On the distance function to the cut locus of a submanifold in Finsler geometry

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

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Tag der mündlichen Prüfung: 19. November 2010

Diese Dissertation ist auf den Internetseiten der Hochschulbibliothek online verfügbar.
Abstract

In the present thesis we study the distance function to the cut locus of a submanifold. We explain the basic notions for a given Finsler manifold \((M, F)\) and a submanifold \(\tilde{M}\). Roughly speaking, for a given point \(x_0\) on the submanifold \(\tilde{M}\) and any unit vector \(y_0\) which is normal to \(\tilde{M}\) we define the cut point as the first point on the geodesic starting at \(x_0\) in direction \(y_0\) such that this geodesic fails to be minimising distance to \(\tilde{M}\) for any point that lies beyond the cut point. The set of all cut points is called the cut locus of \(\tilde{M}\) and is denoted by \(\text{Cut}_{\tilde{M}}\). The distance function to the cut locus, i.e. the function that measures distance from \(x_0\) to the cut point, depends on \((x_0, y_0)\) and is denoted by \(i_{\tilde{M}}\). See Definition 2.15 for precise terminology.

We prove that the distance function to the cut locus of a submanifold \(\tilde{M}\) is locally Lipschitz continuous. For technical reasons, we have to presume the absence of conjugate points, i.e. points at which the derivative of the exponential map is singular. This is the main result of the present thesis and the precise statement can be found in Theorem 3.2. We remark, that the hypothesis on conjugate points is always satisfied in Finsler manifolds of nonpositive flag curvature.

In [LN05] Y.Y. Li and L. Nirenberg establish local Lipschitz continuity of the distance function to the cut locus in a more restrictive setting. They consider a particular geometric situation in which the ambient manifold \(M\) is assumed to be \(\mathbb{R}^N\) and the submanifold \(\tilde{M}\) is the boundary of a \(C^{2,1}\) domain \(\Omega \subset \subset \mathbb{R}^N\). Hence, their setting is of codimension 1 and allows for a distinction between inner and outer normals of \(\tilde{M} = \partial \Omega\). Thus, our result clearly extends the existing theory for Finsler manifolds.
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Introduction

The present thesis is concerned with the distance function to the cut locus of a submanifold.

In order to illustrate the basic notions we study an introductory example. We consider a circular curve in the $x_1 x_2$ plane, centred at the origin, of the Euclidean space $\mathbb{R}^3$ and a straight line segment starting at some fixed point on the curve in some direction normal to the curve. Depending on the starting direction, there might exist a point $z_0$ on this segment at which the segment ends up to be a distance minimising curve joining the point with the curve. We may call this point a cut point of the curve and call the union of all cut points the cut locus. Here, it is easy to observe that the cut locus coincides with the $x_3$ axis. Now we are interested in the distance function to the cut locus. It is the function that measures distance from the starting point to the cut point $z_0$ along the line segment. Clearly, the distance function to the cut locus depends on the starting point and the starting direction. In the situation described above the distance function to the cut locus is constant for any starting direction in the $x_1 x_2$ plane, whereas it is defined as $\infty$ whenever there is no cut point on the segment. In general one might ask for the regularity of this function.

In the present thesis we give an answer to this question in a far more general setting. Instead of a curve in a Euclidean space we consider a submanifold $\tilde{\mathcal{M}}$ of a Finsler manifold $(\mathcal{M},F)$ without restrictions on dimension and codimension. The definition of a cut point of $\tilde{\mathcal{M}}$ corresponds to the definition in the introductory example. Roughly speaking, for a given point $x_0$ on the submanifold $\tilde{\mathcal{M}}$ and any unit vector $y_0$ which is normal to $\tilde{\mathcal{M}}$ we define the cut point as the first point on the geodesic starting at $x_0$ in direction $y_0$ such that this geodesic fails to be minimising distance to $\tilde{\mathcal{M}}$ for any point that lies beyond the cut point. The set of all cut points is called the cut locus of $\tilde{\mathcal{M}}$ and is denoted by $\text{Cut}_{\tilde{\mathcal{M}}}$. The distance function to the cut locus, i.e. the function that measures distance from $x_0$ to the cut point, depends on $(x_0, y_0)$ and is denoted by $i_{\tilde{\mathcal{M}}}$. See Definition 2.15 for precise terminology.

In what follows, we prove that the distance function to the cut locus $i_{\tilde{\mathcal{M}}}$ is locally Lipschitz continuous. For technical reasons, we have to presume the absence of conjugate points, i.e. points at which the derivative of the exponential map is singular. This is the main result of the present thesis and the precise statement can be found in Theorem 3.2. We remark that in a Euclidean setting the derivative of the exponential map is the identity and hence the situation described in the introductory example is clearly covered by our result.

Since it might be inconvenient to verify the absence of conjugate points directly we provide more manageable conditions in two corollaries. Our result is applicable to
Finsler manifolds with nonpositive flag curvature since this condition implies that no geodesic can contain any conjugate point. In a further, less restrictive, corollary we show that a positive upper bound on the flag curvature of \((\mathcal{M}, F)\) yields the existence of a constant such that \(i_{\mathcal{M}}\) is locally Lipschitz continuous whenever it is strictly bounded by this constant.

Regularity results of this type have been proven before. Firstly, in the case of a Riemannian manifold J. Itoh and M. Tanaka established local Lipschitz continuity under general assumptions, see [IT01, Theorem B]. In particular, their result does not require any conditions on the regularity of the derivative of the exponential map. However, a main ingredient for their proof is a corresponding Lipschitz continuity result for the distance function to the focal locus of \(\widetilde{\mathcal{M}}\). In Riemannian geometry, focal points of a submanifold \(\mathcal{M}\) coincide with points at which the derivative of the restriction of the exponential map to the normal bundle of \(\mathcal{M}\) is singular. We remark that whenever the submanifold reduces to a single point the notions of focal points and conjugate points agree. However, to the best of our knowledge the theory of focal points of submanifolds in Finsler geometry is far less developed than in Riemannian geometry.

Secondly, in [LN05] Y.Y. Li and L. Nirenberg derived local Lipschitz continuity for \(i_{\mathcal{M}}\) in a special Finsler setting. They consider a particular geometric situation in which the ambient manifold \(\mathcal{M}\) is assumed to be \(\mathbb{R}^N\) and the submanifold \(\widetilde{\mathcal{M}}\) is the boundary of a \(C^{2,1}\) domain \(\Omega \subset \subset \mathbb{R}^N\). Hence, their setting is of codimension 1 and allows for a distinction between inner and outer normals of \(\widetilde{\mathcal{M}} = \partial \Omega\).

The objective of the present thesis is to analyse to what extent the approach of Y.Y. Li and L. Nirenberg can be extended to a less restrictive setting. We are able to generalise their method to allow for arbitrary dimension and codimension but require a condition that guarantees regularity of the derivative of the exponential map at certain points. Thus, our result is not as general as the corresponding one for Riemannian manifolds, but clearly extends the existing theory for Finsler manifolds.

Before we explain the structure of this thesis we proceed with a few remarks on important differences between Riemannian and Finsler geometry. Although S.S. Shen stated that ‘Finsler geometry is just Riemannian geometry without the quadratic restriction’ there are deep conceptual differences between Riemannian and Finsler geometry. We highlight two of these differences that play a vital role throughout the present thesis.

The most popular difference is probably the fact that the Finsler distance function is not symmetric. Randers spaces provide easy examples of such Finsler manifolds, see Example 1.2. Consequently, for any distance minimising curve \(c : [a, b] \rightarrow \mathcal{M}\) and some fixed \(t_0 \in [a, b]\) the backward curve \(\bar{c}(t) := c(t_0 - t)\) fails to be distance minimising. Accordingly, we have a corresponding statement for geodesics. However, we observe that for any given geodesic \(c : [a, b] \rightarrow \mathcal{M}\) of the Finsler manifold \((\mathcal{M}, F)\) the backward curve is a geodesic with respect to the Finsler manifold \((\mathcal{M}, \bar{F})\) where \(\bar{F} : T\mathcal{M} \rightarrow [0, \infty)\) denotes the Finsler structure defined by \(\bar{F}(x, y) := F(x, -y)\). We will frequently make use of this interrelation in Chapters 2 and 3.

As second difference we note that we merely have \(\exp \in C^1(T\mathcal{M})\) for the exponential map of a Finsler manifold \((\mathcal{M}, F)\) compared to \(\exp \in C^\infty(T\mathcal{M})\) in the Riemannian case. This statement is sharp in the sense that \(\exp \in C^2(T\mathcal{M})\) if and
only if the Finsler structure is of a special type, called Berwald type, see [BCS00, Section 5.3]. Since a main ingredient for our approach is Taylor expansions for the Finsler distance function, the regularity of exp and its inverse are of paramount importance for this thesis.

This thesis is organised as follows. In the first chapter we provide a brief summary of those aspects of Finsler geometry that are relevant for the thesis. We omit a detailed discussion of the various curvature notions of Finsler manifolds and suggest the interested reader to consult [BCS00] or [She01] for more information on this topic. In this chapter, we omit proofs whenever we are able to refer to one of the aforementioned books. However, sometimes we need minor refinements of standard results. In these cases, we include complete proofs. As mentioned before, Taylor expansions of the Finsler distance function are a key tool in the proof of the main theorem. Therefore, we place an emphasis on results concerning the differentiability of the distance function in Section 1.5.

The objective of the second chapter is to give precise definitions of the cut value and the cut locus of a submanifold. Moreover, we provide a characterisation of \( \text{Cut}_{\tilde{M}} \) as the boundary of the so called cut domain \( D_{\tilde{M}} \) and show, by virtue of this characterisation, that under reasonable hypotheses the distance function to the submanifold \( \tilde{M} \) is smooth in \( D_{\tilde{M}} \setminus \tilde{M} \). This will be accomplished in Section 2.3. To this end we require some preparations in the preceding sections of Chapter 2. In Section 2.1 we introduce the notion of a normal vector to a submanifold \( \tilde{M} \) and compute the first variation of the length functional to motivate this definition. Subsequently, we discuss the notion of normal curvature of a submanifold and establish estimates for the normal curvature when the submanifold is touched by a metric sphere. As a special case, we consider a situation in which a metric sphere is touched by another metric sphere with a smaller radius. These considerations are based on results presented in [She01].

The third chapter is dedicated to the proof of the main result of the present thesis. In Section 3.1, we give a precise formulation of our Lipschitz continuity theorem for the distance function to the cut locus and establish the aforementioned corollaries. In the course of the proof of Theorem 3.2 we distinguish between three different cases. Sections 2 and 3 of this chapter are dedicated to these cases. The final proof can be found in Section 3.4.

In the appendix we provide a proof for a merely technical lemma which is needed in Chapter 3.

Acknowledgement First and foremost, it is my pleasure to express my deep gratitude to my advisor Professor Heiko von der Mosel for his constant guidance and patient support. I am indebted to both Professor Heiko von der Mosel and Professor Josef Bemelmans for their trust and support at the very beginning of my stay in Aachen. I would like to thank AOR PD Dr. Alfred Wagner for accepting to be the second referee. Special thanks go to Patrick Overath for proofreading an early version of the present thesis as well as numerous fruitful discussions. I want to thank my friends and former colleagues at the Institut für Mathematik of RWTH Aachen University, especially Dr. Lena Roth, Dr. Frank Roeser, Dr. Norbert Franken and Dr. Armin Schikorra for active listening and taking part in many helpful discussions. Last but not least, I am grateful to my family and Alexandra Albers for their support and motivation.
Finsler Geometry

In the first chapter we provide an introduction to Finsler geometry. The standard reference for this subject is [BCS00] and we will frequently refer to this book. Occasionally, we will also make use of some of the material presented in [She01]. Since general vector bundles play a vital role in Finsler geometry we will commence with a brief review of the basic notions.

Throughout the present thesis $\mathcal{M}^N = \mathcal{M}$ will always be an $N$-dimensional $C^\infty$ manifold. Consider a further $C^\infty$ manifold $E$ and a map $\pi \in C^\infty(E, \mathcal{M})$. The preimage $E_p := \pi^{-1}(p)$ of $p \in \mathcal{M}$ is called the fiber of $\pi$ over $E$. Assuming that the fibers are equipped with an $\mathbb{R}$-vector space structure, the triple $(\pi, E, \mathcal{M})$ is called an $\mathbb{R}$-vector bundle over $\mathcal{M}$ of rank $k$ if for any $p \in \mathcal{M}$ there exists an open neighbourhood $U \subset \mathcal{M}$ of $p$ and a diffeomorphism $\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ such that

(i) $\pi \circ \Phi = \pi_1$, where $\pi_1 : U \times \mathbb{R}^k \to U$ is the projection to the first factor, and

(ii) for $q \in U$ fixed, $\Phi(q, \cdot)$ is a linear isomorphism.

Usually, we speak of the vector bundle $E$ as an abbreviation for the $\mathbb{R}$-vector bundle of rank $k$ $(\pi, E, \mathcal{M})$. A well-known example for a vector bundle is the tangent bundle $E := T\mathcal{M}$ of $\mathcal{M}$, i.e.

$$T\mathcal{M} := \cup_{p \in \mathcal{M}} T_p\mathcal{M}$$

is the union of all tangent spaces $T_p\mathcal{M}$ at $p \in \mathcal{M}$. In that case $\pi : T\mathcal{M} \to \mathcal{M}$ is the projection to the first factor.

A section of a vector bundle $E$ is a map $S : \mathcal{M} \to E$ such that $S \circ \pi = id$, i.e. $S(p) \in E_p$ for $p \in \mathcal{M}$. By $\Gamma(E)$ we denote the space of smooth sections of $E$. In particular, smooth sections of the tangent bundle are called vector fields.

A local frame of $E$ is a $k$-tuple $(S_1, \ldots, S_k)$, $S_i \in \Gamma(E)$, such that for $p \in U \subset \mathcal{M}$ $(S_1(p), \ldots, S_k(p))$ forms a basis of $E_p$. Given a smooth curve $c : I \to \mathcal{M}$, where $I \subset \mathbb{R}$ is some interval, a frame of $E$ along $c$ is a $k$-tuple $(S_1(t), \ldots, S_k(t))$ of sections along $c$ such that for all $t \in I$ $(S_1(t), \ldots, S_k(t))$ is a basis of $E_{c(t)}$.

1.1 Finsler Manifolds

We proceed with some introductory remarks. Given an open subset $U \subset \mathcal{M}$ a chart $x : U \to x(U) \subset \mathbb{R}^N$ is a diffeomorphism whose components define a system of local coordinates $x(p) = (x^i(p))$. Usually, we will identify $p \in \mathcal{M}$ with its coordinate representation $x = x|_p$. Evaluated at $p \in \mathcal{M}$ local coordinates give rise to a basis
\{ \frac{\partial}{\partial \varepsilon} \rvert_p \} \) of \( T_p M \). Thereby, we also obtain coordinates for \( y \in T_p M \) via the formula
\[ y = y^i \frac{\partial}{\partial \varepsilon} \rvert_p. \]
The set
\[ T_M \setminus 0 := T_M \setminus \{(x, 0) ; x \in M \} \]
is called the slit tangent bundle. For \((x, y) \in T_M\), i.e. \( x \in M \) and \( y \in T_p M \), the natural projection \( \pi : T_M \to M \) is given by \( \pi(x, y) := x \). The local coordinates \((x^i, y^i)\) of the manifold \( T_M \setminus 0 \) yield a local frame for \( T(TM \setminus 0) \). We denote the frame in question by \( \{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \} \), knowing full well that the use of the symbol \( \frac{\partial}{\partial y^i} \) is somewhat inaccurate. However, since it both simplifies notation and agrees with the notation of the standard reference [BCS00] we continue using it here.

The dual space of \( T_p M \) is called the cotangent space and denoted by \( T^*_p M \). The dual basis of \( T^*_p M \) is denoted by \( \{ dx^i \} \). The cotangent bundle \( T^* M \) is the union of all cotangent spaces. Local coordinates \((x^i, y^i)\) of the manifold \( T_M \setminus 0 \) give rise to a local coframe of \( T^*(T_M \setminus 0) \) which will be denoted by \( \{ dx^i, dy^i \} \).

We intend to pull back the slit tangent bundle via \( \pi \) and denote the pulled-back bundle by \( \pi^* T M \). More precisely,
\[ \pi^* T M := \bigcup_{(x, y) \in T_M \setminus 0} T_{\pi(x, y)} M \]
which is a vector bundle over the manifold \( T_M \setminus 0 \). The fiber at a point \((x, y) \in T_M \setminus 0\) is \( \pi^* T M \{ (x, y) \} := \{(x, y, v) ; v \in T_x M \} \cong T_x M \).

Following these preparatory remarks we now introduce the notion of a Finsler manifold.

**Definition 1.1.** A function \( F : T M \to [0, \infty) \) is called a Finsler structure of \( M \) if

1. \( F \in C^\infty(T M \setminus 0) \cap C^1(T M) \)
2. \( F(x, \lambda y) = \lambda F(x, y) \) for all \( \lambda > 0 \) and \( (x, y) \in T M \)
3. For each \((x, y) \in T M \setminus 0\), the symmetric bilinear form
\[
g(x, y)(w_1, w_2) := \left. \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \right|_{s,t=0} \{ F^2(x, y + sw_1 + tw_2) \} \]
where \( w_1, w_2 \in T_x M \) is positive definite.

The pair \((M, F)\) is called a Finsler manifold whenever \( M \) is a manifold and \( F \) is a Finsler structure.

**Example 1.2.**

(i) **Riemannian Manifolds:** A family \( \{ g_x \} \) of inner products defines a Finsler structure by \( F(x, y) := \sqrt{g_x(y, y)} \). In this case the Finsler structure is called Riemannian.

(ii) **Randers Spaces:** Provided that we add a \( y \)-linear term to a Riemannian Finsler structure we obtain a so-called Randers metric. A Randers metric \( F(x, y) = \sqrt{g_x(y, y)} + b_i(x) y^i \) is a Finsler structure whenever \( b(x) \in T_x M \) satisfies
\[
\sup_{y \in T_x M} \frac{b_i(x) y^i}{\sqrt{g_x(y, y)}} < 1.
\]
Furthermore, the tuple \((M, F)\) is called Randers space.
A detailed discussion of Randers spaces and further examples of Finsler manifolds can be found in [BCS00]. A Finsler structure is said to be absolutely homogeneous if \( F(x, y) = F(x, -y) \). In general, a Finsler structure does not have this property. Randers metrics provide a family of examples of not absolutely homogeneous Finsler structures since these metrics are absolutely homogeneous if and only if \( b = 0 \), i.e. they are Riemannian.

**Remark 1.3.** Given a Finsler structure \( F \) it is an easy exercise to observe that 
\[
\bar{F} : TM \to [0, \infty), \text{ defined by } \bar{F}(x, y) := F(x, -y),
\]
is again a Finsler structure.

The bilinear form \( g \) satisfies the following Cauchy-Schwarz inequality. See [She01, Lemma 1.2.3] for a proof of the next lemma.

**Lemma 1.4.** Let \((M, F)\) be a Finsler manifold and \( x \in M \). Then
\[
g_{(x,y)}(y,v) \leq F(x,y)F(x,v)
\]
for \( y, v \in T_xM \). Moreover, equality holds if and only if \( v = \lambda y \) for some \( \lambda \geq 0 \).

Distances in the tangent space \( T_xM \) are measured by using the Finsler structure. The Finslerian length of \( y \in T_xM \) is defined by
\[
\| y \|_F := \sqrt{g_{(x,y)}(y,y)} = F(x,y).
\]
Accordingly, we make use of the Finslerian length of the tangent field to measure the length of a curve in a Finsler manifold. Let \( c : [a, b] \to M \) be a (piecewise) \( C^2 \) curve. Its integral length is measured by
\[
L(c) := \int_a^b \| \dot{c}(t) \|_F dt = \int_a^b F(c(t), \dot{c}(t)) dt. \quad (1.1)
\]
For \( x, x_0 \in M \) let the distance \( d : M \times M \to [0, \infty) \) between \( x \) and \( x_0 \) be defined by
\[
d(x,x_0) := \inf \{ L(c); c : [a, b] \to M, \text{ a piecewise } C^\infty \text{ curve, } c(a) = x, c(b) = x_0 \}.
\]
The distance function \( d \) satisfies
\[
\begin{align*}
(i) & \quad d(x_0, x_1) > 0 \text{ provided that } x_0 \neq x_1 \text{ and } d(x_0, x_0) = 0 \\
(ii) & \quad d(x_0, x_1) \leq d(x_0, x_2) + d(x_2, x_1)
\end{align*}
\]
where \( x_0, x_1, x_2 \in M \). However, \( d \) is not a metric in the classic sense since it is, in general, unsymmetric. For example, \( d \) is symmetric if the Finsler structure is absolutely homogeneous. Therefore it is reasonable to distinguish between
\[
\overline{d}_{x_0}(x) := d(x, x_0) \quad \text{and} \quad \overline{d}_{x_0}^*(x) := d(x_0, x) \quad (1.2)
\]
when we are interested in the distance towards or from a fixed point \( x_0 \in M \). Given some subset \( A \subset M \) we set accordingly
\[
\overline{d}_A(x) := \inf_{a \in A} d(x, a) \quad \text{and} \quad \overline{d}_A^*(x) := \inf_{a \in A} d(a, x). \quad (1.3)
\]
Next, we introduce gradients of functions. Let \( f : \mathcal{M} \to \mathbb{R} \) be differentiable. The differential \( df|_x \) of \( f \) at \( x \in \mathcal{M} \) is a linear functional on \( T_x \mathcal{M} \). The gradient \( \text{grad} f|_x \) of \( f \) at \( x \) is the vector that satisfies
\[
df(y) = g(x, \text{grad} f|_x)(\text{grad} f|_x, y).
\]

The existence of \( \text{grad} f \) can be shown by using Legendre transformations, see [She01, Section 3.2] for details. Furthermore, in [She01, Lemma 3.2.3] one finds a proof for the following result on the gradient of the distance function.

**Lemma 1.5.** Let \( (\mathcal{M}, F) \) be a Finsler manifold, \( x_0 \in \mathcal{M} \) and set \( \rho : \mathcal{M} \to \mathbb{R} \) either \( \rho := d^+_{x_0} \) or \( \rho := -d^-_{x_0} \).

Given that \( \rho \) is differentiable at some \( x \in \mathcal{M} \) and under the additional assumption that there exists a neighbourhood \( U \subset \mathcal{M} \) of \( x \) such that for each \( z \in U \) there exists a smooth curve \( c : [0, |\rho(z)|] \to \mathcal{M} \) with \( c(0) = z \) and \( c(|\rho(z)|) = x_0 \) we have
\[
F(x, \text{grad} \rho|_x) = 1.
\]

### 1.2 The Chern Connection and Covariant Derivatives

Given a Finsler manifold \( (\mathcal{M}, F) \) we introduce two important tensors. By applying the bilinear form \( g(x,y) \) induced by \( F \) to standard basis sections we obtain components \( g_{ij} \) of a symmetric covariant 2-tensor \( g = g_{ij} \) on \( T \mathcal{M} \), the fundamental tensor of \( F \).

More precisely, the coefficients \( g_{ij} \) are defined by
\[
g_{ij}(x,y) := \left( \frac{1}{2} F^2(x,y) \right)_{y^iy^j} = g(x,y) \left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \right) .
\]

By \( g^{ij} \) we denote the inverse of \( g_{ij} \). Likewise, the components of the Cartan tensor \( A_{ijk} \), a totally symmetric, covariant 3-tensor are defined by
\[
A_{ijk}(x,y) := \frac{F}{2} \frac{\partial}{\partial y^k} g_{ij}(x,y) = \frac{F}{4} \left( F^2(x,y) \right)_{y^iy^jy^k} .
\]

We collect some properties of the Cartan tensor in the following Lemma.

**Lemma 1.6.** Let \( (\mathcal{M}, F) \) be a Finsler manifold. Then for \( (x,y) \in T\mathcal{M} \setminus 0 \) the components \( A_{ijk} \) of the Cartan tensor have the following properties:

(i) \( A_{ijk}(x,y) \) is symmetric in all its indices.

(ii) \( A_{ijk}(x,y) y^i = A_{ijk}(x,y) y^j = A_{ijk}(x,y) y^k = 0 \)

(iii) \( A_{ijk}(x,y) = 0 \) for all \( (x,y) \in T\mathcal{M} \setminus 0 \) if and only if the Finsler structure \( F \) is Riemannian.

**Proof.** The first assertion is an immediate consequence of the definition of \( A_{ijk} \). Furthermore, the definition implies \( A_{ijk}(x,y) = 0 \) for all \( (x,y) \in T\mathcal{M} \setminus 0 \) if and only if \( g_{ij} \) is independent of \( y \). Thus, \( F \) is Riemannian and the third assertion is proven.
1.2 The Chern Connection and Covariant Derivatives

As to a proof of the second assertion it suffices to consider one of these identities. We compute

\[ y^i A_{ijk}(x, y) = y^i \frac{F}{4} \left( F^2(x, y) \right) y^j y^k = y^i \frac{F}{4} \left( \frac{\partial}{\partial y^j} F \frac{\partial^2}{\partial y^j \partial y^k} F \right. \]

\[ \left. + \frac{\partial}{\partial y^k} F \frac{\partial^2}{\partial y^j \partial y^j} F + F \frac{\partial^3}{\partial y^j \partial y^j \partial y^k} F \right) \]

and observe that the positive homogeneity assumption on the Finsler structure \( F \) yields

\[ F(x, y) = y^i \frac{\partial}{\partial y^i} F(x, y) \]

\[ y^i \frac{\partial^2}{\partial y^i \partial y^j} F(x, y) = 0 \], and

\[ y^i \frac{\partial^3}{\partial y^i \partial y^j \partial y^k} F(x, y) = -\frac{\partial^2}{\partial y^j \partial y^k} F(x, y) \].

Hence, \( y^i A_{ijk}(x, y) = 0 \).

Regarding \((iii)\) in the previous Lemma we obtain a characterisation of Riemannian Finsler structures by vanishing coefficients of the Cartan tensor.

We continue with the introduction of formal Christoffel symbols of the second kind

\[ \gamma^i_{jk} = \left( \gamma^i_{jk} \right)_{(x,y)} := \frac{1}{2} g^{is}(x, y) \left( \frac{\partial}{\partial x^s} g_{sj}(x, y) - \frac{\partial}{\partial x^j} g_{sk}(x, y) + \frac{\partial}{\partial x^s} g_{jk}(x, y) \right) \]

and the coefficients \( N^i_j \) of the Ehresmann connection

\[ N^i_j = (N^i_j)_{(x,y)} := (\gamma^i_{jk})_{(x,y)} y^k - \frac{1}{F} A^i_{jk}(x, y) \left( \gamma^k_{rs} \right)_{(x,y)} y^r y^s, \]

which is a nonlinear connection on \( TM \setminus 0 \).

Remark 1.7. By \( \bar{g}_{ij} \) and \( \bar{A}_{ijk} \) we denote the fundamental tensor and the Cartan tensor related to the Finsler structure \( \bar{F} \) introduced in Remark 1.3. Similarly we denote by \( \bar{\gamma}^i_{jk} \) and \( \bar{N}^i_j \) the corresponding Christoffel symbols of the second kind and the coefficients of the Ehresmann connection respectively. One can quickly verify the following identities for these quantities

\[ \bar{g}_{ij}(x, y) = g_{ij}(x, -y), \]

\[ \bar{A}_{ijk}(x, y) = -A_{ijk}(x, -y), \]

\[ (\bar{\gamma}^i_{jk})_{(x,y)} = (\gamma^i_{jk})_{(x,-y)}, \]

\[ (\bar{N}^i_j)_{(x,y)} = -(N^i_j)_{(x,-y)}. \]

In order to obtain a more convenient behaviour under transformations induced by coordinate changes on \( \mathcal{M} \) we replace \( \frac{\partial}{\partial x^i} \) by

\[ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j} \]

and \( dy^i \) by

\[ \delta y^i := dy^i + N^i_j dx^j. \]

We remark that \( \{ \delta, F \frac{\partial}{\partial y^i} \} \) on \( T(TM \setminus 0) \) is the natural dual basis to \( \{ dx^i, \frac{1}{F} \delta y^i \} \) on \( T^*(TM \setminus 0) \). We also obtain a decomposition of \( T(TM \setminus 0) \) into subbundles, more precisely into a horizontal part spanned by \( \{ \delta \} \), and a vertical part spanned by
\{F, \frac{\partial}{\partial y}\}. More precisely, we have that these subspaces are orthonormal with respect to the metric
\[ g_{ij}(x, y) \, dx^i \otimes dx^j + g_{ij}(x, y) \, \frac{\delta y^j}{F} \otimes \frac{\delta y^i}{F} \] (1.6)
which is a Riemannian metric on \( T(TM) \). The metric (1.6) is called Sasaki metric. See [BCS00, Section 2.3] for further explanation.

The existence of a linear connection on the pull-back bundle was proven by S.-S. Chern in 1948. We state the result in the following theorem whose proof can be found in [BCS00, Theorem 2.4.1].

**Theorem 1.8** (Chern). Let \((\mathcal{M}, F)\) be a Finsler manifold. There exists a unique linear connection \( \nabla \) on the pull-back bundle \( \pi^*TM \) which is determined by
\[ \nabla : \Gamma(T(TM)) \times \Gamma(\pi^*TM) \to \Gamma(\pi^*TM) \]
\[ \nabla(\hat{X}, W) = \nabla_{\hat{X}} W = \left( dW^i(\hat{X}) + W^i j \omega^j(\hat{X}) \right) \otimes \frac{\partial}{\partial x^i}, \]
where \( \hat{X} \in \Gamma(T(TM)) \), \( W = W^i \frac{\partial}{\partial x^i} \) is a smooth section of \( \pi^*TM \) and the connection coefficients \( \omega^i \) are given by
\[ \omega^i = \Gamma^i_{jk} \, dx^k \]
where \( \Gamma^i_{jk} \) are symmetric in the lower indices, i.e. \( \Gamma^i_{jk} = \Gamma^i_{kj} \) and defined by
\[ (\Gamma^i_{jk})(x, y) := (\gamma^i_{jk}(x, y)) - g^{is}(x, y) \left( A_{sjk} \frac{N^l_k}{F} - A_{jk} N^l_s N^l_k + A_{kl} \frac{N^l_s}{F} \right)(x, y), \]
or equivalently
\[ (\Gamma^i_{jk})(x, y) = g^{is}(x, y) \left( \frac{\delta g^{jk}}{\delta x^l} - \frac{\delta g_{sk}}{\delta x^l} + \frac{\delta g^{sk}}{\delta x^l} \right)(x, y). \]

The first condition on the connection coefficients in the previous theorem characterises the connection as being torsion-free whilst the second as being almost g-compatible. In the succeeding Lemma we rephrase the fact that \( \nabla \) is torsion free by using a notation from [She01].

**Lemma 1.9.** Let \((\mathcal{M}, F)\) be a Finsler manifold. Given an arbitrary section \( \hat{X} \in \Gamma(T(TM)) \) we write \( \hat{X} = \hat{X}^i \frac{\partial}{\partial x^i} + X^i \frac{\partial}{\partial y^i} \) and set \( \tilde{X} = \hat{X}^i \frac{\partial}{\partial x^i} \). Then
\[ \nabla_{\hat{X}} \hat{Y} - \nabla_{\hat{Y}} \hat{X} - [\hat{X}, \hat{Y}] = 0 \]
where \( \hat{X}, \hat{Y} \in \Gamma(T(TM)) \), and the Lie bracket \([., .] : \Gamma(T(TM)) \times \Gamma(T(TM)) \to \Gamma(T(TM))\) is defined by \([\hat{X}, \hat{Y}] := \hat{X}\hat{Y} - \hat{Y}\hat{X}\).

**Proof.** Firstly, we compute
\[
\nabla_{\hat{X}} \hat{Y} = \nabla_{\hat{X}} \left( \hat{Y}^j \frac{\partial}{\partial x^j} \right) = \left( d\hat{Y}^j \right)(\hat{X}) \frac{\partial}{\partial x^j} + \hat{Y}^j \nabla_{\hat{X}} \left( \frac{\partial}{\partial x^j} \right) \\
= \left( \hat{X}^i \frac{\partial}{\partial x^i} \hat{Y}^j + X^i \frac{\partial}{\partial y^i} \hat{Y}^j \right) \frac{\partial}{\partial x^j} + \hat{Y}^j \omega^j(\hat{X}) \frac{\partial}{\partial x^j}
\]
1.2 The Chern Connection and Covariant Derivatives

and similarly, \( \nabla_{\dot{Y}} \dot{X} = \left( \dot{Y}^i \frac{\partial}{\partial x^i} \dot{X}^j + Y^i \frac{\partial}{\partial y^i} \dot{X}^j \right) \frac{\partial}{\partial x^j} + \dot{X}^i \omega^j_i (\dot{Y}) \frac{\partial}{\partial x^j} \). Since \( \nabla \) is torsion-free we derive

\[
\dot{X}^j \omega^i_j (\dot{Y}) = \dot{X}^j \left( \dot{Y}^k \omega^i_k (\frac{\partial}{\partial x^k}) + Y^k \omega^i_j (\frac{\partial}{\partial y^k}) \right) = \dot{X}^j \dot{Y}^k \Gamma^i_{jk} = \dot{X}^j \dot{Y}^k \omega^i_k (\frac{\partial}{\partial x^j}) = \dot{Y}^j \omega^i_j (\dot{X})
\]

and conclude

\[
\nabla_{\dot{X}} \dot{Y} - \nabla_{\dot{Y}} \dot{X} = \left( \dot{X}^i \frac{\partial}{\partial x^i} \dot{Y}^j + X^i \frac{\partial}{\partial y^i} \dot{Y}^j - \dot{Y}^i \frac{\partial}{\partial x^i} \dot{X}^j - Y^i \frac{\partial}{\partial y^i} \dot{X}^j \right) \frac{\partial}{\partial x^j} = [\dot{X}, \dot{Y}].
\]

We consider a smooth, regular curve \( c : [a, b] \to \mathcal{M} \) and set \( \dot{c}(t) := \frac{d}{dt} (c(t), \dot{c}(t)) \). For \( W \in \Gamma(T^*(\mathcal{M})) \) we compute

\[
\nabla_{\dot{c}(t)} W \big|_{(c(t), \dot{c}(t))} = \left( d \frac{\partial W^i}{\partial x^i} \big|_{(c(t), \dot{c}(t))} (\dot{c}(t)) + W^j \big|_{(c(t), \dot{c}(t))} \omega^i_j (\dot{c}) \big|_{(c(t), \dot{c}(t))} \right) \frac{\partial}{\partial x^i} \big|_{(c(t), \dot{c}(t))}
\]

\[
= \left( \frac{d}{dt} (W^i \circ (c, \dot{c})) (t) + W^j \big|_{(c(t), \dot{c}(t))} (\Gamma^i_{jk}) \big|_{(c(t), \dot{c}(t))} \dot{c}^k (t) \right) \frac{\partial}{\partial x^i} \big|_{(c(t), \dot{c}(t))}
\]

This observation motivates the subsequent definition of covariant derivatives of vector fields along smooth curves.

**Definition 1.10.** Let \( (\mathcal{M}, F) \) be a Finsler manifold and \( c : [a, b] \to \mathcal{M} \) be a smooth, regular curve. For a given vector field \( W = W^i \frac{\partial}{\partial x^i} \big|_c \) along \( c \) the covariant derivative of \( W \) along \( c \) is defined by

\[
D_c W(t) := \left( \frac{d}{dt} W^i (t) + W^j (t) \dot{c}^k (t) (\Gamma^i_{jk}) \big|_{(c(t), \dot{c}(t))} \right) \frac{\partial}{\partial x^i} \big|_{c(t)}.
\]

**Remark 1.11.** Since one might replace \( (\Gamma^i_{jk}) \big|_{(c(t), \dot{c}(t))} \) in the above definition by \( (\Gamma^i_{jk}) \big|_{(c(t), W(t))} \) there exist two distinct notions of a covariant derivative along a given curve. In order to distinguish between both notions one usually mentions the reference vector \( \dot{c} \) or \( W \) respectively. See [BCS00, Section 5.2] for further details. However, in the present thesis we will only make use of covariant derivatives with reference vector \( \dot{c} \) as defined in Definition 1.10. Consequently, in what follows we will continue to suppress the reference vector.

**Definition 1.12.** Let \( (\mathcal{M}, F) \) be a Finsler manifold and \( c : [a, b] \to \mathcal{M} \) be a smooth, regular curve. A vector field \( V \) along \( c \) is called parallel if

\[
D_c V(t) = 0 \quad \text{for} \quad t \in (a, b).
\]

A map \( P_c : T_{c(a)} \mathcal{M} \to T_{c(b)} \mathcal{M} \) is called parallel transport of \( v \in T_{c(a)} \mathcal{M} \) along \( c \), provided that \( P_c(v) = V(b) \) where \( V \) is a parallel vector field along \( c \) with \( V(a) = v \).
We conclude this section with a brief introduction to Finslerian curvature notions. The curvature tensor $\Omega : \Gamma(T(TM)) \times \Gamma(T(TM)) \times \Gamma(\pi^*TM) \to \Gamma(\pi^*TM)$ of the Chern connection is given by

$$\Omega(\hat{X}, \hat{Y})W := \nabla_{\hat{X}} \nabla_{\hat{Y}} W - \nabla_{\hat{Y}} \nabla_{\hat{X}} W - \nabla_{[\hat{X}, \hat{Y}]} W$$  \hfill (1.7)

where $\hat{X}, \hat{Y} \in \Gamma(T(TM))$, $W \in \Gamma(\pi^*TM)$. For $W = W^j \frac{\partial}{\partial x^j}$ we write

$$\Omega(\hat{X}, \hat{Y})W = W^j \Omega(\hat{X}, \hat{Y}) \frac{\partial}{\partial x^j} =: W^j \Omega_j(\hat{X}, \hat{Y}) \frac{\partial}{\partial x^j}.$$ 

By successively choosing horizontal, vertical and mixed basis sections of $T(TM \setminus \{0\})$ for $\hat{X}, \hat{Y}$ we obtain so called $hh$-, $vv$-, and $hv$-curvature tensors. We refer to [BCS00, Chapter 3] for a detailed discussion of these curvature notions. Here we merely prove a representation for the components $R^i_{j kl}$ of the $hh$-curvature tensor in local coordinates.

**Lemma 1.13.** Let $(M, F)$ be a Finsler manifold and, like in Theorem 1.8 and (1.7), denote the Chern connection by $\nabla$ and its curvature tensor by $\Omega$. In local coordinates, the components $R^i_{j kl}$ of the $hh$-curvature tensor are given by

$$R^i_{j kl} := \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^h_{lj} \Gamma^i_{hk} - \Gamma^i_{hl} \Gamma^h_{jk}. $$

**Proof.** Following (1.7) we initially compute $\nabla_{\frac{\delta}{\delta x^j}} \nabla_{\frac{\delta}{\delta x^k}} \nabla_{\frac{\delta}{\delta x^l}}$ and $\nabla_{\frac{\delta}{\delta x^j}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial x^l}$ by using Theorem 1.8 and the duality of $dx^i$ and $\frac{\delta}{\delta x^j}$. Thereby, we obtain

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^i} = \nabla_{\frac{\delta}{\delta x^j}} \left( \omega^i_j \frac{\delta}{\delta x^j} \right) \frac{\partial}{\partial x^i} = d \left( \omega^i_j \frac{\delta}{\delta x^j} \right) \frac{\partial}{\partial x^i} + \omega^i_j \frac{\delta}{\delta x^i} \nabla_{\frac{\delta}{\delta x^j}} \left( \frac{\delta}{\delta x^i} \right) = d \Gamma^i_{jk} \left( \frac{\delta}{\delta x^j} \right) \frac{\partial}{\partial x^i} + \Gamma^i_{jk} \omega^i_j \frac{\delta}{\delta x^i} \frac{\partial}{\partial x^i}$$

and similarly $\nabla_{\frac{\delta}{\delta x^j}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial x^l} = \left( \frac{\delta}{\delta x^j} \Gamma^i_{jl} + \frac{\delta}{\delta x^j} \Gamma^i_{rl} \right) \frac{\partial}{\partial x^i}$. Hence, it remains to show that $\nabla_{\frac{\delta}{\delta x^j}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial x^l}$ vanishes. In view of Theorem 1.8 this is especially true in case the Lie bracket $[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}]$ is only vertical. From (1.5) we derive

$$[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}] = \delta_j^l \delta_k^i - \delta_j^k \delta_i^l = N^j_k N^i_l \frac{\partial}{\partial y^l} \frac{\partial}{\partial y^i} - N^l_j N^i_k \frac{\partial}{\partial y^l} \frac{\partial}{\partial y^i}$$

which concludes the proof. \qed

Using the coefficients of the $hh$-curvature tensor we define a curvature tensor $R : \Gamma(TM \setminus \{0\}) \times \Gamma(TM \setminus \{0\}) \times \Gamma(TM \setminus \{0\}) \to \Gamma(TM \setminus \{0\})$ for vector fields on $M$ by

$$R(V, T) := (T^j R^i_{j kl} T^l) V^k \frac{\partial}{\partial x^i}$$  \hfill (1.8)

where $V, T \in \Gamma(TM \setminus \{0\})$. We will make use of this tensor in the following section.
1.3 Geodesics, the Exponential Map and Jacobi Fields

We consider a Finsler manifold \((\mathcal{M}, F)\) and \(x, z \in \mathcal{M}\). When seeking for a distance minimising curve from \(x\) to \(z\) one usually performs the first variation of the length functional (1.1). In what follows we work with piecewise smooth, regular curves \(c\), i.e. \(\dot{c} \neq 0\) whenever the derivative exists. Such a curve may be parametrised to have constant Finslerian speed \(F(c, \dot{c}) = \text{const}\) and hence we simply consider constant speed curves. As a consequence of the variational approach mentioned above one obtains the following Proposition.

**Proposition 1.14.** Let \((\mathcal{M}, F)\) be a Finsler manifold, \(x, z \in \mathcal{M}\) and \(c : [a, b] \to \mathcal{M}\) with \(c(a) = x\) and \(c(b) = z\) a constant speed, piecewise \(C^\infty\) curve. If \(c\) minimises distance from \(x\) to \(z\) then \(c \in C^\infty([a, b])\) and \(c\) satisfies the following system of ordinary differential equations

\[
\frac{d^2}{dt^2} c^i(t) + G^i(c(t), \dot{c}(t)) = 0, \quad (1.9)
\]

where \(\dot{c} = \frac{d}{dt} c\) and \(G^i(x, y) := y^j y^k \left(\gamma^i_{jk}\right)(x, y)\).

See [BCS00, Section 5.1] or [She01, Section 5.1] for a proof of the preceding Proposition. In view of this result we give the following definition.

**Definition 1.15.** Let \((\mathcal{M}, F)\) be a Finsler manifold. A smooth, constant speed curve \(c : [a, b] \to \mathcal{M}\) is called a geodesic if

\[
\frac{d^2}{dt^2} c^i(t) + G^i(c(t), \dot{c}(t)) = 0.
\]

The coefficients \(G^i\) are called geodesic coefficients.

Combining the first condition on the connection coefficients in Theorem 1.8 with \((ii)\) in Lemma 1.6 one can show that unit speed geodesics are equivalently characterised by

\[
\frac{d^2}{dt^2} c^i(t) + \frac{d}{dt} c^j(t) \frac{d}{dt} c^k(t) \left(\Gamma^i_{jk}\right)_{(c(t), \dot{c}(t))} = 0,
\]

or equivalently \(D_c \dot{c}(t) = 0\), i.e constant speed geodesics are auto-parallel curves. For later application we prove an equivalent implicit formulation of the system (1.9).

**Lemma 1.16.** Let \((\mathcal{M}, F)\) be a Finsler manifold and \(c : [a, b] \to \mathcal{M}\) be a constant speed geodesic, i.e. a solution of the system (1.9). Then \(c\) is equivalently characterised by the following system of ordinary differential equations

\[
\frac{\partial}{\partial x^j} \left(\frac{1}{2} F^2\right) ((c(t), \dot{c}(t)) = \frac{d}{dt} \left(\frac{\partial}{\partial y^j} \left(\frac{1}{2} F^2\right) (c(t), \dot{c}(t))\right), \quad (1.10)
\]

for \(1 \leq j \leq N\) and \(t \in (a, b)\).

**Proof.** We compute

\[
\frac{\partial}{\partial x^j} \left(\frac{1}{2} F^2\right) ((c(t), \dot{c}(t)) = \frac{d}{dt} \left(\frac{\partial}{\partial y^j} \left(\frac{1}{2} F^2\right) (c(t), \dot{c}(t))\right).
\]
Finsler Geometry

Let $\text{Lemma 1.17.}$ reads as follows. Since $F$ is positively homogeneous we conclude $\frac{\partial}{\partial y^j} \left( \frac{1}{2} F^2(x, y) \right) y^j = F^2(x, y)$ and $F^2(x, y) = \frac{\partial^2}{\partial x^i \partial y^j} \left( \frac{1}{2} F^2(x, y) \right) y^j y^k = g_{ij}(x, y) y^i y^j$ from which we derive

$$\frac{\partial}{\partial x^k} \frac{\partial}{\partial y^j} \left( \frac{1}{2} F^2(x, y) \right) y^k = \frac{\partial}{\partial x^k} g_{ij}(x, y) y^k y^j,$$

$$\frac{\partial}{\partial x^j} \left( \frac{1}{2} F^2(x, y) \right) = \frac{1}{2} \frac{\partial}{\partial x^j} g_{kl}(x, y) y^k y^l.$$

By inserting these identities we obtain

$$\frac{1}{2} \frac{\partial}{\partial x^j} g_{kl}(c(t), \dot{c}(t)) t^k(t) - \frac{\partial}{\partial x^k} g_{ij}(c(t), \dot{c}(t)) t^k(t) t^j(t) = g_{ij}(c(t), \dot{c}(t)) \ddot{c}^j(t).$$

We have $\frac{\partial}{\partial x^j} g_{ij}(c(t), \dot{c}(t)) t^k(t) = \frac{\partial}{\partial x^j} g_{jk}(c(t), \dot{c}(t)) \dot{c}^k(t)$ and hence

$$\frac{\partial}{\partial x^j} g_{ij}(c(t), \dot{c}(t)) \ddot{c}^j(t) = -\frac{1}{2} \left( \frac{\partial}{\partial x^j} g_{ij}(c(t), \dot{c}(t)) - \frac{\partial}{\partial x^j} g_{kl}(c(t), \dot{c}(t)) + \frac{\partial}{\partial x^k} g_{jk}(c(t), \dot{c}(t)) \right) \dot{c}^k(t) \ddot{c}^j(t).$$

Consequently, $\ddot{c}^j(t) = (\gamma_{kl})_{(c(t), \dot{c}(t))} \dot{c}^k(t) \ddot{c}^j(t)$ and the claim is proven. \hfill \Box

Standard existence results for systems of ordinary differential equations yield existence and uniqueness for solutions of (1.9) for given initial data. The precise statement reads as follows.

**Lemma 1.17.** Let $(M, F)$ be a Finsler manifold and $x \in M$. Then there exists $\epsilon > 0$ such that for all $y \in T_x M$ satisfying $|y| < \epsilon$ there exists a unique geodesic $c = c(x, y, \cdot) : [0, 1] \to M$ such that $c(0) = x$ and $\dot{c}(0) = y$.

Given a geodesic $c$ on a Finsler manifold $(M, F)$ one can easily show that the backward curve is a geodesic on the Finsler manifold $(M, \bar{F})$. The precise statement reads as follows.

**Lemma 1.18.** Let $(M, F)$ be a Finsler manifold and $c : [a, b] \to M$ be a geodesic. For $t_0 \in \mathbb{R}$ let $\tilde{c} : [t_0 - b, t_0 - a] \to M, \tilde{c}(s) := c(t_0 - s)$ be the backward curve. Then $\tilde{c}$ is a geodesic with respect to the Finsler structure $\bar{F}$ defined in Remark 1.3. More precisely, one obtains

$$\frac{d^2 \tilde{c}}{ds^2} + \bar{G}^i(\tilde{c}(s), \dot{\tilde{c}}(s)) = 0 \quad \text{for} \quad s \in (t_0 - b, t_0 - a),$$

where $\bar{G}^i(x, y) := y^j y^k (\bar{\gamma}_{jk}^i)_{(x, y)}$ are the geodesic coefficients of $\bar{F}$.

In the subsequent Lemma we provide product rules for derivatives of vector fields along geodesics.
Lemma 1.19. Let \((M, F)\) be a Finsler manifold, \(c : [a, b] \to M\) be a geodesic and \(V, W\) be vector fields along \(c\). Then
\[
\frac{d}{dt} g(c,\dot{c})(V(t), W(t)) = g(c,\dot{c})(D_c V, W) + g(c,\dot{c})(V(t), D_c W(t)),
\]
and
\[
\frac{d}{dt} g(c,V)(V(t), W(t)) = g(c,V)(D_c V(t), W(t)) + g(c,V)(V(t), D_c W(t))
\]
for \(t \in [a, b]\).

Proof. For the sake of improved readability we suppress the \(t\)-dependence of vector fields occurring in the following computations. Given a frame \(\frac{\partial}{\partial x^i}|_c(t)\) along \(c\) we write \(V = V^i \frac{\partial}{\partial x^i}\) and \(\dot{c} = \dot{c}^i \frac{\partial}{\partial x^i}\) and compute
\[
\frac{d}{dt} g(c,\dot{c})(V, W) = \frac{d}{dt} \left( g_{ij}(c, \dot{c}) V^i W^j \right) \\
= \frac{d}{dt} g_{ij}(c, \dot{c}) V^i W^j + g_{ij}(c, \dot{c}) \dot{V}^i W^j + g_{ij}(c, \dot{c}) V^i \dot{W}^j \\
= \frac{\partial}{\partial x^k} g_{ij}(c, \dot{c}) \dot{c}^k V^i W^j + \frac{\partial}{\partial y^k} g_{ij}(c, \dot{c}) \dot{c}^k \dot{V}^i W^j + g_{ij}(c, \dot{c}) \dot{V}^i \dot{W}^j + g_{ij}(c, \dot{c}) V^i \dot{W}^j
\]
and similarly
\[
\frac{d}{dt} g(c,V)(V, W) = \frac{\partial}{\partial x^k} g_{ij}(c, V) \dot{c}^k V^i W^j + \frac{\partial}{\partial y^k} g_{ij}(c, V) \dot{c}^k \dot{V}^i W^j + g_{ij}(c, \dot{c}) \dot{V}^i W^j + g_{ij}(c, \dot{c}) V^i \dot{W}^j
\]
Next, we utilise the identity \(\frac{\partial}{\partial x^k} g_{ij} = g_{ij} \Gamma^s_{ik} + g_{is} \Gamma^s_{jk} + 2A_{ij}^s N^k_{\frac{s}{F}}\) which is taken from [BCS00, Page 40]. Therewith we obtain after relabeling some indices
\[
\frac{d}{dt} g(c,\dot{c})(V, W) = g_{ij}(c, \dot{c}) \Gamma^i_{sk} \dot{c}^k V^s W^j + g_{ij}(c, \dot{c}) \Gamma^j_{sk} \dot{c}^k V^i W^s \\
+ \frac{2}{F(c, \dot{c})} A_{ij}^s(c, \dot{c}) (N^k_{c\dot{c}}) \dot{c}^k V^i W^j + \frac{2}{F(c, \dot{c})} A_{ij}^k(c, \dot{c}) \dot{c}^k V^i W^j \\
+ g_{ij}(c, \dot{c}) V^i \dot{W}^j + g_{ij}(c, \dot{c}) \dot{V}^i W^j
\]
\[
= g_{ij}(c, \dot{c}) (D_c V, W) + g_{ij}(c, \dot{c}) (V, D_c W)
\]
and
\[
\frac{d}{dt} g(c,V)(V, W) = g(c,V)(D_c V, W) + g(c,V)(V, D_c W) + \frac{2}{F(c, V)} A_{ij}^k(c, V) \dot{V}^k V^i W^j
\]
\[
+ \frac{2}{F(c, V)} A_{ij}^s(c, V) (N^k_{c\dot{c}}) \dot{c}^k V^i W^j.
\]
It remains to show that the additional terms in both equations vanish. As to the first equation we initially remark \((N^k_{c\dot{c}})(c, \dot{c}) \dot{c}^k = (\gamma^s_{kl})(c, \dot{c}) \dot{c}^k \dot{c}^l\) which is a straightforward consequence of the definition of \(N^k_{c\dot{c}}\) and (ii) in Lemma 1.6. Since \(c\) is a geodesic we further conclude \((N^k_{c\dot{c}})(c, \dot{c}) \dot{c}^k = -\dot{c}^2\). Regarding the second identity, Lemma 1.6 immediately yields that the additional terms vanish. \(\square\)
Corollary 1.20. Let \((\mathcal{M}, F)\) be a Finsler manifold, \(c : [a, b] \to \mathcal{M}\) be a geodesic and \(V, W\) be parallel vector fields along \(c\). Then \(g_{(c,\dot{c})}(V(t), W(t))\) is constant for \(t \in [a, b]\).

We proceed with the definition of the exponential map and a summary of its properties.

Definition 1.21. Let \((\mathcal{M}, F)\) be a Finsler manifold and \(U \subset T\mathcal{M}\) be an open neighbourhood of the zero section of \(T\mathcal{M}\) such that \(c(1, x, y)\) exists for any \((x, y) \in U \setminus \{0\} := U \setminus \{(x, 0); x \in \mathcal{M}\}\). Then the exponential map \(\exp : U \to \mathcal{M}\) is defined by

\[
\exp(x, y) := \begin{cases} 
  c(x, y, 1) & \text{for } y \neq 0, \\
  x & \text{for } y = 0.
\end{cases}
\]

The existence of the neighbourhood \(U\) of \(T\mathcal{M} \setminus 0\) is assured by Lemma 1.17. We collect main properties of the exponential map in the following proposition.

Proposition 1.22. Let \((\mathcal{M}, F)\) be a Finsler manifold and \(U, U \setminus 0\) be the sets from Definition 1.21. Then

(i) \(\exp \in C^\infty(U \setminus 0, \mathcal{M})\),

(ii) \(D_y \exp(\cdot, 0) = \text{id}\),

(iii) given \(x_0 \in \mathcal{M}\) and an open neighbourhood \(\mathcal{V} \subset \subset \mathcal{M}\) of \(x_0\) there exists a constants \(C_0 = C_0(x_0) > 0\) and \(C_1 = C_1(x_0) > 0\) such that

\[
\left| \frac{\partial}{\partial y^i} \exp^i(x, y) - \delta^i_j \right| \leq C_0 F(x, y) \leq C_1 |y|
\]

and

\[
\left| \frac{\partial}{\partial x^j} \exp^i(x, y) - \delta^i_j \right| \leq C_0 F(x, y) \leq C_1 |y|
\]

for \((x, y) \in U\) with \(x \in \mathcal{V}\),

(iv) \(\exp \in C^1(U, \mathcal{M})\).

Proof. Assertions (i), (ii) and (iv) are standard. Their proofs can be found in [BCS00, Section 5.3]. Moreover, assertion (iii) may be regarded as an intermediate step in the proof of (iv). The following proof of (iii) is just a minor refinement of the proof for (iv) given in [BCS00].

For \(x \in \mathcal{M}\) let \(v \in I_x \mathcal{M} := \{y \in T_x \mathcal{M}; F(x, y) = 1\} \subset T_x \mathcal{M}\). Choose \(\delta > 0\) such that \((x, \lambda v) \in U\) for \(0 \leq \lambda \leq \delta\) and observe \(\exp(x, \lambda v) = c(x, \lambda v, 1) = c(x, v, \lambda)\). Thus, in local coordinates we have

\[
\frac{\partial}{\partial v^j} \exp^i(x, \lambda v) = \frac{\partial}{\partial y^k} \exp^i(x, \lambda v) \lambda \delta^j_k = \frac{\partial}{\partial y^i} \exp^i(x, \lambda v) \lambda = \frac{\partial}{\partial y^i} c^i(x, v, \lambda)
\]

and

\[
\frac{\partial}{\partial x^j} \exp^i(x, \lambda v) = \frac{\partial}{\partial x^j} c^i(x, v, \lambda).
\]
Since \( c(x, y, 0) = x \) for all \((x, y) \in U \setminus 0\) we obtain \( \frac{\partial}{\partial y^j} c^i(x, y, 0) = 0 \) and \( \frac{\partial}{\partial x^j} c^i(x, y, 0) = \delta^i_j \). Consequently, the Taylor expansions for \( \frac{\partial}{\partial y^j} c^i(x, y, \cdot) \) and \( \frac{\partial}{\partial x^j} c^i(x, y, \cdot) \) are

\[
\frac{\partial}{\partial y^j} c^i(x, y, \lambda) = \frac{\partial^2}{\partial t \partial y^j} c^i(x, y, 0) \lambda + O(\lambda^2) = \delta^i_j \lambda + O(\lambda^2)
\]

where we applied Schwartz’s Theorem to derive the last equation and

\[
\frac{\partial}{\partial x^j} c^i(x, y, \lambda) = \delta^i_j + \frac{\partial^2}{\partial t \partial x^j} c^i(x, y, 0) \lambda + O(\lambda^2).
\]

Hence,

\[
\left| \frac{\partial}{\partial y^j} \exp^i(x, \lambda v) - \delta^i_j \right| \leq C(x, v) \lambda \quad \text{and} \quad \left| \frac{\partial}{\partial x^j} \exp^i(x, \lambda v) - \delta^i_j \right| \leq C(x, v) \lambda \quad (1.11)
\]

where the constant \( C \) depends continuously on \((x, v)\). The set \( I_x M \) is compact and hence, the constant in (1.11) can be chosen independently of \( v \).

To conclude the proof we consider \( x_0 \in M \) and an open neighbourhood \( V \subset M \) of \( x_0 \). For \((x, y) \in U \) with \( x \in V \) we derive from (1.11) and the simple fact \( \frac{1}{F(x, y)} y \in I_x M \) that

\[
\left| \frac{\partial}{\partial y^j} \exp^i(x, y) - \delta^i_j \right| \leq C(x) F(x, y) \leq C_0 F(x, y) = C_0 F(x, \frac{y}{|y|}) |y| \leq C_1 |y|
\]

where \( C_1 := C_0 \sup_{z \in V} \sup_{v \in T_{x} M, |v| = 1} F(z, v) \) and \( C_0 := \sup_{z \in V} C(z) \). The second estimate follows similarly. \( \square \)

The remainder of this section is dedicated to the theory of Jacobi fields in Finsler manifolds. To begin with we recall (1.8) in which we defined the R-curvature tensor.

**Definition 1.23.** Let \((M, F)\) be a Finsler manifold and \( c : [0, r] \to M \) be a constant speed geodesic. A vector field \( J \) along \( c \) is called a Jacobi field whenever it satisfies the equation

\[
D_c D_t J(t) + R(J(t), \dot{c}(t)) \dot{c}(t) = 0
\]

for \( t \in [0, r] \).

**Lemma 1.24.** Let \((M, F)\) be a Finsler manifold, \( x \in M \), \( c : [0, r] \to M \) be a constant speed geodesic with \( c(0) = x \), and \( J \) be a Jacobi field along \( c \). Furthermore, let \((E_1(t), \ldots, E_n(t))\) be a \( g_{c, \dot{c}} \) orthonormal frame along \( c \) where each vector field \( E_j \) is assumed to be parallel and set \( J = J^j E_j \). Then the component functions \( J^j \) satisfy the following scalar Jacobi field equation

\[
\ddot{J}^j(t) + R^j_i(t) J^i(t) = 0 \quad (1.12)
\]

where \( R(E_j(t), \dot{c}(t)) \dot{c}(t) = R^j_i(t) E_i(t) \). Provided that we additionally assume \( J(0) = 0 \) the following Taylor expansion holds

\[
J^j(t) = J^j(0) t - \frac{1}{6} R^j_i(0) J^i(0) t^3 + O(t^4).
\]
Proof. Since \( E_i \) is a parallel vector field we have \( D_{E_i} E_i(t) = 0 \). Thus, \( D_{E_i} (J^i(t) E_i(t)) = \dot{J}^i(t) E_i(t) \) and \( D_{E_i} (J^j(t) E_i(t)) = \dot{J}^j(t) E_i(t) \) and the first assertion follows. Furthermore,

\[
J^i(t) = J^i(0) + \dot{J}^i(0)t + \frac{1}{2} \ddot{J}^i(0)t^2 + \frac{1}{6} \frac{d^3}{dt^3} (J^i(0)) t^3 + O(t^4).
\]

From the hypothesis we obtain \( J^i(0) = 0 \) and \( \dot{J}^i(0) = -R^i_j(0)J^j(0) = 0 \). Finally, the second assertion is a consequence of \( \frac{d}{dt} \dot{J}^i(t) = -\frac{d}{dt} \left( R^i_j(t)J^j(t) \right) = -\frac{d}{dt} R^i_j(t)J^j(t) - R^i_j(t) \frac{d}{dt} J^j(t) \).

Again we derive existence and uniqueness for solutions of the system (1.12) for given initial data \( J^i(0) \) and \( \dot{J}^i(0) \) from standard theorems on ordinary differential equations. Natural examples for Jacobi fields arise from geodesic variations.

**Definition 1.25.** Let \((\mathcal{M},F)\) be a Finsler manifold, \( r,\epsilon > 0 \) and \( c : [a,b] \rightarrow \mathcal{M} \) a geodesic. A map \( H : (-\epsilon,\epsilon) \times [a,b] \rightarrow \mathcal{M} \) is called geodesic variation of \( c \) if \( H(0,t) = c(t) \) and \( H(s,\cdot) \) is a geodesic for each \( s \in (-\epsilon,\epsilon) \).

Before we can give the precise statement on Jacobi fields arising from geodesic variations we need the following technical lemma.

**Lemma 1.26.** Let \((\mathcal{M},F)\) be a Finsler manifold, \( c : [a,b] \rightarrow \mathcal{M} \) be a geodesic, and \( H : (-\epsilon,\epsilon) \times [a,b] \rightarrow \mathcal{M} \) be a geodesic variation. Then

\[
D_{\overline{\partial}} H \left( \frac{\partial}{\partial s} H(s,t) \right) = D_{\overline{\partial}} H \left( \frac{\partial}{\partial t} H(s,t) \right).
\]

Proof. We set \( \tilde{T} := \frac{\partial}{\partial \overline{\partial}} H(s,t), \frac{\partial}{\partial \overline{\partial}} H(s,t) \) and \( \tilde{S} := \frac{\partial}{\partial \overline{t}} H(s,t), \frac{\partial}{\partial \overline{t}} H(s,t) \) and obtain from Lemma 1.9

\[
\nabla_{\overline{\partial}} \tilde{T} - \nabla_{\overline{\partial}} \tilde{S} = \widetilde{[\tilde{T},\tilde{S}]} = \left( \frac{\partial^2}{\partial \overline{t}\partial s} H^j(s,t) - \frac{\partial^2}{\partial \overline{s}\partial t} H^j(s,t) \right) \frac{\partial}{\partial \overline{x}^j} = 0
\]

which concludes the proof.

**Lemma 1.27.** Let \((\mathcal{M},F)\) be a Finsler manifold. For \( x_0 \in \mathcal{M}, y \in T_{x_0}\mathcal{M} \) we consider the geodesic \( c : [0,r] \rightarrow \mathcal{M}, c(t) := \exp(x_0, ty) \).

(i) Suppose that \( H : (-\epsilon,\epsilon) \times [0,r] \rightarrow \mathcal{M} \) is an arbitrary geodesic variation of \( c \). Then

\[
J(t) := \left. \frac{\partial}{\partial s} H(s,t) \right|_{s=0}
\]

is a Jacobi field along \( c \).

(ii) Consider a so called `wedge-shaped' geodesic variation \( H : (-\epsilon,\epsilon) \times [0,r] \rightarrow \mathcal{M}, H(s,t) := \exp(x_0, t(y + sv)) \) where \( v \in T_{x_0}\mathcal{M} \). Then the corresponding Jacobi field \( J(t) := \left. \frac{\partial}{\partial s} H(s,t) \right|_{s=0} \) has initial data \( J(0) = 0 \) and \( D_c J(0) = v \).
Proof. For a proof of (i) we refer to [BCS00, Section 5.4]. Regarding (ii) we compute $J(t) = d_y \exp(x_0, ty)(tw)$ and thus $J(0) = 0$. Furthermore, Lemma 1.26 yields

$$D_c J(0) = D_{\frac{\partial}{\partial s}} H(s, t) \bigg|_{s,t=0} = D_{\frac{\partial}{\partial t}} H(s, t) \bigg|_{s,t=0} = 0$$

Proof.

Definition 1.28. Let $(\mathcal{M}, F)$ be a Finsler manifold and $c : [0, r] \to \mathcal{M}$ be a constant speed geodesic. We say that $p := c(0)$ is conjugate to $q := c(t_0)$ along $c$ if there exists a nonzero Jacobi field $J$ along $c$ such that $J(0) = J(t_0) = 0$.

Lemma 1.29. Let $(\mathcal{M}, F)$ be a Finsler manifold. Fix $x \in \mathcal{M}$, $y \in T_x \mathcal{M}$ and set $c(t) := \exp(x, ty)$ for $t \in [0, r]$. Assume that for some $t_0 \in [0, r]$, $z := c(t_0)$ is the first point conjugate to $c(0) = x$ along $c$. Then $D_y \exp(x, ty)$ is nonsingular for $t \in [0, t_0)$. Moreover, $D_y \exp(x, ty)$ is singular at $t_0$.

Proof. Let $w \in T_z \mathcal{M}$ such that $D_y \exp(x, \tilde{t}y)(w) = 0$ for some $0 \leq \tilde{t} < t_0$ and set $J(t) := D_y \exp(x, \tilde{t}y)(tw)$. We have that $J$ is a Jacobi field along $c$ and $J(0) = 0 = J(t_0)$. Since $x$ and $c(t)$ are not conjugate along $c$ we obtain $J(t) = 0$ for all $t \in [0, \tilde{t}]$. Consequently, $D_c(0) = w$ implies $w = 0$.

Given that $x$ and $z$ are conjugate along $c$ there exists a Jacobi field $Y \neq 0$ such that $Y(0) = 0 = Y(t_0)$. Setting $w := D_y Y(0)$ we obtain $Y(t) = J(t) := D_y \exp(x, \tilde{t}y)(tw)$ by uniqueness. Thus, $J(t_0) = D_y \exp(x, \tilde{t}y)(t_0w)$.

1.4 Metric Aspects of Finsler Manifolds

We already introduced the distance function $d$ on a Finsler manifold $(\mathcal{M}, F)$ in Section 1.1 and emphasised that it is in general unsymmetric. In the present section we will compare the metric topology with the topologies induced by $d^+$ and $d^-$ respectively and observe that they coincide. The blue print for this section is [BCS00, Chapter 6].

Of particular interest are metric spheres at which we have to distinguish between forward metric spheres

$$S^+(x_0, r) := \{ x \in \mathcal{M}; d^+_x(x) = r \}$$

and backward metric spheres

$$S^-(x_0, r) := \{ x \in \mathcal{M}; d^-_x(x) = r \}.$$
(i) Every metric ball is an open set.
(ii) Every open set is a union of forward metric balls.

A similar statement holds for the topology generated by backward metric balls.

**Definition 1.31.** Let \((\mathcal{M}, F)\) be a Finsler manifold. A sequence \(\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{M}\) is called a forward Cauchy sequence provided that for \(\epsilon > 0\) there exists \(N = N(\epsilon) \in \mathbb{N}\) such that \(N \leq i < j\) implies \(d(x_i, x_j) < \epsilon\).

A sequence \(\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{M}\) is called a backward Cauchy sequence provided that for \(\epsilon > 0\) there exists \(N = N(\epsilon) \in \mathbb{N}\) such that \(N \leq i < j\) implies \(d(x_j, x_i) < \epsilon\).

**Lemma 1.32.** Let \((\mathcal{M}, F)\) be a Finsler manifold, \(\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{M}\), and \(x_0 \in \mathcal{M}\). Then the following are equivalent

(i) \(\{x_i\}\) converges to \(x_0\) in the manifold topology.
(ii) \(d_{x_0}^+(x_i) \to 0\) as \(i \to \infty\).
(ii) \(d_{x_0}^-(x_i) \to 0\) as \(i \to \infty\).

**Definition 1.33.** A Finsler manifold \((\mathcal{M}, F)\) is called forward complete with respect to \(d\) provided that every forward Cauchy sequence converges in \(\mathcal{M}\).

A Finsler manifold \((\mathcal{M}, F)\) is called backward complete with respect to \(d\) provided that every backward Cauchy sequence converges in \(\mathcal{M}\).

The following Hopf-Rinow theorem provides several characterisations of the completeness of a Finsler manifold. See [BCS00, Theorem 6.6.1] for a proof.

**Theorem 1.34** (Hopf-Rinow). Let \((\mathcal{M}, F)\) be a connected Finsler manifold. Then the following are equivalent.

(i) \(\mathcal{M}\) is forward complete with respect to \(d\).
(ii) \((\mathcal{M}, F)\) is forward geodesically complete, i.e. every constant speed geodesic \(c : [0, r] \to \mathcal{M}\) can be extended to a geodesic defined on \([0, \infty)\).
(iii) For every \(x \in \mathcal{M}\), \(\exp(x, \cdot)\) is defined on all of \(T_x \mathcal{M}\).
(iv) Every closed and forward bounded, i.e. contained in a forward metric ball with finite radius, subset of \(\mathcal{M}\) is compact.

Furthermore, given that the Finsler manifold is complete, each pair of points in \(\mathcal{M}\) can be joined by a minimising geodesic.

Aside from metric balls and spheres we also have tangent balls \(B(x, r) := \{y \in T_x \mathcal{M} : F(x, y) < r\}\) and tangent spheres \(S(x, r) := \{y \in T_x \mathcal{M} : F(x, y) = r\}\). The relationship between these and metric balls (spheres) is explained by the following proposition whose proof can be found in [BCS00, Theorem 6.3.1].

**Proposition 1.35.** Let \((\mathcal{M}, F)\) be a Finsler manifold and fix \(x_0 \in \mathcal{M}\). Suppose that \(\exp_{x_0} := \exp(x_0, \cdot) : T_{x_0} \mathcal{M} \to \mathcal{M}\) is a \(C^1\)-diffeomorphism from \(B(x_0, r + \epsilon)\) onto its image, where \(r, \epsilon > 0\). Then
Lemma 1.37. Consider a forward geodesically complete Finsler manifold \((\mathcal{M}, F)\), \((x, y) \in T\mathcal{M}\) with \(F(x, y) = 1\), and \(c : [0, r] \to \mathcal{M}\) be a unit speed geodesic with \(c(0) = x\) and \(c(0) = y\). The cut value \(i(x, y)\) of \(y\) at \(x\) is defined by

\[
i(x, y) := \sup\{t; t \in [0, r], d^+_x(c(t)) = t\}.
\]

Given that \(i(x, y) < \infty\) the point \(c(i(x, y))\) is called the cut point of \(x\) along \(c\). The union of all cut points of \(x\) is called cut locus and is given by

\[
\text{Cut}_x := \{c(i(x, y)); y \in T_x\mathcal{M}, F(x, y) = 1, i(x, y) < \infty\}.
\]

The two subsequent Lemmata provide crucial properties of \(i\) which are needed to prove smoothness of the distance function. See [BCS00, Proposition 8.2.1] and [BCS00, Proposition 8.4.1] for proofs.

**Lemma 1.37.** Consider a forward geodesically complete Finsler manifold \((\mathcal{M}, F)\), \((x, y) \in T\mathcal{M}\) with \(F(x, y) = 1\), and a unit speed geodesic \(c : [0, r] \to \mathcal{M}\) with \(c(0) = x\) and \(c(0) = y\). Then

(i) assuming that \(x = c(0)\) and \(c(t_0)\) are conjugate and that no point \(c(t)\) for \(t < t_0\)

is conjugate to \(x\) along \(c\) we have \(i(x, y) \leq t_0\), i.e. the cut point comes either

before or precisely at the first conjugate point on \(c\),

(ii) for any \(t_0 < i(x, y)\) the geodesic \(c\) is the unique minimiser of arc length among

all curves with startpoint \(x\) and endpoint \(c(t_0)\),

(iii) given that \(x = c(0)\) and \(c(t_0)\) are conjugate and that no point \(c(t)\) for \(t < t_0\)

is conjugate to \(x\) along \(c\), the inequality \(i(x, y) < t_0\) implies the existence of a

geodesic \(\hat{c}\) which is not a reparametrisation of \(c\), but has startpoint \(x\), endpoint

\(c(t_0)\) and arc length \(t_0\).

**Lemma 1.38.** Let \((\mathcal{M}, F)\) be a forward geodesically complete Finsler manifold. For

\[I\mathcal{M} := \{(x, y) \in T\mathcal{M}; F(x, y) = 1\}\], the function \(i : I\mathcal{M} \to (0, \infty)\) is continuous.
Given that the Finsler manifold is forward geodesically complete the following lemma provides a characterisation of the cut locus as boundary of the cut domain.

**Lemma 1.39.** Let $(M, F)$ be a forward geodesically complete Finsler manifold. Given $x \in M$ we consider the domain

$$D_x := \{ ty; \ y \in T_x M, \ F(x, y) = 1 \text{ and } t \in [0, i(x, y)) \} \subset T_x M.$$  

Then

(i) the exponential map $\exp(x, \cdot)$ is a diffeomorphism from $D_x$ onto the cut domain $D_x := \exp(x, D_x)$,

(ii) $\partial D_x = \text{Cut}_x$,

(iii) given that $M$ is connected, $M$ is the disjoint union of $D_x$ and $\text{Cut}_x$.

See [BCS00, Proposition 8.5.2] for a proof. As an immediate consequence we obtain differentiability of $d^+_x$ in $D_{x_0}$.

**Corollary 1.40.** Let $(M, F)$ be a forward geodesically complete Finsler manifold and $x_0 \in M$. Then $d^+_x \in C^\infty(D_{x_0} \setminus \{x_0\})$.

**Proof.** Let $x \in D_{x_0} \setminus \{x_0\}$. From (i) in Lemma 1.39 we obtain the existence of $y = y(x) \in D_{x_0} \subset T_{x_0} M$ with $\exp(x_0, y(x)) = x$. The definition of $D_{x_0}$ implies $F(x_0, y(x)) < i(x_0, y(x))$ and hence Lemma 1.37 yields that $c : [0, \infty) \to M$, $c(t) := \exp(x_0, ty(x))$ is the unique minimising geodesic from $x_0$ to $x$, i.e $c(0) = x_0$ and $c(1) = x$. Consequently,

$$d^+_x(x) = \int_0^1 F(c(t), \dot{c}(t)) dt = F(x_0, y(x)).$$

Since Lemma 1.39 implies a $C^\infty$ correspondence between $x$ and $y(x)$ the corollary is proven.

Next, we want to establish conditions under which $d^-$ is differentiable. Before we can give the precise statement we need a preparatory lemma.

**Proposition 1.41.** Let $(M, F)$ be a Finsler manifold. Then the map $\text{EXP} : T M \to M \times M$ defined by $\text{EXP}(x, y) := (x, \exp(x, y))$ has the following properties

(i) $\text{EXP} \in C^\infty(U \setminus 0, M \times M) \cap C^1(U, M \times M)$ where $U$ is the set from Definition 1.21,

(ii) $D \text{EXP}$ is nonsingular at $(x_0, y_0) \in T M$ if and only if $D_y \exp(x, \cdot)|_{y_0}$ is nonsingular,

(iii) in local coordinates, the derivative of $\text{EXP}$ at $(x_0, 0) \in T M$ is of the form

$$\begin{pmatrix} \text{id} & 0 \\ \text{id} & \text{id} \end{pmatrix}$$

where id denotes the identity map in $T_{x_0} M$. 

---
(iv) Suppose that for \((x_0,y_0) \in T\mathcal{M}\) we have that \(D_y\exp(x_0,\cdot)\) is nonsingular at \(y_0\). Then there exists a neighbourhood \(V \subset T\mathcal{M}\) of \((x_0,y_0)\) such that \(\exp\) is a diffeomorphism from \(V\) onto its image. Moreover, in local coordinates we have for \((x,z) \in \exp(V), 1 \leq i,j,k \leq N\) and the inverse mapping \(\exp^{-1} : \exp(V) \to V\)

\[
\frac{\partial}{\partial z^j} (\exp^{-1})^k (x,z) = 0 \\
\frac{\partial}{\partial x^j} (\exp^{-1})^k (x,z) = \delta^k_j \\
\frac{\partial}{\partial z^j} (\exp^{-1})^{N+k} (x,z) = E_{kj}(x,z) \\
\frac{\partial}{\partial x^j} (\exp^{-1})^{N+k} (x,z) = -E_{ki}(x,z) \frac{\partial}{\partial x^j} \exp^i(\exp^{-1}(x,z))
\]

where \((E_{lm}(x,z))_{l,m=1}^N\) denotes the inverse of \((\frac{\partial}{\partial y^j} \exp^i(\exp^{-1}(x,z)))_{i,k=1}^N\).

(v) For \(x_0 \in \mathcal{M}\) there exists a neighbourhood \(V \subset T\mathcal{M}\) of \((x_0,0)\) such that \(\exp\) is a diffeomorphism from \(V\) onto its image. Moreover, in local coordinates \(D(\exp^{-1})\) at \((x_0,x_0)\) is of the form

\[
\begin{pmatrix}
\text{id} & 0 \\
-\text{id} & \text{id}
\end{pmatrix}.
\]

(vi) Given \(x_0 \in \mathcal{M}\) there exists an open neighbourhood \(\mathcal{U} \subset \mathcal{M}\) of \(x_0\) and a constant \(C = C(x_0) > 0\) such that

\[
\left| \frac{\partial}{\partial z^j} (\exp^{-1})^{N+k} (x,z) - \delta^k_j \right| \leq C F(\exp^{-1}(x,z)) \\
\left| \frac{\partial}{\partial x^j} (\exp^{-1})^{N+k} (x,z) + \delta^k_j \right| \leq C F(\exp^{-1}(x,z))
\]

for \(x, z \in \mathcal{U}\).

Proof. Proposition 1.22 yields \(\exp \in C^\infty(U \setminus 0, M \times M) \cap C^1(U, M \times M)\) from which we conclude (i). Next, we observe that in local coordinates

\[
\frac{\partial}{\partial x^j} \exp^i(x,y) = \delta^i_j \quad \text{and} \quad \frac{\partial}{\partial y^j} \exp^{N+i}(x,y) = \frac{\partial}{\partial y^j} \exp^i(x,y)
\]

for \(1 \leq i,j \leq N\). The second assertion follows easily. To prove (iii) we initially observe \(\frac{\partial}{\partial y^j} \exp^i(x_0,0) = 0\). Moreover, we utilise that \(\exp(x_0,0) = x_0\) implies \(\frac{\partial}{\partial x^j} \exp^i(x_0,0) = \delta^i_j\) for \(1 \leq i,j \leq N\). Combining these results with (ii) in Proposition 1.22 we derive (iii).

From (ii) we derive that \(D \exp\) is nonsingular at \((x_0,y_0)\) from which we obtain the first part of assertion (iv). Let \((x,z) \in \exp(V)\). From \((\exp^{-1})^k(x,z) = x^k\) for \(1 \leq k \leq N\) we easily derive the first two derivative formulas. Next, we utilise
exp(\(\EXP^{-1}(x, z)) = z\). By differentiating this identity with respect to \(z\) we obtain for \(1 \leq i, j, k \leq N\) and \((x, z) \in \EXP(V)\)

\[
\frac{\partial}{\partial z^j} (\exp^i(\EXP^{-1}(x, z))) = \frac{\partial}{\partial x^k} \exp^i(\EXP^{-1}(x, z)) \frac{\partial}{\partial z^j} (\EXP^{-1})^k(x, z) + \frac{\partial}{\partial y^k} \exp^i(\EXP^{-1}(x, z)) \frac{\partial}{\partial z^j} (\EXP^{-1})^{N+k}(x, z)
\]

\[
= \frac{\partial}{\partial y^k} \exp^i(\EXP^{-1}(x, z)) \frac{\partial}{\partial z^j} (\EXP^{-1})^{N+k}(x, z)
= \delta^i_j.
\]

Similarly, we compute

\[
\frac{\partial}{\partial x^j} (\exp^i(\EXP^{-1}(x, z))) = \frac{\partial}{\partial x^k} \exp^i(\EXP^{-1}(x, z)) \delta^j_k + \frac{\partial}{\partial y^k} \exp^i(\EXP^{-1}(x, z)) \frac{\partial}{\partial x^j} (\EXP^{-1})^{N+k}(x, z)
\]

\[
= 0
\]

and hence

\[
\frac{\partial}{\partial y^k} \exp^i(\EXP^{-1}(x, z)) \frac{\partial}{\partial x^j} (\EXP^{-1})^{N+k}(x, z) = -\frac{\partial}{\partial x^j} \exp^i(\EXP^{-1}(x, z)).
\]

We recall that we denote the inverse of the matrix \(\left(\frac{\partial}{\partial y^k} \exp^i(\EXP^{-1}(x, z))\right)_{i,k=1}^N\) by \((E_{lm}(x, z))_{l,m=1}^N\). Using this notation, the computations above read as follows

\[
\frac{\partial}{\partial z^j} (\EXP^{-1})^{N+k}(x, z) = E_{kj}(x, z)
\]

\[
\frac{\partial}{\partial x^j} (\EXP^{-1})^{N+k}(x, z) = -E_{ki}(x, z) \frac{\partial}{\partial x^j} \exp^i(\EXP^{-1}(x, z))
\]

and \((iv)\) is proven. Assertion \((v)\) is an immediate consequence of \((ii)\) in Proposition 1.22 and \((iv)\).

It remains to prove \((vi)\). For this purpose we firstly observe \(E(x, x) - E(x, z) = id - E(x, z) = E^{-1}(x, z)E(x, z) - E(x, z) = (E^{-1}(x, z) - id)E(x, z)\) for \(x, z \in \mathcal{U}\) where \(E = (E_{lm})_{l,m=1}^N\) is the matrix defined in \((iv)\). Denoting the operator norm by \(\| \cdot \|\) we have that \(E(x, x) = id\) and continuity of \(E\) imply the existence of a constant \(C > 0\) such that \(\| E(x, z) \| \leq C\) for all \(z \in \mathcal{U}\) provided the neighbourhood \(\mathcal{U}\) is chosen sufficiently small. Therewith, we derive the first part of \((vi)\) from \((iii)\) in Proposition 1.22.

In order to prove the second estimate in \((vi)\) we consider the matrix \(B = (B_{lm})_{l,m=1}^N\) where \(B_{lm}(x, z) := \frac{\partial}{\partial x^m} \exp^i(\EXP(x, z))\) and recall \(B(x, x) = id\). Therewith we compute \(E(x, x)B(x, x) - E(x, z)B(x, z) = id - E(x, z)B(x, z) = (id - E(x, z)) +\)
$E(x, z)(id - B(x, z)) = (E^{-1}(x, z) - id)E(x, z) + E(x, z)(id - B(x, z))$. Using this identity and again (iii) in Proposition 1.22 the proof of the second part of (vi) is similar to the proof of the first part.

Lemma 1.42. Let $(M, F)$ be a forward geodesically complete Finsler manifold and $x_0, z_0 \in M$.

Given that $x_0 \in D_{z_0}$ there exists an open neighbourhood $V \subset M$ of $x_0$ such that for each (fixed) $x \in V$ there exists an open neighbourhood $U_x \subset M$ of $z_0$ such that for $z \in U_x$

$$d_{x}^*(z) = F(z, y(z, x))$$

where in local coordinates $y(z, x) \in T_z M$ is given by $y^i(z, x) = (\text{EXP}^{-1})^{N+i}(z, x)$.

Furthermore, $d_{x}^{-1} \in C^\infty(U_x)$.

Proof. Set $r_0 := d(z_0, x_0)$ and let $c : (0, \infty) \rightarrow M$, $c(t) = \text{exp}(z_0, ty_0)$ where $y_0 \in T_{z_0}M$ with $F(z_0, y_0) = 1$ is chosen such that $c(r_0) = x_0$, i.e. $c$ is minimising the distance from $z_0$ towards $x_0$. Consider the map $\text{EXP} : TM \rightarrow M \times M$ from Proposition 1.41. Lemma 1.29 and (i) in Lemma 1.37 yield that $D_y \text{exp}(z_0, \cdot)$ is nonsingular at $r_0 y_0$ and hence (ii) in Proposition 1.41 yields that $D \text{EXP}$ is nonsingular at $(z_0, r_0 y_0) \in TM$. Consequently, there exists an open neighbourhood $\tilde{U} \subset TM$ of $(z_0, r_0 y_0)$ such that $\text{EXP}$ is a diffeomorphism from $\tilde{U}$ onto its image $\tilde{V} := \text{EXP}(\tilde{U})$.

Since $\text{EXP}(z_0, r_0 y_0) = (z_0, x_0)$ we conclude that there exists an open neighbourhood $V \subset M$ of $x_0$ such that for each $x \in V$ there exists an open neighbourhood $U_x \subset M$ of $z_0$ such that for all $z \in U$ there exists $y(x, z) \in T_z M$ with $x = \text{exp}(z, y(x, z))$. Clearly, $y(z, x)$ is the projection of $\text{EXP}^{-1}(z, x)$ onto its second component. For each $x \in V$ we assure $x \not\in U_x$ by choosing $U_x$ sufficiently small and hence $y(x, \cdot) \in C^\infty(U_x, T_x M)$. Furthermore, $(z, y(z, x)) \rightarrow (z_0, y(z_0, x_0))$ as $(z, x) \rightarrow (z_0, x_0)$.

Lemma 1.38 yields continuity of the function $i$ and thus $i(z, y(z, x)) \geq r_0 + \epsilon/2$ after shrinking the neighbourhoods $U, V$ if necessary. By further shrinking of these neighbourhoods we also have $d(z, x) < r_0 + \epsilon/2$ due to the continuity of the distance function. Hence, (ii) in Lemma 1.37 yields that $c(t) := \text{exp}(z, ty(z, x))$ is the unique minimising geodesic which joins $z$ and $x$. Consequently,

$$d_{x}^*(z) = \int_0^1 F(c(t), \dot{c}(t))dt = F(z, y(z, x))$$

from which we conclude the assertion.  

We intend to conclude this section with a third differentiability result for the distance function which is based on [BCS00, Exercise 6.3.3]. However, we need a preparatory Lemma in advance.

Lemma 1.43. Let $(M, F)$ be a Finsler manifold. Then $F^2 \in C^\infty(TM \setminus 0) \cap C^1(TM)$ and $d(\partial (F^2))_{(x, 0)} = 0$ for $x \in M$. Moreover, given any $x_0 \in M$ and an open neighbourhood $V \subset M$ of $x_0$ there exists a constant $C = C(M, F, x_0)$ such that

$$\left| \frac{\partial}{\partial y^i} (F^2) (x, y) \right| \leq C |y|$$

for $(x, y) \in TM$ with $x \in V$.  

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Moreover, for have

Initially, we recall that \( z \in d \) and consequently, \( \partial \) were recall \( \partial \). Let \( \) Proof. 1 Finsler Geometry

1.35 that \( \exp(\) onto its image. \( V \) open neighbourhood that the derivative of \( \text{EXP} \) is nonsingular at \( \). Given \( 1 \) Finsler Geometry

Next, we observe that for each \( \) given \( x, z \in U_r \) we have \( d(x, z) = F(\text{EXP}^{-1}(x, z)) \),

\[ d^2 \in C^1(U_r \times U_r) \cap C^\infty ((U_r \times U_r) \setminus \{(x, x) ; x \in U_r\}) \]

Moreover, for \( x, z \in U_r \) we have for the minimising geodesic \( c : [0, 1] \to M \) from \( x \) to \( z \) that \( c([0, 1]) \subset B^+(x, 3r) \).

Proof. Initially, we recall (ii) in Proposition 1.22 which states that the derivative of \( \exp(x_0, \cdot) \) is nonsingular at \( 0 \in T_{x_0} M \). Consequently, (ii) in Proposition 1.41 yields that the derivative of \( \text{EXP} \) is nonsingular at \( (x_0, 0) \in TM \) and hence there exists an open neighbourhood \( V \subset TM \) of \( (x_0, 0) \) such that \( \text{EXP} \) is a diffeomorphism from \( V \) onto its image.

Next, we observe that for each \( x \in B^+(x_0, r) \cap B^-(x_0, r) \) we have \( (x, B(x, 2r)) \subset V \) provided that \( r > 0 \) is chosen sufficiently small. Thus, we infer from (iii) in Proposition 1.35 that \( \exp(x, \cdot) \) is a diffeomorphism from \( B(x, 2r) \) onto \( B^+(x, 2r) \). Furthermore, the triangle inequality for \( d \) yields

\[ B^+(x_0, r) \cap B^-(x_0, r) \subset B^+(x, 2r). \quad (1.13) \]

Given \( z \in B^+(x_0, r) \cap B^-(x_0, r) \setminus \{x\} \) we obtain the existence of a unique \( y = y(x, z) \in B(x, 2r) \subset T_x M \) such that \( z = \exp(x, y(x, z)) \) and hence, by (ii) in Proposition 1.35,

\[ d(x, z) = \int_0^1 F(c(t), \dot{c}(t))dt = F(x, y(x, z)) \]

where \( c : [0, 1] \to M \) is the unique geodesic with defined by \( c(t) := \exp(x, ty(x, z)) \). Moreover, since \( (x, z) \in \text{EXP}(V) \) we have that \( y \) is the projection of \( \text{EXP}^{-1}(x, z) \) onto its second component. Hence, in view of Lemma 1.43 the differentiability assertions follow easily.

Finally, given \( x, z \in U_r \), (1.13) combined with (ii) in Proposition 1.35 yields that the minimising geodesic from \( x \) to \( z \) is contained in \( B^+(x, 2r) \). Since \( d(x_0, x) < r \) we have \( B^+(x, 2r) \subset B^+(x_0, 3r) \) and thus the lemma is proven.
Chapter 2

The Distance Function From a Submanifold

The objective of this chapter is to analyse the regularity of the distance function \( d^+_{\tilde{M}} \) from a submanifold \( \tilde{M} \). For that purpose we consider geodesics which start on the submanifold in normal direction and therewith generalise the notion of cut points. However, we need some preparations in advance.

We introduce Finsler submanifolds in the first section as well as the notion of a normal direction. Thereafter we define the normal curvature of a submanifold and prove some estimates on the normal curvature that will be needed in the third section. In that section we introduce the cut locus of a submanifold and discuss the aforementioned regularity of \( d^+_{\tilde{M}} \).

2.1 Submanifolds

Let \((\tilde{M}, F)\) be a Finsler manifold, and \(\tilde{M} = \tilde{M}^n\) where \(n \leq N\) be a \(C^{k,\alpha}\) submanifold, i.e. \(\tilde{M}\) is a manifold and the inclusion \(\varphi \in C^{k,\alpha}(\tilde{M}, M)\) is an embedding. The Finsler structure on \(M\) induces a Finsler structure on \(\tilde{M}\).

**Lemma 2.1.** Let \((\tilde{M}, F)\) be a Finsler manifold, \(\tilde{M}\) a \(C^1\) submanifold and \(\varphi : \tilde{M} \to M\) a differentiable embedding. Consider the map \(\varphi_* : T\tilde{M} \to TM\), \(\varphi_*(\tilde{x},\tilde{y}) := (\varphi(\tilde{x}), d\varphi(\tilde{x})\tilde{y})\). Then \(\tilde{F} := F \circ \varphi_*\) is a Finsler structure on \(\tilde{M}\), i.e. \((\tilde{M}, \tilde{F})\) is a Finsler manifold. The function \(\tilde{F}\) is called the Finsler structure induced by \(F\).

**Proof.** Conditions 1 and 2 of Definition 1.1 are immediate consequences of the definition of \(\tilde{F}\). Furthermore,

\[
\tilde{g}(\tilde{x},\tilde{y})(w_1, w_2) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (\tilde{F}^2(\tilde{x}, \tilde{y} + sw_1 + tw_2)) \bigg|_{s,t=0} = g(\varphi(\tilde{x}), d\varphi(\tilde{x})\tilde{y})(d\varphi(\tilde{x})w_1, d\varphi(\tilde{x})w_2)
\]

is positive definite.

Regarding the previous Lemma, we will denote submanifolds by \((\tilde{M}, \varphi)\) and keep in mind that the pair \((\tilde{M}, F)\) is a Finsler manifold. Although this notation is somewhat
Proof. Initially we compute the first derivative of 

\[ L(\tilde{x}) := \int_0^r F(\xi(\varphi(\tilde{x}), t), \dot{\xi}(\varphi(\tilde{x}), t)) \, dt. \]

Recall that the smoothness of \( L \) is a consequence of the smooth dependence of the geodesics \( \xi \) from the initial data. We set \( F(\xi(\varphi(\tilde{x}), t), \dot{\xi}(\varphi(\tilde{x}), t)) = F(\xi, \dot{\xi}) \) as abbreviation in the following computations.

\[
\frac{\partial}{\partial \tilde{x}^\alpha} L(\tilde{x}) = \int_0^r \frac{\partial}{\partial \tilde{x}^i} F(\xi, \dot{\xi}) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i + \frac{\partial}{\partial y^j} F(\xi, \dot{\xi}) \frac{\partial}{\partial \tilde{x}^\alpha} \dot{\xi}^j \, dt \\
= \int_0^r \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial \tilde{x}^i} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt + \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial \tilde{x}^\alpha} \dot{\xi}^j \, dt \\
= \int_0^r \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial \tilde{x}^i} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt - \frac{d}{dt} \left( \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \dot{\xi}^j \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt \\
+ \left[ \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial \tilde{x}^\alpha} \dot{\xi}^j \right]_0^r \\
= \int_0^r \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial \tilde{x}^i} \left( F^2(\xi, \dot{\xi}) \right) \dot{\xi}^j - \frac{d}{dt} \left( \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \dot{\xi}^j \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt \\
- \int_0^r \frac{d}{dt} \left( \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \dot{\xi}^j - 2g_{ij}(\xi, \dot{\xi}) \dot{\xi}^i \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt \\
- \int_0^r \frac{d}{dt} \left( \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^j} \left( F^2(\xi, \dot{\xi}) \right) \dot{\xi}^j - 2g_{ij}(\xi, \dot{\xi}) \dot{\xi}^i \right) \frac{\partial}{\partial \tilde{x}^\alpha} \xi^i \, dt.
\]
\[ + \left[ \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^i} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial x^\alpha} \xi^i \right]^r \]

Now we compute similar to the proof of Lemma 1.16 and obtain

\[
\frac{\partial}{\partial x^\alpha} L(\hat{x}) = - \int_0^r \frac{1}{F(\xi, \dot{\xi})} g_{ij}(\xi, \dot{\xi}) \left( \dot{\xi}^j + G^j(\xi, \dot{\xi}) \right) \frac{\partial}{\partial x^\alpha} \xi^i \, dt \\
- \int_0^r \frac{d}{dt} \left( \frac{1}{2F(\xi, \dot{\xi})} \right) \frac{\partial}{\partial y^i} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial x^\alpha} \xi^i \, dt \\
+ \left[ \frac{1}{2F(\xi, \dot{\xi})} \frac{\partial}{\partial y^i} \left( F^2(\xi, \dot{\xi}) \right) \frac{\partial}{\partial x^\alpha} \xi^i \right]^r \bigg|_0^r
\]

since \( \xi = \xi(\varphi(\hat{x}), \cdot) \) are geodesics. The geodesic \( \xi(\varphi(\hat{x}), \cdot) \) has constant speed and hence \( \frac{d}{dt} \left( \frac{1}{2F(\xi(\varphi(\hat{x}), 0), \cdot(\varphi(\hat{x}), 0))} \right) = 0 \). Moreover, \( \xi(\varphi(\hat{x}), r) = z_0 \) for \( \hat{x} \in \hat{U} \) implies \( \frac{\partial}{\partial y^i} \xi^i(\varphi(\hat{x}), r) = 0 \). Now, since \( L \) attains its minimum at \( \hat{x} = \hat{x}_0 \) we obtain

\[
0 = \frac{\partial}{\partial x^\alpha} L(\hat{x}_0) = \frac{1}{2F(\xi(\varphi(\hat{x}_0), 0), \xi(\varphi(\hat{x}_0), 0))} \frac{\partial}{\partial y^i} \left( F^2(\xi(\varphi(\hat{x}_0), 0), \xi(\varphi(\hat{x}_0), 0)) \right) \frac{\partial}{\partial x^\alpha} \xi^i(\varphi(\hat{x}_0), 0) \\
= \frac{1}{2F(\xi(\varphi(\hat{x}_0), 0), \xi(\varphi(\hat{x}_0), 0))} g_{(x_0, \xi(\varphi(\hat{x}_0), 0))} \left( \dot{\xi}(\varphi(\hat{x}_0), 0), \frac{\partial}{\partial x^\alpha} \xi^i(\varphi(\hat{x}_0), 0) \right)
\]

where we made use of the formula \( g_{(x,y)}(y,v) = \frac{\partial}{\partial y^i}(F^2(x,y))v^i \).

**Definition 2.3.** Let \((\mathcal{M}, F)\) be a Finsler manifold, \((\hat{\mathcal{M}}, \varphi)\) a \(C^1\) submanifold, \(\hat{x} \in \hat{\mathcal{M}}\) and \(x := \varphi(\hat{x})\). A vector \(n \in T_x\mathcal{M}\) is called a normal vector of \(\hat{\mathcal{M}}\) at \(\hat{x}\) provided that \(g_{(x,n)}(n, y) = 0\) for all \(y \in T_x\mathcal{M}\). By \(T^\perp_x \hat{\mathcal{M}} \subset T_x\mathcal{M}\) we denote the space of all normal vectors of \(\hat{\mathcal{M}}\) at \(\hat{x}\) whilst \(I^\perp_x \hat{\mathcal{M}}\) denotes the set of all unit normal vectors, i.e. \(I^\perp_x \hat{\mathcal{M}} := \{ v \in T^\perp_x \hat{\mathcal{M}} ; F(x, v) = 1 \}\). Finally, we define the unit normal bundle by

\[
I^\perp \mathcal{M} := \bigcup_{\hat{x} \in \hat{\mathcal{M}}} I^\perp_{\varphi(\hat{x})} \hat{\mathcal{M}} \subset T\mathcal{M}.
\]

See [She01, Section 2.3] or [Run59, Section 5.2] for more on the notion of a normal vector.

**Remark 2.4.** In the special case of a \(N - 1\) dimensional \(C^1\) submanifold \((\hat{\mathcal{M}}, \varphi)\) we consider \(\hat{x}_0 \in \hat{\mathcal{M}}\) and set \(x_0 = \varphi(\hat{x}_0)\). One can show that there exist precisely two unit normal vectors \(n_1, n_2 \in T_{x_0}\mathcal{M}\) of \(\hat{\mathcal{M}}\). Since the tangent space \(T_{x_0}\hat{\mathcal{M}}\) is a hyperplane in \(T_{x_0}\mathcal{M}\) this statement follows from the considerations in [She01, Section 2.3].

Given that the distance function is differentiable we may easily compute a normal vector of a metric sphere.
Corollary 2.5. Let \((\mathcal{M}, F)\) be a Finsler manifold, \(x_0 \in \mathcal{M}\), and \(r > 0\). Consider the backward metric sphere \(S^-(x_0, r)\) and assume that for \(x_1 \in S^-(x_0, r)\) there exists an open neighbourhood \(U(x_1) \subset \mathcal{M}\) such that \(d^r_{x_0} \in C^1(U)\). Then the vector \(n := \text{grad} (-d_{x_0}^-)\) at \(x_1\) is normal to \(S^-(x_0, r)\) at \(x_1\). Moreover, \(n\) is a unit vector, i.e. \(F(x_1, n) = 1\).

Proof. Since \(d_{x_0}^-\) is constant on \(S^-(x_0, r)\) we conclude \(d(d_{x_0}^-)|_{x_1}(w) = 0\) for \(w \in T_{x_1}S^-(x_0, r)\). Now, the first part of the assertion follows from (1.4) while the second part of the assertion is a consequence of Lemma 1.5. \(\square\)

Lemma 2.6. Let \((\mathcal{M}, F)\) be a Finsler manifold and \(x_0 \in \mathcal{M}\). For \(r > 0\) we consider the backward metric sphere \(S^-(x_0, r)\). Suppose that \(x_1 \in S^-(x_0, r)\) and additionally \(x_0 \in \mathcal{D}_{x_1}\). Then there exists an open neighbourhood \(\mathcal{U} \subset S^-(x_0, r)\) of \(x_1\) such that for \(x \in \mathcal{U}\)

\[
\exp(x, r \ \text{grad} \rho|_x) = x_0
\]

where \(\rho := -d_{x_0}^-\).

Proof. Like in the proof of Lemma 1.42 \(x_0 \in \mathcal{D}_{x_1}\) yields the existence of an open neighbourhood \(\mathcal{U}\) of \(x_1\) such that for each \(x \in \mathcal{U}\) there exists a unique \(y = y(x) \in T_x\mathcal{M}\) such that

\[
\exp(x, y(x)) = x_0.
\]

Consequently, \(c : [0, 1] \rightarrow \mathcal{M}\) defined by \(c(t) := \exp(x, t \ y(x))\) is the unique minimising geodesic with \(c(0) = x\) and \(c(1) = x_0\). We have

\[
r = \int_0^1 F(c(t), \dot{c}(t))dt = F(x, y(x))
\]

and hence \(c_1(t) := c(\frac{t}{r})\) is a unit speed geodesic with \(c_1(0) = x\) and \(c_1(r) = x_0\). Moreover, \(c_1\) satisfies \(\rho(c_1(t)) = t - r\) and by differentiating this identity with respect to \(t\) we obtain

\[
d \rho(\dot{c}_1(t)) = g_{c_1(t)}(\text{grad} \rho|_{c_1(t)} \dot{c}_1(t), \dot{c}_1(t)) = 1
\]

which is, by virtue of Lemma 1.5, equivalent to

\[
g_{c_1(t)}(\text{grad} \rho|_{c_1(t)} \dot{c}_1(t), \dot{c}_1(t)) = 1 = F(c_1(t), \text{grad} \rho|_{c_1(t)} F(c_1(t), \dot{c}_1(t))
\]

Thus,

\[
\dot{c}_1(t) = \text{grad} \rho|_{c_1(t)}
\]

by application of Lemma 1.4 and hence we may replace \(y(x)\) in (2.1) by \(\text{grad} \rho|_x\). \(\square\)

For later reference we show in the subsequent lemma how to compute a normal vector field from a given vector field which is normal in its starting point only. The proof of the following lemma is motivated by the considerations in [Run59, Section 5.2]. However, since the notation in this section is somewhat misleading it should be read with care.
Lemma 2.7. Let \((\mathcal{M}, F)\) be a Finsler manifold and \((\tilde{\mathcal{M}}, \varphi)\) a \(C^1\) submanifold. Given \(\tilde{x}_0 \in \tilde{\mathcal{M}}\) we set \(x_0 := \varphi(\tilde{x}_0)\) and consider a smooth curve \(\tilde{c} : [0, r] \to \tilde{\mathcal{M}}\) starting at \(\tilde{x}_0\) and set \(c := \varphi \circ \tilde{c}\). For any vector field \(V\) along \(c\) with \(g(x_0, V(0))(V(0), y) = 0\) for all \(y \in T_{x_0} \tilde{\mathcal{M}}\) there exists \(t_0 > 0\) and a vector field \(V^\perp\) along \(c\) satisfying \(V^\perp(0) = V(0)\) as well as
\[
g(c(t), V^\perp(t))(V^\perp(t), y) = 0
\]
for all \(y \in T_{c(t)} \tilde{\mathcal{M}}\) and \(0 \leq t < t_0\).

Proof. Let \((E_1, \ldots, E_N)\) be a \(g(c,V)\) orthonormal frame along \(c\) with \(E_1 = \dot{c}\). Moreover, we assume that \((E_1(t), \ldots, E_n(t))\) forms a basis of \(T_{c(t)} \tilde{\mathcal{M}}\). We introduce components of a covariant vector field \(W\) along \(c\) by
\[
W_i(t) := g_{ij}(c(t), V(t)) V^j(t) - \sum_{\alpha=1}^n g_{ij}(c(t), V(t)) g_{ij}(c(t), V(t)) E^j_\alpha(t) \quad (2.2)
\]
for \(0 \leq t < r\) and intend to introduce the vector field \(V^\perp\) as the unique solution of
\[
g_{ij}(c(t), V^\perp(t))(V^\perp(t))^j(t) = W_i(t). \quad (2.3)
\]
In order to assure that this equation is uniquely solvable we initially observe \(W_i(0) = g_{ij}(c(0), V(0)) V^j(0)\) and recall \(g_{ij}(x, y) y^j = \frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (x, y)\) for \((x, y) \in T_x \mathcal{M}\). Furthermore, we observe that the derivative with respect to \(y\) of \(\frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (x, y)\) is non-singular at every \((x, y) \in T \mathcal{M} \setminus 0\) and hence the implicit function theorem is applicable and yields the existence of \(t_0 > 0\) such that (2.3) is uniquely solvable for \(0 \leq t < t_0\). From (2.3), we easily derive that \(V^\perp\) is normal to \(\tilde{\mathcal{M}}\) and \(V^\perp(0) = V(0)\). \(\square\)

### 2.2 Normal Curvature of Submanifolds

In this section we introduce the notion of normal curvature for Finsler submanifolds and prove estimates for the normal curvature in some special geometric situations. Although some results are taken from [She01, Chapter 14] we repeat and sometimes rephrase the proofs here for the sake of completeness. In addition to the notation introduced at the beginning of this section we denote by \(\tilde{\mathcal{G}}^i\) the induced geodesic coefficients. More precisely, \(\tilde{\mathcal{G}}^i(x, y) = y^j y^k \left( \tilde{\gamma}^i_{jk} \right)_{(x,y)}\) where \(\tilde{\gamma}^i_{jk}\) are the formal Christoffel symbols of the induced Finsler structure.

Definition 2.8. Let \((\mathcal{M}, F)\) be a Finsler manifold and \((\tilde{\mathcal{M}}, \varphi)\) be a \(C^2\) submanifold. For \(y \in T_{\tilde{x}} \tilde{\mathcal{M}}\), choose a geodesic \(\tilde{c} : (-\epsilon, \epsilon) \to \tilde{\mathcal{M}}\) such that \(c := \varphi \circ \tilde{c}\) satisfies \(c(0) = x\), and \(\dot{c}(0) = y\). Then
\[
A^\tilde{\mathcal{M}}_x(y) = A_x(y) := -D_c \dot{c}(0)
\]
is called the normal curvature of \(\tilde{\mathcal{M}}\) in the direction \(y \in T_{\tilde{x}} \tilde{\mathcal{M}}\). Given any normal vector \(n\) of \(\mathcal{M}\), the normal curvature in the direction \(n\) at \(\tilde{x} \in \tilde{\mathcal{M}}\) is defined by
\[
\Lambda^\tilde{\mathcal{M}}_{(x,n)} = \Lambda_{(x,n)}(y) := g(x,n)(n, A_x(y))
\]
for \(y \in T_x \tilde{\mathcal{M}}\).
We proceed with the representation of $A$ in local coordinates.

**Lemma 2.9.** Let $(M, F)$ be a Finsler manifold and $(\mathcal{M}, \varphi)$ be a $C^2$ submanifold. For $y \in T_x\tilde{M}$, choose a geodesic $\tilde{c} : (-\epsilon, \epsilon) \to \tilde{M}$ such that $c := \varphi \circ \tilde{c}$ satisfies $c(0) = x$, and $\tilde{c}(0) = y$. Then, in local coordinates, the normal curvature $A_x(y)$ is given by

$$A_x(y) = -\left(\frac{\partial^2 \varphi^i}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}(\tilde{x})\tilde{c}^\alpha(0)\tilde{c}^\beta(0) - \frac{\partial \varphi^i}{\partial \tilde{x}^\alpha}(0)\tilde{G}^\alpha(\tilde{c}(0), \tilde{c}(0)) + G^i(c(0), \tilde{c}(0))\right)\frac{\partial}{\partial x^i}|_x$$

**Proof.** Let $\tilde{x}$ be a system of local coordinates in $\tilde{M}$ and $x = \varphi(\tilde{x})$. We compute

$$D_c\tilde{c}(0) = \left(\frac{d^2 \varphi^i}{dt^2}(0) + G^i(c(0), \tilde{c}(0))\right)\frac{\partial}{\partial x^i}|_x$$

$$= \left(\frac{\partial^2 \varphi^i}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}(\tilde{x})\tilde{c}^\alpha(0)\tilde{c}^\beta(0) + \frac{\partial \varphi^i}{\partial \tilde{x}^\alpha}(0)\tilde{c}^\alpha(0) + G^i(c(0), \tilde{c}(0))\right)\frac{\partial}{\partial x^i}|_x$$

$$= \left(\frac{\partial^2 \varphi^i}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}(\tilde{x})\tilde{c}^\alpha(0)\tilde{c}^\beta(0) - \frac{\partial \varphi^i}{\partial \tilde{x}^\alpha}(0)\tilde{G}^\alpha(\tilde{c}(0), \tilde{c}(0)) + G^i(c(0), \tilde{c}(0))\right)\frac{\partial}{\partial x^i}|_x.$$

\[\square\]

In the case of level hypersurfaces such as metric spheres we may compute the normal curvature with the help of the distance function. Before we can give the precise statement we have to introduce the Hessian of a function $f : M \to \mathbb{R}$.

**Definition 2.10.** Let $(\mathcal{M}, F)$ be a Finsler manifold and $f \in C^2(\mathcal{M})$. For $y \in T_x\mathcal{M}$ let $c : (-\epsilon, \epsilon) \to \mathcal{M}$ be a geodesic such that $c(0) = x$, and $\tilde{c}(0) = y$. The Hessian of $f$ is defined by $D^2 f : TM \to \mathbb{R}$,

$$D^2 f(y) := \frac{d^2}{dt^2}(f \circ c)|_{t=0}.$$

It is easy to check that, in local coordinates, the Hessian is given by

$$D^2 f(y) = \left. \frac{\partial f(x)}{\partial x^i} y^i y^j \frac{\partial^2 f(x)}{\partial x^i \partial x^j} G^i(x, y) \right|_{x=y}.$$  \hspace{1cm} (2.4)

**Lemma 2.11.** Let $(\mathcal{M}, F)$ be a Finsler manifold and $\mathcal{U} \subset \mathcal{M}$ be an open subset. Let $\rho \in C^2(\mathcal{U})$ satisfy $F(x, \rho|_x) = 1$ for $x \in \mathcal{U}$ and set $\tilde{\mathcal{M}} := \rho^{-1}(r) \cap \mathcal{U}$ where $r > 0$. The normal curvature of $\mathcal{M}$ at $x \in \tilde{\mathcal{M}}$ in direction of $n = \text{grad} \rho|_x$ where satisfies

$$A_{(x,n)}(y) = D^2 \rho(y)$$

for $y \in T_x\tilde{\mathcal{M}}$.

**Proof.** Let $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ be an embedding. For $y \in T_x\tilde{\mathcal{M}}$ choose a geodesic $\tilde{c} : (-\epsilon, \epsilon) \to \tilde{\mathcal{M}}$ such that $c := \varphi \circ \tilde{c}$ satisfies $c(0) = x$ and $\tilde{c}(0) = y$. We observe $\rho(c(t)) = r$ for $t \in (-\epsilon, \epsilon)$. Two times differentiation of this identity and evaluation at $t = 0$ yields

$$\frac{\partial^2 \rho(x)}{\partial x^i \partial x^j} y^i y^j + \frac{\partial \rho(x)}{\partial x^i} \tilde{c}^i(0) = 0.$$
Furthermore,
\[ \ddot{c}(0) = \frac{\partial \varphi(x)}{\partial x^i} \dot{\xi}(0) \dot{\xi}(0) + \frac{\partial \varphi(x)}{\partial x^j} \ddot{\xi}(0) \]
\[ = \frac{\partial \varphi(x)}{\partial x^i} \dot{\xi}(0) \dot{\xi}(0) - \frac{\partial \varphi(x)}{\partial x^j} \ddot{\xi}(0). \]

Combining equation (2.4) with the two equations above we finally arrive at
\[ D^2 \rho(y) = d \rho(A_x(y)) = g_{(x,n)}(n, A_x(y)) = \Lambda_{(x,n)}(y). \]

Lemma 2.12. Let \( (\mathcal{M}, F) \) be a Finsler manifold and \( (\tilde{\mathcal{M}}, \varphi) \) be a \( C^2 \) submanifold. Suppose that for some \( x_0 \in \mathcal{M}, r > 0 \) we have \( d_{(x_0)}^2(\varphi(\tilde{x})) \geq r \) for \( \tilde{x} \in \tilde{\mathcal{M}} \) where equality holds for some point \( \tilde{z} \in \tilde{\mathcal{M}} \). We assume \( d_{x_0}^2 \in C^{2,1}(U) \) where \( U \subset \mathcal{M} \) is an open neighbourhood of \( z := \varphi(\tilde{z}) \) and set \( \rho := -d_{x_0}^2 \). Then
\[ \Lambda_{(z,n)}(y) \geq \Lambda_{(x,n)}^S(z) \cup U(y), \]
for \( y \in T_z \tilde{\mathcal{M}} \) where \( n := \text{grad } \rho|_z \).

Proof. Let \( \psi : S^-(x_0, r) \to \mathcal{M} \) be the natural embedding. For \( y \in T_z \tilde{\mathcal{M}} \) choose geodesics \( \tilde{c}_{\tilde{M}} : (-\epsilon, \epsilon) \to \tilde{\mathcal{M}}, \tilde{c}_{S^-} : (-\epsilon, \epsilon) \to S^-(x_0, r) \) such that \( \tilde{c}_{\tilde{M}} := \varphi \circ \tilde{c}_{\tilde{M}} \) satisfies \( \tilde{c}_{\tilde{M}}(0) = z \), and \( \tilde{c}_{S^-}(0) = y \) and \( c_{S^-} := \psi \circ \tilde{c}_{S^-} \) satisfies \( c_{S^-}(0) = z \), and \( \tilde{c}_{S^-}(0) = y \). From the assumptions we deduce
\[ \rho \circ c_{\tilde{M}}(t) < \rho \circ c_{S^-}(t) \quad \text{for } t \neq 0 \quad (2.5) \]
and \( \rho \circ c_{\tilde{M}}(0) = \rho \circ c_{S^-}(0) \). We observe
\[ \frac{d}{dt} (\rho \circ c_{\tilde{M}}(t))|_{t=0} = \frac{d}{dt} (\rho \circ c_{S^-}(t))|_{t=0} \]
and compute
\[ \frac{d^2}{dt^2} (\rho \circ c_{\tilde{M}}(t))|_{t=0} = \frac{\partial^2 \rho(z)}{\partial x^i \partial x^j} y^i y^j + \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{\tilde{M}}(0)^i \]
\[ \frac{d^2}{dt^2} (\rho \circ c_{S^-}(t))|_{t=0} = \frac{\partial^2 \rho(z)}{\partial x^i \partial x^j} y^i y^j + \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{S^-}(0)^i. \]

Using Taylor expansion and inequality (2.5) we get
\[ \frac{1}{2} \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{\tilde{M}}(0)^i t^2 + O(t^3) \leq \frac{1}{2} \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{S^-}(0)^i t^2 + O(t^3) \]
which yields \( \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{S^-}(0)^i \leq \frac{\partial \rho(z)}{\partial x^i} \dot{\xi}_{\tilde{M}}(0)^i \). From this inequality we derive
\[ \frac{\partial}{\partial x^i} \rho(z) \left( \frac{\partial \varphi^i(\tilde{c}_{\tilde{M}}(0))}{\partial x^k \partial x^l} \dot{\xi}_{\tilde{M}}(0)^k \dot{\xi}_{\tilde{M}}(0)^l - \frac{\partial \varphi^i(\tilde{c}_{\tilde{M}}(0))}{\partial x^k} G^k_{\tilde{M}}(\tilde{c}_{\tilde{M}}(0), \dot{c}_{\tilde{M}}(0)) \right) \]
\[ \leq \frac{\partial}{\partial x^i} \rho(z) \left( \frac{\partial \varphi^i(\tilde{c}_{S^-}(0))}{\partial x^k \partial x^l} \dot{\xi}_{S^-}(0)^k \dot{\xi}_{S^-}(0)^l - \frac{\partial \varphi^i(\tilde{c}_{S^-}(0))}{\partial x^k} G^k_{S^-}(\tilde{c}_{S^-}(0), \dot{c}_{S^-}(0)) \right) \]
where \( G_{S^-} \) denotes the geodesic coefficients of the submanifold \( S^-(x_0, r) \). Finally, we add \( d(\rho)|_z G(z, y) \) on both sides of the last inequality to conclude the proof. \( \square \)
In the remainder of this section we prove results on the normal curvature of backward metric spheres. We start with a Taylor expansion for the normal curvature of backward metric spheres. See [She01, Proposition 14.4.4] for the corresponding result for forward metric spheres. For this purpose we recall the notation from Remark 1.3 and Lemma 1.18. Additionally, in what follows we will indicate with an overbar when a certain geometric quantity depends on the Finsler structure $\bar{F}$.

**Proposition 2.13.** Let $(\mathcal{M}, F)$ be a forward geodesically complete Finsler manifold and consider the backward metric sphere $S^-(x_0, r)$ where $x_0 \in \mathcal{M}$, and $r > 0$. Let $x \in S^-(x_0, r)$ and assume additionally that $x_0 \in \mathcal{D}_x := \exp(x, D_x)$ as defined in Lemma 1.39. Suppose that $c : [0, r] \to \mathcal{M}$ is the unit speed geodesic with $c(0) = x$ and $c(r) = x_0$ and define $\bar{c}(t) := c(r - t)$. We set $\rho := -d_{x_0}$.

Then there exists an open neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $x$ such that $\rho \in C^\infty(\mathcal{U})$ and we have the following expansion

$$\Lambda_{(x, n)}(v) = -\frac{1}{r} \tilde{g}(\bar{c}(0), \dot{\bar{c}}(0)) (\bar{v}(0), \ddot{v}(0)) + \tilde{g}(\bar{c}(0), \dot{\bar{c}}(0)) (\dot{T}_{\bar{c}}(0) (-\ddot{v}(0)), \ddot{v}(0))$$

$$+ \left( \tilde{g}(\bar{c}(0), \dot{\bar{c}}(0)) \left( \dot{\bar{T}}_{\bar{c}}(0) (-\ddot{v}(0), \dot{v}(0)) \right) - \frac{1}{3} \tilde{g}(\bar{c}(0), \dot{\bar{c}}(0)) (R(\ddot{v}(0), \dot{\bar{c}}(0)) \dot{\bar{c}}(0), \ddot{v}(0)) \right) r + O(r^2)$$

where $n := \text{grad } \rho|_x$, $v \in T_x S^-(x_0, r)$ and $\ddot{v}(0) \in T_{x_0} \mathcal{M}$ is chosen such that for $\ddot{v}(t) := \bar{P}_{\bar{c}|t}(\ddot{v}(0))$ we have $\ddot{v}(r) = v$. Finally, the vector field $\bar{T}_{\bar{c}}(r)(y) = \bar{T}_{\bar{c}}(r)(y) \frac{\partial}{\partial x^i}$ is given by $\bar{T}_{\bar{c}}(r)(y) = \dot{\bar{c}}(r) y^k \left( (\Gamma^i_{jk})(y^2) - (\bar{\Gamma}^i_{jk})(x, \dot{\bar{c}}(r)) \right)$.

**Proof.** For $y \in T_x S^-(x_0, r)$ we consider a geodesic $\xi : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\xi(0) = x$, $\xi'(0) = -y$ and compute by virtue of Lemma 2.11

$$\Lambda_{(x, n)}(-y) = D^2 \rho(-y) = \frac{d^2}{ds^2} \rho(\xi(s)) \bigg|_{s=0} = \frac{d}{ds} \left( d \rho(\xi(s)) \dot{\xi}(s) \right) \bigg|_{s=0}$$

$$= \frac{d}{ds} \left( g(\xi(s), \text{grad } \rho(\xi(s)) \left( \text{grad } \rho(\xi(s)) \cdot \dot{\xi}(s) \right) \right) \bigg|_{s=0}$$

$$= g(x, n) \left( \dot{\xi} \left( \text{grad } \rho(\xi(s)) \right) \right) \bigg|_{s=0} = -y$$

(2.6)

where we made use of Lemma 1.19 to obtain the last equation. We set $\dot{\xi} W(t) := \left( \frac{d}{dt} W^i(t) + W^j(t) \dot{\xi}^k(t) (\Gamma^i_{jk})(\xi(t), \text{grad } \rho(\xi(t))) \right) \frac{\partial}{\partial x^i} \bigg|_{\xi(t)}$ for a vector field $W = W^i \frac{\partial}{\partial x^i}$ along $\xi$ and decompose equation (2.6) in the following way

$$\Lambda_{(x, n)}(-y) = g(x, n) \left( \dot{\xi} \left( \text{grad } \rho(\xi(s)) \right) \right) \bigg|_{s=0} = -y$$

$$+ g(x, n) \left( \ddot{\xi} \left( \text{grad } \rho(\xi(s)) \right) \right) \bigg|_{s=0} - \dot{\xi} \left( \text{grad } \rho(\xi(s)) \right) \bigg|_{s=0} = -y$$

$$= g(x, n) (\bar{S}(-y), -y) + \bar{T}_{\text{grad } \rho(x)}(y).$$

(2.7)

Here we may regard the previous equations as definitions for the operators $\bar{S}$, and $\bar{T}$ respectively and refer to [She01] for precise definitions and a detailed exposition. For the sake of completeness we remark that $\bar{S}$ is the shape operator and $\bar{T}$ is called
2.2 Normal Curvature of Submanifolds

\textit{T-curvature} or \textit{tangent curvature} in [She01]. Once more we recall that the overbar indicates that these quantities are related to the Finsler structure $\bar{F}$.

In order to analyse the $r$-dependence of $\bar{S}$ and $\bar{T}$ we consider the geodesic $c : [0, r] \to \mathcal{M}$ from the assumptions. From Lemma 1.18 we infer that the backward curve $\bar{c} : [0, r] \to \mathcal{M}, \bar{c}(t) := c(r - t)$ is a geodesic with respect to the Finsler structure $\bar{F}$. Moreover, we have $\rho(\bar{c}(t)) = -t$ and by differentiating this identity with respect to $t$ we obtain

$$-1 = d_\rho(\bar{c}(t)) = g(\bar{c}(t), \text{grad } \rho_{\bar{c}(t)})(\text{grad } \rho_{\bar{c}(t)}, \bar{c}(t))$$

which is equivalent to

$$1 = g(\bar{c}(t), \text{grad } \rho_{\bar{c}(t)})(\text{grad } \rho_{\bar{c}(t)}, -\bar{c}(t)) = \bar{F}(\bar{c}(t), \text{grad } \rho_{\bar{c}(t)}) \bar{F}(\bar{c}(t), -\bar{c}(t)).$$

Thus,

$$-\bar{c}(r - t) = -\bar{c}(t) = \text{grad } \rho_{\bar{c}(t)}$$

by virtue of Lemma 1.4 and in particular, $n = -\bar{c}(r)$. Therewith, we obtain

$$\bar{T}(\text{grad } \rho_x)(y) = g(x, n)((\text{grad } \rho_x)^j \xi^j(0) (\Gamma^i_{jk})(\xi(0), \xi(0)) - (\Gamma^i_{jk})(\xi(0), \text{grad } \rho(\xi(0))) \frac{\partial}{\partial x^i} \bigg|_{\xi(0)}, -y)$$

which enables us to give a Taylor expansion for $\bar{T}$ at $y = -\tilde{v}(r)$

$$\bar{T}(\text{grad } \rho_x)(-\tilde{v}(r)) = \tilde{g}(\tilde{c}(r), \tilde{\tilde{c}}(r))(\bar{T}(\tilde{c}(r), \tilde{c}(r)), \tilde{v}(r))$$

where in the last equation we made use of Lemma 1.19 and the fact that $\tilde{v}$ is parallel along $\tilde{c}$.

Regarding the analysis of the $r$-dependence of $\bar{S}$ we fix $t_0 \in [0, r)$ and consider a geodesic $\eta : (-\epsilon, \epsilon) \to \mathcal{S}^-(x_0, r - t_0)$ with $\eta(0) = c(t_0)$ and $\eta(0) = w$ for some $w \in T_{c(t_0)}\mathcal{S}^-(x_0, r - t_0)$. Let $H : (-\epsilon, \epsilon) \times [t_0, r] \to \mathcal{M}$ be defined by

$$H(s, t) := \exp(\eta(s), (t - t_0) \text{ grad } \rho_{\eta(t)}).$$

Then $H(0, t) = \exp(c(t_0), (t - t_0) \tilde{c}(t_0)) = c(t)$ for $t \in [t_0, r]$ and furthermore for each $s \in (-\epsilon, \epsilon)$ we have that $H(s, \cdot)$ is a geodesic on $\mathcal{M}$. Thus, $H$ is a geodesic variation of $c$. Moreover, $H$ satisfies $H(s, t_0) = \eta(s)$ and since $x_0 \in \mathcal{D}_\varepsilon$ Lemma 2.6 yields

$$H(s, r) = x_0.$$
Consequently, \( d x_\ell (H(s, t)) = r - t \) for \( s \in (-\epsilon, \epsilon) \). One immediately observes that 
\[ \bar{H} : (-\epsilon, \epsilon) \times [0, r - t_0] \to \mathcal{M} \] 
defined by \( \bar{H}(s, t) := H(s, r - t) \) is a geodesic variation of \( \bar{c} \) with \( \bar{H}(s, r - t_0) = \eta(s), \bar{H}(s, 0) = x_0 \) and \( \rho(\bar{H}(s, t)) = -t \) for \( s \in (-\epsilon, \epsilon) \).
Consequently, we obtain similarly to the proof of (2.8)
\[
\text{grad } \rho_{\bar{H}(s, r-t_0)} = -\frac{\partial}{\partial t} \bar{H}(s, r-t_0). 
\tag{2.10}
\]
Furthermore, the corresponding Jacobi field \( J := \frac{\partial}{\partial s} \bar{H}(s, \cdot) \big|_{s=0} \) satisfies, since \( \bar{H}(s, 0) \) is constant,
\[
\bar{J}(0) = \frac{\partial}{\partial s} \bar{H}(s, 0) \bigg|_{s=0} = 0
\]
and \( \bar{J}(r-t_0) = \dot{\eta}(0) \). We compute by virtue of (2.10)
\[
\bar{D}_x \bar{J}(r-t_0) = \left( \dot{\bar{J}}(r-t_0) + \bar{J}(r-t_0) \dot{\bar{c}}^k(r-t_0)(\Gamma^i_{jk})_{\bar{c}(r-t_0)} \right) \frac{\partial}{\partial x^i} 
\]
\[= \left( \frac{\partial}{\partial t} \left( \frac{\partial}{\partial s} \bar{H}(s, t) \right) \bigg|_{s=0} + \dot{\eta}(0) \frac{\partial}{\partial t} \bar{H}^k(0, r-t_0)(\Gamma^i_{jk})(\eta(0), -\dot{\bar{c}}(r-t_0)) \right) \frac{\partial}{\partial x^i} 
\]
\[= \left( -\frac{\partial}{\partial s} \left( \text{grad } \rho_{\bar{H}(s, r-t_0)} \right) \bigg|_{s=0} - \dot{\eta}(0) \left( \text{grad } \rho_{\bar{H}(0, r-t_0)} \right)^k (\Gamma^i_{jk})(\eta(0), \text{grad } \rho_{\eta(0)}) \right) \frac{\partial}{\partial x^i} 
\]
\[= -\bar{D}_\eta \left( \text{grad } \rho_{\eta(0)} \right) \bigg|_{s=0} = -\bar{S}(\bar{J}(r-t_0)). \tag{2.11} \]

Since in the previous computations \( t_0 \in [0, r) \) is arbitrary we have
\[
\bar{D}_x \bar{J}(t) = -\bar{S}(\bar{J}(t)) \tag{2.12} \]
for \( t = r - t_0 \in [0, r) \). We utilise this equation to derive an expansion for the shape operator of \( S^- (x_0, r - t) \) from a Taylor expansion for \( \bar{J} \). For this purpose we consider a \( \tilde{g}_{\bar{c}, \bar{J}} \) orthonormal frame \((\bar{E}_1, \ldots, \bar{E}_N)\) along \( \bar{c} \) spanning \( T_{\bar{c}(t)} \mathcal{M} \) where each vector field \( \bar{E}_i \) is assumed to be parallel. We set \( \bar{J}(t) = \bar{J}^i(t) \bar{E}_i(t) \) and get by linearity of the shape operator and (2.12)
\[
\bar{S}(\bar{J}(t)) = \bar{J}^i(t) \bar{S}(\bar{E}_i(t)) = \bar{J}^i(t) \bar{S}^j_i(t) \bar{E}_j(t) = -\bar{D}_x \bar{J}(t) = -\dot{\bar{J}}^i(t) \bar{E}_i(t) 
\]
and hence
\[
\bar{S}^j_i(t) \bar{J}^i(t) = -\dot{\bar{J}}^j(t). 
\]

By application of Lemma 1.24 we derive
\[
\bar{S}^j_i(t) \left( \dot{\bar{J}}^j(0)t - \frac{1}{6} \bar{R}^j_i(0) \dot{\bar{J}}^k(0)t^3 + O(t^4) \right) = -\left( \dot{\bar{J}}^j(0) - \frac{1}{2} \bar{R}_k^j(0) \dot{\bar{J}}^k(0)t^2 + O(t^3) \right)
\]
which implies
\[ S^i_j(t) = -\frac{1}{t} \delta^i_j - \frac{1}{3} \bar{R}^i_j(0) t + O(t^2) \] (2.13)
for \( t \in (0, r] \). Now we choose \( t = r \) and obtain for \( v = \bar{v}(r) =: \bar{v}^i(r) \bar{E}_i(r) \)
\[ S(v) = S(\bar{v}(r)) = \bar{v}^i(r) S^i_j(r) \bar{E}_j(r) \]
\[ = -\frac{1}{r} \bar{v}(r) - \frac{1}{3} \left( \bar{v}^i(r) \bar{R}^i_j(0) \bar{E}_j(r) \right) r + O(r^2). \] (2.14)

We insert (2.9) and (2.14) into (2.7) and obtain
\[
\Lambda_{(x,n)}(v) = \Lambda_{(x,n)}(\bar{v}(r)) = g_{(x,n)} \left( S(\bar{v}(r)), \bar{v}(r) \right) + \bar{T}_{\left( \text{grad } \rho \right)} \left( -\bar{v}(r) \right)
\]
\[
= \bar{g}(\bar{v}(r), \bar{v}(r)) \left( -\frac{1}{r} \bar{v}(r) - \frac{1}{3} \left( \bar{v}^i(r) \bar{R}^i_j(0) \bar{E}_j(r) \right) \right) r + O(r^2)
\]
\[ = -\frac{1}{r} \bar{g}(\bar{v}(r), \bar{v}(r)) (\bar{v}(r), \bar{v}(r)) - \frac{1}{3} \bar{r} \bar{g}(\bar{v}(r), \bar{v}(r)) \left( \left( \bar{v}^i(r) \bar{R}^i_j(0) \bar{E}_j(r) \right), \bar{v}(r) \right)
\]
\[ + \bar{g}(\bar{v}(r), \bar{v}(r)) \left( \bar{T}_{\bar{v}(r)}(\bar{v}(r), \bar{v}(r)) \right) r + O(r^2)
\]

Since \( \bar{v}, \bar{E} \) are parallel along \( \bar{v} \) we have for \( \bar{v}(t) = \bar{v}^i(t) \bar{E}_i(t) \)
\[ 0 = \bar{D}_t \bar{v}(t) = \bar{D}_t \left( \bar{v}^i(t) \bar{E}_i(t) \right) = \bar{v}^i(t) \bar{E}_i(t) \]
from which we conclude \( \bar{v}^i(t) = 0 \). Thus
\[ \bar{D}_t \left( \bar{v}^i(t) \bar{R}^i_j(0) \bar{E}_j(t) \right) = \frac{d}{dt} \left( \bar{v}^i(t) \bar{R}^i_j(0) \right) \bar{E}_j(t) = 0.
\]

Finally, we obtain by virtue of Corollary 1.20
\[ \Lambda_{(x,n)}(v) = -\frac{1}{r} \bar{g}(\bar{v}(r), \bar{v}(r)) (\bar{v}(r), \bar{v}(r)) + \bar{g}(\bar{v}(r), \bar{v}(r)) \left( \bar{T}_{\bar{v}(r)}(\bar{v}(r), \bar{v}(r)) \right)
\]
\[ + \left( \bar{g}(\bar{v}(r), \bar{v}(r)) \left( \bar{D}_r \bar{T}_{\bar{v}(r)}(\bar{v}(r), \bar{v}(r)) \right) - \frac{1}{3} \bar{g}(\bar{v}(r), \bar{v}(r)) \left( \bar{R}(\bar{v}(r), \bar{v}(r)) \right) \right) r
\]
\[ + O(r^2)
\]
\[ \square
\]

The methods we applied in the previous proposition allow for a stronger version of Lemma 2.12 provided that \( \mathcal{M} \) equals a backward metric sphere with a smaller radius \( r - \epsilon \).

**Proposition 2.14.** Let \((\mathcal{M}, F)\) be a Finsler manifold and consider the backward metric sphere \( S^-(x_0, r) \) where \( x_0 \in \mathcal{M} \), and \( r > 0 \). Let \( x \in S^-(x_0, r) \) and assume that \( x_0 \in D_x \). Set \( n := \text{grad } \rho \vert_x \), where \( \rho := -d_{x_0} \) and let \( c : [0, r] \rightarrow \mathcal{M} \) be the geodesic that joins \( x \) and \( x_0 \).

Then there exist constants \( \epsilon_0 > 0 \) such that for each \( 0 < \epsilon < \epsilon_0 \) there exists a constant \( C(\mathcal{M}, F, \epsilon) > 0 \) such that
\[ \Lambda^{S^-(x, r-\epsilon)}_{(x,n)}(v) - \Lambda^{S^-(x_0, r)}_{(x,n)}(v) \leq -C(\mathcal{M}, F, \epsilon) \]
for \( v \in T_x S^-(x_0, r) \), where \( x_\epsilon := c(r - \epsilon) \).
2 The Distance Function From a Submanifold

Proof. We proceed similar to the proof of Proposition 2.13 and will also make use of the notation introduced in this proof. Initially we recall (2.7) which states for the normal curvature of the sphere $S^-(x_0, r)$

$$\Lambda^{S^-}(x_0, r) (-y) = g(x, n)(\tilde{S}^{S^-}(x_0, r)(y), y) + \tilde{T}_{(\text{grad } \rho|_x)}(y)$$

where $y \in T_xS^-(x_0, r)$ and $\tilde{S}^{S^-}(x_0, r)$ denotes of shape operator of backward metric sphere $S^-(x_0, r)$. In order to derive a similar formula for the normal curvature of the sphere $S^-(x_0, r)$ we consider $\rho_c := -d_{S^c}$ and set $n := \text{grad } \rho_c|_x$. For $y \in T_xS^-(x_0, r)$ and a geodesic $\xi : (-\epsilon, \epsilon) \to S^-(x_0, r)$ such that $\xi(0) = x$, $\dot{\xi}(0) = -y$ we obtain similar to the proof of (2.7)

$$\Lambda^{S^-}(x_0, r) (-y) = g(x, n)(\tilde{S}^{S^-}(x_0, r)(y), y) + \tilde{T}_{(\text{grad } \rho|_x)}(y)$$

where $\tilde{S}^{S^-}(x_0, r)$ denotes of shape operator of $S^-(x_0, r)$.

We recall that the curve $\bar{c}_t : [0, r] \to \mathcal{M}$, $\bar{c}_t(t) := c(r - t - \epsilon)$ is a geodesic with respect to the Finsler structure $\bar{F}$. There holds $\rho_c(\bar{c}_t(t)) = -t$ and by differentiating this identity with respect to $t$ we obtain similar to (2.8)

$$-\dot{\bar{c}}(t) = \text{grad } \rho_c|_{\bar{c}_t(t)}$$

and in particular, we observe $n = -\dot{\bar{c}}(r - \epsilon)$. Hence, since $\bar{c}(r - \epsilon) = \bar{c}(r)$

$$\tilde{T}_{(\text{grad } \rho|_x)}(y) = \tilde{g}(\bar{c}(r - \epsilon), \dot{\bar{c}}(r - \epsilon)) \left( \tilde{T}_{\bar{c}_t(r - \epsilon)}(y), y \right) = \tilde{g}(\bar{c}(r), \dot{\bar{c}}(r)) \left( \tilde{T}_{\bar{c}_t(r)}(y), y \right) = \tilde{T}_{(\text{grad } \rho|_x)}(y).$$

Consequently,

$$\Lambda^{S^-}(x_0, r) (-y) = g(x, n)(\tilde{S}^{S^-}(x_0, r)(y), y) = g(x, n)(\tilde{S}^{S^-}(x_0, r)(y), y),$$

i.e. it remains to analyse the difference between the shape operators $\tilde{S}^{S^-}(x_0, r)$ and $\tilde{S}^{S^-}(x_0, r)$. For this purpose we recall (2.11) which states

$$-\tilde{D}_{\dot{\bar{c}}_t} \tilde{J}_{x_0}(r - t) = \tilde{S}^{S^-}(x_0, r - t)(\tilde{J}_{x_0}(r - t))$$

for $t \in [0, r)$. In order to derive a corresponding formula for $\tilde{S}^{S^-}(x_0, r)$ we proceed similarly to the previous proof and fix $t_0 \in [0, r - \epsilon)$. We consider a geodesic $\eta_c : (-\delta, \delta) \to S^-(x_0, r - \epsilon - t_0)$ with $\eta_c(0) = c(t_0)$ and $\eta_c(0) = w$ for some $w \in T_{c(t_0)}S^-(x_0, r - \epsilon - t_0)$. Let $H_c : (-\delta, \delta) \times [t_0, r - \epsilon - t_0] \to \mathcal{M}$ be defined by

$$H_c(s, t) := \exp(\eta_c(s), (t - t_0) \text{ grad } \rho|_{\eta_c(s)}).$$

Then $H_c(0, t) = \exp(c(t_0), (t - t_0) \dot{c}(t_0)) = c(t)$ for $t \in [t_0, r - \epsilon]$ and furthermore for each $s \in (-\delta, \delta)$ we have that $H_c(s, \cdot)$ is a geodesic on $\mathcal{M}$. Thus, $H_c$ is a geodesic variation of $c$ on $[t_0, r - \epsilon]$. Moreover, $H_c$ satisfies $H_c(s, t_0) = \eta_c(s)$ and since $x_c \in \mathcal{D}_{c(t_0)}$ Lemma 2.6 yields

$$H_c(s, r - \epsilon) = x_c.$$
Consequently, \(\rho_i (H_e(s,t)) = t - (r - \epsilon)\) for \(s \in (-\delta, \delta)\). One immediately observes that \(\bar{H}_e : (-\delta, \delta) \times [0, r - \epsilon - t_0) \to M\) defined by \(\bar{H}_e(s,t) := H_e(s, r - \epsilon - t)\) is a geodesic variation of \(\bar{e}_\epsilon\) with \(\bar{H}_e(s, r - \epsilon - t_0) = \eta_\epsilon(s), \bar{H}_e(s, 0) = x_e\) and \(\rho_i (\bar{H}_e(s,t)) = -t\) for \(s \in (-\delta, \delta)\). Consequently, we obtain similarly to the proof of (2.8)

\[
\text{grad } \rho_i |_{\bar{H}_e(s,r - \epsilon - t_0)} = -\frac{\partial}{\partial t} \bar{H}_e(s, r - \epsilon - t_0).
\]

Furthermore, the corresponding Jacobi field \(\bar{J}_{x_e} := \frac{\partial}{\partial s} \bar{H}_e(s, \cdot)\big|_{s=0}\) satisfies, since \(\bar{H}_e(s, 0)\) is constant,

\[
\bar{J}_{x_e}(0) = \left. \frac{\partial}{\partial s} \bar{H}_e(s, 0) \right|_{s=0} = 0
\]

and \(\bar{J}_{x_e}(r - \epsilon - t_0) = \eta_\epsilon(0)\). By a computation similar to the one in the previous proof we obtain for \(t \in [0, r - \epsilon)\)

\[
-\frac{\partial}{\partial t} \bar{J}_{x_e}(r - \epsilon - t) = S^{-}(x_e, r - \epsilon - t)(\bar{J}_{x_e}(r - \epsilon - t)).
\]

We return to (2.15) and consider a \(g_{(\bar{c}, \bar{e})}\) orthonormal frame \((\bar{E}_1, \ldots, \bar{E}_n)\) along \(\bar{c}\) where each vector field is assumed to be parallel and set \(\bar{J}_{x_{\bar{E}}}(t) = \bar{J}_{x_0}^i(t)\bar{E}_i(t)\).

Therewith we obtain

\[
(S^{-}(x_0, r - t))^{ij}_{x_0}(r - t) = -\bar{J}_{x_0}^j(r - t)
\]

where \(S^{-}(x_0, r - t)(\bar{E}_i(r - t)) = (S^{-}(x_0, r - t))^{ij}(r - t)\bar{E}_j(r - t)\). Using the scalar version of the Jacobi equation from Lemma 1.24 we compute

\[
\bar{R}_j^i(r - t, \bar{J}_{x_0}^j(r - t)) = \frac{\ddot{\bar{J}}_{x_0}^j(r - t)}{(S^{-}(x_0, r - t))^{ij}(r - t)\bar{J}_{x_0}^j(r - t)}
\]

\[
= -(S^{-}(x_0, r - t))^{ij}(r - t)\bar{J}_{x_0}^j(r - t) - (S^{-}(x_0, r - t))^{ij}(r - t)(\bar{J}_{x_0}^j(r - t))
\]

\[
= -(S^{-}(x_0, r - t))^{ij}(r - t)\bar{J}_{x_0}^j(r - t) + (S^{-}(x_0, r - t))^{ij}(r - t)\bar{J}_{x_0}^j(r - t).
\]

We successively choose \(\bar{J}_{x_0}(r - t) = \bar{E}_i(r - t), i.e. \bar{J}_{x_0}^i(r - t) = \delta_i^j\) and obtain the following first order system of ordinary differential equations for \((S^{-}(x_0, r - t))^{ij}\)

\[
\bar{R}_j^i(r - t) = -(S^{-}(x_0, r - t))^{ij}(r - t) + (S^{-}(x_0, r - t))^{ij}(r - t)(S^{-}(x_0, r - t))^{ij}(r - t).
\]

In order to perform a similar computation for \(S^{-}(x, r - \epsilon - t)\) we need a \(g_{(\bar{c}, \bar{e})}\) orthonormal frame \((((\bar{E}_1), \ldots, (\bar{E}_n))\) along \(\bar{c}\) where again each vector field is assumed to be parallel. Since \(\bar{c}_\epsilon(t) = \bar{c}(t + \epsilon)\) for \(t \in [0, r]\) we choose \((\bar{E}_i)(t) := \bar{E}_i(t + \epsilon)\). We set \(S^{-}(x, r - \epsilon - t)(\bar{E}_i)(r - \epsilon - t)) = (S^{-}(x, r - \epsilon - t))^{ij}(r - \epsilon - t)(\bar{E}_j)(r - \epsilon - t)\) and, according to Lemma 1.24, \((\bar{R}_j^i(t)\bar{E}_i)(t) := \bar{R}(\bar{E}_j^i(t), \bar{c}_\epsilon(t))\bar{c}_\epsilon(t)\) and derive

\[
(\bar{R}_j^i)(r - \epsilon - t) = -(S^{-}(x, r - \epsilon - t))^{ij}(r - \epsilon - t) + (S^{-}(x, r - \epsilon - t))^{ij}(r - \epsilon - t)(S^{-}(x, r - \epsilon - t))^{ij}(r - \epsilon - t).
\]
We easily observe \((\bar{R}_\epsilon)^j_i(t)(\bar{E}_\epsilon)_i(t) = \bar{R}_t^j_i(t + \epsilon)\bar{E}_t(t + \epsilon)\) and consequently \(f_j^i(t) := (\bar{S}^{S^-}(x_\epsilon, r - \epsilon - t))^j_i(r - t), \ g_j^i(t) := (\bar{S}^{S^-}(x_\epsilon, r - \epsilon))^j_i(r - t)\) satisfy for \(0 < t < r - \epsilon\)
\[
\frac{f_j^i(t)}{g_j^i(t)} - f_j^i(t) f_j^i(t) = -\frac{1}{r - t} \delta_j^i - \frac{1}{3} \bar{R}_j^i(0)(r - t) + O((r - t)^2)
\]
i.e. \(f_j^i, g_j^i\) satisfy the same initial value problem with initial data \(f_j^i(0)\) and \(g_j^i(0)\) respectively. Given initial data at \(t = 0\) this system is uniquely solvable. However, here the initial data is unknown and moreover we are interested in the difference of the initial data \(f_j^i(0) - g_j^i(0) = (\bar{S}^{S^-}(x_\epsilon, r - \epsilon))^j_i(r - \epsilon) - (\bar{S}^{S^-}(x_\epsilon, r))^j_i(r)\). In order to analyse this difference we recall (2.13) which states
\[
g_j^i(t) = -\frac{1}{r - t} \delta_j^i - \frac{1}{3} \bar{R}_j^i(0)(r - t) + O((r - t)^2)
\]
as \(t \searrow r\). Since Proposition 2.13 is applicable to \(S^-(x_\epsilon, r - \epsilon)\) as well we also have
\[
f_j^i(t) = -\frac{1}{r - \epsilon - t} \delta_j^i - \frac{1}{3} (\bar{R}_\epsilon)^j_i(0)(r - \epsilon - t) + O((r - \epsilon - t)^2)
\]
as \(t \searrow r - \epsilon\). Clearly, the index \(\epsilon\) at \((\bar{R}_\epsilon)^j_i\) indicates the dependence on the centre \(x_\epsilon\) of the sphere \(S^-(x_\epsilon, r - \epsilon)\). Here our notation is a somewhat imprecise since the error term \(O((r - \epsilon - r)^2)\) contains derivatives of \((\bar{R}_\epsilon)^j_i\) and hence it also depends on \(x_\epsilon\). However, since the \(R\)-curvature tensor is smooth, we have \((\bar{R}_\epsilon)^j_i(0) - \bar{R}_j^i(0) = O(\epsilon)\) and we can deal with the remaining terms carrying an \(x_\epsilon\) dependence in a similar way. Consequently,
\[
f_j^i(t) - g_j^i(t) = -\left(\frac{1}{r - \epsilon - t} - \frac{1}{r - t}\right) \delta_j^i + O(1) = -\frac{\epsilon}{(r - \epsilon - t)(r - t)} \delta_j^i + O(1)
\]
as \(t \searrow r - \epsilon\) provided that \(\epsilon < \epsilon_0\) for some \(\epsilon_0 > 0\). By choosing \(r - \epsilon - t\) sufficiently small we get \(f_j^i(t) - g_j^i(t) \leq -C\) for some \(C > 0\). In particular, there exists \(t \in (0, r - \epsilon)\) with \(f_j^i(t) \neq g_j^i(t)\) and hence \(f_j^i(0) \neq g_j^i(0)\). Furthermore, the assumption \(f_j^i(0) > g_j^i(0)\) is a contradiction to Lemma 2.12. Consequently, there exists a constant \(C(\mathcal{M}, F, \epsilon) > 0\) such that
\[
(\bar{S}^{S^-}(x_\epsilon, r - \epsilon))^j_i(r - \epsilon) - (\bar{S}^{S^-}(x_\epsilon, r))^j_i(r) = f_j^i(0) - g_j^i(0) \leq -C(\mathcal{M}, F, \epsilon)
\]
which implies the claim. \(\square\)

### 2.3 The Cut Locus of a Submanifold

Let \((\mathcal{M}, F)\) be a Finsler manifold and \((\tilde{\mathcal{M}}, \bar{\varphi})\) be a submanifold. In order to avoid confusion we recall that our notation of submanifolds is a little inconsistent since \(\varphi : \tilde{\mathcal{M}} \to \mathcal{M}\) is an embedding. We have shown in Lemma 2.1 that the embedding induces a Finsler structure \(\bar{F}\) on \(\tilde{\mathcal{M}}\), i.e. \((\tilde{\mathcal{M}}, \bar{F})\) is a Finsler manifold.

As defined in Section 1.1 we consider the distance function
\[
d_{\tilde{\mathcal{M}}} := \inf_{\tilde{x} \in \tilde{\mathcal{M}}} d(\bar{\varphi}(\tilde{x}), \cdot).
\]
For $\tilde{x} \in \tilde{M}$ we consider unit speed geodesics $c(\varphi(\tilde{x}), y, \cdot) : [0, r] \to M$ starting at $\varphi(\tilde{x}) =: x \in M$ in some unit normal direction $y \in \Gamma_x^{\perp} M$. Henceforth, we denote geodesics starting on $\tilde{M}$ in unit normal direction by $\xi$. More precisely, given $(\tilde{x}, y) \in I^+ \tilde{M}$ we set $\xi(\tilde{x}, y, \cdot) := c(\varphi(\tilde{x}), y, \cdot)$. We generalise the notions 'cut value', 'cut point', and 'cut locus' introduced in Subsection 1.5.

**Definition 2.15.** Let $(M, F)$ be a Finsler manifold, $(\tilde{M}, \varphi)$ be a submanifold, and $(\tilde{x}, y) \in I^+ \tilde{M}$. Then the cut value of $\tilde{M}$ at $(\tilde{x}, y)$ is defined by

$$i_{\tilde{M}}(\tilde{x}, y) := \sup \left\{ t : d_{\tilde{M}}^+(\xi(\tilde{x}, y, t)) = t \right\}.$$ 

Given that $i_{\tilde{M}}^-(\tilde{x}, y) < \infty$ the point $\xi(\tilde{x}, y, i_{\tilde{M}}^-(\tilde{x}, y))$ is called a cut point of $\tilde{M}$. The union of all cut points is called cut locus of $\tilde{M}$ and is given by

$$\text{Cut}_{\tilde{M}} := \left\{ \xi(\tilde{x}, y, i_{\tilde{M}}(\tilde{x}, y)) : (\tilde{x}, y) \in I^+ \tilde{M}, i_{\tilde{M}}(\tilde{x}, y) < \infty \right\}.$$ 

We easily observe the following consequence of the preceding definition.

**Corollary 2.16.** Let $(M, F)$ be a Finsler manifold, $(\tilde{M}, \varphi)$ be a submanifold, and $(\tilde{x}, y) \in I^+ \tilde{M}$. Then

$$i_{\tilde{M}}(\tilde{x}, y) \leq I(\varphi(\tilde{x}), y).$$

By $G \subset M$ we denote the largest open set such that for every $z \in G$ there is a unique closest point $\tilde{x}$ on $\tilde{M}$ in the sense that

$$d_{\tilde{M}}^+(z) = d_{\varphi(\tilde{x})}(z).$$

The following result is a generalisation of [LN05, Lemma 4.10].

**Proposition 2.17.** Let $(M, F)$ be a forward geodesically complete Finsler manifold, $(\tilde{M}, \varphi)$ be a compact $C^2$ submanifold, and $(\tilde{x}_0, y_0) \in I^+ \tilde{M}$. Then

$$\xi(\tilde{x}_0, y_0, t) \in G$$

for $0 \leq t < t_0 := i_{\tilde{M}}(\tilde{x}_0, y_0)$.

**Proof.** Assume that the assertion is false, i.e. there exists $\epsilon_0 > 0$ such that

$$z_0 := \xi(\tilde{x}_0, y_0, t_0 - \epsilon_0) \in M \setminus G.$$ 

Hence, there exists $\{z_k\}_{k \in \mathbb{N}} \subset M \setminus G$ such that $z_k \to z_0$ and $\{(\tilde{x}_1)_k\}_{k \in \mathbb{N}}, \{(\tilde{x}_2)_k\}_{k \in \mathbb{N}} \subset \tilde{M}$ with

$$d(\varphi(\tilde{x}_1)_k, z_k) = d(\varphi(\tilde{x}_2)_k, z_k) = d_{\tilde{M}}^-(z_k) =: t_k.$$ 

Since $\tilde{M}$ is compact there exist $\tilde{x}_1, \tilde{x}_2 \in \tilde{M}$ and subsequences, which we still denote by $\{(\tilde{x}_1)_k\}_{k \in \mathbb{N}}, \{(\tilde{x}_2)_k\}_{k \in \mathbb{N}}$, such that $(\tilde{x}_1)_k \to \tilde{x}_1$ and $(\tilde{x}_2)_k \to \tilde{x}_2$. Clearly, the continuity of the distance function $d_{\tilde{M}}^+$ yields $d(\varphi(\tilde{x}_1), z_0) = d(\varphi(\tilde{x}_2), z_0) = t_0 - \epsilon_0$.

In the next step we show that in fact $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_0$. For $0 < \delta < \epsilon_0$ the triangle inequality for $d$ yields

$$d(\varphi(\tilde{x}_1), \xi(\tilde{x}_0, y_0, t_0 - \epsilon_0 + \delta))$$
\[d(\varphi(\tilde{x}_1), \xi(\tilde{x}_0, y_0, t_0 - \epsilon_0)) + d(\xi(\tilde{x}_0, y_0, t_0 - \epsilon_0), \xi(\tilde{x}_0, y_0, t_0 - \epsilon_0 + \delta)) \quad (2.17)\]

\[= t_0 - \epsilon_0 + \delta.\]

However, from the hypotheses we obtain \(d(\varphi(\tilde{x}_1), \xi(\tilde{x}_0, y_0, t_0 - \epsilon_0 + \delta)) \geq t_0 - \epsilon_0 + \delta\) such that we may replace ‘\(\leq\)’ in (2.17) by ‘\(=\)’. We construct a curve \(\eta\) which consists of a minimising geodesic from \(\varphi(\tilde{x}_1)\) to \(\xi(\tilde{x}_0, y_0, t_0 - \epsilon_0)\) and the arc \(\{\xi(\tilde{x}_0, y_0, t); t \in [t_0 - \epsilon_0, t_0 - \epsilon_0 + \delta]\}\). Clearly, \(\eta\) is a piecewise \(C^\infty\) curve and moreover (2.17) yields that \(\eta\) minimises distance from \(\varphi(\tilde{x}_1)\) to \(\xi(\tilde{x}_0, y_0, t_0 - \epsilon_0 + \delta)\). Thus, Proposition 1.14 yields that \(\eta\) is in fact a geodesic. Since \(\eta\) agrees with \((\xi(\tilde{x}_0, y_0, \cdot), \cdot)\) on the arc \(\{\xi(\tilde{x}_0, y_0, t); t \in [t_0 - \epsilon_0, t_0 - \epsilon_0 + \delta]\}\) standard uniqueness results for solutions of ordinary differential equations yield \(\eta = \xi(\tilde{x}_0, y_0, \cdot)\) everywhere and thus \(\tilde{x}_1 = \tilde{x}_0\). A similar reasoning yields \(\tilde{x}_2 = \tilde{x}_0\).

Let \(g_k : \mathcal{M} \to [0, \infty)\) defined by \(g_k(\tilde{x}) := \rho_k(\varphi(\tilde{x})) := -d(\varphi(\tilde{x}), \tilde{z}_k) = -d^{-1}(\varphi(\tilde{x}))\) for \(k \in \mathbb{N}_0\). Since \(t_0 - \epsilon < i_{M}(\tilde{x}_0, y_0) \leq i(\varphi(\tilde{x}_0), y_0)\) we have \(\tilde{z}_0 \in D_{\tilde{x}_0} \) from Lemma 1.39. Given that \(k\) is sufficiently large, we infer from Lemma 1.42 the existence of a neighbourhood \(U_k \subset M\) of \(x_0 := \varphi(\tilde{x}_0)\) such that \(g_k \in C^\infty(U_k)\). Since \(\varphi \in C^{2,1}(\mathcal{M}, \mathcal{M})\) we get the existence of an open neighbourhood \(\tilde{U}_k \subset \mathcal{M}\) with \(g_k \in C^2(\tilde{U}_k)\). Furthermore, for an arbitrary geodesic \(\tilde{c} : (-\epsilon, \epsilon) \to \mathcal{M}\) with \(\tilde{c}(0) = \tilde{x}_0\) and \(\tilde{c}(0) = \tilde{y} \in T_{\tilde{x}_0} \mathcal{M}\) we set \(y := \frac{d}{dt} \varphi(\tilde{c}(t))\) at \(t_0 = 0\) and compute

\[
\begin{align*}
\frac{d^2}{dt^2} (g_0 \circ \tilde{c}) (t) \bigg|_{t=0} &= \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \rho_0(x_0) \frac{d}{dt} \varphi^i(\tilde{c}(t)) \bigg|_{t=0} \frac{d}{dt} \varphi^j(\tilde{c}(t)) \bigg|_{t=0} \\
&+ \frac{\partial}{\partial x^\alpha} \rho_0(x_0) \frac{\partial^2}{\partial x^\beta \partial x^\gamma} \varphi^j(\tilde{x}_0) \tilde{c}^\gamma(0) \tilde{c}^\beta(0) + \frac{\partial}{\partial x^\alpha} \rho_0(x_0) \frac{\partial^2}{\partial x^\beta} \varphi^j(\tilde{x}_0) \tilde{c}^\gamma(0) \\
&= \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \rho_0(x_0) \frac{d}{dt} \varphi^j(\tilde{c}(t)) \bigg|_{t=0} \frac{d}{dt} \varphi^j(\tilde{c}(t)) \bigg|_{t=0} - \frac{\partial}{\partial x^\alpha} \rho_0(x_0) G^j(x_0, y) \\
&+ \frac{\partial}{\partial x^\alpha} \rho_0(x_0) \left( \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \varphi^j(\tilde{x}_0) \tilde{c}^\gamma(0) \tilde{c}^\beta(0) - \frac{\partial}{\partial x^\alpha} \varphi^j(\tilde{x}_0) \tilde{c}^\alpha(0) + G^j(\tilde{x}_0, \tilde{c}(0)) + G^j(x_0, y) \right) \\
&= D^2 \rho_0(y) - d \rho_0(A(y)) \\
&= \Lambda_n^{\infty}(z_0, t_0 - \epsilon_0)(y) - \Lambda_n^{\infty}(y) \\
&\leq \Lambda_n^{\infty}(z_0, t_0 - \epsilon_0)(y) - \Lambda_n^{\infty}(\xi(\tilde{x}_0, v_0, t_0 - \epsilon_0 + \delta), t_0 - \epsilon_0 + \delta)(y) \\
&\leq -C((\mathcal{M}, F, \delta))
\end{align*}
\]

where we made use of Lemma 2.11, Lemma 2.12, and Proposition 2.14. The last inequality implies that for all \(\tilde{y} \in T_{\tilde{x}_0} \mathcal{M}\) we have \(\tilde{D}^2 \tilde{g}_0(\tilde{y}) \leq -C((\mathcal{M}, F, \delta)) \) at \(\tilde{x}_0\) where \(\tilde{D}^2 \tilde{g}_0\) denotes the Hessian of \(\tilde{g}\) with respect to the Finsler structure \(\tilde{F}\) of \(\tilde{\mathcal{M}}\), see Definition 2.10. We conclude that there exists \(r_0 > 0\) such that for all \(\tilde{x} \in \tilde{B}_+^{\tilde{g}}(\tilde{x}_0, r_0)\) we have \(\tilde{D}^2 \tilde{g}_0(\tilde{y}) \leq -\frac{1}{2}C((\mathcal{M}, F, \delta)) \) at \(\tilde{x}\) for all \(\tilde{y} \in T_{\tilde{x}} \mathcal{M}\). Clearly, \(\tilde{B}_+^{\tilde{g}}\) denotes the forward metric ball in \(\mathcal{M}\).

In order to get the same estimate for \(\tilde{D}^2 \tilde{g}_k\) it is necessary to show \(g_k \to g_0\) in \(C^2(\tilde{B}_+^{\tilde{g}}(\tilde{x}_0, r_0))\). For this purpose we derive from Lemma 1.42 that derivatives of \(\tilde{d}_{zz} = d(\cdot, \cdot)\) of arbitrary order are locally uniformly bounded in both variables and hence \(\tilde{d}_{zz} \to d_{zz}^0\) in \(C^2(U)\) where \(U\) is an open neighbourhood of \(x_0\). Since \(\varphi \in C^{2,1}(\mathcal{M}, \mathcal{M})\) we conclude by first shrinking \(r_0\) if necessary and then choosing \(k\).
sufficiently large \( \tilde{D}^2 g_k(y) < 0 \) (2.18) at \( \tilde{x} \) for all \( \tilde{x} \in \tilde{B}^+(\tilde{x}_0, r_0) \) and \( \tilde{y} \in T_{\tilde{x}} \tilde{M} \). Since

\[ g_k((\tilde{x}_1)_k) = g_k((\tilde{x}_2)_k) = \min_{\tilde{x} \in \tilde{M}} g_k(\tilde{x}) \] (2.19)

we will derive a contradiction from (2.18). For this purpose it is necessary to ensure that any minimising geodesic from \((\tilde{x}_1)_k\) to \((\tilde{x}_2)_k\) is contained in \(\tilde{B}^+(\tilde{x}_0, r_0)\). At first we may choose \(r_0\) smaller such that Lemma 1.44 is applicable. Then we choose \(k\) larger to obtain \((\tilde{x}_1)_k, (\tilde{x}_2)_k \in \tilde{U}_{r_0/3} = \tilde{B}^+(x_0, r_0/3) \bigcap \tilde{B}^-(x_0, r_0/3)\) and hence Lemma 1.44 yields that any minimising geodesic from \((\tilde{x}_1)_k\) to \((\tilde{x}_2)_k\) is contained in \(\tilde{B}^+(x_0, r_0)\). Finally, we consider a minimising geodesic \(\tilde{c}_k\) from \((\tilde{x}_1)_k\) to \((\tilde{x}_2)_k\) and observe that (2.18) implies that \(g_k \circ \tilde{c}_k\) is strictly concave which is a contradiction to (2.19).

In the following lemma we introduce the cut domain of the submanifold \(\tilde{M}\) and characterise its boundary as the cut locus of \(\tilde{M}\). The statement is similar to that of Lemma 1.39 and also the proofs of both results are analogous.

**Lemma 2.18.** Let \((\mathcal{M}, F)\) be a forward geodesically complete, connected Finsler manifold and \((\tilde{\mathcal{M}}, \varphi)\) be a compact submanifold. We consider the domain

\[ D_{\tilde{\mathcal{M}}} := \{ (\tilde{x}, t\tilde{y}); (\tilde{x}, \tilde{y}) \in I^+ \tilde{\mathcal{M}}, 0 \leq t < i_{\tilde{\mathcal{M}}}^{-1}(\tilde{x}, \tilde{y}) \} . \]

Then

(i) the exponential map \(\exp\) is a diffeomorphism from \(D_{\tilde{\mathcal{M}}}\) onto its image \(\tilde{D}_{\tilde{\mathcal{M}}} := \exp(D_{\tilde{\mathcal{M}}})\). The set \(\tilde{D}_{\tilde{\mathcal{M}}}\) is called the cut domain of \(\mathcal{M}\).

(ii) \(\partial D_{\tilde{\mathcal{M}}} = \text{Cut}_{\tilde{\mathcal{M}}}\).

(iii) \(\mathcal{M}\) is the disjoint union of \(D_{\tilde{\mathcal{M}}}\) and \(\text{Cut}_{\tilde{\mathcal{M}}}\).

**Proof.** Regarding the proof of (i) we show that \(\exp\) is injective on \(D_{\tilde{\mathcal{M}}}\). For this purpose we suppose that there exist \((\tilde{x}_1, t_1 y_1), (\tilde{x}_2, t_2 y_2) \in D_{\tilde{\mathcal{M}}}\) with

\[ \exp(\varphi(\tilde{x}_1), t_1 y_1) = z = \exp(\varphi(\tilde{x}_2), t_2 y_2) \].

Since \(t_1 < i_{\tilde{\mathcal{M}}}^{-1}(\tilde{x}_1, y_1)\) and \(t_2 < i_{\tilde{\mathcal{M}}}^{-1}(\tilde{x}_2, y_2)\) we obtain from Definition 2.15

\[ t_1 = d_{\tilde{\mathcal{M}}}(\xi(\tilde{x}_1, y_1), t_1) = d_{\tilde{\mathcal{M}}}(z) = d_{\tilde{\mathcal{M}}}(\xi(\tilde{x}_2, y_2), t_2) = t_2 \]

and hence both \(\tilde{x}_1\) and \(\tilde{x}_2\) minimise distance from \(\tilde{M}\) towards \(z\). However, Proposition 2.17 yields \(\tilde{x}_1 = \tilde{x}_2\) and since \(i_{\tilde{\mathcal{M}}}^{-1}(\tilde{x}_1, y_1) < i(\varphi(\tilde{x}_1), y_1)\) we derive \(y_1 = y_2\) from (ii) in Lemma 1.37.

As to a proof of (ii) we initially observe \(\text{Cut}_{\tilde{\mathcal{M}}} \subset \partial D_{\tilde{\mathcal{M}}}\). Indeed, in each neighbourhood of a cut point \(z := \exp(\varphi(x), \tilde{i}_{\tilde{\mathcal{M}}}(\tilde{x}, y))\) where \((\tilde{x}, \tilde{y}) \in I^+ \tilde{\mathcal{M}}\) there exists a point \(\exp(\varphi(x), \tilde{i}_{\tilde{\mathcal{M}}}(\tilde{x}, y) - \epsilon) \in D_{\tilde{\mathcal{M}}}\) where \(\epsilon > 0\). Since \(z \in \mathcal{M} \setminus D_{\tilde{\mathcal{M}}}\) we obtain \(\text{Cut}_{\tilde{\mathcal{M}}} \subset \partial D_{\tilde{\mathcal{M}}}\). It remains to show the reverse inclusion. For \(z \in \partial D_{\tilde{\mathcal{M}}}\) we consider
Clearly, $\epsilon$ where $\minimises geodesic$ $44$

$\frac{(\tilde{x}, y)}{\partial \frac{r}{\mathcal{M}}} \in \mathcal{M}$ with $r := \frac{d_{\mathcal{M}}^{\pm}(z) = d(\varphi(\tilde{x}), z)}{\minimises distance from}$.

Once again, we obtain from Theorem 1.34 the existence of a minimising geodesic $c(t) := \exp(x, t y z)$ where $c(0) = x$ and $c(r) = z$.

To prove (iii) we consider $z \in \mathcal{M}$. Let $\vec{x} \in \mathcal{M}$, $x := \varphi(\tilde{x})$ with $r := \frac{d_{\mathcal{M}}^{\pm}(z) = d(x, z)}{\minimises distance from}$.

Let $z \in \mathcal{D}_{\mathcal{M}} \setminus \mathcal{M}$. Lemma 1.18 implies that there exists a unique $x = x(z)$ where $x(z) := \varphi(\tilde{x}(z))$ for $\tilde{x}(z) \in \mathcal{M}$ and a unique $y = y(z) \in T_x^\perp \mathcal{M}$, $y(z) \neq 0$, such that $\tilde{x} = \exp(x(z), y(z))$. Moreover, the same lemma yields a $C^\infty$ correspondence between $z$ and $(x(z), y(z))$. Since $i_{\mathcal{M}}(\tilde{x}(z), y(z)) \leq i(x(z), y(z))$ we obtain that the geodesic $c : [0, 1] \rightarrow \mathcal{M}$ defined by $c(t) := \exp(x(z), t y(z))$ is the unique minimising curve from $x$ to $z$.

Thus, $\frac{d_{\mathcal{M}}^{\pm}(z) = \int_0^1 F(c(t), c(t))dt = F(x(z), y(z))}{\minimises distance from}$ from which we conclude the assertion.

Similarly to the definition of $\mathcal{G}$ we now define $\mathcal{G}^* \subset \mathcal{M}$ as the largest open set such that for each $x \in \mathcal{G}$ there exists a unique $(\tilde{z}, y z) \in T_\perp \mathcal{M}$ such that $\xi(\tilde{z}, y z)$ minimises distance from $\mathcal{M}$ to $\xi(\tilde{z}, y z, 1) = x$. Clearly, we have $\mathcal{G}^* \subset \mathcal{G}$ and observe $\xi(\tilde{x}_0, y_0, t) \in \mathcal{G}^*$ for $0 \leq t < t_0 := i_{\mathcal{M}}(\tilde{x}_0, y_0)$ by virtue of (i) in Lemma 1.18. Next, we show $\xi(\tilde{x}_0, y_0, t_0) \in \partial \mathcal{G}^*$. See [LN05, Corollary 4.11] for the corresponding result.

**Lemma 2.20.** Let $(\mathcal{M}, F)$ be a connected, forward geodesically complete Finsler manifold and $(\mathcal{M}, \varphi)$ a compact submanifold. For $(\tilde{x}_0, y_0) \in I^\perp \mathcal{M}$ satisfying $t_0 := i_{\mathcal{M}}(\tilde{x}_0, y_0) < \infty$ we have $\xi(\tilde{x}_0, y_0, t_0) \in \partial \mathcal{G}^*$.

**Proof.** Let $t^* > 0$ be chosen such that $\xi(\tilde{x}_0, y_0, t^*) \in \partial \mathcal{G}^*$ and $\xi(\tilde{x}_0, y_0, t) \in \mathcal{G}^*$ for $0 \leq t < t^*$. As we have already mentioned, (i) in Lemma 1.18 yields $t^* \geq t_0$. Thus, the lemma is proven if we show $t_0 \geq t^*$.

We argue by contradiction and assume $0 < t_0 < t^*$. From the definition of $\mathcal{G}^*$ we obtain the existence of an open neighbourhood $\mathcal{U} \subset \mathcal{G}^*$ of $\xi(\tilde{x}_0, y_0, t_0)$. Moreover, for each $x \in \mathcal{U}$ there exists a unique $(\tilde{z}(x), y z(x)) \in T_\perp \mathcal{M}$ such that $\xi(\tilde{z}(x), y z(x), 1) = x$ and hence we may introduce the map $\Phi : \mathcal{U} \rightarrow T_\perp \mathcal{M}$ defined by $\Phi(x) := (z(x), y z(x))$.

Clearly, $\Phi$ is continuous and one-to-one and hence $\Phi$ is an open map, see [Dei85, Theorem 4.1].

We have $\Phi(\xi(\tilde{x}_0, y_0, t_0)) = (\tilde{x}_0, t_0 y_0)$ and since $\Phi$ is open we conclude $(\tilde{x}_0, (1 + \epsilon) t_0 y_0) \in \Phi(\mathcal{U})$ provided that $\epsilon > 0$ is chosen sufficiently small. Hence, $\xi(\tilde{x}_0, (1 + \epsilon) t_0 y_0, \cdot)$ minimises distance from $\mathcal{M}$ to $\xi(\tilde{x}_0, (1 + \epsilon) t_0 y_0, 1) = \xi(\tilde{x}_0, y_0, (1 + \epsilon) t_0)$ which contradicts $t_0 = i_{\mathcal{M}}(\tilde{x}_0, y_0)$.
Local Lipschitz Continuity of $i\tilde{M}$

In the present chapter we show that, in the absence of conjugate points, the distance function to the cut locus of a submanifold

$$i_\tilde{M} : I^\perp \tilde{M} \to [0, \infty).$$

is locally Lipschitz continuous. Here, $(\mathcal{M}, F)$ is a Finsler manifold and $(\tilde{M}, \varphi)$ is a $C^{2,1}$ submanifold. Therewith we generalise results from [LN05]. The underlying idea of the following proof is taken from the same paper. However, the generalisation of the ideas and techniques in this paper is non-trivial. Before we give the precise formulation of the theorem and explain the idea of its proof we introduce distance functions on the slit tangent bundle $T\mathcal{M} \setminus 0$ of the Finsler manifold $(\mathcal{M}, F)$ and on $I^\perp \tilde{M}$.

For this purpose we recall the Sasaki metric (1.6) on $T(T\mathcal{M} \setminus 0)$ introduced in Section 1.2. The unit normal bundle $I^\perp \tilde{M}$ is understood as a submanifold of $T\mathcal{M} \setminus 0$ and the restriction of the Riemannian metric (1.6) to tangent vectors of $I^\perp \tilde{M}$ yields a Riemannian metric $G_{I^\perp \tilde{M}}$ over $I^\perp \tilde{M}$. Using these metrics we may define the integral length of curves on $T\mathcal{M} \setminus 0$ and $I^\perp \tilde{M}$ in the usual way and therewith Riemannian distance functions on $T\mathcal{M} \setminus 0$ and $I^\perp \tilde{M}$.

**Definition 3.1.** Let $(\mathcal{M}, F)$ be a Finsler manifold. Then the map $D : (T\mathcal{M} \setminus 0) \times (T\mathcal{M} \setminus 0) \to [0, \infty]$ is defined by

$$D((x, y), (x_0, y_0)) := \inf \left\{ \int_a^b g_{ij}(C(t)) \dot{C}^i(t) \dot{C}^j(t) + g_{ij}(C(t)) \dot{C}^{N+i}(t) \dot{C}^{N+j}(t) dt ; \right.$$

$$C : [a, b] \to S^\perp \tilde{M} \text{ a piecewise } C^\infty \text{ curve, } \dot{C} = \dot{C}^i \frac{\delta}{\delta x^i} + \dot{C}^{N+i} F \frac{\partial}{\partial y^i},$$

$$C(a) = (\bar{x}, y), \ C(b) = (\bar{x}_0, y_0) \right\}.$$

**Definition 3.2.** Given a $C^1$ submanifold $(\tilde{M}, \varphi)$, the map $D_{I^\perp \tilde{M}} : I^\perp \tilde{M} \times I^\perp \tilde{M} \to [0, \infty)$ is given by

$$D_{I^\perp \tilde{M}}((\bar{x}, y), (\bar{x}_0, y_0)) := \inf \left\{ \int_a^b G_{I^\perp \tilde{M}}(C(t), \dot{C}(t)) dt ; \ C : [a, b] \to I^\perp \tilde{M} \right.$$ a piecewise $C^\infty$ curve, $C(a) = (\bar{x}, y), \ C(b) = (\bar{x}_0, y_0) \right\}.$$
3.1 Statement of the Main Result

In this section we give a precise formulation of our main result and establish some corollaries.

To begin with, we recall a notion from Section 2.3. Given \((\tilde{x}_0, y_0) \in I^+\tilde{M}\) we denote the geodesic starting at \(\varphi(\tilde{x}_0)\) in direction \(y_0\) by \(\xi(x_0, y_0, \cdot)\). Now, the precise statement reads as follows.

**Theorem 3.2.** Let \((\mathcal{M}, F)\) be a connected, forward geodesically complete Finsler manifold and \((\tilde{\mathcal{M}}, \varphi)\) be a compact \(C^{2,1}\) submanifold. Assume that for \((\tilde{x}_0, y_0) \in I^+\tilde{M}\) we have \(i_{\tilde{\mathcal{M}}}((\tilde{x}_0, y_0)) < \infty\) and that for \(t \in (0, t_0]\) no point \(\xi(\tilde{x}_0, y_0, t)\) is conjugate to \(\varphi(\tilde{x}_0)\) along \(\xi(\tilde{x}_0, y_0, \cdot)\). Then there exist constants \(\delta_0 > 0\) and \(K \geq 1\) such that

\[
|i_{\tilde{\mathcal{M}}}((\tilde{x}_1, y_1)) - i_{\tilde{\mathcal{M}}}((\tilde{x}_2, y_2))| \leq K D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_1, y_1), (\tilde{x}_2, y_2))
\]

for all \((\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^+\tilde{M}\) satisfying \(D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_0, y_0), (\tilde{x}_1, y_1)) < \delta_0\) for \(i \in \{1, 2\}\). The constants \(K\) and \(\delta_0\) depend on \(\mathcal{M}, F, \tilde{\mathcal{M}}, \varphi, \) and \(t_0\) provided that \(\xi(x_0, y_0, \cdot)\) does not contain conjugate points. Otherwise, let \(\xi(\tilde{x}_0, y_0, t_1)\) be the first point which is conjugate to \(\varphi(\tilde{x}_0)\) along \(\xi(x_0, y_0, \cdot)\). In this case, the constant \(\delta_0\) additionally depends on \(t_1 - t_0 > 0\).

Subsequently, we collect conditions which imply the absence of conjugate points required in Theorem 3.2.

**Corollary 3.3.** Let \((\mathcal{M}, F)\) be a connected, forward geodesically complete Finsler manifold whose flag curvature is bounded from above by \(\kappa > 0\) and \((\mathcal{M}, \varphi)\) be a compact \(C^{2,1}\) submanifold. Assume that for \((\tilde{x}_0, y_0) \in I^+\tilde{M}\) we have \(i_{\tilde{\mathcal{M}}}((\tilde{x}_0, y_0)) < \frac{2}{\sqrt{\kappa}}\). Then there exist constants \(\delta_0 = \delta_0(\mathcal{M}, F, \tilde{\mathcal{M}}, \varphi, t_0, t_0 - \frac{\pi}{\sqrt{\kappa}} > 0\) and \(K = K(\mathcal{M}, F, \tilde{\mathcal{M}}, \varphi, t_0) \geq 1\) such that

\[
|i_{\tilde{\mathcal{M}}}((\tilde{x}_1, y_1)) - i_{\tilde{\mathcal{M}}}((\tilde{x}_2, y_2))| \leq K D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_1, y_1), (\tilde{x}_2, y_2))
\]

for all \((\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^+\tilde{M}\) satisfying \(D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_0, y_0), (\tilde{x}_1, y_1)) < \delta_0\) for \(i \in \{1, 2\}\).

**Proof.** We have that for \(t \in (0, \frac{\pi}{\sqrt{\kappa}}]\) no point \(\xi(x_0, y_0, t)\) is conjugate to \(\varphi(\tilde{x}_0)\) along \(\xi(x_0, y_0, \cdot)\) by virtue of [She01, Theorem 13.1.2]. Hence, Theorem 3.2 is applicable and the Corollary is proven. \(\square\)

**Corollary 3.4.** Let \((\mathcal{M}, F)\) be a connected, forward geodesically complete Finsler manifold of nonpositive flag curvature and \((\mathcal{M}, \varphi)\) be a compact \(C^{2,1}\) submanifold. Assume that for \((\tilde{x}_0, y_0) \in I^+\tilde{M}\) we have \(i_{\tilde{\mathcal{M}}}((\tilde{x}_0, y_0)) \leq \infty\). Then there exist constants \(\delta_0 > 0\) and \(K \geq 1\) depending on \(\mathcal{M}, F, \tilde{\mathcal{M}}, \varphi, \) and \(t_0\) such that

\[
|i_{\tilde{\mathcal{M}}}((\tilde{x}_1, y_1)) - i_{\tilde{\mathcal{M}}}((\tilde{x}_2, y_2))| \leq K D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_1, y_1), (\tilde{x}_2, y_2))
\]

for all \((\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^+\tilde{M}\) satisfying \(D_{I^+\tilde{\mathcal{M}}}((\tilde{x}_0, y_0), (\tilde{x}_1, y_1)) < \delta_0\) for \(i \in \{1, 2\}\).

**Proof.** Since the flag curvature of \((\mathcal{M}, F)\) is nonpositive, no geodesic can contain any conjugate points, see [BCS00, Proposition 9.1.2] or [She01, Theorem 13.1.2]. Hence, Theorem 3.2 is applicable and the Corollary is proven. \(\square\)
Clearly, Theorem 3.2 is applicable in a Euclidean setting since there are no conjugate points.

**Corollary 3.5.** Let \( \widetilde{\mathcal{M}} \subset \mathbb{R}^N \) be a compact \( C^{2,1} \) submanifold. Assume that for \((\tilde{x}_0, y_0) \in I^+\widetilde{\mathcal{M}} \) we have \( i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_0, y_0)} < \infty \). Then there exist constants \( \delta_0 > 0 \) and \( K \geq 1 \) depending on \( \mathcal{M}, F, \mathcal{M}, \varphi, \) and \( t_0 \) such that

\[
|i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_1, y_1)} - i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_2, y_2)}| \leq K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_1, y_1, \tilde{x}_2, y_2)
\]

for all \((\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^+\widetilde{\mathcal{M}} \) satisfying \( D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}_1, y_1)) < \delta_0 \) for \( i \in \{1, 2\} \).

Next, we explain the underlying idea of the proof of Theorem 3.2. We fix \((\tilde{x}_0, y_0) \in I^+\widetilde{\mathcal{M}} \), set \( t_0 := i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_0, y_0)} \) and show that there exist constants \( K \geq 1 \) and \( \delta_0 > 0 \) such that for all \((\tilde{x}, y) \in I^+\widetilde{\mathcal{M}} \) satisfying \( D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y)) =: \delta < \delta_0 \) there exists \( \tilde{z} \in \widetilde{\mathcal{M}} \) such that

\[
d(\varphi(\tilde{z}), \xi(\tilde{x}, y, t_0 + K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y))) < t_0 + K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y)).
\]

This inequality implies

\[
i_{\widetilde{\mathcal{M}}}^{(\tilde{x}, y)} \leq t_0 + K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y)) = i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_0, y_0)} + K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y))
\]

and by switching the roles of \((\tilde{x}_0, y_0)\) and \((\tilde{x}, y)\) we obtain

\[
|i_{\widetilde{\mathcal{M}}}^{(\tilde{x}, y)} - i_{\widetilde{\mathcal{M}}}^{(\tilde{x}_0, y_0)}| \leq K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y)).
\]

To prove that a given \( \tilde{z} \) has the desired property we set \( s := K D_{I^+\widetilde{\mathcal{M}}}(\tilde{x}_0, y_0, (\tilde{x}, y)) \) and observe

\[
d(\varphi(\tilde{z}), \xi(\tilde{x}, y, t_0 + s)) \leq d(\varphi(\tilde{z}), \xi(\tilde{z}, y_\tilde{z}, t_0 - s)) + d(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}, y, t_0 + s)) \leq t_0 - s + d(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}, y, t_0 + s))
\]

for a proper choice of \( y_\tilde{z} \in I^+\widetilde{\mathcal{M}} \). Hence, it suffices to show

\[
d(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}, y, t_0 + s)) < 2s \quad (3.1)
\]

for \( s > 0 \) sufficiently small. In the course of the proof we consider three different cases in each of which we prove (3.1) for a certain class of \((\tilde{z}, y_\tilde{z})\) \( I^+\widetilde{\mathcal{M}} \). This will be accomplished in Section 3.2, see Propositions (3.6) and (3.13) and in Section 3.3, see Proposition (3.17).

Before we proceed with the first case we recall \( F \in C^\infty(T\mathcal{M} \setminus 0) \cap C^0(T\mathcal{M}) \) from Definition 1.1. In what follows, this regularity turns out to be insufficient. However, regarding Lemma 1.43 and Lemma 1.44 we consider the squared distance function to overcome this lack of regularity. Thus, we don’t prove (3.1) directly but establish

\[
d^2(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}, y, t_0 + s)) < 4s^2
\]

which implies (3.1).
3 Local Lipschitz Continuity of $i_{\tilde{M}}$

3.2 Cases One and Two

In the present section we work in setting described above and assume that there exists $(\tilde{q}, \tilde{y}) \in I^1 \tilde{M}$, $(\tilde{x}, \tilde{y}) \neq (\tilde{q}, \tilde{y})$ such that $\xi(\tilde{x}, \tilde{y}, t_0) = \xi(\tilde{q}, \tilde{y}, t_0)$. We distinguish between the case in which $(\tilde{q}, \tilde{y})$ is close to $(\tilde{x}, \tilde{y})$ and the case in which $D_{I^1 \tilde{M}}((\tilde{x}, \tilde{y}), (\tilde{q}, \tilde{y}))$ is bounded from below and begin with the latter case.

Proposition 3.6. Let $(\mathcal{M}, F)$ be a connected, forward geodesically complete Finsler manifold and $(\tilde{\mathcal{M}}, \tilde{\varphi})$ a compact $C^{2,1}$ submanifold. Given $q_0 > 0$ and $(\tilde{x}, \tilde{y}) \in I^1 \tilde{M}$ we set $t_0 := i_{\tilde{M}}(\tilde{x}, \tilde{y})$ and assume that there exists $(\tilde{q}, \tilde{y}) \in I^1 \tilde{M}$ which satisfies $D((\tilde{\varphi}(\tilde{x}, \tilde{y}), (\tilde{\varphi}(\tilde{q}, \tilde{y}))) =: q \geq q_0$ and $\xi(\tilde{q}, \tilde{y}, t_0) = \xi(\tilde{x}, \tilde{y}, t_0)$, i.e. we have $d(\varphi(\tilde{q}), (\tilde{x}, \tilde{y}, t_0)) = 0$. Then there exist constants $\delta_0 > 0$ and $K \geq 1$ such that

$$d(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)) < 2s$$

for $D_{I^1 \tilde{M}}((\tilde{x}, \tilde{y}), (\tilde{x}, \tilde{y})) := \delta < \delta_0$ and $s := KD_{I^1 \tilde{M}}((\tilde{x}, \tilde{y}), (\tilde{x}, \tilde{y})) = K \delta$. Thus, in this case we choose $(\tilde{x}, \tilde{y}) = (\tilde{q}, \tilde{y})$.

Proof. Initially we set $q := \varphi(\tilde{q})$. We observe that Lemma 1.44 implies the existence of a positive constant $r = r(\mathcal{M}, F, \xi(\tilde{x}, \tilde{y}, t_0))$ such that the squared distance function satisfies $d^2 \in C^1(\mathcal{U}_r \times \mathcal{U}_r) \cap C^\infty (\mathcal{U}_r \times \mathcal{U}_r) \setminus \{(x, x); x \in \mathcal{U}_r\}$ where \( \mathcal{U}_r := \mathcal{B}^+(\xi(\tilde{x}, \tilde{y}, t_0), r) \cap \mathcal{B}^-(\xi(\tilde{x}, \tilde{y}, t_0), r) \).

From the continuous dependence of the geodesic $\xi$ on its initial values we derive $\xi(\tilde{x}, \tilde{y}, t_0 + s), \xi(\tilde{q}, \tilde{y}, t_0 - s) \in \mathcal{U}_r$ provided that $\delta_0$ is chosen sufficiently small and by Lemma 1.44

$$d^2(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)) = F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s))).$$

Thus, since $\xi$ depends even differentiable on its initial values we furthermore conclude that $d^2(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s))$ is differentiable in all its variables.

Roughly speaking, the remainder of the proof consists of two separate Taylor approximations which we perform in order to analyse the dependence of $d^2(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s))$ on $(\tilde{x}, \tilde{y})$ and $s$. Given that $\delta_0$ is chosen sufficiently small, there exists a minimising geodesic $C : [0, \delta] \to I^1 \tilde{M}$ with $C(0) = (\tilde{x}, \tilde{y}, t_0)$ and $C(\delta) = (\tilde{x}, \tilde{y})$. We compute

$$d^2(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)) = F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)))$$

$$= F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)))$$

$$+ \int_0^s \frac{d}{d\tau} (F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{C}(\tau), t_0 + s)))) d\tau$$

$$= F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s))) + O(\delta) \quad (3.2)$$

as $\delta \to 0$. Next, we analyse the dependence of $F^2(\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)))$ on $s$. Since $\xi(\tilde{q}, \tilde{y}, t_0) = \xi(\tilde{x}, \tilde{y}, t_0)$ we have $\text{EXP}^{-1}(\xi(\tilde{q}, \tilde{y}, t_0), \xi(\tilde{x}, \tilde{y}, t_0)) = (\xi(\tilde{q}, \tilde{y}, t_0), 0)$. We compute by using (v) in Proposition 1.41 for $1 \leq k \leq N$

$$(\text{EXP}^{-1})^k(\xi(\tilde{q}, \tilde{y}, t_0 - s), \xi(\tilde{x}, \tilde{y}, t_0 + s)) = \xi^k(\tilde{q}, \tilde{y}, t_0)$$

$$+ \int_0^s \frac{d}{d\tau} ((\text{EXP}^{-1})^k(\xi(\tilde{q}, \tilde{y}, t_0 - \tau), \xi(\tilde{x}, \tilde{y}, t_0 + \tau))) d\tau$$

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as $s \to 0$. Similarly, we obtain for $\theta \in (0, s)$

\[
(\text{EXP}^{-1})^{N+k}(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}, \bar{y}, t_0 + s))
= \int_0^s \frac{d}{d\tau} (\text{EXP}^{-1})^{N+k}(\xi(\bar{q}, \bar{y}, t_0 - \tau), \xi(\bar{x}, \bar{y}, t_0 + \tau)) d\tau
\]

Next, we obtain by virtue of (vi) in Proposition 1.41 and Lemma 1.44

\[
\left| \frac{\partial}{\partial \bar{z}_j} (\text{EXP}^{-1})^{N+k}(\xi(\bar{q}, \bar{y}, t_0 - \theta), \xi(\bar{x}, \bar{y}, t_0 + \theta)) - \delta_j^k \right|
\leq C F(\text{EXP}^{-1}(\xi(\bar{q}, \bar{y}, t_0 - \theta), \xi(\bar{x}, \bar{y}, t_0 + \theta)))
= C d(\xi(\bar{q}, \bar{y}, t_0 - \theta), \xi(\bar{x}, \bar{y}, t_0 + \theta))
\leq 2C \theta \leq Cs
\]
and similarly \[ \left| \frac{\partial}{\partial s^k} (\exp^{-1})^{N+k}(\xi(\tilde{q}, y_q, t_0 - \theta), \xi(\tilde{x}_0, y_0, t_0 + \theta)) + \delta^k \right| \leq C s. \] Thus,

\[
\begin{aligned}
(\exp^{-1})^{N+k}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \\
= \ s \left( \xi^k(\tilde{q}, y_q, t_0) + \xi^k(\tilde{x}_0, y_0, t_0) \right) + O(s^2)
\end{aligned}
\] (3.4)

for \(1 \leq k \leq N\) and consequently

\[
\lim_{s \to 0} \frac{1}{s} (\exp^{-1})^{N+k}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) = \xi^k(\tilde{q}, y_q, t_0) + \xi^k(\tilde{x}_0, y_0, t_0).
\]

We write \((\exp^{-1})^{N+k}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) = y^k(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))\) as an abbreviation, i.e. \(y = (y^1, \ldots, y^N) \in T_{(\tilde{q}, y_q, t_0 - s)} \mathcal{M}\), and compute

\[
\begin{aligned}
F^2(\xi(\tilde{q}, y_q, t_0 - s), y(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= \ s^2 F^2(\xi(\tilde{q}, y_q, t_0 - s), \frac{1}{s} y(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= \ s^2 \left( F^2(\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0) + O(s) \right) \\
= \ 4 s^2 F^2(\xi(\tilde{q}, y_q, t_0), \frac{1}{2} \xi(\tilde{x}_0, y_0, t_0) + O(s^2) \\
\leq \ 4 s^2 \left( 1 - C F^2(\xi(\tilde{x}_0, y_0, t_0), \frac{1}{2} \xi(\tilde{x}_0, y_0, t_0) \right) + O(s^2)
\end{aligned}
\]

where we applied Lemma A.1 in the last step. Finally, we intend to show that \(F^2(\xi(\tilde{x}_0, y_0, t_0), \frac{1}{2} \xi(\tilde{x}_0, y_0, t_0) - \xi(\tilde{q}, y_q, t_0))\) is bounded from below by a positive constant. For this purpose we recall that the backward curves \(\xi(\tilde{x}, y, t) := \xi(\tilde{x}, y, t_0 - t)\) are geodesics with respect to the Finsler manifold \((\mathcal{M}, F)\). Hence, we derive from the differentiability dependence of geodesics from the initial data

\[
\begin{aligned}
D \left( (\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0)) \right) \\
\leq \ C D \left( (\xi(\tilde{q}, y_q, 0), \xi(\tilde{x}_0, y_0, 0)), (\xi(\tilde{x}_0, y_0, 0), \xi(\tilde{x}_0, y_0, 0)) \right) \\
= \ C D \left( (\xi(\tilde{q}, y_q, t_0), -\xi(\tilde{q}, y_q, t_0)), (\xi(\tilde{x}_0, y_0, t_0), -\xi(\tilde{x}_0, y_0, t_0)) \right)
\end{aligned}
\] (3.5)

where \(C = C(\mathcal{M}, F, t_0)\). A suitable reference for ordinary differential equations in metric spaces is [Tab02] in which the author considers an even more general setting than the tangent bundle. Since \(\xi(\tilde{q}, y_q, t_0) = \xi(\tilde{x}_0, y_0, t_0)\) we further conclude

\[
\begin{aligned}
D \left( (\xi(\tilde{q}, y_q, t_0), -\xi(\tilde{q}, y_q, t_0)), (\xi(\tilde{x}_0, y_0, t_0), -\xi(\tilde{x}_0, y_0, t_0)) \right) \\
\leq \ F(\xi(\tilde{x}_0, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0) - \xi(\tilde{q}, y_q, t_0)).
\end{aligned}
\] (3.6)

Moreover, we have

\[
\begin{aligned}
D \left( (\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0)) \right) \\
= \ D \left( (\xi(\tilde{q}, y_q, 0), -\xi(\tilde{q}, y_q, 0)), (\xi(\tilde{x}_0, y_0, 0), -\xi(\tilde{x}_0, y_0, 0)) \right) \\
= \ D \left( (\varphi(\tilde{q}), -y_q), (\varphi(\tilde{x}_0), -y_0) \right).
\end{aligned}
\] (3.7)
3.2 Cases One and Two

We observe that \( D((\varphi(\bar{q}), -y_\bar{q}), (\varphi(\bar{x}_0), -y_0)) = 0 \) if and only if \((\varphi(\bar{x}_0), y_0) = (\varphi(\bar{q}), y_\bar{q})\) and thus, by compactness

\[
D((\varphi(\bar{q}), -y_\bar{q}), (\varphi(\bar{x}_0), -y_0)) \geq C(g_0)
\]

for any \( g_0 > 0 \) and any \((\varphi(\bar{q}), y_\bar{q})\) satisfying \( D((\varphi(\bar{q}), y_\bar{q}), (\varphi(\bar{x}_0), y_0)) \geq g_0 \).

Altogether, we deduce, \( F^2(\xi(\bar{x}_0, y_0, t_0), \frac{1}{2}(\xi(\bar{x}_0, y_0, t_0) - \xi(\bar{q}, y_\bar{q}, t_0)) \geq C g_0 \). Finally, we obtain by combing the previous estimates with (3.2)

\[
d(\xi(\bar{q}, y_\bar{q}, t_0 - s), \xi(\bar{x}, y, t_0 + s)) \leq 2s \left( 1 - C + O\left( \frac{1}{K} + s \right) \right) < 2s
\]

after first choosing \( K \) larger and then \( \delta_0 \) smaller if necessary. \( \square \)

Next, we provide some preparatory results which are needed for the remaining cases. The following lemma agrees with [LN05, Lemma 2.1]. Moreover, the claim of the subsequent corollary can be found in [LN05, Equation 2.14]. For the readers convenience we repeat the proofs here.

**Lemma 3.7.** Let \((\mathcal{M}, F)\) be a Finsler manifold and \( c : (a, b) \times [c, d] \to \mathcal{M} \) be a map with \( c(\cdot, t) \in \mathcal{C}^1([a, b], \mathcal{M}) \) for \( t \in [c, d] \) and \( c(\tau, \cdot) \) being a unit speed geodesic for each fixed \( \tau \in (a, b) \). Then

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (c(\tau, t), \frac{\partial}{\partial t} c(\tau, t)) \frac{\partial}{\partial \tau} c(\tau, t)^i \right) = 0
\]

for \( (\tau, t) \in (a, b) \times [c, d] \).

**Proof.** By differentiating \( \frac{1}{2} F^2(c(\tau, t), \frac{\partial}{\partial \tau} c(\tau, t)) = \frac{1}{2} \) with respect to \( \tau \) we obtain

\[
\frac{\partial}{\partial x^i} \left( \frac{1}{2} F^2 \right) (c(\tau, t), \frac{\partial}{\partial t} c(\tau, t)) \frac{\partial}{\partial \tau} c^i(\tau, t)
\]

\[
+ \frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (c(\tau, t), \frac{\partial}{\partial t} c(\tau, t)) \frac{\partial^2}{\partial \tau \partial t} c^i(\tau, t) = 0.
\]

By virtue of Lemma 1.16 we derive

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (c(\tau, t), \frac{\partial}{\partial t} c(\tau, t)) \right) \frac{\partial}{\partial \tau} c^i(\tau, t)
\]

\[
+ \frac{\partial}{\partial y^i} \left( \frac{1}{2} F^2 \right) (c(\tau, t), \frac{\partial}{\partial t} c(\tau, t)) \frac{\partial^2}{\partial \tau \partial t} c^i(\tau, t) = 0
\]

which yields the desired assertion. \( \square \)

**Corollary 3.8.** Let \((\mathcal{M}, F)\) be a Finsler manifold and \((\tilde{\mathcal{M}}, \varphi)\) be a \( \mathcal{C}^1 \) submanifold. Given a curve \( C : [a, b] \to I^1 \mathcal{M} \) we have

\[
\frac{\partial}{\partial y^i} (F^2) (\xi(C(\tau), t), \dot{\xi}(C(\tau), t)) \frac{\partial}{\partial \tau} \xi^i(C(\tau), t) = 0
\]

for \( 0 \leq t \leq r \) and \( \tau \in [a, b] \).
Proof. Since $C(\tau) \in \widetilde{I}M$ we have for $\tau \in [a,b]$
\[0 = g_{(C(\tau),0)}(\dot{\xi}(C(\tau),0), \frac{\partial}{\partial \tau} \xi(C(\tau),0))\]
\[= \frac{\partial}{\partial y^i} \left( \frac{1}{2} f^2 \right) (\xi(C(\tau),0), \dot{\xi}(C(\tau),0)) \frac{\partial}{\partial \tau} \xi^i(C(\tau),0).\]
Finally, the claim follows by virtue of Lemma 3.7. \qed

We recall our explanation of the idea of the proof at the beginning of the present chapter. In the second case, we will find \(\tilde{\varepsilon} \in M\) by considering an admissible perturbation of \((\tilde{x}_0, y_0) \in \widetilde{I}M\). In the following lemma we provide a precise description of the class of admissible perturbations and prove that this class is nonempty.

Lemma 3.9. Let \((M, F)\) be a connected, forward geodesically complete Finsler manifold, \((\tilde{M}, \varphi)\) be a compact \(C^{2,1}\) submanifold and \((\tilde{x}_0, y_0) \in \widetilde{I}M\) be fixed. Then there exists \(\varrho_0 > 0\) and an open neighbourhood \(U \subset \widetilde{I}M\) of \((\tilde{x}_0, y_0)\) such that for any given \((\tilde{q}, y_\delta) \in \widetilde{I}M\) with \(D((\tilde{x}_0, y_0), (\tilde{q}, y_\delta)) =: \varrho < \varrho_0\) there exists a map \(Z \in C^\infty(U, \widetilde{I}M)\) with \(Z(\tilde{x}_0, y_0) = (\tilde{q}, y_\delta)\). Moreover, in local coordinates, we have for \(1 \leq \alpha, \beta \leq n\) and \(1 \leq i, k \leq N\)
\[\frac{\partial}{\partial z^\alpha} Z^\beta(\tilde{x}_0, y_0) = \delta^\beta_\alpha + O(\varrho),\]
\[\frac{\partial}{\partial y^k} Z^\beta(\tilde{x}_0, y_0) = \delta^\beta_k + O(\varrho),\]
\[\frac{\partial}{\partial z^\alpha} Z^{n+i}(\tilde{x}_0, y_0) = \delta^i_\alpha + O(\varrho),\]
\[\frac{\partial}{\partial y^k} Z^{n+i}(\tilde{x}_0, y_0) = \delta^i_k + O(\varrho).\]
In what follows, a map \(Z\) with the aforementioned properties is called an admissible perturbation.

Proof. As we have pointed out at the beginning of the present chapter, we regard \(\widetilde{I}M\) as a Riemannian manifold where the metric \(G_{\widetilde{I}M}\) is the restriction of the Sasaki metric of \(TM\) to \(\widetilde{I}M\). We denote the exponential map of the Riemannian manifold \((\widetilde{I}M, G_{\widetilde{I}M})\) by \(\exp_{\widetilde{I}M}\) and define the map \(\exp_{\widetilde{I}M}\) as in Proposition 1.41. From (v) in the same proposition we obtain the existence of an open neighbourhood \(V \subset T(\widetilde{I}M)\) of \(((\tilde{x}_0, y_0), 0) \in T(\widetilde{I}M)\) such that \(\exp_{\widetilde{I}M}\) is a diffeomorphism from \(V\) onto its image \(\exp_{\widetilde{I}M}(V) \subset \widetilde{I}M \times \widetilde{I}M\).

Given that \(\varrho_0\) is chosen sufficiently small, we have \(((\tilde{x}_0, y_0), (\tilde{q}, y_\delta)) \in \exp_{\widetilde{I}M}(V)\). We choose an open neighbourhood \(U \subset \widetilde{I}M\) such that for each \((\tilde{x}, y) \in U\) we have \(((\tilde{x}_0, y_0), (\tilde{x}, y)) \in \exp(V)\). We write
\[\exp_{\widetilde{I}M}^{-1}((\tilde{x}_1, y_1), (\tilde{x}_2, y_2)) = ((\tilde{x}_1, y_1), Y((\tilde{x}_1, y_1), (\tilde{x}_2, y_2)))\]
where \(Y((\tilde{x}_1, y_1), (\tilde{x}_2, y_2)) \in T((\tilde{x}_1, y_1)\widetilde{I}M\) and define
\[Z(\tilde{x}, y) := \exp_{\widetilde{I}M}((\tilde{x}_0, y_0), Y((\tilde{x}_0, y_0), (\tilde{q}, y_\delta)) + Y((\tilde{x}_0, y_0), (\tilde{x}, y)))\]
for \((\bar{x}, y) \in U\). We easily observe \(Z(\bar{x}_0, y_0) = (\bar{q}, y_\bar{q})\). Finally, the statement on the derivative of the map \(Z\) follows from (iii) in Proposition 1.22 and (v) and (vi) in Proposition 1.41 applied to \(\exp_{\bar{t}, \bar{M}}\) and \(\exp_{\bar{t}, \bar{M}}\) respectively.

\textbf{Lemma 3.10.} Let \((\mathcal{M}, F)\) be a connected, forward geodesically complete Finsler manifold, \((\bar{M}, \bar{\varphi})\) be a compact \(C^{2,1}\) submanifold and \((\bar{x}_0, y_0) \in I^{\perp}_{\bar{M}}\). Then there exists a constant \(\varrho_0 > 0\) such that for \(t_0 > 0\) and \((\bar{q}, y_{\bar{q}}) \in I^{\perp}_{\bar{M}}\) with \(D((\bar{x}_0, y_0), (\bar{q}, y_{\bar{q}})) =: \varrho < \varrho_0\) there exists a constant \(K = K(\mathcal{M}, F, \bar{M}, \bar{\varphi}, t_0)\) such that for any smooth curve \(C : (-\epsilon, \epsilon) \to U\) with \(C(0) = (\bar{x}_0, y_0)\) and any admissible perturbation \(Z\) as defined in Lemma 3.9 we have

\[
\left| \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\hat{q}, y_{\hat{q}}, t_0), \hat{\xi}(\hat{q}, y_{\hat{q}}, t_0) \right) \left( \frac{\partial}{\partial \tau^\xi} \xi(C(\tau), t_0) \right) \right|_{\tau=0} - \left. \frac{\partial}{\partial \tau} \xi(Z(C(\tau), t_0) \right|_{\tau=0} \right| \leq K \left( \left| y(\xi(\bar{q}, y_{\bar{q}}, t_0), \xi(\bar{x}_0, y_0, t_0)) \right| + \varrho^2 \right)
\]

where in local coordinates \(y(\xi(\hat{q}, y_{\hat{q}}, t_0), \xi(\bar{x}_0, y_0, t_0)) \in T_{\xi(\bar{q}, y_{\bar{q}}, t_0), \xi(\bar{x}_0, y_0, t_0)} \mathcal{M}\) is given by \(y^i = (\exp^{-1})^{N+i}(\xi(\bar{q}, y_{\bar{q}}, t_0), \xi(\bar{x}_0, y_0, t_0))\) where \(1 \leq i \leq N\).

\textbf{Proof.} Initially, we set \(\imath_0(\tau) := D(C(\tau), Z(C(\tau)))\) and consider a smooth family of unit speed curves \(\Xi(\tau, \cdot) : [0, \imath_0(\tau)] \to I^{\perp}_{\bar{M}}\) with \(\Xi(\tau, 0) = C(\tau)\) and \(\Xi(\tau, \imath_0(\tau)) = Z(C(\tau))\), i.e. \(\Xi(\tau, \cdot)\) joins \(C(\tau)\) with \(Z(C(\tau))\). We infer

\[
\frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(\tau, \imath), t_0), \hat{\xi}(\Xi(\tau, \imath), t_0) \right) \frac{\partial}{\partial \imath} \xi^i(\Xi(\tau, \imath), t_0) = 0
\]

for \(\tau \in (-\epsilon, \epsilon)\) and \(\imath \in [0, \imath_0(\tau)]\) from Corollary 3.8 and hence

\[
\begin{align*}
\frac{\partial}{\partial \tau} \left( \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(\tau, \imath), t_0), \hat{\xi}(\Xi(\tau, \imath), t_0) \right) \frac{\partial}{\partial \imath} \xi^i(\Xi(\tau, \imath), t_0) \right) \Bigg|_{\tau=0} &+ \frac{\partial}{\partial \gamma^2} \left( F^2 \right) \left( \xi(\Xi(\tau, \imath), t_0), \hat{\xi}(\Xi(\tau, \imath), t_0) \right) \frac{\partial^2}{\partial \tau \partial \imath} \xi^i(\Xi(\tau, \imath), t_0) = 0.
\end{align*}
\]

We compute by virtue of the previous identity

\[
- \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(q, y_q, t_0), \hat{\xi}(q, y_Q, t_0) \right) \left( \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0) - \frac{\partial}{\partial \tau} \xi^i(Z(C(\tau), t_0) \right) \right)
\]

\[
= \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(0, \imath_0(0)), t_0), \hat{\xi}(\Xi(0, \imath_0(0)), t_0) \right) \int_{0}^{\imath_0(\tau)} \frac{\partial^2}{\partial \imath \partial \tau} \xi^i(\Xi(\tau, \imath), t_0) \, d\imath
\]

\[
= \int_{0}^{\imath_0(\tau)} \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(\tau, \imath), t_0), \hat{\xi}(\Xi(\tau, \imath), t_0) \right) \frac{\partial^2}{\partial \imath \partial \tau} \xi^i(\Xi(\tau, \imath), t_0) \, d\imath
\]

\[
+ \int_{0}^{\imath_0(\tau)} \left( \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(0, \imath_0(0)), t_0), \hat{\xi}(\Xi(0, \imath_0(0)), t_0) \right) \frac{\partial^2}{\partial \imath \partial \tau} \xi^i(\Xi(\tau, \imath), t_0) \, d\imath
\]

\[
= \int_{0}^{\imath_0(\tau)} \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \xi(\Xi(\tau, \imath), t_0), \hat{\xi}(\Xi(\tau, \imath), t_0) \right) \frac{\partial^2}{\partial \imath \partial \tau} \xi^i(\Xi(\tau, \imath), t_0) \, d\imath
\]

\[
+ O(\tau \imath_0(\tau) + \imath_0(\tau)^2)
\]
Local Lipschitz Continuity of $\varrho$

\[ \frac{\partial}{\partial \varrho} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} = - \frac{\partial}{\partial \varrho} \left( Z(C, t_0) \right)_{\tau=0} \]

as $\tau, \varrho \to 0$. Since $\varrho = u_0(0)$ we obtain at $\tau = 0$

\[ \frac{\partial}{\partial \varrho} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} = - \frac{\partial}{\partial \varrho} \left( Z(C, t_0) \right)_{\tau=0} \]

\[ = \int_0^\varepsilon \frac{\partial}{\partial \tau} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} \]

\[ + \int_0^\varepsilon \frac{\partial}{\partial \tau} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} \]

\[ = \frac{\partial}{\partial \tau} \frac{\partial}{\partial \varrho} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} \]

\[ + \int_0^\varepsilon \frac{\partial}{\partial \tau} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} \]

\[ = \left( \exp^{-1} \right)^{N+i} (\xi(0, q), \xi(0, q)) \]

\[ = \int_0^\varepsilon \frac{\partial}{\partial \tau} \left( \exp^{-1} \right)^{N+i} (\xi(0, q), \xi(0, q)) \]

as $\varrho \to 0$. Consequently,

\[ \frac{\partial}{\partial \varrho} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} = - \frac{\partial}{\partial \varrho} \left( Z(C, t_0) \right)_{\tau=0} \]

\[ = \frac{\partial}{\partial \tau} \frac{\partial}{\partial \varrho} \left( F^2 \right) (\xi(q, q, t_0), \tilde{\xi}(q, q, t_0)) \left( \frac{\partial}{\partial \tau} \xi^i(C, t_0) \right)_{\tau=0} \]

\[ = \left( \exp^{-1} \right)^{N+i} (\xi(q, q, t_0), \xi(\tilde{x}_0, y_0, t_0)) + O(\varrho^2) \]

as $\varrho \to 0$ which yields the claim. \( \square \)

The previous lemma is an adapted version of [LN05, Lemma 5.1] whilst the subsequent lemma generalises [LN05, Proposition 6.1].

**Lemma 3.11.** Let $(\mathcal{M}, F)$ be a connected, forward geodesically complete Finsler manifold and $(\tilde{\mathcal{M}}, \varphi)$ a compact $C^2$ submanifold. Let $(\bar{x}_0, y_0) \in I^+(\tilde{\mathcal{M}})$ be fixed and assume $t_0 := i\bar{x}(\bar{x}_0, y_0) < \infty$. Then there exist constants $\rho_0 \geq 0$, $s_0 \geq 0$ and an open neighbourhood $U \subset I^+(\tilde{\mathcal{M}})$ of $(\bar{x}_0, y_0)$ such that for any $(\bar{q}, \bar{y}) \in I^+(\tilde{\mathcal{M}})$ satisfying
$D((\varphi(\tilde{x}_0), y_0), (\varphi(\tilde{q}), y_0)) = \varrho < \varrho_0$ and any admissible perturbation $Z$ as defined in Lemma 3.9 we have for $(\tilde{x}, y) \in U$, $\delta := D_{I^+}\tilde{M}((\tilde{x}_0, y_0), (\tilde{x}, y))$ and $0 < s < s_0$ the following expansion

$$\left| d^2(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}, y, t_0 + s)) - d^2(\xi(Z(\tilde{x}_0, y_0), t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \right| \leq C(M, F, \tilde{M}, \varphi, t_0) \left( \delta(s\sigma + \varrho^2\sigma + s^2\varrho + s\varrho^2 + s^3) + \delta^2(s^2 + \varrho^2 + \sigma^2) \right.$$ 

$$\left. + \delta^3(s + \varrho + \sigma) + \delta^4 \right).$$

where $\sigma := |y(\xi(\tilde{q}, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0))|$ and $y(\xi(\tilde{q}, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0)) \in T_{\xi(\tilde{q}, y_0, t_0)}M$ is given by $y^i = (\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_0, t_0), \xi(\tilde{x}_0, y_0, t_0))$ where $1 \leq i \leq N$.

**Proof.** We remark that we have already proven the existence of an admissible perturbation $Z$ in Lemma 3.9. Let $\varrho_0$ be the constant and $U \subset I^+\tilde{M}$ be the neighbourhood of $(\tilde{x}_0, y_0)$ from this lemma.

Lemma 1.44 yields the existence of a positive constant $r = r(M, F, \xi(x_0, y_0, t_0))$ such that $d^2 \in C^1(\mathcal{U}_r \times \mathcal{U}_r)$ where $\mathcal{U}_r := B^+(\xi(x_0, y_0, t_0), r) \cap \mathcal{B}(-\xi(x_0, y_0, t_0), r)$.

We recall that geodesics depend differentiable on their initial values. Hence, by choosing $\varrho_0$ smaller if necessary we can ensure $\xi(\tilde{q}, y_0, t_0) \in \mathcal{U}_r$. Furthermore, by shrinking the neighbourhood $U$ if necessary we obtain $\xi(\tilde{x}, y, t_0), \xi(Z(\tilde{x}, y, t_0)) \in \mathcal{U}_r$ for $(\tilde{x}, y) \in U$. Consequently, the map

$$L : \tilde{M} \times \tilde{M} \times [0, s_0] \to [0, \infty)$$

defined by $L((\tilde{x}, y), (\tilde{q}, y_0), s) := d^2(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}, y, t_0 + s))$ satisfies $L \in C^1(I^+\tilde{M} \times I^+\tilde{M} \times [0, s_0]).$ In view of Lemma 1.44 we can write

$$L((\tilde{x}, y), (\tilde{q}, y_0), s) = F^2(\text{EXP}^{-1}(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}, y, t_0 + s))).$$

We consider a minimising geodesic $C : [0, \delta] \to I^+\tilde{M}$ with $C(0) = (\tilde{x}_0, y_0)$ and $C(\delta) = (\tilde{x}, y)$. Therewith we compute

$$L((\tilde{x}, y), (\tilde{q}, y_0), s) = L((\tilde{x}_0, y_0), (\tilde{q}, y_0), s) + \int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_0), s) d\tau$$

$$= L((x_0, y_0), (\tilde{q}, y_0), s) + \left. \delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_0), s) \right|_{\tau = 0}$$

$$+ \int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_0), s) - \left. \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_0), s) \right|_{\tau = 0} d\tau$$

Next, we set $Z(\tau) := Z(C(\tau)) \in I^+\tilde{M}$ as an abbreviation and compute

$$\frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_0), s) = \frac{\partial}{\partial \tau^2} \left( F^2 \right) \left( \text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) \right)$$

$$\left( \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^j(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) \right.$$

$$\left. + \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^j(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \frac{\partial}{\partial \tau} \xi^j(C(\tau), t_0 + s) \right)$$

$$+ \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) \right)$$

$$+ \frac{\partial}{\partial y^i} \left( F^2 \right) \left( \text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) \right)$$

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\[
\begin{align*}
\left( \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+j} (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) & \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \\
+ \frac{\partial}{\partial z^k} (\text{EXP}^{-1})^{N+j} (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) & \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \right).
\end{align*}
\]

Hence, at $\tau = 0$

\[
\frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) \bigg|_{\tau=0} = \frac{\partial}{\partial x^j} (F^2) (\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)))
\]

(3.8)

\[
\begin{align*}
\left( \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^N (\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) & \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \\
+ \frac{\partial}{\partial z^k} (\text{EXP}^{-1})^N (\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) & \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \right)
\end{align*}
\]

We analyse the dependence of each term on $s$ and $\varrho$ separately. Initially, we compute by virtue of (vi) in Proposition 1.41

\[
\begin{align*}
(\text{EXP}^{-1})^{N+k} (\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \\
= (\text{EXP}^{-1})^{N+k} (\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0)) \\
+ \int_0^s \frac{\partial}{\partial \tau} (\text{EXP}^{-1})^{N+k} (\xi(\tilde{q}, y_q, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) d\tau
\end{align*}
\]

(3.9)

We conclude that

\[
\frac{\partial}{\partial x^j} (F^2) (\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= 4s^2 \frac{\partial}{\partial x^j} (F^2) (\xi(\tilde{q}, y_q, t_0 - s), \frac{1}{2s} y(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)))
\]

and

\[
\frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= 2s \frac{\partial}{\partial y^j} (F^2) (\xi(\tilde{q}, y_q, t_0 - s), \frac{1}{2s} y(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)))
\]
as \( \sigma, \varrho, s \to 0 \). Next, (iv) in Proposition 1.41 yields
\[
\frac{\partial}{\partial z^k}(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \bigg|_{\tau=0} = \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) \bigg|_{\tau=0}
\]
and
\[
\frac{\partial}{\partial z^k}(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \bigg|_{\tau=0} = 0.
\]

The most elaborate part of the proof is to find a suitable estimate for the remaining terms of \( \frac{\partial}{\partial \tau} L(C(\tau), (\bar{q}, \bar{y}, s)) \big|_{\tau=0} \) in (3.8) which we abbreviate as follows:
\[
(I) := \frac{\partial}{\partial z^k}(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \bigg|_{\tau=0} = \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0}.
\]

Once again we compute by (iv) in Proposition 1.41:
\[
(I) = E_{jl}(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \left( \delta^l_k \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0} \right.
\]
\[
- \frac{\partial}{\partial \tau} \exp^l(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \bigg|_{\tau=0} \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0}
\]
\[
- \delta^j_k \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0}
\]
\[
+ \left( E_{jl}(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) - \delta^j_l \right) \left( \delta^l_k \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0} \right.
\]
\[
- \frac{\partial}{\partial \tau} \exp^l(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \bigg|_{\tau=0} \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \right).
\]

The last part of the previous equation has the desired order, which we observe as follows. We obtain
\[
\left| E_{jl}(\xi(q, y, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) - \delta^j_l \right| \leq C(M, F, \varphi, \bar{x}_0, y_0, t_0) F(\text{EXP}^{-1})(\xi(\bar{q}, \bar{y}, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \leq C(M, F, \varphi, \bar{x}_0, y_0, t_0) \ (s + \varrho)
\]
as \( \varrho, s \to 0 \) by virtue of (vi) in Proposition 1.41 and Lemma 1.44, and similarly
\[
\left| \delta^l_k \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0} \right|
\]
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\[-\frac{\partial}{\partial x^k} \exp^j (\exp^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \]

\[\leq \delta_k^i \left| \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0} - \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0) \bigg|_{\tau=0} \right. \]

\[+ \delta_k^i \left| \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0) \bigg|_{\tau=0} - \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0) \bigg|_{\tau=0} \right. \]

\[+ \delta_k^i \left| \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0) \bigg|_{\tau=0} - \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \right. \]

\[-\frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \]

\[\leq C(\mathcal{M}, F, \varphi, \tilde{x}_0, y_0, t_0) (s + \varrho) \]

as $\varrho, s \to 0$ where we additionally made use of Lemma 3.9. Consequently,

\[(I) = \delta_k^i \frac{\partial}{\partial \tau} \xi^k(C(\tau), t_0 + s) \bigg|_{\tau=0} \]

\[-\frac{\partial}{\partial \tau} \exp^j (\exp^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \]

\[+ O((s + \varrho)^2) \]

as $\varrho, s \to 0$. Next, we observe

\[2s \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) = (\exp^{-1})^{N+j}(\xi(Z(\tau), t_0 - s), \xi(Z(\tau), t_0 + s)) \]

from which we derive as before

\[2s \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) \bigg|_{\tau=0} = \frac{\partial}{\partial \tau} (\exp^{-1})^{N+j}(\xi(Z(\tau), t_0 - s), \xi(Z(\tau), t_0 + s)) \bigg|_{\tau=0} \]

\[= E_{ij}(\xi(Z(0), t_0 - s), \xi(Z(0), t_0 + s)) \left( \delta_k^i \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 + s) \bigg|_{\tau=0} \right. \]

\[-\frac{\partial}{\partial x^k} \exp^j (\exp^{-1}(\xi(Z(0), t_0 - s), \xi(Z(0), t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0,} \left. \right) \]

\[= \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 + s) \bigg|_{\tau=0} \]

\[-\frac{\partial}{\partial x^k} \exp^j (\exp^{-1}(\xi(Z(0), t_0 - s), \xi(Z(0), t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \]

\[+ O(s^2) \]

as $s \to 0$. Hence,

\[(I) = \frac{\partial}{\partial \tau} \xi^j(C(\tau), t_0 + s) \bigg|_{\tau=0} \left. - \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 + s) \bigg|_{\tau=0} \right. \]

\[+ 2s \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) \bigg|_{\tau=0} \]
\[ \frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{q}, y_q, t_0 + s))) \\
\frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) \bigg|_{\tau=0} \\
+ O((s + \varrho)^2) \]

as \( \varrho, s \to 0 \). We set

\[ y^i(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) := \frac{1}{2s} (\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \]

for \( 1 \leq i \leq N \) and obtain a vector \( y = y^i(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \in T_{(\xi(\tilde{q}, y_q, t_0 - s))} \mathcal{M} \)

via the formula \( y = y^i \frac{\partial}{\partial \tau} \). Now we compute similarly to the proof of (iii) in Proposition 1.41 and get

\[ \frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= \frac{\partial}{\partial x^k} c^j(\xi(\tilde{q}, y_q, t_0 - s), y, 2s) = \delta^j_k + \frac{\partial^2}{\partial t \partial x^k} c^j(\xi(\tilde{q}, y_q, t_0 - s), y, 0) 2s + O(s^2) \]

and

\[ \frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) \\
= \frac{\partial}{\partial x^k} c^j(\xi(\tilde{q}, y_q, t_0 - s), \tilde{\xi}(\tilde{q}, y_q, t_0 - s), 2s) \\
= \delta^j_k + \frac{\partial^2}{\partial t \partial x^k} c^j(\xi(\tilde{q}, y_q, t_0 - s), \tilde{\xi}(\tilde{q}, y_q, t_0 - s), 0) 2s + O(s^2) \]

as \( s \to 0 \). Since Proposition 1.41 and Lemma 1.44 yield

\[ y^i(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) = \frac{1}{2s} (\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0)) \\
+ \frac{1}{2s} \int_0^s \frac{\partial}{\partial u}(\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0 - u), \xi(\tilde{x}_0, y_0, t_0 + u)) \, du \\
= \frac{1}{2s} (\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0), \xi(\tilde{x}_0, y_0, t_0)) \\
+ \frac{1}{2s} \int_0^s \xi^i(\tilde{q}, y_q, t_0 - u) + \tilde{\xi}^i(\tilde{x}_0, y_0, t_0 + u) \, du \\
+ \frac{1}{2s} \int_0^s \left( \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0 - u), \xi(\tilde{x}_0, y_0, t_0 + u)) - \delta^j_k \right) \xi^k(\tilde{x}_0, y_0, t_0 + u) \\
- \left( \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}, y_q, t_0 - u), \xi(\tilde{x}_0, y_0, t_0 + u)) - \delta^j_k \right) \xi^k(\tilde{q}, y_q, t_0 - u) \, du \\
= \xi^i(\tilde{q}, y_q, t_0) + O(s + \varrho + \frac{\sigma}{s}) \]

as \( \varrho, s \to 0 \), we conclude

\[ \frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{q}, y_q, t_0 + s))) \]
By inserting (3.9) - (3.13) in the formula (3.8) for \( s, \varrho, \sigma \) we observe

\[
\frac{\partial}{\partial x^k} \exp^j(\text{EXP}^{-1}(\xi(\tilde{q}, y_q, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))) = O(s^g + s^2 + \sigma)
\]
as \( \varrho, s \to 0 \). Thus,

\[
(I) = \frac{\partial}{\partial \tau} \xi^j(C(\tau), t_0 + s) \bigg|_{\tau=0} - \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 + s) \bigg|_{\tau=0} + 2s \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0 - s) \bigg|_{\tau=0} + O(s^2 + \varrho^2 + \sigma)
\]

where we made use of Lemma 3.10 to obtain the last equation. Finally, we observe

\[
\frac{\partial}{\partial x^k}(F^2)(F(\xi(\tilde{q}, y_q, t_0), \xi(\tilde{q}, y_q, t_0))) \frac{\partial}{\partial \tau} \xi^j(Z(\tau), t_0) = 0
\]

by differentiating \( F^2(\xi(Z(\tau), t_0), \xi(Z(\tau), t_0)) = 1 \) by \( \tau \) and hence

\[
\frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) \big|_{\tau=0} = O(s \varrho + \varrho \sigma + s^2 \varrho + s g^2 + s^3)
\]
as \( s, \varrho, \sigma \to 0 \). It remains to analyse the integral

\[
\int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) - \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) \big|_{\tau=0} \, d\tau
\]

(3.14)
which will be accomplished in the remainder of the proof. Since

\[
\frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_\tilde{q}), s)
= \frac{\partial}{\partial x^j} (F^2) (\text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))
- \frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))
\]

we write

\[
\int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_\tilde{q}), s) - \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_\tilde{q}), s) \bigg|_{\tau=0} d\tau
= \int_0^\delta \left( \frac{\partial}{\partial x^j} (F^2) (\text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))
- \frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))
\right) d\tau
\]

As an abbreviation we set

\[
\int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_\tilde{q}), s) - \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_\tilde{q}), s) \bigg|_{\tau=0} d\tau
=: \int_0^\delta (II) + (III) + (IV) + (V) d\tau
\]
and consider each term separately. We compute
\[
\left| \frac{\partial}{\partial x^j} (F^2) (\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right|
\]
\[
- \frac{\partial}{\partial x^j} (F^2) (\exp^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))
\]
\[
= \left| \int_0^\tau \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x^j} (F^2) (\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right) \, dt \right|
\]
\[
= \int_0^\tau \frac{\partial^2}{\partial x^j \partial x^k} (F^2) (\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))
\]
\[
+ \frac{\partial}{\partial x^j} \left( \exp^{-1} \left( \xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s) \right) \right) \frac{\partial}{\partial t} \xi^i (\xi(Z(\tau), t_0 - s) - \xi^i (\xi(Z(\tau), t_0 + s)))
\]
\[
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \left( (\delta + s + \sigma) \right) \left( (\delta + s + \sigma + \varphi) \right)
\]
and compute by (\textit{vi}) in Proposition 1.41, the positive homogeneity of \( F \) and (3.15)
\[
\left| \frac{\partial}{\partial t} (\exp^{-1})^{N+k} (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right|
\]
\[
= \left| \left( \frac{\partial}{\partial x^j} (\exp^{-1})^{N+k} (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right) + \delta^k \right| \frac{\partial}{\partial t} \xi^i (\xi(Z(\tau), t_0 - s)
\]
\[
+ \frac{\partial}{\partial t} \xi^i (\xi(Z(\tau), t_0 + s)) - \frac{\partial}{\partial t} \xi^k (\xi(Z(\tau), t_0 - s))
\]
\[
= C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \left( (\delta + s + \sigma + \varphi) \right)
\]
\[
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \left( (\delta + s + \sigma + \varphi) \right)
\]
In combination with the positive homogeneity of \( \frac{\partial^2}{\partial x^j \partial x^k} (F^2) \) and \( \frac{\partial^2}{\partial x^j \partial y^k} (F^2) \) as well as (3.15) and Lemma 1.43 we conclude
\[
|\text{II}| \leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \left( (\delta + s + \sigma)^2 + (\delta + s + \sigma)(\delta + s + \sigma + \varphi) \right)
\]
\[
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \left( (\delta + s + \sigma + \varphi) \right)
\]
as \( s, \sigma, \delta \to 0 \). Next, we assume for now that \( \xi(Z(\tau), t_0 - s) \neq \xi(C(\tau), t_0 + s) \) for \( 0 \leq \tau < \delta \), i.e. \( \exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) \in TM \setminus \emptyset \) for \( 0 \leq \tau < \delta \), and compute
\[
\left| \frac{\partial}{\partial y^j} (F^2) (\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right|
\]
where we made use of Lemma 1.43, the fact that $\frac{\partial^2}{\partial y^i \partial \xi^k} (F^2)$ is uniformly bounded on $T \tilde{M} \setminus 0$ as well as (3.15) and (3.16). In case $\text{EXP}^{-1}(\xi(Z(\tau_0), t_0 - s), \xi(C(\tau_0), t_0 + s))$ is an element of the zero section for some $\tau_0 \in (0, \delta)$ we get

$$\frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(Z(\tau_0), t_0 - s), \xi(C(\tau_0), t_0 + s))) = 0$$

from Lemma 1.43 and hence there exists $\epsilon > 0$ such that for $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$

$$\left| \frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \right| \leq \delta^2.$$ 

The existence of a suitable $\epsilon > 0$ is assured by Lemma 1.43. Therewith we derive (3.18) without the previous constraint. Since the term considered before equals the first part in (III) and the second factor in (III) term agrees with the term considered in (3.16) we derive from (3.16) and (3.18)

$$|\text{III}| \leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \delta (\delta + s + \sigma + \varrho)^2 \leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (\delta (s^2 + \sigma^2 + \varrho^2) + \delta^3).$$

The first part of (IV) agrees with the derivative considered in (3.9). Moreover, we observe

$$\frac{\partial}{\partial \tau} (\text{EXP}^{-1})^j (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))$$

$$- \frac{\partial}{\partial \tau} (\text{EXP}^{-1})^j (\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))$$

$$= \frac{\partial}{\partial \tau} \xi^j (Z(\tau), t_0 - s) - \frac{\partial}{\partial \tau} \xi^j (Z(\tau), t_0 - s)$$

and hence

$$|\text{IV}| \leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) \delta (s \sigma + s^2 \varrho + s^2)$$

Finally, we infer from (3.10)

$$\left| \frac{\partial}{\partial y^j} (F^2) (\text{EXP}^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))) \right|$$
Moreover, we compute by using (iv) in Proposition 1.41

\[
\frac{\partial}{\partial \tau} \left((\exp^{-1})^{N+i}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))\right) \\
- \frac{\partial}{\partial \tau} \left((\exp^{-1})^{N+i}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))\right)_{\tau=0}
\]

\[= \left(E_{ji}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) - E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))\right)
\]

\[
\frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s)
\]

\[+ E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))
\]

\[
\left(\frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) - \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s)\right)_{\tau=0}
\]

\[- \left(E_{ji}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) - E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))\right)
\]

\[
\frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s)
\]

\[- E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))
\]

\[
\left(\frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s))) \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s)\right)_{\tau=0}
\]

\[= \left(E_{ji}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) - E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))\right)
\]

\[
\left(\frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) - \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) + \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s)\right)
\]

\[
\left(\delta_k^i - \frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))\right)
\]

\[+ E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))
\]

\[
\left(\frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) - \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s)\right)_{\tau=0}
\]

\[- \left(E_{ji}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) - E_{ji}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))\right)
\]

\[
\left(\delta_k^i - \frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))\right)
\]

\[
\left(\frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s) - \frac{\partial}{\partial \tau} \xi^k(Z(\tau), t_0 - s)\right)_{\tau=0}
\]

\[+ \left(\frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))\right)
\]

\[- \left(\frac{\partial}{\partial t^k} \exp^i(\exp^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))\right)
\]

\[\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (s + m + \sigma).
\]
We have

\[
\left| E_{ji}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)) - \delta_i^j \right|
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) F(\text{EXP}^{-1}(\xi(Z(\tau), t_0 - s), \xi(C(\tau), t_0 + s)))
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (\delta + s + \varrho)
\]

by (vi) in Proposition 1.41,

\[
\left| \frac{\partial}{\partial x^k} \exp^i(\text{EXP}^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s))) - \delta_i^j \right|
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) F(\text{EXP}^{-1}(\xi(Z(0), t_0 - s), \xi(C(0), t_0 + s)))
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (s + \varrho)
\]

by (iii) in Proposition 1.22 and

\[
\left| \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) - \frac{\partial}{\partial \tau} \xi^i(C(\tau), t_0 + s) \right|_{\tau=0}
- \left( \frac{\partial}{\partial \tau} \xi^i(Z(\tau), t_0 - s) - \frac{\partial}{\partial \tau} \xi^i(Z(\tau), t_0 - s) \right)_{\tau=0}
\leq \tau \left| \frac{\partial^2}{\partial \tau^2} \xi^i(C(\tau), t_0 + s) \right|_{\tau=0} - \left| \frac{\partial^2}{\partial \tau^2} \xi^i(Z(\tau), t_0 - s) \right|_{\tau=0} + C(M, F) \tau^2
\]

\leq C(M, F) (\delta(s + \varrho) + \delta^2) .
\]

We deal with the remaining terms in the previous formula in a similar way and thus obtain

\[
|V| \leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (s + s\varrho + \sigma)

\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (s + \varrho + \sigma)
\]

(3.21)

We conclude the proof by inserting (3.17) - (3.21)

\[
\left| \int_0^\delta \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) - \frac{\partial}{\partial \tau} L(C(\tau), (\tilde{q}, y_q), s) \right|_{\tau=0} \text{d}\tau
\leq C(M, F, \varphi, \tilde{x}_0, y_0, t_0) (\delta^2(s + \varrho + \sigma) + \delta^3(s + \varrho + \sigma) + \delta^4) .
\]

Herewith we completed all necessary preparations and proceed with the aforementioned second case. For later reference we initially establish the following lemma which contains the main part of the proof for this case. It corresponds to [LN05, Lemma 7.2].
Lemma 3.12. Let \((\mathcal{M}, F)\) be a connected, forward geodesically complete Finsler manifold and \((\mathcal{M}, \varphi)\) be a compact \(C^{2,1}\) submanifold. Let \((\tilde{x}_0, y_0) \in \mathcal{M}^1\) be fixed and assume \(t_0 := i_{\mathcal{M}}(\tilde{x}_0, y_0) < \infty\). Then there exist constants \(s_0\) and \(\varrho_0 > 0\) such that the existence of \((\tilde{q}_0, y_0) \in \mathcal{M}^1\) satisfying \(D((\varphi(\tilde{x}_0), y_0), (\varphi(\tilde{q}_0), y_0)) =: \varrho < \varrho_0, (\tilde{x}_0, y_0) \neq (\tilde{q}_0, y_0)\) and \(\tilde{\xi}(\tilde{q}_0, y_0, t_0) = \xi(\tilde{x}_0, y_0, t_0)\) yields

\[
\begin{align*}
\text{Lemma 3.12. } & \quad \text{Let } (\mathcal{M}, F) \text{ be a connected, forward geodesically complete Finsler manifold and } (\mathcal{M}, \varphi) \text{ be a compact } C^{2,1} \text{ submanifold. Let } (\tilde{x}_0, y_0) \in \mathcal{M}^1 \text{ be fixed and assume } t_0 := i_{\mathcal{M}}(\tilde{x}_0, y_0) < \infty. \text{ Then there exist constants } s_0 \text{ and } \varrho_0 > 0 \text{ such that the existence of } (\tilde{q}_0, y_0) \in \mathcal{M}^1 \text{ satisfying } D((\varphi(\tilde{x}_0), y_0), (\varphi(\tilde{q}_0), y_0)) =: \varrho < \varrho_0, (\tilde{x}_0, y_0) \neq (\tilde{q}_0, y_0) \text{ and } \\
& \quad \tilde{\xi}(\tilde{q}_0, y_0, t_0) = \xi(\tilde{x}_0, y_0, t_0) \text{ yields}
\end{align*}
\]

\[
d^2(\tilde{\xi}(\tilde{q}_0, y_0, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) \leq 4s^2(F^2(\tilde{\xi}(\tilde{q}_0, y_0, t_0), \frac{1}{2}(\tilde{\xi}(\tilde{q}_0, y_0, t_0) + \tilde{\xi}(\tilde{x}_0, y_0, t_0))) + C\left(g^2s^3 + g^4s + s^5\right)
\]

where \(C = C(\mathcal{M}, F, \tilde{\varphi}, \tilde{q}_0, y_0)\) and \(0 \leq s < s_0\).

Proof. As usual, we denote the projection of \(\text{EXP}^{-1}(x, z)\) onto its second component by \(y(x, z)\), i.e. in local coordinates we have \(y^i(x, z) = (\text{EXP}^{-1})^N(x, z)\). In order to derive (3.22) we consider a minimizing, constant speed geodesic \(c : [0, 1] \to \mathcal{M}\) joining \(\xi(\tilde{q}_0, y_0, t_0 - s) = \xi(\tilde{x}_0, y_0, t_0 + s)\), i.e.

\[
c(r) := \exp(\xi(\tilde{q}_0, y_0, t_0 - s), ry(\xi(\tilde{q}_0, y_0, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)).
\]

We choose to write \(c(s, r)\) instead of \(c(r)\) to emphasize the dependence of \(c\) on \(s\). Since \(c(s, \cdot)\) has constant speed, i.e. \(F(c(s, r), \dot{c}(s, r)) = \text{const}\) for \(s \in (0, s_0), r \in [0, 1]\) we conclude

\[
\begin{align*}
\frac{d^2}{ds^2}(\xi(\tilde{q}_0, y_0, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) & = \left(\int_0^1 F(c(s, r), \dot{c}(s, r)) dr\right)^2 - \frac{1}{4}\left(F(c(s, 0), \dot{c}(s, 0)) + F(c(s, 1), \dot{c}(s, 1))\right)^2
\end{align*}
\]

Next, we derive for \(1 \leq i \leq N\)

\[
\begin{align*}
\dot{c}^i(s, 0) & = \frac{\partial}{\partial y^k} \exp^i(\xi(\tilde{q}_0, y_0, t_0 - s), y(\xi(\tilde{q}_0, y_0, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))
\end{align*}
\]

as \(s \to 0\) similarly to (3.4). Moreover, we have

\[
\begin{align*}
\dot{c}^i(s, 1) & = \frac{\partial}{\partial y^k} \exp^i(\xi(\tilde{q}_0, y_0, t_0 - s), y(\xi(\tilde{q}_0, y_0, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))
\end{align*}
\]
by virtue of (iii) in Proposition 1.22. Consequently,

\[
F^2(c(s, 0), \frac{1}{s} \dot{c}(s, 0)) \bigg|_{s=0} = F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0))
\]

\[
F^2(c(s, 1), \frac{1}{s} \dot{c}(s, 1)) \bigg|_{s=0} = F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0))
\]

and

\[
d^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s))
\]

\[
= s^2 F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0))
\]

\[
+ \frac{1}{2} s^2 \int_0^s \frac{\partial}{\partial \tau} \left( F^2(c(\tau, 0), \frac{1}{\tau} \dot{c}(\tau, 0)) + F^2(c(\tau, 1), \frac{1}{\tau} \dot{c}(\tau, 1)) \right) d\tau
\]

\[
= s^2 F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0))
\]

\[
+ \frac{1}{2} s^2 \int_0^1 \frac{\partial}{\partial \tau} \left( F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} \dot{c}(\tau, 0)) \right) + \frac{1}{2} \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right)
\]

\[
+ \frac{\partial}{\partial y_i} \left( F^2(\xi(\tilde{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1)) \right) \dot{\xi}(\tilde{x}_0, y_0, t_0 + \tau)
\]

\[
+ \frac{\partial}{\partial y_i} \left( F^2(\xi(\tilde{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1)) \right) \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 1) \right) d\tau.
\]

In view of (3.22) we consider the integral in (3.24) as error term and thus it remains to show that it has the correct order. For this purpose we initially evaluate the integrand at \( \tau = 0 \) which requires an examination of \( \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right) \) and \( \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 1) \right) \). We have

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right) = \frac{1}{\tau} \frac{\partial}{\partial \tau} \dot{c}(\tau, 0) - \frac{1}{\tau^2} \dot{c}(\tau, 0)
\]

\[
= \frac{1}{\tau} \left( \frac{\partial}{\partial \tau} \dot{c}(\tau, 0) - \left( \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right)
\]

\[
- \frac{1}{\tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) - \left( \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right)
\]

\[
= \frac{1}{\tau} \left( \frac{\partial}{\partial \tau} \dot{c}(\tau, 0) - \left( \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right)
\]

\[
- \frac{1}{\tau^2} \int_0^\tau \frac{\partial}{\partial \tau} \dot{c}(\tau, 0) - \left( \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) d\tau
\]

and

\[
\frac{\partial}{\partial \tau} \dot{c}(\tau, 0) = \frac{\partial}{\partial \tau} (\text{EXP}^{-1})^{N+1}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau))
\]

\[
= \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau) + \dot{\xi}(\tilde{x}_0, y_0, t_0 + \tau)
\]

\[
- \left( \frac{\partial}{\partial x_k} (\text{EXP}^{-1})^{N+1}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) + \delta^k_0 \right) \dot{c}(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau)
\]

\[
+ \left( \frac{\partial}{\partial y_i} (\text{EXP}^{-1})^{N+1}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) - \delta^i_0 \right) \dot{c}(\tilde{x}_0, y_0, t_0 + \tau)
\]
as $\tau \rightarrow 0$. Next, we focus on the terms containing derivatives of $\text{EXP}^{-1}$ in (3.26). We recall (iv) in Proposition 1.41 and compute

$$
\frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)) + \delta^i_k
$$

and

$$
E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)) \left( \delta^i_k - E_{j,k}^{-1}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)) \right) .
$$

Since $E^{-1}_{j,k}(x, z) = \frac{\partial}{\partial x^j} \exp^j(\text{EXP}^{-1}(x, z))$ we compute like in the proof in (iii) of Proposition 1.22 and get

$$
E^{-1}_{j,k}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau))
$$

$$
= \frac{1}{\tau} \frac{\partial}{\partial y^k} c^j(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)), \tau)
$$

$$
= \delta^j_k + \frac{1}{2} \frac{\partial^2}{\partial y^k \partial y^j} c^j(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)), \tau)
$$

$$
+ O(\tau^2)
$$

and

$$
\frac{\partial}{\partial x^k} \exp^j(\text{EXP}^{-1}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)))
$$

$$
= \frac{\partial}{\partial x^k} c^j(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)), \tau)
$$

$$
= \delta^j_k + \frac{\partial^2}{\partial t \partial x^k} c^j(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)), \tau)
$$

$$
+ O(\tau^2)
$$

as $\tau \rightarrow 0$. We get

$$
\frac{\partial}{\partial \tau} \delta^j (\tau, 0) - \left( \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}^j(\tilde{x}_0, y_{\tilde{x}_0}, t_0) \right)
$$

(3.31)
As to an analysis of the term \( \dot{c}(\tau, 0) \) in (3.24) we prefer to work with the ensuing interpretation. We observe \( \dot{c}(\tau, 1) = -\dot{c}(\tau, 0) \) where \( c(\tau, t) := c(\tau, 1 - t) \) and \( c(\tau, \cdot) \) is the geodesic defined in (3.23). By Lemma 1.18 \( \dot{c} \) is a family of geodesics with respect to the Finsler structure \( \mathcal{F}(x, y) := F(x, -y) \). We denote the exponential map related to the Finsler manifold \( (\mathcal{M}, \mathcal{F}) \) by \( \exp \) and define \( \exp^{-1} \) as in Proposition 1.41 and thus write

\[
\dot{c}(\tau, 1) = -\dot{c}(\tau, 0) = -\dot{y}(\xi(\bar{x}, 0, y_0, t_0 + \tau), \xi(\bar{q}, y_0, t_0 - \tau))
\]

where in local coordinates \( \bar{y}(x, z) \in T_x \mathcal{M} \) is given by \( \bar{y}^i(x, z) := (\exp^{-1})^{N+i}(x, z) \) for \( x, z \in \mathcal{M} \). Hence, we may proceed as before and obtain

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 1) \right) = -\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right)
\]
\[ \frac{1}{\tau} \int_0^\tau \frac{\partial}{\partial \tau} \mathbf{\dot{e}}^i(t,0) + \left( \mathbf{\xi}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) + \mathbf{\mathbf{\dot{\xi}}}^i(\mathbf{\tilde{x}}(0,0), t_0) \right) \, dt - \frac{1}{\tau} \left( \frac{\partial}{\partial \tau} \mathbf{\dot{e}}^i(\tau,0) + \left( \mathbf{\xi}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) + \mathbf{\mathbf{\dot{\xi}}}^i(\mathbf{\tilde{x}}(0,0), t_0) \right) \right) \]

and

\[ \frac{\partial}{\partial \tau} \mathbf{\dot{e}}^i(\tau,0) = - \left( \mathbf{\dot{\xi}}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) + \mathbf{\dot{\xi}}^i(\mathbf{\tilde{x}}(0,0), t_0) \right) - \tau \left( \mathbf{\dot{\xi}}^i(\mathbf{\tilde{x}}(0,0), t_0) - \mathbf{\dot{\xi}}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) \right) \]

\[ - \frac{1}{\tau} \left( \frac{\partial}{\partial \tau} \mathbf{\dot{e}}^i(\tau,0) + \left( \mathbf{\dot{\xi}}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) + \mathbf{\dot{\xi}}^i(\mathbf{\tilde{x}}(0,0), t_0) \right) \right) \]

\[ + \left( \frac{\partial}{\partial x^p} (\text{EXP}^{-1})^{N+i}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) + \delta^i_k \right) \mathbf{\dot{\xi}}^k(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau) \]

\[ - \left( \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) - \delta^i_k \right) \mathbf{\dot{\xi}}^k(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau) \]

for \( 0 < \tau < s \) similarly to (3.25) and (3.26). Moreover, we derive

\[ \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) + \delta^i_k \]

\[ = \mathbf{E}^{-1}_{ij}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) \left( \mathbf{E}^{-1}_{jk}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) \right) \]

\[ - \frac{\partial}{\partial x^k} \text{EXP}^{i} (\text{EXP}^{-1})^{N+i}(\mathbf{\xi}(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0 + \tau), \mathbf{\xi}(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0 - \tau)) \]

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \mathbf{\dot{e}}^i(\tau,1) \right) \bigg|_{\tau=0} = \frac{1}{2} \left( \mathbf{\dot{\xi}}^i(\mathbf{\tilde{x}}, \mathbf{\tilde{y}}, 0, t_0) - \mathbf{\dot{\xi}}^i(\mathbf{\tilde{q}}, \mathbf{\tilde{y}}, t_0) \right) \]
\[ -\frac{1}{2} \left( \frac{1}{2} \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \xi^i(x, y, t) \right) + \left( \frac{1}{2} \frac{\partial^2}{\partial \xi^i \partial \xi^j} \xi^i(x, y, t) \right) + \frac{1}{4} \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \xi^i(x, y, t) \]

similar to (3.32).

We conclude by (3.24), (3.32) and (3.38)

\[ d^2(\xi(q_0, y_0, t - s), \xi(x_0, y_0, t + s)) = \]

\[ s^2 F^2(\xi(q_0, y_0, t), \xi'(q_0, y_0, t) + \xi'(x_0, y_0, t)) + \frac{1}{2} s^3 \frac{\partial}{\partial x^j} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) + \frac{1}{2} s^3 \frac{\partial}{\partial y^j} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) \]

\[ + \frac{1}{4} s^3 \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) \]

\[ + \frac{1}{2} s^3 \frac{\partial}{\partial y^j} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) \]

\[ - \left( \frac{1}{2} \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \xi' (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t), 0) \right) \dot{\xi}^k(q_0, y_0, t) \]

\[ - \frac{1}{2} \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \xi' (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t), 0) \dot{\xi}^k(x_0, y_0, t) \]

\[ - \frac{1}{2} \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k} \xi' (\xi(x_0, y_0, t), -\xi(x_0, y_0, t) + \xi(q_0, y_0, t), 0) \dot{\xi}^k(x_0, y_0, t) \]

\[ + \frac{1}{2} s^2 \int_0^s \frac{\partial}{\partial \xi^j} (F^2) \left( \xi(q_0, y_0, t - \tau), \frac{1}{\tau} \dot{\xi}(\tau, 0) \right) \dot{\xi}^j(q_0, y_0, t - \tau) (-1) \]

\[ + \frac{\partial}{\partial \xi^j} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) \dot{\xi}^j(q_0, y_0, t) \]

\[ + \frac{\partial}{\partial y^j} (F^2) (\xi(q_0, y_0, t), -\xi(x_0, y_0, t) + \xi(q_0, y_0, t)) \dot{\xi}^j(q_0, y_0, t) \]

\[ - \frac{\partial}{\partial y^j} (F^2) (\xi(q_0, y_0, t), \xi(q_0, y_0, t) + \xi(x_0, y_0, t)) \dot{\xi}^j(q_0, y_0, t) \]

\[ \left. \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\tau, 0) \right) \right|_{\tau=0} \]
Local Lipschitz Continuity of $\rho$ and $j$

We recall

\[
\frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1) \right) \right) + \frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0 + \tau) \right)
\]

\[- \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0) \right)
\]

\[+ \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0) \right)
\]

\[- \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\tau, 1) \right) \bigg|_{\tau=0} d\tau.
\]

Below, we analyse terms carrying the factor $s^3$ and intend to show that in fact these terms are of order $s^3 q^2$. For this purpose we initially compute

\[
\frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0) \right) \right) \left( \dot{\xi}^j(\tilde{x}_0, y_0, t_0) - \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right)
\]

\[= 4 \frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \right) \left( \dot{\xi}^j(\tilde{x}_0, y_0, t_0) - \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) + O(q^2)
\]

\[= 4 \frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0)
\]

\[+ 4 \frac{\partial^2}{\partial x \partial y} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0)
\]

\[\left( \dot{\xi}^k(\tilde{q}_0, y_{\tilde{q}_0}, t_0) - \dot{\xi}^k(\tilde{x}_0, y_{\tilde{x}_0}, t_0) \right)
\]

\[-4 \frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \right) \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + O(q^2)
\]

\[\text{and}
\]

\[
\frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0) \right) \right) \left( \dot{\xi}^j(\tilde{x}_0, y_0, t_0) - \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right)
\]

\[= 2 \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \right) \left( \dot{\xi}^j(\tilde{x}_0, y_0, t_0) - \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) + O(q^2)
\]

\[= 2 \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0)
\]

\[+ 2 \frac{\partial^2}{\partial y \partial y} \left( F^2 \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \right) \dot{\xi}^j(\tilde{x}_0, y_0, t_0)
\]

\[\left( \dot{\xi}^k(\tilde{q}_0, y_{\tilde{q}_0}, t_0) - \dot{\xi}^k(\tilde{x}_0, y_{\tilde{x}_0}, t_0) \right)
\]

\[-2 \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \right) \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + O(q^2)
\]

as $q \to 0$. We recall $\frac{\partial}{\partial x} \left( F^2 \left( c(t), \dot{c}(t) \right) \dot{c}(t) \right) = 0$ for any constant speed curve $c$ and hence

\[
\frac{\partial}{\partial x} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0) \right) \right) \left( \dot{\xi}^j(\tilde{x}_0, y_0, t_0) - \dot{\xi}^j(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right)
\]

\[+ \frac{\partial}{\partial y} \left( F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0) \right) \right)
\]

\[\text{for some constant speed curve } c.
\]
as \( \varrho \to 0 \) by virtue of the geodesic equation (1.10).

Next, we observe for \((x,y) \in TM \setminus 0 \) and \( t > 0 \)

\[ c(x, y, -t) = \tau(x, -y, t) \]

and obtain in local coordinates

\[ \frac{\partial^3}{\partial t^2 \partial y^j} c^k(x, y, -t) = - \frac{\partial^3}{\partial t^2 \partial y^j} \tau^k(x, -y, t). \]  \(3.41\)

Moreover, we have

\[ \dot{c}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau), 0)) \]

\[ = \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)) \]

from which we infer by differentiation with respect to \( \tau \)

\[ \dot{c}^{\xi}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau), 0) \hat{\xi}^k(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau)(-1) \]

\[ + \dot{c}^{y^k}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau), 0) \hat{\xi}^k(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau) \]

\[ = \frac{1}{\tau} y^k(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau)) \]
3 Local Lipschitz Continuity of $i_{\bar{M}}$

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} y^j(\xi(\bar{q}, y, t_0 - \tau), \xi(\bar{x}, y, t_0 + \tau)) \right) .
\]

Since \( \frac{\partial}{\partial \tau} c^j(\xi(\bar{q}, y, t_0 - \tau), \frac{1}{\tau} y(\xi(\bar{q}, y, t_0 - \tau), \xi(\bar{x}, y, t_0 + \tau), 0) = \delta_k \) we conclude

\[
\frac{\partial}{\partial x^k} c^j(\xi(\bar{q}, y, t_0 - \tau), \frac{1}{\tau} y(\xi(\bar{q}, y, t_0 - \tau), \xi(\bar{x}, y, t_0 + \tau), 0) \dot{\xi}^k(\bar{q}, y, t_0 - \tau) = 0
\]  

(3.42)

for \( 0 \leq \tau < s_0 \) and in particular

\[
\frac{\partial^2}{\partial t \partial x^k} c^j(\xi(\bar{q}, y, t_0), \xi(\bar{q}, y, t_0) + \xi(\bar{x}, y, t_0), 0) \dot{\xi}^k(\bar{q}, y, t_0) = 0.
\]  

(3.43)

Similarly, we derive

\[
\frac{\partial^2}{\partial t \partial x^k} \tau^j(\xi(\bar{x}, y, t_0), \xi(\bar{q}, y, t_0) + \xi(\bar{x}, y, t_0), 0) \dot{\xi}^k(\bar{x}, y, t_0) = 0.
\]

By inserting the previous equations as well as (3.40) and (3.41) in (3.39) we derive

\[
d^2(\xi(\bar{q}, y, t_0 - s), \xi(\bar{x}, y, t_0 + s))
\]

\[
= s^2 F^2(\xi(\bar{q}, y, t_0), \xi(\bar{q}, y, t_0) + \xi(\bar{x}, y, t_0))
\]

\[
+ \frac{1}{2} s^2 \int_0^s \frac{\partial}{\partial x^j} (F^2) \left( \xi(\bar{q}, y, t_0 - \tau), \frac{1}{\tau} \dot{\xi}^j(\bar{q}, y, t_0 - \tau) \right) (-1)
\]

\[
+ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\bar{q}, y, t_0) \right) |_{\tau = 0}
\]

\[
+ \frac{\partial}{\partial x^j} (F^2) \left( \xi(\bar{x}, y, t_0 + \tau), \frac{1}{\tau} \dot{\xi}^j(\bar{x}, y, t_0 + \tau) \right)
\]

\[
+ \frac{\partial}{\partial y^j} (F^2) \left( \xi(\bar{x}, y, t_0), \frac{1}{\tau} \dot{\xi}^j(\bar{x}, y, t_0) \right)
\]

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\bar{x}, y, t_0) \right) |_{\tau = 0}
\]

\[
+ \frac{\partial}{\partial x^j} (F^2) \left( \xi(\bar{x}, y, t_0 + \tau), \frac{1}{\tau} \dot{\xi}^j(\bar{x}, y, t_0 + \tau) \right)
\]

\[
+ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\bar{x}, y, t_0) \right) |_{\tau = 0} d\tau
\]

\[
+ O(s^3 q^2)
\]

as \( s, q \to 0 \).

It remains to examine the integrand in (3.44). To this end, we compute

\[
\frac{\partial}{\partial x^j} (F^2) \left( \xi(\bar{q}, y, t_0), \xi(\bar{q}, y, t_0) + \xi(\bar{x}, y, t_0) \right) \dot{\xi}^j(\bar{q}, y, t_0)
\]

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{\xi}^j(\bar{q}, y, t_0) \right) |_{\tau = 0}
\]

\[
+ O(s^3 q^2)
\]
3.2 Cases One and Two

\[- \frac{\partial}{\partial x^i} (F^2) (\xi(q_0, y_{q_0}, t_0 - \tau), \frac{1}{\tau} \dot{\xi}(\tau, 0)) \dot{\xi}^i(q_0, y_{q_0}, t_0 - \tau) = \frac{\partial}{\partial x^j} (F^2) (\xi(q_0, y_{q_0}, t_0), \dot{\xi}(q_0, y_{q_0}, t_0) + \dot{\xi}(\bar{x}_0, y_0, t_0))\]

\[\left(\dot{\xi}^i(q_0, y_{q_0}, t_0) - \dot{\xi}^i(q_0, y_{q_0}, t_0 - \tau)\right) + \left(\frac{\partial}{\partial x^j} (F^2) (\xi(q_0, y_{q_0}, t_0), \dot{\xi}(q_0, y_{q_0}, t_0) + \dot{\xi}(\bar{x}_0, y_0, t_0))\right)\]

\[- \frac{\partial}{\partial x^j} (F^2) (\xi(q_0, y_{q_0}, t_0 - \tau), \frac{1}{\tau} \dot{\xi}(\tau, 0)) \dot{\xi}^j(q_0, y_{q_0}, t_0 - \tau) = \tau \frac{\partial}{\partial x^j} (F^2) (\xi(q_0, y_{q_0}, t_0), \dot{\xi}(q_0, y_{q_0}, t_0) + \dot{\xi}(\bar{x}_0, y_0, t_0))\]

\[\dot{\xi}^j(q_0, y_{q_0}, t_0) \frac{\partial}{\partial \tau} \left(\frac{1}{\tau} \dot{\xi}^i(\tau, 0)\right) \bigg|_{\tau=0} + O(\tau^2)\]

as \(\tau \to 0\). We recall (3.32) and (3.43) by which we have

\[\frac{\partial}{\partial \tau} \left(\frac{1}{\tau} \dot{\xi}^i(\tau, 0)\right) \bigg|_{\tau=0} = \frac{1}{2} \left(\dot{\xi}^i(\bar{x}_0, y_0, t_0) - \dot{\xi}^i(q_0, y_{q_0}, t_0)\right)\]

Next, we differentiate

\[2\tau \dot{\xi}^i(q_0, y_{q_0}, t_0 - \tau) = (\text{EXP}^{-1})^{N+i}(\xi(q_0, y_{q_0}, t_0 - \tau), \xi(q_0, y_{q_0}, t_0 + \tau))\]

with respect to \(\tau\) and obtain

\[2 \dot{\xi}^i(q_0, y_{q_0}, t_0 - \tau) - 2\tau \ddot{\xi}^i(q_0, y_{q_0}, t_0 - \tau) = -\frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(q_0, y_{q_0}, t_0 - \tau), \xi(q_0, y_{q_0}, t_0 + \tau)) \dot{\xi}^k(q_0, y_{q_0}, t_0 - \tau)\]

\[+ \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(q_0, y_{q_0}, t_0 - \tau), \xi(q_0, y_{q_0}, t_0 + \tau)) \dot{\xi}^k(q_0, y_{q_0}, t_0 + \tau)\]

which is equivalent to

\[0 = \dot{\xi}^i(q_0, y_{q_0}, t_0 - \tau) - \dot{\xi}^i(q_0, y_{q_0}, t_0 + \tau) - 2\tau \ddot{\xi}^i(q_0, y_{q_0}, t_0 - \tau)\]

\[+ \left(\frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(q_0, y_{q_0}, t_0 - \tau), \xi(q_0, y_{q_0}, t_0 + \tau)) + \delta^i_k\right) \dot{\xi}^k(q_0, y_{q_0}, t_0 - \tau)\]

\[- \left(\frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(q_0, y_{q_0}, t_0 - \tau), \xi(q_0, y_{q_0}, t_0 + \tau)) - \delta^i_k\right) \dot{\xi}^k(q_0, y_{q_0}, t_0 + \tau)\]
Local Lipschitz Continuity of $i_{\tilde{\mathcal{A}}}$}

\[
\begin{align*}
-4\tau \dot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) + 2\tau^2 \ddot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) \\
+ \left( \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) + \delta_k^i \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau) \\
- \left( \frac{\partial}{\partial x^k} (\text{EXP}^{-1})^{N+i}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) - \delta_k^i \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau) \\
+ O(\tau^3)
\end{align*}
\]

\[= \begin{align*}
-4\tau \dot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) + 2\tau^2 \ddot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) \\
+ \tau E_{ij}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) \\
\left( \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \\
- \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau) \\
+ \tau E_{ij}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) \\
\left( \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau) \\
+ O(\tau^3)
\end{align*}
\]

as $\tau \to 0$. Clearly,

\[
\frac{\partial^2}{\partial t \partial x^k} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau) = 0
\]

and hence (3.47) yields

\[
0 = -4\tau \dot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) + 2\tau^2 \ddot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) \\
+ \frac{1}{2} \tau E_{ij}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) \\
\left( \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \\
- \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau) \\
+ \tau E_{ij}(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau)) \\
\left( \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0 - \tau), 0) \right) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau) + O(\tau^3)
\]

from which we derive after dividing by $2\tau$ and passing to the limit as $\tau \to 0$

\[
0 = -2\ddot{\xi}^i(\bar{q}_0, y_{\bar{q}_0}, t_0) + \frac{1}{2} \frac{\partial^3}{\partial t \partial x^k \partial y^l} \xi^j(\xi(\bar{q}_0, y_{\bar{q}_0}, t_0), 2\dot{\xi}(\bar{q}_0, y_{\bar{q}_0}, t_0), 0) \hat{\xi}^k(\bar{q}_0, y_{\bar{q}_0}, t_0).
\]
as \( q \to 0 \). Similarly, we derive

\[
\frac{\partial}{\partial x^j} \left( F^2 \right) \left( \xi(\bar{x}_0, y_0, t_0 + \tau), \frac{1}{\tau} \dot{\xi}(\tau, 1) \right) \frac{\partial^2}{\partial \tau^2} \left( \frac{1}{\tau} \dot{c}^j(\tau, 1) \right) \bigg|_{\tau=0} = 2 \ddot{\xi}^j(\bar{x}_0, y_0, t_0) + O(q)
\]  

(3.50)

as \( \tau \to 0 \) and

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}^j(\tau, 1) \right) \bigg|_{\tau=0} = 2 \ddot{\xi}^j(\bar{x}_0, y_0, t_0) + O(q)
\]  

(3.52)

as \( q \to 0 \). As to the remaining terms of the integrand in (3.44) we compute

\[
\frac{\partial}{\partial y^j} \left( F^2 \right) \left( \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau), \frac{1}{\tau} \dot{\xi}(\tau, 0) \right) \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}^j(\tau, 0) \right)
\]  

(3.53)

and

\[
\frac{\partial}{\partial y^j} \left( F^2 \right) \left( \xi(\bar{q}_0, y_{\bar{q}_0}, t_0 + \tau), \frac{1}{\tau} \dot{c}(\tau, 1) \right) \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}^j(\tau, 1) \right)
\]
The formulas (3.25), (3.31) and (3.32) for these derivatives. Consequently, we obtain

\begin{equation}
\frac{\partial}{\partial y^j} \left( F^2 \right) (\xi(\tilde{x}_0, y_0, t_0), \tilde{\xi}(\tilde{y}_0, y_{\tilde{y}_0}, t_0) + \tilde{\xi}(\tilde{x}_0, y_0, t_0)) \bigg|_{\tau=0}
\end{equation}

Before we proceed with the computation we recall (3.42) by which we may simplify the formulas (3.25), (3.31) and (3.32) for these derivatives. Consequently, we obtain

\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) - \frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) \bigg|_{\tau=0} &= \left( \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) - \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) + \frac{1}{2} \left( \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) \right) \\
&\quad - \frac{1}{2} E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) \\
&\quad - \frac{\partial^3}{\partial^3 t} \partial^j \xi^i(\tilde{y}_0, y_{\tilde{y}_0}, t_0 - \tau) \bigg|_{\tau=0}
\end{align*}

as \( \tau \to 0 \). Next, we analyse

\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) - \frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) \bigg|_{\tau=0} &= \left( \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) - \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) + \frac{1}{2} \left( \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) \right) \\
&\quad - \frac{1}{2} E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) \\
&\quad - \frac{\partial^3}{\partial^3 t} \partial^j \xi^i(\tilde{y}_0, y_{\tilde{y}_0}, t_0 - \tau) \bigg|_{\tau=0}
\end{align*}

Before we proceed with the computation we recall (3.42) by which we may simplify the formulas (3.25), (3.31) and (3.32) for these derivatives. Consequently, we obtain

\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) - \frac{\partial}{\partial t} \left( \frac{1}{\tau} \partial^j (\tau, 0) \right) \bigg|_{\tau=0} &= \left( \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) - \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) + \frac{1}{2} \left( \tilde{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \tilde{\xi}^i(\tilde{x}_0, y_0, t_0) \right) \\
&\quad - \frac{1}{2} E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{x}_0, y_0, t_0 + \tau)) \\
&\quad - \frac{\partial^3}{\partial^3 t} \partial^j \xi^i(\tilde{y}_0, y_{\tilde{y}_0}, t_0 - \tau) \bigg|_{\tau=0}
\end{align*}
\[ + O(\tau) \]
\[ = \frac{1}{3} \tau \left( \dot{\xi}^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \ddot{\xi}^i (\tilde{x}_0, y_0, t_0) \right) \]
\[ + \frac{1}{2} \frac{\partial^3}{\partial y^i \partial y^j} c^j (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi} (\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi} (\tilde{x}_0, y_0, t_0), 0) \]
\[ \left( \dddot{\xi}^k (\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \ddot{\xi}^k (\tilde{x}_0, y_0, t_0) \right) \]
\[ - \frac{1}{2} E_{ij} (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)) \]
\[ - \frac{1}{2} \frac{\partial^3}{\partial y^i \partial y^j} c^j (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \dot{\xi} (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)), 0) \]
\[ \left( \dddot{\xi}^k (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau) + \ddot{\xi}^k (\tilde{x}_0, y_0, t_0 + \tau) \right) \]
\[ = \frac{1}{3} \tau \left( \dot{\xi}^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \ddot{\xi}^i (\tilde{x}_0, y_0, t_0) \right) \]
\[ + \frac{1}{2} \left( \delta^i_j - E_{ij} (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)) \right) \]
\[ - \frac{1}{2} E_{ij} (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)) \]
\[ \int_0^\tau \frac{\partial}{\partial t} \left( \frac{\partial^3}{\partial y^i \partial y^j} c^j (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \frac{1}{\tau} y (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)), 0) \right) \]
\[ \left( \dddot{\xi}^k (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau) + \ddot{\xi}^k (\tilde{x}_0, y_0, t_0 + \tau) \right) \]
\[ - \frac{1}{2} E_{ij} (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi (\tilde{x}_0, y_0, t_0 + \tau)) \]
\[ \int_0^\tau \frac{\partial}{\partial y^i} \left( \frac{\partial^3}{\partial y^j \partial y^k} c^j (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \nu), \frac{1}{\nu} y (\xi (\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \nu), \xi (\tilde{x}_0, y_0, t_0 + \nu)), 0) \right) \]
\[
\left( \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \nu) + \dot{\xi}(\bar{x}_0, \bar{y}_0, t_0 + \nu) \right) \, dv \, dt
\]

From (vi) in Proposition 1.41 and (3.55) we immediately derive

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right) - \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \dot{c}(\tau, 0) \right) \bigg|_{\tau = 0} = O(\tau)
\]

(3.56)
as \( \tau \to 0 \). Next, we divide (3.48) by \( \tau \) and subtract from this equation the expression obtained by integrating (3.48) with respect to \( \tau \). Finally we subtract (3.49) and obtain

\[
0 = -4 \ddot{\xi}(\bar{q}_0, \bar{y}_0, t_0) + 2 \tau \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0)
\]

\[
+ \frac{1}{2} E_{ij}(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau))
\]

\[
- \frac{\partial^3}{\partial t^3 \partial y^k} c^i(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), 2 \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau), 0)
\]

\[
\left( \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau) + \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau) \right)
\]

\[
- \frac{1}{\tau^2} \int_0^\tau -4 \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0) + 2 \tau \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0)
\]

\[
+ \frac{1}{2} \tau E_{ij}(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau))
\]

\[
- \frac{\partial^3}{\partial t^3 \partial y^k} c^i(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), 2 \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau), 0)
\]

\[
\left( \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau) + \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau) \right)
\]

\[
+ 2 \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0) - \frac{1}{2} \frac{\partial^3}{\partial t^3 \partial y^k} c^i(\xi(\bar{q}_0, \bar{y}_0, t_0), 2 \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0), 0) \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0)
\]

\[
+ O(\tau^3)
\]

\[
= \frac{4}{3} \tau \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0)
\]

\[
+ \frac{1}{2} E_{ij}(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau))
\]

\[
- \frac{\partial^3}{\partial t^3 \partial y^k} c^i(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), 2 \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau), 0)
\]

\[
\left( \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau) + \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau) \right)
\]

\[
- \frac{1}{2} \tau^2 \int_0^\tau \dot{E}_{ij}(\xi(\bar{q}_0, \bar{y}_0, t_0 - \tau), \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau))
\]

\[
\left( \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 - \tau) + \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0 + \tau) \right)
\]

\[
+ 2 \dddot{\xi}(\bar{q}_0, \bar{y}_0, t_0) - \frac{1}{2} \frac{\partial^3}{\partial t^3 \partial y^k} c^i(\xi(\bar{q}_0, \bar{y}_0, t_0), 2 \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0), 0) \dot{\xi}(\bar{q}_0, \bar{y}_0, t_0)
\]

\[
+ O(\tau^3)
\]


\[ + O(\tau^3) \]

\[ = \frac{4}{3} \tau \tau^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) \]

\[ - \left( \delta^j_i - E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 + \tau)) \right) \]

\[ + \frac{1}{2} E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 + \tau)) \]

\[ + \frac{1}{2 \tau^2} \int_0^\tau \left( \delta^j_i - E_{ij}(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - \tau), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 + \tau)) \right) dt \]

\[ + O(\tau^3) \]

as \( \tau \to 0 \). We observe that besides the terms \( \frac{1}{3} \tau \tau^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \tau^i (\tilde{x}_0, y_{\tilde{x}_0}, t_0) \) and \( \frac{4}{3} \tau \tau^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) \) the structures of the previous equation and (3.55) agree perfectly. Moreover, we observe that the only difference between these equations is that \( \xi^i(\tilde{x}_0, y_{\tilde{x}_0}, t_0 + \tau) \) has been replaced by \( \xi^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0 + \tau) \). Consequently, by adding (3.57) to (3.55) we obtain

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 0) \right) - \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 0) \right) \bigg|_{\tau=0} = \frac{1}{3} \tau \left( \xi^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \tau^i (\tilde{x}_0, y_{\tilde{x}_0}, t_0) \right) + \frac{4}{3} \tau \tau^i (\tilde{q}_0, y_{\tilde{q}_0}, t_0) + O(\tau \sigma + \tau^3) \]  

as \( \tau, \sigma \to 0 \). We may proceed similarly with

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) - \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) \bigg|_{\tau=0} = O(\tau) \] 

and derive

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) - \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) \bigg|_{\tau=0} = O(\tau) \]

as \( \tau \to 0 \) as well as

\[ \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) - \frac{\partial}{\partial \tau} \left( \frac{1}{\tau} \tau^i(\tau, 1) \right) \bigg|_{\tau=0} \]
\[\frac{1}{3} \tau \left( \dot{\xi}^i(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \ddot{\xi}^i(\tilde{x}_0, y_0, t_0) \right) + \frac{4}{3} \tau \dddot{\xi}^i(\tilde{x}_0, y_0, t_0) + O(\tau^2 + \tau^3) \quad (3.60)\]

as \(\tau, \rho \to 0\). Finally, we insert (3.45) together with (3.50), (3.51) together with (3.52), (3.53) together with (3.58), (3.56), and (3.50), and (3.54) together with (3.60), (3.59) and (3.52) in (3.44) and get

\[
d^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) = s^2 F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0)) \quad (3.61)
\]

\[
+ \frac{1}{4} s^4 \frac{\partial}{\partial x^j} \left( F^2 \right) (\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0))
\]

\[
+ \frac{1}{2} s^4 \frac{\partial}{\partial y^j} \left( F^2 \right) (\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0))
\]

\[
+ \frac{1}{4} s^4 \frac{\partial^2}{\partial x^k \partial x^j} \left( F^2 \right) (\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0))
\]

\[
+ \frac{1}{4} s^4 \frac{\partial^2}{\partial y^k \partial y^j} \left( F^2 \right) (\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0))
\]

\[
+ O(\rho^2 s^3 + \rho s^4 + s^5)
\]

as \(\rho, s \to 0\). We have

\[
\dot{\xi}(\tilde{x}_0, y_0, t_0) + \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) = 2 \dot{\xi}(\tilde{x}_0, y_0, t_0) + O(\rho) = 2 \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + O(\rho)
\]

as \(\rho \to 0\) and thus obtain with regard to the positive homogeneity of \(\frac{\partial^2}{\partial x^k \partial x^j} \left( F^2 \right)\) and \(\frac{\partial^2}{\partial y^k \partial y^j} \left( F^2 \right)\)

\[
d^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) = s^2 F^2(\xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \xi(\tilde{x}_0, y_0, t_0))
\]
\[ +s^4 \frac{\partial}{\partial x^j} (F^2) \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \ddot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \]
\[ +s^4 \frac{\partial}{\partial y^j} (F^2) \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \ddot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \]
\[ +s^4 \frac{\partial^2}{\partial x^k \partial x^j} (F^2) \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \dddot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \]
\[ +2s^4 \frac{\partial^2}{\partial x^k \partial y^j} (F^2) \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \right) \dddot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) \]
\[ +s^4 \frac{\partial}{\partial y^j} (F^2) \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \dddot{\xi}(\tilde{x}_0, y_0, t_0) \]
\[ +s^4 \frac{\partial}{\partial y^j} (F^2) \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \dddot{\xi}(\tilde{x}_0, y_0, t_0) \]
\[ +s^4 \frac{\partial^2}{\partial y^k \partial y^j} (F^2) \left( \xi(\tilde{x}_0, y_0, t_0), \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) \dddot{\xi}(\tilde{x}_0, y_0, t_0) \]
\[ +O(g^2 s^3 + g^3 s^4 + s^5) \]
as \( g, s \to 0 \). Since for any constant speed curve \( c \)
\[ 0 = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x^j} (F^2) (c(t), \dot{c}(t)) \right) \ddot{c}(t) + \frac{\partial}{\partial y^j} (F^2) (c(t), \dot{c}(t)) \dddot{c}(t) \]
\[ + \frac{\partial}{\partial t} \left( \frac{\partial}{\partial y^j} (F^2) (c(t), \dot{c}(t)) \right) \ddot{c}(t) + \frac{\partial}{\partial y^j} (F^2) (c(t), \dot{c}(t)) \dddot{c}(t) \]
by differentiating the constant speed condition twice with respect to \( t \) we conclude
\[ d^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s) \right) \]
\[ = s^2 F^2 \left( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0), \dot{\xi}(\tilde{q}_0, y_{\tilde{q}_0}, t_0) + \dot{\xi}(\tilde{x}_0, y_0, t_0) \right) + O(g^2 s^3 + g^3 s^4 + s^5) \]
as \( g, s \to 0 \).

**Proposition 3.13.** Let \( (\mathcal{M}, F) \) be a connected, forward geodesically complete Finsler manifold and \( (\mathcal{M}, \varphi) \) be a compact \( C^2,1 \) submanifold. Let \( (\tilde{x}_0, y_0) \in I^1 \mathcal{M} \) be fixed and assume \( t_0 := i_{\tilde{\varphi}}(\tilde{x}_0, y_0) < \infty \). Then there exist constants \( \epsilon_0 > 0, \delta_0 > 0, \varrho_0 > 0 \) and \( K > 1 \) depending on \( \mathcal{M}, F, \tilde{\varphi}, \) and \( t_0 \) such that the existence of \( (\tilde{q}_0, y_{\tilde{q}_0}) \in I^1 \tilde{\mathcal{M}} \) satisfying \( D((\varphi(\tilde{x}_0, y_0)), (\varphi(\tilde{q}_0), y_{\tilde{q}_0})) =: g < \varrho_0 \) and \( \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0) = \xi(\tilde{x}_0, y_0, t_0) \) yields
\[ d(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}, y, t_0 + s)) < 2 s \]
for any admissible perturbation \( Z \) as defined in Lemma 3.9 and any \( (\tilde{x}, y) \in I^1 \tilde{\mathcal{M}} \) satisfying \( D_{I^1 \tilde{\mathcal{M}}}((\tilde{x}, y), (\tilde{x}_0, y_0)) =: \delta < \min(\delta_0, \epsilon_0 g) \) where \( s := K\delta \).
Proof. Initially, we remark that the hypotheses of Lemma 3.11 are satisfied provided that \( \delta_0, \rho_0 \) are chosen sufficiently small. Moreover, we recall the definition of the constant \( \sigma := |y(\xi(q_0, y_0, t_0), \xi(x, y_0, t_0))| \) from this lemma and observe \( \sigma = 0 \). Consequently,

\[
d^2(\xi(Z(x, y), t_0 - s), \xi(x, y, t_0 + s))
\]

\[
< d^2(\xi(q_0, y_0, t_0 - s), \xi(x, y, t_0 + s)) + C(\delta(s^2 \rho + s \rho^2 + s^3) + \delta^2(s^2 + \rho^2) + \delta^3(s + \rho) + \delta^4).
\]

Next, we obtain by virtue of Lemma 3.12, Lemma A.1 and \( \frac{K}{K} = \delta < \min\{\delta_0, \epsilon \rho\} \)

\[
d^2(\xi(Z(x, y), t_0 - s), \xi(x, y, t_0 + s)) = 4 s^2 \left(F^2 \right) \left(\xi(q_0, y_0, t_0), \frac{1}{2} \xi(q_0, y_0, t_0) + \xi(x, y, t_0) \right) + C s^2 \left(\frac{1}{K} (s \rho + \rho^2 + s^2) \right)
\]

\[
+ \frac{1}{K^2} (s^2 + \rho^2) + \frac{s}{K^3} (s + \rho) + \frac{s^2}{K^4} + \rho^2 s + s^2 + \rho^2 + s^3 + \delta \rho K + \delta_0 \epsilon K^2 + \delta_0 \epsilon K^3 \right).
\]

Like in the proof of Proposition 3.6 we consider the reverse geodesics \( \tilde{\xi}(\tilde{x}, y, \cdot) \) and \( \tilde{\xi}(\tilde{q}, y_0, \cdot) \) and derive similarly to (3.5) - (3.7)

\[
D \left(\left(\varphi(q_0), -y_0 \right), \left(\varphi(x_0), -y_0 \right) \right) \leq C F(\tilde{\xi}(\tilde{x}, y, t_0), \frac{1}{2} (\tilde{\xi}(\tilde{x}, y, t_0) - \tilde{\xi}(\tilde{q}, y_0, t_0))
\]

where \( C = C(M, F, t_0) \). For the moment, we consider the function \( \Phi : T'M \setminus 0 \to [0, \infty) \) defined by \( \Phi((x, y)) := d^2 \left((\varphi(x_0), -y_0), (x, -y) \right) \) and recall that the squared Riemannian distance function \( d^2 \left((\varphi(x_0), -y_0), \right) \) is \( C^\infty \) in an open neighbourhood of \( (\varphi(x_0), -y_0) \), see [BCS00, Section 6.4 B]. Hence, \( \Phi \) is \( C^\infty \) in an open neighbourhood of \( (\varphi(x_0), -y_0) \). We consider a geodesic \( \Xi : [0, \rho] \to T'M \setminus 0 \) joining \( (\varphi(q_0), y_0) \) with \( (\varphi(x_0), y_0) \) and deduce

\[
\Phi(\varphi(q_0), y_0) = \Phi(\varphi(x_0), y_0) + \rho \frac{d}{dt} \Phi(\Xi(t)) \bigg|_{t=0} + \rho^2 \frac{d^2}{dt^2} \Phi(\Xi(t)) \bigg|_{t=0} + O(\rho^3)
\]

\[
= \rho^2 \frac{d^2}{dt^2} \Phi(\Xi(t)) \bigg|_{t=0} + O(\rho^3)
\]

by virtue of [BCS00, Proposition 6.4.2]. Consequently, \( |\Phi(\varphi(q_0), y_0)| \geq C \rho^2 \) provided that \( \rho_0 \) is chosen sufficiently small and hence

\[
\rho \leq C \rho F(\tilde{\xi}(\tilde{x}, y, t_0), \frac{1}{2} (\tilde{\xi}(\tilde{x}, y, t_0) - \tilde{\xi}(\tilde{q}, y_0, t_0))
\]

and hence

\[
d^2(\xi(Z(x, y), t_0 - s), \xi(x, y, t_0 + s))
\]
\[ \leq 4s^2\left(1 - C\delta^2\left(\epsilon + \frac{1}{K} + \epsilon^2K + \epsilon^2\frac{1}{K} + \epsilon\frac{1}{K^2} + \epsilon^2\frac{1}{K^2} + \delta_0K + \delta_0\epsilon K^2 + \delta_0\epsilon^2K^3\right)\right). \]

Finally, we derive
\[ d^2(\xi(Z(\bar{x}, y), t_0 - s), \xi(\bar{x}, y, t_0 + s)) < 4s^2 \]
by first choosing \( K \) sufficiently large, then \( \epsilon \) sufficiently small and finally \( \delta_0 \) sufficiently small. \( \square \)

### 3.3 The Third Case

In the remaining third case we do no longer presume that for a given \((\bar{x}_0, y_0) \in I^+\bar{M}\) with \( t_0 := i_{\bar{M}}(\bar{x}_0, y_0) < \infty \) there exists \((\bar{q}, \bar{y}_0) \in I^+\bar{M}\) satisfying \( \xi(\bar{x}_0, y_0, t_0) = \xi(\bar{q}, \bar{y}_0, t_0) \). In order to find a suitable estimate for \( d^2(\xi(\bar{q}, \bar{y}_0, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \) we want to make use of the fact that the backward metric sphere \( S^- (\xi(\bar{x}_0, y_0, t_0), t_0) \) touches \( \bar{M} \) at least at \( \varphi(\bar{x}_0) \). Since, by definition, \( d(x, \xi(\bar{x}_0, y_0, t_0)) = t_0 \) for all \( x \in S^- (\xi(\bar{x}_0, y_0, t_0), t_0) \) one might intend to apply Lemma 3.12 to \( S^- (\xi(\bar{x}_0, y_0, t_0), t_0) \) directly.

However, in general we can not eliminate the case \( i(\varphi(\bar{x}_0), y_0) = i_{\bar{M}}(\bar{x}_0, y_0) \) and thus the distance function \( d(\cdot, \xi(\bar{x}_0, y_0, t_0)) \) might not be smooth in an open neighbourhood of \( \varphi(\bar{x}_0) \). Consequently, \( S^- (\xi(\bar{x}_0, y_0, t_0), t_0) \) fails to be a submainfold. In order to overcome this lack of regularity we approximate the aforementioned distance function by \( d(\cdot, \xi(\bar{x}_0, y_0, t_0 - \epsilon)) \) which are smooth functions in an open neighbourhood of \( \varphi(\bar{x}_0) \) and consider \( S^- (\xi(\bar{x}_0, y_0, t_0 - \epsilon), t_0 - \epsilon) \) instead of \( S^- (\xi(\bar{x}_0, y_0, t_0), t_0) \).

In what follows we show that Lemma 3.12 is applicable to \( S^- (\xi(\bar{x}_0, y_0, t_0 - \epsilon), t_0 - \epsilon) \). Thereafter we show that for a suitable choice of \((\bar{q}, y_0) \in I^+\bar{M}\) we can establish an adequate estimate for the difference between \( d^2(\xi(\bar{q}, y_0, t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) \) and the corresponding term for \( S^- (\xi(\bar{x}_0, y_0, t_0 - \epsilon), t_0 - \epsilon) \).

As we have already mentioned, a crucial point in this third case is the choice of a suitable element \((\bar{q}, y_0) \in I^+\bar{M}\). We provide this choice in the following Lemma.

**Lemma 3.14.** Let \((\bar{M}, F)\) be a connected, forward geodesically complete Finsler manifold and \((\bar{M}, \varphi)\) be a compact \( C^{2,1} \) submanifold. Let \((\bar{x}_0, y_0) \in I^+\bar{M}\) be fixed and assume \( t_0 := i_{\bar{M}}(\bar{x}_0, y_0) < \infty \). Then for all \( 0 < \epsilon < t_0 \) there exists an open neighbourhood \( U_\epsilon \subset M \) of \( x_0 := \varphi(\bar{x}_0) \) such that the distance function \( \rho_\epsilon := -d(\cdot, z_\epsilon) \) satisfies \( \rho_\epsilon \in C^\infty(U_\epsilon) \) and hence \( N_\epsilon := S^- (z_\epsilon, t_\epsilon) \cap U_\epsilon \) is a \( C^\infty \) submanifold where the natural inclusion is denoted by \( \psi_\epsilon \).

Moreover, there exists a smooth, regular curve \( c_\bar{M} \) along \( \bar{M} \) satisfying \( c_\bar{M}(0) = x_0 \) and \( c_\bar{M}(0) =: v_1 \in T_{x_0}\bar{M} \subset T_{x_0}N_\epsilon \) and a normal vector field \( V_\bar{M} \) along \( c_\bar{M} \) as well as a corresponding curve \( c_{N_\epsilon} \) along \( N_\epsilon \) and a normal vector field \( V_{N_\epsilon} \) along \( c_{N_\epsilon} \) such that \( V_\bar{M}(0) = V_{N_\epsilon}(0) \) and
\[
\left| V_\bar{M}(0) - V_{N_\epsilon}(0) \right| \leq C\left( A_{(x_0, y_0)}(v_1) - A_{(x_0, y_0)}(v_1) \right)
\]
where $C = C(M, F, \tilde{M}, \varphi)$. Moreover, $v_1$ can be chosen such that

$$\Lambda_{(x_0, y_0)}^M(v_1) - \Lambda_{(x_0, y_0)}^{\tilde{M}}(v_1) \leq \Lambda_{(x_0, y_0)}^M(y) - \Lambda_{(x_0, y_0)}^{\tilde{M}}(y)$$

for all $y \in T_{x_0} \tilde{M}$.

Proof. We fix $\epsilon \in (0, t_0)$ and set $t_\epsilon := t_0 - \epsilon < i_{\tilde{M}}(\tilde{x}_0, y_0)$ as well as $z_\epsilon := \xi(\tilde{x}_0, y_0, t_\epsilon)$ and deduce $d_{z_\epsilon}^-(\xi(\tilde{x}_0, y_0, t_\epsilon)) \geq t_\epsilon$ or equivalently

$$d_{z_\epsilon}^-(\varphi(\tilde{x})) \geq t_\epsilon$$

for all $\tilde{x} \in \tilde{M}$. In particular, $t_\epsilon < t_0 = i(\varphi(\tilde{x}_0), y_0)$ by Corollary 2.16 and thus $z_\epsilon \in D_{\varphi(\tilde{x}_0)}$ by virtue of Lemma 1.39. Consequently, Lemma 1.42 yields the existence of an open neighbourhood $U_\epsilon \subset M$ of $\varphi(\tilde{x}_0)$ such that $\rho \in C^\infty(U)$. We remark that $N_\epsilon := S^2(z_\epsilon, t_\epsilon) \cap U$ is a $C^\infty$ submanifold and observe $(x_0, y_0) \in T_{x_0} N_\epsilon \subset T_{x_0} \tilde{M}$. For $x \in U_\epsilon$ we set $n(x) := \text{grad } \rho_{|x}$ and infer $n(x_0) = y_0$ from Corollary 2.5.

We consider a geodesic $\tilde{c} : [0, r] \to \tilde{M}$ starting at $\tilde{x}_0$ in some direction $\tilde{v}_1 \in T_{\tilde{x}_0} \tilde{M}$, i.e $\tilde{c}(0) = \tilde{x}_0$ and $\tilde{c}(0) = \tilde{v}_1$. The vector $\tilde{v}_1$ is chosen such that $c_{\tilde{M}} := \varphi \circ \tilde{c}$ satisfies $c_{\tilde{M}}(0) = v_1$. We remark that a suitable choice of $v_1$ will be given at the end of the proof. We set $V_{\tilde{M}}(t) := \frac{1}{f^{\perp}(c_{\tilde{M}}(t), n^\perp(c_{\tilde{M}}(t)))} n^\perp(c_{\tilde{M}}(t)) \in T_{c_{\tilde{M}}(t)} \tilde{M}$ where $n^\perp$ is the vector field introduced in Lemma 2.7.

Since $v_1 \in T_{x_0} \tilde{M} \subset T_{x_0} N_\epsilon$ there exists a geodesic $\tilde{c} : [0, r] \to N_\epsilon$ such that $c_{\tilde{N}_\epsilon} := \psi \circ \tilde{c}$ satisfies $c_{\tilde{N}_\epsilon}(0) = x_0$ and $c_{\tilde{N}_\epsilon}(0) = v_1$. We consider the vector field $V_{\tilde{N}_\epsilon}(t) := n(c_{\tilde{N}_\epsilon}(t))$ and infer from Corollary 2.5 that $V_{\tilde{N}_\epsilon}$ is a normal to $N_\epsilon$, i.e. $V_{\tilde{N}_\epsilon}(t) \in T_{c_{\tilde{N}_\epsilon}(t)} N_\epsilon$.

Clearly, we have $V_{\tilde{M}}(0) = V_{\tilde{N}_\epsilon}(0) = n(x_0) = y_0 \in T_{x_0} N_\epsilon \subset T_{x_0} \tilde{M}$. In order to derive an estimate for $\frac{d}{dt} (V_{\tilde{M}}(t) - V_{\tilde{N}_\epsilon}(t))_{|t=0}$ we compute in local coordinates

$$\frac{d}{dt} (V_{\tilde{M}}(t))_{|t=0} = \frac{\partial}{\partial t} \left( \frac{1}{f(c_{\tilde{M}}(t), n^\perp(c_{\tilde{M}}(t)))} \right)_{|t=0} n^\perp(c_{\tilde{M}}(t)) + \frac{d}{dt} \left( n^\perp(c_{\tilde{M}}(t)) \right)_{|t=0}.$$

Since $V_{\tilde{N}_\epsilon}$ is a unit normal field, i.e. $V_{\tilde{N}_\epsilon}(t) = V_{\tilde{N}_\epsilon}(t)$, we obtain similarly to the previous computation

$$\frac{d}{dt} (V_{\tilde{N}_\epsilon}(t))_{|t=0} = -\frac{\partial}{\partial t} (F(c_{\tilde{N}_\epsilon}(t), n(c_{\tilde{N}_\epsilon}(t))))_{|t=0} - \frac{d}{dt} (n^\perp(c_{\tilde{N}_\epsilon}(t))_{|t=0}.$$

Consequently,

$$\dot{V}_{\tilde{M}}(0) - \dot{V}_{\tilde{N}_\epsilon}(0) = \frac{d}{dt} V_{\tilde{M}}(t)_{|t=0} - \frac{d}{dt} V_{\tilde{N}_\epsilon}(t)_{|t=0} = -\frac{\partial}{\partial t} \left( F(c_{\tilde{N}_\epsilon}(t), n^\perp(c_{\til{M}}(t))) \right)_{|t=0} y_0 + \frac{\partial}{\partial t} (F(c_{\til{N}_\epsilon}(t), V_{\til{N}_\epsilon}(t)))_{|t=0}.$$
Next, we obtain by differentiating (2.3) with respect to $t$
\[
\frac{d}{dt} \left( n^{+}(c_{\tilde{\mathcal{M}}}(t)) \right) \bigg|_{t=0} = \frac{d}{dt} \left( n^{+}(c_{\mathcal{N}}(t)) \right) \bigg|_{t=0} - \frac{d}{dt} \left( n^{+}(c_{\tilde{\mathcal{M}}}(t)) \right) \bigg|_{t=0}
\]
\[
= \frac{\partial}{\partial y^k} F(x_0, y_0) \left( \frac{d}{dt} \left( n^{+}(c_{\mathcal{N}}(t)) \right) \right)^k \bigg|_{t=0} - \frac{d}{dt} \left( n^{+}(c_{\tilde{\mathcal{M}}}(t)) \right) \bigg|_{t=0} y_0
\]
\[
+ \frac{d}{dt} \left( n^{+}(c_{\tilde{\mathcal{M}}}(t)) \right) \bigg|_{t=0} - \frac{d}{dt} \left( n^{+}(c_{\mathcal{N}}(t)) \right) \bigg|_{t=0}.
\]

where $(W^m_{\tilde{\mathcal{M}}})^i$ are components of the covariant vector field $W_{\tilde{\mathcal{M}}}$ related to $n(c_{\tilde{\mathcal{M}}})$ and introduced in Lemma 2.7. Once more, we recall that $n(c_{\mathcal{N}}(t)) = n^{+}(c_{\mathcal{N}}(t))$ and hence
\[
\frac{d}{dt} \left( n^{+}(c_{\mathcal{N}}(t)) \right) \bigg|_{t=0} = g^{ik}(x_0, y_0) \left( \frac{\partial^2}{\partial x^j \partial y^k} \left( \frac{1}{2} F^2 \right) (x_0, y_0) \epsilon_{N-1}^{\mathcal{N}}(0) - (W_{\mathcal{N}}(0)) \right)
\]
where here the components of the covariant vector field $W_{\mathcal{N}}$ are related to $n(c_{\mathcal{N}})$. Consequently,
\[
\frac{d}{dt} \left( n^{+}(c_{\tilde{\mathcal{M}}}(t)) \right) \bigg|_{t=0} - \frac{d}{dt} \left( n^{+}(c_{\mathcal{N}}(t)) \right) \bigg|_{t=0}
\]
\[
= g^{ik}(x_0, y_0) \left( (W_{\mathcal{N}}(c_{\tilde{\mathcal{M}}})^i(0) - (W_{\mathcal{N}}(0)) \right).
\]

Before we proceed with the computation we recall the underlying definition of the vector fields $W_{\tilde{\mathcal{M}}}$ and $W_{\mathcal{N}}$ from (2.2) in Lemma 2.7. To this end we introduce an $g(c_{\tilde{\mathcal{M}}}, n(c_{\tilde{\mathcal{M}}}))$ orthonormal frame $((E_{\tilde{\mathcal{M}}}^1), \ldots, (E_{\tilde{\mathcal{M}}}^N))$ along $c_{\tilde{\mathcal{M}}}$ and an $g(c_{\mathcal{N}}, n(c_{\mathcal{N}}))$ orthonormal frame $((E_{\mathcal{N}}^1), \ldots, (E_{\mathcal{N}}^N))$ along $c_{\mathcal{N}}$. Moreover, we may assume that $((E_{\tilde{\mathcal{M}}}^1), \ldots, (E_{\tilde{\mathcal{M}}}^n))$ forms a basis of $T_{c_{\tilde{\mathcal{M}}}(t)} \mathcal{M}$ and $((E_{\mathcal{N}}^1), \ldots, (E_{\mathcal{N}})^N)$ forms a basis of $T_{c_{\mathcal{N}}(t)} \mathcal{N}$. Additionally, we require $(E_{\tilde{\mathcal{M}}})^1(0) = (E_{\mathcal{N}})^1(0) = \hat{c}_{\mathcal{M}}(0) = \hat{c}_{\mathcal{N}}(0) = v_1$ and furthermore $(E_{\tilde{\mathcal{M}}})^i(0) = (E_{\mathcal{N}})^i(0) =: v_i$ for $1 \leq i \leq N$. We remark that these orthonormal frames fit into the setting of Lemma 2.7.

With these notations in mind we obtain by differentiating (2.2) with respect to $t$ and by virtue of $g(x_0, y_0)(y_0, (E_{\mathcal{N}})^i(0)) = g(x_0, y_0)(y_0, (E_{\mathcal{N}})^i(0)) = 0$ for $1 \leq i \leq n$
\[
(W_{\mathcal{N}}(c_{\tilde{\mathcal{M}}})^i(0) - (W_{\mathcal{N}}(0)) = \frac{\partial}{\partial t} g_{kj}(c_{\mathcal{N}}(t), n(c_{\mathcal{N}}(t))) \bigg|_{t=0} y_0^j + g_{kj}(c_{\mathcal{N}}(t), n(c_{\mathcal{N}}(0))) \bigg|_{t=0} n^j(c_{\mathcal{N}}(t)) \bigg|_{t=0}
\]
\[
- \frac{\partial}{\partial t} g_{kj}(c_{\tilde{\mathcal{M}}}(t), n(c_{\tilde{\mathcal{M}}}(t))) \bigg|_{t=0} y_0^j - g_{kj}(c_{\mathcal{N}}(0), n(c_{\tilde{\mathcal{M}}}(0))) \bigg|_{t=0} n^j(c_{\tilde{\mathcal{M}}}(t)) \bigg|_{t=0}
\]
\[
- \sum_{\alpha=1}^{n} \frac{\partial}{\partial t} \left( g(c_{\tilde{\mathcal{M}}}(t), n(c_{\tilde{\mathcal{M}}}(t)))(n(c_{\tilde{\mathcal{M}}}(t)), (E_{\tilde{\mathcal{M}}})^i(0)) \right) \bigg|_{t=0} g_{kj}(x_0, y_0) (E_{\mathcal{N}}^j)^i(0)
\]
Local Lipschitz Continuity of $i_{\tilde{\mathcal{M}}}$

\[ + \sum_{\alpha=1}^{n} \frac{\partial}{\partial t} \left( g(c_{\mathcal{M}},(t,n(c_{\mathcal{M}},(t)),(E_{M}),\alpha(t)) \right) \bigg|_{t=0} g_{kj}(x_0, y_0) (E_{M})_{\alpha}(0) \]

\[ = \sum_{\alpha=1}^{n} \left( -g(c_{\mathcal{M}},(0),n(c_{\mathcal{M}},(0)),(E_{\tilde{\mathcal{M}}}),\alpha(0)) \right. \]

\[ + g(c_{\mathcal{M}},(0),n(c_{\mathcal{M}},(0)),(E_{\tilde{\mathcal{M}}}),\alpha(0)) \]

\[ + g(c_{\mathcal{M}},(0),n(c_{\mathcal{M}},(0)),(E_{\tilde{\mathcal{M}}}),\alpha(0)) \right) g_{kj}(x_0, y_0) (E_{M})_{\alpha}(0) \]

\[ = \sum_{\alpha=1}^{n} g(x_0,y_0) \left( g_{kj}(E_{\mathcal{M}})_{\alpha}(0) - D_{E_{\mathcal{M}}}(E_{\tilde{\mathcal{M}}})_{\alpha}(0) \right) g_{kj}(x_0,y_0) (E_{M})_{\alpha}(0). \quad (3.64) \]

We have $D_{E_{\mathcal{M}}}(E_{\mathcal{M}})_{\alpha}(0) - D_{E_{\mathcal{M}}}(E_{\tilde{\mathcal{M}}})_{\alpha}(0) = (E_{\mathcal{M}})_{\alpha}(0) - (E_{\tilde{\mathcal{M}}})_{\alpha}(0)$ and since $(E_{\tilde{\mathcal{M}}})_{1} = c_{\tilde{\mathcal{M}}}$ we deduce

\[ (E_{\tilde{\mathcal{M}}})_{1}(0) = \frac{d}{dt} \left( \frac{\partial}{\partial \tilde{\mathcal{M}}} \tilde{\mathcal{M}}(t) \right) \bigg|_{t=0} \]

\[ = \frac{\partial^2}{\partial x^\beta \partial x^\gamma} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) + \frac{\partial}{\partial x^\beta} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) \]

\[ = \frac{\partial^2}{\partial x^\beta \partial x^\gamma} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) + \frac{\partial}{\partial x^\beta} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0). \]

By comparing this expression with the one obtained in Lemma 2.9 we observe

\[ (E_{\tilde{\mathcal{M}}})_{1}(0) = -A_{x_{0}}(c_{\tilde{\mathcal{M}}}(0)) + G(c_{\tilde{\mathcal{M}}}(0), \tilde{\mathcal{M}}(0)). \]

A similar reasoning yields $(E_{\mathcal{M}})_{1}(0) = -A_{x_{0}}(c_{\mathcal{M}}(0)) + G(c_{\mathcal{M}}(0), \mathcal{M}(0))$ and hence

\[ g(x_0,y_0) \left( y_0, D_{E_{\mathcal{M}}}(E_{\mathcal{M}})_{1}(0) - D_{E_{\mathcal{M}}}(E_{\tilde{\mathcal{M}}})_{1}(0) \right) = A_{x_{0}}(v_1) - A_{x_{0}}(v_1). \]

In the remainder of the proof we show how to deal with remaining terms in (3.64). To this end, we fix $2 \leq \alpha \leq n$ and consider

\[ g(x_0,y_0) \left( y_0, D_{E_{\mathcal{M}}}(E_{\mathcal{M}})_{\alpha}(0) - D_{E_{\mathcal{M}}}(E_{\tilde{\mathcal{M}}})_{\alpha}(0) \right). \]

Since $(E_{\mathcal{M}})_{\alpha}$ and $(E_{\tilde{\mathcal{M}}})_{\alpha}$ are both tangential there exists a vector field $\tilde{g}_{\alpha}$ along $\tilde{c}$ as well as a vector field $\tilde{g}_{\alpha}$ along $\tilde{c}$ such that $(E_{\mathcal{M}})_{\alpha}(t) = \phi|_{\tilde{c}(t)} \tilde{g}_{\alpha}(t)$ and $(E_{\mathcal{M}})_{\alpha}(t) = d\psi|_{\tilde{c}(t)} \tilde{g}_{\alpha}(t)$. Therewith, we compute

\[ (E_{\tilde{\mathcal{M}}})_{\alpha}(0) = \frac{d}{dt} \left( \frac{\partial}{\partial x^\beta} \tilde{\mathcal{M}}(t) \right) \bigg|_{t=0} \]

\[ = \frac{\partial^2}{\partial x^\beta \partial x^\gamma} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) + \frac{\partial}{\partial x^\beta} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) \]

and similarly

\[ (E_{\mathcal{M}})_{\alpha}(0) = \frac{\partial^2}{\partial x^\beta \partial x^\gamma} \psi|_{\tilde{c}(t)} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) + \frac{\partial}{\partial x^\beta} \psi|_{\tilde{c}(t)} \tilde{\mathcal{M}}(0) \tilde{\mathcal{M}}(0) \]
where $\tilde{x}_0 \in \mathcal{N}_c$ satisfies $\psi_c(\tilde{x}_0) = x_0$. We observe that the terms $\frac{\partial}{\partial \tilde{x}^\gamma} \varphi^j(\tilde{x}_0) \tilde{y}_\alpha^j(0)$ and $\frac{\partial}{\partial \tilde{x}^\gamma} \tilde{y}_\alpha^j(\tilde{x}_0) \tilde{y}_\alpha^j(0)$ are perfectly tangential and hence

$$
g(x_0, y_0) \left( y_0, D_{\psi_c} (E_{\mathcal{N}_c})_\alpha(0) - D_{\psi_c} (E_{\mathcal{M}})_\alpha(0) \right) = g_{ij}(x_0, y_0) y_0^j \left( \frac{\partial}{\partial \tilde{x}^\alpha} \psi_c^i(\tilde{x}_0) \tilde{\dot{\gamma}}^i(0) \tilde{y}_\alpha^j(0) - \frac{\partial^2}{\partial \tilde{x}^\alpha \partial \tilde{x}^\gamma} \varphi^j(\tilde{x}_0) \tilde{\dot{\gamma}}^i(0) \tilde{y}_\alpha^j(0) \right).
$$

We set

$$
\frac{\partial}{\partial \tilde{x}^\beta} \varphi^j(\tilde{x}_0) \tilde{y}_\alpha^j(0) =: X_i \tilde{y}_\alpha^j(0) = v_\alpha = \frac{\partial}{\partial \tilde{x}^\beta} \psi_c^i(\tilde{x}_0) \tilde{y}_\alpha^j(0) =: Y_\beta \tilde{y}_\alpha^j(0)
$$

and write $\tilde{y}_\alpha^j(0) = (X^{-1})^\alpha_i v_\alpha^i$, $\tilde{y}_\beta^j = (Y^{-1})^\beta_i v_\beta^i$, $\tilde{\dot{\gamma}}^i(0) = (X^{-1})^i_\gamma \tilde{\dot{\gamma}}^\gamma(0)$ and $\tilde{\dot{\gamma}}^j(0) = (Y^{-1})^j_\gamma \tilde{\dot{\gamma}}^\gamma(0)$. By virtue of this notation we derive

$$
\left( (Y^{-1})^\alpha_i \frac{\partial^2}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \psi_c^i(\tilde{x}_0) (Y^{-1})^\gamma_\delta - (X^{-1})^i_\gamma \frac{\partial^2}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma} \varphi^j(\tilde{x}_0) (X^{-1})^\gamma_\delta \right) v_\alpha^i \tilde{\dot{\gamma}}^\gamma(0)
$$

where $(\Phi_{rs}^i)$ denotes a symmetric $N \times N$ matrix for each $1 \leq i \leq N$. Altogether, we conclude

$$
g(x_0, y_0) \left( y_0, D_{\psi_c} (E_{\mathcal{N}_c})_\alpha(0) - D_{\psi_c} (E_{\mathcal{M}})_\alpha(0) \right) = \left( g_{ij}(x_0, y_0) y_0^j \Phi_{rs}^i \right) v_\alpha^i v_\beta^j
$$

where the term in brackets is a symmetric $N \times N$ matrix. This matrix has $N$ real eigenvalues and there exists a basis of orthonormal eigenvectors. By virtue of Lemma 2.12 we deduce that the eigenvalues are nonnegative. Now, we choose $\tilde{\dot{\gamma}}(0) = v_1$ to be an eigenvector of the aforesaid matrix corresponding to the smallest eigenvalue $\lambda \geq 0$ and conclude

$$
g(x_0, y_0) \left( y_0, D_{\psi_c} (E_{\mathcal{N}_c})_\alpha(0) - D_{\psi_c} (E_{\mathcal{M}})_\alpha(0) \right) = \lambda \sum_{r=1}^N v_\alpha^i v_1^i \leq \lambda \sum_{r=1}^N (v_1^i)^2.
$$

This estimate retranslates to

$$
g(x_0, y_0) \left( y_0, D_{\psi_c} (E_{\mathcal{N}_c})_\alpha(0) - D_{\psi_c} (E_{\mathcal{M}})_\alpha(0) \right) \leq g_{ij}(x_0, y_0) y_0^j \left( \frac{\partial^2}{\partial \tilde{x}^\alpha \partial \tilde{x}^\gamma} \psi_c^i(\tilde{x}_0) \tilde{\dot{\gamma}}^i(0) \tilde{\dot{\gamma}}^\gamma(0) - \frac{\partial^2}{\partial \tilde{x}^\alpha \partial \tilde{x}^\gamma} \varphi^j(\tilde{x}_0) \tilde{\dot{\gamma}}^i(0) \tilde{\dot{\gamma}}^\gamma(0) \right).
$$

Since $\frac{\partial}{\partial \tilde{x}^\gamma} \psi_c^i(\tilde{x}_0) \tilde{\dot{\gamma}}^\gamma(0)$ and $\frac{\partial}{\partial \tilde{x}^\gamma} \varphi^j(\tilde{x}_0) \tilde{\dot{\gamma}}^\gamma(0)$ are tangential we can add $g_{ij}(x_0, y_0) y_0^j \frac{\partial}{\partial \tilde{x}^\gamma} \psi_c^i(\tilde{x}_0) \tilde{\dot{\gamma}}^\gamma(0)$ to the right hand side of the previous inequality and finally deduce

$$
g(x_0, y_0) \left( y_0, D_{\psi_c} (E_{\mathcal{N}_c})_\alpha(0) - D_{\psi_c} (E_{\mathcal{M}})_\alpha(0) \right) \leq \Lambda_{(x_0, y_0)}(v_1) - \Lambda_{(x_0, y_0)}(v_1).
$$

which yields the first claim. The second claim follows easily from the choice of $v_1$. □
Since the quantity $\Lambda^{I^+(x_0,y_0)}(v_1) - \Lambda^N_{(x_0,y_0)}(v_1)$ plays a vital role in the remainder of this chapter we introduce some abbreviations.

**Definition 3.15.** Under the hypotheses of Lemma 3.14 we set
\[
\Lambda_\epsilon := \Lambda^{I^+(x_0,y_0)}(v_1) - \Lambda^N_{(x_0,y_0)}(v_1)
\]
and
\[
\Lambda_0 := \lim_{\epsilon \to 0} \left( \Lambda^{I^+(x_0,y_0)}(v_1) - \Lambda^N_{(x_0,y_0)}(v_1) \right).
\]

In the following lemma we establish an $\epsilon$ version of the usual estimate in the third case.

**Lemma 3.16.** Let $\mathcal{M}$ be a connected, forward geodesically complete Finsler manifold and $(\tilde{\mathcal{M}}, \varphi)$ be a compact $C^{2,1}$ submanifold. Let $(\tilde{x}_0, y_0) \in I^+ \tilde{\mathcal{M}}$ be fixed and assume $t_0 := i_{\tilde{\mathcal{M}}}(\tilde{x}_0, y_0) < \infty$. For any $\kappa > 0$ and any $0 \leq \epsilon < t_0$ there exists $(\tilde{q}_\epsilon, y_\epsilon) \in I^+ \tilde{\mathcal{M}}$ as well as constants $\delta_0 > 0$ and $K \geq 1$ depending on $\mathcal{M}$, $F$, $\tilde{\mathcal{M}}$, $\varphi$, $t_{\epsilon}$, $\kappa$, $\| \psi \|_{C^{2,1}(\mathcal{N})}$, where $(\mathcal{N}, \psi_{\epsilon})$ is the submanifold introduced in Lemma 3.14, such that
\[
d^2(\xi(Z(\tilde{x}, y), t_{\epsilon} - s), \xi(\tilde{x}_0, y_0, t_{\epsilon} + s)) < 2s
\]
for any admissible perturbation $Z$ as defined in Lemma 3.9 and any $(\tilde{x}, y) \in I^+ \tilde{\mathcal{M}}$ satisfying $\kappa \Lambda_{\epsilon} \leq D_{i_{\tilde{\mathcal{M}}}^{-1}}((\tilde{x}, y), (\tilde{x}_0, y_0)) =: \delta < \delta_0$ where $s := K \delta$ and $t_{\epsilon} := t_0 - \epsilon$.

**Proof.** Initially, we provide a suitable choice of $(\tilde{q}_\epsilon, y_\epsilon) \in I^+ \mathcal{M}$. For this purpose, we recall the definition of the submanifold $\mathcal{N}$ from Lemma 3.14. Let $c_{\tilde{\mathcal{M}}}$, $c_{\mathcal{N}}$, be the curves and $V_{\tilde{\mathcal{M}}}$, $V_{\mathcal{N}}$, be the corresponding vector fields whose existence has been established in the aforementioned lemma.

To simplify notation we introduce $C_{\tilde{\mathcal{M}}}: [0, \varrho_0] \to I^+ \mathcal{M}$ defined by $C_{\tilde{\mathcal{M}}}(\varrho) := (c_{\tilde{\mathcal{M}}}(\varrho), V_{\tilde{\mathcal{M}}}(\varrho))$ and $C_{\mathcal{N}}: [0, \varrho_0] \to I^+ \mathcal{N}$ given by $C_{\mathcal{N}}(\varrho) := (c_{\mathcal{N}}(\varrho), V_{\mathcal{N}}(\varrho))$ and recall
\[
C_{\tilde{\mathcal{M}}}(0) = C_{\mathcal{N}}(0) = (x_0, y_0).
\]

Using this notation, we set
\[
\varrho := K^{3/4} D((\tilde{x}, y), (\tilde{x}_0, y_0))
\]
and introduce $(\tilde{q}_\epsilon, y_\epsilon) := C_{\tilde{\mathcal{M}}}(\varrho)$. Next, we choose $\delta_0$ sufficiently small and get by virtue of Lemma 3.11
\[
d^2(\xi(Z(\tilde{x}, y), t_{\epsilon} - s), \xi(\tilde{x}_0, y_0, t_{\epsilon} + s)) < d^2(\xi(C_{\tilde{\mathcal{M}}}(\varrho), t_0 - s), \xi(\tilde{x}_0, y_0, t_{\epsilon} + s))
\]
\[
+ C \left( \delta(s \sigma + \sigma^2 \varrho + s^2 \varrho + s^2 \sigma + s^3) + \delta^2(s^2 + \sigma^2 + \sigma^2) + \delta^3(s + \sigma + \varrho) + \delta^4 \right)
\]
where $C = C(\mathcal{M}, F, \tilde{\mathcal{M}}, \tilde{x}_0, y_0, t_{\epsilon})$ and $\sigma := |y(\xi(C_{\tilde{\mathcal{M}}}(\varrho), t_{\epsilon}), \xi(\tilde{x}_0, y_0, t_{\epsilon}))|$. We recall that for $x, z \in \mathcal{M}$ we defined $y(x, z) \in T_x \mathcal{M}$ by $y(x, z) = y(x, z) \frac{\partial}{\partial x}$ and $y(x, z) := (\text{EXP}^{-1})^{N+i}(x, z)$ for $1 \leq i \leq N$. We observe $\xi(\tilde{x}_0, y_0, t_{\epsilon}) = \xi(C_{\mathcal{N}}(\varrho), t_{\epsilon})$ and thus
\[
(\text{EXP}^{-1})^{N+i}(\xi(C_{\tilde{\mathcal{M}}}(\varrho), t_{\epsilon}), \xi(\tilde{x}_0, y_0, t_{\epsilon})) = (\text{EXP}^{-1})^{N+i}(\xi(C_{\tilde{\mathcal{M}}}(\varrho), t_{\epsilon}), \xi(C_{\mathcal{N}}(\varrho), t_{\epsilon}))
\]
\[ = (\text{EXP}^{-1})^{N+i}(\xi(C_{\overline{\mathcal{M}}}(0), t_\epsilon), \xi(C_{\mathcal{N}}(0), t_\epsilon)) \]
\[ + \varrho \left. \frac{\partial}{\partial u} \left( (\text{EXP}^{-1})^{N+i}(\xi(C_{\overline{\mathcal{M}}}(u), t_\epsilon), \xi(C_{\mathcal{N}}(u), t_\epsilon)) \right) \right|_{u=0} \]
\[ + \int_0^\varrho \frac{\partial}{\partial u} \left( (\text{EXP}^{-1})^{N+i}(\xi(C_{\overline{\mathcal{M}}}(u), t_\epsilon), \xi(C_{\mathcal{N}}(u), t_\epsilon)) \right) \left. \frac{\partial}{\partial u} (\text{EXP}^{-1})^{N+i}(\xi(C_{\overline{\mathcal{M}}}(u), t_\epsilon), \xi(C_{\mathcal{N}}(u), t_\epsilon)) \right|_{u=0} \, du \]
\[ = \varrho \left. \left( \frac{\partial}{\partial u} \xi^i(C_{\mathcal{N}}(u), t_\epsilon) - \frac{\partial}{\partial u} \xi^i(C_{\overline{\mathcal{M}}}(u), t_\epsilon) \right) \right|_{u=0} + \int_0^\varrho \frac{\partial}{\partial u} \xi^i(C_{\overline{\mathcal{M}}}(u), t_\epsilon) \left. \frac{\partial}{\partial u} \xi^i(C_{\mathcal{N}}(u), t_\epsilon) \right|_{u=0} \, du \]
\[ = \varrho \left. \frac{\partial}{\partial u} \xi^i(\tilde{x}_0, y_0, t_\epsilon) \left( \tilde{V}_{\mathcal{N}}(0) - \tilde{V}_{\overline{\mathcal{M}}}(0) \right) + O(\varrho^2) \right. \]
\[
\text{as } \varrho \to 0 \text{ by virtue of } (vi) \text{ in Proposition 1.41. Consequently, Lemma 3.14 yields }
\]
\[ |\sigma| = |y(\xi(C_{\overline{\mathcal{M}}}(\varrho), t_\epsilon), \xi(\tilde{x}_0, y_0, t_\epsilon))| = O(\varrho \Lambda_\epsilon + \varrho^2) \]  \hfill (3.67)
\[
\text{as } \varrho \to 0. \text{ As we have already indicated, we may not perform a suitable estimate for }
\]
\[ d^2(\xi(C_{\overline{\mathcal{M}}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)) \]
\[
\text{directly but make use of the fact that Lemma 3.12 is applicable to } \mathcal{N}_\epsilon \text{ and any } (\varrho_i, y_i) \in I^+ \mathcal{N}_\epsilon. \text{ Hence, we obtain by virtue of this lemma }
\]
\[ d^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)) \]  \hfill (3.68)
\[
\leq 4s^2 \left( F^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon), \frac{1}{2}(\hat{\xi}(C_{\mathcal{N}}(\varrho), t_\epsilon) + \hat{\xi}(\tilde{x}_0, y_0, t_\epsilon))) + O(s^4 \varrho + s^5) \right). \]
\[
\text{Since it is somewhat hidden in the notation, we point out that the error term in the }
\]
\text{previous estimate carries an } \epsilon \text{ dependence. More precisely, it depends on the } C^{2,1} \text{ norm of the natural embedding } \psi. \]
\[
\text{However, as we are interested in an estimate for } d^2(\xi(C_{\overline{\mathcal{M}}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)) \]
\[
\text{we proceed with an estimate on the difference between } d^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)) \text{ and } d^2(\xi(C_{\overline{\mathcal{M}}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)). \text{ To this end we perform separate expansions of both term with respect to } \varrho. \text{ We recall }
\]
\[ d^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s)) \]
\[ = F^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), y(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s))) \]
\[
\text{where in local coordinates, as usual, } y^i(x, z) := (\text{EXP}^{-1})^{N+i}(x, z) \text{ for } x, z \in \mathcal{M}. \text{ We compute }
\]
\[ F^2(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), y(\xi(C_{\mathcal{N}}(\varrho), t_\epsilon - s), \xi(\tilde{x}_0, y_0, t_\epsilon + s))) \]
\[ F^2(\xi(C_{\mathcal{L}}(0), t_e - s), y(\xi(C_{\mathcal{L}}(0), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \]  

(3.69)

\[ + \frac{\partial}{\partial u} F^2(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \]

\[ + \int_0^e \frac{\partial}{\partial u} (F^2)(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \]

\[ - \frac{\partial}{\partial u} (F^2)(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \text{ du}. \]

Similarly, we have

\[ d^2(\xi(C_{\bar{M}}(0), t_e - s), \xi(\bar{x}_0, y_0, t_e + s)) \]

(3.70)

\[ = \frac{\partial}{\partial u} F^2(\xi(C_{\mathcal{L}}(0), t_e - s), y(\xi(C_{\mathcal{L}}(0), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \]

\[ + \int_0^e \frac{\partial}{\partial u} (F^2)(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \]

\[ - \frac{\partial}{\partial u} (F^2)(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \text{ du}. \]

Next, we observe

\[ F^2(\xi(C_{\mathcal{L}}(0), t_e - s), y(\xi(C_{\mathcal{L}}(0), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \]

(3.71)

\[ = F^2(\xi(\bar{x}_0, y_0, t_e - s), 2s \xi(\bar{x}_0, y_0, t_e - s)) \]

\[ = F^2(\xi(C_{\bar{M}}(0), t_e - s), y(\xi(C_{\bar{M}}(0), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \]

and compute

\[ \frac{\partial}{\partial u} F^2(\xi(C_{\mathcal{L}}(u), t_e - s), y(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \]

\[ = \frac{\partial}{\partial x^j} (F^2)(\xi(\bar{x}_0, y_0, t_e - s), 2s \xi(\bar{x}_0, y_0, t_e - s)) \frac{\partial}{\partial u} \xi^j(C_{\mathcal{L}}(u), t_e - s) \bigg|_{u=0} \]

\[ + \frac{\partial}{\partial y^j} (F^2)(\xi(\bar{x}_0, y_0, t_e - s), 2s \xi(\bar{x}_0, y_0, t_e - s)) \]

\[ \frac{\partial}{\partial u} y^j(\xi(C_{\mathcal{L}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s)) \bigg|_{u=0} \]

as well as

\[ \frac{\partial}{\partial u} F^2(\xi(C_{\bar{M}}(u), t_e - s), y(\xi(C_{\bar{M}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s))) \bigg|_{u=0} \]

\[ = \frac{\partial}{\partial x^j} (F^2)(\xi(\bar{x}_0, y_0, t_e - s), 2s \xi(\bar{x}_0, y_0, t_e - s)) \frac{\partial}{\partial u} \xi^j(C_{\bar{M}}(u), t_e - s) \bigg|_{u=0} \]

\[ + \frac{\partial}{\partial y^j} (F^2)(\xi(\bar{x}_0, y_0, t_e - s), 2s \xi(\bar{x}_0, y_0, t_e - s)) \]

\[ \frac{\partial}{\partial u} y^j(\xi(C_{\bar{M}}(u), t_e - s), \xi(\bar{x}_0, y_0, t_e + s)) \bigg|_{u=0} . \]
We have
\[\begin{align*}
&\frac{\partial}{\partial u}\xi^j(C_{N^e}(u), t_e - s)\bigg|_{u=0} - \frac{\partial}{\partial u}\xi^j(C_{\tilde{M}}(u), t_e - s)\bigg|_{u=0} \\
&= \frac{\partial}{\partial x^k}\xi^j(\tilde{x}_0, y_0, t_e - s) \dot{C}_{N^e}^k(0) + \frac{\partial}{\partial y^k}\xi^j(\tilde{x}_0, y_0, t_e - s) \dot{C}_{N^e}^{N+k}(0) \\
&- \frac{\partial}{\partial x^k}\xi^j(\tilde{x}_0, y_0, t_e - s) \dot{C}_{\tilde{M}}^k(0) - \frac{\partial}{\partial y^k}\xi^j(\tilde{x}_0, y_0, t_e - s) \dot{C}_{N^e}^{N+k}(0) \\
&= \frac{\partial}{\partial y^k}\xi^j(\tilde{x}_0, y_0, t_e - s) \left(\dot{\mathcal{V}}^k_M(0) - \dot{\mathcal{V}}^k_M(0)\right)
\end{align*}\]

and
\[\begin{align*}
&\frac{\partial}{\partial u}y^j(\xi(C_{N^e}(u), t_e - s), \xi(\tilde{x}_0, y_0, t_e + s))\bigg|_{u=0} \\
&- \frac{\partial}{\partial u}y^j(\xi(C_{\tilde{M}}(u), t_e - s), \xi(\tilde{x}_0, y_0, t_e + s))\bigg|_{u=0} \\
&= \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^{N+j}(\xi(C_{N^e}(u), t_e - s), \xi(\tilde{x}_0, y_0, t_e + s))\bigg|_{u=0} \\
&- \frac{\partial}{\partial u}(\text{EXP}^{-1})^{N+j}(\xi(C_{\tilde{M}}(u), t_e - s), \xi(\tilde{x}_0, y_0, t_e + s))\bigg|_{u=0} \\
&= \frac{\partial}{\partial x^k}(\text{EXP}^{-1})^{N+j}(\xi(\tilde{x}_0, y_0, t_e - s), \xi(\tilde{x}_0, y_0, t_e + s)) + \delta^j_k \\
&\left(\frac{\partial}{\partial u}\xi^k(C_{N^e}(u), t_e - s) - \frac{\partial}{\partial u}\xi^k(C_{\tilde{M}}(u), t_e - s)\right)\bigg|_{u=0} \\
&- \left(\frac{\partial}{\partial u}\xi^j(C_{N^e}(u), t_e - s) - \frac{\partial}{\partial u}\xi^j(C_{\tilde{M}}(u), t_e - s)\right)\bigg|_{u=0}.
\end{align*}\]

Since \(C_{\tilde{M}} \in I^1_{\tilde{M}}\) and \(C_{N^e} \in I^1_{N^e}\) we infer from Corollary 3.8
\[0 = \left(\frac{\partial}{\partial y^j}(F^2)(\xi(C_{\tilde{M}}(u), t_e - s), \xi(C_{\tilde{M}}(u), t_e - s)) \frac{\partial}{\partial u}\xi^j(C_{\tilde{M}}(u), t_e - s)\right)\bigg|_{u=0}\]
\[= \frac{\partial}{\partial y^j}(F^2)(\xi(\tilde{x}_0, y_0, t_e - s), \xi(\tilde{x}_0, y_0, t_e - s)) \frac{\partial}{\partial u}\xi^j(C_{\tilde{M}}(u), t_e - s)\bigg|_{u=0}\]
as well as
\[0 = \frac{\partial}{\partial y^j}(F^2)(\xi(\tilde{x}_0, y_0, t_e - s), \xi(\tilde{x}_0, y_0, t_e - s)) \frac{\partial}{\partial u}\xi^j(C_{N^e}(u), t_e - s)\bigg|_{u=0}\]
From the previous computations we derive
\[\frac{\partial}{\partial u}F^2(\xi(C_{N^e}(u), t_e - s), y^j(\xi(C_{N^e}(u), t_e - s), \xi(\tilde{x}_0, y_0, t_e + s)))\bigg|_{u=0}\]
We can deal with the remaining integral in the previous equation similarly to (3.14) and derive from (3.69), (3.70), (3.71) and (3.72)

\[ d^2(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) = O(s^2 \rho \Lambda_\varepsilon) \]

as \( s, \rho \to 0 \). Consequently, we derive from (3.69), (3.70), (3.71) and (3.72)

\[
- \frac{\partial}{\partial u} F^2(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) \bigg|_{u=0} \\
= 4s^2 \frac{\partial}{\partial x} (F^2) (\xi(\tilde{x}_0, y_0, t_\varepsilon - s), \tilde{\xi}(\tilde{x}_0, y_0, t_\varepsilon - s)) \\
+ 2s \frac{\partial}{\partial y} (F^2) (\xi(\tilde{x}_0, y_0, t_\varepsilon - s), \tilde{\xi}(\tilde{x}_0, y_0, t_\varepsilon - s)) \\
+ \left( \frac{\partial}{\partial \xi} (\text{EXP}^{-1})^N (\xi(\tilde{x}_0, y_0, t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) + \delta_k^j \right) \\
+ \frac{\partial}{\partial \xi} (\xi(\tilde{x}_0, y_0, t_\varepsilon - s)) \left( \tilde{V}_N^k(0) - \tilde{V}_M^k(0) \right)
\]

and deduce by virtue of (\( vr \)) in Proposition 1.41 and Lemma 3.14

\[
\frac{\partial}{\partial u} F^2(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) \bigg|_{u=0} \\
- \frac{\partial}{\partial u} F^2(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) \bigg|_{u=0} \\
= O(s^2 \rho \Lambda_\varepsilon) \\
(3.72)
\]

We can deal with the remaining integral in the previous equation similarly to (3.14) and derive

\[
d^2(\xi(C_{\Gamma}(\rho), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) = d^2(\xi(C_{\Gamma}(\rho), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) \\
+ \int_0^\rho \frac{\partial}{\partial u} (F^2) (\xi(C_{\Gamma}(u), t_\varepsilon - s), y(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s))) \\
- \frac{\partial}{\partial u} (F^2) (\xi(C_{\Gamma}(u), t_\varepsilon - s), y(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s))) \\
- \frac{\partial}{\partial u} (F^2) (\xi(C_{\Gamma}(u), t_\varepsilon - s), y(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s))) \\
+ \frac{\partial}{\partial u} (F^2) (\xi(C_{\Gamma}(u), t_\varepsilon - s), y(\xi(C_{\Gamma}(u), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s))) \\
\quad du \\
+ O(s^2 \rho \Lambda_\varepsilon)
\]

We can deal with the remaining integral in the previous equation similarly to (3.14) and derive

\[
d^2(\xi(C_{\Gamma}(\rho), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) = d^2(\xi(C_{\Gamma}(\rho), t_\varepsilon - s), \xi(\tilde{x}_0, y_0, t_\varepsilon + s)) \\
+ O(s^2 \rho \Lambda_\varepsilon + s \rho^2 \Lambda_\varepsilon + \rho^3 \Lambda_\varepsilon + s \rho^3)
\]

Together with (3.66) and (3.68) we derive

\[
d^2(\xi(Z(\tilde{x}, y), t_\varepsilon - s), \xi(\tilde{x}, y, t_\varepsilon + s)) \\
\leq 4s^2 (F^2) (\xi(C_{\Gamma}(\rho), t_\varepsilon), \frac{1}{2} (\xi(C_{\Gamma}(\rho), t_\varepsilon) + \xi(\tilde{x}_0, y_0, t_\varepsilon))) \\
+ C (\delta(s \sigma + \rho^2 \sigma + s^2 \rho + s \rho^2 + s^3) + \delta^2 (s^2 + \rho^2 + \sigma^2) + \delta^3 (s + \sigma + \rho) + \delta^4
\]

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by virtue of Lemma A.1. Next, observe

\begin{align*}
\mathcal{F}(\xi(\tilde{x}_0, y_0, t_\epsilon), \frac{1}{2}(\dot{\xi}(\tilde{x}_0, y_0, t_\epsilon) - \dot{\xi}(C_{N_\epsilon}(\varrho), t_\epsilon))) \geq \varrho
\end{align*}

similarly to the proof of Proposition 3.13 and obtain by inserting \( \delta = \frac{s}{K} \) and (3.67)

\begin{align*}
d^2(\xi(Z(\tilde{x}, y), t_\epsilon - s), \xi(\tilde{x}, y, t_\epsilon + s)) \\
\leq 4s^2 \left(1 - C\varrho^2\right) + s^2\varrho^2 C \left( \frac{\sigma}{K} + \frac{\varrho^2 \sigma}{s K} + \frac{s \varrho}{K} + \frac{s^2 \varrho}{K^2} + \frac{\varrho^2}{K^2} + \frac{s^2}{K^3} + \frac{s \sigma}{K^3} + \frac{\varrho^2}{K} + \frac{s}{K} \right)
\end{align*}

Since \( \kappa \Lambda_\epsilon \leq \delta \) and (3.65) we have

\begin{align*}
\frac{\Lambda_\epsilon}{\varrho} \leq \frac{\delta}{\kappa K^{3/4} \delta} = \frac{1}{\kappa K^{3/4}}, \\
\frac{\Lambda_\epsilon}{s} \leq \frac{\delta}{\kappa K \delta} = \frac{1}{\kappa K}
\end{align*}

and \( \tilde{\delta} = \frac{K^{3/4} \delta}{K} = \frac{1}{K^{7/4}} \). Consequently,

\begin{align*}
d^2(\xi(Z(\tilde{x}, y), t_\epsilon - s), \xi(\tilde{x}, y, t_\epsilon + s)) \\
\leq 4s^2 \left(1 - C\varrho^2\right) + s^2\varrho^2 C \left( \frac{1}{K^{7/4}} + \frac{1}{K} + \frac{\delta}{K^{3/4}} + \frac{1}{K^{3/2}} + \frac{1}{K^{5/4}} + \frac{1}{K^{1/2}} \right)
\end{align*}

Now, let \( \kappa > 0 \) be arbitrary. We finally derive

\begin{align*}
d^2(\xi(Z(\tilde{x}, y), t_\epsilon - s), \xi(\tilde{x}, y, t_\epsilon + s)) < 4s^2
\end{align*}

by first choosing \( K \) sufficiently large and then \( \delta_0 \) sufficiently small.
In order to pass to the limit as $\epsilon \to 0$ in the previous Lemma we additionally assume that $\xi(\tilde{x}_0, y_0, t_0)$ is not conjugate to $\varphi(\tilde{x}_0)$ along $\xi(\tilde{x}_0, y_0, \cdot)$.

**Proposition 3.17.** Let $(\mathcal{M}, F)$ be a connected, forward geodesically complete Finsler manifold and $(\bar{\mathcal{M}}, \varphi)$ be a compact $C^{2,1}$ submanifold. Let $(\tilde{x}_0, y_0) \in \bar{\mathcal{M}}$ be fixed and assume $t_0 := i\bar{\mathcal{M}}(\tilde{x}_0, y_0) < \infty$. Moreover, we suppose that for $t \in (0, t_0]$ no point $\xi(\tilde{x}_0, y_0, t)$ is conjugate to $\varphi(\tilde{x}_0)$ along $\xi(\tilde{x}_0, y_0, \cdot)$. Then, for any $\kappa > 0$ there exists $(\tilde{q}_0, y_0) \in \bar{\mathcal{M}}$ as well as constants $\delta_0 > 0$ and $K \geq 1$ depending on $\mathcal{M}, F, \bar{\mathcal{M}}, \varphi, t_0$ and $\kappa$ such that

$$
d^2(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) < 2s$$

for any admissible perturbation $Z$ as defined in Lemma 3.9 and any $(\tilde{x}, y) \in \bar{\mathcal{M}}$ satisfying $\kappa\Lambda_0 \leq D_{\bar{\mathcal{M}}}(\tilde{x}, y), (\tilde{x}_0, y_0)) =: \delta < \delta_0$ where $s := K\delta$.

**Proof.** We can pass to the limit as $\epsilon \to 0$ in Lemma 3.16 provided that the constants from this lemma can be chosen uniformly in $\epsilon$. Therefore we have to show that the $C^{2,1}$ Norm of $\psi_\epsilon$ is uniformly bounded in $\epsilon$. By virtue of Lemma 1.42 we have $d((\varphi(\cdot), \xi(\tilde{x}_0, y_0, t_0 - \epsilon)) = F(\exp^{-1}(\varphi(\cdot), \xi(\tilde{x}_0, y_0, t_0 - \epsilon))$ for $\epsilon \in (0, t_0)$. Since $\xi(\tilde{x}_0, y_0, t_0)$ is not conjugate to $\varphi(\tilde{x}_0)$ along $\xi(\tilde{x}_0, y_0, \cdot)$ Lemma 1.29 yields that $\mathcal{D}_\varphi \exp(\varphi(\tilde{x}_0), t_0, y_0)$ is nonsingular. Thus, (iii) in Proposition 1.41 yields that $\mathcal{D}\exp$ is nonsingular at $(\varphi(\tilde{x}_0), t_0, y_0)$. Since $\exp(\varphi(\tilde{x}_0), (t_0 - \epsilon), y_0) = (\varphi(\tilde{x}_0), \xi(\tilde{x}_0, y_0, (t_0 - \epsilon))$ we conclude that derivatives of $d((\varphi(\cdot), \xi(\tilde{x}_0, y_0, t_0 - \epsilon))$ are uniformly bounded in $\epsilon$ in an open neighbourhood of $\tilde{x}_0$ which implies the aforementioned uniform bound on the $C^{2,1}$ Norm of $\psi_\epsilon$. \qed

### 3.4 Proof of Theorem 3.2

In the final section we combine the preparatory results from the previous sections and establish the proof of the main result of the present thesis. We will work in our usual setting and distinguish between the case in which there exists $(\tilde{q}, y_0) \in I^1\bar{\mathcal{M}}$ satisfying $\xi(\tilde{x}_0, y_0, (t_0)) = \xi(\tilde{q}, y_0, t_0)$ and the case in which there does not exist an element of $I^1\bar{\mathcal{M}}$ with this property.

Before we start with the proof we show that the latter case has an immediate consequence for the quantity $\Lambda_0$.

**Lemma 3.18.** Let $(\mathcal{M}, F)$ be a connected, forward geodesically complete Finsler manifold and $(\bar{\mathcal{M}}, \varphi)$ be a compact $C^{2,1}$ submanifold. Let $(\tilde{x}_0, y_0) \in \bar{\mathcal{M}}$ be fixed and assume $t_0 := i\bar{\mathcal{M}}(\tilde{x}_0, y_0) < \infty$. Moreover, we suppose that for $t \in (0, t_0]$ no point $\xi(\tilde{x}_0, y_0, t)$ is conjugate to $\varphi(\tilde{x}_0)$ along $\xi(\tilde{x}_0, y_0, \cdot)$. Additionally, we assume $\Lambda_0 > 0$, where $\Lambda_0$ is the constant introduced in Definition 3.15. Then there exists $(\tilde{q}, y_0) \in I^1\bar{\mathcal{M}}, (\tilde{q}, y_0) \neq (\tilde{x}_0, y_0)$, such that $\xi(\tilde{x}_0, y_0, t_0) = \xi(\tilde{q}, y_0, t_0)$.

**Proof.** In case $i\bar{\mathcal{M}}(\tilde{x}_0, y_0) = i(\varphi(\tilde{x}_0), y_0)$ we infer from (iii) in Lemma 1.37 the existence of $(\tilde{x}_0, y_0) \in I^1\bar{\mathcal{M}}, y_0 \neq y_1$ such that $\xi(\tilde{x}_0, y_0, t_0) = \xi(\tilde{x}_0, y_1, t_0)$. Thus, we proceed under the assumption $i\bar{\mathcal{M}}(\tilde{x}_0, y_0) < i(\varphi(\tilde{x}_0), y_0)$.

We set $z_0 := \xi(\tilde{x}_0, y_0, t_0)$ and infer from Lemma 2.20 the existence of a sequence $\{z_i\} \subset \mathcal{M}, z_i \rightharpoonup z_0$ as $i \to \infty$, such that there exist $((\tilde{x}_0), (\tilde{y}_0))$,
$I^+\tilde{M}$ satisfying $((\tilde{x}_1)_i, y(\tilde{x}_1)_i, ) \neq ((\tilde{x}_2)_i, y(\tilde{x}_2)_i)_i$, $t_i := d\tilde{M}(z_i) = d(\varphi((\tilde{x}_1)_i), z_i) = d(\varphi((\tilde{x}_2)_i), z_i)$ and

$$\xi((\tilde{x}_1)_i, y(\tilde{x}_1)_i, t_i) = z_i = \xi((\tilde{x}_2)_i, y(\tilde{x}_2)_i, t_i).$$

By compactness, there exist $(\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^+\tilde{M}$ and subsequences such that $((\tilde{x}_1)_i, y(\tilde{x}_1)_i) \to (\tilde{x}_1, y_1)$ and $((\tilde{x}_2)_i, y(\tilde{x}_2)_i) \to (\tilde{x}_2, y_2)$ as $i \to \infty$. Since the distance function $d\tilde{M}$ is continuous we obtain $d(\varphi(\tilde{x}_1), z_0) = t_0 = d(\varphi(\tilde{x}_2), z_0)$. Thus, if either $(\tilde{x}_1, y_1) \neq (\tilde{x}_0, y_0)$ or $(\tilde{x}_2, y_2) \neq (\tilde{x}_0, y_0)$ the lemma is proven.

Otherwise, we have $((\tilde{x}_1)_i, y(\tilde{x}_1)_i) \to (\tilde{x}_0, y_0)$ and $((\tilde{x}_2)_i, y(\tilde{x}_2)_i) \to (\tilde{x}_0, y_0)$ as $i \to \infty$ and proceed similarly to the proof of Proposition 2.17. Clearly, the distance function $g_0 := -d_{\varphi(\tilde{x}_0)}(\varphi(\cdot))$ is smooth in an open neighbourhood of $\tilde{x}_0$ since $z_0 \in D_{\varphi(\tilde{x}_0)}$. Consequently, we may proceed similarly to the proof of Proposition 2.17 and obtain

$$\frac{d^2}{dt^2} (g_0 \circ \tilde{c}(\tilde{x}_0, \tilde{y}, t)) \bigg|_{t=0} = \Lambda^{S^-}_{\varphi(\tilde{x}_0)}(y) - \Lambda^{\tilde{M}}_{\varphi(\tilde{x}_0, y_0)}(y)$$

where $\tilde{c} : (-\delta, \delta) \to \tilde{M}$ is geodesic with $\tilde{c}(0) = \tilde{x}_0$ and $\dot{\tilde{c}}(0) = \tilde{y} \in T_{\tilde{x}_0}\tilde{M}$. Here, $y := d\varphi(\tilde{x}_0) \tilde{y}$. Since $\Lambda^n_n(y) - \Lambda^N_n(y) \geq \Lambda^n_{\tilde{M}}(v_1) - \Lambda^N_{\tilde{M}}(v_1)$ for $v_1 \in T_{\varphi(\tilde{x}_0)}\tilde{M}$ as defined in Lemma 3.14 and $\Lambda > 0$ we conclude

$$\frac{d^2}{dt^2} (g_0 \circ \tilde{c}(\tilde{x}_0, \tilde{y}, t)) \bigg|_{t=0} \leq -\frac{\Lambda}{2}.$$  

Next, we introduce $g_i := -d_{\varphi(\cdot)}(\varphi(\cdot))$ and deduce similarly to the proof of Proposition 2.17 that there exists $r > 0$ such that

$$\frac{d^2}{dt^2} (g_i \circ \tilde{c}(\tilde{x}, \tilde{y}, t)) \leq -\frac{\Lambda}{4}, \quad (3.73)$$

for all $\tilde{x} \in \tilde{B}(\tilde{x}_0, r), \tilde{y} \in T_{\tilde{x}}\tilde{M}$ and $t \in (-r, r)$.

We have

$$g_i((\tilde{x}_1)_i) = g_i((\tilde{x}_2)_i) = \min_{\tilde{x} \in \tilde{M}} g_i(\tilde{x}) \quad (3.74)$$

and choose a minimising geodesic $\tilde{c}_i : (\tilde{x}_1)_i \to (\tilde{x}_2)_i$. By choosing $i$ sufficiently large we can assure that $\tilde{c}_i$ is contained in $\tilde{B}(\tilde{x}_0, r)$. Finally, we observe that (3.74) contradicts the strict concavity of $g_i \circ \tilde{c}_i$ implied by (3.73).

Therewith we completed all necessary preparations and conclude this thesis with a proof of the main result.

**Proof of Theorem 3.2.** We proceed as described at the beginning of Chapter 3 and fix $(\tilde{x}_0, y_0) \in I^+\tilde{M}$ such that $t_0 := i\tilde{M}(\tilde{x}_0, y_0) < \infty$. We show that there exists $\delta_0 > 0$ such that for each $(\tilde{x}, y) \in I^+\tilde{M}$ satisfying $D_{I^+\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_0$ there exists $(\tilde{z}, y_\tilde{z})$ such that

$$d(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) < 2s$$

where $s = K D_{I^+\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y))$. Therewith we conclude

$$d(\varphi(\tilde{z}), \xi(\tilde{x}, y, t_0 + s)) \leq d(\varphi(\tilde{z}), \xi(\tilde{z}, y_\tilde{z}, t_0 - s)) + d(\xi(\tilde{z}, y_\tilde{z}, t_0 - s), \xi(\tilde{x}, y, t_0 + s))$$

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and hence
\[
i_M(\bar{x}, y) \leq t_0 + s = i_M(\bar{x}_0, y_0) + K D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) \]

We distinguish between two cases. Firstly, we assume that there exists \((\bar{q}_0, y_0) \in I^l M, (\bar{x}_0, y_0) \neq (\bar{q}_0, y_0)\) such that \(\xi(\bar{x}_0, y_0, t_0) = \xi(\bar{q}_0, y_0, t_0)\). Let \(\varrho_0\) be the constant from Proposition 3.13. If \(D((\varphi(\bar{x}_0), y_0), (\varphi(\bar{q}_0), y_0)) \geq \varrho_0\) we set \((\tilde{z}, z) = (\bar{q}_0, y_0)\) and obtain from Proposition 3.6 the existence of constants \(\delta_1 > 0\) and \(K_1 \geq 1\) such that
\[
d((\tilde{z}, z), t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) < 2s \tag{3.75}
\]
and consequently
\[
i_M(\bar{x}, y) \leq t_0 + s = i_M(\bar{x}_0, y_0) + K_1 D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) \]

for \(D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) < \delta_1\). If \(D((\varphi(\bar{x}_0), y_0), (\varphi(\bar{q}_0), y_0)) < \varrho_0\) we apply Proposition 3.13 and obtain the existence of \(\epsilon_0 > 0\), \(\delta_2 > 0\), \(K_2 \geq 1\) such that for any admissible perturbation \(Z\) as defined in Lemma 3.9
\[
d((\xi(\tilde{z}, z), t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) < 2s \tag{3.76}
\]
for \(D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) < \min\{\delta_2, \epsilon_0 \varrho\}\) where \(\varrho := D((\varphi(\bar{x}_0), y_0), (\varphi(\bar{q}_0), y_0))\). Consequently, in case \(\delta_2 < \epsilon_0 \varrho\) we set \((\tilde{z}, z) = Z(\bar{x}, y)\).

In order to derive this estimate for \((\bar{x}, y) \in I^l M\) with \(D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) \geq \epsilon_0 \varrho\) we make use of Proposition 3.17. In order to derive a suitable estimate for \(\Lambda_0\) we consider \(\tilde{c} : [0, \tilde{q}] \to \mathcal{M}\) joining \(\bar{x}_0\) with \(\tilde{q}_0\), i.e. \(\tilde{c}(0) = \bar{x}_0\) and \(\tilde{c}(\tilde{q}) = \tilde{q}_0\). We set \(\dot{c}_M := \varphi \circ \tilde{c}\). Let \((\mathcal{N}_\epsilon, \psi_\epsilon)\) the submanifold from Lemma 3.14 and \(\tilde{c} : [0, \tilde{q}] \to \mathcal{N}_\epsilon\) be a geodesic where \(\tilde{c}(0)\) and \(\tilde{c}(\tilde{q})\) are chosen such that \(c_{\mathcal{N}_\epsilon}\) satisfies \(c_{\mathcal{M}}(0) = c_{\mathcal{N}_\epsilon}\) and \(c_{\mathcal{M}}(0) = y = c_{\mathcal{N}_\epsilon}(0)\). We consider the distance function \(\rho_\epsilon : \mathcal{M} \to [0, \infty)\) defined by \(\rho_\epsilon := d(\cdot, (\tilde{c}(\bar{x}_0, y_0, t_0 - \epsilon))\) and obtain similarly to the proof of Lemma 2.12
\[
\rho_\epsilon \circ c_{\mathcal{M}}(\tilde{q}) - \rho_\epsilon \circ c_{\mathcal{N}_\epsilon}(\tilde{q}) = \tilde{q}^2 \left( \Lambda^M_{(\tilde{c}(\bar{x}_0, y_0)(y) - \Lambda^{\mathcal{N}_\epsilon}_{(\tilde{c}(\bar{x}_0, y_0))}((y)) + O(\tilde{q}^3) \right)
\]

Since derivatives of \(\psi_\epsilon\) are uniformly bounded in \(\epsilon\) we deduce
\[
0 = \rho \circ c_{\mathcal{M}}(\tilde{q}) - \rho \circ c_{\mathcal{N}_\epsilon}(\tilde{q}) \geq \tilde{q}^2 \Lambda_0 + O(\tilde{q}^3)
\]
and hence \(\Lambda_0 \leq C \tilde{q} \leq C \varrho\) by choosing \(\varrho_0\) smaller if necessary. Next, we choose \(\kappa = \frac{\epsilon_0 \varrho}{C}\) where \(\kappa\) is the constant from Proposition 3.17 and obtain by virtue of this proposition the existence of constants \(\delta_3 > 0\), \(K_3 \geq 1\) such that
\[
d((\xi(Z(\tilde{z}, z), t_0 - s), \xi(\bar{x}_0, y_0, t_0 + s)) < 2s \tag{3.77}
\]
for \(\epsilon_0 \varrho < D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) ) < \delta_3\) and \(s = K_3 \delta\). Thus, we choose \((\tilde{z}, z) = Z(\bar{x}, y)\).

Consequently, (3.76), (3.77) yield
\[
i_M(\bar{x}, y) \leq i_M(\bar{x}_0, y_0) + \max\{K_2, K_3\} D_{I^l M}( (\bar{x}_0, y_0), (\bar{x}, y) )
\]
for \((\tilde{x}, y) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \min\{\delta_2, \delta_3\}\).

Secondly, if there does not exist \((\tilde{q}_0, y_{\tilde{q}_0}) \in I^\perp_\tilde{M}\), \((\tilde{x}_0, y_0) \neq (\tilde{q}_0, y_{\tilde{q}_0})\) such that \(\xi(\tilde{x}_0, y_0, t_0) = \xi(\tilde{q}_0, y_{\tilde{q}_0}, t_0)\) we observe \(\Lambda_0 = 0\) by virtue of Lemma 3.18. Thus, Proposition 3.17 yields the existence of \(\delta_4 > 0\) and \(K_4 \geq 1\) such that

\[
d(\xi(Z(\tilde{x}, y), t_0 - s), \xi(\tilde{x}_0, y_0, t_0 + s)) < 2s
\]

for \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_4\) and \(s = K_4\delta\) and hence

\[
i_\tilde{M}(\tilde{x}, y) \leq t_0 + s = i_\tilde{M}(\tilde{x}_0, y_0) + K_4 D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)).
\]

for \((\tilde{x}, y) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_4\). Altogether, we observe

\[
i_\tilde{M}(\tilde{x}, y) \leq i_\tilde{M}(\tilde{x}_0, y_0) + K D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)). \tag{3.78}
\]

for \((\tilde{x}, y) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_5 := \min\{\delta_1, \delta_2, \delta_3, \delta_4\}\) and

\[
K := \max\{K_1, K_2, K_3, K_4\}.
\]

Let \(\xi(\tilde{x}_0, y_0, t_1)\) be the first point which is conjugate to \(\varphi(\tilde{x}_0)\) along \(\xi(\tilde{x}_0, y_0, \cdot)\). We have \(t_1 > t_0\) from the assumptions and set \(\rho := \frac{1}{2}(t_1 - t_0)\). By choosing \(\delta_5\) smaller if necessary we derive from [BCS00, Proposition 8.4.1] that for \((\tilde{x}, y) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_0\) and \(0 < t \leq t_1 - \rho = t_0 + \frac{1}{2}(t_1 - t_0)\) no point \(\xi(\tilde{x}, y, t)\) is conjugate to \(\varphi(\tilde{x})\) along \(\xi(\tilde{x}, y, \cdot)\). Moreover, \(\rho \delta_5\) yields \(i_\tilde{M}(\tilde{x}, y) \leq t_0 + \frac{1}{2}(t_1 - t_0)\) for \((\tilde{x}, y) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_0, y_0), (\tilde{x}, y)) < \delta_0 := \min\{\delta_5, \frac{1}{2K}(t_1 - t_0)\}\).

Finally, we consider \((\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \in I^\perp_\tilde{M}\) satisfying \(D_{I^\perp_\tilde{M}}((\tilde{x}_1, y_1), (\tilde{x}, y)) < \delta_0\) for \(i \in \{1, 2\}\) and infer from (3.78)

\[
i_\tilde{M}(\tilde{x}_i, y_1) \leq i_\tilde{M}(\tilde{x}_2, y_2) + K D_{I^\perp_\tilde{M}}((\tilde{x}_1, y_1), (\tilde{x}_2, y_2)).
\]

We conclude the proof by switching the roles of \((\tilde{x}_1, y_1)\) and \((\tilde{x}_2, y_2)\). \(\square\)
3 Local Lipschitz Continuity of $i_{\mathcal{M}}$
Appendix A

A Technical Lemma

In this appendix we give a proof of the following technical lemma which is crucial for the proof of the main theorem.

Lemma A.1. Given an open set $\Omega \subset \mathbb{R}^N$ we consider $F : \Omega \times \mathbb{R}^N \rightarrow [0, \infty)$ with the following properties

(i) $F \in C^2(\Omega \times (\mathbb{R}^N \setminus \{0\})) \cap C^0(\Omega \times \mathbb{R}^N)$

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$ and $x \in \Omega$, $y \in \mathbb{R}^N \setminus \{0\}$

(iii) For each $x \in \Omega$, $y \in \mathbb{R}^N \setminus \{0\}$ the matrix $\frac{\partial^2}{\partial y^i \partial y^j} (F^2(x, y))$ is positive definite.

Then, for any $\Omega' \subset \subset \Omega$ there exists a constant $C = C(\Omega', F)$ such that for any $x \in \Omega'$ and $y_0, y_1 \in \mathbb{R}^N \setminus \{0\}$ satisfying $F(x, y_0) = 1 = F(x, y_1)$ we have

$$F^2(x, \frac{y_0 + y_1}{2}) \leq 1 - CF^2(x, \frac{1}{2}(y_0 - y_1)).$$

Proof. Let $\Omega' \subset \subset \Omega$, $x \in \Omega'$ and $y_0, y_1 \in \mathbb{R}^N \setminus \{0\}$ with $F(x, y_0) = 1$ and $F(x, y_1) = 1$. Initially, we observe that for $y_1 = -y_0$ the claim follows easily and thus assume $y_1 \neq -y_0$ below.

We set $r := |y_0 - y_1|$ and define $f : [0, r] \rightarrow [0, \infty)$ by $f(t) := F^2(x, y_0 + \frac{t}{r}(y_1 - y_0))$ and infer from (iii) that $f$ is strictly convex. Since $f(0) = 1 = f(r)$ we conclude that there exists precisely one $\tilde{t} \in (0, r)$ such that $f'({\tilde{t}}) = 0$.

In case $\tilde{t} = \frac{r}{2}$ we immediately conclude $f(0) = f\left(\frac{r}{2}\right) + \frac{1}{2} f''(\theta) \frac{r^2}{4}$ for some $\theta \in (0, \frac{r}{2})$ or equivalently

$$f\left(\frac{r}{2}\right) = 1 - f''(\theta) \frac{r^2}{4}.$$ 

We already know $f'' > 0$ and compute

$$f''(t) = \frac{\partial^2}{\partial y^i \partial y^j} (F^2) (x, y_0 + \frac{t}{r}(y_1 - y_0)) \frac{1}{r} (y_1 - y_0)^i \frac{1}{r} (y_1 - y_0)^j.$$

We observe that $\frac{\partial^2}{\partial y^i \partial y^j} (F^2) (x, y) = \frac{\partial^2}{\partial y^i \partial y^j} (F^2) (x, y)$ for $\lambda > 0$ and hence

$$\inf_{y \in \mathbb{R}^N \setminus \{0\}} \frac{\partial^2}{\partial y^i \partial y^j} (F^2) (x, y) = \inf_{F(x,y) = 1} \frac{\partial^2}{\partial y^i \partial y^j} (F^2) (x, y).$$
Consequently,

\[ f''(t) \geq \inf_{x \in \Omega} \inf_{|u|, |v|=1} \frac{\partial^2}{\partial y^i \partial y^j} (F^2)(x, y) u^i v^j =: C > 0 \quad (A.1) \]

which yields \( f(\frac{r}{2}) \leq 1 - C \frac{r^2}{4} \).

For the remaining cases we recall that strict convexity implies

\[ f(t_0) < f(t_0 + h) - f'(t_0) h \]

for \( t_0 \in [0, r] \) and \( h \in \mathbb{R} \) chosen such that \( |h| \) is sufficiently small. In particular for \( h \in \{-t_0, r - t_0\} \) we obtain \( f(t_0) < 1 - f'(t_0) h \) and for precisely one of the admissible choices for \( h \) we have

\[ f(t_0) < 1 - |f'(t_0)| |h|. \]

If \( |\tilde{t} - \frac{r}{2}| \geq \frac{r}{8} \) we choose \( t_0 = \frac{r}{2} \) and get

\[ f\left(\frac{r}{2}\right) < 1 - \left|f'\left(\frac{r}{2}\right)\right| \frac{r}{2}. \]

Next, we observe \( |f'(\frac{r}{2})| = |f'(\tilde{t}) + (\frac{r}{2} - \tilde{t}) f''(\theta)| = |\frac{r}{2} - \tilde{t}| f''(\theta) \geq C \frac{r}{8} \) for some \( \theta \) between \( \frac{r}{2} \) and \( \tilde{t} \) by virtue of (A.1) and thus

\[ f\left(\frac{r}{2}\right) < 1 - \frac{1}{4} C \frac{r^2}{4}. \]

For the remaining case we assume \( |\tilde{t} - \frac{r}{2}| < \frac{r}{8} \) and we observe either \( f(\frac{r}{2}) \leq f(\tilde{t} - \frac{r}{8}) \) or \( f(\frac{r}{2}) \leq f(\tilde{t} + \frac{r}{8}) \). In the first case we choose \( t_0 = \tilde{t} - \frac{r}{8} \) whereas we choose \( t_0 = \tilde{t} + \frac{r}{8} \) in the second case. Since \( |f'(\tilde{t} \pm \frac{r}{8})| \geq C \frac{r}{8} \) we conclude

\[ f\left(\frac{r}{2}\right) \leq f\left(\tilde{t} \pm \frac{r}{8}\right) < 1 - \left|f'(\tilde{t} \pm \frac{r}{8})\right| \frac{r}{4} \leq 1 - \frac{1}{8} C \frac{r^2}{4}. \]

Finally, the claim follows from \( F(x, y) = |y| F\left(x, \frac{y}{|y|}\right) \leq |y| \sup_{|y|=1} F(x, y) \). \( \square \)
Bibliography


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