Nonlinear Dynamic Behaviour of Joined Lightweight Structures
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Von der Fakultät für Maschinenwesen der Rheinisch-Westfälischen Technischen Hochschule Aachen zur Erlangung des akademischen Grades eines Doktors der Ingenieurwissenschaften genehmigte Dissertation

vorgelegt von
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von

Souheib Abdul Karim
Dedication

To my parents

Father and Mother

Who taught me that even the largest task can be accomplished if it is done one step at a time.
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Abstract

This study is primarily focused on investigating the linear and nonlinear behaviour of beams and plates under dynamic loads. Additionally, it aims to provide a thorough understanding of the interaction between the frictional joint and the joined part. To this end, a new model for the joint has been suggested. It is intended to bridge the gap as shown by the literature review. Further, the model is adjusted to mediate a trade-off between low computation cost and acceptable prediction capability of true physics; portability and integrability features are attributed so they can be implemented easily. Practicality of the model is demonstrated by dynamic analysis. In particular, full time series analysis was carried out twice, one time for a beam and one time for a plate. In both cases, the stick-slip mechanism is allowed to interact with the geometrical nonlinearity of the beam or the plate model on a flexible support.

Furthermore, for validation purposes, results from the plate model were compared to results from literature, experiments and some applicable analytical solutions.

In order to make the results available, various computation software packages have been used parallel with some in-house programs coded by MATLAB. Furthermore, a variety of solution methodologies have been adopted such as finite element, finite difference for spatial state variables discretization and a direct time integration scheme for time domain variables. In order to evaluate the joint parameter impact on the built-up structures, the damping due to stick-slip at the joint is estimated by evaluating the dissipated energy.

The final results show that the suggested lumped model is able to capture some realistic phenomena such as stick-slip and the interaction between the joint and the geometrical nonlinearity of the joined components. Obviously, the model is able to accommodate the damping due to the stick-slip at the joints. Hence, this allows for the optimizing process for joint parameters based on damping and stability conditions.

In conclusion, in addition to the well-known vital role of the joint as a major participant in the system damping, the results also show the contribution of the joint to the overall behaviour. More precisely, the joint suppresses the high peaks of the internal, in-plane forces compared with case of fully clamped boundary conditions.
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1 Introduction

1.1 Motivation

Despite the development that has been witnessed recently in composite material technology, metallic components are still preferable in many sensitive applications nowadays. Specifically, metal beams and plates are widely used in various types of machine construction in general and lightweight vehicles such as ships, cars, railways and aeroplanes in particular.

For logistic and technical reasons, the manufacturing process of any machine starts by machining relatively small and simple parts and then assembling them during the last stage. Assembly, in simple terms, is essentially a linking process in which two (or more) components of structural and non-structural elements are joined. Indeed, there are different methods and principles for joining. These joining methods may include welding, brazing, soldering, riveting, bolting or adhesive bonding.

It is a well-known fact that many sheet-metal alloys are not suitable for welding as modification and deformation of material properties can occur. Adhesion bonding materials are also not trusted enough for metals. For these reasons, frictional joints, like bolting and riveting, are the preferable choice for designers.

Joints, based on friction, have been widely used since antiquity and are still predominantly used in very sensitive and modern applications [1]. Many believe that this method will continue to dominate the joining process in the coming years [2].

To illustrate, the knife-shaped stone, joined to a stick (see Figure 1.1) and used as a tool by prehistoric humans, is a good example of the role of friction in joining process.

![Figure 1.1: A prehistoric joint based on friction [1]](image)

Furthermore, the Airbus A380 is just one, very modern example, among many others where aluminium panels with stiffeners form the majority of the wing skin (see Figure 1.2) fuselage and tail skins and the riveting or bolting are adopted for joining process (see Figure 1.3) [3].
Besides practical and safety reasons, the recently discovered fact of being able to dissipate energy is a further important reason to explain why this method of joining is still extensively used, even in very sensitive engineering applications. For instance, in

In 1917, Wilson and Moore [5] were the first who explored the damping in joints due to friction.

Recently, it has also been emphasised that joints are an important source of damping for a built-up structure. The joint contribution has been estimated at about 90% of the overall damping [6]. On the other hand, it has been demonstrated that joints are a dangerous source of crack initiation caused by fatigue and fretting wear [7]. Actually, the effect of dry friction is not only on a damping level; dry friction in joints has a significant effect on the overall dynamic behaviour of built-up structures [8].

The classical design procedure of a riveted or bolted joint relies upon estimating the direct strength of the rivet shank or the bolt stem. This, the oversimplified procedure, may be understood by means of the following reasons: the lack of information surrounding the internal force, the lack of existence of a trusted methodology to estimate the strength based on friction and the type of nonlinearity and the uncertainty. Altogether, these make the prediction of the friction both difficult and poor, if not impossible. However, structural engineers always take the safe side and rely on the classical methods.

The principle of such joints depends on creating a normal clamping force between the connected parts, by the rivet-shank or the bolt-stem pre-tension. As a result, a tangential force, proportional to that force and the friction coefficient, is created to hold the connected components together. However, under dynamic loads, a relative motion between the asperities of the joining surfaces may occur in a stick-slip fashion. The amplitude of the relative sliding motion is often in the order of micrometers, but can also be as low as 3 to 4 nanometers [9].

The disadvantages of this motion are fretting corrosion, the onset of fatigue, noise, the reduction of the structure’s stiffness and less structural strength. Conversely, it has an advantage as an important source of energy dissipation. This is mainly due to the fact that metallic components have a very low material damping and other sources of damping are negligible. However, the role of damping is very important in any transport system; passenger relief is just one of many examples. On the other hand, it makes a significant contribution to defining the dynamic characteristic of structures not only on a local level but also on the dynamic behaviour of the complete built-up structure level [9], [10], [11].

The question remains; Why is damping so important? As previously mentioned, it is not only important for reasons of noise reduction and passenger relief but also to avoid possible catastrophes and disasters. For clarity, a description of the single-degree-of-freedom system is useful. The response in the resonance range is mainly determined by the damping terms. In other words, the response of any system could go to infinity
without damping. Fortunately, all realistic systems have some amount of inherent damping.

Additionally, what is the stick-slip? Stick-slip refers to the alternation between two cases of motion or stopping over a period of time. People in different fields have different attitudes toward this phenomenon. For example, the musician tunes his/her instruments to initiate such phenomenon to generate music using e.g. string instruments like the violin. The strings are sounded by drawing the hair of the bow across them; the stick-slip phenomenon occurs between the two strings. Thus, the strings are vibrated to give the wanted sound. Conversely, researchers in the tire industry constantly improve the profile and the rubber quality of the tires to provide good sticking conditions for improved car safety features. Sometimes the phenomenon is out of human control. To mention just one example, earthquakes are attributed to the slip that occurs in the internal plates of the earth. Civil engineers, on the other hand, develop different techniques to provide passive damping mechanisms on newly designed buildings by relying on slips and dry friction elements.

Over the past century, researchers in the field of elasticity have been concerned with carefully investigating frictional joints in beams and plates and the associated phenomena. In particular, nonlinearity problems associated with such structures have been treated differently. In reality, all of the natural systems behave nonlinearly [9]. However, many nonlinear structural problems have been solved by simply adopting linearization techniques, which mainly depend on assuming that deflection is small. For instance, in civil engineering, large deflection in buildings is not permitted. Therefore, it is kept as small as possible within the design conditions as it is considered negligible when compared to the thickness of the structures.

The contact region is also a significant source of nonlinearity, particularly under dynamic loads where the stick-slip mechanism is inevitable either on the micro-scale or macro-scale level. This mechanism stimulates additional nonlinearity in the system. Also, it is a source of energy dissipation as it converts the stored energy into thermal energy. This is considered an important source of passive damping [11].

Interesting physical phenomena occur in systems in the presence of different types of nonlinearities. In addition to the previously mentioned phenomena, resonance, self-excited oscillation, bifurcation and chaos are also various types of reported phenomena.

In order to interpret such physical phenomena, great efforts have been directed toward the development of a phenomenological model capable of predicting the behaviour of the joined structure under various types of static and dynamic conditions.

Nevertheless, most of commercial modelling tools have extremely limited capability to model joints and interfaces. Geometrical nonlinearity and non-smooth nonlinearity associated with such combinations have also been addressed as unsolved research areas [2].
All these issues motivate this paper, in an attempt to contribute to solving the discussed problems. It focuses particularly on introducing a simple mathematical model for the joined structure that can be integrated into a full-scale analysis. This model is intended to respond to the dynamic loads and to be capable of accommodating the physical phenomenon discussed above with affordable computational effort.

1.2 Literature Review

This chapter is intended to present the literature review that was carried out. Endless lists of publications were found in the area of beam and plate vibrations and frictional joints. As it is not viable to list everything, only the necessary and relevant articles and books have been included.

However, for recent and more detailed reviews concerning the current subject, interested readers are referred to Feeny et al. [1], Berger [6], Ferri, [7], Ibrahim [9], Stein et al. [10], Ibrahim et al. [11], Sudhakar [12]. In 1998, Feeny et al. [1] made a comprehensive historical review of structural and mechanical systems with friction. They contained 304 references. Feeny traced several examples of using friction as a means to connect two different pieces from the earliest prehistoric technologies and introduced systems with friction oscillators and stick-slip.

In the same way, Berger [6] reviewed about 196 references from diverse engineering fields and science disciplines searching the most appropriate way to model friction. He discussed, amongst other issues, stick-slip, friction model and partial slip issues extensively. He concluded in his survey that the system model and the friction model are fundamentally coupled, therefore, they cannot be chosen independently.

Similarly, Ferri [7] surveyed the literature on the use of dry friction in passive damping and vibration isolation. Ferri also discussed several analytical techniques to predict, measure and/or enhance energy dissipation and vibration isolation properties of dry friction in order to improve system performance. He concluded that friction damping will continue to play an important role in many mechanical and structural systems.


Stein et al. [10] reviewed analysis and simulation of a single-degree-of-freedom system with a dry friction element. They compared two models of friction and concluded that the one using the physical and correct stick-slip approach describes the reality better than the simpler signum approach.

On the other hand, Ibrahim et al. [11] introduced a comprehensive review article focused on structural dynamics with bolted joints. They included as many as 516 references and concluded that the behaviour of the joints affects the overall dynamic characteristics of the built-up structures.
In comparison, Sudhakar [12] conducted a literature review with the aim of tracing the development of the nonlinear vibration equation of beams. Sudhakar divided the evolution into four marked phases, namely: earlier continuum solution, debate and discussion, variation correct formulation and application.

However, it is reported that most of the recent studies in the fields of joint investigation have focused on [11]:

1. Energy dissipation of riveted or bolted joints.
2. Identification of the dynamic characteristic of the joints.
3. Relaxation and parameter uncertainties.
4. Preload active control.
5. Fretting wear due to slips at joints.
6. Cracks nucleation due to fatigue at joints.

In the following, for the sake of simplicity, the literature review has been divided into four different parts. The first part presents a brief review of historical development and the most significant contribution in the field of elasticity theory since the eighteenth century. For a comprehensive historical development of plate theory, interested readers are referred to Timoshenko’s textbook [13] and Truesdell’s textbook [14].

The second part is devoted to the contributions in the area of modelling frictional joints. The third part includes publications which are devoted to the stick-slip phenomenon, while the last part of the literature review primarily focuses on publication in the area of damping associated with joints.

### 1.2.1 Elasticity Theory Evolution

Previously, extensive studies were carried in the area of plate bending theory and its application by exceptional scientists such as Euler, Chladni, Bernoulli, Germain, Cauchy, Levy, Timoshenko and many others.

It is worthy to mention their contribution.

The first plate model was attributed to the Swiss mathematician and physicist, Leonhard Euler in 1776. He introduced free vibration analysis of a plate [15]. Some years later, the German physicist and musician, Ernst Florens Friedrich Chladni [16] discovered the various modes of free vibrations by acoustic experiments. He invented a technique to show the various modes of vibration on a mechanical surface. His technique consisted of drawing a bow over a sheet of metal whose surface was uniformly covered by fine sand. The sheet was induced until it reached resonance. The sand accumulated along the nodal lines and formed a pattern showing the nodal regions.

Accordingly, Bernoulli [17] tried to interpret the results of Chladni’s experiment mathematically. He assumed the plate as a group of parallel and perpendicular strips;
every strip works as a beam. The resulted governing differential equation did not contain warping term as it is known in the present equation. Moreover, it is worth mentioning that Bernoulli also introduced the principle of virtual work in 1771.

In 1816, the French mathematician Marie-Sophie Germain was awarded a prize by the Parisian Academy for developing a plate differential equation [18]. Joseph-Louis Lagrange corrected Germain’s equations in 1813 by adding the missing warping term [19].

Augustin-Louis Cauchy [20] and Siméon Denis Poisson [21] also obtained the well-known Germain-Lagrange equation of plate deflection. They were the first to derive the plate equation by expanding it in power series and only the terms of the first order were taken. The solution of the Germain–Lagrange plate equation under static loading was first successfully expanded by Poisson in 1829. However, in his solution, the rigidity D was assumed to be a constant term.

Claude-Louis Navier [22] introduced a satisfactory solution by considering the plate thickness as a function of rigidity. He also transformed the differential equation into algebraic form by using the Fourier trigonometric series.

In 1850, the German physicist Gustav Robert Kirchhoff [23] contributed to the fundamental understanding of elasticity theory. His contributions are now widely known and accepted in the plate bending theory. Kirchhoff simplified the three degree elasticity theory for a bent plate by introducing two assumptions which are also known as Kirchhoff’s hypotheses (1- The stress normal to the plate is zero. 2- The lateral plate deflection is not function of the normal coordinate). Additionally, he introduced the natural frequency equation of plates and the virtual displacement method.

William Thomson Kelvin and Peter Guthrie Tait [24] supplied further insight relative to the boundary conditions by changing twisting moments along the plate boundary to shearing forces. Consequently, shear and moment are applied over the boundaries.

Rudolf Friedrich Alfred Clebsch [25] translated Kirchhoff’s book. Numerous valuable comments made by de Saint-Venant were included in that translation. Saint-Venant extended Kirchhoff’s equation of thin plates and combined the action of bending and stretching in a mathematically correct manner. Also, Saint-Venant indicated that the proposed Cauchy and Poissons series are divergent.

In the late nineteenth century, Levy [26] solved successfully rectangular plates with two simple supports and the other two arbitrary.

Alexei Nikolaevich Krylov [27] and his student Ivan Grigoryevich Bubnov [28] made a significant contribution to the thin plate theory with flexural and extensional rigidities. Bubnov placed the foundation for the theory of flexible plates. He proposed a new method for integration of differential equations of elasticity and tabulated maximum deflections and maximum bending moments for plates of different properties. Later, Galerkin applied this method to plate bending problem. Galerkin [29] gathered many bending problems for plates of arbitrary form in a textbook.
Stephen Timoshenko also contributed extensively to the theory and application of plate bending analysis. The formulation of elastic stability problems and the solution for circular plates with large deflections are among his numerous important contributions [30], [31]. A fundamental monograph that represented an insightful analysis of different plate bending problem was introduced by Timoshenko and Woinowsky-Krieger [32].

Furthermore, Hencky [33], Huber [34], von Karman [35], [36], Nadai [37], Föppl [38] carried out extensive studies in the fields of plate bending theory.

Hencky [33] also contributed to the general theory of elastic and stability of thin plates and the theory of large deflections.

Nadai [37] checked the accuracy of Kirchhoff’s plate theory by extensive theoretical and experimental investigations. He analysed various types of singularities in plates due to the effect of point support and concentrated force application. Föppl simplified the general equations of large deflections of very thin plates. He used the stress function in the middle plane of the plate.

In 1910, von Karman developed the final form of the differential equation of the large-deflection theory or geometrical nonlinear plate theory. He also studied the post buckling of plates.

An approximation of orthotropic plates and the solution for plates subjected to nonsymmetrical distributed loads and edge moments were developed by Huber. Gehring [39] and Boussinesq [40] developed the foundation of the general theory of anisotropic plates. Lekhnitskii [41] contributed an essential development to the theory and the application of anisotropic linear and nonlinear plate analysis. He also presented the method of complex variables as applied to the analysis of anisotropic plates.

The deformation due to transverse shear is included by Reissner [42]. In 1951, Mindlin took into account influence of rotatory inertia and shear deformation in conjunction with a shear correction factor [43]. Work of Panov [44] were dedicated to the solving of nonlinear plate bending problems.

Navier [22] derived the governing equation of a thin rectangular plate subjected to direct compressive forces. Bryan [45] solved the buckling problem for a simply supported plate under direct and constant compressive forces acting in one and two directions.

Solutions of various buckling problems for thin rectangular plates in compression were introduced by Cox [46], Hartmann [47], etc. The buckling problem for circular compressed plates was completed by Dinnik [48], Nadai [49], Meissner [50], etc. Southwell and Skan [51] were the first who studied the effect of the direct shear force on the buckling of a simply supported rectangular plate.
Timoshenko and Gere [52], Bubnov [28] studied the buckling behaviour of a rectangular plate under non-uniform direct compressive forces. Karman et al. [53], Levy [54], Marguerre [55], etc also studied the post buckling behaviour of plates of different shapes.

In addition, Gerard and Becker [56], Volmir [57], Cox [58], etc. presented a comprehensive analysis of linear and nonlinear buckling for thin plates of different shapes under different types of forces, in addition to numerous available results for critical loads and buckling modes which can be used in design.

Voight [59] achieved the first exact solution of the free vibration problem for the two-opposite-sides of simply supported rectangular plates.

The upper bounds on vibration frequencies were obtained by Ritz [60], who extended the Rayleigh principle. Forced and free vibration problem of plates of several shapes were analysed in the monographs written by Timoshenko and Young [61], Den Hartog [62], Thompson [63], etc.

In his textbook, Leissa [64] offered a considerable set of available results for the frequencies and mode shapes of free vibrations of plates.

Timoshenko et al. [65], [66] presented an introduction to elasticity with theory and applications in their textbooks.

Other researchers have focused on the modelling applied to the interaction between aerodynamic forces and elasticity issues such as Vaicaitis, Dowell, etc.

More specifically, Vaicaitis et al. [67] investigated the nonlinear panel response from a turbulent boundary layer. Dowell [68] made a significant contribution to the field of aero-elasticity of plates and shells.

Researchers such as Petrov, Amabili, Nagai, Pradyumma, Abramovich and many others have examined the different nonlinearities with different solution methodologies.

Abramovich et al. [69] explored the influence of rotary inertia and shear deformation on the natural frequencies, `which was introduced by W. Flügge in 1942, of a cantilever beam with rotational and translational springs and a tip mass.

For instance, Petrov et al. [70] investigated the nonlinear theory of plates and shells by the method of successive loadings. Conversely, Amabili reduced the order of the equations by using approximate functions.

Amabili [71] examined the geometrically nonlinear plate with three different boundary conditions, numerically and experimentally. The following boundary conditions were included: simply-supported with immovable edges, simply-supported with movable edges and fully clamped plates. In addition, the imperfection of the plate was also taken into account.
On the contrary, Pradyumma et al. [72] carried out a nonlinear transient analysis of functionally graded shell panels using a higher-order finite element formulation.

Nagai et al. [73] conducted experiments on the chaotic vibrations of a post-buckling beam with an axial elastic constraint. Poincaré’s map methodology is used to examine the chaotic response due to the nonlinearities.

### 1.2.2 Frictional Joints

The recent trend in joint analysis is characterized by a heavy reliance on modern high-speed computers and the development of complete computer-oriented numerical methods. The jump that has been witnessed in the last two decades in the computational capability has encouraged researchers to create very detailed three dimension (3D) solid models for joints. Publications in this area include Ganapathy et al. [74], Fung et al. [75], Jiang et al. [76], Langrand et al. [77], Langrand et al. [78], Langrand et al. [79], Simsons et al. [80], Gaul [81] and Thoppul [82].

Ganapathy et al. [74] modelled typical lap joints using 3D shell elements as well as 2D plane strain quadrilateral finite elements focusing on crack nucleation at the location of the rivet and fretting fatigue.

Fung et al. [75] studied single lap joints using experimental and numerical analysis. The lap joint was modelled by FEM as a stepped plate as shown in Figure 1.4. The full 3D model was used to compare its results with the failure results that were achieved experimentally by fatigue tests and examined metallurgically. He showed the local effect of the clamping force - as the force is increased, the local stress concentration factor increases.

Jiang et al. [76] explored the mechanism of the early stage self-loosening of a bolted joint under dynamic load using a 3D elastic-plastic finite element model in addition to experimental works. Jiang found that the reduction of the clamping force, which varied from 10 % to 40 % of the initial preload, happens after 200 load cycles. Additionally, he concluded that the slip between clamped parts was the main reason for loosening.

Langrand et al. [77] modelled the riveted joint using a 3D-solid finite element model (see Figure 1.5) in order to replace and limit the ARCAN experimental procedure for the assessment of dynamic strength. The results are used to identify the macroscopic failure criteria within the framework of aeronautical crashes.

Similarly, Langrand et al. [78] developed a 3D-solid model focused on a crash loading condition and the riveting process, considering the material nonlinearity.
Langrand et al. [79] introduced a numerical methodology to figure out an equivalent joint element based on a finite-element-database. He presented a 3D-solid detailed finite element model as shown in Figure 1.6 and simulated the elementary riveted joint to predict and to analyse the post-riveting and joint failure.

Simsons et al. [80] described a special 3D finite element model for large-deflection analysis of contact problems with slip capabilities.

Gaul [81] investigated the effects of joints using numerical and experimental investigation tools. He then created a detailed finite element model to simulate the
stick-slip mechanism. He analysed a space platform using a lumped parameter model (Valanis element) with 2D beam elements. In his results, the response was decaying due to damping.

On the other hand, Thoppul et al. [82] studied the effect of the preload and the external static and dynamic load on bolt load relaxation in carbon/epoxy composite bolted joints using a 3D-solid element model as shown in Figure 1.7. He found that increasing the preload, the bolt load relaxation decreases.

![Figure 1.7: Bolted joint using 3D-solid model [82]](image)

Ganapathy et al. [83] modelled a riveted lap joint using a finite element method by implementing contact and dry friction elements with emphasis on fretting and crack nucleation mechanism.

Detailed finite element models are important to predict the local behaviour of individual joints. However, it was found that these models have the disadvantage of being computationally expensive. Further detailed models are, therefore, not useful for the prediction of the joint effects on the overall dynamic behaviour of a built-up structure. Hence, capturing the interaction between the joined element and the joint is not possible because only one joint can be modelled without integration capability within the structural model. Therefore, a reduced-order model for joints is preferred and still needed [2].

Dohner et al. [2], in their papers concerning the modelling development of joints, addressed the problem of the lack of a nonlinear reduced-order model of joints capable of predicing true physics.

Other joint analysis techniques rely on the estimation of the joint parameter by experimental and/or numerical procedures such as those in Song et al. [8], Kim et al. [84], Hanss et al. [85], Shiryayev et al. [86] and Jalali et al. [87].

Song et al. [8] investigated the dynamic behaviour of a beam with a bolted joint using an adjusted Iwan beam model, which consists of springs and frictional sliders. The joint model, as shown in Figure 1.8, included 6 parameters which had to be identified.
A multi-layer feed forward neural network was implemented to determine these parameters.

Figure 1.8: Adjusted Iwan model [8]

Kim et al. [84] developed a method to identify nonlinear joint parameters from measured Frequency Response Function (FRF) using force-state mapping method in frequency domain. The results were tested by comparing with a three-degree-of-freedom system having nonlinear joints.

Hanss et al. [85] modelled two rods, connected by a spring and damper with fuzzy-valued parameters, as a representation for a bolted joint as shown in Figure 1.9. He indentified the joint parameter based on experiments.

Figure 1.9: A model with two parameters of a joint by Hanss [85]

Shiryayev et al. [86] investigated bolted joints by adjusting the Iwan model. The model parameter estimation approach was involved by employing data from oscillatory forcing approach.

Furthermore, Jalali et al. [87] modelled the lap joint using a lateral spring and a rotational one. The joint parameter was identified by experiments using the force-state mapping method in the time domain.

A combination of various types of spring elements is also used in the modelling process of a frictional support both numerically and/or analytically. For instance, Abbas et al. [88] modelled the root fixture by rotational and translational springs as
shown in Figure 1.10 and compared the experimental vibration characteristics of cantilever beams with results obtained from the numerical model.

![Rotational and translational spring model](image)

Figure 1.10: Rotational and translational spring model [88]

In comparison, Abbas [89] analyzed the Timoshenko beam with flexible supports; Knudsen et al. [90] included flexible support with a gap in his model and studied the wear of loosely supported rods due to a harmonic load.

On the contrary, Bedair et al. [91] idealized the riveted joint using a variable circular cross-section beam connected to two separate layers. Rotational spring elements were additionally included to restrain the rivet head in the x-y plane.

Some researchers included nonlinear elements in the modelling process of frictional joints. For instance, Bindemann et al. [92] modelled the fastening arrangement of a beam end using a nonlinear sleeve joint. Transverse and longitudinal restraint, clearance, dry friction and dissipation through impact were all included in the model.

Other researchers have adopted different strategies to capture the stick-slip phenomenon in joints by adding a dry friction element. For instance, Ferri et al. [93] and Ren et al [94] included dry friction elements at the beam or plate supports. Others included the LuGre friction model as shown in Figure 1.11, like Nguyen et al. [95]. Nguyen investigated the best simulation strategies to dynamic systems with the LuGre friction model.

![Description of the frictional contact in the LuGre model](image)

Figure 1.11: Description of the frictional contact in the LuGre model [95]

Michaux et al. [96] studied a mass-on-moving-belt to investigate how high-frequency excitation affects friction-induced oscillations. The study showed how, in many cases, friction-induced oscillations are suppressed by tangential dither.
Agarwal [97] investigated the semi-active damping by modelling friction in the joint and controlling the normal clamping force.

Pereira et al. [98] modelled the pressure distribution of a bolted joint based on statistical data and proposed the Weibull distribution law. His results were supported and confirmed by experiments.

### 1.2.3 Stick-slip

A group of researchers focused primarily on the stick-slip phenomenon in joints. It is well-known that the meeting surfaces in joints start to slip under a certain level of external out-of-plane and, thereby, internal in-plane excitation. The slip starts on a micro-scale level. As the level of excitation increases, it develops until the macro-slip is nucleated [6]. Therefore, in order to capture such phenomenon, it is necessary to consider the geometrical nonlinearity in beams or plates and to develop a specific type of boundary condition so that the stick-slip mechanism is allowed. Such issues have been investigated differently by Ren et al [94], Dowell [99], Tang et al. [100] and Miles et al. [101].

In 1986, Dowell [99] and Tang et al. [100] studied nonlinear beams and plates and investigated the damping due to slipping at the edges. Dowell obtained a formula for an equivalent linear viscous critical damping. Tang compared numerical results with the approximate analytical solution developed by Dowell.

Later, Miles et al. [101] published a study on a randomly excited beam with axial stick-slip end conditions. Miles pointed out that the main weakness is that the normal clamping force is assumed constant, whereby the tangential friction force is also assumed to remain constant. However, Ren et al [94] investigated the effect of the stick-slip phenomenon on the overall damping properties of a nonlinear vibrating beam with dry friction at the supports. Ren also presented a relationship between the equivalent viscous damping ratio and the transverse displacement amplitude.

Correspondingly, in 2006, Somnay et al. [102] studied an externally excited large deflection beam with two-knife edge supports. The beam in this case has the freedom to slip at its edges. This kind of beams is called Gospodinetic-Frisch-Fay beam.

On the contrary, Miles [101] and Tang et al. [100] considered time-independent normal clamping force. Ferri et al. [93] took into account that the normal clamping force consists of two components: a constant preload force and a linearly variable force which is assumed to be proportional to in-plane slippage. Again Ferri [7] argued that models with a constant normal clamping force are oversimplified due to the fact that systems with constant friction forces exhibit significantly different behaviour than the systems with motion-dependent friction force. Berger [6] also explained the limitation in the prediction of the friction model in cases where the normal force variation is not represented.
Ferri et al. [93] studied a beam with a pinned-end and an in-plane frictional interface on the other end. In their model, they included four sets of spring-damper pairs for nonlinear sleeve joints.

A simple case of the stick-slip phenomenon was studied by Chatelet in 2008. Chatelet et al. [103] studied the influence of macroslip-microslip on the dynamic behaviour of a single-degree-of-freedom system. He also presented a technique for selecting the most suitable contact model and categorized contact models into two groups, namely: rheological models and phenomenological model. The first provides damping and stiffness while the second gives a restoring force.

1.2.4 Damping

Damping effects are produced either by internal friction or external friction [15]. In this thesis, internal friction is out of the scope of the paper. However, theoretical studies of vibration show that damping is negligible and has no effect on fundamental frequencies and the steady state response. Thus, it is acceptable to ignore damping in the preliminary treatment of problems for the sake of simplicity. Nowadays, the new computational capability, environmental assessment for noise and vibration control purposes altogether stimulate researchers to explore various procedures to estimate damping and to find the parameters affecting the damping level.

For instance, Scott et al. [4], Nguyen et al. [104] and many others have used different methods to examine how joint parameter influences the damping level. Scott et al. [4] studied the effect of the joint on the damping level and, in addition, demonstrated how to determine the modal loss factor experimentally. He concluded that the location and the stiffness of the joint both affect the modal loss factor. Nguyen et al. [104] investigated the role of the joint in energy dissipation and energy transfer between modes both numerically and experimentally.

Tita et al. [105] considered damping factors based on experiments and by using the program FREQ. These factors were used to update a finite element model for joined beam using the Rayleigh model.

Lee et al. [106] proposed a method for evaluating the equivalent damping ratio of a structure with various supplementary damping elements, involving the analogy with viscous modal damping.

1.3 Objective of the Study

In order to improve the structural design of highly sensitive machines, it is important to improve the performance prediction of the computation tools. Such improvement requires extensive investigation of the behaviour of basic structural elements. Therefore, this study is generally aimed at understanding the nonlinear behaviour of lightweight structural elements such as beams, plates and joints.
Furthermore, to increase damping due to stick-slip phenomenon, it should be noted that damping is produced by the energy dissipated at the joint. The dissipated energy is equal to the work done by the frictional tangential force and the slippage value. Thus, to increase damping, the force and the slippage distance should both be increased. However, an increase in the normal force causes less slippage. Hence, maximum damping requires optimizing the clamping force. It is also necessary to consider the effect of the clamping force on the dynamic of the complete structures. In conclusion, in order to adjust the joints for best performance, it is essential to model the full interaction between the joint itself and the joined part.

For these reasons, the study will introduce a numerical model to such structures (beam and plate) so as to make it possible to investigate two types of nonlinearity, one due to the change in dimension and one due to contact. It also aims to introduce a procedure to estimate the effect of joints on the damping level of the complete structure.

At the beginning of this thesis, the literature review showed the focus on the modelling of frictional joints, on the numerical, analytical and experimental analysis of linear and nonlinear plates and beams and on the stick-slip in joined structures. However, the review showed that structures like beams, plates and joints are strongly coupled. Therefore, they should be simultaneously analysed.

Furthermore, this thesis is focused on modelling a geometrically nonlinear beam with the capability of considering the stick-slip scenario. It also concentrates on performing a full time series analysis on a nano-second level using the finite element method (FEM) and a direct time-integration algorithm. The model is coded using MATLAB.

The investigation is also extended to focus on a more realistic structure. The previous beam will be replaced by a plate. The plate will then be modelled again using FEM and computed using ANSYS Parametric Design Language (APDL). Further, to simulate the joint, the model is provided with some control elements that have the ability to turn on and off during the step-by-step time analysis, in combination with some spring-elements.

For practical reasons, the model is customized between a reasonable computational cost and the capability of capturing geometrical nonlinearity, non-smooth nonlinearity and the expected stick-slip phenomenon at the same time. To this end, the size of the spatial discretization step is selected based on a modal analysis. The size of the integration time step was based on minimizing the spurious energy loss within the accepted computational cost.

Finally, for validation purposes, a plate vibration experiment has been conducted to measure the plate response under various types of excitations and to compare with the mathematical results, compared, in particular, to the modal analysis results. Additionally, the bifurcation response has been captured to show the onset of the stochastic behaviour. Also, a comparison with an example from the literature has been included.
1.4 Outline of the Thesis

This thesis is structured as follows. In the first chapter, a detailed literature survey related to the thesis topic is presented. This is followed by an explanation of the motivation and the purpose of the thesis.

The second chapter is devoted to the theoretical background and a brief introduction to the solution methodology that was implemented in the research. Within the theoretical background, the formulations of linear and geometrically nonlinear beams and plates are briefly demonstrated in addition to the damping theories and the most common damping measurement methods.

The third chapter is intended to present the investigation of nonlinear dynamic systems. The nonlinearities are introduced for many systems (pendulums, beams and plates). Also, modelling the frictional joints is presented.

The forth chapter is dedicated to present the investigation of joined plates. at the beginning, 3D-solid model for joints is introduced. A reduced-order model is then described. Also the stick-slip phenomenon in joints are described using the simple model in the rest of this chapter.

The fifth chapter presents the conducted experiment. The used fixture and implemented devices are described, followed by the presentation of the results of the tested plate. The results include the modal analysis and the transient response of a plate in addition to an experimental bifurcation graphs.

Finally, the sixth chapter introduces the summary and conclusions.
2 Theoretical Background

This chapter is devoted to the theoretical basis of this thesis. At the beginning, the type of nonlinearity in real life and dry friction in practical applications are explained. For the sake of completeness, the linear three-dimensional elasticity formulations are introduced. After this, the simplification assumptions for reduction to obtain linear beam and linear plate theory are discussed. Next, some well-known solution methodologies are briefly introduced and cited. Although, some other important investigation tools were used in this work, they are not presented here. This is mainly due to the limitation of space. Other important methodologies are also briefly mentioned, such as the finite elements method and the finite difference method. Also, some important tools in the field of signal processing and dynamic system investigation are introduced, such as filters, frequency response function, fast Fourier transformation, phase portrait, bifurcation map etc…

2.1 Types of Nonlinearity

Generally, nonlinearities are classified into three different categories, namely: contact nonlinearity, geometric nonlinearity and material nonlinearity. In the following, a brief description for each type of nonlinearity is presented.

2.1.1 Contact Nonlinearity

Boundary nonlinearity takes place in most contact problems when two bodies come in or out of contact (see Figure 2.1 and Figure 2.2). The displacement and stress fields of the two bodies are usually nonlinearly dependent on the applied loads. This type of nonlinearity may happen even if the material behaviour is assumed to be linear and the displacements are extremely small. This is due to the fact that the size of the contact area is usually nonlinearly dependent on the applied loads. If the effect of friction is involved in the analysis, then a stick-slip phenomenon may take place in the contact region which adds further nonlinearity. These types of nonlinearity are normally dependent on the loading history [107], [108], [109], [110].

![Image of a cantilever beam with a nonlinear contact](image-url)

Figure 2.1: A cantilever beam with a nonlinear contact
2.1.2 Geometrical Nonlinearity

Geometrical nonlinearity occurs when the changes in the geometry of a structure are significant due to displacement components under load. The change in geometry affects both equilibrium and kinematic i.e. strain-displacement relationships. In geometric nonlinearity, the equilibrium equations consider the deformed shape, whereas in linear analysis the equilibrium equations are always based on the original (un-deformed) shape. As a consequence of this, the strain-displacement relationships may have to be re-defined to take into account the current deformed shape [107], [108], [109], [110].

Normally, geometrical nonlinearity is divided into two types. This depends on the effect of the nonlinearity in terms of stiffness. In other words, if the nonlinearity increases the stiffness of the structure, then the nonlinearity is called the stiffening type, whereas the softening type of nonlinearity occurs when the nonlinearity decreases the stiffness [107], [108], [109], [110].

2.1.2.1 Softening Type of Nonlinearity

A typical example of the softening type of nonlinearity is a cantilever beam [107] as shown in Figure 2.3. The relationship between load and displacement is not linear. Furthermore, the displacement increases rapidly as the force is increased.
2.1.2.2 Stiffening Type of Nonlinearity

A typical example of a structure subjected to a stiffening type of nonlinearity is shown in Figure 2.4. The figure shows a beam loaded laterally and the beam ends are prevented from moving axially. At low load, the beam deflects slightly. At this level, the bending resistance takes the role of resisting the external load and the behaviour is linear. As the load is increased, the lateral deflection is increased and the beam curves noticeably. This causes a foreshortening in the beam and an axial strain creates a membrane force which causes an additional lateral resistance. This could dominate the behaviour of the beam and introduce the major resistance to any further lateral deflection at a higher load. In this case, the beam behaves non-linearly and the type of nonlinearity is of the stiffening type. Some references estimate that nonlinearity becomes significant when the lateral deflection becomes greater than one third or half of the beam thickness. That corresponds to von Kármán plate theory. For more information, interested readers are referred to the textbook of Doyle [15].

![Figure 2.4: Example for stiffening type of nonlinearity](image)

2.1.3 Material Nonlinearity

Here, the nonlinearity depends only on the material behaviour under load. There are some materials which are considered linear-elastic (or Hookean material) and some other materials which are nonlinear-elastic (non-Hookean). In more detail, the material is assumed linear when the relationship between stress-strain is linear or confined to Hooke’s law. Conversely, the material is assumed nonlinear when that relationship is nonlinear. However, in simple words, Hooke’s law states that the strain is directly proportional to the stress. In fact, this only holds until a certain limit, called yield strength, beyond which the material behaves nonlinearly. This is the case for most metallic materials like steel and aluminium. Conversely, non-metallic material has very low yield strength or it even does not exist. All types of rubber are considered a good example for nonlinear material (often described as Neo-Hookean)[107], [108], [109], [110].

Many factors can influence a material's stress-strain relationship, including load history, environmental conditions and the amount of time that a load is applied. Accordingly, the material nonlinearity is also divided into three categories, namely:
time-independent behaviour, time-dependent behaviour and viscoelastic/viscoplastic
[107], [108], [109], [110].

1. Time-independent behaviour refers to materials, such as metals, in which the
structure is loaded beyond the yield point. When the strain exceeds the yield
limit the material is deformed irreversibly (plastic deformation), so that some
permanent residual strain will remain after unloading. In this case, the specimen
will unload along a line which is shifted to the right and parallel to the loading
curve as shown in Figure 2.5. The unloading line will intersect the strain axis at
a non-zero point. The loading and unloading curves form a loop which is
sometimes called a “hysteresis loop” or “memory effect”.

2. Time-dependent behaviour refers to materials, such as metals, which creep at
high temperatures.

3. Viscoelastic/viscoplastic behaviour refers to materials in which both the effects
of plasticity and creep are exhibited.

![Figure 2.5: Material nonlinearity](image)

### 2.2 Friction

In practical application, there are many types of friction: dry friction, viscous
resistance in fluids, skin friction and internal friction. Regardless of the friction type
and source, all produce a resisting force to the relative motion between contacting
materials. Dry friction resists relative intended motion or intended motion between two
solids in contact and is divided into two types, namely: static friction and dynamic
friction. Viscous resistance is caused by the viscosity of a fluid and resists the relative
motion between fluid particles. Skin friction resists the motion between a solid and
fluid. This resistance is called drag: the force resisting the motion of a solid body
through a fluid. Additionally, when a solid structural element undergoes deformation,
the internal friction produces a resisting force to deformation [111].

#### 2.2.1 Factors Affecting the Friction between Surfaces

Experimentally it has been found that many parameters affect friction such as
atmospheric dust and humidity, oxide films, surface finish, velocity of sliding, critical
sliding distance, acceleration, temperature, normal load, vibration, extent of contamination, temperature, material combination, surface preparation and many more [6], [9].

At low surface pressures, friction is directly proportional to the pressure between the surfaces. As the pressure increases, the friction factor increases slightly. At very high levels of pressure, the friction factor quickly increases to saturation. Also, at low surface pressures, the coefficient of friction is independent of surface area. At low velocities, friction is independent of the relative surface velocity. At higher velocities, the coefficient of friction decreases [1], [6].

2.2.2 Static Coefficient of Friction

The static friction coefficient, $\mu$, between two solid surfaces is defined as the ratio of the tangential force, $F$, required to produce sliding divided by the normal force, $N$, between the surfaces

$$\mu = \frac{F}{N} \quad (2.1)$$

If a body is placed on an inclined flat surface, the frictional tangential force between that body and the surface prevent the body from sliding down. Afterwards, if the angle of the surface is increased until a certain angle, the body begins to slide down. At that limit, the tangent of the angle equals the coefficient of friction [1], [6].

The only way to obtain a sensible value for the friction coefficient is to conduct an experiment. The values available in the literature should, as such, be used with care, because they are just an approximation.

2.2.3 Dynamic Coefficient of Friction

The dynamic frictional force is normally different to the static frictional force. This is due to the fact that the coefficient of friction during motion is not the same as it is before motion. Nonetheless, the dynamic coefficient of friction is also expressed using the same formula as the static coefficient. Generally, the static coefficient of friction is slightly higher than the dynamic coefficient [1], [6].

2.3 Elasticity Theory

The general elasticity theory concerns the relation between loads, stresses, strains and displacements. To summarize, the relations between these physical quantities are illustrated by a flowchart as shown in Figure 2.6. Also, the illustration shows the type of relation between them [112].
Within the next paragraph, different relations are shown in terms of mathematical formulas for general three-dimensional problems.

### 2.3.1 General Strain-Displacements Equations (Geometric-Logic)

It can be shown that the strain components are related to the displacement components by the following equations [113], [15], [114], [112] (see Figure 2.7)

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]
\]  

(2.2)

Figure 2.6: Loads, stresses, strains and displacements relation

Figure 2.7: The coordinate system and an infinitesimal 3D-material element
where $u$, $v$ and $w$ are the displacement components in the $x$, $y$ and $z$ directions, respectively.

### 2.3.2 Constitutive Equations (Stress-Strain Relation)

Using Hooke’s law in the linear elastic range, for a three-dimensional isotropic case, the relation between stresses and strains is given as follows [113], [15], [114], [112]

\[
i_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y - \nu \sigma_z \right) \tag{2.8}
\]

\[
i_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x - \nu \sigma_z \right) \tag{2.9}
\]

\[
i_z = \frac{1}{E} \left( \sigma_z - \nu \sigma_x - \nu \sigma_y \right) \tag{2.10}
\]

\[
\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{2.11}
\]

\[
\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{2.12}
\]

\[
\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{2.13}
\]

\[
G = \frac{E}{2(1+\nu)} \tag{2.14}
\]

where $\nu$, $E$ and $G$ are Poisson’s ratio, Young’s modulus and the shear modulus, respectively.
2.3.3 Equilibrium Equations (Physical Law)

The body forces $B_x$, $B_y$ and $B_z$ are in balance with the internal forces as follows [113], [15], [114], [112] (see Figure 2.8)

![Stress components and body forces on an infinitesimal solid element](image)

Figure 2.8: Stress components and body forces on an infinitesimal solid element

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + B_x = 0 \tag{2.15}
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0 \tag{2.16}
\]

\[
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} + B_z = 0 \tag{2.17}
\]

2.3.4 Compatibility Equations

For a detailed derivation for these equations, interested readers are referred to textbooks [65], [115]. However, the compatibility equations of a body express the continuity of that body. These equations relate the six strain components to the three displacement components as follows [113], [15], [114], [112]
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\[ \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_x}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \]  
(2.18)

\[ \frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_y}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \]  
(2.19)

\[ \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \]  
(2.20)

\[ \frac{\partial}{\partial z} \left[ \frac{\partial \gamma_{xy}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right] = 2 \frac{\partial^2 e_x}{\partial x \partial y} \]  
(2.21)

\[ \frac{\partial}{\partial x} \left[ \frac{\partial \gamma_{yz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} \right] = 2 \frac{\partial^2 e_y}{\partial y \partial z} \]  
(2.22)

\[ \frac{\partial}{\partial y} \left[ \frac{\partial \gamma_{xz}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} \right] = 2 \frac{\partial^2 e_z}{\partial x \partial z} \]  
(2.23)

### 2.4 Beam Theories

#### 2.4.1 Strain-Displacement Equations

Consider the infinitesimal beam element in Figure 2.9, \( u \) and \( w \) respectively correspond to the displacement in the x and z directions of the left end of the element as shown. The right end then displaces correspondingly, as indicated. The element length before and after deformation are \( dx \) and \( ds \) respectively [116].

![Figure 2.9: Translation and local rotation in an infinitesimal beam element](image)

By considering the variation in x-direction only, the general strain-displacement relation is reduced to the following form for the nonlinear beam theory.
\[
\varepsilon_i = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]  

(2.24)

The square of a small quantity is negligible, as compared with its first power. Therefore, equation (2.24) can be simplified to

\[
\varepsilon_i \cong \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]  

(2.25)

This equation contains two terms: the first term expresses the axial strain and the second term expresses the rotation. Therefore, it is possible to distinguish between two cases: first, small strain and moderate rotation and secondly, small strain and small rotation.

### 2.4.1.1 Small Strain and Moderate Rotation

Here, the nonlinear term in equation (2.25) and the strain are of the same order. Therefore, the nonlinear term must be retained. Hence, the infinitesimal strain is given by

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]  

(2.26)

The presence of the rotation term gives rise to a coupling between the axial motion and lateral motion.

### 2.4.1.2 Small Strain and Small Rotation

In this case, the strain and rotation are small. Hence, the higher order of rotation is negligible, as compared with its first power. Therefore, the nonlinear term in equation (2.25) is cancelled out and the infinitesimal strain is given by

\[
\varepsilon_{xx} \cong \frac{\partial u}{\partial x}
\]  

(2.27)

The strain expression in equation (2.27) is used in the linear problem. Here, the length of the infinitesimal element, \(ds\), is approximated by the projection \(dX\) (see Figure 2.9). In this approach, the coupling between the axial and the transverse is ignored.

### 2.4.2 Euler-Bernoulli Beam Theory

Euler-Bernoulli Theory (EBT) is based on the assumption that the plane cross-sections normal to the neutral axis before deformation remains plane and normal to neutral axis during and after deformation. Therefore, the transverse shear-deformation and transverse normal strains can be neglected [112].

Indeed, these approximations are valid only when the beam is slender and made of isotropic material. Conversely, if a thick beam or beam composite-material-made is
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handled or is excited by a high frequency, these assumptions do not hold. Therefore, some correction factors are needed [112].

Igor A. Karnovsky and Olga I. Lebed 2004 [119] wrote a textbook which includes a wide range of elastic theories. Bernoulli-Euler theory uniform beam and analytical solution are demonstrated using a variation of classical boundary conditions such as elastic supports, lumped and rotational masses, elastic linear foundation, compressive and tensile axial loads. Mode shape and eigenvalue for various cases are also tabulated, along with proper sketches clarifying the corresponding boundary condition and the used theory. Bress-Timoshenko theory was also introduced. Two forms of equations that express the Timoshenko theory have been briefly derived. The shear coefficients are given in the table for various cross-sections. However, it is worthy to mention the main difference between the two theories. In the Euler-Bernoulli theory the rotation due to shear is neglected, while in the Timoshenko theory shear force is considered and weighted by a shear coefficient, κ. The main differences between the Euler-Bernoulli and Timoshenko beam theories are shown in the appendix. The equation of motion of Timoshenko beam is presented in two forms as well.

It is not necessary to show the derivation of the EBT, as many textbook have already introduced it. Detailed derivations for the Euler-Bernoulli theory model can be found in [117], [116].

The equation of motion and the Boundary Conditions (BC) are given

\[ \rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = q \]  \hspace{1cm} (2.28)

The Boundary Condition (BC)

\[ \frac{\partial^2 w}{\partial x^2} \delta \left( \frac{\partial w}{\partial x} \right)_{u} = 0 \], \hspace{1cm} (2.29)

\[ \frac{\partial^3 w}{\partial x^3} \delta w \bigg|_{u} = 0 \]

The BC can be translated as follows: w is the displacement, the first derivative \( \partial w / \partial x \) is the slope, the second derivative \( \partial^2 w / \partial x^2 \) is the moment, the third derivative \( \partial^3 w / \partial x^3 \) is the shear. \( \delta w \) is the variation of the displacement.

The equation (2.29) can be satisfied according to one of these four possibilities

1- For hinged end

\[ \frac{\partial^2 w}{\partial x^2} = 0, \; w = 0 \] \hspace{1cm} (2.30)

2- For clamped end
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\[
\frac{\partial^2 w}{\partial x^2} = 0, \quad w = 0
\]  
(2.31)

3- For free end

\[
\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0
\]  
(2.32)

4- For sliding end

\[
\frac{\partial^2 w}{\partial x^2} = 0, \quad w = 0
\]  
(2.33)

Equation (2.28) shows that the external load (the right hand term) is balanced by two internal forces generated by the beam. The first resistance force (the first term on the left-hand) is the inertia force according to Newton’s second law; the second force is the bending force. By solving this equation analytically for clamped-clamped, static uniform distributed load, the maximum deflection is given by

\[
W_{\text{max}} = \frac{q \cdot L^4}{384EI}
\]  
(2.34)

Further solutions for different boundary conditions are found in [118] (page 254).

2.4.3 Geometrically Nonlinear Beam Theory

Geometrical nonlinearity of beams is mostly caused by fixed ends in x and z direction. In order to derive the governing differential equation of motion of a geometrically nonlinear beam, Hamilton’s principle can be used. Applying the balance of forces and moments acting on an infinitesimal beam element can be also used. For detailed derivation, interested readers are referred to [15], [116]. The derivation can be illustrated briefly by inspecting Figure 2.10 and by using the equilibrium of forces and moments on the infinitesimal element.

Figure 2.10: Forces and moments for an infinitesimal beam element.
The sum of the forces in the z-direction is given by [15]

\[ \sum Z = 0 \]  \hspace{1cm} (2.35)

\[ m \frac{d^2w}{dt^2} dx = \left( Q + \frac{\partial Q}{\partial x} dx \right) - Q + qdx + pdx \frac{\partial w}{dx} \]  \hspace{1cm} (2.36)

\[ q + \frac{\partial Q}{\partial x} + p \frac{\partial w}{dx} = m \frac{d^2w}{dt^2} \]  \hspace{1cm} (2.37)

Similarly, the sum of the moments on the y-axis is given by [15]

\[ \sum M = 0 \]  \hspace{1cm} (2.38)

\[ 0 = Qdx + M \left( M + \frac{\partial M}{\partial x} dx \right) + Ndx \frac{\partial w}{dx} \]  \hspace{1cm} (2.39)

\[ Q = \frac{\partial M}{\partial x} + N \frac{\partial w}{dx} \]  \hspace{1cm} (2.40)

\[ Q = \frac{\partial}{\partial x} \left( -EI \frac{\partial^2 w}{dx^2} \right) + N \frac{\partial w}{dx} \]  \hspace{1cm} (2.41)

In addition, the forces equilibrium in the x-direction gives

\[ \sum X = 0 \]  \hspace{1cm} (2.42)

\[ a = 0 \frac{\partial N}{\partial x} = -p \]  \hspace{1cm} (2.43)

Thus, the Partial Differential Equation (PDE) of a geometrically nonlinear beam is given by [15]

\[ m \frac{d^2w}{dt^2} + EI \frac{d^4w}{dx^4} - N \frac{d^2w}{dx^2} = q \]  \hspace{1cm} (2.44)

From this result, it can be seen that a new term appears on the left hand side of the equation. This term adds an additional lateral resistance to those previously mentioned in equation (2.28). The term is resulted by the product of an axial force, \( N \), into a second derivation of displacement. The axial force could be an external load. Or it could be induced by beam stretching (membrane force) which is a result of curvature under lateral deflection. Therefore, this term is a higher power or product of the displacement and it is the only source of nonlinearity in the PDE. Some references refer to this type of nonlinearity as static nonlinearity.

This leads to the fact that beams which are not constrained against axial motion do not develop geometrical nonlinearity if displacements are small compared with the characteristic length of the beam. However, this will be more obvious later, in the numerical results, particularly when the beam end is partially allowed to oscillate in a stick-slip fashion. The axial force \( N \) is then noticeably dropped.
Dowell [68] showed how the lateral motion induces a tension force by considering evaluating the axial strain. This will be discussed in the next paragraph. Correspondingly, the axial deformation and the strain due to curvature are given in [119] (see page 410).

### 2.4.4 Membrane Force Estimation

To show how to estimate the membrane force, consider a beam subjected to a lateral load as shown in Figure 2.11. The membrane force appears when the beam-ends are restricted against the axial movement. Due to the later load, the beam will deflect and the neutral axis will curve, as may be seen in the figure. Due to the restriction of axial degree of freedom, an axial strain, $\varepsilon(x)$, will appear.

The change in length of an infinitesimal beam element of length $dx$ can be estimated according to the strain definition as

$$d\Delta L = \varepsilon(x)dx \quad (2.45)$$

In order to compute the full change in the beam length, $L$, equation (2.45) should be integrated over the beam domain as follows

$$\Delta L = \int_0^L \varepsilon(x)dx \quad (2.46)$$

![Figure 2.11: Membrane force estimation](image)

The length of the infinitesimal beam after the deformation (see the figure) can be estimated by

$$ds^2 = dx^2 + \left(\frac{\partial w}{\partial x} \right)^2 dx \quad (2.47)$$

The length of the curve is given by integrating equation (2.47) over the beam length as follows
The theoretical background involves the analysis of a thin plate under small deflections. The differential equation for the deflection $w(x)$ is given by

$$s = \int_0^L \sqrt{1 + \left(\frac{dw}{dx}\right)^2} \, dx$$  \hspace{1cm} (2.48)

By using series expansion, the root can be approximated resulting in:

$$s = L + \frac{1}{2} \int_0^L \left(\frac{dw}{dx}\right)^2 \, dx$$  \hspace{1cm} (2.49)

The average axial strain can be calculated by

$$\varepsilon_{av} = \frac{s - L}{L}$$  \hspace{1cm} (2.50)

Inserting equation (2.49) in equation (2.50) yields

$$\varepsilon_{av} = \frac{1}{2L} \int_0^L \left(\frac{dw}{dx}\right)^2 \, dx$$  \hspace{1cm} (2.51)

Thereby, the membrane force according to Hooke’s law is given by

$$N = EA \varepsilon_{av}$$  \hspace{1cm} (2.52)

yields

$$N = \frac{EA}{2L} \int_0^L \left(\frac{dw}{dx}\right)^2 \, dx$$  \hspace{1cm} (2.53)

where $E$ and $A$ are Young’s modulus and the area of the cross-section, respectively.

### 2.5 Thin Plate Theories

#### 2.5.1 Small-Deflection Plate Theory

This theory is based on Kirchhoff’s hypothesis. These assumptions lead to the reduction of the three-dimensional equations to two-dimensional ones.

Thus, the small-deflection theory is valid within the following assumptions:

1. The material is assumed elastic, isotropic and homogeneous.
2. The plate is flat before the loading.
3. The deflection of the mid-plane is small compared with the thickness of the plate.
4. Straight lines, normal to the mid-plane, remain straight and normal during the deformation and the lengths of these lines are not altered.
5. The normal component of the stresses is negligible compared with others in stress-strain relations.
6- The middle surface of the plate remains unstrained after bending. From these assumptions, it is possible to reduce the stress components to two-dimensional state of stress

\[ \sigma_z = 0, \ \tau_{xz} = 0, \ \tau_{yz} = 0 \]  

(2.54)

And, also, the two-dimensional state of strain

\[ \varepsilon_z = 0, \ \gamma_{xz} = 0, \ \gamma_{yz} = 0 \]  

(2.55)

2.5.1.1 Strain-Displacement Equations

Integrating strain-displacement equations (2.2), (2.3) and (2.4) and neglecting the second order terms, the following equation is obtained

\[ U = -z \frac{\partial w}{\partial x} + u(x, y) \]  

(2.56)

\[ V = -z \frac{\partial w}{\partial y} + v(x, y) \]  

(2.57)

\[ W = +w(x, y) \]  

(2.58)

Small displacement assumption means that the middle surface remains unstrained after bending. This leads to consider

\[ u(x, y) = 0 \]  

(2.59)

\[ v(x, y) = 0 \]  

(2.60)

Hence, equations (2.56) and (2.57) yield

\[ U = -z \frac{\partial w}{\partial x} \]  

(2.61)

\[ V = -z \frac{\partial w}{\partial y} \]  

(2.62)

\[ W = +w(x, y) \]  

(2.63)

By using these equations, the strain-displacement equation gives

\[ \varepsilon_x = -z \frac{\partial^2 w}{\partial x^2} \]  

(2.64)

\[ \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2} \]  

(2.65)

\[ \gamma_{xy} = -2z \frac{\partial^2 u}{\partial x \partial y} \]  

(2.66)
2.5.1.2 Constitutive Equations (Stress-Strain Relation)

The stress in terms of curvature and deflection are given as follows

\[
\sigma_x = \frac{Ez}{1-\nu^2} \left( \chi_x + \nu \chi_y \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2.67)
\]

\[
\sigma_y = \frac{Ez}{1-\nu^2} \left( \chi_y + \nu \chi_x \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (2.68)
\]

\[
\tau_{xy} = \frac{Ez}{1+\nu} \chi_{xy} = -\frac{Ez}{1+\nu} \frac{\partial^2 w}{\partial x \partial y} \quad (2.69)
\]

The curvatures are defined as follows

\[
\chi_x = \frac{\partial^2 w}{\partial x^2} \quad (2.70)
\]

\[
\chi_y = \frac{\partial^2 w}{\partial y^2} \quad (2.71)
\]

\[
\chi_{xy} = \frac{\partial^2 w}{\partial x \partial y} \quad (2.72)
\]

The stress components are shown in Figure 2.12.

Equivalent forces and moments applied to the middle surface are given by

\[
M_x = \int_{-c}^{c} \sigma_x zdz \quad (2.73)
\]

\[
M_y = \int_{-c}^{c} \sigma_y zdz \quad (2.74)
\]

\[
M_{xy} = \int_{-c}^{c} \sigma_{xy} zdz \quad (2.75)
\]

\[
Q_x = \int_{-c}^{c} \tau_{xy} dz \quad (2.76)
\]

\[
Q_y = \int_{-c}^{c} \tau_{xy} dz \quad (2.77)
\]
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Equivalent forces and moments in terms of deflection are as follows

\[
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2.78)
\]

\[
M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (2.79)
\]

\[
M_{xy} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \quad (2.80)
\]

\[
D = \frac{E h^3}{12(1-\nu^2)} = \frac{2 Ec^3}{3(1-\nu^2)} \quad (2.81)
\]

\[
c = \frac{h}{2} \quad (2.82)
\]

Taking the force summation on the z-axis (see Figure 2.13) yields

\[
\frac{\partial Q_x}{\partial x} \, dx \, dy + \frac{\partial Q_y}{\partial y} \, dx \, dy + q \, dx \, dy = 0, \quad (2.83)
\]

Dividing by \(dx \, dy\)

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \quad (2.84)
\]

Similarly, the moment summation on the x-axis yields

\[
\frac{\partial M_{xy}}{\partial x} \, dx \, dy + \frac{\partial M_y}{\partial y} \, dx \, dy - Q_x \, dx \, dy = 0, \quad (2.85)
\]

or
Theoretical Background

\[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_y = 0, \]  \hspace{1cm} (2.86)

Figure 2.13: Forces and moments on an infinitesimal plate element

In a similar way, the moment summation on the y-axis yields

\[ \frac{\partial M_{yx}}{\partial x} dx dy + \frac{\partial M_{xy}}{\partial y} dx dy - Q_x dx dy = 0, \]  \hspace{1cm} (2.87)

Simplification yields

\[ \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_y = 0, \]  \hspace{1cm} (2.88)

By considering

\[ M_{yx} = M_{xy} \]  \hspace{1cm} (2.89)

The result of simplifying and substituting equations (2.88) and (2.89) into equation (2.84) is

\[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \]  \hspace{1cm} (2.90)
By substituting the moment in terms of deflection from equations (2.78), (2.79) and (2.80) in (2.88) finally yields the governing equation of a thin plate with small deflection, as follows

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} q
\]  

(2.91)

Or in a short form:

\[
\Delta^2 w = \frac{1}{D} q
\]  

(2.92)

where

\[
\Delta^2 (\ ) = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}
\]  

(2.93)

q is the external applied loads.

In case of dynamic problems, the applied loads, q, explicitly include inertia forces in the surface lateral loads according to D’Alambert’s principle.

\[
q(x, y, t) = q - \rho h \frac{\partial^2 w}{\partial t^2}
\]  

(2.94)

where \( \rho \) is the density.

Thus, the differential equation of forced, undamped motion of plates in the linear range has the form

\[
D \Delta^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = q
\]  

(2.95)

where \( \rho \) the mass density of the plate material, and \( h \) is the plate thickness.

### 2.5.2 Large Deflection Thin Plate Bending Theory

The linear plate theory assumptions are valid here, except assumption number six in which the middle surface is assumed unstrained after bending (see paragraph 2.5.1). This means the middle surface of a loaded plate is stretched after bending. More precisely, assumption number six is no longer valid when the lateral deflection is equal to or is greater than 20 \% of the plate thickness. In situations like this, the strains in a layer of a plate parallel to and at a distance z from the middle surface are related to displacement components by

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{\partial^2 w}{\partial x^2} z
\]  

(2.96)
Theoretical Background

\[ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{\partial^2 w}{\partial y^2} z \]  
(2.97)

\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2 \frac{\partial^2 w}{\partial x \partial y} z \]  
(2.98)

Differential equations for large deflections and homogeneous isotropic material thin plates are given by two equations: the equilibrium and the compatibility. They are

\[ \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 y^2} + \frac{\partial^4 \phi}{\partial y^4} = E h \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \]  
(2.99)

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left( q + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial w}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) \]  
(2.100)

where \( q \) is the applied force.

\( \phi \) is the stress function related to the in-plane forces.

\[ N_x = h \frac{\partial^2 \phi}{\partial y^2} \]  
(2.101)

\[ N_y = h \frac{\partial^2 \phi}{\partial x^2} \]  
(2.102)

\[ N_{xy} = h \frac{\partial^2 \phi}{\partial x \partial y} \]  
(2.103)

These equations were first introduced in 1910 by von Kármán.

Analytical solution is possible only in very special conditions using trial function. For example, Ventsel’s textbook [15] introduces an approximate solution for simply supported geometrically nonlinear thin plate using Galerkin’s method.

In case of dynamic problems, the applied loads, \( q \), explicitly include inertia forces in the surface lateral loads according to D’Alambert’s principle.

\[ q(x, y, t) = q - \rho h \frac{\partial^2 w}{\partial t^2} \]  
(2.104)

where \( \rho \) the mass density of the plate material, and \( h \) is the plate thickness.

2.6 Involved Solution Methods

Obviously, the governing differential equations for a beam or a plate include two types of derivatives: spatial derivative terms of the fourth order and time derivative terms from the second order. Therefore, they are linear or nonlinear partial differential
equations (PDE). In fact, a direct analytical solution is not available. Approximate analytical solution is only possible for simple cases within very strict conditions. Alternatively, there are many common methods based on numerical algorithms. These methods are generally divided into two broad categories, direct methods and indirect methods [15], [120].

The direct methods include the Finite Difference Method (FDM), the Boundary Collocation Method (BCM), the Boundary Element Method (BEM) and the Galerkin method. Alternatively, methods such as Ritz and the Finite Element Method (FEM) are considered indirect methods [15], [120].

However, in this thesis, the finite element method was adopted to reduce the order of the PDE by replacing the space derivative terms by algebraic quantities called the spatial discretisation process. The adoption of this method is due to the following advantages [15], [120]:

1- The FEM model may be obtained without using the governing differential equation of the energy methods.
2- The FEM has a clear, easy and understandable physical meaning.
3- Complex geometry or arbitrary boundary can be handled in the same manner as simpler problems.
4- Full automation is possible for the numerical process for solving the boundary value.
5- Various combined structural elements can be analysed by this technique, for example plates, shells and beams.

In contrary, the following issues can be mentioned among the disadvantages [15], [120].

1- High speed processors and high capacity storages are required to deal with problems with FEM.
2- The accuracy of the numerical results is not guaranteed when a large structural system is treated.
3- The performance of this method is poor when processing singular problems such as cracks, corner points, discontinuity, etc…

The resulting equations normally take the following form

\[
[M]\{\ddot{U}\}+[C]\{\dot{U}\}+[K]\{U\} = \{F\}
\]  \hspace{1cm} (2.105)

These set of equations are Ordinary Differential Equations (ODE) because they still contain time derivative terms and need to be processed. The set of equations are written in terms of three matrices; [M], [C], [K] mass matrix, damping matrix and stiffness matrix, respectively.
The stiffness matrix, in the case of linear theories, does not vary during the loading process. Therefore, it depends only on the geometry at the beginning of loading and on the material properties. But in the case of nonlinearity, the stiffness matrix changes during the loading process. In case of the stiffening type of nonlinearity it increases, while in case of softening type it decreases. Therefore, to complete the solution, an iterative scheme is needed to update the stiffness matrix in what is called iterative nonlinear schemes. Actually, there are several common methods to do the iteration process [15], [120].

On the other hand, the damping matrix is approximated in different ways and will be discussed later in this chapter.

Furthermore, the mass matrix expresses the coefficients of the acceleration vector. In static load conditions, this matrix and the damping matrix are neglected. The acceleration terms are the second derivative of the displacement vector. The acceleration vector is important only in dynamic load conditions. Therefore, another set of numerical schemes are commonly used to achieve the direct numerical integration. This set of methods can be also divided into two broad groups: explicit methods and implicit methods.

Each set has both advantages and disadvantages.

Newmark’s method is among the implicit schemes widely used in the literature. In the following paragraph, a brief description will be made to introduce the discretisation process of beam bending equations. The detailed beam discretisation process, in addition to the discretisation with other beam or plate theories, is already described in many textbooks, for example, Kwon & Bang [118], Bathe [120], [121], Zienkiewicz & Taylor [122], David W. Nicholson [123], Smith & Griffiths [125], Hutton [126], Reddy [127] and Liu & Quek [128].

2.6.1 Spatial Discretisation

2.6.1.1 Finite Element Method

In this thesis, first, the finite element method is used to discretise the space derivation. This process will generate a system of equations at discretisation points in terms of time derivative. Or in other words, the PDE is converted to ODE.

For a detailed discretisation for diversity of beam theories, readers are advised to refer to textbook [118]. A general 3D beam element derivation is also illustrated in textbook [126] (see the fourth chapter). Also, the PDE for a beam on elastic foundation are discretized in [125]. However, in the following, the matrix form of the equation terms are shown for a 2-node beam element.

The bending term gives the stiffness matrix as follows:
The theoretical background

\[
[K] = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2 \\
\end{bmatrix}
\]

The acceleration term gives the mass matrix as follows:

\[
[M] = \frac{m}{420} \begin{bmatrix}
156 & 22L & 54 & -13L \\
22L & 4L^2 & 13L & -3L^2 \\
54 & 13L & 156 & -22L \\
-13L & -3L^2 & -22L & 4L^2 \\
\end{bmatrix}
\]

The axial force term gives the geometry stiffness matrix as follows:

\[
[K_{NL}] = \frac{N}{L} \begin{bmatrix}
\frac{36}{L} & L & -\frac{36}{L} & 3 \\
L & 4L & -3 & -L \\
-\frac{36}{L} & -3 & \frac{36}{L} & -3 \\
3 & -L & -3 & 4L \\
\end{bmatrix}
\]

Where L, m and N are the element length, element mass and the axial force, respectively.

2.6.1.2 Finite Difference Method

In this method, the partial differential equations for a small deflection of a thin plate are converted to a system of simultaneous algebraic equations using finite difference operators. These operators can be easily derived using the definition of the derivative. For instance, the first order derivative of y with respect to x is written in terms of algebraic discrete values (see Figure 2.14) as follows [15], [129], [110], [111]

\[
\left( \frac{dy}{dx} \right)_n = \lim_{\Delta x \to 0} \frac{y_{n+1} - y_n}{\Delta x}
\]

To simplify the equation, it could be approximated as follows

\[
\left( \frac{dy}{dx} \right)_n \approx \frac{\Delta y_n}{h} = \frac{y_{n+1} - y_n}{\Delta x}
\]

where \( \Delta y_n \) is the first forward difference of y at point, n, because it uses the difference between the current point, n, and the next point, (n+1).

Rearranging that equation gives
Similarly, the first backward difference at point, $n$, is given

$$
\Delta y_n = y_n - y_{n-1} \approx h \left( \frac{dy}{dx} \right)_n
$$

(2.112)

In the same way, the first central difference approximation at point, $n$, is written as follows

$$
\Delta y_n = \frac{y_{n+1} - y_{n-1}}{2} \approx h \left( \frac{dy}{dx} \right)_n
$$

(2.113)

Furthermore, the similar operator for a higher order derivation is based on the fact that the second order derivation is obtained by deriving the variable twice, one after the other. Therefore, using a similar approximation twice, the higher-order derivatives can be written

$$
(y_{n+1} - 2y_n + y_{n-1}) \approx h^2 \left( \frac{d^2y}{dx^2} \right)_n
$$

(2.114)

$$
\frac{1}{2} (y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}) \approx h^3 \left( \frac{d^3y}{dx^3} \right)_n
$$

(2.115)

$$
(y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2}) \approx h^4 \left( \frac{d^4y}{dx^4} \right)_n
$$

(2.116)

Using the equation (2.113), the mixed derivative can be approximated following the formula shown next:

$$
\left( \frac{\partial^2 w}{\partial x \partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) \approx \frac{\partial}{\partial y} \left( \frac{w_{n+1} - w_{n-1}}{2\Delta_x} \right) \approx \frac{w_{n+1,n+1} - w_{n-1,m+1} - w_{n+1,m-1} - w_{n-1,m-1}}{4\Delta_x \Delta_y}
$$

(2.117)

Where, $\Delta_x$ and $\Delta_y$ are the discretisation steps along $x$ and $y$ directions, respectively.

As a consequence, the finite difference approximation of a Laplace operator at point, 0 (see Figure 2.14), is obtained for $\Delta_x = \Delta_y = h$ as follows

$$
\Delta w = \frac{1}{h^2} (w_1 + w_2 + w_4 + w_5 - 4w_0)
$$

(2.118)

Thus, at the grid point 0 (see Figure 2.14), the biharmonic operator is written in terms of algebraic values using finite difference technique as follows
Figure 2.14: Discretisation using finite difference method [15]

\[ \Delta^2 w = \frac{1}{h^2} \left[ (w_9 + w_{10} + w_{11} + w_{12}) + 2(w_5 + w_6 + w_7 + w_8) - 8(w_i + w_2 + w_3 + w_4) + 20w_o \right] \] (2.119)

For every grid point a similar formula can be written in terms of its adjacent grid points. Thus, a number of algebraic equations are formulated and can all be written in a matrix form to simulate the discretized domain as follows

\[ \Delta^2 w = [A][w] \] (2.120)

By using this equation, the linear thin plate derivative equation (2.92) can be reduced to algebraic equation as follows

\[ [A][w] = \frac{q}{D} \] (2.121)

or

\[ [A][W] = [B] \] (2.122)

This is a system of simultaneous algebraic equations, where \([A]\) is the coefficient matrix, \([B]\) is force vector and \([W]\) is a vector of the unknown displacement at the grid points.

In this system of equations, the number of unknowns is larger than the number of equations. Therefore, boundary conditions are needed to make the number of unknown variables equals to the number of equations.

In conclusion, the displacement values of the uniform grid points of the plate can be estimated by solving the system of equations. This is done by calculating the inverse of the matrix \([A]\). This methodology was tested in this thesis and showed some
agreement with other methods; the MATLAB code involving this method is programmed according to the following procedure

1. Create the coefficient matrix \[ A \] using FDM.
2. Create the right-hand side vector \[ B \].
3. Solve the displacement vector by calculating the inverse of the coefficient matrix.

### 2.6.2 Time Integration Method

After the spatial discretisation process, the governing partial differential equation of a structural dynamical system is normally reduced to an ordinary differential equation containing second-order-time derivative terms. However, it can be solved using a number of methods described in the literature. For nonlinear problems, only numerical integration is possible. There are two numerical integration schemes either implicit or explicit. Central difference is an example for explicit methods. The Newmark’s method is implicit scheme.

#### 2.6.2.1 Central Difference

In this method, the equation of motion is evaluated at the old time step \( t_n \), whereas implicit methods use the equation of motion at the new time step \( t + \Delta t \) [123].

Equation of motion at time \( t \)

\[
[M] \{\dot{U}\} + [C] \{\ddot{U}\} + [K] \{U\} = \{F\} \tag{2.123}
\]

The Finite Difference Method (FDM) can be used to approximate the derivation terms. Thus, the velocity and acceleration at time \( t \) using FDM are approximated as follows

\[
\{\dot{U}\}_i = \frac{1}{2\Delta t}(-U_{i-\Delta t} + U_{i+\Delta t}) \tag{2.124}
\]

\[
\{\ddot{U}\}_i = \frac{1}{\Delta t^2}(U_{i-\Delta t} - 2U_i + U_{i+\Delta t}) \tag{2.125}
\]

Substituting the velocity and acceleration from equations (2.124) and (2.125) in the equation of motion (2.126), it follows:

\[
\left( \frac{1}{\Delta t^2} M + \frac{1}{2\Delta t} C \right) U_{i+\Delta t} = F_i - \left( K - \frac{2}{\Delta t^2} M \right) U_i - \left( \frac{1}{\Delta t^2} M - \frac{1}{2\Delta t} C \right) U_{i-\Delta t} \tag{2.126}
\]

This equation gives the displacement vector in next time step. The main disadvantage of the central difference method is that the scheme is only conditionally stable. For more details, readers are referred to textbook [123].
2.6.2.2 Newmark’s Method

Owing to its improved stability characteristic, there have been a large number of studies involving this method since Nathan M. Newmark introduced it in 1959. For example, A.H. Sheikh, M. Mukhopadhyay 2002 [130] used Newmark’s method for direct time integration and finite element method for space discretisation. For more details, interested readers are referred to textbook [123] p.159.

The Newmark-method is a numerical integration method used to solve differential equations. It is an implicit method, which attempts to satisfy the differential equation at time \( t + \Delta t \) after founding the solution at time \( t \). Considering the linear equilibrium equation [124],

\[
\left[ M \right] \{ \ddot{U} \}_{t+\Delta t} + \left[ C \right] \{ \dot{U} \}_{t+\Delta t} + \left[ K \right] \{ U \}_{t+\Delta t} = \{ F \}_{t+\Delta t} \tag{2.127}
\]

In this equation, for convenience, the vector terms are written without the curly brackets.

Further, Taylor’s series is used to express the derivation as follows

\[
U_{t+\Delta t} = U_t + \Delta t \dot{U}_t + \frac{\Delta t^2}{2} \ddot{U}_t + \frac{\Delta t^3}{6} \dddot{U}_t + ...	ag{2.128}
\]

\[
\dot{U}_{t+\Delta t} = \dot{U}_t + \Delta t \ddot{U}_t + \frac{\Delta t^2}{2} \dddot{U}_t + ...	ag{2.129}
\]

Neglecting the small terms and taking only the first four terms yields

\[
U_{t+\Delta t} = U_t + \Delta t \dot{U}_t + \frac{\Delta t^2}{2} \ddot{U}_t + \beta \Delta t^3 \dddot{U}_t \tag{2.130}
\]

\[
\dot{U}_{t+\Delta t} = \dot{U}_t + \Delta t \ddot{U}_t + \gamma \Delta t^2 \dddot{U}_t \tag{2.131}
\]

In each time step, the acceleration is assumed to be linear. Therefore, it is possible to write

\[
\ddot{U} = \frac{\dddot{U}_{t+\Delta t} - \dddot{U}_t}{\Delta t} \tag{2.132}
\]

The suitable combination and substitution give the standard form for the Newmark’s equations

\[
U_{t+\Delta t} = U_t + \Delta t \dot{U}_t + \left( \frac{1}{2} - \beta \right) \Delta t^2 \ddot{U}_t + \beta \Delta t^3 \dddot{U}_t \tag{2.133}
\]

\[
\dot{U}_{t+\Delta t} = \dot{U}_t + (1 - \gamma) \Delta t \ddot{U}_t + \gamma \Delta t^2 \dddot{U}_t \tag{2.134}
\]

This method is conditionally stable for the zero damping method if
The solution is unconditionally stable if

\[ 2\beta \geq \gamma \geq \frac{1}{2} \]  \hspace{1cm} (2.137)

### 2.6.3 Damping in Built-up Structures

Damping in built-up structures is divided into two big categories: internal damping and external damping. For instance, some external damping arises when a solid body moves in a viscous medium.

Internal damping is also divided into two groups: material damping and damping due to contact areas within the structure.

More specifically, material damping is usually called hysteresis damping. It includes viscous, friction and perhaps yielding, while contact damping is a result of the relative motion between substructures such as joints, bearing … etc.

However, in hysteresis damping, the energy dissipation is calculated by estimating the area inside the hysteresis loop (see Figure 2.15) as follows \[131\], \[132\]

\[ W_D = \int \sigma \, d\varepsilon \]  \hspace{1cm} (2.138)

![Figure 2.15: Strain-stress linear hysteresis loop](image)

The damping factor (or loss factor in some references) of the material is proportional to the ratio of energy dissipation to maximum strain potential energy

\[ \psi = \frac{1}{2\pi} \frac{W_D}{E_{pot}} \]  \hspace{1cm} (2.139)

To obtain the damping factor for the complete structure, the integration over the volume should be considered.
\[ \psi_s = \frac{\int \psi dV}{V} = \frac{1}{2\pi} \frac{W_{DS}}{E_{potS}} \]  

(2.140)

If the structure is a homogeneous solid, then

\[ \psi_s = \psi \]  

(2.141)

If the structure is simplified as a linear oscillatory system excited by a harmonic force, then the governing equation is:

\[ m\ddot{x} + c\dot{x} + kx = F(t) = F\cos(\omega t) \]  

(2.142)

The steady state solution is

\[ x = X\cos(\omega t - \phi) \]  

(2.143)

For this particular system, the dissipated energy per cycle is obtained as

\[ W_{DS} = \int_0^T (c\dot{x})\dot{x} dt = \pi c\omega X^2 \]  

(2.144)

Similarly, the maximum strain energy is

\[ E_{potS} = \frac{1}{2} kX^2 \]  

(2.145)

Substituting equations (2.145) and (2.144) into equation (2.140) yields

\[ \psi_s = 1 \frac{W_{DS}}{2\pi E_{potS}} = \frac{c\omega}{k} \]  

(2.146)

Assuming the excitation frequency equals the natural frequency (at resonance)

\[ \omega = \omega_n = \sqrt{\frac{k}{m}} \]  

(2.147)

The damping factor at resonance becomes

\[ \psi_{sn} = \frac{c}{\sqrt{k m}} \]  

(2.148)

The damping ratio is

\[ \zeta = \frac{c}{2\sqrt{k m}} = \frac{c}{2 m \omega_n} \]  

(2.149)

From this, it is possible to write

\[ \psi_{sn} = 2\zeta \]  

(2.150)
Knowing the critical damping as

\[ c_{crit} = 2m\omega_n \]  \hfill (2.151)

yields

\[ \zeta = \frac{c}{c_{crit}} \]  \hfill (2.152)

when damping equals the critical damping or

\[ \zeta = 1 \]  \hfill (2.153)

the system becomes critically damped and oscillation is damped out immediately. Therefore, equations (2.146) and (2.150) yields

\[ \zeta = \frac{1}{4\pi} \frac{W_{ps}}{E_{ps}S} \]  \hfill (2.154)

For a single-degree of freedom system with mass, spring and viscous damper as seen in Figure 2.16, the damped natural frequency can be calculated as

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]  \hfill (2.155)

![Figure 2.16: Model of a single-degree of freedom oscillator (SDOF model) with viscous damper](image-url)

The logarithmic decrement is the natural logarithm of the ratio of two successive amplitudes one period apart. It can be calculated as

\[ \Lambda = \ln \left( \frac{x(t)}{x \left( t + \frac{2\pi}{\omega_d} \right)} \right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \]  \hfill (2.156)
Paragraph 2.6.7.2 shows more details on the derivation of the logarithmic decrement. However, in case of low damping,

$$\zeta \ll 1,$$ \hfill (2.157)

the high power term of a small quantity is negligible as compared with one. Hence, equation (2.156) can be simplified to

$$\zeta = \frac{\Lambda}{2\pi}$$ \hfill (2.158)

For a viscous damper, the hysteresis loop is an ellipse. However, the transmitted force into the spring and the damper can be calculated by

$$F = kx + c \dot{x} = k X \sin(\omega t) + c X \omega \cos(\omega t)$$ \hfill (2.159)

But, in reality, the structures include different types of damping such as dry friction damping. Knowing that the damping dominates the response at resonance, the actual behaviour can be simulated accurately by an equivalent viscous damper with an equivalent viscous damping ratio, incorporating all kinds of energy dissipation.

This approach can also be used for multi-degrees of freedom systems. An equivalent damper is then associated to each eigenmode. Therefore, any coupling is ignored and priori knowledge of the real damping is assumed for those frequency bands in which damping is significant.

When the limits of such a band are not known, then the damping model must be effective for a large frequency region. However, a frequency-dependent equivalent viscous damping coefficient is introduced

$$m \ddot{x} + c(\omega) \dot{x} + kx = F(t)$$ \hfill (2.160)

$$c(\omega) = \frac{k \psi_s(\omega)}{\omega}$$ \hfill (2.161)

It is found in engineering application that the energy loss is dependent upon the square of the amplitude of the response but for all goals independent of the frequency such that

$$\psi_s = \text{const}$$ \hfill (2.162)

Substituting in equation (2.146) yields

$$c(\omega) = k \frac{\psi_s}{\omega}$$ \hfill (2.163)
2.6.4 Graphical Representation of Energy Loss

The energy loss due to viscous damping can be represented graphically as seen in Figure 2.17. In the graph, the force from equation (2.159) is plotted against the displacement [131], [132].

The lost energy due to the damper is calculated by equation (2.144) and estimated by the area of the dashed ellipse. The area of the ellipse in the graph represents the energy quantity in equation (2.144). Similarly, the strain energy is given by equation (2.145) and equals the area of the triangle as may be seen in the figure.

![Graphical representation of energy loss in a spring and viscous damper in parallel](image)

Figure 2.17: Graphical representation of energy loss in a spring and viscous damper in parallel

2.6.5 Nonlinear Damping

Normally, the damping in a structure is nonlinear. But, practically seen, engineers can safely deal with it as a linear quantity provided that the damping constant is chosen carefully. In this paragraph, two types of nonlinear damping will be analysed, which are extensively found in real application: dry friction damping or Coulomb damping and Square-Law damping (hydraulic damper). The equivalent linear damper will be calculated [131], [132].

2.6.5.1 Coulomb Damping

For dry contact areas, Coulomb friction represents the most significant dissipation mechanism. A simple model of such a mechanism is shown in Figure 2.18.

The equation of motion of that system is

\[ m \ddot{x} + \mu N \text{sgn}(\dot{x}) + k x = F(t) \]  \hspace{1cm} (2.164)

It is assumed here that the displacement, velocity and acceleration time histories remain sinusoidal throughout.

The displacement time history is given by
The first derivative of the displacement function with respect to time gives the velocity time history as

\[ \dot{x} = \omega X \sin(\omega t) \]  

(2.166)

Similarly, the first derivative of the velocity function with respect to time gives the acceleration time history as

\[ \ddot{x} = -\omega^2 X \cos(\omega t) \]  

(2.167)

The applied force is given by

\[ F(t) = \begin{cases} 
  k x + \mu N & \text{if } 0 \leq t < \frac{\pi}{\omega} \\
  k x - \mu N & \text{if } \frac{\pi}{\omega} \leq t < \frac{2\pi}{\omega} 
\end{cases} \]  

(2.168)

where \( \mu \) is the coefficient of friction of the rubbing material.

The dissipated energy per cycle is calculated by integration as follows

\[ W_{DS} = \int_0^{2\pi/\omega} F(t) \dot{x} \, dt \]  

(2.169)

Substituting the force from equation (2.168) and velocity from (2.166) into equation (2.169) yields

\[ W_{DS} = 2 \int_0^{\pi/\omega} [-k X \cos(\omega t) + \mu N] [\omega X \sin(\omega t)] \, dt \]  

(2.170)
By calculating the integrations, the energy dissipation per cycle is given as

\[ W_{DS} = 4\mu N X \quad (2.171) \]

The equivalent damping coefficient can be concluded by comparing this equation with equation (2.144) as

\[ c_{eq} = \frac{4\mu N}{\pi \omega X} \quad (2.172) \]

From this equation, it can be seen that at constant normal force and constant frequency, the equivalent viscous damping drops as the amplitude is increased. Unfortunately, this characteristic is not preferable because the damping is reduced just when it is most wanted.

However, the amplitude ratio for a harmonic excitation force becomes

\[ \frac{X}{F/k} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \sqrt{1 - \left(\frac{4\mu N}{F}\right)} \quad (2.173) \]

By checking equation (2.171), it can be seen that the energy loss is linearly proportional to amplitude. Therefore, the amplitude at resonance grows without limit if

\[ F \pi > 4\mu N \quad (2.174) \]

### 2.6.5.2 Square-Law Damping

This kind of damping can be achieved by enforcing oil through a small orifice by a piston which causes a turbulent flow. This produces a resistance force proportional to the square of the piston velocity. Thus, the resisting force is given by [131], [132]

\[ F = C \dot{x}^2 \text{sgn}(\dot{x}) \quad (2.175) \]

where \( C \) is the square-law damping coefficient. This is, of course, different from the linear viscous coefficient \( c \).

The applied force to a spring parallel with the damper is given by (see Figure 2.19):

\[ F(t) = \begin{cases} kx + C\dot{x}^2 & \text{if } 0 \leq t < \frac{\pi}{\omega} \\ kx - C\dot{x}^2 & \text{if } \frac{\pi}{\omega} \leq t < \frac{2\pi}{\omega} \end{cases} \quad (2.176) \]

The dissipated energy can be calculated by performing the following integration

\[ W_{DS} = \int_{0}^{2\pi/\omega} F(t)\dot{x} \, dt \quad (2.177) \]
Substituting the force from equation (2.176) and velocity from (2.166) into the last equation yields

\[ W_{DS} = 2 \int_{0}^{\pi/\omega} \left[ -k X \cos(\omega t) + C \omega^2 X^2 \sin(\omega t) \right] [\omega X \sin(\omega t)] \, dt \] (2.178)

Simplifying equation (2.178) yields

\[ W_{DS} = 2 \int_{0}^{\pi/\omega} C \omega^3 X^3 \sin(\omega t) \, dt \] (2.179)

\[ W_{DS} = \frac{8}{3} C \omega^2 X^3 \] (2.180)

The equivalent damping coefficient can be calculated again by comparing it with equation (2.144) as

\[ \pi c_e X^2 \omega = \frac{8}{3} C \omega^2 X^3 \] (2.181)

Hence, the equivalent viscous damping coefficient is given by

\[ c_e = \frac{8 C \omega X}{3} \] (2.182)

From this equation, it can be concluded that at constant normal force and frequency, the equivalent viscous damping increased as the amplitude increased.

### 2.6.6 Proportional Damping Formulation, Rayleigh Damping

A system having multi-degrees of freedom, the equation of motion under external forces [131], [132]
The equation of motion of a system without damping and no external force is

\[ [M] \{ \ddot{x} \} + [C] \{ \dot{x} \} + [K] \{ x \} = \{ F \} \]  

(2.183)

The free vibration solution is

\[ \{ x(t) \} = \{ X \} \sin(\omega t) \]  

(2.185)

Substituting equation (2.185) into equation (2.184) yields

\[ (-\omega^2 [M] \{ X \} + [K] \{ X \}) \sin(\omega t) = \{ 0 \} \]  

(2.186)

Simplifying equation (2.186) yields

\[ ([K] - \omega^2 [M]) \{ X \} = \{ 0 \} \]  

(2.187)

There is a solution for this equation. When the determinant equals zero, this gives the characteristic equation in which the solution gives the \( n \) fundamental frequencies associated with \( n \) eigenvectors.

The normalized eigenvector matrix takes the form

\[
[\phi] = \begin{bmatrix}
  x_{11} & x_{12} & \ldots & x_{1n} \\
  x_{21} & x_{22} & \ldots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \ldots & x_{nn}
\end{bmatrix}
\]  

(2.188)

This matrix can be used to make the orthogonal transformation for the equation of motion of the damped system equation (2.183)

\[ \{ \phi \}^T [M] \{ \phi \} \{ \ddot{q} \} + \{ \phi \}^T [C] \{ \dot{q} \} + \{ \phi \}^T [K] \{ q \} = \{ \phi \}^T \{ F \} \]  

(2.189)

Equation (2.189), subsequently reduces to an \( n \)-uncoupled system of equations in the form

\[ \{ \ddot{q}_j \} + 2\zeta_j \omega_j \{ \dot{q}_j \} + \omega_j^2 \{ q_j \} = \{ F_j \} \]  

(2.190)

where \( \{ q \} \) is the displacement of the structure in the transformed coordinate (or generalised coordinate) and \( \zeta \) is the damping ratio in uncoupled mode.

The above orthogonal transformation is valid only when the damping matrix is proportional, i.e. it is some function of the mass and stiffness matrix \([M]\) and \([K]\) as follows

\[ [C] = \alpha [M] + \beta [K] \]  

(2.191)
The damping term in equation (2.189) reduces to

$$\{ \phi \}^T [C] \{ \phi \} = \begin{bmatrix}
\alpha + \beta \omega_1^2 & 0 & \ldots & 0 \\
0 & \alpha + \beta \omega_2^2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha + \beta \omega_n^2 \\
\end{bmatrix}$$  (2.192)

Again from symmetry, it can be inferred that

$$2\zeta_1 \omega_1 = \alpha + \beta \omega_1^2$$
$$2\zeta_2 \omega_2 = \alpha + \beta \omega_2^2$$
...............           
$$2\zeta_n \omega_n = \alpha + \beta \omega_n^2$$  (2.193)

This can be simplified to

$$2\zeta_i \omega_i = \alpha + \beta \omega_i^2$$  (2.194)

For two degrees of freedom, the previous system of equations reduces to

$$2\zeta_1 \omega_1 = \alpha + \beta \omega_1^2$$
$$2\zeta_2 \omega_2 = \alpha + \beta \omega_2^2$$  (2.195)

These two equations have to be solved simultaneously in terms of $\alpha$ and $\beta$. In the case of multi-degrees of freedom, the most two dominant modes could be selected or alternatively the best values are selected based on an iterative solution. This can be obtained possibly from the best-fit values of $\alpha$ and $\beta$ in a particular system.

From equation (2.194) it can be observed that the damping ratio is function to the natural frequencies of the system. It can be simplified to

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta \omega_i}{2}$$  (2.196)

A typical plot of this equation is shown in Figure 2.20.
2.6.7 Methods for Damping Measurement

Any structural oscillatory system contains kinetic and strain energy. In reality, there is also a dissipated energy due to passive or active dissipative elements. In simple terms, damping is the conversion of mechanical energy of a vibrating structure into thermal energy, which is then lost to the structure’s environment.

There are several commonly used methods to estimate the system damping, such as the half-power bandwidth method, the amplification factor method, the log decrement method, the hysteresis loop method, resonant response amplitude and damping interrelationships [129].

In the following, a brief description for the methods used will be introduced. For further details, interested readers are referred to Harris’ handbook [113].

2.6.7.1 Half-Power Bandwidth Method for Viscous Damping

This method is used to calculate the damping ratio $\zeta$ in lightly damped structures. The calculations should be done in the frequency domain.

Knowing the maximum response $A_n$ at resonance frequencies $f_n$ and two frequencies $f_1$ and $f_2$ around the response peak, the response value by definition is $A_n / \sqrt{2}$ as seen in Figure 2.21. The difference between $f_1$ and $f_2$ is defined as half-power bandwidth such that $A_n / \sqrt{2}$.

Damping ratio is related to these frequencies and the resonant frequency by [113], [131], [132]

$$\zeta = \frac{f_2 - f_1}{2f_n}$$  \hspace{1cm} (2.197)
In general, this method is not recommended, particularly not in the case of a nonlinear systems where the peak at the resonance turns to the left (stiffening case) and to the right (softening case). Also, the measurement errors greatly affect the determination of the damping ratio. From the time point of view, it consumes more time while measuring the response in comparison with amplitude decay methods.

### 2.6.7.2 Logarithmic Decrement Method

The logarithmic decrement method is also referred to as the decay curve method. This method is used for viscous damped system. Viscous damping can be estimated from the transient response of the system, as seen in Figure 2.22 [113], [131], [132].

The logarithmic decrement is measured in the time domain where the measurement is carried out on the transient response. This is obtained by applying an external force. The force is then removed and the system is left oscillating freely. The free response for the single-degree of freedom system with viscous damper is plotted, as seen in Figure 2.22. For purely viscous damping, the enveloped curve (the dashed lines) is an exponential decay. For pure dry friction damping, it is straight line decay. However, the envelope curve in real structure lies between those two cases.

The logarithmic decrement is the ratio of decay for two successive amplitudes; it is governed by the following equation

\[
\Lambda = \ln\left(\frac{X_1}{X_2}\right)
\]

\[
\Lambda = \ln(X_1) - \ln(X_2) \tag{2.198}
\]

\[
\Lambda = \ln(e^{-\zeta f_n t_1}) - \ln(e^{-\zeta f_n (t_1 + T_d)}) \tag{2.199}
\]

\[
\Lambda = -\zeta f_n t_1 - (-\zeta f_n (t_1 + T_d)) \tag{2.200}
\]

\[
\Lambda = \zeta f_n T_d \tag{2.201}
\]
Time

Displacement

Figure 2.22: The logarithmic decrement of a transient response

where

\[ T_d = 2\pi f_n \sqrt{1 - \zeta^2} \]  

(2.203)

For low damping, it is possible to assume that

\[ \sqrt{1 - \zeta^2} \approx 1 \]  

(2.204)

Substituting equations (2.204) and (2.203) into (2.202) yields

\[ \Lambda = 2\pi \zeta \]  

(2.205)

This is a simple formula which relates the logarithmic decrement directly to the damping ratio.

2.6.7.3 Dynamic Hysteresis Loop Method

This method is based on the fact that the amount of energy dissipated per cycle of oscillation, due to steady-state harmonic loading, is a measure of the structure’s damping. In this method, the applied force is plotted against the displacement of the application point (see Figure 2.23). A number of hysteresis loops in the displacement-force plane appear. The figure shows the ellipse loop because the material is viscoelastic. In general, the area of one loop equals the work provided to the system per cycle of the applied force. Thus, the applied work is given by the following integration [113], [131], [132]

\[ W_p = \int F(t) \, dx \]  

(2.206)

If the total stored energy of a system is denoted by \( E_{pot} \), then the specific damping capacity is calculated by the ratio
And the damping ratio

\[ \zeta = \frac{1}{4\pi} \frac{W_D}{E_{pot}} \]  

(2.208)

Figure 2.23: A typical hysteresis loop for single-degree of freedom system, ellipse loop because of viscoelastic material

### 2.6.8 Processing the Experimental Results

A large amount of scientific knowledge has been developed over the past two centuries, devoted to the understanding of natural phenomena. Generally, scientists are able to describe many physical systems in terms of equations, particularly in the field of mechanics. The general procedure relies upon neglecting parameters that have only minor effects on the system response and taking into account the predominant parameters. In most of the cases, the resulting equations are complex and hard to tackle. Therefore, further assumptions, less or more confined with reality, are normally needed under what is called the linearization technique. It is then possible to find exact or approximate solutions using some well-known solution methods. Thus, the scientists are able to predict or interpret lots of physical systems [134], [135], [136].

Recently, huge developments in the numerical computational capabilities have made it possible to discover lots of phenomena which were not possible to discover using the classical techniques or linearization methods. For instance, the response of a nonlinear dynamic system was believed to be random. Of course, there are some random components as the situation in any natural dynamical system. But the newly discovered tools make it possible to detect the deterministic behaviour and to capture the period doubling bifurcation. For example, the Lyapunov exponent is devoted to quantify the sensitivity of the dynamical system to the initial conditions [134], [135], [136].
The main purpose of this paragraph is to provide a brief description of the necessary numerical tools that have been used in this work. These are devoted to manipulating the time series analysis, namely: filtering process, discrete fast Fourier and frequency response function. For further details concerning such tools, interested readers are referred to specialized textbooks in the field of time series analysis such as textbook [135].

In the first step after obtaining an experimental time series, analysts normally use filtering process to cancel out any unwanted signals. Several types of filters have been developed over the last two decades. Digital filters are among those which are widely used in signal processing. However, the digital filter, in simple terms, is a mathematical operation carried out on a discrete-time signal. This operation works on eliminating the unwanted frequencies or removing noises. The frequencies of the cancelled out components could be high, low, banded in a range or out of range of frequencies [135], [113].

Generally, filters are divided in two groups, namely: digital filters and analog filters. The analog filter is an electronic circuit processing a continuous-time signal, whereas the digital filters process only digital discrete-time signals. However, there are some electronic circuits which convert the continuous analog signal to digital discrete signal and vice versa. For example, Analog-to-Digital Converter (ADC) or Digital-to-Analog Converter (DAC). However, these types of converters make it possible to use the digital filters to process the analog signals as well [135], [113].

Mathematically, the filters, like any dynamical system, are conveniently represented by a Transfer Function (TF). The TF is a mathematical operator which includes the entire dynamical characteristics. And the mathematical expression of a TF can be expressed in a frequency domain or Z-domain. However, the general format for any filters in terms of Z-domain operator is given [135], [113].

\[
H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}
\]

The coefficients and the order of the numerator or the denominator have a significant effect in characterising the dynamic behaviour of the filter.

However, designing a filter is no more than defining those coefficients and the order required to achieve a specific purpose. But this does not mean it is a straightforward process. Fortunately, many commonly used methods have been developed recently to make the selection based on preferable performance easier, such as the Window Design method, the Frequency Sampling method, the Weighted least squares design and the Parks-McClellan method (also known as the Optimal, Minimax or Equiripple method) [135], [113].

Furthermore, many computer applications introduce comprehensive built-in functions utilizing such methods so as to make the design process easy and effective. Also, these packages present many utilities for checking the performance either in the time domain
Another way of categorizing filters is based on the frequencies that are passed or banned. Accordingly, the filters are divided into four different types: low-pass filter, band-pass filter, high-pass filter and band-cute filter [135], [113].

In addition to the cut-off frequency, the rate of frequency roll-off is also an important characterizing parameter.

Furthermore, the filters can also be divided into two different types:

1- Finite impulse response (FIR):

   The pulse response of the filter is finite because it dies to zero in a finite number of sample intervals. More accurately, the pulse response of an Nth-order FIR lasts for N+1 samples and then settles to zero.

2- Infinite Impulse Response (IIR):

   The impulse response of this type of filter is non-zero over an infinite length of time.

   Example for IIR filters include the Bessel filter, Butterworth filter and the Chebyshev filter.

Some famous filter examples are the Bessel filter, the Butterworth filter, the Elliptical filter (Cauer filter), the Linkwitz-Riley filter, the Chebyshev filter, the Ladder filter, the Biquad filter.

In this thesis, a signal filtering process was extensively required. Therefore, one of those types of filters had to be adopted. The Butterworth filter has shown exceptional performance in comparison with other types of filters (see Figure 2.24).

The most important characteristic of this filter is that it is designed to obtain a frequency response as flat as possible. Therefore, it is termed a maximally flat magnitude filter.

Furthermore, MATLAB introduces many tools required to utilize, design and test such filters. For instance, in order to design low-pass Butterworth filters with normalized cut-off frequency $W_n$, the following MATLAB statement $[b,a] = butter(n,Wn)$ is adequate. It returns the filter coefficients in length $n+1$ vectors $\{b\}$ and $\{a\}$ that are described in equation (2.209).

Another mathematical tool, which is extensively used in the field of time series processing, is the discrete Fourier transform. This is the most useful and powerful tool which is normally used to distinguish between periodic behaviour and stochastic behaviour. The spectral density of a digital signal also reveals frequency contents of a stochastic process and helps identify periodicities [135], [113].
Spectral density, Power Spectral Density (PSD) or Energy Spectral Density (ESD) describe how the variance (or energy) of a time series is distributed with frequency. If the signal is periodic, then the PSD will consist of discrete lines, whereas in a stochastic signal the PSD will be spread over a continuous range of frequencies [135], [113].

The discrete Fourier transform is defined by

\[ Z_j = \frac{1}{N} \sum_{k=0}^{N-1} y_k \exp\left(\frac{-i2\pi(j-1)(k-1)}{N}\right) \]  

where \( y_i \) is a time series, \( N \) is the length of that series.

The power spectrum is defined as

\[ P_j = |Z_j|^2 = X_j^2 + Y_j^2 \]  

where \( X \) is the real part of \( Z \) and \( Y \) is the imaginary part. Each value of \( j \) for which there is a peak in the power spectrum corresponds to a frequency component.

Figure 2.24: A comparison between different types of filters
\[ f_j = \frac{j}{N \Delta t} \]  

(2.212)

in the original time series.

Additionally, the auto-correlation function also introduces an investigative tool for distinguishing between periodic and chaotic response. In stochastic behaviour, the auto-correlation is irregular, whereas in a periodic signal the auto-correlation will be periodic. However, the auto-correlation can be estimated by

\[ c_j = \frac{1}{N} \sum_{i=1}^{N} y_i y_{i+j} \]  

(2.213)

where periodic boundary conditions

\[ y_{N+k} = y_k \]  

(2.214)

are imposed to extend the time series beyond \( y_N \). This function provides a simple quantitative measure of the linear correlation between data points.

Besides this, the Frequency Response Function (FRF) is also a significant tool in the field of digital signal processing. FRF is a mathematical tool to uniquely quantify the relationship between the input and the output for any dynamical system. This relation is formulated differently. Table 2.1 shows commonly used relationships.

For a dynamical system with single input, the general relationship for FRF is written

\[ X_m = H_{uu} F_q \]  

(2.215)

For a multi-input system, \( H \) is a matrix with \( m \) rows and \( n \) columns. While \( F \) is a column vector of length, \( q \). \( X \) is a row vector of length \( m \).

The calculations are usually done in the frequency domain. Therefore, the Fast Fourier Transform (FFT) is used for this purpose. The input signal and output signal are first computed using FFT. Then using any of these formulations the FRF can be found. The problem here is that, due to a number of different reasons, the signal contains different types of noises.

<table>
<thead>
<tr>
<th>Table 2.1: Frequency response function formulations.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Displacement</strong></td>
</tr>
<tr>
<td>Dynamic Stiffness</td>
</tr>
<tr>
<td>Dynamic Compliance</td>
</tr>
<tr>
<td><strong>Velocity</strong></td>
</tr>
<tr>
<td>Force</td>
</tr>
<tr>
<td><strong>Acceleration</strong></td>
</tr>
<tr>
<td>Force</td>
</tr>
</tbody>
</table>
In order to minimize the error due to noises, a number of averages, $N_{\text{avg}}$, is used. For this purpose, the auto and cross power spectrums are used according to the following procedure:

The time series of the displacement, $X$, and force, $F$, are computed in the frequency domain using FFT. The displacement complex conjugate, $X^*$, and the force complex conjugate, $F^*$, are needed. The cross power spectrums are then computed as

$$GXF = \sum_{i=1}^{N_{\text{avg}}} X \cdot F^*$$

$$GFX = \sum_{i=1}^{N_{\text{avg}}} F \cdot X^*$$

And the auto power spectrums are computed as

$$GFF = \sum_{i=1}^{N_{\text{avg}}} F \cdot F^*$$

$$GXX = \sum_{i=1}^{N_{\text{avg}}} X \cdot X^*$$

Once these values are computed, it is possible to use one of two common formulations according to the purpose. The first formulation is called $H1$ and the second is $H2$. These are given as follows

$$H1 = \frac{GXF}{GFF}$$

$$H2 = \frac{GXX}{GFX}$$

The difference between the two formulations is in the capability of cancelling the noise according to its existence. More precisely, the $H1$ formulation is more efficient when the noise exists in the output signal, whereas the $H2$ formulation is more efficient and used when the noise exists in the input signal. Therefore, in this study, the $H1$ formulation is used. This is mainly because the output signal is the important one and has to be used for additional computations. It also contains more noise components due to the large number of electronic devices.

Accordingly, the $H1$-FRF function is extensively used in this study to obtain clean signals which are used particularly for damping estimations in the experimental parts.

### 2.6.9 Phase Portrait and Poincaré Section

The most important question in any time series is how to distinguish between noise and chaos. The noise could be found in any measured data or even in computed data in some cases. The noise should be avoided or cancelled out, because it has no physical meaning. As explained in the previous paragraph, many types of digital filters are
devoted to cancelling out different sources of noise. The question is, however, how to determine whether a given signal is a deterministic response or not. What is the dimension of the phase space? Is it chaotic? The phase portrait introduces the answers to all of these questions [134], [135], [136], [137], [138].

The phase portrait shows a trajectory or more for the relation between selected state variables. More clearly, the circular (or elliptical) orbit reveals linearity or a periodic response (see Figure 2.25). The spiral-in trajectory means that the system is damped and decayed to zero response (see Figure 2.26). The spiral-out trajectory means divergent or instable (see Figure 2.27).

While the phase portrait in Figure 2.28 shows linear, damped and forced, whereas the response in Figure 2.29 is nonlinear, damped and forced. [134], [135], [136].

![Figure 2.25: A typical phase portrait of a linear, un-damped and unforced oscillator](image)

![Figure 2.26: A typical phase portrait of a linear, damped and unforced oscillator.](image)
Additionally, the 3D plots of the state space variable in Figure 2.30 show the chaotic behaviour of the Rössler system [135], which are given by the equations in (2.222)

\[
\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\]
That plot is calculated using MATLAB code. The routine used the following constants: \( a = 0.2, \ b = 0.4, \ c = 5.7, \) number of points = 5000 and sample step size = 0.01 were used.

For more chaotic systems, interested readers are referred to textbook [135]. It gives details of how to obtain the attractor using delay embedding. The book introduces other chaotic systems like Lorenz and Ikeda map system and many others.

**Poincaré Map:**

Poincaré Map has become the most common and the most descriptive method for investigating the behaviour of a dynamical system. This graph was named after Poincaré in 1889 [145]. The Poincaré section shows a group of points. These points are selected stereoscopically of a phase space of a dynamical system. Therefore, if the orbit is circular (or elliptical), the graph shows only one point. While if the orbit is double-periodic the graphs shows two points. and it shows many points for a chaotic behaviour. In general the graph shows a tree-like figure for a nonlinear dynamical system [134], [135], [136]. Chaotic pendulums are investigated by this map in textbook [139]. The Poincaré map for a chaotic pendulum is shown in Figure 2.31
3 Nonlinear Dynamic Behaviour

In this chapter, the investigation of nonlinear dynamic systems is presented. The different effects caused by different types of nonlinearity were investigated gradually.

For the sake of simplicity, it was essential to start with simple cases and isolated problems. Therefore, the investigation was initiated by studying a chaotic pendulum. Then, to explain some causes of nonlinearity and solution algorithms, beam problems were investigated.

Then, to study the plate with different conditions, plate numerical model were first adjusted and then verified. The verification process were demonstrated using different methods. Comparisons with analytical results and with those reported in the literature were also included.

Then, the plate model was studied and the effects of the two types of nonlinearity were demonstrated using different means. The geometrical nonlinearity was demonstrated through the bifurcation map and the backbone curve, while the stick-slip nonlinearity was revealed using hysteresis loop.

Finally, in the last paragraph, models for frictional joints were studied and some solution were suggested.

3.1 Pendulum

Consider a simple pendulum consisting of a negligible-weight rigid rod of length $l$ and a mass, $m$, as shown in Figure 3.1. The rod rotates about a pivot in a medium of viscous, $ν$. This pendulum is periodically driven by an external torque.

![Figure 3.1: A simple Pendulum](image)

The rod is balanced by four moments: the moment due to weight, viscous force, inertia and the driving torque.

Taking the balance of the moment about the pivot point yields the equation of motion as follows
\[ m l \frac{d^2 \theta}{dt^2} + n \frac{d \theta}{dt} + m g \sin \theta = \alpha \cos(\omega t) \quad (3.1) \]

where \( \alpha \) and \( \omega \) are the amplitude and angular frequency of the external driving torque respectively.

The governing equation is a second-order Ordinary Differential Equation (ODE).

For convenience, the equation is normalized by the following conversion:

\[ t = \omega_o t', \]
\[ \omega = \frac{\omega}{\omega_o}, \]
\[ Q = \frac{mg}{\omega_o \nu}, \]
\[ A = \frac{\alpha}{mg} \]

Where

\[ \omega_o = \sqrt{\frac{g}{l}} \quad (3.3) \]

Thus, the governing equation in normalized terms becomes:

\[ \frac{d^2 \theta}{dt'^2} + \frac{1}{Q} \frac{d \theta}{dt} + \sin \theta = A \cos(\omega t) \quad (3.4) \]

The presence of the \( \sin \theta \) term gives rise to nonlinearity. Thus, this pendulum, like most nonlinear systems, is not easy to solve using analytical means.

### 3.1.1 Linearization Process:

Fortunately, the solution is possible using the linearizing process. This process assumes a small amplitude of oscillations. In this case, the angle of the rod is assumed to be small. Therefore, it is possible to use the following approximation:

\[ \sin \theta \approx \theta \quad (3.5) \]

This leads to a linear system as follow:

\[ \frac{d^2 \theta}{dt'^2} + \frac{1}{Q} \frac{d \theta}{dt} + \theta = A \cos(\omega t) \quad (3.6) \]

This can be solved analytically using standard solution methods of differential equation of the second order.

The linear solution gives the time histories of the state variables as follows:
\[
\theta(t) = \left\{ \begin{array}{l}
\theta(0) - \frac{A(1 - \omega^2)}{(1 - \omega^2)^2 + \frac{\omega^2}{Q^2}} e^{-\frac{i}{\omega^2}} \cos(\omega t) + \\
+ \frac{1}{\omega} \left\{ v(0) + \frac{\theta(0)}{2Q} - \frac{A(1 - 3\omega^2)/2Q}{(1 - \omega^2)^2 + \frac{\omega^2}{Q^2}} e^{-\frac{i}{\omega^2}} \sin(\omega t) \right\} \\
+ \frac{A}{(1 - \omega^2)^2 + \frac{\omega^2}{Q^2}} \left\{ (1 - \omega^2) \cos(\omega t) + \left( \frac{\omega}{Q} \right) \sin(\omega t) \right\} \\
+ \left( 1 - \omega^2 \right) \cos(\omega t) + \left( \frac{\omega}{Q} \right) \sin(\omega t) \right\} \\
\end{array} \right.
\]

(3.7)

\[
v(t) = v(0) - \frac{A\omega^2/Q}{(1 - \omega^2)^2 + \frac{\omega^2}{Q^2}} e^{-\frac{i}{\omega^2}} \cos(\omega t)
\]

\[
\left\{ \left( \frac{\omega}{Q} \right) \sin(\omega t) + \left( \frac{\omega}{Q} \right) \cos(\omega t) \right\} \\
+ \left( 1 - \omega^2 \right) \sin(\omega t) + \left( \frac{\omega}{Q} \right) \cos(\omega t) \right\} \\
\right\}
\]

(3.8)

Where,

\[
\omega = \sqrt{1 - \frac{1}{4Q^2}}
\]

(3.9)

Clearly, the solution contains two components, the transient and the steady-state response. After a sufficient amount of time, the transient solution decays away. Only the steady-state solution remains, therefore, the two equations (3.7) and (3.8) become:

\[
\theta(t) = \frac{A\left[ (1 - \omega^2) \cos(\omega t) + \left( \frac{\omega}{Q} \right) \sin(\omega t) \right]}{(1 - \omega^2)^2 + \frac{\omega^2}{Q^2}}
\]

(3.10)
Eliminating the time between equations (3.10) and (3.11) yields the phase-space equations as follows:

\[
\left( \frac{\theta}{A} \right)^2 + \left( \frac{v}{\omega A} \right)^2 = 1,
\]

where

\[
\tilde{A} = \frac{A}{\sqrt{1 - \omega^2} + \frac{\omega^2}{Q^2}}.
\]

As is clear from the equation, the phase-portrait orbit is an ellipse whose principal axes are aligned with the \( \theta \) and \( v \) axes. The orbit is a close curve, which means that the state-space variables repeat themselves at a fixed time. This also means that the motion is periodic in time and the period interval is given by:

\[
\tau = \frac{2\pi}{\omega}.
\]

Clearly, the shape of the orbit does not change (remains ellipse) with the changing of the control parameters, \( A, Q \) and \( \omega \). However, this is true under the assumption of small oscillation (linear solution).

The question remains: what does the phase portrait (phase-space orbit) look like? Does the shape change when the system parameters are changed?

To figure out the answers, the complete system of equations of large oscillation pendulums should be integrated into the time domain. Unfortunately, numerical means is the only possible method to integrate a nonlinear system. The numerical integration for the nonlinear pendulum is presented in the next paragraph.

### 3.1.2 Time Domain Integration, Runge-Kutta

There are many methods to obtain the solution in the time domain such as the Euler, the Runge-Kutta, the Adams-Moulton and the Numerov methods.

In this exercise, the governing nonlinear ordinary differential equation of the pendulum has to be numerically integrated. To this end, the explicit, fourth-order Runge-Kutta method was used, by involving the MATLAB built-in function \texttt{ode45}.

In order to use this function, the ODE should be written in the following first-order state-space format:
\[
\frac{dX}{dt} = AX + B \tag{3.15}
\]

Assuming the first order derivative of the angle as a new state variable gives the angular velocity as follows

\[
\frac{d\theta}{dt} = \nu \tag{3.16}
\]

Deriving equation (3.16) and substituting in equation (3.4) yields

\[
\frac{dv}{dt} = -\frac{\nu}{Q} \sin \theta + A \cos(\omega t) \tag{3.17}
\]

Using state-space format yields

\[
\begin{bmatrix}
\frac{d\theta}{dt} \\
\frac{dv}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & -\frac{1}{Q}
\end{bmatrix}
\begin{bmatrix}
\theta \\
v
\end{bmatrix} +
\begin{bmatrix}
0 \\
-\sin \theta + A \cos(\omega t)
\end{bmatrix} \tag{3.18}
\]

This equation can be translated into MATLAB code as follows

```matlab
function dXdt = Rhs(t,X,Q,A, \omega)
    A=[ 0 1; 0 -1/Q];
    B=[0; -sin(X(1))+A*cos(\omega*t)];
    dXdt =A * X + B
end
```

or it can be rewritten in a simple form as follows:

```matlab
function dXdt = Rhs(t,X,Q,A, \omega)
    dXdt =[X(2); ...
    -X(2)/Q-sin(X(1))+A*cos(\omega*t)];
end
```

where Rhs is a user-defined MATLAB function for the equation (3.18). This function requires 5 parameters: the time vector \( t \); the state-space vector \( X \), and the equation parameters \( Q, A \) and \( \omega \).

The state vector, in this particular application, consists of two components: the first is the angle of the pendulum and the second is the angular velocity. However, it is worth to mentioning here, that this system is a single-degree-of-freedom system (dofs), the length of the state-space vector is two. Therefore, for an \( n \)-dofs system the length of the state-space vector is \( 2^n \).

The following integration parameters were used: the relative tolerance, RelTol, was 1E-8 and the absolute tolerance, AbsTol, was 1E-8, NormControl was off.

This can be defined using the following MATLAB statement:
opts = odeset('RelTol',1e-8,'AbsTol',1e-9,'NormControl','off');

In order to obtain the solution, the initial condition should be defined as follows:

\[ X_0 = [\theta_0 \ v_0] \quad \text{\% initial condition} \]

Knowing the time vector span, the following code can be used to obtain the step-by-step time series of the state vector as follows:

\[ [t, \ dXdt] = \text{ode45}(@rhs,tspan,u0,opts,Q,A,w); \]

The output contains two components: the time vector at which the output is calculated and the solution matrix, \( dXdt \). The solution matrix , \( dXdt \), contains two columns, the first column represents the angle of the rod and the second represents the angular velocity. Both columns are calculated at the corresponding time values given in the first component.

In order to test this procedure, the following value was chosen:

\[ A = 1.5, \ \omega = 2/3, \ \text{initial condition at } t = 0, \ \theta(0) = 0, \ v(0) = 0 \]

Three different values for \( Q \) were selected as follows: \( Q = 1.31, \ 1.36, \ 1.385 \). These solutions were found using the linear-approximation theory (without the linearization). According to these values, the MATLAB code gave the time series of the state variables. Using this data, it was possible to establish the phase-portrait for the six cases. The results are presented in Figure 3.2. This figure shows 6 graphs (3 cases for linear and 3 cases for nonlinear solution).

As it is clear from the graphs, the orbit in the linear cases remains an ellipse whatever the value of \( Q \). On the contrary, in the nonlinear cases the orbit shows an ellipse with some degeneration at low value of \( Q \) (\( Q = 1.31 \)). When the value of \( Q \) rises, the degeneration increases significantly, and the solution repeats itself (periodic motion), until the limit where the orbit splits into two orbits (\( Q = 1.35 \)).

This occurs because the nonlinearity term becomes significant and the small-angle approach becomes invalid. The results show double periodic mode.

At a higher value of \( Q \), the phase portrait shows many orbits. The motion is no longer periodic. The behaviour in this case is chaotic or stochastic. It is worth to mentioning here that the solution does not show any periodicity. Nonetheless, this does not mean that the behaviour of the pendulum is not deterministic.

However, it is important to distinguish between deterministic behaviour and random behaviour. In the former, the behaviour is predictable using some mathematical means. This is because it follows some physical laws, as in the case of the present pendulum. But in the latter, the signal does not obey any physical law. Thus, the value of a random signal is not predictable. However, it is possible to give probability values by studying the time histories of the random signals.

In order to study how the transaction between the different modes of behaviour works, the phase-portraits are summarized in one graph by reducing the order of the orbits.
The results are plotted against a control parameter. The resulting graph is called Poincaré Map. This process is presented in the next paragraph.

![Graphs showing phase portrait trajectory of a linear/nonlinear, damped, periodically driven, pendulum. Results calculated numerically for $A = 1.5; \omega = 2/3$; at $t = 0, \theta(0) = 0, \nu(0) = 0$.](image)

Figure 3.2: The phase-portrait trajectory of a linear/nonlinear, damped, periodically driven, pendulum. Results calculated numerically for $A = 1.5; \omega = 2/3$; at $t = 0, \theta(0) = 0, \nu(0) = 0$.

### 3.1.3 Poincaré Map

However, Figure 3.3 shows the Poincaré section as it is computed according to the following parameters: $A = 1.5, \omega = 2/3$, initial condition at $t = 0$, $\theta(0) = 0, \nu(0) = 0$, number of points taken per cycle = 100. This tree-like graph is in agreement with the one introduced in the previous chapter (see Figure 2.31). A small difference was
noticed at the bifurcation region. The reason may be due to the need for further numerical processing.

Figure 3.3: Poincaré section of a nonlinear pendulum computed by MATLAB code involving Runge-Kutta method

The graph shows the range where the pendulum is fully linear or periodic as well as the region where the behaviour is fully chaotic. In addition, it allows capturing the onset of chaos.

3.2 Beam

To solve a continuous structure using numerical means, it should be discretized. In this thesis, the finite element method is used, as was described briefly in the previous chapter. As a result of the discretization process, a multi-degree of freedom system is obtained. Thus, the nonlinear Partial Differential Equations (PDE) of a beam or a plate are usually reduced to nonlinear Ordinary Differential Equations (ODE) using the discretization process.

Classical analytical solution methods for such nonlinear systems are applicable only in some exceptional cases. Therefore, the numerical iterative procedures are combined with time integration methods in order to tackle such problems.

The successive approximations method, the small parameter method, the method of successive loadings [15], the full and modified Newton-Raphson methods [107], the secant line method, the bisection method and the two-point iterative method are different schemes to deal with nonlinearity problem [133].
3.2.1 Nonlinear Algorithm

In this thesis, the Newton-Raphson iteration was adapted to solve the nonlinearity in beam and plate problems. Therefore, in the next two paragraphs, flowcharts of the adapted schemes are shown to solve the geometrical nonlinearity of a beam subjected to a static load. A similar but more complex flowchart to integrate the dynamic system using an explicit scheme is then presented.

3.2.1.1 Static Nonlinear Algorithm

In the following, a flowchart for a nonlinear iterative algorithm is presented to solve the nonlinear beam bending problem with a static load condition in Figure 3.5. The beam is assumed to be axially constrained so the membrane force is generated (see Figure 3.4). The flowchart clarifies the iterative scheme based on the Newton-Raphson method and is adapted to fit the geometrically nonlinear beam bending problem. Many books explain such methods. Interested readers are referred to textbooks such as [15], [107].

To solve the geometrically nonlinear beam equation from the following form

\[ [K_b] \{u\} + N [K_p] \{u\} = \{F\} \] (3.19)

where \([K_b]\) is flexural stiffness matrix, \([K_p]\) the geometry matrix, \([F]\) the effective force and \(N\) the membrane force. The membrane force is calculated by FDM 1st derivative Matrix, \([D_1]\), as is shown in the flow chart in Figure 3.5.

![Figure 3.4: Axially constrained beam and subjected to distributed load](image-url)
3.2.1.2 Dynamic Nonlinear Algorithm

When inertia effects are significant in nonlinear problems, there is the choice of using either an implicit integration scheme or an explicit scheme. However, it has been proven that the explicit scheme is simpler to utilize, mainly because it does not implement the tangent stiffness matrix [107].

In this scheme, it is assumed that the solution is known at time $t$ and the solution at $t + \Delta t$ is to be found. To make it simpler, it is assumed that the applied loads include the damping and inertia forces [107].

Say the dynamic equilibrium equation at time $t$ is given by

$$[M] \{ \ddot{u} \}_i + [C] \{ \dot{u} \}_i = \{ P \}_i - \{ F \}_i \quad (3.20)$$

where $\{ F \}_i$ is the assembled vector of element nodal forces.

The finite difference expression for the nodal velocities at the current time is

$$\{ \dot{u} \}_i = \frac{1}{2\Delta t} (\{ u \}_{i+\Delta t} - \{ u \}_{i-\Delta t}) \quad (3.21)$$
Similarly, the finite difference expression for the nodal accelerations at the current time is

\[
\{\ddot{u}\}_i = \frac{1}{2\Delta t^2} (\{u\}_{t_i+\Delta t} - 2\{u\}_i + \{u\}_{t_i-\Delta t})
\]  

(3.22)

Substituting equations (3.21) and (3.22) into equation (3.20) and rearranging the equation so that only quantities at the new time are on the left-hand side gives

\[
\begin{bmatrix}
\frac{1}{2\Delta t} C + \frac{1}{\Delta t^2} M
\end{bmatrix} \{u\}_{t_i+\Delta t} = \{P\}_i - \{F\}_i
\]

\[
- \begin{bmatrix}
\frac{2}{\Delta t^2} M
\end{bmatrix} \{u\}_i - \begin{bmatrix}
\frac{1}{\Delta t^2} M - \frac{1}{2\Delta t} C
\end{bmatrix} \{u\}_{t_i-\Delta t}
\]

(3.23)

The initial conditions are given in terms of displacement and velocity vector at the beginning. The initial acceleration is found using the equation of motion at \(t = 0\). These equations also yield the displacements \(\{u\}_{t_i-\Delta t}\)

\[
\{u\}_{t_i-\Delta t} = \{u\}_0 - \Delta t \{\dot{u}\}_0 + \frac{1}{2\Delta t^2} \{\ddot{u}\}_0
\]

(3.24)

After having the state vectors at \(t = 0\), the solution is progressed step-by-step according to the solution procedure which is shown in the flowchart in Figure 3.6.
Specify algorithm parameters such as number and size of time steps

Read the initial geometry and material

Initialize the triads

Assemble the effective inertia matrix

\[
[M_{\text{eff}}] = \frac{1}{2\Delta t} [C] + \frac{1}{\Delta t^2} [M]
\]

Decompose the inertia matrix if necessary

\[
[M_{\text{eff}}] = \{u\}^T \{D\} \{u\}
\]

Specify the initial conditions
For \{u\}_0 and \{u\}'_0
Obtain \{\dot{u}\}_0 from the equations of motion

Read the load vector \{P\}_t at time t
It may be necessary to interpolate this from non-equispaced values.

Begin loop over time increments

Assemble the nodal loads vector \{F\}

Form the effective load vector

\[
\{P_{\text{eff}}\}_t = \{P\}_t - \{F\}_t - \left[ -\frac{2}{\Delta t^2} M \right] \{u\}_t - \left[ -\frac{1}{\Delta t^2} M - \frac{1}{2\Delta t} C \right] \{u\}_{t\Delta t}
\]

Solve for the new displacement increments from

\[
\{u\}_t^T \{D\} \{u\}_{t\Delta t} = \{P_{\text{eff}}\}_t
\]

Update the geometry, stresses and triads

Store results for this time step

End

Figure 3.6: Flowchart for an iterative algorithm for a nonlinear dynamic
3.2.2 Code Validation

In this section, before proceeding with simulation, it is essential that the numerical routines are validated. Therefore, simple cases of beam problems were solved initially using analytical formula and then using the ANSYS Parametric Design Language (APDL). The in-house program was then used to calculate the corresponding results. The comparisons were made for the linear and nonlinear solution in addition to natural frequencies. The validated code was subsequently used to perform the stick-slip boundary conditions and to show the phenomenon of stick-slip.

For the sake of simplicity the same data was used for all analyses. The following parameters were used: beam length \( L = 0.4 \) m, static load \( P = 2000 \) N, material density \( \rho = 2700 \) kg/m\(^3\), Young’s modulus \( E = 7.\times10^{10} \) N/m\(^2\), Possion’s ratio \( \nu = 0.3 \), Shear coefficient \( G = 2.692\times10^{10} \) N/m\(^2\) (calculated value), beam cross-section width \( b = 0.028 \) m, beam cross-section height \( h = 0.004 \) m.

The beam dimensions were deliberately chosen so that the slenderness ratio was reasonably high. The slenderness ratio

\[
0.346 = \frac{I}{ALs} \quad (3.25)
\]

where \( L, A \) and \( I \) are the beam length, beam’s cross-sectional area and the second moment of area of the beam cross-section respectively. Apparently, the slenderness ratio is much greater than 100 (\( s > 100 \)). Therefore, the rotary and the shear effects are negligible. However, for impact loads this is not true. For more details on the significant terms involved in other beam theories, paper [117] is recommended.

3.2.2.1 Calculation of Beam Deflection Using Different Methods

1 - Linear analytical solution: maximum beam deflection under uniform distributed static load for C-C beam using EBT is given in [118] (see Table 8.4.1). For the given data, the maximum deflection at the midpoint was found to be: \( W_{\text{max}} = 0.03189 \) m.

2 - Linear numerical analysis was carried out using commercial package ANSYS. The maximum deflection of the beam at the centre was found to be 0.031766 m. Clearly, this is in agreement with the analytical value.

3 - Nonlinear numerical solution: the same ANSYS model was used, but the large-deflection was considered to include the geometrical nonlinear effects. The result for the maximum deflection of the beam was again calculated at the centre, but the deflection dramatically dropped to 0.008074 m. Also, the reactions at the beam ends were investigated and listed in Table 3.1 along with their summation.

The reaction includes the force components \( F_X, F_Y, F_Z \) and moment components \( M_X, M_Y, M_Z \) at the ends of the beam. Here there are 6 dof at each node.
The axial reaction at the left and right hand ends, unlike the previous case, were found to be nonzero. The only reasonable interpretation for this is the inclusion of the membrane forces due to the large deflection effects. As the deflection is reduced, the system exhibits only stiffening type nonlinearities.

Table 3.1: Reactions at the two ends of the beam obtained using large deflection beam theory.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Left hand end</td>
<td>-7881.900</td>
<td>0.000</td>
<td>-1000.0</td>
<td>0.000</td>
<td>0.000</td>
<td>-29.569</td>
</tr>
<tr>
<td>Right hand end</td>
<td>7881.900</td>
<td>0.000</td>
<td>-1000.0</td>
<td>0.000</td>
<td>0.000</td>
<td>29.569</td>
</tr>
<tr>
<td>Total</td>
<td>0.000</td>
<td>0.000</td>
<td>-2000.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Obviously, the applied distributed load in Z-direction is balanced by the vertical reactions at two ends F_z, but the two opposing horizontal reactions, F_x, of ±7880 N work against each other. In fact, these are similar to a beam case in which an external tension is axially applied. This force resists lateral deflection; therefore, the maximum deflection in the linear theory (31.8E-3 m) is reduced by 4-times in the nonlinear theory (8.07E-3 m). However, the reactions in the Y-direction are still zeros because of the absence of load in the Y-direction.

4 - In-House linear theory based on MATLAB codes: in order to simulate the clamped condition at the left-end of the beam, the parameters of the proposed model should be tuned as follows (see Figure 3.7): the vertical spring constant is set to infinity or $K_v$ is set to $1E16$ N/m and the rotational spring constant is set to infinity or $K_r$ is set to $1E16$ N/m. The other model parameters do not affect the results.

![Figure 3.7: Flexible support – clamped linear beam](image)

The beam deflection results are shown in Figure 3.8. It can be seen that the maximum deflection is at the centre of the beam whose value is found to be 0.032048 m.
Obviously it is in agreement with the results obtained in methods 1 and 2.

5 - In-House nonlinear theory based MATLAB code: the same data was used as the linear case, the final solution was obtained by nonlinear iteration.

The iteration method is done using the procedure as described in the flowchart in Figure 3.5.

The resulting deformed beam axis is plotted in Figure 3.9. To show the efficiency of the solution procedure, the developments of the beam solution during the iteration process are plotted as well. The convergence of the results is depicted in Figure 3.10. It was found that only six iterations are enough to obtain satisfactory results.

The program also gives the following results: the reaction force is 7956.736 N and the nonlinear beam deflection at the centre is 0.0080 m.

Noticeably, these results are in reasonable agreement with those obtained in method 3.

![Figure 3.8: The deformed shape of the beam by linear theory calculated by MATLAB](image)
Figure 3.9: Deformed beam shape during nonlinear iterations performed using the MATLAB code

Figure 3.10: Convergence of results for the iterative nonlinear analysis of beam

### 3.2.2.2 Fundamental Frequencies Comparison

The frequency equations for linear beam theory for diverse boundary conditions are given in handbook [113] and textbook [119]. For clamped-clamped beam, the frequency equation is given by the following formula:

\[
\cos(k\ell)\cos(k\ell) = 1
\]

(3.26)

Where \(k\) is the wave number and \(\ell\) is the frequency parameter in this work.

Unfortunately, analytical solution for this formula is not available. Alternatively, approximation values may be found numerically and given in hand-books. The first
four frequencies for clamped-clamped beams are given by [113] (see Table 7.3). Reference [119] (see Table 5.3) gives more modes with 9 digits after the comma. Nonetheless, the built-in MATLAB function \texttt{fzero} is used to obtain the first ten constants. It is found that there is only a little deviation from those given in [119]. Therefore, the following natural frequency equation in Hz, as given in handbook [113], was used

$$\omega_n = k \ell \sqrt{\frac{E I g}{\gamma A}}$$ \hspace{1cm} (3.27)

The results for the natural frequencies obtained using equation (3.27) are listed in Table 3.2. For the sake of comparison, the natural frequencies for the same beam problem are obtained using the in-house program, whose numerical results are also listed in Table 3.2. The data is arranged as follows: the first column is the mode number, the second column presents the analytical natural frequencies using analytical equation (3.27) and the third column gives the numerical results and, finally, the forth column gives the relative error. Evidently, a reasonable agreement exists until the tenth mode.

Table 3.2: Natural frequencies of the beam under study using analytical and numerical methods.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Natural frequencies Hz</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytic</td>
<td>MATLAB</td>
</tr>
<tr>
<td>1</td>
<td>131.543</td>
<td>130.848</td>
</tr>
<tr>
<td>2</td>
<td>362.602</td>
<td>360.688</td>
</tr>
<tr>
<td>3</td>
<td>710.845</td>
<td>707.092</td>
</tr>
<tr>
<td>4</td>
<td>1175.063</td>
<td>1168.858</td>
</tr>
<tr>
<td>5</td>
<td>1755.342</td>
<td>1746.072</td>
</tr>
<tr>
<td>6</td>
<td>2451.675</td>
<td>2438.728</td>
</tr>
<tr>
<td>7</td>
<td>3264.065</td>
<td>3246.827</td>
</tr>
<tr>
<td>8</td>
<td>4192.510</td>
<td>4170.369</td>
</tr>
<tr>
<td>9</td>
<td>5237.011</td>
<td>5209.354</td>
</tr>
<tr>
<td>10</td>
<td>6397.567</td>
<td>6363.782</td>
</tr>
</tbody>
</table>

### 3.2.3 Modelling Stick-Slip Boundary Conditions

So far the program has been tested and the results are in agreement with those obtained using other methods. In this paragraph an attempt has been made to model a beam with dry-friction element from the left-end while being fully constrained from the other end as shown in Figure 3.11: Stick-slip Beam (joined end).
Accordingly, the parameters of the model are tuned in a way to capture the stick-slip phenomenon. However, to fulfil the full run conditions extra simulation parameters need to be chosen by trial-and-error in addition to those given in the previous sections. The parameters were chosen as follows: number of force increment is 10, time for simulation is 0.1 sec, for excitation: the beam is driven by a uniformly distributed sinusoidal force. The signal frequency is taken as 20% of the first natural frequency; and the signal amplitude is taken as 2000 N. The corresponding plot is shown in Figure 3.12. Simulation time step is selected such that the number of time steps in the integration scheme per cycle of the external force is 200, which is similar to numerical studies in [139].

\[ dt = \frac{2\pi}{\omega} \frac{1}{200} \]  

(3.28)

Additionally, the other values are: stiffness proportional damping matrix \([C] = 0.005[K]\), friction coefficient for Aluminium-Aluminium \(\mu = 0.57\), vertical
spring constant $K_v = 371\text{E}6 \text{ N/m}$, horizontal spring constant $K_h = 19.6\text{E}6 \text{ N/m}$, rotational spring constant $K_r = 10\text{E}3 \text{ N/m}$ and overall simulation time is 0.1 sec.

Finally, for the given data the simulation is possible. The simulation results are shown in Figure 3.13, Figure 3.14, Figure 3.15 and Figure 3.16.

The beam response for that applied force is plotted in Figure 3.13. The figure shows the time history of displacements at the midpoint of the beam. Also, the velocity of the same point is calculated. Hence, the phase portrait for the midpoint of the beam is established and plotted in Figure 3.14 which reveals chaotic behaviour as an indication of nonlinearity.

![Figure 3.13: The beam response using geometrically nonlinear beam theory](image)

![Figure 3.14: The beam phase portrait calculated at the middle using geometrical nonlinear theory](image)
Additionally, the time histories for three forces are plotted in Figure 3.15; these are the real membrane force, the tangential frictional force and the expected membrane force that can be encountered if the beam was clamped.

![Figure 3.15: Forces at Stick-slip Model](image)

Figure 3.15: Forces at Stick-slip Model: —: c-c membrane force, - - - : real membrane force, ○ - ○ : tangential frictional force

Finally, Figure 3.16 presents the time history of the slippage of the left-end of the beam.

Visibly, the plots show that the force and displacement curves are not simple periodic variations.

![Figure 3.16: The time history of the Slippage at the beam end](image)

Figure 3.16: The time history of the Slippage at the beam end
By inspecting the previous plots, some conclusions can be drawn. These are presented in the next paragraph.

3.2.4 Discussion on the Numerical Results

Strong evidence for the stick-slip phenomenon can be detected by inspecting the slippage curve in Figure 3.16. The stick-state is characterised by flattening out the curve (stick-state). Conversely, the slip status is characterised also by the rising and falling of the curve.

The slippage curve explains how the system changes its status. For instance, when the slippage curve rises, it indicates that the left-hand-end moves to the right (slip-right-state). Conversely, when the curve declines, it indicates that the left-hand-end moves to the left (slip-left-state).

In all circumstances, at the beginning of the simulation time, the slippage curve starts from zero (stick-state) and stays for a short period of time. It then rises until the point where the three force curves cross each other as shown in Figure 3.15. At that point, the membrane force equals or exceeds the frictional force (slip-right). The slippage curve continues to rise until a new situation is reached where the horizontal spring develops a restoring force strong enough to resist any further slippage. Therefore, the beam-end sticks again (stick-state); hence, the slippage curve flattens out at sticking conditions. This, however, soon starts to change into a new situation where the membrane force and the frictional force again cross each other but in the case of membrane force there is a declination status. The restoring force of the horizontal spring forces the left-end to move back to the left (slip-left) but not enough to reach the original position. In this case, the slippage curve declines.

Generally, the slippage curve fluctuates upward, downward and flattens, but it does not reach or cross the zero value in spite of the fact that it starts from zero. Thereby, a dead-shot exists in oscillating the left-end of the beam. In other words, the left-end of the beam moves to the right and travels back to the left and, thus, oscillates on a new neutral point in which the beam is shorter. In fact, the neutral point is located slightly towards the right of the initial location. Consequently, this leads to another conclusion that when the beam end moves in a stick-slip fashion, the beam is subjected to axial compressive load (friction load) even if there are no external loads.

Another important conclusion that is revealed in Figure 3.15 is that the real membrane force is considerably less than the clamped-clamped membrane force. It is shifted slightly into a compression range in addition to the tension. Consequently, the absolute amplitude is less. To summarise, the stick-slip phenomenon works on limiting the force (force–limiter) and shifting the load range from pure tension into a new situation of alternating between tension and compression. The tension, however, remains dominant. Therefore, in some cases, the problem of beam buckling may arise due to those shots of the compression.
3.3 Plate

Similar to beam problems, the plate is a continuous domain. In order to solve this problem using numerical means, it was first discretized. The discretization process converts the plate PDE into ODE. The resulted equations still needed to be processed because they contained time derivative terms in addition to nonlinear terms. Therefore, to achieve the integration in the time domain, the Newmark’s method was implemented along with Newton-Raphson iteration to deal with nonlinearity.

In this chapter, the plate model was developed gradually. At the beginning, many comparisons using different methods are presented, in order to justify the selections of many parameters. Then other examples were solved to validate the model. The validation process was achieved by comparing plate case (simply-supported-immovable edges) with results reported in the literature. Additionally the bifurcation map has been presented for the clamped plate as a general descriptive means for any nonlinear dynamics system. After validating the plate model, the plate was again solved with stick-slip boundary conditions. The impact of stick-slip nonlinearity is shown in terms of the hysteresis loop, while the geometrical ones are shown using backbone-curve and bifurcation-map.

3.3.1 Spatial Discretization, dx, dy

Many different methods can be used to discretize a plate. FEM and FDM are common methods for such problems. The step of the discretization process has to be identified. Therefore, in the coming paragraph, some examples are presented to justify the selection of the plate model.

3.3.1.1 Comparing Different Finite Element Methods

There are two types of finite elements that can be used to discretise the spatial domain of the plate either 3D-solid elements or 2D-shell elements. Also the degree of the interpolation used in the element type can be linear, quadratic or higher. For the mid-side node elements, the interpolation is quadratic, while for elements without mid-side node the interpolation is linear. As a rule of thumb, the quadratic elements are more accurate because they have the capability to confine with physical fields using fewer elements. But, on the other hand, discretizing using it requires a higher number of nodes for the same number of elements, thereby it is considered computationally expensive. In order to select the element type and the degree of the interpolation, a case study of a square plate involving different types of elements was made. The investigation includes calculation of the first six natural frequencies under fully clamped conditions. Accordingly, the following data was used: Young’s modulus $E = 7.2E10 \text{ N/m}^2$, material density $\rho = 2.7E3 \text{ kg/m}^3$, plate thickness $h = 0.020 \text{ m}$, edge length $= 1 \text{ m}$, Poisson’s ratio $\nu = 0.3$. The results of the numerical model are tabulated in Table 3.3.
Four different elements were used, namely: a 4-node shell element, an 8-node shell element, an 8-node cubic element and a 20-node cubic element. For each, four levels of mesh density were used on the 25, 100, 400 and 2500 elements. These produced different number of nodes according to type of elements as seen in the table.

Table 3.3: Numerical comparison of a set of the first six natural frequencies of a square plate

<table>
<thead>
<tr>
<th>Element type</th>
<th>Element size mm</th>
<th>Number of nodes</th>
<th>Number of elements</th>
<th>1st Mode Hz</th>
<th>2nd Mode Hz</th>
<th>3rd Mode Hz</th>
<th>4th Mode Hz</th>
<th>5th Mode Hz</th>
<th>6th Mode Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-node shell element</td>
<td>200</td>
<td>36</td>
<td>25</td>
<td>195.84</td>
<td>489.08</td>
<td>692.92</td>
<td>1424.9</td>
<td>1431.6</td>
<td>1496.1</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>121</td>
<td>100</td>
<td>182.22</td>
<td>386.21</td>
<td>564.69</td>
<td>746.65</td>
<td>750.58</td>
<td>902.28</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>441</td>
<td>400</td>
<td>178.68</td>
<td>364.12</td>
<td>535.63</td>
<td>652.36</td>
<td>655.56</td>
<td>815.45</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2601</td>
<td>2500</td>
<td>178.95</td>
<td>364.92</td>
<td>537.86</td>
<td>654.15</td>
<td>657.27</td>
<td>819.87</td>
</tr>
<tr>
<td>8-node shell element</td>
<td>200</td>
<td>96</td>
<td>25</td>
<td>180.88</td>
<td>374.83</td>
<td>579.07</td>
<td>689.81</td>
<td>693.66</td>
<td>910.45</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>341</td>
<td>100</td>
<td>178.28</td>
<td>362.82</td>
<td>533.96</td>
<td>649.32</td>
<td>652.73</td>
<td>813.20</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>1281</td>
<td>400</td>
<td>178.23</td>
<td>362.39</td>
<td>532.76</td>
<td>646.99</td>
<td>650.23</td>
<td>809.13</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>7701</td>
<td>2500</td>
<td>178.22</td>
<td>362.37</td>
<td>532.73</td>
<td>646.85</td>
<td>650.09</td>
<td>808.98</td>
</tr>
<tr>
<td>8-node cubic element</td>
<td>200</td>
<td>72</td>
<td>25</td>
<td>288.89</td>
<td>770.21</td>
<td>1581.5</td>
<td>2100.10</td>
<td>2126.40</td>
<td>3180.80</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>242</td>
<td>100</td>
<td>189.32</td>
<td>406.30</td>
<td>633.98</td>
<td>786.85</td>
<td>787.50</td>
<td>1042.9</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>882</td>
<td>400</td>
<td>179.86</td>
<td>369.81</td>
<td>546.28</td>
<td>672.65</td>
<td>675.79</td>
<td>841.73</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5202</td>
<td>2500</td>
<td>178.48</td>
<td>363.50</td>
<td>534.51</td>
<td>650.70</td>
<td>653.92</td>
<td>813.23</td>
</tr>
<tr>
<td>20-node cubic element</td>
<td>200</td>
<td>228</td>
<td>25</td>
<td>183.64</td>
<td>383.47</td>
<td>609.30</td>
<td>704.72</td>
<td>708.42</td>
<td>953.61</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>803</td>
<td>100</td>
<td>181.08</td>
<td>373.61</td>
<td>551.25</td>
<td>683.28</td>
<td>686.58</td>
<td>852.49</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>3003</td>
<td>400</td>
<td>178.70</td>
<td>363.99</td>
<td>535.31</td>
<td>651.53</td>
<td>654.72</td>
<td>814.34</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>18003</td>
<td>2500</td>
<td>178.42</td>
<td>362.88</td>
<td>533.59</td>
<td>647.91</td>
<td>651.11</td>
<td>810.38</td>
</tr>
</tbody>
</table>

The consumed processing time was also calculated and is listed in Table 3.4.

Table 3.4: The performance of different types of elements in terms of computation costs.

<table>
<thead>
<tr>
<th>Element type</th>
<th>Number of nodes</th>
<th>Number of elements</th>
<th>Processing time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-node shell element</td>
<td>121</td>
<td>100</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>441</td>
<td>400</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>2601</td>
<td>2500</td>
<td>4.0</td>
</tr>
<tr>
<td>8-node shell element</td>
<td>341</td>
<td>100</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>1281</td>
<td>400</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>7701</td>
<td>2500</td>
<td>21.0</td>
</tr>
<tr>
<td>8-node cubic element</td>
<td>242</td>
<td>100</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>882</td>
<td>400</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>5202</td>
<td>2500</td>
<td>7.0</td>
</tr>
<tr>
<td>20-node cubic element</td>
<td>803</td>
<td>100</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>3003</td>
<td>400</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>18003</td>
<td>2500</td>
<td>37.0</td>
</tr>
</tbody>
</table>

In conclusion, the 8-node shell element with rather coarse mesh converged quickly to the final solution without a noticeable impact on the computation costs. This is particularly true for a reasonable number of elements, but with a large number of elements, perhaps unjustified, this element becomes expensive in comparison with the 4-node ones. For instance, 2500 8-node-elements take 21 seconds, whereas the same
number of 4-node elements takes 4 seconds. Conversely, the two types equally take 1 second to process 100 elements. However, the 3D-solid elements show poor performance. This is accepted since the ratio of the element becomes very high in this case.

3.3.1.2 The Step Size

It is well-known that the size of the mesh both in the x and y directions greatly affects the predictability and the performance of the plate model. The finer mesh means better accuracy but on the other hand, it requires extremely high resources. The computation cost, in particular, becomes more questionable when a full transient analysis is needed. Moreover, this becomes more critical when dynamical phenomena have to be studied on a microsecond time scale. For these reasons, a sensitivity analysis of the space mesh size is always necessary. Accordingly, the impact of the mesh size on the modal analysis results was studied. The results of the first ten modes are included in the observations. The main reason for this is that the model is intended to be subjected to different types and levels of excitation; thereby high modes could be excited. Therefore, the quality of the model is assessed by its capability to accommodate all possibly excited modes accurately. In order to simplify the mesh process, the plate was divided equally. Also the ratios of the elements were made equal and adjusted to obtain nearly square elements. Hence, the total number of elements uniquely define the mesh step in x and y directions. In this thesis, the computation was performed by APDL. Thereby, for every mesh density the first ten modes were computed. Hence, it is possible to plot the natural frequencies against the total number of elements. Thus, Figure 3.17 shows the first three-modes and Figure 3.18 shows the next three-modes and Figure 3.19 shows the last modes obtained. To obtain these results for the plate the following data was used: the plate dimensions were 0.5 x 0.2 x 0.0008 m, material density $\rho = 2840 \text{ kg/m}^3$, Poisson’s ratio $\nu = 0.35$ and Young’s modulus $E = 73\cdot10^9 \text{ N/m}^2$.

![Figure 3.17: Convergence the first three natural frequencies](image)
According to the graphs, the low frequency modes are less affected by the mesh density. Even with very few elements satisfactory results were obtained. The higher frequency modes, especially from mode number six, are much more sensitive to the space step size.

In conclusion, it was found that all modes converge to the final solution at 320 elements. For this reason, in this study, the step size that is compatible with this number of elements was used to obtain the numerical results.

### 3.3.2 Time Domain Integration, $dt$

The length of the time step affects the performance of the model in many aspects. The accuracy of the results and the computation time are greatly affected by the length of the time step. The convergence of the solution scheme is also affected.
In this model, Newmark’s method was used. Fortunately, this method can be tuned to guarantee the solution stabilities regardless of the time step. In spite of this, a compromise solution for the time step needs to be found in terms of computation costs and results accuracy.

### 3.3.2.1 The Parameters of Newmark’s Method

In this thesis, the Newmark method was used to integrate the system of equations into the time domain. Therefore, to make this method unconditionally stable, the amplitude decay factor, $\gamma$, was employed as shown in Zienkiewicz [122]. The parameters of the Newmark method were chosen as follows:

$$\gamma = 0.005$$  \hspace{1cm} (3.29)

$$\delta \geq \frac{1}{2} + \gamma$$  \hspace{1cm} (3.30)

$$\delta = 0.505$$  \hspace{1cm} (3.31)

$$\alpha \geq \frac{1}{4}\left(\frac{1}{2} + \delta\right)^2$$  \hspace{1cm} (3.32)

Substituting (3.31) in (3.32) yields

$$\alpha = \frac{1}{4}\left(\frac{1}{2} + 0.505\right)^2 = 0.25250625$$  \hspace{1cm} (3.33)

With these factors, the Newmark method is unconditionally stable [121], as revealed by the results of iteration which will always converge.

### 3.3.2.2 The Time Step Length

It is well-known that the time step highly affects the efficiency of the numerical integration in the time domain. Regardless of many factors, it should be noted that the complete system of equations, whether linear or nonlinear, need to be solved at every new time step. Consequently, compared with a static problem, if a static problem takes only one minute on a normal computer, then the same problem takes roughly 1000 minutes for its one-second-full-transient analysis at the one millisecond time step.

Furthermore, the selection of the time step length is also greatly dependent on the numerical integration scheme. For instance, in the explicit integration schemes there are some conditions that should be met to guarantee the solution stability. While the implicit schemes can be tuned to be unconditionally stable, the level of error is a major concern. One of the disadvantages of implicit schemes is that there are energy losses. As the Newmark integration method was used in this thesis, the time step selection was based on minimizing the spurious energy losses.
Therefore, to estimate the effect of the time step length on the overall computation costs and accuracy of the results, the impact of the time step was studied using the full transient analysis of a plate.

Accordingly, the plate model was analysed without any energy dissipation elements for simplicity reasons. The resulting system contains only mass and stiffness matrices. Generally, systems without damping matrices are conservative. The externally provided work equals the global stored energy. While the external work simply equals the applied force times the displacement, the global stored energy equals the sum of the kinetic energy and the elastic potential energy. In detail, when the kinetic energy is zero the elastic energy is maximum and vice versa. In all instances, the sum of the two types of energy theoretically equals the externally provided work. Practically, the global stored energy decays over time, even in a conservative system. Obviously, the decay is due to the numerical integration scheme. And this is a spurious energy dissipation.

Thus, the error due to the spurious energy loss can be estimated in terms of provided work, \( W \), kinetic energy, \( K \), and strain energy, \( S \), as follows:

\[
Error = \frac{W - (S + K)}{(S + K)}
\]  

(3.34)

To estimate the error, a full nonlinear transient analysis was done using APDL for a clamped-free-clamped-free plate. The plate data used was the same as that of the previous section. The number of elements was taken as 16x20 elements. To avoid expensive computation, the analysis was done only for 0.1 seconds. The applied force was taken half-sine-wave as illustrated in Figure 3.20.

![Figure 3.20: The external applied force](image)
The application time interval of this force was selected to give enough energy as fast as possible and to make the calculation for a free oscillation possible during the rest of the analysis time. The force amplitude was 10 N.

For convenience, in this study the sampling rate in Hz, which is the inverse of the time step, was used instead of the time step itself.

The full transient analysis was performed several times at different sampling rates. For every numerical experiment, the error according to equation (3.34) was estimated.

Thus, the errors are plotted against the sampling rates in Figure 3.21. The curve shows that when the sampling rate is increased the error decreases quickly. However, it also shows that the slope of the curve becomes less as the sampling rate is increased. In other words, at high values of the sampling rate the improvement is very small.

![Figure 3.21: The energy error VS sampling rate](image)

Furthermore, in order to make the selection of the sampling rate more justifiable, the computation costs are also plotted for all numerical experiments against the corresponding sampling rate as in Figure 3.22.

Additionally, the time histories of three types of energies: work, strain and kinetic energies are plotted at a sampling rate of 5000 Hz in Figure 3.23. The graph compares the variation of the two types of energies namely strain and kinetic energy. The stored energy, which is the sum of kinetic and strain ones, is nearly fixed after the transient solution. However, a very small decline exists but is not noticed.
To show this declination, the stored energies for three different numerical experiments are plotted at the sampling rates of 800, 5000 and 15000 Hz in Figure 3.24. This graph reveals clearly the impact of the sampling rate on the stored energy level. In brief, for harsh sampling rates as low as 800 Hz the stored energy is quickly drained out. At very high sampling rates as high as 15000 Hz, the stored energy is well preserved. Also, the case of 5000 Hz is acceptable from an energy preservation point of view and the computation cost is far less. As a consequence, the sampling rate of 5000 Hz was adopted and used in this thesis unless otherwise noted.
3.3.3 Comparing Between Linear and Nonlinear Theory

To show the significance of geometrical nonlinearity, the numerical plate model was solved twice, once using static load, and then again using dynamic load.

3.3.4 Static Analysis

The numerical plate model under static concentrated load was solved twice, once using the linear approach and then again using the nonlinear method. The concentrated load at the centre was changed from 0 to 50 N in steps of 1 N. For every load case, the maximum deflection was noted. Afterwards, the plate model was solved in a similar way using the nonlinear iteration process; the full Newton-Raphson iteration scheme was used. The convergence was always obtained. Noticeably, the plate deflection at the same point is much less than that the linear case, particularly at higher loads. To highlight effectively the differences between these two solutions, the results of the both schemes are plotted in one plane as seen in Figure 3.25. The deflection results are non-dimensionalised by the plate thickness.

The two curves in the graph clearly reveal the differences in the solution obtained by the two methods. The two curves remain nearly identical until the non-dimensional deflection becomes as high as 0.5, after that the two solutions start to deviate. In fact, the nonlinear solution shows significantly lower deflection than for the linear case at higher loads. This indicates that it is a stiffening type of nonlinearity. This also agrees with the fact that the linear solution is acceptable or it nearly equals the nonlinear solution as long as the deflection is less than or equal to half the plate thickness.
3.3.4.1 Dynamic Analysis

In this paragraph, two types of full transient analysis are presented: the first using linear theory and the second using the geometrical nonlinear theory. The system in either case was excited harmonically in the vicinity of the first mode applying a harmonionic force of amplitude 10 N away from the middle (close to the edge). The time histories of the external force for 0.3 sec is depicted in Figure 3.26.

The data for the plate under study was as follows: the plate dimensions were 0.5 x 0.2 x 0.0008 m, material density $\rho = 2840$ kg/m3, Poisson’s ratio $\nu = 0.35$, Young’s modulus $E = 73 \times 10^9$ N/m$^2$. The plate was clamped from the two short sides while the others were kept free. The results of the two methods are presented separately in the following two sections.

However, to make a valid comparison between different analyses, the deflection was always recorded at the middle of the plate. However, reactions were always taken only for one node at the centre of the short edge that was supposed to be clamped. This was later restricted by the joint model.
3.3.4.1.1 Linear Transient Analysis

The results of a full transient analysis for the clamped plate using linear assumption are plotted here. These graphs include the deflection at center in Figure 3.27, the in-plane reaction in Figure 3.28 and the lateral reaction in Figure 3.29.

The results of this case show that the time history of the plate deflection varies harmonically, due to the linearity assumptions. The amplitude increases because there are no damping elements and the frequency of the applied force agrees with the first natural mode.

The reaction to the in-plane-force curve reveals negligible values as indicated in Figure 3.28. This is because the linear plate model is not able to develop a membrane force.
In conclusion, to predict a true interaction between the plate structure and the supposed joint model, the linear theory is not adequate.

3.3.4.1.2 Nonlinear Dynamic Analysis

The results of the full transient analysis for a clamped plate using geometrical nonlinear theory are plotted here. In order to make a valid comparison, all the geometry and material data used here are the same as that of the linear case. In this case, however, nonlinear iteration was involved to figure out the graphs. These graphs include the deflection at center as shown in Figure 3.30, the in-plane reaction and the lateral reaction as shown in Figure 3.31 and Figure 3.32 respectively.

![Figure 3.30: The time histories of the geometrically nonlinear plate deflection at center](image)

![Figure 3.31: The time histories of the in-plane reaction for the geometrically nonlinear plate](image)

Unlike the linear case, the deflection curve shows irregularity. This is observed as long as the system is nonlinear and the response is not necessarily periodic.

Also, it can be seen from the reaction curves that the membrane force is developed by the plate structure. Moreover, the nonlinear plate theory is capable of predicting the in-plane force more effectively.

For this particular case, within the given load, the reaction at the edge-centre due to the membrane force could be as high as 1 N in the tension range, while the normal reaction is still similar to the linear case and varies between ±0.2 N.

Later, these two reactions will be transmitted to the frictional joint. As the membrane force is bigger, it will dominate the frictional joint response.
In conclusion, in-plane reaction does not exist in the linear analysis. In the nonlinear case, the model captures the membrane force reaction.

### 3.3.5 Validation of Numerical Solutions

For validation purposes, the numerical results of the finite element plate model should be compared to other possible methods under similar conditions.

In this thesis, the results are compared with analytical ones. Therefore, in this section, the natural frequencies of the Clamped-Free-Clamped-Free (CFCF) plate are calculated using analytical equations and tabulated along with similar numerical values and relative errors.

Then, the bifurcation graph for a plate is presented in order to highlight the common behaviour of nonlinear dynamic system.

Validation was also performed by comparing simply-supported immovable-edges plate results with similar results reported in the literature.

#### 3.3.5.1 Fundamental Frequencies

**3.3.5.1.1 Numerical Results**

The following data was considered: the plate dimensions were $0.5 \times 0.2 \times 0.0008$ m, material density $\rho = 2840$ kg/m$^3$, Poisson’s ratio $\nu = 0.35$, Young’s modulus $E = 73 \times 10^9$ N/m$^2$.

The first ten modes of the plate were found using numerical calculation and illustrated in Figure 3.33. The mode shapes of the plate show that the centre of the plate coincides with an anti-node in the first, the fifth and the seventh mode. Therefore, measuring the response at this location provides accurate information only for those mentioned modes, as will be seen later. However, corresponding frequencies associated with those ten modes are listed in the next section with analytical values in Table 3.5.
Figure 3.33: The first ten mode shapes of the studied plate
3.3.5.1.2 Analytical Results

A CFCF plate is generally analysed using approximate analytical formulation. The analytical equation of this particular plate case and those of other cases are given in Leissa [64]. The analytical equation to calculate the natural frequencies of CFCF plate is given as follows:

\[ \lambda = \omega_n \cdot a^2 \frac{\rho h}{\sqrt{D}} \]  

(3.35)

where

\[ D = \frac{E h^3}{12(1-\nu^2)} \]  

(3.36)

where E, \( \rho \) and h are Young’s modulus, material density and the plate thickness, respectively. The value of the non-dimensional constant, \( \lambda \), is available in Leissa [64] p.75.

The plate dimensions were 0.5 x 0.2 x 0.0008 m, material density \( \rho = 2840 \text{ kg/m}^3 \), Poisson’s ratio \( \nu = 0.35 \) and Young’s modulus \( E = 73 \cdot 10^9 \text{ N/m}^2 \).

The comparison between numerical and analytical results is shown in Table 3.5, along with the relative errors which show that the two methods are in agreement.

<table>
<thead>
<tr>
<th>No.</th>
<th>Natural frequencies [Hz]</th>
<th>Relative error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical</td>
<td>Numerical</td>
</tr>
<tr>
<td>1</td>
<td>17.456</td>
<td>17.368</td>
</tr>
<tr>
<td>2</td>
<td>33.072</td>
<td>32.120</td>
</tr>
<tr>
<td>3</td>
<td>48.240</td>
<td>47.916</td>
</tr>
<tr>
<td>4</td>
<td>73.106</td>
<td>71.530</td>
</tr>
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<td>165.140</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>187.280</td>
<td>188.670</td>
</tr>
</tbody>
</table>

3.3.5.2 Bifurcation Map

In order to figure out the bifurcation Map for a plate, the simulation for clamped case was performed. the dimensions of the plate studied were 1.0 x 1.0 x 0.005 m and the material properties: Poisson’s ratio = 0.3; Young’s modulus \( E = 7.2 \cdot 10^{10} \text{ N/m}^2 \); the plate was discretized into 8 x 8 elements.
The overall simulation time was 0.2 sec, the sampling rate was 10 KHz. The plate was excited harmonically by one wave of a uniformly-distributed force in the vicinity of the first fundamental frequency (44.85 Hz). The responses were free because the force was acting for only short period of time. The amplitude of the excitation was changed from 100 N/m$^2$ to 3000 N/m$^2$ in steps of 50 N/m$^2$. For every load case, the time histories of the lateral deflection and the velocity at the centre were recorded. Thus, the phase portraits were established. The phase portraits for some of the analyzed cases are shown in Figure 3.34.

![Phase portraits for CCCC-Plate](image)

Figure 3.34: Phase portrait for CCCC-Plate

In order to summarize the phase portraits into one graph, the phase portraits of different cases were reduced by discretizing the orbit stereoscopically. The resulting points were plotted against the amplitude of the excitation for each corresponding case. The resulting map is shown in Figure 3.35.
The Poincaré map for nonlinear plates shows agreement with the nonlinear behaviour of the pendulum. This is the case in most nonlinear dynamical systems, which show similar maps.

With this map, it is possible to distinguish between the periodic behaviour (linear theory is applicable) and the chaotic behaviour (aperiodic) where the linear theory gives poor results. The map also shows the transition between the two modes.

Additionally, the time series of the plate was analysed to investigate other phenomenon. Therefore, the FFT was calculated for the free response obtained in the previous cases. The results were the response in the frequency domain. Therefore, for every load case, there was a corresponding response. These responses were plotted in a 3D graph. The x-axis represents the frequencies (normalized by the first fundamental frequencies using linear theory). The y-axis represents the amplitude in the frequencies domain. The z-axis represents the amplitude of the excitation force. The resulting 3D graph is shown in Figure 3.36.

The graphs show that at low excitation, the FFT response shows one peak, which corresponds to the free response at the first mode.

At a higher level of excitation, the FFT response shows two peaks. It then shows three peaks. This is due to the fact that higher modes were excited. This happens in all nonlinear dynamical systems where the FFT response shows a grassy form.

On the contrary, a linear system shows only one peak. In the free response, it shows one peak corresponding to the excited mode. In the forced response, it shows one peak at the frequency of excitation.

The graph also shows that as the excitation rises, the peaks move to a higher value of frequencies. This means that the natural frequency rises at higher amplitude.
In order to investigate the variation of the natural frequencies with amplitude, the values of the fundamental frequencies and the corresponding lateral deflections (the maximum peaks of the steady-state solution) were recorded for every individual case. Therefore, it was possible to plot the natural frequencies (normalized by linear value of natural frequency) against the non-dimensionalized deflection, as shown in Figure 3.37.

Figure 3.36: Plate response in the frequencies domain at different levels of excitation

Figure 3.37: Normalized natural frequencies against non-dimensional deflection for clamped plate
The graph shows that when the deflection value rises to 140% of the thickness, the natural frequencies increase by 25% of the first mode (using linear approximation).

Additionally, the curve fitting process was carried out for the data obtained. Therefore, the nonlinear natural frequencies can be obtained for this particular case by applying the following empirical equation (in terms of linear natural frequency and the deflection-thickness ratio) as follows:

\[
f_{\text{nonlinear}} = f_{\text{linear}} \left( -0.1318 \left( \frac{\delta}{t} \right)^3 + 0.4108 \left( \frac{\delta}{t} \right)^2 - 0.1341 \left( \frac{\delta}{t} \right) + 1.0129 \right)
\]  

(3.37)

3.3.5.3 Comparison with Literature, Backbone Curve

The case of a simply supported square plate with immovable edges (see Figure 3.38) was studied by Amabili [71], Ribeiro P. [141], Herrmann and Chu [142], El Kadiri and Benamar [143], with the dimensions: 0.3 x 0.3 x 0.001 m, and the material properties: material density \( \rho = 2778 \text{ kg/m}^3 \), Poisson’s ratio \( \nu = 0.3 \), Young’s modulus \( E = 70 \cdot 10^9 \text{ N/m}^2 \).

\[
F(t) = F_0 \sin(2\pi ft)
\]

Figure 3.38: Simply supported plate with immovable edges

The plate was excited at the center by a harmonic force of amplitude 1.74 N and the modal damping was 0.065.

The results given by Amabili [171] are shown in Figure 3.39. Amabili figured out the first mode to be 53.02 Hz.

In order to obtain results for similar cases in the present model, the plate was discretized into 10 x 10 elements. Six degrees of freedom (three rotations and three translation components) \( u, v, w, R_{xy}, R_{yz}, R_{zx} \) were taken at each node.
The boundary conditions were set so that all rotations were allowed but the horizontal and vertical translations were restricted, as given below.

\[
\begin{align*}
    u &= 0, \quad v = 0, \quad w = 0 \\
    R_{xy} \neq 0, \quad R_{yx} \neq 0, \quad R_{z} \neq 0
\end{align*}
\] (3.38)

In order to obtain the first fundamental frequency of the plate, the modal analysis was achieved for the plate. The first fundamental frequency was found to be 52.902 Hz, which was in agreement with that of Amabili (53.02 Hz).

Secondly, step-by-step time nonlinear analysis was done using integration method of Newmark repeatedly. The frequency of the force was always selected between 40 Hz and 90 Hz. The overall simulation time was 0.5 seconds and the sampling rate was 5000 Hz for the all cases.

The simulation time was sufficient to obtain a stable solution and the sampling rate was also adequate for convergence and smooth response as shown in Figure 3.40.
Plate Response due to Lateral Force $F_z=1.74$ [N] at 30 Hz

Plate Response due to Lateral Force $F_z=1.74$ [N] at 35 Hz

Plate Response due to Lateral Force $F_z=1.74$ [N] at 40 Hz

Plate Response due to Lateral Force $F_z=1.74$ [N] at 50 Hz

Plate Response due to Lateral Force $F_z=1.74$ [N] at 80 Hz

Figure 3.40: The time histories of the lateral displacement of a simply-supported plate with immovable edges
Additionally, the plate model was excited by a harmonic force with linear-stepped frequencies. The variation was stepped from 70 Hz to 85 Hz four times. The amplitude of the force was kept constant at 1.74 N. To validate the applied force, the FFT was calculated which is shown in the Figure 3.41.

![Figure 3.41: FFT- for the applied force, stepped 4-times from 70 Hz to 85 Hz]

The graph shows 4 peaks at 70.0000, 73.7500, 77.5000, 81.250 Hz. The simulation time was 5 seconds. Other parameters were kept the same. Thus, the integration was performed and the plate response was obtained. The time history for the plate response is shown in Figure 3.42. The graph shows that the response changes according to the driving frequency-stepped force. The transient solutions decay roughly in 0.2 sec and stay stable until the next change.

Taking the stable solution from all cases yields the variation of the plate response against the excitation frequencies. To make the comparison with previously published results, the results obtained are plotted with that of Amabili [71] as shown in Figure 3.43.

The graph shows excellent agreement. However, there are small differences due to differences in methodology. In Amabili’s model, the plate was simulated with 16 dofs using trigonometric series. In the present model, the plate was modelled using second order finite element method with 6-degree of freedom. (The difference in the fundamental frequency is also due to this reason).
In conclusion, the results gave full validation of the model used. The model is applicable for other cases which only require the changing of the input parameters.

Additionally, for the sake of comparison (with other boundary conditions presented later), the time histories of the membrane forces, measured at the mid-edge of the four-similar-edges, were recorded. For reasons of brevity, the results for three frequencies are presented as shown in Figure 3.44.
These graphs show that the membrane force works in tension mode only and in some cases rises up to 15 N. This, compared with the applied force (1.74 N), is very high. Thus, this gives rise to the question as to what would happen if one edge of these four edges was restricted by dry-frictional elements (instead of in-plane-immovable edge), and to the question of whether the force is sufficient to overcome the friction-resistance? If so, what changes would occur in the plate? This was also investigated and presented in the next paragraph.

### 3.3.6 Modelling Stick-Slip Boundary Conditions

In order to investigate the effects of the stick-slip mechanism on the plate response, a new case of the plate was analysed. The three edges were simply-supported and in-plane-immovable while the fourth edge was restricted by dry-frictional elements which were uniformly distributed as shown in Figure 3.45.
The following parameters were chosen: plate dimension $0.3 \times 0.3 \times 0.001$ m, excitation amplitude $F = 1.74$ N, excitation frequencies $f = [30, 35, 40, 45 \ldots 100$ Hz], coefficient of friction for aluminium-aluminium $\mu = 0.57$, normal clamping force $N = 150$ N, horizontal spring factor $K_h = 1E8$ N/m.

During the simulation process, for this particular case, the results of the time histories of the membrane-force and the in-plane displacement of the centre of the stick-slip-edge were recorded.

In order to compare the present case and the previous case, the membrane force in the both cases were plotted against the frequencies in one plane. To obtain this graph, a similar process to that used in obtaining the backbone-curve was followed again. (The maximum peak of the steady-state solution recorded along with the corresponding frequencies of excitation). The resulting graph shows the membrane forces against the excitation frequencies as shown in Figure 3.46 for the two cases studied. Thus, a comparison was possible.

The figure shows that the maximum values for the immovable edge rise as high as 12 N while this value drops dramatically to 4.2 N. This limitation is due to stick-slip mechanisms. This supports the idea that stick-slip mechanism works as a mechanical filter (remove higher peaks in the membrane force).

It is noticed here that the normal-clamping force applied was 150 N. This force was not enough to prevent slipping, in spite of exciting by a relatively small force (1.74 N). Another numerical experiment was also carried out using a higher normal-clamping force (250 N). The stick-slip again occurred but at a lower scale. Therefore, in conclusion, it is found that in practical application a relatively high clamping force is required to obtain clamped conditions (prevent slipping).
Additionally, the hysteresis loop (force vs. displacement) was plotted. In this particular case, the loop was plotted in terms of the membrane force against the in-plane displacement both taken at the centre of the stick-slip-edge. Loops were possible for the following cases: 30, 40, 45, 60, 70, 80 Hz as shown in Figure 3.47.

The graphs show that at low deflection, the hysteresis loop is just a simple line. This is because there is no slipping (the area of the loop equals zero). When the deflection rises, the membrane force rises, the slipping occurs at 40 Hz (the hysteresis loop shows irreversible behaviour - the area of the loop equals the dissipated energy due to slipping). A higher level of excitation creates a higher value of slipping. Thus, the hysteresis loop becomes bigger (see the 45 Hz case). Thus, as the excitation increases, the area of the hysteresis loop increases. This is valid until the case of 70 Hz. After this, the amplitude of the plate drops suddenly, and the membrane force is not enough to enforce slipping. Therefore, the hysteresis loop is again a linear and similar to the low excitation ones. The area of the loop equals zeros.

Thus, the stick-slip mechanism is very likely to occur in practical application even in cases where the excitation is very small. Occurring stick-slip, as it was seen, gives rise to energy dissipation. These results interpret the fact that joints are responsible for more than 90% of structural damping. However, this will be clearer in the next chapter when the evaluation of the stick-slip measured in terms of its energy-lose-factor is shown.
3.4 Modelling Joined Structures

3.4.1 The Frictional Joint Mechanism

Consider a joined construction as seen in Figure 3.48. The figure shows two panels connected to one support by two rivets.

During the installation process of a bolt or a rivet, the stem or the shank is pretensioned. The heads of the rivet or bolt pressurize the local particles of the two
connected parts due to pre-tension as seen in Figure 3.49. The pressure is not distributed uniformly. Regardless of this, it creates a force normal to the contact area. The resulting force, which equals the integration of the pressure distribution over the pressurized area, equals the axial force in the rivet shank.

![Figure 3.49: Clamping pressure in a riveted joint](image)

The normal force creates an in-plane motion resistance (tangential frictional force). The maximum magnitude of the frictional force equals the product of the acting normal force and the coefficient of friction of the rubbing materials. This is valid regardless of the area of the contact.

The principle of the frictional joint relies upon the idea that the resulting interfacial force prevents any relative in-plane motion between joined parts. This is true as long as the transmitted force is less or equal to the interfacial force. But, in practical application, the transmitted load could exceed the intended level and then a relative slipping may occur.

In order to simplify, a description of the riveted beam example is useful.

In this case, if the beam is subjected to time-dependent lateral loads, then every particle that belongs to the beam will vibrate under the effect of the elastic forces and/or inertia forces. The particles just under the rivet heads will also vibrate. Therefore, the vibration is transmitted to the rivet shank.

In conclusion, the shank is also subjected to an axial oscillation. Hence, the initial value of the pre-tension is modified positively or negatively due to different shots of oscillations. Thereby, the normal clamping force is not a constant value. It is a function of two components, the initial strain (pre-tension) and a new component of strain, which is added due to the new state of vibration.
On the other hand, if the oscillation level of the beam is low enough and the linear assumption is valid, then the membrane force of the beam is small. In other words the effects of the in-plane force on the joint are negligible.

Conversely, if the beam is subjected to a higher level of oscillation, the beam will then develop a higher membrane force. The force causes a separation between the two parts. If the membrane force is not big enough, a micro-scale slipping could be initiated. Only the particle in an adjoining annulus to the shank and away from the rivet hole could initiate a relative in-plane motion on a micro-scale level (the pressure in the annulus area is small and decays to zero). The two parts do not fully slip.

Furthermore, if the oscillation level of the beam is higher (larger deflection), then the membrane force could exceed the interfacial frictional force. In this case, only the slip could extend into a macro-scale level.

In all cases of slip modes, the energy is converted into a thermal one. Furthermore, the relative motion could cause a local corrosion (fretting). This also has many additional effects. All these effects, as well as microslip, are out of scope of the thesis. The thesis is focused on the case of a macro-scale stick-slip case.

### 3.4.2 Idealization in the Literature

The boundary condition for riveted/bolted structures can be classified into four different categories:

1. Beam or plate using classical Boundary Conditions (BC) such as clamped or simply supported, such as Amabili [71].
2. Beam or plate with flexible supports at the boundaries, such as Jalali et al. [87], Abbas et al. [88], Abbas [89] and Knudsen et al. [90].
3. Rivet/bolt joint using 3D or 2D detailed FEM models, such as Ganapathy et al. [74], Fung et al. [75], Jiang et al. [76], Langrand et al. [77], Langrand et al. [78], Langrand et al. [79], Simsons et al. [80], Gaul [81] and Thoppul [82], Ganapathy et al. [83].
4. Beam or plate with flexible and energy dissipation elements, such as Song et al. [8], Kim et al. [84], Hanss et al. [85], Shiryayev et al. [86] and Jalali et al. [87], Ferri et al. [93], Ren et al [94], Dowell [99], Tang [100], Miles et al. [101] and Somnay et al. [102].

These studies show that each of these methods is valid for a certain type of analysis within some very strict conditions. Each method has both advantages and disadvantages.

For example, the first and the second types of modelling both have some deficiencies in capturing true physics. In the first type, the real interaction between in-plane and out-of-plane is oversimplified. The in-plane force is exaggerated in the clamped case or it is lost in the simply supported boundary condition case. This problem, however, is partially solved in the second type by introducing flexible constraints. Of course, in
some cases, it can be acceptable to use classical boundary conditions as shown later in the numerical results. It was found that the clamped condition was an acceptable approximation of the friction joint in some circumstances. In spite of this, the two models still have an obvious limitation in providing a vital source for energy dissipations. As discussed previously, energy dissipation is a key feature for any frictional joint.

Many studies which were done in the last decade have focused on investigating the interaction between fluid and structures. Unfortunately, the general trend in these studies was to adopt some similar strategies by simplifying BC. In spite of achieving step-by-step time analysis on a non-second level and capturing the fluid structure interaction according to well-documented procedures, this oversimplification limited the validity of the results of these studies. A comprehensive revision should be made, incorporating more realistic boundary conditions.

The third group of modelling relies on an extensive use of the numerical capabilities of the newly developed procedures of the finite element method using modern computers.

A large number of papers document this method of modelling such as [74-83].

In fact, the researchers stick to this methodology in this area due to the following reasons: it is easy to develop and to understand, particularly by involving the new commercial packages with such incredibly friendly user-interface.

Also, it introduces an easy method to investigate the behaviour of the material in the local area in the linear and plastic range.

In addition this type of modelling allows studying, fatigue failure, crack, rupture, loosening due to relaxation in joints and many other local behaviour phenomena. In some cases, the scenario of the riveting process itself can be simulated using such schemes.

In spite of all the facilities that were introduced by this method, it is an optimistic view to believe that the riveted/bolted joint is solved.

The most critical reason, which is frequently emphasized in literature in the contest of criticising such trend, is the high cost in terms of computation time which is needed to analysis only one joint. This is, however, more critical under dynamic condition and/or nonlinear computations. A simple numerical 3D-solid model was tested and will be introduced later in this thesis in order to navigate such possibilities. The results show the impracticability of this method.

Most of these studies focus only on modelling and analysing the joint itself, regardless of the structure joined. This kind of analysing, in fact, is meaningless in terms of understanding the mutual relationship between the joint and the structure being joined. The most important feature of the joint is its capability to grant the built-up structure the most important components of damping. Also, it participates in the interaction that
usually develops within the structure under large deflection conditions. These two mutual effects are prejudicially cancelled out within this strategy.

The fourth type of modelling technique for the joined structures, which is believed to be the most promising technique in this work, relies on reduced-order elements. They may include a combination of linear rotational or transitional spring elements and energy dissipation elements. The energy dissipation element was selected differently. Some used dry friction elements. Others used viscous-damper elements.

This strategy of substituting frictional joints is considered more acceptable for many reasons: the source of energy dissipation is regarded, as well as the in-plane degree-of-freedom constraints. However, some limitations were found.

Modelling using viscous elements or equivalent viscous damper is not acceptable in the case of step-by-step time analysis. This is because the behaviour of this type of element is not similar to the friction induced ones.

The modelling using dry friction elements is more acceptable but some deficiencies were noted in a number of studies.

For example: Ren et al. [94], in his modelling process for a beam, considered in-plane flexible constraints in series with a dry friction element as shown in Figure 3.50. In his thesis, he showed some interesting results but the main limitation here was that the normal clamping force was assumed time independent (not function to time).

Figure 3.50: The axial friction constraint work of Ren et al. [94]

In the same way, Dowell [99] neglected the time variation of the normal force in his model as shown in Figure 3.51.
Tang et al. [100] also assumed a constant normal clamping force in an attempt to evaluate the damping due to slipping at the boundaries of beams or plates as shown Figure 3.52.

Similarly, Miles et al. [101] studied a beam with dry friction constrains. He reported that the normal clamping force was also considered constant over time.
The problem of taking the variation in the normal clamping force into account is partially solved in Ferri et al. [93]. He considered a linearly variable force component in addition to a constant preload force as shown in Figure 3.53.


### 3.4.3 Joint Model Features

As a result of the discussion of the different techniques and the mechanism of the joined structures, this original research of this thesis indicates that the model for a joined structure must be characterized by the following features:

1. The joint idealization process relies upon the assumption that the tangential interfacial frictional force in joints plays a crucial role in keeping the connected parts together. This, however, does not agree with the classical design methods for such joints. Actually, the classical design method depends on the direct strength of the shank or the stem, while the frictional force and the normal pretension are considered as an additional safety factor. The reason for such an approach is due to difficulties and uncertainties associated with friction.

2. The capabilities of dissipating energy have to be attributed by using dry friction elements and by allowing the stick-slip mechanism.

3. The capability to interact with the in-plane degree of freedom of the structures joined.

4. The capability to interact with the out-of-plane degree of freedom by introducing rotational and translation flexible elements.

5. The dry friction element has to be provided by a feedback from the vertical degree of freedom of the flexible support. This is the only way to account the time dependency in the normal clamping force, as it is believed in this thesis.

6. Reasonable computational costs by using simple, reduced-order elements.

6. It is proved that the joint model and the structure model are strongly coupled. Therefore, analysing each individually greatly affects the predictability and the performance of each.
3.4.4 Idealisation Using Reduced-Order Elements (Stick-Slip)

Thus, in order to simulate the joint, a setup of four different elements is proposed, three springs and one dry friction element (see Figure 3.54).

The stem/shank is substituted by two springs: a vertical translational spring, $K_v$ and a rotational spring $K_r$. The horizontal resistance is substituted by two elements, dry friction element, $\mu$, and the horizontal translational spring, $K_h$. The normal force of the dry-friction element, $N$, is updated by the vertical reaction transmitted by the vertical spring. Tuning the four parameters to extremely high or low values allows the classical boundary condition clamped or free to be simulated.

![Figure 3.54: Riveted joint idealization](image)

In this study, the two elements (the horizontal spring and the dry friction element) can be connected either in parallel or in series. The difference between the two setups is clarified in terms of hysteresis loops. The parallel connection is depicted in Figure 3.55 and the series connection is depicted in Figure 3.56. The hysteresis loops in the two graphics show some similarities and some differences. They both have the capability to dissipate energy and the capability to accommodate stick-slip mechanism. Also the interaction with structures is possible in both connections by feeding back the lateral deflection to the control parameter of the dry friction element.

On the other hand, the difference lies within the starting phase of the displacement-force relationship. In case of parallel connection, the hysteresis loop shows that the slipping starts at a force that is equal to or greater than the frictional force. While, in the other style of connection, the slipping occurs at a force equal to the frictional force. In the former connection, the transmitted force is unlimited, while in the later one, the force is bounded by the maximum resistance of the friction element.
In practical applications, friction joints are not similar. The number of rows in riveted joints and the overlapping distance and the existence of additional material (rubber or adhesion) are but a few of the many factors that affect the mentioned relationship.

To mention an additional example, in some airplanes (e.g. Airbus A310-200) the wing is used as a fuel tank. Therefore, the riveted joints are tightened in a way to prevent leakage. This is, however, an additional task for the frictional joint.

Therefore, the appropriate setup is strongly dependent on the displacement-force relationship of the joint studied and it could be a combination of the two setups. Therefore, it is not practical to use only one setup for all types of joints.

Figure 3.55: Spring and dry friction in parallel and their hysteresis loop

![Figure 3.55](image)

Figure 3.56: Spring and dry friction in series and their hysteresis loop

![Figure 3.56](image)

In conclusion, the only effective method to properly select the horizontal elements and its’ connection schemes is to conduct an experiment. The objective of the experiment is to figure out the hysteresis loop for the joint studied under in-plane force. Based on the results, the elements could be tuned and selected. But in all cases, the lateral deflection has to be taken as an important component in controlling the dry friction.
element in addition to the initial component which represents the pretension. In addition, the horizontal resistance of the joint model has to be tuned to interact with the membrane force of the joined structure (beams or plates).

### 3.4.5 The Proposed Joint Model Formulations

The definition of strain formula in the rivet shank is given by

\[
\varepsilon = \frac{\Delta a}{a} \tag{3.39}
\]

where \(\Delta a\) is the axial change in the length, \(a\), of the rivet shank. The stress-strain equation is given by Hooke's law for one dimension case as follows

\[
\sigma = E \varepsilon \tag{3.40}
\]

where \(E\) is Young’s modulus and \(\varepsilon\) is the strain.

Using the definition of stress and substituting equation (3.39) into equation (3.40) and assuming the cross section of the rivet is circular yields

\[
F = E \frac{1}{a} \frac{\pi}{4} d^2 \Delta a \tag{3.41}
\]

where \(d\) is the diameter of the rivet shank. Hence, the equivalent stiffness is given by

\[
K_v = E \frac{1}{a} \frac{\pi}{4} d^2 \tag{3.42}
\]

To estimate the real change, \(\Delta a\), in the rivet’s shank length under load condition, suppose \(\delta_{st}\) is the static pretension in the rivet shank. It is also assumed that the change in shank/stem length consists of two components as follows

\[
\Delta a = \delta_{st} - w \big|_{t=0} \tag{3.43}
\]

The change in the rivet shank according to equation (3.43) is due to two components: the first component is the static pretension; the second component is due to the lateral deflection at the left-end of the beam under load condition at time instance, \(t\).

This equation leads to the following,

\[
F = K_v \left( \delta_{st} - w \big|_{t=0} \right) \tag{3.44}
\]

This is a normal force created by a rivet or a bolt and applied to the beam end against lateral motion. However, this normal force generates a pressure between the beam and its support at contact surfaces, where the presence of friction, say the coefficient of friction, \(\mu\), causes a tangential frictional force, \(N_1\), which is supposed to prevent relative axial motion. This force could be estimated using friction Coulomb’s law as follows.
\[ N_1 \leq \mu K_n \left( \delta_u - w_{x=0} \right) \]  

(3.45)

Obviously, the tangential resistance force given in the previous formula is a time-dependent function, because the left hand side of the equation includes the lateral deflection which is also a time-dependent variable.

**Stick-slip State Equations**

To simplify the explanation, a case of initially straight beam is discussed. The beam is subjected to lateral load enough to develop an in-plane force (axial force) due to curvature.

In this study, the beam foreshortening, which is induced due to the curvature, is designated as \( \Delta L \), the beam axial extension as \( \Delta_{bm} \). In order to understand the difference between those two quantities, suppose the beam is clamped-clamped. The beam will then extend as much as the foreshortening value. Hence, an axial strain is developed and this will generate an axial force (membrane force) according to Hooke’s law. This is true as long as the both ends are fully constrained.

Now, suppose, the beam left-end moves to the right. In this case, the beam extension is less than the foreshortening. As a consequence, the resulting axial force is less than the clamped-clamped case. Therefore, two cases of the situation can be distinguished. One, when no movement occurs, stick-state, and another when slippage occurs called slip-state:

1- Stick-state: in this case, the situation is similar to the clamped-clamped beam. Therefore, the beam foreshortening completely converts to beam extension, beam extension = beam foreshortening or mathematically

\[ \Delta L = \Delta_{bm} \]  

(3.46)

The axial force, \( N \), is due to internal beam strain according to Hooke’s law, it could calculated as follows

\[ N = \frac{EA}{L} \Delta_{bm} \]  

(3.47)

2- Slip state: in this case, one of the two beam ends is not constrained, e.g. the left-end. Suppose the slipping value, \( u_o \). The real beam extension is less than the beam foreshortening by the slippage value, \( u_o \), or mathematically

\[ \Delta L = \Delta_{bm} + u_o \]  

(3.48)

Assuming that the horizontal spring works parallel with the dry-friction-element, then the mentioned membrane force is balanced by two forces, the spring force and the tangential friction force \( N_1 \) as follows

\[ N = K_s u_o + N_1 \]  

(3.49)
Substituting equation (3.47) into equation (3.49) yields

\[
\frac{EA}{L} \Delta_{bm} = K_h u_o + N_1
\]  

(3.50)

Equations (3.48) and (3.50) contain two unknowns: the beam extension value, \(\Delta_{bm}\) and the slipping value, \(u_o\), at the beam left-end. Solving those two equations simultaneously yields

\[
\Delta_{bm} = \frac{\Delta L K_h + N_1}{K_h + \frac{EA}{L}}
\]  

(3.51)

\[
u_o = \frac{\Delta L \frac{EA}{L} - N_1}{K_h + \frac{EA}{L}}
\]  

(3.52)

Inserting equation (3.51) in equation (3.47) yields the force as

\[
N = \frac{\Delta L K_h + N_1}{K_h + \frac{EA}{L}} \frac{EA}{L}
\]  

(3.53)

Hence, it is possible to calculate the horizontal spring reaction contribution. The result of the multiplication of the spring coefficient and the slippage value from equation (3.52) is as follows

\[
K_h u_o = K_h \frac{\Delta L \frac{EA}{L} - N_1}{K_h + \frac{EA}{L}}
\]  

(3.54)

Substituting the vertical force from equation (3.45) into the joint resistance force (3.53) yields

\[
N = \frac{\Delta L K_h + \mu K_y \left( \delta_{uv} - w \right)_{x=0} \frac{EA}{L}}{K_h + \frac{EA}{L}}
\]  

(3.55)

As can be seen in equation (3.53), the joint resistance depends on the static pretension; the beam properties \(E, A, L\) and the beam response, \(w\).
4 Numerical Investigations

This chapter is devoted to the description of results that can be obtained by investigating frictional-joined plate.

A 3D-solid FEM model is suggested in the first paragraph of this section in order to study the feasibility as well as the efficiency obtained with other methods. In the second paragraph, another model is suggested using control and spring elements. Later, many different cases involving the simple model are presented in the rest of the thesis.

The feasibility of the joint model was tested in terms of its capabilities to model extreme cases and to capture stick-slip phenomenon and accommodating damping. The onset of stick-slip phenomenon is shown in terms of its energy-loss-factor.

Most of the calculations were carried out using MATLAB and APDL. Dedicated APDL subroutines & MATLAB subroutines have been developed for some specific purposes.

4.1 Modelling Frictional Joined Plates

4.1.1 Use of 3D-Solid Elements

The model is based on the idea that the joined parts are connected by friction. As such, a 3D-solid block is in contact with a part of the plate and normal pressure is applied to that part from the opposite side as shown in Figure 4.1. The pressure works to create the normal clamping force. The friction force is handled using contact elements. The contact element is available in most of the commercial packages.

![Figure 4.1: FEM-solid Model for a joined plate (Clamped-Free-Joined-Free)](image-url)
As done in the clamped plate case, it is possible to theoretically make the full transient analysis. But, practically, it is time consuming as it requires many hours of computation. Therefore, the idea of modelling was abandoned and another method was sought.

In conclusion, without using simple elements it is impossible to make dynamic analysis using the computation resources available. Therefore, a simple model was tested which is presented in the next paragraph.

4.1.2 Use of Reduced-Order Elements

In this thesis, in order to model a frictional joined edge of a plate, three different types of elements were used, namely: vertical spring, horizontal spring and control element (switch on/off). It is also worthwhile mentioning here that the control element as introduced in the ANSYS packages has the capability to model viscous damping as shown in Figure 4.2. This is not considered in this work.

To simulate the continuous joined edge, a setup of the previously mentioned elements is repeated many times and each is connected to one boundary node on that edge.

The compatibility is achieved as follows: every boundary node is connected to a normal spring element and to a control (switch) element. The switch is connected in series with the horizontal spring at one end while the other end of the spring is fully fixed. The other end of the vertical spring is also fixed.

During the step-by-step time analysis, the switch opens and closes at a certain value of a control parameter. To make this capability physically useful, the force produced by the vertical spring is used as a control parameter. This force is supposed to be the frictional force produced by the clamping pressure and it is calculated in equation (4.1).

\[ F_{\text{friction}} = \mu F_{\text{pre}} + C_1 w \]  
\[ C_1 = \mu. K_v \]

where \( K_v, \mu, F_{\text{pre}} \) and \( w \), are the vertical spring constant, coefficient of friction, the rivet pretension and the vertical spring displacement, respectively.
The conditions of stick and slip are as follows:

\[
\begin{align*}
F \leq F_{\text{friction}} & \Rightarrow \text{stick} \\
F > F_{\text{friction}} & \Rightarrow \text{slip}
\end{align*}
\] (4.3)

It can be seen in equation (4.1) that the force has two components. The first component is fixed and is the pre-tension force. The second component is time dependent, as it is related to the vertical spring displacement.

Hence, the switch opens and closes in response to the force produced by the vertical springs. When it closes, the switch fully transmits the membrane force to the horizontal springs and the status is stick. The membrane force is resisted. Thereby, the plate has the opportunity to develop force to a limit at which the switch opens and the status becomes slip.

### 4.1.3 The Reduced-Order Model Capabilities

To test the model capability of simulating the joint, the joined plate model was tested four times. In the first couple tests, high excitation was applied and two extreme cases of clamping force were compared. In the third and fourth test, low excitation was applied and again the two extreme cases were compared. Different parameters are plotted in the same plane.

**Comparing clamping forces at high excitation**: The results of the tested plate in two extreme cases of the clamping force are plotted in Figure 4.3. The plate dimensions were 0.20 x 0.50 x 0.0008 m and Young’s modulus \( E = 73 \times 10^9 \text{ N/m}^2 \), material density \( \rho = 2840 \text{ kg/m}^3 \). The amplitude of the harmonic excitation force was 50 N. For the sake of comparison, the graphs show two cases for two different pretensions of normal clamping force. The first one with pretension as low as 5 N and the second as high as 100 N. The figure includes three graphs showing the time histories until 0.3 seconds. The first one is the deflection. The second is the slippage at the centre of the joined edge and the last is the reaction to the membrane force at the centre of the joined edge.

The graphs show that, in the low pretension case, the deflection and the slippage are larger than in the high pretension case, while the reaction is significantly less. Furthermore, as slippage values are recorded at nonzero, the sliding happens in both cases.
Comparing clamping forces at low excitation: Similarly, to see the effect of the external load, a couple of new numerical experiments were solved. However, in these two experiments, the amplitude of the excitation was less than the previous cases. The amplitude was reduced from 50 N to 5 N, in order to see the differences more clearly. Thus, the results of the two new cases are plotted in Figure 4.4. The three curves in the figure reveal that the differences between the two cases become more dramatic in the sense that the system in the two cases behaves in a completely different way.

In brief, in the high pretension case, the sliding never happens at a low level of excitation. This is attributed to the fact that the plate does not develop enough membrane force to overcome that tangential friction force.
In addition, the reaction graph shows that at low pretension, the reaction is reduced. The range for reaction is shifted from tension range into tension and compressive range. Similarly, the reaction plot shows that in the case of low pretension, the reaction curve is flattened instead of showing sharp peaks as in the case of high pretension force.

4.2 Stick-Slip Phenomenon

In order to study the stick-slip phenomenon, the investigation was divided into three stages: The first stage involved comparing stick-slip and stick only. It aimed to investigate the joint parameters and the reason for starting slipping. Therefore, two cases were compared. In the first case, the joint parameters were selected to prevent slipping (stick only). In the second case, the stick-slip scenario was allowed. Therefore, it was possible to observe the internal forces in the both cases and the energy flow in the system and make the comparison using different parameters.

The second stage was aimed at capturing the onset or the suppressing of the stick-slip. Therefore, two groups of numerical experiments were analysed. Each included many cases with different parameters.
In the first group, pretension was fixed and the excitation amplitude was changed many times. In the second group, pretension was changed many times while the excitation amplitude was fixed.

In the last stage, the impact of the stick-slip on the system was observed in terms of energy. Therefore, the energy-loss factor for the already analysed cases were calculated and plotted. The graphs also show the onset and the suppressing of stick-slip.

For simplicity and to make the comparison more fruitful, the same plate data was used in all of the following experiments: the plate dimensions were $0.20 \times 0.50 \times 0.0008$ m, Young’s modulus $E = 73 \times 10^9$ N/m$^2$ and material density $\rho = 2840$ kg/m$^3$.

The time history of the applied excitation is plotted in Figure 4.5. The overall simulation time was 0.2 seconds.

![Figure 4.5: The time history of the applied force to the plate](image)

**4.2.1 Comparing Stick-Slip and Stick Only**

To see the effects of the slip, a comparison was made between two cases, one without slip (stick only) and the second with slip (stick-slip scenario).

**Stick Condition**

In the first case, the stick condition was allowed and the results obtained are plotted in Figure 4.7, Figure 4.8 and Figure 4.13.

The membrane force and the tangential frictional force are plotted and compared in Figure 4.6.

To clarify the definition of the tangential frictional force, a simple case of a block resting on rough surface can be considered. When a block is to be slid over a rough surface, a force parallel to the surface needs to be applied. This force must be strong enough to overcome the tangential frictional force. The tangential frictional force in
this simple case is equal to the weight times the coefficient of the friction. Once the block moves, the two forces are completely balanced. But when the applied force is less than the limit, the sliding no longer happens (stick conditions). Similarly, the plot here shows time histories for these two forces: the tangential frictional force and the applied force. As it can be seen, the applied force (the membrane force) is always less than the tangential frictional force. Thereby, in this case, the plate is always in stick conditions.

Figure 4.6: Comparison between the tangential frictional and the membrane force at joint (stick)

Furthermore, the three types of displacements are plotted together in Figure 4.7. The first curve expresses the slipping; the second curve is the spring stretching; and the third curve is the control parameter value, $W$. This parameter, in addition to the pretension force, according to equation (4.1) allows a variable tangential frictional force as seen in the previous figure. The variation in this parameter reflects the time-dependency of the normal clamping force.

Additionally, the time histories of the joint reactions are plotted in Figure 4.8. The reactions were in two directions, one normal and the other parallel to the plate; the parallel reaction is attributed to the friction resistance.
Figure 4.7: The time histories of displacements at the joint (stick)

Figure 4.8: The time histories of forces at the joint (stick)

**Stick-slip condition**

In the second case introduced, the slipping is allowed by introducing less pretension force. Similar graphs can be introduced here but only those that reveal a significant difference are plotted in Figure 4.9, Figure 4.10, Figure 4.11 and Figure 4.12.

The graph in Figure 4.9 shows the comparison between the applied membrane force and the tangential frictional force.

As can be seen, these two curves are coupled at time 0.08 sec. In order to see closely what is happening, it was re-plotted using a larger scale as shown in Figure 4.10.
The plot again shows clearly that these two curves are unified at some instances. More accurately, during its climb, the membrane force touches the tangential frictional force during its fall. But, as was previously clarified, there is no possibility for them to cross each other. Therefore, the former curve combines with the latter or the latter curve lowers the former one. However, when the membrane force is declined, the two curves may separate and possibly combine later on etc.

The stage of combined curves means that the membrane force is limited by friction; sliding takes place. Conversely, the separation refers to sticking.

Slipping is also revealed in the slip curve in the next graph shown in Figure 4.11. The slip curve rises dramatically at 0.08 sec where the stretching in the spring is also disturbed. Before this instance, it was oscillating in the positive range only. Afterwards, it started to oscillate in the positive and negative range. Thus, the spring is
stretched and pressed. Also, the graph shows the time-dependency in the control parameter $W$.

The impact of the stick-slip on the reaction levels is shown in the reaction curves in Figure 4.12 in which the membrane force is disturbed and shifted into the negative range. Thereby, the force application becomes tension and compression rather than tension alone.

Additionally, the strain and kinetic energy are plotted in the same plane in Figure 4.13 where the curves alternate. The sum is roughly constant but with a slight graduate drop.
The total energy (which is the sum of both energies together with the provided work), which is computed by the applied force times the displacement is also plotted in Figure 4.14. To make some distinction between these two curves, they were re-plotted using a larger scale and introduced in the same figure.

A clear distinction between the stored energy and the provided work is noticed. The stored energy curve shows a marked declination. Furthermore, the dissipated energy, which equals the difference between the stored energy and the provided work, is also plotted in Figure 4.15
Thus, there is a reason for that energy loss. However, the other graphs show more information.

### 4.2.2 The Onset of Stick-Slip

To show the stick-slip nucleation, a group of six cases was analysed. The pretension force was fixed at 5 N for the six cases, while the amplitude of the excitation force was changed from 5 N to 10 N in steps of 1 N. Thus, it was possible to plot slippage against the amplitude for a certain level of excitation as shown in Figure 4.16.
It was also possible to plot the slippage against the non-dimensional deflection of the plate as shown in Figure 4.17. The deflection results are non-dimensionalized by the plate thickness.

![Graph showing slippage vs non-dimensional deflection](image)

Figure 4.17: The slippage VS the maximum deflection at pretension (F = 5 N)

### 4.2.3 Stick-Slip Suppressing

To show the stick-slip Suppressing, second group of nine cases were analysed. The pretension force was changed from 5 N to 11 N in steps of 1 N, while the amplitude of the excitation force was fixed at 8 N. Thus, it was possible to plot slippage against the joint pretension as shown in Figure 4.18.

![Graph showing slippage vs joint pretension](image)

Figure 4.18: The slippage VS the joint pretension at a certain level of excitation (Amplitude = 8 N)
The results obtained also allowed the slippage to be plotted against the maximum non-dimensional deflection which is shown in Figure 4.19.

![Figure 4.19: The slippage VS the maximum non-dimensional deflection at a certain level of excitation (Amplitude = 8 N)](image)

### 4.2.4 The Impact of Stick-Slip on the Structure

To quantify the impact of stick-slip on the system in terms of energy, the energy loss factor was computed. Thus, the loss factor of a system could be estimated using equation (2.139). According to that equation, the loss factor was plotted for the two previous groups of cases.

For the first group of cases, in which the pretension was fixed and the excitation amplitude was varied, the energy loss factor was plotted against excitation-amplitude as shown in Figure 4.20 and against maximum non-dimensional deflection as shown in Figure 4.21.

Similar plots illustrate the second group of cases. The energy loss factor was plotted against joint pretension as shown in Figure 4.22 and against the maximum non-dimensional deflection as shown in Figure 4.23.

In conclusion, the last set of graphs clearly reveal again the onset of the stick-slip phenomenon in terms of energy loss factor. For instance, the slippage and energy loss factor at stick-slip behave differently and abrupt variation can be observed as an indication for that phenomenon.

Accordingly, stick-slip is initiated when the external excitation level is increased beyond a certain threshold for a certain value of pretension. Conversely, at a certain level of excitation, the stick-slip is initiated when the joint pretension is reduced to a certain limit.
Figure 4.20: The energy loss factor VS the level of excitation at a certain pretension (F = 5 N)

Figure 4.21: The energy loss factor VS the maximum displacement at a certain pretension (F = 5 N)

In conclusion, the plate with a high pretension behaves similarly to a clamped plate and there are no more energy dissipations. However, those small values of dissipation noticed in the plots before stick-slip-initiation are attributed to the numerical errors as is discussed in 3.3.2.2
Figure 4.22: The energy loss factor VS the joint pretension clamping force at a certain level of excitation (Amplitude = 8 N).

Figure 4.23: The energy loss factor VS the maximum non-dimensional deflection at a certain excitation (Amplitude = 8 N).
5 Experimental Works

This chapter is devoted to the study of the dynamic behaviour of a plate using experimental procedure. Measured data will be used to validate the numerical model of the previous chapter.

5.1 Experiments Descriptions

It was initially necessary to design the plate fixture - it was devised to measure a plate with a maximum width of 0.70 m and a maximum height of 0.70 m. In this thesis, two aluminium plates were measured. The dimensions of the first plate were $0.4 \times 0.5 \times 0.0012 \text{ m}$ and the dimensions of the second were $0.2 \times 0.5 \times 0.0008 \text{ m}$.

Due to the fact that the plates studied were very thin (in some cases the thickness was as low as 0.0008 m), they bent under their weight. Therefore, in order to avoid this curvature, it was necessary to place the plate in a vertical position. This direction also enabled the application of the desired boundary condition as Clamped-Free-Clamped-Free as shown in Figure 5.1.

Figure 5.1: The studied plate in the laboratory in a vertical position
5.2 Plate Boundary Conditions

The two long, vertical sides of the plate were free. The upper side of the plate was fixed by two beams that had relatively large cross-sections as depicted in Figure 5.2. (This arrangement was intended to simulate clamped boundary conditions. However, it is important to know that clamping is very theoretical conditions; clamping is only obtainable using rigid supports. As the rigid elements mean infinity of stiffness, it is of course not available in practical application. Therefore, it should be accepted at a certain level of approximation that clamping in that way is occurring).

These two beams were tightened by stiff bolts and worked as two jaws to limit the upper edge of the studied plate. Similarly, the bottom edge was also restricted by two beams. Instead of using bolts, a simple mechanism was used. This mechanism was nothing more than a bolt inside a spring as shown in Figure 5.3. This setup was used to allow an adjustable normal clamping force. The variation in the clamping force was calibrated by changing the active length of the spring.

In this setup, the friction and the normal clamping force were assumed as the key elements in a frictional joint.

To excite the plate with a deterministic and a measurable signal, the plate was rigidly connected to a shaker. The connection point was just 0.02 m down from the upper edge. This was used to reduce the effect of the shaker provision on the dynamic characteristics of the plate.

![Figure 5.2: A schematic illustration for the frictional joined plate experiment](image-url)
5.3 The Used Instruments

The following devices are used in the experimental work:

1- An impedance sensor includes a force-meter and accelerometer from Kistler type: 8770A5.

2- The signal conditioner: PCB Model 482A16.

3- A vibrometer laser head from Polytec OFV 302R.

4- A vibrometer controller from Polytec OF3000.

5- The frequency generator is Tektronix AFG 3022B.

6- The power amplifier is LDS PO300.

7- The shaker is LDS 407.

8- Data acquisition system is a computer built-in card from Microstar lab DAP 5200A/526.

5.4 The Instrument’s Connection

The response of the plate was measured by the vibrometer laser head. The device was located 1.5 meters away from the plate. The device produced a laser beam which was pointed at the center of the plate as shown in Figure 5.4. Thus, the response of the plate was measured remotely and accurately. The velocity of one point was detected. In order to obtain the displacement of that point, a numerical integration of the velocity was involved.

To excite the plate with definite input, the signal generator was first adjusted and triggered. The signal was then transmitted to the power-generator, which fed the power to the shaker according to the wanted excitation. The shaker was connected mechanically to the plate. This connection allowed the impedance to measure the actual applied force and the acceleration of the application point. The measured force and acceleration signal were then transmitted to a signal conditioner which fed them to the data-acquisition element. The vibrometer controller also received the vibrometer head signal and transmitted the results to the data-acquisition element. The generated signal was measured by a direct connection to the data-acquisition element.
Figure 5.4: A schematic illustration for the experiment hardware
5.5 Measuring Natural Frequencies

There are several methods which can be used to measure the natural frequency experimentally. Some of these methods depend on exciting the plate by pulse input, then measuring the transient response in the time domain. The time width of the pulse should be selected depending on the natural frequency that is intended to be measured. As a rule of thumb, the width of the pulse should be at least equal to the time length of one wave of the output of the mode intended to be excited.

Conversely, it is possible to use measurements in the frequencies domain to directly estimate the fundamental frequencies such as the swept-sine method. In this method, the Frequency Response Function (FRF) is plotted, with the peaks expressing the fundamental frequencies.

In this work, these two methods are possible in the experiment which was conducted. It was, however, not possible to detect all of the modes accurately as there was only one sensor to record the response of the plate at only one point. The point selected was in the middle of the plate.

As such, it is only possible to detect the modes which are located on the plate-midpoint on an anti-node line which means in this context an extremum deflection line. These modes produce states of vibration in which the middle point of the plate is on anti-node.

Thus, according to the prescribed setup, the natural frequencies corresponding to the mode shapes 1, 5 and 7 were measured and tabulated in addition to the numerical results in Table 5.1.

<table>
<thead>
<tr>
<th>No.</th>
<th>Natural frequencies [Hz]</th>
<th>Relative error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Numerical</td>
<td>Experimental</td>
</tr>
<tr>
<td>1</td>
<td>17.368</td>
<td>17.67</td>
</tr>
<tr>
<td>5</td>
<td>94.508</td>
<td>95.57</td>
</tr>
<tr>
<td>7</td>
<td>126.100</td>
<td>128.47</td>
</tr>
</tbody>
</table>

By inspecting the error field it can be seen that the error range is between 0.22 and 1.93 %. The increased error at high frequencies may be attributed to the poor performance of the measurement equipment at that height. However, this becomes more evident when comparing the error of the same mode to the analytical results in Table 3.5. The error for the seventh mode is less than 0.566 %.

In summary, as a result of the comparison process in the previous sections, it can be concluded that the numerical model of the plate can be used confidently for further investigations with an acceptable level of errors.
5.6 Comparing Numerical Results with Experiments

For the sake of comparison, the plate was analysed both numerically and experimentally. The plate has the following data: geometries $0.5 \times 0.2 \times 0.0008$ m, material density $\rho = 2840$ kg/m$^3$, Poisson’s ratio $\nu = 0.35$, Young’s modulus $E = 73 \times 10^9$ N/m$^2$. The plate was clamped from the two short sides while the other sides were free.

At the beginning of the experiment, the plate was subjected to a pulse. The pulse was intended to be a square with enough width to excite the first natural mode of the plate at 17.3 Hz, as can be seen in Table 5.1.

The electrical signal generator produced a perfect square pulse, but once it was transmitted by the exciter to the plate it changed. It is generally accepted as a fact that the applied force also reflects the resistance of the destination structure. Therefore, a perfect square force is not obtainable. The applied signal which was recorded can be seen in Figure 5.5.

The figure shows that the real recorded applied force was very different from the intended theoretical force in three aspects: the width of the signal, the shape of the signal and the extra small pulses. This can be seen in the graph.

In order to make a sensible comparison with the theory, the same force must be applied directly to the FEM model using the APDL code.

In brief, all of the requirements needed to make numerical analysis possible are now available. In this context, it is worth mentioning that the sampling rate that was used to record the applied force was 10 KHz. Hence, the numerical solution used the same sampling rate.

Thus, the numerical deflection of the plate at the centre as well as the experimental responses were plotted together and presented on one plane, as can be seen in Figure 5.6.

The plot shows significant differences between the two schemes. The numerical response is larger than the experimental response.
One of the most obvious reasons for this disagreement is attributed to the way that plate was excited. The plate was actually excited by a direct rigid connection to the shaker servo. In addition to changing local stiffening, the exciter servo also added mass to the system. The servo contained a load cell and an accelerometer which weighed 0.140 kg, in addition to the weight of the bolt. Two metallic adapters were necessary to make the connection possible. Thus, the added weight was as high as 0.240 kg. The actual weight of the plate was $0.5 \times 0.2 \times 0.0008 \times 2.84 \times 10^{-6} = 0.227$ kg. Thus, the overall weight of the collection was greater than the weight of the plate.

Unfortunately, that mechanism was the only possible method to excite the studied specimen. Alternatively, updating the model with concentrated masses and stiffness at the excitation point was a good choice to make the comparison more sensible.

Accordingly, the model was updated and re-analysed. The new results of the improved numerical model were compared with the experimental results and plotted in Figure 5.7. The results now show more agreement.

**Figure 5.7: The comparison between the experiment and numeric with added masses**

### 5.7 The Experimental Bifurcation Map

In this experiment, the first plate (described at the beginning of this chapter) was excited in the linear and the nonlinear range to observe the development of the geometrical nonlinearity. Therefore, the plate was experimented on several times. In all of the experiments, a sine signal with a frequency in the vicinity of the first mode of roughly 25 Hz was applied. In order to develop the plate response, the level of excitation was changed from 1 to 50 N and for each experiment the velocity of the
plate at its centre was recorded. The time series of the velocity was then filtered using the well-known Butterworth low pass filter. The filter was tuned according to the following parameters: cut-off frequency 1250 Hz. The order of the filter was 2. For the 10 KHz sampling rate, the built-in MATLAB function gave the coefficients of the filter.

The numerator coefficients were listed in descending powers of $z$ as follows:

$$0.0970 \quad 0 \quad -0.1940 \quad 0 \quad 0.0970$$

In the same way, the coefficients of the denominator were:

$$1.0000 \quad -2.9429 \quad 3.2223 \quad -1.6141 \quad 0.3347.$$ 

Afterwards, the resulted velocity was integrated to obtain the deflection of the plate centre.

Having the velocity and the displacement for the same point, it was then possible to plot the phase portrait. Hence, for convenience, only six phase portraits were plotted to show how the responses developed as shown in Figure 5.8.

In the first plot, the phase portrait shows a rather clean orbit. The orbits in the following cases started to become disturbed as the deflection increased, until case number 5 where the orbit forms two loops rather than one loop. In the corresponding oscillation, the system is a double-period rather than a simple period. This transaction is attributed to geometrical nonlinearity.

To show the transaction more efficiently, the Poincaré section was established. The process of plotting this section was done by reducing the order of the plots. For instance, the orbit in the phase portrait was simulated by only one point in the Poincaré section. This point was actually taken stereoscopically. In practical terms, the orbit was not a simple one path due to imperfect measurements. Hence, the orbit was performed by a band of orbits. Bearing in mind that fact, according to the stereoscopic selection, a cloud of points resulted instead of one point.

A dedicated MATLAB subroutine was developed particularly to achieve this process. Accordingly, the resulting plot is shown in Figure 5.9.
However, to limit the number of resulting points, a tolerance was identified as the radius of the cloud of points. Therefore, a suitable selection for the tolerance converts the cloud into one point. The improved process, with a tolerance of 0.1, resulted in a better Poincaré section, as shown in Figure 5.10.

Figure 5.8: Phase portrait development for six experiments
Figure 5.9: The bifurcation section at 0.0001 tolerance

Figure 5.10: The bifurcation section at 0.1 tolerance
6 Conclusion

This chapter will introduce the main findings of the work on modelling joined structures using frictional based ones. Remarks have also been noted and presented. To make these conclusions more fruitful, they are divided according to the field of application.

The investigation of different types of nonlinearities in pendulums, beams and plates lead to the following conclusions:

It was found that all nonlinear dynamical systems have common phenomenon such as periodic behaviour, bifurcation, period doubling, chaotic behaviour.

To investigate the geometrical nonlinearity, the bifurcation map and backbone curve were plotted. The bifurcation map showed the transition range in which the system converted from linear behaviour into nonlinear one.

To show the effects of the geometrical nonlinearity, the backbone curve was also plotted for the plate model. The graph showed a peak around the linear resonance $\omega_n$. The peak of the vibration amplitude moved to the right of $\omega_n$, indicating a stiffening type of nonlinearity.

Also, the Fast Fourier Transform (FFT) was used to investigate the nonlinearity in the dynamical systems. The FFT graphs showed a grassy form when the behaviour was chaotic. In linear systems, it showed only one peak. In the free response of linear systems, it showed one peak corresponding to the excited mode. In the forced response of linear systems, it showed one peak corresponding to the frequency of the excitation.

To show the nonlinearity due to stick-slip phenomenon, the hysteresis loops were most effective. They showed irreversible behaviour when stick-slip occurred. The area of the loop was equal to the dissipated energy due to stick-slip.

For modelling purposes, it was found that the 8-node elements work very efficiently to capture the linear and nonlinear and static and dynamic behaviour of thin plates.

Additionally, the spatial discretisation process was assessed using the sensitivity analysis of the modal results. The time step length was based on minimizing the error of the stored energy for the complete system.

Modelling frictional-based joined plate:

The joint was modelled after the plate FEM model was made available and verified with some analytical and experimental solutions. Also, for validation purpose, the backbone curve for simply-supported immovable-edges plate was derived. It showed excellent agreement with similar results reported in the literature.

Although there are many modelling techniques for joints, very few are applicable. Moreover, the problem is not to present a model for a quick static analysis. The model
should satisfy the portability requirements and be easy to implement under dynamic analysis.

The 3D-solid FEM model is only acceptable if the local behaviour is sought and only under static condition. But if dynamic analysis has to be performed, such a model is expensive and very unpractical. Hence, 3D-solid models were computationally prohibited. However, a joint model using a 3D-solid element was tested under full transient analysis. The computation time was very long. Therefore, this type of modelling was abandoned. For this reason, another technique is sought. A lumped, parametric and simple model for a joined plate is wanted. In this thesis, it was discovered that the combination of simple spring elements and control (switch on/off) elements work well. This combination, however, is based on the idea that the riveted or bolted joints rely on friction to keep the joined components connected.

Another assumption is that the contribution of joints on the damping level is due to the stick-slip phenomenon that is expected between the contacted flexible parts. The suggested procedure to model the joined structure was computationally experimented using APDL.

The numerical experiments included several cases with different conditions and load levels. The purpose of these experiments was to test the capability of the model to capture true physical behaviour.

The linear and nonlinear full transient analyses were among several investigations which have been carried out in this work. However, the results show the efficiency of such combinations to meet the wanted purpose. An observation of the time series of the introduced and the stored energy shows that the energy is drained out. The only acceptable interpretation for this energy loss is the stick-slip phenomenon that was expected to be inhibited by the suggested model. To clearly prove this interpretation, the time history of the displacements of one node on a nanosecond level was recorded during the step-by-step time integration according to the Newmark method. Records of the acting forces were also saved during the time progress.

The resulting graphs clearly reveal the stick-slip phenomenon. For instance, in the force graphs, two types of time-variable forces were plotted for one node; the node was selected in the centre of the joined-edge. The stick-slip occurred when the applied membrane force curve tried to cross the tangential frictional force curve. The tangential frictional force was generated due to the normal clamping force (the joint pretension) and the friction in the contacted surfaces.

The graphs also show time-dependency for the normal clamping force, unlike most of the studies in the literature in which the normal clamping force was considered to be a constant value. Many other researchers have argued that such an assumption is oversimplification.

Also, the energy loss factor is plotted against the pretension of the normal clamping force and the external amplitude level. The graphs show abrupt variations as a signal
Conclusion

for initiation of stick-slip scenario. Also, the impact of that phenomenon is shown by plotting the lateral deflections and the reaction.

Accordingly, stick-slip mechanism graphs show that the deflection is a bit larger while the reaction to the membrane force is much less.

In the clamped model, the in-plane force always works harmonically in tension mode. Whereas in the stick-slip model, the range of application is a bit shifted. The force now works in the compression and tension range. Thereby, the amplitude is also reduced. From this observation, it can be concluded that the frictional joints work as a limiter or something to remove the force peaks (mechanical filter).

In the experiment part of the thesis, the following conclusions can be drawn from the observed results.

Measuring the natural frequencies:

There are three different methods to measure the natural frequencies of the plate. The first method is to apply a pulse using an external hammer. This method works well because no external interface has to be added to the plate during the free response. The laser vibrometer is also used to measure the plate deflection without touching the system. This is perfect in the sense that the system is not disturbed. The second technique is by applying a Swept-sine input. The essence of this method is to apply a sine signal input with a variable frequency starting from the minimum expected mode until the maximum wanted mode. In this method, the rate of change of applied frequencies has a great effect on the response.

Therefore, several experiments had to be done to define the appropriate rate. The process was made by starting the quickest variation. Then the variation was slowed down until the saturation in the results was achieved. Two drawbacks of this technique were noticed: it is time consuming when compared with the first method and a provision has to be connected to the plate which disturbs the dynamic characteristic of the system. However, to minimize such an impact, the provision is attached to the plate as far as possible away from the anti-node.

The third technique uses a predefined square pulse to excite the plate. The width of the pulse decides the mode that is to be excited. The drawbacks of this technique are similar to those of the second one. But the advantage of the last two techniques is that the measurement can be done on any wanted mode.

Bifurcation plot:

To figure out the bifurcation plot for the plate studied, the following procedure was used in this thesis. First, the plate was excited by sine-signal input. The time series of the velocity of the plate was then recorded at the centre using the laser vibrometer. The Butterworth filter was used to get rid of the noise that was in the measured signal. By using numerical integration, the time series of the plate deflection was obtained. Thus, it was possible to plot the phase portrait as the relation between the velocity on the y-
axis and displacement on the x-axis. However, for linear behaviour as discussed in the theory chapter, it showed one loop and when the nonlinear behaviour became significant it started to show two loops. Thus, the transition to double periodic was captured and the phase portraits showed the two loops as a sign of double periodic.

However, to plot the Poincaré section, the phase portraits were processed in a way for one-loop phase, where only one point was taken stereoscopically. Hence, two loops gave two points. Thus, these points were plotted against the corresponding exciting force. The higher the number of experiments, the clearer the plot became.

The bifurcation plots for the studied plates show that bifurcation occurred at the ratio of displacement to thickness, roughly 0.5. Thus, the bifurcation plot shows the transaction from a periodic behaviour to an aperiodic, as nonlinearity becomes significant.
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# Appendix

## Table 8.1: Comparing between Euler-Bernoulli & Timoshenko beam theories

<table>
<thead>
<tr>
<th></th>
<th>Euler-Bernoulli</th>
<th>Timoshenko</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Displacement</strong></td>
<td>( w )</td>
<td>( w )</td>
</tr>
<tr>
<td><strong>Rotation due to bending</strong></td>
<td>( \varphi )</td>
<td>( \varphi )</td>
</tr>
<tr>
<td><strong>Rotation due to shear</strong></td>
<td>-</td>
<td>( \beta )</td>
</tr>
<tr>
<td><strong>Slope of deflection curve</strong></td>
<td>( \frac{\partial w}{\partial x} = \varphi )</td>
<td>( \frac{\partial w}{\partial x} = \varphi + \beta )</td>
</tr>
<tr>
<td><strong>Moment</strong></td>
<td>( EI \frac{\partial^3 w}{\partial x^2} )</td>
<td>( EI \frac{\partial \varphi}{\partial x} )</td>
</tr>
<tr>
<td><strong>Shear</strong></td>
<td>-</td>
<td>( \kappa AG \left( \frac{\partial w}{\partial x} - \varphi \right) )</td>
</tr>
</tbody>
</table>

## Table 8.2: Energy terms in Euler-Bernoulli & Timoshenko beam theories

<table>
<thead>
<tr>
<th></th>
<th>Euler-Bernoulli</th>
<th>Timoshenko</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Potential energy due to bending</strong></td>
<td>( \frac{1}{2} \int_0^L EI \left( \frac{\partial \varphi}{\partial x} \right)^2 , dx )</td>
<td>( \frac{1}{2} \int_0^L EI \left( \frac{\partial \varphi}{\partial x} \right)^2 , dx )</td>
</tr>
<tr>
<td><strong>Potential energy due to shear</strong></td>
<td>N/A</td>
<td>( \frac{1}{2} \int_0^L \kappa AG \left( \frac{\partial w}{\partial x} - \varphi \right)^2 , dx )</td>
</tr>
<tr>
<td><strong>Translational kinetic energy</strong></td>
<td>( \frac{1}{2} \int_0^L \rho A \left( \frac{\partial w}{\partial t} \right)^2 , dx )</td>
<td>( \frac{1}{2} \int_0^L \rho I \left( \frac{\partial^2 w}{\partial t^2} \right)^2 , dx )</td>
</tr>
<tr>
<td><strong>Rotational kinetic energy</strong></td>
<td>N/A</td>
<td>( \frac{1}{2} \int_0^L \kappa \rho \left( \frac{\partial^2 w}{\partial t^2} \right)^2 , dx )</td>
</tr>
</tbody>
</table>

## Table 8.3: General boundary conditions for Euler-Bernoulli & Timoshenko beam theories.

<table>
<thead>
<tr>
<th><strong>General case</strong></th>
<th>Euler-Bernoulli</th>
<th>Timoshenko</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\partial^2 w}{\partial x^2} \right) \cdot \left( \frac{\partial w}{\partial x} \right) = 0 )</td>
<td>( \left( \frac{\partial \varphi}{\partial x} \right) \cdot (\varphi) = 0 )</td>
<td>( \kappa AG \left( \frac{\partial w}{\partial x} - \varphi \right) \cdot (w) = 0 )</td>
</tr>
<tr>
<td>( \left( \frac{\partial^3 w}{\partial x^3} \right) \cdot (w) = 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Hinged**

\( M = 0 \)

\( w = 0 \)

\( \frac{\partial^2 w}{\partial x^2} = 0 \)

\( \frac{\partial w}{\partial x} = 0 \), \( w = 0 \)

**Clamped**

\( w = 0 \)

\( \varphi = 0 \)

\( \frac{\partial w}{\partial x} = 0 \), \( \varphi = 0 \), \( w = 0 \)
Euler-Bernoulli Timoshenko
Free
\[ M = 0 \]
\[ Q = 0 \]
\[ \frac{\partial^3 w}{\partial x^3} = 0, \]
\[ \frac{\partial^3 w}{\partial x^3} = 0 \]

Timoshenko beam theory for a dynamic case is also introduced into two forms
First, the coupled form is as follow [119]
\[ EI \frac{\partial^2 w}{\partial x^2} + \kappa AG \left( \frac{\partial w}{\partial x} - \phi \right) - \rho I \frac{\partial^2 \phi}{\partial t^2} = 0 \]
\[ (8.1) \]
\[ \kappa AG \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x} \right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \]
\[ (8.2) \]
Secondly, the decoupled form is as follow [119]
\[ EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} - \rho I \left( 1 + \frac{E}{\kappa G} \right) \frac{\partial^4 w}{\partial x^4 \partial t^2} + \frac{\rho^2 I}{\kappa G} \frac{\partial^4 w}{\partial t^4} = 0 \]
\[ (8.3) \]
\[ EI \frac{\partial^4 \phi}{\partial x^4} + \rho A \frac{\partial^2 \phi}{\partial t^2} - \rho I \left( 1 + \frac{E}{\kappa G} \right) \frac{\partial^4 \phi}{\partial x^4 \partial t^2} + \frac{\rho^2 I}{\kappa G} \frac{\partial^4 \phi}{\partial t^4} = 0 \]
while the general boundary conditions constrained by flexible springs are given by
\[ M = \pm \left( K_{rot} \frac{\partial w}{\partial x} + \beta_{rot} \frac{\partial^2 w}{\partial t \partial x} \right) = EI \frac{\partial \phi}{\partial x} \]
\[ (8.4) \]
\[ Q = \pm \left( K_{trans} \frac{\partial w}{\partial t} + \beta_{trans} \frac{\partial w}{\partial t} \right) = \kappa AG \left( \frac{\partial w}{\partial x} - \phi \right) \]