as the "inner" norm on the left-hand side.

**Theorem 5.3.6.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and let $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ be a* tight *semi-structured $L^{p_1}$-decomposition covering of $\mathcal{O}$. Finally, let $\mathcal{P} = \left(P_j\right)_{j \in J}$ be an $L^{p_2}$-decomposition covering of $\mathcal{O}'$, let $u = (u_i)_{i \in I}$ and $v = \left(v_j\right)_{j \in J}$ be moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively and let $\Phi = (\varphi_i)_{i \in I}$ be an $L^{p_1}$-BAPU for $\mathcal{Q}$ and $\Psi = \left(\psi_j\right)_{j \in J}$ be an $L^{p_2}$-BAPU for $\mathcal{P}$.*

*Choose a subset $I_0 \subset I$ such that the restricted family $\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to $\mathcal{P}$, i.e. there is $k \in \mathbb{N}_0$ such that for each $i \in I_0$ the inclusion $Q_i \subset P_{j_i}^{k*}$ is valid for a suitable $j_i \in J$. Define*

$$K := \bigcup_{i \in I_0} Q_i.$$

*Let $q_1, q_2 \in (0, \infty]$ and assume that the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f \tag{5.3.10}$$

*is well-defined and continuous. Then there is a finite constant $C > 0$ depending only on $d, p_1, p_2, q_1, q_2$ and on*

$$k, r_0\left(\mathcal{P}, 2k+3\right), \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, N_{\mathcal{P}}, D_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}$$

*such that*

$$\left\| \left( \left\| (c_i)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq C \left\| \left\| \iota \right\| \right\| \left\| \left( \left|\det\left(T_i\right)\right|^{\frac{1}{p_2} - \frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \tag{5.3.11}$$

*holds for all $(c_i)_{i \in I_0} \in \ell_0\left(I_0\right)$, where $\ell_0\left(I_0\right) := \langle \delta_i \mid i \in I_0 \rangle$ is the space of all finitely supported sequences on $I_0$. The dependence of $C$ on $D_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}$ can be dropped for $p_2 \in [1, \infty]$.*

*As usual, it is assumed that the BAPUs $\Phi, \Psi$ are used to calculate the quasi-norms on $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ and $\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$, respectively, in order to determine the operator norm $\left\| \left\| \iota \right\| \right\|$.*

*Now assume that the map*

$$\iota' : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f \tag{5.3.12}$$

*is well-defined and continuous. Then there is a finite constant $C' > 0$, depending only on*

$$d, p_1, p_2, q_1, k, r_0\left(\mathcal{Q}, 1\right), \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},$$

*such that*

$$\left\| \left( \left|\det\left(T_i\right)\right|^{\frac{1}{p_2} - \frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \leq C' \cdot \left\| \left\| \iota' \right\| \right\| \cdot \left\| \left( \left\| (c_i)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \tag{5.3.13}$$

*holds for all $(c_i)_{i \in I_0} \in \ell_0\left(I_0\right)$. The dependence of $C'$ on $D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}$ can be dropped for $p_1 \in [1, \infty]$, as can the dependence on $R_{\mathcal{P}}, C_{\mathcal{P}}$ for $p_2 \in [1, \infty]$.*

*Remark* 5.3.7. Observe that we can set $c_i = \delta_{i,i_0}$ for any $i_0 \in I_0$. In this case, the estimate in equation (5.3.11) yields

$$v_j \leq \left\| \chi_{J_{i_0}} \right\|_{\ell^{q_2}_v(J)} \leq C \cdot |||\iota||| \cdot u_{i_0} \cdot |\det(T_{i_0})|^{\frac{1}{p_2}-\frac{1}{p_1}}$$

for all $j \in J$ with $Q_{i_0} \cap P_j \neq \varnothing$.

This can be seen as a generalization of a part of Lemma 5.3.1, where the estimate $v_j \lesssim |||\iota||| \cdot u_i$ for all $i \in I$ and $j \in J$ with $K^\circ \cap Q_i^\circ \cap P_j^\circ \neq \varnothing$ was established under the assumption $p_1 = p_2$.

Similiarly, we can choose $c_i = \delta_{i,i_0}$ for $i_0 \in I_0$ in equation (5.3.13) to derive

$$u_{i_0} \cdot |\det(T_{i_0})|^{\frac{1}{p_2}-\frac{1}{p_1}} \leq C' \cdot |||\iota'||| \cdot \left\| \chi_{J_{i_0}} \right\|_{\ell^{q_2}_v(J)}.$$

*To see $J_{i_0} \neq \varnothing$, note $Q_{i_0} \subset P^{k*}_{j_{i_0}}$ by assumption. Furthermore, $Q_{i_0} \neq \varnothing$ by definition of an admissible covering. Hence, $Q_{i_0} \cap P_j \neq \varnothing$ for some $j \in j^{k*}_{i_0}$. In particular, $j \in J_{i_0} \neq \varnothing$.*

But Lemma 3.3.2 shows $\varnothing \neq J_{i_0} \subset j^{(2k+2)*}$ for each $j \in J_{i_0}$ and thus $1 \leq |J_{i_0}| \leq N_{\mathcal{P}}^{2k+2}$. Together with $\mathcal{P}$-moderateness of $v$, this entails

$$u_{i_0} \cdot |\det(T_{i_0})|^{\frac{1}{p_2}-\frac{1}{p_1}} \leq C'' \cdot |||\iota'||| \cdot v_j \tag{5.3.14}$$

for all $i_0 \in I_0$ and $j \in J$ with $Q_{i_0} \cap P_j \neq \varnothing$. Here, $C'' > 0$ is a constant depending only on $d, p_1, p_2, q_1$ and on

$$k, r_0(\mathcal{Q}, 1), \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}.$$

◀

Below, we will give separate proofs for the validity of the two necessary conditions in equations (5.3.11) and (5.3.13). We begin with the proof of equation (5.3.13), as this will turn out to be simpler and thus better suited to illustrate the general idea of the proof.

This idea is to consider functions of the form

$$f = \sum_{i \in I_0} M_{z_i}(c_i \gamma_i),$$

with $(\gamma_i)_{i \in I}$ as in Lemma 5.3.2, to "test" the embedding.

It will then turn out that the quasi-norm of $f$ in $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$ is essentially independent of the choice of the $(z_i)_i$, namely

$$\|f\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)} \asymp \left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} \cdot c_i \right)_{i \in I_0} \right\|_{\ell^{q_1}_u}.$$

The estimate "$\lesssim$" will turn out to be a direct consequence of the definitions and true in general (cf. Lemma 5.3.8). To make the converse estimate rigorous, we need to employ a disjointization argument to modify the definition of $f$ slightly, cf. the proof of equation (5.3.13).

Using Corollary 5.3.5, we will then show that the "modulations" $(z_i)_i$ can be chosen so that

$$\|f\|_{\mathcal{D}(\mathcal{P},L^{p_2},\ell_v^{q_2})} \asymp \left\| \left( \left\| \left( |\det(T_i)|^{1-1/p_2} \cdot c_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}}$$

holds. Here, the estimate "$\lesssim$" will turn out to be a consequence of Corollary 5.3.5 (cf. Lemma 5.3.8 again), whereas the estimate "$\gtrsim$" requires a modified definition of $f$ using a more sophisticated disjointization argument (cf. the proof of equation (5.3.11)).

We begin with a lemma that establishes the "easy" parts of the norm equivalences stated above.

**Lemma 5.3.8.** *Let $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ be open, let $p \in (0,\infty]$ and assume that $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ is a semi-structured $L^p$-decomposition covering of $\mathcal{O}$ with $L^p$-BAPU $\Phi = (\varphi_i)_{i \in I}$. Finally, assume that $u = (u_i)_{i \in I}$ is $\mathcal{Q}$-moderate.*

*For arbitrary $q \in (0,\infty]$ and $k \in \mathbb{N}_0$, there is a constant*

$$C = C\left(k, d, p, q, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}\right) > 0$$

*As usual, the dependence of $C$ on $R_{\mathcal{Q}}, C_{\mathcal{Q}}$ can be dropped for $p \in [1,\infty]$.*

*such that for arbitrary coefficients $(c_i)_{i \in I} \in \ell_0(I)$, functions $(\gamma_i)_{i \in I}$ with $\gamma_i \in C_c^\infty(\mathcal{O})$ and $\gamma_i \equiv 0$ on $\left(Q_i^{k*}\right)^c$ and modulations $(z_i)_{i \in I} \in \left(\mathbb{R}^d\right)^I$, the estimate*

$$\left\| \sum_{i \in I} M_{z_i}(c_i \gamma_i) \right\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell_u^q\right)} \leq C \cdot \left\| (c_i \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p})_{i \in I} \right\|_{\ell_u^q} \tag{5.3.15}$$

*holds.*

*Now choose $(\gamma_i)_{i \in I}$ as in Lemma 5.3.2 and let the general assumptions of Theorem 5.3.6 be satisfied (i.e. everything except for boundedness of $\iota, \iota'$). Then there is a constant*

$$C' = \begin{cases} C'\left(d, p_2, \varepsilon_{\mathcal{Q}}, C_{\mathcal{P},\Psi,p_2}\right), & \text{for } p_2 \in [1,\infty], \\ C'\left(k, d, p_2, \varepsilon_{\mathcal{Q}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right), & \text{for } p_2 \in (0,1), \end{cases}$$

*such that for arbitrary coefficients $c = (c_i)_{i \in I_0} \in \ell_0(I_0)$ there is some $R > 0$ (possibly dependent on $\operatorname{supp} c$) so that for all subsets $S \subset I_0$ and all families of modulations $(z_i)_{i \in I_0} \in \left(\mathbb{R}^d\right)^{I_0}$ that satisfy $|z_i - z_j| \geq R$ for distinct $i, j \in \operatorname{supp} c$, the estimate*

$$\begin{aligned} &\left\| \sum_{i \in S} M_{z_i}(c_i \gamma_i) \right\|_{\mathcal{D}_\Psi\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)} \\ &\leq C' \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-1/p_2} \cdot c_i \right)_{i \in S \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \end{aligned}$$

*holds. Furthermore, such a family of modulations always exists.*

*Proof.* We first prove equation (5.3.15). To this end, let $\ell \in I$ be arbitrary. For $i \in I$, we note that

$$0 \not\equiv \varphi_\ell \cdot M_{z_i}(c_i \gamma_i) = c_i \cdot M_{z_i}(\varphi_\ell \gamma_i)$$

implies $\varnothing \neq Q_\ell \cap Q_i^{k*}$ and thus $i \in \ell^{(k+1)*}$.

As usual, we distinguish the two cases $p \in [1,\infty]$ and $p \in (0,1)$. For $p \in [1,\infty]$, Young's inequality ($L^1 * L^p \hookrightarrow L^p$), together with the identity

$$\mathcal{F}^{-1}[M_z g] = L_{-z}[\mathcal{F}^{-1} g]$$

and with translation invariance of $\|\cdot\|_{L^p}$, implies

$$\begin{aligned}\left\|\mathcal{F}^{-1}(M_{z_i}(\varphi_\ell \cdot \gamma_i))\right\|_{L^p} &= \left\|\mathcal{F}^{-1}(\varphi_\ell \cdot \gamma_i)\right\|_{L^p} \\ &\leq \|\mathcal{F}^{-1}\varphi_\ell\|_{L^1} \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p} \\ &\leq C_{\mathcal{Q},\Phi,p} \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p}.\end{aligned}$$

In case of $p \in (0,1)$, we observe $\operatorname{supp}\varphi_\ell \subset \overline{Q_\ell} \subset \overline{Q_\ell^{(2k+1)*}}$ and $\operatorname{supp}\gamma_i \subset \overline{Q_i^{k*}} \subset \overline{Q_\ell^{(2k+1)*}}$, because of $i \in \ell^{(k+1)*}$, so that Corollary 3.2.15 yields a constant $C_1 = C_1(k,d,p,R_\mathcal{Q},C_\mathcal{Q})$ with

$$\begin{aligned}\left\|\mathcal{F}^{-1}(M_{z_i}(\varphi_\ell \cdot \gamma_i))\right\|_{L^p} &= \left\|\mathcal{F}^{-1}(\varphi_\ell \cdot \gamma_i)\right\|_{L^p} \\ &\leq C_1 |\det(T_\ell)|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_\ell\|_{L^p} \|\mathcal{F}^{-1}\gamma_i\|_{L^p} \\ &\leq C_1 C_{\mathcal{Q},\Phi,p} \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p}.\end{aligned}$$

Thus, if we set $C_1 := 1$ for $p \in [1,\infty]$, this estimate is valid for all $p \in (0,\infty]$.

Since $\|\cdot\|_{L^p}$ is a quasi-norm and because of the uniform bound $|\ell^{(k+1)*}| \leq N_\mathcal{Q}^{k+1}$, which was established in Lemma 3.2.18, there is a constant $C_2 = C_2(p,k,N_\mathcal{Q})$ satisfying

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[\varphi_\ell \cdot \sum_{i\in I} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^p} &\overset{(\dagger)}{\leq} C_2 \sum_{i\in\ell^{(k+1)*}} \left[|c_i| \cdot \left\|\mathcal{F}^{-1}(M_{z_i}(\varphi_\ell\cdot\gamma_i))\right\|_{L^p}\right] \\ &\leq C_1 C_2 C_{\mathcal{Q},\Phi,p} \sum_{i\in\ell^{(k+1)*}} [|c_i| \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p}] \\ &= C_1 C_2 C_{\mathcal{Q},\Phi,p} \cdot (\Theta_{k+1} d)_\ell.\end{aligned}$$

In the last step, we used $d = (d_i)_{i\in I}$ with $d_i := |c_i| \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p}$ for $i \in I$ and the $k+1$-FOLD CLUSTERING MAP $\Theta_{k+1} : \ell_u^q(I) \to \ell_u^q(I)$ as defined in Lemma 3.4.3. At (†), we used that $\varphi_\ell \cdot M_{z_i}(c_i\gamma_i) \not\equiv 0$ can only hold for $i \in \ell^{(k+1)*}$, as observed at the beginning of the proof.

Combining lemmata 3.4.3 and 3.4.2 yields $|||\Theta_{k+1}||| \leq C_3$ for some constant $C_3 = C_3(q, k, N_{\mathcal{Q}}, C_{u,\mathcal{Q}})$. We now define $C_4 := C_1 C_2 C_3 C_{\mathcal{Q},\Phi,p}$ and conclude

$$\begin{aligned}\left\| \sum_{i\in I} M_{z_i}(c_i\gamma_i) \right\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)} &\leq C_1 C_2 C_{\mathcal{Q},\Phi,p} \cdot \|\Theta_{k+1} d\|_{\ell^q_u(I)} \\ &\leq C_4 \cdot \left\| \left( |c_i| \cdot \|\mathcal{F}^{-1}\gamma_i\|_{L^p} \right)_{i\in I} \right\|_{\ell^q_u}.\end{aligned}$$

This completes the proof of equation (5.3.15).

For the second claim, let $j \in J$ be arbitrary. For $i \in I$ with

$$0 \not\equiv \psi_j \cdot M_{z_i}(c_i\gamma_i) = c_i \cdot M_{z_i}(\psi_j\gamma_i),$$

we derive $\varnothing \neq P_j \cap Q_i$, which means nothing but $i \in I_j$. Here, we used that $(\gamma_i)_{i\in I}$ satisfies $\operatorname{supp}\gamma_i \subset Q_i$ for all $i \in I$.

Now, let $c = (c_i)_{i\in I_0} \in \ell_0(I_0)$ be arbitrary, set $M := \operatorname{supp} c$ and define $g_i := c_i\gamma_i \in C_c^\infty(\mathcal{O}) \subset \mathcal{S}(\mathbb{R}^d)$ for $i \in M$. Corollary 5.3.5, applied to the family $(g_i)_{i\in M}$ yields a constant $R = R\left((g_i)_{i\in M}, M, p_2\right) > 0$ such that

$$\begin{aligned}\left\| \mathcal{F}^{-1}\left[ \sum_{i\in S} M_{z_i}(c_i\gamma_i) \right] \right\|_{L^{p_2}} &\leq 2 \cdot \left\| \left( \left\| \mathcal{F}^{-1}(c_i\gamma_i) \right\|_{L^{p_2}} \right)_{i\in S} \right\|_{\ell^{p_2}} \qquad (5.3.16) \\ &= 2C_5 \cdot \varepsilon^{d\left(1-\frac{1}{p_2}\right)} \cdot \left\| \left( c_i \cdot |\det(T_i)|^{1-\frac{1}{p_2}} \right)_{i\in S} \right\|_{\ell^{p_2}}\end{aligned}$$

holds for all $S \subset M$ as soon as $(z_i)_{i\in M}$ satisfies $|z_i - z_\ell| \geq R$ for all $i, \ell \in M$ with $i \neq \ell$. In the last step, we used $\varepsilon = \varepsilon_{\mathcal{Q}}$ and the constant $C_5 = C_5(d, p_2) > 0$ supplied by Lemma 5.3.2, for which

$$\|\mathcal{F}^{-1}\gamma_i\|_{L^{p_2}} = C_5 \cdot \varepsilon^{d\left(1-\frac{1}{p_2}\right)} \cdot |\det(T_i)|^{1-\frac{1}{p_2}}$$

holds for all $i \in I$.

Now, let $S \subset I_0$ be arbitrary. We distinguish the cases $p_2 \in [1,\infty]$ and $p_2 \in (0,1)$. For $p_2 \in [1,\infty]$, Young's inequality ($L^1 * L^{p_2} \hookrightarrow L^{p_2}$), together with $\psi_j \cdot M_{z_i}(c_i\gamma_i) \equiv 0$ for $i \notin I_j$ implies

$$\begin{aligned}\left\| \mathcal{F}^{-1}\left[ \psi_j \cdot \sum_{i\in S} M_{z_i}(c_i\gamma_i) \right] \right\|_{L^{p_2}} &= \left\| \mathcal{F}^{-1}\left[ \psi_j \cdot \sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i) \right] \right\|_{L^{p_2}} \\ &\leq \left\| \mathcal{F}^{-1}\psi_j \right\|_{L^1} \left\| \mathcal{F}^{-1}\left[ \sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i) \right] \right\|_{L^{p_2}} \\ &\leq C_{\mathcal{P},\Psi,p_2} \cdot \left\| \mathcal{F}^{-1}\left[ \sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i) \right] \right\|_{L^{p_2}}.\end{aligned}$$

In case of $p \in (0,1)$, we observe that $\mathcal{P} = (S_j P_j' + e_j)_{j\in J}$ is semi-structured, since it is an $L^{p_2}$-decomposition covering. Furthermore, we have $\operatorname{supp}\psi_j \subset \overline{P_j} \subset \overline{P_j^{(2k+2)*}}$. Finally, for $i \in I_j$, Lemma 3.3.2 implies $Q_i \subset P_j^{(2k+2)*}$ and hence $\operatorname{supp}\gamma_i \subset Q_i \subset \overline{P_j^{(2k+2)*}}$. Hence, Corollary 3.2.15 yields a constant $C_6 = C_6(k,d,p_2,R_{\mathcal{P}},C_{\mathcal{P}}) > 0$ with

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left[\psi_j \cdot \sum_{i\in S} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}} \\
&= \left\| \mathcal{F}^{-1}\left[\psi_j \cdot \sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}} \\
&\leq C_6 \left|\det(S_j)\right|^{\frac{1}{p_2}-1} \left\|\mathcal{F}^{-1}\psi_j\right\|_{L^{p_2}} \cdot \left\| \mathcal{F}^{-1}\left[\sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}} \\
&\leq C_6 C_{\mathcal{P},\Psi,p_2} \cdot \left\| \mathcal{F}^{-1}\left[\sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}}.
\end{aligned}$$

Hence, if we set $C_6 := 1$ in case of $p_2 \in [1,\infty]$, then the above estimate is true for all $p_2 \in (0,\infty]$.

All in all, we conclude

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left[\psi_j \cdot \sum_{i\in S} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}} \\
&\leq C_6 C_{\mathcal{P},\Psi,p_2} \cdot \left\| \mathcal{F}^{-1}\left[\sum_{i\in I_j\cap S\cap M} M_{z_i}(c_i\gamma_i)\right]\right\|_{L^{p_2}} \\
&\overset{(\dagger)}{\leq} 2C_5C_6C_{\mathcal{P},\Psi,p_2} \cdot \varepsilon^{d\left(1-\frac{1}{p_2}\right)} \cdot \left\| \left(c_i \cdot \left|\det(T_i)\right|^{1-\frac{1}{p_2}}\right)_{i\in I_j\cap S\cap M}\right\|_{\ell^{p_2}} \\
&= 2C_5C_6C_{\mathcal{P},\Psi,p_2} \cdot \varepsilon^{d\left(1-\frac{1}{p_2}\right)} \cdot \left\| \left(c_i \cdot \left|\det(T_i)\right|^{1-\frac{1}{p_2}}\right)_{i\in I_j\cap S}\right\|_{\ell^{p_2}}
\end{aligned}$$

for each $j \in J$ as soon as the family $(z_i)_{i\in I_0}$ of modulations satisfies $|z_i - z_\ell| \geq R$ for all $i,\ell \in M$ with $i \neq \ell$. Here, equation (5.3.16) was used at $(\dagger)$. The last step is justified by noting $c_i = 0$ for $i \in S \setminus M$.

This estimate finally yields

$$\begin{aligned}
&\left\| \sum_{i\in S} M_{z_i}(c_i\gamma_i)\right\|_{\mathcal{D}_\Psi\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)} \\
&= \left\| \left( \left\| \mathcal{F}^{-1}\left(\psi_j \cdot \sum_{i\in S} M_{z_i}(c_i\gamma_i)\right)\right\|_{L^{p_2}}\right)_{j\in J}\right\|_{\ell_v^{q_2}} \\
&\leq 2C_5C_6C_{\mathcal{P},\Psi,p_2} \cdot \varepsilon^{d\left(1-\frac{1}{p_2}\right)} \cdot \left\| \left( \left\| \left(c_i \cdot \left|\det(T_i)\right|^{1-\frac{1}{p_2}}\right)_{i\in I_j\cap S}\right\|_{\ell^{p_2}}\right)_{j\in J}\right\|_{\ell_v^{q_2}}
\end{aligned}$$

for all $S \subset I_0$, as soon as the family $(z_i)_{i\in I_0}$ of modulations satisfies $|z_i - z_\ell| \geq R$ for all $i, \ell \in M$ with $i \neq \ell$. Existence of such a family $(z_i)_{i\in I_0}$ was already shown in Corollary 5.3.5. Here, it is worth noting that $M$ is finite and that we can simply take $z_i = 0$ for $i \in I_0 \setminus M$. □

Finally, we come to the proofs of equations (5.3.13) and (5.3.11) of Theorem 5.3.6.

*Proof of equation* (5.3.13). Choose $(\gamma_i)_{i\in I}$ as in Lemma 5.3.2 and define $\varepsilon := \varepsilon_{\mathcal{Q}}$. Hence, there is a constant $C_1 = C_1(d, p_1, \varepsilon) > 0$ with

$$\left\| \mathcal{F}^{-1}\gamma_i \right\|_{L^{p_1}} = C_1 \cdot |\det(T_i)|^{1-\frac{1}{p_1}} \tag{5.3.17}$$

for all $i \in I$.

By definition of the DISJOINTIZATION NUMBER $r_0 := r_0(\mathcal{Q}, 1)$ (cf. Lemma 3.3.5), there is a finite partition $I = \biguplus_{r=1}^{r_0} I^{(r)}$ such that $Q_i^* \cap Q_\ell^* = \varnothing$ holds for all $i, \ell \in I^{(r)}$ with $i \neq \ell$ for every $r \in \underline{r_0}$.

Let $c = (c_i)_{i\in I_0} \in \ell_0(I_0)$ be arbitrary and define

$$c^{(r)} := \left(c_i^{(r)}\right)_{i\in I_0} := \left(c_i \cdot \chi_{I^{(r)}}(i)\right)_{i\in I_0},$$

as well as

$$g_z^{(r)} := \sum_{i\in I_0} M_{z_i}\left(c_i^{(r)}\gamma_i\right) = \sum_{i\in I_0\cap I^{(r)}} M_{z_i}(c_i\gamma_i) \tag{5.3.18}$$

for $r \in \underline{r_0}$ and any family $z = (z_i)_{i\in I_0}$ of modulations. Observe $\operatorname{supp}\gamma_i \subset Q_i \subset K$ for each $i \in I_0$, which implies $g_z^{(r)} \in \mathcal{D}_K$, since it is a *finite* sum of elements of $\mathcal{D}_K$.

For $r \in \underline{r_0}$ and arbitrary $i, m \in I_0 \cap I^{(r)}$, we have

$$\varphi_m^* \cdot M_{z_i}\left(c_i^{(r)}\gamma_i\right) = c_i^{(r)} \cdot M_{z_i}(\varphi_m^*\gamma_i) \equiv 0$$

unless possibly in the case $\varnothing \neq Q_i \cap Q_m^* \subset Q_i^* \cap Q_m^*$ which implies $i = m$ by the choice of the partition $I = \biguplus_{r=1}^{r_0} I^{(r)}$. In this case, Lemma 3.2.12 shows $\varphi_m^* = \varphi_i^* \equiv 1$ on $Q_i$ and thus $\varphi_m^* \cdot \gamma_i = \gamma_i$. In summary, we have established the identity

$$\varphi_m^* \cdot M_{z_i}\left(c_i^{(r)}\gamma_i\right) = \begin{cases} M_{z_i}\left(c_i^{(r)}\gamma_i\right), & \text{if } m = i, \\ 0, & \text{if } m \neq i, \end{cases} \tag{5.3.19}$$

which is valid for all $i, m \in I_0 \cap I^{(r)}$ and $r \in \underline{r_0}$.

A combination of Theorem 3.4.10, Lemma 3.4.2 and Remark 3.4.9 yields a positive constant

$$C_2 = \begin{cases} C_2\left(p_1, q_1, d, N_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}\right), & \text{if } p_1 \in (0,1), \\ C_2\left(p_1, q_1, N_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}\right), & \text{if } p_1 \in [1,\infty] \end{cases}$$

satisfying

$$\begin{aligned} \left\|g_z^{(r)}\right\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} &\geq C_2^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_m^* \cdot g_z^{(r)}\right) \right\|_{L^{p_1}} \right)_{m\in I} \right\|_{\ell_u^{q_1}} \\ &\geq C_2^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_m^* \cdot g_z^{(r)}\right) \right\|_{L^{p_1}} \right)_{m\in I_0\cap I^{(r)}} \right\|_{\ell_u^{q_1}} \\ &\overset{(\dagger)}{=} C_2^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left(M_{z_m}\left(c_m^{(r)}\gamma_m\right)\right) \right\|_{L^{p_1}} \right)_{m\in I_0\cap I^{(r)}} \right\|_{\ell_u^{q_1}} \\ &\overset{(\ddagger)}{=} C_2^{-1} \cdot \left\| \left( |c_m| \cdot \left\| \mathcal{F}^{-1}\gamma_m \right\|_{L^{p_1}} \right)_{m\in I_0\cap I^{(r)}} \right\|_{\ell_u^{q_1}} \\ &= \frac{C_1}{C_2} \cdot \left\| \left( c_i \cdot |\det(T_i)|^{1-\frac{1}{p_1}} \right)_{i\in I_0\cap I^{(r)}} \right\|_{\ell_u^{q_1}}. \end{aligned}$$

Here, we used equation (5.3.19) and the definition of $g_z^{(r)}$ (cf. equation (5.3.18)) at $(\dagger)$. The step marked with $(\ddagger)$ is justified by the identity $\mathcal{F}^{-1}[M_z h] = L_{-z}[\mathcal{F}^{-1}h]$ and by translation invariance of $\|\cdot\|_{L^{p_1}}$. Finally, the last step used equation (5.3.17).

*Observe that the modulations $(z_i)_{i\in I_0}$ possibly depend on the chosen coefficients $(c_i)_{i\in I_0}$, but do not influence the constants $C_1, C_2, C_3$.*

Thanks to Lemma 5.3.8, we can choose the modulations $(z_i)_{i\in I_0}$ for each $r \in \underline{r_0}$ in such a way that

$$\begin{aligned} \left\|g_z^{(r)}\right\|_{\mathcal{D}_\Psi\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)} &\leq C_3 \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} \cdot c_i^{(r)} \right)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ &\leq C_3 \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} \cdot c_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \end{aligned}$$

holds, with $C_3 = C_3\left(k, d, p_2, \varepsilon_{\mathcal{Q}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right)$ for $p_2 \in (0,1)$ and with $C_3 = C_3\left(d, p_2, \varepsilon_{\mathcal{Q}}, C_{\mathcal{P},\Psi,p_2}\right)$ for $p_2 \in [1,\infty]$.

We now use the embedding $\iota'$ to derive

$$\begin{aligned} &\frac{C_1}{C_2} \cdot \left\| \left( c_i \cdot |\det(T_i)|^{1-\frac{1}{p_1}} \right)_{i\in I_0\cap I^{(r)}} \right\|_{\ell_u^{q_1}} \\ &\leq \left\|g_z^{(r)}\right\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} \\ &\leq |||\iota'||| \cdot \left\|g_z^{(r)}\right\|_{\mathcal{D}_\Psi\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)} \\ &\leq |||\iota'||| \cdot C_3 \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} \cdot c_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}}. \end{aligned}$$

Note that the right-hand side is independent of $r \in \underline{r_0}$.

Since $\ell_u^{q_1}$ is a quasi-normed vector space (with triangle constant only depending on $q_1$), there is a constant $C_4 = C_4(q_1, r_0) > 0$ with

$$\left\| \left( c_i \left| \det(T_i) \right|^{1-\frac{1}{p_1}} \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \leq C_4 \sum_{r=1}^{r_0} \left\| \left( c_i \cdot \left| \det(T_i) \right|^{1-\frac{1}{p_1}} \right)_{i \in I_0 \cap I^{(r)}} \right\|_{\ell_u^{q_1}}.$$

This finally leads to

$$\begin{aligned} &\left\| \left( c_i \left| \det(T_i) \right|^{1-\frac{1}{p_1}} \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \\ &\leq \frac{C_2 C_3 C_4}{C_1} r_0 \cdot \left\| \left| \iota' \right| \right\| \cdot \left\| \left( \left\| \left( c_i \left| \det(T_i) \right|^{1-\frac{1}{p_2}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}}. \end{aligned}$$

By moving all factors involving $|\det(T_i)|$ to the left-hand side (i.e. by considering $c_i = |\det(T_i)|^{\frac{1}{p_2}-1} d_i$), this easily implies the claimed inequality (5.3.13), since $(c_i)_{i \in I_0} \in \ell_0(I_0)$ was arbitrary. □

Finally, we come to the proof of equation (5.3.11). The techniques used are similar to the previous proof, but the disjointization argument becomes slightly more complex. The idea is to use the disjointization principle on the index set $J$ of $\mathcal{P}$ and then to transfer this disjointization to one of the index set $I$ of $\mathcal{Q}$.

*Proof of equation* (5.3.11). Choose $(\gamma_i)_{i \in I}$ as in Lemma 5.3.2 and set $\varepsilon := \varepsilon_{\mathcal{Q}}$. By choice of $(\gamma_i)_{i \in I}$, there are constants $C_1 = C_1(d, p_2, \varepsilon) > 0$ and $C_2 = C_2(d, p_1, \varepsilon) > 0$ with

$$\begin{aligned} \left\| \mathcal{F}^{-1} \gamma_i \right\|_{L^{p_2}} &= C_1 \left| \det(T_i) \right|^{1-\frac{1}{p_2}} \\ \text{and } \left\| \mathcal{F}^{-1} \gamma_i \right\|_{L^{p_1}} &= C_2 \left| \det(T_i) \right|^{1-\frac{1}{p_1}} \end{aligned} \tag{5.3.20}$$

for all $i \in I$.

By definition of $r_0 := r_0(\mathcal{P}, 2k+3)$ (cf. Lemma 3.3.5), there is a partition $J = \biguplus_{r=1}^{r_0} J^{(r)}$ such that $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \varnothing$ holds for all $j, \ell \in J^{(r)}$ with $j \neq \ell$ and all $r \in \underline{r_0}$.

We then have

$$\forall j, \ell \in J^{(r)} \text{ with } j \neq \ell : \qquad \left(I_j \cap I_0\right) \cap \left(I_\ell \cap I_0\right) = \varnothing, \tag{5.3.21}$$

because for $i \in I_j \cap I_\ell \cap I_0$, we would have $Q_i \cap P_j \neq \varnothing \neq Q_i \cap P_\ell$, so that Lemma 3.3.2 yields

$$\varnothing \neq Q_i \subset P_j^{(2k+2)*} \cap P_\ell^{(2k+2)*} \subset P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*}$$

in contradition to $j, \ell \in J^{(r)}$ with $j \neq \ell$.

These considerations show that

$$I^{(r)} := \biguplus_{j \in J^{(r)}} (I_j \cap I_0)$$

is well-defined. As a simple consequence of this definition, we observe

$$\forall j \in J^{(r)} : \qquad I^{(r)} \cap I_j = I_0 \cap I_j. \tag{5.3.22}$$

For $i \in I_0$ and $j \in J$ with $Q_i \cap P_j \neq \varnothing$, Lemma 3.3.2 yields

$$\operatorname{supp} \gamma_i \subset Q_i \subset P_j^{(2k+2)*}$$

and Lemma 3.2.12 implies $\psi_j^{(2k+3)*} \equiv 1$ on $P_j^{(2k+2)*}$. Together, this establishes

$$\forall i \in I_0 \, \forall j \in J \text{ with } Q_i \cap P_j \neq \varnothing : \qquad \gamma_i \psi_j^{(2k+3)*} = \gamma_i. \tag{5.3.23}$$

Now, let $c = (c_i)_{i \in I_0} \in \ell_0(I_0)$ be arbitrary, set $M := \operatorname{supp} c \subset I_0$ and define

$$c^{(r)} := \left(c_i^{(r)}\right)_{i \in I} := \left(c_i \cdot \chi_{I^{(r)}}(i)\right)_{i \in I},$$

as well as

$$g_z^{(r)} := \sum_{i \in I^{(r)}} M_{z_i}(c_i \gamma_i) = \sum_{i \in I_0} M_{z_i}\left(c_i^{(r)} \gamma_i\right)$$

for $r \in \underline{r_0}$ and any family $(z_i)_{i \in I_0}$ of modulations. Observe $g_z^{(r)} \in \mathcal{D}_K$ as a *finite* sum of elements of $\mathcal{D}_K$, since each $\gamma_i$ for $i \in I_0$ satisfies $\operatorname{supp} \gamma_i \subset Q_i \subset K$.

Corollary 5.3.5, applied to the family $(h_i)_{i \in M}$ defined by $h_i := c_i^{(r)} \gamma_i$ for $i \in M$, yields a family $z^{(r)} = (z_i^{(r)})_{i \in I_0}$ of modulations so that

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left( \sum_{i \in S} M_{z_i^{(r)}}\left(c_i^{(r)} \gamma_i\right) \right) \right\|_{L^{p_2}} \\ &\geq \frac{1}{2} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left(c_i^{(r)} \gamma_i\right) \right\|_{L^{p_2}} \right)_{i \in S} \right\|_{\ell^{p_2}} \\ &= \frac{C_1}{2} \cdot \left\| \left( c_i \cdot |\det(T_i)|^{1 - \frac{1}{p_2}} \right)_{i \in S \cap I^{(r)}} \right\|_{\ell^{p_2}} \end{aligned} \tag{5.3.24}$$

holds for all subsets $S \subset M$. Here, the last step is justified by equation (5.3.20) and because of $c_i^{(r)} = 0$ for $i \notin I^{(r)}$.

Now, let $j \in J^{(r)}$ be arbitrary. For $i \in I^{(r)} \cap I_j = I_0 \cap I_j$, equation (5.3.23) implies

$$\begin{aligned} \psi_j^{(2k+3)*} \cdot M_{z_i}(c_i \gamma_i) &= M_{z_i}\left(c_i \cdot \psi_j^{(2k+3)*} \gamma_i\right) \\ &= M_{z_i}(c_i \gamma_i). \end{aligned} \tag{5.3.25}$$

If otherwise $i \in I^{(r)} \setminus I_j$ holds, then $i \in I_\ell$ for some $\ell \in J^{(r)} \setminus \{j\}$. Because of $j, \ell \in J^{(r)}$ with $j \neq \ell$, we get $P_j^{(2k+3)*} \cap P_\ell^{(2k+3*)} = \varnothing$ which implies $\psi_j^{(2k+3)*} \psi_\ell^{(2k+3)*} \equiv 0$ and thus (using equation (5.3.25) for $\ell$ instead of $j$)

$$\psi_j^{(2k+3)*} \cdot M_{z_i}(c_i \gamma_i) = \psi_j^{(2k+3)*} \cdot \psi_\ell^{(2k+3)*} \cdot M_{z_i}(c_i \gamma_i) \equiv 0.$$

In summary, these considerations imply

$$\psi_j^{(2k+3)*} \cdot M_{z_i}(c_i \gamma_i) = \begin{cases} M_{z_i}(c_i \gamma_i), & \text{if } i \in I_j, \\ 0, & \text{if } i \notin I_j \end{cases} \tag{5.3.26}$$

for arbitrary $j \in J^{(r)}$, $i \in I^{(r)}$ and $r \in \underline{r_0}$.

A combination of Theorem 3.4.10, Lemma 3.4.2 and Remark 3.4.9 yields a constant

$$C_3 = \begin{cases} C_3\left(p_2, q_2, k, d, N_{\mathcal{P}}, D_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right), & \text{if } p_2 \in (0,1), \\ C_3\left(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right), & \text{if } p_2 \in [1,\infty] \end{cases}$$

satisfying

$$\left\| \left( \left\| \mathcal{F}^{-1}\left( \psi_j^{(2k+3)*} \cdot g \right) \right\|_{L^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq C_3 \cdot \|g\|_{\mathcal{D}_\Psi(\mathcal{P}, L^{p_2}, \ell_v^{q_2})}$$

for all $g \in \mathcal{D}'(\mathcal{O}')$. Applying this to $g = g_{z^{(r)}}^{(r)}$, we derive

$$\begin{aligned}
&\left\| g_{z^{(r)}}^{(r)} \right\|_{\mathcal{D}_\Psi(\mathcal{P}, L^{p_2}, \ell_v^{q_2})} \\
&\geq C_3^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left( \psi_j^{(2k+3)*} \cdot g_{z^{(r)}}^{(r)} \right) \right\|_{L^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \\
&\geq C_3^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left( \psi_j^{(2k+3)*} \cdot g_{z^{(r)}}^{(r)} \right) \right\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}} \\
&= C_3^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left[ \sum_{i \in I^{(r)}} \psi_j^{(2k+3)*} \cdot M_{z_i^{(r)}}(c_i \gamma_i) \right] \right\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}} \\
&\overset{(\dagger)}{=} C_3^{-1} \cdot \left\| \left( \left\| \mathcal{F}^{-1}\left[ \sum_{i \in I^{(r)} \cap I_j} M_{z_i^{(r)}}(c_i \gamma_i) \right] \right\|_{L^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}} \\
&\overset{(\ddagger)}{\geq} \frac{C_1}{2C_3} \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} c_i \right)_{i \in M \cap I^{(r)} \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}} \\
&= \frac{C_1}{2C_3} \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} c_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}}.
\end{aligned}$$

Here, (†) used equation (5.3.26) and (‡) is justified by applying estimate (5.3.24) with $S = M \cap I^{(r)} \cap I_j$. Finally, the last step used equation (5.3.22) as well as $c_i = 0$ for $i \in I_0 \setminus M$.

We now make use of Lemma 5.3.8 and of the embedding $\iota$, to derive

$$\begin{aligned}
&\frac{C_1}{2C_3} \cdot \left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} c_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}} \\
&\leq \left\| g_{z^{(r)}}^{(r)} \right\|_{\mathcal{D}_\Psi\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)} \\
&\leq |||\iota||| \cdot \left\| g_{z^{(r)}}^{(r)} \right\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} \\
&\leq C_4 \, |||\iota||| \cdot \left\| \left( \left\| \mathcal{F}^{-1} \gamma_i \right\|_{L^{p_1}} c_i^{(r)} \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \\
&\leq C_2 C_4 \cdot |||\iota||| \cdot \left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}}, \qquad (5.3.27)
\end{aligned}$$

where the last step is justified by equation (5.3.20) and the constant $C_4 = C_4\left(d, p_1, q_1, N_\mathcal{Q}, R_\mathcal{Q}, C_\mathcal{Q}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}\right)$ is given by Lemma 5.3.8.

Observe that the right-hand side of the above inequality is independent of $r \in \underline{r_0}$. Since $\ell_v^{q_2}$ is a quasi-normed vector space (with triangle constant only depending on $q_2$), there is a constant $C_5 = C_5(q_2, r_0)$ with

$$\left\| (e_j)_{j \in J} \right\|_{\ell_v^{q_2}} \leq C_5 \cdot \sum_{r=1}^{r_0} \left\| (e_j)_{j \in J^{(r)}} \right\|_{\ell_v^{q_2}}$$

for arbitrary sequences $(e_j)_{j \in J}$. By summing the estimate (5.3.27) over $r \in \underline{r_0}$, this finally yields

$$\begin{aligned}
&\left\| \left( \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} c_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \\
&\leq \frac{2C_2 C_3 C_4 C_5 r_0}{C_1} \cdot |||\iota||| \cdot \left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}}.
\end{aligned}$$

By considering $c_i = |\det(T_i)|^{\frac{1}{p_2}-1} d_i$ for an arbitrary finitely supported sequence $(d_i)_{i \in I_0}$, this proves estimate (5.3.11). □

Comparing the necessary conditions established in Theorem 5.3.6 and the sufficient conditions given in Theorems 5.1.6 and 5.1.8 shows the following:

A. The sufficient condition

$$\left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \lesssim \left\| \left( \left\| (c_i)_{i \in I_j} \right\|_{\ell^{p_2}_{w_{\mathcal{Q},p_2}^{\triangle}}} / w_j^{\mathcal{P},p_2} \right)_{j \in J_0} \right\|_{\ell_v^{q_2}} \qquad (5.3.28)$$

given in Theorem 5.1.6 is also necessary (cf. equation (5.3.13)) for the existence of an embedding

$$\left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right),$$

at least in the range $p_2 \in [2,\infty]$, since in this range we have $p_2^{\triangle} = p_2$ as well as $w^{\mathcal{P},p_2} \equiv 1 \equiv w^{\mathcal{Q},p_2}$.

B. The sufficient condition

$$\left\| \left( \left\| (c_i)_{i\in I_j\cap I_0} \right\|_{\ell^{p_2^{\triangledown}}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \lesssim \left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} c_i \right)_{i\in I_0} \right\|_{\ell_u^{q_1}} \tag{5.3.29}$$

given in Theorem 5.1.8 is also necessary (cf. equation (5.3.11)) for the existence of an embedding

$$\left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right),$$

at least in the range $p_2 \in (0,2]$, since in this range we have $p_2^{\triangledown} = p_2$.

In both cases, we note that our sufficient criteria seem to lack sharpness for a certain range of $p_2$. In the next subsection, we will impose more severe restrictions on the relation between the two coverings $\mathcal{Q}, \mathcal{P}$. Under these conditions, we will show that the conditions in equation (5.3.28) and (5.3.29) are always necessary for the existence of the respective embeddings.

But before we do this, we first consider two last theorems which show that sharpness can also be obtained without additional assumptions, at least *for a certain range of exponents* which is slightly larger than it appeared above. The proofs rely heavily on the simplified conditions developed in Section 5.2.

**Theorem 5.3.9.** *Suppose that the assumptions of the second part of Theorem 5.3.6 are satisfied, i.e. that the map*

$$\iota' : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f$$

*is bounded. Furthermore, assume $q_1 > p_2'$ or $q_1 = \infty$ or $p_2 \in [2,\infty]$.*

*Then the conclusion of Theorem 5.3.6 also holds if the "inner" $\ell^{p_2}$-norm is replaced by the $\ell^{p_2^{\triangle}}$-norm in equation* (5.3.13)*, i.e. we have*

$$\left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} c_i \right)_{i\in I_0} \right\|_{\ell_u^{q_1}} \leq C \left\|\left\|\iota'\right\|\right\| \cdot \left\| \left( \left\| (c_i)_{i\in I_j\cap I_0} \right\|_{\ell^{p_2^{\triangle}}} \right)_{j\in J} \right\|_{\ell_v^{q_2}}$$

*for all sequences $(c_i)_{i\in I_0} \in \ell_0(I_0)$ for a constant*

$$C = C\left(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C'\right) > 0,$$

*where $C' > 0$ is as in Theorem 5.3.6.*

*Furthermore, there is a constant*

$$C_0 = C_0\left(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C'\right) > 0$$

*with*

$$\left\| \left( v_j^{-1} \left\| \left( u_i \left| \det\left(T_i\right) \right|^{\frac{1}{p_2} - \frac{1}{p_1}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1 \cdot \left(p_2^{\triangle}/q_1\right)'}} \right)_{j \in J} \right\|_{\ell^{q_1 (q_2/q_1)'}} \leq C_0 \left\| \left| \iota' \right| \right\| . \tag{5.3.30}$$

*Remark.* We remark that we have $p_2' = \infty$ for $p_2 \in (0,1)$, so that the theorem above is only applicable in the case $q_1 = \infty$ for this range of $p_2$. Even in this case, the above embedding for nested sequence spaces is missing the DUALITY DEFECT WEIGHTS $w^{\mathcal{Q},p_2}$ and $w^{\mathcal{P},p_2}$ which appear in the sufficient criterion given in Theorem 5.1.6, cf. equation (5.1.15).

But for $p_2 \in [1,\infty]$, we have $w^{\mathcal{Q},p_2} \equiv 1 \equiv w^{\mathcal{P},p_2}$, so that the above theorem indeed shows sharpness of Theorem 5.1.6, at least for $q_1 > p_2'$ or $q_1 = \infty$.

*Proof.* The idea of the proof is to use the simplification results about embeddings between nested sequence spaces from Section 5.2 to deduce the new estimate from estimate (5.3.13). Thus, we will not actually use continuity of the embedding; instead, we will only use estimate (5.3.13).

As in the assumptions of Lemma 5.2.5, select for each $i \in I_0$ some $j^{(i)} \in J$ with $Q_i \cap P_{j^{(i)}} \neq \varnothing$ and set

$$I^{(j)} := \left\{ i \in I_0 \,\middle|\, j^{(i)} = j \right\} \subset I_j \cap I_0$$

for each $j \in J$. The same lemma yields $C = C\left(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}\right) > 0$ with

$$\begin{aligned} \left\| \left( \left| \det\left(T_i\right) \right|^{\frac{1}{p_2} - \frac{1}{p_1}} c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} &\leq C' \cdot \left\| \left| \iota' \right| \right\| \cdot \left\| \left( \left\| \left(c_i\right)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \\ &\leq CC' \cdot \left\| \left| \iota' \right| \right\| \cdot \left\| \left( \left\| \left(c_i\right)_{i \in I^{(j)}} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \end{aligned}$$

for all sequences $\left(c_i\right)_{i \in I_0} \in \ell_0\left(I_0\right)$. Here, the first estimate is due to equation (5.3.13) from Theorem 5.3.6.

By Lemma 5.2.4, this implies

$$\left\| \left( v_j^{-1} \left\| \left( u_i \left| \det\left(T_i\right) \right|^{\frac{1}{p_2} - \frac{1}{p_1}} \right)_{i \in I^{(j)}} \right\|_{\ell^{q_1 (p_2/q_1)'}} \right)_{j \in J} \right\|_{\ell^{q_1 (q_2/q_1)'}} \leq CC' \cdot \left\| \left| \iota' \right| \right\| .$$

Now, we see how the relation between $q_1$ and $p_2'$ is used: Define $r := q_1 \cdot \left(p_2^{\triangle}/q_1\right)'$ and $s := q_1 \cdot \left(p_2/q_1\right)'$. We want to show $r \geq s$. By assumption, there are only three cases:

*Case* 1. In case of $p_2 \in [2,\infty]$, we have $p_2^{\triangle} = \max\{p_2, p_2'\} = p_2$ and hence $r = s$. Thus, we can additionally assume $p_2 \in (0,2)$ in the remaining two cases, which yields $p_2^{\triangle} = p_2'$.

*Case* 2. In case of $q_1 = \infty$, we have $r = q_1 \cdot (p_2^{\triangle}/q_1)' = \infty \geq s$ by convention for the expression $m \cdot (n/m)'$, cf. Lemma 5.2.2. Thus, we can additionally assume $q_1 < \infty$ in the last case.

*Case* 3. In case of $\infty > q_1 > p_2'$, we have $p_2'/q_1 \in (0,1)$ and hence

$$r = q_1 \cdot \left(p_2'/q_1\right)' = q_1 \cdot \infty = \infty \geq s.$$

We have thus shown $r \geq s$ in all cases, so that we have a norm-decreasing embedding $\ell^s \hookrightarrow \ell^r$ (cf. [33, Proposition 6.11]). Thus, the above estimate yields

$$\begin{aligned}
&\left\| \left( v_j^{-1} \left\| \left( u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'} \\
&\leq \left\| \left( v_j^{-1} \left\| \left( u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I^{(j)}} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'} \\
&\leq CC' \cdot |||\iota'||| \,. \qquad (5.3.31)
\end{aligned}$$

Now, an application of Lemma 5.2.4 shows

$$\begin{aligned}
&\left\| \left( \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} c_i \right)_{i\in I_0} \right\|_{\ell_u^{q_1}} \\
&\leq CC' \cdot |||\iota'||| \cdot \left\| \left( \left\| (c_i)_{i\in I^{(j)}} \right\|_{\ell^{p_2^{\triangle}}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\
&\leq CC' \cdot |||\iota'||| \cdot \left\| \left( \left\| (c_i)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2^{\triangle}}} \right)_{j\in J} \right\|_{\ell_v^{q_2}}
\end{aligned}$$

for all sequences $(c_i)_{i\in I_0} \in \ell_0(I_0)$. Here, we used $I^{(j)} \subset I_0 \cap I_j$ in the last step. This completes the proof of the first part of the theorem.

For the proof of equation (5.3.30), we employ Lemma 5.2.5 and estimate (5.3.31) to obtain a constant $C_1 = C_1(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}})$ with

$$\begin{aligned}
&\left\| \left( v_j^{-1} \left\| \left( u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I_0\cap I_j} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'} \\
&\leq C_1 \left\| \left( v_j^{-1} \left\| \left( u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I^{(j)}} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'} \\
&\leq C_1 CC' \cdot |||\iota'||| \,.
\end{aligned}$$

Because of $r = q_1 \cdot (p_2^{\triangle}/q_1)'$, this proves equation (5.3.30). □

Our final result for this subsection is an analogous theorem for the case of the "reverse" embedding.

**Theorem 5.3.10.** *Suppose that the assumptions of the first part of Theorem 5.3.6 are satisfied, i.e. that the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is bounded. Furthermore, assume $q_1 \leq p_2'$ or $p_2 \in (0, 2]$.*

*Then the conclusion of Theorem 5.3.6 also holds if the "inner" $\ell^{p_2}$-norm is replaced by the $\ell^{p_2^\triangledown}$-norm in equation* (5.3.11)*, i.e., we have*

$$\left\| \left( \left\| (c_i)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2^\triangledown}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \leq C_0 \, |||\iota||| \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}} \cdot c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}}$$

*for all sequences $(c_i)_{i \in I_0} \in \ell_0(I_0)$, with a constant*

$$C_0 = C_0\left(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C\right),$$

*where $C > 0$ is as in Theorem 5.3.6.*

*Furthermore, there is a constant*

$$C_1 = C_1\left(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C\right) > 0$$

*with*

$$\left\| \left( \left\| \left( u_i^{-1} \, |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2^\triangledown \cdot (q_1 / p_2^\triangledown)'}} \right)_{j \in J} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \leq C_1 \, |||\iota|||. \tag{5.3.32}$$

*Proof.* As above, the idea is to use equation (5.3.11) directly, without using boundedness of $\iota$. Let us set $K := C \cdot |||\iota|||$, where $C > 0$ is chosen as in Theorem 5.3.6. As in Lemma 5.2.5, choose for each $i \in I_0$ some $j^{(i)} \in J$ with $Q_i \cap P_{j^{(i)}} \neq \varnothing$ and set

$$I^{(j)} := \left\{ i \in I_0 \,\middle|\, j^{(i)} = j \right\} \subset I_j \cap I_0$$

for each $j \in J$.

Because of $I^{(j)} \subset I_j \cap I_0$, equation (5.3.11) yields

$$\begin{aligned} \left\| \left( \left\| (c_i)_{i \in I^{(j)}} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} &\leq \left\| \left( \left\| (c_i)_{i \in I_j \cap I_0} \right\|_{\ell^{p_2}} \right)_{j \in J} \right\|_{\ell_v^{q_2}} \\ &\leq K \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2} - \frac{1}{p_1}} \, c_i \right)_{i \in I_0} \right\|_{\ell_u^{q_1}} \end{aligned}$$

for arbitrary sequences $(c_i)_{i\in I_0} \in \ell_0(I_0)$. By moving $|\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}}$ to the left-hand side, i.e. by considering $c_i = |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} d_i$ for arbitrary $(d_i)_{i\in I_0} \in \ell_0(I_0)$, we derive

$$\left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} d_i \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \leq K \cdot \left\| (d_i)_{i\in I_0} \right\|_{\ell_u^{q_1}}.$$

Since this holds for every sequence $(d_i)_{i\in I_0} \in \ell_0(I_0)$, Lemma 5.2.2 shows

$$\left\| \left( \left\| \left( u_i^{-1} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2\cdot(q_1/p_2)'}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \leq K.$$

Now, we use our assumptions on the exponents $q_1, p_2$. Let us set $s := p_2^\triangledown \cdot (q_1/p_2^\triangledown)'$ and $t := p_2 \cdot (q_1/p_2)'$. We want to show $s \geq t$. To this end, we distinguish two cases:

*Case* 1. We have $p_2 \in (0,2]$. In this case, $p_2^\triangledown = \min\{p_2, p_2'\} = p_2$ and hence $s = t$.

*Case* 2. We have $p_2 \in (2,\infty]$. Thanks to our assumptions, this implies $q_1 \leq p_2'$. Furthermore, $p_2^\triangledown = p_2'$. In case of $p_2' < \infty$, this yields $q_1/p_2^\triangledown = q_1/p_2' \leq 1$ and hence

$$s = p_2^\triangledown \cdot \left(q_1/p_2^\triangledown\right)' = p_2^\triangledown \cdot \infty = \infty \geq t.$$

In case of $p_2' = \infty$, we get $s = p_2' \cdot (q_1/p_2')' = \infty \geq t$ by convention for the expression $m \cdot (n/m)'$, cf. Lemma 5.2.2.

All in all, we have $s \geq t$ in all cases, so that [33, Proposition 6.11] yields a norm-decreasing embedding $\ell^t \hookrightarrow \ell^s$ and hence

$$\begin{aligned} &\left\| \left( \left\| \left( u_i^{-1} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I^{(j)}} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\ \leq &\left\| \left( \left\| \left( u_i^{-1} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I^{(j)}} \right\|_{\ell^t} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \leq K. \end{aligned} \tag{5.3.33}$$

Another application of Lemma 5.2.2 (marked with (†)) shows that this implies

$$\begin{aligned} &\left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{p_2^\triangledown}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ \leq\ & C_0 \cdot \left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_i \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2^\triangledown}} \right)_{j\in J} \right\|_{\ell_v^{q_2}} \\ \overset{(\dagger)}{\leq}\ & C_0 K \cdot \left\| (c_i)_{i\in I_0} \right\|_{\ell_u^{q_1}} \end{aligned}$$

for arbitrary sequences $(c_i)_{i\in I_0} \in \ell_0(I_0)$. Here, the first estimate is due to Lemma 5.2.5, which yields the constant $C_0 = C_0(p_2, q_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}})$. By moving the factor $|\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}}$ to the right-hand side, this completes the proof of the first part of the theorem.

Finally, equation (5.3.33) – together with Lemma 5.2.5 – yields a constant $C_1 = C_1(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}})$ with

$$\begin{aligned}&\left\| \left( \left\| \left( u_i^{-1} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_0\cap I_j} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\ &\leq C_1 \left\| \left( \left\| \left( u_i^{-1} |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I^{(j)}} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \leq C_1 K.\end{aligned}$$

Because of $s = p_2^{\triangledown}\cdot(q_1/p_2^{\triangledown})'$, this establishes equation (5.3.32). □

### 5.3.2 *The relatively moderate case*

In this subsection, we develop further necessary conditions for embeddings between decomposition spaces. These will show that one has to use the exponents $p_2^{\triangle}$ and $p_2^{\triangledown}$ in general (see e.g. equations (5.3.28) and (5.3.29)) and that the duality defect weights $w^{\mathcal{R},p_1}$ and $w^{\mathcal{S},p_1}$ in equation (5.3.28) can not be omitted. We will give a summary of our collected necessary and sufficient conditions in the next section.

For the proof of our stronger necessary conditions, we impose much stricter conditions on the relation between the two coverings and the respective weights; we will assume that one covering is almost subordinate as well as moderate with respect to the other and that the weight of the "subordinate" covering is also moderate with respect to the "coarse" covering.

The basic idea is to use the BAPU $(\psi_j)_{j\in J}$ of the "coarse" covering $\mathcal{P}$ to construct functions of the form

$$f = \sum_{j\in J} d_j \psi_j^{k*}$$

to "test" the embedding. The moderateness assumptions then imply the conclusion. Despite this simple idea, the proofs will turn out to be quite technical and lengthy, since one has to apply a suitable disjointization to make the construction work.

This method of constructing $f$ is in contrast to the technique used in the proof of Theorem 5.3.6, where we used the family $(\gamma_i)_{i\in I}$, which is adapted to the "fine" covering $\mathcal{Q}$ and not to $\mathcal{P}$, to construct the functions $f$ which are used to "test" the embedding. Finally, we remark that the basic idea for this choice of $f$ is taken from Han and Wang[49], who use a similar construction for the special case of embeddings between $\alpha$-modulation spaces.

As a preparation, we first establish a necessary condition which does *not* assume the covering $\mathcal{Q}$ to be relatively moderate with respect to $\mathcal{P}$. Afterwards, we specialize this to the relatively moderate case. The following lemma will also turn out to be useful in the case of coverings which are not relatively moderate to each other: As we will see in Chapter 6, it implies in many cases that the duality defect index $(p_2)_d$, which is present in the conditions given in Corollary 5.2.9, can not be dropped for $p_2 \in (0,1)$.

**Lemma 5.3.11.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0,\infty]$ and assume that $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ is a tight semi-structured $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = \left(P_j\right)_{j\in J} = (S_j P_j' + c_j)_{j\in J}$ is a semi-structured $L^{p_2}$-decomposition covering of $\mathcal{O}'$. Let $u = (u_i)_{i\in I}$ and $v = \left(v_j\right)_{j\in J}$ be moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.*

*Let $q_1, q_2 \in (0,\infty]$, choose an arbitrary subset $I_0 \subset I$, define*

$$J_0 := \left\{ j \in J \,\middle|\, I_0 \cap I_j \neq \varnothing \right\}$$

*and assume that $\mathcal{Q}_{I_0} := (Q_i)_{i\in I_0}$ is almost subordinate to $\mathcal{P}$.*

*Let $k := k\left(\mathcal{Q}_{I_0}, \mathcal{P}\right)$, define*

$$K := \bigcup_{j\in J_0} \overline{P_j^{(2k+3)*}},$$

*and assume that there is a bounded linear map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right)}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$$

*which satisfies $\langle \iota f,\, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in C_c^\infty\left(\mathcal{O}\cap\mathcal{O}'\right)$.*

*Then there is a constant $C > 0$, depending only on*

$$\begin{aligned}
&d, p_1, p_2, q_1, q_2, k,\\
&\varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}},\\
&r_0\left(\mathcal{P}, 2k+3\right), N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},
\end{aligned}$$

*which satisfies*

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\frac{1}{p_2}-1}}{v_j} \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}} \right)_{j\in J_0} \right\|_{\ell^{q_1\cdot(q_2/q_1)'}} \leq C\, |||\iota|||\,.$$

*Here, the $L^{p_2}$-BAPU $\Psi = \left(\psi_j\right)_{j\in J}$ has to be used to calculate the norm $|||\iota|||$.*

*Proof.* Choose $(\gamma_i)_{i\in I}$ as in Lemma 5.3.2 and let $\Psi = \left(\psi_j\right)_{j\in J}$ be an $L^{p_2}$-BAPU for $\mathcal{P}$. Let $r_0 := r_0\left(\mathcal{P}, 2k+3\right)$, so that Lemma 3.3.5 yields a partition $J = \biguplus_{r=1}^{r_0} J^{(r)}$ for which $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \varnothing$ holds for all $r \in \underline{r_0}$ and all $j, \ell \in J^{(r)}$ with $j \neq \ell$.

We first note that the assumption on $\iota$ implies

$$\gamma_i \cdot \iota f = \gamma_i \cdot f \text{ for all } f \in \mathcal{D}_K \text{ and } i \in I_0, \tag{5.3.34}$$

where the equality has to be understood in the sense of tempered distributions. To see this, let $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ be arbitrary. By definition, we have

$$\langle \gamma_i \cdot \iota f, g \rangle = \langle \iota f, \gamma_i g \rangle \overset{(\dagger)}{=} \langle f, \gamma_i g \rangle = \langle \gamma_i \cdot f, g \rangle,$$

where we used at $(\dagger)$ that we have $\operatorname{supp} \gamma_i \subset Q_i \subset P_{j_i}^{k*} \subset \mathcal{O}'$ for some $j_i \in J$, as well as $\operatorname{supp} \gamma_i \subset Q_i \subset \mathcal{O}$. Altogether, we conclude $\gamma_i g \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, so that

$$\langle \iota f, \gamma_i g \rangle = \langle f, \gamma_i g \rangle$$

holds by assumption on $\iota$.

For $r \in \underline{r_0}$, the family $\left(I_0 \cap I_j\right)_{j \in J^{(r)}}$ is pairwise disjoint, because for $j, \ell \in J^{(r)}$ with $i \in I_0 \cap I_j \cap I_\ell$, Lemma 3.3.2 would imply

*As usual, we have $Q_i \neq \varnothing$, since $\mathcal{Q}$ is an admissible covering. Alternatively, $\mathcal{Q}$ is tight, which necessitates $Q_i \neq \varnothing$ for all $i \in I$.*

$$\varnothing \neq Q_i \subset P_j^{(2k+2)*} \cap P_\ell^{(2k+2)*} \subset P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*}$$

which in turn yields $j = \ell$ by choice of $(J^{(r)})_r$. Thus, we can define

$$I^{(r)} := \biguplus_{j \in J^{(r)}} I_0 \cap I_j.$$

Note that this definition yields

$$I^{(r)} \cap I_j = I_0 \cap I_j \qquad \forall j \in J^{(r)}.$$

As a next step, we note for $i \in I^{(r)}$ and $j \in J^{(r)}$ that

$$\gamma_i \psi_j^{(2k+3)*} = \begin{cases} \gamma_i, & \text{if } i \in I_0 \cap I_j, \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.35}$$

To see this, first assume $i \in I_0 \cap I_j$. Lemma 3.3.2 yields $Q_i \subset P_j^{(2k+2)*}$ and Lemma 3.2.12 implies $\psi_j^{(2k+3)*} \equiv 1$ on $P_j^{(2k+2)*}$. Since $\gamma_i$ vanishes outside of $Q_i$, this establishes the first case. For the case $i \notin I_0 \cap I_j$, observe that we have $i \in I^{(r)}$ and thus $i \in I_0 \cap I_\ell$ for some $\ell \in J^{(r)}$. As we just saw, this implies $\gamma_i = \gamma_i \psi_\ell^{(2k+3)*}$. Now $i \notin I_0 \cap I_j$ yields $j \neq \ell$ and thus $P_j^{(2k+3)*} \cap P_\ell^{(2k+3)*} = \varnothing$ because of $j, \ell \in J^{(r)}$. This leads to

$$\gamma_i \psi_j^{(2k+3)*} = \gamma_i \psi_\ell^{(2k+3)*} \psi_j^{(2k+3)*} \equiv 0,$$

so that equation (5.3.35) is established.

Now, let $d = \left(d_j\right)_{j\in J_0} \in \ell_0\left(J_0\right)$ be arbitrary and consider $d$ as an element of $\ell_0\left(J\right)$ by extending it trivially. For $r \in \underline{r_0}$, define

$$g^{(r)} := \sum_{j\in J_0\cap J^{(r)}} d_j \psi_j^{(2k+3)*}$$

and note that we have $g^{(r)} \in \mathcal{D}_K$ as a *finite* sum of elements of $\mathcal{D}_K$.

For $j \in J_0 \cap J^{(r)}$ and $i \in I_0 \cap I_j \subset I^{(r)}$, equations (5.3.35) and (5.3.34), together with the pairwise disjointness of the family $\left(I_0 \cap I_\ell\right)_{\ell\in J^{(r)}}$, yield

$$\begin{aligned} u_i \cdot \left\|\mathcal{F}^{-1}\left(\gamma_i \cdot \iota g^{(r)}\right)\right\|_{L^{p_1}} &= u_i \cdot d_j \cdot \left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^{p_1}} \\ &= C_1 \cdot u_i d_j \cdot \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}}, \end{aligned} \tag{5.3.36}$$

where the constant $C_1 = C_1\left(d, p_1, \varepsilon_{\mathcal{Q}}\right)$ is taken from Lemma 5.3.2.

Note that the identity

$$\left\|\mathcal{F}^{-1}\gamma_i\right\|_{L^p} = C^{(p)} \cdot \left|\det\left(T_i\right)\right|^{1-\frac{1}{p}},$$

which is valid for all $i \in I$ and $p \in (0,\infty]$, and the inclusion $\operatorname{supp}\gamma_i \subset Q_i$ show that $\Gamma = \left(\gamma_i\right)_{i\in I}$ is an $L^p$-bounded system for $\mathcal{Q}$ for all $p \in (0,\infty]$, with $C_{\mathcal{Q},\Gamma,p} = C\left(d, p, \varepsilon_{\mathcal{Q}}\right)$. By Theorem 3.4.10, together with the estimate for the norm of the clustering map from Lemma 3.4.2, this yields a positive constant

$$C_2 = C_2\left(d, p_1, q_1, \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, C_{u,\mathcal{Q}}\right)$$

with

$$\begin{aligned} &\left\|\iota g^{(r)}\right\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)} \\ &\geq C_2^{-1} \cdot \left\|\left(\left\|\mathcal{F}^{-1}\left(\gamma_i \cdot \iota g^{(r)}\right)\right\|_{L^{p_1}}\right)_{i\in I}\right\|_{\ell_u^{q_1}} \\ &\geq C_2^{-1} \cdot \left\|\left(u_i \cdot \left\|\mathcal{F}^{-1}\left(\gamma_i \cdot \iota g^{(r)}\right)\right\|_{L^{p_1}}\right)_{i\in I^{(r)}}\right\|_{\ell^{q_1}} \\ &\overset{(\dagger)}{=} C_2^{-1} \cdot \left\|\left(\left\|\left(u_i \cdot \left\|\mathcal{F}^{-1}\left(\gamma_i \cdot \iota g^{(r)}\right)\right\|_{L^{p_1}}\right)_{i\in I_0\cap I_j}\right\|_{\ell^{q_1}}\right)_{j\in J^{(r)}}\right\|_{\ell^{q_1}} \\ &\geq C_2^{-1} \cdot \left\|\left(\left\|\left(u_i \cdot \left\|\mathcal{F}^{-1}\left(\gamma_i \cdot \iota g^{(r)}\right)\right\|_{L^{p_1}}\right)_{i\in I_0\cap I_j}\right\|_{\ell^{q_1}}\right)_{j\in J_0\cap J^{(r)}}\right\|_{\ell^{q_1}} \\ &\overset{(*)}{\geq} \frac{C_1}{C_2} \cdot \left\|\left(d_j \cdot \left\|\left(u_i \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}}\right)_{i\in I_0\cap I_j}\right\|_{\ell^{q_1}}\right)_{j\in J_0\cap J^{(r)}}\right\|_{\ell^{q_1}} \\ &= \frac{C_1}{C_2} \cdot \left\|\left(d_j \cdot \left\|\left(u_i \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}}\right)_{i\in I_0\cap I_j}\right\|_{\ell^{q_1}}\right)_{j\in J^{(r)}}\right\|_{\ell^{q_1}} \end{aligned} \tag{5.3.37}$$

Here, the step marked with (†) is justified by $I^{(r)} = \biguplus_{j \in J^{(r)}} I_0 \cap I_j$, which implies

$$\left\| (c_i)_{i \in I^{(r)}} \right\|_{\ell^s} = \left\| \left( \left\| (c_i)_{i \in I_0 \cap I_j} \right\|_{\ell^s} \right)_{j \in J^{(r)}} \right\|_{\ell^s}$$

for arbitrary sequences $(c_i)_{i \in I^{(r)}}$ and any $s \in (0, \infty]$. Furthermore, the step marked with $(*)$ is a consequence of equation (5.3.36). Finally, the last step holds because of $d_j = 0$ for $j \in J \setminus J_0$.

Since the $\ell^{q_1}$-norm is a quasi-norm (with triangle constant only depending on $q_1$), there is a constant $C_3 = C_3(q_1, r_0) > 0$ with

$$\left\| (d_j)_{j \in J} \right\|_{\ell^{q_1}} \leq C_3 \sum_{r=1}^{r_0} \left\| (d_j)_{j \in J^{(r)}} \right\|_{\ell^{q_1}}$$

for arbitrary sequences $(d_j)_{j \in J}$. Using estimate (5.3.37), this implies

$$\left\| \left( d_j \left\| \left( u_i \left| \det(T_i) \right|^{1 - \frac{1}{p_1}} \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J} \right\|_{\ell^{q_1}} \leq C_4 \sum_{r=1}^{r_0} \| \iota g^{(r)} \|_{\mathcal{D}(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u)} \tag{5.3.38}$$

for $C_4 := C_1 C_3 / C_2$.

Equation (5.3.15) of Lemma 5.3.8 (applied to the covering $\mathcal{P}$ instead of $\mathcal{Q}$, with $z_j = 0$ for all $j \in J$ and with $\gamma_j = \psi_j^{(2k+3)*}$) and equation (5.3.7) of Lemma 5.3.2 (applied to $\psi_j^{(2k+3)*}$) guarantee the existence of constants

$$C_5 = C_5\left(k, d, p_2, q_2, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right) > 0$$

and $C_6 = C_6\left(d, p_2, k, N_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right)$ with

$$\begin{aligned}
\| \iota g^{(r)} \|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)} &\leq |||\iota||| \cdot \| g^{(r)} \|_{\mathcal{D}_\Psi\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right)} \\
&\leq C_5 \, |||\iota||| \cdot \left\| \left( d_j \left\| \mathcal{F}^{-1} \psi_j^{(2k+3)*} \right\|_{L^{p_2}} \chi_{J_0 \cap J^{(r)}}(j) \right)_{j \in J} \right\|_{\ell^{q_2}_v} \\
&\leq C_5 C_6 \, |||\iota||| \cdot \left\| \left( \left| \det(S_j) \right|^{1 - \frac{1}{p_2}} d_j \right)_{j \in J} \right\|_{\ell^{q_2}_v} < \infty.
\end{aligned}$$

Combining this with equation (5.3.38) and setting $C_7 := C_4 C_5 C_6 r_0$ leads – for arbitrary sequences $(d_j)_{j \in J_0} \in \ell_0(J_0)$ – to

$$\begin{aligned}
&\left\| \left( d_j \cdot \left\| \left( \left| \det(T_i) \right|^{1 - \frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J} \right\|_{\ell^{q_1}} \\
&\leq C_7 \, |||\iota||| \left\| \left( \left| \det(S_j) \right|^{1 - \frac{1}{p_2}} d_j \right)_{j \in J_0} \right\|_{\ell^{q_2}_v} \\
&= C_7 \, |||\iota||| \left\| \left( \left| \det(S_j) \right|^{1 - \frac{1}{p_2}} v_j d_j \right)_{j \in J_0} \right\|_{\ell^{q_2}} < \infty
\end{aligned}$$

Here, we used that $d_j = 0$ holds for $j \in J \setminus J_0$.

In particular, the estimate above yields

$$\left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} < \infty$$

for all $j \in J_0$. Furthermore, by considering $d_j = \left|\det(S_j)\right|^{\frac{1}{p_2}-1} c_j / v_j$ for an arbitrary sequence $(c_j)_{j \in J_0} \in \ell_0(J_0)$, the above estimate also yields

$$\left\| (w_j c_j)_{j \in J_0} \right\|_{\ell^{q_1}} \leq C_7 \, |||\iota||| \cdot \left\| (c_j)_{j \in J_0} \right\|_{\ell^{q_2}}$$

with

$$w_j = \frac{\left|\det(S_j)\right|^{\frac{1}{p_2}-1}}{v_j} \cdot \left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}}$$

for $j \in J_0$.

Note that we have $I_0 \cap I_j \neq \varnothing$ and hence $w_j > 0$ for all $j \in J_0$. By Lemma 5.2.2 (with the choices $I = J = J_0$, $I^{(j)} = \{j\}$ for all $j \in J$ and $q = q_1$, $p = q_2$ and with weights $v = w$ and $\alpha \equiv u \equiv 1$), this yields

$$\left\| \left( \frac{\left|\det(S_j)\right|^{\frac{1}{p_2}-1}}{v_j} \left\| \left( |\det(T_i)|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1 \cdot (q_2/q_1)'}} \leq C_7 \, |||\iota|||,$$

which completes the proof. □

Now, we specialize the above lemma to the case in which $\mathcal{Q}$ is relatively moderate with respect to $\mathcal{P}$. It is worth noting that the imposed requirements are rather strict. In particular, for $I_0 = I$, they can never be fulfilled in case of $\mathcal{O}' \cap \partial\mathcal{O} \neq \varnothing$, see Lemma 3.3.6.

**Theorem 5.3.12.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q_i' + b_i)_{i \in I}$ is a tight semi-structured $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = (P_j)_{j \in J} = (S_j P_j' + c_j)_{j \in J}$ is a* tight *semi-structured $L^{p_2}$-decomposition covering of $\mathcal{O}'$. Let $u = (u_i)_{i \in I}$ and $v = (v_j)_{j \in J}$ be moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.*

*Let $q_1, q_2 \in (0, \infty]$. Choose an arbitrary subset $I_0 \subset I$ and define*

$$J_0 := \left\{ j \in J \,\middle|\, I_0 \cap I_j \neq \varnothing \right\}.$$

*Assume that*

1. *$\mathcal{Q}_{I_0} := (Q_i)_{i \in I_0}$ is almost subordinate to $\mathcal{P}$.*
2. *$\mathcal{Q}_{I_0}$ is moderate with respect to $\mathcal{P}$.*
3. *The weight $u|_{I_0}$ is relatively $\mathcal{P}$-moderate.*
4. *There is some $\ell \in \mathbb{N}_0$ such that we have $P_j \subset \bigcup_{i \in I_0 \cap I_j} Q_i^{\ell *}$ (up to a nullset) for all $j \in J_0$.*

*Spelled out, assumption 3 means that there is $L = C_{u|_{I_0}, \mathcal{Q}, \mathcal{P}} > 0$ with $u_i \leq L \cdot u_\ell$ for all $i, \ell \in I_0 \cap I_j$ for all $j \in J$.*

*Let $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$, define*

$$K := \bigcup_{j \in J_0} \overline{P_j^{(2k+3)*}},$$

*and assume that the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}(\mathcal{P}, L^{p_2}, \ell_v^{q_2})}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f$$

*is well-defined and bounded. Let $s := \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+$.*

*Then there is a finite constant $C > 0$ which is only dependent on $d$ and on*

$$\begin{aligned}
&k(\mathcal{Q}_{I_0}, \mathcal{P}), C_{\mathrm{mod}}(\mathcal{Q}_{I_0}, \mathcal{P}), C_{u|_{I_0}, \mathcal{Q}, \mathcal{P}},\\
&\ell, r_0(\mathcal{P}, 2k+3), r_0(\mathcal{Q}, 1),\\
&p_1, q_1, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1},\\
&p_2, q_2, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}, \varepsilon_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}
\end{aligned}$$

*and which satisfies*

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det(S_j)\right|^s \left|\det(T_{i_j})\right|^{\frac{1}{p_2} - \frac{1}{p_1} - s} \right)_{j \in J_0} \right\|_{\ell^{q_1}((q_2/q_1)')} \leq C \, |||\iota|||, \quad (5.3.39)$$

*where for each $j \in J_0$, an arbitrary $i_j \in I_0 \cap I_j$ is chosen.*

*Here, the $L^{p_1}/L^{p_2}$-BAPUs $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ have to be used to calculate the norm $|||\iota|||$.*

*Remark.* As in the remark after Lemma 3.3.7, we observe that assumption 4 is automatically satisfied (with $\ell = 0$) if we have $\mathcal{O} = \mathcal{O}'$ and $I_0 = I$.

*Proof.* Lemma 5.3.11 yields a constant $C_1 > 0$ which depends only on quantities mentioned in the current theorem and which satisfies

$$\left\| \left( \frac{\left|\det(S_j)\right|^{\frac{1}{p_2} - 1}}{v_j} \left\| \left( \left|\det(T_i)\right|^{1 - \frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1 \cdot (q_2/q_1)'}} \leq C_1 \, |||\iota|||. \quad (5.3.40)$$

Set $C_2 := C_{\mathrm{mod}}(\mathcal{Q}_{I_0}, \mathcal{P})$ and $L := C_{u|_{I_0}, \mathcal{Q}, \mathcal{P}} > 0$, so that $u_i \leq L \cdot u_\ell$ holds for all $i, \ell \in I_0 \cap I_j$ for all $j \in J$. For each $j \in J_0$, fix some $i_j \in I_0 \cap I_j$. Note that $I_0 \cap I_j \neq \varnothing$ is guaranteed by definition of $J_0$. By choice of $L$, this implies $u_i \geq L^{-1} u_{i_j}$ for all $i \in I_0 \cap I_j$.

By definition of $C_2$, we have

$$C_2^{-1} \left|\det(T_{i_j})\right| \leq \left|\det(T_i)\right| \leq C_2 \left|\det(T_{i_j})\right| \quad (5.3.41)$$

for all $i \in I_0 \cap I_j$, so that there is a constant $C_3 = C_3(C_2, p_1) > 0$ with

$$C_3^{-1} \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}} \leq \left|\det(T_i)\right|^{1-\frac{1}{p_1}} \leq C_3 \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}} \tag{5.3.42}$$

for all $i \in I_0 \cap I_j$ and $j \in J_0$.

Finally note that the assumptions of the theorem include the prerequisites for Lemma 3.3.7. Thus, there is a constant

$$C_4 = C_4\left(d, \varepsilon_{\mathcal{P}}, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right), R_{\mathcal{Q}}, C_{\mathcal{Q}}, \ell\right) > 0$$

with

$$\left|\det(S_j)\right|^{1/q_1} \left|\det(T_{i_j})\right|^{-1/q_1} \leq C_4^{1/q_1} \cdot \left|I_0 \cap I_j\right|^{1/q_1} \tag{5.3.43}$$

for all $j \in J_0$. Together with equation (5.3.42), this implies

$$\begin{aligned}
&\left\| \left( \left|\det(T_i)\right|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \\
&\geq C_3^{-1} L^{-1} \cdot \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}} u_{i_j} \cdot \left\| (1)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \\
&= C_3^{-1} L^{-1} \cdot \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}} u_{i_j} \cdot \left|I_0 \cap I_j\right|^{\frac{1}{q_1}} \\
&\geq \left( C_3 C_4^{1/q_1} L \right)^{-1} \cdot \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}-\frac{1}{q_1}} \left|\det(S_j)\right|^{\frac{1}{q_1}} \cdot u_{i_j}
\end{aligned}$$

for all $j \in J_0$.

Let us set $C_5 := C_1 C_3 C_4^{\frac{1}{q_1}} L$. By combining the estimate above with equation (5.3.40), we conclude

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det(S_j)\right|^{\frac{1}{q_1}+\frac{1}{p_2}-1} \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}-\frac{1}{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1 \cdot (q_2/q_1)'}} \leq C_5 \left|\!\left|\!\left| \iota \right|\!\right|\!\right| . \tag{5.3.44}$$

Finally, we deduce the claim of the theorem. To this end, we distinguish three cases:

CASE 1: $p_2 \in (0,1)$. In this case, we have

$$\frac{1}{p_2^{\pm\triangle}} = \min\left\{ \frac{1}{p_2}, 1 - \frac{1}{p_2} \right\} = 1 - \frac{1}{p_2} < 0$$

and hence

$$s = \left( \frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}} \right)_+ = \frac{1}{q_1} + \frac{1}{p_2} - 1.$$

Thus,

$$\left|\det(S_j)\right|^s \left|\det(T_{i_j})\right|^{\frac{1}{p_2}-\frac{1}{p_1}-s} == \left|\det(S_j)\right|^{\frac{1}{q_1}+\frac{1}{p_2}-1} \cdot \left|\det(T_{i_j})\right|^{1-\frac{1}{p_1}-\frac{1}{q_1}} .$$

Hence, the desired estimate (5.3.39) is a direct consequence of equation (5.3.44).

CASE 2: $p_2 \in [1,2]$ and $q_1 \leq p_2'$. Here, we have

$$\frac{1}{p_2^{\pm\triangle}} = \min\left\{\frac{1}{p_2}, 1-\frac{1}{p_2}\right\} = 1-\frac{1}{p_2} = \frac{1}{p_2'}$$

and hence $\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}} = \frac{1}{q_1} - \frac{1}{p_2'} \geq 0$, which yields

$$\begin{aligned} s &= \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+ \\ &= \frac{1}{q_1} - \frac{1}{p_2'} \\ &= \frac{1}{q_1} + \frac{1}{p_2} - 1. \end{aligned}$$

Exactly as in the first case, we see that the desired estimate is a direct consequence of equation (5.3.44).

CASE 3: We have $p_2 \in [1,2]$ and $q_1 > p_2'$ or we have $p_2 \in [2,\infty]$. Here, we do not use equation (5.3.44). Instead, we first note that for arbitrary $i \in I_0$, we have $\varnothing \neq Q_i \subset P_{j_i}^{k*}$ for some $j_i \in J$. Hence, $Q_i \cap P_j \neq \varnothing$ for some $j \in J$, which implies $j \in J_0$ and

$$Q_i \subset P_j^{(2k+2)*} \subset \overline{P_j^{(2k+3)*}},$$

cf. Lemma 3.3.2. Thus,

$$K = \bigcup_{j\in J_0} \overline{P_j^{(2k+3)*}} \supset \bigcup_{i\in I_0} Q_i.$$

Together with the conditions of the current case, this shows that the prerequisites of Theorem 5.3.9 are satisfied. Thus, there is a constant

$$C_6 = C_6\left(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C'\right),$$

where $C' > 0$ is the constant from Theorem 5.3.6, with

$$\left\| \left( v_j^{-1} \left\| \left( u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}\left(p_2^{\triangle}/q_1\right)'} \right)_{j\in J} \right\|_{\ell^{q_1}(q_2/q_1)'} \leq C_6 \left|\!\left|\!\left| \iota \right|\!\right|\!\right|.$$

Now, equation (5.3.41) yields a constant $C_7 = C_7(C_2, p_1, p_2) > 0$ with

$$u_{i_j} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \leq LC_7 \cdot u_i \left|\det\left(T_i\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}}$$

for all $j \in J_0$ and all $i \in I_0 \cap I_j$. Together with the above estimate, this yields

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}} \left|I_0 \cap I_j\right|^{\frac{1}{q_1\cdot\left(p_2^{\triangle}/q_1\right)'}} \right)_{j\in J_0} \right\|_{\ell^{q_1}(q_2/q_1)'} \leq LC_6C_7 \left|\!\left|\!\left| \iota \right|\!\right|\!\right|.$$

But Remark 5.2.3 shows $\frac{1}{q_1 \cdot \left(p_2^{\triangle}/q_1\right)'} = \left(\frac{1}{q_1} - \frac{1}{p_2^{\triangle}}\right)_+ = s$, because of $p_2 \in [1,\infty]$, which entails $p_2^{\pm\triangle} = p_2^{\triangle}$. Furthermore, as in equation (5.3.43) above, we get

$$\left|\det\left(S_j\right)\right|^s \cdot \left|\det\left(T_{i_j}\right)\right|^{-s} \leq C_4^s \cdot \left|I_0 \cap I_j\right|^s .$$

Altogether, we finally arrive at

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_2}-\frac{1}{p_1}-s} \left|\det\left(S_j\right)\right|^s \right)_{j\in J_0} \right\|_{\ell^{q_1}\left(q_2/q_1\right)'} \leq L C_4^s C_6 C_7 \left|\!\left|\!\left|\iota\right|\!\right|\!\right| ,$$

which easily yields the desired inequality (5.3.39). □

With similar techniques, we will now prove the analogous result for the case of the "reverse" embedding. As above, we first establish an auxiliary result for which we do not assume that $\mathcal{Q}$ is relatively moderate with respect to $\mathcal{P}$.

**Lemma 5.3.13.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0,\infty]$ and assume that $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ is a tight semi-structured $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = \left(P_j\right)_{j\in J} = (S_j P_j' + c_j)_{j\in J}$ is a* tight *semi-structured $L^{p_2}$-decomposition covering of $\mathcal{O}'$. Let $u = (u_i)_{i\in I}$ and $v = \left(v_j\right)_{j\in J}$ be moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.*

*Let $q_1, q_2 \in (0,\infty]$, choose an arbitrary subset $J_0 \subset J$, define*

$$I_0 := \{i \in I \mid J_i \cap J_0 \neq \varnothing\}$$

*and assume that $\mathcal{Q}_{I_0} := (Q_i)_{i\in I}$ is almost subordinate to $\mathcal{P}$ and that $P_j \subset \mathcal{O}$ holds for all $j \in J_0$. Finally, assume*

$$\left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}} < \infty \tag{5.3.45}$$

*for all $j \in J_0$.*

*Furthermore, set*

$$K := \bigcup_{i\in I_0} Q_i$$

*and assume that the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is well-defined and bounded. Then, there is a finite constant $C > 0$ which satisfies*

$$\left\| \left( \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}} \Big/ \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}} \right)_{j\in J_0} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \leq C \left|\!\left|\!\left|\iota\right|\!\right|\!\right| .$$

*Note that each $j \in J_0$ satisfies $\varnothing \neq P_j \subset \mathcal{O}$ by assumption. Hence, there is some $i \in I$ with $Q_i \cap P_j \neq \varnothing$, which yields $i \in I_0 \cap I_j \neq \varnothing$. In particular, the quantity in equation (5.3.45) is always positive.*

*Here, the $L^{p_1}/L^{p_2}$ BAPUs $\Phi = (\varphi_i)_{i\in I}$ and $\Psi = (\psi_j)_{j\in J}$ have to be used to calculate the norm $|||\iota|||$.*

*Finally, $C$ is only dependent on the quantities*

$$\begin{aligned}&d, p_1, p_2, q_2, k(\mathcal{Q}_{I_0}, \mathcal{P})\\ &N_{\mathcal{Q}}, C_{\mathcal{Q}}, R_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1},\\ &r_0(\mathcal{P}, 1), \varepsilon_{\mathcal{P}}, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}.\end{aligned}$$

*Proof.* Define $k := k(\mathcal{Q}_{I_0}, \mathcal{P})$, let $\Phi = (\varphi_i)_{i\in I}$ be an $L^{p_1}$-BAPU for $\mathcal{Q}$ and let $\Psi = (\psi_j)_{j\in J}$ be an $L^{p_2}$-BAPU for $\mathcal{P}$.

Furthermore, choose $(\gamma_j)_{j\in J}$ as in Lemma 5.3.2 (applied to the tight(!) covering $\mathcal{P}$ instead of $\mathcal{Q}$), so that

$$\left\|\mathcal{F}^{-1}\gamma_j\right\|_{L^p} = C_1^{(p)} \cdot \left|\det(S_j)\right|^{1-\frac{1}{p}} \tag{5.3.46}$$

holds for all $j \in J$, $p \in (0,\infty]$ and a constant $C_1^{(p)} = C_1^{(p)}(d, \varepsilon_{\mathcal{P}}) > 0$.

We start with a few useful observations: We first note

$$K' := \bigcup_{j\in J_0} P_j \subset K. \tag{5.3.47}$$

To see this, let $x \in K'$ be arbitrary. Hence, $x \in P_j$ for some $j \in J_0$. By assumption, we have $P_j \subset \mathcal{O}$ for all $j \in J_0$ and hence $K' \subset \mathcal{O}$, which yields $x \in Q_i$ for some $i \in I$. This shows $x \in Q_i \cap P_j \neq \varnothing$ and hence $j \in J_i \cap J_0 \neq \varnothing$, which implies $i \in I_0$ and thus finally $x \in Q_i \subset \bigcup_{\ell\in I_0} Q_\ell = K$.

Furthermore, we have

$$I_j \subset I_0 \text{ for all } j \in J_0, \tag{5.3.48}$$

because for arbitrary $i \in I_j$ we have $j \in J_i \cap J_0 \neq \varnothing$ and hence $i \in I_0$.

Finally, we note the estimate

$$|J_i| \leq N_{\mathcal{P}}^{k+1} \text{ for all } i \in I_0. \tag{5.3.49}$$

To see that this is true, note that $j \in J_i$ entails $\varnothing \neq Q_i \cap P_j \subset P_{j_i}^{k*} \cap P_j$ for some $j_i \in J$ and thus $j \in j_i^{(k+1)*}$, which means $J_i \subset j_i^{(k+1)*}$. Using Lemma 3.2.18, this easily implies that equation (5.3.49) is true.

Now, set $r_0 := r_0(\mathcal{P}, 1)$ so that Lemma 3.3.5 ensures existence of a partition $J = \biguplus_{r=1}^{r_0} J^{(r)}$ with $P_j^* \cap P_m^* = \varnothing$ for all $j, m \in J^{(r)}$ with $j \neq m$ and arbitrary $r \in \underline{r_0}$.

Let $d = (d_j)_{j\in J_0} \in \ell_0(J_0)$ be arbitrary. Fix $r \in \underline{r_0}$ and define

$$g^{(r)} := \sum_{j\in J_0\cap J^{(r)}} d_j\gamma_j.$$

Observe that compactness of $\operatorname{supp}\gamma_j \subset P_j \subset K'$, together with the inclusion (5.3.47), yields $g^{(r)} \in \mathcal{D}_{K'} \subset \mathcal{D}_K$ as a *finite* sum of elements of $\mathcal{D}_{K'}$.

For $j \in J_0 \cap J^{(r)}$ and $m \in J^{(r)}$ with $\gamma_j \psi_m^* \not\equiv 0$, we conclude

$$\varnothing \neq P_j \cap P_m^* \subset P_j^* \cap P_m^*$$

and thus $m = j$ by choice of $J^{(r)}$. But Lemma 3.2.12 shows $\psi_m^* \equiv 1$ on $P_m = P_j \supset \operatorname{supp}\gamma_j$ and thus $\gamma_j \psi_m^* = \gamma_j = \gamma_m$. In summary,

$$\psi_m^* g^{(r)} = d_m \cdot \gamma_m \qquad \forall m \in J^{(r)} \cap J_0. \tag{5.3.50}$$

Remark 3.4.9 shows that $\Gamma = (\psi_m^*)_{m\in J}$ is an $L^{p_2}$-bounded control system for $\mathcal{P}$ with $C_{\mathcal{P},\Gamma,p_2} \leq C_2\left(p_2, N_\mathcal{P}, D_\mathcal{P}, C_{\mathcal{P},\Psi,p_2}\right)$. In combination with Theorem 3.4.10 and with the estimate for the norm of the clustering map given in Lemma 3.4.2, we get

$$\begin{aligned}
C_3 \left\|g^{(r)}\right\|_{\mathcal{D}_\Psi\left(\mathcal{P},L^{p_2},\ell_v^{q_2}\right)} &\geq \left\| \left( \left\| \mathcal{F}^{-1}\left(\psi_m^* g^{(r)}\right)\right\|_{L^{p_2}} \right)_{m\in J} \right\|_{\ell_v^{q_2}} \\
&\geq \left\| \left( \left\| \mathcal{F}^{-1}\left(\psi_m^* g^{(r)}\right)\right\|_{L^{p_2}} \right)_{m\in J^{(r)}\cap J_0} \right\|_{\ell_v^{q_2}} \\
&\overset{(\dagger)}{=} \left\| \left( \left\| \mathcal{F}^{-1}\left(d_m \gamma_m\right)\right\|_{L^{p_2}} \right)_{m\in J^{(r)}\cap J_0} \right\|_{\ell_v^{q_2}} \\
&= C_1^{(p_2)} \left\| \left( |\det(S_m)|^{1-\frac{1}{p_2}} d_m \right)_{m\in J^{(r)}\cap J_0} \right\|_{\ell_v^{q_2}}
\end{aligned} \tag{5.3.51}$$

for some constant

$$C_3 = C_3\left(d, p_2, q_2, N_\mathcal{P}, R_\mathcal{P}, C_\mathcal{P}, D_\mathcal{P}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2}\right) > 0.$$

Here, the step marked with $(\dagger)$ in equation (5.3.51) is justified by identity (5.3.50).

As the next step, we want to estimate $\left\|g^{(r)}\right\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)}$. To this end, consider any $i \in I$ with

$$0 \not\equiv \varphi_i \cdot g^{(r)} = \sum_{j\in J_0\cap J^{(r)}} \left[d_j \cdot \varphi_i \gamma_j\right].$$

Then, there is some $j \in J_0 \cap J^{(r)}$ satisfying $0 \not\equiv \varphi_i\gamma_j$. But for *any* such $j \in J_0 \cap J^{(r)}$, we get $Q_i \cap P_j \neq \varnothing$, which leads to $i \in I_j \subset I_0$ (cf. equation (5.3.48)), or equivalently $i \in I_0$ and $j \in J_i$. Hence,

$$\varphi_i g^{(r)} = \sum_{j\in J_0\cap J^{(r)}\cap J_i} \left[d_j \cdot \varphi_i\gamma_j\right] \tag{5.3.52}$$

and $i \in I_0$.

Now, we distinguish the cases $p_1 \in [1,\infty]$ and $p_1 \in (0,1)$. For $p_1 \in [1,\infty]$, we use Young's inequality ($L^1 * L^{p_1} \hookrightarrow L^{p_1}$), together with the triangle inequality for $\|\cdot\|_{L^{p_1}}$, to derive

$$\begin{aligned} u_i \cdot \left\| \mathcal{F}^{-1}\left(\varphi_i g^{(r)}\right)\right\|_{L^{p_1}} &\overset{(\dagger)}{=} u_i \cdot \left\| \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \mathcal{F}^{-1}\left(d_j \cdot \varphi_i \gamma_j\right)\right\|_{L^{p_1}} \\ &\leq u_i \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \left[ \left|d_j\right| \cdot \left\|\mathcal{F}^{-1}\gamma_j\right\|_{L^1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}}\right] \\ &\overset{(\ddagger)}{\leq} C_1^{(1)} C_4 \cdot u_i \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \left|d_j\right|, \end{aligned}$$

where the left-hand side of the estimate vanishes for $i \in I \setminus I_0$, as the discussion above showed. The step marked with $(\dagger)$ above is justified by equation (5.3.52). Furthermore, at $(\ddagger)$, we used the estimate

$$\left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}} \leq C_4 \cdot \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}}, \tag{5.3.53}$$

where $C_4 = C_4\left(d, p_1, N_{\mathcal{Q}}, R_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}\right)$ is taken from Lemma 5.3.2.

In case of $p_1 \in (0,1)$, we note that $\|\cdot\|_{L^{p_1}}$ is a quasi-norm with triangle constant only depending on $p_1$. Together with the uniform bound $\left|I_0 \cap J_i \cap J^{(r)}\right| \leq |J_i| \leq N_{\mathcal{P}}^{k+1}$ from equation (5.3.49), this yields a constant $C_5 = C_5\left(p_1, k, N_{\mathcal{P}}\right)$ with

$$\begin{aligned} u_i \cdot \left\| \mathcal{F}^{-1}\left(\varphi_i g^{(r)}\right)\right\|_{L^{p_1}} &= u_i \cdot \left\| \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \mathcal{F}^{-1}\left(d_j \cdot \varphi_i \gamma_j\right)\right\|_{L^{p_1}} \\ &\leq C_5 u_i \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \left|d_j\right| \left\|\mathcal{F}^{-1}\left(\varphi_i \gamma_j\right)\right\|_{L^{p_1}}. \end{aligned}$$

Now, we observe that the supports of $\gamma_j$ and $\varphi_i$ satisfy the inclusions $\operatorname{supp}\gamma_j \subset P_j \subset \overline{P_j^{(2k+2)*}}$ and $\operatorname{supp}\varphi_i \subset \overline{Q_i} \subset \overline{P_j^{(2k+2)*}}$ for arbitrary $j \in J_0 \cap J_i \cap J^{(r)}$ and $i \in I_0$. Indeed, $Q_i \subset P_j^{(2k+2)*}$ is a consequence of $Q_i \cap P_j \neq \varnothing$ (since $j \in J_i$) and of Lemma 3.3.2, together with the fact that $\mathcal{Q}_{I_0}$ is almost subordinate to $\mathcal{P}$ with $k = k\left(\mathcal{Q}_{I_0}, \mathcal{P}\right)$. Thus, Corollary 3.2.15 yields a constant $C_6 = C_6\left(k, d, p_1, R_{\mathcal{Q}}, C_{\mathcal{Q}}\right) > 0$ with

$$\begin{aligned} \left\|\mathcal{F}^{-1}\left(\varphi_i \gamma_j\right)\right\|_{L^{p_1}} &\leq C_6 \left|\det\left(S_j\right)\right|^{\frac{1}{p_1}-1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}} \left\|\mathcal{F}^{-1}\gamma_j\right\|_{L^{p_1}} \\ &= C_6 C_1^{(p_1)} \cdot \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^{p_1}} \\ &\leq C_4 C_6 C_1^{(p_1)} \cdot \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}}. \end{aligned}$$

Here, we used equations (5.3.46) and (5.3.53). All in all, we get

$$u_i \cdot \left\| \mathcal{F}^{-1}\left(\varphi_i g^{(r)}\right)\right\|_{L^{p_1}} \leq C_7 \cdot u_i \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \left|d_j\right| \tag{5.3.54}$$

for $C_7 := C_1^{(p_1)} C_4 C_5 C_6$, where the left-hand side vanishes for $i \notin I_0$. Together with the case $p_1 \in [1, \infty]$ considered above, we conclude that this estimate holds for all $p_1 \in (0, \infty]$, with possibly different constants $C_7$.

In case of $q_1 < \infty$, we make use of equation (5.3.49), which yields $\left| J_0 \cap J_i \cap J^{(r)} \right| \leq N_{\mathcal{P}}^{k+1}$, to derive the estimate

$$\left( \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \theta_j \right)^{q_1} \leq \left( N_{\mathcal{P}}^{k+1} \right)^{q_1} \cdot \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \theta_j^{q_1},$$

which is valid for arbitrary non-negative sequences $(\theta_j)_j$. We then use equation (5.3.54) and set $C_8 := C_7 N_{\mathcal{P}}^{k+1}$, to deduce

$$\begin{aligned}
\left\| g^{(r)} \right\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} &= \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_i g^{(r)}\right) \right\|_{L^{p_1}} \right)_{i \in I} \right\|_{\ell_u^{q_1}} \\
&\leq C_8 \cdot \left( \sum_{i \in I_0} \left[ \left( u_i \left| \det\left(T_i\right) \right|^{1 - \frac{1}{p_1}} \right)^{q_1} \sum_{j \in J_0 \cap J_i \cap J^{(r)}} \left| d_j \right|^{q_1} \right] \right)^{\frac{1}{q_1}} \\
&= C_8 \cdot \left( \sum_{j \in J_0 \cap J^{(r)}} \left[ \left| d_j \right|^{q_1} \sum_{\substack{i \in I_0 \\ \text{with } j \in J_i}} \left( u_i \left| \det\left(T_i\right) \right|^{1 - \frac{1}{p_1}} \right)^{q_1} \right] \right)^{\frac{1}{q_1}} \\
&\leq C_8 \cdot \left\| \left( d_j \cdot \left\| \left( \left| \det\left(T_i\right) \right|^{1 - \frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1}}.
\end{aligned}$$

Here, we used the equivalence $j \in J_i \Leftrightarrow i \in I_j$ in the last step. In case of $q_1 = \infty$, one can easily show that the same estimate remains true, either by a similar direct calculation or by using $\|x\|_{\ell^\infty} = \lim_{q \to \infty} \|x\|_{\ell^q}$ for all finitely supported sequences $x$. For this, it is important to observe that none of the constants introduced so far depends on $q_1$. Observe that the right-hand side of the above estimate is finite, since $\left(d_j\right)_{j \in J_0} \in \ell_0\left(J_0\right)$ and because of equation (5.3.45).

We now use estimate (5.3.51) (marked with (†)) and the boundedness of $\iota$, to deduce for arbitrary $d = \left(d_j\right)_{j \in J_0} \in \ell_0\left(J_0\right)$ that

$$\begin{aligned}
&\left\| \left( d_m \cdot \left| \det\left(S_m\right) \right|^{1 - \frac{1}{p_2}} \right)_{m \in J^{(r)} \cap J_0} \right\|_{\ell_v^{q_2}} \\
&\overset{(\dagger)}{\leq} \frac{C_3}{C_1^{(p_2)}} \cdot \left\| g^{(r)} \right\|_{\mathcal{D}_\Psi\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)} \\
&\leq \frac{C_3}{C_1^{(p_2)}} \cdot |||\iota||| \cdot \left\| g^{(r)} \right\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)} \\
&\leq \frac{C_3 C_8}{C_1^{(p_2)}} |||\iota||| \cdot \left\| \left( d_j \cdot \left\| \left( \left| \det\left(T_i\right) \right|^{1 - \frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell^{q_1}}. \qquad (5.3.55)
\end{aligned}$$

Since $\ell_v^{q_2}(J_0)$ is a quasi-normed space with triangle constant only depending on $q_2$, and because of $J_0 = \biguplus_{r=1}^{r_0} J^{(r)} \cap J_0$, there is a constant $C_9 = C_9(q_2, r_0) > 0$ with

$$\left\| (d_m)_{m\in J_0} \right\|_{\ell_v^{q_2}} \leq C_9 \sum_{r=1}^{r_0} \left\| (d_m)_{m\in J^{(r)}\cap J_0} \right\|_{\ell_v^{q_2}}$$

for arbitrary sequences $(d_j)_{j\in J_0}$.

Let $(c_j)_{j\in J_0} \in \ell_0(J_0)$ be arbitrary and choose $d_j = \left|\det(S_j)\right|^{\frac{1}{p_2}-1} c_j$. Setting

$$w_j := \left|\det(S_j)\right|^{\frac{1}{p_2}-1} \left\| \left( \left|\det(T_i)\right|^{1-\frac{1}{p_1}} u_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}}$$

for each $j \in J_0$, equation (5.3.55) yields

$$\left\| (c_j)_{j\in J_0} \right\|_{\ell_v^{q_2}} \leq C_{10} \, |||\iota||| \cdot \left\| (c_j)_{j\in J_0} \right\|_{\ell_w^{q_1}}$$

for all $(c_j)_{j\in J_0} \in \ell_0(J_0)$ and $C_{10} := C_3 C_8 C_9 r_0 / C_1^{(p_2)}$.

We now invoke Lemma 5.2.2 with $I = J = J_0$ and with $I^{(j)} = \{j\}$ for each $j \in J = J_0$, with exponents $q = q_2$, $p = q_1$ and $r = 1$ and with weights $\alpha \equiv 1$, $v = v$ and $u = w$, to conclude

$$\left\| \left( \left|\det(S_j)\right|^{1-\frac{1}{p_2}} \Big/ \left\| \left( \left|\det(T_i)\right|^{1-\frac{1}{p_1}} u_i \right)_{i\in I_0\cap I_j} \right\|_{\ell^{q_1}} \right)_{j\in J_0} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \leq C_{10} \, |||\iota|||.$$

This completes the proof. □

As before, we now specialize the above lemma to the case where $\mathcal{Q}$ is relatively $\mathcal{P}$-moderate.

**Theorem 5.3.14.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open, let $p_1, p_2 \in (0, \infty]$ and assume that $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ is a tight semi-structured $L^{p_1}$-decomposition covering of $\mathcal{O}$ and that $\mathcal{P} = (P_j)_{j\in J} = (S_j P_j' + c_j)_{j\in J}$ is a* tight *semi-structured $L^{p_2}$-decomposition covering of $\mathcal{O}'$. Let $u = (u_i)_{i\in I}$ and $v = (v_j)_{j\in J}$ be moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.*

*Let $q_1, q_2 \in (0, \infty]$, choose an arbitrary subset $J_0 \subset J$, define*

$$I_0 := \{ i \in I \mid J_i \cap J_0 \neq \varnothing \}$$

*and assume that*

1. *$\mathcal{Q}_{I_0} := (Q_i)_{i\in I_0}$ is almost subordinate to $\mathcal{P}$.*
2. *$\mathcal{Q}_{I_0}$ is moderate with respect to $\mathcal{P}$.*
3. *The weight $u|_{I_0}$ is relatively $\mathcal{P}$-moderate.*
4. *We have $\bigcup_{j\in J_0} P_j \subset \mathcal{O}$.*

*Furthermore, set*

$$K := \bigcup_{i \in I_0} Q_i$$

*and assume that the map*

$$\iota : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is well-defined and bounded. Set* $k := k\left(\mathcal{Q}_{I_0}, \mathcal{P}\right)$. *Then, there is a finite constant* $C > 0$ *which is only dependent on* $d$ *and*

$$\begin{aligned}
&C_{u|_{I_0},\mathcal{Q},\mathcal{P}}, k\left(\mathcal{Q}_{I_0}, \mathcal{P}\right), C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right),\\
&r_0\left(\mathcal{P}, 1\right), r_0\left(\mathcal{P}, 2k+3\right),\\
&p_2, q_2, N_{\mathcal{P}}, R_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}, \varepsilon_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},\\
&p_1, q_1, N_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}
\end{aligned}$$

*and which satisfies*

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+}}{u_{i_j}} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+} \right)_{j \in J_0} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \leq C \, |||\iota||| \,, \tag{5.3.56}$$

*where for each* $j \in J_0$, *an arbitrary* $i_j \in I_j$ *is chosen.*

*We observe that existence of* $i_j \in I_j$ *is implied by assumption 4, since this yields* $\varnothing \neq P_j \subset \mathcal{O} = \bigcup_{i \in I} Q_i$.

*Here, the* $L^{p_1}/L^{p_2}$*-BAPUs* $\Phi = (\varphi_i)_{i \in I}$ *and* $\Psi = (\psi_j)_{j \in J}$ *have to be used to calculate the norm* $|||\iota|||$.

*Proof.* Let $L > 0$ with $u_i \leq L u_\ell$ for all $i, \ell \in I_0 \cap I_j$ for some $j \in J$, i.e. $L := C_{u|_{I_0},\mathcal{Q},\mathcal{P}}$. As in Lemma 5.3.13 above, we note $I_j \subset I_0$ for all $j \in J_0$, because for arbitrary $i \in I_j$ we have $j \in J_i \cap J_0 \neq \varnothing$ and hence $i \in I_0$.

Thus, for any $j \in J_0$ and any $i \in I_j$, we have $i, i_j \in I_0 \cap I_j$, and hence

$$u_i \leq L \cdot u_{i_j} \quad \text{and} \quad \left|\det\left(T_i\right)\right|^{1 - \frac{1}{p_1}} \leq C_1 \cdot \left|\det\left(T_{i_j}\right)\right|^{1 - \frac{1}{p_1}} \tag{5.3.57}$$

for some constant $C_1 = C_1\left(p_1, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right)\right) > 0$.

Note that part 4 of the prerequisites implies $P_j \subset \mathcal{O} = \bigcup_{i \in I} Q_i^{0*}$ and hence

$$P_j \subset \bigcup_{i \in I_j} Q_i^{0*} = \bigcup_{i \in I_0 \cap I_j} Q_i^{0*}$$

for each $j \in J_0$. Here, we used $I_j \subset I_0$ for all $j \in J_0$ in the last step. Thus, Lemma 3.3.7 yields constants

$$C_2 = C_2\left(d, k, \varepsilon_{\mathcal{Q}}, N_{\mathcal{Q}}, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right), R_{\mathcal{P}}, C_{\mathcal{P}}\right)$$

and

$$C_3 = C_3\left(d, \varepsilon_{\mathcal{P}}, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right), R_{\mathcal{Q}}, C_{\mathcal{Q}}\right)$$

such that

$$\left|I_0 \cap I_j\right| \leq C_2 \cdot \left|\det\left(S_j\right)\right| \left|\det\left(T_{i_j}\right)\right|^{-1} \tag{5.3.58}$$

and

$$\left|\det\left(T_{i_j}\right)\right|^{-1} \cdot \left|\det\left(S_j\right)\right| \leq C_3 \cdot \left|I_0 \cap I_j\right| \tag{5.3.59}$$

hold for all $j \in J_0$. Here, we used that $I_0 \cap I_j = I_j$ is nonempty for $j \in J$, since $\varnothing \neq P_j \subset \mathcal{O} = \bigcup_{i \in I} Q_i$ by part 4 of the assumptions.

In particular, $I_0 \cap I_j$ is finite for all $j \in J_0$, so that assumption (5.3.45) of Lemma 5.3.13 is satisfied. By comparing our present assumptions with those of Lemma 5.3.13, we thus see that this lemma is applicable. Hence, there is a constant $C_4 > 0$, which only depends on quantities mentioned in the statement of the theorem, and which satisfies

$$\left\| \left( \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}} \Big/ \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \leq C_4 \left|\!\left|\!\left| \iota \right|\!\right|\!\right| . \tag{5.3.60}$$

Using equations (5.3.57) and (5.3.58), we conclude

$$\begin{aligned} \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} &\leq C_1 L \cdot u_{i_j} \left|\det\left(T_{i_j}\right)\right|^{1-\frac{1}{p_1}} \cdot \left\| (1)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \\ &= C_1 L \cdot u_{i_j} \left|\det\left(T_{i_j}\right)\right|^{1-\frac{1}{p_1}} \cdot \left|I_0 \cap I_j\right|^{\frac{1}{q_1}} \\ &\leq C_5 \cdot u_{i_j} \left|\det\left(T_{i_j}\right)\right|^{1-\frac{1}{p_1}-\frac{1}{q_1}} \left|\det\left(S_j\right)\right|^{\frac{1}{q_1}} \end{aligned}$$

for all $j \in J_0$, where we defined $C_5 := C_1 C_2^{\frac{1}{q_1}} L$ for brevity. Together with equation (5.3.60), this implies

$$\begin{aligned} &\left\| \left( \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}-\frac{1}{q_1}} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{q_1}+\frac{1}{p_1}-1} u_{i_j}^{-1} \right)_{j \in J_0} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\leq C_5 \cdot \left\| \left( \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}} \Big/ \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_1}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_1}} \right)_{j \in J_0} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\leq C_4 C_5 \cdot \left|\!\left|\!\left| \iota \right|\!\right|\!\right| . \end{aligned} \tag{5.3.61}$$

To see that this indeed implies the claim, we distinguish two cases:

CASE 1: We have $p_2 \in [2, \infty]$ and $q_1 \geq p_2'$. In this case, we get $p_2^\triangledown = \min\{p_2, p_2'\} = p_2'$ and $\frac{1}{p_2^\triangledown} - \frac{1}{q_1} = \frac{1}{p_2'} - \frac{1}{q_1} \geq 0$. Thus,

$$\left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+ = \frac{1}{p_2'} - \frac{1}{q_1} = 1 - \frac{1}{p_2} - \frac{1}{q_1}$$

and hence

$$\begin{aligned} &\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \cdot \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \\ &= \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}-\frac{1}{q_1}} \cdot \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(1-\frac{1}{p_2}-\frac{1}{q_1}\right)} \\ &= \left|\det\left(S_j\right)\right|^{1-\frac{1}{p_2}-\frac{1}{q_1}} \cdot \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}+\frac{1}{q_1}-1} . \end{aligned}$$

Plugging this into the target inequality (5.3.56), we see that equation (5.3.61) implies the claim.

CASE 2: We have $p_2 \in (0,2]$ or $q_1 < p_2'$. For brevity, define $s := \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+$. Together with boundedness of $\iota$, we see that all assumptions of Theorem 5.3.10 are met. This theorem yields a constant

$$C_6 = C_6\left(q_1, q_2, p_2, k, N_{\mathcal{P}}, C_{v,\mathcal{P}}, C\right),$$

where $C > 0$ is as in Theorem 5.3.6, such that the estimate

$$\begin{aligned}
&\left\| \left( u_{i_j}^{-1} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\nabla}-\frac{1}{q_1}\right)_+} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\nabla}-\frac{1}{q_1}\right)_+} \right)_{j\in J_0} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\
&\overset{(\dagger)}{\leq} C_3^s \left\| \left( u_{i_j}^{-1} \cdot \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot \left\| (1)_{i\in I_j\cap I_0} \right\|_{\ell^{p_2^\nabla\cdot(q_1/p_2^\nabla)'}} \right)_{j\in J_0} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\
&\overset{(*)}{\leq} LC_3^s C_7 \cdot \left\| \left( \left\| \left( u_i^{-1} \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j\cap I_0} \right\|_{\ell^{p_2^\nabla\cdot(q_1/p_2^\nabla)'}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\
&\leq LC_3^s C_6 C_7 \cdot |||\iota|||
\end{aligned}$$

is satisfied. At $(\dagger)$, we made use of equation (5.3.59) and of the identity $\frac{1}{p_2^\nabla\cdot\left(q_1/p_2^\nabla\right)'} = s$, which was shown in Remark 5.2.3. The step marked with $(*)$ is justified by an obvious analog of equation (5.3.57), which provides the constant $C_7 = C_7\left(p_1, p_2, C_{\mathrm{mod}}\left(\mathcal{Q}_{I_0}, \mathcal{P}\right)\right)$.

This completes the proof in the second case. □

## 5.4 SUMMARY

In view of the large number of different sufficient and necessary conditions that we obtained in this chapter, this subsection is devoted to summarizing the most important (special) cases. We begin with the case in which $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and in which we are interested in the existence of a (not necessarily injective) "embedding" $\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \hookrightarrow \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$.

**Theorem 5.4.1.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = \left(T_i Q_i' + b_i\right)_{i\in I}$ and $\mathcal{P} = \left(S_j P_j' + c_j\right)_{j\in J}$ be semi-structured admissible coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = (u_i)_{i\in I}$ be $\mathcal{Q}$-moderate and let $v = \left(v_j\right)_{j\in J}$ be $\mathcal{P}$-moderate.*

*Furthermore, let $p_1, p_2, q_1, q_2 \in (0,\infty]$ such that $\mathcal{Q}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ and $\mathcal{P}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}'$ and assume that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.*

*Let $r := \frac{1}{p_2} - \frac{1}{p_1} + (p_2)_d$. Finally, set*

$$J_{\mathcal{O}} := \left\{ j \in J \,\middle|\, \mathcal{O} \cap P_j \neq \varnothing \right\},$$

*choose for each $j \in J_{\mathcal{O}}$ some $i_j \in I$ with $Q_{i_j} \cap P_j \neq \varnothing$ and define*

$$K_1 := \left\| \left( \left( \left| \det\left(S_j\right) \right|^{(p_2)_d} v_j \right)^{-1} \left\| \left( \left| \det\left(T_i\right) \right|^r u_i \right)_{i \in I_j} \right\|_{\ell^{q_1}\left(p_2^{\triangle}/q_1\right)'} \right)_{j \in J} \right\|_{\ell^{q_1}(q_2/q_1)'}$$

$$K_2 := \left\| \left( v_j^{-1} \cdot \left\| \left( u_i \left| \det\left(T_i\right) \right|^{\frac{1}{p_2} - \frac{1}{p_1}} \right)_{i \in I_j} \right\|_{\ell^{q_1}(p_2/q_1)'} \right)_{j \in J} \right\|_{\ell^{q_1}(q_2/q_1)'}$$

$$K_3 := \left\| \left( \frac{u_{i_j}}{v_j} \left| \det\left(S_j\right) \right|^{\left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+} \left| \det\left(T_{i_j}\right) \right|^{\frac{1}{p_2} - \frac{1}{p_1} - \left(\frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_1}(q_2/q_1)'}$$

*Then, we have $\mathcal{O} \subset \mathcal{O}'$ and the following are true:*

1. *If we have $p_2 \leq p_1$, if $\mathcal{Q}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}$ and if $K_1$ is finite, then*

$$\iota : \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f|_{C_c^\infty(\mathcal{O})}$$

   *is well-defined and bounded with $|||\iota||| \lesssim K_1$.*

2. *Conversely, if $\mathcal{Q}$ is* tight *and if the map*

$$\iota' : \left(C_c^\infty\left(\mathcal{O}\right), \|\cdot\|_{\mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right), f \mapsto f$$

   *is bounded, then we have $p_2 \leq p_1$ and $K_2 \lesssim |||\iota'|||$. Furthermore, we have $K_1 \lesssim |||\iota'|||$ if one of the following conditions are satisfied:*

   a) *$p_2 \in [2, \infty]$,*
   b) *$q_1 > p_2'$,*
   c) *$q_1 = \infty$ and $p_2 \in [1, \infty]$.*

3. *If the following conditions are satisfied:*

   a) *For all $j \in J_{\mathcal{O}}$, we have $P_j \subset \mathcal{O}$,*
   b) *$\mathcal{Q}$ and $\mathcal{P}$ are* tight,
   c) *$\mathcal{Q}$ is* relatively moderate *with respect to $\mathcal{P}$,*
   d) *$u$ is relatively $\mathcal{P}$-moderate,*

   *then we have $K_1 \asymp K_3$ and the following hold:*

   a) *If $p_2 \leq p_1$ and $K_3 < \infty$ hold and if $\mathcal{Q}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}$, then $\iota$ is bounded with $|||\iota||| \lesssim K_3$.*
   b) *If $\iota'$ is bounded, then $p_2 \leq p_1$ and $K_3 \lesssim |||\iota'||| < \infty$.*

*For convenience of the reader, we recall from equation (5.1.3) the notion of the* UPPER CONJUGATE EXPONENT *$p^{\triangle} = \max\{p, p'\}$, where $p'$ is the usual conjugate exponent for $p \in [1, \infty]$ and $p' = \infty$ for $p \in (0, 1)$. Further, the* DUALITY DEFECT *$p_d$ is defined by $p_d = \min\{0, 1 - \frac{1}{p}\}$, cf. equation (5.1.1). Finally, we have $\frac{1}{p_2^{\pm\Delta}} = \min\left\{\frac{1}{p_2}, 1 - \frac{1}{p_2}\right\}$.*

*All implicit constants appearing above only depend on*

$$\begin{aligned}
&d, p_1, p_2, q_1, q_2,\\
&C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), C_{u,\mathcal{Q},\mathcal{P}}, \varepsilon_{\mathcal{P}}\\
&k(\mathcal{Q}, \mathcal{P}), r_0(\mathcal{P}, 2k(\mathcal{Q}, \mathcal{P}) + 3),\\
&N_{\mathcal{P}}, C_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},\\
&r_0(\mathcal{Q}, 1), N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}.
\end{aligned}$$

*Here, $p$ has to be interpreted as $p_2$ in parts 1 and 3 and as $p_1$ in part 2. Furthermore, the $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ and the $L^{p_2}$-BAPU $\Psi = (\psi_j)_{j\in J}$ have to be used to calculate the norms $|||\iota|||$ and $|||\iota'|||$. Finally, the dependence on the quantities in the second line can be dropped, except in part 3.*

*Proof.* First of all, Lemma 3.3.2 implies $\mathcal{O} \subset \mathcal{O}'$, since $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

1. In the notation of Corollary 5.2.9, we have $I^{(j)} \subset I_j$ for all $j \in J$. Thus, boundedness of $\iota$ is a direct consequence of the aforementioned corollary.

2. The inequality $p_2 \leq p_1$ is implied by Lemma 5.3.1, because $K := \mathcal{O}$ satisfies $K^\circ \cap \mathcal{O} \cap \mathcal{O}' = \mathcal{O} \neq \varnothing$.

The estimate $K_2 \lesssim |||\iota'|||$ is a consequence of the second part of Theorem 5.3.6 (with $K = \mathcal{O}$), in combination with Lemma 5.2.5 and Lemma 5.2.4. More precisely, for each $i \in I$, choose some $j^{(i)} \in J_{\mathcal{O}}$ with $Q_i \cap P_{j^{(i)}} \neq \varnothing$. The existence of such an index is guaranteed by

$$\varnothing \neq Q_i \subset \mathcal{O} \subset \mathcal{O}' = \bigcup_{j\in J} P_j.$$

If we set $I^{(j)} := \{i \in I \mid j^{(i)} = j\}$ for $j \in J$, then Theorem 5.3.6 and Lemma 5.2.5 show

$$\begin{aligned}
\left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \cdot c_i \right)_{i\in I} \right\|_{\ell_u^{q_1}} &\lesssim |||\iota'||| \cdot \left\| \left( \|(c_i)_{i\in I_j}\|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2}}\\
&\lesssim |||\iota'||| \cdot \left\| \left( \|(c_i)_{i\in I^{(j)}}\|_{\ell^{p_2}} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell_v^{q_2}}
\end{aligned}$$

for all finitely supported sequences $(c_i)_{i\in I} \in \ell_0(I)$. Here, the last step also used $I^{(j)} = \varnothing$ for all $j \in J \setminus J_{\mathcal{O}}$. By moving the factor $|\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}}$ to the right-hand side, we conclude

$$\left\|(c_i)_{i\in I}\right\|_{\ell_u^{q_1}} \lesssim |||\iota'||| \cdot \left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_i \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2}} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell_v^{q_2}}.$$

Since $I = \biguplus_{j\in J_{\mathcal{O}}} I^{(j)}$, Lemma 5.2.4 and the inequality above imply

$$\left\| \left( v_j^{-1} \cdot \left\| \left( u_i |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \right)_{i\in I^{(j)}} \right\|_{\ell^{q_1(p_2/q_1)'}} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_1(q_2/q_1)'}} \lesssim |||\iota'|||.$$

By applying Lemma 5.2.5 again, this yields $K_2 \lesssim |||\iota'|||$.

If one of the additional conditions is satisfied, then Theorem 5.3.9 yields $K_1 \lesssim |||\iota'|||$, cf. equation (5.3.30). For this, it is important to note that each of the three cases implies $p_2 \in [1,\infty]$ and hence $(p_2)_d = 0$; this is only nontrivial for $q_1 > p_2'$, but $p_2 \in (0,1)$ would imply $p_2' = \infty$, so that $q_1 > p_2'$ is impossible.

3. To see $K_1 \asymp K_3$, note that $\mathcal{P}$-moderateness of $\mathcal{Q}$ and $u$ implies

$$\left\| \left( u_i \left| \det\left(T_i\right) \right|^r \right)_{i \in I_j} \right\|_{\ell^{q_1}\left(p_2^{\triangle}/q_1\right)'} \asymp u_{i_j} \left| \det\left(T_{i_j}\right) \right|^r \cdot \left| I_j \right|^{\frac{1}{q_1 \cdot \left(p_2^{\triangle}/q_1\right)'}}.$$

But since $P_j \subset \mathcal{O}$ holds for all $j \in P_{\mathcal{O}}$, we get $P_j \subset \bigcup_{i \in I_j} Q_i$ for each $j \in \mathcal{P}_{\mathcal{O}}$. Furthermore, $\mathcal{Q}, \mathcal{P}$ are tight and $\mathcal{Q}$ is relatively $\mathcal{P}$-moderate. Thus, Lemma 3.3.7 yields

$$\left| I_j \right| \asymp \left| \det\left(S_j\right) \right| / \left| \det\left(T_{i_j}\right) \right|$$

for all $j \in J_{\mathcal{O}}$, i.e. all $j \in J$ with $I_j \neq \varnothing$. Finally, Remark 5.2.3 shows

$$\frac{1}{q_1 \cdot \left(p_2^{\triangle}/q_1\right)'} = \left( \frac{1}{q_1} - \frac{1}{p_2^{\triangle}} \right)_+$$

and Remark 5.2.10 yields

$$\left( \frac{1}{q_1} - \frac{1}{p_2^{\triangle}} \right)_+ - (p_2)_d = \left( \frac{1}{q_1} - \frac{1}{p_2^{\pm\triangle}} \right)_+.$$

Together, these observations easily yield $K_1 \asymp K_3$.

If $p_2 \leq p_1$ and $K_3 < \infty$ hold, then part 1 yields boundedness of $\iota$ with $|||\iota||| \lesssim K_1 \lesssim K_3$.

Finally, if $\iota'$ is bounded, then part 2 yields $p_2 \leq p_1$ and Theorem 5.3.12 shows $K_3 \lesssim |||\iota'|||$. To this end, we note that $J_{\mathcal{O}}^* = J_{\mathcal{O}}$ holds, since for $x \in P_j \cap P_\ell$ with $j \in J_{\mathcal{O}}$, assumption 3a) implies $x \in P_j \subset \mathcal{O}$ and hence $\ell \in J_{\mathcal{O}}$. This, together with $\mathcal{O} \subset \mathcal{O}'$, easily implies that $\left(P_j\right)_{j \in J_{\mathcal{O}}}$ is an admissible $L^{p_2}$-decomposition covering of $\mathcal{O}$, so that Lemma 3.2.12 yields

$$K \subset \bigcup_{j \in J_{\mathcal{O}}} \overline{P_j} \subset \mathcal{O}$$

with $K$ as in Theorem 5.3.12. Thus, Theorem 5.3.12 is indeed applicable in the present setting. □

An easy, but nevertheless interesting application of the above result is the case where the two coverings $\mathcal{Q}, \mathcal{P}$ coincide.

**Corollary 5.4.2.** *Let $\mathcal{Q} = (T_i Q_i' + b_i)_{i \in I}$ be a* tight *semi-structured admissible covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$ and let $u = (u_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ be $\mathcal{Q}$-moderate.*

*Finally, let* $p_1, p_2, q_1, q_2 \in (0, \infty]$ *such that* $\mathcal{Q}$ *is an* $L^{p_i}$*-decomposition covering of* $\mathcal{O}$ *for* $i \in \underline{2}$*. The map*

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$ *and*

$$C := \left\| \left( \frac{v_i}{u_i} \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty.$$

*For necessity of these conditions, it even suffices if* $\iota|_{C_c^\infty(\mathcal{O})}$ *is bounded.*

*More precisely, if* $p_1 \leq p_2$ *holds, then we have*

$$C \lesssim \left\| \left| \iota|_{C_c^\infty} \right| \right\| \leq |||\iota||| \lesssim C.$$

*The implied constants only depend on*

$$d, p_1, p_2, q_1, q_2, N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{v,\mathcal{Q}}, C_{u,\mathcal{Q}}, r_0(\mathcal{Q}, 3), C_{\mathcal{Q},\Phi,p_2}, C_{\mathcal{Q},\Psi,p_1},$$

*where the* $L^{p_i}$*-BAPUs* $\Psi, \Phi$ *are used to calculate the quasi-norms on the decomposition spaces* $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ *and* $\mathcal{D}\left(\mathcal{Q}, L^{p_2}, \ell_v^{q_2}\right)$*, respectively.*

*Proof.* We apply part 3 of Theorem 5.4.1 with $\mathcal{P} = \mathcal{Q}$ and interchanged roles of $u, v$, of $p_1, p_2$ and of $q_1, q_2$. We remark that the map $\iota$ given in Theorem 5.4.1 satisfies $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ (if it exists), since we have $\mathcal{O} = \mathcal{O}'$.

It is clear that $\mathcal{Q}$ is almost subordinate to and relatively moderate with respect to $\mathcal{P} = \mathcal{Q}$, with $k(\mathcal{Q}, \mathcal{P}) = 0$ and $C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}) \leq D_{\mathcal{Q}}^2$. To see this, let $i, \ell \in I$ and $j \in J = I$ with $Q_i \cap P_j \neq \varnothing \neq Q_\ell \cap P_j$. Because of $P_j = Q_j$, this means $i \in j^*$ and $j \in \ell^*$, which implies

$$|\det(T_i)| \leq D_{\mathcal{Q}} \cdot |\det(T_j)| \leq D_{\mathcal{Q}}^2 \cdot |\det(T_\ell)|$$

by definition of $D_{\mathcal{Q}}$. Using the same argument, we see that $v$ is relatively $\mathcal{P}$-moderate with $C_{v,\mathcal{Q},\mathcal{P}} \leq C_{v,\mathcal{Q}}^2$. Finally, $P_j = Q_j \subset \mathcal{O}$ is trivially satisfied for all $j \in J = I$.

Thus, part 3 of Theorem 5.4.1 is applicable. Recalling the interchanged roles of $u, v$, of $p_1, p_2$ and of $q_1, q_2$, we see that $\iota$ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if

$$C := \left\| \left( \frac{v_{i_j}}{u_j} |\det(T_j)|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \cdot |\det(T_{i_j})|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_2(q_1/q_2)'}}$$

is finite. Here, for each $j \in J_{\mathcal{O}} = J = I$, some $i_j \in I$ with $Q_{i_j} \cap P_j \neq \varnothing$ can be chosen freely. Here, we can choose $i_j = j$, so that

$$C = \left\| \left( \frac{v_i}{u_i} \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}.$$

More precisely, part 3 of Theorem 5.4.1 shows the following: If we have $p_1 \leq p_2$ and if $C$ is finite, then $\iota$ is well-defined and bounded with $|||\iota||| \lesssim C$. Conversely, if $\iota' = \iota|_{C_c^\infty(\mathcal{O})}$ is bounded, then $p_1 \leq p_2$ and $C \lesssim |||\iota'||| < \infty$. Here, all implied constants only depend on

$$\begin{aligned}
&d, p_1, p_2, q_1, q_2,\\
&C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}) \leq D_{\mathcal{Q}}^2,\\
&C_{u,\mathcal{Q},\mathcal{P}} \leq C_{v,\mathcal{Q}}^2,\\
&\varepsilon_{\mathcal{P}} = \varepsilon_{\mathcal{Q}},\\
&k(\mathcal{Q}, \mathcal{P}) = 0,\\
&r_0(\mathcal{P}, 2k(\mathcal{Q}, \mathcal{P}) + 3) = r_0(\mathcal{Q}, 3),\\
&N_{\mathcal{Q}}, C_{\mathcal{Q}}, R_{\mathcal{Q}}, D_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Psi,p_1},\\
&r_0(\mathcal{Q}, 1) \leq r_0(\mathcal{Q}, 3),\\
&N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{v,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_2},
\end{aligned}$$

where the $L^{p_i}$-BAPUs $\Psi, \Phi$ are used to calculate the quasi-norms on the decomposition spaces $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ and $\mathcal{D}\left(\mathcal{Q}, L^{p_2}, \ell_v^{q_2}\right)$, respectively. This completes the proof. □

Together with our interpretation of wavelet coorbit spaces as decomposition spaces from Chapter 4, the above corollary also yields a convenient criterion for checking embeddings of coorbit spaces "living" on the same group. However, it is important to note that this is no new result, but follows already from the general theory of embeddings for coorbit spaces, cf. [30, Theorem 8.4]. Nevertheless, for an application of [30, Theorem 8.4], one would still have to compute the associated discrete sequence spaces $\left(L_{m_i}^{p_i,q_i}\right)_d$ and to characterize the existence of an embedding between these sequence spaces.

**Corollary 5.4.3.** *Let $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$ be a closed dilation group satisfying our standing assumptions from Chapter 4, let $m_1, m_2 : H \to (0, \infty)$ be measurable and assume that each of the two weights $m_i$ is right moderate with respect to a measurable, locally bounded weight $m_i^{(0)} : H \to (0, \infty)$.*

*For $p_1, p_2, q_1, q_2 \in (0, \infty]$, let the coorbit spaces $\mathrm{Co}\left(L_{m_i}^{p_i,q_i}\left(\mathbb{R}^d \rtimes H\right)\right)$ for $i \in \underline{2}$ be defined as in Chapter 4, i.e. with respect to the quasi-regular representation of $G := \mathbb{R}^d \rtimes H$. Then there is an embedding of coorbit spaces*

$$\iota : \mathrm{Co}\left(L_{m_1}^{p_1,q_1}\left(\mathbb{R}^d \rtimes H\right)\right) \to \mathrm{Co}\left(L_{m_2}^{p_2,q_2}\left(\mathbb{R}^d \rtimes H\right)\right)$$

*if and only if we have $p_1 \leq p_2$ and*

$$C := \left\| \left( \frac{|\det(h_i)|^{\frac{1}{p_2} - \frac{1}{q_2}} \cdot m_2(h_i)}{|\det(h_i)|^{\frac{1}{p_1} - \frac{1}{q_1}} \cdot m_1(h_i)} \right)_{i \in I} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} < \infty,$$

*where $(h_i)_{i \in I}$ is an arbitrary family which is well-spread in $H$.*

*Proof.* By Lemma 4.7.2, we know that there is an embedding of coorbit spaces if and only if there is an embedding of decomposition spaces between the associated decomposition spaces. Since $(h_i)_{i\in I}$ is well-spread in $H$, Theorem 4.4.6 shows that there is a precompact set $Q \Subset \mathcal{O}$ such that $\mathcal{Q} = \left(h_i^{-T} Q\right)_{i\in I}$ is a structured admissible covering of $\mathcal{O}$ induced by $H$, where $\mathcal{O} \subset \mathbb{R}^d$ is the open dual orbit of $H$.

For $j \in \underline{2}$, let $u_j : \mathcal{O} \to (0,\infty)$ be a decomposition weight for $\mathrm{Co}\left(L^{p_i,q_i}_{m_i}\left(\mathbb{R}^d \rtimes H\right)\right)$ and let $v_j : I \to (0,\infty)$ be a $\mathcal{Q}$-discretization of $u_j$. By Lemma 4.5.4, we know

$$\left(v_j\right)_i \asymp \left|\det\left(h_i\right)\right|^{\frac{1}{2}-\frac{1}{q_j}} \cdot m_j\left(h_i\right) \text{ for all } i \in I. \tag{5.4.1}$$

All in all, we see that there is an embedding of coorbit spaces as in the statement of the theorem if and only if there is an embedding of decomposition spaces

$$\theta : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_{v_1}\right) \to \mathcal{D}\left(\mathcal{Q}, L^{p_2}, \ell^{q_2}_{v_2}\right).$$

The definition of an embedding of decomposition spaces shows in particular $\theta f = f$ for all $f \in C_c^\infty(\mathcal{O}) \subset L^2\left(\mathbb{R}^d\right) \cap \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_{v_1}\right)$. Thus, Corollary 5.4.2 shows that there is such an embedding of decomposition spaces $\theta$ if and only if we have $p_1 \le p_2$ and if the constant

$$\begin{aligned}
C &= \left\| \left( \frac{(v_2)_i}{(v_1)_i} \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\
&\overset{(\dagger)}{\asymp} \left\| \left( \frac{\left|\det\left(h_i\right)\right|^{\frac{1}{2}-\frac{1}{q_2}} \cdot m_2\left(h_i\right)}{\left|\det\left(h_i\right)\right|^{\frac{1}{2}-\frac{1}{q_1}} \cdot m_1\left(h_i\right)} \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\
&\overset{(\ddagger)}{=} \left\| \left( \frac{\left|\det\left(h_i\right)\right|^{\frac{1}{p_2}-\frac{1}{q_2}} \cdot m_2\left(h_i\right)}{\left|\det\left(h_i\right)\right|^{\frac{1}{p_1}-\frac{1}{q_1}} \cdot m_1\left(h_i\right)} \right)_{i\in I} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}}
\end{aligned}$$

is finite. Here, we used equation (5.4.1) at $(\dagger)$. At $(\ddagger)$, we employed the identity $T_i = h_i^{-T}$, which yields $\left|\det\left(T_i\right)\right| = \left|\det\left(h_i\right)\right|^{-1}$. □

Our next result is concerned with the same setting as in Theorem 5.4.1, but here we consider existence of the reverse embedding, i.e. of $\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right)$.

**Theorem 5.4.4.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let $\mathcal{Q} = \left(T_i Q_i' + b_i\right)_{i\in I}$ and $\mathcal{P} = \left(S_j P_j' + c_j\right)_{j\in J}$ be semi-structured admissible coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = \left(u_i\right)_{i\in I}$ be $\mathcal{Q}$-moderate and let $v = \left(v_j\right)_{j\in J}$ be $\mathcal{P}$-moderate.*

*Further, let $p_1, p_2, q_1, q_2 \in (0,\infty]$ such that $\mathcal{Q}$ is an $L^{p_1}$-decomposition covering of $\mathcal{O}$ and $\mathcal{P}$ is an $L^{p_2}$-decomposition covering of $\mathcal{O}'$ and assume that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.*

*For convenience of the reader, we recall from equation* (5.1.11) *the notion of the* LOWER CONJUGATE EXPONENT $p^\triangledown = \min\{p, p'\}$, *where $p'$ is the usual conjugate exponent for $p \in [1,\infty]$ and $p' = \infty$ for $p \in (0,1)$.*

*Set*

$$C_1 := \left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j} \right\|_{\ell^{p_2^\triangledown\cdot(q_1/p_2^\triangledown)'}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}},$$

$$C_2 := \left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j} \right\|_{\ell^{p_2\cdot(q_1/p_2)'}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}},$$

$$C_3 := \left\| \left( \frac{\left|\det(S_j)\right|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+}}{u_{i_j}} |\det(T_{i_j})|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}}.$$

*Then we have $\mathcal{O} \subset \mathcal{O}'$ and the following are true:*

1. *If we have $p_1 \le p_2$ and if $C_1$ is finite, then there is a bounded linear map*
$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)$$
*with $|||\iota||| \lesssim C_1$ and with the following properties:*
   a) *For $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right) \le \mathcal{D}'(\mathcal{O})$, the distribution $\iota f \in \mathcal{D}'(\mathcal{O}')$ is an extension of $f$.*
   b) *For each distribution $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$ which is given by (integration against) a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ with $f \in L^1_{\mathrm{loc}}(\mathcal{O}')$ and with $f = 0$ almost everywhere on $\mathcal{O}' \setminus \mathcal{O}$, we have $\iota f = f$.*
   c) *We have $\operatorname{supp}(\iota f) \subset \mathcal{O}' \cap \overline{\operatorname{supp} f}$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$.*
   d) *Explicitly, $\iota$ is given by*
$$\langle \iota f, \varphi\rangle = \sum_{i\in I} \langle f, \varphi_i \varphi\rangle \quad \text{for } \varphi \in C_c^\infty(\mathcal{O}'),$$
   *where $\Phi = (\varphi_i)_{i\in I}$ is an $L^{p_1}$-BAPU for $\mathcal{Q}$.*
2. *Conversely, if $\mathcal{Q}$ is* tight *and if the map*
$$\iota' : \left(C_c^\infty(\mathcal{O}), \|\cdot\|_{\mathcal{D}\left(\mathcal{Q},L^{p_1},\ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$
   *is bounded, then we have $p_1 \le p_2$ and $C_2 \lesssim |||\iota'|||$. Furthermore, we have $C_1 \lesssim |||\iota'|||$ if one of the following conditions is satisfied:*
   a) *$q_1 \le p_2'$,*
   b) *$p_2 \in (0,2]$.*
3. *If the following conditions are satisfied:*
   a) *For all $j \in J_{\mathcal{O}}$, we have $P_j \subset \mathcal{O}$,*
   b) *$\mathcal{Q}$ and $\mathcal{P}$ are* tight,
   c) *$\mathcal{Q}$ is* relatively moderate *with respect to $\mathcal{P}$,*

*d) $u$ is relatively $\mathcal{P}$-moderate,*

*then we have $C_1 \asymp C_3$. Furthermore, the following are true:*

*a) If $p_1 \leq p_2$ and $C_3 < \infty$ hold, then the map $\iota$ as above is well-defined and bounded with $|||\iota||| \lesssim C_3$.*

*b) If $\iota'$ is bounded, then $p_1 \leq p_2$ and $C_3 \lesssim |||\iota'||| < \infty$.*

*All implicit constants appearing above only depend on the following quantities:*

$$\begin{aligned}
&d, p_1, p_2, q_1, q_2,\\
&C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}), C_{u,\mathcal{Q},\mathcal{P}}, \varepsilon_{\mathcal{P}},\\
&k(\mathcal{Q}, \mathcal{P}), r_0(\mathcal{P}, 2k(\mathcal{Q}, \mathcal{P}) + 3), r_0(\mathcal{P}, 1),\\
&N_{\mathcal{P}}, C_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Psi,p_2},\\
&N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p_1}.
\end{aligned}$$

*Here, the $L^{p_1}$-BAPU $\Phi = (\varphi_i)_{i\in I}$ and the $L^{p_2}$-BAPU $\Psi = (\psi_j)_{j\in J}$ are used to calculate the norms $|||\iota|||$ and $|||\iota'|||$. Finally, the dependence on the quantities in the second line can be dropped, except in part 3.*

*Proof.* First of all, Lemma 3.3.2 implies $\mathcal{O} \subset \mathcal{O}'$, since $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$.

1. In the notation of Corollary 5.2.8, we have $I^{(j)} \subset I_j$ for all $j \in J$. Thus, boundedness of $\iota$ is a direct consequence of the aforementioned corollary.

2. The inequality $p_1 \leq p_2$ is implied by Lemma 5.3.1, because $K := \mathcal{O}$ satisfies $K^\circ \cap \mathcal{O} \cap \mathcal{O}' = \mathcal{O} \neq \varnothing$.

The estimate $C_2 \lesssim |||\iota'|||$ is a consequence of Theorem 5.3.6, in combination with Lemma 5.2.5 and Lemma 5.2.2. More precisely, for each $i \in I$, choose some $j^{(i)} \in J_{\mathcal{O}}$ with $Q_i \cap P_{j^{(i)}} \neq \varnothing$. The existence of such an index is guaranteed by

$$\varnothing \neq Q_i \subset \mathcal{O} \subset \mathcal{O}' = \bigcup_{j\in J} P_j.$$

If we set $I^{(j)} := \{i \in I \mid j^{(i)} = j\}$ for $j \in J$, then Theorem 5.3.6 and the inclusion $I^{(j)} \subset I_j$ yield

$$\begin{aligned}
\left\| \left( \| (c_i)_{i\in I^{(j)}} \|_{\ell^{p_2}} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2}_v} &\leq \left\| \left( \| (c_i)_{i\in I_j} \|_{\ell^{p_2}} \right)_{j\in J} \right\|_{\ell^{q_2}_v}\\
&\lesssim |||\iota'||| \cdot \left\| \left( |\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}} \cdot c_i \right)_{i\in I} \right\|_{\ell^{q_1}_u}
\end{aligned}$$

for all finitely supported sequences $(c_i)_{i\in I} \in \ell_0(I)$. By moving the factor $|\det(T_i)|^{\frac{1}{p_2}-\frac{1}{p_1}}$ to the left-hand side, we conclude

$$\left\| \left( \left\| \left( |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} c_i \right)_{i\in I^{(j)}} \right\|_{\ell^{p_2}} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2}_v} \lesssim |||\iota'||| \cdot \left\| (c_i)_{i\in I} \right\|_{\ell^{q_1}_u}.$$

Because of $I = \biguplus_{j \in J_{\mathcal{O}}} I^{(j)}$, Lemma 5.2.2 shows that the above inequality implies

$$\left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} \right)_{i \in I^{(j)}} \right\|_{\ell^{p_2 \cdot (q_1/p_2)'}} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \lesssim |||\iota'|||.$$

Using Lemma 5.2.5, this yields $C_2 \lesssim |||\iota'|||$. Here, we also used $I_j = \varnothing$ for $j \in J \setminus J_{\mathcal{O}}$.

If one of the additional conditions is satisfied, then Theorem 5.3.10 yields $C_1 \lesssim |||\iota'|||$, cf. equation (5.3.32).

3. To see $C_1 \asymp C_3$, note that $\mathcal{P}$-moderateness of $\mathcal{Q}$ and $u$ implies

$$\left\| \left( |\det(T_i)|^{\frac{1}{p_1} - \frac{1}{p_2}} u_i^{-1} \right)_{i \in I_j} \right\|_{\ell^{p_2^\nabla \cdot (q_1/p_2^\nabla)'}} \asymp |\det(T_{i_j})|^{\frac{1}{p_1} - \frac{1}{p_2}} u_{i_j}^{-1} \left| I_j \right|^{\frac{1}{p_2^\nabla \cdot (q_1/p_2^\nabla)'}}.$$

But because of $P_j \subset \mathcal{O}$ for all $j \in J_{\mathcal{O}}$ (cf. assumption 3a), we get $P_j \subset \bigcup_{i \in I_j} Q_i$, since $\mathcal{Q}$ covers $\mathcal{O}$. Furthermore, $\mathcal{Q}$ and $\mathcal{P}$ are tight and $\mathcal{Q}$ is relatively $\mathcal{P}$-moderate. Hence, all assumptions of Lemma 3.3.7 are satisfied, which yields

$$\left| I_j \right| \asymp \left| \det(S_j) \right| / \left| \det(T_{i_j}) \right|$$

for all $j \in J_{\mathcal{O}}$, i.e. all $j \in J$ with $I_j \neq \varnothing$. Finally, Remark 5.2.3 shows

$$\frac{1}{p_2^\nabla \cdot (q_1/p_2^\nabla)'} = \left( \frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+.$$

Together, we easily see $C_1 \asymp C_3$.

If $p_1 \leq p_2$ and $C_3 < \infty$ hold, then part 1 yields boundedness of $\iota$ with $|||\iota||| \lesssim C_1 \lesssim C_3$.

Finally, if $\iota'$ is bounded, then part 2 yields $p_1 \leq p_2$ and Theorem 5.3.14 shows $C_3 \lesssim |||\iota'|||$. □

As our final result in this section, we give a similar simplification – including a detailed discussion of sharpness – for the general result in Corollary 5.1.11, in which the covering $\mathcal{R}$ is almost subordinate to $\mathcal{S}$ "near $A$" and vice versa "near $B$".

**Theorem 5.4.5.** *Let $\varnothing \neq \mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$ be open and let*

$$\mathcal{R} = (R_\lambda)_{\lambda \in \Lambda} = \left( T_\lambda R'_\lambda + b_\lambda \right)_{\lambda \in \Lambda}$$

*and*

$$\mathcal{S} = (S_\omega)_{\omega \in \Omega} = \left( M_\omega S'_\omega + c_\omega \right)_{\omega \in \Omega}$$

*be semi-structured coverings of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let $u = (u_\lambda)_{\lambda \in \Lambda}$ and $v = (v_\omega)_{\omega \in \Omega}$ be $\mathcal{R}$- and $\mathcal{S}$-moderate, respectively.*

*Finally, assume that there are $k, \ell \in \mathbb{N}_0$ and sets $A, B \subset \mathcal{O} \cap \mathcal{O}'$ with $\mathcal{O} \cap \mathcal{O}' = A \cup B$ and with the following two properties:*

1. *for each* $\lambda \in \Lambda_A := \{\lambda \in \Lambda \mid R_\lambda \cap A \neq \varnothing\}$, *there is some* $\omega_\lambda \in \Omega$ *with* $R_\lambda \subset S^{k*}_{\omega_\lambda}$,
2. *for each* $\omega \in \Omega_B := \{\omega \in \Omega \mid S_\omega \cap B \neq \varnothing\}$, *there is some* $\lambda_\omega \in \Lambda$ *with* $S_\omega \subset R^{\ell*}_{\lambda_\omega}$.

*Let* $r \in \mathbb{R}$ *and* $t_1, t_2, t_3, t_4 \in (0, \infty]$ *be arbitrary and set*

$$L_1 := \left\| \left( \left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^r \right)_{\lambda \in \Lambda_A \cap \Lambda_\omega} \right\|_{\ell^{t_3}\cdot(t_1/t_3)'} \right)_{\omega \in \Omega} \right\|_{\ell_v^{t_2}\cdot(t_1/t_2)'}$$

$$L_2 := \left\| \left( u_\lambda^{-1} \cdot \left\| \left( v_\omega \cdot |\det(M_\omega)|^r \right)_{\omega \in \Omega_B \cap \Omega_\lambda} \right\|_{\ell^{t_2}\cdot(t_4/t_2)'} \right)_{\lambda \in \Lambda} \right\|_{\ell^{t_2}\cdot(t_1/t_2)'}$$

*Denote these constants by* $L_1(t_1, t_2, t_3, r, u, v)$ *and* $L_2 = L_2(t_1, t_2, t_4, r, u, v)$.

*Let* $p_1, p_2, q_1, q_2 \in (0, \infty]$ *and assume that* $\mathcal{R}$ *is an* $L^{p_1}$*-decomposition covering of* $\mathcal{O}$ *and that* $\mathcal{S}$ *is an* $L^{p_2}$*-decomposition covering of* $\mathcal{O}'$. *Then the following are true:*

1. *If we have* $p_1 \leq p_2$ *as well as*

$$M_1 := L_1\left(q_1, q_2, p_2^{\triangledown}, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right) < \infty$$

*and*

$$M_2 := L_2\left(q_1, q_2, p_1^{\triangle}, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, \left(u_\lambda \left|\det(T_\lambda)\right|^{(p_1)_d}\right)_\lambda, v\right) < \infty,$$

*and if* $\mathcal{S}$ *is an* $L^{p_1}$*-decomposition covering of* $\mathcal{O}'$, *then there is a bounded linear map*

$$\iota : \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell_v^{q_2}\right)$$

*with* $|||\iota||| \lesssim M_1 + M_2$ *and with the following properties:*

   a) *For* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right)$ *and* $\gamma \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$, *we have*

$$\langle f, \gamma \rangle = \langle \iota f, \gamma \rangle .$$

   b) *For* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right)$, *the inclusion*

$$\operatorname{supp} \iota f \subset \mathcal{O}' \cap \overline{\operatorname{supp} f}$$

   *is valid.*

   c) *If* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right)$ *is given by integration against a measurable function* $f : \mathbb{R}^d \to \mathbb{C}$ *with* $f \in L^1_{\mathrm{loc}}(\mathcal{O}) \cap L^1_{\mathrm{loc}}(\mathcal{O}')$ *and with* $f(x) = 0$ *a.e. on* $\mathcal{O}' \setminus \mathcal{O}$, *then* $\iota f = f$.

   d) *In particular, we have* $\iota f = f$ *for all* $f \in C_c^\infty(\mathcal{O} \cap \mathcal{O}')$.

*If* $\mathcal{O} \subset \mathbb{R}^d$ *is of full measure, then* $f = 0$ *a.e. on* $\mathcal{O}' \setminus \mathcal{O}$ *holds. In this case,* $\iota f = f$ *holds for every* $f \in \mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right)$ *which is given by (integration against) some* $f \in L^1_{\mathrm{loc}}(\mathcal{O}) \cap L^1_{\mathrm{loc}}(\mathcal{O}')$.

2. *Assume that* $\mathcal{R}, \mathcal{S}$ *are* tight *and that* $\mathcal{O} \cap \mathcal{O}' \neq \varnothing$. *If*

$$\theta : \left(C_c^\infty\left(\mathcal{O} \cap \mathcal{O}'\right), \|\cdot\|_{\mathcal{D}\left(\mathcal{R}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{S}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

*is bounded, then we have* $p_1 \leq p_2$ *and*

$$L_1\left(q_1, q_2, p_2, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right) \lesssim |||\theta|||,$$
$$L_2\left(q_1, q_2, p_1, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right) \lesssim |||\theta|||.$$

*Furthermore, we have* $L_1\left(q_1, q_2, p_2^\nabla, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right) \lesssim |||\iota|||$ *if one of the following conditions are satisfied:*

a) $q_1 \leq p_2'$,
b) $p_2 \in (0, 2]$.

*Finally, we have*

$$L_2\left(q_1, q_2, p_1^\triangle, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, \left(u_\lambda \left|\det\left(T_\lambda\right)\right|^{(p_1)_d}\right)_\lambda, v\right) \lesssim |||\iota|||$$

*if one of the following conditions hold:*

a) $p_1 \in [2, \infty]$,
b) $q_2 > p_1'$,
c) $q_2 = \infty$ *and* $p_1 \in [1, \infty]$.

3. *Set*

$$\Lambda_1 := \left\{\lambda \in \Lambda \,|\, \exists \omega \in \Omega_B : \, S_\omega \cap R_\lambda \neq \varnothing\right\}.$$

*Assume that*

a) $\mathcal{S}_{\Omega_B}$ *is moderate with respect to* $\mathcal{R}$,
b) $v|_{\Omega_B}$ *is* $\mathcal{R}$*-moderate,*
c) *we have*

$$\bigcup_{\lambda \in \Lambda_1} \overline{R_\lambda^{(2\ell+3)*}} \subset \mathcal{O}', \tag{5.4.2}$$

d) *there is* $m \in \mathbb{N}_0$ *such that for all* $\lambda \in \Lambda_1$, *we have*

$$R_\lambda \subset \bigcup_{\omega \in \Omega_B \cap \Omega_\lambda} S_\omega^{m*}. \tag{5.4.3}$$

*Then*

$$\begin{aligned} &L_2\left(q_1, q_2, p_1^\triangle, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, \left(u_\lambda \left|\det\left(T_\lambda\right)\right|^{(p_1)_d}\right)_\lambda, v\right) \\ &\asymp \left\| \left( \frac{v_{\omega_\lambda}}{u_\lambda} \left|\det\left(T_\lambda\right)\right|^t \left|\det\left(M_{\omega_\lambda}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - t} \right)_{\lambda \in \Lambda_1} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\ &\lesssim |||\theta||| \end{aligned}$$

*holds, with* $t = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+$, *where for each* $\lambda \in \Lambda_1$, *the index* $\omega_\lambda \in \Omega_B$ *can be chosen arbitrarily with* $S_{\omega_\lambda} \cap R_\lambda \neq \varnothing$.

4. *Set*

$$\Omega_1 := \{\omega \in \Omega \mid \exists \lambda \in \Lambda_A : R_\lambda \cap S_\omega \neq \varnothing\}$$

*and*

$$\Lambda_0 := \{\lambda \in \Lambda \mid \exists \omega \in \Omega_1 : S_\omega \cap R_\lambda \neq \varnothing\}.$$

*Then* $\Lambda_A \subset \Lambda_0 \subset \Lambda_A^{(2\ell+3)*}$ *holds.*

*Assume that*

*a)* $\mathcal{R}_{\Lambda_0}$ *is almost subordinate to and moderate with respect to* $\mathcal{S}$,

*b)* $u|_{\Lambda_0}$ *is* $\mathcal{S}$*-moderate,*

*c) we have*

$$\bigcup_{\omega\in\Omega_1} S_\omega \subset \mathcal{O}. \tag{5.4.4}$$

*Then*

$$L_1\left(q_1, q_2, p_2^\triangledown, \frac{1}{p_1} - \frac{1}{p_2}, u, v\right)$$
$$\asymp \left\| \left( \frac{|\det(M_\omega)|^{\left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+}}{u_{\lambda_\omega}} |\det(T_{\lambda_\omega})|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+} \right)_{\omega\in\Omega_1} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}}$$
$$\lesssim |||\theta|||.$$

*The reason for the slightly modified assumptions (in comparison with part 3) is that the assumptions of theorems 5.3.12 and 5.3.14 are slightly different. In 5.3.14, we can choose the set* $J_0 \subset J$ *freely and this determines the set* $I_0$, *whereas in 5.3.12, we can choose* $I_0$ *and this determines* $J_0$. *But in the present case, we need to use Theorem 5.3.14 and want to choose* $I_0$.

*The implied constants in parts 1 and 2 only depend on*

$$d, p_1, p_2, q_1, q_2$$
$$k, \ell,$$
$$\varepsilon_{\mathcal{R}}, r_0(\mathcal{R}, 2\ell+3), N_{\mathcal{R}}, C_{\mathcal{R}}, R_{\mathcal{R}}, D_{\mathcal{R}}, C_{u,\mathcal{R}}, C_{\mathcal{R},\Phi,p_1}$$
$$\varepsilon_{\mathcal{S}}, r_0(\mathcal{S}, 2k+3), N_{\mathcal{S}}, R_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}, C_{v,\mathcal{S}}, C_{\mathcal{S},\Psi,p},$$

*where the* $L^{p_1}$*-BAPU* $\Phi$ *for* $\mathcal{R}$ *and the* $L^p$*-BAPU* $\Psi$ *for* $\mathcal{S}$ *are used to calculate the norms* $|||\iota|||$ *and* $|||\theta|||$. *Here, we use the convention* $p = p_1$ *in part 1 and* $p = p_2$ *in the remaining parts.*

*The implied constants in part 3 additionally depend on*

$$C_{\mathrm{mod}}(\mathcal{S}_{\Omega_B}, \mathcal{R}),\ C_{v|_{\Omega_B},\mathcal{S},\mathcal{R}},\ m,$$

*and the implied constants in part 4 additionally depend on*

$$C_{\mathrm{mod}}(\mathcal{R}_{\Lambda_0}, \mathcal{S}),\ C_{u|_{\Lambda_0},\mathcal{R},\mathcal{S}},\ k(\mathcal{R}_{\Lambda_0}, \mathcal{S}).$$

*Remark* 5.4.6. It is important to observe that the additional conditions (5.4.2) and (5.4.3) in part 3 of the theorem are always satisfied in case

of $\mathcal{O} = \mathcal{O}'$. For condition (5.4.2), this is clear, because Lemma 3.2.12, together with finiteness of $\lambda^{(2\ell+3)*}$, shows

$$\overline{R_\lambda^{(2\ell+3)*}} = \bigcup_{\theta \in \lambda^{(2\ell+3)*}} \overline{R_\theta} \subset \mathcal{O} = \mathcal{O}'.$$

For condition (5.4.3), let $\lambda \in \Lambda_1$ be arbitrary. By definition of $\Lambda_1$, this means $R_\lambda \cap S_{\omega_0} \neq \varnothing$ for some $\omega_0 \in \Omega_B$. There are now two cases:

1. We have $\lambda \in \Lambda_A$. Here, Lemma 3.3.2 implies $R_\lambda \subset S_{\omega_0}^{(2k+2)*}$, because $\mathcal{R}_{\Lambda_A}$ is almost subordinate to $\mathcal{S}$. But $\omega_0 \in \Omega_B \cap \Omega_\lambda$ because of $R_\lambda \cap S_{\omega_0} \neq \varnothing$. Hence,

$$R_\lambda \subset S_{\omega_0}^{(2k+2)*} \subset \bigcup_{\omega \in \Omega_B \cap \Omega_\lambda} S_\omega^{(2k+2)*}.$$

2. We have $\lambda \notin \Lambda_A$. For arbitrary $x \in R_\lambda \subset \mathcal{O} = \mathcal{O} \cap \mathcal{O}' = A \cup B$ this implies $x \notin A$ and hence $x \in B \subset \mathcal{O}' = \bigcup_{\omega \in \Omega} S_\omega$, so that we get $x \in S_\omega$ for some $\omega \in \Omega$. But $x \in B$ implies $\omega \in \Omega_B$. Further, $x \in S_\omega \cap R_\lambda$ yields $\omega \in \Omega_B \cap \Omega_\lambda$ and hence

$$R_\lambda \subset \bigcup_{\omega \in \Omega_B \cap \Omega_\lambda} S_\omega \subset \bigcup_{\omega \in \Omega_B \cap \Omega_\lambda} S_\omega^{(2k+2)*}.$$

All in all, this shows that $m = 2k + 2$ is a valid choice. ◀

*Proof of Theorem 5.4.5.* In the following, we will make use of the constants $C_1, C_2, C_2''$ and $K_1, K_1', K_2, K_2', K_2''$ as defined in Corollary 5.2.6. With this notation, we have $L_1 = K_1'$ and $L_2 = K_2''$.

1. This is an immediate consequence of Corollary 5.2.7 together with the last part of the ensuing remark and with $\Omega_0 \subset \Omega_B$, which was shown in Corollary 5.2.6.

2. First note that Lemma 5.3.1 (with $K = \mathcal{O} \cap \mathcal{O}'$) yields $p_1 \leq p_2$.

An application of equation (5.3.11) from Theorem 5.3.6 (where we set $\mathcal{Q} = \mathcal{R}$, $\mathcal{P} = \mathcal{S}$ and $I_0 = \Lambda_A$) yields

$$C_1 \left( q_1, q_2, p_2, \frac{1}{p_1} - \frac{1}{p_2}, u, v \right) \lesssim |||\theta|||\,.$$

But Corollary 5.2.6 shows $L_1 = K_1' \asymp C_1$, which proves the first claim.

An application of equation (5.3.13) from Theorem 5.3.6, (this time with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$ and $I_0 = \Omega_B$ and with interchanged roles of $u, v$ and $p_1, p_2$, as well as $q_1, q_2$) shows (in the notation of Corollary 5.2.6) that

$$C_2'' \left( q_1, q_2, p_2, \frac{1}{p_1} - \frac{1}{p_2}, u, v \right) \lesssim |||\theta|||\,.$$

Finally, Corollary 5.2.6 implies $C_2'' \asymp K_2'' = L_2$, which proves the second claim.

Furthermore, if one of the first set of additional assumptions is satisfied, we can apply Theorem 5.3.10 (with $\mathcal{Q} = \mathcal{R}$, $\mathcal{P} = \mathcal{S}$ and $I_0 = \Lambda_A$), which implies (in the notation of Corollary 5.2.6) that

$$L_1 = K_1' \left( q_1, q_2, p_2^{\triangledown}, \frac{1}{p_1} - \frac{1}{p_2}, u, v \right) \lesssim |||\theta||| \,.$$

Finally, each of the conditions in the second set of additional assumptions entails $p_1 \in [1, \infty]$. This is clear for the first and last conditions. For the second condition note that $p_1 \in (0,1)$ would imply $p_1' = \infty$, in contradiction to $q_2 > p_1'$. But $p_1 \in [1, \infty]$ entails $(p_1)_d = 0$. Furthermore, since one of the three conditions holds, all assumptions of Theorem 5.3.9 (with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$ and $I_0 = \Omega_B$ and with interchanged roles of $u, v$ and $p_1, p_2$, as well as $q_1, q_2$) are satisfied. But with this choice of parameters, equation (5.3.30) of Theorem 5.3.9 implies

$$L_2 \left( q_1, q_2, p_1^{\triangle}, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d \,, \left( u_\lambda \left| \det \left( T_\lambda \right) \right|^{(p_1)_d} \right)_\lambda, v \right) \lesssim |||\theta||| \,,$$

as desired. Here, we used $(p_1)_d = 0$.

3. We observe that the assumptions stated in this part imply the assumptions required in Theorem 5.3.12, with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$ and $I_0 = \Omega_B$ and with interchanged roles of $p_1, p_2$ and $q_1, q_2$, as well as $u, v$. These choices imply (in the notation of Theorem 5.3.12)

$$\begin{aligned} J_0 &= \left\{ j \in J \,\middle|\, I_0 \cap I_j \neq \varnothing \right\} \\ &= \left\{ \lambda \in \Lambda \,|\, \Omega_B \cap \Omega_\lambda \neq \varnothing \right\} \\ &= \left\{ \lambda \in \Lambda \,|\, \exists \omega \in \Omega_B : \, S_\omega \cap R_\lambda \neq \varnothing \right\} = \Lambda_1. \end{aligned}$$

Using this identity, we see that the set $K$ as in Theorem 5.3.12 satisfies

$$\begin{aligned} K &:= \bigcup_{j \in J_0} \overline{P_j^{(2k+3)*}} \\ &= \bigcup_{\lambda \in \Lambda_1} \overline{R_\lambda^{(2k+3)*}} \\ &\subset \mathcal{O} \cap \mathcal{O}', \end{aligned}$$

due to equation (5.4.2). We also used that $\overline{R_\lambda^{(2k+3)*}} \subset \mathcal{O}$ holds due to Lemma 3.2.12.

Thus, boundedness of $\theta$ implies boundedness of

$$\gamma : \left( \mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{P}, L^{p_1}, \ell_u^{q_1}\right)} \right) \to \mathcal{D}\left( \mathcal{Q}, L^{p_2}, \ell_v^{q_2} \right), f \mapsto f$$

with $|||\gamma||| \leq |||\theta|||$.

Finally, equation (5.4.3) shows that part 4 of the assumptions of Theorem 5.3.12 is indeed satisfied. The remaining assumptions are easily seen to hold as well, so that we get

$$\left\| \left( \frac{v_{\omega_\lambda}}{u_\lambda} \left|\det\left(T_\lambda\right)\right|^t \left|\det\left(M_{\omega_\lambda}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-t} \right)_{\lambda\in\Lambda_1} \right\|_{\ell^{q_2}(q_1/q_2)'} \lesssim |||\gamma||| \leq |||\theta|||\,, \tag{5.4.5}$$

with $t = \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+$ and where the index $\omega_\lambda \in \Omega_B \cap \Omega_\lambda$ can be chosen arbitrarily for each $\lambda \in \Lambda_1$. Here, it is important to recall the interchanged roles of $p_1, p_2$ and $q_1, q_2$ as well as $u, v$.

Next, we observe that relative $\mathcal{R}$-moderateness of $\mathcal{S}_{\Omega_B}$, together with equation (5.4.3), ensures that all assumptions of Lemma 3.3.7 (with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{R}$, $I_0 = \Omega_B$ and $J_0 = \Lambda_1$) are satisfied. The only nontrivial part of the assumptions is part 3, which is a direct consequence of equation (5.4.3). This implies

$$|\Omega_B \cap \Omega_\lambda| \asymp \frac{|\det\left(T_\lambda\right)|}{|\det\left(M_{\omega_\lambda}\right)|}$$

for all $\lambda \in \Lambda_1$, since we have $\Omega_B \cap \Omega_\lambda \neq \varnothing$ by definition of $\Lambda_1$.

For brevity, set $r := \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d$. Together with $\mathcal{R}$-moderateness of $v|_{\Omega_B}$, we conclude

$$\begin{aligned}
&\left\| \left(v_\omega \left|\det\left(M_\omega\right)\right|^r\right)_{\omega\in\Omega_B\cap\Omega_\lambda} \right\|_{\ell^{q_2\cdot\left(p_1^\triangle/q_2\right)'}} \\
&\asymp v_{\omega_\lambda} \left|\det\left(M_{\omega_\lambda}\right)\right|^r \cdot |\Omega_B \cap \Omega_\lambda|^{\frac{1}{q_2\cdot\left(p_1^\triangle/q_2\right)'}} \\
&\asymp v_{\omega_\lambda} \left|\det\left(M_{\omega_\lambda}\right)\right|^{r-s} \cdot \left|\det\left(T_\lambda\right)\right|^s,
\end{aligned}$$

for

$$\begin{aligned}
s &:= \frac{1}{q_2 \cdot \left(p_1^\triangle/q_2\right)'} \\
&\overset{(\dagger)}{=} \left(\frac{1}{q_2} - \frac{1}{p_1^\triangle}\right)_+ \\
&\overset{(\ddagger)}{=} t + (p_1)_d\,,
\end{aligned}$$

where Remark 5.2.3 was used at (†) and Remark 5.2.10 was used at (‡).

In summary, we conclude

$$\begin{aligned}
&L_2\left(q_1, q_2, p_1^\triangle, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d\,, \left(u_\lambda \left|\det\left(T_\lambda\right)\right|^{(p_1)_d}\right)_\lambda, v\right) \\
&= \left\| \left( u_\lambda^{-1} \left|\det\left(T_\lambda\right)\right|^{-(p_1)_d} \cdot \left\| \left(v_\omega \left|\det\left(M_\omega\right)\right|^r\right)_{\omega\in\Omega_B\cap\Omega_\lambda} \right\|_{\ell^{q_2\cdot\left(p_1^\triangle/q_2\right)'}} \right)_{\lambda\in\Lambda} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(\dagger)}{=} \left\| \left( u_\lambda^{-1} \left|\det\left(T_\lambda\right)\right|^{-(p_1)_d} \cdot \left\| \left( v_\omega \left|\det\left(M_\omega\right)\right|^r \right)_{\omega\in\Omega_B\cap\Omega_\lambda} \right\|_{\ell^{q_2}\cdot\left(p_1^{\triangle}/q_2\right)'} \right)_{\lambda\in\Lambda_1} \right\|_{\ell^{q_2}\cdot(q_1/q_2)'} \\
&\asymp \left\| \left( \frac{v_{\omega_\lambda}}{u_\lambda} \left|\det\left(T_\lambda\right)\right|^{s-(p_1)_d} \left|\det\left(M_{\omega_\lambda}\right)\right|^{r-s} \right)_{\lambda\in\Lambda_1} \right\|_{\ell^{q_2}\cdot(q_1/q_2)'} \\
&= \left\| \left( \frac{v_{\omega_\lambda}}{u_\lambda} \left|\det\left(T_\lambda\right)\right|^{t} \left|\det\left(M_{\omega_\lambda}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-t} \right)_{\lambda\in\Lambda_1} \right\|_{\ell^{q_2}\cdot(q_1/q_2)'} \\
&\lesssim |||\theta|||\,.
\end{aligned}
$$

Here, the last step is justified by equation (5.4.5). The step marked with (†) used that $\Omega_B \cap \Omega_\lambda = \varnothing$ holds for all $\lambda \in \Lambda \setminus \Lambda_1$, by definition of $\Lambda_1$.

4. Let us first show $\Lambda_A \subset \Lambda_0 \subset \Lambda_A^{(2\ell+3)*}$. For $\lambda \in \Lambda_A$, there is some $x \in R_\lambda \cap A$. Hence,

$$
x \in A \subset \mathcal{O} \cap \mathcal{O}' \subset \mathcal{O}' = \bigcup_{\omega\in\Omega} S_\omega,
$$

which yields $x \in S_\omega$ for some $\omega \in \Omega$ and hence $x \in S_\omega \cap R_\lambda \neq \varnothing$, i.e. $\omega \in \Omega_1$ because of $\lambda \in \Lambda_A$. By definition of $\Lambda_0$, this implies $\lambda \in \Lambda_0$.

For the reverse inclusion, let $\lambda \in \Lambda_0$ be arbitrary. This furnishes an index $\omega \in \Omega_1$ and some $x \in R_\lambda \cap S_\omega$. By definition of $\Omega_1$, there is a suitable $\gamma \in \Lambda_A$ as well as some $y \in R_\gamma \cap S_\omega \neq \varnothing$. There are two cases:

*Case* 1. $S_\omega \cap B \neq \varnothing$, i.e. $\omega \in \Omega_B$. In this case, assumption 2 from the beginning of the theorem, together with Lemma 3.3.2 yields $S_\omega \subset R_\lambda^{(2\ell+2)*}$ and hence

$$
y \in R_\gamma \cap S_\omega \subset R_\gamma \cap R_\lambda^{(2\ell+2)*},
$$

which implies $\lambda \in \gamma^{(2\ell+3)*} \subset \Lambda_A^{(2\ell+3)*}$.

*Case* 2. $S_\omega \cap B = \varnothing$. In this case, $x \in R_\lambda \cap S_\omega \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$. But $S_\omega \cap B = \varnothing$ yields $x \in A$ and hence $x \in R_\lambda \cap A$, which entails $\lambda \in \Lambda_A \subset \Lambda_A^{(2\ell+3)*}$.

Note that

$$
\begin{aligned}
\Lambda_0 &= \{\lambda \in \Lambda \mid \exists \omega \in \Omega_1 : \omega \in \Omega_\lambda\} \\
&= \{\lambda \in \Lambda \mid \Omega_1 \cap \Omega_\Lambda \neq \varnothing\}\,.
\end{aligned}
$$

With this identity, we see that our assumptions precisely ensure that Theorem 5.3.14 is applicable with $\mathcal{Q} = \mathcal{R}$, $\mathcal{P} = \mathcal{S}$, $J_0 = \Omega_1$ and $I_0 = \Lambda_0$. For this, we observe that we have $R_\lambda \subset \bigcup_{\omega\in\Omega} S_\omega = \mathcal{O}'$

for each $\lambda \in \Lambda_0$, since $\mathcal{R}_{\Lambda_0}$ is almost subordinate to $\mathcal{S}$. Furthermore, $R_\lambda \subset \mathcal{O}$ is trivial. With $K$ as in Theorem 5.3.14, this yields

$$K = \bigcup_{\lambda \in \Lambda_0} R_\lambda \subset \mathcal{O} \cap \mathcal{O}',$$

so that the map

$$\gamma : \left(\mathcal{D}_K, \|\cdot\|_{\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)}\right) \to \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right), f \mapsto f$$

is well-defined and bounded with $|||\gamma||| \leq |||\theta|||$, since $\gamma$ is a restriction of $\theta$.

Hence, Theorem 5.3.14 yields

$$\begin{aligned} &\left\| \left( \frac{|\det(M_\omega)|^{\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+}}{u_{\lambda_\omega}} |\det(T_{\lambda_\omega})|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+} \right)_{\omega \in \Omega_1} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\lesssim |||\gamma||| \leq |||\theta|||, \end{aligned} \tag{5.4.6}$$

where for each $\omega \in \Omega_1$, an arbitrary $\lambda_\omega \in \Lambda_\omega$ is chosen.

Let us now check that Lemma 3.3.7 is applicable with $\mathcal{Q} = \mathcal{R}$, $\mathcal{P} = \mathcal{S}$, $I_0 = \Lambda_A$ and $J_0 = \Omega_1$. First note that $\mathcal{Q}_{I_0} = \mathcal{R}_{\Lambda_A}$ is almost subordinate to $\mathcal{S}$, since $\mathcal{R}_{\Lambda_0}$ is almost subordinate to $\mathcal{S}$ and because of $\Lambda_A \subset \Lambda_0$. With the same argument, we see that $\mathcal{Q}_{I_0} = \mathcal{R}_{\Lambda_A}$ is relatively moderate with respect to $\mathcal{P}$ and hence with respect to $\mathcal{P}_{J_0}$.

Finally, for $\omega \in \Omega_1$, we have $S_\omega \subset \mathcal{O}$ by equation (5.4.4) and hence $S_\omega \subset \mathcal{O} \cap \mathcal{O}'$. There are now two cases:

*Case* 1. We have $S_\omega \cap B = \varnothing$. Because of $S_\omega \subset \mathcal{O} \cap \mathcal{O}' = A \cup B$, this implies $S_\omega \subset A$. But since $A \subset \mathcal{O}$ and $(R_\lambda)_{\lambda \in \Lambda}$ covers $\mathcal{O}$, it is easy to see that $(R_\lambda)_{\lambda \in \Lambda_A}$ covers $A$. Hence,

$$S_\omega \subset A \subset \bigcup_{\lambda \in \Lambda_A} R_\lambda \subset \bigcup_{\lambda \in \Lambda_A} R_\lambda^{(2\ell+2)*}.$$

*Case* 2. We have $S_\omega \cap B \neq \varnothing$. By assumption, this yields some $\lambda_\omega \in \Lambda$ with $S_\omega \subset R_{\lambda_\omega}^{\ell *}$. Furthermore, since $\omega \in \Omega_1$, there is some $\lambda \in \Lambda_A$ with $R_\lambda \cap S_\omega \neq \varnothing$. By Lemma 3.3.2, this implies $S_\omega \subset R_\lambda^{(2\ell+2)*}$, so that the inclusion from the first case also holds here.

Together, we have verified all assumptions of Lemma 3.3.7, so that

$$|\Lambda_A \cap \Lambda_\omega| \asymp \frac{|\det(M_\omega)|}{|\det(T_{\lambda_\omega})|}$$

holds for all $\omega \in \Omega_1$, where $\lambda_\omega \in \Lambda_A \cap \Lambda_\omega$ is chosen arbitrarily. This choice is possible, since

$$\begin{aligned}\Omega_1 &= \{\omega \in \Omega \,|\, \exists \lambda \in \Lambda_A : \lambda \in \Lambda_\omega\} \\ &= \{\omega \in \Omega \,|\, \Lambda_A \cap \Lambda_\omega \neq \varnothing\}.\end{aligned}$$

Together with $\mathcal{S}$-moderateness of $u|_{\Lambda_0}$ and hence of $u|_{\Lambda_A}$, we conclude

$$\begin{aligned}&\left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{\lambda \in \Lambda_A \cap \Lambda_\omega} \right\|_{\ell^{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'}} \\ &\asymp u_{\lambda_\omega}^{-1} \, |\det(T_{\lambda_\omega})|^{\frac{1}{p_1}-\frac{1}{p_2}} \cdot |\Lambda_A \cap \Lambda_\omega|^{\frac{1}{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'}} \\ &\asymp u_{\lambda_\omega}^{-1} \, |\det(T_{\lambda_\omega})|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \, |\det(M_\omega)|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+}\end{aligned}$$

for all $\omega \in \Omega_1$. In the last step, we used the identity

$$\frac{1}{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'} = \left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+,$$

which was shown in Remark 5.2.3. Together, we conclude

$$\begin{aligned}&L_1\left(q_1, q_2, p_2^\triangledown, \frac{1}{p_1}-\frac{1}{p_2}, u, v\right) \\ &= \left\| \left( \left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{\lambda \in \Lambda_A \cap \Lambda_\omega} \right\|_{\ell^{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'}} \right)_{\omega \in \Omega} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\overset{(\dagger)}{=} \left\| \left( \left\| \left( u_\lambda^{-1} \cdot |\det(T_\lambda)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{\lambda \in \Lambda_A \cap \Lambda_\omega} \right\|_{\ell^{p_2^\triangledown \cdot (q_1/p_2^\triangledown)'}} \right)_{\omega \in \Omega_1} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\asymp \left\| \left( \frac{|\det(M_\omega)|^{\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+}}{u_{\lambda_\omega}} \cdot |\det(T_{\lambda_\omega})|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\triangledown}-\frac{1}{q_1}\right)_+} \right)_{\omega \in \Omega_1} \right\|_{\ell_v^{q_2 \cdot (q_1/q_2)'}} \\ &\lesssim |||\theta|||.\end{aligned}$$

Here, the last step is justified by equation (5.4.6). The step marked with (†) used that $\Lambda_A \cap \Lambda_\omega = \varnothing$ holds for all $\omega \in \Omega \setminus \Omega_1$, by definition of $\Omega_1$. □

## 5.5 DECOMPOSITION SPACES AS SPACES OF TEMPERED DISTRIBUTIONS

In this last section of the chapter on embeddings for decomposition spaces, we study embeddings of decomposition spaces into the space of tempered distributions.

This question is nontrivial, since the (Fourier-side) decomposition spaces in this thesis are defined as subspaces of $\mathcal{D}'(\mathcal{O})$ instead of

$\mathcal{S}'\left(\mathbb{R}^d\right)$. As noted in Chapter 3 (see in particular Remark 3.4.7 and Example 3.4.14), there are two reasons for this:

1. We want to allow the case $\mathcal{O} \subsetneq \mathbb{R}^d$. In this case, one would have to factor out a certain subspace of $\mathcal{S}'\left(\mathbb{R}^d\right)$ to obtain a positive definite quasi-norm $\|\cdot\|_{\mathcal{D}\left(\mathcal{Q},L^p,\ell^q_u\right)}$.

2. Even in case of $\mathcal{O} = \mathbb{R}^d$, using $\mathcal{S}'\left(\mathbb{R}^d\right)$ as the reservoir can lead to incomplete spaces $\mathcal{D}_{\mathcal{S}'}\left(\mathcal{Q}, L^p, \ell^q_u\right)$.

The second reason can be seen as just a technical nuisance in most cases. That is why we now provide a readily verifiable sufficient criterion which ensures that each element $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \subset \mathcal{D}'\left(\mathcal{O}\right)$ of a decomposition space has an extension to a tempered distribution. In case of $\mathcal{O} = \mathbb{R}^d$, this implies that the two spaces $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ and $\mathcal{D}_{\mathcal{S}'}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ coincide (up to obviuous identifications). Our criterion reads as follows:

**Lemma 5.5.1.** *Assume that $\mathcal{Q} = (Q_i)_{i\in I} = (T_iQ' + b_i)_{i\in I}$ is a semi-structured admissible covering of the open set $\varnothing \neq \mathcal{O} \subset \mathbb{R}^d$. Let $p \in (0,\infty]$ and assume that there is an $L^p$-BAPU $\Phi = (\varphi_i)_{i\in I}$ for $\mathcal{Q}$ which satisfies*

$$C_\beta := \sup_{i\in I} \left\| \partial^\beta \left(\varphi_i \circ S_i\right) \right\|_{\sup} < \infty \tag{5.5.1}$$

*for all multiindices $\beta \in \mathbb{N}_0^d$ and $S_i : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto T_ix + b_i$.*

*Let $q \in (0,\infty]$ and assume that $u = (u_i)_{i\in I}$ is $\mathcal{Q}$-moderate. For each $N \in \mathbb{N}_0$, define $w^{(N)} = (w_i^{(N)})_{i\in I}$ by*

$$w_i^{(N)} := \left|\det\left(T_i\right)\right|^{1/p} \cdot \max\left\{1, \|T_i^{-1}\|^{d+1}\right\} \cdot \left[\inf_{y\in Q_i^*} \left(1+|y|\right)\right]^{-N}.$$

*If there is some subset $I_0 \subset I$ and some $N \in \mathbb{N}_0$ for which we have $w^{(N)}|_{I_0} \in \ell^{q'}_{1/u}\left(I_0\right)$, then*

$$\Phi : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{S}'\left(\mathbb{R}^d\right), f \mapsto \left[g \mapsto \sum_{i\in I_0} \left(\varphi_i f\right)(g)\right]$$

*is well-defined and continuous with respect to the weak-$*$-topology on $\mathcal{S}'\left(\mathbb{R}^d\right)$. Furthermore, the series defining $\left(\Phi f\right)(g)$ converges absolutely for every $g \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

*For $I_0 = I$, the tempered distribution $\Phi f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is an extension of $f : C_c^\infty\left(\mathcal{O}\right) \to \mathbb{C}$ to $\mathcal{S}\left(\mathbb{R}^d\right)$ for each $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$.*

*Remark.* The proof of [5, Proposition 1] (see in particular equation (3.3) of that proof) shows that an $L^p$-BAPU $(\varphi_i)_{i\in I}$ for which equation (5.5.1) is satisfied always exists in the case that $\mathcal{Q} = (T_iQ + b_i)_{i\in I}$ is a *structured* admissible covering of $\mathcal{O}$.

*Proof.* Let $N := N_{\mathcal{Q}}$, $R := R_{\mathcal{Q}}$ and $K := C_{\mathcal{Q}}$. For brevity, let us also set

$$c_i := \inf_{y \in Q_i^*} (1 + |y|)$$

for $i \in I$. Finally, let

$$\|g\|_{k,N} := \max_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^d} \left[ (1 + |x|)^k \cdot |(\partial^\alpha g)(x)| \right]$$

for $g \in \mathcal{S}(\mathbb{R}^d)$ and $k, N \in \mathbb{N}_0$.

Lemma 5.1.3 furnishes a constant $C^{(1)} = C^{(1)}(R, d, p) > 0$ such that

$$\left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^\infty} \leq C^{(1)} \cdot |\det(T_i)|^{1/p} \cdot \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \tag{5.5.2}$$

holds for all $i \in I$ and all $f \in \mathcal{D}'(\mathcal{O}) \supset \mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$.

For $\ell \in \underline{d}$, $k \in \mathbb{N}_0$, and $j \in I$, the chain rule implies (for $h_j := \varphi_j \circ S_j$)

$$\begin{aligned}
\left| \left( \partial_\ell^k \varphi_j \right)(x) \right| &= \left| \left( \partial_\ell^k \left( h_j \circ S_j^{-1} \right) \right)(x) \right| \\
&= \left| \left( \partial_\ell^k \left[ y \mapsto h_j \left( T_j^{-1}(y - b_j) \right) \right] \right)(x) \right| \\
&= \left| \left( \partial_\ell^k \left[ h_j \circ T_j^{-1} \right] \right)(x - b_j) \right| \\
&\leq C_k \cdot \|T_j^{-1}\|^k \cdot \max_{|\alpha| \leq k} \left\| \partial^\alpha h_j \right\|_{\sup} \\
&\leq C_k \cdot \|T_j^{-1}\|^k \cdot \max_{|\alpha| \leq k} C_\alpha \\
&\leq C_k' \cdot \|T_j^{-1}\|^k
\end{aligned}$$

for suitable constants $C_k, C_k'$ depending only on $k \in \mathbb{N}_0$ and on the constants $(C_\beta)_{\beta \in \mathbb{N}_0^d}$ (and on $d \in \mathbb{N}$).

If we additionally assume $j \in i^*$, we get

$$\|T_j^{-1}\| = \|T_j^{-1} T_i T_i^{-1}\| \leq \|T_j^{-1} T_i\| \cdot \|T_i^{-1}\| \leq K \cdot \|T_i^{-1}\|,$$

which leads to

$$\begin{aligned}
|(\partial_\ell^k \varphi_i^*)(x)| &\leq \sum_{j \in i^*} \left| \left( \partial_\ell^k \varphi_j \right)(x) \right| \\
&\leq C_k' \cdot \sum_{j \in i^*} \|T_j^{-1}\|^k \\
&\leq C_k' \cdot N \cdot K^k \cdot \|T_i^{-1}\|^k \\
&=: K_k \cdot \|T_i^{-1}\|^k .
\end{aligned}$$

Because of $\varphi_i^* \equiv 0$ on $\mathbb{R}^d \setminus Q_i^*$, we arrive at

$$|(\partial_\ell^k \varphi_i^*)(x)| \leq K_k \cdot \|T_i^{-1}\|^k \cdot \chi_{\overline{Q_i^*}}(x) \quad \text{for } x \in \mathbb{R}^d \text{ and } i \in I. \tag{5.5.3}$$

Let $g \in \mathcal{S}(\mathbb{R}^d)$ be arbitrary. Using Leibniz' rule we derive

$$\left|\left(\partial_\ell^k [\varphi_i^* \cdot g]\right)(x)\right| \leq \sum_{j=0}^{k} \left[\binom{k}{j} \cdot |(\partial_\ell^{k-j} \varphi_i^*)(x)| \cdot |(\partial_\ell^j g)(x)|\right].$$

But using $c_i = \inf_{y \in Q_i^*} (1+|y|) = \min_{y \in \overline{Q_i^*}} (1+|y|)$, we see

$$\begin{aligned}(1+|x|)^{-N} |(\partial_\ell^{k-j} \varphi_i^*)(x)| &\leq K_{k-j} \cdot \|T_i^{-1}\|^{k-j} \cdot \chi_{\overline{Q_i^*}}(x) (1+|x|)^{-N} \\ &\leq K_{k-j} \cdot \|T_i^{-1}\|^{k-j} \cdot \left[\inf_{y \in Q_i^*} (1+|y|)\right]^{-N} \\ &\leq K_{k-j} \cdot c_i^{-N} \cdot \max\left\{1, \|T_i^{-1}\|^k\right\}\end{aligned}$$

for all $x \in \mathbb{R}^d$. Furthermore,

$$(1+|x|)^{N+d+1} |(\partial_\ell^j g)(x)| \leq \|g\|_{N+d+1,k}$$

also holds for all $x \in \mathbb{R}^d$ and $j \in \{0, \dots, k\}$.

By setting $L_k := \sum_{j=0}^{k} \binom{k}{j} K_{k-j}$, a combination of the three estimates above shows

$$\begin{aligned}&\left|\left(\partial_\ell^k [\varphi_i^* \cdot g]\right)(x)\right| \\ &\leq L_k \|g\|_{N+d+1,k} \cdot c_i^{-N} \cdot \max\left\{1, \|T_i^{-1}\|^k\right\} \cdot (1+|x|)^{-d-1}\end{aligned}$$

for arbitrary $x \in \mathbb{R}^d$. For $L_k' := L_k \cdot \left\|(1+|x|)^{-d-1}\right\|_{L^1}$, this implies

$$\left\|\partial_\ell^k (\varphi_i^* \cdot g)\right\|_{L^1} \leq L_k' \cdot \|g\|_{N+d+1,k} \cdot c_i^{-N} \cdot \max\left\{1, \|T_i^{-1}\|^k\right\}. \quad (5.5.4)$$

It is easy to see that we have

$$(1+|x|)^{d+1} \leq M \cdot \left(1 + \sum_{\ell=1}^{d} |x_\ell|^{d+1}\right)$$

for all $x \in \mathbb{R}^d$ with a suitable constant $M > 0$ which only depends on $d \in \mathbb{N}$. Employing the well-known identity (see e.g. [33, Theorem 8.22(e)])

$$\xi^\alpha \cdot \widehat{g}(\xi) = \left(\frac{1}{2\pi i}\right)^{|\alpha|} \cdot \widehat{\partial^\alpha g}(\xi) \qquad \forall \xi \in \mathbb{R}^d$$

for $g \in \mathcal{S}(\mathbb{R}^d)$, we derive

$$\begin{aligned}&\left\|\widehat{\varphi_i^* \cdot g}\right\|_{L^1} \\ &\leq M \cdot \left\|(1+|\xi|)^{-d-1}\right\|_{L^1} \cdot \left\|\left(1 + \sum_{\ell=1}^{d} |\xi_\ell|^{d+1}\right) \cdot \widehat{\varphi_i^* \cdot g}\right\|_{\sup}\end{aligned}$$

$$
\begin{aligned}
&\leq M' \cdot \left[ \left\| \widehat{\varphi_i^* \cdot g} \right\|_{\sup} + \sum_{\ell=1}^{d} \left\| \xi_\ell^{d+1} \cdot \widehat{\varphi_i^* \cdot g} \right\|_{\sup} \right] \\
&= M' \cdot \left[ \left\| \widehat{\varphi_i^* \cdot g} \right\|_{\sup} + \sum_{\ell=1}^{d} \left| (2\pi i)^{-(d+1)} \right| \left\| \mathcal{F}\left[ \partial_\ell^{d+1} \left( \varphi_i^* \cdot g \right) \right] \right\|_{\sup} \right] \\
&\leq M' \cdot \left[ \left\| \varphi_i^* \cdot g \right\|_{L^1} + \sum_{\ell=1}^{d} \left\| \partial_\ell^{d+1} \left( \varphi_i^* \cdot g \right) \right\|_{L^1} \right] \qquad (5.5.5) \\
&\leq M' \max\left\{ L_0', L_{d+1}' \right\} \cdot \|g\|_{N+d+1,d+1} \cdot c_i^{-N} \cdot \max\left\{ 1, \|T_i^{-1}\|^{d+1} \right\},
\end{aligned}
$$

where the last step used estimate (5.5.4) and that the quantity

$$
\|g\|_{N+d+1,k} \cdot \max\left\{ 1, \|T_i^{-1}\|^{k} \right\}
$$

is monotonically increasing as a function of $k$ (because both factors are). The step before used boundedness of $\mathcal{F} : L^1\left(\mathbb{R}^d\right) \to C_0\left(\mathbb{R}^d\right)$.

Setting $v_i^{(N)} := c_i^{-N} \cdot \max\left\{ 1, \|T_i^{-1}\|^{d+1} \right\}$ for each $i \in I$, we have thus shown

$$
\left\| \widehat{\varphi_i^* \cdot g} \right\|_{L^1} \leq M'' \cdot \|g\|_{N+d+1,d+1} \cdot v_i^{(N)} \qquad \forall i \in I.
$$

Recall from Lemma 3.2.12 that $\varphi_i^* \equiv 1$ holds on $Q_i$. Consequently, $\varphi_i^* \varphi_i = \varphi_i$. But for $d_N := N + d + 1$ and $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_u^q\right)$, this implies

$$
\begin{aligned}
&\sum_{i \in I_0} \left| \left(\varphi_i f\right)(g) \right| \\
&= \sum_{i \in I_0} \left| \left(\varphi_i f\right)\left(\varphi_i^* g\right) \right| \\
&= \sum_{i \in I_0} \left| \left\langle \mathcal{F}^{-1}\left(\varphi_i f\right), \widehat{\varphi_i^* g} \right\rangle \right| \\
&\leq \sum_{i \in I_0} \left[ u_i \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^\infty} \cdot u_i^{-1} \left\| \widehat{\varphi_i^* g} \right\|_{L^1} \right] \\
&\overset{(\dagger)}{\leq} C^{(1)} M'' \|g\|_{d_N,d+1} \sum_{i \in I_0} \left[ u_i \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \cdot \left|\det\left(T_i\right)\right|^{\frac{1}{p}} \frac{v_i^{(N)}}{u_i} \right] \\
&\overset{(\ddagger)}{\leq} C^{(1)} M'' \|g\|_{d_N,d+1} \left\| \left( \left\| \mathcal{F}^{-1}(\varphi_i f) \right\|_{L^p} \right)_{i \in I_0} \right\|_{\ell_u^q} \left\| \left( \left|\det\left(T_i\right)\right|^{\frac{1}{p}} v_i^{(N)} \right)_{i \in I_0} \right\|_{\ell_{1/u}^{q'}} \\
&= C^{(1)} M'' \left\| \left( w_i^{(N)} \right)_{i \in I_0} \right\|_{\ell_{1/u}^{q'}} \|g\|_{d_N,d+1} \|f\|_{\mathcal{D}\left(\mathcal{Q},L^p,\ell_u^q\right)} < \infty.
\end{aligned}
$$

Here, the step marked with $(\dagger)$ is a consequence of equation (5.5.2) and $(\ddagger)$ is justified by Lemma 5.2.1. The above estimate proves that $\Phi f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is well-defined with absolute convergence of the series defining $\left(\Phi f\right)(g)$ for every $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Note that the term $\left\| \left(w_i^{(N)}\right)_{i \in I_0} \right\|_{\ell_{1/u}^{q'}}$ is finite by assumption.

Additionally, we see $(\Phi f)(g) \to 0$ for $f \to 0$ in $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ for every $g \in \mathcal{S}\left(\mathbb{R}^d\right)$, so that $\Phi$ is continuous.

Finally, Lemma 3.2.11 shows that $(\varphi_i)_{i\in I}$ is a locally finite partition of unity on $\mathcal{O}$, which implies $g = \sum_{i\in I} \varphi_i g$ for every $g \in C_c^\infty(\mathcal{O})$, where only finitely many terms of the sum do not vanish. In case of $I_0 = I$, this implies

$$(\Phi f)(g) = \sum_{i\in I} (\varphi_i f)(g) = \sum_{i\in I} f(\varphi_i g) = f\left(\sum_{i\in I} \varphi_i g\right) = f(g),$$

for all $g \in C_c^\infty(\mathcal{O})$, so that $\Phi f$ is indeed an extension of $f$. □

# 6

# APPLICATIONS

In this chapter, we illustrate the power and generality of our approach to embeddings for decomposition spaces by considering a number of concrete examples.

This chapter is structured as follows: In the first section, we study embeddings $M^{\gamma_1,\alpha_1}_{p_1,q_1} \hookrightarrow M^{\gamma_2,\alpha_2}_{p_2,q_2}$ between $\alpha$-MODULATION SPACES for different values of $\alpha$. For the case $(p_1, q_1) = (p_2, q_2)$, the existence of this embedding was completely characterized by Han and Wang in [49]. We will also obtain a complete characterization, even for $(p_1, q_1) \neq (p_2, q_2)$. In particular, the results of Han and Wang will turn out to be special cases of our approach. In this context it should be mentioned that the paper [49] was important inspiration for parts of our treatment of embeddings for decomposition spaces, in particular for the proofs of Theorems 5.3.12 and 5.3.14.

Nevertheless, our approach for proving existence of embeddings is quite different from the arguments in [49]. Han and Wang establish embeddings for some special cases and then proceed by interpolation of the whole modulation spaces $M^{s,\alpha}_{p,q}$. In contrast to this, our approach is based on a local interpolation result (cf. Lemmata 5.1.1 and 5.1.2). As seen in the previous chapter, we then lift the local statement to a global one by requiring certain embeddings for nested sequence spaces.

We finally mention that the results in this section shed new light on why the results in [49] actually hold. The main geometric fact that is used is that the $\alpha$-covering $\mathcal{O}^{(\alpha)}$ of $\mathbb{R}^d$ (to be defined below) is almost subordinate to the $\beta$-covering $\mathcal{O}^{(\beta)}$ for $\alpha \leq \beta$. Furthermore, $\mathcal{O}^{(\alpha)}$ is relatively moderate with respect to $\mathcal{O}^{(\beta)}$ and the same is true of the weights used to define the $\alpha$-modulation spaces.

In the second section, we consider homogeneous and inhomogeneous BESOV SPACES. We first show that our "Fourier-side" versions of these spaces are canonically isomorphic to the usual "space-side" versions. Afterwards, we consider embeddings of homogeneous Besov spaces into inhomogeneous Besov spaces and vice versa. Finally, we also characterize the existence of embeddings between $\alpha$-modulation spaces and (inhomogeneous) Besov spaces.

The interpretation of wavelet coorbit spaces as decomposition spaces is put to use in the third section. Here, we study the SHEARLET-TYPE

COORBIT SPACES $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$, where the shearlet-type group $H_c$ is given by

$$H_c = \left\{ \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \,\middle|\, \varepsilon \in \{\pm 1\}\,,\, a \in (0,\infty)\,,\, b \in \mathbb{R} \right\}.$$

For $c = \frac{1}{2}$, this yields the usual class of (two-dimensional) shearlet coorbit spaces as considered in [13, 15].

We consider embeddings between these coorbit spaces for different values of $c \in \mathbb{R}$. Furthermore, we study the existence of embeddings between shearlet-type coorbit spaces and (inhomogeneous) Besov spaces.

The fourth section is devoted to studying the recently introduced SHEARLET SMOOTHNESS SPACES[53]. We will see that our embeddings results yield a complete characterization of embeddings between these spaces and (inhomogeneous) Besov spaces, as well as $\alpha$-modulation spaces, at least for $\alpha \in \left[0, \frac{1}{2}\right]$. This improves (and corrects) the results in [53].

In the last section, we consider the question of dilation invariance for shearlet-type coorbit spaces and $\alpha$-modulation spaces. For certain weights $m$ and $c \in (-\infty, 1]$, we will asymptotically calculate the operator norm of the isotropic dilation as an operator on the shearlet-type coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. In particular, this includes the class of shearlet coorbit spaces as considered in [15].

Furthermore, we will precisely characterize the class of UNIVERSAL DILATIONS for the shearlet-type groups $H_c$. Here, we call a matrix $g \in \mathrm{GL}\left(\mathbb{R}^2\right)$ a universal dilation for $H_c$, if *all* shearlet coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ are invariant under dilation by $g$.

Finally, we calculate the asymptotics of the operator norm of the isotropic dilation as an operator on the $\alpha$-modulation spaces. For essentially all cases, this has already been done in [49]. Our findings will slightly extend and correct these results.

## 6.1 EMBEDDINGS BETWEEN $\alpha$-MODULATION SPACES

The class of $\alpha$-modulation spaces was introduced by Gröbner in his thesis [43]. A more recent treatment of these spaces is [3], in which Borup and Nielsen construct Banach frames for the $\alpha$-modulation spaces. An earlier construction of such Banach frames was given in [34]. Finally, in [64] and [49], the respective authors establish embedding statements between $\alpha$-modulation spaces for different values of $\alpha$.

Related to coorbit theory, the paper [12] is worth mentioning. There, the authors show that the $\alpha$-modulation spaces can be obtained as coorbit spaces of certain *quotients* of the affine Weyl-Heisenberg group.

We begin our study of $\alpha$-modulation spaces by describing the construction of the associated covering $\mathcal{O}^{(\alpha)}$. Our treatment is based on that of Borup and Nielsen[3].

**Theorem 6.1.1.** *(cf. [3, Theorem 2.6]) Let $0 \leq \alpha < 1$ be arbitrary and set $\alpha_0 := \frac{\alpha}{1-\alpha}$. Then there is a constant $r_1 = r_1(\alpha, d) > 0$ such that the family*

$$\mathcal{O}^{(\alpha)} := \mathcal{O}_r^{(\alpha)} := \left(O_{r,k}^{(\alpha)}\right)_{k\in\mathbb{Z}^d\setminus\{0\}} := \left(B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)\right)_{k\in\mathbb{Z}^d\setminus\{0\}}$$

*is an admissible covering of $\mathbb{R}^d$ for every $r > r_1$.*

Our plan is to examine the geometry of the coverings $\mathcal{O}^{(\alpha)}$ by

1. showing that the covering $\mathcal{O}^{(\alpha)}$ is a *structured* admissible covering of $\mathbb{R}^d$,
2. showing that the weight $\mathbb{Z}^d \setminus \{0\} \to (0,\infty), k \mapsto \langle k\rangle$ is moderate with respect to $\mathcal{O}^{(\alpha)}$,
3. showing that $\mathcal{O}^{(\alpha)}$ is almost subordinate to and relatively moderate with respect to $\mathcal{O}^{(\beta)}$ for $\alpha \leq \beta$.

*For convenience of the reader, we recall the definition $\langle x\rangle = \sqrt{1+|x|^2}$.*

We begin with establishing the second point.

**Lemma 6.1.2.** *Let $d \in \mathbb{N}$, $0 \leq \alpha < 1$ and $r > 0$ so that $\mathcal{O}_r^{(\alpha)}$ is an admissible covering of $\mathbb{R}^d$. We then have*

$$\langle x\rangle \asymp \langle k\rangle^{\frac{1}{1-\alpha}} \qquad \text{for all } k \in \mathbb{Z}^d \setminus \{0\} \text{ and } x \in O_{r,k}^{(\alpha)},$$

*where the implied constant only depends on $r, \alpha$.*

*In particular, the weight $w^{(\gamma)} = \left(\langle k\rangle^{\gamma}\right)_{k\in\mathbb{Z}^d\setminus\{0\}}$ is moderate with respect to $\mathcal{O}_r^{(\alpha)}$ for all $\gamma \in \mathbb{R}$.*

*Proof.* Define $\alpha_0 := \frac{\alpha}{1-\alpha} \in [0,\infty)$. For $k \in \mathbb{Z}^d \setminus \{0\}$ and arbitrary $x \in O_{r,k}^{(\alpha)} = B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)$, we have

$$\begin{aligned}
|x| &\leq \left|x - |k|^{\alpha_0} k\right| + \left||k|^{\alpha_0} k\right| \\
&< r\,|k|^{\alpha_0} + |k|^{\alpha_0}\,|k| \\
&\leq \langle k\rangle^{\alpha_0} \cdot [r + \langle k\rangle] \\
&\overset{(\dagger)}{\leq} (1+r) \cdot \langle k\rangle^{\alpha_0} \langle k\rangle \\
&= (1+r) \cdot \langle k\rangle^{\frac{1}{1-\alpha}}.
\end{aligned}$$

Here, $\langle k\rangle \geq 1$ was used at $(\dagger)$. Because of $\frac{1}{1-\alpha} > 0$, we have $\langle k\rangle^{\frac{1}{1-\alpha}} \geq 1$. Together, this implies

$$\langle x\rangle \leq 1 + |x| \leq \langle k\rangle^{\frac{1}{1-\alpha}} + |x| \leq (2+r) \cdot \langle k\rangle^{\frac{1}{1-\alpha}}.$$

For the other direction, we distinguish two cases:

*Case* 1. We have $|k| \geq 2r$. This implies $|k| - r \geq |k| - \frac{|k|}{2} = \frac{|k|}{2}$ and hence

$$\begin{aligned} |x| &= \left| |k|^{\alpha_0} k - \left( |k|^{\alpha_0} k - x \right) \right| \\ &\geq \left| |k|^{\alpha_0} k \right| - \left| |k|^{\alpha_0} k - x \right| \\ &> |k|^{\alpha_0} |k| - r |k|^{\alpha_0} \\ &= |k|^{\alpha_0} \cdot (|k| - r) \\ &\geq \frac{1}{2} \cdot |k|^{\alpha_0} |k| = \frac{1}{2} \cdot |k|^{\frac{1}{1-\alpha}} . \end{aligned}$$

Because of $k \in \mathbb{Z}^d \setminus \{0\}$, we have $|k| \geq 1$. But this implies $\langle k \rangle \leq 1 + |k| \leq 2 |k|$ and hence

$$\langle k \rangle^{\frac{1}{1-\alpha}} \leq 2^{\frac{1}{1-\alpha}} \cdot |k|^{\frac{1}{1-\alpha}} \leq 2^{1+\frac{1}{1-\alpha}} \cdot |x| \leq 2^{\frac{2-\alpha}{1-\alpha}} \cdot \langle x \rangle .$$

*Case* 2. We have $|k| \leq 2r$. In this case, we use the positivity of $\frac{1}{1-\alpha}$ together with $\langle x \rangle \geq 1$ and $\langle k \rangle \leq 1 + |k| \leq 1 + 2r$ to derive

$$\langle k \rangle^{\frac{1}{1-\alpha}} \leq (1 + 2r)^{\frac{1}{1-\alpha}} \leq (1 + 2r)^{\frac{1}{1-\alpha}} \cdot \langle x \rangle .$$

Together, this proves $\langle k \rangle^{\frac{1}{1-\alpha}} \leq C_0 \cdot \langle x \rangle$ with

$$C_0 := \max \left\{ (1 + 2r)^{\frac{1}{1-\alpha}} , 2^{\frac{2-\alpha}{1-\alpha}} \right\} .$$

We have thus established the first part of the lemma.

To prove moderateness of $w^{(\gamma)}$, note that $x \in O_{r,k}^{(\alpha)} \cap O_{r,\ell}^{(\alpha)}$ implies

$$\langle k \rangle \asymp \langle x \rangle^{1-\alpha} \asymp \langle \ell \rangle$$

and hence $1 \lesssim \langle k \rangle / \langle \ell \rangle \lesssim 1$, which yields

$$1 \lesssim \left( \langle k \rangle / \langle \ell \rangle \right)^{\gamma} \lesssim 1$$

and thus finally $w_k^{(\gamma)} = \langle k \rangle^{\gamma} \asymp \langle \ell \rangle^{\gamma} = w_\ell^{(\gamma)}$, where the implied constants only depend on $r, \alpha, \gamma$. □

Using this moderateness, we can now show that $\mathbb{O}_r^{(\alpha)}$ is indeed a structured admissible covering of $\mathbb{R}^d$.

**Theorem 6.1.3.** *Let $d \in \mathbb{N}$, $\alpha \in [0, 1)$ and set $\alpha_0 := \frac{\alpha}{1-\alpha}$. Let $r_1 = r_1(d, \alpha)$ as in Theorem 6.1.1. Then*

$$\mathbb{O}_r^{(\alpha)} = \left( O_{r,k}^{(\alpha)} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} = \left( T_k^{(\alpha)} Q^{(r)} + b_k \right)_{k \in \mathbb{Z}^d \setminus \{0\}}$$

*is a structured admissible covering of $\mathbb{R}^d$ for each $r > r_1$ with*

$$T_k^{(\alpha)} := |k|^{\alpha_0} \operatorname{id} \quad \text{and} \quad b_k := |k|^{\alpha_0} k \quad \text{as well as} \quad Q^{(r)} := B_r(0)$$

*for each $k \in \mathbb{Z}^d \setminus \{0\}$.*

*Proof.* Let $s := \frac{r_1+r}{2}$, so that we have $r_1 < s < r$. Define $P := Q^{(s)}$. This implies $\overline{P} := \overline{Q^{(s)}} \subset Q^{(r)} =: Q$. Using Theorem 6.1.1, we see that $\mathcal{O}_s^{(\alpha)} = (T_k^{(\alpha)} P + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ and $\mathcal{O}_r^{(\alpha)} = (T_k^{(\alpha)} Q + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ are both admissible coverings of $\mathbb{R}^d$.

It remains to show that $C_{\mathcal{O}_r^{(\alpha)}}$ is finite. To see this, let $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ be arbitrary with $\varnothing \neq O_{r,k}^{(\alpha)} \cap O_{r,\ell}^{(\alpha)}$. Using $\langle k \rangle \leq 1 + |k| \leq 2\,|k|$ as well as $|\ell| \leq \langle \ell \rangle$ and $\alpha_0 \geq 0$, we see

$$\begin{aligned}\left\| (T_k^{(\alpha)})^{-1} T_\ell^{(\alpha)} \right\| &= \left( \frac{|\ell|}{|k|} \right)^{\alpha_0} \leq \left( \frac{\langle \ell \rangle}{|k|} \right)^{\alpha_0} \leq \left( 2 \frac{\langle \ell \rangle}{\langle k \rangle} \right)^{\alpha_0} \\ &= 2^{\alpha_0} \cdot \frac{w_\ell^{(\alpha_0)}}{w_k^{(\alpha_0)}} \leq 2^{\alpha_0} \cdot C_{w^{(\alpha_0)}, \mathcal{O}_r^{(\alpha)}}.\end{aligned}$$

But $C_{w^{(\alpha_0)}, \mathcal{O}_r^{(\alpha)}}$ is finite by Lemma 6.1.2. This completes the proof. □

Theorem 6.1.3 (together with Theorem 3.2.17) implies in particular that the $\alpha$-covering $\mathcal{O}_r^{(\alpha)}$ admits a family $(\varphi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ which is an $L^p$-BAPU for $\mathcal{O}_r^{(\alpha)}$ for every $p \in (0, \infty]$. Hence, the associated decomposition spaces are well-defined.

**Definition 6.1.4.** Let $d \in \mathbb{N}$ and $0 \leq \alpha < 1$ be arbitrary. Choose $r_1 = r_1(d, \alpha)$ as in Theorem 6.1.1. For each $r > r_1$, and $p, q \in (0, \infty]$, as well as $\gamma \in \mathbb{R}$, we define the FOURIER-SIDE $\alpha$-MODULATION SPACE with integrability exponents $p, q$ and weight-exponent $\gamma$ as

$$\mathcal{F}M_{p,q}^{\gamma,\alpha} := \mathcal{D}\left( \mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma/(1-\alpha))}} \right),$$

where the weight $w^{(\gamma/(1-\alpha))}$ is defined as in Lemma 6.1.2.

*Remark.* Observe that the parameter $r$ is suppressed on the left-hand side of the definition above. But since any two coverings $\mathcal{O}_r^{(\alpha)}, \mathcal{O}_s^{(\alpha)}$ (with $r, s > r_1(d, \alpha)$) use the *same* family of normalizations, it follows that every $L^p$-BAPU $\Phi$ for $\mathcal{O}_r^{(\alpha)}$ is also an $L^p$-BAPU for $\mathcal{O}_s^{(\alpha)}$, at least for $s \geq r$. We can always assume this by symmetry. With this choice of the BAPUs for both spaces, we get

$$\|\cdot\|_{\mathcal{D}_\Phi\left(\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma/(1-\alpha))}}\right)} = \|\cdot\|_{\mathcal{D}_\Phi\left(\mathcal{O}_s^{(\alpha)}, L^p, \ell^q_{w^{(\gamma/(1-\alpha))}}\right)},$$

which also implies that both spaces coincide. Since different choices of the BAPU yield the same space with equivalent quasi-norms (cf. Corollary 3.4.11), we see that the right-hand side of the above definition is independent of the choice of $r > r_1(d, \alpha)$, with equivalent quasi-norms for different choices.

We will also show below (cf. Lemma 6.1.9) that the Fourier transform restricts to an isomorphism

$$\mathcal{F} : M_{p,q}^{\gamma,\alpha}(\mathbb{R}^d) \to \mathcal{F}M_{p,q}^{\gamma,\alpha},$$

where the (space-side) $\alpha$-modulation space on the left-hand side is defined as in [3, Definition 2.4]. This justifies the notation $\mathcal{F}M^{\gamma,\alpha}_{p,q}$.

As the next step in our program, we will estimate the cardinalities of the "intersection sets". As a byproduct, we will then see (using Lemma 3.3.3) that $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{O}_s^{(\beta)}$ for $\alpha \leq \beta$.

**Lemma 6.1.5.** *Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0,1)$ with $\alpha \leq \beta$. Let $r_1 = r_1(d,\alpha)$ and $r_2 = r_1(d,\beta)$ as in Theorem 6.1.1 and let $r > r_1$ and $s > r_2$.*

*There is a constant $C = C(d,\alpha,\beta,r,s) > 0$ with*

$$1 \leq \left| \left\{ \ell \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; O^{(\alpha)}_{r,k} \cap O^{(\beta)}_{s,\ell} \neq \varnothing \right\} \right| \leq C \tag{6.1.1}$$

*for all $k \in \mathbb{Z}^d \setminus \{0\}$.*

*Proof.* Set

$$J_k := \left\{ \ell \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; O^{(\alpha)}_{r,k} \cap O^{(\beta)}_{s,\ell} \neq \varnothing \right\}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$ and let $\alpha_0 := \frac{\alpha}{1-\alpha}$ and $\beta_0 := \frac{\beta}{1-\beta}$.

Lemma 6.1.2 implies

$$\begin{aligned} \langle k \rangle^{\frac{1}{1-\alpha}} &\asymp \langle x \rangle \qquad \forall k \in \mathbb{Z}^d \setminus \{0\} \text{ and } x \in O^{(\alpha)}_{r,k}, \\ \langle \ell \rangle^{\frac{1}{1-\beta}} &\asymp \langle x \rangle \qquad \forall \ell \in \mathbb{Z}^d \setminus \{0\} \text{ and } x \in O^{(\beta)}_{s,\ell}, \end{aligned} \tag{6.1.2}$$

where the implied constants only depend on $d, \alpha, \beta, r, s$.

The lower estimate in equation (6.1.1) is trivial, because of

$$\varnothing \neq O^{(\alpha)}_{r,k} \subset \mathbb{R}^d = \bigcup_{\ell \in \mathbb{Z}^d \setminus \{0\}} O^{(\beta)}_{s,\ell}.$$

To prove the upper estimate, fix $k \in \mathbb{Z}^d \setminus \{0\}$, some $\ell_0 \in J_k$ and some $x_0 \in O^{(\beta)}_{s,\ell_0} \cap O^{(\alpha)}_{r,k}$. Using equation (6.1.2), this yields

$$|k|^{\alpha_0} \leq \left( \langle k \rangle^{\frac{1}{1-\alpha}} \right)^{\alpha} \lesssim \langle x_0 \rangle^{\alpha} \lesssim \left( \langle \ell_0 \rangle^{\frac{1}{1-\beta}} \right)^{\alpha} \overset{(\dagger)}{\leq} \langle \ell_0 \rangle^{\frac{\beta}{1-\beta}} = \langle \ell_0 \rangle^{\beta_0}$$

where the step marked with $(\dagger)$ is justified by $\langle \ell_0 \rangle \geq 1$ and $\alpha \leq \beta$. As usual, the implied constants only depend on $d, \alpha, \beta, r, s$.

Note that for each $\ell \in J_k$, there is some $x \in O^{(\alpha)}_{r,k} \cap O^{(\beta)}_{s,\ell}$, so that an application of equation (6.1.2), together with $\langle \ell \rangle \leq 1 + |\ell| \leq 2\,|\ell|$, leads to

$$\langle \ell_0 \rangle^{\beta_0} \lesssim \langle x \rangle^{\beta} \lesssim \left( \langle \ell \rangle^{\frac{1}{1-\beta}} \right)^{\beta} \lesssim |\ell|^{\beta_0} \tag{6.1.3}$$

and

$$|\ell|^{\beta_0} \leq \left( \langle \ell \rangle^{\frac{1}{1-\beta}} \right)^{\beta} \lesssim \langle x \rangle^{\beta} \lesssim \left( \langle \ell_0 \rangle^{\frac{1}{1-\beta}} \right)^{\beta} = \langle \ell_0 \rangle^{\beta_0}.$$

For $m \in \mathbb{Z}^d \setminus \{0\}$ and $\gamma \in [0,1)$, let us denote the $\gamma$-NORMALIZED VERSION of $m$ by $m^{(\gamma)} := |m|^{\gamma_0} m$ with $\gamma_0 := \frac{\gamma}{1-\gamma}$. By definition,

we have $O_{t,m}^{(\gamma)} = B_{t|m|^{\gamma_0}}(m^{(\gamma)})$ for all $t > 0$ and $m \in \mathbb{Z}^d \setminus \{0\}$. For arbitrary $y \in O_{s,\ell}^{(\beta)}$, a combination of the estimates from the previous two paragraphs shows

$$\begin{aligned}
&\left|y - |\ell_0|^{\beta_0}\,\ell_0\right| \\
&\leq \left|y - \ell^{(\beta_0)}\right| + \left|\ell^{(\beta_0)} - x\right| + \left|x - k^{(\alpha_0)}\right| + \left|k^{(\alpha_0)} - x_0\right| + \left|x_0 - \ell_0^{(\beta_0)}\right| \\
&< s \cdot |\ell|^{\beta_0} + s \cdot |\ell|^{\beta_0} + r \cdot |k|^{\alpha_0} + r \cdot |k|^{\alpha_0} + s \cdot |\ell_0|^{\beta_0} \\
&\lesssim (s+r) \cdot \langle \ell_0 \rangle^{\beta_0}
\end{aligned}$$

and thus $O_{s,\ell}^{(\beta)} \subset B_{C_1\langle \ell_0\rangle^{\beta_0}}(|\ell_0|^{\beta_0}\,\ell_0)$ for all $\ell \in J_k$ and some constant $C_1 = C_1(d, \alpha, \beta, r, s)$.

But admissibility of $\mathcal{O}_s^{(\beta)}$ yields a constant $C_2 = C_2(d, \beta, s) > 0$ with

$$\sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \chi_{O_{s,\ell}^{(\beta)}} \leq C_2.$$

In fact, we can take $C_2 = N_{\mathcal{O}_s^{(\beta)}}$ by the remark after Definition 3.2.3. Together with the above inclusion, we conclude

$$\sum_{\ell \in J_k} \chi_{O_{s,\ell}^{(\beta)}} \leq C_2 \cdot \chi_{B_{C_1\langle \ell_0\rangle^{\beta_0}}(|\ell_0|^{\beta_0}\ell_0)}.$$

Finally, integration of this estimate yields

$$\begin{aligned}
s^d \cdot |J_k| \cdot \langle \ell_0 \rangle^{d\beta_0} &\lesssim \sum_{\ell \in J_k} s^d\,|\ell|^{d\beta_0} \\
&\asymp \sum_{\ell \in J_k} \lambda\left(O_{s,\ell}^{(\beta)}\right) \\
&\leq C_2 \cdot \lambda\left(B_{C_1\langle \ell_0\rangle^{\beta_0}}(|\ell_0|^{\beta_0}\,\ell_0)\right) \\
&\asymp C_1^d C_2 \cdot \langle \ell_0 \rangle^{d\beta_0},
\end{aligned}$$

where the first step is justified by equation (6.1.3). This establishes the upper estimate in equation (6.1.1). □

We are now in a position to show that the $\alpha$-covering $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to the $\beta$-covering $\mathcal{O}_s^{(\beta)}$ for $\alpha \leq \beta$.

**Theorem 6.1.6.** *Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0,1)$ with $\alpha \leq \beta$ be arbitrary. Choose $r_1 = r_1(d, \alpha)$ and $r_2 = r_1(d, \beta)$ as in Theorem 6.1.1 and let $r > r_1$ as well as $s > r_2$.*

*Then $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to and moderate with respect to $\mathcal{O}_s^{(\beta)}$. More precisely,*

$$C^{-1} \cdot \langle \ell \rangle^{d\frac{\alpha}{1-\beta}} \leq |\det(T_k^{(\alpha)})| \leq C \cdot \langle \ell \rangle^{d\frac{\alpha}{1-\beta}}$$

*holds for all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap O_{s,\ell}^{(\beta)} \neq \varnothing$ and some constant $C = C(d, \alpha, \beta, r, s) > 0$.*

*Finally, let $\gamma \in \mathbb{R}$ be arbitrary. Then, there is $L = L(d, \alpha, \beta, \gamma, r, s) > 0$ with*

$$L^{-1} \cdot w_\ell^{\left(\gamma \cdot \frac{1-\alpha}{1-\beta}\right)} \leq w_k^{(\gamma)} \leq L \cdot w_\ell^{\left(\gamma \cdot \frac{1-\alpha}{1-\beta}\right)} \tag{6.1.4}$$

*for all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap O_{s,\ell}^{(\beta)} \neq \varnothing$. Here, $w^{(\gamma)}$ is defined as in Lemma 6.1.2.*

*Proof.* The coverings $\mathcal{O}_r^{(\alpha)}, \mathcal{O}_s^{(\beta)}$ are families of open balls, which are in particular path connected. Equation (6.1.1) of Lemma 6.1.5 shows that $\mathcal{O}_r^{(\alpha)}$ is *weakly* subordinate to $\mathcal{O}_s^{(\beta)}$, so that Corollary 3.3.4 implies that $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{O}_s^{(\beta)}$.

Now, let $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $\varnothing \neq O_{r,k}^{(\alpha)} \cap O_{s,\ell}^{(\beta)} \ni x$. Lemma 6.1.2 allows us to conclude

$$\langle k \rangle^{\frac{1}{1-\alpha}} \asymp \langle x \rangle \asymp \langle \ell \rangle^{\frac{1}{1-\beta}},$$

where the implied constants only depend on $d, \alpha, \beta, r, s$. If implied constants also depend on $\gamma$, this is indicated by $\asymp_\gamma$ and $\lesssim_\gamma$. The above implies $\langle k \rangle^{\frac{1}{1-\alpha}} / \langle \ell \rangle^{\frac{1}{1-\beta}} \asymp 1$ and hence

$$1 \lesssim_\gamma \left( \frac{\langle k \rangle^{\frac{1}{1-\alpha}}}{\langle \ell \rangle^{\frac{1}{1-\beta}}} \right)^{(1-\alpha)\gamma} \lesssim_\gamma 1,$$

which means $\langle k \rangle^\gamma \asymp_\gamma \langle \ell \rangle^{\gamma \frac{1-\alpha}{1-\beta}}$.

This also shows that $\mathcal{O}_r^{(\alpha)} = (T_k^{(\alpha)} Q^{(r)} + b_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is relatively moderate with respect to $\mathcal{O}_s^{(\beta)}$, because of

$$\begin{aligned} |\det(T_k^{(\alpha)})| &= \left|\det\left(|k|^{\alpha_0} \operatorname{id}\right)\right| = |k|^{d \cdot \alpha_0} \\ &\asymp \langle k \rangle^{d \cdot \alpha_0} = w_k^{(d \cdot \alpha_0)} \asymp \langle \ell \rangle^{d \cdot \alpha_0 \cdot \frac{1-\alpha}{1-\beta}} = \langle \ell \rangle^{d \cdot \frac{\alpha}{1-\beta}} \end{aligned}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap O_{s,\ell}^{(\beta)} \neq \varnothing$. □

After the above analysis of the geometric relationships between the different $\alpha$-coverings, it is now straightforward to derive the announced embeddings between $\alpha$-modulation spaces with different values of $\alpha$.

The "limit case" $\alpha = 1$ – which corresponds to (inhomogeneous) Besov spaces – is treated below, cf. Theorem 6.2.8.

**Theorem 6.1.7.** *Let $d \in \mathbb{N}$ and $\alpha, \beta \in [0, 1)$ with $\alpha \leq \beta$. Finally, let $p_1, p_2, q_1, q_2 \in (0, \infty]$, as well as $\gamma_1, \gamma_2 \in \mathbb{R}$. Define*

$$\begin{aligned} \gamma^{(0)} &:= \alpha \left( \frac{1}{p_2} - \frac{1}{p_1} \right) + (\alpha - \beta) \left( \frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+, \\ \gamma^{(1)} &:= \alpha \left( \frac{1}{p_2} - \frac{1}{p_1} \right) + (\alpha - \beta) \left( \frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}} \right)_+. \end{aligned}$$

*For convenience of the reader, we recall the notations $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$ and $p^\nabla = \min\{p, p'\}$.*

*The map*

$$\iota : \mathcal{F}M^{\gamma_1,\alpha}_{p_1,q_1} \to \mathcal{F}M^{\gamma_2,\beta}_{p_2,q_2}, f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$ *as well as*

$$\begin{cases} \gamma_2 \leq \gamma_1 + d\gamma^{(0)}, & \text{for } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d\left(\gamma^{(0)} + (1-\beta)\left(\frac{1}{q_1} - \frac{1}{q_2}\right)\right), & \text{for } q_1 > q_2. \end{cases}$$

*Conversely, the map*

$$\theta : \mathcal{F}M^{\gamma_1,\beta}_{p_1,q_1} \to \mathcal{F}M^{\gamma_2,\alpha}_{p_2,q_2}, f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$*, as well as*

$$\begin{cases} \gamma_2 \leq \gamma_1 + d\gamma^{(1)}, & \text{for } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + d\left(\gamma^{(1)} + (1-\beta)\left(\frac{1}{q_1} - \frac{1}{q_2}\right)\right), & \text{for } q_1 > q_2. \end{cases}$$

*Remark* 6.1.8. In case of $q_1 = q_2 = q$ and $p_1 = p_2 = p$, the first condition reduces to

$$\gamma_2 + d\,(\beta - \alpha)\left(\frac{1}{p^{\triangledown}} - \frac{1}{q}\right)_+ \leq \gamma_1,$$

which is identical to the condition given by Han and Wang in [49, Theorem 4.1].

Analogously, the second condition reduces to

$$\gamma_2 + d\,(\beta - \alpha)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+ \leq \gamma_1,$$

which also coincides with the condition stated in [49, Theorem 4.1]. The verification that the stated conditions are equivalent to the ones given by Han and Wang is straightforward, but slightly tedious, since Han and Wang use a different notation. ◀

*Proof.* Let $r_1 = r_1(d, \alpha)$ and $r_2 = r_1(d, \beta)$ as in Theorem 6.1.1, choose $r > r_1$, as well as $s > r_2$ and recall the definition of the spaces $\mathcal{F}M^{\gamma,\alpha}_{p,q}$ from Definition 6.1.4.

Theorem 6.1.6 shows that all assumptions of part 3 of Theorem 5.4.4 (with $\mathcal{Q} = \mathbb{O}^{(\alpha)}_r$, $\mathcal{P} = \mathbb{O}^{(\beta)}_s$ and $u = w^{(\gamma_1/(1-\alpha))}$, $v = w^{(\gamma_2/(1-\beta))}$) are satisfied. Because of $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$, we see that the map $\iota$ given in Theorem 5.4.4 satisfies $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$ (if it exists), since $\iota f$ extends $f$.

Thus, part 3 of Theorem 5.4.4 shows that the map $\iota$ as defined above is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if additionally the (quasi)-norm

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}}{u_{i_j}} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}_v}$$

is finite. Here, we have $J_{\mathcal{O}} = J = \mathbb{Z}^d \setminus \{0\}$, because of $\mathcal{O} = \mathcal{O}' = \mathbb{R}^d$. Finally, for each $j \in J_{\mathcal{O}}$, some $i_j \in I_j$ is chosen arbitrarily. This means $i_j \in \mathbb{Z}^d \setminus \{0\}$ with $O^{(\alpha)}_{r,i_j} \cap O^{(\beta)}_{s,j} \neq \varnothing$.

But with our choices of $\mathcal{Q}, \mathcal{P}$ and $u, v$, we have

$$\begin{aligned} \left|\det\left(S_j\right)\right| &= \left|\det\left(T_j^{(\beta)}\right)\right| \\ &= \left|\det\left(|j|^{\beta_0}\mathrm{id}\right)\right| \\ &= |j|^{d\beta_0} \asymp \langle j\rangle^{d\beta_0} \end{aligned}$$

and

$$v_j = w_j^{(\gamma_2/(1-\beta))} = \langle j\rangle^{\frac{\gamma_2}{1-\beta}}$$

for all $j \in J = \mathbb{Z}^d \setminus \{0\}$, where the implied constants only depend on $d$ and $\beta$.

Furthermore, Theorem 6.1.6 yields

$$\left|\det\left(T_{i_j}\right)\right| = \left|\det\left(T_{i_j}^{(\alpha)}\right)\right| \asymp \langle j\rangle^{d\frac{\alpha}{1-\beta}}$$

and

$$u_{i_j} = w_{i_j}^{(\gamma_1/(1-\alpha))} \asymp \langle j\rangle^{\frac{\gamma_1}{1-\alpha}\cdot\frac{1-\alpha}{1-\beta}} = \langle j\rangle^{\frac{\gamma_1}{1-\beta}},$$

with implied constants only depending on $d, \alpha, \beta, \gamma_1, \gamma_2, r, s$.

All in all, this shows that finiteness of the quasi-norm above is equivalent to finiteness of

$$\begin{aligned} &\left\| \left( \frac{\langle j\rangle^{\frac{\gamma_2}{1-\beta}}}{\langle j\rangle^{\frac{\gamma_1}{1-\beta}}} \cdot \langle j\rangle^{d\beta_0\cdot\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+} \cdot \langle j\rangle^{d\frac{\alpha}{1-\beta}\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right]} \right)_{j\in\mathbb{Z}^d\setminus\{0\}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\ &= \left\| \left( \langle j\rangle^{\frac{\gamma_2-\gamma_1}{1-\beta}+d\frac{\beta}{1-\beta}\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+ + d\frac{\alpha}{1-\beta}\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right]} \right)_{j\in\mathbb{Z}^d\setminus\{0\}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\ &= \left\| \left( \langle j\rangle^{\frac{1}{1-\beta}\left(\gamma_2-\gamma_1+d(\beta-\alpha)\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+ + d\alpha\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)} \right)_{j\in\mathbb{Z}^d\setminus\{0\}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\ &= \left\| \left( \langle j\rangle^{\frac{1}{1-\beta}\left(\gamma_2-\gamma_1-d\gamma^{(0)}\right)} \right)_{j\in\mathbb{Z}^d\setminus\{0\}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}}. \end{aligned}$$

There are now two cases. For $q_1 \leq q_2$, we have $q_2 \cdot (q_1/q_2)' = \infty$, so that the above quasi-norm is finite if and only if we have

$$\gamma_2 - \gamma_1 - d\gamma^{(0)} \leq 0.$$

In case of $q_1 > q_2$, we have $\frac{1}{q_2 \cdot (q_1/q_2)'} = \frac{1}{q_2} - \frac{1}{q_1}$ by Remark 5.2.3. An easy argument using (e.g.) integral comparison shows that the quasi-norm $\left\| (\langle j \rangle^{\delta})_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^p}$ is finite for $p \in (0, \infty)$ if and only if we have $p\delta < -d$. Thus, the above quasi-norm is finite if and only if we have

*A formal proof of the characterization of finiteness of $\left\| (\langle j \rangle^{\delta})_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^p}$ is given in Lemma 6.5.11*

$$\begin{aligned} &\frac{1}{\frac{1}{q_2} - \frac{1}{q_1}} \cdot \frac{1}{1-\beta} \left( \gamma_2 - \gamma_1 - d\gamma^{(0)} \right) < -d \\ \iff &\gamma_2 - \gamma_1 - d\gamma^{(0)} < -d\,(1-\beta) \left( \frac{1}{q_2} - \frac{1}{q_1} \right). \end{aligned}$$

Together, this proves the first claim.

For the second claim, we note that Theorem 6.1.6 implies that all assumptions of part 3 of Theorem 5.4.1 (again with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$ and $\mathcal{P} = \mathbb{O}_s^{(\beta)}$, but with $u = w^{(\gamma_2/(1-\alpha))}$ and $v = w^{(\gamma_1/(1-\beta))}$ and with interchanged roles of $p_1, p_2$ and $q_1, q_2$) are satisfied. Since the map $\iota$ from Theorem 5.4.1 is given explicitly by $\iota f = f|_{C_c^\infty(\mathcal{O})}$ and because of $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$, we see $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{P}, L^{p_1}, \ell_v^{q_1}\right)$, if the map $\iota$ from Theorem 5.4.1 is well-defined.

Thus, part 3 of Theorem 5.4.1 shows that the map $\theta$ as defined above is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the (quasi)-norm

$$\left\| \left( \frac{u_{i_j}}{v_j} \left| \det\left(S_j\right) \right|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \left| \det\left(T_{i_j}\right) \right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_2(q_1/q_2)'}}$$

is finite. As above, we have $J_{\mathcal{O}} = J = \mathbb{Z}^d \setminus \{0\}$, because of $\mathcal{O} = \mathcal{O}'$. Finally, for each $j \in J_{\mathcal{O}}$, some $i_j \in I_j$ is chosen as above.

Using the same arguments as before, we see that finiteness of the above quasi-norm is equivalent to finiteness of

$$\begin{aligned} &\left\| \left( \frac{\langle j \rangle^{\frac{\gamma_2}{1-\beta}}}{\langle j \rangle^{\frac{\gamma_1}{1-\beta}}} \langle j \rangle^{d\beta_0 \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \langle j \rangle^{d\frac{\alpha}{1-\beta} \left[ \frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ \right]} \right)_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2(q_1/q_2)'}} \\ &= \left\| \left( \langle j \rangle^{\frac{1}{1-\beta} \left( \gamma_2 - \gamma_1 + d(\beta-\alpha) \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + d\alpha \left(\frac{1}{p_1} - \frac{1}{p_2}\right) \right)} \right)_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2(q_1/q_2)'}} \\ &= \left\| \left( \langle j \rangle^{\frac{1}{1-\beta}\left(\gamma_2 - \gamma_1 - d\gamma^{(1)}\right)} \right)_{j \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^{q_2(q_1/q_2)'}}. \end{aligned}$$

Exactly as above, we see for $q_1 \leq q_2$ that $q_2 \cdot (q_1/q_2)' = \infty$, so that the above quasi-norm is finite if and only if we have

$$\gamma_2 - \gamma_1 - d\gamma^{(1)} \leq 0.$$

Finally, if $q_1 > q_2$, similar arguments as before show that finiteness of the above quasi-norm is equivalent to

$$\frac{1}{\frac{1}{q_2} - \frac{1}{q_1}} \frac{1}{1-\beta} \left( \gamma_2 - \gamma_1 - d\gamma^{(1)} \right) < -d$$

and hence to

$$\gamma_2 - \gamma_1 - d\gamma^{(1)} < -d\,(1-\beta) \left( \frac{1}{q_2} - \frac{1}{q_1} \right).$$

This is easily seen to be equivalent to the condition given in the theorem. □

Our final result for this section is to show that the Fourier transform yields an isomorphism between the "classical" $\alpha$-modulation spaces and our Fourier-side $\alpha$-modulation spaces. The main point here consists in showing that each element of $\mathcal{F}M^{\gamma,\alpha}_{p,q}$ is in fact (the restriction of) a *tempered* distribution and not just an arbitrary distribution.

**Lemma 6.1.9.** *Let $d \in \mathbb{N}$, $\alpha \in [0,1)$ and $p, q \in (0,\infty]$, as well as $\gamma \in \mathbb{R}$. The Fourier transform of tempered distributions restricts to an isomorphism*

$$\mathcal{F} : M^{\gamma,\alpha}_{p,q}(\mathbb{R}^d) \to \mathcal{F}M^{\gamma,\alpha}_{p,q}, f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d)}$$

*where the $\alpha$-modulation space $M^{\gamma,\alpha}_{p,q}(\mathbb{R}^d)$ is defined as in [3, Definition 2.4].*

*Hence, the characterization of embeddings given in Theorem 6.1.7 and Remark 6.1.8 is also valid for the conventional $\alpha$-modulation spaces $M^{\gamma,\alpha}_{p,q}$ and not only for the "Fourier side" $\alpha$-modulation spaces $\mathcal{F}M^{\gamma,\alpha}_{p,q}$.*

*Proof.* In [3, Definition 2.4], the quasi-norm on the $\alpha$-modulation space $M^{\gamma,\alpha}_{p,q}$ is defined as

$$\|f\|_{M^{\gamma,\alpha}_{p,q}} = \left\| \left( \langle \xi_Q \rangle^\gamma \cdot \left\| \mathcal{F}^{-1}\left( \psi_Q \widehat{f} \right) \right\|_{L^p} \right)_{Q \in \mathcal{Q}} \right\|_{\ell^q},$$

where $\mathcal{Q}$ is an (arbitrary) $\alpha$-COVERING of $\mathbb{R}^d$, $(\psi_Q)_{Q\in\mathcal{Q}}$ is an $L^p$-BAPU for $\mathcal{Q}$, and $\xi_Q \in Q$ is chosen arbitrarily for every $Q \in \mathcal{Q}$. The notion of an $\alpha$-covering is defined in [3, Definition 2.1], but we will not actually need this definition here. The only property which matters for us is that the covering $\mathcal{O}_r^{(\alpha)}$, for $r > r_1 = r_1(\alpha, d)$ as in Theorem 6.1.1, is indeed an $\alpha$-covering of $\mathbb{R}^d$. This is shown in [3, Theorem 2.6].

Hence, we can take $\mathcal{Q} = \mathcal{O}_r^{(\alpha)}$ and we can pick an arbitrary $\xi_Q \in Q$ for each $Q = O_{r,k}^{(\alpha)} \in \mathcal{Q}$. Lemma 6.1.2 shows that this entails

$$\langle \xi_Q \rangle^\gamma \asymp \left( \langle k \rangle^{\frac{1}{1-\alpha}} \right)^\gamma = w_k^{(\gamma/(1-\alpha))}.$$

Together, we have shown

$$\begin{aligned} \|f\|_{M_{p,q}^{\gamma,\alpha}} &\asymp \left\| \left( w_k^{(\gamma/(1-\alpha))} \cdot \left\| \mathcal{F}^{-1}\left(\psi_k \widehat{f}\right) \right\|_{L^p} \right)_{k \in \mathbb{Z}^d \setminus \{0\}} \right\|_{\ell^q} \\ &= \|\widehat{f}\|_{\mathcal{D}\left(\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma/(1-\alpha))}}\right)} \\ &= \|\widehat{f}\|_{\mathcal{F}M_{p,q}^{\gamma,\alpha}}, \end{aligned}$$

where $(\psi_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ is an $L^p$-BAPU for $\mathcal{O}_r^{(\alpha)}$. It is thus sufficient to prove that the restricted Fourier transform $\mathcal{F} : M_{p,q}^{\gamma,\alpha}(\mathbb{R}^d) \to \mathcal{F}M_{p,q}^{\gamma,\alpha}$ is surjective. To this end, it suffices to show that each $f \in \mathcal{F}M_{p,q}^{\gamma,\alpha}$ extends to a tempered distribution $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^d)$, since this entails $g := \mathcal{F}^{-1}\widetilde{f} \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|g\|_{M_{p,q}^{\gamma,\alpha}} \asymp \|\widehat{g}\|_{\mathcal{F}M_{p,q}^{\gamma,\alpha}} = \|f\|_{\mathcal{F}M_{p,q}^{\gamma,\alpha}} < \infty,$$

and hence $g \in M_{p,q}^{\gamma,\alpha}$ with $\mathcal{F}g = \widehat{g}|_{C_c^\infty(\mathbb{R}^d)} = \widetilde{f}|_{C_c^\infty(\mathbb{R}^d)} = f$.

We use Lemma 5.5.1 to prove the existence of such an extension. We first note that $y \in (O_{r,k}^{(\alpha)})^*$ for some $k \in \mathbb{Z}^d \setminus \{0\}$ yields $y \in O_{r,\ell}^{(\alpha)}$ for some $\ell \in \mathbb{Z}^d \setminus \{0\}$ for which there is some $x \in O_{r,\ell}^{(\alpha)} \cap O_{r,k}^{(\alpha)}$. By Lemma 6.1.2, this implies

$$1 + |y| \asymp \langle y \rangle \asymp \langle \ell \rangle^{\frac{1}{1-\alpha}} \asymp \langle x \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}}.$$

Recalling the definition of the weight $w^{(N)}$ from Lemma 5.5.1 and of the normalizations $T_k^{(\alpha)} = |k|^{\frac{\alpha}{1-\alpha}} \mathrm{id}$ for $\mathcal{O}_r^{(\alpha)}$, we see

$$\begin{aligned} &\frac{1}{w_k^{(\gamma/(1-\alpha))}} \cdot w_k^{(N)} \\ &= \langle k \rangle^{-\frac{\gamma}{1-\alpha}} |\det(T_k^{(\alpha)})|^{\frac{1}{p}} \max\left\{1, \left\|(T_k^{(\alpha)})^{-1}\right\|^{d+1}\right\} \cdot \left[\inf_{y \in (O_{r,k}^{(\alpha)})^*} (1+|y|)\right]^{-N} \\ &\lesssim \langle k \rangle^{-\frac{\gamma}{1-\alpha}} \cdot \langle k \rangle^{d\frac{\alpha/p}{1-\alpha}} \cdot \max\left\{1, \langle k \rangle^{-(d+1)\frac{\alpha}{1-\alpha}}\right\} \cdot \langle k \rangle^{-\frac{N}{1-\alpha}} \\ &= \langle k \rangle^{\frac{1}{1-\alpha}\left(\frac{d\alpha}{p} - \gamma - N\right)}. \end{aligned}$$

But $\left(\langle k \rangle^\varrho\right)_{k \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ if we have $\varrho < -d$. This implies

$$\left(w_k^{(N)}\right)_{k \in \mathbb{Z}^d \setminus \{0\}} \in \ell^1_{1/w^{(\gamma/(1-\alpha))}}(\mathbb{Z}^d \setminus \{0\}) \hookrightarrow \ell^{q'}_{1/w^{(\gamma/(1-\alpha))}}(\mathbb{Z}^d \setminus \{0\})$$

as soon as $\frac{1}{1-\alpha}\left(\frac{d\alpha}{p}-\gamma-N\right) < -d$, which holds for large $N \in \mathbb{N}$. Here, we used $q' \in [1,\infty]$ (even for $q \in (0,1)$), so that $\ell^1 \hookrightarrow \ell^{q'}$ indeed holds. Thus, Lemma 5.5.1 is applicable and yields the desired extension $\widetilde{f} \in \mathcal{S}'\left(\mathbb{R}^d\right)$ of $f \in \mathcal{F}M^{\gamma,\alpha}_{p,q}$. Here, it is important to observe that existence of the special BAPU required in Lemma 5.5.1 is guaranteed, since $\mathbb{O}_r^{(\alpha)}$ is a *structured* admissible covering, see the remark after Lemma 5.5.1. □

In summary, we have shown that the embedding results for $\alpha$-modulation spaces obtained in [49, Theorem 4.1] could be derived with ease using our more general approach. We remark that our approach can also handle the case $(p_1,q_1) \neq (p_2,q_2)$ which is not considered in [49, Theorem 4.1]. Furthermore, the most tedious parts of our derivation, namely the estimates for the cardinality of the intersection sets in Lemma 6.1.5 and the proof (given in Lemma 6.1.2) that the weight $w^{(\gamma)}$ is moderate with respect to $\mathbb{O}_r^{(\alpha)}$, are used in [49] without much justification, see equations (4.4) and (4.6) of that paper.

In the next section, we will begin with the study of homogeneous and inhomogeneous Besov spaces. In particular, we will show that the results for embeddings between $\alpha$-modulation spaces and inhomogeneous Besov spaces obtained in [49, Theorem 4.2] are also special cases of our approach.

## 6.2 EMBEDDINGS FOR BESOV SPACES

In this section, we will study embeddings for Besov spaces. These spaces come in two different flavors, the *homogeneous* and the *inhomogeneous* Besov spaces. Our first aim is thus to compare these two different types of Besov spaces, i.e. to investigate the existence of embeddings from homogeneous into inhomogeneous Besov spaces or vice versa. The derivations of these will be relatively short, thus illustrating the power of our approach. It might be that the embedding results given below are folklore, but I am not aware of any other source in which these results have been established.

*One case in which results* similar *(but not identical) to those derived here have been known is the embedding of inhomogeneous into homogeneous Besov spaces. This is discussed in the remark after Theorem 6.2.6 below.*

Finally, we characterize the existence of embeddings between $\alpha$-modulation spaces and (inhomogeneous) Besov spaces, thereby closing the gap that was left in the previous section.

**Definition 6.2.1.** For $d \in \mathbb{N}$ and $n \in \mathbb{Z}$, define

$$\dot{B}_n := 2^n \cdot \left[B_4(0) \setminus \overline{B_{\frac{1}{4}}}(0)\right] = B_{2^{n+2}}(0) \setminus \overline{B_{2^{n-2}}}(0) \subset \mathbb{R}^d.$$

Then $\dot{\mathcal{B}} := (\dot{B}_n)_{n\in\mathbb{Z}}$ is called the (HOMOGENEOUS) BESOV COVERING of $\mathbb{R}^d \setminus \{0\}$.

Furthermore, let $B_n := \dot{B}_n$ for $n \in \mathbb{N}$ and define $B_0 := B_8(0)$. Then $\mathcal{B} := (B_n)_{n\in\mathbb{N}_0}$ is called the (INHOMOGENEOUS) BESOV COVERING of $\mathbb{R}^d$.

Our first aim is to show that the usual Besov spaces are isomorphic (by virtue of the Fourier transform) to certain (Fourier-side) decomposition spaces which are formed using the two coverings above. For this, we also show that the two coverings defined above are (semi)-structured decomposition coverings, so that decomposition spaces with respect to these coverings can be formed.

The main technical difficulty is again that the usual (homogeneous) Besov spaces are defined using the space of *tempered* distributions (modulo polynomials) as the reservoir, whereas the Fourier-side decomposition spaces are defined using the space of distributions on $\mathbb{R}^d$ or $\mathbb{R}^d \setminus \{0\}$.

**Lemma 6.2.2.** *The following are true:*

1. *The homogeneous Besov covering*

$$\dot{\mathcal{B}} = \left(2^n \mathrm{id} \cdot \dot{B}\right)_{n\in\mathbb{Z}} \quad \text{with} \quad \dot{B} := B_4(0) \setminus \overline{B_{\frac{1}{4}}}(0)$$

   *is a structured admissible covering of* $\mathbb{R}^d \setminus \{0\}$.

2. *The inhomogeneous Besov covering*

$$\mathcal{B} = \left(2^n \mathrm{id} \cdot B'_n\right)_{n\in\mathbb{N}_0}$$

   *with* $B'_n := B_4(0) \setminus \overline{B_{\frac{1}{4}}}(0)$ *for* $n \in \mathbb{N}$ *and* $B'_0 := B_8(0)$ *is a tight semi-structured admissible covering of* $\mathbb{R}^d$. *Furthermore, there is a family* $(\varphi_n)_{n\in\mathbb{N}_0}$ *which is an* $L^p$*-BAPU for* $\mathcal{B}$ *for all* $p \in (0,\infty]$.

3. $\dot{\mathcal{B}}$ *is almost subordinate to* $\mathcal{B}$. *For* $j \in \mathbb{N}$, *we have* $\dot{B}_j \cap B_j \neq \varnothing$, *whereas* $\dot{B}_j \cap B_0 \neq \varnothing$ *holds for* $j \in \mathbb{Z}_{\leq 0}$.

4. *The weights* $u^{(\alpha)} = \left(2^{j\alpha}\right)_{j\in\mathbb{Z}}$ *and* $v^{(\alpha)} = \left(2^{i\alpha}\right)_{i\in\mathbb{N}_0}$ *are moderate with respect to* $\dot{\mathcal{B}}$ *and* $\mathcal{B}$, *respectively.*

*Finally, let* $p,q \in (0,\infty]$ *and* $\alpha \in \mathbb{R}$. *Let* $B^p_{\alpha,q} \subset \mathcal{S}'\left(\mathbb{R}^d\right)$ *denote the* INHOMOGENEOUS BESOV SPACE *on* $\mathbb{R}^d$ *and let* $\dot{B}^p_{\alpha,q} \subset \mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P}$ *denote the* HOMOGENEOUS BESOV SPACE *on* $\mathbb{R}^d$ *(cf. [42, Definition 6.5.1]). Then the following statements hold:*

*We remark that* $\mathcal{P} \leq \mathcal{S}'(\mathbb{R}^d)$ *is the space of polynomials.*

A. *The Fourier transform* $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{S}'\left(\mathbb{R}^d\right) \subset \mathcal{D}'\left(\mathbb{R}^d\right)$ *restricts to an isomorphism of (Quasi)-Banach spaces*

$$\mathcal{F} : B^p_{\alpha,q} \to \mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}}\right).$$

B. *The Fourier transform* $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{S}'\left(\mathbb{R}^d\right)$ *composed with the restriction*

$$\varrho : \mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d \setminus \{0\}\right), f \mapsto f|_{C_c^\infty\left(\mathbb{R}^d\setminus\{0\}\right)}$$

*yields a well-defined map*

$$[\varrho \circ \mathcal{F}] : \mathcal{S}'(\mathbb{R}^d)/\mathcal{P} \to \mathcal{D}'(\mathbb{R}^d \setminus \{0\}), f + \mathcal{P} \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d \setminus \{0\})}$$

*and the restricted map*

$$[\varrho \circ \mathcal{F}] : \dot{B}^p_{\alpha,q} \to \mathcal{D}(\dot{\mathcal{B}}, L^p, \ell^q_{u^{(\alpha)}})$$

*is an isomorphism of (Quasi)-Banach spaces.*

*Proof.* 1. Lemma 4.7.5 shows that $\dot{\mathcal{B}}$ is a structured admissible covering of $\mathbb{R}^d \setminus \{0\}$. To see that $u^{(\alpha)}$ is moderate with respect to $\dot{\mathcal{B}}$, let $i, j \in \mathbb{Z}$ with $x \in \dot{B}_i \cap \dot{B}_j \neq \varnothing$. This entails $2^{i-2} < |x| < 2^{j+2}$ and $2^{j-2} < |x| < 2^{i+2}$, which in turn implies $i - 2 < j + 2$ and $j - 2 < i + 2$. Hence, $i < j + 4$ and $j < i + 4$. Since these are integer quantities, we conclude $i \leq j + 3$ and $j \leq i + 3$, i.e. $|i - j| \leq 3$. This implies $i^* \subset \{i - 3, \dots, i + 3\}$ and

$$\frac{u_i^{(\alpha)}}{u_j^{(\alpha)}} = 2^{\alpha(i-j)} \leq 2^{|\alpha| \cdot |i-j|} \leq 2^{3|\alpha|}$$

for all $j \in i^*$.

2. Let $i \in \mathbb{N}$ with $x \in B_0 \cap B_i$. This implies $2^{i-2} < |x| < 8 = 2^3$ and hence $i \leq 4$, which means $i \in \{1, \dots, 4\}$. We deduce $0^* \subset \{0, \dots, 4\}$ and hence $|0^*| \leq 5$, where the cluster is taken with respect to $\mathcal{B}$ . We also get

$$\left\| \left(2^0 \mathrm{id}\right)^{-1} \cdot 2^i \mathrm{id} \right\| = 2^i \leq 2^4 = 16 \quad \text{and} \quad \left\| (2^i \mathrm{id})^{-1} \cdot 2^0 \mathrm{id} \right\| = 2^{-i} \leq 1,$$

as well as

$$\frac{v_0^{(\alpha)}}{v_i^{(\alpha)}} = 2^{-\alpha i} \leq 2^{4|\alpha|} \quad \text{and} \quad \frac{v_i^{(\alpha)}}{v_0^{(\alpha)}} = 2^{\alpha i} \leq 2^{4|\alpha|}.$$

Analogous to the case of $\dot{\mathcal{B}}$ above, we see

$$i^* \subset \{0\} \cup [\{i - 3, \dots, i + 3\} \cap \mathbb{N}] \text{ for all } i \in \mathbb{N},$$

where the cluster is taken with respect to $\mathcal{B}$. Hence, $|i^*| \leq 8$.

For $j \in i^* \cap \mathbb{N}$, we get $\left\| \left(2^i \mathrm{id}\right)^{-1} \cdot 2^j \mathrm{id} \right\| \leq 2^3 = 8$ as well as $v_i^{(\alpha)}/v_j^{(\alpha)} \leq 2^{3|\alpha|}$ and we have already seen $v_i^{(\alpha)}/v_0^{(\alpha)} \leq 2^{4|\alpha|}$ as well as $\left\| \left(2^i \mathrm{id}\right)^{-1} \cdot 2^0 \mathrm{id} \right\| \leq 1$ in the case $j = 0 \in i^*$.

Finally, for $x \in \mathbb{R}^d$, there are two possibilities: For $|x| < 8$, we get $x \in B_0$. Otherwise,

$$|x| \in [8, \infty) \subset \left(\frac{1}{2}, \infty\right) = \bigcup_{n \in \mathbb{N}} \left(2^{n-2}, 2^{n+2}\right)$$

and thus $x \in B_n$ for some $n \in \mathbb{N}$.

In summary, we have shown that $\mathcal{B}$ is a semi-structured admissible covering of $\mathbb{R}^d$ and that $v^{(\alpha)}$ is $\mathcal{B}$-moderate. Tightness of $\mathcal{B}$ is immediate, because the family $(B'_n)_{n\in\mathbb{N}_0}$ consists only of finitely many distinct open sets. Existence of the family $(\varphi_n)_{n\in\mathbb{N}_0}$ will be shown below.

3. It is easy to see that $\dot{\mathcal{B}}$ is almost subordinate to $\mathcal{B}$: Indeed, for $i \in \mathbb{N}$, we have $\dot{B}_i = B_i = B_i^{0*}$, whereas for $i \leq 0$, the inclusion

$$\dot{B}_i \subset B_{2^{i+2}}(0) \subset B_4(0) \subset B_0 = B_0^*$$

is satisfied. These inclusions also imply $\dot{B}_i \cap B_i \neq \varnothing$ for $i \in \mathbb{N}$ as well as $\dot{B}_i \cap B_0 \neq \varnothing$ for $i \in \mathbb{Z}_{\leq 0}$.

Let us now construct the family $(\varphi_n)_{n\in\mathbb{N}_0}$ which is an $L^p$-BAPU for $\mathcal{B}$ for all $p \in (0,\infty]$. By [51, Lemma 2.3], there is a function $\widehat{\Psi} \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\widehat{\Psi} \subset B_2(0) \setminus \overline{B_{1/2}}(0)$ and with

$$\sum_{j=-\infty}^{\infty} \widehat{\Psi}(2^{-j}\xi) = 1 \qquad \forall \xi \in \mathbb{R}^d \setminus \{0\}. \tag{6.2.1}$$

Let $\Psi := \mathcal{F}^{-1}\widehat{\Psi} \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi_j(x) := 2^{dj} \cdot \Psi(2^j x)$ for $j \in \mathbb{Z}$. This implies the identity $\widehat{\Psi_j}(\xi) = \widehat{\Psi}(2^{-j}\xi) =: \psi_j(\xi)$ for all $\xi \in \mathbb{R}^d$, so that Lemma 4.5.5 yields

$$\begin{aligned}\left\|\mathcal{F}^{-1}\psi_j\right\|_{L^p} &= \|\Psi_j\|_{L^p} = 2^{dj} \cdot \|\Delta_{2^j\mathrm{id}}\Psi\|_{L^p} \\ &= 2^{dj\left(1-\frac{1}{p}\right)} \|\Psi\|_{L^p} = |\det(2^j\mathrm{id})|^{1-\frac{1}{p}} \cdot \|\Psi\|_{L^p}\end{aligned}$$

for all $j \in \mathbb{Z}$ and $p \in (0,\infty]$. Using $\operatorname{supp}\widehat{\Psi} \subset B_2(0) \setminus \overline{B_{1/2}}(0)$, we see $\operatorname{supp}\psi_j = 2^j \cdot \operatorname{supp}\widehat{\Psi} \subset \dot{B}_j$ for all $j \in \mathbb{Z}$. Together with equation (6.2.1), it follows that $(\psi_j)_{j\in\mathbb{Z}}$ is an $L^p$-BAPU for $\dot{\mathcal{B}}$ for all $p \in (0,\infty]$.

Grafakos (cf. [42, equation (6.5.2)]) states that there is a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\widehat{\Phi}(\xi) = \begin{cases} \sum_{j\leq 0} \widehat{\Psi}(2^{-j}\xi) = \sum_{j\leq 0} \psi_j(\xi), & \text{if } \xi \neq 0, \\ 1, & \text{if } \xi = 0 \end{cases}$$

and $\widehat{\Phi}(\xi) = 1$ for $|\xi| \leq 1$, as well as $\widehat{\Phi}(\xi) = 0$ for $|\xi| \geq 2$.

Set $\varphi_i := \psi_i$ for $i \in \mathbb{N}$, and let $\varphi_0 := \widehat{\Phi}$. Then $\sum_{i\in\mathbb{N}_0} \varphi_i(\xi) = 1$ holds for all $\xi \in \mathbb{R}^d$. Furthermore, $\varphi_i \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi_i \subset \dot{B}_i = B_i$ for $i \in \mathbb{N}$ and with $\operatorname{supp}\varphi_0 \subset \overline{B_2}(0) \subset B_0$. Finally,

$$\begin{aligned}&|\det(2^i\mathrm{id})|^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\varphi_i\right\|_{L^p} \\ &\leq \max\left\{\|\Phi\|_{L^p}, \sup_{j\in\mathbb{Z}} |\det(2^j\mathrm{id})|^{\frac{1}{p}-1} \left\|\mathcal{F}^{-1}\psi_j\right\|_{L^p}\right\} < \infty\end{aligned}$$

holds for all $i \in \mathbb{N}_0$. Together, we conclude that $(\varphi_i)_{i\in\mathbb{N}_0}$ is an $L^p$-BAPU for $\mathcal{B}$ for all $p \in (0,\infty]$. In particular, $\mathcal{B}$ is an $L^p$-decomposition covering of $\mathbb{R}^d$ for arbitrary $p \in (0,\infty]$.

It remains to prove parts A and B. Following Grafakos[42, Definition 6.5.1], we define

$$\Delta_j : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), f \mapsto \mathcal{F}^{-1}\left(\widehat{\Psi_j} \cdot \widehat{f}\right) = \mathcal{F}^{-1}\left(\psi_j \cdot \widehat{f}\right)$$

for $j \in \mathbb{Z}$ as well as

$$S_0 : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), f \mapsto \Phi * f = \mathcal{F}^{-1}\left(\widehat{\Phi} \cdot \widehat{f}\right) = \mathcal{F}^{-1}\left(\varphi_0 \widehat{f}\right)$$

and let

$$\begin{aligned}
\|f\|_{\dot{B}^p_{\alpha,q}} &:= \left\| \left(2^{j\alpha} \left\|\Delta_j f\right\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^q} \\
&= \left\| \left( \left\| \mathcal{F}^{-1}\left(\psi_j \cdot \widehat{f}\right) \right\|_{L^p} \right)_{j\in\mathbb{Z}} \right\|_{\ell^q_{u(\alpha)}} \\
&= \|\varrho \widehat{f}\|_{\mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{u(\alpha)}\right)}
\end{aligned} \tag{6.2.2}$$

for $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Here, we used the BAPU $\left(\psi_j\right)_{j\in\mathbb{Z}}$ to calculate the (quasi)-norm on $\mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{u(\alpha)}\right)$.

We note that $f + \mathcal{P} = g + \mathcal{P}$ for arbitrary $f, g \in \mathcal{S}'\left(\mathbb{R}^d\right)$ implies $f - g \in \mathcal{P}$ and thus (cf. [33, Proposition 9.12]) $\operatorname{supp} \widehat{f-g} \subset \{0\}$, which entails $\varrho\widehat{f} = \varrho\widehat{g}$, where $\varrho$ is the restriction map as defined in the statement of the lemma.

Hence, $[\varrho \circ \mathcal{F}] : \mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P} \to \mathcal{D}'\left(\mathbb{R}^d \setminus \{0\}\right)$ is well-defined. Together with equation (6.2.2), we conclude that

$$[\varrho \circ \mathcal{F}] : \dot{B}^p_{\alpha,q} \to \mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{u(\alpha)}\right)$$

is indeed well-defined and even an isometry (if the quasi-norms on both spaces are chosen accordingly).

For the inhomogeneous Besov spaces, we observe that Grafakos defines the corresponding quasi-norm as

$$\begin{aligned}
\|f\|_{B^p_{\alpha,q}} &:= \|S_0 f\|_{L^p} + \left\| \left(2^{j\alpha} \cdot \left\|\Delta_j f\right\|_{L^p}\right)_{j\in\mathbb{N}} \right\|_{\ell^q} \\
&= \left\| \mathcal{F}^{-1}\left(\varphi_0 \cdot \widehat{f}\right) \right\|_{L^p} + \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_i \widehat{f}\right) \right\|_{L^p} \right)_{j\in\mathbb{N}} \right\|_{\ell^q_{v(\alpha)}},
\end{aligned}$$

which easily implies

$$\|\widehat{f}\|_{\mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{v(\alpha)}\right)} = \left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_i \widehat{f}\right) \right\|_{L^p} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^q_{v(\alpha)}} \asymp \|f\|_{B^{p,q,\alpha}} \tag{6.2.3}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)$, where the implied constants only depend on $q$. In the above equation, $\widehat{f} \in \mathcal{S}'(\mathbb{R}^d)$ is considered as an element of $\mathcal{D}'(\mathbb{R}^d) \supset \mathcal{S}'(\mathbb{R}^d)$. All in all, we see that $\mathcal{F} : B^p_{\alpha,q} \to \mathcal{D}(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}})$ is well-defined and bounded and an isomorphism of Quasi-Banach spaces *if* it is surjective.

It remains to prove that $[\varrho \circ \mathcal{F}]$ and $\mathcal{F}$ as defined above are surjective. To this end, we first show that Lemma 5.5.1 is applicable to $\mathcal{D}(\dot{\mathcal{B}}, L^p, \ell^q_{u^{(\alpha)}})$ with $I_0 = \mathbb{N}$ and to $\mathcal{D}(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}})$ with $I_0 = \mathbb{N}_0$. To see this, we set $T_j := 2^j \mathrm{id}$ for $j \in \mathbb{Z}$ and note (with $S_j = T_j$ as in Lemma 5.5.1) that we have

$$\begin{aligned}\left\| \partial^\beta \left( \psi_j \circ S_j \right) \right\|_{\sup} &= \| \partial^\beta \widehat{\Psi} \|_{\sup} \\ &\leq \max \left\{ \left\| \partial^\beta \varphi_0 \right\|_{\sup}, \| \partial^\beta \widehat{\Psi} \|_{\sup} \right\} =: L_\beta\end{aligned}$$

for all $j \in \mathbb{Z}$ and $\beta \in \mathbb{N}_0^d$, where the right-hand side is independent of $j$. Because of $\varphi_j = \psi_j$ for $j \in \mathbb{N}$, we also get

$$\left\| \partial^\beta \left( \varphi_j \circ S_j \right) \right\|_{\sup} \leq L_\beta$$

for all $j \in \mathbb{N}_0$, since this estimate is trivial for $j = 0$ by definition of $L_\beta$.

For $i \in \mathbb{N}_{\geq 5}$, we have $i \notin 0^* \subset \{0, \ldots, 4\}$ and hence $0 \notin i^*$, where the cluster is formed with respect to $\mathcal{B}$. We conclude $B_i^* \subset \dot{B}_i^*$, so that $y \in B_i^* \subset \dot{B}_i^*$ implies $y \in \dot{B}_{i-3} \cup \cdots \cup \dot{B}_{i+3}$ as seen above and hence $|y| > 2^{(i-3)-2} = 2^{i-5}$. This inequality allows us to estimate

$$\begin{aligned}\sigma_i^{(N)} :&= \left[ \inf_{y \in B_i^*} (1 + |y|) \right]^{-N} \\ &\leq \left[ \inf_{y \in \dot{B}_i^*} (1 + |y|) \right]^{-N} \\ &\leq 2^{-N(i-5)} = 32^N \cdot 2^{-iN}\end{aligned}$$

for all $N \in \mathbb{N}_0$ and $i \in \mathbb{N}_{\geq 5}$. With $w_i^{(N)}$ defined as in Lemma 5.5.1, we derive

$$\begin{aligned}\frac{1}{v_i^{(\alpha)}} \cdot w_i^{(N)} = \frac{1}{u_i^{(\alpha)}} \cdot w_i^{(N)} &= 2^{-\alpha i} \cdot |\det(T_i)|^{\frac{1}{p}} \cdot \max\left\{1, \|T_i^{-1}\|^{d+1}\right\} \cdot \sigma_i^{(N)} \\ &\leq 2^{i\left(\frac{d}{p} - \alpha\right)} \cdot \max\left\{1, \|2^{-i}\mathrm{id}\|^{d+1}\right\} \cdot 32^N \cdot 2^{-iN} \\ &= 32^N \cdot \left(2^{N + \alpha - \frac{d}{p}}\right)^{-i}\end{aligned}$$

for $i \in \mathbb{N}_{\geq 5}$. For $N + \alpha - \frac{d}{p} > 0$, we observe that the last sequence is in $\ell^{q'}(\mathbb{N}_{\geq 5})$ for all $q \in (0, \infty]$. Since the finite set of numbers

$\left\{w_0^{(N)},\dots,w_4^{(N)}\right\}$ is irrelevant for the membership of $(w_i^{(N)})_{i\in\mathbb{N}_0}$ in $\ell^{q'}_{1/v^{(\alpha)}}(\mathbb{N}_0)$ or of $(w_i^{(N)})_{i\in\mathbb{N}}$ in $\ell^{q'}_{1/u^{(\alpha)}}(\mathbb{N})$, we see that Lemma 5.5.1 is indeed applicable in both cases.

We conclude that each $f \in \mathcal{D}(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}})$ extends to a tempered distribution $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^d)$, which we again denote by $f$. Equation (6.2.3) shows that this extension satisfies

$$\left\|\mathcal{F}^{-1}f\right\|_{B^p_{\alpha,q}} \asymp \left\|\widehat{\mathcal{F}^{-1}f}\right\|_{\mathcal{D}(\mathcal{B},L^p,\ell^q_{v^{(\alpha)}})} = \|f\|_{\mathcal{D}(\mathcal{B},L^p,\ell^q_{v^{(\alpha)}})} < \infty,$$

which means $\mathcal{F}^{-1}f \in B^p_{\alpha,q}$ and thus $f = \mathcal{F}\left(\mathcal{F}^{-1}f\right) \in \mathcal{F}\left(B^p_{\alpha,q}\right)$. We have thus shown that the Fourier transform $\mathcal{F} : B^p_{\alpha,q} \to \mathcal{D}(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}})$ is indeed surjective.

For the homogeneous Besov space, Lemma 5.5.1 implies that

$$\widetilde{f} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}, g \mapsto \sum_{j=1}^{\infty} \left(\psi_j f\right)(g)$$

is a well-defined, tempered distribution for every $f \in \mathcal{D}(\dot{\mathcal{B}}, L^p, \ell^q_{u^{(\alpha)}})$. It remains to prove that $f$ admits an extension "near the origin".

To this end, let us denote the Taylor approximation of $\varphi \in C^\infty\left(\mathbb{R}^d\right)$ of degree $N-1$ around 0 by

$$\left(P_N\varphi\right)(\eta) := \sum_{|\gamma|<N} \frac{\left(\partial^\gamma\varphi\right)(0)}{\gamma!}\cdot\eta^\gamma \quad \text{for } \eta\in\mathbb{R}^d$$

for each $N \in \mathbb{N}$, where the sum is taken over all multiindices $\gamma \in \mathbb{N}_0^d$ with $|\gamma| = \gamma_1 + \cdots + \gamma_d < N$. We observe the identity

$$\partial^\beta \eta^\gamma = \begin{cases} 0, & \text{for } \beta \not\leq \gamma, \\ \frac{\gamma_1!}{(\gamma-\beta)_1!}\cdots\frac{\gamma_d!}{(\gamma-\beta)_d!}\eta^{\gamma-\beta} = \frac{\gamma!}{(\gamma-\beta)!}\eta^{\gamma-\beta}, & \text{for } \beta \leq \gamma, \end{cases}$$

where we write $\beta \leq \gamma$ for two multiindices $\beta, \gamma \in \mathbb{N}_0^d$ iff $\beta_i \leq \gamma_i$ holds for all $i \in \underline{d}$. This identity implies

$$\begin{aligned}\left(\partial^\beta\left(P_N\varphi\right)\right)(\eta) &= \sum_{\substack{|\gamma|<N\\ \beta\leq\gamma}} \frac{\left(\partial^\gamma\varphi\right)(0)}{\gamma!}\cdot\frac{\gamma!}{(\gamma-\beta)!}\cdot\eta^{\gamma-\beta} \\ &\overset{(\dagger)}{=} \sum_{|\delta|<N-|\beta|} \frac{\left(\partial^\delta\left(\partial^\beta\varphi\right)\right)(0)}{\delta!}\cdot\eta^\delta \\ &= \left(P_{N-|\beta|}\left(\partial^\beta\varphi\right)\right)(\eta)\end{aligned}$$

for all multiindices $\beta \in \mathbb{N}_0^d$ satisfying $|\beta| < N$. Here, the change of variables $\delta = \gamma - \beta$ was used at $(\dagger)$.

Using the multi-dimensional Taylor theorem (see e.g. [68, Theorem 6.3]), we conclude

$$\begin{aligned}
&\left|\left(\partial^\beta\left(\varphi - P_N\varphi\right)\right)(\eta)\right| \\
&= \left|\left(\partial^\beta\varphi - P_{N-|\beta|}\left(\partial^\beta\varphi\right)\right)(\eta)\right| \\
&= \left|(N-|\beta|)\cdot\sum_{|\gamma|=N-|\beta|}\frac{\eta^\gamma}{\gamma!}\cdot\int_0^1(1-\theta)^{N-|\beta|-1}\cdot\left[\partial^\gamma\left(\partial^\beta\varphi\right)\right](\theta\eta)\,\mathrm{d}\theta\right| \\
&\leq N\cdot\left[\sum_{|\gamma|\leq N}\frac{1}{\gamma!}\right]\cdot\left[\max_{|\gamma|=N}\left\|\partial^\gamma\varphi\right\|_{\sup}\right]\cdot|\eta|^{N-|\beta|} \\
&= C_N\cdot\left[\max_{|\gamma|=N}\left\|\partial^\gamma\varphi\right\|_{\sup}\right]\cdot|\eta|^{N-|\beta|} \qquad (6.2.4)
\end{aligned}$$

for all multiindices $\beta\in\mathbb{N}_0^d$ with $|\beta|<N$ and all $\eta\in\mathbb{R}^d$.

Now, fix any $N\in\mathbb{N}$ with $N>|\alpha|+d+1$ and define

*Recall that $\alpha$ is used to define the weight $u^{(\alpha)}$, i.e. $\alpha\in\mathbb{R}$ is* not *a multiindex.*

$$\check{f}:\mathcal{S}\left(\mathbb{R}^d\right)\to\mathbb{C},\varphi\mapsto\sum_{i\leq 0}f\left(\psi_i\cdot\left(\varphi-P_N\varphi\right)\right).$$

Observe that this is well-defined (up to convergence issues which will be considered below), because of $\psi_i\left(\varphi-P_N\varphi\right)\in C_c^\infty\left(\mathbb{R}^d\setminus\{0\}\right)$ for all $i\in\mathbb{Z}$ and $\varphi\in\mathcal{S}\left(\mathbb{R}^d\right)$.

For $i\leq 0$, we have $\psi_i=\psi_i\psi_i^*$ (cf. Lemma 3.2.12), and thus

$$\begin{aligned}
&\left|f\left(\psi_i\cdot\left(\varphi-P_N\varphi\right)\right)\right| \\
&= \left|\left\langle\psi_i f,\,\psi_i^*\left(\varphi-P_N\varphi\right)\right\rangle\right| \\
&= \left|\left\langle\mathcal{F}^{-1}\left(\psi_i f\right),\,\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\rangle\right| \\
&\leq \left\|\mathcal{F}^{-1}\left(\psi_i f\right)\right\|_{L^\infty}\cdot\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1} \\
&\overset{(\dagger)}{\leq} C\cdot\left|\det\left(2^i\mathrm{id}\right)\right|^{\frac{1}{p}}\cdot\left\|\mathcal{F}^{-1}\left(\psi_i f\right)\right\|_{L^p}\cdot\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1} \\
&= C\cdot 2^{\frac{id}{p}}\cdot\left\|\mathcal{F}^{-1}\left(\psi_i f\right)\right\|_{L^p}\cdot\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1},
\end{aligned}$$

where the constant $C=C\left(d,p,R_{\dot{\mathcal{B}}}\right)$ used at $(\dagger)$ is from Lemma 5.1.3.

The above estimate, together with Lemma 5.2.1, entails

$$\begin{aligned}
&\left|\check{f}\left(\varphi\right)\right| \\
&\leq C\sum_{i\leq 0}\left[u_i^{(\alpha)}\left\|\mathcal{F}^{-1}\left(\psi_i f\right)\right\|_{L^p}\cdot 2^{\frac{id}{p}}u_i^{(-\alpha)}\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1}\right] \\
&\leq \left\|\left(u_i^{(\alpha)}\left\|\mathcal{F}^{-1}\left(\psi_i f\right)\right\|_{L^p}\right)_{i\leq 0}\right\|_{\ell^q}\cdot\left\|\left(2^{\frac{id}{p}}u_i^{(-\alpha)}\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1}\right)_{i\leq 0}\right\|_{\ell^{q'}} \\
&\leq \|f\|_{\mathcal{D}\left(\dot{\mathcal{B}},L^p,\ell^q_{u^{(\alpha)}}\right)}\cdot\left\|\left(2^{i\left(\frac{d}{p}-\alpha\right)}\left\|\mathcal{F}\left[\psi_i^*\left(\varphi-P_N\varphi\right)\right]\right\|_{L^1}\right)_{i\leq 0}\right\|_{\ell^{q'}}. \qquad (6.2.5)
\end{aligned}$$

We now use equation (5.5.5) from the proof of Lemma 5.5.1 (with $\psi_i$ instead of $\varphi_i$) to estimate

$$\begin{aligned}&\|\mathcal{F}\left[\psi_i^*\left(\varphi - P_N\varphi\right)\right]\|_{L^1}\\&\leq M' \cdot \left[\|\psi_i^*\left(\varphi - P_N\varphi\right)\|_{L^1} + \sum_{\ell=1}^{d}\left\|\partial_\ell^{d+1}\left[\psi_i^*\left(\varphi - P_N\varphi\right)\right]\right\|_{L^1}\right] \qquad (6.2.6)\end{aligned}$$

for some constant $M' = M'(d)$.

We observe $\psi_i^* \equiv 0$ on $\mathbb{R}^d \setminus \dot{B}_i^{\,*}$ and

$$\dot{B}_i^{\,*} \subset \dot{B}_{i-3} \cup \cdots \cup \dot{B}_{i+3} \subset B_{2^{i+5}}(0) \subset B_{32}(0)$$

for all $i \leq 0$, which implies $\left(\partial_\ell^k \psi_i^*\right)(\xi) = 0$ for $|\xi| \geq 2^{i+5}$. Now,

$$\begin{aligned}\left|\left(\partial_\ell^k \psi_i^*\right)(\xi)\right| &\leq \sum_{j=i-3}^{i+3}\left|\partial_\ell^k\left[\widehat{\Psi}\left(2^{-j}\xi\right)\right]\right|\\&\leq \left\|\partial_\ell^k \widehat{\Psi}\right\|_{\sup} \cdot \chi_{\overline{B_{2^{i+5}}(0)}}(\xi) \cdot \sum_{j=i-3}^{i+3} 2^{-kj}\\&\leq 7 \cdot 8^k \cdot \left[\max_{\substack{m\in\underline{d}\\ n\in\underline{d+1}}}\left\|\partial_m^n \widehat{\Psi}\right\|_{\sup}\right] \cdot \chi_{\overline{B_{2^{i+5}}(0)}}(\xi) \cdot 2^{-ki}\\&\leq C_{\Psi,d} \cdot \chi_{\overline{B_{2^{i+5}}(0)}}(\xi) \cdot 2^{-ki} \qquad (6.2.7)\end{aligned}$$

for all $0 \leq k \leq d+1$. Together with Leibniz' formula and with the estimate $\binom{d+1}{k} \leq 2^{d+1}$, we derive

$$\begin{aligned}&\left|\left[\partial_\ell^{d+1}\left(\psi_i^* \cdot \left(\varphi - P_N\varphi\right)\right)\right](\xi)\right|\\&\leq \sum_{k=0}^{d+1}\left[\binom{d+1}{k} \cdot \left|\left(\partial_\ell^{d+1-k}\psi_i^*\right)(\xi)\right| \cdot \left|\left(\partial_\ell^k\left(\varphi - P_N\varphi\right)\right)(\xi)\right|\right]\\&\overset{(\dagger)}{\leq} \left[\max_{|\gamma|=N}\|\partial^\gamma\varphi\|_{\sup}\right] 2^{d+1} C_N C_{\Psi,d} \cdot \chi_{\overline{B_{2^{i+5}}(0)}}(\xi) \cdot \sum_{k=0}^{d+1} 2^{-(d+1-k)i}\,|\xi|^{N-k}\\&\overset{(\ddagger)}{\leq} \left[\max_{|\gamma|=N}\|\partial^\gamma\varphi\|_{\sup}\right] 2^{d+1} C_N C_{\Psi,d} \cdot \chi_{\overline{B_{32}(0)}}(\xi) \cdot \sum_{k=0}^{d+1} 2^{-(d+1-k)i} 2^{(i+5)(N-k)}\\&\leq 2^{5N}(d+1)\,2^{d+1} C_N C_{\Psi,d} \cdot \left[\max_{|\gamma|=N}\|\partial^\gamma\varphi\|_{\sup}\right] \cdot 2^{i(N-d-1)} \cdot \chi_{\overline{B_{32}(0)}}(\xi),\end{aligned}$$

where we used $N > |\alpha| + d + 1 \geq d + 1$ at $(\ddagger)$. Furthermore, equation (6.2.4) was used at $(\dagger)$. The above estimate easily implies

$$\left\|\partial_\ell^{d+1}\left(\psi_i^* \cdot \left(\varphi - P_N\varphi\right)\right)\right\|_{L^1} \leq K \cdot \left[\max_{|\gamma|=N}\|\partial^\gamma\varphi\|_{\sup}\right] \cdot 2^{i(N-d-1)}$$

for some constant $K = K(d, N, \Psi)$ which is independent of $i \leq 0$, and of $f$ and $\varphi$.

A similar, but easier calculation employing the estimates (6.2.4) and (6.2.7) (with $k = 0$) yields

$$\begin{aligned}
&\left|\left[\psi_i^* \left(\varphi - P_N\varphi\right)\right](\xi)\right| \\
&\leq \left[\max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}\right] C_N C_{\Psi,d} \cdot \chi_{\overline{B_{2^{i+5}}(0)}}(\xi) \cdot |\xi|^N \\
&\leq \left[\max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}\right] 2^{5N} C_N C_{\Psi,d} \cdot \chi_{\overline{B_{32}(0)}}(\xi) \cdot 2^{Ni} \\
&\overset{(\dagger)}{\leq} \left[\max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}\right] 2^{5N} C_N C_{\Psi,d} \cdot \chi_{\overline{B_{32}(0)}}(\xi) \cdot 2^{i(N-d-1)}
\end{aligned}$$

and thus also $\left\|\psi_i^* \left(\varphi - P_N\varphi\right)\right\|_{L^1} \leq K \cdot \left[\max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}\right] \cdot 2^{i(N-d-1)}$ with the same constant as above. In the step marked with $(\dagger)$, we used $i \leq 0$.

We finally employ estimate (6.2.6) to derive

$$\begin{aligned}
&\left\| \left( 2^{i\left(\frac{d}{p}-\alpha\right)} \left\|\mathcal{F}\left(\psi_i^* \left(\varphi - P_N\varphi\right)\right)\right\|_{L^1} \right)_{i\leq 0} \right\|_{\ell^{q'}} \\
&\leq (1+d)\, KM' \cdot \left[\max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}\right] \cdot \left\| \left( 2^{i\left(N+\frac{d}{p}-d-1-\alpha\right)} \right)_{i\leq 0} \right\|_{\ell^{q'}} \\
&= K' \cdot \max_{|\gamma|=N} \left\|\partial^\gamma \varphi\right\|_{\sup}.
\end{aligned}$$

The last step is justified by $N + \frac{d}{p} - d - 1 - \alpha \geq N - d - 1 - \alpha > 0$ which holds by choice of $N$.

Together with estimate (6.2.5) this shows that $\check{f} \in \mathcal{S}'\left(\mathbb{R}^d\right)$ is a well-defined tempered distribution with absolute convergence of the series for every $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. It remains to note that $P_N\varphi = 0$ as well as $\varphi = \sum_{j=-\infty}^{\infty} \psi_j\varphi$ hold for all $\varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$, where only finitely many terms do not vanish. This implies

$$\begin{aligned}
\left(\widetilde{f} + \check{f}\right)(\varphi) &= \sum_{j=1}^{\infty} \left(\psi_j f\right)(\varphi) + \sum_{i\leq 0} f\left(\psi_i \cdot \left(\varphi - P_N\varphi\right)\right) \\
&= \sum_{j=-\infty}^{\infty} f\left(\psi_j\varphi\right) = f\left(\sum_{j=-\infty}^{\infty} \psi_j\varphi\right) = f(\varphi)
\end{aligned}$$

and thus $\varrho\left(\widetilde{f} + \check{f}\right) = f$. Equation (6.2.2) also yields

$$\left\|\mathcal{F}^{-1}\left(\widetilde{f} + \check{f}\right)\right\|_{\dot{B}^p_{\alpha,q}} = \left\|\varrho\left(\widetilde{f} + \check{f}\right)\right\|_{\mathcal{D}\left(\dot{\mathcal{B}},L^p,\ell^q_{u(\alpha)}\right)} = \|f\|_{\mathcal{D}\left(\dot{\mathcal{B}},L^p,\ell^q_{u(\alpha)}\right)} < \infty,$$

which means $\mathcal{F}^{-1}\left(\widetilde{f} + \check{f}\right) + \mathcal{P} \in \dot{B}^p_{\alpha,q}$ and hence

$$f = \left[\varrho \circ \mathcal{F}\right]\left(\mathcal{F}^{-1}\left(\widetilde{f} + \check{f}\right) + \mathcal{P}\right) \in \left[\varrho \circ \mathcal{F}\right]\left(\dot{B}^p_{\alpha,q}\right).$$

We have thus shown that $[\varrho \circ \mathcal{F}]$ is surjective. □

Now that we have established the isomorphism between the usual Besov spaces and our "Fourier side" decomposition space version of Besov spaces, we use our general results from the previous chapter to establish embeddings between homogeneous and inhomogeneous Besov spaces. We start with embeddings from homogeneous into inhomogeneous Besov spaces.

It is important to observe that $\dot{\mathcal{B}}$ is *not* moderate with respect to $\mathcal{B}$ (at least not "near the origin"), so that the conditions requiring moderateness of the whole covering are not applicable in the following theorem.

Nevertheless, the conditions given in the theorem below almost characterize the existence of embeddings from homogeneous into inhomogeneous Besov spaces. The only difference between the sufficient and the necessary conditions is that in some cases the sufficient condition requires a *strict* inequality, whereas the necessary condition only guarantees a *non*-strict inequality.

The reason for this (perhaps unexpected) sharpness of our criteria, expressed in terms of the geometry of the coverings and in terms of the decay behaviour of the weight, is the following: For all sets $B_n$ of the "coarse" covering $\mathcal{B}$, except for $B_0$, there is only a *uniformly bounded* number of sets $\dot{B}_m$ of the "fine" covering $\dot{\mathcal{B}}$ which intersect $B_n$.

In sharp contrast, for the set $B_0$, there is even an *infinite* number of "small" sets $\dot{B}_i$ which "lie below $B_0$". But this is remedied by the fact that the weight $2^{\alpha i}$ (together with a suitable power of the determinants $\left|\det\left(2^i \mathrm{id}\right)\right|$) is of exponential nature. Hence, there are only three cases:

1. The weight decays exponentially. In this case, it lies in *all* $\ell^p$-spaces, independently of $p \in (0, \infty]$.
2. The weight grows exponentially and is hence unbounded, so that it lies in *none* of the $\ell^p$-spaces for any $p \in (0, \infty]$.
3. The weight is constant with respect to $i$. This is the case which produces the "gap", because the weight lies in $\ell^\infty$, but not in any other $\ell^p$-space.

Since there is only the single set $B_0$ for which this occurs, the *magnitude* of the respective $\ell^p$-norm is not relevant, as long as it is finite. For a more detailed and formal analysis, see the proof below.

**Theorem 6.2.3.** *Let $p_1, p_2, q_1, q_2 \in (0, \infty]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Assume that the following conditions are satisfied:*

1. *We have $p_1 \leq p_2$,*
2. *We have*

$$\begin{cases} \alpha_1 \leq d \cdot \left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } q_1 \leq p_2^\triangledown, \\ \alpha_1 < d \cdot \left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } q_1 > p_2^\triangledown. \end{cases} \tag{6.2.8}$$

3. *We have*

$$\begin{cases} \alpha_2 \leq \alpha_1 + d \cdot \left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{if } q_1 \leq q_2, \\ \alpha_2 < \alpha_1 + d \cdot \left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{if } q_1 > q_2. \end{cases} \tag{6.2.9}$$

*Then, there is a bounded, linear map*

$$\iota : \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{\left(2^{j\alpha_1}\right)_j}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{\left(2^{i\alpha_2}\right)_i}\right)$$

*that satisfies*

$$\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle \text{ for all } \varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right).$$

*Thus, for arbitrary $f \in \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{\left(2^{j\alpha_1}\right)_j}\right) \subset \mathcal{D}'\left(\mathbb{R}^d \setminus \{0\}\right)$, the distribution $\iota f$ is an extension of $f$ to $C_c^\infty\left(\mathbb{R}^d\right)$.*

*In particular, we have $\dot{B}^{p_1}_{\alpha_1,q_1} \hookrightarrow B^{p_2}_{\alpha_2,q_2}$ with the interpretation that there is a bounded linear map $\eta : \dot{B}^{p_1}_{\alpha_1,q_1} \to B^{p_2}_{\alpha_2,q_2}$ which satisfies*

$$\eta\left(f + \mathcal{P}\right) + \mathcal{P} = f + \mathcal{P}$$

*for all $f + \mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1} \subset \mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P}$. Furthermore, one can choose $\eta$ such that $\eta\left(f + \mathcal{P}\right) = f$ holds for all $f + \mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1}$ with $\operatorname{supp} \widehat{f} \subset \mathbb{R}^d \setminus \{0\}$ or with $\widehat{f} \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$.*

*Conversely, assume that*

$$\theta : \left(C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right), \|\cdot\|_{\mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{\left(2^{j\alpha_1}\right)_j}\right)}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{\left(2^{i\alpha_2}\right)_i}\right), f \mapsto f$$

*is bounded or that there is a bounded linear map $\eta : \dot{B}^{p_1}_{\alpha_1,q_1} \to B^{p_2}_{\alpha_2,q_2}$ with $\eta\left(f + \mathcal{P}\right) = f$ for all $f \in L^2\left(\mathbb{R}^d\right)$ with $f + \mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1}$. Then the following hold:*

1. *We have $p_1 \leq p_2$,*
2. *we have*

$$\begin{cases} \alpha_1 \leq d\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{in case of } q_1 \leq p_2, \\ \alpha_1 < d\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{in case of } q_1 > p_2, \end{cases} \tag{6.2.10}$$

3. *the condition in equation* (6.2.9) *is satisfied.*

*Remark* 6.2.4. Our necessary conditions for the existence of an embedding always entail $\alpha_2 \leq 0$, so that an embedding as above can only hold for non-positive values of $\alpha_2$, which correspond to a "low smoothness".

The reason for these restrictive conditions is the form of the weight $\left(2^{j\alpha_1}\right)_{j\in\mathbb{Z}}$ used for the homogeneous Besov decomposition space, i.e. the fact that the weight decays near the origin if it grows away from the origin and vice versa. Let us give a short geometric argument

showing this, which avoids most opaque computations. To this end, we assume $p_1 = p_2 = p$. Under this assumption, the simple necessary criterion from Lemma 5.3.1 shows that we can only have an embedding of $\mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^{q_1}_{(2^{i\alpha_1})}\right)$ into $\mathcal{D}\left(\mathcal{B}, L^p, \ell^{q_2}_{(2^{i\alpha_2})}\right)$ if $2^{i\alpha_2} \leq 2^{j\alpha_1}$ holds for all $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$ with $B_i \cap \dot{B}_j \neq \varnothing$. But for $i = j \in \mathbb{N}$, we have $\dot{B}_i = B_i$, so that we must have $2^{i\alpha_2} \leq 2^{i\alpha_1}$ for all $i \in \mathbb{N}$, i.e. $\alpha_2 \leq \alpha_1$.

But finally, we have $B_0 \cap \dot{B}_j \neq \varnothing$ for all $j \in \mathbb{Z}_{\leq 0}$, so that we also get $1 = 2^{0\alpha_2} \leq 2^{j\alpha_1}$ for all $j \in \mathbb{Z}_{\leq 0}$. This clearly entails $\alpha_1 \leq 0$ and hence $\alpha_2 \leq \alpha_1 \leq 0$.

Thus, it would be worthwile to consider slightly modified homogeneous Besov decomposition spaces, for example using the weight

$$v_j^{(\gamma_1,\gamma_2)} = \begin{cases} 2^{j\gamma_1}, & \text{if } j \geq 0, \\ 2^{j\gamma_2}, & \text{if } j < 0. \end{cases}$$

This weight allows for *different* growth/decay behaviours near the origin and away from the origin. A slight modification of the proof below will show that this modified homogeneous Besov decomposition space admits embeddings into $\mathcal{D}\left(\mathcal{B}, L^p, \ell^{q_2}_{(2^{i\alpha})_i}\right)$ also for positive values of $\alpha$, as long as $\gamma_1 > 0$ and $\gamma_2 < 0$ hold. For this choice of $\gamma_1, \gamma_2$ it is not hard to see

$$\mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{w^{(\gamma_1,\gamma_2)}}\right) = \mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{w^{(\gamma_1)}}\right) \cap \mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^q_{w^{(\gamma_2)}}\right),$$

which establishes the connection to the "traditional" homogeneous Besov spaces.

Now, let us discuss the sharpness of our criteria. The only case in which the conditions given above do *not* completely characterize the existence of the desired embedding occurs if we have $p_2' < q_1 \leq p_2$ and

$$\alpha_1 = d \cdot \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$$

as well as

$$\begin{cases} \alpha_2 \leq 0, & \text{if } q_1 \leq q_2, \\ \alpha_2 < 0, & \text{if } q_1 > q_2. \end{cases}$$

In this case, our sufficient criteria do not yield existence of the desired embedding, and our necessary criteria do not disprove existence of such an embedding. In *all* other cases, the theorem provides a convenient and complete characterization for deciding the existence of the desired embedding.

Finally, we remark that $p_2' < q_1 \leq p_2$ can only hold for $p_2 > 2$, since we have $p_2 \leq 2 \leq p_2'$ for all $p_2 \in (0, 2]$. Thus, we achieve a *complete characterization* for the range $p_2 \in (0, 2]$. ◀

*Proof.* We apply Theorem 5.4.4 with $\mathcal{Q} = \dot{\mathcal{B}} = \left(2^k \mathrm{id} \cdot \dot{B}\right)_{k\in\mathbb{Z}}$, $\mathcal{P} = \mathcal{B}$ and $u = \left(2^{\alpha_1 k}\right)_{k\in\mathbb{Z}}$ as well as $v = \left(2^{\alpha_2 j}\right)_{j\in\mathbb{N}_0}$. Lemma 6.2.2 implies that $\mathcal{Q}, \mathcal{P}$ are indeed tight semi-structured $L^p$-decomposition coverings of $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$ and of $\mathcal{O}' = \mathbb{R}^d$, respectively, for all $p \in (0, \infty]$. The same lemma also shows that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and that the weights $u, v$ are moderate with respect to $\mathcal{Q}, \mathcal{P}$, respectively.

Hence, parts 1 and 2 of Theorem 5.4.4 are indeed applicable. Thus, we are led to investigating finiteness of the quasi-norm

$$\begin{aligned} K_r &:= \left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j} \right\|_{\ell^r} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\ &= \left\| \left( 2^{\alpha_2 j} \cdot \left\| \left( 2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{k\in I_j} \right\|_{\ell^r} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \end{aligned} \tag{6.2.11}$$

for different values of $r \in (0, \infty]$, with $J = \mathbb{N}_0$, $I = \mathbb{Z}$, $T_i = 2^i \mathrm{id}$ for $i \in I$ and

$$I_j = \left\{ k \in \mathbb{Z} \,\middle|\, \dot{B}_k \cap B_j \neq \varnothing \right\}.$$

For $j = 0$, we have $\dot{B}_k \cap B_j = \dot{B}_k \cap B_0 \neq \varnothing$ for all $k \in \mathbb{Z}_{\leq 0}$, thanks to Lemma 6.2.2. This means $I_0 \supset \mathbb{Z}_{\leq 0}$. Conversely, $B_0 \cap \dot{B}_k \neq \varnothing$ implies

$$\varnothing \neq B_8(0) \cap \mathbb{R}^d \setminus \overline{B_{2^{k-2}}(0)}$$

and hence $2^{k-2} < 8 = 2^3$, i.e. $k \leq 4$. We conclude $\mathbb{Z}_{\leq 0} \subset I_0 \subset \mathbb{Z}_{\leq 4}$. This yields the equivalence

$$\begin{aligned} &\left\| \left( 2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{k\in I_0} \right\|_{\ell^r} < \infty \\ \iff &\begin{cases} \left(\frac{1}{p_1}-\frac{1}{p_2}\right)d - \alpha_1 \geq 0, & \text{in case of } r = \infty, \\ \left(\frac{1}{p_1}-\frac{1}{p_2}\right)d - \alpha_1 > 0, & \text{in case of } r < \infty, \end{cases} \end{aligned} \tag{6.2.12}$$

due to the exponential nature of $\left( 2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{k\in\mathbb{Z}_{\leq 0}}$.

In case of $j \in \mathbb{N}$, we have $B_j = \dot{B}_j$, which yields $I_j = j^{*_{\mathcal{B}}}$, where the cluster is formed with respect to $\dot{\mathcal{B}}$. But Lemma 6.2.2 shows that the weight $\left( 2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{k\in\mathbb{Z}}$ is $\dot{\mathcal{B}}$-moderate. We conclude $2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \asymp 2^{j\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]}$ for all $k \in I_j = j^{*_{\mathcal{B}}}$. Because of $1 \leq |j^{*_{\mathcal{B}}}| \leq N_{\dot{\mathcal{B}}}$, these considerations imply

$$\left\| \left( 2^{k\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{k\in I_j} \right\|_{\ell^r} \asymp 2^{j\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]}$$

for all $j \in \mathbb{N}$. The implied constants only depend on $r, p_1, p_2, d, \alpha_1, N_{\dot{\mathcal{B}}}$.

Hence, under the assumption that the condition in equation (6.2.12) is fulfilled, we see that finiteness of $K_r$ is equivalent to finiteness of

$$\begin{aligned}&\left\| \left( 2^{\alpha_2 j} \cdot 2^{j\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d-\alpha_1\right]} \right)_{j\in\mathbb{N}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} \\ &= \left\| \left( 2^{j\left[\left(\frac{1}{p_1}-\frac{1}{p_2}\right)d+\alpha_2-\alpha_1\right]} \right)_{j\in\mathbb{N}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}}.\end{aligned}$$

As above, we see that this is equivalent to

$$\begin{cases} \left(\frac{1}{p_1}-\frac{1}{p_2}\right)d+\alpha_2-\alpha_1 \leq 0, & \text{in case of } q_2\cdot(q_1/q_2)' = \infty, \\ \left(\frac{1}{p_1}-\frac{1}{p_2}\right)d+\alpha_2-\alpha_1 < 0, & \text{in case of } q_2\cdot(q_1/q_2)' < \infty. \end{cases}$$

But Remark 5.2.3 shows that $q_2\cdot(q_1/q_2)'$ is finite if and only if $q_2 < q_1$.

Let us first establish the necessary conditions. To this end, we will first show that existence of a bounded map $\eta : \dot{B}^{p_1}_{\alpha_1,q_1} \to B^{p_2}_{\alpha_2,q_2}$ that satisfies $\eta(f+\mathcal{P}) = f$ for all $f \in L^2(\mathbb{R}^d)$ with $f+\mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1}$ implies boundedness of the map $\theta$ as in the statement of the theorem. Then, we will show that boundedness of $\theta$ implies the stated conditions.

So, assume that $\eta$ as above is bounded. By Lemma 6.2.2, we know that the Fourier transform induces isomorphisms

$$\begin{aligned}\Phi :& B^{p_2}_{\alpha_2,q_2} \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{(2^{i\alpha_2})_i}\right), f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d)}, \\ \Psi :& \dot{B}^{p_1}_{\alpha_1,q_1} \to \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{(2^{j\alpha_1})_j}\right), (f+\mathcal{P}) \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}.\end{aligned}$$

Composition yields a bounded linear map $\theta := \Phi\circ\eta\circ\Psi^{-1}$.

For $f \in C_c^\infty(\mathbb{R}^d\setminus\{0\})$, let $g := \mathcal{F}^{-1}f \in \mathcal{S}(\mathbb{R}^d)$. As in the proof of Lemma 6.2.2, we see

$$\|g+\mathcal{P}\|_{\dot{B}^{p_1}_{q_1,\alpha_1}} = \|\widehat{g}\|_{\mathcal{D}(\dot{\mathcal{B}},L^{p_1},\ell^{q_1}_{(2^{j\alpha_1})_j})} = \|f\|_{\mathcal{D}(\dot{\mathcal{B}},L^{p_1},\ell^{q_1}_{(2^{j\alpha_1})_j})} < \infty.$$

Further, we have $\Psi(g+\mathcal{P}) = f|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}$, where $f$ is considered as a tempered distribution. Hence, we conclude $\Psi^{-1}f = g+\mathcal{P}$ with $g \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and $g+\mathcal{P} \in \dot{B}^{p_1}_{q_1,\alpha_1}$. This implies $\eta(g+\mathcal{P}) = g$ by assumption on $\eta$ and allows us to conclude

$$\theta f = \Phi(\eta(\Psi^{-1}f)) = \Phi(\eta(g+\mathcal{P})) = \Phi g = \widehat{g} = f,$$

so that $\theta|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}$ has the required properties.

Let us now assume that the map $\theta$ as in the statement of the theorem is bounded. By part 2 of Theorem 5.4.4, this yields $p_1 \leq p_2$ as well as $K_{p_2\cdot(q_1/p_2)'} \lesssim |||\theta||| < \infty$, where the constant $K_{p_2\cdot(q_1/p_2)'}$ is defined as in equation (6.2.11) above. But Remark 5.2.3 shows that $p_2\cdot(q_1/p_2)' < \infty$ is equivalent to $p_2 < q_1$. Together with our char-

acterization of finiteness of $K_r$ from the beginning of the proof, this establishes the second part of the theorem.

Now assume that $p_1 \leq p_2$ holds and that the conditions given in equations (6.2.8) and (6.2.9) are satisfied. By the considerations at the beginning of the proof, this ensures finiteness of $K_{p_2^\nabla \cdot \left(q_1/p_2^\nabla\right)'}$, so that part 1 of Theorem 5.4.4 implies existence of a bounded linear map $\iota$ as in the statement of the theorem.

It remains to translate this to an embedding of the usual Besov spaces. To this end, define

$$\eta : \dot{B}^{p_1}_{\alpha_1,q_1} \to B^{p_2}_{\alpha_2,q_2}, f + \mathcal{P} \mapsto \Phi^{-1}\left(\iota\left(\Psi\left(f + \mathcal{P}\right)\right)\right),$$

where $\Phi, \Psi$ are defined as above.

For $f + \mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1}$ (which implies $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$), we first note

$$\begin{aligned}\left(\mathcal{F}\left[\eta\left(f + \mathcal{P}\right)\right]\right)|_{C_c^\infty\left(\mathbb{R}^d\right)} &= \Phi\left[\eta\left(f + \mathcal{P}\right)\right] \\ &= \iota\left(\Psi\left(f + \mathcal{P}\right)\right). \end{aligned} \tag{6.2.13}$$

For arbitrary $\varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ this yields

$$\begin{aligned}\left\langle \mathcal{F}\left[\eta\left(f + \mathcal{P}\right)\right], \varphi\right\rangle &= \left\langle \iota\left(\Psi\left(f + \mathcal{P}\right)\right), \varphi\right\rangle \\ &\overset{(\dagger)}{=} \left\langle \Psi\left(f + \mathcal{P}\right), \varphi\right\rangle \\ &= \left\langle \widehat{f}, \varphi\right\rangle,\end{aligned}$$

where we used $\langle \iota g, \varphi\rangle = \langle g, \varphi\rangle$ for all $\varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ at $(\dagger)$. Since $\varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ was arbitrary, we conclude

$$\operatorname{supp}\left(\mathcal{F}\left[f - \eta\left(f + \mathcal{P}\right)\right]\right) \subset \{0\}. \tag{6.2.14}$$

Here, we note that $\mathcal{F}\left[\eta\left(f + \mathcal{P}\right)\right]$ is a tempered distribution, whereas $\mathcal{F}\left[\eta\left(f + \mathcal{P}\right)\right]|_{C_c^\infty\left(\mathbb{R}^d\right)}$ is an "ordinary" distribution on $\mathbb{R}^d$. But since a tempered distribution is already determined by its behaviour on $C_c^\infty\left(\mathbb{R}^d\right)$ and since the support of a tempered distribution remains the same when considered as an "ordinary" distribution, we will not distinguish between these two objects in the following.

Using equation (6.2.14) and [42, Proposition 2.4.1], we see that $\mathcal{F}\left[f - \eta\left(f + \mathcal{P}\right)\right]$ is a linear combination of derivatives of $\delta_0$. Thus, $f - \eta\left(f + \mathcal{P}\right) \in \mathcal{P}$, which means $\eta\left(f + \mathcal{P}\right) + \mathcal{P} = f + \mathcal{P}$, as claimed.

*Here, $\delta_0$ is the Dirac delta distribution at the origin, i.e., $\langle \delta_0, \varphi\rangle = \varphi(0)$.*

Regarding the remaining two properties of $\eta$, let us first assume $f + \mathcal{P} \in \dot{B}^{p_1}_{\alpha_1,q_1}$ with $\operatorname{supp} \widehat{f} \subset \mathbb{R}^d \setminus \{0\}$. Define $g := \mathcal{F}\left[\eta\left(f + \mathcal{P}\right)\right]$ and note $g = \iota\left(\Psi\left(f + \mathcal{P}\right)\right)$ by equation (6.2.13). Equation (6.2.14) yields

$$\operatorname{supp}\left(\widehat{f} - g\right) = \operatorname{supp}\left(\mathcal{F}\left[f - \eta\left(f + \mathcal{P}\right)\right]\right) \subset \{0\}.$$

But property 1c of Theorem 5.4.4 implies

$$\begin{aligned}\operatorname{supp} g &= \operatorname{supp}\left(\iota\left(\widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}\right)\right)\\ &\subset \overline{\operatorname{supp}\widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}}\\ &\subset \overline{\operatorname{supp}\widehat{f}}\\ &\overset{(\dagger)}{=} \operatorname{supp}\widehat{f}\\ &\subset \mathbb{R}^d\setminus\{0\}\end{aligned}$$

and hence $\operatorname{supp}(\widehat{f}-g)\subset\mathbb{R}^d\setminus\{0\}$. Here the step marked with $(\dagger)$ is justified, since $\widehat{f}$ is a distribution on $\mathbb{R}^d$, so that $\operatorname{supp}\widehat{f}\subset\mathbb{R}^d$ is closed.

Together, we conclude $\operatorname{supp}(\widehat{f}-g)=\varnothing$ and hence

$$\mathcal{F}[\eta(f+\mathcal{P})]=g=\widehat{f},$$

which entails $\eta(f+\mathcal{P})=f$.

Finally, let $f+\mathcal{P}\in\dot{B}^{p_1}_{\alpha_1,q_1}$ with $\widehat{f}\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Using property 1b of Theorem 5.4.4, we derive

$$\mathcal{F}[\eta(f+\mathcal{P})]=\iota\left(\widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}\right)=\widehat{f},$$

because $\mathbb{R}^d\setminus\{0\}$ is of full measure, so that $\widehat{f}(x)=0$ holds for almost every $x\in\mathbb{R}^d\setminus\left[\mathbb{R}^d\setminus\{0\}\right]$. As above, this yields $\eta(f+\mathcal{P})=f$. □

*Remark* 6.2.5. The embedding $\iota$ is obtained from Theorem 5.4.4, which implies that $\iota$ is given by

$$\langle \iota f, g\rangle=\sum_{k\in\mathbb{Z}}\langle\varphi_k f, g\rangle$$

for all $g\in C_c^\infty(\mathbb{R}^d)$, where $(\varphi_\lambda)_{\lambda\in\Lambda}=(\varphi_k)_{k\in\mathbb{Z}}$ is an $L^{p_1}$-BAPU for $\mathcal{Q}=\dot{\mathcal{B}}$. This in turn shows that the map $\eta$ constructed in Theorem 6.2.3 above satisfies

$$\begin{aligned}\langle\eta[f+\mathcal{P}],h\rangle &= \left\langle\mathcal{F}^{-1}\left[\iota\left(\widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}\right)\right],h\right\rangle\\ &= \left\langle\iota\left(\widehat{f}|_{C_c^\infty(\mathbb{R}^d\setminus\{0\})}\right),\mathcal{F}^{-1}h\right\rangle\\ &= \sum_{k\in\mathbb{Z}}\left\langle\widehat{f},\varphi_k\cdot\mathcal{F}^{-1}h\right\rangle \qquad (6.2.15)\end{aligned}$$

for all $h\in\mathcal{S}(\mathbb{R}^d)$ with $\mathcal{F}^{-1}h\in C_c^\infty(\mathbb{R}^d)$, i.e. for all $h\in\mathcal{F}\left[C_c^\infty(\mathbb{R}^d)\right]$.

We now want to show that this implies that $\eta$ commutes with translations. To this end, first note that the homogeneous Besov space $\dot{B}^p_{\alpha,q}$ is (isometrically) invariant under translations. Indeed, for $f+\mathcal{P}\in\dot{B}^p_{\alpha,q}$, we have

$$\begin{aligned}\mathcal{F}^{-1}\left(\varphi_j\widehat{L_x f}\right) &= \mathcal{F}^{-1}\left(\varphi_j M_{-x}\widehat{f}\right)\\ &= \mathcal{F}^{-1}\left[M_{-x}\left(\varphi_j\widehat{f}\right)\right]\\ &= L_x\left[\mathcal{F}^{-1}\left(\varphi_j\widehat{f}\right)\right],\end{aligned}$$

where $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$ denotes the modulation of $f$ as usual. Since $L^p$ is isometrically translation invariant, this yields

$$\begin{aligned}\|L_x f\|_{\dot{B}^p_{\alpha,q}} &= \left\|\left(2^{j\alpha}\left\|\mathcal{F}^{-1}\left(\varphi_j\widehat{L_x f}\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^q}\\ &= \left\|\left(2^{j\alpha}\left\|L_x\left[\mathcal{F}^{-1}\left(\varphi_j\widehat{f}\right)\right]\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^q}\\ &= \|f\|_{\dot{B}^p_{\alpha,q}} < \infty.\end{aligned}$$

To see that $\eta$ commutes with translations, first note

$$\begin{aligned}\left\langle\widehat{L_x f},\, \varphi_k\cdot\mathcal{F}^{-1}h\right\rangle &= \left\langle L_x f,\, \mathcal{F}\left[\varphi_k\cdot\mathcal{F}^{-1}h\right]\right\rangle\\ &= \left\langle f,\, L_{-x}\left(\mathcal{F}\left[\varphi_k\cdot\mathcal{F}^{-1}h\right]\right)\right\rangle\\ &= \left\langle f,\, \mathcal{F}\left[M_{-x}\left(\varphi_k\cdot\mathcal{F}^{-1}h\right)\right]\right\rangle\\ &= \left\langle\widehat{f},\, \varphi_k\cdot M_{-x}\left(\mathcal{F}^{-1}h\right)\right\rangle\\ &= \left\langle\widehat{f},\, \varphi_k\cdot\mathcal{F}^{-1}\left(L_{-x}h\right)\right\rangle\end{aligned}$$

for all $k\in\mathbb{Z}$ and $h\in\mathcal{F}\left(C_c^\infty\left(\mathbb{R}^d\right)\right)$. But by equation (6.2.15) above, this yields

$$\begin{aligned}\langle\eta\left[L_x f+\mathcal{P}\right], h\rangle &= \sum_{k\in\mathbb{Z}}\left\langle\widehat{L_x f},\, \varphi_k\cdot\mathcal{F}^{-1}h\right\rangle\\ &= \sum_{k\in\mathbb{Z}}\left\langle\widehat{f},\, \varphi_k\cdot\mathcal{F}^{-1}\left(L_{-x}h\right)\right\rangle\\ &= \langle\eta\left[f+\mathcal{P}\right], L_{-x}h\rangle\\ &= \langle L_x\left[\eta\left(f+\mathcal{P}\right)\right], h\rangle\end{aligned}$$

for all $h\in\mathcal{F}\left[C_c^\infty\left(\mathbb{R}^d\right)\right]$. But it is well-known (see e.g. [33, Proposition 9.9] that $C_c^\infty\left(\mathbb{R}^d\right)$ is dense in $\mathcal{S}\left(\mathbb{R}^d\right)$. Since the Fourier transform is a homeomorphism of $\mathcal{S}\left(\mathbb{R}^d\right)$ with itself (see [33, Corollary 8.28]), this implies that $\mathcal{F}\left[C_c^\infty\left(\mathbb{R}^d\right)\right]$ is also dense in $\mathcal{S}\left(\mathbb{R}^d\right)$.

Since $\eta\left[f+\mathcal{P}\right]\in\mathcal{B}^{p_2}_{\alpha_2,q_2}\subset\mathcal{S}'\left(\mathbb{R}^d\right)$ is a tempered distribution, the same is true of $L_x\left[\eta\left(f+\mathcal{P}\right)\right]$, so that the above equation yields

$$\eta\left[L_x f+\mathcal{P}\right] = L_x\left[\eta\left(f+\mathcal{P}\right)\right]$$

for all $f\in\dot{\mathcal{B}}^{p_1}_{\alpha_1,q_1}$ and all $x\in\mathbb{R}^d$.

In summary, we have shown that the map

$$\eta : \dot{B}^{p_1}_{\alpha_1,q_1} \to B^{p_2}_{\alpha_2,q_2} \leq \mathcal{S}'\left(\mathbb{R}^d\right)$$

constructed in Theorem 6.2.3 yields a TRANSLATION COMMUTING REALIZATION IN $\mathcal{S}'\left(\mathbb{R}^d\right)$ of the homogeneous Besov space $\dot{B}^{p_1}_{\alpha_1,q_1}$. Here, the term "realization" is used to refer to the fact that the map $\eta$ picks one representative $\eta\left(f+\mathcal{P}\right)$ of the equivalence class $f+\mathcal{P}$ of $f$. This is the case, since we saw above that $\eta\left(f+\mathcal{P}\right)+\mathcal{P} = f+\mathcal{P}$ holds for all $f$.

By choosing $p_2 = q_2 = \infty$ (which yields $p_2^\nabla = 1$) and $\alpha_2 < \alpha_1 - \frac{d}{p_1}$, Theorem 6.2.3 thus implies that the space $\dot{B}^{p_1}_{\alpha_1,q_1}$ admits a translation commuting realization in $\mathcal{S}'\left(\mathbb{R}^d\right)$ as soon as the condition

$$\begin{cases} \alpha_1 \leq \frac{d}{p_1}, & \text{if } q_1 \leq 1, \\ \alpha_1 < \frac{d}{p_1}, & \text{if } q_1 > 1 \end{cases}$$

is fulfilled.

Now, our results can be compared to [7]: In [7, Theorem 4.2], Bourdaud shows that for arbitrary $p, q \in [1,\infty]$, the homogeneous Besov space $\dot{B}^p_{\alpha,q}$ admits a translation commuting realization with values in

$$\mathcal{S}'_\nu\left(\mathbb{R}^d\right) = \mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P}_\nu,$$

where $\mathcal{P}_\nu$ denotes the space of all polynomials of degree *less than* $\nu$. Here, $\nu$ has to be chosen as

$$\nu := \begin{cases} \alpha - \frac{d}{p}, & \text{if } q = 1 \text{ and } \alpha \in \mathbb{N}_0 + \frac{d}{p}, \\ \left(\left\lfloor \alpha - \frac{d}{p} \right\rfloor + 1\right)_+, & \text{otherwise.} \end{cases}$$

We are interested in the case $\mathcal{S}'_\nu\left(\mathbb{R}^d\right) \cong \mathcal{S}'\left(\mathbb{R}^d\right)$, i.e. $\nu = 0$. This is fulfilled precisely for

$$q = 1 \text{ and } \alpha \leq \frac{d}{p}$$

or for

$$q > 1 \text{ and } \alpha < \frac{d}{p}.$$

Thus, we see that the *existence* part of [7, Theorem 4.2] is a special case of our embedding results for decomposition spaces, but only for the case $\nu = 0$.

We remark that [7, Theorem 4.2] also contains the statement that the number $\nu$ above is optimal, i.e., there is no translation commuting realization of $\dot{B}^p_{\alpha,q}$ with values in $\mathcal{S}'_{\nu-1}\left(\mathbb{R}^d\right)$ if $\nu \geq 1$. This shows that

the map $\iota$ defined by $\langle \iota f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, \varphi_k g\rangle$ does *not* yield a bounded linear map

$$\iota : \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{(2^{j\alpha_1})_j}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{(2^{i\alpha_2})_i}\right),$$

for $p_2 = q_2 = \infty$, $q_1 > 1$ and $\alpha_1 = \frac{d}{p_1}$, since otherwise the same arguments as above would yield a translation commuting realization of $\dot{B}^p_{\alpha,q}$ with values in $\mathcal{S}'\left(\mathbb{R}^d\right)$. Thus, in this case the sufficient condition given in Theorem 6.2.3 is sharp, whereas the necessary condition in equation (6.2.10) lacks sharpness.

Note, however, that we assumed a special form of $\iota$ for this argument. It could still be possible that there is an "embedding"

$$\iota : \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_1}, \ell^{q_1}_{(2^{j\alpha_1})_j}\right) \to \mathcal{D}\left(\mathcal{B}, L^{\infty}, \ell^{\infty}_{(2^{i\alpha_2})_i}\right)$$

with $\langle \iota f, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$, even for $q_1 > 1$ and $\alpha_1 = \frac{d}{p_1}$. But the above argument shows that this map can not commute with modulations, since the corresponding map between the "space side" Besov spaces would then be invariant under translations. ◀

In our next theorem, we study the existence of the "reversed" embedding, i.e., of an embedding of inhomogeneous Besov spaces into homogeneous Besov spaces. As in the theorem above, we will see that our sufficient and necessary criteria are basically the same, with the single exception that in some cases the sufficient criteria demand a strict inequality, but the necessary criteria only yield a non-strict inequality.

We remark that in the present setting, the necessary criteria in Theorem 5.4.1 do not suffice to establish all conditions given below. Instead, the proof will make use of Lemma 5.3.11.

**Theorem 6.2.6.** *Let $p_1, p_2, q_1, q_2 \in (0,\infty]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Assume that the following conditions are satisfied:*

1. *We have $p_1 \leq p_2$,*
2. *We have*

$$\begin{cases} \alpha_2 \geq d\left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{if } q_2 \geq p_1^{\triangle} \text{ and } p_1 \in [1,\infty], \\ \alpha_2 \geq d\left(\frac{1}{p_2} - 1\right), & \text{if } q_2 = \infty \text{ and } p_1 \in (0,1), \\ \alpha_2 > d\left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{if } q_2 < p_1^{\triangle} \text{ and } p_1 \in [1,\infty], \\ \alpha_2 > d\left(\frac{1}{p_2} - 1\right), & \text{if } q_2 < \infty \text{ and } p_1 \in (0,1), \end{cases} \tag{6.2.16}$$

3. *We have*

$$\begin{cases} \alpha_1 \geq \alpha_2 + d\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{in case of } q_2 \geq q_1, \\ \alpha_1 > \alpha_2 + d\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{in case of } q_2 < q_1. \end{cases} \tag{6.2.17}$$

*Then, the map*

$$\iota : \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{\left(2^{j\alpha_1}\right)_j}\right) \to \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_2}, \ell^{q_2}_{\left(2^{j\alpha_2}\right)_j}\right), f \mapsto f|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)} \tag{6.2.18}$$

*is well-defined and continuous. In particular, we have* $B^{p_1}_{\alpha_1,q_1} \hookrightarrow \dot{B}^{p_2}_{\alpha_2,q_2}$, *which means that the map*

$$\nu : B^{p_1}_{\alpha_1,q_1} \to \dot{B}^{p_2}_{\alpha_2,q_2}, f \mapsto f + \mathcal{P}$$

*is well-defined and bounded.*

*Conversely, assume that one of the two maps* $\iota$ *or* $\nu$ *given above is bounded. This implies* $p_1 \leq p_2$ *and validity of equation* (6.2.17). *Furthermore, boundedness of one of these maps also implies*

$$\begin{cases} \alpha_2 \geq d\left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{in case of } q_2 \geq p_1, \\ \alpha_2 > d\left(\frac{1}{p_2} - \frac{1}{p_1}\right), & \text{in case of } q_2 < p_1 \end{cases} \tag{6.2.19}$$

*and*

$$\begin{cases} \alpha_2 \geq d\left(\frac{1}{p_2} - 1\right), & \text{in case of } q_2 = \infty, \\ \alpha_2 > d\left(\frac{1}{p_2} - 1\right), & \text{in case of } q_2 < \infty. \end{cases} \tag{6.2.20}$$

*Remark.* We remark that the above theorem yields a *complete characterization* in the range $p_1 \in [2, \infty] \cup (0, 1]$. Indeed, $p_1 \in [2, \infty]$ implies $p_1^{\triangle} = p_1$, so that the conditions in equation (6.2.16) coincide with those in equation (6.2.19). Furthermore, for $p_1 \in (0, 1]$, we have $p_1^{\triangle} = p_1' = \infty$, so that the conditions in equation (6.2.16) are easily seen to be indentical to those in equation (6.2.20).

It is worth noting that for $p_1 = p_2$, the sufficient conditions above reduce to

1. $\alpha_2 \geq d\left(\frac{1}{p_1} - 1\right)_+$ or $\alpha_2 > d\left(\frac{1}{p_1} - 1\right)_+$ in the cases $q_2 \geq p_1^{\triangle}$ and $q_2 < p_1^{\triangle}$, respectively.
2. $\alpha_1 \geq \alpha_2$ or $\alpha_1 > \alpha_2$ in the cases $q_2 \geq q_1$ and $q_2 < q_1$ , respectively.

In particular, $\nu$ is always well-defined and bounded for $p_1 = p_2$ and $\alpha_1 > \alpha_2 > d\left(\frac{1}{p_1} - 1\right)_+$.

Finally, we remark that [67, Theorem in §2.3.3] implies that

$$f \mapsto \|f\|_{L^p} + \|f\|_{\dot{B}^p_{\alpha,q}}$$

is an equivalent quasi-norm on the space $B^p_{\alpha,q}$ for $\alpha > \sigma_p = d\left(\frac{1}{p}-1\right)_+$. In particular, this implies

$$B^p_{\alpha,q} \hookrightarrow \dot{B}^p_{\alpha,q}$$

for $\alpha > d\left(\frac{1}{p}-1\right)_+$. Our results above show that this embedding is even true for $\alpha \geq d\left(\frac{1}{p}-1\right)_+$, as long as $q \geq p^{\triangle}$ holds.

*Proof.* The proof is similar to that of Theorem 6.2.3. One difference is that the embedding is in this case simply given by a restriction mapping.

We apply Theorem 5.4.1 with $\mathcal{Q} = \dot{\mathcal{B}} = \left(2^k \mathrm{id} \cdot \dot{B}\right)_{k\in\mathbb{Z}}$, $\mathcal{P} = \mathcal{B}$ and $u = \left(2^{\alpha_2 k}\right)_{k\in\mathbb{Z}}$ as well as $v = \left(2^{\alpha_1 j}\right)_{j\in\mathbb{N}_0}$ and with interchanged roles of $p_1, p_2$ and $q_1, q_2$. Lemma 6.2.2 implies that $\mathcal{Q}, \mathcal{P}$ are indeed tight semi-structured $L^p$-decomposition coverings of $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$ and of $\mathcal{O}' = \mathbb{R}^d$, respectively, for all $p \in (0,\infty]$. The same lemma also show that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and that the weights $u, v$ are moderate with respect to $\mathcal{Q}, \mathcal{P}$, respectively.

Hence, parts 1 and 2 of Theorem 5.4.1 are indeed applicable. Thus, we are led to investigating finiteness of the quasi-norm

$$\begin{aligned} K_{s,r,t} &:= \left\| \left( \left( \left|\det\left(S_j\right)\right|^s v_j \right)^{-1} \left\| \left( \left|\det\left(T_i\right)\right|^r u_i \right)_{i\in I_j} \right\|_{\ell^t} \right)_{j\in J} \right\|_{\ell^{q_2(q_1/q_2)'}} \\ &= \left\| \left( 2^{-j(ds+\alpha_1)} \left\| \left( 2^{k(\alpha_2+rd)} \right)_{k\in I_j} \right\|_{\ell^t} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q_2(q_1/q_2)'}} \end{aligned} \tag{6.2.21}$$

for different values of $r, s \in \mathbb{R}$ and $t \in (0,\infty]$, with $J = \mathbb{N}_0$, $I = \mathbb{Z}$, $T_i = 2^i \mathrm{id}$ for $i \in I$ and

$$I_j = \left\{ k \in \mathbb{Z} \,\middle|\, \dot{B}_k \cap B_j \neq \varnothing \right\} \qquad \text{for } j \in J = \mathbb{N}_0.$$

We recall the inclusion $\mathbb{Z}_{\leq 0} \subset I_0 \subset \mathbb{Z}_{\leq 4}$ from the proof of Theorem 6.2.3. This yields the equivalence

$$\left\| \left( 2^{k(rd+\alpha_2)} \right)_{k\in I_0} \right\|_{\ell^t} < \infty \iff \begin{cases} rd + \alpha_2 \geq 0, & \text{if } t = \infty, \\ rd + \alpha_2 > 0, & \text{if } t < \infty, \end{cases} \tag{6.2.22}$$

due to the exponential nature of $\left(2^{k(rd+\alpha_2)}\right)_{k\in\mathbb{Z}_{\leq 0}}$.

For $j \in \mathbb{N}$, we recall from the proof of Theorem 6.2.3 that we have $I_j = j^{*_{\mathcal{B}}}$, where the cluster is formed with respect to $\dot{\mathcal{B}}$. But the weight $\left(2^{k(rd+\alpha_2)}\right)_{k\in\mathbb{Z}}$ is $\dot{\mathcal{B}}$-moderate by Lemma 6.2.2. Hence,

$2^{k(rd+\alpha_2)} \asymp 2^{j(rd+\alpha_2)}$ for all $k \in I_j = j^{*_{\mathcal{B}}}$. Using $1 \leq |j^{*_{\mathcal{B}}}| \leq N_{\mathcal{B}}$, these considerations imply

$$\left\| \left( 2^{k(rd+\alpha_2)} \right)_{k \in I_j} \right\|_{\ell^t} \asymp 2^{j(rd+\alpha_2)}$$

for all $j \in \mathbb{N}$, where the implied constants only depend on $t, r, d, \alpha_2, N_{\mathcal{B}}$.

Hence, under the assumption that the condition in equation (6.2.22) is fulfilled, we see that finiteness of $K_{s,r,t}$ is equivalent to finiteness of

$$\left\| \left( 2^{j[d(r-s)+\alpha_2-\alpha_1]} \right)_{j \in \mathbb{N}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}.$$

As above, we see that this is equivalent to

$$\begin{cases} d\,(r-s) + \alpha_2 - \alpha_1 \leq 0, & \text{in case of } q_2 \cdot (q_1/q_2)' = \infty, \\ d\,(r-s) + \alpha_2 - \alpha_1 < 0, & \text{in case of } q_2 \cdot (q_1/q_2)' < \infty. \end{cases}$$

But Remark 5.2.3 shows that $q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_2 < q_1$.

Let us first show sufficiency of the given conditions. Part 1 of Theorem 5.4.1 shows that $\iota$ is well-defined and bounded if we have $p_1 \leq p_2$ and

$$K_{(p_1)_d, \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d, q_2 \left(p_1^{\triangle}/q_2\right)'} < \infty,$$

where we recall the interchanged roles of $p_1, p_2$ and of $q_1, q_2$. By virtue of the considerations above, and since Remark 5.2.3 shows that finiteness of $q_2 \cdot (p_1^{\triangle}/q_2)'$ is equivalent to $q_2 < p_1^{\triangle}$, the above constant is finite if and only if we have

$$\begin{cases} \left[\frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d\right] d + \alpha_2 \geq 0, & \text{if } q_2 \geq p_1^{\triangle}, \\ \left[\frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d\right] d + \alpha_2 > 0, & \text{if } q_2 < p_1^{\triangle} \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \alpha_2 - \alpha_1 \leq 0, & \text{in case of } q_2 \geq q_1, \\ d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \alpha_2 - \alpha_1 < 0, & \text{in case of } q_2 < q_1. \end{cases}$$

We finally recall from equation (5.1.1) that we have $(p_1)_d = 0$ for $p_1 \in [1, \infty]$ and $(p_1)_d = 1 - \frac{1}{p_1}$ for $p_1 \in (0, 1]$. Furthermore, for $p_1 \in (0, 1]$, we have $p_1^{\triangle} = p_1' = \infty$.

It remains to translate the embedding for decomposition spaces to an embedding statement for Besov spaces. To this end, define

$$\nu : B^{p_1}_{\alpha_1, q_1} \to \dot{B}^{p_2}_{\alpha_2, q_2}, f \mapsto \left[\varrho \circ \mathcal{F}\right]^{-1} \left(\iota \left[\mathcal{F}_B f\right]\right),$$

where

$$[\varrho \circ \mathcal{F}] : \dot{B}^{p_2}_{\alpha_2,q_2} \to \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_2}, \ell^{q_2}_{u^{(\alpha_2)}}\right), f + \mathcal{P} \mapsto \widehat{f}|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)}$$

and $\mathcal{F}_B : B^{p_1}_{\alpha_1,q_1} \to \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}}\right), f \mapsto \widehat{f}|_{C_c^\infty\left(\mathbb{R}^d\right)}$ are the isomorphisms constructed in Lemma 6.2.2. As seen above, $\iota$ is bounded, so that $\nu$ is also well-defined and bounded.

For $f \in B^{p_1}_{\alpha_1,q_1}$, we see

$$[\varrho \circ \mathcal{F}](\nu f) = \iota(\mathcal{F}_B f) = \widehat{f}|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)} = \varrho \widehat{f} = [\varrho \circ \mathcal{F}](f + \mathcal{P})$$

and thus $\nu f = f + \mathcal{P}$, since $[\varrho \circ \mathcal{F}]$ is injective on $\dot{B}^{p_2}_{\alpha_2,q_2}$ by Lemma 6.2.2. Here, we note (cf. equation (6.2.2))

$$\begin{aligned}\|f + \mathcal{P}\|_{\dot{B}^{p_2}_{\alpha_2,q_2}} &= \|\widehat{f}|_{C_c^\infty(\mathbb{R}^d \setminus \{0\})}\|_{\mathcal{D}(\dot{\mathcal{B}}, L^{p_2}, \ell^{q_2}_{u^{(\alpha_2)}})} = \|\iota \widehat{f}\|_{\mathcal{D}(\dot{\mathcal{B}}, L^{p_2}, \ell^{q_2}_{u^{(\alpha_2)}})} \\ &\lesssim \|\widehat{f}\|_{\mathcal{D}(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}})} = \|\mathcal{F}_B f\|_{\mathcal{D}(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}})} \lesssim \|f\|_{B^{p_1}_{\alpha_1,q_1}} < \infty,\end{aligned}$$

so that $f + \mathcal{P}$ is indeed an element of $\dot{B}^{p_2}_{\alpha_2,q_2}$. Thus, we have shown $\nu f = f + \mathcal{P}$ for all $f \in B^{p_1}_{\alpha_1,q_1}$, so that the map $\nu$ given in the theorem is indeed well-defined and bounded.

Finally, let us consider the necessary conditions. For this, we will show that boundedness of $\nu$ entails boundedness of $\iota$ and finally that boundedness of $\iota$ implies the stated conditions. If $\nu$ is bounded, define

$$\theta := [\varrho \circ \mathcal{F}] \circ \nu \circ \mathcal{F}_B^{-1}.$$

Then $\theta$ is well-defined and bounded. For arbitrary $f \in \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}}\right)$, Lemma 6.2.2 shows that we have $f = \widehat{g}|_{C_c^\infty\left(\mathbb{R}^d\right)} = \mathcal{F}_B g$ for some $g \in B^{p_1}_{\alpha_1,q_1} \subset \mathcal{S}'\left(\mathbb{R}^d\right)$. We conclude

$$\begin{aligned}\theta f &= [\varrho \circ \mathcal{F}]\left(\nu\left(\mathcal{F}_B^{-1} \mathcal{F}_B g\right)\right) = [\varrho \circ \mathcal{F}](\nu g) \\ &= [\varrho \circ \mathcal{F}](g + \mathcal{P}) = \widehat{g}|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)} \\ &= f|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)} = \iota f,\end{aligned}$$

so that $\iota = \theta$ is a bounded linear map. In particular, this implies that

$$\left(C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right), \|\cdot\|_{\mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_v\right)}\right) \to \mathcal{D}\left(\dot{\mathcal{B}}, L^{p_2}, \ell^{q_2}_u\right), f \mapsto f$$

is well-defined and bounded, with $v = \left(2^{\alpha_1 j}\right)_{j \in \mathbb{Z}_0}$ and $u = \left(2^{\alpha_2 k}\right)_{k \in \mathbb{Z}}$ as above.

Hence, part 2 of Theorem 5.4.1 yields $p_1 \leq p_2$ and finiteness of

$$K_{0, \frac{1}{p_1} - \frac{1}{p_2}, q_2 \cdot (p_1/q_2)'} \quad ,$$

where we recall the interchanged roles of $p_1, p_2$ and of $q_1, q_2$. With the considerations from the beginning of the proof, this yields

$$\begin{cases} \left(\frac{1}{p_1} - \frac{1}{p_2}\right) d + \alpha_2 \geq 0, & \text{if } q_2 \geq p_1, \\ \left(\frac{1}{p_1} - \frac{1}{p_2}\right) d + \alpha_2 > 0, & \text{if } q_2 < p_1 \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \alpha_2 - \alpha_1 \leq 0, & \text{in case of } q_2 \geq q_1, \\ d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \alpha_2 - \alpha_1 < 0, & \text{in case of } q_2 < q_1. \end{cases}$$

Here, we also used that $q_2 \cdot (p_1/q_2)'$ is finite if and only if $q_2 < p_1$ holds, see Remark 5.2.3. We have thus shown that equations (6.2.17) and (6.2.19) are satisfied if $\iota$ or $\nu$ are bounded.

For the proof of equation (6.2.20), we apply Lemma 5.3.11 with $\mathcal{Q} = \dot{\mathcal{B}}$, $\mathcal{P} = \mathcal{B}$ and $I_0 := \mathbb{Z}$, with $u = \left(2^{\alpha_2 i}\right)_{i \in \mathbb{Z}}$ and $v = \left(2^{\alpha_1 j}\right)_{j \in \mathbb{N}_0}$ and with interchanged roles of $p_1, p_2$ and of $q_1, q_2$. In the notation of Lemma 5.3.11, this implies $J_0 = J = \mathbb{N}_0$, since each $j \in \mathbb{N}_0$ satisfies $B_j \cap \left(\mathbb{R}^d \setminus \{0\}\right) \neq \varnothing$ and hence $B_j \cap \dot{B}_i \neq \varnothing$ for some $i \in \mathbb{Z} = I_0$. This implies (again in the notation of Lemma 5.3.11)

$$\mathbb{R}^d \supset K = \overline{\bigcup_{j \in J_0} P_j^{(2k+3)*}} \supset \bigcup_{j \in \mathbb{N}_0} B_j = \mathbb{R}^d \text{ with } k = k(\mathcal{Q}, \mathcal{P})$$

and hence $K = \mathbb{R}^d$.

We note that Lemma 6.2.2 shows that $\mathcal{Q}, \mathcal{P}$ are indeed tight semi-structured $L^p$-decomposition coverings for every $p \in (0, \infty]$ and that $u, v$ are moderate with respect to $\mathcal{Q}, \mathcal{P}$, respectively. Finally, as discussed above, our assumptions (boundedness of $\nu$ or $\iota$) imply that the map $\iota$ as defined in equation (6.2.18) is bounded. But for any $f \in \mathcal{D}_K = C_c^\infty\left(\mathbb{R}^d\right)$ and $\varphi \in C_c^\infty\left(\mathcal{O} \cap \mathcal{O}'\right) = C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$, this implies

$$\langle \iota f, \varphi \rangle = \left\langle f|_{C_c^\infty\left(\mathbb{R}^d \setminus \{0\}\right)}, \varphi \right\rangle = \langle f, \varphi \rangle .$$

Hence, $\iota|_{\mathcal{D}_K}$ yields a bounded linear map with the properties required in Lemma 5.3.11. By virtue of this lemma (and recalling the interchanged roles of $p_1, p_2$ and of $q_1, q_2$), we conclude

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\frac{1}{p_1} - 1}}{v_j} \left\| \left( |\det(T_i)|^{1 - \frac{1}{p_2}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_2}} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \lesssim \left\| \left| \iota|_{\mathcal{D}_K} \right| \right\| .$$

In particular, this implies

$$\left\| \left( |\det(T_i)|^{1 - \frac{1}{p_2}} u_i \right)_{i \in I_j} \right\|_{\ell^{q_2}} = \left\| \left( |\det(T_i)|^{1 - \frac{1}{p_2}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_2}} < \infty$$

for all $j \in J_0 = \mathbb{N}_0$. In the following, we will take $j = 0$. Above, we saw $\mathbb{Z}_{\leq 0} \subset I_j \subset \mathbb{Z}_{\leq 4}$ for $j = 0$. This implies

$$\left\| \left( 2^{i\left(\alpha_2 + d\left(1-\frac{1}{p_2}\right)\right)} \right)_{i \in \mathbb{Z}_{\leq 0}} \right\|_{\ell^{q_2}} = \left\| \left( |\det(T_i)|^{1-\frac{1}{p_2}} u_i \right)_{i \in \mathbb{Z}_{\leq 0}} \right\|_{\ell^{q_2}} < \infty,$$

because of $T_i = 2^i \cdot \mathrm{id}$. We conclude

$$\begin{cases} \alpha_2 + d\left(1 - \frac{1}{p_2}\right) > 0, & \text{in case of } q_2 < \infty, \\ \alpha_2 + d\left(1 - \frac{1}{p_2}\right) \geq 0, & \text{in case of } q_2 = \infty, \end{cases}$$

due to the exponential nature of $\left( 2^{i\left(\alpha_2 - d\left(1-\frac{1}{p_2}\right)\right)} \right)_{i \in \mathbb{Z}_{\leq 0}}$. This proves validity of equation (6.2.20). □

As our final topic in this section, we will establish embeddings between inhomogeneous Besov spaces and $\alpha$-modulation spaces. As a preparation for the application of our results from Chapter 5, we first show that the coverings $\mathcal{O}_r^{(\alpha)}$ are almost subordinate to and relatively moderate with respect to the Besov covering $\mathcal{B}$. This is the content of the next lemma, which should be compared to Theorem 6.1.6, since $\mathcal{B}$ can be considered as the "limit case" of the coverings $\mathcal{O}^{(\beta)}$ as $\beta \uparrow 1$.

**Lemma 6.2.7.** *Let $d \in \mathbb{N}$ and $\alpha \in [0,1)$. Choose $r > r_1(d,\alpha)$ with $r_1(d,\alpha)$ as in Theorem 6.1.1.*

*The covering $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to and relatively moderate with respect to $\mathcal{B}$. More precisely, we have*

$$|k| \asymp \langle k \rangle \asymp 2^{n(1-\alpha)} \quad \text{and} \quad |\det(T_k^{(\alpha)})| \asymp 2^{n\alpha d} \tag{6.2.23}$$

*for all $k \in \mathbb{Z}^d \setminus \{0\}$ and $n \in \mathbb{N}_0$ with $O_{r,k}^{(\alpha)} \cap B_n \neq \varnothing$, where all implied constants only depend on $r, \alpha$.*

*Proof.* Let us first show that $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{B}$. Observe that each of the sets $O_{r,k}^{(\alpha)}$ is convex and hence path-connected. Moreover, each of the sets $B_n$ is open, so that Corollary 3.3.4 implies that it suffices to show that the cardinality of the sets

$$J_k := \left\{ n \in \mathbb{N}_0 \,\middle|\, B_n \cap O_{r,k}^{(\alpha)} \neq \varnothing \right\}$$

is uniformly bounded with respect to $k \in \mathbb{Z}^d \setminus \{0\}$.

To see this, let $k \in \mathbb{Z}^d \setminus \{0\}$ be arbitrary and choose $n \in J_k$. Hence, there is some $x \in B_n \cap O_{r,k}^{(\alpha)}$. Lemma 6.1.2 implies

$$\langle x \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}}$$

where the implied constants only depend on $r, \alpha$.

Let us assume $n \geq 2$ for the moment. This yields $|x| \geq 2^{n-2} \geq 1$ because of $x \in B_n$ and thus $|x| \leq \langle x \rangle \leq 1 + |x| \leq 2|x|$. Together with $2^{n-2} < |x| < 2^{n+2}$, i.e. $\langle x \rangle \asymp |x| \asymp 2^n$, we arrive at

$$2^n \asymp \langle x \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}} \qquad \text{for } k \in \mathbb{Z}^d \setminus \{0\} \text{ and } n \in J_k \cap \mathbb{N}_{\geq 2}. \tag{6.2.24}$$

This yields

$$2^{\frac{\log_2 \langle k \rangle}{1-\alpha} + \log_2 C_0} = C_0 \cdot \langle k \rangle^{\frac{1}{1-\alpha}} \leq 2^n \leq C_1 \cdot \langle k \rangle^{\frac{1}{1-\alpha}} = 2^{\frac{\log_2 \langle k \rangle}{1-\alpha} + \log_2 C_1}$$

for suitable constants $C_0, C_1 > 0$ which only depend on $r, \alpha$. We conclude

$$\log_2 C_0 \leq n - \frac{\log_2 \langle k \rangle}{1-\alpha} \leq \log_2 C_1$$

and hence $n \in \mathbb{N}_0 \cap \overline{B_R}\left(\frac{\log_2 \langle k \rangle}{1-\alpha}\right)$ for $R := \max\{-\log_2 C_0, \log_2 C_1\}$.

By dropping the assumption $n \geq 2$, we finally arrive at

$$J_k \subset \{0,1\} \cup \left[\mathbb{N}_0 \cap \overline{B_R}\left(\frac{\log_2 \langle k \rangle}{1-\alpha}\right)\right],$$

where the latter set has cardinality at most

$$2 + 2\lceil R \rceil + 1 = 2\lceil R \rceil + 3 =: N.$$

As observed above, this implies that $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{B}$. More precisely, Lemma 3.3.3 shows that $O_{r,k}^{(\alpha)} \subset B_n^{N*}$ holds for all $n \in J_k$ and all $k \in \mathbb{Z}^d \setminus \{0\}$.

Now, we prove the first part of equation (6.2.23). For $n \geq 2$, and $k \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap B_n \neq \varnothing$, we get $|k| \asymp \langle k \rangle \asymp 2^{n(1-\alpha)}$ from equation (6.2.24). This establishes the first half of equation (6.2.23) for $n \geq 2$. In case of $n \leq 1$, we note that Lemma 6.1.2 yields, for arbitrary $x \in O_{r,k}^{(\alpha)} \cap B_n$, the estimate

$$1 \leq |k| \leq \langle k \rangle \asymp \langle x \rangle^{1-\alpha} \leq (1 + |x|)^{1-\alpha} < \left(1 + 2^{n+2}\right)^{1-\alpha} \leq 9^{1-\alpha} \lesssim 1$$

as well as $1 \leq 2^{n(1-\alpha)} \leq 2^{1-\alpha} \lesssim 1$, where all implied constants only depend on $r$ and $\alpha \in [0,1)$. Thus, the first part of equation (6.2.23) holds in all cases.

Finally, the first half of equation (6.2.23) also implies the second half and that $\mathcal{O}_r^{(\alpha)}$ is moderate with respect to $\mathcal{B}$, because of

$$|\det(T_k^{(\alpha)})| = \left|\det\left(|k|^{\alpha_0} \operatorname{id}\right)\right| = |k|^{\alpha_0 d} \asymp 2^{n(1-\alpha)\cdot\alpha_0 d} = 2^{n\alpha d}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$. Here, $\alpha_0 = \frac{\alpha}{1-\alpha}$ as usual. □

With these preparations, we can characterize the embeddings between inhomogeneous Besov spaces and $\alpha$-modulation spaces.

**Theorem 6.2.8.** *Let $d \in \mathbb{N}$, $\alpha \in [0,1)$ and $p_1, p_2, q_1, q_2 \in (0,\infty]$ as well as $\gamma_1, \gamma_2 \in \mathbb{R}$.*

*The map*

$$\iota : \mathcal{F}M^{\gamma_1,\alpha}_{p_1,q_1} \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{\left(2^{\gamma_2 i}\right)_i}\right), f \mapsto f$$

*is well-defined and bounded if and only if we have $p_1 \leq p_2$ as well as*

$$\begin{cases} \gamma_2 \leq \gamma_1 + \alpha d\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + d\left(\alpha - 1\right)\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+, & \text{if } q_1 \leq q_2, \\ \gamma_2 < \gamma_1 + \alpha d\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + d\left(\alpha - 1\right)\left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+, & \text{if } q_1 > q_2. \end{cases}$$

*Conversely, the map*

$$\theta : \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{\left(2^{\gamma_1 i}\right)_i}\right) \to \mathcal{F}M^{\gamma_2,\alpha}_{p_2,q_2}, f \mapsto f$$

*is well-defined and bounded if and only if we have $p_1 \leq p_2$, as well as*

$$\begin{cases} \gamma_2 < \gamma_1 + \alpha d\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + d\left(\alpha - 1\right)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+, & \text{if } q_1 > q_2, \\ \gamma_2 \leq \gamma_1 + \alpha d\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + d\left(\alpha - 1\right)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+, & \text{if } q_1 \leq q_2. \end{cases}$$

*For convenience of the reader, we recall the notation $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$ from Remark 5.2.10.*

*Remark.* In view of the isomorphism of the "usual" $\alpha$-modulation spaces $M^{\gamma,\alpha}_{p,q}$ with the "Fourier side" $\alpha$-modulation spaces $\mathcal{F}M^{\gamma,\alpha}_{p,q}$ (cf. Lemma 6.1.9) and of the Besov spaces $B^p_{\gamma,q}$ with $\mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{\left(2^{\gamma i}\right)_i}\right)$ (cf. Lemma 6.2.2), it is easy to see that the characterizations above are also valid for these spaces, i.e. on the "space side".

It is worth noting that the conditions given in Theorem 6.1.7 exactly yield the present conditions if $\beta = 1$ is plugged in.

We finally observe that the first condition reduces to

$$\gamma_2 + d\left(1 - \alpha\right)\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+ \leq \gamma_1$$

in case of $p_1 = p_2 = p$ and $q_1 = q_2 = q$. This inequality is identical to the respective condition stated in [49, Theorem 4.2].

Similarly, the second condition reduces to

$$\gamma_2 + d\left(1 - \alpha\right)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+ \leq \gamma_1$$

which is also identical to the respective condition given in [49, Theorem 4.2].

*Proof.* Choose $r_1 = r_1\left(d, \alpha\right)$ as in Theorem 6.1.1 and let $r > r_1$ be arbitrary. For $\gamma \in \mathbb{R}$, let $w^{(\gamma)}$ be defined as in Lemma 6.1.2.

For the proof of the first part of the theorem, we invoke Theorem 5.4.4 with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \mathcal{B}$, $u = w^{(\gamma_1/(1-\alpha))}$ and $v = \left(2^{i\gamma_2}\right)_{i \in \mathbb{N}_0}$. We observe that Lemma 6.1.2, Theorem 6.1.3 and Lemma 6.2.2 indeed

imply that $\mathcal{Q}, \mathcal{P}$ are tight, semi-structured $L^p$-decomposition coverings for all $p \in (0, \infty]$ and that $u$ is $\mathcal{Q}$-moderate and $v$ is $\mathcal{P}$-moderate. Finally, Lemma 6.2.7 shows that $\mathcal{Q}$ is almost subordinate to and relatively moderate with respect to $\mathcal{P}$ and that $u$ is relatively $\mathcal{P}$-moderate.

All in all, we have verified all prerequisites for part 3 of Theorem 5.4.4. We note that the properties of the map $\iota$ in Theorem 5.4.4 imply that $\iota f$ is an extension of $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$ to $C^\infty_c(\mathcal{O}')$. But using $\mathcal{O} = \mathbb{R}^d = \mathcal{O}'$, this implies $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right)$. Conversely, if the (identity) map $\iota$ as given in the present theorem is bounded, this easily implies boundedness of $\iota'$ as in Theorem 5.4.4. In summary, we conclude that the map $\iota$ from above is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+}}{u_{i_j}} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}_v}$$

is finite, where for each $j \in J_{\mathcal{O}} = J = \mathbb{N}_0$, an arbitrary $i_j \in \mathbb{Z}^d \setminus \{0\}$ with $B_j \cap O^{(\alpha)}_{r, i_j} \neq \varnothing$ is chosen and where we have $S_j = 2^j \mathrm{id}$ and $T_i = T^{(\alpha)}_i = |i|^{\alpha_0} \mathrm{id}$ for all $j \in J = \mathbb{N}_0$ and $i \in I = \mathbb{Z}^d \setminus \{0\}$, with $\alpha_0 = \frac{\alpha}{1-\alpha}$.

But equation (6.2.23) from Lemma 6.2.7 yields

$$\left|\det\left(T_{i_j}\right)\right| \asymp 2^{j\alpha d}$$

and

$$u_{i_j} = w^{(\gamma_1/(1-\alpha))}_{i_j} = \left\langle i_j \right\rangle^{\frac{\gamma_1}{1-\alpha}} \asymp \left[2^{j(1-\alpha)}\right]^{\frac{\gamma_1}{1-\alpha}} = 2^{j\gamma_1},$$

where the implied constants only depend on $r, \alpha, \gamma_1$. Thus, $\iota$ is well-defined and bounded if and only we have $p_1 \leq p_2$ and if the quasi-norm

$$\begin{aligned}
&\left\| \left( 2^{jd\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+} \cdot 2^{-j\gamma_1} \cdot 2^{j\alpha d\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+\right]} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}_v} \\
&= \left\| \left( 2^{jd\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+} \cdot 2^{j(\gamma_2 - \gamma_1)} \cdot 2^{j\alpha d\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+\right]} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}} \\
&= \left\| \left( 2^{j\left[(\gamma_2 - \gamma_1) + d\left[\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + (1-\alpha)\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+\right]\right]} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}}
\end{aligned}$$

is finite.

By Remark 5.2.3, we know that $q_2 \cdot (q_1/q_2)' < \infty$ is equivalent to $q_2 < q_1$. Thus finiteness of the above quasi-norm is equivalent to

$$\begin{cases} (\gamma_2 - \gamma_1) + d\left[\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + (1-\alpha)\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+\right] < 0, & \text{if } q_2 < q_1, \\ (\gamma_2 - \gamma_1) + d\left[\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + (1-\alpha)\left(\frac{1}{p_2^\vee} - \frac{1}{q_1}\right)_+\right] \leq 0, & \text{if } q_2 \geq q_1. \end{cases}$$

This completes the proof of the first part of the theorem.

For the proof of the first part, we use Theorem 5.4.1 with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \mathcal{B}$, $u = w^{(\gamma_2/(1-\alpha))}$, $v = \left(2^{\gamma_1 j}\right)_{j\in\mathbb{N}_0}$ and with interchanged roles of $p_1, p_2$ and of $q_1, q_2$. With the same arguments as above, we see that all assumptions of part 3 of that theorem are satisfied. This shows that $\theta$ is well-defined and bounded if and only we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2}(q_1/q_2)'}$$

is finite. For the precise form of this quasi-norm, it is important to recall the interchanged roles of $p_1, p_2$ and of $q_1, q_2$.

Using the same arguments as above, we see that finiteness of this quasi-norm is equivalent to finiteness of

$$\left\| \left( 2^{j(\gamma_2-\gamma_1)} 2^{jd(1-\alpha)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} 2^{j\alpha d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q_2}\cdot(q_1/q_2)'},$$

which in turn is equivalent to

$$\begin{cases} \gamma_2 - \gamma_1 + d\,(1-\alpha)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + \alpha d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) < 0, & \text{if } q_2 < q_1, \\ \gamma_2 - \gamma_1 + d\,(1-\alpha)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + \alpha d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \leq 0, & \text{if } q_2 \geq q_1. \end{cases}$$

This completes the proof. □

## 6.3 EMBEDDINGS FOR SHEARLET TYPE COORBIT SPACES

In this section, we will combine our embedding results with the theory developed in Chapter 4, where we saw that certain wavelet coorbit spaces are canonically isomorphic to (Fourier-side) decomposition spaces. This will yield new embeddings between wavelet coorbit spaces. Specifically, we consider the family of two-dimensional SHEARLET-TYPE GROUPS.

The structure of this section is as follows: In the first subsection, we define the shearlet-type groups $H_c$ and compute the covering $\mathcal{S}^{(c)}$ of the dual orbit $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$ induced by $H_c$. Furthermore, we apply the results of Chapter 4 to conclude that the Fourier transform yields an isomorphism of certain coorbit spaces of the group $\mathbb{R}^2 \rtimes H_c$ with suitable decomposition spaces which are defined using the covering $\mathcal{S}^{(c)}$.

This isomorphism is applied in the second subsection to establish embeddings for coorbit spaces "living" on $\mathbb{R}^2 \rtimes H_c$ for *distinct* values of $c$. This uses our embedding results for decomposition spaces.

Finally, embeddings between shearlet-type coorbit spaces and (inhomogeneous) Besov spaces are studied in the last subsection.

### 6.3.1 *Shearlet type coorbit spaces as decomposition spaces*

We begin by defining the class of shearlet type groups which are the main object of interest in this section.

**Definition 6.3.1.** The TWO-DIMENSIONAL SHEARLET-TYPE GROUP WITH PARAMETER $c$, for arbitary $c \in \mathbb{R}$, is defined as

$$H_c := \left\{ \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \,\middle|\, a \in (0,\infty)\,,\ b \in \mathbb{R},\ \varepsilon \in \{\pm 1\} \right\}.$$

In order to apply the theory developed in Chapter 4, we observe the following:

1. $H_c \leq \mathrm{GL}\left(\mathbb{R}^2\right)$ is a closed subgroup.
2. The dual orbit

   $$\mathcal{O} = H_c^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ \varepsilon \begin{pmatrix} a \\ b \end{pmatrix} \,\middle|\, a \in (0,\infty)\,,\ b \in \mathbb{R},\ \varepsilon \in \{\pm 1\} \right\} = \mathbb{R}^* \times \mathbb{R}$$

   is the single open orbit of the dual action of $H_c$ on $\mathbb{R}^2$ and is of full measure.
3. The isotropy group

   $$(H_c)_{(1,0)^T} = \left\{ \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \in H_c \,\middle|\, \varepsilon \begin{pmatrix} a \\ b \end{pmatrix} = \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

   is a compact subgroup of $H_c$. In fact, $(H_c)_{(1,0)^T} = \{\mathrm{id}\}$.

These two conditions guarantee that the results from Chapter 4 are indeed applicable. The next step is to find a "nice" well-spread family $(h_i)_{i\in I}$ in $H_c$ and to compute an induced covering of the dual orbit $\mathcal{O}$ of the form $\mathcal{Q} = (Q_i)_{i\in I} = \left(h_i^{-T} Q\right)_{i\in I}$.

The family $(h_i)_{i\in I}$ is constructed in the following lemma. Here, we use the matrix norm

$$\|A\|_\infty = \max_{\substack{x\in\mathbb{R}^2\\ \|x\|_\infty=1}} \|Ax\|_\infty = \max_{i=1,2} \sum_{j=1}^{2} |A_{i,j}|$$

*Note that $\|\cdot\|_\infty$ is submultiplicative, since it is a matrix norm induced by the $\|\cdot\|_\infty$-norm on $\mathbb{R}^2$.*

for $A \in \mathbb{R}^{2\times 2}$.

**Lemma 6.3.2.** *Let $c \in \mathbb{R}$. For $m, n \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$, define*

$$B^{(c)}_{n,m,\varepsilon} := \varepsilon \cdot \begin{pmatrix} 2^n & 0 \\ 0 & 2^{nc} \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} 2^n & 2^n m \\ 0 & 2^{nc} \end{pmatrix} \in H_c,$$

$$A^{(c)}_{n,m,\varepsilon} := \left(B^{(c)}_{-n,-m,\varepsilon}\right)^{-T} = \varepsilon \cdot \begin{pmatrix} 2^n & 0 \\ 2^{nc} m & 2^{nc} \end{pmatrix} = \varepsilon \begin{pmatrix} 2^n & 0 \\ 0 & 2^{nc} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}.$$

*With these definitions, the family*

$$\Gamma := \left(B^{(c)}_{n,m,\varepsilon}\right)_{n,m\in\mathbb{Z},\varepsilon\in\{\pm 1\}}$$

*is $U$-dense and $V$-separated for the compact unit-neighborhoods*

$$U := \left\{ A \in H_c \,\middle|\, \|A\|_\infty \le 2 + 2^{|c|} \text{ and } \|A^{-1}\|_\infty \le 2 + 2^{|c|} \right\},$$
$$V := \left\{ A \in H_c \,\middle|\, \|A - \mathrm{id}\|_\infty \le \frac{1}{4} \right\}.$$

*In particular, $\Gamma$ is well-spread in $H_c$.*

*Proof.* The set

$$K := \left\{ A \in \mathrm{GL}\left(\mathbb{R}^2\right) \,\middle|\, \|A\|_\infty \le 2 + 2^{|c|} \text{ and } \|A^{-1}\|_\infty \le 2 + 2^{|c|} \right\}$$

is a compact subset of $\mathrm{GL}\left(\mathbb{R}^2\right)$, because every sequence $(A_n)_{n\in\mathbb{N}}$ in $K$ admits a subsequence $(A_{n_k})_k$, for which $(A_{n_k})_k$ and $\left(A^{-1}_{n_k}\right)_k$ converge to $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$. It is easy to see $\|A\|_\infty, \|B\|_\infty \le 2 + 2^{|c|}$ and $AB = BA = \mathrm{id}$. Hence, $A \in \mathrm{GL}\left(\mathbb{R}^2\right)$ with $B = A^{-1}$, which entails $A \in K$. Thus, $U = K \cap H_c$ is a compact subset of $H_c$.

Further, $U$ is a unit neighbourhood, because of $\|\mathrm{id}\|_\infty = 1 < 2 + 2^{|c|}$ and because $U$ contains the open set

$$\left\{ A \in H_c \,\middle|\, \|A\|_\infty < 2 + 2^{|c|} \text{ and } \|A^{-1}\|_\infty < 2 + 2^{|c|} \right\}.$$

It is clear that $K' := \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, \|A - \mathrm{id}\|_\infty \le \frac{1}{4} \right\}$ is compact. By a standard argument using the Neumann-series, we see $K' \subset \mathrm{GL}\left(\mathbb{R}^2\right)$. Since $H_c \subset \mathrm{GL}\left(\mathbb{R}^2\right)$ is closed, this shows that $V = K' \cap H_c$ is a compact subset of $H_c$. Finally, $V$ is a unit neighborhood, because it contains the open set $\left\{ A \in H_c \,\middle|\, \|A - \mathrm{id}\|_\infty < \frac{1}{4} \right\}$, which also contains id.

It remains to prove that $\Gamma$ is $U$-dense and $V$-separated. To this end, let $\varepsilon \in \{\pm 1\}$, $a \in (0,\infty)$ and $b \in \mathbb{R}$ be arbitrary and set $A := \varepsilon \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)$. Let $n \in \mathbb{Z}$ be maximal with the property $2^n \leq a$. Then $2^n \leq a < 2^{n+1}$, which entails $1 \leq 2^{-n}a < 2$. Let $\widetilde{b} := b/\left(2^{-n}a\right)^c \in \mathbb{R}$ and define $m := \lfloor 2^{-n}\widetilde{b} \rfloor \in \mathbb{Z}$. We then have

$$\begin{aligned}\left(B^{(c)}_{n,m,\varepsilon}\right)^{-1} \cdot A &= \begin{pmatrix} 2^{-n} & -2^{-nc}m \\ 0 & 2^{-nc} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \\ &= \begin{pmatrix} 2^{-n}a & 2^{-n}b - 2^{-nc}a^c m \\ 0 & \left(2^{-n}a\right)^c \end{pmatrix}.\end{aligned}$$

As seen above, $|2^{-n}a| = 2^{-n}a < 2$. This inequality, together with $2^{-n}a \geq 1$ yields $0 < \left(2^{-n}a\right)^c \leq \left(2^{-n}a\right)^{|c|} \leq 2^{|c|}$. Furthermore,

$$\begin{aligned}\left|2^{-n}b - a^c 2^{-nc} m\right| &= \left(2^{-n}a\right)^c \cdot \left|2^{-n}\widetilde{b} - m\right| \\ &= \left(2^{-n}a\right)^c \cdot \left|2^{-n}\widetilde{b} - \left\lfloor 2^{-n}\widetilde{b} \right\rfloor\right| \leq 2^{|c|}.\end{aligned}$$

Altogether, we conclude

$$\begin{aligned}\left\| \left(B^{(c)}_{n,m,\varepsilon}\right)^{-1} A \right\|_\infty &= \max\left\{ \left|2^{-n}a\right| + \left|2^{-n}b - 2^{-nc}a^c m\right|, \left|\left(2^{-n}a\right)^c\right| \right\} \\ &\leq \max\left\{2 + 2^{|c|}, 2^{|c|}\right\} = 2 + 2^{|c|}.\end{aligned}$$

For the opposite ordering of factors, we calculate

$$\begin{aligned}A^{-1} \cdot B^{(c)}_{n,m,\varepsilon} &= \begin{pmatrix} a^{-1} & -a^{-(c+1)}b \\ 0 & a^{-c} \end{pmatrix} \begin{pmatrix} 2^n & 2^n m \\ 0 & 2^{nc} \end{pmatrix} \\ &= \begin{pmatrix} 2^n a^{-1} & 2^n a^{-1} m - 2^{nc} a^{-(c+1)} b \\ 0 & \left(2^n a^{-1}\right)^c \end{pmatrix}.\end{aligned}$$

We know $\left|2^n a^{-1}\right| = \left(2^{-n}a\right)^{-1} \leq 1$ because of $2^{-n}a \geq 1$. The inequality $2^{-n}a \geq 1$ also implies $\left|\left(2^n a^{-1}\right)^c\right| = \left(2^{-n}a\right)^{-c} \leq \left(2^{-n}a\right)^{|-c|} \leq 2^{|c|}$ because of $2^{-n}a \leq 2$. Finally, we have

$$\begin{aligned}\left|2^n a^{-1} m - 2^{nc} a^{-(c+1)} b\right| &= \left|2^n a^{-1}\right| \cdot \left|m - 2^{-n}\widetilde{b}\right| \\ &= \left|2^n a^{-1}\right| \cdot \left|\lfloor 2^{-n}\widetilde{b} \rfloor - 2^{-n}\widetilde{b}\right| \leq \left|2^n a^{-1}\right| \leq 1.\end{aligned}$$

Similar to the estimate above, we arrive at

$$\left\| \left[ \left(B^{(c)}_{n,m,\varepsilon}\right)^{-1} \cdot A \right]^{-1} \right\|_\infty = \left\| A^{-1} \cdot B^{(c)}_{n,m,\varepsilon} \right\|_\infty \leq 2 + 2^{|c|}.$$

All in all, we have shown $\left(B^{(c)}_{n,m,\varepsilon}\right)^{-1} \cdot A \in U$, i.e. $A \in B^{(c)}_{n,m,\varepsilon} \cdot U$. Hence, $\Gamma$ is $U$-dense in $H_c$.

To prove that $\Gamma$ is $V$-separated, we first note that $A \in V$ implies

$$\begin{aligned}\left\|A^{-1} - \mathrm{id}\right\|_\infty &= \left\|(\mathrm{id} - (\mathrm{id} - A))^{-1} - \mathrm{id}\right\|_\infty \\ &= \left\|\sum_{n=0}^{\infty} (\mathrm{id} - A)^n - \mathrm{id}\right\|_\infty \leq \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3}\end{aligned}$$

and thus also $\left\|A^{-1}\right\|_\infty \leq \left\|A^{-1} - \mathrm{id}\right\|_\infty + \|\mathrm{id}\|_\infty \leq \frac{4}{3}$. For $A, B \in H_c$ with $AV \cap BV \neq \varnothing$, we have $Au = Bv$ for suitable $u, v \in V$, so that the above estimates imply

$$\begin{aligned}\left\|B^{-1}A - \mathrm{id}\right\|_\infty &= \left\|vu^{-1} - \mathrm{id}\right\|_\infty \\ &= \left\|(v - \mathrm{id})\, u^{-1} + u^{-1} - \mathrm{id}\right\|_\infty \\ &\leq \|v - \mathrm{id}\|_\infty \left\|u^{-1}\right\|_\infty + \left\|u^{-1} - \mathrm{id}\right\|_\infty \\ &\leq \frac{1}{4} \cdot \frac{4}{3} + \frac{1}{3} = \frac{2}{3} < 1.\end{aligned}$$

Since the condition $AV \cap BV \neq \varnothing$ is symmetric in $A, B$, we also get $\left\|A^{-1}B - \mathrm{id}\right\|_\infty < 1$.

To prove that $\Gamma$ is $V$-separated, it is thus sufficient to show

$$\left\|\left(B^{(c)}_{n_1,m_1,\varepsilon_1}\right)^{-1} B^{(c)}_{n_2,m_2,\varepsilon_2} - \mathrm{id}\right\|_\infty \geq 1 \text{ or } \left\|\left(B^{(c)}_{n_2,m_2,\varepsilon_2}\right)^{-1} B^{(c)}_{n_1,m_1,\varepsilon_1} - \mathrm{id}\right\|_\infty \geq 1$$

for any two distinct indices $(n_1, m_1, \varepsilon_1), (n_2, m_2, \varepsilon_2) \in \mathbb{Z}^2 \times \{\pm 1\}$. By symmetry, we can assume $n_1 \geq n_2$ without loss of generality. We distinguish three cases:

CASE 1: We have $\varepsilon_1 \neq \varepsilon_2$ and thus $\varepsilon_1 \varepsilon_2^{-1} = -1$. Using $n_1 \geq n_2$, we see $2^{n_1 - n_2} \geq 1$. This implies (for suitable $*_1, *_2 \in \mathbb{R}$):

$$\begin{aligned}\left\|\left(B^{(c)}_{n_2,m_2,\varepsilon_2}\right)^{-1} B^{(c)}_{n_1,m_1,\varepsilon_1} - \mathrm{id}\right\|_\infty &= \left\|\varepsilon_2^{-1}\varepsilon_1 \cdot \begin{pmatrix} 2^{n_1-n_2} & *_1 \\ 0 & *_2 \end{pmatrix} - \mathrm{id}\right\|_\infty \\ &= \left\|\begin{pmatrix} -2^{n_1-n_2} - 1 & -*_1 \\ 0 & -*_2 - 1 \end{pmatrix}\right\|_\infty \\ &\geq \left|-2^{n_1-n_2} - 1\right| = 1 + 2^{n_1-n_2} \geq 2 \geq 1.\end{aligned}$$

CASE 2: We have $\varepsilon_1 = \varepsilon_2$, but $n_1 \neq n_2$ and thus $n_1 \geq n_2 + 1$, which entails $2^{n_1 - n_2} \geq 2$. This yields (again for suitable $*_1, *_2 \in \mathbb{R}$):

$$\begin{aligned}\left\|\left(B^{(c)}_{n_2,m_2,\varepsilon_2}\right)^{-1} B^{(c)}_{n_1,m_1,\varepsilon_1} - \mathrm{id}\right\|_\infty &= \left\|\begin{pmatrix} 2^{n_1-n_2} - 1 & *_1 \\ 0 & *_2 \end{pmatrix}\right\|_\infty \\ &\geq \left|2^{n_1-n_2} - 1\right| = 2^{n_1-n_2} - 1 \geq 1.\end{aligned}$$

CASE 3: We have $\varepsilon_1 = \varepsilon_2$ and $n_1 = n_2$. This implies $m_1 \neq m_2$ and

$$\left\| \left( B^{(c)}_{n_2,m_2,\varepsilon_2} \right)^{-1} B^{(c)}_{n_1,m_1,\varepsilon_1} - \mathrm{id} \right\|_\infty = \left\| \begin{pmatrix} 1 & m_1 - m_2 \\ 0 & 1 \end{pmatrix} - \mathrm{id} \right\|_\infty \geq |m_1 - m_2| \geq 1.$$

Together with the considerations above, this completes the proof. □

To study the covering $\mathcal{Q}$ of $\mathcal{O}$ induced by $H_c$, it turns out to be easier to consider the image $\Phi\mathcal{Q}$ of $\mathcal{Q}$ under the map $\Phi$ defined below. Furthermore, we will see that the covering $\mathcal{Q}$ can be conveniently described in terms of the sets $U^{(\gamma,\delta)}_{(a,b)}$ that we define now.

**Definition 6.3.3.** Let

$$\Phi : \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}^* \times \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y/x \end{pmatrix}.$$

For $a, b \in \mathbb{R}$ with $a < b$ and $\gamma, \delta \in (0, \infty)$ with $\gamma < \delta$, define

$$U^{(\gamma,\delta)}_{(a,b)} := \Phi^{-1}\left((\gamma,\delta) \times (a,b)\right) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\gamma,\delta) \times \mathbb{R} \;\middle|\; \frac{y}{x} \in (a,b) \right\}.$$

The following lemma indicates why the sets $U^{(\gamma,\delta)}_{(a,b)}$ are well-suited to build an induced covering of $H_c$: The family of all sets $U^{(\gamma,\delta)}_{(a,b)}$ is closed under application of arbitrary elements of $H_c$ and the transformation law is easily understood.

**Lemma 6.3.4.** *Let $a, b \in \mathbb{R}$ with $a < b$ and $\gamma, \delta \in (0, \infty)$ with $\gamma < \delta$.*

*Then $U^{(\gamma,\delta)}_{(a,b)}$ is an open, convex subset of $(0, \infty) \times \mathbb{R}$. Furthermore,*

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} U^{(\gamma,\delta)}_{(a,b)} &= U^{(\gamma,\delta)}_{(m+a,m+b)}, \\ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U^{(\gamma,\delta)}_{(a,b)} &= U^{(\alpha\gamma,\alpha\delta)}_{\left(\frac{\beta}{\alpha}a, \frac{\beta}{\alpha}b\right)} \end{aligned} \tag{6.3.1}$$

*holds for all $m \in \mathbb{R}$ and $\alpha, \beta > 0$. Thus,*

$$A^{(c)}_{n,m,\varepsilon} U^{(\gamma,\delta)}_{(a,b)} = \varepsilon \cdot U^{(2^n\gamma, 2^n\delta)}_{(\varrho_c^n(m+a), \varrho_c^n(m+b))} \tag{6.3.2}$$

*for $\varrho_c := 2^{c-1}$ and $A^{(c)}_{n,m,\varepsilon}$ as in Lemma 6.3.2.*

*In case of $b - a > 1$, we have*

$$\bigcup_{m \in \mathbb{Z}} A^{(c)}_{n,m,\varepsilon} U^{(\gamma,\delta)}_{(a,b)} = (\varepsilon \cdot (2^n\gamma, 2^n\delta)) \times \mathbb{R} \tag{6.3.3}$$

*for all $n \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$.*

*If in addition $\delta/\gamma > 2$ holds, then $\left( A^{(c)}_{n,m,\varepsilon} U^{(\gamma,\delta)}_{(a,b)} \right)_{(n,m,\varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}}$ is a covering of $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$.*

*Proof.* With the diffeomorphism $\Phi$ introduced in Definition 6.3.3, we have $U_{(a,b)}^{(\gamma,\delta)} = \Phi^{-1}((\gamma,\delta)\times(a,b))$, so that $U_{(a,b)}^{(\gamma,\delta)}$ is open.

To prove convexity, let $t \in (0,1)$ and $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in U_{(a,b)}^{(\gamma,\delta)}$. This means $\frac{y_1}{x_1}, \frac{y_2}{x_2} \in (a,b)$ and $x_1, x_2 \in (\gamma,\delta) \subset (0,\infty)$ and thus

$$ax_1 < y_1 < bx_1 \qquad \text{and} \qquad ax_2 < y_2 < bx_2.$$

Since $(\gamma,\delta)$ is convex, we also get $tx_1 + (1-t)\,x_2 \in (\gamma,\delta) \subset (0,\infty)$. Furthermore, we can estimate

$$\begin{aligned} a\,(tx_1 + (1-t)\,x_2) &= tax_1 + (1-t)\,ax_2 \\ &< ty_1 + (1-t)\,y_2 \\ &< tbx_1 + (1-t)\,bx_2 \\ &= b\,(tx_1 + (1-t)\,x_2)\,, \end{aligned}$$

which means nothing but

$$\frac{ty_1 + (1-t)\,y_2}{tx_1 + (1-t)\,x_2} \in (a,b)\,.$$

All in all, we have shown $t\binom{x_1}{y_1} + (1-t)\binom{x_2}{y_2} \in U_{(a,b)}^{(\gamma,\delta)}$.

To prove equation (6.3.1), we observe the equivalences

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} \in U_{(m+a,m+b)}^{(\gamma,\delta)} &\iff x \in (\gamma,\delta) \text{ and } m + a < \frac{y}{x} < m + b \\ &\iff x \in (\gamma,\delta) \text{ and } a < \frac{y - mx}{x} < b \\ &\iff \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y - mx \end{pmatrix} \in U_{(a,b)}^{(\gamma,\delta)} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} \in U_{\left(\frac{\beta}{\alpha}a, \frac{\beta}{\alpha}b\right)}^{(\alpha\gamma,\alpha\delta)} &\iff x \in (\alpha\gamma,\alpha\delta) \text{ and } \frac{\beta}{\alpha}a < \frac{y}{x} < \frac{\beta}{\alpha}b \\ &\iff \alpha^{-1}x \in (\gamma,\delta) \text{ and } a < \frac{\beta^{-1}y}{\alpha^{-1}x} < b \\ &\iff \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha^{-1}x \\ \beta^{-1}y \end{pmatrix} \in U_{(a,b)}^{(\gamma,\delta)}. \end{aligned}$$

Using the factorization $A_{n,m,\varepsilon}^{(c)} = \varepsilon \left(\begin{smallmatrix} 2^n & 0 \\ 0 & 2^{nc} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$, this also establishes equation (6.3.2).

Let us now consider equation (6.3.3). We first assume $\varepsilon = 1$. Using $\varrho_c = 2^{c-1} > 0$ and the transformation laws from above, we derive

$$\begin{aligned}
\Phi\left(\bigcup_{m\in\mathbb{Z}} A^{(c)}_{n,m,1} U^{(\gamma,\delta)}_{(a,b)}\right) &= \bigcup_{m\in\mathbb{Z}} \Phi\left(U^{(2^n\gamma,2^n\delta)}_{(\varrho_c^n(m+a),\varrho_c^n(m+b))}\right) \\
&= \bigcup_{m\in\mathbb{Z}} \left[(2^n\gamma, 2^n\delta) \times \left(\varrho_c^n\,(m+a)\,, \varrho_c^n\,(m+b)\right)\right] \\
&= (2^n\gamma, 2^n\delta) \times \left[\varrho_c^n \cdot \bigcup_{m\in\mathbb{Z}} (m+a, m+b)\right] \\
&\overset{(\dagger)}{=} (2^n\gamma, 2^n\delta) \times [\varrho_c^n \cdot \mathbb{R}] = (2^n\gamma, 2^n\delta) \times \mathbb{R},
\end{aligned}$$

where we used $b - a > 1$ at $(\dagger)$.

Now, we note $\Phi^{-1}(\{x\} \times \mathbb{R}) = \{x\} \times \mathbb{R}$ for all $x \in (0, \infty)$. Since $\Phi$ is bijective, this implies

$$\bigcup_{m\in\mathbb{Z}} A^{(c)}_{n,m,1} U^{(\gamma,\delta)}_{(a,b)} = \Phi^{-1}\left((2^n\gamma, 2^n\delta) \times \mathbb{R}\right) = (2^n\gamma, 2^n\delta) \times \mathbb{R}.$$

The case $\varepsilon = -1$ follows easily from this using $A^{(c)}_{n,m,-1} = -A^{(c)}_{n,m,1}$.

Finally, observe that $\delta/\gamma > 2$ implies

$$\log_2(\delta) - \log_2(\gamma) = \log_2(\delta/\gamma) > 1$$

and hence

$$\begin{aligned}
(\log_2 \times \mathrm{id})\left(\bigcup_{n,m\in\mathbb{Z}} A^{(c)}_{n,m,1} U^{(\gamma,\delta)}_{(a,b)}\right) &= (\log_2 \times \mathrm{id})\left(\bigcup_{n\in\mathbb{Z}} (2^n\gamma, 2^n\delta) \times \mathbb{R}\right) \\
&= \left(\bigcup_{n\in\mathbb{Z}} n + (\log_2\gamma, \log_2\delta)\right) \times \mathbb{R} \\
&= \mathbb{R} \times \mathbb{R},
\end{aligned}$$

which implies

$$\bigcup_{n,m\in\mathbb{Z}} A^{(c)}_{n,m,1} U^{(\gamma,\delta)}_{(a,b)} = (0, \infty) \times \mathbb{R}.$$

Using $A^{(c)}_{n,m,-1} = -A^{(c)}_{n,m,1}$, we conclude that the given family indeed covers $\mathcal{O}$. □

Using the previous results as well as the theory developed in Chapter 4, we will now see that the family $\mathcal{S}^{(c)} = \left(A^{(c)}_{n,m,\varepsilon} U^{(\gamma,\delta)}_{(a,b)}\right)_{n,m,\varepsilon}$ is a covering of $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$ induced by $H_c$. Once we have shown that $\mathcal{S}^{(c)}$ is an induced covering of $H_c$, this also yields the desired isomorphism of the coorbit space $\mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ with the decomposition space $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_w\right)$ for suitable weights $v, w$.

**Corollary 6.3.5.** *Let $c \in \mathbb{R}$ and $a, b \in \mathbb{R}$ with $b - a > 1$. Furthermore, let $\gamma, \delta \in (0, \infty)$ with $\delta/\gamma > 2$. Set $I := \mathbb{Z}^2 \times \{\pm 1\}$. Then the family*

$$\mathcal{S}^{(c)}_{a,b,\gamma,\delta} := \left(A^{(c)}_{n,m,\varepsilon} U^{(\gamma,\delta)}_{(a,b)}\right)_{(n,m,\varepsilon)\in I}$$

*is a covering of $\mathcal{O} = H_c^T \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \mathbb{R}^* \times \mathbb{R}$ induced by $H_c$.*

*Furthermore, $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$ is a structured admissible covering of $\mathcal{O}$ and in particular an $L^p$-decomposition covering for all $p \in (0, \infty]$.*

*Let $p, q \in (0, \infty]$ and assume that $v : H_c \to (0, \infty)$ is $v_0$-moderate for some measurable, locally bounded weight $v_0 : H_c \to (0, \infty)$. Define $u^{(q)}$ by*

$$u^{(q)}_{n,m,\varepsilon} := 2^{-n(1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \cdot v\left(\left(A^{(c)}_{n,m,\varepsilon}\right)^{-T}\right) \qquad \text{for } (n, m, \varepsilon) \in I.$$

*Then $u^{(q)}$ is a $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$-discretization of a decomposition weight for the space $\mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. In particular, $u^{(q)}$ is $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$-moderate.*

*For the definition of a decomposition weight, see Definition 4.5.3.*

*Finally, the Fourier transform as defined in Theorem 4.3.1 restricts to an isomorphism*

$$\mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}_{a,b,\gamma,\delta}, L^p, \ell^q_{u^{(q)}}\right),$$

*where the coorbit space is defined with respect to the quasi-regular representation of $\mathbb{R}^2 \rtimes H_c$. In particular, the coorbit space is well-defined.*

*On $L^2\left(\mathbb{R}^2\right) \cap \mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$, the Fourier transform defined above coincides with the usual Fourier transform. Similarly, the inverse map of the Fourier transform above coincides with the usual inverse Fourier transform on $L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}_{a,b,\gamma,\delta}, L^p, \ell^q_{u^{(q)}}\right)$.*

*Proof.* Recall the definitions of the matrices $A^{(c)}_{n,m,\varepsilon}$ and $B^{(c)}_{n,m,\varepsilon}$ from Lemma 6.3.2 and set $h_i := B^{(c)}_{-n,-m,\varepsilon}$ for $i = (n, m, \varepsilon) \in I$. This yields $A^{(c)}_i = h_i^{-T}$ for all $i \in I$. Lemma 6.3.2 implies that $(h_i)_{i\in I}$ is a well-spread family in $H_c$.

Define $Q := U^{(\gamma,\delta)}_{(a,b)}$. We then have

$$Q = \Phi^{-1}\left((\gamma, \delta) \times (a, b)\right) \subset \Phi^{-1}\left([\gamma, \delta] \times [a, b]\right)$$

with $\Phi$ as in Definition 6.3.3. The latter set is a compact subset of $\mathcal{O}$ because $\Phi : \mathcal{O} \to \mathbb{R}^* \times \mathbb{R}$ is a diffeomorphism. This shows that $Q \subset \mathcal{O}$ is precompact with $\overline{Q} \subset \mathcal{O}$. Since $\left(h_i^{-T} Q\right)_{i\in I}$ is a covering of $\mathcal{O}$ by Lemma 6.3.4, Definition 4.4.3 shows that $\left(h_i^{-T} Q\right)_{i\in I} = \mathcal{S}^{(c)}_{a,b,\gamma,\delta}$ is a covering of $\mathcal{O}$ induced by $H_c$. By Lemma 4.4.4, this implies in particular that $\mathcal{Q}$ is a semi-structured, admissible covering of $\mathcal{O}$.

It is easy to see that there exist $a', b' \in (a, b)$ with $b' - a' > 1$ and $\delta', \gamma' \in (\gamma, \delta)$ with $\delta'/\gamma' > 2$. Set $P := U^{(\gamma',\delta')}_{(a',b')}$. As above, we see

$$\overline{P} \subset \Phi^{-1}\left(\left[\gamma', \delta'\right] \times \left[a', b'\right]\right) \subset \Phi^{-1}\left((\gamma, \delta) \times (a, b)\right) = Q \subset \mathcal{O}.$$

Further, the argument just given shows that $\left(h_i^{-T}P\right)_{i\in I} = \left(A_i^{(c)}P\right)_{i\in I}$ is still a covering of $\mathcal{O}$. Using the definition of (semi)-structured admissible coverings, this easily shows that $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$ is indeed a structured admissible covering of $\mathcal{O}$. By Theorem 3.2.17, this shows that $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$ is an $L^p$-decomposition covering of $\mathcal{O}$ for all $p \in (0,\infty]$.

Let

$$v^{(q)} : H \to (0,\infty), h \mapsto |\det(h)|^{\frac{1}{2}-\frac{1}{q}} \cdot v(h)$$

as in Definition 4.5.3. By choosing $\xi_0 \in Q$ and $\xi_i = h_i^{-T}\xi_0$ and by combining Definition 4.4.7 and Definition 3.2.4, we see that

$$\begin{aligned} u_i^{(q)} &:= v^{(q)}(h_i) = |\det(h_i)|^{\frac{1}{2}-\frac{1}{q}} \cdot v(h_i) \\ &= 2^{-n(1+c)\left(\frac{1}{2}-\frac{1}{q}\right)} \cdot v\left(\left(A^{(c)}_{n,m,\varepsilon}\right)^{-T}\right) \end{aligned}$$

for $i = (n,m,\varepsilon) \in I$ is a valid $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$-discretization of a decomposition weight for the space $\mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. By Lemma 3.2.5, this implies in particular that $u^{(q)}$ is moderate with respect to $\mathcal{S}^{(c)}_{a,b,\gamma,\delta}$.

Finally, Theorem 4.6.3 shows that the Fourier transform $\mathcal{F}_{\mathcal{R}}$ yields the stated isomorphism. By Theorem 4.3.1, we know that $\mathcal{F}_{\mathcal{R}}$ coincides with the usual Fourier transform on $L^2\left(\mathbb{R}^2\right) \cap \mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ and Theorem 4.6.3 shows that the inverse Fourier transform $\mathcal{F}_{\mathcal{R}}^{-1}$ coincides on $L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}_{a,b,\gamma,\delta}, L^p, \ell^q_{u^{(q)}}\right)$ with the usual inverse Fourier transform. □

### 6.3.2 *Embeddings between shearlet type coorbit spaces*

We now want to apply the embedding results developed in Chapter 5 to the setting of the decomposition spaces given by the coverings $\mathcal{S}^{(c)} = \mathcal{S}^{(c)}_{a,b,\gamma,\delta}$ associated to the shearlet-type groups $H_c$. Together with Corollary 6.3.5, this will yield embeddings between the coorbit spaces $\mathrm{Co}\left(L^{p,q}_v\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ for *distinct* values of $c$.

The first step is to determine which "parts" of the covering $\mathcal{S}^{(c_1)}$ are (almost) subordinate to $\mathcal{S}^{(c_2)}$ for different values of $c_1, c_2$ (cf. the assumptions of Theorem 5.4.5). This is done in the following lemma. For a graphical intuition, see also Figure 1. Roughly speaking, $\mathcal{S}^{(c_1)}$ is almost subordinate to $\mathcal{S}^{(c_2)}$ "away from the $y$-axis" for $c_1 \leq c_2$. Conversely, the covering $\mathcal{S}^{(c_2)}$ is almost subordinate to $\mathcal{S}^{(c_1)}$ "near the $y$-axis".

**Lemma 6.3.6.** *Let $c_1, c_2 \in \mathbb{R}$ with $c_1 \leq c_2$. Let $a, b, \alpha, \beta \in \mathbb{R}$ and $c, d, \gamma, \delta \in (0,\infty)$ with $b - a > 1$, $\beta - \alpha > 1$ and $\delta/\gamma > 2$ as well as $d/c > 2$.*

*Define $\mathcal{S}^{(c_1)} := \mathcal{S}^{(c_1)}_{a,b,c,d}$ and $\mathcal{S}^{(c_2)} := \mathcal{S}^{(c_2)}_{\alpha,\beta,\gamma,\delta}$, write $\mathcal{S}^{(c_1)} = \left(S_i^{(c_1)}\right)_{i\in I}$ and $\mathcal{S}^{(c_2)} = \left(S_j^{(c_2)}\right)_{j\in J}$ with $I := J := \mathbb{Z}^2 \times \{\pm 1\}$ and let $n_0 \in \mathbb{Z}$ be arbitrary.*

*Then there is a constant* $N = N(c_1, c_2, n_0, a, b, c, d, \alpha, \beta, \gamma, \delta) \in \mathbb{N}$ *with*

$$S_i^{(c_1)} \subset \left(S_j^{(c_2)}\right)^{N*} \qquad \text{for all } i \in \mathbb{Z}_{\geq n_0} \times \mathbb{Z} \times \{\pm 1\} \text{ and } j \in J_i,$$
$$S_j^{(c_2)} \subset \left(S_i^{(c_1)}\right)^{N*} \qquad \text{for all } j \in \mathbb{Z}_{\leq n_0} \times \mathbb{Z} \times \{\pm 1\} \text{ and } i \in I_j.$$

*Finally, there is* $R_1 = R_1(c, d, \gamma, \delta) > 0$ *so that* $S_{n,m,\varepsilon}^{(c_1)} \cap S_{k,\ell,\theta}^{(c_2)} \neq \varnothing$ *implies* $\varepsilon = \theta$ *and* $|n - k| \leq R_1$.

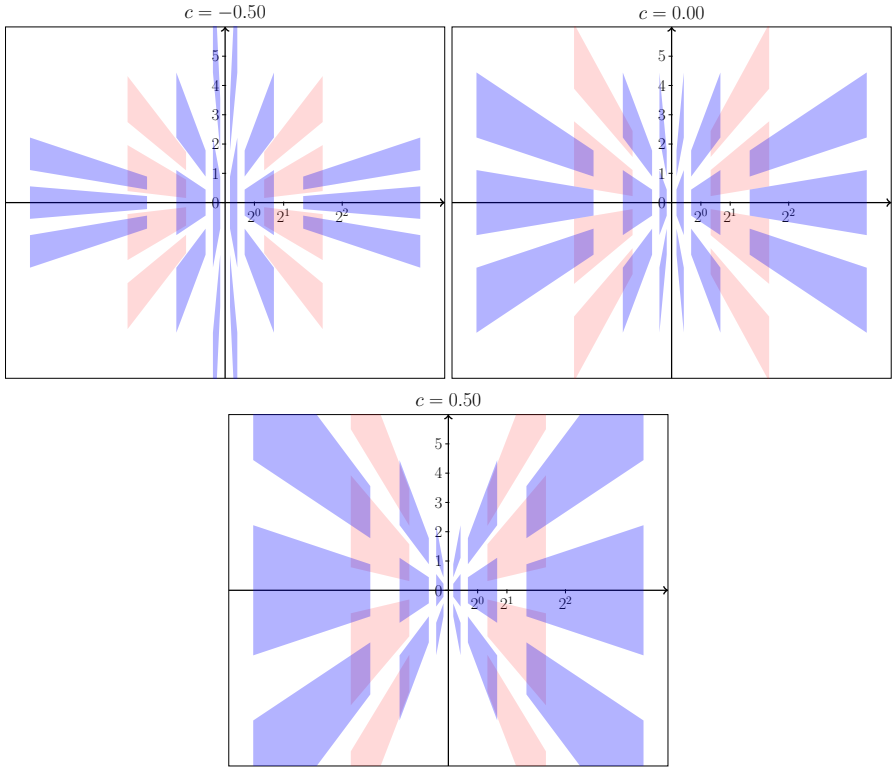


Figure 1: The three figures show (parts of) the coverings $\mathcal{S}^{(c)}$ for different values of $c \in \mathbb{R}$, namely for $c = -\frac{1}{2}$, $c = 0$ and $c = \frac{1}{2}$. These three choices of $c$ show the qualitatively different behaviour of the covering for different values of $c$. For $c_1 < c_2$, the covering $\mathcal{S}^{(c_1)}$ is "larger/coarser" near the $y$-axis, whereas the covering $\mathcal{S}^{(c_2)}$ is "larger/coarser" away from the $y$-axis (cf. also Lemma 6.3.6).

*Proof.* By Lemma 6.3.4, all of the sets $S_i^{(c_1)}, S_j^{(c_2)}$ are open and convex (hence path-connected). Thus, Lemma 3.3.3 shows that it suffices to prove

$$|J_i| \leq N \qquad \text{for all } i \in \mathbb{Z}_{\geq n_0} \times \mathbb{Z} \times \{\pm 1\},$$
$$\text{and } |I_j| \leq N \qquad \text{for all } j \in \mathbb{Z}_{\leq n_0} \times \mathbb{Z} \times \{\pm 1\}$$

for a fixed constant $N \in \mathbb{N}$ as in the statement of the lemma.

To this end, let $i = (n, m, \varepsilon) \in I$ and $j = (k, \ell, \theta) \in J$ be arbitrary with $S_i^{(c_1)} \cap S_j^{(c_2)} \neq \varnothing$. Using the diffeomorphism $\Phi$ from Definition 6.3.3 and the transformation behaviour of the sets $U_{(*,*)}^{(*,*)}$ as described in Lemma 6.3.4, we see

$$\begin{aligned}
\varnothing &\neq \Phi\left(S_i^{(c_1)} \cap S_j^{(c_2)}\right) \\
&= \Phi\left(\varepsilon \cdot U^{(2^n c, 2^n d)}_{\left(\varrho^n_{c_1}(m+a), \varrho^n_{c_1}(m+b)\right)}\right) \cap \Phi\left(\theta \cdot U^{\left(2^k\gamma, 2^k\delta\right)}_{\left(\varrho^k_{c_2}(\ell+\alpha), \varrho^k_{c_2}(\ell+\beta)\right)}\right) \\
&= \left[(\varepsilon \cdot (2^n c, 2^n d)) \times \left(\varrho^n_{c_1}(m+a), \varrho^n_{c_1}(m+b)\right)\right] \\
&\cap \left[(\theta \cdot (2^k\gamma, 2^k\delta)) \times \left(\varrho^k_{c_2}(\ell+\alpha), \varrho^k_{c_2}(\ell+\beta)\right)\right]. \qquad (6.3.4)
\end{aligned}$$

Here, we also used

$$\Phi(-(x,y)) = \left(-x, \frac{-y}{-x}\right) = \left(-x, \frac{y}{x}\right)$$

for $(x, y) \in \mathbb{R}^* \times \mathbb{R}$.

Using $(2^n c, 2^n d), \left(2^k\gamma, 2^k\delta\right) \subset (0, \infty)$, the above considerations yield $\varepsilon = \theta$ and thus $(2^n c, 2^n d) \cap \left(2^k\gamma, 2^k\delta\right) \neq \varnothing$. Hence,

$$2^n c < 2^k \delta \qquad \text{and} \qquad 2^k\gamma < 2^n d.$$

Taking logarithms, we arrive at $n + \log_2 c < k + \log_2 \delta$, as well as $k + \log_2 \gamma < n + \log_2 d$.

We conclude

$$\begin{aligned}
-\left|\log_2\left(\frac{\gamma}{d}\right)\right| \leq \log_2\left(\frac{\gamma}{d}\right) &= \log_2\gamma - \log_2 d \\
&< n - k \\
&< \log_2\delta - \log_2 c = \log_2\left(\frac{\delta}{c}\right) \leq \left|\log_2\left(\frac{\delta}{c}\right)\right|,
\end{aligned}$$

which means $|n-k| \leq \left\lfloor \max\left\{\left|\log_2\left(\frac{\delta}{c}\right)\right|, \left|\log_2\left(\frac{\gamma}{d}\right)\right|\right\}\right\rfloor =: R_1$. In particular, we have established the last statement of the lemma.

Equation (6.3.4) also yields

$$\left(\varrho^n_{c_1}(m+a), \varrho^n_{c_1}(m+b)\right) \cap \left(\varrho^k_{c_2}(\ell+\alpha), \varrho^k_{c_2}(\ell+\beta)\right) \neq \varnothing,$$

which implies

$$\varrho^n_{c_1}(m+a) < \varrho^k_{c_2}(\ell+\beta) \quad \text{and} \quad \varrho^k_{c_2}(\ell+\alpha) < \varrho^n_{c_1}(m+b). \qquad (6.3.5)$$

We now distinguish the two cases for which we want to show the claim:

CASE 1: We have $n \geq n_0$. By assumption, we have $c_1 - c_2 \leq 0$ and thus $(c_1 - c_2)\, n \leq (c_1 - c_2)\, n_0$. We recall the notation $\varrho_c = 2^{c-1}$ from Lemma 6.3.4 and employ the estimate $|n-k| \leq R_1$ to derive

$$\begin{aligned}\frac{\varrho_{c_1}^n}{\varrho_{c_2}^k} &= \frac{\left(2^{c_1-1}\right)^n}{\left(2^{c_2-1}\right)^k} = 2^{c_1 n - n - c_2 k + k} \\ &= 2^{(c_1-c_2)n} \cdot 2^{k-n} \cdot 2^{c_2(n-k)} \\ &\leq 2^{(c_1-c_2)n_0} \cdot 2^{R_1} \cdot 2^{|c_2| R_1} =: R_2.\end{aligned}$$

Set $\xi_{k,n,m} := \frac{\varrho_{c_1}^n}{\varrho_{c_2}^k} m \in \mathbb{R}$. In view of estimate (6.3.5), we conclude

$$\ell - \xi_{k,n,m} > \frac{\varrho_{c_1}^n}{\varrho_{c_2}^k} a - \beta \geq - \left| \frac{\varrho_{c_1}^n}{\varrho_{c_2}^k} a - \beta \right| \geq - \left[ R_2 \, |a| + |\beta| \right]$$

and

$$\ell - \xi_{k,n,m} < \frac{\varrho_{c_1}^n}{\varrho_{c_2}^k} b - \alpha \leq R_2 \, |b| + |\alpha| \, .$$

For brevity, set $R_3 := \lceil \max \{ R_2 \, |a| + |\beta| \, , R_2 \, |b| + |\alpha| \} \rceil$. With this notation, our findings simply mean $\ell \in B_{R_3}(\xi_{k,n,m}) \cap \mathbb{Z}$. We observe $|B_{R_3}(\xi_{k,n,m}) \cap \mathbb{Z}| \leq 2R_3 + 1$.

All in all, we arrive at

$$j = (k, \ell, \theta) \in \left[ \bigcup_{\kappa = n - R_1}^{n + R_1} \{\kappa\} \times \left( B_{R_3}(\xi_{\kappa,n,m}) \cap \mathbb{Z} \right) \right] \times \{\varepsilon\} \, ,$$

where the latter set has cardinality at most $(2R_1 + 1) \cdot (2R_3 + 1) =: R_4$. This proves $|J_i| \leq R_4$.

CASE 2: We have $k \leq n_0$. The argument is analogous to the one above. Here, we set $\eta_{n,k,\ell} := \frac{\varrho_{c_2}^k}{\varrho_{c_1}^n} \ell \in \mathbb{R}$ and use the inequalities in equation (6.3.5) together with

$$\frac{\varrho_{c_2}^k}{\varrho_{c_1}^n} = \frac{2^{(c_2-1)k}}{2^{(c_1-1)n}} = 2^{(c_2-c_1)k} \cdot 2^{c_1(k-n)} \cdot 2^{n-k} \overset{(\dagger)}{\leq} 2^{(c_2-c_1)n_0} \cdot 2^{|c_1| R_1} \cdot 2^{R_1} =: R_5$$

to derive

$$- \left[ R_5 \, |\alpha| + |b| \right] \leq \frac{\varrho_{c_2}^k}{\varrho_{c_1}^n} \alpha - b < m - \eta_{n,k,\ell} < \frac{\varrho_{c_2}^k}{\varrho_{c_1}^n} \beta - a \leq R_5 \, |\beta| + |a| \, .$$

In the step marked with (†), we used $|n-k| \leq R_1$ and $k \leq n_0$ as well as $c_2 - c_1 \geq 0$. As above, we conclude $m \in B_{R_6}(\eta_{n,k,\ell}) \cap \mathbb{Z}$ for $R_6 := \lceil \max \{ R_5 \, |\beta| + |a| \, , R_5 \, |\alpha| + |b| \} \rceil$ and thus

$$i = (n, m, \varepsilon) \in \left[ \bigcup_{\kappa = k - R_1}^{k + R_1} \{\kappa\} \times \left( B_{R_6}(\eta_{\kappa,k,\ell}) \cap \mathbb{Z} \right) \right] \times \{\theta\} \, ,$$

where the latter set has at most $(2R_1+1)\cdot(2R_6+1) =: R_7$ elements.

Setting $N := \max\{R_4, R_7\}$ completes the proof, since $R_1, \dots, R_7$ only depend on $c_1, c_2, n_0, a, b, c, d, \alpha, \beta, \gamma, \delta$. □

We will now apply the above subordinateness properties to derive embeddings between decomposition spaces with respect to the coverings $\mathcal{S}^{(c_1)}$ and $\mathcal{S}^{(c_2)}$ for different values of $c_1, c_2$. Here, we make the specific choice $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ for definiteness. By Lemma 6.3.6, we know that different (admissible) choices of the parameters $a, b, \gamma, \delta$ (for the same value $c$) lead to coverings which are mutually almost subordinate, so that Lemma 5.1.9 shows that the induced decomposition spaces (with equivalent weights) are the same.

In analogy to [15, Page 11], we only consider weights of the form $(n, m, \varepsilon) \mapsto 2^{\alpha n}$ for $\alpha \in \mathbb{R}$. Note that this is indeed analogous to the weight

$$\mathbb{R}^2 \rtimes H_c \to (0,\infty), \left(x, \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)\right) \mapsto |a|^{-r}$$

considered in [15, Page 11], because of $A^{(c)}_{n,m,\varepsilon} = \varepsilon \left(\begin{smallmatrix} 2^n & 0 \\ 2^{nc}m & 2^{nc} \end{smallmatrix}\right)$, so that $|a|$ corresponds to $2^{\pm n}$. See also Theorem 6.3.9 for a formal statement.

As the proof shows, considering weights of this form has the advantage that the weight is moderate with respect to the covering $\mathcal{S}^{(c)}$ for *arbitrary* choices of $c$, so that the sharpness results in Theorem 5.4.5 become available and so that the calculation of the quantities $L_1$ and $L_2$ in that theorem are simplified.

But the missing dependence of the weight $(n, m, \varepsilon) \mapsto 2^{\alpha n}$ on $m$ implies that the weight has no decay with respect to $m$. We will see that this prevents the existence of embeddings for $q_1 > q_2$.

**Theorem 6.3.7.** *For $c, \alpha \in \mathbb{R}$, define*

$$v^{(\alpha)} : \mathbb{Z}^2 \times \{\pm 1\} \to (0,\infty), (n, m, \varepsilon) \mapsto 2^{\alpha n}.$$

*Then $v^{(\alpha)}$ is $\mathcal{S}^{(c)}$-moderate for $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ (with $\mathcal{S}^{(c)}_{-1,1,\frac{1}{2},1}$ as in Corollary 6.3.5).*

*Let $c_1, c_2 \in \mathbb{R}$ with $c_1 \le c_2$ and let $p_1, p_2, q_1, q_2 \in (0,\infty]$, as well as $\alpha_1, \alpha_2 \in \mathbb{R}$.*

*The map*

$$\iota : \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}}\right) \to \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_2}, \ell^{q_2}_{v^{(\alpha_2)}}\right), f \mapsto f$$

*is well-defined and bounded if and only if we have $p_1 \le p_2$ and $q_1 \le q_2$, as well as*

$$\alpha_2 \ge \alpha_1 + (1+c_2)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + (c_2 - c_1)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+$$

$$\text{and } \alpha_2 \le \alpha_1 + (1+c_1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + (c_1 - c_2)\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+.$$

*For convenience of the reader, we recall the notation $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}$ from Remark 5.2.10.*

*In the reverse direction, the map*

$$\theta : \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}}\right) \to \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_2}, \ell^{q_2}_{v^{(\alpha_2)}}\right), f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$ *and* $q_1 \leq q_2$*, as well as*

$$\alpha_2 \geq \alpha_1 + (1+c_2)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + (c_2 - c_1)\left(\frac{1}{p_2^{\nabla}} - \frac{1}{q_1}\right)_+$$
$$\text{and } \alpha_2 \leq \alpha_1 + (1+c_1)\left(\frac{1}{p_2} - \frac{1}{p_1}\right) + (c_1 - c_2)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+.$$

*For necessity of the given conditions, it even suffices if the maps* $\iota$ *or* $\theta$*, respectively, are bounded when restricted to* $C_c^\infty(\mathbb{R}^* \times \mathbb{R})$*.*

*Remark* 6.3.8. The conditions given above are quite restrictive. To see this, let us consider the case $p_1 = p_2 = p$. In this case, the conditions for the existence of the first embedding reduce to $q_1 \leq q_2$ as well as

$$\alpha_1 + (c_2 - c_1)\left(\frac{1}{q_2} - \frac{1}{p^{\pm\triangle}}\right)_+ \leq \alpha_2 \leq \alpha_1 + (c_1 - c_2)\left(\frac{1}{p^{\nabla}} - \frac{1}{q_1}\right)_+. \tag{6.3.6}$$

In particular, this implies

$$(c_2 - c_1)\left[\left(\frac{1}{q_2} - \frac{1}{p^{\pm\triangle}}\right)_+ + \left(\frac{1}{p^{\nabla}} - \frac{1}{q_1}\right)_+\right] \leq 0.$$

Hence, there are two cases:

*Case* 1. $c_1 = c_2$. In this case, eq. (6.3.6) is equivalent to $\alpha_1 = \alpha_2$.

*Case* 2. $c_1 < c_2$. This implies $\left(\frac{1}{q_2} - \frac{1}{p^{\pm\triangle}}\right)_+ = 0 = \left(\frac{1}{p^{\nabla}} - \frac{1}{q_1}\right)_+$, because both terms are non-negative. Equivalently we get,

$$\frac{1}{q_2} \leq \frac{1}{p^{\pm\triangle}} \qquad \text{and} \qquad \frac{1}{p^{\nabla}} \leq \frac{1}{q_1}.$$

Furthermore, equation (6.3.6) is easily seen to be equivalent to $\alpha_1 = \alpha_2$ also in this case. Finally, observe that the above conditions imply

$$\frac{1}{q_2} \leq \frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\} \leq \frac{1}{2} \leq \frac{1}{p^{\nabla}} \leq \frac{1}{q_1},$$

so that the condition $q_1 \leq q_2$ is automatically implied.

In summary, existence of the embedding

$$\mathcal{D}\left(\mathcal{S}^{(c_1)}, L^p, \ell^{q_1}_{v^{(\alpha_1)}}\right) \hookrightarrow \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^p, \ell^{q_2}_{v^{(\alpha_2)}}\right)$$

for $c_1 < c_2$ is equivalent to $\alpha_1 = \alpha_2$ and $\frac{1}{q_2} \leq \frac{1}{p^{\pm\triangle}} \leq \frac{1}{p^{\nabla}} \leq \frac{1}{q_1}$. For $c_1 = c_2$, the embedding is valid if and only if $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$. ◀

*Note that we do* not *assume $c_1 \leq c_2$ for proving relative moderateness of $v^{(\alpha)}$ with respect to $\mathcal{S}^{(c_2)}$.*

*Proof of Theorem 6.3.7.* We start by proving that $v^{(\alpha)}$ (considered as a weight for $\mathcal{S}^{(c_1)}$) is relatively $\mathcal{S}^{(c_2)}$-moderate for arbitrary $c_1, c_2 \in \mathbb{R}$. To this end, we note that Lemma 6.3.6 yields an absolute constant $R \in \mathbb{N}$ (*not* depending on $c_1, c_2$) so that $S^{(c_i)}_{k,\ell,\delta} \cap S^{(c_j)}_{n,m,\varepsilon} \neq \varnothing$ implies $|n-k| \leq R$ for all $i, j \in \{1,2\}$ and $(n,m,\varepsilon), (k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}$. This yields

$$\begin{aligned} v^{(\alpha)}_{n,m,\varepsilon} &= 2^{\alpha n} = 2^{\alpha k} \cdot 2^{\alpha(n-k)} \\ &\leq 2^{|\alpha||n-k|} \cdot 2^{\alpha k} \\ &\leq 2^{|\alpha| R} \cdot 2^{\alpha k} \\ &=: C_1 \cdot 2^{\alpha k}. \end{aligned}$$

An analogous estimate shows $v^{(\alpha)}_{n,m,\varepsilon} \geq C_1^{-1} \cdot 2^{\alpha k}$. Hence,

$$C_1^{-1} \cdot 2^{\alpha k} \leq v^{(\alpha)}_{n,m,\varepsilon} \leq C_1 \cdot 2^{\alpha k} \quad \text{for} \quad S^{(c_1)}_{n,m,\varepsilon} \cap S^{(c_2)}_{k,\ell,\delta} \neq \varnothing, \tag{6.3.7}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary (not necessarily with $c_1 \leq c_2$).

Using this, we see

$$\frac{v^{(\alpha)}_{n,m,\varepsilon}}{v^{(\alpha)}_{k,\ell,\delta}} \leq \frac{C_1 \cdot 2^{\alpha\kappa}}{C_1^{-1} \cdot 2^{\alpha\kappa}} = C_1^2$$

for $S^{(c_1)}_{n,m,\varepsilon} \cap S^{(c_2)}_{\kappa,\nu,\gamma} \neq \varnothing \neq S^{(c_1)}_{k,\ell,\delta} \cap S^{(c_2)}_{\kappa,\nu,\gamma}$, so that $v^{(\alpha)}$ (considered as a weight for $\mathcal{S}^{(c_1)}$) is indeed $\mathcal{S}^{(c_2)}$-moderate for arbitrary $c_1, c_2 \in \mathbb{R}$. In particular, we see that $\mathcal{S}^{(c_1)}$ is relatively moderate with respect to $\mathcal{S}^{(c_2)}$ for arbitrary $c_1, c_2 \in \mathbb{R}$, since we have $\mathcal{S}^{(c_1)} = \left(A^{(c_1)}_{n,m,\varepsilon} U^{(1/2,2)}_{(-1,1)}\right)_{n,m,\varepsilon}$ with

$$|\det(A^{(c_1)}_{n,m,\varepsilon})| = 2^{n(1+c_1)} = v^{(1+c_1)}_{n,m,\varepsilon}$$

and the weight $v^{(1+c_1)}$, considered as a weight for $\mathcal{S}^{(c_1)}$, is relatively $\mathcal{S}^{(c_2)}$-moderate, as we just saw.

We now invoke Theorem 5.4.5 with $\mathcal{R} = (R_\lambda)_{\lambda\in\Lambda} := \mathcal{S}^{(c_1)}$ and $\mathcal{S} = (S_\omega)_{\omega\in\Omega} := \mathcal{S}^{(c_2)}$ as well as $u = v^{(\alpha_1)}$ and $v = v^{(\alpha_2)}$ and with

$$A := [(-\infty, -1) \cup (1, \infty)] \times \mathbb{R} \quad \text{and} \quad B := [[-1,1] \setminus \{0\}] \times \mathbb{R}$$

to show that the existence of the first embedding is equivalent to the stated conditions (where the necessity of the conditions is already implied by boundedness of $\iota$ on $C_c^\infty(\mathbb{R}^* \times \mathbb{R})$).

For this, we first note that the map $\iota$ constructed in Theorem 5.4.5 satisfies $\langle \iota f, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in C_c^\infty(\mathcal{O} \cap \mathcal{O}') = C_c^\infty(\mathbb{R}^* \times \mathbb{R})$ and hence $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_1}, \ell^{q_1}_{v^{(\alpha_1)}}\right)$, as long as that theorem is applicable.

For brevity, let us set $Z := \mathbb{Z} \times \{\pm 1\}$. Next, recall that (by taking the maximum of the individual constants) Lemma 6.3.6 yields a constant $N = N(c_1, c_2, R)$ with

*We recall from above that the constant $R \in \mathbb{N}$ is chosen so that $S_{k,\ell,\delta}^{(c_1)} \cap S_{n,m,\varepsilon}^{(c_2)} \neq \varnothing$ implies $|n-k| \leq R$ for arbitrary $c_1, c_2 \in \mathbb{R}$.*

$$\begin{aligned} S_i^{(c_1)} \subset (S_j^{(c_2)})^{N*} \quad & \forall i \in \mathbb{Z}_{\geq -R} \times Z \, \forall j \in \mathbb{Z} \times Z \text{ with } S_i^{(c_1)} \cap S_j^{(c_2)} \neq \varnothing, \\ S_j^{(c_2)} \subset (S_i^{(c_1)})^{N*} \quad & \forall j \in \mathbb{Z}_{\leq R} \times Z \, \forall i \in \mathbb{Z} \times Z \text{ with } S_i^{(c_1)} \cap S_j^{(c_2)} \neq \varnothing. \end{aligned} \tag{6.3.8}$$

We now want to calculate the sets $\Lambda_A$ and $\Omega_B$ (in the notation of Theorem 5.4.5). To this end, note that $\lambda = (n, m, \varepsilon) \in \Lambda_A$ implies that there is some

$$\begin{pmatrix} x \\ y \end{pmatrix} \in A \cap S_{n,m,\varepsilon}^{(c_1)} \subset A \cap \left[ \varepsilon \cdot \left( 2^{n-1}, 2^{n+1} \right) \times \mathbb{R} \right],$$

which implies $2^0 = 1 < |x| < 2^{n+1}$ and hence $n \geq 0$. This shows $\Lambda_A \subset \mathbb{N}_0 \times \mathbb{Z} \times \{\pm 1\}$. Furthermore, because of $n \geq 0 \geq -R$, equation (6.3.8) yields

$$R_\lambda = S_{n,m,\varepsilon}^{(c_1)} \subset (S_{k,\ell,\delta}^{(c_2)})^{N*} = S_{\omega_\lambda}^{N*}$$

for an arbitrary $\omega_\lambda = (k, \ell, \delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with $S_{n,m,\varepsilon}^{(c_1)} \cap S_{k,\ell,\delta}^{(c_2)} \neq \varnothing$. Thus, the first part of the prerequisites of Theorem 5.4.5 is satisfied (with $k = N$).

Conversely, for $n \geq 0$, we have $\varepsilon \cdot \begin{pmatrix} 2^{n+1/2} \\ \varrho_{c_1}^n m \cdot 2^{n+1/2} \end{pmatrix} \in A \cap S_{n,m,\varepsilon}^{(c_1)} \neq \varnothing$ and hence $(n, m, \varepsilon) \in \Lambda_A$. This shows

$$\Lambda_A = \mathbb{N}_0 \times \mathbb{Z} \times \{\pm 1\}, \tag{6.3.9}$$

which implies

$$\begin{aligned} \bigcup_{\lambda \in \Lambda_A} S_\lambda^{(c_1)} &= \bigcup_{\varepsilon \in \{\pm 1\}} \left[ \varepsilon \cdot \bigcup_{n \geq 0} \left( 2^{n-1}, 2^{n+1} \right) \times \mathbb{R} \right] \\ &= \left[ \left( -\infty, -\frac{1}{2} \right) \cup \left( \frac{1}{2}, \infty \right) \right] \times \mathbb{R}, \end{aligned} \tag{6.3.10}$$

thanks to Lemma 6.3.4.

Now, for $\omega = (k, \ell, \delta) \in \Omega_B$, there exists some

$$\begin{pmatrix} x \\ y \end{pmatrix} \in B \cap S_{k,\ell,\delta}^{(c_2)} \subset B \cap \left[ \delta \cdot \left( 2^{k-1}, 2^{k+1} \right) \times \mathbb{R} \right].$$

This implies $2^{k-1} < |x| \leq 1 = 2^0$ and hence $k \leq 0$. We have thus shown $\Omega_B \subset \mathbb{Z}_{\leq 0} \times \mathbb{Z} \times \{\pm 1\}$. Finally, we have $k \leq 0 \leq R$, so that equation (6.3.8) yields $S_\omega \subset R_{\lambda_\omega}^{N*}$ for an arbitrary $\lambda_\omega \in \mathbb{Z}^2 \times \{\pm 1\}$

with $S^{(c_1)}_{\lambda_\omega} \cap S^{(c_2)}_{\omega} \neq \varnothing$. Hence, the second part of the prerequisites of Theorem 5.4.5 is also satisfied (with $\ell = N$).

Similar to the above, we have $\delta \cdot \begin{pmatrix} 2^k \\ \varrho_{c_2}^k \ell \cdot 2^k \end{pmatrix} \in B \cap S^{(c_2)}_{k,\ell,\delta} \neq \varnothing$ for $k \leq 0$ and hence $(k, \ell, \delta) \in \Omega_B$. Together with the preceding paragraph, this yields

$$\Omega_B = \mathbb{Z}_{\leq 0} \times \mathbb{Z} \times \{\pm 1\}\,, \tag{6.3.11}$$

so that Lemma 6.3.4 implies

$$\begin{aligned} \bigcup_{\omega \in \Omega_B} S^{(c_2)}_{\omega} &= \bigcup_{\delta \in \{\pm 1\}} \left[ \delta \cdot \bigcup_{k \leq 0} \left( 2^{k-1}, 2^{k+1} \right) \times \mathbb{R} \right] \\ &= [(-2, 0) \cup (0, 2)] \times \mathbb{R}. \end{aligned} \tag{6.3.12}$$

Now, we check the remaining prerequisites for parts 4 and 3 of Theorem 5.4.5. Recall that $R \in \mathbb{N}$ is chosen so that $S^{(c_1)}_{n,m,\varepsilon} \cap S^{(c_2)}_{k,\ell,\delta} \neq \varnothing$ implies $|n - k| \leq R$ for arbitrary $c_1, c_2 \in \mathbb{R}$. Inductively, this entails $|n - k| \leq MR$ for $(k, \ell, \delta) \in (n, m, \varepsilon)^{M*}$, where the cluster is formed with respect to $\mathcal{S}^{(c)}$ for an arbitrary $c \in \mathbb{R}$. Together with equation (6.3.9), we conclude

$$\Lambda_A^{(2N+3)*} \subset \mathbb{Z}_{\geq -(2N+3)R} \times \mathbb{Z} \times \{\pm 1\}\,. \tag{6.3.13}$$

Hence, Lemma 6.3.6 yields a constant $M = M(R, N, c_1, c_2) \in \mathbb{N}$ with

$$R_\lambda = S^{(c_1)}_{\lambda} \subset (S^{(c_2)}_{\omega_\lambda})^{M*}$$

for all $\lambda \in \Lambda_A^{(2N+3)*}$ and suitable $\omega_\lambda \in \Omega = \mathbb{Z}^2 \times \{\pm 1\}$. Together with the $\mathcal{S}$-moderateness of $u = v^{(\alpha_1)}$ and of $\mathcal{R}$ (which were shown at the beginning of the proof), we see that all assumptions of part 4 of Theorem 5.4.5 are satisfied, since $\mathcal{O} = \mathbb{R}^* \times \mathbb{R} = \mathcal{O}'$ clearly implies that equation (5.4.4) holds.

Furthermore, $S_{\Omega_B}$ and $v|_{\Omega_B}$ are $\mathcal{R}$-moderate (this also holds without restricting to $\Omega_B$), so that all the assumptions of part 3 of Theorem 5.4.5 are valid. Equations (5.4.2) and (5.4.3) are fulfilled by Remark 5.4.6 because of $\mathcal{O} = \mathcal{O}'$.

By applying parts 1 and 3,4 of Theorem 5.4.5, we thus see that the map $\iota$ is well-defined and bounded if and only if we have $p_1 \leq p_2$, as well as

$$\left\| \left( \frac{v_{\omega_\lambda}}{u_\lambda} \left| \det\left(T_\lambda\right) \right|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \left| \det\left(M_{\omega_\lambda}\right) \right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{\lambda \in \Lambda_1} \right\|_{\ell^t} < \infty$$

and

$$\left\| \left( \frac{|\det(M_\omega)|^{\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}}{u_{\lambda_\omega}} |\det(T_{\lambda_\omega})|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+} \right)_{\omega\in\Omega_1} \right\|_{\ell^t_v} < \infty.$$

Here, we defined $t := q_2 \cdot (q_1/q_2)'$ for brevity.

Furthermore, we observe that finiteness of these constants is already implied by boundedness of $\iota|_{C_c^\infty(\mathbb{R}^*\times\mathbb{R})}$. Finally, the sets $\Lambda_1, \Omega_1$ are given by

$$\begin{aligned}\Lambda_1 &= \{\lambda \in \Lambda \,|\, \exists \omega \in \Omega_B : \, S_\omega \cap R_\lambda \neq \varnothing\}, \\ \Omega_1 &= \{\omega \in \Omega \,|\, \exists \lambda \in \Lambda_A : \, R_\lambda \cap S_\omega \neq \varnothing\}\end{aligned}$$

and for each $\lambda \in \Lambda_1$, some $\omega_\lambda \in \Omega_B$ with $S_{\omega_\lambda} \cap R_\lambda \neq \varnothing$ is selected. Analogously, for each $\omega \in \Omega_1$, some $\lambda_\omega \in \Lambda_A$ with $R_\lambda \cap S_\omega \neq \varnothing$ is chosen.

Using equations (6.3.12) and (6.3.10), we see

$$\begin{aligned}\Lambda_1 &= \{\lambda \in \Lambda \,|\, R_\lambda \cap ([(-2,0)\cup(0,2)]\times\mathbb{R}) \neq \varnothing\}, \\ \Omega_1 &= \left\{\omega \in \Omega \,\middle|\, S_\omega \cap \left(\left[\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)\right]\times\mathbb{R}\right) \neq \varnothing\right\}.\end{aligned}$$

But using considerations entirely analogous to those leading to equations (6.3.9) and (6.3.11), this implies

$$\Lambda_1 = \mathbb{Z}_{\leq 1} \times \mathbb{Z} \times \{\pm 1\} \quad \text{and} \quad \Omega_1 = \mathbb{Z}_{\geq -1} \times \mathbb{Z} \times \{\pm 1\}.$$

Finally, equation (6.3.7) yields

$$v_{\omega_\lambda} = v_{\omega_\lambda}^{(\alpha_2)} \asymp 2^{\alpha_2 k} \text{ and } |\det(M_{\omega_\lambda})| = |\det(A_{\omega_\lambda}^{(c_2)})| = v_{\omega_\lambda}^{(1+c_2)} \asymp 2^{(1+c_2)k}$$

for $\lambda = (k,\ell,\delta) \in \Lambda_1$ and furthermore

$$u_{\lambda_\omega} = v_{\lambda_\omega}^{(\alpha_1)} \asymp 2^{\alpha_1 n} \text{ and } |\det(T_{\lambda_\omega})| = |\det(A_{\lambda_\omega}^{(c_1)})| = v_{\lambda_\omega}^{(1+c_1)} \asymp 2^{(1+c_1)n}$$

for $\omega = (n,m,\varepsilon) \in \Omega_1$, where the implied constants depend only on $\alpha_1, \alpha_2, c_1, c_2, R$.

Recall the abbreviation $Z := \mathbb{Z} \times \{\pm 1\}$ from above. All in all, we see that the conditions derived above are equivalent to $p_1 \leq p_2$ and to finiteness of

$$\left\| \left( 2^{k(\alpha_2-\alpha_1)} 2^{k(1+c_2)\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+\right]} 2^{k(1+c_1)\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{(k,\ell,\delta)\in\mathbb{Z}_{\leq 1}\times Z} \right\|_{\ell^t}$$

and of

$$\left\| \left( 2^{n(\alpha_2-\alpha_1)} 2^{n(1+c_1)\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^{\triangledown}}-\frac{1}{q_1}\right)_+\right]} 2^{n(1+c_2)\left(\frac{1}{p_2^{\triangledown}}-\frac{1}{q_1}\right)_+} \right)_{(n,m,\varepsilon)\in\mathbb{Z}_{\geq -1}\times Z} \right\|_{\ell^t}.$$

We observe that the two sequences above do not depend on $\ell$ or $m$, respectively. Hence, none of the two quasi-norms above is finite for $t < \infty$. But Remark 5.2.3 shows that $t = \infty$ is equivalent to $q_2 \geq q_1$. In this case, it is easy to see that the first quasi-norm above is finite if and only if we have

$$\alpha_2 - \alpha_1 + (1+c_1)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + (1+c_2)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+\right] \geq 0$$

and the second quasi-norm is finite if and only if we have

$$\alpha_2 - \alpha_1 + (1+c_2)\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+ + (1+c_1)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+\right] \leq 0.$$

It is straightforward to see that these conditions are equivalent to those stated in the theorem.

The second part is largely analogous. Here, we apply Theorem 5.4.5 with interchanged roles of $\mathcal{R}, \mathcal{S}$ and $A, B$ compared to the first case. Precisely, we use $\mathcal{R}' = (R'_\lambda)_{\lambda\in\Lambda'} := \mathcal{S}^{(c_2)}$ and $\mathcal{S}' = (S'_\omega)_{\omega\in\Omega'} := \mathcal{S}^{(c_1)}$ as well as $u' = v^{(\alpha_1)}$ and $v' = v^{(\alpha_2)}$ and finally

$$B' := [(-\infty,-1)\cup(1,\infty)]\times\mathbb{R} \quad \text{and} \quad A' := [[-1,1]\setminus\{0\}]\times\mathbb{R}.$$

Observe

$$\Lambda'_{A'} = \Omega_B = \mathbb{Z}_{\leq 0}\times\mathbb{Z}\times\{\pm 1\} \quad \text{and} \quad \Omega'_{B'} = \Lambda_A = \mathbb{Z}_{\geq 0}\times\mathbb{Z}\times\{\pm 1\},$$

as well as $\mathcal{R}'_{\Lambda'_{A'}} = \mathcal{S}_{\Omega_B}$ and $\mathcal{S}'_{\Omega'_{B'}} = \mathcal{R}_{\Lambda_A}$ with $\mathcal{R}, \mathcal{S}$ and $A, B$ as in the proof of the first embedding. Hence, $\mathcal{R}'_{\Lambda'_{A'}}$ is almost subordinate to and moderate with respect to $\mathcal{S}'$ and $\mathcal{S}'_{\Omega'_{B'}}$ is almost subordinate to and moderate with respect to $\mathcal{R}'$, and the same constant $N \in \mathbb{N}$ that was used in equation (6.3.8) can be used here. We have thus shown that the general assumptions of Theorem 5.4.5 are fulfilled.

Analogously to the derivation of equation (6.3.13), we see

$$\left(\Lambda'_{A'}\right)^{(2N+3)*} \subset \mathbb{Z}_{\leq (2N+3)R}\times\mathbb{Z}\times\{\pm 1\},$$

so that Lemma 6.3.6 shows that $\mathcal{R}'_{\left(\Lambda'_{A'}\right)^{(2N+3)*}}$ is almost subordinate to $\mathcal{S}'$. Relative moderateness of $\mathcal{R}'$ and of $u'$ (considered as a weight on $\mathcal{R}'$) with respect to $\mathcal{S}'$ was already shown at the beginning of the

proof. Thus, the prerequisites of part 4 of Theorem 5.4.5 are satisfied; as above, equation (5.4.4) holds because of $\mathcal{O} = \mathcal{O}'$.

Finally, the requirements of part 3 of Theorem 5.4.5 also hold because of $\mathcal{O} = \mathcal{O}'$, see Remark 5.4.6.

Just as above, this shows that $\theta$ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norms

$$\left\| \left( \frac{v'_{\omega_\lambda}}{u'_\lambda} \left|\det\left(T'_\lambda\right)\right|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \left|\det\left(M'_{\omega_\lambda}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{\lambda \in \Lambda'_1} \right\|_{\ell^t}$$

and

$$\left\| \left( \frac{\left|\det\left(M'_\omega\right)\right|^{\left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+}}{u'_{\lambda_\omega}} \left|\det\left(T'_{\lambda_\omega}\right)\right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+} \right)_{\omega \in \Omega'_1} \right\|_{\ell^t_{v'}}$$

are finite. Here, we have

$$\begin{aligned} \Lambda'_1 &= \left\{ \lambda \in \Lambda' \,\middle|\, \exists \omega \in \Omega'_{B'} : \, S'_\omega \cap R'_\lambda \neq \varnothing \right\} \\ &= \left\{ \lambda \in \Omega \,\middle|\, \exists \omega \in \Lambda_A : \, R_\omega \cap S_\lambda \neq \varnothing \right\} \\ &= \Omega_1 = \mathbb{Z}_{\geq -1} \times \mathbb{Z} \times \{\pm 1\} \end{aligned}$$

and analogously $\Omega'_1 = \Lambda_1 = \mathbb{Z}_{\leq 1} \times \mathbb{Z} \times \{\pm 1\}$.

Recall the abbreviation $Z := \mathbb{Z} \times \{\pm 1\}$ from above. With precisely the same arguments as in the proof of the first part of the theorem, we see that finiteness of the above quasi-norms is equivalent to finiteness of

$$\left\| \left( 2^{k(\alpha_2 - \alpha_1)} 2^{k(1+c_1)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+\right]} 2^{k(1+c_2)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{(k,\ell,\delta) \in \mathbb{Z}_{\geq -1} \times Z} \right\|_{\ell^t}$$

and of

$$\left\| \left( 2^{n(\alpha_2 - \alpha_1)} 2^{n(1+c_2)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+\right]} 2^{n(1+c_1)\left(\frac{1}{p_2^\triangledown} - \frac{1}{q_1}\right)_+} \right)_{(n,m,\varepsilon) \in \mathbb{Z}_{\leq 1} \times Z} \right\|_{\ell^t}.$$

As above, we see that the two sequences do not depend on $\ell$ or on $m$, respectively, so that the above quasi-norms are both infinite unless $t = \infty$, which is equivalent to $q_2 \geq q_1$. In this case, finiteness of the first quasi-norm is equivalent to

$$\alpha_2 - \alpha_1 + (1 + c_2)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + (1 + c_1)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+\right] \leq 0$$

and finiteness of the second quasi-norm is equivalent to

$$\alpha_2 - \alpha_1 + (1+c_1)\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+ + (1+c_2)\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+\right] \geq 0.$$

It is easy to see that these conditions are equivalent to those stated in the theorem. □

Finally, we use Corollary 6.3.5 to translate the above embedding statements for decomposition spaces to embedding statements for coorbit spaces.

**Theorem 6.3.9.** *For $\alpha, c \in \mathbb{R}$, define the (multiplicative, continuous) weight*

$$w^{(\alpha)} : H_c \to (0,\infty), \varepsilon \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right) \mapsto a^{\alpha},$$

*where as usual $\varepsilon \in \{\pm 1\}$, $a \in (0,\infty)$ and $b \in \mathbb{R}$.*

*Let $c_1, c_2 \in \mathbb{R}$ with $c_1 \leq c_2$ and $p_1, p_2, q_1, q_2 \in (0,\infty]$. Finally, let $\alpha_1, \alpha_2 \in \mathbb{R}$.*

*There is a bounded linear map*

$$\iota : \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right) \to \mathrm{Co}\left(L^{p_2,q_2}_{w^{(\alpha_2)}}\left(\mathbb{R}^2 \rtimes H_{c_2}\right)\right)$$

*which satisfies $\iota f = f$ for all $f \in L^2\left(\mathbb{R}^2\right) \cap \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right)$ if and only if we have $p_1 \leq p_2$ and $q_1 \leq q_2$ as well as*

$$\begin{aligned}\alpha_2 \leq \alpha_1 &+ (1+c_1)\left[\frac{1}{2} - \frac{1}{q_1}\right] + (1+c_2)\left[\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{2}\right] \\ &+ (c_1 - c_2)\left[\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right]_+\end{aligned}$$

*and*

$$\begin{aligned}\alpha_2 \geq \alpha_1 &+ (1+c_2)\left[\frac{1}{q_2} - \frac{1}{2}\right] + (1+c_1)\left[\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{2} - \frac{1}{q_1}\right] \\ &+ (c_2 - c_1)\left[\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right]_+.\end{aligned}$$

*Conversely, there is a bounded linear map*

$$\theta : \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_2}\right)\right) \to \mathrm{Co}\left(L^{p_2,q_2}_{w^{(\alpha_2)}}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right)$$

*which satisfies $\iota f = f$ for all $f \in L^2\left(\mathbb{R}^2\right) \cap \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_2}\right)\right)$ if and only if we have $p_1 \leq p_2$ and $q_1 \leq q_2$ as well as*

$$\begin{aligned}\alpha_2 \leq \alpha_1 &+ (1+c_1)\left[\frac{1}{q_2} - \frac{1}{2}\right] + (1+c_2)\left[\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{2} - \frac{1}{q_1}\right] \\ &+ (c_1 - c_2)\left[\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right]_+\end{aligned}$$

*and*

$$\alpha_2 \geq \alpha_1 + (1+c_2)\left[\frac{1}{2} - \frac{1}{q_1}\right] + (1+c_1)\left[\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{2}\right] + (c_2 - c_1)\left[\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right]_+.$$

*Proof.* We only show the claim for the map $\iota$. The proof for $\theta$ is completely analogous. Let us define weights $u, v$ by

$$\begin{aligned} u_{n,m,\varepsilon} &:= 2^{-n(1+c_1)\left(\frac{1}{2}-\frac{1}{q_1}\right)} \cdot w^{(\alpha_1)}\left(\left(A_{n,m,\varepsilon}^{(c_1)}\right)^{-T}\right) \\ &= 2^{-n\left[\alpha_1+(1+c_1)\left(\frac{1}{2}-\frac{1}{q_1}\right)\right]} = v_{n,m,\varepsilon}^{\left((1+c_1)\left(\frac{1}{q_1}-\frac{1}{2}\right)-\alpha_1\right)} \end{aligned}$$

and

$$\begin{aligned} v_{k,\ell,\delta} &:= 2^{-k(1+c_2)\left(\frac{1}{2}-\frac{1}{q_2}\right)} \cdot w^{(\alpha_2)}\left(\left(A_{k,\ell,\delta}^{(c_2)}\right)^{-T}\right) \\ &= 2^{-k\left[\alpha_2+(1+c_2)\left(\frac{1}{2}-\frac{1}{q_2}\right)\right]} = v_{k,\ell,\delta}^{\left((1+c_2)\left(\frac{1}{q_2}-\frac{1}{2}\right)-\alpha_2\right)}, \end{aligned}$$

where the weights $v^{(\alpha)}$ for $\alpha \in \mathbb{R}$ are defined as in Theorem 6.3.7.

With these definitions, Corollary 6.3.5 shows that the Fourier transform as defined in Theorem 4.3.1 restricts to isomorphisms

$$\begin{aligned} \Phi := \mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right) &\to \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_1}, \ell^{q_1}_u\right), \\ \Psi := \mathcal{F}_{\mathcal{R}} : \mathrm{Co}\left(L^{p_2,q_2}_{w^{(\alpha_2)}}\left(\mathbb{R}^2 \rtimes H_{c_2}\right)\right) &\to \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_2}, \ell^{q_2}_v\right). \end{aligned}$$

The same corollary also shows that $\Phi, \Psi$ coincide with the usual Fourier transform on the intersection of $L^2\left(\mathbb{R}^2\right)$ with the respective coorbit space and that the inverse maps $\Phi^{-1}, \Psi^{-1}$ coincide with the usual inverse Fourier transform on the intersection of $L^2\left(\mathbb{R}^2\right)$ with the respective decomposition space.

By substituting the present parameters into the conditions given in Theorem 6.3.7 (and simplifying), we see that

$$\varrho : \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_1}, \ell^{q_1}_u\right) \to \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_2}, \ell^{q_2}_v\right), f \mapsto f$$

is well-defined and bounded as long as the condition given in the statement of the present theorem is satisfied. We can then define the bounded linear map $\iota := \Psi^{-1} \circ \varrho \circ \Phi$. It remains to note that

$$\iota f = \Psi^{-1}\left(\varrho\left(\mathcal{F}_{\mathcal{R}} f\right)\right) = \Psi^{-1}\left(\mathcal{F}_{\mathcal{R}} f\right) = \Psi^{-1}\left(\widehat{f}\right) = \mathcal{F}^{-1}\left(\widehat{f}\right) = f$$

holds for all $f \in \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}\left(\mathbb{R}^2 \rtimes H_{c_1}\right)\right) \cap L^2\left(\mathbb{R}^2\right)$, since we have

$$\widehat{f} = \varrho\left(\mathcal{F}_{\mathcal{R}} f\right) = \mathcal{F}_{\mathcal{R}} f \in L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_2}, \ell^{q_2}_v\right)$$

by Plancherel's theorem and by definition of $\mathcal{F}_{\mathcal{R}}$.

For the necessity of the given condition, assume that there is a bounded linear map $\iota$ as in the statement of the theorem. Define $\sigma := \Psi \circ \iota \circ \Phi^{-1}$ and note that this is a bounded linear map

$$\sigma : \mathcal{D}\left(\mathcal{S}^{(c_1)}, L^{p_1}, \ell_u^{q_1}\right) \to \mathcal{D}\left(\mathcal{S}^{(c_2)}, L^{p_2}, \ell_v^{q_2}\right).$$

Now, let $f \in C_c^\infty(\mathbb{R}^* \times \mathbb{R}) \subset L^2(\mathbb{R}^2) \cap \mathcal{D}(\mathcal{S}^{(c_1)}, L^{p_1}, \ell_u^{q_1})$ be arbitrary. Because $\Phi^{-1}, \Psi$ coincide with the usual (inverse) Fourier transform and because of $\iota g = g$ for elements $g$ of the coorbit space/decomposition space that are also $L^2$-functions, we get

$$\sigma f = \Psi(\iota(\Phi^{-1} f)) = \Psi(\iota(\mathcal{F}^{-1} f)) = \Psi(\mathcal{F}^{-1} f) = \mathcal{F}\mathcal{F}^{-1} f = f.$$

Here, we also used

$$\mathcal{F}^{-1} f = \Phi^{-1} f \in L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L^{p_1,q_1}_{w^{(\alpha_1)}}(\mathbb{R}^2 \rtimes H_{c_1})\right),$$

which holds by Plancherel's theorem and by definition of $\Phi$.

Now, recall that the conditions given in Theorem 6.3.7 already hold if the map the $\iota$ given there is bounded when restricted to $C_c^\infty(\mathbb{R}^* \times \mathbb{R})$. But we just showed that this is the case. □

### 6.3.3 *Shearlet type coorbit spaces and Besov spaces*

In this subsection, we investigate the existence of embeddings

$$\mathrm{Co}\left(L_v^{p_1,q_1}(\mathbb{R}^2 \rtimes H_c)\right) \hookrightarrow B^{p_2}_{\alpha,q_2},$$

i.e., embeddings of coorbit spaces of shearlet-type groups into (inhomogeneous) Besov spaces. The existence of the reverse embedding is also treated.

We will first approach this problem in the setting of decomposition spaces. After that, we transfer these results to the setting of coorbit spaces using Corollary 6.3.5 and Lemma 6.2.2.

In the following, we will only consider the case $c \in [0,1]$. Note that this includes the important case $c = 1/2$ of the "ordinary" shearlet group. We restrict our attention to this range, because the shearlet-type covering $\mathcal{S}^{(c)}$ in this case is almost subordinate to the (inhomogeneous) Besov-covering $\mathcal{B}$, as the next lemma shows.

**Lemma 6.3.10.** *Let $c \in [0,1]$ and $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2} = (S^{(c)}_{n,m,\varepsilon})_{(n,m,\varepsilon)\in I}$ with $I = \mathbb{Z}^2 \times \{\pm 1\}$ as defined in Corollary 6.3.5. Then $\mathcal{S}^{(c)}$ is almost subordinate to the covering $\mathcal{B}$ given in Definition 6.2.1.*

*More precisely, define*

$$r^{(c)}_{n,m} := \max\{2^n, 2^{cn}|m|\} \qquad \text{for } (n,m,\varepsilon) \in I.$$

*For arbitrary $(n,m,\varepsilon) \in I$, set*

$$k^{(n,m,\varepsilon)} := \begin{cases} \max\{0,k\} \text{ for } k \in \mathbb{Z} \text{ with } 2^{k-1} < r_{n,m}^{(c)} \leq 2^k, & \text{if } |m| > 2, \\ \max\{0,n\}, & \text{if } |m| \leq 2. \end{cases}$$

*Note that for each $x \in (0,\infty)$, there is a* unique *$k \in \mathbb{Z}$ with $2^{k-1} < x \leq 2^k$. Hence, $k^{(n,m,\varepsilon)}$ is well-defined.*

*These definitions entail $S_{n,m,\varepsilon}^{(c)} \cap B_{k^{(n,m,\varepsilon)}} \neq \varnothing$ as well as*

$$S_{n,m,\varepsilon}^{(c)} \subset B^*_{k^{(n,m,\varepsilon)}}$$

*for all $(n,m,\varepsilon) \in I$.*

*Proof.* Let us first show $S_{n,m,\varepsilon}^{(c)} \subset B^*_{k^{(n,m,\varepsilon)}}$. To this end, let $(n,m,\varepsilon) \in I$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in S_{n,m,\varepsilon}^{(c)}$. By Lemma 6.3.4, this means

$$\begin{pmatrix} w \\ z \end{pmatrix} := \varepsilon \begin{pmatrix} x \\ y \end{pmatrix} \in U^{\left(2^{n-1},2^{n+1}\right)}_{\left(\varrho_c^n(m-1),\varrho_c^n(m+1)\right)}$$

with $\varrho_c = 2^{c-1}$. Hence,

$$w \in (2^{n-1},2^{n+1}) \quad \text{and} \quad \frac{z}{w} \in \left(\varrho_c^n(m-1), \varrho_c^n(m+1)\right),$$

which entails

$$|z| = \left|w \cdot \frac{z}{w}\right| < 2^{n+1}\varrho_c^n(|m|+1) = 2 \cdot 2^{cn} \cdot (|m|+1).$$

Now, we consider two cases:

1. $|m| \leq 2$. In this case, we again split into two subcases:

   a) $n \leq -1$. In this case, we get $0 < w < 2^{n+1} \leq 1$ and $|z| < 6 \cdot 2^{cn} \leq 6$, where we used $c \geq 0$ in the last step. This shows $\left|\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right| = \left|\left(\begin{smallmatrix} w \\ z \end{smallmatrix}\right)\right| < 7$ and thus

   $$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in B_7(0) \subset B_8(0) = B_0 \subset B_0^*.$$

   Finally, observe that we have $k^{(n,m,\varepsilon)} = 0$ in this case.

   b) $n \geq 0$. In this case, we use $0 \leq c \leq 1$ to derive $cn \leq n$ and thus

   $$\begin{aligned} 2^{n-2} &\leq 2^{n-1} < |w| = |x| \\ &\leq \left|\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right| = \left|\left(\begin{smallmatrix} w \\ z \end{smallmatrix}\right)\right| \\ &\leq |w| + |z| < 2^{n+1} + 2 \cdot 2^{cn}(|m|+1) \\ &\leq 2^{n+1} + 6 \cdot 2^{cn} \leq 2 \cdot 2^n + 6 \cdot 2^n = 2^{n+3}, \end{aligned}$$

   which implies $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in B_n \cup B_{n+1}$. But $\left(\begin{smallmatrix} 2^n \\ 0 \end{smallmatrix}\right) \in B_n \cap B_{n+1} \neq \varnothing$ and thus $n+1 \in n^*$. Hence, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in B_n^*$. Note that we have $k^{(n,m,\varepsilon)} = n$ in this case.

2. $|m| \geq 3$. In this case, we have $1 + |m| \leq \frac{4}{3}|m| < 2|m|$ as well as $|m| - 1 \geq \frac{2}{3}|m| > \frac{1}{2}|m|$. Furthermore,

$$\left|\frac{z}{w}\right| > \frac{1}{2}\varrho_c^n |m| .$$

To show this last estimate, we distinguish two cases:

a) $m \leq -3$. In this case, we have

$$\left|\frac{z}{w}\right| \geq -\frac{z}{w} > -\varrho_c^n (m+1) = \varrho_c^n (|m| - 1) > \frac{1}{2}\varrho_c^n |m| .$$

b) $m \geq 3$. Here, we estimate

$$\left|\frac{z}{w}\right| \geq \frac{z}{w} > \varrho_c^n (m-1) = \varrho_c^n (|m| - 1) > \frac{1}{2}\varrho_c^n |m| .$$

Now, let $k \in \mathbb{Z}$ with $2^{k-1} < r_{n,m}^{(c)} \leq 2^k$. The existence of such a $k$ is guaranteed by $r_{n,m}^{(c)} \in (0, \infty)$. We can estimate

$$\begin{aligned}
\left|\begin{pmatrix} w \\ z \end{pmatrix}\right| &\leq 2 \max\{|w|, |z|\} \\
&= 2|w| \cdot \max\left\{1, \left|\frac{z}{w}\right|\right\} \\
&< 2^{n+2} \cdot \max\{1, \varrho_c^n (|m| + 1)\} \\
&\leq 2^{n+2} \cdot \max\{1, 2\varrho_c^n |m|\} \\
&\leq 8 \cdot \max\left\{2^n, 2^n \cdot 2^{(c-1)n} |m|\right\} \\
&= 8 \cdot r_{n,m}^{(c)} \leq 8 \cdot 2^k = 2^{k+3},
\end{aligned}$$

as well as

$$\begin{aligned}
\left|\begin{pmatrix} w \\ z \end{pmatrix}\right| &\geq \max\{|w|, |z|\} \\
&= |w| \cdot \max\left\{1, \left|\frac{z}{w}\right|\right\} \\
&> 2^{n-1} \cdot \max\left\{1, \frac{1}{2}\varrho_c^n |m|\right\} \\
&\geq \frac{1}{4} \cdot \max\left\{2^n, 2^n \cdot 2^{(c-1)n} |m|\right\} \\
&= \frac{1}{4} r_{n,m}^{(c)} > 2^{k-3}.
\end{aligned}$$

Thus, we have shown

$$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \left|\begin{pmatrix} w \\ z \end{pmatrix}\right| \in \bigcup_{\ell=k-1}^{k+1} \left(2^{\ell-2}, 2^{\ell+2}\right).$$

Now, we distinguish two cases:

a) $k \leq 0$. In this case, we get $|\binom{x}{y}| < 2^3 = 8$ and hence $\binom{x}{y} \in B_0 \subset B_0^*$ as above. Note that we also have $k^{(n,m,\varepsilon)} = 0$ in this case.

b) $k > 0$. Here, $\binom{2^k}{0} \in B_{k-1} \cap B_k \cap B_{k+1} \neq \varnothing$, which implies $\binom{x}{y} \in B_{k-1} \cup B_k \cup B_{k+1} \subset B_k^*$. Furthermore, we have $k^{(n,m,\varepsilon)} = k$ in this case.

All in all, we have established that $\mathcal{S}^{(c)}$ is almost subordinate to $\mathcal{B}$, specifically $S^{(c)}_{n,m,\varepsilon} \subset B^*_{k^{(n,m,\varepsilon)}}$ for all $(n,m,\varepsilon) \in I$.

It remains to prove $S^{(c)}_{n,m,\varepsilon} \cap B_{k^{(n,m,\varepsilon)}} \neq \varnothing$. To this end, we again consider two cases:

1. $|m| > 2$. Let $k \in \mathbb{Z}$ with $2^{k-1} < r^{(c)}_{n,m} \leq 2^k$. Lemma 6.3.4 implies

$$\varepsilon \begin{pmatrix} 2^n \\ 2^{cn} m \end{pmatrix} = \varepsilon \begin{pmatrix} 2^n \\ 2^n \varrho_c^n m \end{pmatrix} \in \varepsilon \cdot U^{\left(2^{n-1}, 2^{n+1}\right)}_{\left(\varrho_c^n(m-1), \varrho_c^n(m+1)\right)} = S^{(c)}_{n,m,\varepsilon}$$

with

$$\begin{aligned} 2^{k-2} < 2^{k-1} \leq r^{(c)}_{n,m} &= \max\left\{2^n, 2^{cn}|m|\right\} \\ &\leq \left|\varepsilon \begin{pmatrix} 2^n \\ 2^{cn} m \end{pmatrix}\right| \\ &\leq 2 \max\left\{2^n, 2^{cn}|m|\right\} = 2 r^{(c)}_{n,m} \leq 2^{k+1} < 2^{k+2}. \end{aligned}$$

For $k \geq 1$, this yields $\varepsilon \binom{2^n}{2^{cn} m} \in S^{(c)}_{n,m,\varepsilon} \cap B_k \neq \varnothing$, where we also note $k = k^{(n,m,\varepsilon)}$. In case of $k \leq 0$, we derive

$$\left|\varepsilon \begin{pmatrix} 2^n \\ 2^{cn} m \end{pmatrix}\right| \leq 2^{k+1} \leq 2 < 8$$

and hence $\varepsilon \binom{2^n}{2^{cn} m} \in S^{(c)}_{n,m,\varepsilon} \cap B_0 \neq \varnothing$, which shows the claim in this case because of $k^{(n,m,\varepsilon)} = 0$.

2. $|m| \leq 2$. For $n \geq 1$, we use $0 \leq c \leq 1$ and $n \geq 1 > 0$ to derive $cn \leq n$ and hence

$$2^{n-2} < 2^n \leq \left|\varepsilon \binom{2^n}{2^{cn} m}\right| \leq 2^n + 2^{cn}|m| \leq 2^n (1+2) < 2^{n+2},$$

which shows $\varepsilon \binom{2^n}{2^{cn} m} \in S^{(c)}_{n,m,\varepsilon} \cap B_n \neq \varnothing$. Because of $k^{(n,m,\varepsilon)} = n$, this proves the claim in this case.

Finally, for $n \leq 0$, we have $2^n \leq 1$ and $2^{cn} \leq 2^0 = 1$ because of $c \geq 0$. Hence,

$$\left|\varepsilon \binom{2^n}{2^{cn} m}\right| \leq \sqrt{1 + |m|^2} \leq \sqrt{5} < 3 < 8$$

and thus $\varepsilon \binom{2^n}{2^{cn} m} \in S^{(c)}_{n,m,\varepsilon} \cap B_0 \neq \varnothing$, which completes the proof because of $k^{(n,m,\varepsilon)} = 0$. □

We now use this almost subordinateness to prove embeddings of decomposition spaces with respect to the shearlet-type coverings $\mathcal{S}^{(c)}$ into the Fourier-side Besov spaces and vice versa. For reasons to be explained later, we use weights of the form

$$w_{n,m,\varepsilon}^{(\alpha,\beta)} := \|A_{n,m,\varepsilon}^{(c)}\|^{\alpha} \cdot |\det(A_{n,m,\varepsilon}^{(c)})|^{\beta} = \|A_{n,m,\varepsilon}^{(c)}\|^{\alpha} \cdot 2^{n(1+c)\beta} \qquad (6.3.14)$$

for the shearlet-type covering $\mathcal{S}^{(c)}$. Since all norms on the finite dimensional vector space $\mathbb{R}^{2\times 2}$ are equivalent and equivalent weights yield equivalent quasi-norms, we are free to choose our favourite matrix norm in equation (6.3.14).

We remark that the above weight depends (at least for $\alpha \neq 0$) also on the "shearing parameter" $m$ of the matrix $A_{n,m,\varepsilon}^{(c)}$. This is in sharp contrast to the weight $v^{(\alpha)}$ considered in Theorem 6.3.7, which only depends on the parameter $n$.

We now want to show that the above weight is a discretization of a transplant of some right-moderate weight on the shearlet-type group $H_c$. For this, we first need the following lemma.

**Lemma 6.3.11.** *Let $H$ be a topological group. If $v : H \to (0,\infty)$ is right-moderate with respect to the measurable, [submultiplicative], locally bounded weight $v_0 : H \to (0,\infty)$, then*

$$v^{-1} : H \to (0,\infty), h \mapsto (v(h))^{-1}$$

*is right-moderate with respect to $v_0^{\vee} : H \to (0,\infty), h \mapsto v_0\left(h^{-1}\right)$, which is a measurable, [submultiplicative] and locally bounded weight.*

*Proof.* Let $x, y \in G$ be arbitrary. Set $g := xy$ and $h := y^{-1}$. Right-moderateness of $v$ with respect to $v_0$ yields

$$v(x) = v(xyy^{-1}) = v(gh) \leq v(g) \cdot v_0(h) = v(xy) \cdot v_0(y^{-1}).$$

By rearranging, this implies

$$v^{-1}(xy) = (v(xy))^{-1} \leq (v(x))^{-1} \cdot v_0(y^{-1}) = v^{-1}(x) \cdot v_0^{\vee}(y).$$

Hence, $v^{-1}$ is right-moderate with respect to $v_0^{\vee}$.

It is clear that $v_0^{\vee}$ is measurable and locally bounded. Finally, if $v_0$ is submultiplicative, then the same holds for $v_0^{\vee}$, because of

$$\begin{aligned} v_0^{\vee}(xy) &= v_0\left((xy)^{-1}\right) = v_0(y^{-1}x^{-1}) \\ &\leq v_0(y^{-1})\, v_0(x^{-1}) = v_0^{\vee}(x)\, v_0^{\vee}(y). \end{aligned}$$

□

The above lemma shows that the weight $H_c \to (0,\infty), h \mapsto \|h\|^{\alpha}$ is moderate with respect to some measurable, submultiplicative, locally bounded weight on $H_c$. For $\alpha \geq 0$, this follows by submultiplicativity

of $\|\cdot\|$ and for $\alpha < 0$, we use the above lemma. Since the determinant is multiplicative, the weight

$$m^{(\alpha,\beta)} : H_c \to (0,\infty), h \mapsto \|h^{-1}\|^{\alpha} \cdot |\det(h^{-1})|^{\beta} \tag{6.3.15}$$

is also moderate with respect to some measurable, submultiplicative, locally bounded weight on $H_c$.

Now, fix some $\xi_0 \in U_{(-1,1)}^{\left(\frac{1}{2},2\right)}$. We have

$$\left(B_{-n,-m,\varepsilon}^{(c)}\right)^{-T} \xi_0 = A_{n,m,\varepsilon}^{(c)} \xi_0 \in S_{n,m,\varepsilon}^{(c)} \quad ,$$

so that Definitions 4.4.7 and 3.2.4 imply that

$$\begin{aligned}
(n,m,\varepsilon) &\mapsto m^{(\alpha,\beta)}\left(B_{-n,-m,\varepsilon}^{(c)}\right) \\
&= m^{(\alpha,\beta)}\left((A_{n,m,\varepsilon}^{(c)})^{-T}\right) \\
&= \left\|(A_{n,m,\varepsilon}^{(c)})^{T}\right\|^{\alpha} \cdot \left|\det\left((A_{n,m,\varepsilon}^{(c)})^{T}\right)\right|^{\beta} \\
&\asymp \left\|A_{n,m,\varepsilon}^{(c)}\right\|^{\alpha} \cdot |\det(A_{n,m,\varepsilon}^{(c)})|^{\beta} = w_{n,m,\varepsilon}^{(\alpha,\beta)}
\end{aligned}$$

is (equivalent to) an $\mathcal{S}^{(c)}$-discretization of a $\xi_0$-transplant of $m^{(\alpha,\beta)}$ onto $\mathcal{O}$. Here, the last step used $\|A^T\| \asymp \|A\|$ for all $A \in \mathbb{R}^{2\times 2}$, which is true, since all norm on the finite dimensional space $\mathbb{R}^{2\times 2}$ are equivalent.

Now, a combination of the Lemmata 4.4.8 and 3.2.5, together with the fact that $\mathcal{S}^{(c)}$ is an induced covering of the group $H_c$ (which was shown in Corollary 6.3.5), shows that the weight $w^{(\alpha,\beta)}$ given in equation (6.3.14) is indeed $\mathcal{S}^{(c)}$-moderate.

The next theorem gives sufficient conditions for embeddings between shearlet-type decomposition spaces and Besov-type decomposition spaces. Necessary conditions for the existence of the embedding are also provided, but due to missing moderateness of the covering $\mathcal{S}^{(c)}$ and of the weight $w^{(\alpha,\beta)}$, these conditions are (in some cases) weaker than the corresponding sufficient conditions.

**Theorem 6.3.12.** *Let $c \in (0,1]$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0,\infty]$.*

*Let the weight $w^{(\alpha,\beta)} := (w_i^{(\alpha,\beta)})_{i\in I}$ on $I = \mathbb{Z}^2 \times \{\pm 1\}$ be given by equation (6.3.14). Let $v^{(\gamma)}$ be the weight on $\mathbb{N}_0$ as defined in Lemma 6.2.2.*

*Set*

$$\begin{aligned}
\gamma^{(1)} &:= \alpha + (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \beta\right) + (1-c)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+, \\
\gamma^{(2)} &:= \alpha - (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} - \beta\right) + (c-1)\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+
\end{aligned}$$

*and*

$$\alpha^{(1)} := \frac{1+c}{c} \cdot \left( \frac{1}{p_2} - \frac{1}{p_1} - (p_1)_d - \beta \right),$$
$$\alpha^{(2)} := \frac{1+c}{c} \cdot \left( \frac{1}{p_1} - \frac{1}{p_2} - \beta \right).$$

*Then,*

$$\iota : \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\gamma)}}\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{w^{(\alpha,\beta)}}\right), f \mapsto f|_{C_c^\infty(\mathbb{R}^* \times \mathbb{R})}$$

*is well-defined and bounded if we have $p_1 \leq p_2$, as well as*

$$\begin{cases} \min\left\{\alpha, \frac{1}{p_1^\triangle} - \frac{1}{q_2}\right\} > \alpha^{(1)}, & \text{if } p_1^\triangle > q_2, \\ \min\{\alpha, 0\} \geq \alpha^{(1)}, & \text{if } p_1^\triangle \leq q_2 \end{cases} \tag{6.3.16}$$

*and*

$$\begin{cases} \gamma \geq \gamma^{(1)}, & \text{if } q_1 \leq q_2, \\ \gamma > \gamma^{(1)}, & \text{if } q_1 > q_2. \end{cases} \tag{6.3.17}$$

*Note that we get a true characterization of the existence of an embedding for $p_1 \in (0,1] \cup [2,\infty]$. For $p_1 \in (0,1]$ this is clear and for $p_1 \in [2,\infty]$, we have $p_1^\triangle = p_1^{\pm\triangle} = p_1$.*

*These conditions are also necessary for boundedness of the map*

$$\left(C_c^\infty(\mathbb{R}^2), \|\cdot\|_{\mathcal{D}(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\gamma)}})}\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{w^{(\alpha,\beta)}}\right), f \mapsto f \tag{6.3.18}$$

*as long as we have $p_1 \in (0,1]$ or as long as all occurrences of $p_1^\triangle$ and of $p_1^{\pm\Delta}$ are replaced by $p_1$ (also in the definition of $\gamma^{(1)}$).*

*Conversely, there is a bounded linear map*

$$\vartheta : \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{w^{(\alpha,\beta)}}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\gamma)}}\right)$$

*with the three properties given in part 1 of Theorem 5.4.4, as long as we have $p_1 \leq p_2$ and*

$$\begin{cases} \max\left\{\alpha, \frac{1}{p_2^\nabla} - \frac{1}{q_1}\right\} < \alpha^{(2)}, & \text{if } q_1 > p_2^\nabla, \\ \max\{\alpha, 0\} \leq \alpha^{(2)}, & \text{if } q_1 \leq p_2^\nabla, \end{cases} \tag{6.3.19}$$

*as well as*

$$\begin{cases} \gamma \leq \gamma^{(2)}, & \text{if } q_1 \leq q_2, \\ \gamma < \gamma^{(2)}, & \text{if } q_1 > q_2. \end{cases} \tag{6.3.20}$$

*Note that we get a true characterization of the existence of an embedding for the range $p_2 \in (0,2]$, since we have $p_2^\nabla = p_2$ in this case.*

*Again, the conditions given in equations* (6.3.19) *and* (6.3.20) *as well as $p_1 \leq p_2$ are also necessary for boundedness of the map*

$$\left(C_c^\infty(\mathbb{R}^* \times \mathbb{R}), \|\cdot\|_{\mathcal{D}(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{w^{(\alpha,\beta)}})}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\gamma)}}\right), f \mapsto f, \tag{6.3.21}$$

*as long as all occurrences of $p_2^\nabla$ in equations* (6.3.19) *and* (6.3.20) *and in the definition of $\gamma^{(2)}$ are replaced by $p_2$.*

*Proof.* To establish the second claim, we use parts 1 and 2 of Theorem 5.4.4, with $\mathcal{Q} = \mathcal{S}^{(c)}$, $\mathcal{P} = \mathcal{B}$, as well as $u = w^{(\alpha,\beta)}$ and $v = v^{(\gamma)}$. Lemma 6.3.10 shows that the requirement of Theorem 5.4.4 that $\mathcal{Q}$ be almost subordinate to $\mathcal{P}$ is satisfied. Furthermore, Lemma 6.2.2 and Corollary 6.3.5, together with the considerations regarding the weight $w^{(\alpha,\beta)}$ from above show that $\mathcal{Q}, \mathcal{P}$ are both tight semi-structured $L^p$-decomposition coverings of $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$ and $\mathcal{O}' = \mathbb{R}^2$, respectively, for all $p \in (0, \infty]$ and that $u$ and $v$ are moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.

Thus, Theorem 5.4.4 shows that we are interested in finiteness of the quasi-norm

$$\begin{aligned}
&\left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\theta} \right)_{i \in I_j} \right\|_{\ell^s} \right)_{j \in J} \right\|_{\ell^t_v} \\
&= \left\| \left( \left\| \left( \left( w_i^{(\alpha,\beta)} \right)^{-1} \cdot |\det(T_i)|^{\theta} \right)_{i \in I_j} \right\|_{\ell^s} \right)_{j \in J} \right\|_{\ell^t_{v^{(\gamma)}}}
\end{aligned} \tag{6.3.22}$$

for $t = q_2 \cdot (q_1/q_2)'$, $\theta = \frac{1}{p_1} - \frac{1}{p_2}$ and varying choices of $s \in (0, \infty]$. For later use, we will consider this quasi-norm for general $\theta \in \mathbb{R}$ instead for the specific choice $\theta = \frac{1}{p_1} - \frac{1}{p_2}$.

Lemma 5.2.5 (with $I_0 = I = \mathbb{Z}^2 \times \{\pm 1\}$) and the definition of $u = w^{(\alpha,\beta)}$ in equation (6.3.14) show that finiteness of the above quasi-norm is equivalent to finiteness of

$$\begin{aligned}
&\left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\theta} \right)_{i \in I^{(j)}} \right\|_{\ell^s} \right)_{j \in J} \right\|_{\ell^t_v} \\
&= \left\| \left( \left\| \left( \left\| A^{(c)}_{n,m,\varepsilon} \right\|^{-\alpha} \cdot 2^{n(1+c)(\theta-\beta)} \right)_{(n,m,\varepsilon) \in I^{(k)}} \right\|_{\ell^s} \right)_{k \in \mathbb{N}_0} \right\|_{\ell^t_v} \\
&=: \left\| \left( \gamma_k^{(s,\theta)} \right)_{k \in \mathbb{N}_0} \right\|_{\ell^t_v},
\end{aligned} \tag{6.3.23}$$

where for each $i \in I = \mathbb{Z}^2 \times \{\pm 1\}$, some index $j^{(i)} \in J = \mathbb{N}_0$ with $\varnothing \neq Q_i \cap P_{j^{(i)}} = S_i^{(c)} \cap B_{j^{(i)}}$ is chosen and where

$$I^{(j)} = \{ i \in I \mid j^{(i)} = j \} \text{ for } j \in J = \mathbb{N}_0.$$

Lemma 6.3.10 shows that we can choose $j^{(i)} := k^{(n,m,\varepsilon)}$ for arbitrary $i = (n, m, \varepsilon) \in I = \mathbb{Z}^2 \times \{\pm 1\}$. More explicitly, this means

$$j^{(i)} = \begin{cases} \max\{0, k\} \text{ for } k \in \mathbb{Z} \text{ with } 2^{k-1} < r^{(c)}_{n,m} \leq 2^k, & \text{if } |m| > 2, \\ \max\{0, n\}, & \text{if } |m| \leq 2, \end{cases}$$

with $r^{(c)}_{n,m} := \max\{2^n, 2^{cn}\,|m|\}$. Thus, our first goal is to calculate the sets

$$I^{(k)} = \left\{(n,m,\varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\} \,\middle|\, j^{(n,m,\varepsilon)} = k\right\}$$

for $k \in \mathbb{N}_0$.

We claim

$$I^{(0)} = \left\{ \begin{array}{l} \{(n,m,\varepsilon) \in I \,|\, n \leq 0 \text{ and } 2 < |m| \leq 2^{-cn}\} \\ \uplus\, [\mathbb{Z}_{\leq 0} \times \{-2,\ldots,2\} \times \{\pm 1\}] \end{array} \right. \tag{6.3.24}$$

and

$$I^{(k)} = \left\{ \begin{array}{l} [\{k\} \times \{-2,\ldots,2\} \times \{\pm 1\}] \\ \uplus \left\{(n,m,\varepsilon) \in I \,\middle|\, n = k \text{ and } 2 < |m| \leq 2^{(1-c)n}\right\} \\ \uplus \left\{(n,m,\varepsilon) \in I \,\middle|\, n < k \text{ and } \frac{1}{2}\max\left\{4, \gamma^{(c)}_{k,n}\right\} < |m| \leq \gamma^{(c)}_{k,n}\right\} \end{array} \right. \tag{6.3.25}$$

for $k \geq 1$, with $\gamma^{(c)}_{k,n} := 2^{k-cn}$. It is easy to see that all unions indicated above are indeed disjoint, so that we only have to prove the claimed equality.

Now, let $(n,m,\varepsilon) \in I$ with $|m| > 2$. Using $r^{(c)}_{n,m} = \max\{2^n, 2^{cn}\,|m|\}$, we see that the following are equivalent for each $k \in \mathbb{Z}$:

*Here, $\wedge$ denotes the logical "and" and $\vee$ denotes the logical "or".*

$$\begin{aligned}
& 2^{k-1} < r^{(c)}_{n,m} \leq 2^k \\
\Leftrightarrow\ & \left[2^{k-1} < 2^n \,\vee\, 2^{k-1} < 2^{cn}\,|m|\right] \,\wedge\, \left[2^n \leq 2^k \,\wedge\, 2^{cn}\,|m| \leq 2^k\right] \\
\Leftrightarrow\ & \left[2^{k-1} < 2^n \,\wedge\, 2^n \leq 2^k \,\wedge\, 2^{cn}\,|m| \leq 2^k\right] \\
& \quad \vee\, \left[2^{k-1} < 2^{cn}\,|m| \,\wedge\, 2^n \leq 2^k \,\wedge\, 2^{cn}\,|m| \leq 2^k\right] \\
\Leftrightarrow\ & \left[k = n \,\wedge\, |m| \leq 2^{k-cn}\right] \,\vee\, \left[n \leq k \,\wedge\, 2^{k-cn-1} < |m| \leq 2^{k-cn}\right] \\
\overset{(\dagger)}{\Leftrightarrow}\ & \left[k = n \,\wedge\, 2 < |m| \leq 2^{(1-c)n}\right] \\
& \quad \vee\, \left[n \leq k \,\wedge\, \frac{1}{2}\max\left\{4, \gamma^{(c)}_{k,n}\right\} < |m| \leq \gamma^{(c)}_{k,n}\right] \\
\overset{(\ddagger)}{\Leftrightarrow}\ & \left[k = n \,\wedge\, 2 < |m| \leq 2^{(1-c)n}\right] \\
& \quad \vee\, \left[n < k \,\wedge\, \frac{1}{2}\max\left\{4, \gamma^{(c)}_{k,n}\right\} < |m| \leq \gamma^{(c)}_{k,n}\right].
\end{aligned} \tag{6.3.26}$$

The step marked with (†) used $|m| > 2$. Finally, to see that (‡) is true, observe that $n = k$ together with $\frac{1}{2}\max\left\{4, \gamma^{(c)}_{k,n}\right\} < |m| \leq \gamma^{(c)}_{k,n}$ already implies $2 < |m| \leq 2^{(1-c)n}$.

Using the equivalence (6.3.26), we will now prove the above expressions for $I^{(k)}$. To this end, we distinguish two cases:

CASE 1: We have $k \geq 1$. In this case, elementary considerations and an application of equivalence (6.3.26) (marked with (†)) yield

$$\begin{aligned}
& k^{(n,m,\varepsilon)} = k \\
\iff & \left[|m| > 2 \text{ and } 2^{k-1} < r_{n,m}^{(c)} \leq 2^k\right] \text{ or } [|m| \leq 2 \text{ and } n = k] \\
\overset{(\dagger)}{\iff} & \left[k = n \text{ and } 2 < |m| \leq 2^{(1-c)n}\right] \\
& \text{or } \left[n < k \text{ and } \frac{1}{2}\max\left\{4, \gamma_{k,n}^{(c)}\right\} < |m| \leq \gamma_{k,n}^{(c)}\right]. \\
& \text{or } [(n,m,\varepsilon) \in \{k\} \times \{-2,\dots,2\} \times \{\pm 1\}]
\end{aligned}$$

for all $(n,m,\varepsilon) \in I$. This completes the proof for the claimed form of $I^{(k)}$ in this case.

CASE 2: We have $k = 0$. By definition of $j^{(i)} = k^{(n,m,\varepsilon)}$, the following are equivalent for all $(n,m,\varepsilon) \in I$:

$$\begin{aligned}
& k^{(n,m,\varepsilon)} = 0 \\
\iff & \left[|m| > 2 \text{ and } 2^{\ell-1} < r_{n,m}^{(c)} \leq 2^{\ell} \text{ for some } \ell \leq 0\right] \\
& \text{or } [|m| \leq 2 \text{ and } n \leq 0] \\
\iff & \left[|m| > 2 \text{ and } r_{n,m}^{(c)} \leq 1\right] \\
& \text{or } [(n,m,\varepsilon) \in \mathbb{Z}_{\leq 0} \times \{-2,\dots,2\} \times \{\pm 1\}] \\
\iff & [|m| > 2 \text{ and } 2^n \leq 1 \text{ and } 2^{cn}\,|m| \leq 1] \\
& \text{or } [(n,m,\varepsilon) \in \mathbb{Z}_{\leq 0} \times \{-2,\dots,2\} \times \{\pm 1\}] \\
\iff & \left[|m| > 2 \text{ and } n \leq 0 \text{ and } |m| \leq 2^{-cn}\right] \\
& \text{or } [(n,m,\varepsilon) \in \mathbb{Z}_{\leq 0} \times \{-2,\dots,2\} \times \{\pm 1\}].
\end{aligned}$$

This establishes the claimed form of $I^{(k)}$ in this case.

Now, we estimate the sequence $(\gamma_k^{(s,\theta)})_{k \in \mathbb{N}_0}$ introduced in equation (6.3.23). As a first step, we estimate $\|A_{n,m,\varepsilon}^{(c)}\|$ for $(n,m,\varepsilon) \in I^{(k)}$ for each $k \in \mathbb{N}_0$. In the following, we write $A \lesssim B$ if we have $A \leq CB$, where $C$ only depends upon $\alpha, \beta, c, p_1, p_2, q_1, s, \theta$. We will write $A \asymp B$ for $A \lesssim B \lesssim A$. We recall the definition

$$A_{n,m,\varepsilon}^{(c)} = \varepsilon \cdot \begin{pmatrix} 2^n & 0 \\ 2^{nc} m & 2^{nc} \end{pmatrix}$$

given in Lemma 6.3.2. In order to estimate the norm $\|A_{n,m,\varepsilon}^{(c)}\|$, we distinguish two cases:

CASE 1: We have $k \in \mathbb{N}$. Using equation (6.3.25), we see that there are three subcases:

1. $n = k \geq 1$ and $|m| \leq 2$. Using $0 \leq nc \leq n$, this implies

$$2^k = 2^n \lesssim \|A_{n,m,\varepsilon}^{(c)}\| \lesssim \max\left\{2^n, 2^{nc}\,|m|\,, 2^{nc}\right\} \lesssim 2^n = 2^k.$$

2. $n = k$ and $2 < |m| \leq 2^{(1-c)n}$. In this case, we use $0 \leq nc \leq n$ to derive

$$\begin{aligned} 2^k = 2^n &\lesssim \|A^{(c)}_{n,m,\varepsilon}\| \lesssim \max\left\{2^n, 2^{nc}\,|m|\,, 2^{nc}\right\} \\ &\leq \max\left\{2^n, 2^{nc} \cdot 2^{(1-c)n}\right\} = 2^n = 2^k. \end{aligned}$$

3. $n < k$ and $\frac{1}{2}\max\left\{4, \gamma^{(c)}_{k,n}\right\} < |m| \leq \gamma^{(c)}_{k,n}$. Here, we first remark that $n \leq 0$ implies $nc \leq 0$ and hence $2^{nc} \leq 1 \leq 2^k$. Similarly, $n \geq 0$ yields $cn \leq n < k$ and hence also $2^{nc} \leq 2^k$. Using this, as well as $\gamma^{(c)}_{k,n} = 2^{k-nc}$, we arrive at

$$\begin{aligned} 2^k \lesssim \frac{2^k}{2} = 2^{nc} \cdot \frac{\gamma^{(c)}_{k,n}}{2} &< 2^{nc}\,|m| \lesssim \|A^{(c)}_{n,m,\varepsilon}\| \\ &\lesssim \max\left\{2^n, 2^{nc}\,|m|\,, 2^{nc}\right\} \leq \max\left\{2^k, 2^{nc}2^{k-nc}, 2^k\right\} = 2^k. \end{aligned}$$

We have thus shown

$$\|A^{(c)}_{n,m,\varepsilon}\| \asymp 2^k \text{ for all } (n,m,\varepsilon) \in I^{(k)} \text{ and } k \in \mathbb{N}. \tag{6.3.27}$$

CASE 2: We have $k = 0$. Here, equation (6.3.24) shows that we must have $n \leq 0$ and thus $n \leq nc$, which yields $2^n \leq 2^{nc}$. Appealing to equation (6.3.24) again, we see that there are two cases:

1. $n \leq 0$ and $2 < |m| \leq 2^{-nc}$. In this case, we get

$$2^{nc}\,|m| \lesssim \|A^{(c)}_{n,m,\varepsilon}\| \lesssim \max\left\{2^n, 2^{nc}\,|m|\,, 2^{nc}\right\} \leq 2^{nc}\,|m|\,. \tag{6.3.28}$$

2. $n \leq 0$ and $|m| \leq 2$. Here, we derive

$$2^{nc} \lesssim \|A^{(c)}_{n,m,\varepsilon}\| \lesssim \max\left\{2^n, 2^{nc}\,|m|\,, 2^{nc}\right\} \lesssim 2^{nc}. \tag{6.3.29}$$

Using these estimates for the norm $\|A^{(c)}_{n,m,\varepsilon}\|$, we will now (asymptotically) evaluate $(\gamma^{(s,\theta)}_k)_{k\in\mathbb{N}_0}$. For brevity, we set $\nu := \theta - \beta$. Let us start with $\gamma^{(s,\theta)}_0$. Since this is a single term, we only characterize finiteness of $\gamma^{(s,\theta)}_0$ instead of calculating the asymptotic behaviour. In case of $s < \infty$, we use equations (6.3.24), (6.3.28) and (6.3.29) to derive

$$\begin{aligned} &\left(\gamma^{(s,\theta)}_0\right)^s \\ &\asymp \sum_{n=-\infty}^{0} \left[2^{-nc\alpha} \cdot 2^{n(1+c)\nu}\right]^s + \sum_{n=-\infty}^{0} \sum_{m=3}^{\lfloor 2^{-cn}\rfloor} \left[2^{-nc\alpha} m^{-\alpha} 2^{n(1+c)\nu}\right]^s \\ &= \sum_{n=-\infty}^{0} 2^{(-n)\cdot s[c\alpha-(1+c)\nu]} + \sum_{n=-\infty}^{0} \left[2^{(-n)\cdot s[c\alpha-(1+c)\nu]} \sum_{m=3}^{\lfloor 2^{-cn}\rfloor} m^{-\alpha s}\right]. \end{aligned} \tag{6.3.30}$$

Here, the first series is finite if and only if we have $c\alpha - (1+c)\,\nu < 0$.

The second series is finite if and only if the same is true for the series $\sum_{n=-\infty}^{-\lceil 2/c \rceil}$. In the range of this sum, we have $n \leq -\lceil 2/c \rceil \leq -2/c$ and thus $2^{-cn} \geq 2^{c\frac{2}{c}} = 4$.

Let us now study the series $\sum_{m=3}^{\ell} m^{\varrho}$ for $\ell \geq 4$ and $\varrho := -\alpha s \in \mathbb{R}$. There are two main cases:

1. $\varrho < -1$. This implies $3^{\varrho} \leq \sum_{m=3}^{\ell} m^{\varrho} \leq \sum_{m=3}^{\infty} m^{\varrho} < \infty$ and hence $\sum_{m=3}^{\ell} m^{\varrho} \asymp 1$.

2. $\varrho \geq -1$. For $m \geq 3$ and $m-1 \leq x \leq m$, we have

$$\frac{2}{3}m \leq m-1 \leq x \leq m$$

and thus $m^{\varrho} \asymp x^{\varrho}$. Using an integral comparison, this yields

$$\sum_{m=3}^{\ell} m^{\varrho} = \sum_{m=3}^{\ell} \int_{m-1}^{m} m^{\varrho} \, \mathrm{d}x \asymp \sum_{m=3}^{\ell} \int_{m-1}^{m} x^{\varrho} \, \mathrm{d}x = \int_{2}^{\ell} x^{\varrho} \, \mathrm{d}x.$$

In case of $\varrho = -1$, we get

$$\int_{2}^{\ell} x^{\varrho} \, \mathrm{d}x = \ln(\ell) - \ln(2) = \ln\left(\frac{\ell}{2}\right).$$

In contrast, for $\varrho > -1$, we have $2^{1+\varrho} \leq \left(\frac{\ell}{2}\right)^{1+\varrho}$ (recall $\ell \geq 4$) and $(1/2)^{1+\varrho} < 1^{\varrho} = 1$. Thus,

$$\ell^{1+\varrho} \geq \ell^{1+\varrho} - 2^{1+\varrho} \geq \ell^{1+\varrho} \cdot \left[1 - (1/2)^{1+\varrho}\right] \gtrsim \ell^{1+\varrho},$$

which implies

$$\int_{2}^{\ell} x^{\varrho} \, \mathrm{d}x = \frac{x^{1+\varrho}}{1+\varrho}\bigg|_{2}^{\ell} \asymp \ell^{1+\varrho} - 2^{1+\varrho} \asymp \ell^{1+\varrho}.$$

We saw above that $4 \leq 2^{-cn} < \lfloor 2^{-cn} \rfloor + 1$ holds for all $n$ in the range of the series $\sum_{n=-\infty}^{-\lceil 2/c \rceil}$. But this entails

$$2^{-cn} \geq \lfloor 2^{-cn} \rfloor \geq 2^{-cn} - 1 \geq \frac{3}{4}2^{-cn} > 0,$$

which means $2^{-cn} \asymp \lfloor 2^{-cn} \rfloor$. Furthermore, we also get

$$\frac{4}{5}\ln\left(2^{-cn}\right) - \ln\frac{4}{3} - \ln 2 \geq \frac{4}{5}\ln 4 - \ln\frac{4}{3} - \ln 2 \geq \frac{1}{10} > 0$$

and thus

$$\frac{\ln(2^{-cn})}{5} \leq \ln\left(2^{-cn}\right) - \ln\left(\frac{4}{3}\right) - \ln 2 = \ln\left(\frac{3}{4}2^{-cn}\right) - \ln 2$$

$$\leq \ln\left(\frac{\lfloor 2^{-cn}\rfloor}{2}\right) \leq \ln\left(\lfloor 2^{-cn}\rfloor\right) \leq \ln\left(2^{-cn}\right).$$

In summary, we arrive at

$$\sum_{m=3}^{\lfloor 2^{-cn}\rfloor} m^{-\alpha s} \asymp \begin{cases} 1, & \text{if } \alpha s > 1, \\ \ln\left(2^{-cn}\right) \asymp -cn, & \text{if } \alpha s = 1, \\ \left(2^{-cn}\right)^{1-\alpha s}, & \text{if } \alpha s < 1 \end{cases}$$

for all $n \leq -\lceil 2/c \rceil$.

Thus, the second series in equation (6.3.30) is finite if and only if we have

$$\begin{cases} \sum_{n=-\infty}^{-\lceil 2/c\rceil} 2^{(-n)s[c\alpha-(1+c)\nu]} < \infty, & \text{if } \alpha s > 1, \\ \sum_{n=-\infty}^{-\lceil 2/c\rceil} (-n)\, 2^{(-n)s[c\alpha-(1+c)\nu]} < \infty, & \text{if } \alpha s = 1, \\ \sum_{n=-\infty}^{-\lceil 2/c\rceil} 2^{(-n)s\left[c\left(\frac{1}{s}-\alpha\right)+c\alpha-(1+c)\nu\right]} < \infty, & \text{if } \alpha s < 1. \end{cases}$$

Because of $s > 0$, this holds if and only if we have

$$\begin{cases} c\alpha - (1+c)\,\nu < 0 & \text{if } \alpha s > 1, \\ c\alpha - (1+c)\,\nu < 0, & \text{if } \alpha s = 1, \\ \frac{c}{s} - (1+c)\,\nu < 0, & \text{if } \alpha s < 1, \end{cases}$$

which in turn is equivalent to

$$c \cdot \max\left\{s^{-1}, \alpha\right\} < (1+c)\,\nu. \tag{6.3.31}$$

In particular, this inequality implies the condition $c\alpha - (1+c)\,\nu < 0$ which was shown above to be equivalent to the convergence of the first series in equation (6.3.30). In summary, for $s < \infty$, finiteness of $\gamma_0^{(s,\theta)}$ is equivalent to the condition given in equation (6.3.31).

In case of $s = \infty$, equations (6.3.24), (6.3.28) and (6.3.29) yield

$$\gamma_0^{(s,\theta)} \asymp \max\left\{\sup_{n\leq 0}\left[2^{-nc\alpha}\cdot 2^{n(1+c)\nu}\right], \sup_{n\leq 0}\sup_{2<|m|\leq 2^{-nc}} 2^{-nc\alpha}\,|m|^{-\alpha}\,2^{n(1+c)\nu}\right\}.$$

The first term of the maximum is finite if and only if $c\alpha - (1+c)\,\nu \leq 0$ holds. For the analysis of the second term, we note that we only want to know whether $\gamma_0^{(s,\theta)}$ is finite or infinite. Thus, we can discard finitely many terms and only consider the supremum $\sup_{n\leq -\lceil 2/c\rceil}$ instead of $\sup_{n\leq 0}$. As seen above, we have $2^{-nc} \geq 4$ for $n \leq -\lceil 2/c\rceil$. Now, we distinguish two cases:

1. $\alpha \geq 0$. In this case, $\sup_{2<|m|\leq 2^{-nc}} |m|^{-\alpha} = 3^{-\alpha} \asymp 1$, so that the second (and also the first) term of the maximum is finite if and only if we have

$$c \cdot \max\{s^{-1}, \alpha\} = c\alpha \overset{!}{\leq} (1+c)\,\nu.$$

2. $\alpha < 0$. In this case, we have

$$\sup_{2<|m|\leq 2^{-nc}} |m|^{-\alpha} = \left\lfloor 2^{-nc} \right\rfloor^{-\alpha} \asymp 2^{nc\alpha}$$

for small $n \in \mathbb{Z}_{\leq 0}$ (because of $c > 0$), which shows that the second term of the maximum is bounded above and below by $\sup_{n\leq 0} 2^{(-n)\cdot[-(1+c)\nu]}$. This is a finite number if and only if we have

$$c \cdot \max\{s^{-1}, \alpha\} = 0 \overset{!}{\leq} (1+c)\,\nu,$$

which also implies $c\alpha - (1+c)\,\nu < -(1+c)\,\nu \leq 0$ and thus finiteness of the first term of the maximum.

In summary, we have shown

$$\gamma_0^{(s,\theta)} < \infty \quad \Longleftrightarrow \quad \begin{cases} c \cdot \max\left\{\alpha, \frac{1}{s}\right\} < (1+c)\,\nu, & \text{if } s < \infty, \\ c \cdot \max\left\{\alpha, \frac{1}{s}\right\} \leq (1+c)\,\nu, & \text{if } s = \infty. \end{cases} \tag{6.3.32}$$

It remains to estimate $\gamma_k^{(s,\theta)}$ for $k \in \mathbb{N}$. Here, we are interested in the asymptotic behaviour and not only in finiteness. Let us first consider the case $s < \infty$. Set

$$\kappa(k,n) := \left\lfloor \frac{1}{2} \max\left\{4, \gamma_{k,n}^{(c)}\right\} \right\rfloor + 1 \geq 3$$

for $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Equations (6.3.25) and (6.3.27) imply

$$\begin{aligned} &\left(\gamma_k^{(s,\theta)}\right)^s \\ &\asymp \left(2^{k((1+c)\nu-\alpha)}\right)^s + \sum_{m=3}^{\left\lfloor 2^{(1-c)k} \right\rfloor} \left(2^{k((1+c)\nu-\alpha)}\right)^s + \sum_{n=-\infty}^{k-1} \sum_{m=\kappa(k,n)}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} \left(2^{-k\alpha} 2^{n(1+c)\nu}\right)^s. \end{aligned} \tag{6.3.33}$$

Let us first estimate the last series. For $n \leq \frac{k-2}{c}$, we have

$$\gamma_{k,n}^{(c)} = 2^{k-nc} \geq 2^{k-c\frac{k-2}{c}} = 4,$$

which implies

$$\frac{1}{2} \max\left\{4, \gamma_{k,n}^{(c)}\right\} = \frac{\gamma_{k,n}^{(c)}}{2},$$

so that the sum

$$\sum_{m=\kappa(k,n)}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} = \sum_{m=\left\lfloor \frac{1}{2}\max\left\{4,\gamma_{k,n}^{(c)}\right\}\right\rfloor+1}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} = \sum_{m=\left\lfloor \gamma_{k,n}^{(c)}/2\right\rfloor+1}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor}$$

has exactly

$$\begin{aligned}\lfloor \gamma_{k,n}^{(c)} \rfloor - \lfloor \gamma_{k,n}^{(c)}/2 \rfloor &\geq \lfloor \gamma_{k,n}^{(c)} \rfloor - \frac{\gamma_{k,n}^{(c)}}{2} > (\gamma_{k,n}^{(c)} - 1) - \frac{\gamma_{k,n}^{(c)}}{2} \\ &\geq \frac{3}{4}\gamma_{k,n}^{(c)} - \frac{\gamma_{k,n}^{(c)}}{2} = \frac{\gamma_{k,n}^{(c)}}{4}\end{aligned}$$

terms. Together with the trivial estimate $\lfloor \gamma_{k,n}^{(c)} \rfloor - \lfloor \gamma_{k,n}^{(c)}/2 \rfloor \leq \gamma_{k,n}^{(c)}$, we see that $n \leq \frac{k-2}{c}$ entails

$$\begin{aligned}\sum_{m=\kappa(k,n)}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s &\asymp \gamma_{k,n}^{(c)} \cdot \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s \\ &= 2^{k(1-\alpha s)}\cdot 2^{(-n)[c-(1+c)\nu s]}\end{aligned}$$

The inequalities $\frac{k-2}{c} < n \leq \frac{k-\log_2 3}{c}$ and $\log_2 3 \leq k - nc < 2$, as well as $3 \leq \gamma_{k,n}^{(c)} < 2^2$ are all equivalent. If they hold, we have

$$\begin{aligned}\sum_{m=\kappa(k,n)}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s &\overset{(\dagger)}{=} \sum_{m=3}^{3} \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s \\ &= \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s \\ &\overset{(\dagger)}{\asymp} \gamma_{k,n}^{(c)} \cdot \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s \\ &= 2^{k(1-\alpha s)}\cdot 2^{(-n)[c-(1+c)\nu s]},\end{aligned}$$

where we used $3 \leq \gamma_{k,n}^{(c)} < 4$ at $(\dagger)$.

Set $n_k := \min\{k-1, \lfloor (k-\log_2 3)/c \rfloor\}$ and note that $n > \frac{k-\log_2 3}{c}$ implies $\gamma_{k,n}^{(c)} = 2^{k-nc} < 3$. Together with $\kappa(k,n) \geq 3$, this shows that the sum above is empty for $n > \frac{1}{c}(k-\log_2 3)$. Thus, we arrive at

$$\begin{aligned}\gamma_k^{(s,\theta,3)} &:= \sum_{n=-\infty}^{k-1} \sum_{m=\kappa(k,n)}^{\left\lfloor \gamma_{k,n}^{(c)} \right\rfloor} \left[2^{-k\alpha}\cdot 2^{n(1+c)\nu}\right]^s \\ &\asymp \sum_{n=-\infty}^{n_k} 2^{k(1-\alpha s)}\cdot 2^{(-n)[c-(1+c)\nu s]} \\ &= 2^{k(1-\alpha s)}\cdot \sum_{n=-\infty}^{n_k} 2^{(-n)[c-(1+c)\nu s]} \\ &\overset{(\ddagger)}{\asymp} 2^{k(1-\alpha s)}\cdot 2^{(-n_k)[c-(1+c)\nu s]}. \qquad (6.3.34)\end{aligned}$$

Here, the step marked with (‡) uses a geometric series argument and is thus only valid in the case $c - (1+c)\,\nu s < 0$. But because of $s \in (0,\infty)$, this is equivalent to $c/s < (1+c)\,\nu$ and is thus (again because of $s < \infty$) implied by the condition on the right-hand side of equation (6.3.32), and thus by $\gamma_0^{(s,\theta)} < \infty$.

In case of $c = 1$, we note $n_k = \lfloor k - \log_2 3 \rfloor = k - 2$, which yields

$$\gamma_k^{(s,\theta,3)} \asymp 2^{k(1-\alpha s)} \cdot 2^{(-k)[c-(1+c)\nu s]} = 2^{sk\left[\frac{1-c}{s} - \alpha + (1+c)\nu\right]}.$$

In case of $0 < c < 1$, we have $\frac{k - \log_2 3}{c} \geq k - 1$ and hence $n_k = k - 1$ for sufficiently large $k \in \mathbb{N}_0$, i.e. for all $k \geq k_0$, with $k_0 = k_0\,(c)$. Since each of the terms $\gamma_k^{(s,\theta)}$ is finite if $\gamma_k^{(s,\theta,3)}$ is finite and because this is the case under the condition given in equation (6.3.32), which is necessary for finiteness of $\gamma_0^{(s,\theta)}$, we can discard finitely many of the $(\gamma_k^{(s,\theta)})_{k\in\mathbb{N}}$ without affecting finiteness of $\|(\gamma_k^{(s,\theta)})_k\|_{\ell_v^t(\mathbb{N}_0)}$. Thus, we only need to consider $\gamma_k^{(s,\theta)}$ for $k \geq k_0$. In this case, equation (6.3.34) yields

$$\gamma_k^{(s,\theta,3)} \asymp 2^{sk\left[\frac{1-c}{s} - \alpha + (1+c)\nu\right]},$$

also in case of $0 < c < 1$.

Using $\frac{1-c}{s} \geq 0$ and the above estimate for $\gamma_k^{(s,\theta,3)}$, it is easy to see that the first term in equation (6.3.33), namely $\left(2^{k((1+c)\nu-\alpha)}\right)^s$, can be estimated by $\gamma_k^{(s,\theta,3)}$ for $k \geq k_0$. It remains to consider the middle term in equation (6.3.33), that is

$$\begin{aligned}
\sum_{m=3}^{\lfloor 2^{(1-c)k} \rfloor} \left[2^{-k\alpha} \cdot 2^{k(1+c)\nu}\right]^s &\leq \max\left\{0, \left\lfloor 2^{(1-c)k} \right\rfloor - 2\right\} \cdot \left[2^{-k\alpha} \cdot 2^{k(1+c)\nu}\right]^s \\
&\leq 2^{(1-c)k} \cdot \left[2^{-k\alpha} \cdot 2^{k(1+c)\nu}\right]^s \\
&= 2^{sk\left[\frac{1-c}{s} - \alpha + (1+c)\nu\right]} \\
&\asymp \gamma_k^{(s,\theta,3)}.
\end{aligned}$$

All in all, we have shown

$$\gamma_k^{(s,\theta)} \asymp 2^{k\left[\frac{1-c}{s} - \alpha + (1+c)\nu\right]} \qquad \text{for all } k \geq k_0 = k_0\,(c)$$

under the conditions $s < \infty$ and assuming that $\gamma_0^{(s,\theta)} < \infty$ holds (cf. equation (6.3.32)).

In case of $s = \infty$, equations (6.3.25) and (6.3.27) yield

$$\gamma_k^{(s,\theta)} \asymp \max\left\{2^{k[(1+c)\nu-\alpha]}, \sup_{n<k} 2^{-k\alpha} \cdot 2^{n(1+c)\nu}\right\}.$$

Under the condition $0 = c \cdot 0 \leq c \cdot \max\left\{\alpha, \frac{1}{s}\right\} \leq (1+c)\,\nu$ of equation (6.3.32), this yields

$$\gamma_k^{(s)} \asymp 2^{k[(1+c)\nu-\alpha]} = 2^{k\left[\frac{1-c}{s}-\alpha+(1+c)\nu\right]}.$$

All in all, we have shown that condition (6.3.32) is necessary for finiteness of $C_0^{(s,\theta)} := \left\|(\gamma_k^{(s,\theta)})_{k\in\mathbb{N}_0}\right\|_{\ell_v^t}$ and that, given this condition, finiteness of $C_0^{(s,\theta)}$ is equivalent to finiteness of

$$C_1^{(s,\theta)} := \left\|\left(2^{k\left[\frac{1-c}{s}+\gamma-\alpha+(1+c)\nu\right]}\right)_{k\geq k_0}\right\|_{\ell^t}.$$

But Remark 5.2.3 shows that $t = q_2 \cdot (q_1/q_2)'$ is finite if and only if $q_2 < q_1$. Hence, $C_1^{(s,\theta)} < \infty$ is equivalent to

$$\begin{cases} \gamma \leq \alpha - (1+c)\,\nu + \frac{c-1}{s}, & \text{if } q_1 \leq q_2, \\ \gamma < \alpha - (1+c)\,\nu + \frac{c-1}{s}, & \text{if } q_1 > q_2. \end{cases}$$

With these considerations, it is easy to complete the proof of the second part of the theorem: If we have $p_1 \leq p_2$ and if the conditions in equations (6.3.19) and (6.3.20) are fulfilled, then the above considerations, together with the identity $\frac{1}{s} = \left(\frac{1}{p_2^\nabla} - \frac{1}{q_1}\right)_+$ for $s = p_2^\nabla \cdot (q_1/p_2^\nabla)'$ and with the equivalence $s < \infty \Leftrightarrow p_2^\nabla < q_1$, show $C_0^{(s,\theta)} < \infty$ for $s = p_2^\nabla \cdot (q_1/p_2^\nabla)'$ and $\theta = \frac{1}{p_1} - \frac{1}{p_2}$, so that all assumptions of part 1 of Theorem 5.4.4 are fulfilled. This implies existence of a bounded linear map $\vartheta$ with the desired properties.

Conversely, if the map given in equation (6.3.21) is bounded, then part 2 of Theorem 5.4.4 yields $p_1 \leq p_2$ and shows that the constant $C_0^{(s,\theta)}$ is finite, for $s = p_2 \cdot (q_1/p_2)'$ and $\theta = \frac{1}{p_1} - \frac{1}{p_2}$. Using the equivalence $s < \infty \Leftrightarrow p_2 < q_1$ and the considerations above, this easily implies that the conditions given in equations (6.3.19) and (6.3.20) are fulfilled, if $p_2^\nabla$ is replaced by $p_2$ everywhere.

It remains to consider the first part of the theorem. The proof is largely analogous. Precisely, we apply parts 1 and 2 of Theorem 5.4.1 with $\mathcal{Q} = \mathcal{S}^{(c)}$, $\mathcal{P} = \mathcal{B}$, as well as $u = w^{(\alpha,\beta)}$ and $v = v^{(\gamma)}$ and with interchanged roles of $p_1, p_2$ and $q_1, q_2$. Lemma 6.3.10 shows that the requirement of Theorem 5.4.1 that $\mathcal{Q}$ be almost subordinate to $\mathcal{P}$ is satisfied. Furthermore, Lemma 6.2.2 and Corollary 6.3.5, together with the considerations regarding the weight $w^{(\alpha,\beta)}$ from above show that $\mathcal{Q}, \mathcal{P}$ are both tight semi-structured $L^p$-decomposition coverings of $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$ and $\mathcal{O}' = \mathbb{R}^2$, respectively, for all $p \in (0, \infty]$ and that $u$ and $v$ are moderate with respect to $\mathcal{Q}$ and $\mathcal{P}$, respectively.

Thus, Theorem 5.4.1 shows (if we recall the interchanged roles of $p_1, p_2$ and of $q_1, q_2$) that we are interested in finiteness of the quasi-norm

$$\begin{aligned}&\left\| \left( \left( \left|\det\left(S_j\right)\right|^r v_j \right)^{-1} \left\| \left( \left|\det\left(T_i\right)\right|^\theta u_i \right)_{i\in I_j} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell^t} \\ &= \left\| \left( \left\| \left( \left( w_i^{(-\alpha,-\beta)} \right)^{-1} \cdot \left|\det\left(T_i\right)\right|^\theta \right)_{i\in I_j} \right\|_{\ell^s} \right)_{j\in J} \right\|_{\ell^t_{v^{(-(\gamma+2r))}}} \end{aligned} \tag{6.3.35}$$

for $t = q_2 \cdot (q_1/q_2)'$ as above and different choices of $r, \theta \in \mathbb{R}$ and of $s \in (0, \infty]$. Here, $\left|\det\left(S_j\right)\right| = \left|\det\left(2^j \mathrm{id}\right)\right| = 2^{2j}$ for $j \in J = \mathbb{N}_0$.

By comparing this with equation (6.3.22), we see that we already characterized finiteness of this quasi-norm above. All we have to do is to replace $\alpha$ by $-\alpha$, $\beta$ by $-\beta$ and $\gamma$ by $-(\gamma + 2r)$.

Thus, the above quasi-norm is finite if and only if we have

$$\begin{aligned} &\begin{cases} c \cdot \max\left\{-\alpha, \frac{1}{s}\right\} < (1+c)\,\nu, & \text{if } s < \infty, \\ c \cdot \max\left\{-\alpha, \frac{1}{s}\right\} \leq (1+c)\,\nu, & \text{if } s = \infty \end{cases} \\ \iff &\begin{cases} -c \cdot \min\left\{\alpha, -\frac{1}{s}\right\} < (1+c)\,\nu, & \text{if } s < \infty, \\ -c \cdot \min\left\{\alpha, 0\right\} \leq (1+c)\,\nu, & \text{if } s = \infty \end{cases} \\ \iff &\begin{cases} \min\left\{\alpha, -\frac{1}{s}\right\} > -\frac{1+c}{c}\nu, & \text{if } s < \infty, \\ \min\left\{\alpha, 0\right\} \geq -\frac{1+c}{c}\nu, & \text{if } s = \infty \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\begin{cases} -(\gamma+2r) \leq -\alpha - (1+c)\,\nu + \frac{c-1}{s}, & \text{if } q_1 \leq q_2, \\ -(\gamma+2r) < -\alpha - (1+c)\,\nu + \frac{c-1}{s}, & \text{if } q_1 > q_2 \end{cases} \\ \iff &\begin{cases} \gamma + 2r \geq \alpha + (1+c)\,\nu + \frac{1-c}{s}, & \text{if } q_1 \leq q_2, \\ \gamma + 2r > \alpha + (1+c)\,\nu + \frac{1-c}{s}, & \text{if } q_1 > q_2. \end{cases} \end{aligned}$$

Here, $\nu = \theta - (-\beta) = \theta + \beta$.

Now, part 1 of Theorem 5.4.1 (with interchanged roles of $p_1, p_2$ and of $q_1, q_2$) requires us to check finiteness of the quasi-norm in equation (6.3.35) for $\theta = \frac{1}{p_1} - \frac{1}{p_2} + (p_1)_d$ and $r = (p_1)_d$, as well as $s = q_2 \cdot (p_1^\triangle / q_2)'$. Recall that Remark 5.2.3 shows

$$\frac{1}{s} = \left(\frac{1}{q_2} - \frac{1}{p_1^\triangle}\right)_+ \qquad \text{and the equivalence} \qquad s < \infty \Leftrightarrow q_2 < p_1^\triangle$$

and that we have

$$\left(\frac{1}{q_2}-\frac{1}{p_1^{\triangle}}\right)_+ - (p_1)_d = \left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+,$$

thanks to Remark 5.2.10. Using these identities, it is easy to see that the above conditions are equivalent to those given in equations (6.3.16) and (6.3.17).

Thus, if these conditions are satisfied and if furthermore $p_1 \leq p_2$ holds, then all assumptions of part 1 of Theorem 5.4.1 are fulfilled, since we are considering $p_1, p_2$ and $q_1, q_2$ with interchanged roles. This yields boundedness of $\iota$.

Conversely, if the map given in equation (6.3.18) is bounded, then part 2 of Theorem 5.4.1 (still with interchanged roles of $p_1, p_2$ and of $q_1, q_2$) implies finiteness of the quasi-norm in equation (6.3.35) for $r = 0$, $\theta = \frac{1}{p_1} - \frac{1}{p_2}$ and $s = q_2 \cdot (p_1/q_2)'$. Using the equivalence $s < \infty \Leftrightarrow q_2 < p_1$ and the identity $\frac{1}{s} = \left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+$, together with the above characterization of finiteness of the quasi-norm in equation (6.3.35), this implies

$$\begin{cases} \min\left\{\alpha, \frac{1}{p_1} - \frac{1}{q_2}\right\} > \frac{1+c}{c}\left(\frac{1}{p_2} - \frac{1}{p_1} - \beta\right), & \text{if } p_1 > q_2, \\ \min\{\alpha, 0\} \geq \frac{1+c}{c}\left(\frac{1}{p_2} - \frac{1}{p_1} - \beta\right), & \text{if } p_1 \leq q_2, \end{cases}$$

as well as

$$\begin{cases} \gamma \geq \alpha + (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \beta\right) + (1-c)\left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+, & \text{if } q_1 \leq q_2, \\ \gamma > \alpha + (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \beta\right) + (1-c)\left(\frac{1}{q_2} - \frac{1}{p_1}\right)_+, & \text{if } q_1 > q_2. \end{cases}$$

Observe that these two conditions are precisely the conditions given in equations (6.3.16) and (6.3.17), with all occurences of $p_1^{\triangle}$ and $p_1^{\pm\triangle}$ replaced by $p_1$ and with all occurences of $(p_1)_d$ replaced by 0.

Thus, we have established the claim for $p_1 \in [1, \infty]$, since we have $(p_1)_d = 0$ in this case. For $p_1 \in (0, 1)$, we employ Lemma 5.3.11 with interchanged roles of $p_1, p_2$ and of $q_1, q_2$ and with $\mathcal{Q} = \mathcal{S}^{(c)}$, $\mathcal{P} = \mathcal{B}$, $u = w^{(\alpha,\beta)}$, $v = v^{(\gamma)}$ and $I_0 = I = \mathbb{Z}^2 \times \{\pm 1\}$. This easily implies (in the notation of Lemma 5.3.11) that we have $J_0 = J = \mathbb{N}_0$. The set $K$ from Lemma 5.3.11 trivially satisfies $K \subset \mathbb{R}^2$, so that the assumptions of the current theorem imply that all assumptions of Lemma 5.3.11 are satisfied, since $\mathcal{Q} = \mathcal{S}^{(c)}$ is almost subordinate to $\mathcal{P} = \mathcal{B}$ by Lemma 6.3.10.

Hence, Lemma 5.3.11 yields finiteness of the quasi-norm

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\frac{1}{p_1}-1}}{v_j} \left\| \left( \left|\det\left(T_i\right)\right|^{1-\frac{1}{p_2}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^{q_2}} \right)_{j \in J_0} \right\|_{\ell^{q_2 \cdot (q_1/q_2)'}},$$

which is nothing but the quasi-norm given in equation (6.3.35), with the choices $r = 1 - \frac{1}{p_1}$, $\theta = 1 - \frac{1}{p_2}$ and $s = q_2$. As above, this implies

$$\begin{cases} \min\left\{\alpha, -\frac{1}{q_2}\right\} > \frac{1+c}{c}\left(\frac{1}{p_2} - 1 - \beta\right), & \text{if } q_2 < \infty, \\ \min\{\alpha, 0\} \geq \frac{1+c}{c}\left(\frac{1}{p_2} - 1 - \beta\right), & \text{if } q_2 = \infty \end{cases} \tag{6.3.36}$$

and

$$\begin{cases} \gamma + 2\left(1 - \frac{1}{p_1}\right) \geq \alpha + (1+c)\left(1 - \frac{1}{p_2} + \beta\right) + \frac{1-c}{q_2}, & \text{if } q_1 \leq q_2, \\ \gamma + 2\left(1 - \frac{1}{p_1}\right) > \alpha + (1+c)\left(1 - \frac{1}{p_2} + \beta\right) + \frac{1-c}{q_2}, & \text{if } q_1 > q_2. \end{cases} \tag{6.3.37}$$

But since we are in the case $p_1 \in (0,1)$, we have $(p_1)_d = 1 - \frac{1}{p_1}$ and $p_1^{\triangle} = \infty$, as well as

$$\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ = \left(\frac{1}{q_2} - 1 + \frac{1}{p_1}\right)_+ = \frac{1}{q_2} - 1 + \frac{1}{p_1},$$

so that the conditions in equations (6.3.16) and (6.3.17) are easily seen to be equivalent to

$$\begin{cases} \min\left\{\alpha, -\frac{1}{q_2}\right\} > \frac{1+c}{c} \cdot \left(\frac{1}{p_2} - 1 - \beta\right), & \text{if } q_2 < \infty, \\ \min\{\alpha, 0\} \geq \frac{1+c}{c} \cdot \left(\frac{1}{p_2} - 1 - \beta\right), & \text{if } q_2 = \infty \end{cases}$$

and

$$\begin{cases} \gamma \geq \alpha + (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \beta\right) + (1-c)\left(\frac{1}{q_2} - 1 + \frac{1}{p_1}\right), & \text{if } q_1 \leq q_2, \\ \gamma > \alpha + (1+c)\left(\frac{1}{p_1} - \frac{1}{p_2} + \beta\right) + (1-c)\left(\frac{1}{q_2} - 1 + \frac{1}{p_1}\right), & \text{if } q_1 > q_2. \end{cases}$$

But these conditions are equivalent to those derived in equations (6.3.36) and (6.3.37). □

*Remark* 6.3.13. In view of the above theorem, we can now explain why we used the weight as given in equation (6.3.14) and not only a power of the determinant. To this end, let us consider the case $\alpha = 0$, which corresponds to taking only a power of the determinant as the weight. Furthermore, let us assume $p_1 = p_2 = p$ and $q_1 = q_2 = q$ for simplicity. If we consider the sufficient conditions given in the above theorem for the existence of an embedding $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_{w^{(0,\beta)}}\right) \hookrightarrow \mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{v^{(\gamma)}}\right)$, we get

$$\begin{cases} 0 \leq \max\left\{0, \frac{1}{p^{\triangledown}} - \frac{1}{q}\right\} < -\frac{1+c}{c}\beta, & \text{if } q > p^{\triangledown}, \\ 0 \leq -\frac{1+c}{c}\beta & \text{if } q \leq p^{\triangledown} \end{cases}$$

and

$$\gamma \leq (1+c)\,\beta + (c-1)\left(\frac{1}{p^{\nabla}} - \frac{1}{q}\right)_+ .$$

Hence, we have $(1+c)\,\beta \leq 0$ by the first condition and $c-1 \leq 0$ by our restriction on $c$. Thus, we must necessarily have $\gamma \leq 0$ to be able to apply the above theorem. Of course, the theorem still gives rise to an embedding statement for suitable $\gamma \leq 0$, but the Besov spaces with negative values of $\gamma$ correspond to a low smoothness.

We now provide a *short* proof which shows that we can not hope to get an embedding of $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^{q_1}_{w^{(0,\beta)}}\right)$ into $\mathcal{D}\left(\mathcal{B}, L^p, \ell^{q_2}_{v^{(\gamma)}}\right)$ for positive $\gamma$. This of course follows from the necessary conditions derived above, but our geometric proof here will provide more intuition than the relatively opaque manipulations in the proof above.

If there was such an embedding, Lemma 5.3.1 would yield a constant $C > 0$ such that

$$1 \leq 2^{k\gamma} = v_k^{(\gamma)} \leq C \cdot w_{n,m,\varepsilon}^{(0,\beta)} = C \cdot 2^{n(1+c)\beta} \tag{6.3.38}$$

holds for all $(n,m,\varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}$ and $k \in \mathbb{N}$ with $S^{(c)}_{n,m,\varepsilon} \cap B_k \neq \varnothing$. Here, we implicitly used that the sets $S^{(c)}_{n,m,\varepsilon}$ and $B_k$ are open.

But each of the "dyadic rings" $B_k$ intersects the $y$-axis and thus also intersects suitable sets $S^{(c)}_{n,m,1} = U^{\left(2^{n-1},2^{n+1}\right)}_{(*,*)}$ for *arbitrarily small* values of $n \in \mathbb{Z}$. Using equation (6.3.38), this implies $(1+c)\,\beta \leq 0$. Thus, the existence of such an embedding necessitates $(1+c)\,\beta \leq 0$.

Furthermore, each of the dyadic rings $B_k = B_{2^{k+2}}(0) \setminus \overline{B_{2^{k-2}}}(0)$ also intersects $S^{(c)}_{k,0,1} = U^{\left(2^{k-1},2^{k+1}\right)}_{\left(-\varrho_c^k,\varrho_c^k\right)}$, for example in $\binom{2^k}{0}$. Thus, equation (6.3.38) yields

$$2^{k\gamma} \leq C \cdot 2^{k(1+c)\beta} \tag{6.3.39}$$

for all $k \in \mathbb{N}$, which entails $\gamma \leq (1+c)\,\beta \leq 0$.

Note that we did not even have to apply the more sophisticated necessary conditions derived in Theorem 5.3.6, but only the simple Lemma 5.3.1 to derive $\gamma \leq 0$, at least in the case $p_1 = p_2$.

Finally, we remark that a slight variant of the above discussion shows that an embedding of $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^{q_1}_{w^{(0,\beta)}}\right)$ into the *homogeneous* Besov decomposition space $\mathcal{D}\left(\dot{\mathcal{B}}, L^p, \ell^{q_2}_{w^{(\gamma)}}\right)$, with $w_k^{(\gamma)} = 2^{k\gamma}$ for $k \in \mathbb{Z}$, is only possible for very specific choices of $\beta, \gamma$. More precisely, the estimate

$$2^{k\gamma} \leq C \cdot w_{n,m,\varepsilon}^{(0,\beta)} = C \cdot 2^{n(1+c)\beta} \tag{6.3.40}$$

from equation (6.3.38) also holds in this case, for all $k \in \mathbb{Z}$ and $(n,m,\varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}$ with $S^{(c)}_{n,m,\varepsilon} \cap \dot{B}_k \neq \varnothing$.

But as above, we see that each "dyadic ring" $\dot{B}_k$ intersects the $y$-axis and thus also intersects suitable sets $S^{(c)}_{n,m,1} = U^{\left(2^{n-1},2^{n+1}\right)}_{(*,*)}$ for *arbitrarily small* values of $n \in \mathbb{Z}$. Together with estimate (6.3.40), we conclude $(1+c)\,\beta \leq 0$.

Furthermore, since we are considering $\dot{\mathcal{B}}$ instead of $\mathcal{B}$, equation (6.3.39) even has to hold for all $k \in \mathbb{Z}$ (not only for $k \in \mathbb{N}$), which is only possible for $\gamma = (1+c)\beta$.

In summary, an embedding of $\mathcal{D}(\mathcal{S}^{(c)}, L^p, \ell^{q_1}_{w^{(0,\beta)}})$ into the *homogeneous* Besov decomposition space $\mathcal{D}(\dot{\mathcal{B}}, L^p, \ell^{q_2}_{w^{(\gamma)}})$ can only hold for $\gamma = (1+c)\beta$. This very restrictive condition is one of the reasons why we only considered embeddings into *inhomogeneous* Besov decomposition spaces in the theorem above. ◀

As the final result in this section, we translate the above results for embeddings between decomposition spaces to the setting of coorbit spaces.

**Theorem 6.3.14.** *Let $c \in (0,1]$, $p_1, p_2, q_1, q_2 \in (0,\infty]$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Consider the weight*

$$u^{(\alpha,\beta)} : H_c \to (0,\infty), h \mapsto \|h^{-1}\|^\alpha \cdot |\det(h)|^\beta .$$

*By the discussion around equation (6.3.15), this weight is right-moderate with respect to some measurable, locally bounded, submultiplicative weight on $H_c$.*

*Define*

$$\alpha^{(1)} := \frac{1+c}{c} \cdot \left( \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{q_1} + \frac{1}{2} + \beta \right),$$
$$\alpha^{(2)} := \frac{1+c}{c} \cdot \left( \frac{1}{p_2} - \frac{1}{p_1} - (p_1)_d - \frac{1}{q_2} + \frac{1}{2} + \beta \right)$$

*and*

$$\gamma^{(1)} := -(1+c)\left( \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{q_1} + \frac{1}{2} + \beta \right) + (c-1)\left( \frac{1}{p_2^\nabla} - \frac{1}{q_1} \right)_+ ,$$
$$\gamma^{(2)} := (1+c)\left( \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{2} - \beta \right) + (1-c)\left( \frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}} \right)_+ .$$

*If $p_1 \le p_2$, as well as*

$$\begin{cases} \max\left\{\alpha, \frac{1}{p_2^\nabla} - \frac{1}{q_1}\right\} < \alpha^{(1)}, & \text{if } q_1 > p_2^\nabla, \\ \max\{0, \alpha\} \le \alpha^{(1)}, & \text{if } q_1 \le p_2^\nabla, \end{cases} \tag{6.3.41}$$

*and*

$$\begin{cases} \gamma \le \alpha + \gamma^{(1)}, & \text{if } q_1 \le q_2, \\ \gamma < \alpha + \gamma^{(1)}, & \text{if } q_1 > q_2. \end{cases} \tag{6.3.42}$$

*holds, there is a bounded linear map*

$$\iota : \mathrm{Co}\left( L^{p_1,q_1}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to B^{p_2}_{\gamma,q_2}$$

*which satisfies $\iota f = f$ for all $f \in L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L^{p,q}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$.*

*Conversely, if such a bounded linear map exists, then we have* $p_1 \leq p_2$ *and equations* (6.3.41) *and* (6.3.42) *are satisfied, as long as* $p_2^{\triangledown}$ *is replaced by* $p_2$ *everywhere, including the definition of* $\gamma^{(1)}$.

*Furthermore, there is a bounded linear map*

$$\theta : B^{p_1}_{\gamma,q_1} \to \mathrm{Co}\left(L^{p_2,q_2}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$$

*with* $\theta f = f$ *for all* $f \in L^2\left(\mathbb{R}^2\right) \cap B^{p_1}_{\gamma,q_1}$ *if we have* $p_1 \leq p_2$, *as well as*

$$\begin{cases} \min\left\{\alpha, \frac{1}{p_1^{\triangle}} - \frac{1}{q_2}\right\} > \alpha^{(2)}, & \text{if } p_1^{\triangle} > q_2, \\ \min\{\alpha, 0\} \geq \alpha^{(2)}, & \text{if } p_1^{\triangle} \leq q_2 \end{cases} \tag{6.3.43}$$

*and*

$$\begin{cases} \gamma \geq \alpha + \gamma^{(2)}, & \text{if } q_1 \leq q_2, \\ \gamma > \alpha + \gamma^{(2)}, & \text{if } q_1 > q_2. \end{cases} \tag{6.3.44}$$

*Conversely, the existence of such a bounded linear map* $\theta$ *implies that we have* $p_1 \leq p_2$ *and that equations* (6.3.43) *and* (6.3.44), *with* $p_1^{\triangle}$ *and* $p_1^{\pm\triangle}$ *replaced by* $p_1$ *(also in the definition of* $\gamma^{(2)}$*), are satisfied. For* $p_1 \in (0, 1]$, *boundedness of* $\theta$ *even implies that equations* (6.3.43) *and* (6.3.44) *are satisfied in their verbatim form (i.e.* without *replacing* $p_1^{\triangle}$ *and* $p_1^{\pm\triangle}$ *by* $p_1$*).*

*Remark* 6.3.15. This embedding result is superficially similar to [15, Theorem 4.7]. Nevertheless, the two results are very different, since Dahlke et al. only consider embeddings of the space $\mathcal{SCC}_{p,r}$ into the sum of *homogeneous* Besov spaces $\dot{B}^p_{\sigma_1,p}\left(\mathbb{R}^2\right) + \dot{B}^p_{\sigma_2,p}\left(\mathbb{R}^2\right)$ for certain values of $\sigma_1, \sigma_2$, as opposed to an embedding of the *whole* shearlet coorbit space into a *single inhomogeneous* Besov space. Here, the space $\mathcal{SCC}_{p,r}$ is a proper subspace of the whole shearlet coorbit space, defined by considering only those elements which admit an atomic decomposition using only a restricted subset of atoms, cf. [15, equation (20)].

More precisely, Dahlke et al. use the weight $m(a, s, t) = |a|^{-r}$ for $r \geq 0$. In our notation, this means $\mathcal{SCC}_{p,r} \leq \mathrm{Co}\left(L^{p,p}_{u^{(0,-2r/3)}}\left(\mathbb{R}^2 \rtimes H_{1/2}\right)\right)$. In particular, we have $\alpha = 0$ in the setting of theorems 6.3.12 and 6.3.14. Thus, Remark 6.3.13 shows that an embedding of the *whole* shearlet coorbit space into a single *homogeneous* Besov space is almost never possible. ◀

*Proof.* Set $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ with $\mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ as in Corollary 6.3.5. Furthermore, choose

$$\mathcal{F}^{(1)}_{\text{shear}} : \mathrm{Co}\left(L^{p_1,q_1}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{u_1}\right),$$
$$\mathcal{F}^{(2)}_{\text{shear}} : \mathrm{Co}\left(L^{p_2,q_2}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{u_2}\right)$$

and

$$\begin{aligned}(u_i)_{n,m,\varepsilon} &:= 2^{-n(1+c)\left(\frac{1}{2}-\frac{1}{q_i}\right)} \cdot u^{(\alpha,\beta)}\left(\left(A^{(c)}_{n,m,\varepsilon}\right)^{-T}\right)\\ &= 2^{-n(1+c)\left(\frac{1}{2}-\frac{1}{q_i}\right)} \cdot \left\|A^{(c)}_{n,m,\varepsilon}\right\|^{\alpha} \cdot \left|\det\left(A^{(c)}_{n,m,\varepsilon}\right)\right|^{-\beta}\\ &= \left|\det\left(A^{(c)}_{n,m,\varepsilon}\right)\right|^{\frac{1}{q_i}-\frac{1}{2}-\beta} \cdot \left\|A^{(c)}_{n,m,\varepsilon}\right\|^{\alpha}\\ &= w^{\left(\alpha,\frac{1}{q_i}-\frac{1}{2}-\beta\right)}_{n,m,\varepsilon}\end{aligned}$$

as in Corollary 6.3.5 for $i \in \underline{2}$ and $(n,m,\varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}$. Here, $w^{\left(\alpha,\frac{1}{q_i}-\frac{1}{2}-\beta\right)}$ is as in equation (6.3.14) and hence as in Theorem 6.3.12. Recall from Corollary 6.3.5 that we have

$$\mathcal{F}^{(i)}_{\text{shear}} f = \widehat{f} \text{ for all } f \in L^2\left(\mathbb{R}^2\right) \cap \text{Co}\left(L^{p_i,q_i}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \text{ and } i \in \underline{2},$$

as well as $\left(\mathcal{F}^{(i)}_{\text{shear}}\right)^{-1} f = \mathcal{F}^{-1} f$ for all $f \in L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_i}, \ell^{q_i}_{u_i}\right)$ and $i \in \underline{2}$.

Under the respective conditions stated above, Theorem 6.3.12 implies that

$$\Gamma_1 : \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\gamma)}}\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{w^{(\alpha,\frac{1}{q_2}-\frac{1}{2}-\beta)}}\right), f \mapsto f|_{C_c^\infty(\mathbb{R}^* \times \mathbb{R})}$$

is bounded or that there is a bounded linear map

$$\Gamma_2 : \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{w^{(\alpha,\frac{1}{q_1}-\frac{1}{2}-\beta)}}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\gamma)}}\right)$$

which satisfies

$$\Gamma_2 f = f \quad \text{for all} \quad f \in L^1_{\text{loc}}\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{u_1}\right), \tag{6.3.45}$$

see the properties of $\theta$ given in part 1 of Theorem 5.4.4. Here, we used the fact that $\mathbb{R}^* \times \mathbb{R}$ is of full measure in $\mathbb{R}^2$.

Now, define

$$\iota := \left(\mathcal{F}^{(2)}_{\text{Besov}}\right)^{-1} \circ \Gamma_2 \circ \mathcal{F}^{(1)}_{\text{shear}}$$

or

$$\theta := \left(\mathcal{F}^{(2)}_{\text{shear}}\right)^{-1} \circ \Gamma_1 \circ \mathcal{F}^{(1)}_{\text{Besov}},$$

where

$$\mathcal{F}^{(i)}_{\text{Besov}} : B^{p_i}_{\gamma,q_i} \to \mathcal{D}\left(\mathcal{B}, L^{p_i}, \ell^{q_i}_{v^{(\gamma)}}\right), f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^2)}$$

is an isomorphism, as shown in Lemma 6.2.2.

Now, let $f \in L^2\left(\mathbb{R}^2\right) \cap \text{Co}\left(L^{p_1,q_1}_{u^{(\alpha,\beta)}}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. Then,

$$\widehat{\iota f}|_{C_c^\infty(\mathbb{R}^2)} = \mathcal{F}^{(2)}_{\text{Besov}}(\iota f) = \left(\Gamma_2 \circ \mathcal{F}^{(1)}_{\text{shear}}\right) f = \Gamma_2 \widehat{f} = \widehat{f},$$

where the last step used $\widehat{f} \in L^2(\mathbb{R}^2) \subset L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and equation (6.3.45). This shows $\widehat{f} = \iota\widehat{f}$ on $C_c^\infty(\mathbb{R}^2)$ (i.e. as distributions) and hence $f = \iota f$, since $C_c^\infty(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$ is dense by [33, Proposition 9.9] and because of $f \in L^2(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$ and $\iota f \in B^{p_2}_{\gamma,q_2} \subset \mathcal{S}'(\mathbb{R}^2)$.

Now, let $f \in L^2(\mathbb{R}^2) \cap B^{p_1}_{\gamma,q_1}$. We have

$$\begin{aligned}\theta f &= \left(\mathcal{F}^{(2)}_{\mathrm{shear}}\right)^{-1}\left(\Gamma_1\left(\mathcal{F}^{(1)}_{\mathrm{Besov}} f\right)\right)\\ &= \left(\mathcal{F}^{(2)}_{\mathrm{shear}}\right)^{-1}\left(\widehat{f}|_{C_c^\infty(\mathbb{R}^*\times\mathbb{R})}\right) = \mathcal{F}^{-1}\widehat{f} = f.\end{aligned}$$

Here, we used that

$$\widehat{f}|_{C_c^\infty(\mathbb{R}^*\times\mathbb{R})} = \Gamma_1\left(\mathcal{F}^{(1)}_{\mathrm{Besov}} f\right) \in L^2(\mathbb{R}^2) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{u_2}\right)$$

is given by integration against the $L^2$-function $\widehat{f}$ and that $(\mathcal{F}^{(2)}_{\mathrm{shear}})^{-1}$ coincides with the usual inverse Fourier transform on those distributions, as noted above.

For the two converse statements, assume that bounded linear maps $\iota$ or $\theta$ as above exist. Define

$$\Gamma_2 := \mathcal{F}^{(2)}_{\mathrm{Besov}} \circ \iota \circ \left(\mathcal{F}^{(1)}_{\mathrm{shear}}\right)^{-1} : \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{u_1}\right) \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\gamma)}}\right)$$

or

$$\Gamma_1 := \mathcal{F}^{(2)}_{\mathrm{shear}} \circ \theta \circ \left(\mathcal{F}^{(1)}_{\mathrm{Besov}}\right)^{-1} : \mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\gamma)}}\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell^{q_2}_{u_2}\right).$$

These are well-defined bounded linear maps.

For $f \in C_c^\infty(\mathbb{R}^* \times \mathbb{R}) \subset L^2(\mathbb{R}^2) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_1}, \ell^{q_1}_{u_1}\right)$, we have

$$\left(\mathcal{F}^{(1)}_{\mathrm{shear}}\right)^{-1} f = \mathcal{F}^{-1} f \in L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L^{p_1,q_1}_{u^{(\alpha,\beta)}}(\mathbb{R}^2 \rtimes H_c)\right)$$

as noted above and hence

$$\Gamma_2 f = \mathcal{F}^{(2)}_{\mathrm{Besov}}\left(\iota\left(\mathcal{F}^{-1} f\right)\right) = \mathcal{F}^{(2)}_{\mathrm{Besov}}\left(\mathcal{F}^{-1} f\right) = \widehat{\mathcal{F}^{-1} f}|_{C_c^\infty(\mathbb{R}^2)} = f,$$

where $f$ is considered as a distribution $f \in \mathcal{D}'(\mathbb{R}^2)$ on the right-hand side.

Similarly, for $f \in C_c^\infty(\mathbb{R}^2) \subset \mathcal{D}(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\gamma)}})$ there is some $g \in B^{p_1}_{\gamma,q_1}$ with $f = \mathcal{F}^{(1)}_{\mathrm{Besov}} g = \widehat{g}|_{C_c^\infty(\mathbb{R}^2)}$, by surjectivity of $\mathcal{F}^{(1)}_{\mathrm{Besov}}$. This implies that $\widehat{g} \in \mathcal{S}'(\mathbb{R}^2)$ is given by integration against the $L^2$-function $f$ and thus $g = \mathcal{F}^{-1} f \in L^2(\mathbb{R}^2) \cap B^{p_1}_{\gamma,q_1}$ as tempered distributions. We conclude $\theta g = g$ and hence

$$\Gamma_1 f = \mathcal{F}^{(2)}_{\mathrm{shear}}\left(\theta\left[\left(\mathcal{F}^{(1)}_{\mathrm{Besov}}\right)^{-1} f\right]\right) = \mathcal{F}^{(2)}_{\mathrm{shear}}(\theta g) = \mathcal{F}^{(2)}_{\mathrm{shear}} g = \widehat{g}|_{C_c^\infty(\mathbb{R}^*\times\mathbb{R})},$$

where $\widehat{g}$ is interpreted as a distribution

$$\widehat{g} \in \mathcal{D}\left(\mathcal{S}^{(c)}, L^{p_2}, \ell_{u_2}^{q_2}\right) \subset \mathcal{D}'\left(\mathbb{R}^* \times \mathbb{R}\right)$$

on the right-hand side. But $\widehat{g}|_{C_c^\infty(\mathbb{R}^2)} = f$ and hence $\Gamma_1 f = f|_{C_c^\infty(\mathbb{R}^* \times \mathbb{R})}$, where $f$ is interpreted as a distribution on $\mathbb{R}^* \times \mathbb{R}$ on the right-hand side.

This shows that the maps $\Gamma_1$ or $\Gamma_2$ satisfy the properties of the maps in equations (6.3.18) (with $\frac{1}{q_2} - \frac{1}{2} - \beta$ instead of $\beta$) or (6.3.21) (with $\frac{1}{q_1} - \frac{1}{2} - \beta$ instead of $\beta$), respectively. Thus, an application of Theorem 6.3.12 completes the proof. □

## 6.4 EMBEDDINGS FOR SHEARLET SMOOTHNESS SPACES

In this section, we consider the class of shearlet smoothness spaces which is essentially the same as in [53]. Using our method, one can completely characterize the embeddings between these shearlet smoothness spaces and (inhomogeneous) Besov spaces. The same is true for embeddings between shearlet smoothness spaces and $\alpha$-modulation spaces for $\alpha \in \left[0, \frac{1}{2}\right]$.

In comparison to [53], we use a slightly different parametrization for the covering used to define the shearlet smoothness spaces. The main difference is that the constituents of our covering are connected, whereas all but one set of the covering considered in [53] consist of two components. We will make use of the family $\left(A_{(n,m,\varepsilon)}^{(1/2)}\right)_{n,m,\varepsilon}$ of matrices which we used in the previous section to describe the induced covering of the shearlet group $H_{1/2}$ (cf. Lemma 6.3.2 and Corollary 6.3.5). Precisely, we use the following covering:

**Definition 6.4.1.** Let

$$I_0 := \left\{(n, m, \varepsilon, \delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{\pm 1\} \times \{0, 1\} \,|\, |m| \leq 2^n\right\},$$

as well as $I := I_0 \cup \{0\}$. Define $P := U_{\left(-\frac{3}{4}, \frac{3}{4}\right)}^{\left(\frac{1}{2}, \frac{5}{2}\right)}$ and $Q := U_{(-1,1)}^{\left(\frac{1}{3}, 3\right)}$, where the sets $U_{(a,b)}^{(\gamma,\delta)} \subset \mathbb{R}^2$ are chosen as in Definition 6.3.3.

Define $T_0 := 4 \cdot \mathrm{id}$ and $b_0 := -\left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right)$, as well as

$$T_{n,m,\varepsilon,\delta} := R^\delta \cdot A_{2n,m,\varepsilon}^{(1/2)} \quad \text{and} \quad b_{n,m,\varepsilon,\delta} := 0 \quad \text{for } (n, m, \varepsilon, \delta) \in I_0,$$

with $R := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where the matrix

$$A_{2n,m,\varepsilon}^{(1/2)} = \varepsilon \cdot \begin{pmatrix} 2^{2n} & 0 \\ 2^n m & 2^n \end{pmatrix}$$

is defined as in Lemma 6.3.2.

We call the family $\mathcal{S} := (S_i)_{i \in I} := (T_i Q + b_i)_{i \in I}$ the SHEARLET SMOOTHNESS COVERING OF $\mathbb{R}^2$.

*Remark.* We will see below that the (seemingly unmotivated) choice of $T_0, b_0$ from above ensures that $\mathcal{S}$ indeed covers $\mathbb{R}^2$. An alternative would be to define $S_i' := T_i Q + b_i$ for $i \in I_0$ and to set $S_0' := B_R(0)$ for a suitable $R > 0$.

This will yield a covering $\mathcal{S}' = (S_i')_{i \in I}$ which is easily seen to be equivalent to $\mathcal{S}$, in the sense that $\mathcal{S}$ is almost subordinate to $\mathcal{S}'$ and vice versa. But $\mathcal{S}'$ will only be a *semi*-structured admissible covering instead of a structured admissible covering, so that existence of an accompanying BAPU is stricly speaking not ensured by Theorem 3.2.17.

To avoid the technical issue of constructing a BAPU for $\mathcal{S}'$ , we adopt the definition as above, which ensures that $\mathcal{S}$ is structured admissible.

In [53, Proposition 4.1] it was claimed that the shearlet smoothness covering *as defined in that paper* is indeed a structured admissible covering. In the next lemma, we prove this for our shearlet smoothness covering. We provide the full proof for two reasons:

- In [53], the authors invoke the "symmetry of construction" in order to restrict attention only to "the cone-shaped region $\mathcal{P}_1$". But this ignores the transition between the two cone-shaped regions $\mathcal{P}_1, \mathcal{P}_2$, i.e. the case of intersections between elements of the covering, where only *some* of the two elements belong to $\mathcal{P}_1$. This is especially important with regard to the estimate

$$\left\| T_i^{-1} T_j \right\| \overset{!}{\leq} C$$

  for $i, j \in I$ with $S_i \cap S_j \neq \varnothing$.

- As explained above, we use a slightly different covering consisting solely of connected sets, so that we are not completely in the setting of [53]. Nevertheless, we will see in Remark 6.4.4 that the resulting shearlet smoothness spaces are the same.

**Lemma 6.4.2.** *The family $\mathcal{S} = (T_i Q + b_i)_{i \in I}$ defined above is a structured admissible covering of $\mathbb{R}^2$ which is almost subordinate to the Besov covering $\mathcal{B}$ of $\mathbb{R}^2$. More precisely, we have*

$$T_i Q + b_i \subset \begin{cases} B_1^*, & \text{if } i = 0, \\ B_{2n+1}^*, & \text{if } i = (n, m, \varepsilon, \delta) \in I_0, \end{cases}$$

*Finally, $\mathcal{S}$ is relatively moderate with respect to $\mathcal{B}$. In fact, the estimate*

$$\frac{1}{512} \cdot 2^{\frac{3}{2} j} \leq |\det(T_i)| \leq 64 \cdot 2^{\frac{3}{2} j} \tag{6.4.1}$$

*holds for all $i \in I$ and $j \in \mathbb{N}_0$ with $(T_i Q + b_i) \cap B_j \neq \varnothing$.*

*Proof.* We first show that $\mathcal{S}$ is indeed a structured admissible covering of $\mathbb{R}^2$. To this end, we will use that the shearlet covering

$$\mathcal{S}^{(1/2)} := \mathcal{S}^{(1/2)}_{-1,1,\frac{1}{3},3} = \left(S^{(1/2)}_{n,m,\varepsilon}\right)_{(n,m,\varepsilon)\in\mathbb{Z}^2\times\{\pm 1\}} = \left(A^{(1/2)}_{n,m,\varepsilon}Q\right)_{(n,m,\varepsilon)\in\mathbb{Z}^2\times\{\pm 1\}}$$

is a structured admissible covering, as shown in Corollary 6.3.5. Let us write $(n,m,\varepsilon)^*$ for the index-cluster of $(n,m,\varepsilon)\in\mathbb{Z}^2\times\{\pm 1\}$ with respect to the covering $\mathcal{S}^{(1/2)}$. Since the family $\mathcal{S}$ uses a different (even disjoint) index set, this will cause no confusion. Choose $K>0$ and $N\in\mathbb{N}$ with $\left|(n,m,\varepsilon)^*\right| \leq N$ for all $(n,m,\varepsilon)\in\mathbb{Z}^2\times\{\pm 1\}$ and with

$$\left\| \left(A^{(1/2)}_{n,m,\varepsilon}\right)^{-1} A^{(1/2)}_{k,\ell,\delta} \right\| \leq K \qquad \text{for all } (k,\ell,\delta)\in(n,m,\varepsilon)^*.$$

Finally, let us write

$$(n,m,\varepsilon)^+ := \left\{(k,\ell,\alpha)\in\mathbb{Z}^2\times\{\pm 1\} \,\middle|\, (2k,\ell,\alpha)\in(n,m,\varepsilon)^*\right\}.$$

Note that the map

$$(n,m,\varepsilon)^+ \to (n,m,\varepsilon)^*, (k,\ell,\alpha)\mapsto(2k,\ell,\alpha)$$

is injective. This implies $|(n,m,\varepsilon)^+| \leq \left|(n,m,\varepsilon)^*\right| \leq N$ for arbitrary $(n,m,\varepsilon)\in\mathbb{Z}^2\times\{\pm 1\}$.

We start with a few auxiliary observations: We have

$$|y| \leq \left(1+2^{-n}\right)|x| \leq 2|x| \;\; \forall \binom{x}{y} \in S_i \, \forall i=(n,m,\varepsilon,0)\in I_0. \tag{6.4.2}$$

Indeed, since we are only interested in absolute values and because of $T_{n,m,-1,\delta} = -T_{n,m,1,\delta}$, we can assume $\varepsilon = 1$ without loss of generality. Lemma 6.3.4 implies

$$\begin{aligned}\binom{x}{y} \in S_i = A^{(1/2)}_{2n,m,1}Q &= U^{\left(2^{2n}/3,3\cdot 2^{2n}\right)}_{\left(\varrho_{1/2}^{2n}(m-1),\varrho_{1/2}^{2n}(m+1)\right)} \\ &= U^{\left(2^{2n}/3,3\cdot 2^{2n}\right)}_{\left(2^{-n}(m-1),2^{-n}(m+1)\right)}\end{aligned} \tag{6.4.3}$$

because of $\varrho_{1/2} = 2^{\frac{1}{2}-1} = 2^{-\frac{1}{2}}$. Hence,

$$\frac{y}{x} \in \left(2^{-n}(m-1), 2^{-n}(m+1)\right) \subset \left(-2^{-n}(|m|+1), 2^{-n}(|m|+1)\right).$$

But this, together with $|m|\leq 2^n$ (cf. the definition of the index set $I_0$), yields

$$|y| = |x|\cdot\left|\frac{y}{x}\right| \leq |x|\cdot 2^{-n}(|m|+1) \leq |x|\cdot\left(1+2^{-n}\right) \leq 2|x|,$$

which proves equation (6.4.2). In contrast, we have

$$|x| \leq \left(1+2^{-n}\right)|y| \leq 2\,|y| \;\forall \binom{x}{y} \in S_i \,\forall i = (n,m,\varepsilon,1) \in I_0, \quad (6.4.4)$$

because of

$$\begin{pmatrix} y \\ x \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} \in RT_iQ = RRT_{(n,m,\varepsilon,0)} = T_{(n,m,\varepsilon,0)},$$

so that $|x| \leq |y|\,(1+2^{-n}) \leq 2\,|y|$ is implied by equation (6.4.2). Here, we used the matrix $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ from Definition 6.4.1.

Our first goal is the following: For $i = (n,m,\varepsilon,\delta) \in I_0$, we let

$$\begin{aligned} M_i &:= \left\{ j \in I_0 \,\middle|\, S_i \cap S_j \neq \varnothing \right\} = i^* \cap I_0, \\ C_i &:= \sup_{j \in M_i} \left\| T_i^{-1} T_j \right\|. \end{aligned} \quad (6.4.5)$$

We want to show that there are constants $K_1, K_2$ with $|M_i| \leq K_1$ and $C_i \leq K_2$ for all $i \in I_0$. This will in particular imply $i^* \subset M_i \cup \{0\}$ for all $i \in I_0$ and hence $|i^*| \leq 1 + K_1$.

Furthermore, for $i \in I_0$ with $0 \notin i^*$, this will yield

$$\sup_{j \in i^*} \left\| T_i^{-1} T_j \right\| \leq C_i \leq K_2.$$

Hence, once these claims are established, we only need to show that $0^*$ is finite, since this will show that

$$\sup_{j \in 0^*} \left\| T_0^{-1} T_j \right\| =: K_3$$

is finite. Furthermore, for $i \in I_0$ with $0 \in i^*$, i.e. for $i \in 0^*$, we will have

$$\sup_{j \in i^*} \left\| T_i^{-1} T_j \right\| \leq \sup_{\ell \in 0^*} \left[ \max \left\{ \left\| T_\ell^{-1} T_0 \right\|, \max_{j \in M_\ell} \left\| T_\ell^{-1} T_j \right\| \right\} \right] =: K_4,$$

where $K_4$ is finite since $0^*$ is. All in all, this will show

$$\sup_{i \in I} \sup_{j \in i^*} \left\| T_i^{-1} T_j \right\| \leq \max\{K_2, K_3, K_4\}$$

and $|i^*| \leq \max\{1 + K_1, |0^*|\}$ for all $i \in I$.

Before we estimate the sets $M_i$, we first reduce to the case $\delta = 0$. To this end, define

$$(k,\ell,\alpha,\beta)' := \begin{cases} (k,\ell,\alpha,1), & \text{if } \beta = 0, \\ (k,\ell,\alpha,0), & \text{if } \beta = 1 \end{cases}$$

for each $(k, \ell, \alpha, \beta) \in I_0$. Observe $RR = \mathrm{id}$, which yields $T_{i'} = RT_i$ for all $i \in I_0$. Furthermore, we have

$$\begin{aligned} \varnothing \neq S_i \cap S_j &\iff \varnothing \neq T_i Q \cap T_j Q \\ &\iff \varnothing \neq R\left[T_i Q \cap T_j Q\right] \\ &\iff \varnothing \neq T_{i'} Q \cap T_{j'} Q \\ &\iff \varnothing \neq S_{i'} \cap S_{j'} \end{aligned}$$

and

$$T_i^{-1} T_j = (RT_i)^{-1} (RT_j) = T_{i'}^{-1} T_{j'}.$$

Hence, $M_{i'} = \{j' \mid j \in M_i\}$ and $C_i = C_{i'}$ for all $i \in I_0$. Thus, it suffices to consider the case $i = (n, m, \varepsilon, \delta) \in I_0$ with $\delta = 0$.

Now, we have $M_i = M_i^{(0)} \cup M_i^{(1)}$ and $C_i = \max\{C_i^{(0)}, C_i^{(1)}\}$ with

$$M_i^{(\gamma)} = \{(k, \ell, \alpha, \beta) \in M_i \mid \beta = \gamma\}$$

and $C_i^{(\gamma)} = \sup_{j \in M_i^{(\gamma)}} \left\| T_i^{-1} T_j \right\|$. Thus, we can consider the two cases $\gamma = 0$ and $\gamma = 1$ separately.

CASE 1: Let $i = (n, m, \varepsilon, \delta) \in I_0$ with $\delta = 0$ and $j = (k, \ell, \alpha, \beta) \in I_0$ with $\beta = 0$ and with $S_i \cap S_j \neq \varnothing$. This means

$$\varnothing \neq S_i \cap S_j = A_{2n,m,\varepsilon}^{(1/2)} Q \cap A_{2k,\ell,\alpha}^{(1/2)} Q$$

and hence $(2k, \ell, \alpha) \in (2n, m, \varepsilon)^*$, which yields $(k, \ell, \alpha) \in (2n, m, \varepsilon)^+$ and

$$\left\| T_i^{-1} T_j \right\| = \left\| \left( A_{2n,m,\varepsilon}^{(1/2)} \right)^{-1} A_{2k,\ell,\alpha}^{(1/2)} \right\| \leq K.$$

Hence, $M_i^{(0)} \subset (2n, m, \varepsilon)^+$, which yields $\left| M_i^{(0)} \right| \leq N$. Furthermore, we have $C_i^{(0)} \leq K$ for all $i = (n, m, \varepsilon, \delta) \in I_0$ with $\delta = 0$.

CASE 2: Let $i = (n, m, \varepsilon, \delta) \in I_0$ with $\delta = 0$ and $j = (k, \ell, \alpha, \beta) \in I_0$ with $\beta = 1$ and with $S_i \cap S_j \neq \varnothing$. Because of $S_i \cap S_j \neq \varnothing$, we can choose some $\binom{x}{y} \in S_i \cap S_j$. Using Lemma 6.3.4, we get

$$\binom{x}{y} \in S_i = A_{2n,m,\varepsilon}^{(1/2)} Q = \varepsilon \cdot U_{(2^{-n}(m-1), 2^{-n}(m+1))}^{\left(2^{2n}/3, 3 \cdot 2^{2n}\right)},$$

which implies $\varepsilon x \in \left(2^{2n}/3, 3 \cdot 2^{2n}\right)$ and thus $2^{2n}/3 < |x| < 3 \cdot 2^{2n}$. Using equation (6.4.2), we derive $|y| \leq 2|x| < 6 \cdot 2^{2n}$. But we also have

$$\binom{y}{x} = R \binom{x}{y} \in RS_j = A_{2k,\ell,\alpha}^{(1/2)} Q = \alpha \cdot U_{\left(2^{-k}(\ell-1), 2^{-k}(\ell+1)\right)}^{\left(2^{2k}/3, 3 \cdot 2^{2k}\right)}$$

and hence $\alpha y \in \left(2^{2k}/3, 3 \cdot 2^{2k}\right)$, which implies $2^{2k}/3 < |y| < 3 \cdot 2^{2k}$. Recalling the estimate $|y| \leq 2|x| < 6 \cdot 2^{2n}$ from above, we derive

$2^{2k}/3 < |y| \leq 2|x| < 6 \cdot 2^{2n}$. Furthermore, an invocation of equation (6.4.4) shows

$$2^{2n}/3 \leq |x| \leq 2|y| < 6 \cdot 2^{2k}.$$

Together, we conclude $2^{2n} < 2^5 \cdot 2^{2k}$, as well as $2^{2k} < 2^5 \cdot 2^{2n}$ and thus finally

$$|k - n| \leq 2. \tag{6.4.6}$$

In case of $n < 6$, this implies $k < 8$, which yields $M_i^{(1)} \subset M$, where

$$M := \bigcup_{\kappa=0}^{7} \left[\{\kappa\} \times \{-2^\kappa, \ldots, 2^\kappa\} \times \{\pm 1\} \times \{0,1\}\right]$$

is a fixed finite set. Furthermore, we get

$$C_i^{(1)} = \sup_{j \in M_i^{(1)}} \left\| T_i^{-1} T_j \right\| \leq \max_{\ell,\lambda \in M} \left\| T_\lambda^{-1} T_\ell \right\|.$$

It thus suffices to consider the case $n \geq 6$ in the following. By equation (6.4.6), this yields $k \geq 4$. As the next step, we compute the transition matrix which we want to control. We have

$$\begin{aligned} T_i^{-1} T_j &= \left(A_{2n,m,\varepsilon}^{(1/2)}\right)^{-1} R A_{2k,\ell,\alpha}^{(1/2)} \\ &= \alpha\varepsilon \cdot \begin{pmatrix} 2^{-2n} & 0 \\ -2^{-2n} m & 2^{-n} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2^{2k} & 0 \\ 2^k \ell & 2^k \end{pmatrix} \\ &= \alpha\varepsilon \cdot \begin{pmatrix} 2^{k-2n}\ell & 2^{k-2n} \\ 2^{2k-n} - 2^{k-2n}\ell m & -2^{k-2n} m \end{pmatrix}, \end{aligned} \tag{6.4.7}$$

As seen above, we have $0 < |x| \leq 2|y|$ and $0 < |y| \leq 2|x|$. This implies $\frac{1}{2} \leq \left|\frac{x}{y}\right| \leq 2$. The inequality $|m| \leq 4$ would entail

$$\begin{aligned} \frac{y}{x} = \frac{\varepsilon y}{\varepsilon x} &\in \left(2^{-n}(m-1), 2^{-n}(m+1)\right) \\ &\subset \left(-\frac{5}{2^n}, \frac{5}{2^n}\right) \\ &\overset{(\dagger)}{\subset} \left(-\frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

in contradiction to $\left|\frac{y}{x}\right| \geq \frac{1}{2}$. In the step marked with $(\dagger)$, we used $n \geq 6$.

By the previous contradiction, we must have $|m| \geq 5$. An analogous argument using $\frac{x}{y} \in \left(2^{-k}(\ell-1), 2^{-k}(\ell+1)\right)$ and $k \geq 4$ shows $|\ell| \geq 5$. Hence, there are two cases:

*Case* 1. $m \geq 5$. Then, $\frac{y}{x} > 2^{-n}(m-1) > 0$ and thus

$$0 < \frac{x}{y} = \frac{\alpha x}{\alpha y} < 2^{-k}(\ell+1),$$

which implies $\ell > -1$. Because of $|\ell| \geq 5$, we conclude $\ell \geq 5$. In case of $m < 2^n - 4$, we would have (because of $2^n - 4 > 0$ thanks to $n \geq 6$)

$$\begin{aligned} 1 + \frac{4}{2^n} \leq 1 + \frac{4}{2^n - 4} = \frac{2^n}{2^n - 4} \leq \frac{2^n}{m+1} &= \left(2^{-n}(m+1)\right)^{-1} \\ < \left(\frac{y}{x}\right)^{-1} = \frac{x}{y} < 2^{-k}(\ell+1) &\overset{(\dagger)}{\leq} \frac{2^k+1}{2^k} = 1 + 2^{-k}, \end{aligned}$$

where the step marked with $(\dagger)$ used that $\ell \leq 2^k$ holds by definition of $I_0$. But the above estimate yields $2^2 = 4 < 2^{n-k}$, which implies $n - k > 2$ in contradiction to $|n-k| \leq 2$ (cf. equation (6.4.6)). Hence, $m \geq 2^n - 4 > 0$.

Analogously, $\ell < 2^k - 4$ would imply (because of $2^k - 4$, thanks to $k \geq 4$)

$$\begin{aligned} 1 + \frac{4}{2^k} \leq 1 + \frac{4}{2^k - 4} = \frac{2^k}{2^k - 4} \leq \frac{2^k}{\ell+1} &= \left(2^{-k}(\ell+1)\right)^{-1} \\ < \left(\frac{x}{y}\right)^{-1} = \frac{y}{x} < 2^{-n}(m+1) &\leq \frac{2^n+1}{2^n} = 1 + 2^{-n} \end{aligned}$$

and thus $2^2 = 4 < 2^{k-n}$, which implies $k - n > 2$ in contradiction to $|n-k| \leq 2$. Hence, $\ell \geq 2^k - 4 > 0$.

Using these estimate together with equation (6.4.6) and with the inequalities $|\ell| \leq 2^k$ and $|m| \leq 2^n$, as well as $n, k \geq 0$, which hold by definition of $I_0$, we arrive at

$$\begin{aligned} \left|2^{k-2n}\ell\right| &\leq 2^{k-2n} \cdot 2^k = 2^{2(k-n)} \leq 2^4 = 16, \\ \left|2^{k-2n}\right| &= 2^{-n} \cdot 2^{k-n} \leq 2^{|k-n|} \leq 4, \\ \left|2^{2k-n} - 2^{k-2n}\ell m\right| &= 2^{k-n} \cdot \left|2^k - \frac{\ell m}{2^n}\right| \leq 4 \cdot \frac{2^{k+n} - \ell m}{2^n} \\ &\leq 4 \cdot \frac{2^{k+n} - \left(2^k - 4\right)\left(2^n - 4\right)}{2^n} \\ &= 4 \cdot \frac{4 \cdot 2^k + 4 \cdot 2^n - 16}{2^n} \\ &\leq 4 \cdot \frac{4 \cdot 2^n \cdot \left(1 + 2^{k-n}\right)}{2^n} \leq 5 \cdot 16, \\ \left|-2^{k-2n}m\right| &\leq 2^{k-2n} \cdot 2^n \leq 2^{|k-n|} \leq 4. \end{aligned}$$

Together with the equation (6.4.7), we derive $\left\|T_i^{-1}T_j\right\| \leq C$ for some fixed constant $C$ and thus $C_i^{(1)} \leq C$.

*Case* 2. $m \leq -5$. In this case, we have $\frac{y}{x} < 2^{-n}(m+1) < 0$ and thus $2^{-k}(\ell - 1) < \frac{x}{y} < 0$, which implies $\ell < 1$ and thus $\ell \leq -5$, because of $|\ell| \geq 5$. Setting $\widetilde{m} := -m$ and $\widetilde{\ell} := -\ell$, we see

$$0 < 2^{-k}(\widetilde{\ell} - 1) < -\frac{x}{y} < 2^{-k}(\widetilde{\ell} + 1)$$

and

$$0 < 2^{-n}(\widetilde{m} - 1) < -\frac{y}{x} < 2^{-n}(\widetilde{m} + 1).$$

Using exactly the same arguments as in the first case (with $\widetilde{m}$ and $\widetilde{\ell}$ instead of $m$ and $\ell$, with $-\frac{y}{x}$ instead of $\frac{y}{x}$ and with $-\frac{x}{y}$ instead of $\frac{x}{y}$), this implies $\widetilde{m} \geq 2^n - 4$ and $\widetilde{\ell} \geq 2^k - 4$, which means $m \leq -2^n + 4$ and $\ell \leq -2^k + 4$.

Because of $m\ell = \widetilde{m}\widetilde{\ell}$, $|m| = |\widetilde{m}|$ and $|\ell| = |\widetilde{\ell}|$, we also get exactly the same bounds on the entries of $T_i^{-1}T_j$ as in the first case.

Apart from the estimate $\left\|T_i^{-1}T_j\right\| \lesssim 1$, the considerations in the two cases ($m \geq 5$ and $m \leq -5$) above also show

$$M_i^{(1)} \subset \bigcup_{k=n-2}^{n+2} \left[\{k\} \times \left(\left\{2^k - 4, \ldots, 2^k\right\} \cup \left\{-2^k, \ldots, -2^k + 4\right\}\right) \times \{\pm 1\} \times \{1\}\right],$$

where the set on the right-hand side has cardinality at most 100.

As note above, it remains to show that $0^*$ is finite. To this end, let $i = (n, m, \varepsilon, \delta) \in I_0$ and assume that there is some $\binom{x}{y} \in S_0 \cap S_i \neq \varnothing$. As shown (independently) below (cf. equations (6.4.8) and (6.4.9)), this implies $2^{2n-2} < \left|\binom{x}{y}\right| < 16 = 2^4$ and hence $n \leq 2$. Thus, we have shown

$$0^* \subset \{0\} \cup \bigcup_{n=0}^{2} \{n\} \times \{-2^n, \ldots, 2^n\} \times \{\pm 1\} \times \{0, 1\},$$

so that $0^*$ is finite.

To complete the proof that $\mathcal{S}$ is a structured admissible covering of $\mathbb{R}^2$, it suffices to show that the (smaller) family $(T_i P + b_i)_{i \in I}$ (with $P = U_{\binom{-\frac{3}{4}, \frac{3}{4}}}^{\binom{\frac{1}{2}, \frac{5}{2}}}$ as in Definition 6.4.1) is a covering of $\mathbb{R}^2$. To this end, we first show $T_0 P + b_0 \supset B_1(0)$. For this, suppose $\left|\binom{x}{y} - \binom{1}{0}\right| < \frac{1}{4}$. We get $x \in \left(\frac{3}{4}, \frac{5}{4}\right) \subset \left(\frac{1}{2}, \frac{5}{2}\right)$ as well as $\left|\frac{y}{x}\right| < \frac{1/4}{3/4} = \frac{1}{3} < \frac{3}{4}$ and thus $\binom{x}{y} \in P$. Hence, $B_{1/4}\left(\binom{1}{0}\right) \subset P$, which implies

$$T_0 P + b_0 = 4\left[P - \binom{1}{0}\right] \supset 4 \cdot B_{1/4}(0) = B_1(0).$$

Next, observe that Lemma 6.3.4 (see also equation (6.4.3)) yields

$$\begin{aligned}\bigcup_{j=0}^{\infty}\bigcup_{\ell=-2^j}^{2^j} T_{j,\ell,1,0}P &= \bigcup_{j=0}^{\infty}\bigcup_{\ell=-2^j}^{2^j} U^{\left(2^{2j}/2,\frac{5}{2}\cdot 2^{2j}\right)}_{\left(2^{-j}\left(\ell-\frac{3}{4}\right),2^{-j}\left(\ell+\frac{3}{4}\right)\right)}\\ &\overset{(\dagger)}{=} \bigcup_{j=0}^{\infty} U^{\left(4^j/2,\frac{5}{2}\cdot 4^j\right)}_{\left(-1-\frac{3}{4}\cdot 2^{-j},1+\frac{3}{4}\cdot 2^{-j}\right)}\\ &\supset \bigcup_{j=0}^{\infty}\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\left(\frac{4^j}{2},\frac{5}{2}\cdot 4^j\right)\times\mathbb{R}\;\middle|\;\frac{y}{x}\in[-1,1]\right\}\\ &\supset \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\left(\frac{1}{2},\infty\right)\times\mathbb{R}\;\middle|\;|y|\leq|x|\right\}.\end{aligned}$$

At $(\dagger)$, we used the definition of the sets $U^{(\gamma,\delta)}_{(a,b)}$, together with

$$\begin{aligned}\bigcup_{\ell=-2^j}^{2^j}\left(2^{-j}\left(\ell-\frac{3}{4}\right),2^{-j}\left(\ell+\frac{3}{4}\right)\right) &= 2^{-j}\cdot\bigcup_{\ell=-2^j}^{2^j}\left(\ell-\frac{3}{4},\ell+\frac{3}{4}\right)\\ &= 2^{-j}\left(-2^j-\frac{3}{4},2^j+\frac{3}{4}\right)\\ &= \left(-1-\frac{3}{4}2^{-j},1+\frac{3}{4}2^{-j}\right).\end{aligned}$$

Using the identity $T_{j,\ell,-1,0} = -T_{j,\ell,1,0}$, we conclude

$$\bigcup_{j=0}^{\infty}\bigcup_{\ell=-2^j}^{2^j}\bigcup_{\varepsilon\in\{\pm 1\}} T_{j,\ell,\varepsilon,0}P \supset \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2\;\middle|\;|x|>\frac{1}{2}\text{ and }|y|\leq|x|\right\}.$$

Finally, we use $R\binom{x}{y} = \binom{y}{x}$, as well as $T_{n,m,\varepsilon,1} = RT_{n,m,\varepsilon,0}$ to arrive at

$$\begin{aligned}\bigcup_{i\in I_0} T_iP &\supset \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2\;\middle|\;|x|>\frac{1}{2}\text{ and }|y|\leq|x|\right\}\\ &\quad\cup\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2\;\middle|\;|y|>\frac{1}{2}\text{ and }|x|\leq|y|\right\}.\end{aligned}$$

Now, let $\binom{x}{y}\in\mathbb{R}^2$ be arbitrary. There are two cases:

*Case* 1. $|x|\leq|y|$. Then, there are again two cases:

- *Case* i. $|y|\leq\frac{1}{2}$. Here, $\left|\binom{x}{y}\right|\leq\sqrt{2^{-2}+2^{-2}}=\sqrt{2^{-1}}<1$ and hence $\binom{x}{y}\in B_1(0)\subset T_0P+b_0$.
- *Case* ii. $|y|>\frac{1}{2}$. As seen above, this implies $\binom{x}{y}\in\bigcup_{i\in I_0}T_iP$.

*Case* 2. $|y|\leq|x|$: The proof is completely analogous to the case above.

Thus, we have shown that $(T_i P + b_i)_{i \in I}$ covers all of $\mathbb{R}^2$. As seen above, this completes the proof that $\mathcal{S}$ is a structured admissible covering of $\mathbb{R}^2$.

Finally, we show that $\mathcal{S}$ is almost subordinate to and moderate with respect to $\mathcal{B}$. To this end, let $i = (n, m, \varepsilon, \delta) \in I_0$ and $\binom{x}{y} \in S_i Q$ be arbitrary. By Lemma 6.3.4, this means

$$R^\delta \binom{x}{y} \in R^\delta T_{n,m,\varepsilon,\delta} Q = \varepsilon \cdot U^{(4^n/3, 3\cdot 4^n)}_{(2^{-n}(m-1), 2^{-n}(m+1))}.$$

We now distinguish two cases:

*Case 1.* $\delta = 0$: This implies $\varepsilon x \in (4^n/3, 3 \cdot 4^n)$, i.e. $\frac{4^n}{3} < |x| < 3 \cdot 4^n$. Furthermore, equation (6.4.2) yields $|y| \leq 2|x|$. Consequently,

$$\begin{aligned} 2^{2n-2} \leq \frac{4^n}{3} < |x| \leq \left|\binom{x}{y}\right| &\leq |x| + |y| \\ &\leq 3|x| < 9 \cdot 4^n \leq 2^{2n+4}, \end{aligned} \tag{6.4.8}$$

and thus $\left|\binom{x}{y}\right| \in \bigcup_{\ell=2n}^{2n+2} \left(2^{\ell-2}, 2^{\ell+2}\right)$. Clearly, $k+1 \in k^*$ for all $k \in \mathbb{N}_0$, where the cluster is formed with respect to the Besov covering $\mathcal{B}$. For $n \geq 1$, this shows

$$\binom{x}{y} \in \bigcup_{\ell=2n}^{2n+2} B_\ell \subset B^*_{2n+1}.$$

For $n = 0$, this remains true, because equation (6.4.8) implies

$$\binom{x}{y} \in B_{16}(0) \subset B_8(0) \cup \left[B_{16}(0) \setminus \overline{B_1}(0)\right] \subset B^*_1.$$

*Case 2.* $\delta = 1$: Here, $\varepsilon y \in (4^n/3, 3 \cdot 4^n)$, i.e. $\frac{4^n}{3} < |y| < 3 \cdot 4^n$. Analogous to the derivation above, but using equation (6.4.4) instead of (6.4.2), we get $2^{2n-2} < \left|\binom{x}{y}\right| < 2^{2n+4}$, which easily entails $\binom{x}{y} \in B^*_{2n+1}$.

To conclude the proof that $\mathcal{S}$ is almost subordinate to $\mathcal{B}$, we note $Q \subset (0,3) \times (-3,3)$, which implies

$$\begin{aligned} S_0 = T_0 Q + b_0 &\subset [(0,12) \times (-12,12)] + \binom{-4}{0} \\ &= (-4,8) \times (-12,12) \end{aligned} \tag{6.4.9}$$

and thus $S_0 \subset B_{16}(0) \subset B^*_1$.

To show moderateness of $\mathcal{S}$ with respect to $\mathcal{B}$, let $j \in \mathbb{N}$ and assume

$$\binom{x}{y} \in T_i Q \cap B_j \neq \varnothing$$

for some $i = (n, m, \varepsilon, \delta) \in I_0$.

As seen in equation (6.4.8), this implies

$$2^{j-2} < |\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| < 2^{2n+4} \quad \text{and} \quad 2^{2n-2} < |\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| < 2^{j+2},$$

and thus $\frac{j}{2} - 3 \leq n \leq \frac{j}{2} + 2$. Hence,

$$\frac{2^{\frac{3}{2}j}}{512} \leq 2^{3n} = |\det(T_i)| = 2^{2n} \cdot 2^n = 2^{3n} \leq 64 \cdot 2^{\frac{3}{2}j},$$

which also proves equation (6.4.1) for $i \in I_0$ and $j \in \mathbb{N}$.

In case of $j = 0$, we see $2^{2n-2} < |\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| < 8 = 2^3$ and hence $0 \leq n \leq 2$, which yields

$$\frac{2^{\frac{3}{2}j}}{512} \leq 1 \leq 2^{3n} = |\det(T_i)| = 2^{2n} \cdot 2^n = 2^{3n} \leq 2^6 = 64 = 64 \cdot 2^{\frac{3}{2}j}.$$

Finally, if $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in S_0 \cap B_j \neq \varnothing$ holds for some $j \in \mathbb{N}$, then equation (6.4.9) yields

$$2^{j-2} < |\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| < 16 = 2^4$$

and hence $0 \leq j \leq 5$. For $j = 0$ this is trivial, so that $0 \leq j \leq 5$ holds for all $j \in \mathbb{N}_0$ with $S_0 \cap B_j \neq \varnothing$. Thus, we get

$$\frac{2^{\frac{3}{2}j}}{512} \leq 2^4 = |\det(T_0)| = 4^2 = 2^4 \leq 64 \cdot 2^{\frac{3}{2}j},$$

so that equation (6.4.1) is established in all cases.

Finally, for $i, \ell \in I$ with $S_i \cap B_j \neq \varnothing \neq S_\ell \cap B_j$, equation (6.4.1) yields

$$\frac{|\det(T_i)|}{|\det(T_\ell)|} \leq \frac{64 \cdot 2^{\frac{3}{2}j}}{2^{\frac{3}{2}j}/512} = 2^{15},$$

so that $\mathcal{S}$ is indeed relatively moderate with respect to $\mathcal{B}$. □

Since the shearlet smoothness covering $\mathcal{S}$ is almost subordinate to and relatively moderate with respect to $\mathcal{B}$, we can now easily characterize the existence of embeddings between shearlet smoothness spaces and inhomogeneous Besov spaces.

**Theorem 6.4.3.** *Let $I_0$, $I$ and $\mathcal{S}$ be as in Definition 6.4.1. For $\beta \in \mathbb{R}$, define the weight $w^{(\beta)} = (w_i^{(\beta)})_{i \in I}$ by $w_0^{(\beta)} := 1$ and $w_i^{(\beta)} := 2^{2n\beta}$ for $i = (n, m, \varepsilon, \delta) \in I_0$.*

*Then $w^{(\beta)}$ is $\mathcal{S}$-moderate and $\mathcal{S}$ is an $L^p$-decomposition covering of $\mathbb{R}^2$ for all $p \in (0, \infty]$, so that the* FOURIER-SIDE SHEARLET SMOOTHNESS SPACE

$$\mathcal{FS}_{\beta,q}^p := \mathcal{D}\left(\mathcal{S}, L^p, \ell^q_{w^{(\beta)}}\right)$$

*is well-defined for $p, q \in (0, \infty]$ and $\beta \in \mathbb{R}$.*

*Let $\alpha, \beta \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in (0, \infty]$. Define $v^{(\alpha)} = (2^{n\alpha})_{n \in \mathbb{N}_0}$ as in Lemma 6.2.2.*

*The map*

$$\mathcal{FS}^{p_1}_{\beta,q_1} \to \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\alpha)}}\right), f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$ *and*

$$\begin{cases} \alpha < \beta - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right] - \frac{1}{2}\left(\frac{1}{p_2^{\vee}} - \frac{1}{q_1}\right)_+, & \text{if } q_2 < q_1, \\ \alpha \leq \beta - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right] - \frac{1}{2}\left(\frac{1}{p_2^{\vee}} - \frac{1}{q_1}\right)_+, & \text{if } q_2 \geq q_1. \end{cases} \tag{6.4.10}$$

*Conversely, the map*

$$\mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha)}}\right) \to \mathcal{FS}^{p_2}_{\beta,q_2}, f \mapsto f$$

*is well-defined and bounded if and only if we have* $p_1 \leq p_2$ *and*

*For convenience of the reader, we recall* $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$ *from Remark 5.2.10.*

$$\begin{cases} \beta \leq \alpha - \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right], & \text{if } q_1 \leq q_2, \\ \beta < \alpha - \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right], & \text{if } q_1 > q_2. \end{cases} \tag{6.4.11}$$

*Remark* 6.4.4. A few remarks are in order:

1. Using the criterion given in equation (6.4.10), we see that we can always choose $\alpha \in \mathbb{R}$ in such a way that the identity map

$$\mathcal{FS}^{p}_{\beta,q} \to \mathcal{D}\left(\mathcal{B}, L^{p}, \ell^{q}_{v^{(\alpha)}}\right), f \mapsto f$$

is well-defined and bounded. But Lemma 6.2.2 implies that each distribution $f \in \mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}}\right)$ admits an extension $\widetilde{f} \in \mathcal{S}'\left(\mathbb{R}^2\right)$ of $f$ to a *tempered* distribution. Hence, the same is true for every $f \in \mathcal{FS}^p_{\beta,q}$.

2. The weight $w^{(\beta)}$ as defined above is natural in connection with the definition of Besov spaces, because the proof of Lemma 6.4.2 shows $\langle x \rangle \asymp 2^{2n}$ for $x \in S_{n,m,\varepsilon,\delta}$. For $x \in B_k$, we likewise have $\langle x \rangle \asymp 2^k$, so that the value of the weight $v_k^{(\alpha)}$, which is usually used to define Besov spaces (cf. Lemma 6.2.2), is equivalent to $\langle x \rangle^{\alpha}$ for each $x \in B_k$. With our definition, the same is true for the weight $w^{(\beta)}$ and the covering $\mathcal{S}$ instead of $\mathcal{B}$.

The correct choice of the weight seems to be a delicate matter: In [53, before Proposition 4.3], the authors claim that the Besov quasi-norm $\|f\|_{B^p_{\beta,q}(\mathbb{R}^2)}$ is equivalent to

$$\left[\sum_{j \in \mathbb{N}} \left(2^{\beta j} \left\|\omega_j(D) f\right\|_{L^p}\right)^q\right]^{1/q},$$

where $\left(\omega_j\right)_{j\in\mathbb{N}}$ is a partition of unity subordinate to the family $\left(C_j\right)_j$ with $C_j = \left[-2^{2j+2}, 2^{2j+2}\right]^2 \setminus \left[-2^{2j}, 2^{2j}\right]^2$. Apart from the fact that this "covering" misses a neighborhood of the origin, this quasi-norm is

actually equivalent to $\|f\|_{B^p_{\beta/2,q}(\mathbb{R}^2)}$, since the covering $\left(C_j\right)_j$ does *not* partition $\mathbb{R}^2$ in the usual, "dyadic" way, but rather in a "quaternary" way, so that the weight should be given by $2^{2\beta j}$ instead of $2^{\beta j}$.

3. We remark that the covering in [53, Proposition 4.1] is obtained essentially, with slightly changed notation, as follows: The index set used is $J = \{0\} \cup J_0$ with

$$J_0 = \{(n, m, \delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0, 1\} \,|\, |m| \leq 2^n - 1\}, \tag{6.4.12}$$

and the resulting NON-CONNECTED SHEARLET SMOOTHNESS COVERING is given by

$$\mathcal{S}^{\mathrm{nc}} = (S_j^{\mathrm{nc}})_{j \in J} = \left(P_j Q' + c_j\right)_{j \in J}$$

with

$$P_{n,m,\delta} = A^{(1/2)}_{2n,m,\delta} = T_{n,m,1,\delta} \quad \text{and} \quad c_{n,m,\delta} = 0 = b_{n,m,1,\delta}$$

for $(n, m, \delta) \in J_0$. Finally, the set $Q' = V \cup (-V)$ is the union of a trapezoid $V$ and its inverse $-V$. Essentially, this means

$$Q' \approx Q \cup (-Q),$$

where $Q = U^{\left(\frac{1}{3}, 3\right)}_{-1,1}$ is the "base set" used to define the covering $\mathcal{S}$. Hence, the covering $\mathcal{S}^{\mathrm{nc}}$ is obtained by subsuming the sets $S_{n,m,\pm 1,\delta}$ of the covering $\mathcal{S}$ into the single set $S^{\mathrm{nc}}_{n,m,\delta}$.

Note, however that the actual choice of $V$ used in [53] is $V = U^{\left(\frac{1}{8}, \frac{1}{2}\right)}_{-1,1}$. We remark that with this choice, contrary to [53, Proposition 4.1], the resulting covering is *not* a structured admissible covering of $\mathbb{R}^2$, since not all of $\mathbb{R}^2$ is covered. To see this, note that every

$$\begin{pmatrix} x \\ y \end{pmatrix} \in [(0, \infty) \times \mathbb{R}] \cap A^{(1/2)}_{2n,m,0} Q' = U^{\left(\frac{2^n}{8}, \frac{2^n}{2}\right)}_{(2^{-n}(m-1), 2^{-n}(m+1))}$$

satisfies

$$-1 \leq 2^{-n}(m-1) < \frac{y}{x} < 2^{-n}(m+1) \leq 1,$$

because of $|m| \leq 2^n - 1$. In particular $y \neq x$. It is easy to see that $x \neq y$ also remains true for $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in [(0, \infty) \times \mathbb{R}] \cap A^{(1/2)}_{2n,m,1} Q'$, so that the diagonal of $\mathbb{R}^2$ is not covered by $\mathcal{S}^{\mathrm{nc}}$ (except for the part covered by $S_0^{\mathrm{nc}}$, which is a precompact set). We remark that in [53], the set $V$ is only described through its vertices, so that it could be that actually $V = \overline{U^{\left(\frac{1}{8}, \frac{1}{2}\right)}_{-1,1}}$ is meant. But since any structured admissible covering has to consist of *open* sets, this is not possible either.

Thus, let us assume in the following that the set $J_0$ is given by

$$J_0 = \{(n, m, \delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0, 1\} \,|\, |m| \leq 2^n\}$$

instead of the choice made in [53] (cf. equation (6.4.12) above). With this modification, $\mathcal{S}^{\mathrm{nc}}$ indeed is a structured admissible covering of $\mathbb{R}^2$ if $P_0, c_0$ are chosen appropriately.

Now, although the covering $\mathcal{S} = (S_i)_{i\in I} = (T_iQ + b_i)_{i\in I}$ constructed here is not completely the same as (the corrected version of) the covering $\mathcal{S}^{\mathrm{nc}}$ considered in [53, Proposition 4.1], it is easy to see that $\mathcal{S}$ is almost subordinate to and relatively moderate with respect to $\mathcal{S}^{\mathrm{nc}}$. Furthermore, it is not hard to see that $\mathcal{S}$ and $\mathcal{S}^{\mathrm{nc}}$ are *weakly* equivalent and that we have

$$|\det T_i| \asymp \left|\det\left(P_j\right)\right|$$

for $S_j^{\mathrm{nc}} \cap S_i \neq \varnothing$. Thus, Lemma 5.1.9 shows

$$\mathcal{D}\left(\mathcal{S}, L^p, \ell_u^q\right) = \mathcal{D}\left(\mathcal{S}^{\mathrm{nc}}, L^p, \ell_v^q\right)$$

for all $p, q \in (0, \infty]$ and all weights $u, v$ which are moderate with respect to $\mathcal{S}$ and $\mathcal{S}_{\mathrm{nc}}$, respectively, and which satisfy

$$u_i \asymp v_j \quad \text{for} \quad S_j^{\mathrm{nc}} \cap S_i \neq \varnothing.$$

In particular, we saw in the previous part of this remark that we have $\langle x\rangle \asymp 2^{2n}$ on $S_{n,m,\varepsilon,\delta}$. An analogous estimate also holds for $\mathcal{S}^{\mathrm{nc}}$. Thus, if the weights $u, v$ are selected as

$$u_i = \langle y_i\rangle^{\beta} \quad \text{and} \quad v_j = \langle x_j\rangle^{\beta}$$

for arbitrary $y_i \in S_i$ and $x_j \in S_j^{\mathrm{nc}}$, it is easy to see that the above condition is satisfied. The resulting weight $u$ for $\mathcal{S}$ is equivalent to the weight $w^{(\beta)}$ considered above. The weight $v$ obtained in this way seems to be the weight which is intended in [53, Section 4.3], although the actual definition of the weight is somewhat unclear: In the footnote explaining the choice of the weight $w(j, \ell) = 2^j$, it is claimed that the weight $w$ is given by the projection onto the first axis. But in general, this is not the case, since we have $|x| \asymp 2^{2j} \not\asymp 2^j$ for $\binom{x}{y} \in S_{j,\ell,0} = Q_{j,\ell}$, where $Q_{j,\ell}$ is defined as in [53, Section 4.3 and Proposition 4.1].

Under the assumption that the weights describe above are used, a combination the equality $\mathcal{D}\left(\mathcal{S}, L^p, \ell_u^q\right) = \mathcal{D}\left(\mathcal{S}^{\mathrm{nc}}, L^p, \ell_v^q\right)$ with the embedding $\mathcal{FS}_{\beta,q}^p \hookrightarrow \mathcal{S}'\left(\mathbb{R}^2\right)$ obtained above, shows that the Fourier transform restricts to an isomorphism $\mathcal{F} : S_{p,q}^{\beta} \to \mathcal{FS}_{\beta,q}^p$, where the space-sided shearlet smoothness space $S_{p,q}^{\beta}$ is defined as in [53, Section 4.3].

Since Lemma 6.2.2 shows that the Fourier transform also yields an isomorphism $\mathcal{F} : B_{\alpha,q}^p \to \mathcal{D}\left(\mathcal{B}, L^p, \ell^q_{v^{(\alpha)}}\right)$, we see that the conditions given in equations (6.4.10) and (6.4.11) also characterize existence of the embeddings $S_{p,q}^{\beta} \hookrightarrow B_{\alpha,q_2}^{p_2}$ and $B_{\alpha,q_1}^{p_1} \hookrightarrow S_{p_2,q_2}^{\beta}$, respectively.

4. The theorem above is an improvement of [53, Proposition 4.3], where embeddings between shearlet smoothness spaces and inhomogeneous Besov spaces are considered. In comparison, our result has the following advantages:

A. The present theorem completely characterizes the existence of embeddings, whereas in [53], only sufficient conditions are given.

B. The present theorem also allows the case $(p_1, q_1) \neq (p_2, q_2)$, whereas in [53] only $(p_1, q_1) = (p_2, q_2)$ is considered. ◀

*Proof of Theorem 6.4.3.* We first show that $w^{(\beta)}$ (considered as a weight for $\mathcal{S}$) is $\mathcal{S}$-moderate and relatively $\mathcal{B}$-moderate. To this end, note

$$|\det(T_i)| = \begin{cases} 2^{3n} \asymp w_i^{(3/2)}, & \text{if } i = (n, m, \varepsilon, \delta) \in I_0, \\ 4^2 = 2^{16} \asymp 1 = w_i^{(3/2)}, & \text{if } i = 0. \end{cases}$$

But Lemma 6.4.2 shows that the weight $(|\det(T_i)|)_{i \in I}$ is $\mathcal{B}$-moderate, so that the same is true of the (equivalent) weight $w^{(3/2)}$.

Hence, there are absolute constants $0 < C_0 \leq 1 \leq C_1$ such that

$$C_0 \leq w_i^{(3/2)} / w_j^{(3/2)} \leq C_1$$

holds for all $i, j \in I$ with $S_i \cap B_n \neq \varnothing \neq S_j \cap B_n$ for any $n \in \mathbb{N}_0$. We conclude

$$\min\left\{C_0^{\frac{2}{3}|\beta|}, C_1^{-\frac{2}{3}|\beta|}\right\} \leq \frac{w_i^{(\beta)}}{w_j^{(\beta)}} = \left[\frac{w_i^{(3/2)}}{w_j^{(3/2)}}\right]^{\frac{2}{3}\beta} \leq \max\left\{C_0^{-\frac{2}{3}|\beta|}, C_1^{\frac{2}{3}|\beta|}\right\}$$

for the same indices $i, j$. Hence, $w^{(\beta)}$ is relatively $\mathcal{B}$-moderate.

This also yields $\mathcal{S}$-moderateness of $w^{(\beta)}$: For $x \in S_i \cap S_j \neq \varnothing$, there is some $n \in \mathbb{N}_0$ with $x \in B_n$, so that we have

$$S_i \cap B_n \neq \varnothing \neq S_j \cap B_n$$

and hence $w_i^{(\beta)} \asymp w_j^{(\beta)}$, as we just saw. Finally, Lemma 6.4.2 shows that $\mathcal{S}$ is a structured admissible covering of $\mathbb{R}^2$ and thus an $L^p$-decomposition covering of $\mathbb{R}^2$ for every $p \in (0, \infty]$ by Theorem 3.2.17. This shows that the Fourier-side shearlet smoothness space $\mathcal{S}_{\beta,p}^q$ is well-defined.

To characterize existence of the embedding $\mathcal{FS}_{\beta,q_1}^{p_1} \hookrightarrow \mathcal{D}(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\alpha)}})$, we use part 3 of Theorem 5.4.4, with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{B}$, $u = w^{(\beta)}$ and $v = v^{(\alpha)}$. By Lemma 6.4.2, we know that $\mathcal{Q}$ is indeed almost subordinate to $\mathcal{P}$ and that $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate. Finally, we note that $\mathcal{Q}, \mathcal{P}$ both cover the same set $\mathcal{O} = \mathbb{R}^2 = \mathcal{O}'$. By property 1a of the map $\iota$ constructed in Theorem 5.4.4, this implies $\iota f = f$ for all $f \in \mathcal{D}(\mathcal{Q}, L^{p_1}, \ell^q_u)$. Furthermore, $J_{\mathcal{O}} = J = \mathbb{N}_0$

In summary, part 3 of Theorem 5.4.4 shows that the embedding $\mathcal{FS}^{p_1}_{\beta,q_1} \hookrightarrow \mathcal{D}\left(\mathcal{B}, L^{p_2}, \ell^{q_2}_{v^{(\alpha)}}\right), f \mapsto f$ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}}{u_{i_j}} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}_v}$$

is finite. Here, for each $j \in J_{\mathcal{O}} = \mathbb{N}_0$, an arbitrary index $i_j \in I$ with $\varnothing \neq P_j \cap Q_{i_j} = B_j \cap S_{i_j}$ is chosen. But by Lemma 6.4.2, in particular by equation (6.4.1), this implies

$$w^{(3/2)}_{i_j} \asymp \left|\det\left(T_{i_j}\right)\right| \asymp 2^{\frac{3}{2}j}$$

and

$$u_{i_j} = w^{(\beta)}_{i_j} \asymp \left(w^{(3/2)}_{i_j}\right)^{\frac{2}{3}\beta} \asymp \left(2^{\frac{3}{2}j}\right)^{\frac{2}{3}\beta} = 2^{\beta j},$$

where the implied constants only depend on $\beta \in \mathbb{R}$. Furthermore, $S_j = 2^j \cdot \mathrm{id}$.

Hence, finiteness of the above quasi-norm is equivalent to

$$\begin{aligned} &\left\| \left( 2^{j\cdot\frac{3}{2}\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right]} \cdot 2^{-\beta j} \cdot 2^{j\cdot 2\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}_v} \\ &= \left\| \left( 2^{j\cdot\left(\alpha-\beta+\frac{3}{2}\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right]+2\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right)} \right)_{j\in\mathbb{N}_0} \right\|_{\ell^{q_2\cdot(q_1/q_2)'}} . \end{aligned}$$

But Remark 5.2.3 shows $q_2 \cdot (q_1/q_2)' < \infty \Leftrightarrow q_2 < q_1$, so that finiteness of the above quasi-norm is equivalent to

$$\begin{cases} \alpha < \beta - \frac{3}{2}\left[\frac{1}{p_1}-\frac{1}{p_2}\right] - \frac{1}{2}\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+, & \text{if } q_2 < q_1, \\ \alpha \leq \beta - \frac{3}{2}\left[\frac{1}{p_1}-\frac{1}{p_2}\right] - \frac{1}{2}\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+, & \text{if } q_2 \geq q_1. \end{cases}$$

This proves the first claim.

To characterize existence of the embedding $\mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha)}}\right) \hookrightarrow \mathcal{FS}^{p_2}_{\beta,q_2}$, we use part 3 of Theorem 5.4.1, with $\mathcal{Q} = \mathcal{S}$, $\mathcal{P} = \mathcal{B}$, $u = w^{(\beta)}$ and $v = v^{(\alpha)}$, but with interchanged roles of $p_1, p_2$ and of $q_1, q_2$. The prerequisites are verified just as in the case of the reverse embedding. Thus, we conclude that the map

$$\mathcal{D}\left(\mathcal{B}, L^{p_1}, \ell^{q_1}_{v^{(\alpha)}}\right) \to \mathcal{FS}^{p_2}_{\beta,q_2}, f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\| \left( \frac{u_{i_j}}{v_j} \left| \det\left(S_j\right) \right|^{\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \left| \det\left(T_{i_j}\right) \right|^{\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell^{q_2}(q_1/q_2)'}$$

is finite. As above, we see that finiteness of the above quasi-norm is equivalent to finiteness of

$$\left\| \left( 2^{j(\beta-\alpha)} \cdot 2^{2j\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+} \cdot 2^{j \cdot \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+\right]} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^{q_2}(q_1/q_2)'}$$

and hence to

$$\begin{cases} \beta \leq \alpha - \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right], & \text{if } q_1 \leq q_2, \\ \beta < \alpha - \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ - \frac{3}{2}\left[\frac{1}{p_1} - \frac{1}{p_2}\right], & \text{if } q_1 > q_2. \end{cases}$$

This completes the proof. □

As a final application of our embedding results to shearlet smoothness spaces, we want to characterize existence of embeddings between certain $\alpha$-modulation spaces and shearlet smoothness spaces. To this end, we first have to establish suitable subordinateness results. This is done in the following lemma.

**Lemma 6.4.5.** *Let $\alpha \in \left[0, \frac{1}{2}\right]$ and choose $r_1 = r_1(\alpha, 2)$ as in Theorem 6.1.1. If $r > r_1$, then $\mathcal{O}_r^{(\alpha)}$ is almost subordinate to and moderate with respect to the shearlet smoothness covering $\mathcal{S} = (S_i)_{i \in I}$ as given in Definition 6.4.1.*

*The weight $w^{(\gamma)} : \mathbb{Z}^2 \setminus \{0\} \to (0, \infty), k \mapsto \langle k \rangle^{\gamma}$ given in Lemma 6.1.2, considered as a weight for $\mathcal{O}_r^{(\alpha)}$, is relatively moderate with respect to $\mathcal{S}$. More precisely, we have*

$$w_k^{(\gamma)} \asymp 2^{2n(1-\alpha)\gamma} \text{ for } O_{r,k}^{(\alpha)} \cap S_i \neq \varnothing \text{ with } i = (n, m, \varepsilon, \delta) \in I_0, \quad (6.4.13)$$

*where the implied constants only depend on $r, \alpha, \gamma$.*

*Proof.* Let $\alpha_0 := \frac{\alpha}{1-\alpha}$. We start by (asymptotically) calculating the bracket $\langle x \rangle$ for $x \in S_i$ and $x \in O_{r,k}^{(\alpha)}$, respectively. Lemma 6.1.2 implies

$$\langle x \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}} \quad \text{for all } k \in \mathbb{Z}^2 \setminus \{0\} \text{ and } x \in O_{r,k}^{(\alpha)}. \quad (6.4.14)$$

Regarding the covering $\mathcal{S}$, recall that Lemma 6.4.2 yields $S_i \subset B^*_{2n+1}$ for arbitary $i = (n, m, \varepsilon, \delta) \in I_0$. But it is easy to see $\langle x \rangle \asymp 2^k$ on $B_k$ for $k \in \mathbb{N}_0$. This yields

$$\langle x \rangle \asymp 2^{2n+1} \asymp 2^{2n} \text{ on } S_i \text{ for } i = (n, m, \varepsilon, \delta) \in I_0. \tag{6.4.15}$$

Using these estimates, we can show that the weight $w^{(\gamma)}$, considered as a weight for $\mathbb{O}_r^{(\alpha)}$, is moderate with respect to $\mathcal{S}$. Indeed, let $x \in O_{r,k}^{(\alpha)} \cap S_i \neq \varnothing$ for some $i = (n, m, \varepsilon, \delta) \in I_0$. Using equations (6.4.14) and (6.4.15), we see

$$2^{2n} \asymp \langle x \rangle \asymp \langle k \rangle^{\frac{1}{1-\alpha}}$$

and hence

$$w_k^{(\gamma)} = \langle k \rangle^{\gamma} = \left( \langle k \rangle^{\frac{1}{1-\alpha}} \right)^{\gamma(1-\alpha)} \asymp 2^{2n(1-\alpha)\gamma},$$

which in particular establishes equation (6.4.13). Here, the implied constants are independent of $i, k$.

In particular, we see that $O_{r,k}^{(\alpha)} \cap S_i \neq \varnothing \neq O_{r,\ell}^{(\alpha)} \cap S_i$ for some index $i = (n, m, \varepsilon, \delta) \in I_0$ and $k, \ell \in \mathbb{Z}^2 \setminus \{0\}$ implies

$$w_\ell^{(\gamma)} \asymp 2^{2n(1-\alpha)\gamma} \asymp w_k^{(\gamma)}$$

and hence

$$\frac{w_k^{(\gamma)}}{w_\ell^{(\gamma)}} \leq C_1$$

for a constant $C_1 > 0$ which is independent of $k, \ell \in \mathbb{Z}^2 \setminus \{0\}$ and of $i \in I_0$.

Finally, $\overline{S_0}$ is a compact subset of $\mathbb{R}^2$. Since $\mathbb{O}_r^{(\alpha)}$ is a locally finite covering of $\mathbb{R}^2$, we see that

$$L_0 := \left\{ k \in \mathbb{Z}^2 \setminus \{0\} \;\middle|\; O_{r,k}^{(\alpha)} \cap S_0 \neq \varnothing \right\}$$

is a finite set, so that

$$\frac{w_k^{(\gamma)}}{w_\ell^{(\gamma)}} \leq \frac{\max_{\kappa \in L_0} w_\kappa^{(\gamma)}}{\min_{\lambda \in L_0} w_\lambda^{(\gamma)}} =: C_2$$

holds for all $k, \ell \in \mathbb{Z}^2 \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap S_0 \neq \varnothing \neq O_{r,\ell}^{(\alpha)} \cap S_0$.

In summary, we have shown that $w^{(\gamma)}$ is $\mathcal{S}$-moderate. Because of $O_{r,k}^{(\alpha)} = \left( |k|^{\alpha_0} \cdot \mathrm{id} \right) (B_r(0)) + |k|^{\alpha_0} k$, and

$$\left| \det \left( |k|^{\alpha_0} \mathrm{id} \right) \right| = |k|^{2\alpha_0} \asymp \langle k \rangle^{2\alpha_0} = w_k^{(2\alpha_0)} \text{ for } k \in \mathbb{Z}^2 \setminus \{0\},$$

this also shows that $\mathbb{O}_r^{(\alpha)}$ is relatively $\mathcal{S}$-moderate.

It remains to show that $\mathbb{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{S}$. Since each set $O_{r,k}^{(\alpha)} = B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)$ is convex and hence path-connected, since each of the sets $S_i$ of the covering $\mathcal{S}$ is open, and because $\mathcal{S}$ and $\mathbb{O}_r^{(\alpha)}$ both cover all of $\mathbb{R}^2$, Corollary 3.3.4 implies that it suffices to show that the cardinality of the sets

$$I_k := \left\{ i \in I \,\middle|\, O_{r,k}^{(\alpha)} \cap S_i \neq \varnothing \right\},$$

with $I$ as in Definition 6.4.1, is uniformly bounded, independent of $k \in \mathbb{Z}^2 \setminus \{0\}$.

To this end, note that equations (6.4.13) and (6.4.15) yield constants $C_1, C_2 \geq 1$ with

$$C_1^{-1} \cdot 2^{2n} \leq \langle k \rangle^{\frac{1}{1-\alpha}} \leq C_1 \cdot 2^{2n}, \tag{6.4.16}$$

for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and $i = (n, m, \varepsilon, \delta) \in I_0$ with $S_i \cap O_{r,k}^{(\alpha)} \neq \varnothing$ and with

$$C_2^{-1} \cdot 2^{2n} \leq \langle x \rangle \leq C_2 \cdot 2^{2n} \tag{6.4.17}$$

for all $x \in S_i$ for $i = (n, m, \varepsilon, \delta) \in I_0$.

Define $N_0 := \lceil \log_2 (12 \cdot r \cdot C_1^{\alpha}) \,/\, [2\,(1-\alpha)] \rceil$ and observe that the set

$$\begin{aligned} M &:= \{0\} \cup (I_0 \cap [\{0, \dots, N_0\} \times \{-2^{N_0}, \dots, 2^{N_0}\} \times \{\pm 1\} \times \{0,1\}]) \\ &= \{0\} \cup (I_0 \cap [\{0, \dots, N_0\} \times \mathbb{Z} \times \{\pm 1\} \times \{0,1\}]) \subset I \end{aligned}$$

is a fixed finite set, so that it suffices to uniformly bound the cardinality of the sets $I_k \setminus M$.

Now, let $k \in \mathbb{Z}^2 \setminus \{0\}$ and $i = (n, m, \varepsilon, \delta) \in I_k \setminus M \subset I_0$. This implies $|m| \leq 2^n$ and yields existence of

$$\begin{pmatrix} x \\ y \end{pmatrix} \in O_{r,k}^{(\alpha)} \cap S_i = B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right) \cap \left[ \varepsilon R^{\delta} U_{(2^{-n}(m-1), 2^{-n}(m+1))}^{\left(2^{2n}/3, 3 \cdot 2^{2n}\right)} \right].$$

Furthermore, we have $n \geq N_0$ because of $i \notin M$. We conclude $2n\,(1-\alpha) \geq \log_2 (12 \cdot r \cdot C_1^{\alpha})$ and hence $2n \geq 2\alpha n + \log_2 (12 \cdot r \cdot C_1^{\alpha})$, which means

$$2^{2\alpha n} \cdot 12 \cdot r \cdot C_1^{\alpha} \leq 2^{2n}.$$

Together with equation (6.4.16), we derive

$$\begin{aligned} r\,|k|^{\alpha_0} \leq r\,\langle k \rangle^{\alpha_0} &= r\,\langle k \rangle^{\frac{\alpha}{1-\alpha}} \leq r \cdot C_1^{\alpha} \cdot 2^{2n\alpha} \\ &\leq \frac{2^{2n}}{12} = 2^{2n}\left(\frac{1}{3} - \frac{1}{4}\right) \leq 2^{2n}. \end{aligned} \tag{6.4.18}$$

Let $\binom{\ell}{t} := |k|^{\alpha_0} k$. We distinguish two cases:

CASE 1: $\delta = 0$. This yields $\varepsilon x \in \left(2^{2n}/3, 3 \cdot 2^{2n}\right)$ and hence $2^{2n}/3 \leq |x| \leq 3 \cdot 2^{2n}$. Recalling equation (6.4.18), we conclude

$$|\ell| \leq |x| + \left|\binom{\ell}{t} - \binom{x}{y}\right| < 3 \cdot 2^{2n} + r\,|k|^{\alpha_0} \leq 4 \cdot 2^{2n}.$$

Similarly

$$\begin{aligned} |\ell| &\geq |x| - |\ell - x| > \frac{2^{2n}}{3} - \left|\binom{\ell}{t} - \binom{x}{y}\right| \\ &\geq \frac{2^{2n}}{3} - r\,|k|^{\alpha_0} \geq \frac{2^{2n}}{3} - 2^{2n}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2^{2n}}{4} > 0. \end{aligned}$$

Together, these estimates yield $2^{2n-2} \leq |\ell| \leq 2^{2n+2}$ and hence also $-1 \leq n - \frac{1}{2}\log_2|\ell| \leq 1$, which in particular implies

$$n \in \overline{B_1}\left(\frac{1}{2}\log_2|\ell|\right) \cap \mathbb{Z} =: M_\ell,$$

where the latter set has cardinality at most $|M_\ell| \leq 3$.

Furthermore, equations (6.4.17) and (6.4.18) imply

$$|t| \leq \left|\binom{\ell}{t}\right| \leq \left|\binom{x}{y}\right| + r\,|k|^{\alpha_0} \leq C_2 \cdot 2^{2n} + 2^{2n} = (1 + C_2) \cdot 2^{2n}.$$

Finally, we note that equation (6.4.16) yields

$$|k|^{\alpha_0} \leq \langle k \rangle^{\frac{\alpha}{1-\alpha}} \leq C_1^\alpha \cdot 2^{2n\alpha}.$$

Recall from above $2^{2n}/3 \leq |x| \leq 3 \cdot 2^{2n}$ and $2^{2n}/4 \leq |\ell| \leq 4 \cdot 2^{2n}$. Altogether, we arrive at

$$\begin{aligned} \left|\frac{y}{x} - \frac{t}{\ell}\right| = \left|\frac{y\ell - tx}{x\ell}\right| &\leq \frac{|(y-t)\,\ell + t\,(\ell - x)|}{\frac{2^{2n}}{3} \cdot \frac{2^{2n}}{4}} \\ &\leq \frac{12}{2^{4n}} \cdot \left[\left|\binom{x}{y} - \binom{\ell}{t}\right| \cdot |\ell| + |t| \cdot \left|\binom{\ell}{t} - \binom{x}{y}\right|\right] \\ &\leq \frac{12}{2^{4n}} \cdot r\,|k|^{\alpha_0}\left(4 \cdot 2^{2n} + (1 + C_2) \cdot 2^{2n}\right) \\ &\leq \frac{12 \cdot (5 + C_2)}{2^{2n}} \cdot r \cdot C_1^\alpha \cdot 2^{2n\alpha} \\ &\leq \frac{12 \cdot (5 + C_2)}{2^n} \cdot r \cdot C_1^\alpha, \end{aligned}$$

where $\alpha \leq 1/2$ was used in the last step.

But we have $\frac{y}{x} = \frac{\varepsilon y}{\varepsilon x} \in (2^{-n}(m-1), 2^{-n}(m+1))$ and hence

$$\begin{aligned} \left|m - 2^n\frac{t}{\ell}\right| &\leq \left|m - 2^n\frac{y}{x}\right| + \left|2^n\frac{y}{x} - 2^n\frac{t}{\ell}\right| \\ &< 1 + 12 \cdot (5 + C_2) \cdot C_1^\alpha r =: K. \end{aligned}$$

This means $m \in B_K\left(2^n \frac{t}{\ell}\right) \cap \mathbb{Z} =: M_{t,\ell,n}$, where $M_{t,\ell,n}$ has cardinality at most $|M_{t,\ell,n}| \leq 2K+1$. All in all, we arrive at

$$i = (n, m, \varepsilon, \delta) \in \bigcup_{\lambda \in M_\ell} [\{\lambda\} \times M_{t,\ell,\lambda} \times \{\pm 1\} \times \{0,1\}] =: M^{(k)},$$

where the set on the right-hand side only depends on $\binom{\ell}{t} = |k|^{\alpha_0} k$, i.e. only on $k$. Furthermore, $M^{(k)}$ has cardinality at most $3 \cdot 4 \cdot (2K+1)$, because of $|M_\ell| \leq 3$.

CASE 2: $\delta = 1$. We observe

$$R \cdot O_{r,k}^{(\alpha)} = R\left(B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)\right) = B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} Rk\right) = B_{r|k'|^{\alpha_0}}\left(\left|k'\right|^{\alpha_0} \cdot k'\right)$$

for $k' := Rk \in \mathbb{Z}^2 \setminus \{0\}$, where we used the fact that $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{Z}^2$ is orthogonal. But this implies

$$\begin{aligned} \varnothing \neq R \cdot \left(O_{r,k}^{(\alpha)} \cap S_i\right) &= RO_{r,k}^{(\alpha)} \cap RT_i Q \\ &= O_{r,k'}^{(\alpha)} \cap T_{(n,m,\varepsilon,0)} Q. \end{aligned}$$

As seen in the first case, this entails $(n, m, \varepsilon, 0) \in M^{(k')}$ and hence also $i = (n, m, \varepsilon, \delta) \in M^{(k')}$. Here, we used that the form of $M^{(k)}$ easily implies that $(n, m, \varepsilon, 0) \in M^{(k)}$ is equivalent to $(n, m, \varepsilon, \theta) \in M^{(k)}$, for arbitrary $\theta \in \{0, 1\}$.

Together, our findings in the two cases imply $I_k \setminus M \subset M^{(k)} \cup M^{(k')}$, where the set on the right has cardinality at most $24 \cdot (2K+1)$. This completes the proof. □

Using these subordinateness and moderateness statements, we can now specialize our general embedding results to characterize the embeddings between shearlet smoothness spaces and $\alpha$-modulation spaces, at least in the case $\alpha \in \left[0, \frac{1}{2}\right]$.

**Theorem 6.4.6.** *Let $\alpha \in \left[0, \frac{1}{2}\right]$, and $\beta, \gamma \in \mathbb{R}$. For $p, q \in (0, \infty]$, let the Fourier side shearlet smoothness spaces $\mathcal{FS}_{\beta,q}^{p}$ be defined as in Theorem 6.4.3 and let the Fourier side $\alpha$-modulation space $\mathcal{F}M_{p,q}^{\gamma,\alpha}$ be as in Definition 6.1.4.*

*Let $p_1, p_2, q_1, q_2 \in (0, \infty]$ and define*

$$\begin{aligned} \beta^{(1)} &:= \left(\frac{3}{2} - 2\alpha\right)\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + 2\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right), \\ \beta^{(2)} &:= \left(\frac{3}{2} - 2\alpha\right)\left(\frac{1}{p_2^{\triangledown}} - \frac{1}{q_1}\right)_+ + 2\alpha\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \end{aligned}$$

*Then,*

$$\mathcal{FS}_{\beta,q_1}^{p_1} \to \mathcal{F}M_{p_2,q_2}^{\gamma,\alpha}, f \mapsto f$$

*is well-defined and bounded if and only if we have $p_1 \leq p_2$ as well as*

$$\begin{cases} \beta \geq \gamma + \beta^{(1)}, & \text{if } q_1 \leq q_2, \\ \beta > \gamma + \beta^{(1)} + \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{q_1}\right), & \text{if } q_1 > q_2. \end{cases}$$

*Conversely,*

$$\mathcal{F}M^{\gamma,\alpha}_{p_1,q_1} \to \mathcal{FS}^{p_2}_{\beta,q_2}, f \mapsto f$$

*is well-defined and bounded if and only if we have $p_1 \leq p_2$ as well as*

$$\begin{cases} \gamma \geq \beta + \beta^{(2)}, & \text{if } q_1 \leq q_2, \\ \gamma > \beta + \beta^{(2)} + \frac{1}{2}\left(\frac{1}{q_2} - \frac{1}{q_1}\right), & \text{if } q_1 > q_2. \end{cases}$$

*Remark.* As seen in Remark 6.4.4 and Lemma 6.1.9, the Fourier transform yields isomorphisms between the Fourier-side shearlet smoothness and $\alpha$-modulation spaces with their space-sided versions. Thus, the above characterization also remains true for the respective embeddings for the space-sided versions of $\alpha$-modulation and shearlet smoothness spaces.

*Proof.* Let $r_1 = r_1(\alpha, d)$ as in Theorem 6.1.1 and choose an arbitrary $r > r_1$. Set $\alpha_0 := \frac{\alpha}{1-\alpha}$. Note that we have

$$\mathcal{F}M^{\gamma,\alpha}_{p,q} = \mathcal{D}\left(\mathcal{O}^{(\alpha)}_r, L^p, \ell^q_{w^{(\gamma/(1-\alpha))}}\right),$$

where the weight $w^{(\gamma/(1-\alpha))}$ is defined as in Lemma 6.1.2.

Furthermore, note that we have $\mathcal{FS}^p_{\beta,q} = \mathcal{D}(\mathcal{S}, L^p, \ell^q_{w^{(\beta)}})$, where the weight $w^{(\beta)}$ is defined as in Theorem 6.4.3. To avoid confusion between the weights $w^{(\gamma/(1-\alpha))}$ and $w^{(\beta)}$, we will write $w^{(\beta)}_0 = w^{(\beta)}$ for the weight which is used for the shearlet smoothness covering $\mathcal{S}$.

To characterize boundedness of the map

$$\mathcal{FS}^{p_1}_{\beta,q_1} \to \mathcal{F}M^{\gamma,\alpha}_{p_2,q_2}, f \mapsto f,$$

we use part 3 of Theorem 5.4.1 with $\mathcal{Q} := \mathcal{O}^{(\alpha)}_r$, $\mathcal{P} = \mathcal{S}$, $u = w^{(\gamma/(1-\alpha))}$ and $v = w^{(\beta)}_0$, but with interchanged roles of $p_1, p_2$ and of $q_1, q_2$. Note that $\mathcal{Q}, \mathcal{P}$ are indeed structured (and hence tight) admissible coverings of $\mathcal{O} = \mathbb{R}^2 = \mathcal{O}'$ by Theorem 6.1.3 and Lemma 6.4.2. Furthermore, $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and relatively $\mathcal{P}$-moderate, as shown in Lemma 6.4.5. The same lemma also shows that the weight $u = w^{(\gamma/(1-\alpha))}$ is relatively $\mathcal{P}$-moderate.

Thus, all assumptions of 3 of Theorem 5.4.1 are fulfilled. Here, we also used $\mathcal{O} = \mathcal{O}'$ to ensure $P_j \subset \mathcal{O}$ for all

$$j \in J_{\mathcal{O}} = \left\{ j \in J \,\middle|\, P_j \cap \mathcal{O} \neq \varnothing \right\} = J.$$

Using Theorem 5.4.1, we conclude that $\mathcal{FS}^{p_1}_{\beta,q_1} \hookrightarrow \mathcal{FM}^{\gamma,\alpha}_{p_2,q_2}, f \mapsto f$ is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+} \left|\det\left(T_{i_j}\right)\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q_2}\left(q_1/q_2\right)'}$$

is finite. Here, for each $j \in J_{\mathcal{O}} = J$ (where $J$ denotes the index set of $\mathcal{P} = \mathcal{S}$), some $i_j \in I = \mathbb{Z}^2 \setminus \{0\}$ is chosen with $\varnothing \neq O^{(\alpha)}_{r,i_j} \cap S_j$. Note that the single term for $j = \{0\}$ does not affect finiteness of this norm. Thus, we can restrict ourselves to $J_{\mathcal{O}} \setminus \{0\} = J \setminus \{0\} = I_0$, with $I_0$ as in Definition 6.4.1

Because of $\varnothing \neq O^{(\alpha)}_{r,i_j} \cap S_j$, Lemma 6.4.5 yields

$$u_{i_j} = w_{i_j}^{(\gamma/(1-\alpha))} \asymp 2^{2n(1-\alpha)\frac{\gamma}{1-\alpha}} = 2^{2\gamma n}$$

for all $j = (n, m, \varepsilon, \delta) \in I_0$. Because of

$$Q_{i_j} = O^{(\alpha)}_{r,i_j} = B_{r\left|i_j\right|^{\alpha_0}}\left(\left|i_j\right|^{\alpha_0} i_j\right) = \left(\left|i_j\right|^{\alpha_0} \cdot \mathrm{id}\right)\left(B_r\left(0\right)\right) + \left|i_j\right|^{\alpha_0} i_j,$$

we also get

$$\left|\det\left(T_{i_j}\right)\right| = \left|\det\left(\left|i_j\right|^{\alpha_0} \cdot \mathrm{id}\right)\right| = \left|i_j\right|^{2\alpha_0} \asymp w_{i_j}^{(2\alpha_0)} \asymp 2^{2n(1-\alpha)\cdot 2\alpha_0} = 2^{4n\alpha}.$$

Finally, we have

$$\left|\det\left(S_j\right)\right| = \left|\det\left(R^{\delta} A^{(1/2)}_{2n,m,\varepsilon}\right)\right| = 2^{3n}$$

for $j = (n, m, \varepsilon, \delta) \in I_0$, because of

$$A^{(1/2)}_{2n,m,\varepsilon} = \varepsilon \cdot \begin{pmatrix} 2^{2n} & 0 \\ 2^n m & 2^n \end{pmatrix}.$$

Thus, finiteness of the above quasi-norm is equivalent to finiteness of

$$\left\| \left( \frac{2^{2\gamma n}}{2^{2n\beta}} 2^{3n\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+ + 4n\alpha\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{q_2}-\frac{1}{p_1^{\pm\triangle}}\right)_+\right]} \right)_{(n,m,\varepsilon,\delta)\in I_0} \right\|_{\ell^{q_2}\left(q_1/q_2\right)'}. \tag{6.4.19}$$

Here, we recall

$$I_0 = \left\{(n, m, \varepsilon, \delta) \in \mathbb{N}_0 \times \mathbb{Z} \times \{\pm 1\} \times \{0, 1\} \mid |m| \leq 2^n\right\}.$$

There are now two cases: For $q_1 \leq q_2$, Remark 5.2.3 shows that we have $q_2 \cdot (q_1/q_2)' = \infty$, so that the above quasi-norm is finite if and only if we have

$$\theta := 2(\gamma - \beta) + 3\left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+ + 4\alpha\left[\frac{1}{p_1} - \frac{1}{p_2} - \left(\frac{1}{q_2} - \frac{1}{p_1^{\pm\triangle}}\right)_+\right] \leq 0,$$

which is equivalent to the condition given in the theorem for this case.

For $q_1 > q_2$, the same remark yields $\frac{1}{s} = \frac{1}{q_2} - \frac{1}{q_1} \in (0, \infty)$ for $s := q_2 \cdot (q_1/q_2)'$ and hence

$$\begin{aligned}\left\|(2^{n\theta})_{(n,m,\varepsilon,\delta)\in I_0}\right\|_{\ell^s} &= \left\|\left(\left\|(2^{n\theta})_{(m,\varepsilon,\delta)\in\{-2^n,\dots,2^n\}\times\{\pm1\}\times\{0,1\}}\right\|_{\ell^s}\right)_{n\in\mathbb{N}_0}\right\|_{\ell^s} \\ &= \left\|\left(2^{n\theta}\cdot\left[4\cdot(2\cdot 2^n+1)\right]^{\frac{1}{s}}\right)_{n\in\mathbb{N}_0}\right\|_{\ell^s} \\ &\asymp \left\|\left(2^{n\left(\theta+\frac{1}{s}\right)}\right)_{n\in\mathbb{N}_0}\right\|_{\ell^s}.\end{aligned}$$

This shows that the quasi-norm in equation (6.4.19) is finite if and only if we have $\theta + \frac{1}{s} < 0$. This is again equivalent to the condition given in the statement of the theorem.

For the reverse embedding, we employ part 3 of Theorem 5.4.4 with $\mathcal{Q} := \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \mathcal{S}$, $u = w^{(\gamma/(1-\alpha))}$ and $v = w_0^{(\beta)}$. Exactly as in the case above, we see that all assumptions of this theorem are satisfied. Because of $\mathcal{O} = \mathcal{O}' = \mathbb{R}^2$ and because of property 1a of Theorem 5.4.4, we see that the map $\iota$ constructed in this theorem statisfies $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_u^{q_1}\right)$. Hence, the map

$$\mathcal{F}M_{p_1,q_1}^{\gamma,\alpha} \to \mathcal{FS}_{\beta,q_2}^{p_2}, f \mapsto f$$

is well-defined and bounded if and only if we have $p_1 \leq p_2$ and if the quasi-norm

$$\left\|\left(\frac{\left|\det(S_j)\right|^{\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}}{u_{i_j}}\left|\det(T_{i_j})\right|^{\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}\right)_{j\in J_{\mathcal{O}}}\right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}}$$

is finite.

As above, we see that this is equivalent to finiteness of

$$\left\|\left(2^{2\beta n}2^{-2\gamma n}2^{3n\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+}2^{4n\alpha\left[\frac{1}{p_1}-\frac{1}{p_2}-\left(\frac{1}{p_2^\vee}-\frac{1}{q_1}\right)_+\right]}\right)_{(n,m,\varepsilon,\delta)\in I_0}\right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}}.$$

With the same arguments as for the first embedding, we see that this is equivalent to the condition in the statement of theorem. □

## 6.5 DILATIONS OF DECOMPOSITION SPACES AND COORBIT SPACES

In this section, we will use our embedding results for decomposition spaces to asymptotically calculate the operator norm of the isotropic dilation operator

$$f \mapsto f \circ \lambda \mathrm{id}$$

as a function of $\lambda \in (0, \infty)$, considered on different decomposition spaces.

Precisely, we will consider the coverings $\mathcal{S}^{(c)}$ for various values of $c \in \mathbb{R}$, as well as the $\alpha$-modulation coverings $\mathcal{O}^{(\alpha)}$. This will yield an asymptotic calculation of the norm of the dilation operator for certain shearlet-type coorbit spaces and for $\alpha$-modulation spaces.

We first remark that calculation of the operator norm $|||f \mapsto f \circ \lambda \mathrm{id}|||$ for a fixed $\lambda \in (0, \infty)$ on a fixed decomposition space does not make sense, since the quasi-norm on $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ is only well-defined up to constant factors. But if we *fix* one BAPU $\Phi = (\varphi_i)_{i \in I}$ for $\mathcal{Q}$, we can calculate the behaviour of

$$|||f \mapsto f \circ \lambda \mathrm{id}|||_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)}$$

as a function of $\lambda$, where the same BAPU $\Phi$ is used for all values of $\lambda \in (0, \infty)$. A different BAPU $\Psi$ will yield the same *asymptotic* behaviour (for $\lambda \downarrow 0$ and $\lambda \to \infty$), since we have

$$\begin{aligned} &|||f \mapsto f \circ \lambda \mathrm{id}|||_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}_\Phi\left(\mathcal{Q}, L^p, \ell^q_u\right)} \\ &\leq C_{\Phi,\Psi} \cdot |||f \mapsto f \circ \lambda \mathrm{id}|||_{\mathcal{D}_\Psi\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}_\Psi\left(\mathcal{Q}, L^p, \ell^q_u\right)}, \end{aligned}$$

with a constant $C_{\Phi,\Psi}$ which does not depend on $\lambda$.

To see how our embedding results can be used to calculate this asymptotic behaviour of the operator norm of the dilation operator, recall the identity

$$\|f_A\|_{\mathcal{D}_\Phi\left(\mathcal{Q}, L^p, \ell^q_u\right)} = |\det(A)|^{\frac{1}{p}-1} \cdot \|f\|_{\mathcal{D}_{\Phi_A}\left(A\mathcal{Q}, L^p, \ell^q_u\right)}$$

from Lemma 3.5.3, where $A \in \mathrm{GL}\left(\mathbb{R}^d\right)$ is an invertible matrix and $f \in \mathcal{D}'(A\mathcal{O})$.

Now, suppose $A\mathcal{O} = \mathcal{O}$ for some $A \in \mathrm{GL}\left(\mathbb{R}^d\right)$. Then

$$\Gamma_A : \mathcal{D}'(\mathcal{O}) \to \mathcal{D}'(\mathcal{O}), f \mapsto f_A = f \circ A$$

is well-defined and the above identity implies that $\Gamma_A$ restricts to a bounded operator on $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ if and only if the embedding

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right) \to \mathcal{D}\left(A\mathcal{Q}, L^p, \ell^q_u\right), f \mapsto f$$

is well-defined and bounded. In that case, we have

$$|||\Gamma_A|||_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)} = |\det(A)|^{\frac{1}{p}-1} \cdot |||\iota|||_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)\to\mathcal{D}_{\Phi_A}\left(A\mathcal{Q},L^p,\ell^q_u\right)}.$$

Here, the subscripts $\Phi$ and $\Phi_A$ indicate the BAPU which is used to calculate the (quasi)-norm on the respective decomposition spaces.

In the present context, it is important to take the BAPU $\Phi_A$ for $A\mathcal{Q}$, where $\Phi$ is a fixed BAPU for $\mathcal{Q}$, since we are interested in the behaviour of $|||\Gamma_A|||$ for varying $A$. Of course, we have norm-equivalence

$$\|f_A\|_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)} \asymp \|f\|_{\mathcal{D}_{\Psi^{(A)}}\left(A\mathcal{Q},L^p,\ell^q_u\right)}$$

for every choice of the $L^p$-BAPU $\Psi^{(A)}$ for $A\mathcal{Q}$, but the constant of norm equivalence might heavily depend on the choice of $\Psi^{(A)}$, which would make it impossible to determine the asymptotics of $|||\Gamma_A|||$. But with the choice $\Psi^{(A)} = \Phi_A = \left(\varphi_i \circ A^{-1}\right)_{i\in I}$, the above considerations reduce the calculation of $|||\Gamma_A|||$ to calculation of the norm

$$|||\iota|||_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)\to\mathcal{D}_{\Phi_A}\left(A\mathcal{Q},L^p,\ell^q_u\right)},$$

i.e. to calculation of the operator norm of an *embedding for decomposition spaces*.

Using the results from Chapter 5, this norm can be asymptotically computed in many cases. This also justifies the close attention which we gave to the dependence of the constants in Chapter 5. We recall from Lemma 3.5.1 that all constans $N_{\mathcal{Q}}, C_{\mathcal{Q}}$, etc. remain unchanged when we consider the covering $A\mathcal{Q}$ instead of $\mathcal{Q}$.

Thanks to our careful handling of the constants in Chapter 5, we know that the constant of equivalence between the norm

$$|||\iota|||_{\mathcal{D}_\Phi\left(\mathcal{Q},L^p,\ell^q_u\right)\to\mathcal{D}_{\Phi_A}\left(A\mathcal{Q},L^p,\ell^q_u\right)}$$

and between the nested sequence norms used to characterize existence of this embedding only depend on these constants, and on very few other quantities like $k(\mathcal{Q}, A\mathcal{Q})$, which we will control uniformly.

### 6.5.1 *Isotropic dilations of shearlet-type decomposition spaces*

As our first application of the considerations above, we consider the case of the covering $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$, which is induced by the shearlet-type group $H_c$, cf. Corollary 6.3.5. Eventually, this will allow us to calculate the asymptotic behaviour of the norm of the dilation operator on the shearlet-type coorbit spaces $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$, with weight given by $m\left(\varepsilon\left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)\right) = a^\alpha$ for some $\alpha \in \mathbb{R}$.

We begin by analyzing the relation between the covering $\mathcal{S}^{(c)}$ and the dilated covering $\lambda\mathcal{S}^{(c)}$, at least for $c \in (-\infty, 1]$. Note that this

includes the important special case $c = \frac{1}{2}$, which corresponds to the usual shearlet group $H_{1/2}$.

**Lemma 6.5.1.** *Let $c \in (-\infty, 1]$ and $\lambda \geq 1$. Then $\mathcal{S}^{(c)}$ is almost subordinate to and relatively moderate with respect to the covering $\lambda\mathcal{S}^{(c)}$. For $\alpha \in \mathbb{R}$, define*

$$w^{(\alpha)} : \mathbb{Z}^2 \times \{\pm 1\} \to (0, \infty), (n, m, \varepsilon) \mapsto 2^{n\alpha}.$$

*Then $w^{(\alpha)}$ is moderate with respect to $\mathcal{S}^{(c)}$. Furthermore, $w^{(\alpha)}$ (considered as a weight on $\mathcal{S}^{(c)}$) is relatively moderate with respect to $\lambda\mathcal{S}^{(c)}$.*

*In fact, there are constants $N = N(c) \in \mathbb{N}$ and $C = C(\alpha, c) > 0$, which are independent of $\lambda \geq 1$, such that*

$$\begin{aligned} C_{w^{(\alpha)}, \mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}} &\leq C, \\ C_{\mathrm{mod}}(\mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}) &\leq C, \\ k(\mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}) &\leq N, \end{aligned} \tag{6.5.1}$$

*holds for all $\lambda \geq 1$.*

*More precisely, for $\lambda \geq 1$ and $(n, m, \varepsilon), (k, \ell, \delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with $S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing$, we have*

$$4^{-|\alpha|} \cdot \lambda^{\alpha} 2^{k\alpha} \leq w^{(\alpha)}_{n,m,\varepsilon} \leq 4^{|\alpha|} \cdot \lambda^{\alpha} 2^{k\alpha}. \tag{6.5.2}$$

*Now, let $\lambda \in (0, 1)$. Then $\lambda\mathcal{S}^{(c)}$ is almost subordinate to and moderate with respect to $\mathcal{S}^{(c)}$ and $w^{(\alpha)}$, considered as a weight on $\lambda\mathcal{S}^{(c)}$, is relatively moderate with respect to $\mathcal{S}^{(c)}$. More precisely, we have*

$$\begin{aligned} C_{w^{(\alpha)}, \lambda\mathcal{S}^{(c)}, \mathcal{S}^{(c)}} &\leq C, \\ C_{\mathrm{mod}}(\lambda\mathcal{S}^{(c)}, \mathcal{S}^{(c)}) &\leq C, \\ k(\lambda\mathcal{S}^{(c)}, \mathcal{S}^{(c)}) &\leq N \end{aligned}$$

*for all $\lambda \in (0, 1)$, with the same constants $C, N$ as above.*

*Finally,*

$$4^{-|\alpha|} \cdot \lambda^{-\alpha} 2^{k\alpha} \leq w^{(\alpha)}_{n,m,\varepsilon} \leq 4^{|\alpha|} \cdot \lambda^{-\alpha} 2^{k\alpha} \tag{6.5.3}$$

*holds for all $(n, m, \varepsilon), (k, \ell, \delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with $S^{(c)}_{k,\ell,\delta} \cap \lambda S^{(c)}_{n,m,\varepsilon} \neq \varnothing$ and $\lambda \in (0, 1)$.*

*Proof.* Let us first consider the case $\lambda \geq 1$. To this end, we employ Corollary 3.3.4 to show that $\mathcal{S}^{(c)}$ is almost subordinate to $\lambda\mathcal{S}^{(c)}$ with $k(\mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}) \leq 5 \cdot \left\lceil 2 \cdot 4^{|1-c|} + 3 \right\rceil$. For this, we note that each of the sets $S^{(c)}_{n,m,\varepsilon} = \varepsilon \cdot U^{(2^{n-1}, 2^{n+1})}_{(\varrho^n_c(m-1), \varrho^n_c(m+1))}$ is convex and thus path-connected by Lemma 6.3.4. By the same lemma, each of the sets $\lambda S^{(c)}_{n,m,\varepsilon}$ is open. Finally,

*We remark that the assumption $c \leq 1$ yields $|1 - c| = 1 - c$, but we will not need this fact in the following.*

$$S^{(c)}_{n,m,\varepsilon} \subset \mathbb{R}^* \times \mathbb{R} = \lambda \cdot [\mathbb{R}^* \times \mathbb{R}] = \bigcup_{(k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}} \lambda S^{(c)}_{k,\ell,\delta}$$

holds for all $(n, m, \varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}$. Thus, Corollary 3.3.4 implies that it is enough to show

$$\begin{aligned} &\left|\left\{(k, \ell, \delta) \in \mathbb{Z}^2 \times \{\pm 1\} \,\middle|\, S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing\right\}\right| \\ &\leq 5 \cdot \left\lceil 2 \cdot 4^{|1-c|} + 3 \right\rceil =: N \end{aligned} \tag{6.5.4}$$

for each $(n, m, \varepsilon) \in \mathbb{Z}^2 \times \{\pm 1\}$.

For the proof of equation (6.5.4), we first note that the identities $S^{(c)}_{n,m,\varepsilon} = \varepsilon \cdot U^{\left(2^{n-1}, 2^{n+1}\right)}_{\left(\varrho_c^n(m-1), \varrho_c^n(m+1)\right)}$ and

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U^{(\gamma,\delta)}_{(a,b)} = U^{(\alpha\gamma, \alpha\delta)}_{\left(\frac{\beta}{\alpha}a, \frac{\beta}{\alpha}b\right)}$$

from Lemma 6.3.4 imply

$$\lambda S^{(c)}_{k,\ell,\delta} = \delta \cdot U^{\left(\lambda \cdot 2^{k-1}, \lambda \cdot 2^{k+1}\right)}_{\left(\varrho_c^k(\ell-1), \varrho_c^k(\ell+1)\right)}$$

for all $\lambda \in (0, \infty)$ and $(k, \ell, \delta) \in \mathbb{Z}^2 \times \{\pm 1\}$. Here, $\varrho_c = 2^{c-1}$.

Now, assume that there is some

$$\begin{pmatrix} x \\ y \end{pmatrix} \in S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing.$$

Note that we have $x \in (0, \infty)$ in case of $\varepsilon = 1$ and $x \in (-\infty, 0)$ in case of $\varepsilon = -1$. Since an analogous statement holds for $\delta$ instead of $\varepsilon$, we get $\varepsilon = \delta$. Hence,

$$\varepsilon x \in \left(2^{n-1}, 2^{n+1}\right) \cap \left(\lambda \cdot 2^{k-1}, \lambda \cdot 2^{k+1}\right),$$

which implies

$$2^{n-1} < \varepsilon x < \lambda \cdot 2^{k+1} = 2^{k + \log_2 \lambda + 1},$$

as well as $2^{k+\log_2 \lambda - 1} < \varepsilon x < 2^{n+1}$. We conclude

$$k \in (n - \log_2 \lambda - 2, n - \log_2 \lambda + 2) \cap \mathbb{Z} := N_{n,\lambda}.$$

But by definition of the sets $U^{(\gamma,\delta)}_{(a,b)}$, we also have

$$\frac{\varepsilon y}{\varepsilon x} \in \left(\varrho_c^n(m-1), \varrho_c^n(m+1)\right) \cap \left(\varrho_c^k(\ell-1), \varrho_c^k(\ell+1)\right),$$

which yields

$$\varrho_c^n(m-1) < \frac{\varepsilon y}{\varepsilon x} < \varrho_c^k(\ell+1) \text{ and hence } \ell > \varrho_c^{n-k}(m-1) - 1,$$

as well as

$$\varrho_c^k(\ell-1) < \frac{\varepsilon y}{\varepsilon x} < \varrho_c^n(m+1) \text{ and hence } \ell < \varrho_c^{n-k}(m+1)+1.$$

Thus,

$$\ell \in \left(\varrho_c^{n-k}(m-1)-1, \varrho_c^{n-k}(m+1)+1\right) \cap \mathbb{Z} =: M_{n,m,k,\lambda}.$$

We have thus shown

$$\begin{aligned}&\left\{(k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\} \;\middle|\; S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing\right\}\\ &\subset \bigcup_{k' \in N_{n,\lambda}} \left[\{k'\} \times M_{n,m,k',\lambda} \times \{\varepsilon\}\right].\end{aligned}$$

We first note that $N_{n,\lambda}$ has at most $|N_{n,\lambda}| \leq 5$ elements. In addition to that, $M_{n,m,k',\lambda}$ satisfies

$$\begin{aligned}|M_{n,m,k',\lambda}| &\leq \left(\varrho_c^{n-k'}(m+1)+1\right) - \left(\varrho_c^{n-k'}(m-1)-1\right)+1\\ &= 2\cdot\varrho_c^{n-k'}+3\\ &= 2\cdot\varrho_c^{n-k'-\log_2\lambda}\cdot\varrho_c^{\log_2\lambda}+3\\ &\leq 2\cdot 2^{|c-1|\cdot|n-k'-\log_2\lambda|}\cdot 2^{(c-1)\cdot\log_2\lambda}+3\\ &\overset{(\dagger)}{\leq} 2\cdot 2^{2|c-1|}\cdot\lambda^{c-1}+3\\ &\overset{(\ddagger)}{\leq} 2\cdot 4^{|c-1|}+3 \leq \frac{N}{5}\end{aligned}$$

for all $k' \in N_{n,\lambda}$. The assumption $k' \in N_{n,\lambda}$ is actually only used at $(\dagger)$. Furthermore, the step marked with $(\ddagger)$ is justified since we have $\lambda \geq 1$ and $c-1 \leq 0$.

All in all, this proves equation (6.5.4) and hence establishes the "uniform" almost subordinateness of $\mathcal{S}^{(c)}$ to $\lambda\mathcal{S}^{(c)}$, i.e. $k(\mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}) \leq N$.

Now, let us prove equation (6.5.2). To this end, recall from above that $S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing$ implies $k \in N_{n,\lambda}$ and thus

$$|n-k-\log_2\lambda| < 2.$$

This yields

$$\begin{aligned}w^{(\alpha)}_{n,m,\varepsilon} &= 2^{n\alpha} = 2^{(n-k-\log_2\lambda)\alpha}\cdot 2^{(k+\log_2\lambda)\alpha}\\ &\leq 2^{|n-k-\log_2\lambda||\alpha|}\cdot\lambda^{\alpha}\cdot 2^{k\alpha}\\ &\leq 4^{|\alpha|}\cdot\lambda^{\alpha}\cdot 2^{k\alpha}\end{aligned}$$

as well as

$$\begin{aligned}w^{(\alpha)}_{n,m,\varepsilon} &= 2^{(n-k-\log_2\lambda)\alpha}\cdot 2^{(k+\log_2\lambda)\alpha}\\ &\geq 2^{-|n-k-\log_2\lambda||\alpha|}\cdot\lambda^{\alpha}\cdot 2^{k\alpha}\end{aligned}$$

$$\geq \frac{\lambda^\alpha \cdot 2^{k\alpha}}{4^{|\alpha|}}.$$

Hence, we have established equation (6.5.2). Furthermore, we get

$$\frac{w^{(\alpha)}_{n,m,\varepsilon}}{w^{(\alpha)}_{n',m',\varepsilon'}} \leq \frac{4^{|\alpha|} \cdot \lambda^\alpha 2^{k\alpha}}{\lambda^\alpha 2^{k\alpha}/4^{|\alpha|}} = 16^{|\alpha|}$$

in case of $S^{(c)}_{n,m,\varepsilon} \cap \lambda S^{(c)}_{k,\ell,\delta} \neq \varnothing \neq S^{(c)}_{n',m',\varepsilon'} \cap \lambda S^{(c)}_{k,\ell,\delta}$. Hence,

$$C_{w^{(\alpha)},\mathcal{S}^{(c)},\lambda\mathcal{S}^{(c)}} \leq 16^{|\alpha|}.$$

Finally, observe $|\det(A^{(c)}_{n,m,\varepsilon})| = 2^{(1+c)n} = w^{(1+c)}_{n,m,\varepsilon}$ to conclude that

$$C_{\mathrm{mod}}(\mathcal{S}^{(c)}, \lambda\mathcal{S}^{(c)}) = C_{w^{(1+c)},\mathcal{S}^{(c)},\lambda\mathcal{S}^{(c)}} \leq 16^{|1+c|}$$

is uniformly bounded, independently of $\lambda \geq 1$.

Finally, we consider the case $\lambda \in (0,1)$. Let $\gamma := \lambda^{-1} \in (1,\infty)$. We note that $S^{(c)}_{k,\ell,\delta} \cap \lambda S^{(c)}_{n,m,\varepsilon} \neq \varnothing$ is equivalent to $S^{(c)}_{n,m,\varepsilon} \cap \gamma S^{(c)}_{k,\ell,\delta} \neq \varnothing$, which yields

$$4^{-|\alpha|} \cdot \lambda^{-\alpha} 2^{k\alpha} = 4^{-|\alpha|} \cdot \gamma^\alpha 2^{k\alpha} \leq w^{(\alpha)}_{n,m,\varepsilon} \leq 4^{|\alpha|} \cdot \gamma^\alpha 2^{k\alpha} = 4^{|\alpha|} \cdot \lambda^{-\alpha} 2^{k\alpha}$$

by equation (6.5.2). This establishes equation (6.5.3).

Using the case $\lambda \geq 1$ together with part 7 of Lemma 3.5.1 (with the linear map $A = \lambda \cdot \mathrm{id}$) shows that $\lambda\mathcal{S}^{(c)}$ is almost subordinate to $\lambda\gamma\mathcal{S}^{(c)} = \mathcal{S}^{(c)}$, since $\mathcal{S}^{(c)}$ is almost subordinate to $\gamma\mathcal{S}^{(c)}$. The same lemma also yields

$$\begin{aligned} C_{w^{(\alpha)},\lambda\mathcal{S}^{(c)},\mathcal{S}^{(c)}} &= C_{w^{(\alpha)},\lambda\mathcal{S}^{(c)},\lambda\gamma\mathcal{S}^{(c)}} = C_{w^{(\alpha)},\mathcal{S}^{(c)},\gamma\mathcal{S}^{(c)}} \leq C \\ C_{\mathrm{mod}}(\lambda\mathcal{S}^{(c)}, \mathcal{S}^{(c)}) &= C_{\mathrm{mod}}(\lambda\mathcal{S}^{(c)}, \lambda\gamma\mathcal{S}^{(c)}) = C_{\mathrm{mod}}(\mathcal{S}^{(c)}, \gamma\mathcal{S}^{(c)}) \leq C, \\ k(\lambda\mathcal{S}^{(c)}, \mathcal{S}^{(c)}) &= k(\lambda\mathcal{S}^{(c)}, \lambda\gamma\mathcal{S}^{(c)}) = k(\mathcal{S}^{(c)}, \gamma\mathcal{S}^{(c)}) \leq N, \end{aligned}$$

which completes the proof. □

As a last step for the estimation of the operator norm of the dilation, we establish the following embedding. It then just remains to apply Lemma 3.5.3 and to translate the result to the setting of coorbit spaces using Corollary 6.3.5.

**Lemma 6.5.2.** *Let $c \in (-\infty, 1]$ and $p, q \in (0,\infty]$. Choose a fixed $L^p$-BAPU $\Phi = (\varphi_i)_{i \in \mathbb{Z}^2 \times \{\pm 1\}}$ for $\mathcal{S}^{(c)}$.*

*Lemma 3.5.1 shows that $\Phi^{(\gamma)} := \left(\varphi_i \circ \gamma^{-1}\mathrm{id}\right)_{i \in \mathbb{Z}^2 \times \{\pm 1\}}$ is an $L^p$-BAPU for $\gamma\mathcal{S}^{(c)}$ for all $\gamma \in (0,\infty)$. In the present lemma, we will use these BAPUs for the calculation of the quasi-norm on the decomposition spaces $\mathcal{D}(\mathcal{S}^{(c)}, L^p, \ell^q_{w^{(\alpha)}})$ and $\mathcal{D}(\gamma\mathcal{S}^{(c)}, L^p, \ell^q_{w^{(\alpha)}})$.*

*Let $\gamma \geq 1$ and $\alpha \in \mathbb{R}$. Then*

$$\iota : \mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_{w^{(\alpha)}}\right) \hookrightarrow \mathcal{D}\left(\gamma\mathcal{S}^{(c)}, L^p, \ell^q_{w^{(\alpha)}}\right), f \mapsto f$$

*is well-defined with*

$$|||\iota||| \asymp \gamma^{-\alpha+(1-c)\left(\frac{1}{p^\vee}-\frac{1}{q}\right)_+}, \tag{6.5.5}$$

*where the implied constants are independent of $\gamma \geq 1$.*

*For $\gamma \in (0,1)$, the embedding $\iota$ defined as above is also well-defined and bounded with*

$$|||\iota||| \asymp \gamma^{-\alpha+(c-1)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+}. \tag{6.5.6}$$

*Again, the implied constants are independent of $\gamma \in (0,1)$.*

*Proof.* For the proof of equation (6.5.5), we use part 3 of Theorem 5.4.4, with $\mathcal{Q} = \mathcal{S}^{(c)}$, $\mathcal{P} = \gamma\mathcal{S}^{(c)}$ and $u = v = w^{(\alpha)}$ and with exponents $p_1 = p_2 = p$ and $q_1 = q_2 = q$. We note that $\mathcal{Q}, \mathcal{P}$ are indeed tight semi-structured $L^p$-decomposition coverings, thanks to Corollary 6.3.5 and Lemma 3.5.1. Furthermore, Lemma 6.5.1 shows that $\mathcal{Q}$ is almost subordinate to and moderate with respect to $\mathcal{P}$ and that $u$ is relatively $\mathcal{P}$-moderate. Finally, $\mathcal{Q}$ and $\mathcal{P}$ both cover the same set $\mathcal{O} = \mathbb{R}^2 = \mathcal{O}'$, so that condition 3a of Theorem 5.4.4 is trivial and so that property 1a of the map $\iota$ constructed in Theorem 5.4.4 yields $\iota f = f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$.

Finally, Lemma 6.5.1 and Lemma 3.5.1 yield the identities

$$\begin{aligned}
N_\mathcal{P} &= N_\mathcal{Q} = N_{\mathcal{S}^{(c)}} \\
C_\mathcal{P} &= C_\mathcal{Q} = C_{\mathcal{S}^{(c)}}, \\
R_\mathcal{P} &= R_\mathcal{Q} = C_{\mathcal{S}^{(c)}}, \\
D_\mathcal{P} &= D_\mathcal{Q} = D_{\mathcal{S}^{(c)}}, \\
\varepsilon_\mathcal{P} &= \varepsilon_\mathcal{Q} = \varepsilon_{\mathcal{S}^{(c)}}, \\
C_{v,\mathcal{P}} &= C_{u,\mathcal{Q}} = C_{w^{(\alpha)},\mathcal{S}^{(c)}}, \\
C_{\mathcal{P},\Phi^{(\gamma)},p} &= C_{\mathcal{Q},\Phi,p} = C_{\mathcal{S}^{(c)},\Phi,p}
\end{aligned}$$

and the estimates

$$\begin{aligned}
C_{\mathrm{mod}}\left(\mathcal{Q},\mathcal{P}\right) &\leq C, \\
C_{u,\mathcal{Q},\mathcal{P}} &\leq C, \\
k\left(\mathcal{Q},\mathcal{P}\right) &\leq N, \\
r_0\left(\mathcal{P}, 2k\left(\mathcal{Q},\mathcal{P}\right)+3\right) &\leq r_0\left(\mathcal{P}, 2N+3\right) \leq N', \\
r_0\left(\mathcal{P},1\right) &\leq r_0\left(\mathcal{P}, 2N+3\right) \leq N',
\end{aligned}$$

for absolute constants $N = N(c)$ and $N' = N'(N,c) = N'(c)$, as well as $C = C(\alpha, c) > 0$.

Note that the above constants are the only quantities (except for $d, p, q$) on which the implied constants in Theorem 5.4.4 depend. For us, the only important fact is that none of the bounds above involves the number $\gamma \geq 1$. Thus, the implied constants provided by Theorem 5.4.4 can be chosen independently of $\gamma \geq 1$. Invoking part 3 of Theorem 5.4.4, we conclude

*Here, it is important to recall that the "linear part" of the normalizations for $\mathcal{P} = \gamma\mathcal{S}^{(c)}$ is given by $S_j = \gamma A_j^{(c)}$ for $j \in J = \mathbb{Z}^2 \times \{\pm 1\}$ and* not *by $S_j = A_j^{(c)}$, cf. Lemma 3.5.1.*

$$|||\iota||| \asymp \left\| \left( u_{i_j}^{-1} \cdot \left|\det\left(S_j\right)\right|^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \cdot \left|\det\left(T_{i_j}\right)\right|^{-\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \right)_{j \in J_{\mathcal{O}}} \right\|_{\ell_v^{q \cdot (q/q)'}},$$

where for each $j \in J_{\mathcal{O}} = J = \mathbb{Z}^2 \times \{\pm 1\}$, some $i_j \in I = \mathbb{Z}^2 \times \{\pm 1\}$ is chosen with

$$\varnothing \neq Q_{i_j} \cap P_j = S_{i_j}^{(c)} \cap \gamma S_j^{(c)}.$$

Here and in the following, all implied constants will be independent of $\gamma \geq 1$.

For $i_j = (n, m, \varepsilon)$ and $j = (k, \ell, \delta)$, equation (6.5.2) yields

$$4^{-|\alpha|} \cdot \gamma^\alpha 2^{k\alpha} \leq w_{i_j}^{(\alpha)} = u_{i_j} = w_{i_j}^{(\alpha)} \leq 4^{|\alpha|} \cdot \gamma^\alpha 2^{k\alpha}$$

and

$$\left|\det\left(T_{i_j}\right)\right| = \left|\det\left(A_{n,m,\varepsilon}^{(c)}\right)\right| = 2^{(1+c)n} = w_{i_j}^{(1+c)} \asymp \gamma^{1+c} 2^{k(1+c)}.$$

Furthermore, Remark 5.2.3 shows $q \cdot (q/q)' = \infty$ and we have

$$\left|\det\left(S_j\right)\right| = \left|\det\left(\gamma A_j^{(c)}\right)\right| = \gamma^2 \cdot 2^{(1+c)k} \quad \text{for } j = (k, \ell, \delta) \in J.$$

Altogether, we arrive at

$$\begin{aligned} |||\iota||| &\asymp \left\| \left( \frac{2^{k\alpha}}{\gamma^\alpha 2^{k\alpha}} \left(\gamma^2 \cdot 2^{(1+c)k}\right)^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \cdot \left(\gamma^{1+c} 2^{k(1+c)}\right)^{-\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \right)_{(k,\ell,\delta) \in J} \right\|_{\ell^\infty} \\ &= \gamma^{-\alpha - (1+c)\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+ + 2\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \cdot \left\| (1)_{(k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}} \right\|_{\ell^\infty} \\ &= \gamma^{-\alpha + (1-c)\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}. \end{aligned}$$

In case of $\gamma \in (0, 1)$, the proof is almost identical. In this case, we use part 3 of Theorem 5.4.1, with $\mathcal{Q} = \gamma\mathcal{S}^{(c)}$ and $\mathcal{P} = \mathcal{S}^{(c)}$, as well as $u = v = w^{(\alpha)}$ and with $p_1 = p_2 = p$ and $q_1 = q_2 = q$. In this case, Lemma 6.5.1 shows that $\mathcal{Q}$ is indeed almost subordinate to $\mathcal{P}$ with

$$\begin{aligned} k(\mathcal{Q}, \mathcal{P}) &\leq N, \\ C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}) &\leq C, \\ C_{u,\mathcal{Q},\mathcal{P}} &\leq C, \end{aligned}$$

for absolute constants $C = C(c, \alpha)$ and $N = N(c)$.

Analogous to the arguments above, we see that Lemma 3.5.1 shows that the remaining quantities on which the implied constants in Theorem 5.4.1 depend, can also be estimated independently of $\gamma \in (0,1)$.

By part 3 of Theorem 5.4.1, this yields

*As above, it is important to recall that the linear part of the normalizations for $\mathcal{Q} = \gamma\mathcal{S}^{(c)}$ is given by $T_i = \gamma A_i^{(c)}$ for $i \in I = \mathbb{Z}^2 \times \{\pm 1\}$.*

$$|||\iota||| \asymp \left\| \left( \frac{u_{i_j}}{v_j} \left|\det\left(S_j\right)\right|^{\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \left|\det\left(T_{i_j}\right)\right|^{-\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \right)_{j\in J_{\mathcal{O}}} \right\|_{\ell^{q\cdot(q/q)'}},$$

where for each $j \in J_{\mathcal{O}} = J = \mathbb{Z}^2 \times \{\pm 1\}$, some $i_j \in I = \mathbb{Z}^2 \times \{\pm 1\}$ with

$$\varnothing \neq Q_{i_j} \cap P_j = \gamma S_{i_j}^{(c)} \cap S_j^{(c)}$$

is chosen. As for the case $\gamma \geq 1$, the implied constants do not depend on $\gamma \in (0,1)$.

For $i_j = (n,m,\varepsilon)$ and $j = (k,\ell,\delta)$, equation (6.5.3) yields

$$4^{-|\alpha|} \cdot \gamma^{-\alpha} 2^{k\alpha} \leq w_{n,m,\varepsilon}^{(\alpha)} = u_{i_j} = w_{n,m,\varepsilon}^{(\alpha)} \leq 4^{|\alpha|} \cdot \gamma^{-\alpha} 2^{k\alpha}$$

and

$$\begin{aligned} \left|\det\left(T_{i_j}\right)\right| &= \left|\det\left(\gamma A_{n,m,\varepsilon}^{(c)}\right)\right| \\ &= \gamma^2 2^{n(1+c)} \\ &= \gamma^2 w_{n,m,\varepsilon}^{(1+c)} \\ &\asymp \gamma^{2-(1+c)} 2^{k(1+c)} \\ &= \gamma^{1-c} 2^{k(1+c)}, \end{aligned}$$

where the implied constants are independent of $\gamma \in (0,1)$. Together with $q \cdot (q/q)' = \infty$, we conclude

$$\begin{aligned} |||\iota||| &\asymp \left\| \left( \frac{\gamma^{-\alpha} 2^{k\alpha}}{2^{k\alpha}} 2^{k(1+c)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \left(\gamma^{1-c} 2^{k(1+c)}\right)^{-\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \right)_{(k,\ell,\delta)\in J} \right\|_{\ell^\infty} \\ &= \gamma^{-\alpha+(c-1)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \left\| (1)_{(k,\ell,\delta)\in\mathbb{Z}^2\times\{\pm1\}} \right\|_{\ell^\infty} \\ &= \gamma^{-\alpha+(c-1)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+}. \end{aligned}$$

This completes the proof. □

We can now easily prove the following theorem which provides the asymptotic operator norm of the isotropic dilation on shearlet-type coorbit spaces.

**Theorem 6.5.3.** *Let $c \in (-\infty, 1]$ and $p, q \in (0, \infty]$, as well as $\lambda \in (0, \infty)$ and $\alpha \in \mathbb{R}$. Let*

$$v_\alpha : H_c \to (0, \infty), \left[ \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \text{ with } \varepsilon \in \{\pm 1\}, a > 0, b \in \mathbb{R} \right] \mapsto a^\alpha.$$

*Then there is a linear operator*

$$D_\lambda : \mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$$

*which satisfies $D_\lambda f = f \circ \lambda \mathrm{id}$ for all $f \in \mathrm{Co}\left(L^{p,q}_{v}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \cap L^2\left(\mathbb{R}^2\right)$.*

*Furthermore, the operator norm of $D_\lambda$ can be estimated by*

$$|||D_\lambda||| \asymp \lambda^{-\alpha - \frac{2}{p}} \cdot \lambda^{(1+c)\left(\frac{1}{q} - \frac{1}{2}\right) + (1-c)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+}$$

*for $\lambda > 1$ and by*

$$|||D_\lambda||| \asymp \lambda^{-\alpha - \frac{2}{p}} \cdot \lambda^{(1+c)\left(\frac{1}{q} - \frac{1}{2}\right) - (1-c)\left(\frac{1}{p^{\triangledown}} - \frac{1}{q}\right)_+}$$

*for $\lambda \in (0, 1]$.*

*Here, all implied constants are independent of $\lambda \in (0, \infty)$.*

*Proof.* Let $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ and

$$\mathcal{F}_{\mathrm{shear}} : \mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)$$

as well as

$$\begin{aligned} u_{n,m,\varepsilon} &:= 2^{-n(1+c)\left(\frac{1}{2} - \frac{1}{q}\right)} \cdot v\left(\left(A^{(c)}_{n,m,\varepsilon}\right)^{-T}\right) \\ &= 2^{-n(1+c)\left(\frac{1}{2} - \frac{1}{q}\right)} \cdot 2^{-n\alpha} \\ &= 2^{-n\left(\alpha + (1+c)\left(\frac{1}{2} - \frac{1}{q}\right)\right)} \end{aligned}$$

for $(n, m, \varepsilon) \in I := \mathbb{Z}^2 \times \{\pm 1\}$ be as in Corollary 6.3.5.

We have $u := (u_{n,m,\varepsilon})_{(n,m,\varepsilon)\in I} = w^{\left(-\left[\alpha + (1+c)\left(\frac{1}{2} - \frac{1}{q}\right)\right]\right)}$ in the notation of Lemma 6.5.1. Hence, the Lemmata 3.5.3 and 6.5.2 show that the map

$$\Gamma_\lambda : \mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right), f \mapsto f_{\lambda^{-1}\mathrm{id}}$$

is well-defined with

$$\begin{aligned} \left\|\Gamma_\lambda f\right\|_{\mathcal{D}_\Phi\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)} &= \left\|f_{\lambda^{-1}\mathrm{id}}\right\|_{\mathcal{D}_\Phi\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)} \\ &= \left|\det\left(\lambda^{-1}\mathrm{id}\right)\right|^{\frac{1}{p}-1} \cdot \|f\|_{\mathcal{D}_{\Phi_{\lambda^{-1}\mathrm{id}}}\left(\lambda^{-1}\mathcal{S}^{(c)}, L^p, \ell^q_u\right)} \\ &= \lambda^{2-\frac{2}{p}} \cdot \left\|\iota_\lambda f\right\|_{\mathcal{D}\left(\lambda^{-1}\mathcal{S}^{(c)}, L^p, \ell^q_u\right)}. \end{aligned} \tag{6.5.7}$$

Here, $\iota_\lambda$ is the embedding from Lemma 6.5.2, where that lemma is invoked for $\gamma := \lambda^{-1}$ and with $-\left[\alpha + (1+c)\left(\frac{1}{2} - \frac{1}{q}\right)\right]$ instead of $\alpha$. Further, $\Phi = (\varphi_i)_{i \in \mathbb{Z}^2 \times \{\pm 1\}}$ is a fixed $L^p$-BAPU for $\mathcal{S}^{(c)}$ and the dilated BAPU $\Phi_{\lambda^{-1}\mathrm{id}} = (\varphi_i \circ \lambda \mathrm{id})_{i \in \mathbb{Z}^2 \times \{\pm 1\}}$ is as in Lemma 3.5.3.

Define

$$D_\lambda f := \lambda^{-2} \cdot \mathcal{F}_{\mathrm{shear}}^{-1}\left(\left(\mathcal{F}_{\mathrm{shear}} f\right)_{\lambda^{-1}\mathrm{id}}\right) = \lambda^{-2} \cdot \left(\mathcal{F}_{\mathrm{shear}}^{-1} \circ \Gamma_\lambda \circ \mathcal{F}_{\mathrm{shear}}\right) f$$

for $f \in \mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ with $g_{\lambda^{-1}\mathrm{id}}$ as in Definition 3.5.2 for each $g \in \mathcal{D}'\left(\mathbb{R}^* \times \mathbb{R}\right)$.

Corollary 6.3.5 shows that the map $\mathcal{F}_{\mathrm{shear}}$ coincides with the usual Fourier transform on $\mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \cap L^2\left(\mathbb{R}^2\right)$ and that the same is true for the inverse map $\mathcal{F}_{\mathrm{shear}}^{-1}$ on $L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)$. This, together with the remark before Definition 3.5.2 (i.e. that $f_A = f \circ A$ holds for $f \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$) shows

$$D_\lambda f = \lambda^{-2} \cdot \mathcal{F}^{-1}\left[\widehat{f} \circ (\lambda^{-1}\mathrm{id})\right] = \left(\mathcal{F}^{-1}\widehat{f}\right) \circ \lambda \mathrm{id} = f \circ \lambda \mathrm{id}$$

for each $f \in \mathrm{Co}\left(L^{p,q}_{v_\alpha}\left(\mathbb{R}^2 \rtimes H_c\right)\right) \cap L^2\left(\mathbb{R}^2\right)$.

Finally, the fact that $\mathcal{F}_{\mathrm{shear}}$ is an isomorphism of Quasi-Banach spaces, equation (6.5.7) and Lemma 6.5.2 imply

$$\begin{aligned}
|||D_\lambda||| &\asymp \lambda^{-2} \cdot |||\Gamma_\lambda||| \\
&\asymp \lambda^{-2} \cdot \lambda^{2-\frac{2}{p}} \cdot |||\iota_\lambda||| \\
&\asymp \lambda^{-\frac{2}{p}} \cdot \gamma^{(c-1)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+ - \left[-\left(\alpha + (1+c)\left(\frac{1}{2} - \frac{1}{q}\right)\right)\right]} \\
&\asymp \lambda^{-\frac{2}{p}} \cdot \lambda^{-\alpha - (1+c)\left(\frac{1}{2} - \frac{1}{q}\right) - (c-1)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+}
\end{aligned}$$

in the case $\lambda > 1$ (which yields $\gamma \in (0,1)$) and

$$\begin{aligned}
|||D_\lambda||| &\asymp \lambda^{-2} \cdot |||\Gamma_\lambda||| \\
&\asymp \lambda^{-2} \cdot \lambda^{2-\frac{2}{p}} \cdot |||\iota_\lambda||| \\
&\asymp \lambda^{-\frac{2}{p}} \cdot \gamma^{-\left[-\left(\alpha + (1+c)\left(\frac{1}{2} - \frac{1}{q}\right)\right)\right] + (1-c)\left(\frac{1}{p^\vee} - \frac{1}{q}\right)_+} \\
&\asymp \lambda^{-\frac{2}{p}} \cdot \lambda^{-\alpha - (1+c)\left(\frac{1}{2} - \frac{1}{q}\right) - (1-c)\left(\frac{1}{p^\vee} - \frac{1}{q}\right)_+}
\end{aligned}$$

in case of $\lambda \leq 1$ (which yields $\gamma \geq 1$). □

### 6.5.2 *Universal dilations for shearlet-type coorbit spaces*

For the similitude group $H$, we saw in Section 4.7 that every coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is invariant under dilation by *arbitrary* matrices $g \in \mathrm{GL}\left(\mathbb{R}^d\right)$, at least if the weight $m$ only depends on the second factor of the product $\mathbb{R}^d \rtimes H$.

Our goal in this section is similar: We want to characterize the set of UNIVERSAL DILATIONS for each of the shearlet-type groups $H_c$, with $c \in \mathbb{R} \setminus \{1\}$. Note that this includes the important special case $c = \frac{1}{2}$.

**Definition 6.5.4.** Let $c \in \mathbb{R}$ be arbitrary. We say that $g \in \mathrm{GL}\left(\mathbb{R}^2\right)$ is a UNIVERSAL DILATION for the shearlet-type group $H_c$ if the coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ is invariant under dilation by $g$ for *all* choices of $p, q \in (0, \infty]$ and for *all* weights $m : H_c \to (0, \infty)$ for which there is a submultiplicative, measurable weight $m_0 : H_c \to (0, \infty)$ such that $m$ is $m_0$-moderate.

*Here, we say that $m$ is $m_0$-moderate, if $m(xyz) \leq m_0(x)\, m(y)\, m_0(z)$ holds for all $x, y, z \in H_c$. In particular, we get $m(xy) = m(1_H xy) \leq m_0(1_H)\, m(x)\, m_0(y)$, so that $m$ is right-moderate with respect to the weight $y \mapsto m_0(1_H)\, m_0(y)$.*

We write

$$\mathrm{UD}_c := \left\{g \in \mathrm{GL}\left(\mathbb{R}^2\right) \mid g \text{ universal dilation for } H_c\right\}.$$

*Remark.* It is important to note that the condition of being a universal dilation is rather strong. It is not sufficient if some coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ is invariant under $g$, but this has to hold for (essentially) *all* choices of $m, p, q$. Thus, it can still be the case that a specific coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ is invariant under dilation by $g$, even though $g \notin \mathrm{UD}_c$. In fact, for $m \equiv 1$ and $p = q = 2$, we always have $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right) = L^2\left(\mathbb{R}^2\right)$, which is invariant under arbitrary dilations.

An easy, but important observation is that $\mathrm{UD}_c$ is closed under matrix multiplication and contains the identity, i.e. it is a monoid.

The following lemma clarifies why we require the weight $m$ to be $m_0$-moderate instead of just right-moderate with respect to $m_0$; this ensures that $L^{p,q}_m(G)$ is left-invariant.

**Lemma 6.5.5.** *We have $H_c \subset \mathrm{UD}_c$ for every $c \in \mathbb{R}$.*

*Proof.* For brevity, set $G_c := \mathbb{R}^2 \rtimes H_c$. Let $p, q \in (0, \infty]$ and assume that $m : H_c \to (0, \infty)$ is $m_0$-moderate for some submultiplicative weight $m_0 : H_c \to (0, \infty)$. In particular, we get

$$\begin{aligned} m(xy) &= m(xy1_H) \\ &\leq m_0(x)\, m(y)\, m_0(1_H) \end{aligned}$$

for all $x, y \in H_c$, so that $m$ is left-moderate with respect to the measurable, locally bounded weight $x \mapsto m_0(1_H)\, m_0(x)$. Local boundedness of $m_0$ is a consequence of Theorem 2.2.22, since $m_0$ is submultiplicative.

Using Lemma 4.2.6, we conclude that $L^{p,q}_m(G_c)$ is invariant under left translations, so that Lemma 2.3.18 shows that $W\left(L^\infty, L^{p,q}_m\right)$ is also left invariant.

Now, fix $g \in H_c$ and let $f \in L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L_m^{p,q}(G_c)\right)$ be arbitrary. Using the fact that the wavelet transform intertwines $\pi$ and $L_x$ (cf. equation (2.4.5)), we get (for an arbitrary analyzing vector $\psi$)

$$\begin{aligned}
\|\sigma(0,g) f\|_{\mathrm{Co}\left(L_m^{p,q}\right)} &= \|\pi(0,g) f\|_{\mathrm{Co}\left(L_m^{p,q}\right)} \\
&= \left\|W_\psi\left[\pi(0,g) f\right]\right\|_{W\left(L^\infty, L_m^{p,q}\right)} \\
&= \left\|L_{(0,g)}\left[W_\psi f\right]\right\|_{W\left(L^\infty, L_m^{p,q}\right)} \\
&\leq \left|\left|\left|L_{(0,g)}\right|\right|\right|_{W\left(L_m^{p,q}\right) \to W\left(L_m^{p,q}\right)} \cdot \left\|W_\psi f\right\|_{W\left(L^\infty, L_m^{p,q}\right)} \\
&= \left|\left|\left|L_{(0,g)}\right|\right|\right|_{W\left(L_m^{p,q}\right) \to W\left(L_m^{p,q}\right)} \cdot \|f\|_{\mathrm{Co}\left(L_m^{p,q}\right)} < \infty,
\end{aligned}$$

so that

$$L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L_m^{p,q}(G_c)\right) \to \mathrm{Co}\left(L_m^{p,q}(G_c)\right), f \mapsto \sigma(0,g) f$$

is well-defined and bounded. By definition (cf. Remark 4.7.4), this means that $\mathrm{Co}\left(L_m^{p,q}(G_c)\right)$ is invariant under dilation by $g$. □

Our next result establishes a convenient necessary condition that every universal dilation for $H_c$ has to satisfy. Here, we crucially use the isomorphism between the coorbit space $\mathrm{Co}\left(L_m^{p,q}\right)$ and the decomposition space $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell_u^q\right)$ for a suitable weight $u$.

The geometric intuition is that since the covering $\mathcal{S}^{(c)}$ "clusters" at the "blind spot" $\mathcal{O}^c = \{0\} \times \mathbb{R}$, we can choose the weight $u$ in such a way that it blows up near $\mathcal{O}^c$. It is then a consequence of Lemma 5.3.1 that the embedding of decomposition spaces which is induced by dilation invariance of $\mathrm{Co}\left(L_m^{p,q}\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ under $g$ can not hold if $g^T$ does not leave $\mathcal{O}$ invariant.

**Lemma 6.5.6.** *Let $c \in \mathbb{R}$ be arbitrary and let $g \in \mathrm{UD}_c$. Then $e_2 = (0,1)$ is an eigenvector of $g^T$.*

*Proof.* For brevity, set $G_c := \mathbb{R}^2 \rtimes H_c$. Let

$$v : H_c \to (0,\infty), \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mapsto a,$$

where as usual $\varepsilon \in \{\pm 1\}$, $a \in (0,\infty)$ and $b \in \mathbb{R}$. Note that $v$ is multiplicative and hence moderate with respect to the (sub)multiplicative, locally bounded weight $v$.

Because of $g \in \mathrm{UD}_c$, we see that the map

$$\iota : L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L_v^{2,2}(G_c)\right) \to \mathrm{Co}\left(L_v^{2,2}(G_c)\right), f \mapsto \sigma(0,g) f$$

is well-defined and bounded. Note that we chose $p = q = 2$. Let

$$u_{n,m,\varepsilon}^{(2)} := v\left(\left(A_{n,m,\varepsilon}^{(c)}\right)^{-T}\right) = 2^{-n} \qquad \text{for } (n,m,\varepsilon) \in I := \mathbb{Z}^2 \times \{\pm 1\}$$

be defined as in Corollary 6.3.5. That corollary yields an isomorphism

$$\mathcal{F} : \mathrm{Co}\left(L^{2,2}_v(G_c)\right) \to \mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)$$

which coincides on $L^2\left(\mathbb{R}^2\right) \cap \mathrm{Co}\left(L^{2,2}_v(G_c)\right)$ with the usual Fourier transform and whose inverse coincides on $L^2\left(\mathbb{R}^2\right) \cap \mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)$ with the usual inverse Fourier transform. Here, we use the convention $\mathcal{S}^{(c)} := \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$, cf. Corollary 6.3.5.

Now, let $f \in C^\infty_c\left(\mathbb{R}^* \times \mathbb{R}\right) \subset \mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)$ be arbitrary. We get

$$\begin{aligned}\left|\det(g)\right|^{\frac{1}{2}} \cdot \left(f \circ g^T\right) &= \mathcal{F}\left[\left|\det(g)\right|^{-\frac{1}{2}} \cdot \left(\mathcal{F}^{-1} f\right) \circ g^{-1}\right] \\ &= \mathcal{F}\left[\iota\left(\mathcal{F}^{-1} f\right)\right] \in \mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)\end{aligned}$$

with

$$\left|\det(g)\right|^{\frac{1}{2}} \cdot \left\|f \circ g^T\right\|_{\mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)} \lesssim |||\iota||| \cdot \|f\|_{\mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)},$$

where the implied constants may depend on $g$, but not on $f$.

But Lemma 3.5.3 shows

$$\begin{aligned}\left\|f \circ g^T\right\|_{\mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)} &= \left\|f_{g^T}\right\|_{\mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)} \\ &\asymp \left|\det(g)\right|^{\frac{1}{2}-1} \cdot \|f\|_{\mathcal{D}\left(g^T \mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)}.\end{aligned}$$

All in all, we have shown

$$\|f\|_{\mathcal{D}\left(g^T \mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)} \lesssim \|f\|_{\mathcal{D}\left(\mathcal{S}^{(c)}, L^2, \ell^2_{u^{(2)}}\right)}$$

for all $f \in C^\infty_c\left(\mathbb{R}^* \times \mathbb{R}\right)$, where the implied constant is independent of $f$.

Using Lemma 5.3.1, we conclude

$$u^{(2)}_{n,m,\varepsilon} \lesssim u^{(2)}_{k,\ell,\delta} \tag{6.5.8}$$

for all $(n,m,\varepsilon), (k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with

$$\varnothing \neq \left(\mathbb{R}^* \times \mathbb{R}\right) \cap g^T S^{(c)}_{n,m,\varepsilon} \cap S^{(c)}_{k,\ell,\delta}.$$

Here, the implied constant is independent of $(n,m,\varepsilon)$ and $(k,\ell,\delta)$. Note that $g^T S^{(c)}_{n,m,\varepsilon} \cap S^{(c)}_{k,\ell,\delta}$ is an open set, so that the above intersection is nontrivial as soon as $g^T S^{(c)}_{n,m,\varepsilon} \cap S^{(c)}_{k,\ell,\delta} \neq \varnothing$ holds.

Now, let us assume towards a contradiction that $e_2$ is *not* an eigenvector of $g^T$. This means $g^T e_2 \notin \langle e_2 \rangle = \{0\} \times \mathbb{R}$, i.e.

$$x := g^T e_2 \in \mathbb{R}^* \times \mathbb{R} = \mathcal{O}.$$

Since $\mathcal{S}^{(c)}$ is a covering of $\mathcal{O}$, there is some $(k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with $x \in S^{(c)}_{k,\ell,\delta}$. This yields $e_2 = g^{-T}x \in g^{-T} S^{(c)}_{k,\ell,\delta}$. Since $g^{-T} S^{(c)}_{k,\ell,\delta}$ is open, there is some $\varepsilon > 0$ with $B_\varepsilon(e_2) \subset g^{-T} S^{(c)}_{k,\ell,\delta}$.

Hence, there is some $n_0 \in \mathbb{N}$ with $\binom{2^{-n}}{1} \in g^{-T} S^{(c)}_{k,\ell,\delta}$ for all $n \geq n_0$. But equation (6.3.3) of Lemma 6.3.4 implies

$$\begin{pmatrix} 2^{-n} \\ 1 \end{pmatrix} \in (2^{-n-1}, 2^{-n+1}) \times \mathbb{R} = \bigcup_{m \in \mathbb{Z}} S^{(c)}_{-n,m,1},$$

so that there is some $m_n \in \mathbb{Z}$ with $\binom{2^{-n}}{1} \in S^{(c)}_{-n,m_n,1} \cap g^{-T} S^{(c)}_{k,\ell,\delta}$ for all $n \geq n_0$. Hence, $g^T S^{(c)}_{-n,m_n,1} \cap S^{(c)}_{k,\ell,\delta} \neq \varnothing$. As seen above, this entails

$$2^n = 2^{-(-n)} = u^{(c)}_{-n,m_n,1} \lesssim u^{(c)}_{k,\ell,\delta} = 2^{-k}$$

for all $n \geq n_0$, which is impossible. □

Now we want to show that the property from the previous lemma is almost the only restriction on $g$. The first step in this direction is the following lemma.

Note that we have to assume $c \neq 1$ for the proof to work. I do not know whether the result is true in case of $c = 1$. The only part of the proof which does not work "as is" for $c = 1$ is the estimate

$$m \begin{pmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{pmatrix} \asymp m \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^c \end{pmatrix},$$

where $m : H_c \to (0,\infty)$ is an $m_0$-moderate weight and where $b$ is arbitrary with $0 < C_0 \leq b \leq C_1 < \infty$. Here, the implied constants are not allowed to depend on $\alpha, \beta, b$, but on $m_0, m, C_0, C_1$.

**Lemma 6.5.7.** *Let $c \in \mathbb{R} \setminus \{1\}$ and $\mathcal{O} := \mathbb{R}^* \times \mathbb{R}$. Let $u : \mathcal{O} \to (0,\infty)$ be a transplant of a weight $m : H_c \to (0,\infty)$ which is $m_0$-moderate for some locally bounded weight $m_0$. Finally, let $v : \mathbb{Z}^2 \times \{\pm 1\} \to (0,\infty)$ be an $\mathcal{S}^{(c)}$-discretization of $u$, for $\mathcal{S}^{(c)} = \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$.*

*Let $a \in (0,\infty)$ and $p, q \in (0,\infty]$ be arbitrary. We then have*

$$\mathcal{D}(\mathcal{S}^{(c)}, L^p, \ell^q_v) = \mathcal{D}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \mathcal{S}^{(c)}, L^p, \ell^q_v\right)$$

*with equivalent quasi-norms.*

*Proof.* Let $A := \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ and $I := \mathbb{Z}^2 \times \{\pm 1\}$ and observe $A\mathcal{O} = \mathbb{R}^* \times \mathbb{R}$. Lemma 4.4.8 (together with Corollary 6.3.5) implies that $u$ is $\mathcal{S}^{(c)}$-moderate, so that the same is true of $v$ by Lemma 3.2.5. Finally, Lemma 3.5.1 shows that $v$ (considered as a weight for $A\mathcal{S}^{(c)}$) is also $A\mathcal{S}^{(c)}$-moderate.

Now, we show that we can assume $v$ to be of a specific form: Since any two transplants of $m$ are equivalent by Lemma 4.4.8, we can as-

sume without loss of generality that $u$ is the $\binom{1}{0}$-transplant of $m$ given by

$$u\left(\varepsilon\binom{a}{b}\right) = m\left(\left[\varepsilon\cdot\begin{pmatrix}a & b\\ 0 & a^c\end{pmatrix}\right]^{-1}\right) = m^\vee\left(\varepsilon\cdot\begin{pmatrix}a & b\\ 0 & a^c\end{pmatrix}\right)$$

for $\varepsilon\in\{\pm 1\}$, $a\in(0,\infty)$ and $b\in\mathbb{R}$. Here, we used

$$\left(\left[\varepsilon\cdot\begin{pmatrix}a & b\\ 0 & a^c\end{pmatrix}\right]^{-1}\right)^{-T}\binom{1}{0} = \varepsilon\binom{a}{b}.$$

Furthermore, since any two $\mathcal{S}^{(c)}$-discretizations of $u$ are equivalent (cf. Lemma 3.2.5), we can without loss of generality assume

$$v_{k,\ell,\delta} = u\left(\delta\binom{2^k}{2^k\varrho_c^{k\ell}}\right) = m^\vee\left(\delta\cdot\begin{pmatrix}2^k & 2^k\varrho_c^{k\ell}\\ 0 & 2^{kc}\end{pmatrix}\right) \tag{6.5.9}$$

for all $(k,\ell,\delta)\in I$, with $\varrho_c = 2^{c-1}$.

Next, we prove that $A\mathcal{S}^{(c)}$ is almost subordinate to $\mathcal{S}^{(c)}$. To this end, note that equation (6.3.2) of Lemma 6.3.4 yields

$$S^{(c)}_{n,m,\varepsilon} = \varepsilon U^{\left(2^{n-1},2^{n+1}\right)}_{\left(\varrho_c^n(m-1),\varrho_c^n(m+1)\right)} \quad\text{and}\quad AS^{(c)}_{n,m,\varepsilon} = \varepsilon U^{\left(2^{n-1},2^{n+1}\right)}_{\left(a\varrho_c^n(m-1),a\varrho_c^n(m+1)\right)}$$

for all $(n,m,\varepsilon)\in I$. This was also used in equation (6.5.9) above.

Fix $i=(n,m,\varepsilon)\in I$ and let $j=(k,\ell,\delta)\in I$ with $S^{(c)}_j\cap AS^{(c)}_i\neq\varnothing$ be arbitrary. Fix $\binom{x}{y}\in S^{(c)}_j\cap AS^{(c)}_i$. Because of

$$S^{(c)}_{\alpha,\beta,\theta}\subset[\theta\cdot(0,\infty)]\times\mathbb{R},$$

this yields $\varepsilon=\delta$ and $\varepsilon x\in\left(2^{n-1},2^{n+1}\right)\cap\left(2^{k-1},2^{k+1}\right)$. As usual, we derive $2^{n-1}<2^{k+1}$ and $2^{k-1}<2^{n+1}$, which implies

$$n-2<k<n+2,$$

i.e. $k\in\{n-1,n,n+1\}$. Furthermore, we get

$$\frac{\varepsilon y}{\varepsilon x} = \left(\varrho_c^k(\ell-1),\varrho_c^k(\ell+1)\right)\cap\left(a\varrho_c^n(m-1),a\varrho_c^n(m+1)\right)$$

and hence

$$\varrho_c^k(\ell-1)<a\varrho_c^n(m+1)\quad\text{and}\quad a\varrho_c^n(m-1)<\varrho_c^k(\ell+1).$$

Rearranging yields

$$\ell<a\varrho_c^{n-k}(m+1)+1 = a\varrho_c^{n-k}m+a\varrho_c^{n-k}+1$$

and

$$a\varrho_c^{n-k}m-\left(a\varrho_c^{n-k}+1\right) = a\varrho_c^{n-k}(m-1)-1<\ell.$$

But we have $\varrho_c^{n-k}=2^{(c-1)(n-k)}\leq 2^{|1-c||n-k|}\leq 2^{|1-c|}=:\gamma$. Hence,

$$\ell \in \mathbb{Z} \cap B_{1+a\gamma}\left(a\varrho_c^{n-k} m\right) =: M_{n,m,k}.$$

Note that $M_{n,m,k}$ has at most $|M_{n,m,k}| \leq 1 + 2\lceil 1 + a\gamma \rceil =: N$ elements. All in all, we conclude

$$j = (k,\ell,\delta) \in \bigcup_{\kappa=n-1}^{n+1} \left[\{\kappa\} \times M_{n,m,\kappa} \times \{\varepsilon\}\right],$$

where the set on the right-hand side has at most $3N$ elements.

Hence, we have shown

$$\left|\left\{ j \in I \;\middle|\; S_j^{(c)} \cap AS_i^{(c)} \neq \varnothing \right\}\right| \leq 3N$$

for all $i \in I$. Since all of the sets $S_{n,m,\varepsilon}^{(c)}$ and $AS_{n,m,\varepsilon}^{(c)}$ are open and path-connected by Lemma 6.3.4 and because $\mathcal{S}^{(c)}$ and $A\mathcal{S}^{(c)}$ are both coverings of the same set $\mathcal{O}$, Corollary 3.3.4 shows that $A\mathcal{S}_i^{(c)}$ is almost subordinate to $\mathcal{S}^{(c)}$.

Now, recall from above that $AS_{n,m,\varepsilon}^{(c)} \cap S_{k,\ell,\delta}^{(c)} \neq \varnothing$ implies $|n-k| \leq 1$ and hence

$$\left|\det\left(AA_{n,m,\varepsilon}^{(c)}\right)\right| = a \cdot 2^{n(1+c)} \asymp 2^{k(1+c)} = \left|\det\left(A_{k,\ell,\delta}^{(c)}\right)\right|. \tag{6.5.10}$$

In particular, $A\mathcal{S}^{(c)}$ is relatively moderate with respect to $\mathcal{S}^{(c)}$. Thus, if we set

$$I_j = \left\{ i \in I \;\middle|\; AS_i^{(c)} \cap S_j^{(c)} \neq \varnothing \right\},$$

then Lemma 3.3.7 (with $\mathcal{Q} = A\mathcal{S}^{(c)}$, $\mathcal{P} = \mathcal{S}^{(c)}$, $J_0 = J = I$ and $I_0 = I$) yields

$$\left|I_j\right| \asymp \left|\det\left(AA_{i_j}^{(c)}\right)\right|^{-1} \cdot \left|\det\left(A_j^{(c)}\right)\right| \asymp 1$$

for all $j \in J = I$, where $i_j \in I$ with $AS_{i_j}^{(c)} \cap S_j^{(c)} \neq \varnothing$ is chosen arbitrarily. Hence, another application of Corollary 3.3.4 shows that $\mathcal{S}^{(c)}$ is almost subordinate to $A\mathcal{S}^{(c)}$.

It remains to show $v_{n,m,\varepsilon} \asymp v_{k,\ell,\delta}$ for all $(n,m,\varepsilon), (k,\ell,\delta) \in I$ with $AS_{n,m,\varepsilon}^{(c)} \cap S_{k,\ell,\delta}^{(c)} \neq \varnothing$. Indeed, once this is shown, the desired equality is a direct consequence of Lemma 5.1.9. For $p \in (0,1)$, note that validity of the second assumption of Lemma 5.1.9 is implied by equation (6.5.10).

Let $k := \max\left\{k\left(A\mathcal{S}^{(c)}, \mathcal{S}^{(c)}\right), k\left(\mathcal{S}^{(c)}, A\mathcal{S}^{(c)}\right)\right\}$ and $N := 2k+2$. Let $(n,m,\varepsilon) \in I$ be arbitrary. We want to find some $t = t(n,m) \in \mathbb{Z}$ with $AS_{n,m,\varepsilon}^{(c)} \cap S_{n,t,\varepsilon}^{(c)} \neq \varnothing$ and which satisfies $v_{n,m,\varepsilon} \asymp v_{n,t,\varepsilon}$, where the implied constant is independent of $(n,m,\varepsilon)$ and of $t$.

Once this is shown, any $(k,\ell,\delta) \in I$ with $AS_{n,m,\varepsilon}^{(c)} \cap S_{k,\ell,\delta}^{(c)} \neq \varnothing$ already satisfies $(k,\ell,\delta) \in (n,t,\varepsilon)^{N*}$, thanks to Lemma 3.3.2. Hence,

$$v_{n,m,\varepsilon} \asymp v_{n,t,\varepsilon} \asymp v_{k,\ell,\delta},$$

where the last estimate is due to $\mathcal{S}^{(c)}$-moderateness of $v$.

*Here, the cluster is fomed with respect to $\mathcal{S}^{(c)}$, but the cluster for $A\mathcal{S}^{(c)}$ would actually be the same, cf. Lemma 3.5.1.*

It remains to construct $t = t(n,m) \in \mathbb{Z}$ as described above and to show $v_{n,m,\varepsilon} \asymp v_{n,t,\varepsilon}$. To this end, note that if $\alpha < \beta$ and $\eta < \theta$ hold, then $(\alpha,\beta) \cap (\eta,\theta) \neq \varnothing$ is true as soon as

$$\alpha < \theta \text{ and } \eta < \beta$$

is fulfilled. Indeed, these inequalities imply $\max\{\alpha,\eta\} < \min\{\beta,\theta\}$ and thus

$$\varnothing \neq (\max\{\alpha,\eta\}, \min\{\beta,\theta\}) \subset (\alpha,\beta) \cap (\eta,\theta).$$

But using the map

$$\Phi : \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}^* \times \mathbb{R}, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x \\ y/x \end{smallmatrix}\right)$$

from Definition 6.3.3, we get

$$\begin{aligned} \Phi\left(AS^{(c)}_{n,m,\varepsilon}\right) &= \left[\varepsilon\left(2^{n-1}, 2^{n+1}\right)\right] \times \left(a\varrho_c^n(m-1), a\varrho_c^n(m+1)\right), \\ \Phi\left(S^{(c)}_{n,t,\varepsilon}\right) &= \left[\varepsilon\left(2^{n-1}, 2^{n+1}\right)\right] \times \left(\varrho_c^n(t-1), \varrho_c^n(t+1)\right), \end{aligned}$$

so that it suffices (by bijectivity of $\Phi$) to ensure

$$\left(\varrho_c^n(t-1), \varrho_c^n(t+1)\right) \cap \left(a\varrho_c^n(m-1), a\varrho_c^n(m+1)\right) \neq \varnothing.$$

As seen above, it is thus enough to choose $t \in \mathbb{Z}$ with

$$\begin{aligned} &\varrho_c^n(t-1) \overset{!}{<} a\varrho_c^n(m+1) \\ \iff &t \overset{!}{<} a(m+1)+1 = am+1+a \end{aligned}$$

and with

$$\begin{aligned} &a\varrho_c^n(m-1) \overset{!}{<} \varrho_c^n(t+1) \\ \iff &am-(1+a) \overset{!}{<} t, \end{aligned}$$

i.e. with $t \in B_{1+a}(am)$. Because of $1+a > 1$, we can thus choose

$$t := \begin{cases} \lceil am \rceil, & \text{if } m \geq 1, \\ 0, & \text{if } m = 0, \\ \lfloor am \rfloor, & \text{if } m \leq -1. \end{cases} \tag{6.5.11}$$

It remains to prove $v_{n,m,\varepsilon} \asymp v_{n,t,\varepsilon}$. By virtue of equation (6.5.9) above, we we thus want to show

$$m^\vee\left(\varepsilon \cdot \left(\begin{smallmatrix} 2^n & 2^n\varrho_c^n t \\ 0 & 2^{nc} \end{smallmatrix}\right)\right) = v_{n,t,\varepsilon} \overset{!}{\asymp} v_{n,m,\varepsilon} = m^\vee\left(\varepsilon \cdot \left(\begin{smallmatrix} 2^n & 2^n\varrho_c^n m \\ 0 & 2^{nc} \end{smallmatrix}\right)\right).$$

To this end, we will show below that we can write $t = bm$ for some $b \in (0, \infty)$ with $a \leq b \leq a+1$. Using the notation $\alpha := 2^n$ and $\beta := 2^n \varrho_c^n m$, we thus want to show

$$m^\vee \left( \varepsilon \cdot \left( \begin{smallmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right) \asymp m^\vee \left( \varepsilon \cdot \left( \begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right).$$

But for $\gamma \in (0, \infty)$, we have

$$\begin{aligned} \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^c \end{pmatrix} \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma^{-c} \end{pmatrix} &= \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^c \end{pmatrix} \begin{pmatrix} \alpha/\gamma & \gamma^{-c}\beta \\ 0 & (\alpha/\gamma)^c \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \gamma^{1-c}\beta \\ 0 & \alpha^c \end{pmatrix} = \begin{pmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{pmatrix}, \end{aligned}$$

if we choose $\gamma = b^{\frac{1}{1-c}}$. Note that we assume $c \neq 1$, so that $\frac{1}{1-c} \in \mathbb{R}$ is well-defined. Because of $a \leq b \leq a+1$, we get

$$\begin{cases} 0 < \gamma_0 := (a+1)^{\frac{1}{1-c}} \leq \gamma = b^{\frac{1}{1-c}} \leq a^{\frac{1}{1-c}} =: \gamma_1, & \text{if } c > 1, \\ 0 < \gamma_0 := a^{\frac{1}{1-c}} \leq \gamma = b^{\frac{1}{1-c}} \leq (a+1)^{\frac{1}{1-c}} =: \gamma_1, & \text{if } c < 1, \end{cases}$$

and hence $B := \left( \begin{smallmatrix} \gamma & 0 \\ 0 & \gamma^c \end{smallmatrix} \right) \in K$ for a fixed compact set $K = K(a, c) \subset H_c$. Since $m_0^\vee$ is locally bounded, the constant $C := \sup_{h \in K \cup K^{-1}} m_0^\vee(h)$ is finite.

Now observe

$$\begin{aligned} m^\vee(xyz) &= m(z^{-1}y^{-1}x^{-1}) \\ &\leq m_0(z^{-1})\, m(y^{-1})\, m_0(x^{-1}) \\ &= m_0^\vee(x)\, m^\vee(y)\, m_0^\vee(z) \end{aligned}$$

for all $x, y, z \in H_c$, so that $m^\vee$ is $m_0^\vee$-moderate. Hence, we see

$$\begin{aligned} m^\vee \left( \varepsilon \left( \begin{smallmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right) &= m^\vee \left( B \cdot \varepsilon \left( \begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix} \right) \cdot B^{-1} \right) \\ &\leq m_0^\vee(B)\, m^\vee \left( \varepsilon \left( \begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right) m_0^\vee(B^{-1}) \\ &\leq C^2 \cdot m^\vee \left( \varepsilon \left( \begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right). \end{aligned}$$

Similarly, we get

$$m^\vee \left( \varepsilon \left( \begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right) = m^\vee \left( B^{-1} \left( \begin{smallmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{smallmatrix} \right) B \right) \leq C^2 \cdot m^\vee \left( \varepsilon \left( \begin{smallmatrix} \alpha & b\beta \\ 0 & \alpha^c \end{smallmatrix} \right) \right).$$

As seen above, the only thing missing is to show that we can write $t = bm$ with $t$ as in equation (6.5.11) and with $a \leq b \leq a+1$. To see this, we distinguish three cases:

*Case* 1. $m = 0$. Here, we have $t = 0$ and hence $t = bm$ for $b = a$.

*Case* 2. $m \geq 1$. Here, we have $t = \lceil am \rceil$ with

$$am \leq \lceil am \rceil = t = \lceil am \rceil \leq am + 1 \leq am + m = m(a+1).$$

Using $m \geq 1 > 0$, we conclude

$$a = \frac{am}{m} \leq \frac{t}{m} = b = \frac{t}{m} \leq \frac{m(a+1)}{m} = a + 1.$$

*Case* 3. $m \leq -1$. Here, we have $t = \lfloor am \rfloor$ and

$$m(a+1) = am + m \leq am - 1 \leq \lfloor am \rfloor = t = \lfloor am \rfloor \leq am.$$

Using $m \leq -1 < 0$, we get

$$a = \frac{am}{m} \leq \frac{t}{m} = b = \frac{t}{m} \leq \frac{m(a+1)}{m} = a + 1,$$

as desired. □

Using the above auxiliary result, we can now show that $\mathrm{UD}_c$ contains a large set of dilations. In fact, we will see below that the inclusion which we prove now is actually an equality.

**Lemma 6.5.8.** *Let $c \in \mathbb{R} \setminus \{1\}$. Then*

$$\begin{aligned}\mathrm{UD}_c &\supset \left\{ \varepsilon \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \,\middle|\, \varepsilon \in \{\pm 1\},\ a, b \in (0,\infty),\ m \in \mathbb{R} \right\} \\ &= \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, A \text{ upper triangular matrix with } \det A > 0 \right\}.\end{aligned}$$

*Remark.* An alternative proof method to the one used below (avoiding the interpretation of coorbit spaces as decomposition spaces) is as follows: We first observe that each invertible upper triangular matrix $A \in \mathbb{R}^{2\times 2}$ leaves the group $H_c$ invariant under conjugation. Indeed, we have

$$\begin{aligned}&\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}^{-1} \cdot \varepsilon \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^c \end{pmatrix} \cdot \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \\ &= \varepsilon \cdot \begin{pmatrix} a^{-1} & *_1 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} a\alpha & *_2 \\ 0 & b\alpha^c \end{pmatrix} = \varepsilon \begin{pmatrix} \alpha & *_3 \\ 0 & \alpha^c \end{pmatrix} \in H_c\end{aligned}$$

for arbitrary $a, b \in \mathbb{R}^*$ and $m \in \mathbb{R}$ and suitable $*_1, *_2, *_3 \in \mathbb{R}$.

But Remark 4.7.4 shows that invariance of $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ under dilation by $g$ is equivalent to boundedness of the map

$$\iota : L^2(\mathbb{R}^2) \cap \mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right) \to \mathrm{Co}\left(L^{p,q}_{m^g}\left(\mathbb{R}^2 \rtimes H_c^g\right)\right), f \mapsto f$$

with $H_c^g := g^{-1} H_c g$ and $m^g : H_c^g \to (0,\infty), h \mapsto m\left(ghg^{-1}\right)$. If $g$ is an invertible upper triangular matrix, the considerations above yield

$H_c^g = H_c$, so that it is clearly sufficient if we have $m^g \lesssim m$. For $g \in H_c$, this follows by $m_0$-moderateness of $m$ and for $c \neq 1$ and $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a > 0$, it can be verified as in the proof of Lemma 6.5.7 above. Since $\mathrm{UD}_c$ is closed under matrix multiplication, we get the claim of the present lemma.

We remark however that the view of coorbit spaces as decomposition spaces is very helpful in showing that certain dilations are *not* universal dilations for $H_c$, see the proofs of Lemma 6.5.6 and Theorem 6.5.9.

*Proof.* Let $p, q \in (0, \infty]$ and let $m : H_c \to (0, \infty)$ be $m_0$-moderate for some submultiplicative, measurable weight $m_0 : H_c \to (0, \infty)$. Remark 4.7.4 shows that $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ is invariant under dilation by $g \in \mathrm{GL}\left(\mathbb{R}^2\right)$ if there is an embedding of decomposition spaces

$$\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right) \to \mathcal{D}\left(g^T \mathcal{S}^{(c)}, L^p, \ell^q_u\right), \tag{6.5.12}$$

where $u$ is an $\mathcal{S}^{(c)}$-discretization of a decomposition weight $v$ for the coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. Here, we used that $\mathcal{S}^{(c)}$ is an induced $L^p$-decomposition covering of $H_c$, see Corollary 6.3.5.

By Definition 4.5.3, any transplant $v$ of

$$m^{(q)} : H_c \to (0, \infty), h \mapsto |\det(h)|^{\frac{1}{2} - \frac{1}{q}} \cdot m(h)$$

is a decomposition weight for $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$. But since the determinant is multiplicative, it is easy to see that $m^{(q)}$ is $m_1$-moderate for the submultiplicative, measurable weight

$$m_1 : H_c \to (0, \infty), h \mapsto |\det(h)|^{\frac{1}{2} - \frac{1}{q}} \cdot m_0(h).$$

Hence, Lemma 6.5.7 shows $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right) = \mathcal{D}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \mathcal{S}^{(c)}, L^p, \ell^q_u\right)$ for all $a \in (0, \infty)$, since $m_1$ is locally bounded by Theorem 2.2.22.

This show that an embedding of decomposition spaces as in equation (6.5.12) is true for $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)^T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ for all $a \in (0, \infty)$. As noted above, this shows that $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ is invariant under dilation by $g$. Since $p, q, m$ were chosen arbitrarily, we get $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{UD}_c$ for all $a \in (0, \infty)$.

Now, let $\varepsilon \in \{\pm 1\}$, $a, b \in (0, \infty)$ and $m \in \mathbb{R}$ be arbitrary. Choose $\gamma := \frac{b}{a^c} \in (0, \infty)$ and $k = \frac{a^c m}{b} = \frac{m}{\gamma} \in \mathbb{R}$. As just seen, we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right) \in \mathrm{UD}_c$. Furthermore, Lemma 6.5.5 yields $\varepsilon \left(\begin{smallmatrix} a & k \\ 0 & a^c \end{smallmatrix}\right) \in H_c \subset \mathrm{UD}_c$. Since $\mathrm{UD}_c$ is closed under matrix multiplication, we arrive at

$$\varepsilon \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \varepsilon \begin{pmatrix} a & k\gamma \\ 0 & a^c \gamma \end{pmatrix} = \varepsilon \begin{pmatrix} a & k \\ 0 & a^c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \in \mathrm{UD}_c.$$

This completes the proof. □

As promised, we finally show that the inclusion from above is actually an equality.

**Theorem 6.5.9.** *Let $c \in \mathbb{R} \setminus \{1\}$ be arbitrary. Then*

$$\mathrm{UD}_c = \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, A \text{ upper triangular matrix with } \det(A) > 0 \right\}.$$

*Proof.* By Lemma 6.5.8, we only have to prove "$\subset$". To this end, let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{UD}_c$ be arbitrary. By Lemma 6.5.6, this implies that $e_2 = (0,1)$ is an eigenvector of $A^T$, i.e.

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = A^T e_2 = \theta e_2$$

for some $\theta \in \mathbb{R}$. Hence, $\gamma = 0$, so that $A$ is indeed an upper triangular matrix.

We will show below $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \notin \mathrm{UD}_c$. Now, let us prove that this already implies $\det A > 0$. Assume towards a contradiction that this is not the case. Hence, we have $\alpha < 0$ and $\delta > 0$ or we have $\alpha > 0$ and $\delta < 0$. Note that Lemma 6.5.5 yields $-\mathrm{id} \in H_c \subset \mathrm{UD}_c$ and thus also $-A \in \mathrm{UD}_c$, since $\mathrm{UD}_c$ is closed under multiplication. Hence, we can assume $\alpha > 0$ and $\delta < 0$. But Lemma 6.5.8 yields

$$B := \begin{pmatrix} \alpha^{-1} & -\frac{1}{\delta}\frac{\beta}{\alpha} \\ 0 & |\delta|^{-1} \end{pmatrix} \in \mathrm{UD}_c.$$

Since $\mathrm{UD}_c$ is closed under multiplication, we conclude

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\beta}{\alpha} - \frac{\delta}{\delta}\frac{\beta}{\alpha} \\ 0 & \delta/|\delta| \end{pmatrix} = BA \in \mathrm{UD}_c.$$

In the remainder of the proof, we will show $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \notin \mathrm{UD}_c$, which yields the desired contradiction. This will then imply $\det A > 0$ and thus complete the proof.

Assume towards a contradiction that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{UD}_c$. We proceed analogously to the proof of Lemma 6.5.6, but this time, we use the weight

$$v : H_c \to (0,\infty), \varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \mapsto a + a^c + b_+$$

with $b_+ = \max\{0, b\}$. Here, we assume as usual $a \in (0,\infty)$, $b \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$.

We first need to show that $v$ is $v_0$-moderate for some submultiplicative weight $v_0$. To this end, define

$$\|A\| := \sum_{i,j=1}^{2} \left|A_{i,j}\right| \text{ for } A \in \mathbb{R}^{2\times 2}$$

and note that this defines a norm on $\mathbb{R}^{2\times 2}$. Since matrix multiplication is continuous, there is a constant $L \geq 1$ with

$$\|AB\| \leq L \cdot \|A\| \cdot \|B\|$$

for all $A, B \in \mathbb{R}^{2\times 2}$. Now, set $v_0 : H_c \to (0,\infty), h \mapsto 2L\|A\|$. Note that we have

$$v_0(AB) = 2L\|AB\| \leq 2L^2 \|A\| \|B\| \leq v_0(A)\, v_0(B)$$

for $A, B \in H_c$, so that $v_0$ is indeed submultiplicative.

Furthermore, we have

$$\begin{aligned} v\left(\varepsilon \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} \delta \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^c \end{pmatrix}\right) &= v\left(\varepsilon\delta \begin{pmatrix} a\alpha & a\beta + b\alpha^c \\ 0 & (a\alpha)^c \end{pmatrix}\right) \\ &= a\alpha + (a\alpha)^c + (a\beta + b\alpha^c)_+ \\ &\leq a\alpha + a^c\alpha^c + \beta_+ a + b_+\alpha^c. \end{aligned}$$

In the last step, we used that $x \mapsto x_+$ is nondecreasing with $\beta \leq \beta_+$ and $b \leq b_+$, as well as $a, \alpha > 0$.

But on the one hand, we have

$$\begin{aligned} a\alpha + a^c\alpha^c + \beta_+ a + b_+\alpha^c &\leq (a + a^c + a + b_+) \cdot \max\{\alpha, \alpha^c, \beta_+, \alpha^c\} \\ &\leq 2\left\|\varepsilon \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)\right\| \cdot (\alpha + \alpha^c + \beta_+) \\ &\leq v_0\left(\varepsilon \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)\right) \cdot v\left(\delta \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix}\right)\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} a\alpha + a^c\alpha^c + \beta_+ a + b_+\alpha^c &\leq (\alpha + \alpha^c + \beta_+ + \alpha^c) \cdot \max\{a, a^c, a, b_+\} \\ &\leq \left\|\delta \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix}\right)\right\| \cdot (a + a^c + b_+) \\ &\leq v_0\left(\delta \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^c \end{smallmatrix}\right)\right) \cdot v\left(\varepsilon \left(\begin{smallmatrix} a & b \\ 0 & a^c \end{smallmatrix}\right)\right). \end{aligned}$$

Together, we easily see that $v$ is $v_0$-moderate.

With this modified version of $v$, we have

$$\begin{aligned} u^{(2)}_{n,m,\varepsilon} &= v\left(\left(A^{(c)}_{n,m,\varepsilon}\right)^{-T}\right) \\ &= v\left(\varepsilon \cdot \begin{pmatrix} 2^{-n} & -2^{-n}m \\ 0 & 2^{-nc} \end{pmatrix}\right) = 2^{-n} + 2^{-nc} + \left(-2^{-n}m\right)_+ \end{aligned}$$

in the notation of the proof of Lemma 6.5.6. Now, literally the same argument as in the proof of that lemma (up to equation (6.5.8)), but with the modified version of $v$ and with $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, implies

$$u^{(2)}_{n,m,\varepsilon} \lesssim u^{(2)}_{k,\ell,\delta}$$

for all $(n,m,\varepsilon),(k,\ell,\delta) \in \mathbb{Z}^2 \times \{\pm 1\}$ with

$$\varnothing \neq (\mathbb{R}^* \times \mathbb{R}) \cap g^T S^{(c)}_{n,m,\varepsilon} \cap S^{(c)}_{k,\ell,\delta}.$$

Now, let $m \in \mathbb{N}$ be arbitrary. Using the explicit description of the sets $S^{(c)}_{n,m,\varepsilon}$ of the covering $\mathcal{S}^{(c)} = \mathcal{S}^{(c)}_{-1,1,\frac{1}{2},2}$ developed in Lemma 6.3.4, we see

$$x := \begin{pmatrix} 1 \\ m \end{pmatrix} \in U^{\left(2^{0-1},2^{0+1}\right)}_{\left(\varrho^0_c(m-1),\varrho^0_c(m+1)\right)} = S^{(c)}_{0,m,1}.$$

Furthermore,

$$g^{-T}x = g^T x = \begin{pmatrix} 1 \\ -m \end{pmatrix} \in U^{\left(2^{0-1},2^{0+1}\right)}_{\left(\varrho^0_c(-m-1),\varrho^0_c(-m+1)\right)} = S^{(c)}_{0,-m,1}.$$

All in all, we see $x \in (\mathbb{R}^* \times \mathbb{R}) \cap g^T S^{(c)}_{0,-m,1} \cap S^{(c)}_{0,m,1}$, so that the above estimate yields

$$2 + m = 2 + (-(-m))_+ = u^{(2)}_{0,-m,1} \lesssim u^{(2)}_{0,m,1} = 2 + (-m)_+ = 2$$

for all $m \in \mathbb{N}$. This is the desired contradiction. □

*Remark.* It is worth noting that the only results used in this subsection are

1. The isomorphism of $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^2 \rtimes H_c\right)\right)$ with $\mathcal{D}\left(\mathcal{S}^{(c)}, L^p, \ell^q_u\right)$ developed in Chapter 4 and
2. the (relatively) elementary results from Lemmas 5.3.1 and 5.1.9.

In particular, none of the more sophisticated embedding results using the connection with discrete nested sequence spaces was used. This shows that the viewpoint of wavelet coorbit spaces as decomposition spaces is very powerful, even without much of the theory of embeddings for decomposition spaces. I am not aware of any results similar to those developed in this subsection, not even for the special case $c = \frac{1}{2}$.

### 6.5.3 *Isotropic dilations of α-modulation spaces*

In this subsection, we will asymptotically calculate the operator norm of the isotropic dilation operator on $\alpha$-modulation spaces. For this, we first recall the definition of the covering

$$\mathcal{O}^{(\alpha)} := \mathcal{O}^{(\alpha)}_r := \left(O^{(\alpha)}_{r,k}\right)_{k\in\mathbb{Z}^d\setminus\{0\}} := \left(B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)\right)_{k\in\mathbb{Z}^d\setminus\{0\}}$$

for $\alpha \in [0,1)$ with $\alpha_0 = \frac{\alpha}{1-\alpha}$ and suitable $r > 0$, which is used for the definition of $\alpha$-modulation spaces, cf. Theorem 6.1.1.

As in the case of the shearlet covering $\mathcal{S}^{(c)}$ in the preceding subsection, we first have to study the subordinateness relation between the covering $\mathbb{O}^{(\alpha)}$ and the dilated covering $\lambda\mathbb{O}^{(\alpha)}$. In this case, the almost subordinateness works just as well as in the case of the covering $\mathcal{S}^{(c)}$, but relative moderateness of the weight $k \mapsto \langle k \rangle$ is only given "away from the origin". This will complicate the analysis of the norm of the dilation operator.

**Lemma 6.5.10.** *Let $\alpha \in [0,1)$ and let $r_1 = r_1(\alpha, d) > 0$ as in Theorem 6.1.1. For each $r > r_1$, there is a constant $K = K(r, \alpha, d) \in \mathbb{N}$ such that the covering $\mathbb{O}_r^{(\alpha)}$ is almost subordinate to $\lambda\mathbb{O}_r^{(\alpha)}$ and such that we have*

$$k(\mathbb{O}_r^{(\alpha)}, \lambda\mathbb{O}_r^{(\alpha)}) \leq K$$

*for all $\lambda \geq 1$.*

*Conversely, the covering $\lambda\mathbb{O}_r^{(\alpha)}$ is almost subordinate to $\mathbb{O}_r^{(\alpha)}$ with*

$$k(\lambda\mathbb{O}_r^{(\alpha)}, \mathbb{O}_r^{(\alpha)}) \leq K$$

*for all $\lambda \in (0,1)$, with the same constant $K$ as above.*

*Furthermore, there is a constant $C = C(r, \alpha) \geq 1$ such that*

$$C^{-1} \cdot \lambda^{1-\alpha} \langle \ell \rangle \leq \langle k \rangle \leq C \cdot \lambda^{1-\alpha} \langle \ell \rangle \tag{6.5.13}$$

*holds for all $\lambda \geq 1$ and all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)} \neq \varnothing$, as long as $|k| \geq \max\{2r, (2\lambda)^{1-\alpha}\}$ holds.*

*Finally, there is a constant $M = M(r, \alpha) \in \mathbb{N}$ with the following property: If $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ satisfy $O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)} \neq \varnothing$ for some $\lambda \geq 1$ and if $|k| \leq \max\{2r, (2\lambda)^{1-\alpha}\}$ holds, then we have $\ell \in \{-M, \dots, M\}^d$.*

*Proof.* We consider the case $\lambda \geq 1$ first. Observe that all sets of the coverings $\mathbb{O}_r^{(\alpha)}$ and $\lambda\mathbb{O}_r^{(\alpha)}$ consist of open balls, which are trivially open and path-connected. Hence, Corollary 3.3.4 shows that in order to establish the first part of the claim, it is sufficient to show that there is a constant $N = N(r, \alpha, d) \in \mathbb{N}$ such that $|J_k^{(\lambda)}| \leq N$ holds for all $k \in \mathbb{Z}^d \setminus \{0\}$ and $\lambda \geq 1$, with

$$J_k^{(\lambda)} := \left\{ \ell \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)} \neq \varnothing \right\}.$$

To see this, recall that Lemma 6.1.2 yields a constant $C_{r,\alpha} \geq 1$ such that

$$C_{r,\alpha}^{-1} \cdot \langle m \rangle^{\frac{1}{1-\alpha}} \leq \langle x \rangle \leq C_{r,\alpha} \cdot \langle m \rangle^{\frac{1}{1-\alpha}} \tag{6.5.14}$$

holds for all $m \in \mathbb{Z}^d \setminus \{0\}$ and all $x \in O_{r,m}^{(\alpha)}$.

We distinguish two cases:

CASE 1: We have $|k| \leq \max\{2r, (2\lambda)^{1-\alpha}\}$. Using $\lambda^{1-\alpha} \geq 1$, we see $2r \leq 2r \cdot \lambda^{1-\alpha} \leq (1+2r) \cdot \lambda^{1-\alpha}$ and hence

$$\begin{aligned}|k| &\leq \max\left\{2r, (2\lambda)^{1-\alpha}\right\} \\ &\leq (1+2r) \cdot \max\left\{\lambda^{1-\alpha}, 2^{1-\alpha} \cdot \lambda^{1-\alpha}\right\} \\ &= 2^{1-\alpha}(1+2r) \cdot \lambda^{1-\alpha}.\end{aligned}$$

Now, let $\ell \in J_k^{(\lambda)}$ be arbitrary. By definition of $J_k^{(\lambda)}$, there is some $x \in O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)}$. This yields

$$\begin{aligned}|x| \leq \langle x \rangle &\leq C_{r,\alpha} \cdot \langle k \rangle^{\frac{1}{1-\alpha}} \leq C_{r,\alpha} \cdot (1+|k|)^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}} C_{r,\alpha} \cdot \left(1 + |k|^{\frac{1}{1-\alpha}}\right) \\ &\leq 2^{\frac{1}{1-\alpha}} C_{r,\alpha} \cdot \left(1 + 2 \cdot (1+2r)^{\frac{1}{1-\alpha}} \cdot \lambda\right) \\ &\leq C'_{r,\alpha} \cdot \lambda,\end{aligned}$$

where we used $\lambda \geq 1$ in the last step.

But $\lambda^{-1}x \in O_{r,\ell}^{(\alpha)}$ and hence

$$\begin{aligned}\langle \ell \rangle &\leq C_{r,\alpha}^{1-\alpha} \cdot \left\langle \lambda^{-1}x \right\rangle^{1-\alpha} \leq C_{r,\alpha}^{1-\alpha} \cdot \left(1 + \left|\lambda^{-1}x\right|\right)^{1-\alpha} \\ &\leq C_{r,\alpha}^{1-\alpha} \cdot \left(1 + C'_{r,\alpha}\right)^{1-\alpha} =: C''_{r,\alpha}.\end{aligned}$$

For $M_{r,\alpha} := \lceil C''_{r,\alpha} \rceil$, this yields $\ell \in \{-M_{r,\alpha}, \dots, M_{r,\alpha}\}^d$ and hence $J_k^{(\lambda)} \subset \{-M_{r,\alpha}, \dots, M_{r,\alpha}\}^d$ for all $|k| \leq \max\{2r, (2\lambda)^{1-\alpha}\}$. In particular, we have established the last claim of the lemma.

CASE 2: We have $|k| \geq \max\{2r, (2\lambda)^{1-\alpha}\}$. We will first establish $|x| \geq \lambda$ for all $x \in O_{r,k}^{(\alpha)}$. To see this, note $|k| - r \geq |k|/2$ because of $|k| \geq 2r$. For $x \in O_{r,k}^{(\alpha)} = B_{r|k|^{\alpha_0}}\left(|k|^{\alpha_0} k\right)$, this implies

$$\begin{aligned}|x| &\geq \left||k|^{\alpha_0} k\right| - \left||k|^{\alpha_0} k - x\right| \\ &> |k|^{\alpha_0} |k| - r |k|^{\alpha_0} \\ &= |k|^{\alpha_0} \cdot (|k| - r) \\ &\geq \frac{1}{2} \cdot |k|^{\alpha_0+1} = \frac{1}{2} \cdot |k|^{\frac{1}{1-\alpha}} \geq \frac{1}{2} \cdot (2\lambda) = \lambda.\end{aligned}$$

We will now first establish estimate (6.5.13) for all $\ell \in J_k^{(\lambda)}$. To this end, let $x \in O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)}$ be arbitrary. The above shows that this yields $|x| \geq \lambda$ and hence

$$|\lambda^{-1}x| \leq \langle \lambda^{-1}x \rangle \leq 1 + |\lambda^{-1}x| \leq 2 \cdot |\lambda^{-1}x|.$$

But $\lambda^{-1}x \in O_{r,\ell}^{(\alpha)}$, so that equation (6.5.14) implies

$$\begin{aligned}
\langle \ell \rangle &\leq \left[C_{r,\alpha} \cdot \langle \lambda^{-1}x \rangle\right]^{1-\alpha} \\
&\leq \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot |\lambda^{-1}x|^{1-\alpha} \\
&= \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot \frac{|x|^{1-\alpha}}{\lambda^{1-\alpha}} \\
&\leq \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot \frac{\langle x \rangle^{1-\alpha}}{\lambda^{1-\alpha}} \\
&\leq \frac{\left(2C_{r,\alpha}\right)^{1-\alpha}}{\lambda^{1-\alpha}} \cdot \left[C_{r,\alpha} \cdot \langle k \rangle^{\frac{1}{1-\alpha}}\right]^{1-\alpha} \\
&\leq C_{r,\alpha}''' \cdot \frac{\langle k \rangle}{\lambda^{1-\alpha}}.
\end{aligned}$$

The proof of the reverse inequality is similar: We first recall the estimate $|x| \geq \lambda \geq 1$ from above , which implies $\langle x \rangle \leq 1 + |x| \leq 2\,|x|$. Hence,

$$\begin{aligned}
\frac{\langle k \rangle}{\lambda^{1-\alpha}} &\leq \frac{\left(C_{r,\alpha} \cdot \langle x \rangle\right)^{1-\alpha}}{\lambda^{1-\alpha}} \\
&\leq C_{r,\alpha}^{1-\alpha} \cdot \frac{\left(2\,|x|\right)^{1-\alpha}}{\lambda^{1-\alpha}} \\
&= \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot \left|\lambda^{-1}x\right|^{1-\alpha} \\
&\leq \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot \left\langle \lambda^{-1}x \right\rangle^{1-\alpha} \\
&\leq \left(2C_{r,\alpha}\right)^{1-\alpha} \cdot \left[C_{r,\alpha} \cdot \langle \ell \rangle^{\frac{1}{1-\alpha}}\right]^{1-\alpha} \\
&= \left(2C_{r,\alpha}^2\right)^{1-\alpha} \cdot \langle \ell \rangle\,.
\end{aligned}$$

Now fix some $m = m_k \in J_k^{(\lambda)}$. We will show $J_k^{(\lambda)} \subset m_k^*$, where the cluster is formed with respect to the covering $\mathcal{O}_{\tilde{r}}^{(\alpha)}$ for some $\tilde{r} = \tilde{r}\,(r, \alpha)$ with $\tilde{r} \geq r > r_1$. This will imply

$$\left|J_k^{(\lambda)}\right| \leq |m_k^*| \leq N_{\mathcal{O}_{\tilde{r}}^{(\alpha)}},$$

where the right-hand side is a finite constant (which only depends on $\tilde{r} = \tilde{r}\,(r, \alpha)$ and $\alpha$ and on the dimension $d$), since $\mathcal{O}_{\tilde{r}}^{(\alpha)}$ is an admissible covering (cf. Theorem 6.1.1).

For the proof of $J_k^{(\lambda)} \subset m_k^*$, let $\ell \in J_k^{(\lambda)}$ be arbitrary and choose $x \in O_{r,k}^{(\alpha)} \cap \lambda O_{r,m}^{(\alpha)}$ as well as $y \in O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)}$. This yields

$$\begin{aligned}
&\left||m|^{\alpha_0}\, m - |\ell|^{\alpha_0}\, \ell\right| \\
&\leq \left||m|^{\alpha_0}\, m - \lambda^{-1}x\right| + \left|\lambda^{-1}x - \lambda^{-1}y\right| + \left|\lambda^{-1}y - |\ell|^{\alpha_0}\, \ell\right| \\
&< r \cdot |m|^{\alpha_0} + \frac{2r \cdot |k|^{\alpha_0}}{\lambda} + r \cdot |\ell|^{\alpha_0}\,.
\end{aligned}$$

Using equation (6.5.13) twice (first for $\ell$ and then for $m$), we see

$$\begin{aligned}|\ell|^{\alpha_0} \leq \langle \ell \rangle^{\alpha_0} &\leq \left( C \cdot \frac{\langle k \rangle}{\lambda^{1-\alpha}} \right)^{\alpha_0} = C^{\alpha_0} \cdot \frac{\langle k \rangle^{\alpha_0}}{\lambda^{\alpha}} \\ &\leq C^{\alpha_0} \cdot \frac{\left( C \cdot \lambda^{1-\alpha} \cdot \langle m \rangle \right)^{\alpha_0}}{\lambda^{\alpha}} \\ &= C^{2\alpha_0} \cdot \langle m \rangle^{\alpha_0} .\end{aligned}$$

Similarly, using $\lambda \geq 1$ and $\alpha - 1 \leq 0$, we derive

$$\frac{|k|^{\alpha_0}}{\lambda} \leq \frac{\left( C \cdot \lambda^{1-\alpha} \cdot \langle m \rangle \right)^{\alpha_0}}{\lambda} = C^{\alpha_0} \cdot \lambda^{\alpha-1} \cdot \langle m \rangle^{\alpha_0} \leq C^{\alpha_0} \cdot \langle m \rangle^{\alpha_0} .$$

Combining these estimates, we arrive at

$$\left| |m|^{\alpha_0} m - |\ell|^{\alpha_0} \ell \right| < 4r \cdot C^{2\alpha_0} \cdot \langle m \rangle^{\alpha_0} \leq 2^{\alpha_0} \cdot 4r \cdot C^{2\alpha_0} |m|^{\alpha_0} ,$$

where we used $\langle m \rangle \leq 1 + |m| \leq 2|m|$, because of $m \in \mathbb{Z}^d \setminus \{0\}$, in the last step. Setting $\tilde{r} := 2^{\alpha_0} \cdot C^{2\alpha_0} \cdot 4r$, we have thus shown

$$|\ell|^{\alpha_0} \ell \in O^{(\alpha)}_{\tilde{r},\ell} \cap O^{(\alpha)}_{\tilde{r},m_k} \neq \varnothing$$

for each $\ell \in J_k^{(\lambda)}$. Hence, $J_k^{(\lambda)} \subset m_k^*$, where the cluster is taken with respect to $\mathcal{O}^{(\alpha)}_{\tilde{r}}$. As observed above, this completes the proof in the case $\lambda \geq 1$.

Finally, we consider the case $0 < \lambda < 1$. Set $\gamma := \lambda^{-1} > 1$. Applying the above, we see that $\mathcal{O}^{(\alpha)}_r$ is almost subordinate to $\gamma \mathcal{O}^{(\alpha)}_r$, so that Lemma 3.5.1 (using $A = \lambda \cdot \mathrm{id}$) shows that $\lambda \mathcal{O}^{(\alpha)}_r$ is almost subordinate to $\lambda\gamma \mathcal{O}^{(\alpha)}_r = \mathcal{O}^{(\alpha)}_r$, with

$$k\left(\lambda \mathcal{O}^{(\alpha)}_r, \mathcal{O}^{(\alpha)}_r\right) \leq k\left(\mathcal{O}^{(\alpha)}_r, \gamma \mathcal{O}^{(\alpha)}_r\right) \leq N.$$

This establishes the claim for $0 < \lambda < 1$. □

Having clarified the subordinateness relation, we can now apply the general theory of embeddings for decomposition spaces, in conjunction with Lemma 3.5.3, to asymptotically calculate the norm of the (isotropic) dilation on the $\alpha$-modulation spaces.

The only problem is that even though the covering $\mathcal{O}^{(\alpha)}_r$ is moderate with respect to the dilated covering $\lambda \mathcal{O}^{(\alpha)}_r$, the "moderateness constant" $C_{\mathrm{mod}}\left(\mathcal{O}^{(\alpha)}_r, \lambda \mathcal{O}^{(\alpha)}_r\right)$ can *not* be chosen independently of $\lambda \geq 1$. This makes the calculation of the discrete norms that occur in theorems 5.4.1 and 5.4.4 more difficult to evaluate exactly. Furthermore, we can not invoke the sharpness results which are dependent on relative moderateness, at least not "near the origin". The remedying fact is that the "moderateness" *is* uniform far enough away from the origin. But the required "distance" to the origin depends on $\lambda$.

To simplify the calculation, we first establish the following elementary estimate:

**Lemma 6.5.11.** *Let $d \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $R \geq 2$. Then we have*

$$\sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq R}} \langle k \rangle^\theta \asymp \begin{cases} R^{\theta+d}, & \text{if } \theta + d > 0, \\ \ln R, & \text{if } \theta + d = 0, \\ 1, & \text{if } \theta + d < 0, \end{cases}$$

*where the implied constants depend on $\theta$ and $d$, but* not *on $R \geq 2$.*

*Proof.* Let $r > 0$ be arbitrary. First note that $|x - y| \leq r$ implies

$$\langle x \rangle \leq 1 + |x| \leq 1 + r + |y| \leq (1 + r) \cdot (1 + |y|) \leq 2\,(1 + r) \cdot \langle y \rangle .$$

Here, we used $\langle y \rangle \geq 1$ and $\langle y \rangle \geq |y|$ in the last step. By symmetry, we see $\langle x \rangle \asymp_r \langle y \rangle$, where the implied constant depends (only) on $r > 0$.

Hence,

$$\begin{aligned} \sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq R}} \langle k \rangle^\theta &\asymp_{r,d} \sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq R}} \int_{B_r(k)} \langle k \rangle^\theta \, \mathrm{d}x \\ &\asymp_{r,\theta} \sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq R}} \int_{B_r(k)} \langle x \rangle^\theta \, \mathrm{d}x. \end{aligned} \tag{6.5.15}$$

For the upper part of the estimate stated in the lemma, note that the balls $(B_r(k))_{k \in \mathbb{Z}^d \setminus \{0\}}$ are pairwise disjoint for $r = \frac{1}{2}$ and that we have $B_{1/2}(k) \subset B_{R+\frac{1}{2}}(0)$ for $|k| \leq R$. Hence, integration in polar coordinates yields

$$\begin{aligned} \sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq R}} \int_{B_{1/2}(k)} \langle x \rangle^\theta \, \mathrm{d}x &\leq \int_{B_{R+\frac{1}{2}}(0)} \langle x \rangle^\theta \, \mathrm{d}x \\ &\asymp_d \int_0^{R+\frac{1}{2}} r^{d-1} \cdot \langle r \rangle^\theta \, \mathrm{d}r \\ &\lesssim_{\theta,d} 1 + \int_1^{R+\frac{1}{2}} r^{\theta+d-1} \, \mathrm{d}r \\ &\lesssim_{\theta,d} \begin{cases} 1 + \left(R + \frac{1}{2}\right)^{\theta+d} \lesssim_{\theta,d} R^{\theta+d}, & \text{if } \theta + d > 0, \\ 1 + \ln\left(R + \frac{1}{2}\right) \lesssim \ln(R), & \text{if } \theta + d = 0, \\ 1, & \text{if } \theta + d < 0. \end{cases} \end{aligned}$$

Here, we used $R \geq 2$ in the last step. More precisely, we made use of $1 \lesssim \ln(2) \leq \ln(R)$ and of

$$\ln\left(R + \frac{1}{2}\right) \leq \ln(2R) = \ln(2) + \ln(R) \leq 2 \ln(R).$$

For the reverse estimate, we first show

$$B_R(0) \subset \bigcup_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ |k|\le R}} B_{1+\sqrt{d}}(k). \tag{6.5.16}$$

To see this, observe that for each $x \in \mathbb{R}^d$ with $\sqrt{d} \le |x| \le R$ and $x_i \ge 0$ for all $i \in \underline{d}$, we have

$$k := (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor) \in \mathbb{N}_0^d \subset \mathbb{Z}^d$$

with $|k| \le |x| \le R$. Furthermore, $k \neq 0$, since $k = 0$ would imply $0 \le x_i < 1$ for all $i \in \underline{d}$ and hence $|x| < \sqrt{d}$. Finally, we have $|x-k| \le \sqrt{d} \cdot \|x-k\|_\infty < \sqrt{d}$ and hence $x \in B_{\sqrt{d}}(k) \subset B_{1+\sqrt{d}}(k)$.

If $|x| < \sqrt{d}$ holds, we get $x \in B_{1+\sqrt{d}}(e_1)$, where $e_1$ is the first vector of the standard basis of $\mathbb{R}^d$, i.e. $e_1 = (1,0,\dots,0) \in \mathbb{Z}^d \setminus \{0\}$ with $|e_1| = 1 \le R$.

Finally, note that each $x \in \mathbb{R}^d$ with $|x| \le R$ can be written as $x = \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_d)\cdot y$ with $|y| \le R$ and $y_i \ge 0$ as well as $\varepsilon_i \in \{\pm 1\}$ for all $i \in \underline{d}$. By the previous paragraph, there is some $k \in \mathbb{Z}^d \setminus \{0\}$ with $|k| \le R$ and with

$$|x - \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_d)\,k| = |\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_d)\,x - k| = |y-k| < 1+\sqrt{d}.$$

Because of $\ell := \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_d)\,k \in \mathbb{Z}^d \setminus \{0\}$ with $|\ell| = |k| \le R$, we have established equation (6.5.16).

Employing estimate (6.5.15) with $r = 1+\sqrt{d}$, we conclude

$$\begin{aligned}
\sum_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ |k|\le R}} \langle k\rangle^\theta &\asymp_{r,d,\theta} \sum_{\substack{k\in\mathbb{Z}^d\setminus\{0\}\\ |k|\le R}} \int_{B_r(k)} \langle x\rangle^\theta \,\mathrm{d}x \\
&\ge \int_{B_R(0)} \langle x\rangle^\theta \,\mathrm{d}x \\
&\asymp_d \int_0^R r^{d-1}\cdot\langle r\rangle^\theta \,\mathrm{d}r \\
&\gtrsim_{\theta,d} 1 + \int_1^R r^{d+\theta-1}\,\mathrm{d}r \\
&\gtrsim_{\theta,d} \begin{cases} 1+R^{d+\theta} \ge R^{d+\theta}, & \text{if } \theta+d>0,\\ 1+\ln(R) \gtrsim \ln(R), & \text{if } \theta+d=0,\\ 1, & \text{if } \theta+d<0. \end{cases}
\end{aligned}$$

Here, the last step for $\theta + d > 0$ is not completely obvious. But $\theta + d > 0$ yields $c := 1 - 2^{-(d+\theta)} > 0$ and $R \ge 2$ implies

$$\begin{aligned}
& c = 1 - 2^{-(d+\theta)} \le 1 - R^{-(d+\theta)} = \frac{R^{d+\theta}-1}{R^{d+\theta}} \\
\iff & R^{d+\theta} - 1 \ge cR^{d+\theta}.
\end{aligned}$$

Since $c > 0$ only depends on $t, \theta$, the proof is complete. □

Now we are ready to calculate the operator norm of the isotropic dilations on the Fourier-side $\alpha$-modulation spaces.

**Theorem 6.5.12.** *Let $d \in \mathbb{N}$, $p, q \in (0, \infty]$ and $\alpha \in [0, 1)$, as well as $s \in \mathbb{R}$. For each $\lambda \in (0, \infty)$, define the* (ISOTROPIC) DILATION OPERATOR $U_\lambda$ *by*

$$U_\lambda : \mathcal{D}\left(\mathbb{O}^{(\alpha)}, L^p, \ell^q_{w^{(s/(1-\alpha))}}\right) \to \mathcal{D}\left(\mathbb{O}^{(\alpha)}, L^p, \ell^q_{w^{(s/(1-\alpha))}}\right), f \mapsto f_{\lambda \cdot \mathrm{id}},$$

*where $f_{\lambda \cdot \mathrm{id}}$ is as in Definition 3.5.2. Then $U_\lambda$ is well-defined and bounded for each $\lambda \in (0, \infty)$.*

*Define*

$$s_0 := d\,(1-\alpha)\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+,$$
$$s_1 := -d\left(\frac{1}{q} - \frac{1}{p^\triangle}\right)_+ + \alpha d\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+.$$

*There are constants $C_1, C_2 > 0$ such that*

$$\lambda^{d\left(1-\frac{1}{p}\right)} |||U_\lambda||| \leq C_1 \cdot \begin{cases} \lambda^{s_0 - s}, & \text{if } s < s_0, \\ 1 + (\ln(\lambda))^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}, & \text{if } s = s_0, \\ 1, & \text{if } s > s_0 \end{cases} \tag{6.5.17}$$

*and*

$$\lambda^{d\left(1-\frac{1}{p}\right)} |||U_\lambda||| \geq C_2 \cdot \begin{cases} \lambda^{s_0 - s}, & \text{if } s > s_0, \\ 1 + (\ln(\lambda))^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}, & \text{if } s = s_0 \text{ and } 0 < p \leq 2, \\ 1, & \text{otherwise} \end{cases} \tag{6.5.18}$$

*hold for all $\lambda \geq 1$.*

*Similarly, there are constants $C_3, C_4 > 0$ such that*

$$\lambda^{d\left(1-\frac{1}{p}-p_d\right)} |||U_\lambda||| \leq C_3 \cdot \begin{cases} \lambda^{s_1 - s}, & \text{if } s > s_1, \\ 1 + |\ln \lambda|^{\left(\frac{1}{q} - \frac{1}{p^\triangle}\right)_+}, & \text{if } s = s_1, \\ 1, & \text{if } s < s_1, \end{cases} \tag{6.5.19}$$

*and*

$$\lambda^{d\left(1-\frac{1}{p}-p_d\right)} |||U_\lambda||| \geq C_4 \cdot \begin{cases} \lambda^{s_1 - s}, & \text{if } s > s_1, \\ 1 + |\ln \lambda|^{\left(\frac{1}{q} - \frac{1}{p^\triangle}\right)_+}, & \text{if } s = s_1 \text{ and } p \notin (1, 2), \\ 1, & \text{otherwise}, \end{cases} \tag{6.5.20}$$

*hold for all $\lambda \in (0, 1)$.*

*Proof.* Let $\alpha \in [0,1)$ and $r_1 = r_1(\alpha, d) > 0$ as in Theorem 6.1.1. Choose $r > r_1$. Further, for $\gamma \in \mathbb{R}$, choose

$$w^{(\gamma)} : \mathbb{Z}^d \setminus \{0\} \to (0,\infty), k \mapsto \langle k \rangle^{\gamma}$$

as in Lemma 6.1.2. This lemma shows that $w^{(\gamma)}$ is moderate with respect to $\mathcal{O}_r^{(\alpha)}$. Finally, let $\gamma := \frac{s}{1-\alpha}$.

Theorem 6.1.3 shows that the covering $\mathcal{O}_r^{(\alpha)}$ is a structured admissible covering. By Theorem 3.2.17, this implies that there is an $L^p$-BAPU $\Phi = (\varphi_k)_{k\in\mathbb{Z}^d\setminus\{0\}}$ for $\mathcal{O}_r^{(\alpha)}$. In the following, the quasi-norm on $\mathcal{D}\left(\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma)}}\right)$ will always be calculated with respect to this fixed BAPU.

Likewise, we will use the dilated BAPU $\Phi^{(\lambda)} := \left(\varphi_k \circ \lambda^{-1}\mathrm{id}\right)_{k\in\mathbb{Z}^d\setminus\{0\}}$ for the covering $\lambda\mathcal{O}_r^{(\alpha)}$. Lemma 3.5.1 shows that this is indeed an $L^p$-BAPU for $\lambda\mathcal{O}_r^{(\alpha)}$, with $C_{\lambda\mathcal{O}_r^{(\alpha)},\Phi^{(\lambda)},p} = C_{\mathcal{O}_r^{(\alpha)},\Phi,p}$.

Lemma 3.5.3 yields

$$\begin{aligned}\|f_{\lambda\cdot\mathrm{id}}\|_{\mathcal{D}_\Phi\left(\mathcal{O}_r^{(\alpha)},L^p,\ell^q_{w^{(\gamma)}}\right)} &= |\det(\lambda\cdot\mathrm{id})|^{\frac{1}{p}-1}\cdot\|f\|_{\mathcal{D}_{\Phi^{(\lambda)}}\left(\lambda\mathcal{O}_r^{(\alpha)},L^p,\ell^q_{w^{(\gamma)}}\right)}\\ &= \lambda^{d\left(\frac{1}{p}-1\right)}\cdot\|f\|_{\mathcal{D}_{\Phi^{(\lambda)}}\left(\lambda\mathcal{O}_r^{(\alpha)},L^p,\ell^q_{w^{(\gamma)}}\right)}\end{aligned}$$

for all $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$. We conclude

$$|||U_\lambda||| = \lambda^{d\left(\frac{1}{p}-1\right)}\cdot|||\iota_\lambda|||$$

for

$$\iota_\lambda : \mathcal{D}\left(\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma)}}\right) \to \mathcal{D}\left(\lambda\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma)}}\right), f \mapsto f, \tag{6.5.21}$$

where exact equality holds as long as the BAPU $\Phi^{(\lambda)}$ is used to calculate the quasi-norm on $\mathcal{D}\left(\lambda\mathcal{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(\gamma)}}\right)$. We will always do this in the following. We are thus reduced to estimating the asymptotics of $|||\iota_\lambda|||$.

Recall from Lemma 3.5.1 that

$$\lambda\mathcal{O}_r^{(\alpha)} = \left(\lambda|k|^{\alpha_0}\,\mathrm{id}\cdot B_r(0) + \lambda|k|^{\alpha_0}k\right)_{k\in\mathbb{Z}^d\setminus\{0\}} \tag{6.5.22}$$

is a tight semi-structured admissible covering of $\mathbb{R}^d$ and that $w^{(\gamma)}$ is moderate with respect to $\lambda\mathcal{O}_r^{(\alpha)}$ with

$$\begin{aligned}\varepsilon_{\lambda\mathcal{O}_r^{(\alpha)}} &= \varepsilon_{\mathcal{O}_r^{(\alpha)}} =: \varepsilon,\\ R_{\lambda\mathcal{O}_r^{(\alpha)}} &= R_{\mathcal{O}_r^{(\alpha)}} =: R,\\ D_{\lambda\mathcal{O}_r^{(\alpha)}} &= D_{\mathcal{O}_r^{(\alpha)}} =: D,\\ C_{\lambda\mathcal{O}_r^{(\alpha)}} &= C_{\mathcal{O}_r^{(\alpha)}} =: C_1,\end{aligned}$$

$$\begin{aligned} N_{\lambda \mathbb{O}_r^{(\alpha)}} &= N_{\mathbb{O}_r^{(\alpha)}} =: N, \\ C_{w^{(\gamma)}, \lambda \mathbb{O}_r^{(\alpha)}} &= C_{w^{(\gamma)}, \mathbb{O}_r^{(\alpha)}} =: C_2, \\ C_{\lambda \mathbb{O}_r^{(\alpha)}, \Phi^{(\lambda)}, p} &= C_{\mathbb{O}_r^{(\alpha)}, \Phi, p} =: C_3, \end{aligned} \tag{6.5.23}$$

where the constants on the right-hand side are all independent of $\lambda \in (0, \infty)$. For the above estimates to hold, it is important to use the family of standardizations as indicated by equation (6.5.22), cf. Lemma 3.5.1.

Let us first consider the case $\lambda \geq 1$. Here, we use Theorem 5.4.4 with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \lambda \mathbb{O}_r^{(\alpha)}$ and $u = v = w^{(\gamma)}$, as well as $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Lemma 6.5.10 shows that the assumption that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ is satisfied, with $k(\mathcal{Q}, \mathcal{P}) \leq K$ for some constant $K = K(r, \alpha, d) \in \mathbb{N}$ which is independent of $\lambda \geq 1$. Note that property 1a of the map $\iota$ constructed in Theorem 5.4.4 implies $\iota_\lambda = \iota$ with our choice of $\mathcal{Q}, \mathcal{P}$. Furthermore, the map $\iota'$ from that theorem satisfies $\iota' = \iota_\lambda|_{C_c^\infty(\mathbb{R}^d)}$.

Hence, parts 1 and 2 of Theorem 5.4.4 yield

$$\begin{aligned} &\left\| \left( \left\| (u_i^{-1})_{i \in I_j} \right\|_{\ell^{p \cdot (q/p)'}} \right)_{j \in J} \right\|_{\ell_v^{q \cdot (q/q)'}} \\ &\lesssim |||\iota_\lambda||| \\ &\lesssim \left\| \left( \left\| (u_i^{-1})_{i \in I_j} \right\|_{\ell^{p^\nabla \cdot (q/p^\nabla)'}} \right)_{j \in J} \right\|_{\ell_v^{q \cdot (q/q)'}}, \end{aligned} \tag{6.5.24}$$

where the sets $I_j$ for $j \in J = \mathbb{Z}^d \setminus \{0\}$ are given by

$$I_j^{(\lambda)} := I_j = \left\{ i \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; O_{r,i}^{(\alpha)} \cap \lambda O_{r,j}^{(\alpha)} \neq \varnothing \right\}. \tag{6.5.25}$$

The implied constants in equation (6.5.24) only depend on $d, p, q$, on $k(\mathcal{Q}, \mathcal{P}) \leq K$, on

$$r_0(\mathcal{P}, 1) \leq r_0(\mathcal{P}, 2k(\mathcal{Q}, \mathcal{P}) + 3) \leq r_0(\mathcal{P}, 2K + 3) \overset{(\dagger)}{=} r_0(\mathbb{O}_r^{(\alpha)}, 2K + 3),$$

and on $N_{\mathcal{P}}, C_{\mathcal{P}}, R_{\mathcal{P}}, D_{\mathcal{P}}, C_{v,\mathcal{P}}, C_{\mathcal{P},\Phi^{(\lambda)},p}, N_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}, C_{\mathcal{Q}}, D_{\mathcal{Q}}, R_{\mathcal{Q}}, C_{u,\mathcal{Q}}, C_{\mathcal{Q},\Phi,p}$. Here, Lemma 3.5.1 was used at $(\dagger)$. But we saw in equation (6.5.23) that all of the last-mentioned constants are independent of $\lambda \geq 1$, even though $\mathcal{P} = \lambda \mathbb{O}_r^{(\alpha)}$ depends on $\lambda$.

Let us set $\theta_1 := p \cdot (q/p)'$ and $\theta_2 := p^\nabla \cdot (q/p^\nabla)'$ and let $\theta \in (0, \infty]$. Since Remark 5.2.3 shows $q \cdot (q/q)' = \infty$, we get

$$\left\| \left( \left\| (u_i^{-1})_{i \in I_j^{(\lambda)}} \right\|_{\ell^\theta} \right)_{j \in J} \right\|_{\ell_v^{q \cdot (q/q)'}} = \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \left[ \langle j \rangle^\gamma \cdot \left\| (\langle i \rangle^{-\gamma})_{i \in I_j^{(\lambda)}} \right\|_{\ell^\theta} \right].$$

Choose $M = M(r, \alpha) \in \mathbb{N}$ as in Lemma 6.5.10, i.e. such that for every $\ell \in \mathbb{Z}^d \setminus \{0\}$, if there is $k \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)} \neq \varnothing$ and

with $|k| \leq \max\{2r, (2\lambda)^{1-\alpha}\}$, then $\ell \in \{-M, \dots, M\}^d$. Since we are assuming $\lambda \geq 1$, this is indeed possible.

For $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$ and $i \in I_j^{(\lambda)}$, this choice of $M$ implies $|i| > \max\{2r, (2\lambda)^{1-\alpha}\}$, so that an application of equation (6.5.13) from Lemma 6.5.10 yields

$$C^{-1} \cdot \lambda^{1-\alpha} \langle j \rangle \leq \langle i \rangle \leq C \cdot \lambda^{1-\alpha} \langle j \rangle \tag{6.5.26}$$

for $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$ and $i \in I_j^{(\lambda)}$. The constant $C = C(r, \alpha) \geq 1$ appearing in the above estimate is independent of $\lambda \geq 1$. For arbitrary $\theta \in (0, \infty]$, this yields

$$\begin{aligned} \langle j \rangle^{\gamma} \cdot \left\| \left( \langle i \rangle^{-\gamma} \right)_{i \in I_j^{(\lambda)}} \right\|_{\ell^\theta} &\asymp \langle j \rangle^{\gamma} \cdot \left\| \left( \left( \lambda^{1-\alpha} \cdot \langle j \rangle \right)^{-\gamma} \right)_{i \in I_j^{(\lambda)}} \right\|_{\ell^\theta} \\ &= \lambda^{(\alpha-1)\gamma} \cdot \left| I_j^{(\lambda)} \right|^{1/\theta} = \lambda^{-s} \cdot \left| I_j^{(\lambda)} \right|^{1/\theta} \end{aligned} \tag{6.5.27}$$

for all $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$. Here, the implied constants are independent of $\lambda \geq 1$ and of $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$.

But equation (6.5.26) shows that $\mathcal{Q}$ is relatively moderate with respect to $\mathcal{P}_{J_0}$, with $J_0 := \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$ and with

$$C_{\mathrm{mod}}(\mathcal{Q}, \mathcal{P}_{J_0}) \lesssim C = C(d, r, \alpha),$$

since we have

$$\left| \det\left( |i|^{\alpha_0} \cdot \mathrm{id} \right) \right| = |i|^{\alpha_0 d} \asymp_{\alpha, d} \langle i \rangle^{\alpha_0 d} \asymp_C \left( \lambda^{1-\alpha} \langle j \rangle \right)^{\alpha_0 d} = \lambda^{\alpha d} \langle j \rangle^{\alpha_0 d}$$

for all $j \in J_0$ and $i \in I$ with $Q_i \cap P_j \neq \varnothing$. Again, the implied constants are independent of $\lambda \geq 1$.

Hence, we can invoke Lemma 3.3.7 with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \lambda \mathbb{O}_r^{(\alpha)}$ and $I_0 = \mathbb{Z}^d \setminus \{0\}$, as well as $J_0 = \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$, to derive

*Here, it is important to recall that the standardization for $\mathcal{P} = \lambda \mathbb{O}_r^{(\alpha)}$ are given by $S_j = \lambda |j|^{\alpha_0} \mathrm{id}$ and not by $S_j = |j|^{\alpha_0} \mathrm{id}$, cf. Lemma 3.5.1.*

$$\begin{aligned} \left| I_j^{(\lambda)} \right| &\asymp \frac{\left| \det\left( \lambda \cdot |j|^{\alpha_0} \cdot \mathrm{id} \right) \right|}{\left| \det\left( |i|^{\alpha_0} \cdot \mathrm{id} \right) \right|} \asymp \lambda^d \cdot \left( \frac{\langle j \rangle}{\langle i \rangle} \right)^{d\alpha_0} \\ &\asymp \lambda^d \cdot \left( \lambda^{\alpha - 1} \right)^{d\alpha_0} = \lambda^{d(1-\alpha)} \end{aligned} \tag{6.5.28}$$

for all $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$ and any $i \in I_j^{(\lambda)}$. Here, the implied constant in the first step comes from Lemma 3.3.7 and is hence (thanks to equations (6.5.26) and (6.5.23) and because of $k(\mathcal{Q}, \mathcal{P}) \leq K$) only dependent on $d, \varepsilon, N, C_1, C_2, C_3, C, R, K$, but not on $\lambda$. The implied constants in the remaining steps are only dependent on $d, \alpha, C$ and are thus also independent of $\lambda \geq 1$. Finally, it follows from the remark after Lemma 3.3.7 that this lemma is indeed applicable in the present setting, since we have $I_0 = I$ and $\mathcal{O} = \mathcal{O}'$.

Together with $\gamma = \frac{s}{1-\alpha}$ and with equation (6.5.27), we conclude

$$\begin{aligned}\langle j\rangle^{\gamma} \cdot \left\| \left(\langle i\rangle^{-\gamma}\right)_{i\in I_j^{(\lambda)}} \right\|_{\ell^\theta} &\asymp \lambda^{-s} \cdot \left|I_j^{(\lambda)}\right|^{1/\theta} \\ &\asymp \lambda^{(1-\alpha)d/\theta} \lambda^{-s} \end{aligned} \tag{6.5.29}$$

for all $j \in \mathbb{Z}^d \setminus \{-M,\dots,M\}^d$, where the implied constants are independent of $\lambda \geq 1$ and of $j \in \mathbb{Z}^d \setminus \{-M,\dots,M\}^d$.

For the proof of equation (6.5.17), it remains to consider the case $j \in \{-M,\dots,M\}^d \setminus \{0\}$. Here, let $i \in I_j^{(\lambda)}$ be arbitrary. This yields some $x \in O_{r,i}^{(\alpha)} \cap \lambda O_{r,j}^{(\alpha)}$, whence $\lambda^{-1}x \in O_{r,j}^{(\alpha)}$. The triangle inequality for $|\cdot|$ yields $\langle j\rangle \leq 1 + |j| \leq 1 + dM$, because of $j \in \{-M,\dots,M\}^d$. Together with Lemma 6.1.2, we conclude

$$|\lambda^{-1}x| \leq \langle \lambda^{-1}x\rangle \asymp_{r,\alpha} \langle j\rangle^{\frac{1}{1-\alpha}} \leq (1+dM)^{\frac{1}{1-\alpha}}$$

and hence

$$\langle x\rangle \leq 1 + |x| \lesssim_{r,d,\alpha,M} 1 + \lambda \lesssim \lambda,$$

where $\lambda \geq 1$ was used in the last step. Recall that $M$ only depends on $r, \alpha$. Thus, another application of Lemma 6.1.2 yields

$$|i| \leq \langle i\rangle \lesssim_{r,\alpha} \langle x\rangle^{1-\alpha} \lesssim_{d,r,\alpha} \lambda^{1-\alpha}, \tag{6.5.30}$$

i.e. $|i| \leq \kappa \cdot \lambda^{1-\alpha}$ for some constant $\kappa = \kappa(d,r,\alpha) \geq 2$.

We first assume $\theta < \infty$. Using Lemma 6.5.11, the identity $\gamma = \frac{s}{1-\alpha}$, as well as the estimate $1 \leq |j| \leq dM$ with $M = M(r,\alpha)$, we arrive at

$$\begin{aligned}&\langle j\rangle^{\gamma} \cdot \left\| \left(\langle i\rangle^{-\gamma}\right)_{i\in I_j^{(\lambda)}} \right\|_{\ell^\theta} \\ &\lesssim_{\gamma,d,r,\alpha} \left[ \sum_{\substack{i\in\mathbb{Z}^d\setminus\{0\} \\ |i|\leq\kappa\lambda^{1-\alpha}}} \langle i\rangle^{-\theta\gamma} \right]^{1/\theta} \\ &\lesssim_{\gamma,\theta,\kappa} \begin{cases} \left[\left(\kappa\cdot\lambda^{1-\alpha}\right)^{d-\theta\gamma}\right]^{1/\theta} \asymp_{d,r,\alpha,\theta,\gamma} \lambda^{-s+\frac{d}{\theta}(1-\alpha)}, & \text{if } d > \theta\gamma, \\ \left[\ln\left(\kappa\cdot\lambda^{1-\alpha}\right)\right]^{1/\theta} \lesssim_{d,r,\alpha,\theta} 1 + (\ln(\lambda))^{1/\theta}, & \text{if } d = \theta\gamma, \\ 1, & \text{if } d < \theta\gamma \end{cases}\end{aligned}$$

for $j \in \{-M,\dots,M\}^d \setminus \{0\}$. Now, observe

$$\frac{d}{\theta}(1-\alpha) - s = \frac{d(1-\alpha)}{\theta} - (1-\alpha)\gamma = \frac{(1-\alpha)}{\theta}\left[d - \theta\gamma\right].$$

Using this and $\frac{1-\alpha}{\theta} > 0$, the above estimate can be reformulated as

$$\langle j\rangle^{\gamma} \cdot \left\| \left(\langle i\rangle^{-\gamma}\right)_{i \in I_j^{(\lambda)}} \right\|_{\ell^\theta} \lesssim \begin{cases} \lambda^{-s+\frac{d}{\theta}(1-\alpha)}, & \text{if } s < \frac{d}{\theta}(1-\alpha), \\ 1 + (\ln(\lambda))^{1/\theta}, & \text{if } s = \frac{d}{\theta}(1-\alpha), \\ 1, & \text{if } s > \frac{d}{\theta}(1-\alpha) \end{cases}$$

for $j \in \{-M, \dots, M\}^d \setminus \{0\}$, where the implied constants are independent of $\lambda \geq 1$ and of $j$. Using $\gamma = \frac{s}{1-\alpha}$ and $\langle j\rangle^{\gamma} \asymp 1$ together with equation (6.5.30), it is easy to see that this estimate also remains true for $\theta = \infty$.

If we set $\theta = \theta_2 = p^{\triangledown} \cdot (q/p^{\triangledown})'$, then a combination of equation (6.5.24) with the estimate from the previous paragraph and with equation (6.5.29) yields

$$|||\iota_\lambda||| \lesssim \begin{cases} \lambda^{s_0 - s}, & \text{if } s < s_0, \\ 1 + (\ln(\lambda))^{\left(\frac{1}{p^{\triangledown}} - \frac{1}{q}\right)_+}, & \text{if } s = s_0, \\ 1, & \text{if } s > s_0, \end{cases} \tag{6.5.31}$$

where the implied constant does not depend on $\lambda$. In particular, Theorem 5.4.4 shows that $\iota_\lambda$ is well-defined and bounded. This proves equation (6.5.17).

We will now establish equation (6.5.18). The easiest case is the last one: First observe that the simple necessary condition in Lemma 5.3.1 yields a constant $C_4 > 0$ depending only on $d, p, q$ and on

$$\begin{aligned} &N_{\mathcal{O}_r^{(\alpha)}}, R_{\mathcal{O}_r^{(\alpha)}}, C_{\mathcal{O}_r^{(\alpha)}}, D_{\mathcal{O}_r^{(\alpha)}}, C_{w^{(\gamma)}, \mathcal{O}_r^{(\alpha)}}, C_{\mathcal{O}_r^{(\alpha)}, \Phi, p}, \\ &N_{\lambda\mathcal{O}_r^{(\alpha)}}, R_{\lambda\mathcal{O}_r^{(\alpha)}}, C_{\lambda\mathcal{O}_r^{(\alpha)}}, D_{\lambda\mathcal{O}_r^{(\alpha)}}, C_{w^{(\gamma)}, \lambda\mathcal{O}_r^{(\alpha)}}, C_{\lambda\mathcal{O}_r^{(\alpha)}, \Phi^{(\lambda)}, p}, \end{aligned}$$

such that $w_\ell^{(\gamma)} \leq C_4 \cdot |||\iota_\lambda||| \cdot w_k^{(\gamma)}$ is satisfied for all $k, \ell \in \mathbb{Z}^d \setminus \{0\}$ with $O_{r,k}^{(\alpha)} \cap \lambda O_{r,\ell}^{(\alpha)} \neq \varnothing$. Here, we used that each of the sets $O_{r,k}^{(\alpha)}, \lambda O_{r,\ell}^{(\alpha)}$ is open. But equation (6.5.23) shows that all of the above constants only depend on $d, p, q, N, R, D, C_1, C_2, C_3$ and are hence independent of $\lambda \in (0, \infty)$, so that $C_4$ is also independent of $\lambda \in (0, \infty)$.

Observe that there is some $\ell_0 \in \mathbb{Z}^d \setminus \{0\}$ with $0 \in O_{r,\ell_0}^{(\alpha)}$. This yields $0 \in O_{r,\ell_0}^{(\alpha)} \cap \lambda O_{r,\ell_0}^{(\alpha)}$ and hence $w_{\ell_0}^{(\gamma)} \lesssim C_4 \cdot |||\iota_\lambda||| \cdot w_{\ell_0}^{(\gamma)}$, which means $|||\iota_\lambda||| \gtrsim C_4^{-1} \gtrsim 1$, independently of $\lambda \in (0, \infty)$. Note that we did not use $\lambda \geq 1$ in this argument.

Now, let us consider the first two cases of equation (6.5.18), but only for $0 < p \leq 2$. Note that this implies $p^{\triangledown} = p$ and hence $\theta_1 = \theta_2$. The lower bound for $|||\iota_\lambda|||$ from equation (6.5.24) shows

$$|||\iota_\lambda||| \gtrsim \left\| \left( \left\| (u_i^{-1})_{i \in I_j} \right\|_{\ell^{\theta_1}} \right)_{j \in J} \right\|_{\ell_v^{q \cdot (q/q)'}}$$

$$= \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \left[ \langle j \rangle^{\gamma} \cdot \left\| \left( \langle i \rangle^{-\gamma} \right)_{i \in I_j^{(\lambda)}} \right\|_{\ell^{\theta_2}} \right], \tag{6.5.32}$$

where the implied constant is independent of $\lambda \geq 1$. Now, choose any $\ell \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$. Together with equation (6.5.29), we conclude

$$\begin{aligned} |||\iota_\lambda||| &\gtrsim \langle \ell \rangle^{\gamma} \cdot \left\| \left( \langle i \rangle^{-\gamma} \right)_{i \in I_\ell^{(\lambda)}} \right\|_{\ell^{\theta_2}} \\ &\gtrsim \lambda^{(1-\alpha)d/\theta_2} \lambda^{-s} \\ &= \lambda^{s_0 - s}. \end{aligned}$$

This establishes the first case of equation (6.5.18), but only under the assumption $0 < p \leq 2$. The case $p > 2$ is considered below.

Now, we consider the second case of equation (6.5.18), i.e. $s = s_0$ and $0 < p \leq 2$. As above, this yields $\theta_1 = \theta_2$. In case of $\theta_1 = \infty$, we have $\left( \frac{1}{p^\vee} - \frac{1}{q} \right)_+ = \frac{1}{\theta_1} = 0$, so that the claim of equation (6.5.18) is already implied by the estimate $|||\iota_\lambda||| \gtrsim 1$ from above. Hence, we can assume $0 < \theta_1 < \infty$.

Now, choose $j_0 \in \mathbb{Z}^d \setminus \{0\}$ with $0 \in O_{r,j_0}^{(\alpha)}$. Since $O_{r,j_0}^{(\alpha)}$ is open, there is some $\delta > 0$ (which is independent of $\lambda \geq 1$) with $\overline{B_\delta}(0) \subset O_{r,j_0}^{(\alpha)}$. Since we have already shown $|||\iota_\lambda||| \gtrsim 1$, it suffices to consider the case $\lambda \geq \frac{2^{1/(1-\alpha)}}{\delta}$. This yields $(\delta\lambda)^{1-\alpha} \geq 2$. Now, let $i \in \mathbb{Z}^d \setminus \{0\}$ be arbitrary with $|i| \leq (\delta\lambda)^{1-\alpha}$. We have $\left| |i|^{\alpha_0} i \right| = |i|^{\frac{1}{1-\alpha}} \leq \delta\lambda$, which means

$$|i|^{\alpha_0} i \in O_{r,i}^{(\alpha)} \cap \lambda \cdot \overline{B_\delta}(0) \subset O_{r,i}^{(\alpha)} \cap \lambda O_{r,j_0}^{(\alpha)},$$

i.e. $i \in I_{j_0}^{(\lambda)}$. Note that $s = s_0 = d(1-\alpha)\frac{1}{\theta_1}$ implies $d = \theta_1 \frac{s}{1-\alpha} = \theta_1 \gamma$. Together with Lemma 6.5.11 (used at (†)) and equation (6.5.32), we conclude

$$\begin{aligned} |||\iota_\lambda||| &\gtrsim \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \langle j \rangle^{\gamma} \cdot \left\| \left( \langle i \rangle^{-\gamma} \right)_{i \in I_j^{(\lambda)}} \right\|_{\ell^{\theta_1}} \\ &\geq \langle j_0 \rangle^{\gamma} \cdot \left\| \left( \langle i \rangle^{-\gamma} \right)_{i \in I_{j_0}^{(\lambda)}} \right\|_{\ell^{\theta_1}} \\ &\gtrsim_{j_0} \left[ \sum_{\substack{k \in \mathbb{Z}^d \setminus \{0\} \\ |k| \leq (\delta\lambda)^{1-\alpha}}} \langle k \rangle^{-\gamma\theta_1} \right]^{1/\theta_1} \\ &\overset{(\dagger)}{\gtrsim} \left[ \ln\left( (\lambda\delta)^{1-\alpha} \right) \right]^{1/\theta_1} \\ &\gtrsim 1 + [\ln(\lambda)]^{1/\theta_2} \\ &= 1 + [\ln(\lambda)]^{\left( \frac{1}{p^\vee} - \frac{1}{q} \right)_+} \end{aligned}$$

for $\lambda \geq \frac{2^{1/(1-\alpha)}}{\delta}$. Here, we used $\theta_1 = \theta_2$, since $0 < p \leq 2$. In summary, we have established equation (6.5.18) in the second case.

It remains to establish the first case of equation (6.5.18) under the additional assumption $2 < p \leq \infty$. For this, the important observation is (cf. equation (6.5.26)) that $\mathbb{O}_r^{(\alpha)}$ (and the weight $w^{(\gamma)}$ considered as a weight on $\mathbb{O}_r^{(\alpha)}$) are not moderate (at least not uniformly with respect to $\lambda \geq 1$) with respect to $\lambda\mathbb{O}_r^{(\alpha)}$ *near* the origin, but they are moderate *far away* from the origin. This behaviour will suffice to obtain the claimed estimate.

To see this, we employ Theorem 5.3.14 with $\mathcal{Q} = \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \lambda\mathbb{O}_r^{(\alpha)}$ and $u = v = w^{(\gamma)}$, as well as $J_0 := \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$. As seen above (around equation (6.5.26)), we then have (in the notation of Theorem 5.3.14) the inclusion

$$\begin{aligned} I_0 = I_0^{(\lambda)} &= \left\{ i \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; O_{r,i}^{(\alpha)} \cap \lambda O_{r,j}^{(\alpha)} \neq \varnothing \text{ for some } j \in J_0 \right\} \\ &\subset \left\{ k \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; |k| > \max\left\{ 2r, (2\lambda)^{1-\alpha} \right\} \right\}. \end{aligned}$$

Thus, equation (6.5.13) shows that $\mathcal{Q}_{I_0}$ and $w^{(\gamma)}|_{I_0}$ are relatively $\mathcal{P}$-moderate, with "moderateness constants" independent of $\lambda \geq 1$.

With similar reasoning as above, we see that the constant supplied by Theorem 5.3.14 can be chosen independently of $\lambda \geq 1$. Hence, that theorem and equation (6.5.13) (used at (†)) yield

$$\begin{aligned} |||\iota_\lambda||| &\gtrsim \left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}}{u_{i_j}} \left|\det\left(T_{i_j}\right)\right|^{-\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \right)_{j \in J_0} \right\|_{\ell_v^{q \cdot (q/q)'}} \\ &= \sup_{j \in J_0} \frac{\left(\lambda^d \cdot |j|^{\alpha_0 d}\right)^{\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}}{\left|i_j\right|^{\alpha_0 d \left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+}} \cdot \frac{\langle j \rangle^\gamma}{\langle i_j \rangle^\gamma} \\ &\asymp \sup_{j \in J_0} \lambda^{d\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \left( \frac{\langle j \rangle}{\langle i_j \rangle} \right)^{\alpha_0 d \left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \cdot \left( \frac{\langle j \rangle}{\langle i_j \rangle} \right)^\gamma \\ &\overset{(\dagger)}{\gtrsim} \lambda^{d\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \cdot \lambda^{(\alpha-1)\gamma + (\alpha-1)d\alpha_0\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} \\ &= \lambda^{-s + d(1-\alpha)\left(\frac{1}{p^\nabla} - \frac{1}{q}\right)_+} = \lambda^{s_0 - s}. \end{aligned}$$

In this calculation, it is crucial to note that the family of standardizations for $\mathcal{P} = \lambda\mathbb{O}_r^{(\alpha)}$ is given by $\left(\lambda \cdot |j|^{\alpha_0} \operatorname{id}\right)_{j \in \mathbb{Z}^d \setminus \{0\}}$ and not by $\left(|j|^{\alpha_0} \operatorname{id}\right)_{j \in \mathbb{Z}^d \setminus \{0\}}$, cf. Lemma 3.5.1. This completes the proof of equation (6.5.18).

Finally, we consider the case $\lambda \in (0, 1)$. Here, $\mathcal{Q} := \lambda\mathbb{O}_r^{(\alpha)}$ is almost subordinate to $\mathcal{P} := \mathbb{O}_r^{(\alpha)}$, with $k(\mathcal{Q}, \mathcal{P}) \leq K$ with the same constant $K = K(r, \alpha, d)$ as in the case $\lambda \geq 1$, cf. Lemma 6.5.10.

Hence, we can apply Theorem 5.4.1, with weights $u = v = w^{(\gamma)}$ and exponents $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Precisely as above, we

see that all assumptions of parts 1 and 2 of Theorem 5.4.1 are satisfied and that all quantities on which the constants in these parts depend can be bounded independently of $\lambda \in (0,1)$. Since $\mathcal{Q}$ and $\mathcal{P}$ both cover $\mathcal{O} = \mathbb{R}^d$, we see that the map $\iota$ from Theorem 5.4.1 satisfies $\iota f = f$ for all $f$ and hence $\iota = \iota_\lambda$ with $\iota_\lambda$ as in equation (6.5.21). Furthermore, the map $\iota'$ from that theorem is given by $\iota' = \iota_\lambda|_{C_c^\infty(\mathbb{R}^d)}$.

Using $q \cdot (q/q)' = \infty$, we see that parts 1 and 2 of Theorem 5.4.1 yield

$$\begin{aligned}
&\left\| \left( v_j^{-1} \cdot \left\| (u_i)_{i \in I_j} \right\|_{\ell^{q(p/q)'}} \right)_{j \in J} \right\|_{\ell^\infty} \\
&\lesssim |||\iota_\lambda||| \\
&\lesssim \left\| \left( \left( \left| \det(S_j) \right|^{p_d} v_j \right)^{-1} \left\| \left( |\det(T_i)|^{p_d} u_i \right)_{i \in I_j} \right\|_{\ell^q\left(p^\triangle/q\right)'} \right)_{j \in J} \right\|_{\ell^\infty},
\end{aligned} \tag{6.5.33}$$

where the implied constants are independent of $\lambda \in (0,1)$. Here, we have $p_d = \left(1 - \frac{1}{p}\right)_-$ and

$$\begin{aligned}
\tilde{I}_j^{(\lambda)} := I_j &= \left\{ i \in \mathbb{Z}^d \setminus \{0\} \,\middle|\, \lambda O_{r,i}^{(\alpha)} \cap O_{r,j}^{(\alpha)} \neq \varnothing \right\} \\
&= \left\{ i \in \mathbb{Z}^d \setminus \{0\} \,\middle|\, O_{r,i}^{(\alpha)} \cap \lambda^{-1} O_{r,j}^{(\alpha)} \neq \varnothing \right\} = I_j^{(\lambda^{-1})}
\end{aligned} \tag{6.5.34}$$

for all $j \in J = \mathbb{Z}^d \setminus \{0\}$, where $I_j^{(\lambda^{-1})}$ is defined as in equation (6.5.25).

For brevity, set $\theta := q \cdot \left(p^\triangle/q\right)'$ and

$$\nu := \gamma + d\alpha_0 \cdot p_d = \frac{1}{1-\alpha}\left[s + d\alpha \cdot p_d\right].$$

With these abbreviations, we have

*Recall once again the we have $\mathcal{Q} = \lambda \mathbb{O}_r^{(\alpha)}$ and hence $T_i = \lambda\,|i|^{\alpha_0}\,\mathrm{id}$ and not $T_i = |i|^{\alpha_0}\,\mathrm{id}$, cf. Lemma 3.5.1.*

$$\begin{aligned}
&\left\| \left( \left( \left| \det(S_j) \right|^{p_d} v_j \right)^{-1} \left\| \left( |\det(T_i)|^{p_d} u_i \right)_{i \in I_j} \right\|_{\ell^q\left(p^\triangle/q\right)'} \right)_{j \in J} \right\|_{\ell^\infty} \\
&= \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \left( |j|^{d\alpha_0 \cdot p_d} \langle j \rangle^\gamma \right)^{-1} \cdot \left\| \left( \left( \lambda^d\, |i|^{d\alpha_0} \right)^{p_d} \langle i \rangle^\gamma \right)_{i \in I_j^{(\lambda^{-1})}} \right\|_{\ell^\theta} \\
&\asymp \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \lambda^{d \cdot p_d} \langle j \rangle^{-\nu} \cdot \left\| \left( \langle i \rangle^\nu \right)_{i \in I_j^{(\lambda^{-1})}} \right\|_{\ell^\theta}.
\end{aligned} \tag{6.5.35}$$

Note that we have $\lambda^{-1} \geq 1$, so that we can make use of the estimates from the first part. More precisely, for $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$, equations (6.5.26) and (6.5.28) imply

$$\langle j \rangle^{-\nu} \cdot \left\| \left( \langle i \rangle^\nu \right)_{i \in I_j^{(\lambda^{-1})}} \right\|_{\ell^\theta} \asymp \langle j \rangle^{-\nu} \cdot \left[ \left( \lambda^{-1} \right)^{1-\alpha} \langle j \rangle \right]^\nu \cdot \left| I_j^{(\lambda^{-1})} \right|^{\frac{1}{\theta}}$$

$$\begin{aligned}
&\asymp \left[\lambda^{\alpha-1}\right]^{\nu} \cdot \lambda^{-\frac{d}{\theta}(1-\alpha)} \\
&= \lambda^{-s-d\alpha\cdot p_d-\frac{d}{\theta}(1-\alpha)} \\
&= \lambda^{-s+d\alpha\left(\frac{1}{\theta}-p_d\right)-\frac{d}{\theta}}. \qquad (6.5.36)
\end{aligned}$$

For $j \in \{-M,\dots,M\}^d \setminus \{0\}$ and $\theta \in (0,\infty)$, we can use the constant $\kappa = \kappa(d,r,\alpha) \geq 2$ from the discussion after equation (6.5.30), which satisfies

$$\begin{aligned}
I_j^{(\lambda^{-1})} &\subset \left\{ i \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; |i| \leq \kappa \left(\lambda^{-1}\right)^{1-\alpha} \right\} \\
&= \left\{ i \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; |i| \leq \kappa\lambda^{\alpha-1} \right\}, \qquad (6.5.37)
\end{aligned}$$

to conclude

$$\begin{aligned}
&\langle j\rangle^{-\nu} \cdot \left\| \left(\langle i\rangle^{\nu}\right)_{i\in I_j^{(\lambda^{-1})}} \right\|_{\ell^\theta} \\
&\overset{(\dagger)}{\lesssim} \left[ \sum_{\substack{i\in\mathbb{Z}^d\setminus\{0\} \\ |i|\leq\kappa\lambda^{\alpha-1}}} \langle i\rangle^{\theta\nu} \right]^{1/\theta} \\
&\overset{(\ddagger)}{\lesssim} \begin{cases} \left[\left(\kappa\lambda^{\alpha-1}\right)^{d+\theta\nu}\right]^{1/\theta} \asymp \lambda^{-s+d\alpha\left(\frac{1}{\theta}-p_d\right)-\frac{d}{\theta}}, & \text{if } \theta\nu+d>0, \\ \left[\ln\left(\kappa\lambda^{\alpha-1}\right)\right]^{1/\theta} \lesssim \left(1+|\ln\lambda|^{1/\theta}\right), & \text{if } \theta\nu+d=0, \\ 1, & \text{if } \theta\nu+d<0, \end{cases}
\end{aligned}$$

where Lemma 6.5.11 was used at (‡). Note that $\lambda \in (0,1)$ and $\alpha \in [0,1)$ imply $\kappa\lambda^{\alpha-1} \geq 2\lambda^{\alpha-1} \geq 2$, so that Lemma 6.5.11 is indeed applicable. Furthermore, we used $|j| \leq dM$, with $M = M(r,\alpha)$, at (†).

Now observe

$$\begin{aligned}
\theta\nu+d>0 &\iff \frac{\theta}{1-\alpha}\left[s+d\alpha\cdot p_d\right]+d>0 \\
&\iff s > -d\alpha\cdot p_d - d\frac{1-\alpha}{\theta} \\
&\iff s > -\frac{d}{\theta} + \alpha d\left(\frac{1}{\theta}-p_d\right),
\end{aligned}$$

so that the above estimate can be rewritten as

$$\langle j\rangle^{-\nu} \left\| \left(\langle i\rangle^{\nu}\right)_{i\in I_j^{(\lambda^{-1})}} \right\|_{\ell^\theta} \lesssim \begin{cases} \lambda^{-s+d\alpha\left(\frac{1}{\theta}-p_d\right)-\frac{d}{\theta}}, & \text{if } s > -\frac{d}{\theta}+\alpha d\left(\frac{1}{\theta}-p_d\right), \\ 1+|\ln\lambda|^{1/\theta}, & \text{if } s = -\frac{d}{\theta}+\alpha d\left(\frac{1}{\theta}-p_d\right), \\ 1, & \text{if } s < -\frac{d}{\theta}+\alpha d\left(\frac{1}{\theta}-p_d\right) \end{cases}$$

for $j \in \{-M,\dots,M\}^d \setminus \{0\}$ and $\theta \in (0,\infty)$. For $\theta = \infty$, the three conditions reduce to $s + d\alpha\cdot p_d \gtrless 0$. Using $\nu = \frac{1}{1-\alpha}\left[s+d\alpha\cdot p_d\right]$ and

$\langle j \rangle^{-\nu} \asymp 1$, together with equation (6.5.37), it is thus easy to see that the above estimate also remains true for $\theta = \infty$.

Together with $\frac{1}{\theta} = \left(\frac{1}{q} - \frac{1}{p^{\triangle}}\right)_+$ and with the identity

$$\left(\frac{1}{q} - \frac{1}{p^{\triangle}}\right)_+ - p_d = \left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+$$

from Remark 5.2.10 and by applying equations (6.5.33), (6.5.35) and (6.5.36), we conclude

$$\lambda^{-d\cdot p_d} \cdot |||\iota_\lambda||| \lesssim \begin{cases} \lambda^{s_1 - s}, & \text{if } s > s_1, \\ 1 + |\ln \lambda|^{\left(\frac{1}{q} - \frac{1}{p^{\triangle}}\right)_+}, & \text{if } s = s_1, \\ 1, & \text{if } s < s_1, \end{cases}$$

which establishes equation (6.5.19).

It remains to prove equation (6.5.20). The estimate $|||\iota_\lambda||| \gtrsim 1$ was already shown above (even for $\lambda \in (0,1)$), so that the "otherwise" case is already established for $p \in [1,\infty]$, since we have $p_d = 0$ in this range.

To prove the first case of equation (6.5.20) under the assumptions $p \in (0,2]$ and $\frac{1}{q} - \frac{1}{p^{\pm\triangle}} \geq 0$, we use Lemma 5.3.11 with $\mathcal{Q} = \lambda \mathbb{O}_r^{(\alpha)}$, $\mathcal{P} = \mathbb{O}_r^{(\alpha)}$, $u = v = w^{(\gamma)}$ and $p_1 = p_2 = p$ as well as $q_1 = q_2 = q$ and with $I_0 := I = \mathbb{Z}^d \setminus \{0\}$, which easily implies (in the notation of that lemma) $J_0 = J = \mathbb{Z}^d \setminus \{0\}$.

*We emphasize once more that the family of standardizations for $\mathcal{Q} = \lambda \mathbb{O}_r^{(\alpha)}$ is given by $\left(\lambda \cdot |i|^{\alpha_0} \operatorname{id}\right)_{i \in \mathbb{Z}^d \setminus \{0\}}$ and not by $\left(|i|^{\alpha_0} \operatorname{id}\right)_{i \in \mathbb{Z}^d \setminus \{0\}}$.*

Note that $\mathcal{Q}, \mathcal{P}$ are structured admissible coverings (and hence tight) and that Lemma 6.5.10 shows that $\mathcal{Q} = \mathcal{Q}_{I_0}$ is almost subordinate to $\mathcal{P}$ with $k(\mathcal{Q}, \mathcal{P}) \leq K$ for an absolute constant $K = K(r, \alpha, d)$ independent of $\lambda \in (0,1)$. Together with equation (6.5.23) and with Lemma 3.5.1, it is easy to see that all quantities on which the constant provided by Lemma 5.3.11 depends can be estimated independently of $\lambda \in (0,1)$.

Hence, we get

$$\begin{aligned} |||\iota_\lambda||| &\gtrsim \left\| \left( \frac{\left|\det\left(S_j\right)\right|^{\frac{1}{p}-1}}{v_j} \left\| \left( |\det(T_i)|^{1-\frac{1}{p}} u_i \right)_{i \in I_0 \cap I_j} \right\|_{\ell^q} \right)_{j \in J_0} \right\|_{\ell^{q \cdot (q/q)'}} \\ &= \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{|j|^{\alpha_0 d\left(\frac{1}{p}-1\right)}}{\langle j \rangle^{\gamma}} \cdot \left\| \left( \left(\lambda^d |i|^{\alpha_0 d}\right)^{1-\frac{1}{p}} \langle i \rangle^{\gamma} \right)_{i \in I_j} \right\|_{\ell^q} \\ &\asymp \sup_{j \in \mathbb{Z}^d \setminus \{0\}} \lambda^{d\left(1-\frac{1}{p}\right)} \langle j \rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \cdot \left\| \left( \langle i \rangle^{\gamma - \alpha_0 d\left(\frac{1}{p}-1\right)} \right)_{i \in I_j} \right\|_{\ell^q}, \end{aligned} \tag{6.5.38}$$

where we have $I_j = \tilde{I}_j^{(\lambda)} = I_j^{(\lambda^{-1})}$ as in equation (6.5.34).

Now, choose any $j \in \mathbb{Z}^d \setminus \{-M, \dots, M\}^d$. Equation (6.5.28) yields

$$\left|I_j\right| = \left|I_j^{(\lambda^{-1})}\right| \asymp (\lambda^{-1})^{d(1-\alpha)} = \lambda^{d(\alpha-1)}, \tag{6.5.39}$$

whereas equation (6.5.26) implies

$$\langle i \rangle \asymp (\lambda^{-1})^{1-\alpha} \langle j \rangle = \lambda^{\alpha-1} \langle j \rangle \tag{6.5.40}$$

for all $i \in I_j = I_j^{(\lambda^{-1})}$. Together, with equation (6.5.38), we conclude

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \lambda^{d\left(1-\frac{1}{p}\right)} \langle j \rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \left\| \left( \langle i \rangle^{\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)} \right)_{i \in I_j} \right\|_{\ell q} \\
&\asymp \lambda^{d\left(1-\frac{1}{p}\right)} \langle j \rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \cdot \left|I_j\right|^{\frac{1}{q}} \cdot \left( \lambda^{\alpha-1} \langle j \rangle \right)^{\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)} \\
&\asymp \lambda^{d\left(1-\frac{1}{p}\right)} \lambda^{\frac{d}{q}(\alpha-1)} \cdot \lambda^{(\alpha-1)\left[\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)\right]} \\
&= \lambda^{-s+d\alpha\left(\frac{1}{p}-1\right)+\frac{d}{q}(\alpha-1)+d\left(1-\frac{1}{p}\right)} \\
&= \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)+d\alpha\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)},
\end{aligned}$$

where we used $p \in (0,2]$ in the last step, which yields

$$\frac{1}{p^{\pm\Delta}} = \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\} = 1-\frac{1}{p}.$$

Now note that equation (6.5.20) for the case

$$s > s_1 \quad \text{and} \quad p \in (0,2] \quad \text{and} \quad \frac{1}{q} - \frac{1}{p^{\pm\triangle}} \geq 0. \tag{6.5.41}$$

is equivalent to

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \lambda^{d \cdot p_d} \cdot \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ + \alpha d \cdot \left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \\
&= \lambda^{-s-d\left[\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ - p_d\right]+\alpha d \cdot \left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \\
&\overset{(\dagger)}{=} \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ + \alpha d \cdot \left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+} \\
&= \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)+\alpha d \cdot \left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)}.
\end{aligned}$$

Here, Remark 5.2.10 was used at (†). Hence, we have established the first case of equation (6.5.20) under the assumption $p \in (0,2]$ and $\frac{1}{q} - \frac{1}{p^{\pm\Delta}} \geq 0$. Note that we have $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\} = 1-\frac{1}{p} \leq 0$ for $p \in (0,1]$, so that $\frac{1}{q} - \frac{1}{p^{\pm\triangle}} \geq \frac{1}{q} \geq 0$ is always satisfied in this case. Thus, we have in particular established the first case of equation (6.5.20) for $p \in (0,1]$.

Now, we can also establish the "otherwise" case of equation (6.5.20) for $p \in (0,1]$. To this end, choose $j_0 \in \mathbb{Z}^d \setminus \{0\}$ with $0 \in O^{(\alpha)}_{r,j_0}$. This implies $0 \in \lambda O^{(\alpha)}_{r,j_0} \cap O^{(\alpha)}_{r,j_0}$ for all $\lambda \in (0,1)$ and hence $j_0 \in I_{j_0}$. Together with equation (6.5.38), we conclude

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \lambda^{d\left(1-\frac{1}{p}\right)} \langle j_0\rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \cdot \left\| \left( \langle i\rangle^{\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)} \right)_{i\in I_{j_0}} \right\|_{\ell^q} \\
&\geq \lambda^{d\left(1-\frac{1}{p}\right)} \cdot \langle j_0\rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \cdot \langle j_0\rangle^{\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)} \\
&= \lambda^{d\left(1-\frac{1}{p}\right)}. \qquad (6.5.42)
\end{aligned}$$

In case of $p \in (0,1]$, we have $p_d = 1 - \frac{1}{p}$, so that the above estimate yields $\lambda^{-d\cdot p_d} |||\iota_\lambda||| \gtrsim 1$, where the implied constant is independent of $\lambda \in (0,1)$. Thus, we have established the "otherwise" case of equation (6.5.20) also for $p \in (0,1]$. Recall that the case $p \in [1,\infty]$ was already taken care of above, so that the proof of the "otherwise" case is complete.

To prove equation (6.5.20) in the remaining cases, we use the lower bound from equation (6.5.33). With $\theta_2 := q \cdot (p/q)'$, this lower bound yields

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \left\| \left( v_j^{-1} \cdot \left\| (u_i)_{i\in I_j} \right\|_{\ell^{q(p/q)'}} \right)_{j\in J} \right\|_{\ell^\infty} \\
&= \sup_{j\in\mathbb{Z}^d\setminus\{0\}} \langle j\rangle^{-\gamma} \cdot \left\| \left( \langle i\rangle^{\gamma} \right)_{i\in I_j^{(\lambda^{-1})}} \right\|_{\ell^{\theta_2}}. \qquad (6.5.43)
\end{aligned}$$

But for $j \in \mathbb{Z}^d \setminus \{-M,\dots,M\}^d$, equations (6.5.39) and (6.5.40) yield

$$\begin{aligned}
\langle j\rangle^{-\gamma} \cdot \left\| \left( \langle i\rangle^{\gamma} \right)_{i\in I_j^{(\lambda^{-1})}} \right\|_{\ell^{\theta_2}} &\gtrsim \lambda^{(\alpha-1)\gamma} \left| I_j^{(\lambda^{-1})} \right|^{\frac{1}{\theta_2}} \\
&\gtrsim \lambda^{(\alpha-1)\gamma} \lambda^{d(\alpha-1)\left(\frac{1}{q}-\frac{1}{p}\right)_+} \\
&= \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p}\right)_+ + \alpha d\left(\frac{1}{q}-\frac{1}{p}\right)_+}.
\end{aligned}$$

In view of equation (6.5.43), this establishes the first case of equation (6.5.20) under the additional assumption $p \in [2,\infty]$, since we have $p_d = 0$ and $p^\triangle = p^{\pm\triangle} = p$ for this range. Furthermore, we get

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \lambda^{-s-d\left(\frac{1}{q}-\frac{1}{p}\right)_+ + \alpha d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \\
&= \lambda^{-s} \cdot \lambda^{d(\alpha-1)\left(\frac{1}{q}-\frac{1}{p}\right)_+} \\
&\geq \lambda^{-s},
\end{aligned}$$

because of $\lambda \in (0,1)$ and $\alpha - 1 < 0$.

Hence, the proof of the first case of equation (6.5.33) is also complete for $p \in (0,2]$, since the only case which is still left is $\frac{1}{q} - \frac{1}{p^{\pm\triangle}} < 0$. But as seen above, this entails $p \notin (0,1)$, i.e. $p \in [1,2]$. Hence, we have $p_d = 0$ and $p^{\pm\triangle} = p^{\triangle} = p'$ as well as

$$\frac{1}{q} - \frac{1}{p^{\triangle}} = \frac{1}{q} - \frac{1}{p^{\pm\triangle}} < 0.$$

Furthermore, we have $s > 0$, since we are in the first case of equation (6.5.20). Thus, equation (6.5.20) is equivalent to $|||\iota_\lambda||| \gtrsim \lambda^{-s}$ in this case.

The only thing left to prove is the second case of equation (6.5.20). In this case, there are two subcases, namely $p \in [2,\infty]$ and $p \in (0,1]$.

CASE 1: We have $p \in [2,\infty]$ and hence $p_d = 0$ and $p^{\pm\triangle} = p^{\triangle} = p$, as well as

$$\left(\frac{1}{q} - \frac{1}{p^{\triangle}}\right)_+ = \left(\frac{1}{q} - \frac{1}{p}\right)_+ = \frac{1}{\theta_2}.$$

In case of $\theta_2 = \infty$, the desired estimate is a trivial consequence of $|||\iota_\lambda||| \gtrsim 1$, so that we can assume $\theta_2 < \infty$. This yields

$$\begin{aligned}(1-\alpha)\,\gamma = s &= s_1 \\ &= -d\left(\frac{1}{q} - \frac{1}{p^{\triangle}}\right)_+ + \alpha d\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+ \\ &= -\frac{d}{\theta_2} + \alpha\frac{d}{\theta_2} \\ &= -\frac{d}{\theta_2}(1-\alpha)\end{aligned}$$

and hence $\theta_2\gamma = -d$.

As above, let $j_0 \in \mathbb{Z}^d \setminus \{0\}$ with $0 \in O^{(\alpha)}_{r,j_0}$. Note that $O^{(\alpha)}_{r,j_0}$ is open, so that there is some $\varepsilon > 0$ (independent of $\lambda \in (0,1)$) with $\overline{B_\varepsilon}(0) \subset O^{(\alpha)}_{r,j_0}$. Since we already know $|||\iota_\lambda||| \gtrsim 1$, we can without loss of generality assume $0 < \lambda < \frac{\varepsilon}{2^{1/(1-\alpha)}}$. Now, let $i \in \mathbb{Z}^d \setminus \{0\}$ be arbitrary with $|i| \le (\varepsilon/\lambda)^{1-\alpha}$. This implies

$$\left|\lambda\,|i|^{\alpha_0}\,i\right| = \lambda\,|i|^{\frac{1}{1-\alpha}} \le \lambda\frac{\varepsilon}{\lambda} = \varepsilon$$

and hence

$$\lambda\,|i|^{\alpha_0}\,i \in \lambda O^{(\alpha)}_{r,i} \cap O^{(\alpha)}_{r,j_0} \neq \varnothing,$$

i.e. $i \in I^{(\lambda^{-1})}_{j_0}$. By equation (6.5.43), this implies

$$\begin{aligned}|||\iota_\lambda||| &\gtrsim \langle j_0\rangle^{-\gamma} \cdot \left\|\left(\langle i\rangle^{\gamma}\right)_{i\in I^{(\lambda^{-1})}_{j_0}}\right\|_{\ell^{\theta_2}} \\ &\gtrsim_{j_0,\gamma} \left\|\left(\langle i\rangle^{\gamma}\right)_{i\in\mathbb{Z}^d\setminus\{0\} \text{ with } |i|\le(\varepsilon/\lambda)^{1-\alpha}}\right\|_{\ell^{\theta_2}}\end{aligned}$$

$$
\begin{aligned}
&= \left[ \sum_{\substack{i \in \mathbb{Z}^d \setminus \{0\} \\ |i| \leq (\varepsilon/\lambda)^{1-\alpha}}} \langle i \rangle^{\theta_2 \gamma} \right]^{1/\theta_2} \\
&\overset{(\dagger)}{\gtrsim} \left[ \ln \left( (\varepsilon/\lambda)^{1-\alpha} \right) \right]^{\left(\frac{1}{q} - \frac{1}{p}\right)_+} \\
&\overset{(\ddagger)}{\gtrsim}_{\varepsilon,\alpha,p,q} |\ln \lambda|^{\left(\frac{1}{q} - \frac{1}{p}\right)_+} \\
&\overset{(\ddagger)}{\gtrsim} 1 + |\ln \lambda|^{\left(\frac{1}{q} - \frac{1}{p}\right)_+}.
\end{aligned}
$$

Here, we used Lemma 6.5.11 together with $\theta_2 \gamma = -d$ at $(\dagger)$ and the assumption $0 < \lambda < \frac{\varepsilon}{2^{1(1-\alpha)}}$ at $(\ddagger)$.

CASE 2: We have $p \in (0,1]$ and hence $p_d = 1 - \frac{1}{p}$ and $p^{\triangle} = \infty$, as well as

$$
\begin{aligned}
\frac{1}{p^{\pm\triangle}} &= \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\} \\
&= 1 - \frac{1}{p} < 0.
\end{aligned}
$$

Thus,

$$
\left( \frac{1}{q} - \frac{1}{p^{\triangle}} \right)_+ = \frac{1}{q}
$$

and

$$
\left( \frac{1}{q} - \frac{1}{p^{\pm\triangle}} \right)_+ = \left( \frac{1}{q} + \frac{1}{p} - 1 \right)_+ = \frac{1}{q} + \frac{1}{p} - 1.
$$

Since we are in the second case of equation (6.5.20), this yields

$$
\begin{aligned}
s = s_1 &= -d \left( \frac{1}{q} - \frac{1}{p^{\triangle}} \right)_+ + \alpha d \left( \frac{1}{q} - \frac{1}{p^{\pm\triangle}} \right)_+ \\
&= -\frac{d}{q} + \alpha d \left( \frac{1}{q} + \frac{1}{p} - 1 \right) \\
&= -d\,(1-\alpha)\,\frac{1}{q} + \alpha d \left( \frac{1}{p} - 1 \right)
\end{aligned}
$$

and hence

$$
\gamma = \frac{s}{1-\alpha} = -\frac{d}{q} + \alpha_0 d \left( \frac{1}{p} - 1 \right).
$$

Note that equation (6.5.42) yields $\|\!|\!| \iota_\lambda |\!|\!| \gtrsim \lambda^{d \cdot p_d}$, so that the desired estimate is trivial in case of $q = \infty$, since this entails $\left( \frac{1}{q} - \frac{1}{p^{\triangle}} \right)_+ = 0$. Hence, we can assume $0 < q < \infty$.

Choose $\varepsilon$ and $j_0$ as in the previous case. Exactly as in that case, we see that it suffices to assume $0 < \lambda < \frac{\varepsilon}{2^{1/(1-\alpha)}}$ and that this implies

$$
I_j = I_{j_0}^{(\lambda^{-1})} \supset \left\{ i \in \mathbb{Z}^d \setminus \{0\} \;\middle|\; |i| \leq (\varepsilon/\lambda)^{1-\alpha} \right\}.
$$

In tandem with equation (6.5.38), we derive

$$\begin{aligned}
|||\iota_\lambda||| &\gtrsim \sup_{j\in\mathbb{Z}^d\setminus\{0\}} \lambda^{d\left(1-\frac{1}{p}\right)} \langle j\rangle^{\alpha_0 d\left(\frac{1}{p}-1\right)-\gamma} \cdot \left\| \left( \langle i\rangle^{\gamma-\alpha_0 d\left(\frac{1}{p}-1\right)} \right)_{i\in I_j} \right\|_{\ell^q} \\
&\gtrsim_{j_0} \lambda^{d\cdot p_d} \cdot \left\| \left( \langle i\rangle^{-\frac{d}{q}} \right)_{i\in\mathbb{Z}^d\setminus\{0\} \text{ with } |i|\leq(\varepsilon/\lambda)^{1-\alpha}} \right\|_{\ell^q} \\
&= \lambda^{d\cdot p_d} \cdot \left[ \sum_{\substack{i\in\mathbb{Z}^d\setminus\{0\} \\ |i|\leq(\varepsilon/\lambda)^{1-\alpha}}} \langle i\rangle^{-q\cdot\frac{d}{q}} \right]^{1/q} \\
&\overset{(\dagger)}{\gtrsim} \lambda^{d\cdot p_d} \cdot \left[ \ln\left( (\varepsilon/\lambda)^{1-\alpha} \right) \right]^{1/q} \\
&\gtrsim \lambda^{d\cdot p_d} \cdot |\ln\lambda|^{1/q} \\
&\gtrsim \lambda^{d\cdot p_d} \cdot (1+|\ln\lambda|)^{\left(\frac{1}{q}-\frac{1}{p^\triangle}\right)_+}.
\end{aligned}$$

Thus, equation (6.5.20) is established also in this case. The step marked with (†) is justified by Lemma 6.5.11 and the two steps afterwards use $0<\lambda\leq\min\left\{1,2^{-1/(1-\alpha)}\varepsilon\right\}$ and $1-\alpha>0$. □

Using Lemma 6.1.9, we can now transfer our results to the setting of the "usual" (space-sided) $\alpha$-modulation spaces.

**Corollary 6.5.13.** *Let $d\in\mathbb{N}$, $p,q\in(0,\infty]$, $\alpha\in[0,1)$ and $s\in\mathbb{R}$. For each $\lambda\in(0,\infty)$, define the* (ISOTROPIC) DILATION OPERATOR $D_\lambda$ *by*

$$D_\lambda : M^{s,\alpha}_{p,q}(\mathbb{R}^d) \to M^{s,\alpha}_{p,q}(\mathbb{R}^d), f\mapsto f_{\lambda\cdot\mathrm{id}},$$

*where $f_{\lambda\cdot\mathrm{id}}$ is as in Definition 3.5.2 (with the obvious modification for tempered distributions). Then $D_\lambda$ is a well-defined and bounded linear opeartor for each $\lambda\in(0,\infty)$.*

*Define*

$$\begin{aligned}
s_0 &:= d(1-\alpha)\left(\frac{1}{p^\triangledown}-\frac{1}{q}\right)_+, \\
s_1 &:= -d\left(\frac{1}{q}-\frac{1}{p^\triangle}\right)_+ + \alpha d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+, \\
s_2 &:= d(1-\alpha)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+.
\end{aligned}$$

*We remark* $\frac{1}{p}+p_d=\min\left\{\frac{1}{p},1\right\}$.

*There are constants $C_1,C_2>0$ such that*

$$|||D_\lambda||| \leq C_1 \cdot \begin{cases} \lambda^{-d\left(\frac{1}{p}+p_d\right)}\cdot\lambda^{s-s_1} = \lambda^{-\frac{d}{p}+s+s_2}, & \text{if } s>s_1, \\ \lambda^{-d\left(\frac{1}{p}+p_d\right)}\cdot\left[1+(\ln\lambda)^{\left(\frac{1}{q}-\frac{1}{p^\triangle}\right)_+}\right], & \text{if } s=s_1, \\ \lambda^{-d\left(\frac{1}{p}+p_d\right)}, & \text{if } s<s_1, \end{cases}$$

*and*

$$|||D_\lambda||| \geq C_2 \cdot \begin{cases} \lambda^{-d\left(\frac{1}{p}+p_d\right)} \cdot \lambda^{s-s_1} = \lambda^{-\frac{d}{p}+s+s_2}, & \text{if } s > s_1, \\ \lambda^{-d\left(\frac{1}{p}+p_d\right)} \left(1 + (\ln \lambda)^{\left(\frac{1}{q}-\frac{1}{p^\triangle}\right)_+}\right), & \text{if } s = s_1 \text{ and } p \notin (1,2), \\ \lambda^{-d\left(\frac{1}{p}+p_d\right)}, & \text{otherwise} \end{cases}$$

*hold for all* $\lambda > 1$.

*Similarly, there are constants* $C_3, C_4 > 0$ *such that*

$$|||D_\lambda||| \leq C_3 \cdot \begin{cases} \lambda^{-\frac{d}{p}} \cdot \lambda^{s-s_0}, & \text{if } s < s_0, \\ \lambda^{-\frac{d}{p}} \cdot \left[1 + |\ln \lambda|^{\left(\frac{1}{p^\triangledown}-\frac{1}{q}\right)_+}\right], & \text{if } s = s_0, \\ \lambda^{-\frac{d}{p}}, & \text{if } s > s_0, \end{cases}$$

*and*

$$|||D_\lambda||| \geq C_4 \cdot \begin{cases} \lambda^{-\frac{d}{p}} \cdot \lambda^{s-s_0}, & \text{if } s < s_0, \\ \lambda^{-\frac{d}{p}} \cdot \left[1 + |\ln \lambda|^{\left(\frac{1}{p^\triangledown}-\frac{1}{q}\right)_+}\right], & \text{if } s = s_0 \text{ and } 0 < p \leq 2, \\ \lambda^{-\frac{d}{p}}, & \text{otherwise}, \end{cases}$$

*hold for all* $\lambda \in (0,1]$.

*Proof.* Lemma 6.1.9 shows that the Fourier transform

$$\mathcal{F} : M^{s,\alpha}_{p,q}(\mathbb{R}^d) \to \mathcal{F}M^{s,\alpha}_{p,q}, f \mapsto \widehat{f}|_{C_c^\infty(\mathbb{R}^d)}$$

induces an isomorphism of (Quasi)-Banach spaces. Here, the Fourier-side $\alpha$-modulation space is defined by

$$\mathcal{F}M^{s,\alpha}_{p,q} := \mathcal{D}\left(\mathbb{O}_r^{(\alpha)}, L^p, \ell^q_{w^{(s/(1-\alpha))}}\right),$$

see Definition 6.1.4.

Let $\lambda \in (0,\infty)$ and let $U_{\lambda^{-1}}$ be as in Theorem 6.5.12. We first compute the conjugation of this operator under the Fourier transform. Let $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ and set

$$g := \mathcal{F}^{-1}U_{\lambda^{-1}}\mathcal{F}f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d).$$

We then have

$$\begin{aligned} h &:= U_{\lambda^{-1}}\mathcal{F}f \\ &= \mathcal{F}g \\ &= \widehat{g}|_{C_c^\infty(\mathbb{R}^d)}. \end{aligned}$$

For $\varphi \in \mathcal{F}\left[C_c^\infty\left(\mathbb{R}^d\right)\right]$, this implies

$$\begin{aligned}
\langle g, \varphi\rangle &= \left\langle \widehat{g}, \mathcal{F}^{-1}\varphi\right\rangle = \left\langle h, \mathcal{F}^{-1}\varphi\right\rangle \\
&= \left\langle U_{\lambda^{-1}}\mathcal{F}f, \mathcal{F}^{-1}\varphi\right\rangle \\
&\overset{(\dagger)}{=} \left|\det\left(\lambda^{-1}\mathrm{id}\right)\right|^{-1} \cdot \left\langle \mathcal{F}f, \left(\mathcal{F}^{-1}\varphi\right)\circ\lambda\mathrm{id}\right\rangle \\
&= \lambda^d \cdot \left\langle \mathcal{F}f, \mathcal{F}^{-1}\left[\lambda^{-d}\cdot\varphi\circ\lambda^{-1}\mathrm{id}\right]\right\rangle \\
&= \left\langle f, \varphi\circ\lambda^{-1}\mathrm{id}\right\rangle \\
&= \lambda^d \cdot \left|\det\left(\lambda\mathrm{id}\right)\right|^{-1}\left\langle f, \varphi\circ\lambda^{-1}\mathrm{id}\right\rangle \\
&\overset{(\dagger)}{=} \lambda^d \cdot \langle D_\lambda f, \varphi\rangle .
\end{aligned}$$

Here, the definition of $f_{\lambda^{\pm 1}\mathrm{id}}$ (i.e. Definition 3.5.2) was used at the steps marked with $(\dagger)$. Since $C_c^\infty\left(\mathbb{R}^d\right)$ is dense in $\mathcal{S}\left(\mathbb{R}^d\right)$ by [33, Proposition 9.9] and since the Fourier transform is a homeomorphism of $\mathcal{S}\left(\mathbb{R}^d\right)$ with itself by [33, Corollary 8.28], we know that $\mathcal{F}\left[C_c^\infty\left(\mathbb{R}^d\right)\right]$ is dense in $\mathcal{S}\left(\mathbb{R}^d\right)$.

Hence, the above calculation allows us to conclude $g = \lambda^d \cdot D_\lambda f$. We have thus shown

$$D_\lambda = \lambda^{-d}\cdot\mathcal{F}^{-1}\circ U_{\lambda^{-1}}\circ\mathcal{F},$$

as expected. Since the Fourier transform is independent of $\lambda \in (0,\infty)$, we thus see that each of the operators $D_\lambda$ is bounded with

$$|||D_\lambda||| \asymp \lambda^{-d}\cdot|||U_{\lambda^{-1}}||| .$$

The stated result is now a direct consequence of Theorem 6.5.12. We only remark that the alternate form using the exponent $s_2$ instead of $s_1$ follows from Remark 5.2.10, which yields the identity

$$\begin{aligned}
-s_1 - d\cdot p_d &= d\left(\frac{1}{q}-\frac{1}{p^{\triangle}}\right)_+ - \alpha d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ - d\cdot p_d \\
&= d\left[\left(\frac{1}{q}-\frac{1}{p^{\triangle}}\right)_+ - p_d\right] - \alpha d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ \\
&= d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ - \alpha d\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ \\
&= d\,(1-\alpha)\left(\frac{1}{q}-\frac{1}{p^{\pm\triangle}}\right)_+ = s_2. \qquad (6.5.44)
\end{aligned}$$

This completes the proof. □

*Remark* 6.5.14. In [49, Theorem 3.1], Han and Wang claim the estimate

$$\|D_\lambda f\|_{M^{s,\alpha}_{p,q}} \lesssim \lambda^{-\frac{d}{p}}\cdot\left[(1\vee\lambda)^{s_p}\vee\lambda^{s+s_c}\right]\cdot\|f\|_{M^{s,\alpha}_{p,q}}$$

for all $f \in M_{p,q}^{s,\alpha}\left(\mathbb{R}^d\right)$, but under the assumption $s + s_c \neq 0$. Han and Wang use (in our notation) the conventions $a \vee b := \max\{a, b\}$ and

$$\begin{aligned} s_p &= d \cdot \left[\frac{1}{1 \wedge p} - 1\right] = d \cdot \left[\frac{1}{\min\{1, p\}} - 1\right] \\ &= \begin{cases} d\left(\frac{1}{p} - 1\right), & \text{if } p \in (0,1), \\ 0, & \text{if } p \in [1, \infty], \end{cases} \\ &= -d \cdot p_d, \end{aligned}$$

as well as

$$s_c = \begin{cases} R(p, q; 1, \alpha), & \text{if } \lambda > 1, \\ -R(p, q; \alpha, 1), & \text{if } \lambda \leq 1, \end{cases}$$

where the quantity $R(p, q; \alpha_1, \alpha_2)$ is defined in [49, equations (3.5) and (3.6)].

A straightforward case distinction shows

$$R(p, q; 1, \alpha) = d(1 - \alpha)\left(\frac{1}{q} - \frac{1}{p^{\pm\triangle}}\right)_+ = s_2$$

and

$$R(p, q; \alpha, 1) = d(1 - \alpha)\left(\frac{1}{p^{\triangledown}} - \frac{1}{q}\right)_+ = s_0.$$

Note that we have

$$(1 \vee \lambda)^{s_p} = \begin{cases} 1, & \text{if } \lambda \leq 1, \\ \lambda^{s_p} = \lambda^{-d \cdot p_d}, & \text{if } \lambda > 1. \end{cases}$$

We first consider the case $\lambda > 1$. Here, $s_c = R(p, q; 1, \alpha) = s_2$, which yields (see also equation (6.5.44) from the proof above) the equivalence

$$s + s_c \overset{!}{\geq} s_p \iff s \overset{!}{\geq} s_p - s_c = -(s_2 + d \cdot p_d) = s_1.$$

In summary, in our notation, the result from [49, Theorem 3.1] takes the form

$$|||D_\lambda||| \lesssim \lambda^{-\frac{d}{p}} \cdot \lambda^{\max\{s_p, s + s_c\}} = \begin{cases} \lambda^{-\frac{d}{p}} \cdot \lambda^{s + s_2}, & \text{if } s \geq s_1, \\ \lambda^{-\frac{d}{p}} \cdot \lambda^{s_p} = \lambda^{-d\left(\frac{1}{p} + p_d\right)}, & \text{if } s < s_1 \end{cases}$$

for $\lambda > 1$ and $s \neq -s_c = -s_2$.

We observe that this upper bound for the operator norm is identical with our bound from Corollary 6.5.13, except in case of $-s_2 \neq s = s_1$. Now, observe that $p \in [1, \infty]$ implies $p^{\triangle} = p^{\pm\triangle}$ and hence $-s_2 = s_1$.

Thus, $-s_2 \neq s = s_1$ can only occur for $p \in (0,1)$. In this case, we have (cf. equation (6.5.44))

$$s_2 = -s_1 - d \cdot p_d = -s_1 + d \cdot \left(\frac{1}{p} - 1\right).$$

Hence, for $s = s_1 = d\left(\frac{1}{p} - 1\right) - s_2 \neq -s_2$, the results of Han and Wang yield

$$|||D_\lambda||| \lesssim \lambda^{-\frac{d}{p}} \cdot \lambda^{s+s_2} = \lambda^{-\frac{d}{p}} \cdot \lambda^{d\left(\frac{1}{p}-1\right)} = \lambda^{-d},$$

whereas Corollary 6.5.13 shows

$$|||D_\lambda||| \gtrsim \lambda^{-d\left(\frac{1}{p}+p_d\right)} \cdot \left(1 + (\ln \lambda)^{\frac{1}{q}}\right) = \lambda^{-d} \cdot \left(1 + (\ln \lambda)^{\frac{1}{q}}\right).$$

Thus, [49, Theorem 3.1] is incorrect in this case. We remark that Han and Wang also state (without proof) the bound

$$|||D_\lambda||| \lesssim \lambda^{-d} \cdot (\ln \lambda)^{\frac{1}{q}}$$

in [49, Remark 3.1], but under the assumption $s = -s_c$ instead of $s = s_1$.

In summary, for $\lambda > 1$, the *form of the bound* for the operator norm of $|||D_\lambda|||$ is completely identical to the bound derived in Corollary 6.5.13, but Han and Wang state that the special case where the logarithmic factor occurs is given by $s = -s_c = -s_2$, whereas our result shows that this logarithmic factor occurs for $s = s_1$. For the range $p \in [1,\infty]$, we have $s_1 = -s_2$, so that these conditions coincide, but for $p \in (0,1)$ and $s = s_1$, the result of Han and Wang is incorrect.

For $\lambda \in (0,1]$, we have $s_c = -R(p,q;\alpha,1) = -s_0$ so that the bound given in [49, Theorem 3.1] takes the form

$$|||D_\lambda||| \lesssim \lambda^{-\frac{d}{p}} \max\left\{1, \lambda^{s+s_c}\right\} = \begin{cases} \lambda^{-\frac{d}{p}+s+s_c}, & \text{if } s + s_c < 0, \\ \lambda^{-\frac{d}{p}}, & \text{if } s + s_c > 0 \end{cases}$$

$$= \begin{cases} \lambda^{-\frac{d}{p}+s-s_0}, & \text{if } s < s_0, \\ \lambda^{-\frac{d}{p}}, & \text{if } s > s_0, \end{cases}$$

which is identical to the bound derived in Corollary 6.5.13.

In summary, we conclude that the findings of Han and Wang are identical with our results for $0 < \lambda \leq 1$ and also for $\lambda > 1$, as long as $p \in [1,\infty]$. But for $\lambda > 1$ and $p \in (0,1)$ and certain values of $s$, the bound given by Han and Wang misses a logarithmic factor. Although [49, Remark 3.1] shows that Han and Wang are aware of the occurence of this factor, the exact case in which this happens is not identified correctly.

Finally, our sharpness results also show that this logarithmic factor *has* to occur (at least if certain values of $p$ are excluded), while the findings in [49] only establish sharpness in those cases in which no logarithmic factor occurs. ◀

*Remark* 6.5.15. A further paper in which the norm of the dilation operator was studied is [10]. More precisely, the authors estimate the operator norm of the isotropic dilation on weighted (classical) modulation spaces. In our setting, this corresponds to the case $\alpha = 0$.

In [10, Theorem 3.2] it is claimed (in our notation) that the dilation operator

$$D_\lambda : M^{s,0}_{p,q}(\mathbb{R}^d) \to M^{s,0}_{p,q}(\mathbb{R}^d), f \mapsto f_{\lambda \mathrm{id}}$$

is well-defined and bounded with

$$\begin{aligned} |||D_\lambda||| &\lesssim \begin{cases} \lambda^{d\cdot\mu_1} \max\{1, \lambda^s\}, & \text{if } \lambda \geq 1, \\ \lambda^{d\cdot\mu_2} \max\{1, \lambda^s\}, & \text{if } 0 < \lambda \leq 1 \end{cases} \\ &= \begin{cases} \lambda^{s+d\cdot\mu_1}, & \text{if } \lambda \geq 1 \text{ and } s \geq 0, \\ \lambda^{d\cdot\mu_1}, & \text{if } \lambda \geq 1 \text{ and } s < 0, \\ \lambda^{d\cdot\mu_2}, & \text{if } 0 < \lambda < 1 \text{ and } s \geq 0, \\ \lambda^{s+d\cdot\mu_2}, & \text{if } 0 < \lambda < 1 \text{ and } s < 0. \end{cases} \end{aligned}$$

Here, a straightforward case distinction shows that the constants $\mu_1, \mu_2$ defined in [10] satisfy

$$\begin{aligned} \mu_1 &= \left(\frac{1}{q} - \frac{1}{p^\triangle}\right)_+ - \frac{1}{p}, \\ \mu_2 &= -\frac{1}{p} - \left(\frac{1}{p^\triangledown} - \frac{1}{q}\right)_+. \end{aligned}$$

We note $p^\triangle = p^{\pm\triangle}$, since in [10] only the case $p, q \in [1, \infty]$ is considered.

For the setting $\alpha = 0$ and $p, q \in [1, \infty]$, the constants $s_0, s_1, s_2$ from Corollary 6.5.13 satisfy

$$\begin{aligned} s_0 &= d\left(\frac{1}{p^\triangledown} - \frac{1}{q}\right)_+ = -d\left(\mu_2 + \frac{1}{p}\right), \\ s_1 = -s_2 &= -d\left(\frac{1}{q} - \frac{1}{p^\triangle}\right)_+ = -d\left(\mu_1 + \frac{1}{p}\right). \end{aligned}$$

For $\lambda > 1$, Corollary 6.5.13 thus yields

$$|||D_\lambda||| \asymp \begin{cases} \lambda^{-\frac{d}{p}+s+d\left(\mu_1+\frac{1}{p}\right)} = \lambda^{s+d\mu_1}, & \text{if } s > s_1, \\ \lambda^{-\frac{d}{p}}, & \text{if } s < s_1. \end{cases}$$

For $0 < \lambda \leq 1$, we get

$$|||D_\lambda||| \asymp \begin{cases} \lambda^{-\frac{d}{p}+s+d\left(\mu_2+\frac{1}{p}\right)} = \lambda^{s+d\mu_2}, & \text{if } s < s_0, \\ \lambda^{-\frac{d}{p}}, & \text{if } s > s_0. \end{cases}$$

We now indicate with two concrete cases that the bounds obtained in Corollary 6.5.13 are (in some cases) strictly better than those in [10, Theorem 3.2]. Sadly, this implies that [10, Theorem 4.13] fails in general, since this theorem claims sharpness of the bounds obtained in [10, Theorem 3.2].

For $\lambda > 1$, we take $p = 2$ and $q = 1$. This implies $s_1 = -\frac{d}{2}$ and $\mu_1 = 0$. Now, if we take e.g. $s = -\frac{d}{4}$, we have $s = -\frac{d}{4} > -\frac{d}{2} = s_1$ and hence

$$|||D_\lambda||| \asymp \lambda^{s+d\mu_1} = \lambda^s = \lambda^{-d/4}$$

by Corollary 6.5.13, but the results in [10, Theorem 3.2] yield (because of $s < 0$) that

$$|||D_\lambda||| \lesssim \lambda^{d\mu_1} = 1.$$

Since we have $\lambda > 1$, we have $\lambda^{-d/4} \ll 1$ for large $\lambda$, so that the estimate in [10, Theorem 3.2] is indeed suboptimal.

For $0 < \lambda < 1$, we take $q = \infty$ and $p = 2$. This implies $s_0 = \frac{d}{2}$ and $\mu_2 = -1$. Now, if we take e.g. $s = \frac{d}{4}$, we have $s = \frac{d}{4} < \frac{d}{2} = s_0$ and hence

$$|||D_\lambda||| \asymp \lambda^{s+d\mu_2} = \lambda^{\frac{d}{4}-d} = \lambda^{-\frac{3}{4}d}$$

by Corollary 6.5.13. But the findings in [10, Theorem 3.2] yield (because of $s \geq 0$) that

$$|||D_\lambda||| \lesssim \lambda^{d\mu_2} = \lambda^{-d}.$$

Because of $0 < \lambda < 1$, we have $\lambda^{-\frac{3}{4}d} \ll \lambda^{-d}$ for small $\lambda > 0$, so that the estimate derived in [10, Theorem 3.2] is again not optimal, in contradiction to [10, Theorem 4.13]. ◀

# 7

# CONCLUSION AND OUTLOOK

In this chapter, we briefly review our results and outline possible directions for further research.

WAVELET COORBIT SPACES AS DECOMPOSITION SPACES Around the time of their inception, decomposition spaces mainly had the following applications or connections to more classical spaces:

A. They provided a common framework for homogeneous and inhomogeneous BESOV SPACES,

B. they yielded the class of WIENER AMALGAM SPACES as a special case,

C. they could be used to describe the MODULATION SPACES and were then also used for the definition of the more general $\alpha$-MODULATION SPACES.

Thus, the main types of coverings considered were the uniform covering, the dyadic (Besov-type) covering and the family of $\alpha$-coverings, which provided an intermediate family of coverings between these two extremes.

Of course, a large number of different, more exotic coverings was also studied. As two selected examples, we mention [26] and [65, Section 2.5]. More recently, decomposition spaces have received new attention. We mention in particular the construction of anisotropic Besov spaces[8], the recently defined shearlet smoothness spaces[53] and the work of Borup and Nielsen[3] on frames for $\alpha$-modulation spaces.

Nevertheless, apart from the above-mentioned connections to the (classical) modulation and Besov spaces, the main use case of decomposition spaces was to start with a new covering (like the $\alpha$-covering or the shearlet smoothness covering) and to use this covering to construct *new* spaces.

In contrast, we started with the *existing* class of wavelet coorbit spaces and used the dual action of the dilation group $H$ to obtain an induced covering $\mathcal{Q}$ such that the coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is isomorphic to the decomposition space $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_u\right)$ for a certain weight $u$. This result indicates the richness and flexibility of the class of decomposition spaces by showing that every wavelet coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ is isomorphic to a certain decomposition space. Thus, the theory of wavelet coorbit spaces becomes a branch of decomposition space theory.

This point of view opens the door to a Fourier analytic investigation of wavelet coorbit spaces. Roughly speaking, the decomposition point of view shows that a large part of the structure of the dilation group $H$ is inconsequential for the resulting coorbit spaces. More precisely, the (approximation theoretic) properties of the coorbit space $\mathrm{Co}\left(L^{p,q}_m\left(\mathbb{R}^d \rtimes H\right)\right)$ are predominantly determined by the way in which the dual action

$$H \times \mathbb{R}^d \to \mathbb{R}^d, (h, \xi) \mapsto h^{-T}\xi$$

partitions/covers the frequency space $\mathbb{R}^d$.

The power of this point of view became apparent in Chapter 6. Although coorbit spaces have been known since the 80s and even though especially the class of shearlet coorbit spaces recently received increased attention[13, 15, 14, 17, 16], the decomposition space point of view allowed us to derive many nontrivial and novel results, in particular

A. Embeddings between shearlet-type coorbit spaces and (inhomogeneous) Besov spaces.

B. Characterization of the class of matrices for which every shearlet coorbit space is invariant under conjugation with the respective matrix.

C. Asymptotic evaluation of the operator norm of the isotropic dilation operator.

D. Embeddings of shearlet coorbit spaces into coorbit spaces defined using the shearlet-*type* groups $H_c$ for $c \neq \frac{1}{2}$.

Due to space constraints, we only considered the case of two dimensional shearlet coorbit spaces in this thesis. A detailed investigation of shearlet coorbit spaces in higher dimensions using the decomposition space point of view promises to be very interesting. In particular, since the decomposition space point of view allows to compare coorbit spaces "living on different groups", it should be possible to compare the two types of constructions used for higher-dimensional shearlet groups/coorbit spaces, namely the block form used in [14] and the Toeplitz form considered in [16].

EMBEDDINGS FOR DECOMPOSITION SPACES In view of the huge variety of coverings obtained as induced coverings from different dilation groups, it becomes relevant to classify the resulting decomposition spaces, i.e. to derive embeddings of the form

$$\iota : \mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right).$$

Since the two decomposition spaces in question are defined using only the two coverings $\mathcal{Q}, \mathcal{P}$, the weights $u, v$ and the exponents

$p_1, p_2, q_1, q_2$, the existence or non-existence of the above embedding has to be determined by these quantities alone.

Our results in Chapter 5 make this abstract argument concrete by providing systematic, readily verifiable criteria, expressed only in terms of the above quantities, for the (non)-existence of an embedding of the above form. More precisely, our criteria reduce the question of existence of embeddings to a purely combinatorial, discrete calculation for which no knowledge of Fourier analysis is needed.

For concreteness, let us recall the form of our results: The most important quantities are the INTERSECTION SETS

$$I_j := \left\{ i \in I \,\middle|\, Q_i \cap P_j \neq \varnothing \right\},$$

at least if the covering $\mathcal{Q} = (Q_i)_{i\in I} = (T_i Q_i' + b_i)_{i\in I}$ is almost subordinate to $\mathcal{P} = \left(P_j\right)_{j\in J} = (S_j P_j' + c_j)_{j\in J}$. Once these sets have been computed, and if $p_1 \leq p_2$ holds, our results imply an equivalence

$$\begin{aligned}
&\left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j} \right\|_{\ell^{\theta_1}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}} \\
&\lesssim |||\iota||| \\
&\lesssim \left\| \left( \left\| \left( u_i^{-1} \cdot |\det(T_i)|^{\frac{1}{p_1}-\frac{1}{p_2}} \right)_{i\in I_j} \right\|_{\ell^{\theta_2}} \right)_{j\in J} \right\|_{\ell_v^{q_2\cdot(q_1/q_2)'}}
\end{aligned}$$

for different values $\theta_1, \theta_2$, only depending on $p_1, p_2, q_1, q_2$. Here, we have $\theta_1 = \theta_2$ for $p_2 \in (0, 2]$, so that our criteria yield a complete characterization in this range. Even for $p_2 \in (2, \infty]$, our results turn out to be sharp under additional assumptions, which are restrictive, but frequently satisfied, namely if $\mathcal{Q}$ and $u$ are relatively $\mathcal{P}$-moderate.

Thus, the only relevant properties of the coverings $\mathcal{Q}, \mathcal{P}$ are

A. The size/measure of the individual sets $Q_i, P_j$, as encoded by the determinants $|\det(T_i)|$ and $\left|\det\left(S_j\right)\right|$,

B. The *relative* geometry of $\mathcal{Q}, \mathcal{P}$, as encoded by the intersection sets $I_j$.

In particular, the *absolute* position of the sets $Q_i, P_j$ or quantities like $\sup_{x\in Q_i} |x|$ play no role.

The power of this characterization is demonstrated by our results in Chapter 6. The embedding (and scaling) results from [43, 10, 64, 49, 53] turn out to be special cases of our general criteria, which often yield stronger results than those previously known.

Furthermore, we are able to handle the case $(p_1, q_1) \neq (p_2, q_2)$, which to my knowledge has only been considered for special cases and then only for $\mathcal{Q} = \mathcal{P}$. Finally, our approach is general enough to handle the case $\mathcal{O} \neq \mathcal{O}'$, in which some of the intersection sets $I_j$ may be infinite. Applications of this include our embedding results

between homogeneous and inhomogeneous Besov spaces, but also between shearlet coorbit spaces and Besov spaces.

Last, but not least, our systematic method has the advantage of being less error-prone than the ad hoc approach as used before. In fact, we saw in Chapter 6 that parts of the results in [10, 49, 53] are actually incorrect. Hence, it might be useful to check other existing results using our new approach.

OUTLOOK As seen in Chapter 6, our method for studying embeddings between decomposition spaces is applicable in many concrete cases and often yields sharp results or even complete characterizations. Nevertheless,

A. The most general setting handled in this thesis is the case in which $\mathcal{O} \cap \mathcal{O}'$ can be written as $\mathcal{O} \cap \mathcal{O}' = A \cup B$, where $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ "near $A$" and vice versa "near $B$". It would be desirable to have even more general embedding results, in particular since the coverings induced by different dilation groups $H$ often have very different geometries, so that no such sets $A, B$ can be found.

   Of course, in most cases, it is possible to choose a *common refinement* $\mathcal{R}$ which is almost subordinate to $\mathcal{Q}$ and to $\mathcal{P}$ and then to compose the embeddings

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell^{q_1}_u\right) \hookrightarrow \mathcal{D}\left(\mathcal{R}, L^{p_3}, \ell^{q_3}_w\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell^{q_2}_v\right),$$

   but in general this will lose sharpness of the embedding results.

B. If the covering $\mathcal{Q}$ or the weight $u$ are not relatively $\mathcal{P}$-moderate, our results only characterize the existence of an embedding for certain ranges of $p_1, p_2$. Outside of this range, the results in Chapter 5 still provide sufficient conditions, as well as necessary conditions, but there is a gap between the two.

   It would be worthwile to study whether our sufficient conditions are also necessary for existence of an embedding in these cases. Proving this necessity would probably need a new idea for "testing" the embedding. On the other hand, in order to *disprove* necessity, one would need a different way of proving existence of an embedding, at least in some concrete special case. Another approach would be to explicitly compute the operator norm of the isotropic dilation on the $\alpha$-modulation spaces for the critical range $\lambda > 1$, $s = s_1$ and $p \in (1, 2)$, or $\lambda < 1$, $s = s_0$ and $p > 2$, cf. Corollary 6.5.13.

Apart from these open problems which pertain to the class of Fourier-side decomposition spaces as defined in Chapter 3, there are still different classes of smoothness spaces for which a similar embedding theory would be useful: The decomposition spaces considered

in this thesis belong to the class of BESOV-LIPSCHITZ TYPE SPACES. Another important class of smoothness spaces is given by TRIEBEL-LIZORKIN TYPE SPACES. Here, the (quasi)-norm

$$\left\| \left( \left\| \mathcal{F}^{-1}\left(\varphi_i f\right) \right\|_{L^p} \right)_{i\in I} \right\|_{\ell^q_u}$$

is replaced by a quasi-norm of the form

$$\left\| x \mapsto \left\| \left( \left[ \mathcal{F}^{-1}\left(\varphi_i f\right) \right](x) \right)_{i\in I} \right\|_{\ell^q_u} \right\|_{L^p\left(\mathbb{R}^d\right)},$$

i.e. the "continuous" and "discrete" parts of the norm change places. General embedding results for this class of spaces would be interesting, but probably harder to develop. In fact, even well-definedness of these spaces is non-trivial: In [63], only coverings consisting of (higher-dimensional) rectangles with sides parallel to the coordinate axes are considered. Without this assumption, or at least without assuming that the sides of the rectangles are parallel to some system of orthogonal coordinates, the proof of well-definedness given in [63] is no longer applicable, since it is based on the sequence-valued version of the HARDY-LITTLEWOOD MAXIMAL THEOREM, cf. [63, Section 3.6]. At least for certain types of coverings, one can nevertheless prove well-definedness: For example in [6], Borup and Nielsen use estimates for *parabolic maximal functions* to show well-definedness of Triebel-Lizorkin type spaces which are defined using a covering generated by a one-parameter group of dilations. Nevertheless, this type of covering is much less general than the class of (semi) structured admissible coverings.

Finally, an important property of modulation spaces is that they are suitable for the analysis of PSEUDODIFFERENTIAL OPERATORS. An overview of such results is given in [46]; original publications include[47, 9, 52]. A typical result is that a pseudodifferential operator with symbol in a certain modulation space, e.g. $\sigma \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$, is bounded on certain (different) modulation spaces, in the present case for example on all spaces $M^{p,q}\left(\mathbb{R}^d\right)$, cf. [45, Theorem 14.5.2]. Other results pertain to stronger properties than boundedness, for example membership in certain SCHATTEN CLASSES.

Boundedness results have also been obtained for the class of $\alpha$-modulation spaces, cf. [4]. It should be mentioned, however, that the symbol class considered in [4] is *not* a $2d$-dimensional ($\alpha$-)modulation space, but the usual Hörmander class $S^b_{\varrho,0}$.

In this thesis, we studied the simplest of all nontrivial pseudodifferential operators, the identity operator. The natural next step would be to consider Fourier multipliers (for this, see also [27, Theorem 2.11]) and more general pseudodifferential operators. An important special case are partial differential operators with constant coefficients. One

of the questions here is whether the symbol class should be some kind of decomposition space (as in the case of modulation spaces) or rather a "classical" symbol class as in the case of $\alpha$-modulation spaces. These questions could be possible topics for future research.

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