Topological aspects of nonsmooth optimization

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Chapter 1

Introduction

1.1 Nonsmooth optimization framework

We consider the following nonsmooth optimization framework:

$$P(f, F) : \min f(x) \text{ s.t. } x \in M[F], \quad (1.1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-valued objective function, $F : \mathbb{R}^n \rightarrow \mathbb{R}^l$ is a vector-valued function and $M[F] \subset \mathbb{R}^n$ is a feasible set defined by F in some structured way.

Within this general framework the nonsmoothness might be caused by:

- (a) the objective function f ,
- (b) the defining function F ,
- (c) the structure according to which F defines $M[F]$.

Here, we assume functions f, F to be sufficiently smooth and we restrict our study to the nonsmoothness given by (c). Thus, we focus rather on the underlying nonsmooth structures which fit the smooth function F to define the feasible set $M[F]$. We give some examples on particular optimization problems of type (1.1) to illustrate possible nonsmooth structures arise.

Example 1.1.1 (MPCC)

The mathematical programming problem with complementarity constraints (MPCC) is defined as follows:

$$MPCC: \min f(x) \text{ s.t. } x \in M[h, g, F_1, F_2]$$

with

$$M[h, g, F_1, F_2] := \{x \in \mathbb{R}^n \mid \begin{aligned} &F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, \\ &F_{1,m}(x)F_{2,m}(x) = 0, \quad m = 1, \dots, k, \\ &h_i(x) = 0, \quad i \in I, \quad g_j(x) \geq 0, \quad j \in J \end{aligned}\},$$

where $f, h_i, i \in I, g_j, j \in J, F_{1,i}, F_{2,i}, i = 1, \dots, k$ are real-valued and smooth functions.

Here, the nonsmoothness comes into play due to the complementarity constraints:

$$F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, F_{1,m}(x)F_{2,m}(x) = 0, \quad m = 1, \dots, k.$$

Indeed, the basic complementarity relation

$$u \geq 0, v \geq 0, u \cdot v = 0$$

defines the boundary of the non-negative orthant in $\mathbb{R}^2$.

Example 1.1.2 (GSIP)

Generalized semi-infinite optimization problems have the form

$$\text{GSIP:} \quad \text{minimize } f(x) \text{ s.t. } x \in M$$

with

$$M := \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y(x)\}$$

and

$$Y(x) := \{y \in \mathbb{R}^m \mid g_k(x, y) \leq 0, \quad k = 1, \dots, s\}.$$

All defining functions $f, g_k, k = 1, \dots, s$, are assumed to be real-valued and smooth on their respective domains.

Note that testing feasibility for x means that $\inf_{y \in Y(x)} g_0(x, y) \geq 0$. The appearance of the optimal value function $\inf_{y \in Y(x)} g_0(x, y)$ causes nonsmoothness.

Example 1.1.3 (MPVC)

We consider the following mathematical programming problem with vanishing constraints (MPVC):

$$\text{MPVC:} \quad \min f(x) \text{ s.t. } x \in M[h, g, H, G]$$

with

$$M[h, g, H, G] := \{x \in \mathbb{R}^n \mid \begin{array}{l} H_m(x) \geq 0, H_m(x)G_m(x) \leq 0, m = 1, \dots, k, \\ h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J \end{array}\},$$

where $f, h_i, i \in I, g_j, j \in J, H_m, G_m, m = 1, \dots, k$ are real-valued and smooth functions.

Here, the difficulty is due to the vanishing constraints:

$$H_m(x) \geq 0, H_m(x)G_m(x) \leq 0, m = 1, \dots, k.$$

Note that for those x with $H_m(x) = 0$ the sign of $G_m(x)$ is not restricted.

Example 1.1.4 (Bilevel optimization)

We consider bilevel optimization from the optimistic point of view:

$$U : \min_{(x,y)} f(x,y) \quad \text{s.t.} \quad y \in \text{Argmin } L(x),$$

where

$$L(x) : \min_y g(x,y) \quad \text{s.t.} \quad h_j(x,y) \geq 0, j \in J.$$

Above we have $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and the real-valued mappings $f, g, h_j, j \in J$ are smooth. $\text{Argmin } L(x)$ denotes the solution set of the optimization problem $L(x)$.

Here, the nonsmoothness comes from the fact that we deal with a parametric nonlinear programming problem $L(x)$ at the lower level. Moreover, to insure feasibility for (x, y) at the upper level U , the problem $L(x)$ should be solved up to global optimality.

1.2 Topological approach

The main goal of our study is an attempt to understand and classify nonsmooth structures arising in (1.1) within the optimization setting. The basis of such comparison is the topological approach. It encompasses two objects of study:

- the feasible set $M[F]$ and
- the lower level sets $M[f, F]^a := \{x \in M[F] \mid f(x) \leq a\}$, $a \in \mathbb{R}$.

These objects are considered along the levels of study due to topology, optimization and stability issues as outlined in the following scheme (see Figure 1):

Figure 1: Topological approach

On the topology and stability level we deal with topological invariants of $M[F]$ and $M[f, F]^a$, $a \in \mathbb{R}$. Here the questionings mainly arise from. They lead to establishing of an adequate theory on the optimization level. It is worth to point out that the same topological questionings provide different (analytical) optimization concepts while applied to particular problems (e.g. MPCC, GSIP, MPVC and Bilevel optimization). The difference between these analytically described optimization concepts is a key point in understanding and comparing different kinds of nonsmoothness. In what follows we introduce the notions from the above scheme in detail.

For the **structure of $M[F]$** it is crucial to study under which conditions on F the feasible set is a **topological** or **Lipschitz manifold** (with boundary) of an appropriate dimension.

Definition 1.2.1 (Topological and Lipschitz manifold , cf. [102])

A subset $\mathcal{M} \subseteq \mathbb{R}^n$ is called *topological* (resp., *Lipschitz*) *manifold* (with boundary) of dimension $m \geq 0$, if for each $\bar{x} \in \mathcal{M}$ there exist open neighborhoods $U \subseteq \mathbb{R}^n$ of $\bar{x}$ and $V \subseteq \mathbb{R}^n$ of 0 and a homeomorphism $H : U \rightarrow V$ (resp., with H, H^{-1} being Lipschitz continuous) such that

$$(i) \quad H(\bar{x}) = 0,$$

(ii) either

$$H(\mathcal{M} \cap U) = (\mathbb{R}^m \times \{0_{n-m}\}) \cap V,$$

or

$$H(\mathcal{M} \cap U) = (\mathbb{H} \times \mathbb{R}^{m-1} \times \{0_{n-m}\}) \cap V,$$

In the latter case $\bar{x}$ is said to be a *boundary point* of $\mathcal{M}$.

If for all $x \in \mathcal{M}$ the first case in (ii) holds, then $\mathcal{M}$ is called *topological* (resp. *Lipschitz*) *manifold of dimension m* .

We shall use the **tools of nonsmooth and variational analysis** to tackle the above question on $M[F]$ being a Lipschitz manifold. In particular, the application of nonsmooth versions of **Implicit Function Theorem** plays a major role.

Another issue on the structure of $M[F]$ is the **(topological) stability** of the feasible set under smooth perturbations of F .

Definition 1.2.2 (Topological stability)

The feasible set $M[F]$ from (1.1) is called (topologically) *stable* at $\bar{x} \in M[F]$ if there exists a C^1 -neighborhood U of F in $C^1(\mathbb{R}^n, \mathbb{R}^l)$ (w.r.t. the strong or Whitney topology, cf. [42, 61] and Section 1.3) such that for every $\tilde{F} \in U$, the corresponding feasible set $M[\tilde{F}]$ is homeomorphic with $M[F]$.

The stability of the feasible set is tightly connected with its Lipschitz manifold property. Addressing both of them will immediately lead us to suitable **constraint qualifications** for $M[F]$.

Actually, the list of topological invariants worth to study for $M[F]$ usually depends on particular problem realization. E.g., having in mind GSIP and Bilevel optimization, an important issue for the description of the feasible set $M[F]$ becomes the so-called **Reduction Ansatz**. It deals with possibly infinite index sets which can be equivalently reduced to their finite subsets

at least at stationary points. Moreover, the feasible set in GSIP need not to be closed in general. This fact leads to the topological study of its closure instead. Next, the MPVC feasible set is not a Lipschitz manifold, but a set glued together from manifold pieces of different dimensions along their strata.

Regarding the **behavior of the lower level sets** $M[f, F]^a$ we study changes of their topological properties as $a \in \mathbb{R}$ varies. For that, an adequate stationarity concept of **(topologically) stationary points** will be introduced. The analytical description of this concept depends certainly on a particular realization of (1.1). The definition of stationary points will be given in **dual terms** using Lagrange multipliers. Additionally, it will be shown that local minimizers are stationary points under some suitable constraint qualifications.

Within this context, two basic theorems from **Morse theory** (cf. [61, 91]) are crucial.

Theorem 1.2.3 (Deformation Theorem)

If for $a < b$ the (compact) set $M[f, F]_a^b := \{x \in M \mid a \leq f(x) \leq b\}$ does not contain stationary points, then the set $M[f, F]^a$ is a strong deformation retract of $M[f, F]^b$.

As a consequence, the homotopy type of the lower level sets $M[f, F]^a$ and $M[f, F]^b$ are equal. This means that the connectedness structure of the lower level sets does not change when passing from level a to level b . In particular, the number of connected (path)components remains invariant.

For the second result a notion of a nondegenerate stationary point, along with its index, will be introduced. Note that a nondegenerate stationary point is a local minimizer if and only if its index vanishes.

Theorem 1.2.4 (Cell-attachment Theorem)

If $M[f, F]_a^b$ contains exactly one nondegenerate stationary point, then $M[f, F]^b$ is homotopy equivalent to $M[f, F]^a$ with a q -cell attached. Here, the dimension q is the so-called index of the nondegenerate stationary point.

The latter two theorems on homotopy equivalence show that **Morse relations**, such as Morse inequalities (cf. [61]), are valid. Roughly speaking, Morse relations relate the existence of stationary points of various index with the topology of the feasible set. A **global interpretation** of Deformation and Cell-attachment Theorems is the following. Suppose that the feasible set is compact and connected and that all stationary points are nondegenerate with pairwise different functional values. Then, passing a level corresponding to a local minimizer, a connected component of the lower level set is created.

Different components can only be connected by attaching 1-cells. This shows the existence of at least $(k - 1)$ stationary points with index equal to one, where k is the number of local minimizers; see also [27, 61]. This issue is closely related to the global aspects of optimization theory, in particular, to the existence of $0 - 1 - 0$ and $0 - n - 0$ graphs. The latter connect local minimizers with stationary points having index equal to one, resp. with local maximizers [61]. Finally, we refer to [2, 6, 90] for the results on Morse theory for piecewise smooth functions.

The **structural stability w.r.t. lower level sets** is defined via special equivalence relation on $P(f, F)$ as follows.

Definition 1.2.5 (Equivalence relation for optimization problems)

Two optimization problems $P(f, F)$ and $P(\tilde{f}, \tilde{F})$ are called equivalent if there exist continuous mappings $\varphi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that:

1. The mapping $\varphi(a, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism for each $a \in \mathbb{R}$.
2. The mapping ψ is a homeomorphism and monotonically increasing.
3. For all $a \in \mathbb{R}$ we have $\varphi(a, M[f, F]^a) = M[\tilde{f}, \tilde{F}]^{\psi(a)}$.

The latter concept of equivalence was introduced in [32], and it was shown that it is indeed an equivalence relation.

Definition 1.2.6 (Structural stability)

The optimization problem $P(f, F)$ is called structurally stable if there exists a C^2 -neighborhood U of (f, F) in $C^2(\mathbb{R}^n, \mathbb{R}) \times C^2(\mathbb{R}^n, \mathbb{R}^l)$ (w.r.t. the strong or Whitney topology) such that for every $(\tilde{f}, \tilde{F}) \in U$, $P(\tilde{f}, \tilde{F})$ is equivalent to $P(f, F)$.

The characterization of structural stability is tightly related to both the stability of the feasible set (cf. Definition 1.2.2) and the **strong stability of stationary points** (in the sense of M. Kojima). The latter concept enlightens **parametric aspects** within the optimization context.

Definition 1.2.7 (Strong stability, cf. [80])

A stationary point $\bar{x} \in M[F]$ for $P(f, F)$ is called (C^2) -strongly stable if for some $r > 0$ and each $\varepsilon \in (0, r]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that whenever $(\tilde{f}, \tilde{F}) \in C^2$ and $\left\| (f - \tilde{f}, F - \tilde{F}) \right\|_{B(\bar{x}, r)}^{C^2} \leq \delta$, the ball $B(\bar{x}, \varepsilon)$ contains a stationary point $\tilde{x}$ for $P(\tilde{f}, \tilde{F})$ which is unique in $B(\bar{x}, r)$.

1.3 Genericity and stability issues

Our goal is to justify the assumptions made on the optimization level such as constraint qualifications, Reduction Ansatz, nondegeneracy of stationary points etc. For that, genericity and stability issues w.r.t. the strong or Whitney topology on defining functions (f, F) come into play.

Let $C^k(\mathbb{R}^n, \mathbb{R})$, $k = 0, 1, \dots$, denote the space of k -times continuously differentiable real-valued functions. Let $C^k(\mathbb{R}^n, \mathbb{R})$ be endowed with the strong (or Whitney) C^k -topology, denoted by C_s^k (cf. [42, 61]). The C_s^k -topology is generated by allowing perturbations of the functions and their derivatives up to k -th order which are controlled by means of continuous positive functions. The product space of continuously differentiable functions will be topologized with the corresponding product topology. Note that the space of continuously differentiable functions endowed with the strong C_s^k -topology constitutes a Baire space. We say that a set is C_s^k -generic if it contains a countable intersection of C_s^k -open and C_s^k -dense subsets. Generic sets in a Baire space are dense as well.

Next, we explain a typical application of C_s^k -topology in the optimization context. Let $\mathfrak{A}$ be an assumption involving derivatives of (f, F) up to k -th order in its formulation (e.g. constraint qualification, conditions in Reduction Ansatz, nondegeneracy etc.) We are interested in the following type of results.

Theorem 1.3.1 (Assumption $\mathfrak{A}$ is generic and stable)

Let $\mathcal{A}$ denote the set of problem data $(f, F) \in C^k(\mathbb{R}^n, \mathbb{R}) \times C^k(\mathbb{R}^n, \mathbb{R}^l)$ such that the assumption $\mathfrak{A}$ is satisfied. Then, $\mathcal{A}$ is C_s^k -generic and C_s^k -open.

The above genericity and stability theorem can be interpreted as follows. If Assumption $\mathfrak{A}$ does not hold for the concrete problem data (f, F) , then we may find arbitrary small perturbed problem $(\tilde{f}, \tilde{F})$ with assumption $\mathfrak{A}$ fulfilled. Moreover, if Assumption $\mathfrak{A}$ holds for (f, F) , then it also holds for sufficiently small perturbations of (f, F) . We point out that the proofs of such results are mainly based on **transversality theory**, in particular, on Thom's Transversality Theorem (cf. [42, 61, 88]).

The justification of assumptions w.r.t. genericity and stability issues does not certainly exclude the study of so called singular situation. The latter are characterized by the fact that certain assumption is not fulfilled. One speaks also of problem data being not in general position. This phenomena are studied within the scope of **singularity theory** (cf. [3, 13, 28, 29]). It remains very challenging to apply ideas from singularity theory in the

optimization context (cf. [4, 59, 55]). We will touch this topic studying Bilevel optimization.

Chapter 2

Mathematical programming problems with complementarity constraints

2.1 Applications and examples

We consider the following mathematical programming problem with complementarity constraints (MPCC):

$$\text{MPCC: } \min f(x) \text{ s.t. } x \in M[h, g, F_1, F_2] \quad (2.1)$$

with

$$M[h, g, F_1, F_2] := \{x \in \mathbb{R}^n \mid \begin{aligned} &F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, \\ &F_{1,m}(x)F_{2,m}(x) = 0, m = 1, \dots, k, \\ &h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J \}, \end{aligned}$$

where $h := (h_i, i \in I)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|I|})$, $g := (g_j, j \in J)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|J|})$, $F_1 := (F_{1,i}, i = 1, \dots, k)^T, F_2 := (F_{2,i}, i = 1, \dots, k)^T \in C^2(\mathbb{R}^n, \mathbb{R}^k)$, $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $k + |I| \leq n$, $|J| < \infty$. For simplicity, we write M for $M[h, g, F_1, F_2]$ if no confusion is possible.

For $m = 1, \dots, k$ the constraint " $F_{1,m}(x) \geq 0, F_{2,m}(x) \geq 0, F_{1,m}(x)F_{2,m}(x) = 0$ " is called a complementarity constraint. Note that it can be equivalently written as " $\min \{F_{1,m}(x), F_{2,m}(x)\} = 0$ ".

MPCC is a special case of the so called mathematical programming problem with equilibrium constraints (MPEC) (cf. [86]). In what follows we show that MPCC's appear quite naturally in bilevel optimization (via Karush-Kuhn-Tucker or Fritz John conditions at the lower level) and by solving nonlinear complementarity problems. Moreover, complementarity constraints

arise in the context of variational inequalities. For other applications we refer to [24, 86, 96].

Bilevel optimization with convexity at the lower level

We model the bilevel optimization problem in the so-called optimistic formulation. To this aim, assume that the follower solves the parametric optimization problem (lower level problem L)

$$L(x) : \min_y g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, j \in J$$

and that the leader's optimization problem (upper level problem U) is the following

$$U : \min_{(x,y)} f(x, y) \quad \text{s.t.} \quad y \in \text{Argmin } L(x).$$

Above we have $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and the real valued mappings $f, g, h_j, j \in J$ belong to $C^2(\mathbb{R}^n \times \mathbb{R}^m)$. $\text{Argmin } L(x)$ denotes the solution set of the optimization problem $L(x)$. For simplicity, additional (in)equality constraints in defining U are omitted.

We assume convexity at the lower level $L(\cdot)$, i.e. for all $x \in \mathbb{R}^n$ let the functions $g(x, \cdot)$, $-h_j(x, \cdot)$, $j \in J$ be convex. Let e.g. Slater Constraint Qualification (CQ) hold for $L(\cdot)$. Then, it is well-known that $y \in \text{Argmin } L(x)$ if and only if there exist Lagrange multipliers $\mu_j \in \mathbb{R}$, $j \in J$ such that:

$$D_y g(x, y) = \sum_{j \in J} \mu_j D_y h_j(x, y), \mu_j \geq 0, h_j \geq 0, \mu_j h_j(x, y) = 0. \quad (2.2)$$

Hence, we can write the corresponding MPCC:

$$U\text{-KKT} : \min_{(y,\mu) \in \mathbb{R}^m \times \mathbb{R}^{|J|}} g(x, y) \quad \text{s.t.}$$

$$D_y g(x, y) = \sum_{j \in J} \mu_j D_y h_j(x, y), \mu_j \geq 0, h_j \geq 0, \mu_j h_j(x, y) = 0.$$

Here, complementarity constraints are $\mu_j \geq 0$, $h_j \geq 0$, $\mu_j h_j(x, y) = 0$.

The links between U and $U\text{-KKT}$ were elaborated in [18]. It turns out that global solutions of U and $U\text{-KKT}$ coincide. But, due to the possible non-uniqueness of Lagrange multipliers in (2.2) local solutions of U and $U\text{-KKT}$ may differ.

Note that it is very restrictive to assume Slater CQ in $L(x)$ for all $x \in \mathbb{R}^n$. Hence, one may try to assume Slater CQ only at the point of interest $\bar{x}$.

However, in that case even global solutions of U and U -KKT may differ (as shown in [18]).

Without assuming Slater CQ we arrive at the MPCC-relaxation of U :

$$\begin{aligned}
 U\text{-JOHN : } & \min_{(y, \delta, \mu) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{|J|}} g(x, y) \quad \text{s.t.} \\
 & \delta D_y g(x, y) = \sum_{j \in J} \mu_j D_y h_j(x, y), \\
 & \mu_j D_y h_j(x, y), \mu_j \geq 0, h_j \geq 0, \mu_j h_j(x, y) = 0, \delta \geq 0.
 \end{aligned} \tag{2.3}$$

Here, we use the fact that $y \in \text{Argmin } L(x)$ fulfills the Fritz John condition. In fact, generically one can not exclude the violation of Linear Independence or even Mangasarian-Fromovitz Constraint Qualification at the lower level. Thus, the case of vanishing δ in (2.3) can not be omitted (see Chapter 5 for details).

Solving nonlinear complementarity problems

We consider a nonlinear complementarity problem (NCP) of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0,$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. Such problems appear in many applications such as equilibria models of economics, contact and structural mechanics problems, obstacle problems (cf. also [97]).

Setting $H(x) := \min \{x, F(x)\}$ componentwise, we obtain a residual optimization problem

$$\text{RES : } \min_x \vartheta(x) := \frac{1}{2} H(x)^T H(x) \quad \text{s.t. } x \geq 0.$$

Obviously, if $\bar{x}$ a solution of NCP then $\bar{x}$ is a solution of RES with $\vartheta(\bar{x}) = 0$. Moreover, ϑ is nonnegative and vanishes exactly at solutions of NCP.

With $y := x - \min \{x, F(x)\}$ it is easy to see that RES can be equivalently written as MPCC:

$$\text{RES-MPCC : } \min_{(x, y)} \frac{1}{2} (x - y)^T (x - y) \quad \text{s.t.}$$

$$x \geq 0, y \geq 0, F(x) - x - y \geq 0, y^T (F(x) - x - y) = 0.$$

The latter problem is used to solve NCP numerically (cf. [86]).

Variational inequalities setting

Let $K \subset \mathbb{R}^n$ and $F : K \longrightarrow \mathbb{R}^n$ be given. The variational inequality $VI(K, F)$ is the following problem:

$$VI(K, F) : \quad \text{Find } x \in \mathbb{R}^n \text{ such that } (y - x)^T F(x) \geq 0 \text{ for all } y \in K.$$

Clearly, $\bar{x}$ is a solution of $VI(K, F)$ if and only if

$$0 \in F(\bar{x}) + N(\bar{x}, K), \tag{2.4}$$

where $N(\bar{x}, K)$ is a normal cone of K at $\bar{x}$:

$$N(\bar{x}, K) := \{d \in \mathbb{R}^n \mid d^T(\bar{x} - y) \leq 0 \text{ for all } y \in K\}.$$

Equation (2.4) can be seen as a generalization of the first order optimality conditions to minimize a differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ on a convex set K .

Further, if K is a cone we may link variational inequalities with so-called complementarity problems:

$$CP(K, F) : \quad \text{Find } x \in \mathbb{R}^n \text{ such that } x \in K, F(x) \in K^*, x^T F(x) = 0$$

where $K^* := \{d \in \mathbb{R}^n \mid v^T d \geq 0 \text{ for all } v \in K\}$ is a dual cone of K .

It can be shown (see e.g. [23]) that in case of a cone K solutions of $VI(K, F)$ and $CP(K, F)$ coincide. Moreover, let

$$K := \{x \in \mathbb{R}^n \mid Ax \leq b, Cx = d\}$$

with matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{l \times n}$ and vectors $b \in \mathbb{R}^m$, $d \in \mathbb{R}^l$. Then, $\bar{x}$ solves $VI(K, F)$ if and only if there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^l$ such that

$$F(x) + A^T \lambda + C^T \mu = 0, Cx - d = 0,$$

$$\lambda \geq 0, b - Ax \geq 0, \lambda^T(b - Ax) = 0.$$

The latter system exhibits complementarity constraints and, hence, fits in the context of MPCC.

Note that setting $K := \mathbb{H}^n$ in $CP(K, F)$ we obtain the usual nonlinear complementarity problem NCP.

2.2 Stability and structure of the feasible set

In this section we concentrate only on the substantial new case of complementarity constraints. Hence, we omit smooth equality and inequality constraints and consider the following mathematical programming problem with complementarity constraints (MPCC):

$$\text{MPCC: } \min f(x) \text{ s.t. } x \in M[F_1, F_2] \quad (2.5)$$

with

$$M[F_1, F_2] := \{x \in \mathbb{R}^n \mid F_1(x) \geq 0, F_2(x) \geq 0, F_1(x)^T F_2(x) = 0\},$$

where $F_1 := (F_{1,i}, i = 1, \dots, k)^T, F_2 := (F_{2,i}, i = 1, \dots, k)^T \in C^1(\mathbb{R}^n, \mathbb{R}^k)$, $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $k \leq n$.

Note that $M[F_1, F_2]$ can be written as follows:

$$M[F_1, F_2] = \{x \in \mathbb{R}^n \mid \min\{F_{1,i}(x), F_{2,i}(x)\} = 0, i = 1, \dots, k\}.$$

Here, we deal with the local stability property of the feasible set $M[F_1, F_2]$ with respect to C^1 -perturbations of the defining functions F_1 and F_2 . Under C^1 -neighborhood of a function $g \in C^1(\mathbb{R}^n, \mathbb{R}^l)$ we understand a subset of $C^1(\mathbb{R}^n, \mathbb{R}^l)$, which contains for some $\varepsilon > 0$ the set

$$\left\{ \tilde{g} \in C^1(\mathbb{R}^n, \mathbb{R}^l) \mid \sum_{i=1}^l \sup_{x \in \mathbb{R}^n} (|\tilde{g}_i(x) - g_i(x)| + \|\nabla \tilde{g}_i(x) - \nabla g_i(x)\|) \leq \varepsilon \right\}.$$

Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$ and ∇g_i stands for the gradient of g_i as a column vector.

Definition 2.2.1 *The feasible set $M[F_1, F_2]$ from (2.5) is called locally stable at $\bar{x} \in M[F_1, F_2]$ if there exists a $\mathbb{R}^n$ -neighborhood V of $\bar{x}$ and a C^1 -neighborhood U of (F_1, F_2) in $C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k)$ such that for every $(\tilde{F}_1, \tilde{F}_2) \in U$, the corresponding feasible set $M[\tilde{F}_1, \tilde{F}_2] \cap V$ is homeomorphic with $M[F_1, F_2] \cap V$.*

Our main goal is to characterize the local stability property of the feasible set $M[F_1, F_2]$ in terms of the gradients of F_1 and F_2 . In case of standard nonlinear programming (local) stability of the feasible set was studied in [33, 65]. In fact, for the feasible set

$$M_{NLP}[h, g] := \{x \in \mathbb{R}^n \mid h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J\}$$

$$\text{with } h_i, g_j \in C^1(\mathbb{R}^n, \mathbb{R}), |I| < n, |J| < \infty,$$

the local stability property at $\bar{x} \in M_{NLP}[h, g]$ is characterized by the Mangasarian-Fromovitz Constraint Qualification (MFCQ), i.e.:

- (1) $\nabla h_i(\bar{x})$, $i \in I$ are linearly independent,
- (2) there exists a $\xi \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \nabla h_i(\bar{x})\xi &= 0, \quad i \in I, \\ \nabla g_j(\bar{x})\xi &> 0, \quad j \in J_0(\bar{x}) := \{j \in J \mid g_j(\bar{x}) = 0\}. \end{aligned}$$

For stability in MPCC setting we propose a kind of Mangasarian-Fromovitz Condition (MFC) and its stronger version (SMFC). Section 2.2.1 will be devoted to MFC and SMFC, their relations to other Constraint Qualifications (Linear Independent CQ, Mordukhovich's extremal principle, metric regularity, Generalized Mangasarian-Fromovitz CQ and standard subdifferential qualification condition). The conjectured equivalence of MFC and SMFC is discussed. In Section 2.2.2 we prove that SMFC implies local stability and insures that the MPCC feasible set is a Lipschitz manifold. Here, the application of nonsmooth versions of Implicit Function Theorems (due to Clarke and Kummer) is crucial.

We refer to [20, 68] for details.

2.2.1 Constraint Qualifications MFC and SMFC

Definition of MFC and SMFC

Assume that the following Assumption A holds throughout.

Assumption A For every $\bar{x} \in M[F_1, F_2]$ and $i \in \{1, \dots, k\}$ the set of vectors $\{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = 0, j = 1, 2\}$ is linearly independent.

Further, we define for $\bar{x} \in M[F_1, F_2]$ and $i = 1, \dots, k$ the (non-empty) convex hull

$$C_i(\bar{x}) := \text{conv}\{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = 0\}.$$

Note that $C_i(\bar{x}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x})$ the Clarke's subdifferential of the function $\min\{F_{1,i}(\cdot), F_{2,i}(\cdot)\}$ (cf. [15]).

Definition 2.2.2 (MFC and SMFC)

The Mangasarian-Fromovitz Condition (MFC) is said to hold at $\bar{x} \in M[F_1, F_2]$ if any k vectors $(w_1, \dots, w_k) \in C_1(\bar{x}) \times \dots \times C_k(\bar{x})$ are linearly independent.

The Strong Mangasarian-Fromovitz Condition (SMFC) is said to hold at $\bar{x} \in M[F_1, F_2]$ if there exists a k -dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1, \dots, u_k) \in P_E(C_1(\bar{x})) \times \dots \times P_E(C_k(\bar{x}))$ are linearly independent, where $P_E : \mathbb{R}^n \rightarrow E$ denotes the orthogonal projection.

Remark 2.2.3 In presence of additional C^1 -equality and -inequality constraints in the description of the MPCC feasible set, MFC will be enlarged by the standard MFCQ formulation with respect to these constraints.

We give some equivalent reformulations of MFC and SMFC.

Lemma 2.2.4 (MFC and SMFC via Clarke's subdifferentials)

(a) MFC at $\bar{x} \in \mathbb{R}^n$ means that the Clarke's subdifferentials

$$\partial \min\{F_{1,i}, F_{2,i}\}(\bar{x}), i = 1, \dots, k \text{ are linearly independent.}$$

(b) SMFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if there exists a basis decomposition of $\mathbb{R}^n$ given by a nonsingular $n \times n$ matrix A such that after the linear coordinate transformation $y := Ax$ the Clarke's subdifferentials of the functions $h_i(y) := \min\{F_{1,i}(A^{-1}(y)), F_{2,i}(A^{-1}(y))\}$ w.r.t. $z := (y_{n-k+1}, \dots, y_n)$ are linearly independent, i.e.

$$\partial_z h_i(\bar{y}), i = 1, \dots, k \text{ are linearly independent,}$$

$$\text{where } \partial_z h_i(\bar{y}) := \{\eta \in \mathbb{R}^k \mid \text{there exists } \xi \in \mathbb{R}^{n-k} \text{ with } [\xi, \eta] \in \partial h_i(\bar{y})\}.$$

Proof. For (a) we only recall that $C_i(\bar{x}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x})$. To prove (b) we first calculate:

$$\partial h_i(\bar{y}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x}) \cdot A^{-1} = C_i(\bar{x}) \cdot A^{-1}.$$

Hence, if SMFC holds at $\bar{x}$ we take as columns of A^{-1} any orthogonal bases of $E^\perp$ and E . Conversely, given A we set the linear subspace E be spanned by the k last columns of A^{-1} . $\square$

Remark 2.2.5 (SMFC as a maximal rank condition)

From Lemma 2.2.4 (b) we see that SMFC is the so-called maximal rank condition (in terms of Clarke, cf. [15]) w.r.t. some basis decomposition of $\mathbb{R}^n$. It turns out that the concrete choice of such a basis decomposition may effect the validity of maximal rank condition (see Example 2.2.28 for details). It means that the property of maximal rank is not basis independent. This observation is crucial and motivates SMFC (see also Section 2.2.2).

Lemma 2.2.6 (MFC and SMFC via basis enlargement)

- (a) MFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if for any $w_i \in C_i(\bar{x}), i = 1, \dots, k$ there exist $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that the vectors $w_1, \dots, w_k, \xi_1, \dots, \xi_{n-k}$ are linearly independent.
- (b) SMFC holds at $\bar{x} \in \mathbb{R}^n$ if and only if there exist $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that for any $w_i \in C_i(\bar{x}), i = 1, \dots, k$ the vectors $w_1, \dots, w_k, \xi_1, \dots, \xi_{n-k}$ are linearly independent.

Proof. (a) follows immediately from the definition of linear independence. To prove (b): if SMFC holds we choose $\xi_1, \dots, \xi_{n-k}$ being a basis of $E^\perp$. Conversely, we set $E := (\text{span}\{\xi_1, \dots, \xi_{n-k}\})^\perp$ in SMFC. $\square$

Further, we notice that MFC is a natural Constraint Qualification for the Clarke stationarity.

Definition 2.2.7 (cf. [23, 106])

A point $\bar{x} \in M[F_1, F_2]$ is called Clarke stationary (C-stationary) if there exist real numbers $\lambda_{j,i}, j = 1, 2, i = 1, \dots, k$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^k (\lambda_{1,i} \nabla F_{1,i}(\bar{x}) + \lambda_{2,i} \nabla F_{2,i}(\bar{x})) = 0,$$

$$F_{j,i}(\bar{x}) \lambda_{j,i} = 0 \text{ for every } j = 1, 2, i = 1, \dots, k,$$

$$\lambda_{1,i} \lambda_{2,i} \geq 0 \text{ for every } i \in \{1, \dots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.$$

Proposition 2.2.8 (MFC and C-stationarity)

If $\bar{x}$ is a local minimizer of the MPCC and MFC holds at $\bar{x}$, then $\bar{x}$ is C-stationary.

Proof. Due to Lemma 1 in [106] if $\bar{x}$ is a local minimizer of the MPCC then there exist real numbers λ , $\lambda_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ (not all vanishing) such that

$$\lambda \nabla f(\bar{x}) + \sum_{i=1}^k (\lambda_{1,i} \nabla F_{1,i}(\bar{x}) + \lambda_{2,i} \nabla F_{2,i}(\bar{x})) = 0,$$

$$F_{j,i}(\bar{x}) \lambda_{j,i} = 0 \text{ for every } j = 1, 2, i = 1, \dots, k,$$

$$\lambda_{1,i} \lambda_{2,i} \geq 0 \text{ for every } i \in \{1, \dots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.$$

Clearly, if $\lambda = 0$ then MFC is violated at $\bar{x}$. Hence, $\bar{x}$ is C-stationary. $\square$

For more details on C-stationarity and other stationarity concepts, such as W-, A-, M-, and S-stationarity, see [25], [86], [95], [106], [119] and Sections 2.2.1, 2.4.

Conceptual relations to other CQ

We recall the well-known **LICQ for MPCC** (e.g. [106, 107]), which is said to hold at $\bar{x} \in M[F_1, F_2]$ if

$$\{\nabla F_{i,j}(\bar{x}) \mid F_{i,j}(\bar{x}) = 0, i = 1, \dots, k, j = 1, 2\} \text{ are linearly independent.}$$

LICQ can be equivalently formulated in terms of transversal intersection of stratified sets (see [61]). As shown in [107], LICQ is a generic constraint qualification. However, LICQ is not necessary for local stability as one can see from the following Example 2.2.9. In this and all further examples only the local stability in 0 is of interest.

Example 2.2.9 (2D, stable: one point $\longrightarrow$ one point)

The set $M^{2.2.9} := \{(x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{x - y, 2x - y\} = 0\}$ is a singleton and it is locally stable at 0 (see Figure 2). However, LICQ does not hold at 0.

Figure 2: Illustration of Example 2.2.9

In this sense, LICQ appears to be too restrictive. It comes from the fact that LICQ does not impose the combinatorial structure of the complementarity constraints. Additionally, we notice that LICQ implies MFC.

Another condition, we intend to discuss, comes from the **exact Mordukhovich's extremal principle** (cf. [26, 92]).

Let $\Omega \subset \mathbb{R}^n$ be any arbitrary closed set and $\bar{x} \in \Omega$. The nonempty cone

$$T(\bar{x}, \Omega) := \limsup_{\tau \searrow 0} \frac{\Omega - \bar{x}}{\tau} = \left\{ d \in \mathbb{R}^n \mid \text{there exist } x_k \longrightarrow \bar{x}, x_k \in \Omega, \tau_k \searrow 0 \text{ such that } \frac{x_k - \bar{x}}{\tau_k} \longrightarrow d \right\}$$

is called the contingent (also Bouligand or tangent) cone to Ω at x .

The Fréchet normal cone is defined via polarization as follows:

$$\hat{N}(\bar{x}, \Omega) := (T(\bar{x}, \Omega))^\circ.$$

Finally, the limiting normal cone (also called Mordukhovich normal cone) is defined by

$$N(\bar{x}, \Omega) := \limsup_{x' \xrightarrow{\Omega} \bar{x}} \hat{N}(x', \Omega) = \left\{ \lim_{k \rightarrow \infty} w_k \mid \text{there exist } x_k \longrightarrow \bar{x}, x_k \in \Omega, w_k \in \hat{N}(x_k, \Omega) \right\}.$$

Definition 2.2.10 (local extremal point of set systems, cf. [92])

Let $\Omega_i, i = 1, \dots, k$ be nonempty subsets of $\mathbb{R}^n$ and $\bar{x} \in \bigcap_{i=1}^k \Omega_i$. We say that $\bar{x}$ is a local extremal point of the set system $\{\Omega_1, \dots, \Omega_k\}$ if there are sequences $\{a_{ij}\} \subset \mathbb{R}^n, i = 1, \dots, k$, and a neighborhood V of $\bar{x}$ such that $a_{ij} \longrightarrow 0$ as $j \longrightarrow \infty$ and

$$\bigcap_{i=1}^k (\Omega_i - a_{ij}) \cap V = \emptyset \text{ for all large } j \in \mathbb{N}.$$

We recall the finite-dimensional version of the exact Mordukhovich's extremal principle.

Theorem 2.2.11 (Exact Extremal Principle in finite dimensions, cf. [92])

Let Ω_i , $i = 1, \dots, k$ be nonempty closed subsets of $\mathbb{R}^n$ and $\bar{x} \in \bigcap_{i=1}^k \Omega_i$ be an extremal point of the set system $\{\Omega_1, \dots, \Omega_k\}$. Then there are $x_i^* \in N(\bar{x}, \Omega_i)$, $i = 1, \dots, k$ (not all vanishing) such that $\sum_{i=1}^k x_i^* = 0$.

Actually, Theorem 2.2.11 provides a sufficient condition for the property that the intersection of nonempty closed subsets Ω_i , $i = 1, \dots, k$ of $\mathbb{R}^n$ remains locally nonempty with respect to translations. This sufficient condition can be formulated as follows:

(Δ) For all $x_i^* \in N(\bar{x}, \Omega_i)$, $i = 1, \dots, k$:

$$\sum_{i=1}^k x_i^* = 0 \text{ implies } x_i^* = 0, i = 1, \dots, k.$$

In order to refer to the foregoing discussion in our setting, we set from now on $\Omega_i := M_i$, $i = 1, \dots, k$, where

$$M_i := \{x \in \mathbb{R}^n \mid F_{1,i}(x) \geq 0, F_{2,i}(x) \geq 0, F_{1,i}(x)F_{2,i}(x) = 0\}.$$

Proposition 2.2.12 (MFC implies Δ)

If MFC holds at $\bar{x} \in M[F_1, F_2]$ then Δ also holds at $\bar{x}$.

Proof. Let $i \in \{1, \dots, k\}$ be fixed. We provide a representation formula for $N(\bar{x}, M_i)$. We restrict ourselves to the interesting case that $F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0$. Due to Assumption A we choose vectors $\xi_1, \dots, \xi_{n-2} \in \mathbb{R}^n$ which form - together with the vectors $\nabla F_{1,i}(\bar{x}), \nabla F_{2,i}(\bar{x})$ - a basis for $\mathbb{R}^n$. Next we put $y = \Phi(x)$ as follows:

$$y_1 := F_{1,i}(x), y_2 := F_{2,i}(x), y_3 := \xi_1^T(x - \bar{x}), \dots, y_n := \xi_{n-2}^T(x - \bar{x}).$$

Note that $\Phi(\bar{x}) = 0$ and $D\Phi(\bar{x})$ is nonsingular. Therefore, Φ maps M_i diffeomorphically to $K := \{y \in \mathbb{R}^n \mid y_1 \geq 0, y_2 \geq 0, y_1 y_2 = 0\}$ locally at $\bar{x}$. Setting $L := \{y \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0, y_1 y_2 = 0\}$ Proposition 6.41 from [103] yields:

$$N(0, K) = N(0, L \times \mathbb{R}^{n-2}) = N(0, L) \times N(0, \mathbb{R}^{n-2}).$$

From [26] and [95] we conclude that $N(0, L) = \mathbb{R}_-^2 \cup L$. Clearly, $N(0, \mathbb{R}^{n-2}) = \{0_{n-2}\}$. Altogether, we get

$$N(0, K) = \mathbb{R}_-^2 \cup L \times \{0_{n-2}\}.$$

Using Exercise 6.7 (change of coordinates) from [103] we get:

$$N(\bar{x}, M_i) = \{\beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1 \beta_2 = 0\}. \quad (2.6)$$

Analogously, we obtain:

$$\widehat{N}(\bar{x}, M_i) = \{\beta_1 \nabla F_{1,i}(\bar{x}) + \beta_2 \nabla F_{2,i}(\bar{x}) \mid \beta_1 \leq 0, \beta_2 \leq 0\}. \quad (2.7)$$

The representation (2.7) yields that MFC is equivalent to the following condition:

$$\begin{aligned} &\text{For all } x_i^* \in \pm \widehat{N}(\bar{x}, M_i), i = 1, \dots, k : \\ &\sum_{i=1}^k x_i^* = 0 \text{ implies } x_i^* = 0, i = 1, \dots, k. \end{aligned}$$

Since $N(\bar{x}, M_i) \subset \pm \widehat{N}(\bar{x}, M_i)$, $i = 1, \dots, k$, (cf. (2.6) and (2.7)), the proposition follows immediately. $\square$

Corollary 2.2.13 (MFC via Fréchet normal cones)

MFC is equivalent to the following condition:

$$\begin{aligned} &\text{For all } x_i^* \in \pm \widehat{N}(\bar{x}, M_i), i = 1, \dots, k : \\ &\sum_{i=1}^k x_i^* = 0 \text{ implies } x_i^* = 0, i = 1, \dots, k, \end{aligned}$$

where $M_i = \{x \in \mathbb{R}^n \mid F_{1,i}(x) \geq 0, F_{2,i}(x) \geq 0, F_{1,i}(x)F_{2,i}(x) = 0\}$.

As we show by Example 2.2.14, $\triangle$ is not sufficient for $M[F_1, F_2]$ to be locally stable at 0. In this and all further examples in 3D we understand under "two-star", "three-star" and "four-star" subsets of $\mathbb{R}^3$ as depicted in Figure 3 up to a homeomorphism.

Figure 3: "two-star"

"three-star"

"four-star"

Example 2.2.14 (3D, nonstable: "four-star" $\longrightarrow$ 2 "two-stars")

Consider the "four-star" subset

$$M^{2.2.14} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0, \min\{x+y-\sqrt{2}z, x+y+\sqrt{2}z\} = 0\}$$

(see Figure 4). After an appropriate perturbation the resulting set would have two path-connected components. Therefore, $M^{2.2.14}$ is not locally stable at 0.

Figure 4: Illustration of Example 2.2.14

To show that Δ holds at 0 we set

$$M_1^{2.2.14} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0\},$$

$$M_2^{2.2.14} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x+y-\sqrt{2}z, x+y+\sqrt{2}z\} = 0\}.$$

and obtain due to (2.6) from the proof of Proposition 2.2.12:

$$N(0, M_1^{2.2.14}) = \{(\beta_1, \beta_2, 0)^T \in \mathbb{R}^3 \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0\}.$$

$$N(0, M_2^{2.2.14}) = \left\{ \beta_1 \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \mid \text{either } \beta_1 < 0, \beta_2 < 0 \text{ or } \beta_1\beta_2 = 0 \right\}.$$

From the above representations of $N(0, M_1^{2.2.14})$ and $N(0, M_2^{2.2.14})$ it is easy to see that Δ (but not MFC) is satisfied at $0 \in M^{2.2.14}$. $\square$

The next stability concept we would like to discuss here is **metric regularity**. We recall that a multi-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ is called metrically regular (with rank $L > 0$) at $(\bar{x}, \bar{y}) \in \text{gph } T$ if, for certain neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, it holds:

$$\text{dist}(x, T^{-1}(y)) \leq L \text{dist}(y, T(x)) \text{ for all } x \in U, y \in V.$$

Further, a multi-valued map $S : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$ is called pseudo-Lipschitz (with rank $L > 0$) at $(\bar{y}, \bar{x}) \in \text{gph } S$ if there are neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, such that, given any points $(y, x) \in (V \times U) \cap \text{gph } S$, it holds:

$$\text{dist}(x, S(y')) \leq L \|y' - y\| \text{ for all } y' \in V.$$

(see e.g. [50], [78]).

It holds (cf. [49]) that T is metrically regular at $(\bar{x}, \bar{y}) \in \text{gph } T$ if and only if T^{-1} is pseudo-Lipschitz at $(\bar{y}, \bar{x})$.

It is well-known from [99] that the solution map

$$S(y, z) := \{x \in \mathbb{R}^n \mid h(x) = y, g(x) \leq z\}, (g, h) \in C^1(\mathbb{R}^n, \mathbb{R}^{k+m}),$$

is pseudo-Lipschitz at $(0, 0, \bar{x})$ if and only if MFCQ is satisfied at $\bar{x} \in S(0, 0)$. It means, therefore, that the local stability of $M_{NLP}[h, g]$ ($= S(0, 0)$) at $\bar{x} \in M_{NLP}[h, g]$ is equivalent to the metric regularity of

$$S^{-1}(x) = \{(h(x), z) \mid g(x) \leq z\}$$

at $(\bar{x}, 0, 0)$.

To apply this idea in our setting we say that the Metric Regularity Condition (MRC) holds at $\bar{x} \in M[F_1, F_2]$ if and only if

$$G : \begin{cases} \mathbb{R}^n & \longrightarrow \mathbb{R}^k, \\ x & \longmapsto (\min\{F_{1,i}(x), F_{2,i}(x)\})_{i=1,\dots,k} \end{cases}$$

is metrically regular at $(\bar{x}, 0)$.

Next proposition can be derived with the aim of Proposition 3.3, [53]. For the sake of completeness we present its proof.

Proposition 2.2.15 (MRC is equivalent to Δ)

MRC holds at $\bar{x} \in M[F_1, F_2]$ if and only if Δ holds at $\bar{x}$.

Proof. MRC holds at $\bar{x} \in M[F_1, F_2]$ if and only if the solution map $S(y) := \{x \in \mathbb{R}^n \mid G(x) = y\}$, $y \in \mathbb{R}^k$, is pseudo-Lipschitz at $(0, \bar{x})$. Setting

$$F : \begin{cases} \mathbb{R}^n & \longrightarrow \mathbb{R}^{2k}, \\ x & \longmapsto (F_{1,i}(x), F_{2,i}(x))_{i=1,\dots,k}. \end{cases}$$

and $D_i := \{(a_i, b_i) \in \mathbb{R}^2 \mid a_i \geq 0, b_i \geq 0, ab = 0\}$, $i = 1, \dots, k$ we obtain:

$$S(y) = \{x \in \mathbb{R}^n \mid F(x) - y \in D_1 \times \dots \times D_k\},$$

$$S^{-1}(x) = F(x) - D_1 \times \dots \times D_k.$$

Therefore, MRC holds at $\bar{x} \in M[F_1, F_2]$ if and only if $F(\cdot) - D_1 \times \dots \times D_k$ is metrically regular at $(\bar{x}, 0)$. Since $F \in C^1(\mathbb{R}^n, \mathbb{R}^{2k})$ and $D_1 \times \dots \times D_k$ is closed, we can apply Example 9.44 from [103]. Due to this Example 9.44 the constraint qualification

$$u \in N(F(\bar{x}), D_1 \times \dots \times D_k), \nabla^T F(\bar{x})u = 0 \implies u = 0 \quad (2.8)$$

is equivalent to the metric regularity of $F(\cdot) - D_1 \times \cdots \times D_k$ at $(\bar{x}, 0)$. Since $N(F(\bar{x}), D_1 \times \cdots \times D_k) = N(F_{1,1}(\bar{x}), F_{2,1}(\bar{x}), D_1) \times \cdots \times N(F_{1,k}(\bar{x}), F_{2,k}(\bar{x}), D_k)$ and $N(0, D_i) = \mathbb{R}^2 \cup D_i$, the formula (2.6) allows to conclude that the constraint qualification (2.8) is equivalent to Δ . $\square$

We mention some valuable remarks on the previously discussed constraint qualifications (they were pointed out by an anonymous referee).

Remark 2.2.16 (Standard subdifferential qualification condition Δ)

Δ is the standard subdifferential qualification condition for the system $\Omega_i, i = 1, \dots, k$ at $\bar{x} \in \bigcap_{i=1}^k \Omega_i$ (cf. [53, 92, 103]). Moreover, Δ means that the multi-valued map $M(z) := \{x \in \mathbb{R}^n \mid x + z_i \in \Omega_i, i = 1, \dots, k\}, z = (z_1, \dots, z_k) \in \mathbb{R}^{nk}$, is pseudo-Lipschitz (has the Aubin property) at $(0, \dots, 0, \bar{x})$. It means that its inverse $M^{-1}(x) := (\Omega_1 - x) \times \dots \times (\Omega_k - x), x \in \mathbb{R}^n$, is metrically regular at $(\bar{x}, 0, \dots, 0)$ (e.g. Proposition 3.3, [53]).

Remark 2.2.17 (MFC and GMFCQ)

The generalized Mangasarian-Fromovitz Constraint Qualification (GMFCQ) can be related to MFC. Indeed, GMFCQ for the constraint set $M = \{x \in \mathbb{R}^n \mid F(x) \in D_1 \times \cdots \times D_k\}$ is exactly (2.8) (cf. [53]). Thus, it is clear from the proof of Proposition 2.2.15 that Δ is equivalent to GMFCQ. Hence, MFC implies MRC, as well as GMFCQ. Moreover, Example 2.2.14 shows that neither MRC nor GMFCQ is sufficient for $M[F_1, F_2]$ being locally stable.

Remark 2.2.18 (Constraint Qualifications for M-stationarity)

It is well known that under Δ (or, equivalently, MRC and GMFCQ) a local minimum for (2.5) is M-stationary. It means that in addition to C-stationarity in Definition 2.2.7 it holds:

$$\text{either } \lambda_{1,i}, \lambda_{2,i} < 0 \text{ or } \lambda_{1,i}\lambda_{2,i} = 0$$

$$\text{for every } i \in \{1, \dots, k\} \text{ with } F_{1,i}(\bar{x}) = F_{2,i}(\bar{x}) = 0.$$

On equivalence of MFC and SMFC

It is clear that SMFC implies MFC. Moreover, these two conditions coincide for $n = k$. The question, whether SMFC is equivalent to MFC in general, is highly nontrivial.

First, we show that SMFC implies MFC at least in the following cases:

- $k = 2$,
- LICQ is fulfilled.

It follows mainly from the following (linear-algebraic) Lemma 2.2.19.

Lemma 2.2.19 *Let $C_i := \text{conv}\{v_{j,i} \in \mathbb{R}^n \mid j = 1, 2\}$, $i = 1, \dots, k$ and for every $i \in \{1, \dots, k\}$ let $v_{1,i}, v_{2,i}$ be linearly independent. Let assertions (A) and (B) be given as follows:*

- (A) *any k vectors $(w_1, \dots, w_k) \in C_1 \times \dots \times C_k$ are linearly independent.*
- (B) *there exists a k -dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1, \dots, u_k) \in P_E(C_1) \times \dots \times P_E(C_k)$ are linearly independent, where $P_E : \mathbb{R}^n \rightarrow E$ denotes the orthogonal projection.*

Then, (A) and (B) are equivalent in the following cases:

- 1) *the vectors $v_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ are linearly independent.*
- 2) *$k = 2$.*

Proof. The nontrivial part is to prove that (A) implies (B) for $n > k$.

Firstly, we claim that (B) is equivalent to the following condition (C) (cf. Lemma 2.2.6):

- (C) *there exist $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that for any $w_i \in C_i$, $i = 1, \dots, k$ the vectors $w_1, \dots, w_k, \xi_1, \dots, \xi_{n-k}$ are linearly independent.*

Indeed, if (B) holds we choose $\xi_1, \dots, \xi_{n-k}$ being a basis of $E^\perp$ in (C). If (C) holds we set $E := (\text{span}\{\xi_1, \dots, \xi_{n-k}\})^\perp$ in (B).

Case 1)

Let the vectors $v_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ be linearly independent. Then, $n \geq 2k$ and $v_{j,i}$, $j = 1, 2$, $i = 1, \dots, k$ span a $2k$ -dimensional linear subspace of $\mathbb{R}^n$. Hence, w.l.o.g. we may assume that $n = 2k$.

Define a linear coordinate transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows:

$$L(v_{1,i}) = e_{2i-1} + e_{2i}, L(v_{2,i}) = e_{2i-1}, i = 1, \dots, k,$$

whereby e_m denotes the m -th standard basis vector for $1 \leq m \leq n$.

It holds: $L(C_i) = \{e_{2i-1} + \lambda_i e_{2i} \mid \lambda_i \in [0, 1]\}$, $i = 1, \dots, k$.

Setting $T := \text{span}\{e_{2i-1}, i = 1, \dots, k\}$ we, obviously, obtain:

- (*) *any k vectors $(v_1, \dots, v_k) \in P_T(L(C_1)) \times \dots \times P_T(L(C_k))$ are linearly independent, where $P_T : \mathbb{R}^n \rightarrow T$ denotes the orthogonal projection.*

As above, $(\star)$ is equivalent to the following condition:

- $(\star\star)$ there exist $\gamma_1, \dots, \gamma_{n-k} \in \mathbb{R}^n$ such that for any $v_i \in L(C_i), i = 1, \dots, k$ the vectors $v_1, \dots, v_k, \gamma_1, \dots, \gamma_{n-k}$ are linearly independent.

Setting $\xi_i := L^{-1}(\gamma_i), i = 1, \dots, n - k$ we conclude that (C) is fulfilled due to $(\star\star)$. Thus, B is proved.

Case 2)

Let $k = 2$. It is clear that the vectors $v_{j,i}, j = 1, 2, i = 1, 2$ span at most a 4-dimensional linear subspace S of $\mathbb{R}^n$ and, hence, $\dim S \leq 4$. If $\dim S = 4$, then (B) holds as in Case 2). If $\dim S < 4$, we may assume w.l.o.g. that $n = 3$.

For $a \in \{-1, 1\}^2$ we set $K_a := \text{cone}\{a_i v_{1,i}, a_i v_{2,i} \mid i = 1, 2\}$. Due to the theorem about alternatives (e.g. [101]) we claim that (A) is equivalent to the following condition:

$$\text{int}(K_a^\circ) \neq \emptyset \text{ for all } a \in \{-1, 1\}^2.$$

Here, $\text{int}(K_a^\circ)$ denotes the interior of the polar cone of K_a .

Due to this fact K_a properly lies in a half-space for all $a \in \{-1, 1\}^2$. Setting $\{-1, 1\}^2 =: \{a^1, -a^1, a^2, -a^2\}$ we can strictly separate K_{a^l} and K_{-a^l} by a plane $\beta_l \ni 0, l = 1, 2$. Since $0 \in \beta_1 \cap \beta_2$, there exists $\xi \in \beta_1 \cap \beta_2, \xi \neq 0$ such that $\xi \notin \bigcup_{a \in \{-1, 1\}^2} K_a$ by construction. It means that (C) is fulfilled.

Thus, (B) is proved. $\square$

Theorem 2.2.20 (MFC implies SMFC for $k=2$ and under LICQ)

Let $k = 2$ or LICQ be fulfilled. Then, SMFC is equivalent to MFC.

Proof. It is straightforward to see that the conclusion can be obtained by application of Lemma 2.2.19. We have to adjust the proof of Lemma 2.2.19 only for the case, that only one constraint in $\min\{F_{1,i}(\bar{x}), F_{2,i}(\bar{x})\} = 0$ is active (i.e. $F_{1,i}(\bar{x}) = 0, F_{2,i}(\bar{x}) > 0$ or vice versa). For that we define C_i from Lemma 2.2.19 just to be $C_i(\bar{x})$. The respective change in the proof of Lemma 2.2.19 is straightforward. In fact, only the so-called biactive set of constraints is crucial (cf. [86], [119]). $\square$

Remark 2.2.21 (MFC implies SMFC for $k=3$, [105])

Recently it was proven that MFC implies SMFC in case $k = 3$. The proof uses a kind of dual description of SMFC and MFC.

In what follows, we discuss the difficulties by proving that MFC implies SMFC for general n and k . They arise not so much because of linear-algebraic, but rather than because of combinatorial and topological matter of the problem. In fact, using the notation from Lemma 2.2.19 we set for $a \in \{-1, 1\}^k$

$$K_a := \text{cone}\{a_i v_{1,i}, a_i v_{2,i} \mid i = 1, \dots, k\}.$$

Condition (A) means that all cones K_a are pointed, i.e.

if $x_1 + \dots + x_p = 0$, $x_s \in K_a$, $s = 1, \dots, p$ then $x_s = 0$ for all $s = 1, \dots, p$.

Condition (B) means that there exist $n - k$ linearly independent vectors $\xi_1, \dots, \xi_{n-k} \in \mathbb{R}^n$ such that

$$\xi_j \notin \bigcup_{a \in \{-1, 1\}^k} K_a \text{ for all } j = 1, \dots, n - k.$$

Thus, for proving "(A) implies (B)" we need to show that (for all k and n)

$$\bigcup_{a \in \{-1, 1\}^k} K_a \neq \mathbb{R}^n. \quad (2.9)$$

Here, we deal with a union of pointed cones with additional property:

$$K_{-a} = -K_a \text{ for all } a \in \{-1, 1\}^k.$$

Moreover, 2^k - the number of these cones - grows exponentially in k . It is clear that for proving (2.9) topological properties of $\bigcup_{a \in \{-1, 1\}^k} K_a$ (such as e.g.

Euler characteristic) are crucial.

We conclude that the conjectured equivalence of MFC and SMFC is very sophisticated and is a topic of current research.

2.2.2 SMFC implies stability and Lipschitz manifold

We intend to prove that SMFC implies local stability of the feasible set $M[F_1, F_2]$ (cf. Theorem 2.2.29). The main idea is to show that under SMFC $M[F_1, F_2]$ appears to be an $(n-k)$ -dimensional Lipschitz manifold (cf. Corollary 2.2.30 and Definition 1.2.1).

Guiding examples

First, we briefly mention 2- and 3-dimensional examples with 2 linear constraints respectively. These examples illustrate which phenomena might occur in general. They mainly highlight the possibilities arising with respect to the stability property of the feasible set $M[F_1, F_2]$ in low dimensions.

Example 2.2.22 (2D, nonstable: one point $\longrightarrow$ empty, two points)

The set $M^{2.2.22} := \{(x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{-x, -y\} = 0\}$ is a singleton (see Figure 5 a)). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.2.22}$ either becomes empty or contains at least two points.

Example 2.2.23 (2D, nonstable: one point $\longrightarrow$ two points)

The set $M^{2.2.23} := \{(x, y) \in \mathbb{R}^2 \mid \min\{x, y\} = 0, \min\{-x + y, x + y\} = 0\}$ is a singleton (see Figure 5 b)). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.2.23}$ contains at least two points.

Figure 5: a) Illustration of Example 2.2.22 b) Illustration of Example 2.2.23

Example 2.2.24 (3D, nonstable: "three-star" $\longrightarrow$ 1 or 2 "two-stars")

The set $M^{2.2.24} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0, \min\{y - z, y + z\} = 0\}$ is a "three-star" (see Figure 6 a)). Note that MFC is not satisfied at 0. After an appropriate perturbation $M^{2.2.24}$ either has two path-connected components or is a "two-star".

Example 2.2.25 (3D, stable: "two-star" $\longrightarrow$ "two-star")

The set $M^{2.2.25} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0, \min\{x - y + z, -x + y + z\} = 0\}$ is a "two-star" (see Figure 6 b)). Note that MFC holds at 0. After any sufficiently small perturbation $M^{2.2.25}$ remains to be a "two-star".

Figure 6: a) Illustration of Example 2.2 b) Illustration of Example 2.2.25

It is easy to see that in all these examples MFC holds at 0 if and only if the corresponding feasible set is locally stable. Moreover, these examples emphasize that the locally stable case corresponds to a feasible set being a Lipschitz manifold (see Corollary 2.2.30 below).

Main results via Clarke's IFT

We recall briefly the notion of the Clarke's generalized Jacobian and the corresponding Inverse and Implicit Function Theorems (cf. [15]).

For a vector-valued function $G = (g_1, \dots, g_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ with g_i being Lipschitz near $\bar{x} \in \mathbb{R}^n$, the set

$$\partial G(\bar{x}) := \text{conv}\{\lim DG(x_i) \mid x_i \longrightarrow \bar{x}, x_i \notin \Omega_G\}$$

is called the Clarke's generalized Jacobian, where $\Omega_G \subset \mathbb{R}^n$ denotes the set of points at which G fails to be differentiable.

Theorem 2.2.26 (Clarke's Inverse Function Theorem, [15])

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be Lipschitz near $\bar{x}$. If all matrices in $\partial F(\bar{x})$ are nonsingular, then F has the unique Lipschitz inverse function F^{-1} locally around $\bar{x}$.

Theorem 2.2.27 (Clarke's Implicit Function Theorem, [15])

Let $G : \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be Lipschitz near $(\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $G(\bar{y}, \bar{z}) = 0$. Suppose that

$$\pi_z \partial G(\bar{y}, \bar{z}) := \{M \in \mathbb{R}^{k \times k} \mid \text{there exists } N \in \mathbb{R}^{k \times n} \text{ with } [N, M] \in \partial G(\bar{y}, \bar{z})\}$$

is of maximal rank, i.e. contains merely nonsingular matrices. Then there exist a $\mathbb{R}^{n-k}$ -neighborhood Y of $\bar{y}$, a $\mathbb{R}^k$ -neighborhood Z of $\bar{z}$ and a Lipschitz function $\zeta : Y \longrightarrow Z$ such that $\zeta(\bar{y}) = \bar{z}$ and for every $(y, z) \in Y \times Z$ it holds:

$$G(y, z) = 0 \text{ if and only if } z = \zeta(y).$$

However, Example 2.2.28 illustrates that Theorem 2.2.27 can not be applied directly in general just for the linear case of a stable $M[F_1, F_2]$.

Example 2.2.28 (3D, stable: IFT is not applicable)

Consider the set $M^{2.2.28} := \{(x, y, z) \in \mathbb{R}^3 \mid \min\{x, y\} = 0, \min\{-y + z, z\} = 0\}$ (see Figure 7). This example shows that although $M[F_1, F_2]$ is a Lipschitz manifold, it can not be parameterized by means of any splitting of $\mathbb{R}^3$ in the *standard* basis. Therefore, Theorem 2.2.27 (and, actually, any Implicit Function Theorem) can not be applied directly.

Figure 7: Illustration of Example 2.2.28

Indeed, Example 2.2.28 suggests to firstly perform a linear coordinate transformation in order to make Theorem 2.2.27 applicable. Exactly this idea is incorporated in SMFC and allows to prove the following result.

Theorem 2.2.29 (Local stability under SMFC)

If SMFC holds at $x \in M[F_1, F_2]$, then the feasible set $M[F_1, F_2]$ is locally stable at $\bar{x}$.

Proof. Let $\bar{x} \in M[F_1, F_2]$. Since SMFC holds at $\bar{x}$, there exists a k -dimensional linear subspace E of $\mathbb{R}^n$ such that any k vectors $(u_1, \dots, u_k) \in$

$P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x}))$ are linearly independent. W.l.o.g., we may assume that $E = \{0_{n-k}\} \times \mathbb{R}^k$.

Setting $g_i := \min\{F_{1,i}, F_{2,i}\}$, $i = 1, \dots, k$ we define

$$G : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^k, \\ (y, z) & \longmapsto & (g_1(y, z), \dots, g_k(y, z)). \end{cases}$$

Let $\bar{x} = (\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. We obtain from $\partial g_i(\bar{x}) = C_i(\bar{x})$, $i = 1, \dots, k$, and the choice of E that

$$\pi_z \partial G(\bar{y}, \bar{z}) \subset P_E(C_1(\bar{x})) \times \cdots \times P_E(C_k(\bar{x})).$$

Hence, due to SMFC $\pi_z \partial G(\bar{y}, \bar{z})$ is of maximal rank and Theorem 2.2.27 can be applied. Then there exist a compact $\mathbb{R}^{n-k}$ -neighborhood Y of $\bar{y}$, a $\mathbb{R}^k$ -neighborhood Z of $\bar{z}$ and a Lipschitz function $\zeta : Y \longrightarrow Z$ such that $\zeta(\bar{y}) = \bar{z}$ and for every $(y, z) \in Y \times Z$ it holds:

$$G(y, z) = 0 \text{ if and only if } z = \zeta(y).$$

For $\varepsilon > 0$ we set

$$K_\varepsilon := (\{(y, \zeta(y)) \mid y \in Y\} + \bar{B}_{\mathbb{R}^n}(0, \varepsilon)) \cap (Y \times Z)$$

an ε -tube around $M[F_1, F_2] \cap (Y \times Z)$. Due to the compactness of Y , continuity reasonings and stability of SMFC within the space of C^1 -functions (taking Y smaller if needed) there exists $\varepsilon > 0$ such that:

- (•) $K_\varepsilon \subset Y \times Z$ and K_ε is compact,
- (••) there exists a C^1 -neighborhood U of (F_1, F_2) in $C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k)$ such that for every $(\tilde{F}_1, \tilde{F}_2) \in U$ it holds:

$$M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) \subset K_\varepsilon.$$

We assume U to be a ball of radius $r > 0$ in $C^1(\mathbb{R}^n, \mathbb{R}^k) \times C^1(\mathbb{R}^n, \mathbb{R}^k)$.

- (•••) SMFC is fulfilled at every $x \in M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z)$ for every $(\tilde{F}_1, \tilde{F}_2) \in U$ with the same k -dimensional linear subspace E .

Let now $(\tilde{F}_1, \tilde{F}_2) \in U$ be arbitrary, but fixed. Setting $\tilde{g}_i := \min\{\tilde{F}_{1,i}, \tilde{F}_{2,i}\}$, $i = 1, \dots, k$ we define

$$\tilde{G} : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^k, \\ (y, z) & \longmapsto & (\tilde{g}_1(y, z), \dots, \tilde{g}_k(y, z)). \end{cases}$$

Our aim is to show that for every fixed $\tilde{y} \in Y$ the equation $\tilde{G}(\tilde{y}, z) = 0$ is uniquely solvable with $(\tilde{y}, z) \in K_\varepsilon$. For that, we set for $(t, y, z) \in [0, 1] \times \mathbb{R}^{n-k} \times \mathbb{R}^k$

$$H_{1,i}(t, y, z) := (1 - t)F_{1,i}(y, z) + t\tilde{F}_{1,i}(y, z),$$

$$H_{2,i}(t, y, z) := (1 - t)F_{2,i}(y, z) + t\tilde{F}_{2,i}(y, z),$$

$$g_i(t, y, z) := \min\{H_{1,i}(t, z), H_{2,i}(t, z)\}.$$

Further, we construct a homotopy mapping

$$H : \begin{cases} [0, 1] \times \mathbb{R}^{n-k} \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^k, \\ (t, y, z) & \longmapsto & (g_1(t, y, z), \dots, g_k(t, y, z)). \end{cases}$$

We keep in mind that $H(0, y, z) = G(y, z)$ and $H(1, y, z) = \tilde{G}(y, z)$, moreover, $(H_1(t, \cdot, \cdot), H_2(t, \cdot, \cdot)) \in U$ for every $t \in [0, 1]$.

Next, we fix $\tilde{y} \in Y$ and consider the equation $H(t, \tilde{y}, z) = 0$ near its solution $(0, \tilde{y}, \zeta(\tilde{y}))$. Since $(\tilde{y}, \zeta(\tilde{y})) \in M[F_1, F_2] \cap (Y \times Z)$ we obtain from $(\bullet \bullet \bullet)$ that SMFC holds at $(\tilde{y}, \zeta(\tilde{y}))$. It means that

$$\pi_z \partial H(0, \tilde{y}, \zeta(\tilde{y})) = \pi_z \partial G(\tilde{y}, \zeta(\tilde{y}))$$

is of maximal rank and Theorem 2.2.27 can be applied for $H(t, \tilde{y}, z) = 0$ near its solution $(0, \tilde{y}, \zeta(\tilde{y}))$. Thus, we obtain for every $t \in [0, \delta]$, $0 < \delta \leq 1$ a solution $z(t)$ such that $H(t, \tilde{y}, z(t)) = 0$. Since $(H_1(t, \cdot, \cdot), H_2(t, \cdot, \cdot)) \in U$, $(\bullet \bullet)$ yields that $(\tilde{y}, z(t)) \in K_\varepsilon$ for every $t \in [0, \delta]$. Hereby, δ is taken smaller if needed.

These considerations allow us to claim, that

$$\tilde{t} := \sup\{\bar{t} \in [0, 1) \mid \text{for every } t \in [0, \bar{t}) \text{ there exists at least one } (\tilde{y}, z(t)) \in K_\varepsilon \text{ such that } H(t, \tilde{y}, z(t)) = 0\}$$

is well-defined.

Assume that $\tilde{t} \neq 1$. Then, there is a sequence of solutions $z(t_m), t_m \in [0, \tilde{t})$, $t_m \rightarrow \tilde{t}$ such that $(\tilde{y}, z(t_m)) \in K_\varepsilon$ and $H(t_m, \tilde{y}, z(t_m)) = 0$. We use the compactness of K_ε from $(\bullet)$ to obtain the existence of $\tilde{z}$ with $(\tilde{y}, \tilde{z}) \in K_\varepsilon$ and $z_m \rightarrow \tilde{z}$. Hence, due to the continuity we get in the limit $H(\tilde{t}, \tilde{y}, \tilde{z}) = 0$. This conclusion allows us to apply Theorem 2.2.27 for the equation $H(t, \tilde{y}, z) = 0$ near $(\tilde{t}, \tilde{y}, \tilde{z})$ to extend the solution for $t > \tilde{t}$. This yields a contradiction with the definition of $\tilde{t}$.

So, we claim that $\tilde{t} = 1$ and as above we obtain: $\tilde{G}(\tilde{y}, z) \equiv H(1, \tilde{y}, z) = 0$ is solvable with $(\tilde{y}, z) \in K_\varepsilon$.

The unique solvability of $\tilde{G}(\tilde{y}, z) = 0$ for $(\tilde{y}, z) \in K_\varepsilon$ can be proven by contradiction using analogous arguments. One has only to follow different solutions by applying Theorem 2.2.27 successively until the unique solution $(0, \tilde{y}, \zeta(\tilde{y}))$ of $G(\tilde{y}, z) \equiv H(t, \tilde{y}, z) = 0$ will be reached.

Altogether, it is proven: For every $\tilde{y} \in Y$ the equation $\tilde{G}(\tilde{y}, z) = 0$ is uniquely solvable with $(\tilde{y}, z(\tilde{y})) \in K_\varepsilon$. From $(\bullet\bullet)$ one can immediately see that $\tilde{G}(\tilde{y}, z) = 0$ is uniquely solvable, actually, in Z . Therefore, $M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z) = \{(y, z(y)) \mid y \in Y\}$. Hereby, $z : Y \rightarrow Z$ is Lipschitz due to $(\bullet\bullet\bullet)$ and Theorem 2.2.27, which is applicable locally around every $\tilde{x} \in M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z)$.

It remains to add that $M[F_1, F_2] \cap (Y \times Z)$ and $M[\tilde{F}_1, \tilde{F}_2] \cap (Y \times Z)$, both being Lipschitz graphs on Y , are homeomorphic with $\mathbb{R}^{n-k}$ and, thus, with each other. $\square$

From the proof of Theorem 2.2.29 we deduce the following Corollary 2.2.30.

Corollary 2.2.30 *If SMFC holds at every $\bar{x} \in M[F_1, F_2]$, then the feasible set $M[F_1, F_2]$ is a $(n - k)$ -dimensional Lipschitz manifold.*

Proof. We use notations as in Theorem 2.2.29. Due to SMFC at $\bar{x} \in M[F_1, F_2]$ we may assume that after an appropriate linear coordinate transformation it holds:

$$(Y \times Z) \cap M[F_1, F_2] = \{(y, \zeta(y)) \mid y \in Y\},$$

where $\bar{x} = (\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$, Y is a $\mathbb{R}^{n-k}$ -neighborhood of $\bar{y}$, Z is a $\mathbb{R}^k$ -neighborhood of $\bar{z}$ and $\zeta : Y \rightarrow Z$ is Lipschitz. Hence, $M[F_1, F_2]$ being locally the graph of a Lipschitz function ζ fits Definition 1.2.1 and is a $(n - k)$ -dimensional Lipschitz manifold. $\square$

On application of Kummer's IFT

In this section we link SMFC with the so-called Thibault limiting sets (or strict graphical derivatives) via Kummer's Implicit Function Theorem.

For a vector-valued function $G = (g_1, \dots, g_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the mapping $TG(\bar{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ with

$$TG(\bar{x})(\bar{u}) := \left\{ v \in \mathbb{R}^k \left| \begin{array}{l} v = \lim_{k \rightarrow \infty} \frac{f(x_k + t_k u_k) - f(x_k)}{t_k} \\ \text{for certain } t_k \downarrow 0, x_k \rightarrow \bar{x}, u_k \rightarrow \bar{u} \end{array} \right. \right\}$$

is called Thibault derivative at $\bar{x}$ (cf. [115, 116]) or strict graphical derivative (cf. [103]).

If, additionally, g_i are Lipschitz near $\bar{x} \in \mathbb{R}^n$, then we may omit the sequence $u_k \rightarrow \bar{u}$ in the definition of $TG(\bar{x})(\bar{u})$ and we get:

$$TG(\bar{x})(\bar{u}) = \left\{ v \in \mathbb{R}^k \left| v = \lim_{k \rightarrow \infty} \frac{f(x_k + t_k \bar{u}) - f(x_k)}{t_k} \right. \right. \\ \left. \left. \text{for certain } t_k \downarrow 0, x_k \rightarrow \bar{x} \right. \right\}.$$

Necessary and sufficient conditions for local invertability of Lipschitz functions can be given in terms of Thibault derivatives.

Theorem 2.2.31 (Kummer's Inverse Function Theorem, [78, 83])

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be Lipschitz near $\bar{x}$. Then the following statements are equivalent:

- (i) F has the locally unique Lipschitz inverse function F^{-1} .
- (ii) There exists $c > 0$ such that

$$\|F(x) - F(x')\| \geq c\|x - x'\| \text{ for all } x, x' \text{ with } \|\bar{x} - x\| \leq c, \|\bar{x} - x'\| \leq c.$$

- (iii) $TF(\bar{x})$ is injective, i.e. $0 \notin TF(\bar{x})(u)$ for all $u \neq 0$.

Remark 2.2.32 Note that the injectivity of $TF(\bar{x})$ in Theorem 2.2.31 is in general weaker than the Clarke's requirement of all matrices in $\partial F(\bar{x})$ being nonsingular. In fact, there exists a Lipschitz homeomorphism F of $\mathbb{R}^2$ such that $\partial F(\bar{x})$ contains the zero matrix (see Example BE.3 in [78]).

Remark 2.2.33 We point out that (iii) from Theorem 2.2.31 implies the existence of the unique Lipschitz inverse of F w.r.t. Lipschitz perturbations of F performed locally. It means that there exists a $\mathbb{R}^n$ -neighborhood U of $\bar{x}$ and a neighborhood V of F in the space $C^{0,1}(U, \mathbb{R}^n)$ of Lipschitz functions such that for all $\hat{F} \in V$ and $\hat{x} \in U$, $\hat{F}$ has the locally unique Lipschitz inverse function $\hat{F}^{-1}$ around $\hat{x}$. Note that we equip $F \in C^{0,1}(U, \mathbb{R}^n)$ with the norm

$$|F| := \max \left\{ \sup_{x \in U} \|F(x)\| + \text{Lip}(F, U) \right\},$$

where

$$\text{Lip}(F, U) := \inf \{ r > 0 \mid \|F(x) - F(x')\| \leq r\|x - x'\| \text{ for all } x, x' \in U \}.$$

For details we refer to Theorem 5.14 and Corollary 4.4 in [78].

Theorem 2.2.34 (Kummer's Implicit Function Theorem, [78, 82])

Let $G : \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be Lipschitz near $(\bar{y}, \bar{z}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $G(\bar{y}, \bar{z}) = 0$. Then, the following statements are equivalent:

- (i) There exist $\mathbb{R}^{n-k}$ -neighborhoods Y of $\bar{y}$ and W of 0 , a $\mathbb{R}^k$ -neighborhood Z of $\bar{z}$ and a Lipschitz function $\zeta : Y \times W \longrightarrow Z$ such that $\zeta(\bar{y}, 0) = \bar{z}$ and for every $(y, z, w) \in Y \times Z \times W$ it holds:

$$G(y, z) = w \text{ if and only if } z = \zeta(y, w).$$

- (ii) $0 \notin TG(\bar{y}, \bar{z})(0, u)$ for all $u \neq 0$.

Remark 2.2.35 We point out that Theorem 2.2.34 gives necessary and sufficient condition for the existence of implicit functions. Recall that Clarke's IFT (cf. Theorem 2.2.27) gives only sufficient condition for that fact. Moreover, it is important to note that in Theorem 2.2.34 the implicit function ζ depends Lipschitz also on the right-hand side perturbations w . This issue was used extensively in the proof of Stability Theorem 2.2.29.

Now, we turn our attention to the case of min-functions. Let a basis decomposition of $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ be fixed. It turns out that the assumptions of Clarke's and Kummer's Implicit Function Theorems coincide. Moreover, they are also equivalent with SMFC w.r.t. the subspace $E := \{0_{n-k}\} \times \mathbb{R}^k$ (cf. [84, 112]).

Lemma 2.2.36 Setting $g_i := \min\{F_{1,i}, F_{2,i}\}$, $i = 1, \dots, k$ we define

$$G : \begin{cases} \mathbb{R}^{n-k} \times \mathbb{R}^k & \longrightarrow & \mathbb{R}^k, \\ (y, z) & \longmapsto & (g_1(y, z), \dots, g_k(y, z)). \end{cases}$$

Then, the following conditions are equivalent for $\bar{x} = (\bar{y}, \bar{z})$:

- (i) $\pi_z \partial G(\bar{y}, \bar{z})$ is of maximal rank, i.e

$$\pi_z \partial G(\bar{y}, \bar{z}) := \{M \in \mathbb{R}^{k \times k} \mid \text{there exists } N \in \mathbb{R}^{k \times n} \text{ with } [N, M] \in \partial G(\bar{y}, \bar{z})\}$$

contains merely nonsingular matrices.

- (ii) All matrices in $\partial_z g_1(\bar{x}) \times \partial_z g_2(\bar{x}) \times \dots \times \partial_z g_k(\bar{x})$ are nonsingular.

- (iii) $0 \notin TG(\bar{y}, \bar{z})(0, u)$ for all $u \neq 0$.

Proof. ”(i) $\implies$ (iii)”: Due to (i) we may apply Clarke’s Implicit Function Theorem. Hence, the implicit function $\zeta(y)$ exists. It is not hard to see that ζ depends uniquely and Lipschitz on the w -values of G . Hence, we obtain, in fact, $\zeta(y, w)$. Applying the Kummer’s Implicit Function Theorem we get (iii).

”(ii) $\implies$ (i)”: In general, it holds (cf. e.g. [24]):

$$\pi_z \partial G(\bar{y}, \bar{z}) \subset \partial_z g_1(\bar{x}) \times \partial_z g_2(\bar{x}) \times \dots \times \partial_z g_k(\bar{x}).$$

This inclusion shows the assertion.

”(iii) $\implies$ (ii)”: Let $0 \notin TG(\bar{y}, \bar{z})(0, u)$ for all $u \neq 0$. For $q \in \mathbb{R}^k$ we set

$$q^+ := (q_1^+, \dots, q_k^+), \text{ where } q_i^+ := \max\{q, 0\}, i = 1, \dots, k,$$

$$q^- := (q_1^-, \dots, q_k^-), \text{ where } q_i^- := \min\{q, 0\}, i = 1, \dots, k.$$

We define the mapping $\widehat{G} : \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^{2k}$ as follows:

$$\widehat{G}(y, z, q) = \begin{pmatrix} F_1(y, z) - q^+ \\ -F_2(y, z) - q^- \end{pmatrix}.$$

The zeros of G and $\widehat{G}$ correspond as follows. If $G(x) = 0$ then $\widehat{G}(x, q) = 0$ with $q := F_1 - F_2$. If $\widehat{G}(x, q) = 0$ then $G(x) = 0$.

Setting $\bar{q} = F_1(\bar{y}, \bar{z}) - F_2(\bar{y}, \bar{z})$, we claim that $0 \notin T\widehat{G}(\bar{x}, \bar{q})(0, u, p)$ for all $(u, p) \neq 0$. In fact, due to Kummer’s IFT the latter is equivalent to the existence of Lipschitz implicit functions $\zeta(y, w_1, w_2)$ and $q(y, w_1, w_2)$ for the system:

$$F_1(y, z) - q^+ = w_1, \quad -F_2(y, z) - q^- = w_2, \quad (2.10)$$

The system (2.10) can be equivalently written as:

$$F_1(y, z) - w_1 - q^+ = 0, \quad -F_2(y, z) - w_2 - q^- = 0.$$

Hence, we need to find the implicit function $\zeta(y, w_1, w_2)$ for

$$\min \{F_{1,i}(y, z) - w_{1,i}, F_{2,i}(y, z) + w_{2,i}\} = 0, \quad i = 1, \dots, k \quad (2.11)$$

and afterwards to set

$$q(y, w_1, w_2) := F_1(y, \zeta(y, w_1, w_2)) - w_1 - F_2(y, \zeta(y, w_1, w_2)) - w_2.$$

Note that (2.11) is a Lipschitz perturbed version of $G(x) = 0$. Remark 2.2.33 and (iii) justify then the application of Kummer’s IFT for the perturbed system (2.11). Hence, $0 \notin T\widehat{G}(\bar{x}, \bar{q})(0, u, p)$ for all $(u, p) \neq 0$.

Now, we compute $T\widehat{G}(\bar{x}, \bar{q})(0, u, p)$ using results from [78] on the so-called Kojima-functions. For that, we set

$$N(q) := (1, q^+, q^-) \text{ and } M(x) := \begin{pmatrix} F_1(x) & F_2(x) \\ -I_k & 0 \\ 0 & -I_k \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix. It holds:

$$\widehat{F}(x, q) = N(q) \cdot M(x).$$

Applying the product rule (cf. Theorem 7.5 in [78]) we get:

$$T\widehat{F}(\bar{x}, \bar{q})(0, u, p) = N(\bar{q})TM(\bar{x})(0, u) + TN(\bar{q})(p)M(\bar{x}).$$

We compute $TM(\bar{x})$ and $TN(\bar{q})$ (cf. Lemma 7.3 in [78]):

$$TM(\bar{x})(0, u) = \begin{pmatrix} D_z F_1(\bar{x})u & D_z F_2(\bar{x})u \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$TN(\bar{q})(p) = \{(0, \lambda, p - \lambda) \mid \lambda_i = r_i p_i, r_i \in \mathfrak{R}(\bar{q}), i = 1, \dots, k\}, \text{ where}$$

$$\mathfrak{R}(\bar{q}) := \{r \in [0, 1]^m \mid r_i = 1 \text{ if } \bar{q}_i > 0, r_i = 0 \text{ if } \bar{q}_i < 0\}.$$

Altogether,

$$T\widehat{G}(\bar{x}, \bar{q})(0, u, p) = \left\{ \begin{pmatrix} D_z F_{1,i}(\bar{x})u - r_i p_i, i = 1, \dots, k \\ -D_z F_{2,i}(\bar{x})u - (1 - r_i)p_i, i = 1, \dots, k \end{pmatrix} \mid r \in \mathfrak{R}(\bar{q}) \right\}.$$

Next, let (ii) assume to fail. We show that it contradicts the fact

$$0 \notin T\widehat{G}(\bar{x}, \bar{q})(0, u, p) \text{ for all } (u, p) \neq 0.$$

Indeed, if (ii) does not hold we obtain $u \in \mathbb{R}^k$, $u \neq 0$ and $r \in [0, 1]^k$ such that:

$$[(1 - r_i)D_z F_{1,i}(\bar{x}) + r_i D_z F_{2,i}(\bar{x})]u = 0, i = 1, \dots, k. \quad (2.12)$$

Note that $r \in \mathfrak{R}(\bar{q})$ due to the definition of Clarke's subdifferentials.

Case $r_i \neq 0$: Then, we set $p_i := \frac{1}{r_i} D_z F_{1,i}(\bar{x})u$ and obtain:

$$D_z F_{1,i}(\bar{x})u - r_i p_i = 0.$$

Further, we get due to (2.12):

$$\begin{aligned} r_i [-D_z F_{2,i}(\bar{x})u - (1 - r_i)p_i] &= r_i \left[-D_z F_{2,i}(\bar{x})u - (1 - r_i) \frac{1}{r_i} D_z F_{1,i}(\bar{x})u \right] = \\ &= -r_i D_z F_{2,i}(\bar{x})u - (1 - r_i) D_z F_{1,i}(\bar{x})u = 0. \end{aligned}$$

Hence, $-D_z F_{2,i}(\bar{x})u - (1 - r_i)p_i = 0$.

Case $r_i = 0$: Then, we set $p_i := -D_z F_{2,i}u$ and obtain due to (2.12):

$$D_z F_{1,i}(\bar{x})u - r_i p_i = 0.$$

Moreover,

$$-D_z F_{2,i}(\bar{x})u - (1 - r_i)p_i = -D_z F_{2,i}(\bar{x})u + D_z F_{2,i}u = 0.$$

Altogether, we see that for $(u, p) \neq 0$ defined as above it holds:

$$0 \in T\widehat{G}(\bar{x}, \bar{q})(0, u, p). \square$$

From Lemma 2.2.36 we deduce the following result.

Theorem 2.2.37 (SMFC and Kummer's Implicit Function Theorem)

SMFC holds if and only if Kummer's Implicit Function Theorem is applicable w.r.t. some basis decomposition of $\mathbb{R}^n$.

Proof. The equivalence of (ii) and (iii) from Lemma 2.2.36 implies immediately the result. In fact, we only need to use the chain rule from [78]:

$$T(G \circ A)(x)(u) = TG(Ax)(Au),$$

where A is a nonsingular $(n \times n)$ -matrix. Confer also the characterization of SMFC in terms of Clarke's subdifferentials in Lemma 2.2.4. $\square$

Theorem 2.2.37 shows that the remaining difficulty concerning topological stability of the MPCC feasible set lies in conjectured equivalence between MFC and SMFC rather than in an application of different Implicit Function Theorems.

2.3 Critical point theory

We study the behavior of the topological properties of lower level sets

$$M^a := \{x \in M \mid f(x) \leq a\}$$

as the level $a \in \mathbb{R}$ varies. It turns out that the concept of C-stationarity is the adequate stationarity concept. In fact, we present two basic theorems from Morse Theory (cf. [61, 91]). First, we show that, for $a < b$, the set M^a is a strong deformation retract of M^b if the (compact) set

$$M_a^b := \{x \in M \mid a \leq f(x) \leq b\}$$

does not contain C-stationary points (see Theorem 2.3.13(a)). Second, if M_a^b contains exactly one (nondegenerate) C-stationary point, then M^b is shown to be homotopy equivalent to M^a with a q -cell attached (see Theorem 2.3.13(b)). Here, the dimension q is the so-called C-index. It depends on both the restricted Hessian of the Lagrangian and the Lagrange multipliers related to bi-active complementarity constraints. The latter fact is the main difference with respect to the well-known case where feasible set is described only by equality and finitely many inequality constraints (cf. [61]).

We would like to refer to some related papers. In [107] the concept of a non-degenerate feasible point for MPCC is introduced. Some genericity results are obtained. In [98] the concepts of a non-degenerate C-stationary point and its stationary C-index are introduced for quadratic programs with complementarity constraints (QPCC). The generic structure of the C-stationary point set for non-parametric and one-parametric QPCCs is discussed and some homotopy methods for QPCC are developed.

We refer to [67] for details.

Notations and Auxiliary Results

Given $\bar{x} \in M$, we define the following index sets:

$$\begin{aligned} J_0(\bar{x}) &:= \{j \in J \mid g_j(\bar{x}) = 0\}, \\ \alpha(\bar{x}) &:= \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) > 0\}, \\ \beta(\bar{x}) &:= \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) = 0\}, \\ \gamma(\bar{x}) &:= \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) > 0, F_{2,m}(\bar{x}) = 0\}. \end{aligned}$$

We call $J_0(\bar{x})$ the active inequality index set and $\beta(\bar{x})$ the bi-active index set at $\bar{x}$.

Without loss of generality (w.l.o.g.), we assume that at the particular point of interest $\bar{x} \in M$ it holds:

$$J_0(\bar{x}) = \{1, \dots, |J_0(\bar{x})|\}, \alpha(\bar{x}) = \{1, \dots, |\alpha(\bar{x})|\},$$

$$\gamma(\bar{x}) = \{|\alpha(\bar{x})| + 1, \dots, |\alpha(\bar{x})| + |\gamma(\bar{x})|\}.$$

We put $s := |I| + |\alpha(\bar{x})| + |\gamma(\bar{x})|$, $q := s + |J_0(\bar{x})|$, $p := n - q - 2|\beta(\bar{x})|$.

Further, we recall the well-known Linear Independence Constraint Qualification (LICQ) for MPCC (e.g. [106]), which is said to hold at $\bar{x} \in M$ if the set of vectors

$$\{D^T h_i(\bar{x}), i \in I, D^T F_{1,m_\alpha}(\bar{x}), m_\alpha \in \alpha(\bar{x}), D^T F_{2,m_\gamma}(\bar{x}), m_\gamma \in \gamma(\bar{x}), D^T g_j(\bar{x}), j \in J_0(\bar{x}), D^T F_{1,m_\beta}(\bar{x}), D^T F_{2,m_\beta}(\bar{x}), m_\beta \in \beta(\bar{x})\}$$

is linearly independent.

Definition 2.3.1 (C-stationary point, cf. [23, 106])

A point $\bar{x} \in M$ is called *Clarke stationary (C-stationary)* for MPCC if there exist real numbers $\bar{\lambda}_i$, $i \in I$, $\bar{\varrho}_{m_\alpha}$, $m_\alpha \in \alpha(\bar{x})$, $\bar{\vartheta}_{m_\gamma}$, $m_\gamma \in \gamma(\bar{x})$, $\bar{\mu}_j$, $j \in J_0(\bar{x})$, $\bar{\sigma}_{1,m_\beta}$, $\bar{\sigma}_{2,m_\beta}$, $m_\beta \in \beta(\bar{x})$, (Lagrange multipliers), such that:

$$\begin{aligned} Df(\bar{x}) &= \sum_{i \in I} \bar{\lambda}_i D h_i(\bar{x}) + \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\varrho}_{m_\alpha} D F_{1,m_\alpha}(\bar{x}) + \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} D F_{2,m_\gamma}(\bar{x}) \\ &+ \sum_{j \in J_0(\bar{x})} \bar{\mu}_j D g_j(\bar{x}) + \sum_{m_\beta \in \beta(\bar{x})} (\bar{\sigma}_{1,m_\beta} D F_{1,m_\beta}(\bar{x}) + \bar{\sigma}_{2,m_\beta} D F_{2,m_\beta}(\bar{x})), \end{aligned} \quad (2.13)$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0(\bar{x}), \quad (2.14)$$

$$\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}). \quad (2.15)$$

In the case where LICQ holds at $\bar{x} \in M$, the Lagrange multipliers in (2.13) are uniquely determined.

Given a C-stationary point $\bar{x} \in M$ for MPCC, we set:

$$\begin{aligned} M(\bar{x}) := \{x \in \mathbb{R}^n \mid & h_i(x) = 0, i \in I, F_{1,m_\alpha}(x) = 0, m_\alpha \in \alpha(\bar{x}), \\ & F_{2,m_\gamma}(x) = 0, m_\gamma \in \gamma(\bar{x}), g_j(x) = 0, j \in J_0(\bar{x}), \\ & F_{1,m_\beta}(x) = 0, F_{2,m_\beta}(x) = 0, m_\beta \in \beta(\bar{x})\}. \end{aligned}$$

Obviously, $M(\bar{x}) \subset M$ and, in the case where LICQ holds at $\bar{x}$, $M(\bar{x})$ is locally a p -dimensional C^2 -manifold.

Definition 2.3.2 (Nondegenerate C-stationary point, cf. [98, 107])

A C-stationary point $\bar{x} \in M$ with Lagrange multipliers as in Definition 2.3.1 is called nondegenerate if the following conditions are satisfied:

ND1: LICQ holds at $\bar{x}$,

ND2: $\bar{\mu}_j > 0$ for all $j \in J_0(\bar{x})$,

ND3: $D^2L(\bar{x})|_{T_{\bar{x}}M(\bar{x})}$ is nonsingular,

ND4: $\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} > 0$ for all $m_\beta \in \beta(\bar{x})$.

Here, the matrix D^2L stands for the Hessian of the Lagrange function L ,

$$\begin{aligned} L(x) := & f(x) - \sum_{i \in I} \bar{\lambda}_i h_i(x) - \sum_{m_\alpha \in \alpha(\bar{x})} \bar{\varrho}_{m_\alpha} F_{1,m_\alpha}(x) - \sum_{m_\gamma \in \gamma(\bar{x})} \bar{\vartheta}_{m_\gamma} F_{2,m_\gamma}(x) \\ & - \sum_{j \in J_0(\bar{x})} \bar{\mu}_j g_j(x) - \sum_{m_\beta \in \beta(\bar{x})} (\bar{\sigma}_{1,m_\beta} F_{1,m_\beta}(x) + \bar{\sigma}_{2,m_\beta} F_{2,m_\beta}(x)) \end{aligned} \quad (2.16)$$

and $T_{\bar{x}}M(\bar{x})$ denotes the tangent space of $M(\bar{x})$ at $\bar{x}$,

$$\begin{aligned} T_{\bar{x}}M(\bar{x}) := \{ \xi \in \mathbb{R}^n \mid & Dh_i(\bar{x}) \xi = 0, i \in I, \\ & DF_{1,m_\alpha}(\bar{x}) \xi = 0, m_\alpha \in \alpha(\bar{x}), \\ & DF_{2,m_\gamma}(\bar{x}) \xi = 0, m_\gamma \in \gamma(\bar{x}), \\ & Dg_j(\bar{x}) \xi = 0, j \in J_0(\bar{x}) \\ & DF_{1,m_\beta}(\bar{x}) \xi = 0, DF_{2,m_\beta}(\bar{x}) \xi = 0, m_\beta \in \beta(\bar{x}) \}. \end{aligned}$$

Condition ND3 means that the matrix $V^T D^2L(\bar{x})V$ is nonsingular, where V is some matrix whose columns form a basis for the tangent space $T_{\bar{x}}M(\bar{x})$.

Definition 2.3.3 (C-index, cf. [98])

Let $\bar{x} \in M$ be a nondegenerate C-stationary point with Lagrange multipliers as in Definition 2.3.2. The number of negative/positive eigenvalues of $D^2L(\bar{x})|_{T_{\bar{x}}M(\bar{x})}$ is called the quadratic index (QI)/quadratic coindex (QCI) of $\bar{x}$. The number of pairs $(\bar{\sigma}_{1,m_\beta}, \bar{\sigma}_{2,m_\beta})$, $m_\beta \in \beta(\bar{x})$ with both $\bar{\sigma}_{1,m_\beta}$ and $\bar{\sigma}_{2,m_\beta}$ negative/positive is called the bi-active index (BI)/bi-active coindex (BCI) of $\bar{x}$. The number $(QI + BI)/(QCI + BCI)$ is called the Clarke index (C-index)/Clarke coindex (C-coindex) of $\bar{x}$.

Note that in the absence of complementarity constraints, the C-index has only the QI-part and coincides with the well-known quadratic index of a nondegenerate Karush-Kuhn-Tucker-point in nonlinear programming or, equivalently, with the Morse index (cf. [61, 80, 91]).

The following proposition uses the C-index for the characterization of a local minimizer. Its proof is omitted since it can be easily seen (see also [98, 106]).

Proposition 2.3.4 *(i) Assume that $\bar{x}$ is a local minimizer for MPCC and that LICQ holds at $\bar{x}$. Then, $\bar{x}$ is a C-stationary point for MPCC.*

(ii) Let $\bar{x}$ be a nondegenerate C-stationary point for MPCC. Then, $\bar{x}$ is a local minimizer for MPCC if and only if its C-index is equal to zero.

The next proposition concerning genericity results for LICQ and for non-degeneracy of C-stationary points mainly follows from [61]. It was shown in [107] and for the special case of QPCC in [98].

Proposition 2.3.5 (Genericity and Stability, cf. [98, 107])

(i) Let $\mathcal{F}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those (h, g, F_1, F_2) for which LICQ holds at all points $x \in M[h, g, F_1, F_2]$. Then, $\mathcal{F}$ is C_s^2 -open and -dense.

(ii) Let $\mathcal{D}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those (f, h, g, F_1, F_2) for which each C-stationary point of MPCC with data functions (f, h, g, F_1, F_2) is nondegenerate. Then, $\mathcal{D}$ is C_s^2 -open and -dense.

Morse Lemma for MPCC

For the proof of deformation and cell-attachment results we locally describe the MPCC feasible set under the Linear Independence Constraint Qualification (see Lemma 2.3.7). Moreover, an equivariant Morse Lemma for MPCC is derived in order to obtain suitable normal forms for the objective function at C-stationary points (see Theorem 2.3.10).

Definition 2.3.6 *The feasible set M admits a local C^r -coordinate system of $\mathbb{R}^n$ ($r \geq 1$) at $\bar{x}$ by means of a C^r -diffeomorphism $\Phi : U \rightarrow V$ with open $\mathbb{R}^n$ -neighborhoods U and V of $\bar{x}$ and 0, respectively, if it holds:*

(i) $\Phi(\bar{x}) = 0$,

(ii) $\Phi(M \cap U) = \left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p \right) \cap V$.

Lemma 2.3.7 (cf. also [107])

Suppose that LICQ holds at $\bar{x} \in M$. Then M admits a local C^2 -coordinate system of $\mathbb{R}^n$ at $\bar{x}$.

Proof. Choose vectors $\xi_l \in \mathbb{R}^n$, $l = 1, \dots, p$, which form - together with the vectors

$$\{D^T h_i(\bar{x}), i \in I, D^T F_{1,m_\alpha}(\bar{x}), m_\alpha \in \alpha(\bar{x}), D^T F_{2,m_\gamma}(\bar{x}), m_\gamma \in \gamma(\bar{x}), D^T g_j(\bar{x}), j \in J_0(\bar{x}), D^T F_{1,m_\beta}(\bar{x}), D^T F_{2,m_\beta}(\bar{x}), m_\beta \in \beta(\bar{x})\}$$

- a basis for $\mathbb{R}^n$. Next we put

$$\left. \begin{aligned} y_i &:= h_i(x), i \in I \\ y_{|I|+m_\alpha} &:= F_{1,m_\alpha}(x), m_\alpha \in \alpha(\bar{x}) \\ y_{|I|+m_\gamma} &:= F_{2,m_\gamma}(x), m_\gamma \in \gamma(\bar{x}) \\ y_{s+j} &:= g_j(x), j \in J_0(\bar{x}) \\ y_{s+|J_0(\bar{x})|+2m_\beta-1} &:= F_{1,m_\beta}(x) \\ y_{s+|J_0(\bar{x})|+2m_\beta} &:= F_{2,m_\beta}(x), m_\beta = 1, \dots, |\beta(\bar{x})| \\ y_{n-p+l} &:= \xi_l^T(x - \bar{x}), l = 1, \dots, p. \end{aligned} \right\} \quad (2.17)$$

or, shortly,

$$y = \Phi(x). \quad (2.18)$$

Note that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\Phi(\bar{x}) = 0$ and the Jacobian matrix $D\Phi(\bar{x})$ is nonsingular (in virtue of LICQ and the choice of ξ_l , $l = 1, \dots, p$). By means of the Implicit Function Theorem there exist open neighborhoods U of $\bar{x}$ and V of 0 such that $\Phi : U \rightarrow V$ is a C^2 -diffeomorphism. By shrinking U , if necessary, we can guarantee that $J_0(x) \subset J_0(\bar{x})$ and $\beta(x) \subset \beta(\bar{x})$ for all $x \in M \cap U$. Thus, the property (ii) in Definition 2.3.6 follows directly from the definition of Φ . $\square$

Definition 2.3.8 We will refer to the C^2 -diffeomorphism Φ defined by (2.17), (2.18) as standard diffeomorphism.

Remark 2.3.9 From the proof of Lemma 2.3.7 it follows that the Lagrange multipliers at a nondegenerate C -stationary point are the corresponding partial derivatives of the objective function in new coordinates given by the standard diffeomorphism (cf. [63], Lemma 2.2.1). Moreover, the Hessian with respect to the last p coordinates corresponds to the restriction of the Lagrange function's Hessian on the respective tangent space (cf. [63], Lemma 2.2.10).

Theorem 2.3.10 (Morse Lemma for MPCC)

Suppose that $\bar{x}$ is a nondegenerate C -stationary point for MPCC with quadratic index QI , bi-active index BI and C -index $= QI + BI$. Then, there exists a local C^1 -coordinate system $\Psi : U \rightarrow V$ of $\mathbb{R}^n$ around $\bar{x}$ (according to Definition 2.3.6) such that:

$$f \circ \Psi^{-1}(0_s, y_{s+1}, \dots, y_n) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_{i+s} + \sum_{j=1}^{|\beta(\bar{x})|} \pm (y_{2j+q-1} + y_{2j+q}) + \sum_{k=1}^p \pm y_{k+n-p}^2, \quad (2.19)$$

where $y \in \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$. Moreover, in (2.19) there are exactly BI negative linear pairs and QI negative squares.

Proof. W.l.o.g., we may assume $f(\bar{x}) = 0$. Let $\Phi : U \rightarrow V$ be a standard diffeomorphism according to Definition (2.3.8). We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p\right) \cap V$. From now on we may assume $s = 0$. In view of Remark 2.3.9 we have at the origin:

- (i) $\frac{\partial \bar{f}}{\partial y_i} > 0, i \in J_0(\bar{x}),$
- (ii) $\frac{\partial \bar{f}}{\partial y_{2j+q-1}} \cdot \frac{\partial \bar{f}}{\partial y_{2j+q}} > 0, j = 1, \dots, |\beta(\bar{x})|,$
- (iii) $\frac{\partial \bar{f}}{\partial y_{2j+q-1}} < 0$ for exactly BI indices $j \in \{1, \dots, |\beta(\bar{x})|\},$
- (iv) $\frac{\partial \bar{f}}{\partial y_{k+n-p}} = 0, k = 1, \dots, p$ and $\left(\frac{\partial^2 \bar{f}}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}}\right)_{1 \leq k_1, k_2 \leq p}$ is a non-singular matrix with QI negative eigenvalues.

From now on we denote $\bar{f}$ by f . Under the following coordinate transformations the set $\mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ will be transformed in itself (equivariance). As an abbreviation we put $y = (Y_{n-p}, Y^p)$, where $Y_{n-p} = (y_1, \dots, y_{n-p})$ and $Y^p = (y_{n-p+1}, \dots, y_n)$. We write

$$f(Y_{n-p}, Y^p) = f(0, Y^p) + \int_0^1 \frac{d}{dt} f(tY_{n-p}, Y^p) dt = f(0, Y^p) + \sum_{i=1}^{n-p} y_i d_i(y),$$

where $d_i \in C^1, i = 1, \dots, n-p$.

In view of (iv) we may apply the Morse Lemma on the C^2 -function $f(0, Y^p)$ (cf. [61], Theorem 2.8.2) without affecting the coordinates Y_{n-p} . The corresponding coordinate transformation is of class C^1 . Denoting the transformed functions f, d_j again by f, d_j , we obtain:

$$f(y) = \sum_{i=1}^{n-p} y_i d_i(y) + \sum_{k=1}^p \pm y_{k+n-p}^2.$$

Note that $d_i(0) = \frac{\partial f}{\partial y_i}(0)$, $i = 1, \dots, n-p$. Recalling (i)-(iii), we have

$$y_i |d_i(y)|, i = 1, \dots, n-p, \quad y_j, j = n-p+1, \dots, n \quad (2.20)$$

as new local C^1 -coordinates. Denoting the transformed function f again by f and, recalling the signs in (i)-(iii), we obtain (2.19). Here, the coordinate transformation Ψ is understood as the composition of all previous ones. $\square$

Theorem 2.3.10 allows us to provide two other local representations (normal forms) of the objective function on the MPCC feasible set with respect to Lipschitz and Hölder coordinate systems.

Recall that the set $\partial\mathbb{H}^2$ represents the complementarity relations

$$u \geq 0, v \geq 0, u \cdot v = 0.$$

Define the mapping $\varphi : \partial\mathbb{H}^2 \longrightarrow \mathbb{R}^1 \times 0_1$ as follows:

$$\varphi(u, 0) := (u, 0), \quad \varphi(0, v) := (-v, 0). \quad (2.21)$$

Coordinatewise extension of φ on $(\partial\mathbb{H}^2)^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (2.21) induces the Lipschitz coordinate transformation Φ ,

$$\Phi : \{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \quad (2.22)$$

In the right-hand side of (2.22) the zeros $\{0_s\}$ and $\{0_1\}$ ($|\beta(\bar{x})|$ -times) are deleted. The proof of the following corollary is now straightforward.

Corollary 2.3.11 (Normal forms in Lipschitz coordinates)

Let f have the normal form as in (2.19) and let Φ be the Lipschitz coordinate transformation (2.22). Then, we have:

$$f \circ \Phi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{J_0(\bar{x})+|\beta(\bar{x})|} \pm |y_j| + \sum_{k=J_0(\bar{x})+|\beta(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y_{k+n-p}^2. \quad (2.23)$$

In (2.23) there are exactly BI negative absolute value terms and QI negative squares.

On $\mathbb{R}^1$ we introduce the transformation ψ :

$$\psi(y) := \operatorname{sgn}(y)\sqrt{|y|}. \quad (2.24)$$

Note that the function $\pm|y|$ transforms into $\pm y^2$ under ψ^{-1} . Coordinate-wise extension of ψ on $\mathbb{R}^{|\beta(\bar{x})|}$ and leaving the other coordinates invariant, (2.24) induces the Hölder coordinate transformation Ψ ,

$$\Psi : \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \longrightarrow \mathbb{H}^{|J_0(\bar{x})|} \times \mathbb{R}^{|\beta(\bar{x})|} \times \mathbb{R}^p \quad (2.25)$$

The proof of the following corollary is again straightforward.

Corollary 2.3.12 (Normal forms in Hölder coordinates)

Let f have the normal form as in (2.23) and let Ψ be the Hölder coordinate transformation (2.25). Then, we have:

$$f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i=1}^{|J_0(\bar{x})|} y_i + \sum_{j=|J_0(\bar{x})|+1}^{n-|\beta(\bar{x})|+s} \pm y_j^2. \quad (2.26)$$

The number of negative squares in (2.26) equals the C-index $BI+QI$.

Deformation and Cell-Attachment

We state and prove the main deformation and cell-attachment theorem for MPCC. Recall that for $a, b \in \mathbb{R}$, $a < b$ the sets M^a and M_a^b are defined as follows:

$$M^a := \{x \in M \mid f(x) \leq a\}$$

and

$$M_a^b := \{x \in M \mid a \leq f(x) \leq b\}.$$

Theorem 2.3.13 *Let M_a^b be compact and suppose that LICQ is satisfied at all points $x \in M_a^b$.*

- (a) **(Deformation Theorem)** *If M_a^b does not contain any C-stationary point for MPCC, then M^a is a strong deformation retract of M^b .*
- (b) **(Cell-attachment Theorem)** *If M_a^b contains exactly one C-stationary point for MPCC, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the C-index of $\bar{x}$ is equal to q , then M^b is homotopy-equivalent to M^a with a q -cell attached.*

Proof. (a) Due to LICQ at all $x \in M_a^b$ there exist real numbers $\lambda_i(x)$, $i \in I$, $\varrho_{m_\alpha}(x)$, $m_\alpha \in \alpha(x)$, $\vartheta_{m_\gamma}(x)$, $m_\gamma \in \gamma(x)$, $\mu_j(x)$, $j \in J_0(x)$, $\sigma_{1,m_\beta}(x)$, $\sigma_{2,m_\beta}(x)$, $m_\beta \in \beta(x)$, $\nu_l(x)$, $l = 1, \dots, p$ such that:

$$\begin{aligned} Df(x) = & \sum_{i \in I} \lambda_i(x) Dh_i(x) + \sum_{m_\alpha \in \alpha(x)} \varrho_{m_\alpha}(x) DF_{1,m_\alpha}(x) + \sum_{m_\gamma \in \gamma(x)} \vartheta_{m_\gamma}(x) DF_{2,m_\gamma}(x) \\ & + \sum_{j \in J_0(x)} \mu_j(x) Dg_j(x) + \sum_{m_\beta \in \beta(x)} (\sigma_{1,m_\beta}(x) DF_{1,m_\beta}(x) + \sigma_{2,m_\beta}(x) DF_{2,m_\beta}(x)) + \sum_{l=1}^p \nu_l(x) \xi_l, \end{aligned}$$

where vectors ξ_l , $l = 1, \dots, p$ are chosen as in Lemma 2.3.7. We set:

$A := \{x \in M_a^b \mid \text{there exists } l \in \{1, \dots, p\} \text{ with } \nu_l(x) \neq 0\},$

$B := \{x \in M_a^b \mid \text{there exists } j \in J_0(x) \text{ with } \mu_j(x) < 0\},$

$C := \{x \in M_a^b \mid \text{there exists } m_\beta \in \beta(x) \text{ with } \sigma_{1,m_\beta}(x) \cdot \sigma_{2,m_\beta}(x) < 0\}.$

Since each $\bar{x} \in M_a^b$ is not C-stationary for MPCC, we get $\bar{x} \in A \cup B \cup C$.

The proof consists of a local argument and its globalization.

First, we show the **local argument**:

For each $\bar{x} \in M_a^b$ there exist an $(\mathbb{R}^n)$ -neighborhood $U_{\bar{x}}$ of $\bar{x}$, $t_{\bar{x}} > 0$ and a mapping

$$\Psi^{\bar{x}} : \begin{cases} [0, t_{\bar{x}}) \times (M^b \cap U_{\bar{x}}) \\ (t, x) \end{cases} \begin{array}{c} \longrightarrow M \\ \mapsto \Psi^{\bar{x}}(t, x) \end{array} \quad \text{such that:}$$

- (i) $\Psi^{\bar{x}}(t, M^b \cap U_{\bar{x}}) \subset M^{b-t}$ for all $t \in [0, t_{\bar{x}})$,
- (ii) $\Psi^{\bar{x}}(t_1 + t_2, \cdot) = \Psi^{\bar{x}}(t_1, \Psi^{\bar{x}}(t_2, \cdot))$ for all $t_1, t_2 \in [0, t_{\bar{x}})$ with $t_1 + t_2 \in [0, t_{\bar{x}})$,
- (iii) if $\bar{x} \in A \cup B$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a C^1 -flow corresponding to a C^1 -vector field $F^{\bar{x}}$.
- (iv) if $\bar{x} \in C$, then $\Psi^{\bar{x}}(\cdot, \cdot)$ is a Lipschitz flow.

Obviously, the level sets of f are locally mapped onto the level sets of $f \circ \Phi^{-1}$, where Φ is a C^1 -diffeomorphism according to Definition 2.3.6. Applying the standard diffeomorphism Φ from Definition 2.3.8, we consider $f \circ \Phi^{-1}$ (denoted by f again). Thus, we have $\bar{x} = 0$ and f is given on the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$.

Case a) $\bar{x} \in A$

Then, due to Remark 2.3.9 there exists $l \in \{1, \dots, p\}$ with $\frac{\partial f}{\partial x_l}(\bar{x}) \neq 0$. Define a local C^1 -vector field $F^{\bar{x}}$ as follows:

$$F^{\bar{x}}(x_1, \dots, x_l, \dots, x_n) := \left(0, \dots, -\frac{\partial f}{\partial x_l}(x) \cdot \left(\frac{\partial f}{\partial x_l}(x) \right)^{-2}, \dots, 0 \right)^T.$$

After respective inverse changes of local coordinates $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see [61], Theorem 2.7.6 for details).

Case b) $\bar{x} \in B$

Then, due to Remark 2.3.9 there exists $j \in J_0(x)$ with $\frac{\partial f}{\partial x_j}(\bar{x}) < 0$. By means of a C^1 -coordinate transformation (along the lines of [61], Theorem 3.2.26) in the j -th coordinate on $\mathbb{H}$, leaving the other coordinates unchanged, we obtain locally for f :

$$f(x_1, \dots, x_j, \dots, x_n) = -x_j + f(x_1, \dots, \bar{x}_j, \dots, x_n).$$

Define a local C^1 -vector field $F^{\bar{x}}$ as follows:

$$F^{\bar{x}}(x_1, \dots, x_j, \dots, x_n) := (0, \dots, 1, \dots, 0)^T.$$

After respective inverse changes of local coordinates $F^{\bar{x}}$ induces the flow $\Psi^{\bar{x}}$, which fits the local argument (see [61], Theorem 3.3.25 for details).

Case c) $\bar{x} \in C$

Then, due to Remark 2.3.9 there exists $m_\beta \in \beta(x)$ with

$$\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) \cdot \frac{\partial f}{\partial x_{2,m_\beta}}(\bar{x}) < 0.$$

W.l.o.g., we assume that $\frac{\partial f}{\partial x_{1,m_\beta}}(\bar{x}) < 0$ and $\frac{\partial f}{\partial x_{2,m_\beta}}(\bar{x}) > 0$.

From the proof of Theorem 2.3.10, Formula (2.20) we can obtain for f in new C^1 -coordinates the representation:

$$f(x_1, \dots, x_j, \dots, x_n) = -x_{1,m_\beta} + x_{2,m_\beta} + f(x_1, \dots, \bar{x}_{1,m_\beta}, \bar{x}_{2,m_\beta}, \dots, x_n).$$

Define the mapping $\Psi^{\bar{x}}$ locally as follows:

$$\begin{aligned} \Psi^{\bar{x}}(t, x_1, \dots, x_{1,m_\beta}, x_{2,m_\beta}, \dots, x_n) := \\ (x_1, \dots, x_{1,m_\beta} + \max\{0, t - x_{2,m_\beta}\}, \max\{0, x_{2,m_\beta} - t\}, \dots, x_n)^T. \end{aligned}$$

After respective inverse changes of local coordinates $\Psi^{\bar{x}}$ fits the local argument.

Note that in all the Cases a)-c) $\Psi^{\bar{x}}(t, \cdot)$ leaves the feasible set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ invariant.

Globalization.

Consider the open covering $\{U_x | x \in C\} \cup \{U_{\bar{x}} | \bar{x} \in M_a^b \setminus \{U_x | x \in C\}\}$ of M_a^b . Due to continuity arguments $U_{\bar{x}}$, $\bar{x} \in M_a^b \setminus \{U_x | x \in C\}$ can be taken smaller, if necessary, to be disjoint with C . Since M_a^b is compact, we get a finite open subcovering $\{U_{x_i} | x_i \in C\} \cup \{U_{\bar{x}_j} | \bar{x}_j \in M_a^b \setminus \{U_x | x \in C\}\}$ of M_a^b . Using a C^∞ -partition of unity $\{\phi_j\}$ subordinate to $\{U_{\bar{x}_j} | \bar{x}_j \in M_a^b \setminus \{U_x | x \in C\}\}$ we define with $F^{\bar{x}_j}$ (cf. Cases a), b)) a C^1 -vector field $F := \sum_j \phi_j F^{\bar{x}_j}$.

The last induces a flow Ψ on $\{U_{\bar{x}_j} | \bar{x}_j \in M_a^b \setminus \{U_x | x \in C\}\}$ (see [61], Theorem 3.3.14 for details). Note that in each nonempty overlapping region $U_{x_i} \cap U_{x_j}$, $x_i \in C$, $x_j \in M_a^b \setminus \{U_x | x \in C\}$ the flow Ψ^{x_i} induces exactly the vector field F (cf. Case c)). Hence, local trajectories can be glued together on M_a^b , named by Ψ again. Moreover, moving along the local pieces of the trajectories $\Psi(\cdot, x)$, $x \in M_a^b$ reduces the level of f at least by a positive real number

$$\frac{\min\{t_{x_i}, t_{x_j} | x_i \in C, x_j \in M_a^b \setminus \{U_x | x \in C\}\}}{2}.$$

Thus, we obtain for $x \in M_a^b$ a unique $t_a(x) > 0$ with $\Psi(t_a(x), x) \in M^a$. It is not hard (but technical) to realize that $t_a : x \rightarrow t_a(x)$ is Lipschitz. Finally, we define $r : [0, 1] \times M^b \rightarrow M^b$ as follows:

$$r(\tau, x) : \begin{cases} x & \text{for } x \in M^a, \quad \tau \in [0, 1] \\ \Psi(\tau t_a(x), x) & \text{for } x \in M_a^b, \quad \tau \in [0, 1]. \end{cases}$$

The mapping r provides that M^a is a strong deformation retract of M^b .

(b) In virtue of the Deformation Theorem and the normal forms (2.19), (2.23), (2.26), the proof of the Cell-attachment part becomes standard. In fact, the Deformation Theorem allows deformations up to an arbitrarily small neighborhood of the C-stationary point $\bar{x}$. In such a neighborhood we can work in continuous local coordinates, and use the explicit normal form (2.26). In the normal form (2.26) the origin is a non-degenerate KKT-point and the cell-attachment can be performed as in [61], Theorem 3.3.33. $\square$

Remark 2.3.14 *We emphasize that the linear terms y_i , $i \in J_0(\bar{x})$, in (2.26) do not contribute to the dimension of the cell to be attached. In fact, w.r.t.*

lower level sets, the 1-dim. constrained singularity y , $y \geq 0$, plays the same role as the unconstrained singularity y^2 . In this sense the constrained linear terms in (2.26) do not contribute to the number of negative squares.

Remark 2.3.15 Another way of looking at the cell-attachment part is via stratified Morse Theory ([30], Section 3.7). In fact, recall the normal form (2.19). The set $\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$ can be interpreted as the product of the "tangential part" $\{0_s\} \times \mathbb{R}^p$ and the "normal part" $\mathbb{H}^{|J_0(\bar{x})|} \times (\partial\mathbb{H}^2)^{|\beta(\bar{x})|}$. The main theorem in [30] states that the local "Morse data" is the product of the tangential "Morse data" with the normal "Morse data". The tangential Morse index equals QI and, in view of Remark 2.3.14, the normal Morse index equals BI . In the product, the index then becomes the sum $QI+BI$, what is precisely the C -index.

Remark 2.3.16 As it was pointed out by an anonymous referee, Theorem 2.3.13 can be interpreted as follows. The complementarity constraints can be reformulated as Lipschitzian equality constraints of the minimum type. For $u, v \in \mathbb{R}$ we have:

$$u \geq 0, v \geq 0, u \cdot v = 0 \iff \min\{u, v\} = 0$$

Regarding this issue Corollary 2.3.11 provides a normal form of f in Lipschitzian coordinates. Finally, Theorem 2.3.13 shows why the Morse index from the smooth mathematical programming has to be modified into the Clarke index for MPCC.

Discussion of different stationarity concepts

We briefly review well-known definitions of various stationarity concepts and connections between them (cf. [25], [95], [106]).

Definition 2.3.17 Let $\bar{x} \in M$.

(i) $\bar{x}$ is called W -stationary if (2.13), (2.14) hold.

(ii) $\bar{x}$ is called A -stationary if (2.13), (2.14) hold and

$$\bar{\sigma}_{1,m_\beta} \geq 0 \text{ or } \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}).$$

(iii) $\bar{x}$ is called M -stationary if (2.13), (2.14) hold and

$$(\bar{\sigma}_{1,m_\beta} > 0 \text{ and } \bar{\sigma}_{2,m_\beta} > 0) \text{ or } \bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} = 0 \text{ for all } m_\beta \in \beta(\bar{x}).$$

(iv) $\bar{x}$ is called *S-stationary* if (2.13), (2.14) hold and

$$\bar{\sigma}_{1,m_\beta} \geq 0, \bar{\sigma}_{2,m_\beta} \geq 0 \quad \text{for all } m_\beta \in \beta(\bar{x}).$$

(v) $\bar{x}$ is called *B-stationary* if $d = 0$ is a local solution of the linearized problem:

$$\min f(\bar{x}) + Df(\bar{x})d \quad \text{s.t.}$$

$$\begin{cases} F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d \geq 0, F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d \geq 0, \\ (F_{1,m}(\bar{x}) + DF_{1,m}(\bar{x})d) \cdot (F_{2,m}(\bar{x}) + DF_{2,m}(\bar{x})d) = 0, \quad m = 1, \dots, k, \\ h(\bar{x}) + Dh(\bar{x})d = 0, g(\bar{x}) + Dg(\bar{x})d \geq 0. \end{cases}$$

The following diagram (see Figure 8) summarizes the relations between mentioned stationarity concepts (e.g. [119]):

Figure 8: Stationarity concepts in MPCC

Assuming nondegeneracy (as in Definition 2.3.2) we see that A-, M-, S-, B-stationary points describe local minima tighter than C-stationary points. However, they exclude C-stationary points with $BI > 0$. These points are also crucial for the topological structure of MPCC (cf. Cell-attachment Theorem). For global optimization, points of *C-index* = 1 play an important role, see also the Section 1.2. We emphasize that among the points of *C-index* = 1 from a topological point of view there is no substantial difference between the points with $BI = 1, QI = 0$ and $BI = 0, QI = 1$. It is worth to mention that a linear descent direction might exist in a nondegenerate C-stationary point with positive *C-index* (see [85] and [106] for examples and the following discussion). However, at points with $BI = 1, QI = 0$ there are exactly two directions of linear decrease. Both of them are important from a global point of view. In turn, W-stationary points contain those with negative and positive Lagrange multipliers corresponding to the same complementarity constraint. Due to Deformation Theorem such points are irrelevant for the topological structure of MPCC.

Further, we illustrate the foregoing considerations by the following Example 2.3.18 due to [106] (c.f. also [85]).

Example 2.3.18

$$\min (x-1)^2 + (y-1)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.27)$$

It is clear that C -stationary points for (2.27) are $(1,0)$, $(0,1)$ and $(0,0)$. Moreover, $(1,0)$ and $(0,1)$ are local (and global) minimizers with C -index 0. Bi-active Lagrange multipliers for $(0,0)$ are both -2 , hence, its C -index is 1. One might think that the C -stationary point $(0,0)$ is irrelevant for numerical purposes, since it posses linear descent directions. However, globally it precisely connects the local minima. Moreover, if we consider the problem (2.27) with smoothed complementarity constraints:

$$\min (x-1)^2 + (y-1)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = \varepsilon, \quad (2.28)$$

where $\varepsilon > 0$ is sufficiently small. Then, it is easily seen that the critical points for (2.28) are:

$$(x_1, y_1) = \left(\frac{1 + \sqrt{1-4\varepsilon}}{2}, \frac{1 - \sqrt{1-4\varepsilon}}{2} \right),$$

$$(x_2, y_2) = \left(\frac{1 - \sqrt{1-4\varepsilon}}{2}, \frac{1 + \sqrt{1-4\varepsilon}}{2} \right),$$

$$(x_3, y_3) = (\sqrt{\varepsilon}, \sqrt{\varepsilon}).$$

Obviously, $(x_1, y_1) \rightarrow (1,0)$, $(x_2, y_2) \rightarrow (0,1)$ and $(x_3, y_3) \rightarrow (0,0)$ as $\varepsilon \rightarrow 0$. Moreover, (x_1, y_1) and (x_2, y_2) are local (and global) minimizers for (2.28) with quadratic index 0, the quadratic index of (x_3, y_3) is 1 (local maximum). Hence, by the smoothing procedure the C -stationary point $(0,0)$ with C -index 1 corresponds to the critical point (x_3, y_3) with quadratic index 1. In particular, the smoothed version preserves the global topological structure.

We notice that adding positive squares to the objective function in (2.27) provides a more dimensional example with the same features.

2.4 Parametric aspects

The aim of this section is the introduction and characterization of the strong stability notion in MPCC (cf. Definition 2.4.2). In 1980, M. Kojima introduced in [80] the (topological) concept of strong stability for stationary solutions (Karush-Kuhn-Tucker points) for nonlinear optimization problems (see also [100] by Robinson). This concept plays an important role in optimization theory, for example in sensitivity and parametric optimization [62, 81], and structural stability [76]. It turns out that the concept of C-stationarity is the adequate stationarity concept regarding possible bifurcations.

We characterize of strong stability for C-stationary points by means of first and second order information of the defining functions f, h, g, F_1, F_2 under Linear Independence Constraint Qualification (LICQ) (see Theorem 2.4.6). The main issue in strong stability of C-stationary points is related to the so-called bi-active Lagrange multipliers (see also Corollary 2.4.7). A bi-active pair of Lagrange multipliers corresponds to such complementarity constraints which both vanish at a C-stationary point. There are three (degeneracy-)possibilities for bi-active multipliers:

- (a) both bi-active Lagrange multipliers do not vanish (nondegenerate case),
- (b) only one bi-active Lagrange multiplier vanishes (first degenerate case),
- (c) both bi-active Lagrange multipliers vanish (second degenerate case).

Depending on the kind of possible degeneracy we use corresponding ideas on strong stability of Kojima (cases (a) and (b)). Moreover, we describe new unstable phenomena (case (c)).

We would like to refer to some related papers. In [106] an extension of the stability results of Kojima and Robinson to MPCC is presented. It refers to the nondegenerate case (a) of nonvanishing bi-active Lagrange multipliers. In [98] the concept of the so-called co-1-singularity for quadratic programs with complementarity constraints (QPCC) is studied. In our terms they refer to the first degenerate case (b).

We refer to [74] for details.

Notations and Auxiliary Results

From Section 2.3 we recall the following index sets for given $\bar{x} \in M$:

$$J_0(\bar{x}) := \{j \in J \mid g_j(\bar{x}) = 0\},$$

$$\alpha(\bar{x}) := \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) > 0\},$$

$$\beta(\bar{x}) := \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) = 0, F_{2,m}(\bar{x}) = 0\},$$

$$\gamma(\bar{x}) := \{m \in \{1, \dots, k\} \mid F_{1,m}(\bar{x}) > 0, F_{2,m}(\bar{x}) = 0\}.$$

Without loss of generality (w.l.o.g.), we assume that at the particular point of interest $\bar{x} \in M$ it holds:

$$J_0(\bar{x}) = \{1, \dots, |J_0(\bar{x})|\}, \alpha(\bar{x}) = \{1, \dots, |\alpha(\bar{x})|\},$$

$$\gamma(\bar{x}) = \{|\alpha(\bar{x})| + 1, \dots, |\alpha(\bar{x})| + |\gamma(\bar{x})|\}.$$

We put $s := |I| + |\alpha(\bar{x})| + |\gamma(\bar{x})|$, $p := n - s - |J_0(\bar{x})| - 2|\beta(\bar{x})|$.

We also recall the notions of LICQ and C-stationarity (cf. Section 2.3).

The Linear Independence Constraint Qualification (LICQ) for MPCC is said to hold at $\bar{x} \in M$ if the vectors

$$\begin{aligned} Dh_i(\bar{x}), i \in I, DF_{1,m_\alpha}(\bar{x}), m_\alpha \in \alpha(\bar{x}), DF_{2,m_\gamma}(\bar{x}), m_\gamma \in \gamma(\bar{x}), \\ Dg_j(\bar{x}), j \in J_0(\bar{x}), DF_{1,m_\beta}(\bar{x}), DF_{2,m_\beta}(\bar{x}), m_\beta \in \beta(\bar{x}) \end{aligned}$$

are linearly independent.

A point $\bar{x} \in M$ is called Clarke stationary (C-stationary) for MPCC (cf. Definition 2.3.1) if there exist real numbers $\bar{\lambda}_i, i \in I, \bar{\mu}_j, j \in J, \bar{\sigma}_{1,m}, \bar{\sigma}_{2,m}, m = 1, \dots, k$ (Lagrange multipliers), such that:

$$\begin{aligned} Df(\bar{x}) &= \sum_{i \in I} \bar{\lambda}_i Dh_i(\bar{x}) + \sum_{j \in J} \bar{\mu}_j Dg_j(\bar{x}) + \\ &+ \sum_{m=1}^k (\bar{\sigma}_{1,m} DF_{1,m}(\bar{x}) + \bar{\sigma}_{2,m} DF_{2,m}(\bar{x})), \end{aligned} \quad (2.29)$$

$$\bar{\mu}_j \cdot g_j(\bar{x}) = 0, j \in J \quad (2.30)$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0(\bar{x}) \quad (2.31)$$

$$\bar{\sigma}_{j,m} \cdot F_{j,m}(\bar{x}) = 0, j = 1, 2, m = 1, \dots, k \quad (2.32)$$

$$\bar{\sigma}_{1,m_\beta} \cdot \bar{\sigma}_{2,m_\beta} \geq 0 \text{ for all } m_\beta \in \beta(\bar{x}). \quad (2.33)$$

The Lagrange function L is defined as follows:

$$\begin{aligned} L(x, \lambda, \mu, \sigma) &:= f(x) - \sum_{i \in I} \lambda_i h_i(\bar{x}) - \sum_{j \in J} \mu_j g_j(\bar{x}) - \\ &- \sum_{m=1}^k (\sigma_{1,m} F_{1,m}(\bar{x}) + \sigma_{2,m} F_{2,m}(\bar{x})). \end{aligned} \quad (2.34)$$

Definition 2.4.1 (C-stationary pair)

A vector $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma}) \in M \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} \times \mathbb{R}^{2k}$ satisfying (2.29)-(2.33) is called a C-stationary pair for MPCC.

The concept of strong stability is defined by means of an appropriate semi-norm. To this aim let be $\bar{x} \in \mathbb{R}^n$, $r > 0$. For defining functions (f, h, g, F_1, F_2) from (2.5) the seminorm $\|(f, h, g, F_1, F_2)\|_{B(\bar{x}, r)}^{C^2}$ is defined to be the maximum modulus of the function values and partial derivatives up to order two of f, h, g, F_1, F_2 .

Definition 2.4.2 (Strong Stability, cf. [80])

A C-stationary point $\bar{x} \in M$, resp. a C-stationary pair $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$, for $MPCC[f, g, h, F_1, F_2]$ is called (C^2) -strongly stable if for some $r > 0$ and each $\varepsilon \in (0, r]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that whenever $(\tilde{f}, \tilde{h}, \tilde{g}, \tilde{F}_1, \tilde{F}_2) \in C^2$ and

$$\left\| (f - \tilde{f}, h - \tilde{h}, g - \tilde{g}, F_1 - \tilde{F}_1, F_2 - \tilde{F}_2) \right\|_{B(\bar{x}, r)}^{C^2} \leq \delta,$$

$B(\bar{x}, \varepsilon)$, resp. $B((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma}), \varepsilon)$, contains a C-stationary point $\tilde{x}$, resp. a C-stationary pair $(\tilde{x}, \tilde{\lambda}, \tilde{\mu}, \tilde{\sigma})$, for $MPCC[\tilde{f}, \tilde{h}, \tilde{g}, \tilde{F}_1, \tilde{F}_2]$ which is unique in $B(\bar{x}, r)$, resp. unique in $B((\tilde{x}, \tilde{\lambda}, \tilde{\mu}, \tilde{\sigma}), r)$.

The following lemma establishes the connection between both definitions just introduced (cf. [79]).

Lemma 2.4.3 (C-stationary points and pairs)

The following assertions are equivalent:

- (a) $\bar{x}$ is a strongly stable C-stationary point for MPCC which satisfies LICQ, and $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$ is the associated Lagrange multiplier vector.
- (b) $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$ is a strongly stable C-stationary pair for MPCC.

Proof. : (a) $\implies$ (b) LICQ remains valid under small perturbations of the defining functions. Hence, the corresponding Lagrange multipliers are unique. Moreover, Remark 2.3.9 provides the continuity of Lagrange multipliers w.r.t. perturbations under consideration.

(b) $\implies$ (a) The nontrivial part is to prove that LICQ holds at $\bar{x}$. The proof goes along the lines of Theorem 2.3 from [79]. To stress the new aspects here we assume that there are only bi-active constraints (i.e. $I = \emptyset$, $J = \emptyset$, $\alpha(\bar{x}) = \emptyset$ and $\gamma(\bar{x}) = \emptyset$). Let $(\bar{x}, \bar{\sigma})$ be a strongly stable C-stationary pair for

MPCC and let LICQ be not fulfilled at $\bar{x}$. Then, there exist real numbers δ_{1,m_β} , δ_{2,m_β} , $m_\beta \in \beta(\bar{x})$ (not all vanishing) such that:

$$\sum_{m_\beta \in \beta(\bar{x})} (\delta_{1,m} DF_{1,m}(\bar{x}) + \delta_{2,m} DF_{2,m}(\bar{x})) = 0. \quad (2.35)$$

Setting

$$m_\beta^+(\bar{x}) := \{m_\beta \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m}, \bar{\sigma}_{2,m} \geq 0\},$$

$$m_\beta^-(\bar{x}) := \{m_\beta \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m}, \bar{\sigma}_{2,m} \leq 0\}$$

we define

$$c := - \left[\sum_{m_\beta \in m_\beta^+(\bar{x})} (DF_{1,m}(\bar{x}) + DF_{2,m}(\bar{x})) - \sum_{m_\beta \in m_\beta^-(\bar{x})} (DF_{1,m}(\bar{x}) + DF_{2,m}(\bar{x})) \right].$$

For $\varepsilon > 0$ let

$$\sigma_{1,m}(\varepsilon) := \bar{\sigma}_{1,m} + \varepsilon, \sigma_{2,m}(\varepsilon) := \bar{\sigma}_{2,m} + \varepsilon \text{ for all } m \in m_\beta^+(\bar{x}),$$

$$\sigma_{1,m}(\varepsilon) := \bar{\sigma}_{1,m} - \varepsilon, \sigma_{2,m}(\varepsilon) := \bar{\sigma}_{2,m} - \varepsilon \text{ for all } m \in m_\beta^-(\bar{x}).$$

Putting $\varphi(x) := c \cdot x$ we obtain that $(\bar{x}, \sigma(\varepsilon))$ is a C-stationary pair for $\text{MPCC}[f + \varepsilon \cdot \varphi, F_1, F_2]$. Moreover, due to the strong stability of $(\bar{x}, \bar{\sigma})$ for $\text{MPCC}[f, F_1, F_2]$ we claim that for each sufficiently small $\varepsilon > 0$ the C-stationary pair $(\bar{x}, \sigma(\varepsilon))$ is unique for $\text{MPCC}[f + \varepsilon \cdot \varphi, F_1, F_2]$ in some neighborhood U of $(\bar{x}, \bar{\sigma})$.

However, (2.35) and $\sigma_{i,m}(\varepsilon) \neq 0$ for $m \in m_\beta(\bar{x})$, $i = 1, 2$ ensure that for any sufficiently small real number t , the pair $(\bar{x}, v(\varepsilon, \delta, t))$ with

$$v_{1,m}(\varepsilon, \delta, t) := \sigma_{1,m}(\varepsilon) + \delta_{1,m}t, v_{2,m}(\varepsilon, \delta, t) := \sigma_{2,m}(\varepsilon) + \delta_{2,m}t \text{ for all } m \in m_\beta(\bar{x})$$

belongs to U and is a C-stationary pair for $\text{MPCC}[f + \varepsilon \cdot \varphi, F_1, F_2]$. Hence, necessarily $\delta = 0$, and LICQ is shown. $\square$

Now we give two guiding examples for instability which may occur at C-stationary points in the second degenerate case (c).

Example 2.4.4 (Unstable minimum/maximum, cf. [106])

Consider the MPCC:

$$\min x^2 + y^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.36)$$

Obviously, $(0, 0)$ is the unique C -stationary point for (2.36) with both vanishing bi-active Lagrange multipliers. Consider the following perturbation of (2.36) w.r.t. parameter $t > 0$:

$$\min (x - t)^2 + (y - t)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.37)$$

It is easy to see that $(0, 0)$, $(0, t)$ and $(t, 0)$ are C -stationary points for (2.37). It means that $(0, 0)$ is not a strongly stable C -stationary point for (2.36). Analogously we can treat $-x^2 - y^2$ on $\partial\mathbb{H}^2$ at the origin.

Example 2.4.5 (Unstable saddle point)

Consider the MPCC:

$$\min x^2 - y^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.38)$$

Obviously, $(0, 0)$ is the unique C -stationary point for (2.38) with both vanishing bi-active Lagrange multipliers. Consider the following perturbation of (2.38) w.r.t. parameter $t > 0$:

$$\min (x - t)^2 - (y - t)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.39)$$

It is easy to see that $(0, t)$ and $(t, 0)$ are C -stationary points for (2.39). It means that $(0, 0)$ is not a strongly stable C -stationary point for (2.38).

Characterization of strong stability for C -stationary points

Before stating the main result we define the following index sets at a C -stationary point $\bar{x} \in M$ with Lagrange multipliers $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$ (cf. Definition 2.3.1):

$$J_+ := \{j \in J_0(\bar{x}) \mid \bar{\mu}_j > 0\},$$

$$p(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} \cdot \bar{\sigma}_{2,m} > 0\},$$

$$q(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} > 0, \bar{\sigma}_{2,m} = 0\},$$

$$r(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} = 0, \bar{\sigma}_{2,m} > 0\},$$

$$s(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} < 0, \bar{\sigma}_{2,m} = 0\},$$

$$w(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} = 0, \bar{\sigma}_{2,m} < 0\},$$

$$u(\bar{x}) := \{m \in \beta(\bar{x}) \mid \bar{\sigma}_{1,m} = 0, \bar{\sigma}_{2,m} = 0\}.$$

Obviously, $p(\bar{x})$, $q(\bar{x})$, $r(\bar{x})$, $s(\bar{x})$, $w(\bar{x})$, $u(\bar{x})$ constitute a partition of $\beta(\bar{x})$.

For $\bar{J} \subset J$, $K \subset \{1, \dots, k\}$, $j = 1, 2$ we write h , $g_{\bar{J}}$, $F_{j,K}$ for $(h_i \mid i \in I)$, $(g_j \mid j \in \bar{J})$, $(F_{j,m} \mid m \in K)$, respectively.

Furthermore, for $J_+ \subset \bar{J} \subset J_0(\bar{x})$, $\bar{q} \subset q(\bar{x})$, $\bar{r} \subset r(\bar{x})$, $\bar{s} \subset s(\bar{x})$, $\bar{w} \subset w(\bar{x})$ we define $M_{\bar{J}, \bar{q}, \bar{r}, \bar{s}, \bar{w}}(\bar{x})$ to be the block matrix $\begin{pmatrix} C & X \\ Y & 0 \end{pmatrix}$ with

$$X = \begin{pmatrix} H^T & G_J^T & A^T & \Gamma^T & P^T & Q^T & \bar{Q}^T & R^T & \bar{R}^T & S^T & \bar{S}^T & W^T & \bar{W}^T \end{pmatrix},$$

$$Y^T = \begin{pmatrix} H^T & -G_J^T & A^T & \Gamma^T & P^T & Q^T & -\bar{Q}^T & R^T & -\bar{R}^T & S^T & \bar{S}^T & W^T & \bar{W}^T \end{pmatrix},$$

where

$$C = D_{xx}^2 L(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma}), \quad H = Dh(\bar{x}), \quad G_J = Dg_{\bar{J}}(\bar{x}),$$

$$A = DF_{1,\alpha(\bar{x})}(\bar{x}), \quad \Gamma = DF_{2,\gamma(\bar{x})}(\bar{x}), \quad P = (DF_{1,p(\bar{x})}, DF_{2,p(\bar{x})})(\bar{x}),$$

$$Q = DF_{1,q(\bar{x})}(\bar{x}), \quad \bar{Q} = DF_{2,\bar{q}}(\bar{x}), \quad R = DF_{2,r(\bar{x})}(\bar{x}), \quad \bar{R} = DF_{1,\bar{r}}(\bar{x}),$$

$$S = DF_{1,s(\bar{x})}(\bar{x}), \quad \bar{S} = DF_{2,\bar{s}}(\bar{x}), \quad W = DF_{2,w(\bar{x})}(\bar{x}), \quad \bar{W} = DF_{1,\bar{w}}(\bar{x}).$$

Theorem 2.4.6 (Characterization of strong stability)

Suppose that LICQ holds at a C-stationary point $\bar{x} \in M$ with Lagrange multipliers $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$ (cf. Definition 2.3.1). Then, $\bar{x}$ is a strongly stable C-stationary point for MPCC (cf. Definition 2.4.2) if and only if

$$(i) \quad u(\bar{x}) = \emptyset \text{ and}$$

$$(ii) \quad \text{all matrices } M_{\bar{J}, \bar{q}, \bar{r}, \bar{s}, \bar{w}}(\bar{x}) \text{ with}$$

$$J_+ \subset \bar{J} \subset J_0(\bar{x}), \quad \bar{q} \subset q(\bar{x}), \quad \bar{r} \subset r(\bar{x}), \quad \bar{s} \subset s(\bar{x}), \quad \bar{w} \subset w(\bar{x})$$

are nonsingular with the same determinant sign.

Proof. In virtue of LICQ at $\bar{x}$, Lemma 2.4.3 allows us to deal equivalently with the strong stability of the C-stationary pair $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$.

Case 1: $u(\bar{x}) = \emptyset$

We consider the following mapping $\mathcal{T} : \mathbb{R}^n \times \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} \times \mathbb{R}^{2k} \longrightarrow \mathbb{R}^{n+|I|+|J|+2k}$ locally at its zero $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$:

$$\mathcal{T}(x, \lambda, \mu, \sigma) := \begin{pmatrix} D_x L(x, \lambda, \mu, \sigma) \\ h(x) \\ \min \{ \mu, g(x) \} \\ F_{1, \alpha(\bar{x})}(x) \\ F_{2, \gamma(\bar{x})}(x) \\ F_{1, p(\bar{x})}(x) \\ F_{2, p(\bar{x})}(x) \\ F_{1, q(\bar{x})}(x) \\ \min \{ \sigma_{2, q(\bar{x})}, F_{2, q(\bar{x})}(x) \} \\ F_{2, r(\bar{x})} \\ \min \{ \sigma_{1, r(\bar{x})}, F_{1, r(\bar{x})}(x) \} \\ F_{1, s(\bar{x})} \\ \min \{ -\sigma_{2, s(\bar{x})}, F_{2, s(\bar{x})}(x) \} \\ F_{2, w(\bar{x})} \\ \min \{ -\sigma_{1, w(\bar{x})}, F_{1, w(\bar{x})}(x) \} \end{pmatrix}.$$

Note that C-stationary pairs for MPCC - in a sufficiently small neighborhood of $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$ - are precisely the zeros of $\mathcal{T}$. Moreover, the only difference in $\mathcal{T}$ compared with the case of standard nonlinear optimization programs is the appearing minus sign in

$$\min \{ -\sigma_{2, s(\bar{x})}, F_{2, s(\bar{x})}(x) \}, \min \{ -\sigma_{1, w(\bar{x})}, F_{1, w(\bar{x})}(x) \}.$$

Since we deal with equality constraints of minimum-type, Theorem 4.3 from [79] (characterization of strong stability for KKT-points) can be simply adapted here. Indeed, as in Theorem 4.3 from [79], the strong stability for $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$ can be characterized by the fact that all matrices in the Clarke's subdifferential $\partial \mathcal{T}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\sigma})$ are nonsingular. The latter can be equivalently rewritten as condition (ii) (cf. also [77] for the case of nonlinear optimization programs).

Case 2: $u(\bar{x}) \neq \emptyset$

Let $\Phi : U \longrightarrow V$ be a standard diffeomorphism according to Definition 2.3.8. We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p \right) \cap V$. From now on we may assume $s = 0$. In view of Remark 2.3.9 we have at the origin:

$$(i) \quad \frac{\partial \bar{f}}{\partial y_j} \geq 0, \quad j \in J_0(\bar{x}),$$

$$(ii) \frac{\partial \bar{f}}{\partial y_{|J_0|+2m-1}} \cdot \frac{\partial \bar{f}}{\partial y_{|J_0|+2m}} \geq 0, m = 1, \dots, |\beta(\bar{x})|,$$

$$(iii) \frac{\partial \bar{f}}{\partial y_{l+n-p}} = 0, l = 1, \dots, p.$$

Moreover, due to condition $u(\bar{x}) \neq \emptyset$ we may assume w.l.o.g. that

$$(iv) \frac{\partial \bar{f}}{\partial y_{|J_0|+1}} = 0, \frac{\partial \bar{f}}{\partial y_{|J_0|+2}} = 0.$$

From now on we denote $\bar{f}$ again by f .

In what follows, we successively perform arbitrarily small perturbations of f such that the origin remains a C-stationary point on $\mathbb{H}^{|J_0(\bar{x})|} \times (\partial \mathbb{H}^2)^{|\beta(\bar{x})|} \times \mathbb{R}^p$.

1) As a stabilization step we add to f an arbitrarily small linear-quadratic term

$$\sum_{j=1}^{|J_0(\bar{x})|} c_j \cdot y_j + \sum_{m=2}^{|\beta(\bar{x})|} (c_{|J_0|+2m-1} \cdot y_{|J_0|+2m-1} + c_{|J_0|+2m} \cdot y_{|J_0|+2m}) + \sum_{l=1}^p c_{l+n-p} y_{l+n-p}^2,$$

such that it holds for the perturbed function (denoted again by f):

$$(i) \frac{\partial f}{\partial y_j} > 0, j \in J_0(\bar{x}),$$

$$(ii) \frac{\partial f}{\partial y_{|J_0|+2m-1}} \cdot \frac{\partial f}{\partial y_{|J_0|+2m}} > 0, m = 2, \dots, |\beta(\bar{x})|,$$

$$(iii) \frac{\partial f}{\partial y_{l+n-p}} = 0, l = 1, \dots, p \text{ and } \left(\frac{\partial^2 f}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}} \right)_{1 \leq k_1, k_2 \leq p} \text{ is a nonsingular matrix.}$$

$$(iv) \frac{\partial f}{\partial y_{|J_0|+1}} = 0, \frac{\partial f}{\partial y_{|J_0|+2}} = 0.$$

2) We approximate f by means of a C^∞ -function in a small C^2 -neighborhood of f such leaving its value, first and second order derivatives at the origin invariant. This can be done since C^∞ -functions lie C^2 -dense in C^2 -functions. We denote the latter C^∞ -approximation again by f .

Due to the stabilization step 1) and step 2) we can restrict our considerations to the following case:

$$f \in C^\infty(\mathbb{R}^2, \mathbb{R}), 0 \text{ is a C-stationary point for } f|_{\partial \mathbb{H}^2} \text{ and } \frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0.$$

Now we can write $f(x, y)$ as follows:

$$f(x, y) = g_{1,1}(x, y)x^2 + 2g_{1,2}(x, y)xy + g_{2,2}(x, y)y^2$$

with $g_{1,1}, g_{1,2}, g_{2,2} \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

Adding to f an arbitrarily small quadratic term $ax^2 + by^2$, $a, b \in \mathbb{R}$ we can ensure that

$$g_{1,1}(0, 0) \neq 0 \text{ and } g_{2,2}(0, 0) \neq 0.$$

Hence, $\Psi(x, y) := \begin{pmatrix} x \cdot \sqrt{|g_{1,1}(x, y)|} \\ y \cdot \sqrt{|g_{2,2}(x, y)|} \end{pmatrix}$ is a local C^∞ -diffeomorphism leaving $\partial\mathbb{H}^2$ invariant. In new local coordinates induced by Ψ we obtain:

$$f(x, y) = \varepsilon_1 x^2 + G(x, y)xy + \varepsilon_2 y^2,$$

where $\varepsilon_1 = \text{sign}(g_{1,1}(0, 0))$, $\varepsilon_2 = \text{sign}(g_{2,2}(0, 0))$.

Since $G(x, y)xy = 0$ on $\partial\mathbb{H}^2$ we can perturb f by means of a real parameter as in Examples 2.4.4 or 2.4.5 to get a bifurcation of 0 as a C-stationary point.

Finally, performing all perturbations described above we ensure that 0 is not a strongly stable C-stationary point. $\square$

The main new issue in the characterization of strong stability of C-stationary points in MPCC will be clarified in the following Corollary. Its proof follows from Theorem 2.4.6 by means of few elementary calculations.

Corollary 2.4.7 *Let $f \in C^2(\mathbb{R}^2, \mathbb{R})$ and suppose that 0 is a C-stationary point for $f|_{\partial\mathbb{H}^2}$. Then, 0 is a strongly stable C-stationary point if and only if*

$$\text{either } \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} > 0 \text{ or } \frac{\partial f}{\partial x} = 0, \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial f}{\partial y} > 0 \text{ or } \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial f}{\partial x} > 0 \text{ at } 0.$$

Now, we relate the notion of a nondegenerate C-stationary point to the results in Theorem 2.4.6 and Corollary 2.4.7.

Corollary 2.4.8 *Let $\bar{x} \in M$ be a nondegenerate C-stationary point as in Definition 2.3.2. Then, $\bar{x}$ is a strongly stable C-stationary point for MPCC.*

In the situation of Corollary 2.4.7 we claim that 0 is a nondegenerate C-stationary point for $f|_{\partial\mathbb{H}^2}$ if and only if

$$\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} > 0 \text{ at } 0.$$

On stability w.r.t. different stationarity concepts

For different stationarity concepts (such as A-, M-, S- and B-stationarity) we refer to Definition 2.3.17. Strong stability for A-, M-, S- and B-stationary points can be defined analogously as in Definition 2.4.2. From now on we assume that LICQ holds at all points of interest.

It is clear that strongly stable S-stationary points can be characterized by means of Theorem 2.4.6. Indeed, each (not) strongly stable S-stationary point corresponds to a (not) strongly stable C-stationary point.

However, the issue is different as soon as we consider M-stationary points (or A-stationary points).

Example 2.4.9 *Consider the MPCC:*

$$\min -x - y^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.40)$$

Obviously, $(0,0)$ is the unique C-stationary point for (2.40) with bi-active Lagrange multipliers $(-1,0)$. Hence, $(0,0)$ is also M-stationary. Moreover, due to Corollary 2.4.7 $(0,0)$ is a strongly stable C-stationary point. Consider the following perturbation of (2.40) w.r.t. parameter $t > 0$:

$$\min -x - (y+t)^2 \text{ s.t. } x \geq 0, y \geq 0, x \cdot y = 0. \quad (2.41)$$

It is easy to see that $(0,0)$ is the unique C-stationary point for (2.41) with both bi-active Lagrange multipliers $(-1, -2t)$ negative. It means that $(0,0)$ is not a strongly stable M-stationary point for (2.40).

Remark 2.4.10 *We consider once more Example 2.4.4. We recall that $(0,0)$ is the unique C-stationary point for (2.36) with both vanishing bi-active Lagrange multipliers. Hence, $(0,0)$ is also M-stationary. For the perturbed program (2.37) we have that $(0,0)$, $(0,t)$ and $(t,0)$ are C-stationary. It is easy to see that $(0,0)$ is not M-stationary for (2.37).*

We note that adding positive or negative squares to the objective functions above provides higher dimensional examples with the similar features.

Finally, we point out some issues on MPCC motivated by the strong stability results.

Remark 2.4.11 *Due to Example 2.4.9 and Remark 2.4.10 the concept of C-stationarity is crucial for the numerical treatment of MPCC via homotopy based methods. Further, Theorem 2.4.6 provides a characterization of the strongly stable C-stationary points. These are solutions of certain stable equations involving first and second order information of the defining functions. This fact might be used to establish some nonsmooth versions of Newton Method for MPCC (cf. [54, 58]). This is an issue of current research.*

Remark 2.4.12 *In Theorem 2.4.6 the strong stability of C -stationary points is characterized under LICQ. Its characterization in absence of LICQ is still open. We point out that this issue might be related to the version of Mangasarian-Fromovitz Condition (MFC) as introduced in [68]. The Constraint Qualification MFC has been introduced in [68] in connection with topological stability of the MPCC feasible set. This is still an issue of current research.*

Chapter 3

General semi-infinite programming

3.1 Applications and examples

Generalized semi-infinite optimization problems have the form

$$\text{GSIP: } \text{minimize } f(x) \text{ s.t. } x \in M \tag{3.1}$$

with

$$M := \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y(x)\}$$

and

$$Y(x) := \{y \in \mathbb{R}^m \mid g_k(x, y) \leq 0, k = 1, \dots, s\}.$$

All defining functions $f, g_k, k = 0, \dots, s$, are assumed to be real-valued and continuously differentiable on their respective domains. In case of a constant mapping $Y(\cdot) = Y$, we refer to semi-infinite optimization problems (SIP).

We present the well-known applications of GSIP in the area of Chebyshev approximation, design centering and robust optimization from a survey [40]. Further, we give some examples of GSIP which illustrate two main new features of GSIP (in addition to SIP):

- M need not to be a closed set,
- M might exhibit so-called "re-entrant" corners.

Chebyshev and Reverse Chebyshev Approximation

We approximate a given continuous function F on a nonempty and compact set $Y \subset \mathbb{R}^m$ by a function $f(x, \cdot)$. The latter comes from a parameterized family of continuous functions $\{f(x, \cdot) \mid x \in X\}$ with some parameter set $X \subset \mathbb{R}^n$. The problem of Chebyshev approximation is as follows:

$$\text{CA: minimize } \|F(\cdot) - f(x, \cdot)\|_{\infty, Y} \text{ s.t. } x \in X, \quad (3.2)$$

where

$$\|F(\cdot) - f(x, \cdot)\|_{\infty, Y} := \max_{y \in Y} \|F(y) - f(x, y)\|.$$

This problem can be rewritten as

$$\text{CA-SIP: minimize } z \text{ s.t. } -z \leq F(y) - f(x, y) \leq z \text{ for all } y \in Y.$$

CA-SIP is a standard semi-infinite optimization problem. Note that CA-SIP is a smooth optimization problem in addition to CA which is nonsmooth due to the maximum norm. Here, the difficulty is shifted into infinitely many inequality constraints.

Next, we formulate the problem of reverse Chebyshev approximation. Let F be a real-valued continuous function on a nonempty and compact set $Y(x) \subset \mathbb{R}^m$ which depends on a parameter $x \in X \subset \mathbb{R}^n$. Given an approximating family of functions $f(z, \cdot)$, $z \in Z \subset \mathbb{R}^k$ and a desired precision $e(z, x)$, the aim is to find parameter vectors z and x such that the domain $Y(x)$ is as large as possible without exceeding the approximation error $e(z, x)$. This yields the problem

$$\text{RCA: maximize } \text{vol}(Y(x)) \text{ s.t. } \|F(\cdot) - f(z, \cdot)\|_{\infty, Y(x)} \leq e(z, x).$$

Again, this nonsmooth optimization problem can be reformulated with semi-infinite constraints. However, we now obtain a generalized semi-infinite optimization problem:

$$\text{RCA-GSIP: maximize } \text{vol}(Y(x))$$

$$\text{s.t. } -e(z, x) \leq F(y) - f(z, y) \leq e(z, x) \text{ for all } y \in Y(x).$$

Design Centering

In a design centering problem we try to maximize the size of a parameterized body $B(x)$ contained in a container set C :

$$\text{DC: } \underset{x \in \mathbb{R}^n}{\text{maximize}} \text{ vol}(B(x)) \text{ s.t. } B(x) \subset C.$$

Let the container C be given by inequality constraints:

$$C = \{y \in \mathbb{R}^m \mid g(y) \leq 0\}.$$

Then, DC can be equivalently written as:

$$\text{DC-GSIP: } \underset{x \in \mathbb{R}^n}{\text{maximize}} \text{ vol}(B(x)) \text{ s.t. } g(y) \leq 0 \text{ for all } y \in B(x).$$

Robust Optimization

Robust optimization deals with an a priori analysis of optimization problems depending on uncertain data (cf. [7]). The so-called robust counterparts of finite optimization problems fit in the context of GSIP. Let an inequality constraint function $G(x, y)$ depend on some uncertain parameter vector y from a so-called uncertainty set $Y \subset \mathbb{R}^m$. Then the pessimistic way to deal with this constraint is to use its worst case reformulation

$$G(x, y) \leq 0 \text{ for all } y \in Y.$$

The latter inequality system is of semi-infinite type. Let now the uncertainty set Y also depend on the decision variable x . We obtain a generalized semi-infinite constraint

$$G(x, y) \leq 0 \text{ for all } y \in Y(x).$$

Next, let an objective function $F(x, y)$ depend on the unknown parameter $y \in Y(x)$. In the worst case one has to minimize the maximal objective value, that is, one considers the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \underset{y \in Y(x)}{\text{maximize}} F(x, y).$$

Hence, we are ready to write down a robust counterpart for the following parametric optimization problem:

$$\text{NLP: } \underset{x \in \mathbb{R}^n}{\text{minimize}} F(x, y) \text{ s.t. } G_i(x, y) \leq 0, i \in I$$

with an unknown parameter $y \in Y(x)$. The robust counterpart is

$$\begin{aligned} \text{ROBUST-GSIP: } & \underset{(x, z) \in \mathbb{R}^n \times \mathbb{R}}{\text{maximize}} z \\ \text{s.t. } & F(x, y) \leq z, G_i(x, y) \leq 0 \text{ for all } y \in Y(x). \end{aligned}$$

Nonclosedness and Re-entrant Corners

We present two illustrative examples which show the intrinsic difficulties of GSIP in addition to SIP.

Example 3.1.1 (Nonclosedness of the feasible set)

Let $m = 1$, $s = 1$ and the GSIP feasible set M be given by

$$M = \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y(x)\} \text{ with}$$

$$Y(x) = \{y \in \mathbb{R} \mid g_1(x, y) \leq 0\}.$$

For $\bar{x} \in \mathbb{R}^n$ the graphs of $g_0(\bar{x}, \cdot)$ and $g_1(\bar{x}, \cdot)$ are depicted in Figure 9.

Figure 9: Graphs of $g_0(\bar{x}, \cdot)$ and $g_1(\bar{x}, \cdot)$ in Example 3.1.1

It is clear that $\bar{x} \in M$. Let $\bar{y}_1$ and $\bar{y}_2$ be local minimizers of $g_0(\bar{x}, \cdot)$ and $g_1(\bar{x}, \cdot)$, respectively (see Figure 9). In a neighborhood of $\bar{x}$ we parametrize by $y_1(x)$ and $y_2(x)$ the local minimizers of $g_0(x, \cdot)$ and $g_1(x, \cdot)$ such that $y_1(\bar{x}) = \bar{y}_1$ and $y_2(\bar{x}) = \bar{y}_2$. A moment of reflection shows that locally around $\bar{x}$ the feasible set M is given as follows:

$$M_{loc} = \{x \mid g_0(x, y_1(x)) \geq 0, g_1(x, y_2(x)) > 0\}.$$

We conclude that M is locally nonclosed.

Example 3.1.2 (Re-entrant Corners, cf. [40])

Let $n = 2$, $m = 1$, $s = 2$ and the GSIP feasible set M be given by

$$M = \{x \in \mathbb{R}^2 \mid g_0(x, y) := y \geq 0 \text{ for all } y \in Y(x)\} \text{ with}$$

$$Y(x) = \{y \in \mathbb{R} \mid g_1(x, y) := x_1 - y \leq 0, g_2(x, y) := y - x_2 \leq 0\}.$$

It is clear that $x \in M$ if and only if $y \geq 0$ for all $x_1 \leq y \leq x_2$. The feasible set M is depicted in Figure 10.

Figure 10: Feasible set M from Example 3.1.2

Note that M is nonclosed and exhibits a re-entrant corner at the origin.

3.2 Structure of the feasible set

3.2.1 Closure of the feasible set and Sym-MFCQ

The feasible set M in General Semi-Infinite Programming (GSIP) need not to be closed. We introduce a natural constraint qualification, called Symmetric Mangasarian Fromovitz Constraint Qualification (Sym-MFCQ). The Sym-MFCQ is a non-trivial extension of the well-known (Extended) MFCQ for the special case of Semi-Infinite Programming (SIP) and Disjunctive Optimization. Under the Sym-MFCQ the closure $\overline{M}$ has an easy and also natural description. As a consequence, we get a description of the interior and boundary of M . The Sym-MFCQ is shown to be generic and stable under C^1 -perturbations of the defining functions. For the latter stability the consideration of the closure of M is essential. We introduce an appropriate notion of Karush-Kuhn-Tucker (KKT) points. We show that local minimizers are KKT-points under the Sym-MFCQ.

We refer to [39] for details.

Sym-MFCQ and its Consequences

Recall that the set-valued mapping $Y : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called locally bounded if for each $\bar{x} \in \mathbb{R}^n$ there exists a neighborhood U of $\bar{x}$ such that $\bigcup_{x \in U} Y(x)$ is bounded in $\mathbb{R}^m$.

We state the following standard assumption in context of GSIP.

Assumption B *The mapping $Y : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded.*

It is well-known that the feasible set M need not to be closed. Moreover, the local nonclosedness of the feasible set M is stable (e.g. [40]). Therefore, one considers the topological closure $\overline{M}$ of M instead. In [36] an explicit description of $\overline{M}$ is provided. In fact, under Assumption B and additional generic assumptions (see [36] for details) the closure of the feasible set is given by

$$\overline{M} = \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y^<(x)\}$$

with

$$Y^<(x) = \{y \in \mathbb{R}^m \mid g_k(x, y) < 0, k = 1, \dots, s\}.$$

Using the function

$$\sigma(x, y) := \max_{0 \leq k \leq s} g_k(x, y)$$

the latter can be equivalently written as follows (cf. [37]):

$$\overline{M} = \{x \in \mathbb{R}^n \mid \sigma(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m\}. \quad (3.3)$$

Note that the description of $\overline{M}$ is symmetric in the functions $g_k, k = 0, \dots, s$. This means that the function g_0 does not play any special role.

The main goal is to present a stable and generic constraint qualification for GSIP (see Definition 3.2.1), which provides the foregoing description of $\overline{M}$ as in (3.3). We need some auxiliary notations for its formulation.

We denote the right-hand side of (3.3) as

$$M^{max} := \{x \in \mathbb{R}^n \mid \sigma(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m\}.$$

Note that M^{max} is a closed set due to the continuity of σ .

We set $M(\bar{x}) := \{y \in \mathbb{R}^m \mid \sigma(\bar{x}, y) = 0\}$ for $\bar{x} \in M^{max}$ and empty otherwise. Note that every $y \in M(\bar{x})$ is a global minimizer of $\sigma(\bar{x}, \cdot)$ with the vanishing optimal value. Further, we set for $y \in M(\bar{x})$

$$K_0(\bar{x}, y) := \{k \in \{0, \dots, s\} \mid g_k(\bar{x}, y) = 0\}.$$

Obviously, $K_0(\bar{x}, y) \neq \emptyset$ for all $y \in M(\bar{x})$. Finally, $V(\bar{x}, y)$ is defined as a compact convex subset of $\mathbb{R}^n$ by the following equality in $\mathbb{R}^n \times \mathbb{R}^m$:

$$V(\bar{x}, y) \times \{0\} = (\mathbb{R}^n \times \{0\}) \cap \text{conv} \{Dg_k(\bar{x}, y) \mid k \in K_0(\bar{x}, y)\}, \text{ i.e.}$$

$$V(\bar{x}, y) = \left\{ \sum_{k \in K_0(\bar{x}, y)} \gamma_k D_x g_k(\bar{x}, y) \mid \begin{array}{l} \sum_{k \in K_0(\bar{x}, y)} \gamma_k D_y g_k(\bar{x}, y) = 0, \\ \sum_{k \in K_0(\bar{x}, y)} \gamma_k = 1, \\ \gamma_k \geq 0 \end{array} \right\}.$$

It is clear that $V(\bar{x}, y) \neq \emptyset$ for all $y \in M(\bar{x})$. Moreover, we put:

$$V(\bar{x}) := \bigcup_{y \in M(\bar{x})} V(\bar{x}, y). \quad (3.4)$$

In order to indicate the dependence on the data functions $g := (g_0, \dots, g_s)$ we write M_g^{max} , $M_g(\bar{x})$, $K_0^g(\bar{x}, y)$ and $V_g(\bar{x}, y)$, if needed.

Definition 3.2.1 (Sym-MFCQ)

Let $\bar{x} \in M^{max}$. The Symmetric Mangasarian-Fromovitz Constraint Qualification (Sym-MFCQ) is said to hold at $\bar{x}$ if there exists a vector $\xi \in \mathbb{R}^n$ such that for all $v \in V(\bar{x})$ it holds:

$$v \cdot \xi > 0.$$

Remark 3.2.2 *It is worth to mention that Sym-MFCQ was indicated already in [36]. Sym-MFCQ is also connected with a constraint qualification for GSIP proposed in [70]. Indeed, it is not difficult to see that $M \subset M^{\max}$. Then, for $\bar{x} \in M$, Sym-MFCQ coincides with the extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) for GSIP as proposed in the final remarks in [70]. In addition to the latter, Sym-MFCQ provides also a condition for the points from $M^{\max} \setminus M$. We emphasize that these points have to be regarded because of the possible nonclosedness of the feasible set M .*

The following example shows that a "naive" generalization of the standard MFCQ fails here.

Example 3.2.3 *With $g_1(x, y) := -2x + y$ and $g_2(x, y) := x - y$ we consider the set*

$$M^{\max} := \{x \in \mathbb{R} \mid \max \{g_1(x, y), g_2(x, y)\} \geq 0 \text{ for all } y \in \mathbb{R}\}.$$

It is easy to see that

$$\psi(x) := \min_{y \in \mathbb{R}} \max \{g_1(x, y), g_2(x, y)\} = -\frac{1}{2}x \text{ and}$$

$$M^{\max} = \{x \in \mathbb{R} \mid \psi(x) \geq 0\} = (-\infty, 0].$$

For the boundary point of $M^{\max}$, $\bar{x} = 0$ we have $M(\bar{x}) = \{0\}$. With $D_x g_1(0, 0) = -2$ and $D_x g_2(0, 0) = 1$ the "naive" generalization of the standard MFCQ at $\bar{x}$ fails. Namely, there does not exist a real number ξ such that

$$D_x g_1(0, 0)\xi > 0 \text{ and } D_x g_2(0, 0)\xi > 0.$$

Nevertheless, it is easy to see that $V(0, 0) = \left\{-\frac{1}{2}\right\}$. Hence, Sym-MFCQ holds at 0 and $\{D_x \psi(0)\} = V(0, 0)$. Moreover, the zero of ψ remains stable under small C^1 -perturbations of g_1 and g_2 .

The following simple reformulation of Sym-MFCQ was indicated by one of the anonymous referees.

Proposition 3.2.4 *Let Assumption B be satisfied. Sym-MFCQ holds at $\bar{x} \in M^{\max}$ if and only if there exists a vector $\eta \in \mathbb{R}^{n+m}$ such that for all $y \in M(\bar{x})$ and $k \in K_0(\bar{x}, y)$ it holds:*

$$Dg_k(\bar{x}, y) \cdot \eta > 0.$$

The proof of Proposition 3.2.4 and the subsequent theorems are given below.

We state the main results concerning Sym-MFCQ and its impacts on the feasible set M .

Let $\mathcal{A}$ denote the set of problem data $(f, g_0, \dots, g_s) \in C^1(\mathbb{R}^n) \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$ such that Assumption B is satisfied. The set $\mathcal{A}$ is C_s^0 -open (cf. [66]).

Theorem 3.2.5 (Sym-MFCQ is stable and generic)

Let $\mathcal{F}$ denote the subset of $\mathcal{A}$ consisting of those problem data $(f, g_0, \dots, g_s)$ for which Sym-MFCQ holds at all points $\bar{x} \in M^{max}$. Then, $\mathcal{F}$ is C_s^1 -open and C_s^1 -dense in $\mathcal{A}$.

Theorem 3.2.6 (Closure Theorem)

Let Sym-MFCQ hold at all points $\bar{x} \in M^{max}$ and Assumption B be satisfied. Then, $\overline{M} = M^{max}$.

Theorem 3.2.7 (Topological properties of M)

Let Sym-MFCQ hold at all points $\bar{x} \in M^{max}$ and Assumption B be satisfied. Then:

- (i) $\text{int}(\overline{M}) = \text{int}(M)$,
- (ii) $\partial \overline{M} = \partial M$.

Proofs of Main Results

First, we provide a local description of M^{max} which is crucial for the following.

Lemma 3.2.8 (Local description of M^{max} , cf. [37])

Let Assumption B be satisfied. For $\bar{x} \in M^{max}$ there exist some neighborhood U of $\bar{x}$ and a nonempty compact set $V \subset \mathbb{R}^m$ such that

$$M^{max} \cap U = \{x \in U \mid \sigma(x, y) \geq 0 \text{ for all } y \in V\} = \{x \in U \mid \psi(x) \geq 0\}$$

with the well-defined continuous function $\psi(x) := \min_{y \in V} \sigma(x, y)$.

If additionally $\psi(\bar{x}) = 0$ then $M(\bar{x}) = \{y \in V \mid \sigma(\bar{x}, y) = 0\}$.

Proof. One chooses as V the closure of the bounded set $\bigcup_{x \in U} Y(x)$ for the neighborhood U of $\bar{x}$ from Assumption B. $\square$

Proof of Proposition 3.2.4: a) Let $\eta \in \mathbb{R}^{n+m}$ be a vector such that:

$$Dg_k(\bar{x}, y) \cdot \eta > 0 \text{ for all } y \in M(\bar{x}) \text{ and } k \in K_0(\bar{x}, y). \quad (3.5)$$

Putting $\eta = (\eta_1, \eta_2) \in \mathbb{R}^{n+m}$ we show that for all $v \in V(\bar{x})$ it holds:

$$v \cdot \eta_1 > 0.$$

Indeed, let $v = \sum_{k \in K_0(\bar{x}, y)} \gamma_k D_x g_k(\bar{x}, y)$ with

$$\sum_{k \in K_0(\bar{x}, y)} \gamma_k D_y g_k(\bar{x}, y) = 0, \quad \sum_{k \in K_0(\bar{x}, y)} \gamma_k = 1, \quad \gamma_k \geq 0.$$

Multiplying (3.5) by γ_k and summing up w.r.t $k \in K_0(\bar{x}, y)$ we get:

$$\sum_{k \in K_0(\bar{x}, y)} \gamma_k D_x g_k(\bar{x}, y) \cdot \eta_1 + \sum_{k \in K_0(\bar{x}, y)} \gamma_k D_y g_k(\bar{x}, y) \cdot \eta_2 = v \cdot \eta_1 > 0.$$

The latter means that Sym-MFCQ holds at $\bar{x}$.

b) Assume that there does not exist a vector $\eta \in \mathbb{R}^{n+m}$ such that:

$$Dg_k(\bar{x}, y) \cdot \eta > 0 \text{ for all } y \in M(\bar{x}) \text{ and } k \in K_0(\bar{x}, y).$$

Due to the compactness of $M(\bar{x})$ (cf. Lemma 3.2.8), a separation argument can be used as in [111] and we obtain:

$$0 \in \text{conv} \{ D^T g_k(\bar{x}, y) \mid y \in M(\bar{x}), k \in K_0(\bar{x}, y) \}.$$

Hence, there exist $y_i \in M(\bar{x})$ and $\gamma_k^i \geq 0$, $k \in K_0(\bar{x}, y_i)$, $i = 1, \dots, l$ such that

$$\sum_{i=1}^l \sum_{k \in K_0(\bar{x}, y_i)} \gamma_k^i Dg_k(\bar{x}, y_i) = 0. \quad (3.6)$$

We put $\gamma^i := \sum_{k \in K_0(\bar{x}, y_i)} \gamma_k^i$, $i = 1, \dots, l$. Without loss of generality, we may assume $\gamma^i > 0$ for all i . Further, (3.6) can be written as:

$$\sum_{i=1}^l \gamma^i \cdot \begin{pmatrix} \sum_{k \in K_0(\bar{x}, y_i)} \frac{\gamma_k^i}{\gamma^i} D_x g_k(\bar{x}, y_i) \\ \sum_{k \in K_0(\bar{x}, y_i)} \frac{\gamma_k^i}{\gamma^i} D_y g_k(\bar{x}, y_i) \end{pmatrix} = 0.$$

It means, in particular, that

$$\sum_{k \in K_0(\bar{x}, y_i)} \frac{\gamma_k^i}{\gamma^i} D_y g_k(\bar{x}, y_i) = 0, \quad \sum_{k \in K_0(\bar{x}, y_i)} \frac{\gamma_k^i}{\gamma^i} = 1, \quad \frac{\gamma_k^i}{\gamma^i} \geq 0$$

and, hence, $v^i := \sum_{k \in K_0(\bar{x}, y_i)} \frac{\gamma_k^i}{\gamma^i} D_x g_k(\bar{x}, y_i) \in V(\bar{x}, y_i)$. Moreover, we conclude that

$$\sum_{i=1}^l \gamma^i \cdot v^i = 0 \text{ with } \gamma^i > 0, v_i \in V(\bar{x}).$$

The latter shows that Sym-MFCQ does not hold at $\bar{x}$. $\square$

For the proof of Theorem 3.2.5 we need some upper semicontinuity properties of the set-valued mappings $(x, g) \rightrightarrows M_g(x)$ and $(x, y, g) \rightrightarrows V_g(x, y)$. Recall that a set-valued mapping $\mathcal{M}$ from a topological space T into a family of all subsets of $\mathbb{R}^n$ is said to be upper semicontinuous at $\bar{v} \in T$ if, for any open set $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{M}(\bar{v}) \subset \mathcal{U}$, there exists a neighborhood $\mathcal{V}$ of $\bar{v}$ such that $\mathcal{M}(v) \subset \mathcal{U}$ whenever $v \in \mathcal{V}$.

Lemma 3.2.9 (Upper semicontinuity of $M_g(x)$ and $V_g(x, y)$)

Let Assumption B be satisfied. For $\bar{x} \in M_{\bar{g}}^{max}$ and $\bar{y} \in M_{\bar{g}}(\bar{x})$ it holds:

- a) *the set-valued mapping $(x, g) \rightrightarrows M_g(x)$ is upper semicontinuous at $(\bar{x}, \bar{g})$ w.r.t. the topology in $\mathbb{R}^n \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$,*
- b) *the set-valued mapping $(x, y, g) \rightrightarrows V_g(x, y)$ is upper semicontinuous at $(\bar{x}, \bar{y}, \bar{g})$ w.r.t. the topology in $\mathbb{R}^n \times \mathbb{R}^m \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$.*

Proof. a) We assume that $(x, g) \rightrightarrows M_g(x)$ is not upper semicontinuous at $(\bar{x}, \bar{g})$. Then, there exist an open set $\mathcal{U} \subset \mathbb{R}^n$ with $M_{\bar{g}}(\bar{x}) \subset \mathcal{U}$ and $(x^i, g^i) \in \mathbb{R}^n \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$, $i \in \mathbb{N}$ such that

$$(x^i, g^i) \xrightarrow{i} (\bar{x}, \bar{g}) \text{ and } y^i \in M_{g^i}(x^i), y^i \notin \mathcal{U}.$$

(Strictly speaking, we should use net convergence instead of sequential convergence in the C_s^1 -topology. However, the argumentation will essentially be the same.)

Now we use the representation of $M_{\bar{g}}^{max}$ and $M_{\bar{g}}(\bar{x})$ from Lemma 3.2.8 using the neighborhoods U and V as defined there. For sufficient large $i \in \mathbb{N}$, we have $x^i \in U$ and, moreover, $y^i \in V$. Indeed, otherwise we get $y^i \notin V$ for some subsequence, denoted again by y^i . It means, that there exists $k^i \in \{1, \dots, s\}$ such that $g_{k^i}(x^i, y^i) > 0$ (cf. definition of V from Lemma 3.2.8). After taking an appropriate subsequence, if needed, we obtain for some $k \in \{1, \dots, s\}$

$$g_k(x^i, y^i) > 0. \tag{3.7}$$

Due to $y^i \in M_{g^i}(x^i)$ we also obtain

$$g_k^i(x^i, y^i) \leq 0 \quad (3.8)$$

Together (3.7) and (3.8) contradict the fact that $g^i \xrightarrow{i} \bar{g}$ in the C_s^1 -topology. Further, since V is compact, we assume, w.l.o.g, that $y^i \xrightarrow{i} \bar{y} \in V$. Thus, from $y^i \in M_{g^i}(x^i)$, i.e. $\sigma_{g^i}(x^i, y^i) = 0$, $i \in \mathbb{N}$, it follows:

$$\sigma_{\bar{g}}(\bar{x}, \bar{y}) = 0 \text{ and } \bar{y} \in M_{\bar{g}}(\bar{x}).$$

From $M_{\bar{g}}(\bar{x}) \subset \mathcal{U}$ we obtain that $\bar{y} \in \mathcal{U}$. This contradicts the fact that $y^i \xrightarrow{i} \bar{y}$ and $y^i \notin \mathcal{U}$.

b) We assume that $(x, y, g) \rightrightarrows V_g(x, y)$ is not upper semicontinuous at $(\bar{x}, \bar{y}, \bar{g})$. Then, there exist an open set $\mathcal{U} \subset \mathbb{R}^n$ with $V_{\bar{g}}(\bar{x}, \bar{y}) \subset \mathcal{U}$ and $(x^i, y^i, g^i) \in \mathbb{R}^n \times \mathbb{R}^m \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$, $i \in \mathbb{N}$ such that

$$(x^i, y^i, g^i) \xrightarrow{i} (\bar{x}, \bar{y}, \bar{g}) \text{ and } v^i \in V_{g^i}(x^i, y^i), v^i \notin \mathcal{U}.$$

The latter means

$$v^i = \sum_{k \in K_0^{g^i}(x^i, y^i)} \gamma_k^i D_x g_k^i(x^i, y^i) \text{ with} \quad (3.9)$$

$$\sum_{k \in K_0^{g^i}(x^i, y^i)} \gamma_k^i D_y g_k^i(x^i, y^i) = 0, \quad \sum_{k \in K_0^{g^i}(x^i, y^i)} \gamma_k^i = 1, \gamma_k^i \geq 0. \quad (3.10)$$

Since $(x^i, y^i, g^i) \xrightarrow{i} (\bar{x}, \bar{y}, \bar{g})$, we obtain: $K_0^{g^i}(x^i, y^i) \subset K_0^{\bar{g}}(\bar{x}, \bar{y})$ for sufficient large $i \in \mathbb{N}$. We enlarge the sum in (3.9, 3.10) up to $K_0^{\bar{g}}(\bar{x}, \bar{y})$ with respective $\gamma_k^i = 0$. Since the sum of γ_k^i , $k \in K_0^{\bar{g}}(\bar{x}, \bar{y})$ is one, we may assume, w.l.o.g., that $(\gamma_k^i, k \in K_0^{\bar{g}}(\bar{x}, \bar{y})) \xrightarrow{i} (\bar{\gamma}_k, k \in K_0^{\bar{g}}(\bar{x}, \bar{y}))$. Taking the limes $i \rightarrow \infty$ in (3.9, 3.10), we conclude

$$\bar{v} \in V_{\bar{g}}(\bar{x}, \bar{y}) \text{ with } v^i \xrightarrow{i} \bar{v}.$$

From $V_{\bar{g}}(\bar{x}, \bar{y}) \subset \mathcal{U}$ we obtain that $\bar{v} \in \mathcal{U}$. This contradicts the fact that $v^i \xrightarrow{i} \bar{v}$ and $v^i \notin \mathcal{U}$. $\square$

The description of $M(\bar{x})$ in Lemma 3.2.8 easily provides the following result.

Lemma 3.2.10 *Let Assumption B be satisfied. For $\bar{x} \in M^{max}$ with $\psi(\bar{x}) = 0$ the set $V(\bar{x})$ is compact.*

We state the Symmetric Linear Independence Constraint Qualification (Sym-LICQ) for GSIP which is shown to be stronger than Sym-MFCQ.

Definition 3.2.11 (Sym-LICQ, cf. [38])

Let $\bar{x} \in M^{max}$. The Symmetric Linear Independence Constraint Qualification (Sym-LICQ) is said to hold at $\bar{x}$ if for any finite subset $\{y_1, \dots, y_p\} \subset M(\bar{x})$ and any choice of $v_i \in V(\bar{x}, y_i)$, $i = 1, \dots, p$ the vectors $\{v_1, \dots, v_p\}$ are linearly independent.

Lemma 3.2.12 (Sym-LICQ implies Sym-MFCQ)

Let Assumption B be satisfied. If Sym-LICQ holds at $\bar{x} \in M^{max}$ then Sym-MFCQ holds as well.

Proof. Let Sym-LICQ hold at $\bar{x} \in M^{max}$. W.l.o.g., we may assume that $\psi(\bar{x}) = 0$ and, hence, $M(\bar{x}) \neq \emptyset$ (otherwise, Sym-MFCQ holds trivially). In particular, Sym-LICQ implies that $M(\bar{x})$ is finite and we have

$$M(\bar{x}) = \{y_1, \dots, y_l\}, l \in \mathbb{N}.$$

Now, we assume that Sym-MFCQ does not hold at $\bar{x}$. Since $V(\bar{x})$ is compact (see Lemma 3.2.10) a separation argument can be used as in [111] and we obtain:

$$0 \in \text{conv}(V(\bar{x})).$$

Thus, with some finite index sets J_i , $i = 1, \dots, l$:

$$\sum_{i=1}^l \sum_{v_j \in V(\bar{x}, y_i), j \in J_i} \lambda_{i,j} v_j = 0, \sum_{i,j} \lambda_{i,j} = 1, \lambda_{i,j} \geq 0.$$

For $i \in \{1, \dots, l\}$ we assume, w.l.o.g., that $\lambda_{i,j} \neq 0$ for at least one $j \in J_i$. We set for $i \in \{1, \dots, l\}$

$$b_i := \sum_{j \in J_i} \lambda_{i,j} > 0. \tag{3.11}$$

Further, we write for $v_j \in V(\bar{x}, y_i)$, $j \in J_i$:

$$v_j = \sum_{k \in K_0(\bar{x}, y_i)} \mu_{i,j}^k D_x g_k(\bar{x}, y_i) \text{ with}$$

$$\sum_{k \in K_0(\bar{x}, y_i)} \mu_{i,j}^k D_y g_k(\bar{x}, y_i) = 0, \sum_{k \in K_0(\bar{x}, y_i)} \mu_{i,j}^k = 1, \mu_{i,j}^k \geq 0.$$

Setting $a_i^k := \sum_{v_j \in V(\bar{x}, y_i), j \in J_i} \frac{1}{b_i} \lambda_{i,j} \mu_{i,j}^k \geq 0$ and $\tilde{v}_i := \sum_{k \in K_0(\bar{x}, y_i)} a_i^k D_x g_k(\bar{x}, y_i)$ we

obtain: $\tilde{v}_i \in V(\bar{x}, y_i)$ and $\sum_{i=1}^l b_i \tilde{v}_i = 0$. Thus, Sym-LICQ at $\bar{x}$ implies that $b_i = 0$ for all $i \in \{1, \dots, l\}$. This provides a contradiction to (3.11). $\square$

Proof of Theorem 3.2.5: a) We prove that $\mathcal{F}$ is C_s^1 -open in $\mathcal{A}$.

1) **Local argument.** First we prove the following assertion:

Let Sym-MFCQ hold at $\bar{x} \in M_{\bar{g}}^{max}$ (with the vector $\xi \in \mathbb{R}^n$ as in Definition 3.2.1). Then, there exist an open neighborhood $U_{\bar{x}}$ of $\bar{x}$ and a C^1 -neighborhood $W_{\bar{g}}$ of $\bar{g}$, such that

$$v^T \xi > 0 \text{ for all } v \in V_g(x, y), g \in W_{\bar{g}}, x \in U_{\bar{x}} \cap M_g^{max}, y \in M_g(x). \quad (3.12)$$

The assertion (3.12) is of local nature. Therefore, we may use the representations of $M_{\bar{g}}^{max}$ and $M_{\bar{g}}(\bar{x})$ from Lemma 3.2.8 with the neighborhoods U and V as defined there.

Due to the compactness of $V(\bar{x})$ (cf. Lemma 3.2.10), Sym-MFCQ at $\bar{x}$ provides the existence of an open set $\tilde{V}$ such that

- $V(\bar{x}, y) \subset \tilde{V}$ for all $y \in M_{\bar{g}}(\bar{x})$ and
- $v^T \xi > 0$ for all $v \in \tilde{V}$.

We apply to $\tilde{V}$ the upper semicontinuity property of $(x, y, g) \Rightarrow V_g(x, y)$ at $(\bar{x}, \bar{y}, \bar{g})$, $\bar{y} \in M_{\bar{g}}(\bar{x})$ (cf. Lemma 3.2.9). We obtain the existence of an open neighborhood $U_{\bar{x}}(\bar{y}) \times V_{\bar{y}} \times W_{\bar{g}}(\bar{y})$ of $(\bar{x}, \bar{y}, \bar{g})$ such that

$$V_g(x, y) \subset \tilde{V} \text{ for all } (x, y, g) \in U_{\bar{x}}(\bar{y}) \times V_{\bar{y}} \times W_{\bar{g}}(\bar{y}).$$

The family of sets $\{V_{\bar{y}}, \bar{y} \in M_{\bar{g}}(\bar{x})\}$ constitutes an open covering of a compact set $M_{\bar{g}}(\bar{x})$ (cf. Lemma 3.2.8). Taking its finite subcovering $\{V_{\bar{y}_i}, i = 1, \dots, p\}$, we obtain

$$M_{\bar{g}}(\bar{x}) \subset \widetilde{M} \text{ with } \widetilde{M} := \bigcup_{i=1}^p V_{\bar{y}_i}.$$

We apply to $\widetilde{M}$ the upper semicontinuity property of $(x, g) \Rightarrow M_g(x)$ at $(\bar{x}, \bar{g})$ (cf. Lemma 3.2.9). We obtain the existence of an open neighborhood $\tilde{U}_{\bar{x}} \times W_{\bar{g}}$ of $(\bar{x}, \bar{g})$ such that

$$M_g(x) \subset \widetilde{M} \text{ for all } (x, g) \in \tilde{U}_{\bar{x}} \times W_{\bar{g}}.$$

Finally, we define the open neighborhoods of $\bar{x}$ and $\bar{g}$, respectively:

$$U_{\bar{x}} := \bigcap_{i=1}^p U_{\bar{x}}(\bar{y}_i) \cap \tilde{U}_{\bar{x}}, \quad W_{\bar{g}} := \bigcap_{i=1}^p W_{\bar{g}}(\bar{y}_i) \cap \tilde{W}_{\bar{g}}.$$

These neighborhoods fit the Local argument (3.12).

2) **Global argument:** The Global argument is standard. Due to the Local argument we define a global vector field $\xi(\cdot)$ via a C^∞ -partition of unity. Since the set of Sym-MFCQ vectors is convex, this vector field $\xi(\cdot)$ fits Sym-MFCQ under C^1 -perturbations of defining functions in C_s^1 -topology.

b) We prove that $\mathcal{F}$ is C_s^1 -generic in $C^1(\mathbb{R}^n) \times [C^1(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$. This implies that $\mathcal{F}$ is C_s^1 -dense and, hence, $\mathcal{F} \cap \mathcal{A}$ C_s^1 -dense in $\mathcal{A}$.

Let $\mathcal{G}$ denote the subset of $\mathcal{A}$ consisting of those problem data $(f, g_0, \dots, g_s)$ for which Sym-LICQ holds at all points $\bar{x} \in M^{max}$. Due to Lemma 3.2.12, it suffices to prove that $\mathcal{G}$ is C_s^1 -generic. The latter result is given in [36]. We shortly recapitulate its proof for the sake of completeness.

The proof is based on an application of the Jet Transversality Theorem, for details see e.g. [61]. For $p \in \mathbb{N}$, $K_i \subset \{0, \dots, s\}$ and $r_i \in \mathbb{N}$, $r_i \leq m$, $i = 1, \dots, p$, we consider the set Γ of $(x, y_1, \dots, y_p, v_1, \dots, v_p)$ such that the following conditions are satisfied:

- (i) $x \in \mathbb{R}^n$, $y_i \in \mathbb{R}^m$ (pairwise different), $v_i \in \mathbb{R}^n$,
- (ii) $K_i = K_0(x, y_i)$,
- (iii) $\text{span} \{D_y g_k(x, y_i), k \in K_i\}$ has dimension r_i ,
- (iv) $(v_i, 0) \in \text{span} \{Dg_k(x, y_i), k \in K_i\}$,
- (v) $\|v_i\| = 1$,
- (vi) the vectors $\{v_1, \dots, v_p\}$ are linearly dependent.

Now, it suffices to prove that Γ is generically empty. In fact, the available degrees of freedom of the variables involved in Γ are $n + pm + pn$ (see (i)). The loss of freedom, caused by independent equations given in (ii)-(vi), can be counted as follows. For each $i \in \{1, \dots, p\}$ condition (ii) generates a loss of $|K_i|$ degrees of freedom, and condition (iii) reduces the freedom by $(|K_i| - r_i)(m - r_i) \geq (m - r_i)$ degrees. Since the subspace of $\mathbb{R}^n$ formed by those vectors v_i , satisfying (iv), has at most dimension $|K_i| - r_i$, thus (iv) causes the loss of at least $n - |K_i| + r_i$ degrees of freedom. Condition (v) reduces the freedom by 1 degree per index i . Condition (vi) defines the loss

of freedom by at least $n - p + 1$. Altogether, we claim that the loss of freedom is at least

$$p(|K_i| + (m - r_i) + (n - |K_i| + r_i) + 1) + n - p + 1 = pm + pn + n + 1.$$

degrees. This exceeds the total available freedom $n + pm + pn$ by 1. In virtue of the Jet Transversality Theorem, generically the set Γ must be empty. $\square$

Lemma 3.2.13 *Let Sym-MFCQ hold at $\bar{x} \in M^{max}$ with the Sym-MFCQ vector ξ as in Definition 3.2.1. Moreover, let Assumption B be satisfied.*

Define the function $\varphi : \begin{cases} M(\bar{x}) & \longrightarrow \mathbb{R}, \\ y & \longrightarrow \min_{v \in V(\bar{x}, y)} v^T \xi \end{cases}$.
 φ is well-defined and $\inf_{y \in M(\bar{x})} \varphi(y) > 0$.

Proof. Due to the compactness of $V(\bar{x}, y)$, φ is well-defined. Moreover, $\varphi(y) > 0, y \in M(\bar{x})$ due to Sym-MFCQ. We assume that $\inf_{y \in M(\bar{x})} \varphi(y) = 0$.

Then, w.l.o.g. we may assume that there exist $y_i \xrightarrow{i} \bar{y} \in M(\bar{x})$ such that $\varphi(y_i) \xrightarrow{i} 0$ (recall that $M(\bar{x})$ is compact as in Lemma 3.2.8). We choose a vector $v_i \in V(\bar{x}, y_i)$ with $\varphi(y_i) = v_i^T \xi$. Using particular representations of $v_i \in V(\bar{x}, y_i)$ and taking a subsequence, if needed, we may assume that $v_i \xrightarrow{i} \bar{v} \in V(\bar{x}, \bar{y})$. Finally we obtain that $\bar{v}^T \xi = 0$ for $\bar{v} \in V(\bar{x}, \bar{y}), \bar{y} \in M(\bar{x})$. This fact contradicts Sym-MFCQ at $\bar{x}$. $\square$

Proof of Theorem 3.2.6:

Using the notations from [36], we define the following sets:

$$N^{\leq} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g_i(x, y) \leq 0, i = 0, \dots, s\},$$

$$N := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g_i(x, y) < 0, i = 0, \dots, s\},$$

$$N^M := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g_0(x, y) < 0, g_i(x, y) \leq 0, i = 1, \dots, s\}.$$

Let $\Pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ denote the projection on $\mathbb{R}^n$.

Obviously, it holds:

$$CM = \Pi N^M, C\Pi N = M^{max}.$$

Since $N \subset N^M \subset N^{\leq}$, we obtain

$$C\Pi N^{\leq} \subset M \subset C\Pi N.$$

Thus, due to the closedness of $C\Pi N$ and $C\Pi N = M^{max}$, it suffices to show:

$$\overline{C\Pi N^{\leq}} = M^{max}.$$

Equivalently, we prove:

$$\text{for each } \bar{x} \in \Pi N^\leq \cap M^{\max} \text{ there exists } \bar{z} \in C\Pi N^\leq \text{ arbitrary close to } \bar{x}. \quad (3.13)$$

Let $\bar{x} \in \Pi N^\leq \cap M^{\max}$. We use the local representation of $M^{\max}$ from Lemma 3.2.8 with the neighborhoods U and V as defined there. Moreover, the following representation of $C\Pi N^\leq$ is valid with the same neighborhoods U and V :

$$C\Pi N^\leq \cap U = \{x \in U \mid \sigma(x, y) > 0 \text{ for all } y \in V\}.$$

Thus, we get from $\bar{x} \in \Pi N^\leq \cap M^{\max}$ that $\min_{y \in V} \sigma(\bar{x}, y) \leq 0$. Recall that Sym-MFCQ holds at $\bar{x}$. With the Sym-MFCQ vector ξ as in Definition 3.2.1 we set $z_t := \bar{x} + t\xi$. For (3.13) it suffices to show that

$$\text{there exists } \varepsilon > 0 \text{ such that } z_t \in C\Pi N^\leq \text{ for all } t \in (0, \varepsilon). \quad (3.14)$$

Using the above representation of $C\Pi N^\leq$ we consider two cases for $y \in V$.

Case 1: Let $y \notin M(\bar{x})$. Thus, we obtain the existence of $k \in \{0, \dots, s\}$ such that $g_k(\bar{x}, y) > 0$.

Case 2: Let $y \in M(\bar{x})$. We write the Taylor expansion for $g_k(\cdot, y)$, $k \in K_0(\bar{x}, y)$ at $\bar{x}$:

$$g_k(z_t, y) = t \left[D_x g_k(\bar{x}, y) \xi + \frac{o_k(t, y)}{t} \right]. \quad (3.15)$$

We choose a vector $v \in V(\bar{x}, y)$ written as

$$v = \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) D_x g_k(\bar{x}, y) \text{ with}$$

$$\sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) D_y g_k(\bar{x}, y) = 0, \quad \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) = 1, \quad \gamma^k(y) \geq 0.$$

Multiplying (3.15) by $\gamma_k(y)$ and summing up, we obtain:

$$\begin{aligned} \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) g_k(z_t, y) &= t \left[v^T \xi + \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) \frac{o_k(t, y)}{t} \right] \geq \\ &t \left[\min_{v \in V(\bar{x}, y)} v^T \xi + \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) \frac{o_k(t, y)}{t} \right] \geq \end{aligned}$$

$$t \left[\inf_{y \in M(\bar{x})} \min_{v^T \in V(\bar{x}, y)} v^T \xi + \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) \frac{o_k(t, y)}{t} \right].$$

Due to Lemma 3.2.13, $\inf_{y \in M(\bar{x})} \min_{v^T \in V(\bar{x}, y)} v^T \xi > 0$. Moreover,

$$\sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) \frac{o_k(t, y)}{t} \longrightarrow 0, \text{ (as } t \longrightarrow 0 \text{) uniformly on } M(\bar{x}).$$

The latter comes from the fact that

$$M(\bar{x}) \text{ is compact, } \sum_{k \in K_0(\bar{x}, y)} \gamma^k(y) = 1 \text{ and}$$

$$\frac{o_k(t, y)}{t} = \int_0^1 [D_x g_k(\bar{x} + st\xi, y) - D_x g_k(\bar{x}, y)] \xi ds \text{ is continuous w.r.t. } (t, y).$$

Altogether, we obtain the existence of an $\varepsilon > 0$ (which is independent from $y \in M(\bar{x})$) such that for all $t \in (0, \varepsilon)$ there exists an index $k \in 0, \dots, s$ such that $g_k(z_t, y) > 0$.

Cases 1 and 2 provide (3.14) and, hence, the assertion. $\square$

To prove Theorem 3.2.7 we need the following description of ∂M^{max} .

Lemma 3.2.14 *Let Sym-MFCQ hold at $\bar{x} \in M^{max}$ and Assumption B be satisfied. Then,*

$$\bar{x} \in \partial M^{max} \text{ if and only if } \psi(\bar{x}) = 0,$$

where ψ is defined as in Lemma 3.2.8.

Proof. We use the local representation of M^{max} from Lemma 3.2.8 with the neighborhoods U and V as defined there:

$$M^{max} \cap U = \{x \in U \mid \psi(x) \geq 0\},$$

where $\psi(x) := \min_{y \in V} \sigma(x, y)$.

Due to the continuity of ψ on U , if $\psi(\bar{x}) > 0$ then $\bar{x} \in \text{int}(M^{max})$. Hence, we restrict our considerations to the case $\psi(\bar{x}) = 0$ and prove that $\bar{x} \in \partial M^{max}$. We use the Sym-MFCQ vector ξ at $\bar{x}$ as in Definition 3.2.1. Putting $x(t) := \bar{x} - t\xi$, $t > 0$ we show that for sufficiently small $t > 0$ we have $\psi(x(t)) < 0$, thus, $x(t) \notin M^{max}$. It implies that $\bar{x} \notin \text{int}(M^{max})$ and, since M^{max} is a closed set, we get: $\bar{x} \in M^{max} \setminus \text{int}(M^{max}) = \partial M^{max}$.

Let $\bar{y} \in M(\bar{x}) \neq \emptyset$, since $\psi(\bar{x}) = 0$.

Case a): $k \notin K_0(\bar{x}, \bar{y})$

For those k we have $g_k(\bar{x}, \bar{y}) < 0$. Because of the continuity of $g_k(\cdot, \cdot)$, it means that

$$g_k(x, y) < 0 \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y}). \quad (3.16)$$

Case b): $k \in K_0(\bar{x}, \bar{y})$

We set $\tilde{g}_k(t, y) := g_k(x(t), y)$ and $D\tilde{g}_k(t, y) = (-D_x g_k(\bar{x}, \bar{y})\xi, D_y g_k(\bar{x}, \bar{y}))$. We claim that there exists a vector $w \in \mathbb{R}^{m+1}$ such that

$$D\tilde{g}_k(0, \bar{y})w > 0 \text{ for all } k \in K_0(\bar{x}, \bar{y}). \quad (3.17)$$

Otherwise, due to the Gordan's Theorem, we obtain the existence of γ_k , $k \in K_0(\bar{x}, \bar{y})$ such that:

$$\sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k D_x g_k(\bar{x}, \bar{y})\xi = 0, \quad \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k D_y g_k(\bar{x}, \bar{y}) = 0, \quad \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k = 1, \quad \gamma_k \geq 0.$$

This fact contradicts Sym-MFCQ at $\bar{x}$. Moreover, setting $w = (w_1, w_2) \in \mathbb{R} \times \mathbb{R}^m$ we get $w_1 < 0$ in (3.17). To see the latter, we recall that $\bar{y} \in M(\bar{x})$. Hence, we multiply the inequalities in (3.17) by appropriate γ_k , $k \in K_0(\bar{x}, \bar{y})$ (see Definition of $V(\bar{x}, \bar{y})$) and sum up w.r.t. k afterwards. Sym-MFCQ at $\bar{x}$ insures that $w_1 < 0$. W.l.o.g., we assume, $w_1 = -1$.

Further, due to the continuity of $D\tilde{g}_k(\cdot, \cdot)$, we obtain from (3.17)

$$D\tilde{g}_k(t, y)w > 0 \text{ for all } t \in [0, \varepsilon), y \in \tilde{V}, k \in K_0(\bar{x}, \bar{y}), \quad (3.18)$$

where $\varepsilon > 0$ and $\tilde{V}$ is a convex neighborhood of $\bar{y}$.

We set $y(t) := \bar{y} - tw_2$. Thus, $y(t) \in \tilde{V}$, for sufficiently small t . Moreover,

$$(t, y(t)) = (0, \bar{y}) - tw, \text{ (recall } w_1 = -1).$$

We apply the mean-value theorem and get for $k \in K_0(\bar{x}, \bar{y})$:

$$\tilde{g}_k(t, y(t)) = \tilde{g}_k(0, \bar{y}) - tD\tilde{g}_k(\tilde{t}, \tilde{y}) \cdot w \text{ with some } \tilde{t} \in [0, \varepsilon], \tilde{y} \in \tilde{V}.$$

Due to $D\tilde{g}_k(\tilde{t}, \tilde{y})w > 0$ (cf. (3.18)) and $\tilde{g}_k(0, \bar{y}) = g_k(\bar{x}, \bar{y}) = 0$ for $k \in K_0(\bar{x}, \bar{y})$, it holds:

$$g_k(x(t), y(t)) = \tilde{g}_k(t, y(t)) < 0 \text{ for all } k \in K_0(\bar{x}, \bar{y}). \quad (3.19)$$

Altogether, (3.16) and (3.19) provide that

$$\sigma(x(t), y(t)) < 0 \text{ for arbitrary small } t > 0,$$

since $(x(t), y(t)) \xrightarrow{t \rightarrow 0} (\bar{x}, \bar{y})$. It shows that $x(t) \notin M^{max}$ for arbitrary small $t > 0$. $\square$

Proof of Theorem 3.2.7:

Due to Theorem 3.2.6, $\bar{M} = M^{max}$.

(i) Let $\bar{x} \in \text{int}(\bar{M})$. We use the local representation of $M^{max} = \bar{M}$ from Lemma 3.2.8 with the neighborhoods U and V as defined there:

$$\bar{M} \cap U = \{x \in U \mid \psi(x) \geq 0\}$$

Moreover, from the proof of Theorem 3.2.6 we obtain:

$$C\Pi N^{\leq} \cap U = \{x \in U \mid \psi(x) > 0\}.$$

Lemma 3.2.14 implies that $\psi(\bar{x}) \neq 0$. Thus, $\bar{x} \in C\Pi N^{\leq}$ and, due to the continuity of ψ , there exists a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that $U_{\bar{x}} \subset C\Pi N^{\leq}$. Obviously, $C\Pi N^{\leq} \subset M$ and, hence, $\bar{x} \in \text{int}(M)$. Note that $M \subset \bar{M}$ implies $\text{int}(M) \subset \text{int}(\bar{M})$. This shows assertion (i).

Finally, assertion (ii) is just a consequence of assertion (i) and the fact that $\partial M = \bar{M} \setminus \text{int}(M)$. $\square$

Sym-MFCQ and KKT points

Further, we enlighten the consequences of Sym-MFCQ for GSIP regarding local minima and Karush-Kuhn-Tucker (KKT) points.

Note that $\bar{x} \in M$ is a local minimizer of the continuous function f on M if and only if it is a local minimizer of f on $\bar{M}$. Hence, we consider the optimization problem

$$\overline{\text{GSIP}} : \quad \text{minimize } f(x) \text{ s.t. } x \in \bar{M}$$

and introduce the notion of KKT points for $\overline{\text{GSIP}}$.

The following definition is motivated by the description (3.3) of $\bar{M}$ (which is valid under Sym-MFCQ in virtue of Theorem 3.2.6).

Definition 3.2.15 (KKT point, cf. [37])

$\bar{x} \in \bar{M}$ is called a KKT point if there exist $y_1, \dots, y_l \in M(\bar{x})$, $v_i \in V(x, y_i)$ and $\mu_i \geq 0, i = 1, \dots, l$ such that

$$Df(\bar{x}) = \sum_{i=1}^l \mu_i v_i.$$

Theorem 3.2.16 (Local minimum is a KKT point)

Let Sym-MFCQ hold at a local minimum $\bar{x} \in \bar{M}$ for $\overline{\text{GSIP}}$ and Assumption B be satisfied. Then, $\bar{x}$ is a KKT point.

Proof. Here, we use some ideas from [70]. Let $\bar{x} \in \overline{M}$ be a local minimum for $\overline{\text{GSIP}}$. Moreover, let Sym-MFCQ hold at $\bar{x}$ with the Sym-MFCQ vector ξ as in Definition 3.2.1. Recall that $\overline{M} \subset M^{max}$ and, hence, $\bar{x} \in M^{max}$. Due to the Local argument (see proof of the Theorem 3.2.5), there exists a neighborhood $\tilde{U}$ of $\bar{x}$ such that Sym-MFCQ holds at all $M^{max} \cap \tilde{U}$ with the same vector ξ . Thus, from the proof of Theorem 3.2.6 it follows:

$$\overline{M} \cap \tilde{U} = M^{max} \cap \tilde{U}.$$

Now, we use the local representation of M^{max} from Lemma 3.2.8 with the neighborhoods U and V as defined there. Shrinking the neighborhood U , if needed, we get:

$$\overline{M} \cap U = \{x \in U \mid \sigma(x, y) \geq 0 \text{ for all } y \in V\} = \{x \in U \mid \psi(x) \geq 0\}$$

with the well-defined continuous function $\psi(x) := \min_{y \in V} \sigma(x, y)$.

Case 1: $\psi(\bar{x}) > 0$

Here, the local minimum $\bar{x}$ lies in the interior of $\overline{M} \cap U$ and, hence, $Df(\bar{x}) = 0$. We see that $\bar{x}$ is a KKT point as in Definition 3.2.15.

Case 2: $\psi(\bar{x}) = 0$

Then, $M(\bar{x}) \neq \emptyset$ and Lemma 3.2.8 provides the representation

$$M(\bar{x}) = \{y \in V \mid \sigma(\bar{x}, y) = 0\}.$$

We assume that

$$0 \notin \text{conv}(\{-D^T f(\bar{x})\} \cup V(\bar{x})). \quad (3.20)$$

Due to the compactness of $V(\bar{x})$ (cf. Lemma 3.2.10), a separation argument can be used as in [111]. From (3.20) we obtain the existence of a vector $\xi \in \mathbb{R}^n$ such that

$$D^T f(\bar{x})\xi < 0 \text{ and}$$

$$v^T \xi > 0 \text{ for all } v \in V(\bar{x}). \quad (3.21)$$

Since $\bar{x}$ is a local minimum of f on $\overline{M} \cap U$ and $D^T f(\bar{x})\xi < 0$, there exists $\varepsilon > 0$ such that for all $t \in (0, \varepsilon]$:

$$f(\bar{x} + t\xi) < f(\bar{x}) \text{ and } \bar{x} + t\xi \notin \overline{M}.$$

We choose a sequence $(t_i)_{i \in \mathbb{N}} \subset (0, \varepsilon)$ with $t_i \xrightarrow{i} 0$ and set $x_i := \bar{x} + t_i \xi$, $i \in \mathbb{N}$ with $x_i \notin \overline{M}$. Hence, there exist $y_i \in V$, $i \in \mathbb{N}$ such that

$$\sigma(x_i, y_i) < 0 \text{ for all } i \in \mathbb{N}. \quad (3.22)$$

W.l.o.g., we assume that $y_i \xrightarrow{i} \bar{y} \in V$ and, thus,

$$\sigma(\bar{x}, \bar{y}) \leq 0. \quad (3.23)$$

It means, together with $\bar{x} \in M^{max}$, that $\sigma(\bar{x}, \bar{y}) = 0$ and $\bar{y} \in M(\bar{x})$.

To produce a contradiction with $\bar{x} \in M^{max}$, we show that there exists $\tilde{y} \in \mathbb{R}^m$ such that $\sigma(\bar{x}, \tilde{y}) < 0$.

Case a): $k \notin K_0(\bar{x}, \bar{y})$

Due to (3.23), $g_k(\bar{x}, \bar{y}) < 0$. Because of the continuity of $g_k(\bar{x}, \cdot)$, it means that

$$g_k(\bar{x}, y) < 0 \text{ for all } y \text{ close to } \bar{y}. \quad (3.24)$$

Case b): $k \in K_0(\bar{x}, \bar{y})$

We set $\tilde{g}_k(t, y) := g_k(\bar{x} + t\xi, y)$ and $D\tilde{g}_k(t, y) = (D_x g_k(\bar{x}, \bar{y})\xi, D_y g_k(\bar{x}, \bar{y}))$. We claim that there exists a vector $w \in \mathbb{R}^{m+1}$ such that

$$D\tilde{g}_k(0, \bar{y})w > 0 \text{ for all } k \in K_0(\bar{x}, \bar{y}). \quad (3.25)$$

Otherwise, due to the Farkas Lemma, we obtain the existence of γ_k , $k \in K_0(\bar{x}, \bar{y})$ such that:

$$\sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k D_x g_k(\bar{x}, \bar{y})\xi = 0, \quad \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k D_y g_k(\bar{x}, \bar{y}) = 0, \quad \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k = 1, \quad \gamma_k \geq 0.$$

This fact contradicts (3.21). Moreover, setting $w = (w_1, w_2) \in \mathbb{R} \times \mathbb{R}^m$ we get $w_1 > 0$ in (3.25). To see the latter, we recall that $\bar{y} \in M(\bar{x})$. Hence, we multiply the inequalities in (3.25) by appropriate γ_k , $k \in K_0(\bar{x}, \bar{y})$ (see Definition of $V(\bar{x}, \bar{y})$) and sum up w.r.t. k afterwards. The inequalities in (3.21) insures that $w_1 > 0$. W.l.o.g., we assume, $w_1 = 1$.

Further, due to the continuity of $D\tilde{g}_k(\cdot, \cdot)$, we obtain from (3.25)

$$D\tilde{g}_k(t, y)w > 0 \text{ for all } t \in [0, \tilde{\varepsilon}], y \in \tilde{V}, k \in K_0(\bar{x}, \bar{y}), \quad (3.26)$$

where $\tilde{V}$ is a convex neighborhood of $\bar{y}$.

We set $\tilde{y}_i := y_i - t_i w_2$, $i \in \mathbb{N}$. Thus, $\tilde{y}_i \in \tilde{V}$ and $y_i \in \tilde{V}$, for sufficiently large i , because $y_i \xrightarrow{i} \bar{y}$ and $t_i \xrightarrow{i} 0$. Moreover,

$$(t_i, y_i) = (0, \tilde{y}_i) + t_i w, \text{ (recall } w_1 = 1).$$

We apply the mean-value theorem and get for $k \in K_0(\bar{x}, \bar{y})$:

$$\tilde{g}_k(t_i, y_i) - \tilde{g}_k(0, \tilde{y}_i) = t_i D\tilde{g}_k(\tilde{t}, \tilde{y}) \cdot w \text{ with some } \tilde{t} \in [0, \tilde{\varepsilon}], \tilde{y} \in \tilde{V}.$$

Due to $D\tilde{g}_k(\tilde{t}, \tilde{y})w > 0$ (cf. (3.26)) and $\tilde{g}_k(t_i, y_i) = g_k(x_i, y_i) < 0$ (cf. 3.22), it holds:

$$g_k(\bar{x}, \tilde{y}_i) = \tilde{g}_k(0, \tilde{y}_i) < 0 \text{ for all } k \in K_0(\bar{x}, \bar{y}). \quad (3.27)$$

Altogether, (3.24) and (3.27) provide a contradiction with $\bar{x} \in \overline{M}$, since $\tilde{y}_i \xrightarrow{i} \bar{y}$.

Finally, we obtain that (3.20) is not valid. Hence, there exist $y_1, \dots, y_l \in M(\bar{x})$, $v_i \in V(x, y_i)$, $\mu_i \geq 0$, $i = 1, \dots, l$, $\mu \geq 0$ (not all vanishing) such that

$$\mu Df(\bar{x}) = \sum_{i=1}^l \mu_i v_i. \quad (3.28)$$

Assume that $\mu = 0$. Then, multiplying (3.28) by a Sym-MFCQ vector ξ , we obtain:

$$0 = \sum_{i=1}^l \mu_i v_i^T \xi.$$

Since $v_i^T \xi > 0$, we get $\mu_i = 0$, $i = 1, \dots, l$. This contradiction shows that $\bar{x}$ is a KKT point. $\square$

The following corollary is an easy consequence of Theorem 3.2.16.

Corollary 3.2.17 *Let Sym-MFCQ hold at a local minimum $\bar{x} \in M$ for GSIP and Assumption B be satisfied. Then, $\bar{x}$ is a KKT point.*

Sym-MFCQ in SIP and Disjunctive Optimization

We consider the special case of standard SIP, characterized by a constant set $Y := Y(x)$:

$$\text{SIP: } \text{minimize } f(x) \text{ s.t. } x \in M \quad (3.29)$$

with

$$M := \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y\}$$

and a compact set

$$Y := \{y \in \mathbb{R}^m \mid g_k(y) \leq 0, k = 1, \dots, s\}.$$

We recall the well-known Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) for SIP.

Definition 3.2.18 (EMFCQ for SIP)

Let M be given as in (3.29) and $\bar{x} \in M$. The Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) is said to hold at $\bar{x}$ if there exists a vector $\xi \in \mathbb{R}^n$ such that it holds:

$$D_x g_0(\bar{x}, y) \cdot \xi > 0 \text{ for all } y \in E_{g_0}(\bar{x}) := \{y \in Y \mid g_0(\bar{x}, y) = 0\}.$$

The following Lemma clarifies the relationship between EMFCQ and Sym-MFCQ in the case of SIP.

Lemma 3.2.19 (Sym-MFCQ vs. EMFCQ in SIP)

Let M be given as in (3.29) and $\bar{x} \in M$. Then, Sym-MFCQ holds at $\bar{x}$ if and only if the following conditions are both fulfilled:

- (i) EMFCQ holds at $\bar{x}$,
- (ii) the standard MFCQ holds for Y at all $y \in E_{g_0}(\bar{x})$.

Proof. It is easy to see that $M(\bar{x}) = E_{g_0}(\bar{x})$. We obtain for $y \in E_{g_0}(\bar{x})$ and $v \in V(\bar{x}, y)$:

$$v = \gamma_0 D_x g_0(\bar{x}, y) \text{ with}$$

$$\gamma_0 D_y g_0(\bar{x}, y) + \sum_{k \in K(\bar{x}, y)} \gamma_k D_y g_k(y) = 0, \gamma_0 + \sum_{k \in K(\bar{x}, y)} \gamma_k = 1, \gamma_k \geq 0.$$

If $\gamma_0 = 0$, then the standard MFCQ is violated for Y at y . Conversely, if the standard MFCQ is violated for Y at $y \in E_{g_0}(\bar{x})$, then $0 \in V(\bar{x}, y)$. The proof follows directly with the aid of these considerations. $\square$

Example 3.2.20 ($M \neq M^{max}$ in SIP)

Lemma 3.2.19 shows that in the case of SIP Sym-MFCQ incorporates not only the usual EMFCQ, but also the standard MFCQ for Y . The lack of the latter is closely related to the fact that M , being closed, need not to be equal M^{max} , even under EMFCQ.

We consider the following example of SIP from [36]:

$$n = m = 1, s = 1, g_0(x, y) := x - y \text{ and } g_1(y) := y(y - 1)^2.$$

It can be easily seen that $Y = (-\infty, 0] \cup \{1\}$ and, therefore, $M = [1, \infty)$. Moreover, $M^{max} = [0, \infty]$. Setting $\bar{x} := 1$, we have $E_{g_0}(\bar{x}) = M(\bar{x}) = \{1\}$ and EMFCQ holds at $\bar{x}$. Nevertheless, $M \neq M^{max}$. It is due to the violation of the standard MFCQ for Y at 1 and, hence, due to the violation of Sym-MFCQ.

In some situations GSIP can be equivalently rewritten as a so-called Disjunctive Optimization Problem (cf. [69]):

$$\text{DISJ: } \text{minimize } f(x) \text{ s.t. } x \in M_{disj} \quad (3.30)$$

with

$$M_{disj} := \left\{ x \in \mathbb{R}^n \mid \min_{j \in J} \left\{ \max_{\nu_j \in J^j} g_{\nu_j}^j(x) \right\} \geq 0 \right\}$$

and

$$J := \{1, \dots, s\}, \quad J^j = \{1, \dots, k_j\}, \quad k_j \geq 1, \quad j \in J.$$

The Mangasarian-Fromovitz Constraint Qualification (MFCQ) for DISJ is defined as follows.

Definition 3.2.21 (MFCQ for DISJ)

Let M_{disj} be given as in (3.30) and $\bar{x} \in M_{disj}$. The Mangasarian-Fromovitz Constraint Qualification (MFCQ) is said to hold at $\bar{x}$ if there exists a vector $\xi \in \mathbb{R}^n$ such that it holds:

$$Dg_{\nu_j}^j(\bar{x}) \cdot \xi > 0 \text{ for all } \nu_j \in J_0^j(\bar{x}), \quad j \in J.$$

$$\text{Here, } J_0^j(\bar{x}) := \left\{ \tilde{\nu} \in J^j \mid g_{\tilde{\nu}}^j = \max_{\nu_j \in J^j} g_{\nu_j}^j = 0 \right\}.$$

Remark 3.2.22 (Sym-MFCQ and MFCQ for DISJ)

We compare M_{disj} and the local representation of M^{max} (cf. Lemma 3.2.8). The only difference is that in the description of M_{disj} the minimum of finitely many maximum functions is taken over a discrete set, whereas in the description of M^{max} it is minimized over a subset of $\mathbb{R}^m$. This new issue leads to a certain modification of MFCQ for DISJ and results in Sym-MFCQ. In fact, the derivatives of defining functions w.r.t. y -coordinates play an important role in Sym-MFCQ and the whole analysis above.

3.2.2 Feasible set as a Lipschitz manifold

We examine the topological structure of $\overline{M}$ - the closure of the GSIP feasible set. For that, we set

$$M^{max} = \{x \in \mathbb{R}^n \mid \max_{0 \leq k \leq s} g_k(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m\}.$$

Recall that M^{max} is proven to be the topological closure of the GSIP feasible set under Assumption B and Sym-MFCQ (see Theorem 3.2.6). Hence, we concentrate on the upper-level set M^{max} given by a min-max function φ :

$$M^{max} = \{x \in \mathbb{R}^n \mid \varphi(x) \geq 0\},$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as

$$\varphi(x) := \inf_{y \in \mathbb{R}^m} \max_{0 \leq k \leq s} g_k(x, y).$$

We establish assumptions (Compactness Condition CC and Sym-MFCQ) which guarantee that

$$\partial M^{max} = \{x \in \mathbb{R}^n \mid \varphi(x) = 0\},$$

and, moreover, that ∂M^{max} is a Lipschitz manifold of dimension $n - 1$.

The Compactness Condition is shown to be stable under C^0 -perturbations of the defining functions of φ . Sym-MFCQ can be seen as a constraint qualification in terms of Clarke's subdifferential of the min-max function φ . Finally, we conclude that generically the closure of the GSIP feasible set is a Lipschitz manifold (with boundary).

Compactness Condition

We define the Compactness Condition and describe its impacts.

Definition 3.2.23 (Compactness Condition CC)

We say that the Compactness Condition CC is fulfilled, if for all sequences $(x_k, y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^m$ with

$$\left\{ \begin{array}{l} \bullet x_k \rightarrow x \in \mathbb{R}^n, k \rightarrow \infty \\ \bullet \text{ either } \sigma(x_k, y_k) \rightarrow a, k \rightarrow \infty, \text{ and } a \leq \varphi(x) \\ \quad \text{or } \sigma(x_k, y_k) \rightarrow -\infty, k \rightarrow \infty \end{array} \right.$$

the sequence $(y_k)_{k \in \mathbb{N}}$ contains a convergent subsequence.

An essential implication of Condition CC is that we have a local description of the function φ around a given point $\bar{x} \in \mathbb{R}^n$:

$$\varphi(x) = \min_{y \in W} \sigma(x, y) \quad \text{for all } x \in U_{\bar{x}}.$$

Here, $W \subset \mathbb{R}^m$ is a compact set and $U_{\bar{x}} \subseteq \mathbb{R}^n$ is an open neighborhood of $\bar{x}$. This local description can be obtained by another, slightly weaker assumption which we will refer to as Condition C*. However, Condition C* is not stable w.r.t. C^0 -perturbations of the defining functions. A counterexample then motivates the consideration of the more restrictive Condition CC. It turns out that Condition CC is stable and, moreover, implies Condition C*.

Definition 3.2.24 (Condition C*)

We say that Condition C is fulfilled if*

- (C1) *for all $x \in \mathbb{R}^n$ and sequences $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^m$ with $\sigma(x, y_k) \rightarrow \varphi(x)$, $k \rightarrow \infty$, there exists a convergent subsequence of $(y_k)_{k \in \mathbb{N}}$ and*
- (C2) *the mapping $x \mapsto M(x)$ is locally bounded, i.e., for all $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that $\bigcup_{x \in U_{\bar{x}}} M(x)$ is bounded.*

Note that (C1) is a kind of Palais-Smale Condition. Together with the standard assumption (C2) it implies the desired local description of φ .

Lemma 3.2.25 (Condition C* implies the local description of φ)

Let Condition C be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then, there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a compact subset $W \subset \mathbb{R}^m$ such that:*

$$\varphi(x) = \min_{y \in W} \sigma(x, y) \quad \text{for all } x \in U_{\bar{x}}.$$

Proof. Take the neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ from property (C2) of Condition C*. Then

$$W := \overline{\bigcup_{x \in U_{\bar{x}}} M(x)}$$

is obviously a compact set. Now, let $(y_k)_{k \in \mathbb{N}}$ be a minimizing sequence for $\sigma(x, \cdot)$ with $x \in U_{\bar{x}}$, i.e.,

$$\sigma(x, y_k) \rightarrow \varphi(x), \quad k \rightarrow \infty.$$

(C1) implies the existence of a subsequence $(y_{k_l})_{l \in \mathbb{N}}$ of $(y_k)_{k \in \mathbb{N}}$ with

$$y_{k_l} \rightarrow \bar{y} \in \mathbb{R}^m, \quad l \rightarrow \infty.$$

σ is continuous, so we have

$$\sigma(x, y_{k_l}) \rightarrow \sigma(x, \bar{y}), \quad l \rightarrow \infty.$$

Since $(y_{k_l})_{l \in \mathbb{N}}$ is a minimizing sequence it holds:

$$\varphi(x) = \sigma(x, \bar{y}).$$

By definition $\bar{y} \in M(x) \subseteq W$. $\square$

We give two examples showing that (C1) and (C2) are independent.

Example 3.2.26 ($C1 \not\Rightarrow C2$)

Take $n = m = 1$, $s = 0$ and let $\eta \in C^\infty(\mathbb{R})$ be the smooth function with

$$\eta(y) := \begin{cases} \exp(\frac{1}{y^2-1}), & |y| < 1 \\ 0, & |y| \geq 1 \end{cases}.$$

Now, define

$$g_0(x, y) := \begin{cases} -\eta(y), & x \leq 0 \\ -\eta(y) - \eta(y - \frac{1}{x}), & x > 0 \end{cases}.$$

Note that g_0 is differentiable. (C1) is fulfilled since for $x \in \mathbb{R}$ fixed, clearly, $\varphi(x) < 0$ and there are at most two compact sets containing all $y \in \mathbb{R}$ with $g_0(x, y) < 0$. But (C2) is not fulfilled at $\bar{x} = 0$ since for $x > 0$ the minimizers which are induced by the term $\eta(y - \frac{1}{x})$ are arbitrarily far away from 0.

Example 3.2.27 ($C2 \not\Rightarrow C1$)

Take $n = m = 1$, $s = 0$ and $g_0(x, y) := e^{-y^2}$. Then $\varphi(x) = 0$ for all $x \in \mathbb{R}$ and, obviously, $M(x) = \emptyset$ for all $x \in \mathbb{R}$. So (C2) is trivially fulfilled. But for $\bar{x} \in \mathbb{R}$ fixed and a sequence $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ with $\sigma(\bar{x}, y_k) \rightarrow \varphi(\bar{x}) = 0$, $k \rightarrow \infty$, we have that $\|y_k\| \rightarrow \infty$, $k \rightarrow \infty$. So (C1) is not fulfilled.

We now give an example showing that Condition C* is not stable.

Example 3.2.28 (Condition C* is not stable)

We set $\bar{g}_0, \bar{g}_1 \in C^1(\mathbb{R}^2)$:

$$\bar{g}_0 := \begin{cases} x(y+1)^2 + 1, & y \leq -1 \\ 1 - \frac{\eta(y)}{\eta(0)}, & |y| < 1 \\ x(y-1)^2 + 1, & y \geq 1 \end{cases} \quad \bar{g}_1 := \begin{cases} \bar{g}_0(x, y) - 1 + x^2, & |y| < 1 \\ x^2, & |y| \geq 1 \end{cases},$$

where η is the smooth function from Example 3.2.26. We get for x near 0 the following representation for $\sigma(x, y)$:

$$\sigma(x, y) = \begin{cases} \bar{g}_0(x, y), & x \geq 0 \text{ or } x < 0, |y| < 1 \\ \bar{g}_1(x, y), & x < 0, |y| \geq 1 + \sqrt{\frac{x^2-1}{x}}. \end{cases}$$

Since $y = 0$ is the unique minimizer of $\sigma(x, \cdot)$ for all x near 0 we clearly see that Condition C^* is fulfilled.

Figure 11: Illustration of Example 3.2.28.

Now, suppose that Condition C^* is stable w.r.t. the C_s^0 -topology. Then, there exists an open neighborhood $\mathcal{O} \subseteq C^0(\mathbb{R}^2)^2$ of $(\bar{g}_0, \bar{g}_1)$, such that Condition C^* holds for all $(g_0, g_1) \in \mathcal{O}$. Now, fix such an open neighborhood $\mathcal{O}$. We will construct a pair $(g_0, g_1) \in \mathcal{O}$ which does not fulfill Condition C^* . We set

$$g_0(x, y) := \bar{g}_0(x, y) + C \cdot \eta\left(\frac{x}{\varepsilon_x}\right) \eta\left(\frac{y}{\varepsilon_y}\right), \quad C, \varepsilon_x, \varepsilon_y > 0,$$

and

$$g_1(x, y) := \bar{g}_1(x, y).$$

We now choose $C, \varepsilon_x, \varepsilon_y$ sufficiently small such that $(g_0, g_1) \in \mathcal{O}$.

Then, there exists $g_{\min} \in \mathbb{R}$ and $R > 0$ with

$$g_0(x, y) \geq g_{\min} > 0, \quad \text{for all } (x, y) \in B_R(0) \times B_1(0).$$

Now, we can find an $x < 0$ with $|x| < R$ and $\varphi_{(g_0, g_1)}(x) < g_{\min}$. Here, the minimum is attained by $\sigma(x, \cdot)$ for $|y| \geq 1 + \sqrt{\frac{x^2-1}{x}}$ and it is produced by g_1 (see Figure 11). This is a contradiction to (C2).

Note that for a given $x < 0$ near 0 the existence of the minima y with $|y| \geq 1 + \sqrt{\frac{x^2-1}{x}}$ motivates Definition 3.2.23.

Lemma 3.2.29 *Condition CC implies Condition C^* .*

Proof. Let Condition CC be fulfilled. Then, trivially (C1) holds, since all minimizing sequences from the definition of (C1) are admissible sequences in definition of Condition CC and, therefore, the compactness of $(y_k)_{k \in \mathbb{N}}$ is implied.

We assume that (C2) does not hold. Then, for fixed $\bar{x} \in \mathbb{R}^n$ we have:

For all open neighborhoods $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ the set $\bigcup_{x \in U_{\bar{x}}} M(x)$ is not bounded. Hence, there exists a sequence $(x_k, y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^m$ with

$$\left. \begin{aligned} &\bullet x_k \rightarrow \bar{x}, k \rightarrow \infty \\ &\bullet \sigma(x_k, y_k) = \varphi(x_k) \text{ for all } k \in \mathbb{N} \\ &\bullet \|y_k\| \rightarrow \infty, k \rightarrow \infty \end{aligned} \right\}. \quad (3.31)$$

Since (C1) holds, we get the existence of a minimizer for $\sigma(\bar{x}, \cdot)$, i.e., there exists $\bar{y} \in \mathbb{R}^m$ with $\sigma(\bar{x}, \bar{y}) = \varphi(\bar{x})$.

It holds:

$$\sigma(x_k, y_k) = \varphi(x_k) \leq \sigma(x_k, \bar{y}) \quad \text{for all } k \in \mathbb{N}.$$

Since σ is continuous we get that $\sigma(x_k, \bar{y}) \rightarrow \sigma(\bar{x}, \bar{y}) = \varphi(\bar{x})$, $k \rightarrow \infty$.

Further, either $(\sigma(x_k, y_k))_{k \in \mathbb{N}}$ is bounded and, therefore, there exists $a \in \mathbb{R}$ with $a \leq \varphi(\bar{x})$ and (regarding a subsequence if needed)

$$\sigma(x_k, y_k) \rightarrow a, \quad k \rightarrow \infty,$$

or

$$\sigma(x_k, y_k) \rightarrow -\infty, \quad k \rightarrow \infty.$$

In both cases $(x_k, y_k)_{k \in \mathbb{N}}$ is an admissible sequence in the definition of Condition CC. Together with the fact that $\|y_k\| \rightarrow \infty, k \rightarrow \infty$, we have a contradiction to Condition CC. $\square$

We will now prove that Condition CC is in fact stable under C^0 -perturbations. We set

$$\mathcal{C} := \{g = (g_0, \dots, g_s) \in C^1(\mathbb{R}^n \times \mathbb{R}^m)^{s+1} \mid \text{Condition CC is fulfilled for } g\},$$

and from now on the notations $\varphi_g(x)$, $M_g(x)$ and $\sigma_g(x, y)$ indicate the dependence on the defining functions $g = (g_0, \dots, g_s)$.

Theorem 3.2.30 *The set $\mathcal{C}$ is C_s^0 -open.*

Proof. Let $\bar{g} \in \mathcal{C}$. We show that there exists an open neighborhood $\mathcal{V}_{\bar{g}} \subseteq C^0(\mathbb{R}^n \times \mathbb{R}^m)^{s+1}$ of $\bar{g}$ such that $\mathcal{V}_{\bar{g}} \subseteq \mathcal{C}$.

The proof consists of a local part and a globalization step. For the local part let $\bar{x} \in \mathbb{R}^n$ be fixed. We show:

There exist open neighborhoods $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and $\mathcal{U}_{\bar{g}} \subseteq C^0(\mathbb{R}^n \times \mathbb{R}^m)^{s+1}$ of $\bar{g}$ such that

$$\text{Condition CC holds at } (x, g) \quad \text{for all } (x, g) \in U_{\bar{x}} \times \mathcal{U}_{\bar{g}}. \quad (3.32)$$

Here, we say that Condition CC holds at (x, g) for a pair (x, g) , if the sequence $(x_k)_{k \in \mathbb{N}}$ from Definition 3.2.23 converges to x and σ, φ is replaced by σ_g, φ_g .

Now, assume that (3.32) does not hold, i.e.:

For all open neighborhoods $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and $\mathcal{U}_{\bar{g}} \subseteq C^0(\mathbb{R}^n \times \mathbb{R}^m)^{s+1}$ of $\bar{g}$ there exists $(x, g) \in U_{\bar{x}} \times \mathcal{U}_{\bar{g}}$ with:

$$\text{Condition CC fails at } (x, g). \quad (3.33)$$

The failure of Condition CC at (x, g) is by definition equivalent to the existence of a sequence $(x^{(k)}, y^{(k)})_{k \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^m$ with the properties

$$\left. \begin{aligned} &\bullet x^{(k)} \rightarrow x, k \rightarrow \infty \\ &\bullet \text{either } \sigma_g(x^{(k)}, y^{(k)}) \rightarrow a, k \rightarrow \infty, \text{ and } a \leq \varphi_g(x) \\ &\quad \text{or } \sigma_g(x^{(k)}, y^{(k)}) \rightarrow -\infty, k \rightarrow \infty \\ &\bullet \|y^{(k)}\| \rightarrow \infty, k \rightarrow \infty \end{aligned} \right\}. \quad (3.34)$$

From Lemmata 3.2.25 and 3.2.29 we get the existence of a minimizer for $\sigma(\bar{x}, \cdot)$, i.e., there exists $\bar{y} \in \mathbb{R}^m$ with

$$\sigma(\bar{x}, \bar{y}) = \varphi(\bar{x}). \quad (3.35)$$

We now construct a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ constituting a contradiction to Condition CC at $(\bar{x}, \bar{g})$.

For that, let $n \in \mathbb{N}$ be fixed. Choose an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ with

$$\|x - \bar{x}\| < \frac{1}{n} \text{ and } \|\sigma(x, \bar{y}) - \sigma(\bar{x}, \bar{y})\| < \frac{1}{n}, \quad \text{for all } x \in U_{\bar{x}}, \quad (3.36)$$

and an open neighborhood $\mathcal{U}_{\bar{g}} \subseteq C^0(\mathbb{R}^n \times \mathbb{R}^m)^{s+1}$ of $\bar{g}$ such that for all $g \in \mathcal{U}_{\bar{g}}$ and for all $k \in \{0, \dots, s\}$ it holds:

$$|g_k(x, y) - \bar{g}_k(x, y)| < \frac{1}{n}, \quad \text{for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (3.37)$$

Then (3.33) gives us a pair $(x, g) \in U_{\bar{x}} \times \mathcal{U}_{\bar{g}}$ and a sequence $(x^{(k)}, y^{(k)})_{k \in \mathbb{N}}$ with (3.34). Thus, for k sufficiently large, we can define the n -th sequence element $(x_n, y_n) := (x^{(k)}, y^{(k)})$ and get

$$\left. \begin{aligned} &\bullet \|x_n - x\| < \frac{1}{n} \\ &\bullet \sigma_g(x_n, y_n) < \varphi_g(x) + \frac{1}{n} \\ &\bullet \|y_n\| > n \end{aligned} \right\}. \quad (3.38)$$

By construction, i.e., (3.36), (3.37) and (3.38), it holds

$$\|x_n - \bar{x}\| \leq \|x_n - x\| + \|x - \bar{x}\| < \frac{2}{n}$$

and

$$\begin{aligned} \sigma(x_n, y_n) &< \sigma_g(x_n, y_n) + \frac{1}{n} < \varphi_g(x) + \frac{2}{n} \leq \sigma_g(x, \bar{y}) + \frac{2}{n} \\ &< \sigma(x, \bar{y}) + \frac{3}{n} < \sigma(\bar{x}, \bar{y}) + \frac{4}{n} \\ &= \varphi(\bar{x}) + \frac{4}{n}, \end{aligned}$$

which implies that $(x_n, y_n)_{n \in \mathbb{N}}$ is an admissible sequence in the definition of Condition CC and together with the property $\|y_n\| > n$ we have a contradiction to Condition CC at $(\bar{x}, \bar{g})$.

The globalization step is standard. From (3.32) we get a family of neighborhoods $\{U_{\bar{x}} \times \mathcal{U}_{\bar{g}}\}_{\bar{x} \in \mathbb{R}^n}$. Then there exists a locally finite C^∞ -partition of unity subsequent to the covering $\{U_{\bar{x}}\}_{\bar{x} \in \mathbb{R}^n}$ which enables us to construct a positive function $\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ inducing the desired open neighborhood $\mathcal{V}_{\bar{g}} \subseteq C^0(\mathbb{R}^n \times \mathbb{R}^m)^{s+1}$ of $\bar{g}$ from the assertion (cf. [61], [94] for details on this procedure). $\square$

Topological Properties of M^{max}

We will prove that under the Condition CC and Sym-MFCQ the upper-level set M^{max} is a Lipschitz manifold with boundary (see Definition 1.2.1).

In what follows, we use a result from [15] regarding the Clarke's subdifferential of a certain optimal-value function. For that, we consider the following nonlinear optimization problem:

$$\text{NLP:} \quad \text{minimize } F(z) \text{ s.t. } z \in N, \quad (3.39)$$

where

$$N := \{z \in \mathbb{R}^n \mid H(z) = 0, G(z) \leq 0\},$$

and $F \in C^1(\mathbb{R}^n)$, $H \in C^1(\mathbb{R}^n)^k$, $G \in C^1(\mathbb{R}^n)^l$.

Now, we define $\psi : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R} \cup \{-\infty\}$ as the optimal-value function of the right-hand side perturbations of (3.39):

$$\psi(v, w) := \inf\{F(z) \mid z \in \mathbb{R}^n, H(z) + v = 0, G(z) + w \leq 0\} \quad (3.40)$$

Definition 3.2.31 (Hypothesis H, cf. Hypothesis 6.5.1 in [15])

We say that Hypothesis H holds if $\psi(0, 0)$ is finite and there exists a compact subset $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ such that for all $(v, w) \in \mathbb{R}^k \times \mathbb{R}^l$ with $\|(v, w)\| < \varepsilon$ and $\psi(v, w) < \psi(0, 0) + \varepsilon$ we have that $\psi(v, w)$ is finite and has a solution in K .

The proof of the following result is given in [15].

Theorem 3.2.32 (cf. Theorem 6.5.2 in [15])

Let Hypothesis H be fulfilled. Then, it holds:

$$\partial\psi(0, 0) \subseteq \overline{\text{conv}(\Delta^1(M_0) + \Delta^0(M_0))},$$

where

$$\begin{aligned} \Delta^\delta(z) &:= \{(\lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^l \mid D_z L(z, \delta, \lambda, \mu) = 0, \mu_j \geq 0, \mu_j G_j(z) = 0\}, \\ L(z, \delta, \lambda, \mu) &:= \delta F(z) + \sum_{i=1}^k \lambda_i H_i(z) + \sum_{j=1}^l \mu_j G_j(z), \\ K_0(z) &:= \{j \in \{1, \dots, l\} \mid G_j(z) = 0\}, \\ M_0 &:= \{z \in \mathbb{R}^n \mid F(z) = \psi(0, 0), H(z) = 0, G(z) \leq 0\}. \end{aligned}$$

We apply Theorem 3.2.32 to obtain an inclusion for Clarke's subdifferential of φ . For that, we write φ as an optimal-value function of the type (3.40). Note that

$$\varphi_{g+w}(x) = \psi(x - \bar{x}, w) \quad \text{for all } x \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^{s+1}, \quad (3.41)$$

where in (3.40) we define $z := (u, y, a) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, $F(z) := a$, $H(z) := -u$ and $G(z) := (g_0(u + \bar{x}, y) - a, \dots, g_s(u + \bar{x}, y) - a)^T$.

To see that the identity (3.41) is in fact valid we have to check:

$$\inf_{y \in \mathbb{R}^m} \sigma_{g+w}(x, y) = \inf\{F(z) \mid z \in \mathbb{R}^{n+m+1}, H(z) + (x - \bar{x}) = 0, G(z) + w \leq 0\}$$

$$(3.42)$$

But, since $H(z) + (x - \bar{x}) = 0$ together with $G(z) + w \leq 0$ is equivalent to the fact that $u = x - \bar{x}$ and $g_j(x, y) + w_j \leq a$, for all $j \in \{0, \dots, s\}$, it follows that (3.42) can equivalently be written as

$$\inf_{y \in \mathbb{R}^m} \sigma_{g+w}(x, y) = \inf\{a \mid (y, a) \in \mathbb{R}^{m+1}, \sigma_{g+w}(x, y) \leq a\}.$$

The last equality holds, and hence (3.41) is valid.

From now on let z, F, H, G, ψ be defined as in (3.41).

In what follows we need the notion of upper semicontinuity. Recall that a set-valued mapping $\mathcal{M}$ from a topological space T into a family of all subsets of $\mathbb{R}^n$ is said to be upper semicontinuous at $\bar{v} \in T$ if, for any open set $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{M}(\bar{v}) \subset \mathcal{U}$, there exists an open neighborhood $\mathcal{V}_{\bar{v}} \subset T$ of $\bar{v}$ such that $\mathcal{M}(v) \subset \mathcal{U}$ whenever $v \in \mathcal{V}_{\bar{v}}$.

Lemma 3.2.33 *Condition CC implies Hypothesis H.*

Proof. We prove first that the mapping $(x, w) \mapsto M_{g+w}(x)$ is upper semicontinuous at all $(\bar{x}, 0) \in \mathbb{R}^n \times \mathbb{R}^{s+1}$.

If not, there exists an open set $O \subset \mathbb{R}^m$ with $M(\bar{x}) \subset O$ such that for all open neighborhoods $U_{(\bar{x}, 0)} \subset \mathbb{R}^n \times \mathbb{R}^{s+1}$ of $(\bar{x}, 0)$ there exists $(x, w) \in U_{(\bar{x}, 0)}$ with:

$$M_{g+w}(x) \not\subset O. \quad (3.43)$$

Now, (3.43) directly implies the existence of sequences $(x_k, w_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^{s+1}$ and $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^m$ with

$$\left. \begin{aligned} &\bullet (x_k, w_k) \rightarrow (\bar{x}, 0), \quad k \rightarrow \infty \\ &\bullet y_k \in M_{g+w_k}(x_k), \quad \text{for all } k \in \mathbb{N} \\ &\bullet y_k \notin O, \quad \text{for all } k \in \mathbb{N} \end{aligned} \right\}. \quad (3.44)$$

For $k \in \mathbb{N}$ we have:

$$\begin{aligned} |\sigma(\bar{x}, y_k) - \varphi(\bar{x})| &\leq |\sigma(\bar{x}, y_k) - \sigma_{g+w_k}(x_k, y_k)| \\ &\quad + |\sigma_{g+w_k}(x_k, y_k) - \varphi_{g+w_k}(x_k)| \\ &\quad + |\varphi_{g+w_k}(x_k) - \varphi(\bar{x})|. \end{aligned} \quad (3.45)$$

Here, the second term on the right-hand side is zero since by (3.44) we have $y_k \in M_{g+w_k}(x_k)$. The first and the last term converge to zero for $k \rightarrow \infty$ since

$(x_k, w_k) \rightarrow (\bar{x}, 0)$, $k \rightarrow \infty$ (note that $x \mapsto \varphi(x)$ is continuous by Condition CC). This implies that $(y_k)_{k \in \mathbb{N}}$ is a minimizing sequence for $\sigma(\bar{x}, \cdot)$, i.e.,

$$\sigma(\bar{x}, y_k) \rightarrow \varphi(\bar{x}), \quad k \rightarrow \infty.$$

Due to Lemma 3.2.29 Condition CC implies Condition C*. Hence, by (C2), w.l.o.g. (without loss of generality), $y_k \rightarrow \bar{y} \in M(\bar{x})$, $k \rightarrow \infty$. Since $y_k \notin O$ for all $k \in \mathbb{N}$, we see that $\bar{y} \notin O$ (recall that O is open). This is a contradiction to $M(\bar{x}) \subset O$. We conclude that $(x, w) \rightrightarrows M_{g+w}(x)$ is upper semicontinuous at $(\bar{x}, 0)$.

Now, choose an arbitrary open and bounded set $O \subset \mathbb{R}^m$ with $M(\bar{x}) \subset O$ (note that by (C2) the set $M(\bar{x})$ is bounded). Then, the upper semicontinuity of $(x, w) \rightrightarrows M_{g+w}(x)$ at $(\bar{x}, 0)$ gives us a neighborhood $U_{(\bar{x}, 0)}^1 \subseteq \mathbb{R}^n \times \mathbb{R}^{s+1}$ of $(\bar{x}, 0)$ such that

$$M_{g+w}(x) \subset O \quad \text{for all } (x, w) \in U_{(\bar{x}, 0)}^1.$$

It means that for small perturbations all minimizers stay in the compact closure $\bar{O}$.

For the existence of a minimizer note that due to the openness of Condition CC (see Lemma 3.2.30) we find an open neighborhood $U_{(\bar{x}, 0)}^2 \subseteq \mathbb{R}^n \times \mathbb{R}^{s+1}$ of $(\bar{x}, 0)$ such that Condition CC holds for all $(x, w) \in U_{(\bar{x}, 0)}^2$ (we can easily modify the proof of Lemma 3.2.30 to see that the openness property also holds for this special class of perturbations - note that we only use the uniform estimate (3.37)).

Now, we can define $\varepsilon > 0$ such that $B_\varepsilon((\bar{x}, 0)) \subseteq U_{(\bar{x}, 0)}^1 \cap U_{(\bar{x}, 0)}^2$. Finally, we obtain that for all (x, w) with $\|(x - \bar{x}, w)\| < \varepsilon$ the value $\varphi_{g+w}(x)$ is finite and is attained by $\sigma_{g+w}(x, \cdot)$ in O , i.e., $\psi(x - \bar{x}, w)$ is finite and the corresponding NLP has a solution $(x - \bar{x}, y, \varphi_{g+w}(x)) \in K$ with

$$K := \overline{B_\varepsilon(\bar{x})} \times \bar{O} \times \bigcup_{(x, w) \in \overline{B_\varepsilon((\bar{x}, 0))}} \varphi_{g+w}(x).$$

Due to the continuity of $(x, w) \rightrightarrows \varphi_{g+w}(x)$ the set K is compact. This implies Hypothesis H.

Recall the following notation:

$$\sigma(x, y) := \max_{0 \leq k \leq s} g_k(x, y).$$

Moreover, for $x \in \mathbb{R}^n$ let

$$M(x) := \{y \in \mathbb{R}^m \mid \sigma(x, y) = \varphi(x)\},$$

and for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ let

$$K(x, y) := \{k \in \{0, \dots, s\} \mid g_k(x, y) = \sigma(x, y)\}.$$

At a given point $\bar{x} \in \mathbb{R}^n$ we set

$$V(\bar{x}) := \bigcup_{y \in M(\bar{x})} V(\bar{x}, y) \subseteq \mathbb{R}^n,$$

where

$$V(\bar{x}, y) := \left\{ \sum_{k \in K(\bar{x}, y)} \mu_k D_x g_k(\bar{x}, y) \mid \begin{array}{l} \sum_{k \in K(\bar{x}, y)} \mu_k D_y g_k(\bar{x}, y) = 0, \\ \sum_{k \in K(\bar{x}, y)} \mu_k = 1, \\ \mu_k \geq 0 \end{array} \right\}. \quad (3.46)$$

Lemma 3.2.34 (Clarke's subdifferential of φ)

Let Condition CC be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then it holds:

$$\partial\varphi(\bar{x}) \subseteq \text{conv}(V(\bar{x})) \quad (3.47)$$

Proof. Since Condition CC holds we can apply Lemma 3.2.33. Using Theorem 3.2.32 and (3.41) we obtain the formula for $\partial\varphi(\bar{x})$:

$$\partial\varphi(\bar{x}) = \partial_v \psi(0, 0) = \Pi \partial\psi(0, 0) \subseteq \Pi \overline{\text{conv}(\Delta^1(M_0) + \Delta^0(M_0))}. \quad (3.48)$$

Here, $\Pi : \mathbb{R}^n \times \mathbb{R}^{s+1} \rightarrow \mathbb{R}^n$, $(x, w) \mapsto x$, denotes the projection on the first n variables. Note that the identity $\partial_v \psi(0, 0) = \Pi \partial\psi(0, 0)$ holds due to the Clarke's chain rule (cf. Theorem 2.3.10 in [15]).

We calculate the right-hand side of (3.48).

$$D_z L(z, \delta, \lambda, \mu) = \delta \begin{pmatrix} 0_n \\ 0_{s+1} \\ 1 \end{pmatrix} - \sum_{i=1}^n \lambda_i e_i + \sum_{j=1}^{s+1} \mu_j \begin{pmatrix} D_x g_j(u + \bar{x}, y) \\ D_y g_j(u + \bar{x}, y) \\ -1 \end{pmatrix}$$

If $z = (u, y, a) \in M_0$ we get by $H(z) = -u = 0$ that $u = 0$ and, therefore,

$$\Delta^\delta(z) = \left\{ (\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^{s+1} \mid \begin{array}{l} \sum_{j \in K_0(z)} \mu_j = \delta \\ \sum_{j \in K_0(z)} \mu_j D_y g_j(\bar{x}, y) = 0 \\ \sum_{j \in K_0(z)} \mu_j D_x g_j(\bar{x}, y) = \lambda \\ \mu_j \geq 0 \end{array} \right\}.$$

Furthermore, $z \in M_0$ implies that $K_0(z) = K(\bar{x}, y)$, hence

$$\Pi \Delta^1(z) = \left\{ \sum_{j \in K(\bar{x}, y)} \mu_j D_x g_j(\bar{x}, y) \left| \begin{array}{l} \sum_{j \in K(\bar{x}, y)} \mu_j D_y g_j(\bar{x}, y) = 0, \\ \sum_{j \in K(\bar{x}, y)} \mu_j = 1, \\ \mu_j \geq 0 \end{array} \right. \right\}.$$

Note, that by definition of $V(x, y)$ (see (3.46)) and $\Delta^\delta(z)$ we have:

$$\Pi \Delta^1(z) = V(\bar{x}, y) \text{ and (trivially) } \Pi \Delta^0(z) = \{0\}.$$

Now, it holds

$$\Pi \text{conv}(\Delta^1(M_0) + \Delta^0(M_0)) = \text{conv}(\Pi \Delta^1(M_0)) = \text{conv} \left(\bigcup_{z \in M_0} \Pi \Delta^1(z) \right).$$

Together with the fact that $\Delta^1(M_0)$ is bounded we have

$$\Pi \overline{\text{conv}(\Delta^1(M_0))} = \overline{\Pi \text{conv}(\Delta^1(M_0))}.$$

We obtain:

$$\begin{aligned} \Pi \overline{\text{conv}(\Delta^1(M_0) + \Delta^0(M_0))} &= \overline{\text{conv} \left(\bigcup_{z \in M_0} \Pi \Delta^1(z) \right)} \\ &= \overline{\text{conv} \left(\bigcup_{y \in M(\bar{x})} V(\bar{x}, y) \right)} \\ &= \overline{\text{conv}(V(\bar{x}))}. \end{aligned}$$

Since Condition CC holds we know that $M(\bar{x})$ is compact. This fact implies that $V(\bar{x})$ is compact. Finally, recalling the inclusion in (3.48) we get the assertion. $\square$

Remark 3.2.35 *It is not known whether the inclusion (3.47) is in fact an equality. This is a topic of future research (cf. [11, 93]).*

Theorem 3.2.36 ($M^{\max}$ is a Lipschitz manifold with boundary)

Let Condition CC and Sym-MFCQ at all points $\bar{x} \in M^{\max}$ be fulfilled. Then $M^{\max}$ is a Lipschitz manifold (with boundary) of dimension n .

Proof. Since by Lemma 3.2.29 Condition C^* holds, we get from Lemma 3.2.25 that φ is continuous on $\mathbb{R}^n$. Hence, if $\bar{x} \in M^{max} = \{x \in \mathbb{R}^n \mid \varphi(x) \geq 0\}$ with $\varphi(\bar{x}) > 0$, then there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$ such that $\varphi(x) > 0$ for all $x \in U$. Setting $H(x) := x - \bar{x}$ and $V := H(U)$ we obtain

$$H(\bar{x}) = 0 \text{ and } H(M^{max} \cap U) = H(U) = \mathbb{R}^n \cap V.$$

Now, let $\bar{x} \in M^{max} = \{x \in \mathbb{R}^n \mid \varphi(x) \geq 0\}$ with $\varphi(\bar{x}) = 0$. Since Sym-MFCQ holds there exists $\xi \in \mathbb{R}^n$ with $v \cdot \xi > 0$ for all $v \in V(\bar{x})$. By applying Lemma 3.2.34 we conclude:

$$v \cdot \xi > 0 \text{ for all } v \in \partial\varphi(\bar{x}).$$

W.l.o.g. we assume that $\xi = e_1$ and $\bar{x} = 0$ (otherwise we introduce new coordinates $y := P^{-1} \cdot (x - \bar{x})$, with $P \in \mathbb{R}^{n \times n}$ being a rotation matrix with $P \cdot e_1 = \xi$; using Clarke's chain rule (see Thm. 2.3.10 in [15]) we obtain the desired properties for φ in new coordinates). So now we have:

$$v \cdot e_1 > 0 \text{ for all } v \in \partial\varphi(0).$$

We set H as

$$H(x) := \begin{pmatrix} \varphi(x) \\ x_2 \\ \vdots \\ x_n \end{pmatrix};$$

then the generalized Jacobian (according to Clarke, see [15]) at $\bar{x} = 0$ is given by

$$\partial H(0) = \left(\begin{array}{c|ccc} \frac{\partial_{x_1}\varphi(0)}{0} & \frac{\partial_{x_2}\varphi(0)}{1} & \dots & \frac{\partial_{x_n}\varphi(0)}{1} \\ \hline \vdots & & \ddots & \\ 0 & & & 1 \end{array} \right).$$

From Clarke's Inverse Function Theorem (cf. Thm. 7.1.1 in [15]) we get that there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$ such that the inverse H^{-1} exists on U . Moreover we have with $V := H(U)$

$$H(\bar{x}) = 0 \text{ and } H(M^{max} \cap U) = (\mathbb{H} \times \mathbb{R}^{n-1}) \cap V.$$

This finishes the proof. $\square$

Corollary 3.2.37 (∂M^{max} is a Lipschitz manifold)

Let the Condition CC and Sym-MFCQ be fulfilled. Then, ∂M^{max} is a Lipschitz manifold of dimension $n - 1$. Moreover, it holds:

$$\partial M^{max} = \{x \in \mathbb{R}^n \mid \varphi(x) = 0\}.$$

Proof. The assertion follows directly from the proof of Theorem 3.2.36. $\square$

Application to GSIP

We apply our results on the topological properties of M^{max} in the context of GSIP. It turns out that Assumption B might be replaced by Condition CC to obtain the same results as in Section 3.2.1.

Theorem 3.2.38 (Sym-MFCQ is generic and stable under CC)

Let $\mathcal{F}$ denote the subset of $\mathcal{C}$ (the set of functions which fulfill Condition CC) consisting of those defining functions $(g_0, \dots, g_s)$ for which Sym-MFCQ holds. Then, $\mathcal{F}$ is C_s^1 -open and C_s^1 -dense in $\mathcal{C}$.

Proof. The proof runs along the same lines as the proof of Theorem 3.2.5. Condition CC implies a local description of φ as it is shown to hold in Lemma 3.2.8. Furthermore, the set-valued mappings $(x, g) \rightrightarrows M_g(x)$ and $(x, y, g) \rightrightarrows V_g(x, y)$ can be proven to be upper semicontinuous with the same arguments as used in the proof of Lemma 3.2.9. This, together with the compactness of $M_g(x)$ for $(x, g) \in \mathbb{R}^n \times \mathcal{C}$, implies that Sym-MFCQ is locally stable. The globalization procedure is standard. $\square$

Theorem 3.2.39 (Closure Theorem under CC)

Let Condition CC and Sym-MFCQ at all points $\bar{x} \in M^{max}$ hold. Then,

$$\overline{M} = M^{max}.$$

Proof. Confer with the proof of Theorems 3.2.6. Again, the main property which is used there is the local description of M^{max} . As we have seen above this description is also valid under Condition CC. $\square$

As a direct consequence of Theorems 3.2.36 and 3.2.39 we describe the topological structure of $\overline{M}$ in a generic situation.

Theorem 3.2.40 (Closure of the GSIP feasible set is Lipschitz)

Let Condition CC and Sym-MFCQ at all points $\bar{x} \in M^{max}$ be fulfilled. Then, the closure of the GSIP feasible set $\overline{M}$ is a Lipschitz manifold (with boundary) of dimension n .

The main reason to introduce Condition CC instead of Assumption B is the result in Theorem 3.2.40. Moreover, Assumption B is not symmetric w.r.t. defining functions $g_0, \dots, g_s$. It does not involve the function g_0 . This issue may cause some undesirable effects as the following example shows.

Example 3.2.41 (Assumption B does not imply Hypothesis H)

Let

$m = n = 1$ and $s = 1$. Consider a function $g_1 \in C^1(\mathbb{R} \times \mathbb{R})$ with $g_1(x, y) \leq C$, $C > 0$, which fulfills Assumption B, i.e., the set-valued mapping $x \rightrightarrows \{y \in \mathbb{R} \mid g_1(x, y) \leq 0\}$ is locally bounded. Then let $g_0 \in C^1(\mathbb{R} \times \mathbb{R})$ be a function with the following properties:

- $g_0(x, y) > \max\{g_1(x, y), C, 0\}$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$
- $g_0(x, y) \rightarrow C$, $|y| \rightarrow \infty$, for all $x \in \mathbb{R}$.

Now, we have

$$\varphi(x) = \inf_{y \in \mathbb{R}} \sigma(x, y) = \inf_{y \in \mathbb{R}} g_0(x, y) = C,$$

and since the infimum is not attained Hypothesis H does not hold.

We conclude that Condition CC is a natural symmetric assumption for the related optimization problem on the closure of the GSIP feasible set:

$$\overline{\text{GSIP}} : \quad \text{minimize } f(x) \text{ s.t. } x \in \overline{M}.$$

Nonsmooth Analysis Perspective

Lemma 3.2.34 and Theorem 3.2.38 can be interpreted in terms of nonsmooth analysis. In fact, under CC we get:

$$\partial\varphi(\bar{x}) \subseteq \text{conv}(V(\bar{x})).$$

Generic and stable Sym-MFCQ provides the existence of a vector $\xi \in \mathbb{R}^n$ such that

$$V(\bar{x}) \cdot \xi > 0.$$

Altogether, there exists generically a vector $\xi \in \mathbb{R}^n$ such that

$$\partial\varphi(\bar{x}) \cdot \xi > 0.$$

It means that the Clarke's subdifferential of a min-max function is generically regular. Moreover, the zero-set of a min-max function defined on $\mathbb{R}^n$

is Lipschitz homeomorphic to $\mathbb{R}^{n-1}$. This is a considerable generalization of corresponding results from transversality theory and smooth analysis (see e.g. [61]), where it is for instance commonly known that under generic assumptions the zero-set of a smooth function defined on $\mathbb{R}^n$ is locally diffeomorphic to $\mathbb{R}^{n-1}$. It motivates a further investigation of other types of nonsmoothness in order to derive similar results.

3.3 Nonsmooth Symmetric Reduction Ansatz

As we have seen in Section 3.2.1, the feasible set M in Generalized Semi-Infinite Programming (GSIP) need not be closed. Under the so-called Symmetric Mangasarian-Fromovitz Constraint Qualification (Sym-MFCQ) its closure $\overline{M}$ can be described by means of infinitely many inequality constraints of maximum-type. In this section we introduce the Nonsmooth Symmetric Reduction Ansatz (NSRA). Under NSRA we prove that the set $\overline{M}$ can locally be described as the feasible set of a so-called Disjunctive Optimization Problem defined by finitely many inequality constraints of maximum type. This also shows the appearance of re-entrant corners in $\overline{M}$. Under Sym-MFCQ all local minimizers of GSIP are KKT-points for GSIP. We show that NSRA is generic and stable at all KKT-points and that all KKT-points are nondegenerate. The concept of (nondegenerate) KKT-points as well as a corresponding GSIP-index are introduced in this paper. In particular, a nondegenerate KKT-point is a local minimizer if and only if its GSIP-index vanishes. At local minimizers NSRA coincides with the Symmetric Reduction Ansatz (SRA) as introduced in [37]. In comparison with SRA, the main new issue in NSRA is the following. At KKT-points different from local minimizers the Lagrange polytope at the lower level generically need not be a singleton anymore. In fact, it will be a full dimensional simplex. This fact is crucial to provide the above mentioned local reduction to a Disjunctive Optimization Problem.

We refer to [71] for details.

Formulation of NSRA

Recall that due to Assumption B and Sym-MFCQ on M^{max} , we obtain:

$$\overline{M} = M^{max}, \quad (3.49)$$

where

$$M^{max} = \{x \in \mathbb{R}^n \mid \sigma(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m\}.$$

We consider the relaxed problem

$$\overline{\text{GSIP}} : \quad \text{minimize } f(x) \text{ s.t. } x \in \overline{M}. \quad (3.50)$$

Its feasible set $\overline{M}$ is given by infinitely many constraints of maximum-type (cf. (3.49)).

Our main goal is to provide a reduced local description of $\overline{M}$. To this aim, the Nonsmooth Symmetric Reduction Ansatz (NSRA) will be introduced.

Let $\bar{x} \in \overline{M}$. We set as before

$$M(\bar{x}) := \{y \in \mathbb{R}^m \mid \sigma(\bar{x}, y) = 0\}.$$

Note that $M(\bar{x})$ consists of the global minimizers of $\sigma(\bar{x}, \cdot)$ with vanishing optimal value. We consider the well-known epigraph reformulation: $\bar{y}$ is a global minimizer of $\sigma(\bar{x}, \cdot)$ with vanishing optimal value if and only if $(\bar{y}, 0)$ is a global minimizer of

$$Q(\bar{x}) : \min_{(y,z) \in \mathbb{R}^m \times \mathbb{R}} z \quad \text{s.t.} \quad z - g_k(\bar{x}, y) \geq 0, \quad k = 0, \dots, s.$$

From the first-order optimality condition for $(\bar{y}, 0)$ we obtain that the corresponding polytope of Lagrange multipliers $\Delta(\bar{x}, \bar{y})$ is nonempty:

$$\Delta(\bar{x}, \bar{y}) := \left\{ (\gamma_k)_{k \in K_0(\bar{x}, \bar{y})} \in \mathbb{R}^{|K_0(\bar{x}, \bar{y})|} \left| \begin{array}{l} \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k D_y g_k(\bar{x}, \bar{y}) = 0, \\ \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k = 1, \quad \gamma_k \geq 0, \quad k \in K_0(\bar{x}, \bar{y}) \end{array} \right. \right\}.$$

$K_0(\bar{x}, \bar{y}) := \{k \in \{0, \dots, s\} \mid g_k(\bar{x}, \bar{y}) = 0\}$ is the active index set for $(\bar{y}, 0)$.

For $\gamma \in \Delta(\bar{x}, \bar{y})$ we set

$$K_+(\gamma) := \{k \in K_0(\bar{x}, \bar{y}) \mid \gamma_k > 0\}.$$

For $\bar{x} \in \overline{M}$ and $\bar{y} \in M(\bar{x})$ we define the finite set

$$\mathcal{E}(\bar{x}, \bar{y}) := \{\gamma \mid \gamma \text{ is a vertex of the polytope } \Delta(\bar{x}, \bar{y})\}.$$

Now, we are ready to state the Nonsmooth Symmetric Reduction Ansatz.

Definition 3.3.1 (NSRA)

The Nonsmooth Symmetric Reduction Ansatz (NSRA) is said to hold at $\bar{x} \in \overline{M}$ if for every $\bar{y} \in M(\bar{x})$ either Case I or Case II occurs:

Case I: $(\bar{y}, 0)$ is a nondegenerate minimizer for $Q(\bar{x})$,

Case II: $\Delta(\bar{x}, \bar{y})$ is not a singleton and $|K_+(\gamma)| = m + 1$ for all vertices $\gamma \in \mathcal{E}(\bar{x}, \bar{y})$.

We give three guiding remarks on the Cases I and II.

Remark 3.3.2 (Case I in NSRA)

Case I means that Linear Independence Constraint Qualification (LICQ), Strict Complementarity Slackness (SC) and Second Order Sufficiency Condition (SOSC) hold at the solution $(\bar{y}, 0)$ of $Q(\bar{x})$. It corresponds to the Symmetric Reduction Ansatz as introduced in [37]. In Case I we obtain, in particular, that $\Delta(\bar{x}, \bar{y}) = \{\bar{\gamma}\}$ is a singleton (due to LICQ) and $K_+(\bar{\gamma}) = K_0(\bar{x}, \bar{y})$ (due to SC).

Remark 3.3.3 (Case II in NSRA)

In Case II, the polytope $\Delta(\bar{x}, \bar{y})$ is not a singleton due to the possible violation of LICQ for $Q(\bar{x})$. Nevertheless, for a vertex $\gamma \in \mathcal{E}(\bar{x}, \bar{y})$ we may define the following truncated optimization problem

$$Q^\gamma(\bar{x}) : \min_{(y,z) \in \mathbb{R}^m \times \mathbb{R}} z \quad \text{s.t.} \quad z - g_k(\bar{x}, y) \geq 0, \quad k \in K_+(\gamma).$$

The crucial fact here is that $(\bar{y}, 0)$ is a nondegenerate local minimizer of $Q^\gamma(\bar{x})$ (see Lemma 3.3.14 for details). Indeed, for $Q^\gamma(\bar{x})$ LICQ follows from the fact that $K_+(\gamma)$ is minimal w.r.t. inclusion among all the sets $K_+(\delta)$, $\delta \in \Delta(\bar{x}, \bar{y})$. SC holds due to the definition of $K_+(\gamma)$. The "full-dimensionality" condition $|K_+(\gamma)| = m + 1$ implies that the corresponding tangent space shrinks to a point (the origin), hence, SOSC holds trivially. Moreover, the family of parametrized nondegenerate optimization problems

$$\{Q^\gamma(\bar{x}) \mid \gamma \in \mathcal{E}(\bar{x}, \bar{y})\}$$

will lead us to the reduction result on the local description of $\overline{M}$ as a so called Disjunctive Optimization Problem (see Theorem 3.3.6).

Remark 3.3.4 (SRA)

In [37] the so-called Symmetric Reduction Ansatz (SRA) is introduced. The main difference between SRA and NSRA is that SRA only focuses on Case I in the Definition of NSRA. In [38] it is shown that SRA generically holds at all local minimizers. In order to extend the idea of reduction to all KKT points (see Definition 3.2.15 below), Case II in NSRA is crucial and its appearance cannot be avoided (see Example 3.3.13).

Next, we recall the notion of a Karush-Kuhn-Tucker point for $\overline{\text{GSIP}}$ from Definition 3.2.15.

The main results concerning NSRA and its impacts on the local description of $\overline{M}$ are the following:

- (i) Under NSRA, the set $\overline{M}$ can be locally described as in Disjunctive Optimization (see Theorem 3.3.6). Using the corresponding results on Disjunctive Optimization Problems (cf. [69]) we introduce the notions of a nondegenerate KKT point for $\overline{\text{GSIP}}$ and its GSIP-index (see Definitions 3.3.9, 3.3.10). In particular, the GSIP-index vanishes if and only if the corresponding point is a local minimizer for $\overline{\text{GSIP}}$.
- (ii) NSRA is proven to hold generically at all KKT points. Moreover, all KKT points are proven to be generically nondegenerate (see Theorem 3.3.11).

- (iii) NSRA is shown to be stable under C^2 -perturbations of the defining functions at nondegenerate KKT points (see Theorem 3.3.11).

Remark 3.3.5 *The main result (iii) above shows in particular that the concept of SRA as introduced in [37] is stable at local minimizers.*

Reduction under NSRA

First, we state Theorem 3.3.6 on the Local Reduction of $\overline{M}$ under NSRA.

Theorem 3.3.6 (Local Reduction)

Let Sym-MFCQ hold at all points of $M^{\max}$ and Assumption B be satisfied. Let NSRA hold at $\bar{x} \in \overline{M}$. Then,

- (i) $M(\bar{x})$ is finite, w.l.o.g. $M(\bar{x}) = \{\bar{y}_1, \dots, \bar{y}_p, \bar{y}_{p+1}, \dots, \bar{y}_l\}$ where for $\bar{y}_1, \dots, \bar{y}_p$ Case I and for $\bar{y}_{p+1}, \dots, \bar{y}_l$ Case II from NSRA occurs,
- (ii) *there exist an open neighborhood U of $\bar{x}$ and implicit functions $(y_i, z_i) : U \rightarrow \mathbb{R}^m \times \mathbb{R}$, $i = 1, \dots, p$ and $(y_j^\gamma, z_j^\gamma) : U \rightarrow \mathbb{R}^m \times \mathbb{R}$, $\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)$, $j = p+1, \dots, l$ such that*
 - (a) $(y_i, z_i)(\bar{x}) = (\bar{y}_i, 0)$ and $(y_j^\gamma, z_j^\gamma)(\bar{x}) = (\bar{y}_j, 0)$,
 - (b) $(y_i(x), z_i(x))$ is the locally unique local minimizer of $Q(x)$,
 $(y_j^\gamma(x), z_j^\gamma(x))$ is the locally unique local minimizer of $Q^\gamma(x)$,
 - (c) $y_i(\cdot), y_j^\gamma(\cdot)$ are at least once and $z_i(\cdot), z_j^\gamma(\cdot)$ twice continuously differentiable.

Moreover, it holds:

$$\overline{M} \cap U = \left\{ x \in U \left| \begin{array}{l} z_i(x) \geq 0, i = 1, \dots, p \\ \max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} z_j^\gamma(x) \geq 0, j = p+1, \dots, l \end{array} \right. \right\}. \quad (3.51)$$

Remark 3.3.7 *Note that the "max-inequalities" in (3.51) give rise to the well-known appearance of re-entrant corners in $\overline{M}$ (cf. [111]). The representation (3.51) of $\overline{M}$ is due to the special structure of the Lagrange polytope $\Delta(\bar{x}, \bar{y})$ in Case II. The observation that the description of the GSIP feasible set is connected with the lower level multipliers, is due to [110].*

From Theorem 3.3.6 we see that under NSRA at $\bar{x}$, the optimization problem $\overline{\text{GSIP}}$ is locally equivalent to the following reduced problem:

$$\overline{\text{GSIP}}_{\text{red}} : \quad \text{minimize } f(x) \text{ s.t. } \quad \begin{array}{l} z_i(x) \geq 0, i = 1, \dots, p, \\ \max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} z_j^\gamma(x) \geq 0, j = p+1, \dots, l \end{array} \quad (3.52)$$

The feasible set of $\overline{\text{GSIP}}_{red}$ is given by the finite number of inequality constraints and the finite number of maximum-type constraints with twice continuously differentiable data functions. $\overline{\text{GSIP}}_{red}$ is well-known to be referred to as a Disjunctive Optimization Problem (cf. [69]). Although its data functions $z_i(\cdot)$, $z_j^\gamma(\cdot)$ are defined implicitly, we can explicitly obtain their first and second order derivatives at the point of interest $\bar{x}$.

Remark 3.3.8 *We recall that $(\bar{y}_i, 0)$ is a nondegenerate minimizer of $Q(\bar{x})$ (see Case I). Moreover, $(\bar{y}_j, 0)$ is a nondegenerate minimizer of $Q^\gamma(\bar{x})$, $\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)$ (see Lemma 3.3.14). Hence, implicit functions $z_i(\cdot)$, $z_j^\gamma(\cdot)$ in Theorem 3.3.6 can be obtained by standard results on parametric nonlinear optimization. In fact, $z_i(\cdot)$ and $z_j^\gamma(\cdot)$ are local optimal value functions for $Q(\bar{x})$ and $Q^\gamma(\bar{x})$, respectively. Thus, we can explicitly obtain their first and second order derivatives at $\bar{x}$ (cf. e.g. [9, 63]):*

$$\begin{aligned} Dz_i(\bar{x}) &= \sum_{k \in K_0(\bar{x}, \bar{y}_i)} \bar{\gamma}_k^i D_x g_k(\bar{x}, \bar{y}_i) \quad \text{with } \Delta(\bar{x}, \bar{y}_i) = \{\bar{\gamma}^i\} \text{ for } i = 1, \dots, p, \\ Dz_j^\gamma(\bar{x}) &= \sum_{k \in K_+(\gamma)} \gamma_k D_x g_k(\bar{x}, \bar{y}_j) \quad \text{for } \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l. \end{aligned}$$

Note that $Dz_i(\bar{x}) \in V(\bar{x}, \bar{y}_i)$ and $Dz_j^\gamma(\bar{x}) \in V(\bar{x}, \bar{y}_j)$.

Setting $K_0 := K_0(\bar{x}, \bar{y}_i)$ and evaluating at $(\bar{x}, \bar{y}_i)$ we get

$$\begin{aligned} D^2 z_i(\bar{x}) &= \sum_{k \in K_0} \bar{\gamma}_k^i D_{xx}^2 g_k - A^T \cdot B^{-1} \cdot A, \quad \text{where} \\ A &:= \begin{pmatrix} \sum_{k \in K_0} \bar{\gamma}_k^i D_{xy} g_k \\ 0 \\ -D_x g_k, k \in K_0 \end{pmatrix}, \quad B := \begin{pmatrix} \sum_{k \in K_0} \bar{\gamma}_k^i D_{yy}^2 g_k & 0 & -D_y^T g_k, k \in K_0 \\ 0 & 0 & 1 \\ -D_y g_k, k \in K_0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

For $D^2 z_j^\gamma(\bar{x})$ a similar formula holds, where K_0 has to be replaced by $K_+(\gamma)$.

The above formulas can be used for explicit formulations of optimality criteria for GSIP (see [41]).

We consider the notion of a stationary point for the Disjunctive Optimization Problem $\overline{\text{GSIP}}_{red}$ as defined in [69]. The point $\bar{x}$ is called stationary for $\overline{\text{GSIP}}_{red}$ if there exist $\bar{\lambda}_i \geq 0, i = 1, \dots, p$ and $\bar{\lambda}_j^\gamma \geq 0, \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l$ such that

$$Df(\bar{x}) = \sum_{i=1}^p \bar{\lambda}_i Dz_i(\bar{x}) + \sum_{j=p+1}^l \sum_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} \bar{\lambda}_j^\gamma Dz_j^\gamma(\bar{x}). \quad (3.53)$$

Note that all constraints in (3.52) are active at $\bar{x}$ due to Theorem 3.3.6 (i).

The following crucial observation is due to Remark 3.3.8. The point $\bar{x}$ is stationary for $\overline{\text{GSIP}}_{red}$ if and only if it is a KKT point for $\overline{\text{GSIP}}$ according to Definition 3.2.15. This fact gives rise to introduce the notions of a nondegenerate KKT point and its GSIP-index along the lines in [69].

Definition 3.3.9 (Nondegenerate KKT point)

Let $\bar{x} \in \bar{M}$ be a KKT point for $\overline{\text{GSIP}}$ according to Definition 3.2.15.

Then, $\bar{x}$ is called nondegenerate if the following conditions are satisfied:

ND1: NSRA holds at $\bar{x}$;

ND2: the vectors

$$Dz_i(\bar{x}), i = 1, \dots, p, Dz_j^\gamma(\bar{x}), \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l$$

are linearly independent (i.e. LICQ for $\overline{\text{GSIP}}_{red}$);

ND3: the uniquely determined (due to ND2) multipliers in (3.53)

$$\bar{\lambda}_i, i = 1, \dots, p, \bar{\lambda}_j^\gamma, \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l$$

are positive (i.e. SC for $\overline{\text{GSIP}}_{red}$);

ND4: the matrix

$$V^T \cdot D^2L(\bar{x}) \cdot V$$

is nonsingular, where $D^2L(\bar{x})$ stands for the Hessian of the Lagrange function

$$L(x) = f(x) - \sum_{i=1}^p \bar{\lambda}_i z_i(x) + \sum_{j=p+1}^l \sum_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} \bar{\lambda}_j^\gamma z_j^\gamma(x) \quad (3.54)$$

and V is a matrix whose columns form a basis of the tangent space

$$\{\xi \in \mathbb{R}^n \mid \begin{aligned} Dz_i(\bar{x}) \cdot \xi &= 0, i = 1, \dots, p, \\ Dz_j^\gamma(\bar{x}) \cdot \xi &= 0, \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l \end{aligned}\}. \quad (3.55)$$

Definition 3.3.10 (GSIP-index, cf. [69, Definition 2.3])

Let $\bar{x} \in \bar{M}$ be a nondegenerate KKT point for $\overline{\text{GSIP}}$. The number of negative eigenvalues of the matrix $V^T \cdot D^2L(\bar{x}) \cdot V$ from ND4 we call the quadratic index of $\bar{x}$ and denote it by QI . Further, we call the number

$$QI + \sum_{j=p+1}^l [|\mathcal{E}(\bar{x}, \bar{y}_j)| - 1]$$

the GSIP-index of $\bar{x}$.

In particular, the point $\bar{x} \in \overline{M}$ is a local minimizer for GSIP if and only if its GSIP-index vanishes (cf. [69]).

Let $\mathcal{A}$ denote the set of problem data $(f, g_0, \dots, g_s) \in C^2(\mathbb{R}^n) \times [C^2(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$ such that Assumption B is satisfied. The set $\mathcal{A}$ is C_s^0 -open (cf. [66]).

Let $\mathcal{B}$ denote the set of problem data $(f, g_0, \dots, g_s) \in C^2(\mathbb{R}^n) \times [C^2(\mathbb{R}^n \times \mathbb{R}^m)]^{s+1}$ such that Sym-MFCQ is satisfied at all points of M^{max} . The set $\mathcal{B}$ is C_s^1 -open and C_s^1 -dense in $\mathcal{A}$ (cf. Theorem 3.2.5).

Theorem 3.3.11 (Nondegenerate KKT points are generic and stable)

Let $\mathcal{F}$ denote the subset of $\mathcal{A} \cap \mathcal{B}$ consisting of those problem data $(f, g_0, \dots, g_s)$ for which all KKT points are nondegenerate. Then, $\mathcal{F}$ is C_s^2 -open and C_s^2 -dense in $\mathcal{A}$.

Recall that NSRA holds at a nondegenerate KKT point according to Definition 3.3.9. Hence, in particular, NSRA is a generic and stable condition at KKT points.

As an illustration we give two examples on Cases I and II in NSRA. First, we provide Example 3.3.12 for which NSRA does not hold. By special perturbations of the defining functions one can attain Case I. Here, for the sake of explanation we restrict our considerations to the particular case of SIP (see also the discussion in Section 4). Secondly, in Example 3.3.13 (from [40, 111]) Case II turns out to be stable under arbitrary small C^1 -perturbations of defining functions.

Example 3.3.12 (Density of Case I in NSRA)

Let $n = 0$, $m = 2$, $s = 2$ and GSIP be given by

$$g_0(y_1, y_2) = -y_1 + y_2^2, \quad g_1(y_1, y_2) = y_1 + y_2^2, \quad g_2(y_1, y_2) = 2y_1 + y_2^2.$$

We consider the global minimizer $(0, 0)$ of $\sigma(y_1, y_2) = \max_{k \in \{0, 1, 2\}} g_k(y_1, y_2)$. The vectors $(-Dg_k(0, 0), 1)^T$, $k \in \{0, 1, 2\}$ are linearly dependent. Hence, Case I in NSRA does not hold at $(0, 0)$. The vertices of $\Delta(0, 0)$ are $\gamma_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\gamma_2 = (\frac{2}{3}, 0, \frac{1}{3})$. Thus, $|K_+(\gamma_1)| = |K_+(\gamma_2)| = 2 \neq 3 = m + 1$, and Case II in NSRA does not hold at $(0, 0)$.

For sufficiently small $\varepsilon > 0$ we perturb the functions g_0, g_1, g_2 as follows:

$$g_0^\varepsilon(y_1, y_2) = -y_1 + y_2^2 - \varepsilon y_2, \quad g_1^\varepsilon(y_1, y_2) = y_1 + y_2^2 + \varepsilon y_2, \quad g_2^\varepsilon(y_1, y_2) = 2y_1 + y_2^2 + 3\varepsilon y_2.$$

We consider $GSIP^\varepsilon$ with defining functions $g_0^\varepsilon, g_1^\varepsilon, g_2^\varepsilon$. The vectors $(-Dg_k^\varepsilon(0, 0), 1)^T$, $k \in \{0, 1, 2\}$ become now linearly independent. Moreover, $\Delta^\varepsilon(0, 0) = \{(\frac{5}{8}, \frac{1}{8}, \frac{2}{8})\}$. It implies that $(0, 0)$ is the nondegenerate global minimizer of $\sigma^\varepsilon(y_1, y_2) = \max_{k \in \{0, 1, 2\}} g_k^\varepsilon(y_1, y_2)$. Hence, Case I in NSRA occurs at $(0, 0)$ for $GSIP^\varepsilon$.

Example 3.3.13 (Stability of Case II in NSRA)

Let $n = 2$, $m = 1$, $s = 2$ and GSIP be given by

$$g_0(x_1, x_2, y) = y, \quad g_1(x_1, x_2, y) = x_1 - y, \quad g_2(x_1, x_2, y) = x_2 - y.$$

We consider for $(\bar{x}_1, \bar{x}_2) = (0, 0)$ the global minimizer 0 of the function $\sigma(\bar{x}_1, \bar{x}_2, y) = \max_{k \in \{0, 1, 2\}} g_k(\bar{x}_1, \bar{x}_2, y)$. The vertices of $\Delta(0, 0, 0)$ are $\gamma_1 = (\frac{1}{2}, 0, \frac{1}{2})$ and $\gamma_2 = (\frac{1}{2}, \frac{1}{2}, 0)$. Thus, $|K_+(\gamma_1)| = |K_+(\gamma_2)| = 2 = m + 1$, and Case II in NSRA holds at 0. Note that the feasible set M of GSIP is given by

$$M = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \max \{x_1, x_2\} \geq 0 \right\}$$

and posses the disjunctive structure. We point out that the validity of Case II here is stable under arbitrary small C^1 -perturbations of the defining functions g_0, g_1, g_2 .

Proofs of Main Results

Lemmas 3.3.14, 3.3.15, 3.3.16 deal with the parametric problems $Q(\cdot)$ and $Q^\gamma(\cdot)$ corresponding to Case II in NSRA.

Lemma 3.3.14 (Nondegeneracy for $Q^\gamma(\bar{x})$ in Case II)

Let NSRA hold at $\bar{x} \in \overline{M}$ and for $\bar{y} \in M(\bar{x})$ let Case II occur. Given a vertex $\gamma \in \mathcal{E}(\bar{x}, \bar{y})$, $(\bar{y}, 0)$ is a nondegenerate local minimizer of the following truncated optimization problem

$$Q^\gamma(\bar{x}) : \min_{(y, z) \in \mathbb{R}^m \times \mathbb{R}} z \quad \text{s.t.} \quad z - g_k(\bar{x}, y) \geq 0, \quad k \in K_+(\gamma).$$

Proof. Note that $(\bar{y}, 0)$ is a KKT point for $Q^\gamma(\bar{x})$ with the Lagrange multiplier vector γ . We show that LICQ, SC and SOSC are fulfilled at $(\bar{y}, 0)$.

a) Assume that LICQ does not hold at $(\bar{y}, 0)$. Then, there exist real numbers β_k , $k \in K_+(\gamma)$ (not all vanishing) such that

$$\sum_{k \in K_+(\gamma)} \beta_k D_y g_k(\bar{x}, \bar{y}) = 0, \quad \sum_{k \in K_+(\gamma)} \beta_k = 0. \quad (3.56)$$

We claim that there exists a real number $a \neq 0$ such that

$$\begin{aligned} \gamma_k + a\beta_k &\geq 0 \text{ for all } k \in K_+(\gamma) \text{ and} \\ \gamma_i + a\beta_i &= 0 \text{ for at least one } i \in K_+(\gamma). \end{aligned} \quad (3.57)$$

Indeed, put $\tau := \min \left\{ \frac{\gamma_k}{|\beta_k|} \mid \beta_k \neq 0 \right\} > 0$ and let the minimum be attained at some index l . We define

$$a := \begin{cases} \tau & \text{if } \beta_l < 0, \\ -\tau & \text{if } \beta_l > 0. \end{cases}$$

From (3.56) and $\gamma \in \Delta(\bar{x}, \bar{y})$ we obtain

$$\sum_{k \in K_+(\gamma)} (\gamma_k + a\beta_k) D_y g_k(\bar{x}, \bar{y}) = 0, \quad \sum_{k \in K_+(\gamma)} \gamma_k + a\beta_k = 1.$$

Hence, $\gamma_k + a\beta_k \in \Delta(\bar{x}, \bar{y})$. Moreover, $K_+(\gamma_k + a\beta_k)$ is a proper subset of $K_+(\gamma)$ due to (3.57). However, since $\gamma \in \mathcal{E}(\bar{x}, \bar{y})$ is a vertex of $\Delta(\bar{x}, \bar{y})$, $K_+(\gamma)$ is minimal w.r.t. inclusion among all the sets $K_+(\delta)$, $\delta \in \Delta(\bar{x}, \bar{y})$. This fact provides a contradiction.

b) SC holds due to the definition of $K_+(\gamma)$.

c) Since $|K_+(\gamma)| = m + 1$ and all constraints $z - g_k(\bar{x}, y)$, $k \in K_+(\gamma)$ are active at $(\bar{y}, 0)$, the corresponding tangent space for $Q^\gamma(\bar{x})$ vanishes. Hence, SOSC holds trivially. $\square$

For the following we need the concept of a strongly stable (in the sense of Kojima, [80] and Definition 1.2.7) KKT point for nonlinear optimization problems. We refer e.g. to [35] for the definition of a strongly stable KKT point and its characterization.

Lemma 3.3.15 (Strong Stability for $Q(\bar{x})$ in Case II)

Let NSRA hold at $\bar{x} \in \bar{M}$ and for $\bar{y} \in M(\bar{x})$ let Case II occur. Then, $(\bar{y}, 0)$ is a strongly stable (local) minimizer of $Q(\bar{x})$.

Proof. We use the characterization of a strongly stable KKT point given in [35, Theorem 6]. First, we note that the standard Mangasarian-Fromovitz Constraint Qualification (MFCQ) is fulfilled at $(\bar{y}, 0)$ due to the epigraph reformulation $Q(\bar{x})$. Since $\Delta(\bar{x}, \bar{y})$ is not a singleton in Case II, LICQ fails to hold at $(\bar{y}, 0)$. Thus, $(\bar{y}, 0)$ is a strongly stable minimizer of $Q(\bar{x})$ if and only if

$$D_{(y,z)}^2 L(\bar{y}, 0, \delta) \text{ is positive definite on } T^{K_+(\delta)} \text{ for all } \delta \in \Delta(\bar{x}, \bar{y}), \quad (3.58)$$

where $D_{(y,z)}^2 L(\bar{y}, 0, \delta)$ stands for the Hessian of the Lagrange function

$$L(y, z, \delta) = z - \sum_{k \in K_0(\bar{x}, \bar{y})} \delta_k (z - g_k(\bar{x}, y)) = \sum_{k \in K_0(\bar{x}, \bar{y})} \delta_k g_k(\bar{x}, y)$$

and $T^{K_+(\delta)}$ is the subspace

$$\left\{ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R} \mid \begin{pmatrix} -D_y g_k(\bar{x}, \bar{y}) & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0, k \in K_+(\delta) \right\}.$$

For a vertex $\gamma \in \mathcal{E}(\bar{x}, \bar{y})$ the vectors

$$\begin{pmatrix} -D_y^T g_k(\bar{x}, \bar{y}) \\ 1 \end{pmatrix}, k \in K_+(\gamma)$$

are linearly independent (see part a) in the proof of Lemma 3.3.14). Hence, $T^{K_+(\gamma)} = \{0\}$, since $|K_+(\gamma)| = m + 1$. Moreover, $K_+(\gamma)$ is minimal w.r.t. inclusion among all the sets $K_+(\delta)$, $\delta \in \Delta(\bar{x}, \bar{y})$. Thus, we have that $T^{K_+(\delta)} \subset T^{K_+(\gamma)} = \{0\}$ for all $\delta \in \Delta(\bar{x}, \bar{y})$ and (3.58) is trivially satisfied. $\square$

Lemma 3.3.16 (Active index set for $Q(\cdot)$ in Case II)

Let NSRA hold at $\bar{x} \in \bar{M}$ and for $\bar{y} \in M(\bar{x})$ let Case II occur. Let $(y(x), z(x))$ be the locally unique local minimizer of $Q(x)$ (existing due to Lemma 3.3.15), with the active index set

$$K_0(x) := \{k \in \{0, \dots, s\} \mid z(x) - g_k(x, y(x)) = 0\}.$$

Then, for all x sufficiently close to $\bar{x}$ it holds: $K_0(x) \in \mathcal{K}(\bar{x}, \bar{y})$, where

$$\mathcal{K}(\bar{x}, \bar{y}) := \{K \subset K_0(\bar{x}, \bar{y}) \mid \text{there exists } \gamma \in \Delta(\bar{x}, \bar{y}) \text{ with } K_+(\gamma) \subset K\}.$$

Proof. Note that the function $(y(\cdot), z(\cdot))$ is locally Lipschitz continuous due to the strong stability of $(\bar{y}, 0)$ for $Q(\bar{x})$. Moreover, $K_0(\bar{x}) = K_0(\bar{x}, \bar{y})$. Hence, for x sufficiently close to $\bar{x}$ it holds: $K_0(x) \subset K_0(\bar{x}, \bar{y})$. In particular, for x there exists a sequence $(x_i)_{i \in \mathbb{N}}$ such that

$$x_i \xrightarrow{i} \bar{x} \text{ and } K_0(x_i) = K_0(x).$$

Let $\gamma(x_i)$ be a vector of Lagrange multipliers for $(y(x_i), z(x_i))$, i.e.

$$\sum_{k \in K_0(x)} \gamma_k(x_i) D_y g_k(x_i, y(x_i)) = 0, \quad \sum_{k \in K_0(x)} \gamma_k(x_i) = 1, \quad \gamma_k(x_i) \geq 0, \quad k \in K_0(x). \quad (3.59)$$

We set $\gamma_k(x_i) = 0$ for $k \notin K_0(x)$. The corresponding polytopes of Lagrange multipliers $\Delta(x_i, y(x_i))$ (consisting of those $\gamma(x_i)$ in (3.59)) are compact.

Moreover, since their vertices are depending continuously on x_i and $x_i \xrightarrow{i} \bar{x}$, there exists a compact set $V \subset \mathbb{R}^{|K_0(\bar{x}, \bar{y})|}$ such that

$$\bigcup_{i \in \mathbb{N}} \Delta(x_i, y(x_i)) \subset V.$$

Without loss of generality, we may assume that $\gamma(x_i) \xrightarrow{i} \bar{\gamma} \in V$. Obviously, $K_+(\bar{\gamma}) \subset K_0(x)$. Letting $i \rightarrow \infty$ in (3.59), we obtain that $\bar{\gamma} \in \Delta(\bar{x}, \bar{y})$. $\square$

Proof of Theorem 3.3.6. Let NSRA hold at $\bar{x} \in \bar{M}$.

(1) For $\bar{y} \in M(\bar{x})$ we claim that $(\bar{y}, 0)$ is a nondegenerate minimizer for $Q(\bar{x})$ if Case I occurs and a strongly stable minimizer if Case II occurs (see Lemma 3.3.15). In both cases $(\bar{y}, 0)$ is an isolated minimizer for $Q(\bar{x})$. Moreover, due to Lemma 3.2.8, the set $M(\bar{x})$ is compact. Hence, $M(\bar{x})$ is finite. W.l.o.g., $M(\bar{x}) = \{\bar{y}_1, \dots, \bar{y}_p, \bar{y}_{p+1}, \dots, \bar{y}_l\}$ where for $\bar{y}_1, \dots, \bar{y}_p$ Case I and for $\bar{y}_{p+1}, \dots, \bar{y}_l$ Case II from NSRA occurs.

(2) The existence of locally defined implicit functions

$$(y_i(\cdot), z_i(\cdot)), i = 1, \dots, p \text{ and } (y_j^\gamma(\cdot), z_j^\gamma(\cdot)), \gamma \in \mathcal{E}(\bar{x}, \bar{y}_j), j = p+1, \dots, l$$

with (a), (b), (c) is due to the standard results on parametric nonlinear optimization problems (recall also Lemma 3.3.14).

From Lemma 3.3.15 we obtain also the locally defined implicit functions

$$(y_j(\cdot), z_j(\cdot)), j = p+1, \dots, l \text{ such that}$$

- $(y_j, z_j)(\bar{x}) = (\bar{y}_j, 0)$,
- $(y_j(x), z_j(x))$ is the locally unique local minimizer of $Q(x)$,
- the mapping $x \mapsto (y_j(x), z_j(x))$ is continuous.

Thus, we obtain that, locally around $\bar{x}$, the set $\bar{M}$ can be described as follows:

$$\left\{ x \left| \begin{array}{l} z_i(x) \geq 0, i = 1, \dots, p \\ z_j(x) \geq 0, j = p+1, \dots, l \end{array} \right. \right\}. \quad (3.60)$$

For the desired description (3.51) it is sufficient to prove that locally around $\bar{x}$ the following nontrivial representation is valid:

$$z_j(x) = \max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} z_j^\gamma(x), j = p+1, \dots, l. \quad (3.61)$$

Let $j \in \{j = p+1, \dots, l\}$ be arbitrary, but fixed.

(i) $z_j(x) \geq \max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} z_j^\gamma(x)$ since $K_+(\gamma) \subset K_0(\bar{x}, \bar{y})$ for all $\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)$.

(ii) For $z_j(x) \leq \max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y}_j)} z_j^\gamma(x)$ we find a vertex $\bar{\gamma} \in \mathcal{E}(\bar{x}, \bar{y}_j)$ (in general depending on x) with $z_j(x) = z_j^{\bar{\gamma}}(x)$.

Due to Lemma 3.3.16 there exists $\delta \in \Delta(\bar{x}, \bar{y})$ with $K_+(\delta) \subset K_0(x)$. We obtain the existence of a vertex $\bar{\gamma} \in \mathcal{E}(\bar{x}, \bar{y})$ such that $K_+(\bar{\gamma}) \subset K_+(\delta)$ and, hence, $K_+(\bar{\gamma}) \subset K_0(x)$.

Further, the vectors $\begin{pmatrix} -D_y^T g_k(\bar{x}, \bar{y}) \\ 1 \end{pmatrix}$, $k \in K_+(\bar{\gamma})$ are linearly independent (see Lemma 3.3.14 a)). Since $y_j(\cdot)$ depends continuously on x , the vectors

$$\begin{pmatrix} -D_y^T g_k(x, y_j(x)) \\ 1 \end{pmatrix}, k \in K_+(\bar{\gamma})$$

are also linearly independent for x sufficiently close to $\bar{x}$. Moreover (and this is crucial here), they form a basis for $\mathbb{R}^m \times \mathbb{R}$ due to the fact that $|K_+(\bar{\gamma})| = m + 1$ in Case II. Hence, we write

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{k \in K_+(\bar{\gamma})} \gamma_k(x) \begin{pmatrix} -D_y^T g_k(x, y_j(x)) \\ 1 \end{pmatrix} \text{ with } \gamma_k(x) \in \mathbb{R}, k \in K_+(\bar{\gamma}). \quad (3.62)$$

From (3.62) we see that $(y_j(x), z_j(x))$ is a critical point of the following optimization problem with equality constraints only:

$$Q_{=}^{\bar{\gamma}}(x) : \min_{(y, z) \in \mathbb{R}^m \times \mathbb{R}} z \quad \text{s.t.} \quad z - g_k(x, y) = 0, k \in K_+(\bar{\gamma}).$$

The feasibility of $(y_j(x), z_j(x))$ for $Q_{=}^{\bar{\gamma}}(x)$ is provided by $K_+(\bar{\gamma}) \subset K_0(x)$. The vector of Lagrange multipliers for $(y_j(x), z_j(x))$ is given by $\gamma(x)$.

As in Lemma 3.3.14 it can be seen that $(\bar{y}, 0)$ is a nondegenerate critical point of $Q_{=}^{\bar{\gamma}}(\bar{x})$ with unique positive Lagrange multipliers $\bar{\gamma}_k$, $k \in K_+(\bar{\gamma})$. Thus, $\gamma_k(\bar{x}) = \bar{\gamma}_k > 0$, $k \in K_+(\bar{\gamma})$ and $\gamma_k(\cdot)$ depends at least continuously on x . We obtain that $\gamma_k(x)$, $k \in K_+(\bar{\gamma})$ in (3.62) are positive for x sufficiently close to $\bar{x}$. It means that $(y_j(x), z_j(x))$ is a KKT point for $Q^{\bar{\gamma}}(x)$. However, $(y_j^{\bar{\gamma}}(x), z_j^{\bar{\gamma}}(x))$ is the locally unique KKT point for $Q^{\bar{\gamma}}(x)$ due to (2b). Hence, $z_j(x) = z_j^{\bar{\gamma}}(x)$. $\square$

For the proof of Theorem 3.3.11, in particular inequality (3.65), we need some results on the geometry of the polytope $\Delta(\bar{x}, \bar{y})$ given in Lemmas 3.3.17, 3.3.18.

Lemma 3.3.17 (Dimension at a vertex of $\Delta(\bar{x}, \bar{y})$)

Let $\bar{x} \in \bar{M}$, $\bar{y} \in M(\bar{x})$ and $\bar{\gamma}$ be a vertex of $\Delta(\bar{x}, \bar{y})$. Then,

$$\dim \{ \text{span} \{ (D_y^T g_k(\bar{x}, \bar{y})) , k \in K_+(\bar{\gamma}) \} \} = |K_+(\bar{\gamma})| - 1.$$

Proof. As a direct consequence of part a) in the proof of Lemma 3.3.14 a) we obtain first that the vectors $(D_y^T g_k(\bar{x}, \bar{y}))$, $k \in K_+(\bar{\gamma})$ are affine independent, i.e.

$$\text{If } \sum_{k \in K_+(\bar{\gamma})} \beta_k D_y g_k(\bar{x}, \bar{y}) = 0 \text{ and } \sum_{k \in K_+(\bar{\gamma})} \beta_k = 0 \text{ then } \beta_k = 0, k \in K_+(\bar{\gamma}). \quad (3.63)$$

Further, assume that for all $K \subset K_+(\bar{\gamma})$ with $|K| = |K_+(\bar{\gamma})| - 1$ the vectors $(D_y^T g_k(\bar{x}, \bar{y}))$, $k \in K$ are linearly dependent. There is $j \in K_+(\bar{\gamma})$ such that $\bar{\gamma}_j \neq 0$. Setting $K_j := K_+(\bar{\gamma}) \setminus \{j\}$ we get

$$\sum_{k \in K_j} \alpha_k D_y g_k(\bar{x}, \bar{y}) = 0 \text{ with } \alpha_k \in \mathbb{R} \text{ (not all vanishing).}$$

We set $a := \sum_{k \in K_j} \alpha_k \neq 0$ due to affine independence in (3.63). Thus,

$$\bar{\gamma}_j D_y g_j(\bar{x}, \bar{y}) + \sum_{k \in K_j} \left(\bar{\gamma}_k - \frac{\alpha_k}{a} \right) D_y g_k(\bar{x}, \bar{y}) = 0 \text{ and } \bar{\gamma}_j + \sum_{k \in K_j} \left(\bar{\gamma}_k - \frac{\alpha_k}{a} \right) = 0.$$

(3.63) provides, in particular, $\bar{\gamma}_j = 0$, hence, a contradiction. $\square$

Lemma 3.3.18 (Number of vertices of the simplex $\Delta(\bar{x}, \bar{y})$)

Let $\bar{x} \in \bar{M}$ and $\bar{y} \in M(\bar{x})$. Then, $\Delta(\bar{x}, \bar{y})$ is a simplex.

Moreover, let $\bar{\gamma}$ be a vertex of $\Delta(\bar{x}, \bar{y})$. Then,

$$|\mathcal{E}(\bar{x}, \bar{y})| \leq |K_0| - |K_+(\bar{\gamma})| + 1.$$

Proof. Note that the vectors $\begin{pmatrix} -D_y^T g_k(\bar{x}, \bar{y}) \\ 1 \end{pmatrix}$, $k \in K_+(\bar{\gamma})$ are linearly independent (see part a) in the proof of Lemma 3.3.14). Hence, $\Delta(\bar{x}, \bar{y})$ lies in the affine subspace

$$\left\{ \gamma \in \mathbb{R}^{|K_0(\bar{x}, \bar{y})|} \left| \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k \begin{pmatrix} -D_y^T g_k(\bar{x}, \bar{y}) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right. \right\}$$

of dimension at most $|K_0| - |K_+(\bar{\gamma})|$. In fact, $\Delta(\bar{x}, \bar{y})$ is the intersection of the latter affine subspace and the simplex

$$\left\{ \gamma \in \mathbb{R}^{|K_0(\bar{x}, \bar{y})|} \left| \sum_{k \in K_0(\bar{x}, \bar{y})} \gamma_k = 1, \gamma_k \geq 0, k \in K_0(\bar{x}, \bar{y}) \right. \right\}.$$

Consequently, $\Delta(\bar{x}, \bar{y})$ is a simplex itself and the inequality on the number of vertices $\mathcal{E}(\bar{x}, \bar{y})$ of $\Delta(\bar{x}, \bar{y})$ follows. $\square$

Proof of Theorem 3.3.11: a) We prove that $\mathcal{F}$ is C_s^2 -dense in $\mathcal{A}$.

We use the ideas from [38] to prove that $\mathcal{F}$ is C_s^2 -generic. The proof is based on an application of the Structured Jet Transversality Theorem, for details see e.g. [34, 61].

We consider a KKT point $x \in \overline{M}$ and write $M(x) = \{y_1, \dots, y_l\}$. There exist $v_i \in V(x, y_i)$ and $\mu_i \geq 0, i = 1, \dots, l$ such that $Df(x) = \sum_{i=1}^l \mu_i v_i$. We represent any of $v_i \in V(x, y_i)$ by means of a linear combination with strictly positive multipliers of a minimal number of vectors $v_{i,j}, j = 1, \dots, q_i$ forming vertices of the polytope $V(x, y_i)$. Note that

$$v_{i,j} = \sum_{k \in K_0(x, y_i)} \gamma_k^j D_x g_k(x, y_i) \text{ with a vertex } \gamma^j \in \mathcal{E}(x, y_i). \quad (3.64)$$

For $q_i, r_i \in \mathbb{N}$ and index sets $K_i, Q_{i,j} \subset K_i, i = 1, \dots, l, j = 1, \dots, q_i$ we consider the set Γ of $(x, y_1, \dots, y_l, v_{1,1}, \dots, v_{l,q_l})$ such that the following conditions are satisfied:

- (i) $x \in \mathbb{R}^n, y_i \in \mathbb{R}^m$ (pairwise different),
 $v_{i,j} \in \mathbb{R}^n$ (uniquely determined as vertices of $V(x, y_i)$ with (3.64)),
- (ii) $K_i = K_0(x, y_i)$,
- (iii) $\text{span} \{D_y g_k(x, y_i), k \in K_i\}$ has dimension r_i ,
- (iv) $Q_{i,j} = K_+(\gamma^j)$ and $\text{span} \{D_y g_k(x, y_i), k \in Q_{i,j}\}$ has dimension $|Q_{i,j}| - 1$ (cf. Lemma 3.3.17). Moreover, $(v_{i,j}, 0) \in \text{span} \{Dg_k(x, y_i), k \in Q_{i,j}\}$,
- (v) $Df(x) \in \text{span} \{v_{i,j} \mid i = 1, \dots, l, j = 1, \dots, q_i\}$.

Γ constitutes a stratified manifold. Generically (due to Structured Jet Transversality Theorem), its dimension coincides with the difference between the amount of available degrees of freedom and the number of independent equations representing (i)-(v). Setting $q := \sum_{i=1}^l q_i$ we see that the ambient space of Γ has dimension $nq + ml + n$. Now, we count the loss of freedom $Loss_i$ caused by (ii)-(iv):

- (i) nq_i , since $v_{i,j}$ are uniquely determined
- (ii) $|K_i|$,

(iii) $(m - r_i)(|K_i| - r_i)$, since $r_i \leq m$ and $r_i < |K_i|$,

(iv) $(r_i - (|Q_{i,j}| - 1))(|Q_{i,j}| - (|Q_{i,j}| - 1)) = r_i + 1 - |Q_{i,j}|$,

Hence,

$$Loss_i = nq_i + |K_i| + (m - r_i)(|K_i| - r_i) + \sum_{j=1}^{q_i} (r_i + 1 - |Q_{i,j}|).$$

Setting $M_i := \max_{j=1, \dots, q_i} |Q_{i,j}|$ we get (due to Lemma 3.3.18)

$$|K_i| \geq M_i + |\mathcal{E}(x, y_i)| - 1 \geq M_i + q_i - 1 \text{ and also } |Q_{i,j}| \leq M_i. \quad (3.65)$$

For estimating $Loss_i$ by means of (3.65), we distinguish two cases:

Case a) ($r_i = m$):

$$\begin{aligned} Loss_i &= nq_i + |K_i| + \sum_{j=1}^{q_i} (m + 1 - |Q_{i,j}|) \\ &\geq nq_i + (M_i + q_i - 1) + mq_i + q_i - M_i q_i \\ &= nq_i + (q_i - 1)(m + 1 - M_i) + m + q_i \geq nq_i + m + q_i. \end{aligned} \quad (3.66)$$

In the last inequality we have $q_i - 1 \geq 0$ due to $\Delta(x, y_i) \neq \emptyset$ and also $m + 1 - M_i \geq 0$ due to Lemma 3.3.17.

Case b) ($r_i < m$): Setting for (convenience) $\beta_i := |K_i| - r_i - 1 \geq 0$

$$\begin{aligned} Loss_i &= nq_i + |K_i| + (m - r_i)(|K_i| - r_i) + \sum_{j=1}^{q_i} (r_i + 1 - |Q_{i,j}|) \\ &\geq nq_i + (M_i + q_i - 1) + (m - r_i)(1 + \beta_i) + r_i q_i + q_i - M_i q_i \\ &= nq_i + m + q_i + (m - r_i)\beta_i + (q_i - 1)(r_i + 1 - M_i) \geq nq_i + m + q_i. \end{aligned} \quad (3.67)$$

In the last inequality we have $m - r_i \geq 0$, $q_i - 1 \geq 0$ and also $r_i + 1 - M_i \geq 0$ due to Lemma 3.3.17.

Finally, (v) reduces the freedom by $n - d$ degrees, where

$$d := \dim \{ \text{span} \{ v_{i,j} \mid i = 1, \dots, l, j = 1, \dots, q_i \} \} \leq q.$$

Summing up over $i = 1, \dots, l$, (i)-(v) cause a loss of at least

$$\sum_{i=1}^l (nq_i + m + q_i) + n - d = nq + ms + q + n - d \geq nq + mq + n \quad (3.68)$$

degrees of freedom. This means that for C_s^2 -generic defining functions the set Γ has dimension at most 0. Moreover, another loss of freedom would cause Γ to be empty. Thus, all inequalities in (3.65), (3.66), (3.67) and (3.68) turn to equalities to avoid the emptiness of Γ .

The equalities in (3.65) read

$$|K_i| = M_i + q_i - 1, |\mathcal{E}(x, y_i)| = q_i \text{ and } |Q_{i,j}| = M_i. \quad (3.69)$$

We consider the Cases a) and b) again letting (3.66), (3.67) turn to equalities.

Case a) ($r_i = m$) with equalities:

Here, we get additionally to (3.69): $q_i = 1$ or $M_i = m + 1$.

If $q_i = 1$, then $\Delta(x, y_i)$ is a singleton and $|K_i| = M_i = |Q_{i,1}|$. Hence, $K_i = Q_{i,1} = K_+(\gamma_i)$ and we see that LICQ and SC in the Case I from NSRA hold. Moreover, the violation of SOSC reduces the freedom and can be generically avoided here. Case I from NSRA occurs.

If $q_i \neq 1$, then $\Delta(x, y_i)$ is not a singleton. From $|\mathcal{E}(x, y_i)| = q_i$ we see that for every vertex $\gamma \in \mathcal{E}(x, y_i)$ there exists $j \in \{1, \dots, q_i\}$ with $Q_{i,j} = K_+(\gamma)$. Moreover, $|K_+(\gamma)| = |Q_{i,j}| = M_i = m + 1$. Case II from NSRA occurs. We point out that in this case $\Delta(\bar{x}, \bar{y})$ is a $(|K_i| - m - 1)$ -dimensional simplex in $\mathbb{R}^{|K_i| - m}$, hence with $|K_i| - m$ vertices.

Case b) ($r_i < m$) with equalities:

Here, we get additionally to (3.69): $|K_i| = r_i + 1$ and $[q_i = 1 \text{ or } M_i = r_i + 1]$.

If $q_i = 1$, then Case I from NSRA occurs (as above in Case a)).

If $q_i \neq 1$, then $M_i = r_i + 1$. Together with $|K_i| = r_i + 1$ we obtain $|K_i| = M_i$ and, hence from (3.69), $q_i = |K_i| - M_i + 1 = 1$, a contradiction.

Altogether, we claim that either Case I or Case II from NSRA occurs, i.e. ND1 in Definition 3.3.9 is generically fulfilled.

We examine the generic validity of ND2, ND3, ND4 in Definition 3.3.9. The equality in (3.68) means that $d = q$ and, hence, the vectors

$$v_{i,j}, i = 1, \dots, l, j = 1, \dots, q_i \quad (3.70)$$

are linearly independent. We see that ND2 is generically fulfilled. Further, if $Df(x)$ does not belong to the relative interior of the cone generated by the vectors from (3.70), this causes additional loss of freedom in (v). Thus, ND3 is generically fulfilled. Moreover, the possible singularity of the matrix from ND4 also reduces the freedom and, hence, can be generically avoided. Finally, every KKT point x is shown to be generically nondegenerate. $\square$

Remark 3.3.19 *From the above proof we see that in Case II from NSRA the Lagrange polytope $\Delta(\bar{x}, \bar{y})$ is generically a full dimensional simplex (as it is claimed in the abstract). Here, the full dimensionality refers to the fact*

that $\Delta(\bar{x}, \bar{y})$ is a $(|K_0(\bar{x}, \bar{y})| - m - 1)$ -dimensional simplex, hence with exactly $|K_0(\bar{x}, \bar{y})| - m$ vertices $\mathcal{E}(\bar{x}, \bar{y})$. Recalling Definition 3.3.10 we have under NSRA at $\bar{x}$:

$$\text{GSIP-index} = QI + \sum_{j=p+1}^l [|K_0(\bar{x}, \bar{y}_j)| - m - 1].$$

Proof of Theorem 3.3.11: b) We prove that $\mathcal{F}$ is C_s^2 -open in $\mathcal{A}$.

1) **Local argument:** First we construct for every nondegenerate KKT point $\bar{x} \in \bar{M}$ a system of equations whose locally unique zero corresponds exactly to $\bar{x}$. We show that such a system of equations is stable, in the sense that the usual Implicit Function Theorem can be applied to follow KKT points w.r.t. local C^2 -perturbations of the defining functions. Moreover, these KKT points for perturbed problems remain nondegenerate and locally unique.

To avoid unnecessary technicalities we assume that there is only one implicit constraint in the local description (3.51) of $\bar{M}$: either $z(\cdot)$ (Case I in NSRA) or $\max_{\gamma \in \mathcal{E}(\bar{x}, \bar{y})} z^\gamma(\cdot)$ (Case II in NSRA). Here, we set $M(\bar{x}) = \{\bar{y}\}$.

Case I occurs for $\bar{y}$:

Let $\bar{x} \in \bar{M}$ be a nondegenerate KKT point with Lagrange multiplier $\bar{\lambda} > 0$ as in Definition (3.3.9). W.l.o.g, we assume that $\bar{\lambda} = 1$. Since Case I occurs in $\bar{y}$, let $\Delta(\bar{x}, \bar{y}) = \{\bar{\gamma}\}$. We set $K_0 := K_0(\bar{x}, \bar{y})$.

We consider the following mapping $\mathcal{T} : \mathbb{R}^{n+m+1+|K_0|+1} \longrightarrow \mathbb{R}^{n+m+1+|K_0|+1}$ locally at its zero $(\bar{x}, \bar{y}, 0, \bar{\gamma}, \bar{\lambda})$:

$$\mathcal{T}(x, y, z, \gamma, \lambda) := \begin{pmatrix} D_x f(x) - \lambda \sum_{k \in K_0} \gamma_k D_x g_k(x, y) \\ \sum_{k \in K_0} \gamma_k D_y g_k(x, y) \\ \sum_{k \in K_0} \gamma_k - 1 \\ z - g_k(x, y), k \in K_0 \\ \sum_{k \in K_0} \gamma_k g_k(x, y) \end{pmatrix}.$$

Note that $\sum_{k \in K_0} \gamma_k g_k(x, y) = 0$ ensures feasibility for the reduced problem (3.51). In fact, let $(y(x), z(x))$ be the local minimizer of $Q(x)$. We obtain

$$\sum_{k \in K_0} \gamma_k g_k(x, y(x)) = \sum_{k \in K_0} \gamma_k z(x) = z(x) = 0.$$

Further, we prove that $D\mathcal{T}(\bar{x}, \bar{y}, 0, \bar{\gamma}, \bar{\lambda})$ is nonsingular.

$$D\mathcal{T}(\bar{x}, \bar{y}, 0, \bar{\gamma}, \bar{\lambda}) = \begin{pmatrix} A & B & D \\ B^T & C & 0 \\ D^T & 0 & 0 \end{pmatrix},$$

where

$$A = D_x^2 f(\bar{x}) - \bar{\lambda} \sum_{k \in K_0} \bar{\gamma}_k D_{xx}^2 g_k(\bar{x}, \bar{y}),$$

$$B = \left(-\bar{\lambda} \sum_{k \in K_0} \bar{\gamma}_k D_{xy} g_k(\bar{x}, \bar{y}) \mid 0 \mid -\bar{\lambda} D_x^T g_k, k \in K_0 \right),$$

$$C = \begin{pmatrix} \sum_{k \in K_0} \bar{\gamma}_k D_{yy}^2 g_k(\bar{x}, \bar{y}) & 0 & -D_y^T g_k(\bar{x}, \bar{y}), k \in K_0 \\ 0 & 0 & 1 \\ -D_y g_k(\bar{x}, \bar{y}), k \in K_0 & 1 & 0 \end{pmatrix},$$

$$D = - \sum_{k \in K_0} \bar{\gamma}_k D_x^T g_k.$$

Note that C is nonsingular since $(\bar{y}, 0)$ is a nondegenerate minimizer of $Q(\bar{x})$ (cf. Theorem 2.3.2, [63]). The Schur-complement of the submatrix C in $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is the Hessian $D^2 L(\bar{x})$ of the Lagrange function (3.54) for the reduced problem (cf. Definition 3.3.9 and Remark 3.3.8). Due to ND4, $D^2 L(\bar{x})$ is nonsingular on the tangent space (3.55). However, the columns of D span the orthogonal complement of (3.55) due to ND1. It provides, again by means of Theorem 2.3.2, [63] that the matrix $\begin{pmatrix} D^2 L(\bar{x}) & D \\ D^T & 0 \end{pmatrix}$ is nonsingular, hence, also $D\mathcal{T}(\bar{x}, \bar{y}, 0, \bar{\gamma}, \bar{\lambda})$.

Case II occurs for $\bar{y}$:

Let $\bar{x} \in \bar{M}$ be a nondegenerate KKT point with the vector of Lagrange multipliers $(\bar{\lambda}^\gamma, \gamma \in \mathcal{E}(\bar{x}, \bar{y}))$ as in Definition (3.3.9). We put $K_\gamma := K_+(\gamma)$ and $\mathcal{E} := \mathcal{E}(\bar{x}, \bar{y})$. For $\Gamma = (\Gamma_k^\gamma, k \in K_\gamma, \gamma \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}| \cdot (m+1)}$ we set $\bar{\Gamma} := (\gamma_k, k \in K_\gamma, \gamma \in \mathcal{E})$. Moreover, $Y := (y^\gamma, \gamma \in \mathcal{E}) \in \mathbb{R}^{m \cdot |\mathcal{E}|}$, $Z := (z^\gamma, \gamma \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}|}$ and $\Lambda := (\lambda^\gamma, \gamma \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}|}$.

We consider the mapping $\mathfrak{R} : \mathbb{R}^{n+m \cdot |\mathcal{E}| + |\mathcal{E}| + |\mathcal{E}| \cdot (m+1) + |\mathcal{E}|} \longrightarrow \mathbb{R}^{n+m \cdot |\mathcal{E}| + |\mathcal{E}| + |\mathcal{E}| \cdot (m+1) + |\mathcal{E}|}$ locally at its zero $(\bar{x}, \bar{Y}, 0, \bar{\Gamma}, \bar{\Lambda})$:

$$\mathfrak{R}(x, Y, Z, \Gamma, \Lambda) := \begin{pmatrix} D_x f(x) - \sum_{\gamma \in \mathcal{E}} \lambda^\gamma \sum_{k \in K_\gamma} \Gamma_k^\gamma D_x g_k(x, y^\gamma) \\ \sum_{k \in K_\gamma} \Gamma_k^\gamma D_y g_k(x, y^\gamma), \gamma \in \mathcal{E} \\ \sum_{k \in K_\gamma} \Gamma_k^\gamma - 1, \gamma \in \mathcal{E} \\ z^\gamma - g_k(x, y^\gamma), k \in K_\gamma, \gamma \in \mathcal{E} \\ \sum_{k \in K_\gamma} \Gamma_k^\gamma g_k(x, y^\gamma) \end{pmatrix}.$$

Note that $\sum_{k \in K_\gamma} \Gamma_k^\gamma g_k(x, y^\gamma) = 0$ ensures feasibility for the reduced problem (3.51). In fact, let $(y^\gamma(x), z^\gamma(x))$ be the local minimizers of $Q^\gamma(x)$. We obtain

$$\sum_{k \in K_\gamma} \Gamma_k^\gamma g_k(x, y^\gamma(x)) = \sum_{k \in K_\gamma} \Gamma_k^\gamma z^\gamma(x) = z^\gamma(x) = 0.$$

The nonsingularity of $D\mathfrak{R}(\bar{x}, \bar{Y}, 0, \bar{\Gamma}, \bar{\Lambda})$ can be proved analogously as in Case I. To obtain the corresponding Schur-complement one uses the fact that $(\bar{y}, 0)$ is a nondegenerate minimizer of $Q^\gamma(\bar{x})$ (cf. Lemma 3.3.14). This Schur-complement is exactly the Hessian $D^2 L(\bar{x})$ of the Lagrange function (3.54) for the reduced problem (cf. Definition 3.3.9). Here, the formula for the second derivative of the implicit constraint $z^\gamma(\cdot)$ is used (cf. Remark 3.3.8). Finally, ND1 and ND4 imply the nonsingularity of $D\mathfrak{R}(\bar{x}, \bar{Y}, 0, \bar{\Gamma}, \bar{\Lambda})$.

Altogether, in both Cases I and II the Implicit Function Theorem can be applied to follow KKT points w.r.t. local C^2 -perturbations of the defining functions. By means of continuity arguments in ND1-ND4, these KKT points for perturbed problems remain nondegenerate and locally unique.

2) **Global argument:** The Global argument is standard. We only stress that under Assumption B the set-valued mapping $(x, g) \rightrightarrows M_g(x)$ is upper semicontinuous w.r.t. the topology in $\mathbb{R}^n \times [C^2(\mathbb{R}^n \times \mathbb{R}^m)]^{k+1}$. Hence, $M_g(x)$ is locally bounded w.r.t. C^2 -perturbations of the defining functions (see also Lemma 3.2.8). The global issue is due to the strong C_s^2 -topology. $\square$

Remark 3.3.20 *The above proof provides a description of nondegenerate KKT points as solutions of certain stable equations involving first and second*

order information of the defining functions. (We refer to mappings $\mathcal{T}$ and $\mathfrak{R}$ from the proof of Theorem 3.3.11). This fact might be used to establish some (nonsmooth) versions of Newton Method for GSIP (cf. [14, 78, 114]). This issue is a topic of current research.

Application to SIP

We consider the special case of standard SIP, characterized by a constant set $Y := Y(x)$:

$$\text{SIP:} \quad \text{minimize } f(x) \text{ s.t. } x \in M \quad (3.71)$$

with

$$M := \{x \in \mathbb{R}^n \mid g_0(x, y) \geq 0 \text{ for all } y \in Y\}$$

and a compact set

$$Y := \{y \in \mathbb{R}^m \mid g_k(y) \leq 0, k = 1, \dots, s\}.$$

For $\bar{x} \in M$ we denote by $E_{g_0}(\bar{x})$ the active index set $\{y \in Y \mid g_0(\bar{x}, y) = 0\}$. It consists of all global minimizers for $g_0(\bar{x}, \cdot)|_Y$.

Let Sym-MFCQ be fulfilled for SIP to provide the description

$$M = \{x \in \mathbb{R}^n \mid \sigma(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m\}.$$

Sym-MFCQ is equivalent here to the well-known Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) for SIP and the standard MFCQ for Y at all $y \in E_{g_0}(\bar{x})$ (cf. Lemma 3.2.19).

The well-known Reduction Ansatz for SIP (cf. [37]) states that all $y \in E_{g_0}(\bar{x})$ are nondegenerate minimizers for $g_0(\bar{x}, \cdot)|_Y$. It turns out that Reduction Ansatz for SIP corresponds exactly to Case I in NRSA for GSIP. The proof of the Theorem 3.3.21 is straightforward.

Theorem 3.3.21 (Reduction Ansatz for SIP vs. Case I in NSRA)

Let M be given as in (3.71) and $\bar{x} \in M$. Then, $\bar{y} \in E_{g_0}(\bar{x})$ is a nondegenerate minimizer for $g_0(\bar{x}, \cdot)|_Y$ if and only if $(\bar{y}, 0)$ with $\bar{y} \in M(\bar{x})$ is a nondegenerate minimizer for $Q(\bar{x})$.

From Theorem 3.3.21 we see that, compared to SIP, a main new issue for GSIP is Case II in NSRA.

Remark 3.3.22 It is worth to mention that the well-known corresponding genericity results for KKT points in SIP for the reduced feasible set (i.e. ND2-ND4) can be achieved by means of the perturbations of $g_0(\cdot, \cdot)$ only. The set Y remains unchanged.

3.4 Critical point theory

Under the Symmetric Mangasarian-Fromovitz Constraint Qualification (Sym-MFCQ) two basic theorems from Morse theory (deformation theorem and cell-attachment theorem) are proved. Outside the set of Karush-Kuhn-Tucker (KKT) points, continuous deformation of lower level sets can be performed. As a consequence, the topological data (such as the number of connected components) then remain invariant. However, when passing a KKT level, the topology of the lower level set changes via the attachment of a q -dimensional cell. The dimension q equals the so-called GSIP-index of the (nondegenerate) KKT-point. Here, the Nonsmooth Symmetric Reduction Ansatz (NSRA) allows to perform a local reduction of GSIP to a Disjunctive Optimization Problem. The GSIP-index then coincides with the stationary index from the corresponding Disjunctive Optimization Problem.

We refer to [72] for details.

Deformation and Cell-Attachment

For $a, b \in \mathbb{R}$, $a < b$ define the sets

$$M^a := \{x \in M \mid f(x) \leq a\}, \quad \overline{M}^a := \{x \in \overline{M} \mid f(x) \leq a\},$$

$$\overline{M}_a^b := \{x \in \overline{M} \mid a \leq f(x) \leq b\},$$

$$\text{lev}(\overline{M}, a) := \{x \in \overline{M} \mid f(x) = a\}.$$

Our aim is to prove the following result.

Theorem 3.4.1 (Critical point theory on M)

Let Condition CC be fulfilled, Sym-MFCQ hold at all points $x \in M^{\max}$ and M be bounded. Then, the following results are valid:

- (a) **(Deformation Theorem on M)** *If $\overline{M}_a^b$ does not contain any KKT-point, then M^a is homotopy-equivalent to M^b .*
- (b) **(Cell-attachment Theorem on M)** *If $\overline{M}_a^b$ contains exactly one non-degenerate KKT-point, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the GSIP-index of $\bar{x}$ is equal to q , then M^b is homotopy-equivalent to M^a with a q -cell attached.*

We describe the idea behind the proof of the main deformation and cell-attachment Theorem 3.4.1. For that, we consider an explicit description of $\overline{M}$. In Theorem 3.2.39 it is shown that under the Compactness Condition

(CC) and the Symmetric Mangasarian-Fromovitz Constraint Qualification (Sym-MFCQ) (see Definitions 3.2.1 and 3.2.23) the closure of the feasible set is given by

$$\overline{M} = M^{max}, \quad (3.72)$$

where

$$M^{max} = \left\{ x \in \mathbb{R}^n \mid \max_{0 \leq k \leq s} g_k(x, y) \geq 0 \text{ for all } y \in \mathbb{R}^m \right\}.$$

Having description (3.72) in mind, we consider the relaxed problem

$$\overline{\text{GSIP}} : \quad \text{minimize } f(x) \text{ s.t. } x \in M^{max}. \quad (3.73)$$

We prove the corresponding deformation and cell-attachment results for $\overline{\text{GSIP}}$.

Theorem 3.4.2 (Critical point theory on $\overline{M}$)

Let Condition CC be fulfilled, Sym-MFCQ hold at all points $x \in M^{max}$ and M be bounded. Then, the following results are valid:

- (a) **(Deformation Theorem on $\overline{M}$)** *If $\overline{M}_a^b$ does not contain KKT-points, then $\overline{M}^a$ is a strong deformation retract of $\overline{M}^b$.*
- (b) **(Cell-attachment Theorem on $\overline{M}$)** *If $\overline{M}_a^b$ contains exactly one non-degenerate KKT-point, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the GSIP-index of $\bar{x}$ is equal to q , then $\overline{M}^b$ is homotopy-equivalent to $\overline{M}^a$ with a q -cell attached.*

For proving Theorem 3.4.2 Sym-MFCQ becomes crucial. Moreover, we use the fact that under Condition CC and Sym-MFCQ the set $\overline{M}$ is a Lipschitz manifold (cf. Theorem 3.2.36). Furthermore, we link the topology of the lower level set for GSIP and $\overline{\text{GSIP}}$, respectively.

Theorem 3.4.3 (Topology of M^a vs. $\overline{M}^a$)

If the set $\text{lev}(\overline{M}, a)$ does not contain any KKT-points (i.e. the level $a \in \mathbb{R}$ is regular), M^a is homotopy-equivalent to $\overline{M}^a$.

Finally, the main deformation and cell-attachment Theorem 3.4.1 follows easily from these both results.

Proofs of Main Results

For the proof of Theorem 3.4.2 we need the following simple, but crucial lemma.

Lemma 3.4.4 (Local descriptions of M^{max} and φ)

Let Condition CC be fulfilled and let $\bar{x} \in M^{max}$. Then, there exist some neighborhood $U_{\bar{x}}$ of $\bar{x}$ and a nonempty compact set $W \subset \mathbb{R}^m$ such that

$$M^{max} \cap U_{\bar{x}} = \{x \in U_{\bar{x}} \mid \sigma(x, y) \geq 0 \text{ for all } y \in W\}$$

and $\varphi(x) = \min_{y \in W} \sigma(x, y)$, $x \in U_{\bar{x}}$.

If additionally $\varphi(x) = 0$ then $M(x) = \{y \in W \mid \sigma(x, y) = 0\}$, $x \in U_{\bar{x}}$.

Proof. Due to Lemma 3.2.29, Condition CC implies that

- (C1) for all $x \in \mathbb{R}^n$ and sequences $(y_k)_{k \in \mathbb{N}} \subset \mathbb{R}^m$ with $\sigma(x, y_k) \rightarrow \varphi(x)$, $k \rightarrow \infty$, there exists a convergent subsequence of $(y_k)_{k \in \mathbb{N}}$ and
- (C2) the mapping $x \mapsto M^\varphi(x) := \{y \in \mathbb{R}^m \mid \sigma(x, y) = \varphi(x)\}$ is locally bounded, i.e. for all $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that $\bigcup_{x \in U_{\bar{x}}} M^\varphi(x)$ is bounded.

Let $\bar{x} \in M^{max}$. Using the neighborhood $U_{\bar{x}} \subset \mathbb{R}^n$ of $\bar{x}$ from (C2) we set

$$W := \overline{\bigcup_{x \in U_{\bar{x}}} M^\varphi(x)}$$

W is a compact set to (C2).

Now, let $(y_k)_{k \in \mathbb{N}}$ be a minimizing sequence for $\sigma(x, \cdot)$ with $x \in U_{\bar{x}}$, i.e.

$$\sigma(x, y_k) \rightarrow \varphi(x), \quad k \rightarrow \infty.$$

(C1) implies the existence of a subsequence $(y_{k_l})_{l \in \mathbb{N}}$ of $(y_k)_{k \in \mathbb{N}}$ with

$$y_{k_l} \rightarrow \bar{y} \in \mathbb{R}^m, \quad l \rightarrow \infty.$$

Due to the continuity of $\sigma(x, \cdot)$, we have

$$\sigma(x, y_{k_l}) \rightarrow \sigma(x, \bar{y}), \quad l \rightarrow \infty.$$

Since $(y_{k_l})_{l \in \mathbb{N}}$ is a minimizing sequence, it holds:

$$\varphi(x) = \sigma(x, \bar{y}).$$

Thus, we get $\bar{y} \in M^\varphi(x) \subseteq W$.

Note that for $x \in U_{\bar{x}}$ with $\varphi(x) = 0$ it holds: $M^\varphi(x) = M(x)$. $\square$

Proof of Theorem 3.4.2

(a) We show that $\overline{M}^a$ is a strong deformation retract of $\overline{M}^b$ in several steps.

Step 1: Existence of Sym-MFCQ vectors $\xi_{\bar{x}}, \bar{x} \in \overline{M}_a^b$

Since every $\bar{x} \in \overline{M}_a^b$ is not a KKT-point, we obtain

$$0 \notin \text{conv}(\{-D^T f(\bar{x})\} \cup V(\bar{x})). \quad (3.74)$$

Due to the compactness of $V(\bar{x})$ (which easily follows from the compactness of $M(\bar{x})$, cf. Lemma 3.4.4), a separation argument can be used as in [111]. From (3.74) we obtain the existence of a vector $\xi_{\bar{x}} \in \mathbb{R}^n$ such that

$$D^T f(\bar{x}) \cdot \xi_{\bar{x}} < 0 \text{ and}$$

$$v \cdot \xi_{\bar{x}} > 0 \text{ for all } v \in V(\bar{x}).$$

The latter means, in particular, that $\xi_{\bar{x}}$ is a Sym-MFCQ vector at $\bar{x}$.

Step 2: Localization of Sym-MFCQ vectors $\xi_{\bar{x}}, \bar{x} \in \overline{M}_a^b$

Let $\bar{x} \in \overline{M}_a^b$. We claim that there exist an open neighborhood $O_{\bar{x}}$ of $\bar{x}$, such that

$$D^T f(x) \cdot \xi_{\bar{x}} < 0 \text{ and}$$

$$v \cdot \xi_{\bar{x}} > 0 \text{ for all } v \in V(x), x \in O_{\bar{x}} \cap \overline{M}. \quad (3.75)$$

We refer to Theorem 3.2.5 for details on the proof of (3.75). It follows mainly from the fact that the mappings $x \mapsto M(x)$ and $(x, y) \mapsto V(x, y)$ are upper-semicontinuous due to the local representation of $\overline{M}$ from Lemma 3.4.4. Moreover, $M(\bar{x})$ and $V(\bar{x})$ are compact sets.

Step 3: Globalization of Sym-MFCQ vectors $\xi_{\bar{x}}, \bar{x} \in \overline{M}_a^b$

Due to the boundedness of M , $\overline{M}_a^b$ is a compact set. Hence, we get from the open covering $\{O_{\bar{x}} \mid \bar{x} \in \overline{M}_a^b\}$ of $\overline{M}_a^b$ a subcovering $\{O_{\bar{x}_j} \mid \bar{x}_j \in \overline{M}_a^b, j \in J\}$ with a finite set J .

Using a C^∞ -partition of unity $\{\phi_j\}$ subordinate to $\{O_{\bar{x}_j} \mid \bar{x}_j \in \overline{M}_a^b, j \in J\}$ we define with $\xi_{\bar{x}_j}$ from Step 2 a C^∞ -vector field

$$\xi(x) := \sum_{j \in J} \phi_j(x) \xi_{\bar{x}_j} \text{ for } x \in \overline{M}_a^b.$$

The last induces a flow $\Psi(t, \cdot)$ on $\overline{M}_a^b$ (see [61], Theorem 3.3.14 for details).

Since $\xi(x)$ is a convex combination of the vectors $\{\xi_{\bar{x}_j} \mid x \in O_{\bar{x}_j}, j \in J\}$, we obtain:

$$\begin{aligned} D^T f(x) \cdot \xi(x) &< 0 \text{ and} \\ v \cdot \xi(x) &> 0 \text{ for all } v \in V(x). \end{aligned} \quad (3.76)$$

Step 4: Feasibility and descent behavior of $\Psi(t, \cdot)$ on $\overline{M}_a^b$

Our aim is to show that there exist $\varepsilon > 0$ and $\bar{t} > 0$ such that

- $\Psi(t, x) \in \overline{M}$ for all $t \in [0, \bar{t}]$, $x \in \overline{M}_a^b$ and
- $\Psi(\bar{t}, x) \in \overline{M}^{f(x)-\varepsilon}$ for all $x \in \overline{M}_a^b$.

Step 4a: Feasibility of $\Psi(t, \cdot)$ on $\overline{M}_a^b$

For $\bar{x} \in \overline{M}_a^b$, we consider the local description of $\overline{M}$ with $U_{\bar{x}}$ and W as given in Lemma 3.4.4. By shrinking $U_{\bar{x}}$ we may assume that it is a compact neighborhood of $\bar{x}$.

First, we show the local feasibility of $\Psi(t, \cdot)$ on $\overline{M}_a^b \cap U_{\bar{x}}$. Namely, that there exists $t_{\bar{x}} > 0$ such that

$$\Psi(t, x) \in \overline{M} \text{ for all } t \in [0, t_{\bar{x}}], x \in \overline{M}_a^b \cap U_{\bar{x}}. \quad (3.77)$$

Let $x \in \overline{M}_a^b \cap U_{\bar{x}}$ and $y \in W$.

Case 1: $y \notin M(x)$. Then, $\max_{0 \leq k \leq s} g_k(x, y) > 0$.

Case 2: $y \in M(x)$. We write the Taylor expansion for $g_k(\Psi(\cdot, x), y)$, $k \in K_0(\bar{x}, y)$ at 0:

$$g_k(\Psi(t, x), y) = t \left[D_x g_k(x, y) \cdot \xi(x) + \frac{o_k(t, x, y)}{t} \right]. \quad (3.78)$$

We choose a vector $v \in V(x, y) \subset V(x)$ written as

$$v = \sum_{k \in K_0(x, y)} \gamma^k(x, y) D_x g_k(x, y) \text{ with}$$

$$\sum_{k \in K_0(x, y)} \gamma^k(x, y) D_y g_k(x, y) = 0, \quad \sum_{k \in K_0(x, y)} \gamma^k(x, y) = 1, \quad \gamma^k(x, y) \geq 0.$$

Multiplying (3.78) by $\gamma_k(x, y)$ and summing up, we obtain:

$$\sum_{k \in K_0(x, y)} \gamma^k(x, y) g_k(\Psi(t, x), y) = t \left[v \cdot \xi(x) + \sum_{k \in K_0(x, y)} \gamma^k(x, y) \frac{o_k(t, x, y)}{t} \right] \geq$$

$$t \left[\min_{v \in V(x)} v \cdot \xi(x) + \sum_{k \in K_0(x,y)} \gamma^k(x,y) \frac{o_k(t,x,y)}{t} \right] \geq$$

$$t \left[\inf_{x \in U_{\bar{x}}} \min_{v \in V(x)} v \cdot \xi(x) + \sum_{k \in K_0(x,y)} \gamma^k(x,y) \frac{o_k(t,x,y)}{t} \right].$$

Hence, we get:

$$\max_{k=0,\dots,s} g_k(\Psi(t,x), y) \geq \max_{k \in K_0(x,y)} g_k(\Psi(t,x), y) \geq$$

$$\sum_{k \in K_0(x,y)} \gamma^k(x,y) \max_{k \in K_0(x,y)} g_k(\Psi(t,x), y) \geq \sum_{k \in K_0(x,y)} \gamma^k(x,y) g_k(\Psi(t,x), y) \geq$$

$$t \left[\inf_{x \in U_{\bar{x}}} \min_{v \in V(x)} v \cdot \xi(x) + \sum_{k \in K_0(x,y)} \gamma^k(x,y) \frac{o_k(t,x,y)}{t} \right].$$

We claim that

$$\inf_{x \in U_{\bar{x}}} \min_{v \in V(x)} v \cdot \xi(x) > 0. \quad (3.79)$$

In fact, the description of $M(x)$ in Lemma 3.4.4 easily provides that $V(x)$ is compact. Hence, the set $\bigcup_{x \in U_{\bar{x}}} V(x)$ is also compact. Finally, for the validity of (3.79) it is crucial to note that $\xi(\cdot)$ is continuous (see Steps 2 and 3).

Moreover,

$$\sum_{k \in K_0(x,y)} \gamma^k(x,y) \frac{o_k(t,x,y)}{t} \longrightarrow 0, \text{ (as } t \longrightarrow 0) \text{ uniformly on } (x,y) \in U_{\bar{x}} \times \overline{\bigcup_{x \in U_{\bar{x}}} M(x)}.$$

The latter comes from the fact that

$$\overline{U_{\bar{x}} \times \bigcup_{x \in U_{\bar{x}}} M(x)} \subset U_{\bar{x}} \times W \text{ is compact, } \sum_{k \in K_0(x,y)} \gamma^k(x,y) = 1 \text{ and}$$

$$\frac{o_k(t,x,y)}{t} = \int_0^1 [D_x g_k(\Psi(st,x), y) - D_x g_k(x,y)] \cdot \xi(x) ds$$

is continuous w.r.t. (t,x,y) .

Altogether, we obtain the existence of a real number $t_{\bar{x}} > 0$ (which is independent from $x \in U_{\bar{x}}$) such that (3.77) holds.

Now, it is straightforward to see that there exists $\bar{t}_1 > 0$ such that

$$\Psi(t, x) \in \overline{M} \text{ for all } t \in [0, \bar{t}_1], x \in \overline{M}_a^b.$$

In fact, we consider the covering of a compact set $\overline{M}_a^b$ by the compact neighborhoods $\{U_{\bar{x}} \mid \bar{x} \in \overline{M}_a^b\}$. Then, using a subcovering $\{U_{\bar{x}_i} \mid \bar{x}_i \in \overline{M}_a^b, i \in I\}$ with a finite set I , we define

$$\bar{t}_1 := \min_{i \in I} \{t_{\bar{x}_i}\} > 0.$$

Step 4b: Descent behavior of $\Psi(t, \cdot)$ on $\overline{M}_a^b$

We show that there exist $\varepsilon > 0$ and $\bar{t}_2 > 0$ such that

$$\Psi(\bar{t}_2, x) \in \overline{M}^{f(x)-\varepsilon} \text{ for all } x \in \overline{M}_a^b. \quad (3.80)$$

Having proved the feasibility of $\Psi(t, \cdot)$ on $\overline{M}_a^b$ in Step 4a, we only need to consider the function $f(\Psi(\cdot, x))$ for $x \in \overline{M}_a^b$.

We write the Taylor expansion for $f(\Psi(\cdot, x))$ at 0:

$$f(\Psi(t, x)) = f(x) + t \left[Df(x) \cdot \xi(x) + \frac{o(t, x)}{t} \right].$$

We get

$$f(\Psi(t, x)) \leq f(x) + t \left[\sup_{x \in \overline{M}_a^b} Df(x) \cdot \xi(x) + \frac{o(t, x)}{t} \right].$$

Due to (3.76) and the continuity of $\xi(\cdot)$, we have:

$$\sup_{x \in \overline{M}_a^b} Df(x) \cdot \xi(x) = \max_{x \in \overline{M}_a^b} Df(x) \cdot \xi(x) < 0.$$

Moreover,

$$\frac{o(t, x)}{t} \longrightarrow 0, \text{ (as } t \longrightarrow 0) \text{ uniformly on } \overline{M}_a^b.$$

The latter comes from the fact that $\overline{M}_a^b$ is compact and

$$\frac{o(t, x)}{t} = \int_0^1 [Df(\Psi(st, x)) - Df(x)] \cdot \xi(x) ds \text{ is continuous w.r.t. } (t, x).$$

Thus, we conclude the existence of real numbers $\varepsilon > 0$ and $\bar{t}_2 > 0$ such that (3.80) holds.

Altogether, we obtain the validity of Step 4 putting $\bar{t} := \min \{\bar{t}_1, \bar{t}_2\}$.

Step 5: Deformation via Ψ

Due to Step 4 we obtain for $x \in \bar{M}_a^b$ a unique $t_a(x) \geq 0$ with $\Psi(t_a(x), x) \in \bar{M}^a$. It is not hard (but technical) to realize that $t_a : x \rightarrow t_a(x)$ is Lipschitz. It follows mainly from the application of the standard Implicit Function Theorem and the fact that $\bar{M}_a^b$ is a Lipschitz manifold (cf. Theorem 3.2.36).

Finally, we define $r : [0, 1] \times \bar{M}^b \rightarrow \bar{M}^b$ as follows:

$$r(\tau, x) := \begin{cases} x & \text{for } x \in \bar{M}^a, \quad \tau \in [0, 1] \\ \Psi(\tau t_a(x), x) & \text{for } x \in \bar{M}_a^b, \quad \tau \in [0, 1]. \end{cases}$$

The mapping r provides that $\bar{M}^a$ is a strong deformation retract of $\bar{M}^b$.

(b) In virtue of the Deformation part (a) and the Local Reduction Theorem 3.3.6 the proof of the Cell-attachment Theorem becomes standard. In fact, the Deformation Theorem allows deformations up to an arbitrarily small neighborhood U of the nondegenerate KKT-point $\bar{x}$. In such a neighborhood U we can use the Local Reduction Theorem 3.3.6 to apply the corresponding Cell-attachment Theorem for Disjunctive Optimization Problems (see Theorem 3.2 in [69]). Hence, we obtain that $\bar{M}^b \cap U$ is homotopy-equivalent to $\bar{M}^a \cap U$ with a q -cell attached. This provides the validity of (b). $\square$

Proof of Theorem 3.4.3

For $\delta > 0$ we set

$$\bar{M}^a(\delta) := \{x \in \bar{M} \mid f(x) \leq a, \varphi(x) \geq \delta\} \text{ and}$$

$$(\bar{M}^a)_0^\delta := \{x \in \bar{M} \mid f(x) \leq a, 0 \leq \varphi(x) \leq \delta\}.$$

First, we prove that there exists a real number $\bar{\delta} > 0$ such that

$$\bar{M}^a(\bar{\delta}) \text{ is a strong deformation retract of } \bar{M}^a. \quad (3.81)$$

Since $\text{lev}(\bar{M}, a)$ does not contain any KKT-points, we obtain due to the continuity and compactness arguments that there exists $\bar{\delta} > 0$ such that every $\bar{x} \in \text{lev}(\bar{M}, a) \cap (\bar{M}^a)_0^\delta$ is not a KKT-point for the following optimization problem:

$$\overline{\text{GSIP}}(\bar{x}) : \text{ minimize } f(x) \text{ s.t. } x \in \bar{M}(\bar{x})$$

with

$$\bar{M}(\bar{x}) := \{x \in \mathbb{R}^n \mid \varphi(x) - \varphi(\bar{x}) \geq 0\}.$$

Note that $\overline{M}(\bar{x})$ is the feasible set of $\overline{\text{GSIP}}$ w.r.t. the perturbed data functions $g_k^{\bar{x}}(x, y) := g_k(x, y) - \varphi(\bar{x})$. We write $V^{\bar{x}}(\cdot)$ for the corresponding set $V(\cdot)$ as in (3.4) w.r.t. $g^{\bar{x}}(\cdot, \cdot)$.

Thus, for $\bar{x} \in \text{lev}(\overline{M}, a) \cap (\overline{M}^a)_\delta^0$ we obtain the existence of a vector $\xi_{\bar{x}} \in \mathbb{R}^n$ such that

$$D^T f(\bar{x}) \cdot \xi_{\bar{x}} < 0 \text{ and}$$

$$v \cdot \xi_{\bar{x}} > 0 \text{ for all } v \in V^{\bar{x}}(\bar{x}).$$

Moreover, for $\bar{x} \in (\overline{M}^a)_0^\delta$ with $f(x) < a$ we also obtain the existence of a vector $\xi_{\bar{x}} \in \mathbb{R}^n$ such that

$$v \cdot \xi_{\bar{x}} > 0 \text{ for all } v \in V^{\bar{x}}(\bar{x}).$$

It is due to the stability property of Sym-MFCQ w.r.t. C_s^1 -topology (cf. Theorem 3.2.5). Note that C_s^1 -topology coincides with the usual C^1 -topology on the compact set $\overline{M}$. Here, $\bar{\delta}$ can be taken smaller, if needed.

Further, we proceed analogously as in the proof of Deformation Theorem 3.4.2(b) to construct a C^∞ -flow $\Psi(t, \cdot)$ on $(\overline{M}^a)_0^\delta$ such that:

- $\Psi(t, x) \in \overline{M}^a$ for all $t \in [0, \bar{t}]$, $x \in (\overline{M}^a)_0^\delta$ and
- $\Psi(\bar{t}, x) \in \overline{M}^a(\varphi(x) + \varepsilon)$ for all $x \in (\overline{M}^a)_0^\delta$

with some $\varepsilon > 0$ and $\bar{t} > 0$.

As in the proof of Deformation Theorem 3.4.2(b) we see that the flow Ψ induces a strong retraction mapping between $\overline{M}^a(\bar{\delta})$ and $\overline{M}^a$.

We claim that the same flow Ψ induces a strong retraction mapping between $\overline{M}^a(\bar{\delta})$ and M^a itself. In fact, from the estimation of o-terms in the proof of Deformation Theorem 3.4.2(b) we see that

$$\Psi(t, x) \in \text{int}(\overline{M}) \text{ for all } t \in (0, \bar{t}], x \in (\overline{M}^a)_0^\delta.$$

Moreover, Theorem 3.2.36 implies that $\text{int}(\overline{M}) = \text{int}M$. Hence,

$$\Psi(t, x) \in \text{int}(M) \text{ for all } t \in (0, \bar{t}], x \in (\overline{M}^a)_0^\delta.$$

This shows that $\overline{M}^a(\bar{\delta})$ is a strong deformation retract of M^a .

Altogether, we have: $\overline{M}^a(\bar{\delta})$ is a strong deformation retract of both $\overline{M}^a$ and M^a . Consequently, $\overline{M}^a(\bar{\delta})$ is homotopy-equivalent to $\overline{M}^a$ and M^a . Since homotopy-equivalence is an equivalence relation, we obtain that $\overline{M}^a$ and M^a are homotopy-equivalent. $\square$

Proof of Theorem 3.4.1

The assertions follow directly from Theorems 3.4.2 and 3.4.3. We only note that the cell attachment on a homotopy-equivalent space is induced via the corresponding homotopy mapping. $\square$

Chapter 4

Mathematical programming problems with vanishing constraints

4.1 Applications and examples

We consider the following mathematical programming problem with vanishing constraints (MPVC):

$$\text{MPVC: } \min f(x) \text{ s.t. } x \in M[h, g, H, G] \quad (4.1)$$

with

$$M[h, g, H, G] := \{x \in \mathbb{R}^n \mid H_m(x) \geq 0, H_m(x)G_m(x) \leq 0, m = 1, \dots, k, \\ h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J\},$$

where $h := (h_i, i \in I)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|I|})$, $g := (g_j, j \in J)^T \in C^2(\mathbb{R}^n, \mathbb{R}^{|J|})$, $H := (H_m, m = 1, \dots, k)^T, G := (G_m, m = 1, \dots, k)^T \in C^2(\mathbb{R}^n, \mathbb{R}^k)$, $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $|I| \leq n$, $k \geq 0$, $|J| < \infty$. For simplicity, we write M for $M[h, g, H, G]$ if no confusion is possible.

MPVC was introduced in [1] as a model for structural and topology optimization. It is motivated by the fact that the constraint G_m does not play any role whenever H_m is active. We refer to [43, 44, 45, 46, 57, 56] for more details on optimality conditions, constraint qualifications, sensitivity and numerical methods for MPVC. Note that additional constraints $G_m(x) \geq 0, m = 1, \dots, k$ would restrict MPVC to a so-called mathematical program with complementarity constraints (MPCC). In addition to MPCC feasible set, M is glued together from manifold pieces of **different dimensions** along their strata. Indeed, a typical MPVC feasible set

$$\mathbb{V} := \{(x, y) \mid x \geq 0, xy \geq 0\}$$

is depicted in Figure 12.

Figure 12: $\mathbb{V}$ solution set of the basic vanishing constraint relation

It represents the solution set of the basic vanishing constraint relations and exhibits 1- and 2-dimensional parts glued together at $(0, 0)$.

Truss topology optimization

The following application of truss topology optimization is from [1].

The problem is to construct the optimal design of a truss structure. Let us consider a set of potential bars which are defined by the coordinates of their end nodes. For each potential bar, material parameters are given (Young's modulus E_i , relative moment of inertia s_i , stress bounds $\sigma_i^t > 0$ $\sigma_i^c < 0$ for tension and compression, respectively). These parameters are needed for the formulation of constraints preventing structural failure in the case when the potential bar is realized as a real bar. The latter is the case if the calculated cross-sectional area a_i is positive. Finally, boundary conditions and external loads at some of the nodes are given. The problem is now to find crosssectional areas a_i for each potential bar such that failure of the whole structure is prevented, the external load is carried by the structure, and a suitable objective function is minimal. The latter is usually the total weight of the structure or its deformation energy. In view of a practical realization of the calculated structure after optimization, one hopes that the optimal design will make use of only a few of the potential bars. Such a behavior is typical in applied truss topology optimization problems. The main difficulty in formulating (and solving) the problem lies in the fact that constraints on structural failure can be formulated in a well-defined way only if there is some material giving mechanical response. However, most potential bars will possess a zero cross-section at the optimizer. Hence, the truss topology optimization problem might be formulated as MPVC:

$$\text{TRUSS-TOP: } \underset{(a \times u) \in \mathbb{R}^M \times \mathbb{R}^d}{\text{minimize}} f(a, u) \text{ s.t.}$$

$$g(a, u) \leq 0, K(a)u = f^{ext},$$

$$\begin{aligned}
a_i &\geq 0, \quad i = 1, \dots, M, \\
\sigma_i^c &\leq \sigma(a, u) \leq \sigma_i^t \text{ if } a_i > 0, \quad i = 1, \dots, M, \\
f_i^{int}(a, u) &\geq f_i^{buck}(a, u) \text{ if } a_i > 0, \quad i = 1, \dots, M.
\end{aligned}$$

Here, the vector $a \in \mathbb{R}^M$ contains the vector of cross-sectional areas of the potential bars. $u \in \mathbb{R}^d$ denotes the vector of nodal displacements of the structure under load, where d is the so-called degree of freedom of the structure, i.e. the number of free nodal displacement coordinates. The state variable u serves as an auxiliary variable. The objective function f expresses structural weight. The nonlinear system of equations $K(a)u = f^{ext}$ symbolizes force equilibrium of (given) external loads $f^{ext} \in \mathbb{R}^d$ and internal forces expressed via Hooke's law in terms of displacements and cross-sections. The matrix $K(a) \in \mathbb{R}^{d \times d}$ is the global stiffness matrix corresponding to the structure a . The constraint $g(a, u) \leq 0$ is a resource constraint. If $a_i > 0$, then $\sigma_i(a, u) \in \mathbb{R}$ is the stress along the i -th bar. Similarly, if $a_i > 0$, $f_i^{int}(a, u) \in \mathbb{R}$ denotes the internal force along the i -th bar, and $f_i^{buck}(a)$ corresponds to the permitted Euler buckling force. Then, the constraints on stresses and on local buckling make sense only if $a_i > 0$. Therefore, they must vanish from the problem if $a_i = 0$.

4.2 Critical point theory

Our goal is the investigation of MPVC from a topological point of view. To this end we introduce the new notion of a T-stationary point for MPVC (see Definition 4.2.1). It turns out that the concept of T-stationarity is the adequate stationarity concept for topological considerations. In fact, we introduce the letter 'T' for a stationarity concept which is **topologically** relevant, rather than giving a tight first order condition for local minimizers (see also discussion below).

Further, We study the behavior of the topological properties of lower level sets

$$M^a := \{x \in M \mid f(x) \leq a\}$$

for MPVC, as the level $a \in \mathbb{R}$ varies. In particular, within this context, we present two basic theorems from Morse Theory (cf. [61, 91]). First, we show that, for $a < b$, the set M^a is a strong deformation retract of M^b if the (compact) set

$$M_a^b := \{x \in M \mid a \leq f(x) \leq b\}$$

does not contain T-stationary points (see Theorem 4.2.11(a)). Second, if M_a^b contains exactly one (nondegenerate) T-stationary point, then M^b is shown to be homotopy equivalent to M^a with a q -cell attached (see Theorem 4.2.11(b)). Here, the dimension q is the T-index (cf. Definition 4.2.1, 4.2.3).

We refer to [21] for details.

T-stationarity

Given $\bar{x} \in M$, we define the following (active) index sets:

$$J_0 = J_0(\bar{x}) := \{j \in J \mid g_j(\bar{x}) = 0\},$$

$$I_{0+} = I_{0+}(\bar{x}) := \{m \in \{1, \dots, k\} \mid H_m(\bar{x}) = 0, G_m(\bar{x}) > 0\},$$

$$I_{0-} = I_{0-}(\bar{x}) := \{m \in \{1, \dots, k\} \mid H_m(\bar{x}) = 0, G_m(\bar{x}) < 0\},$$

$$I_{+0} = I_{+0}(\bar{x}) := \{m \in \{1, \dots, k\} \mid H_m(\bar{x}) > 0, G_m(\bar{x}) = 0\},$$

$$I_{00} = I_{00}(\bar{x}) := \{m \in \{1, \dots, k\} \mid H_m(\bar{x}) = 0, G_m(\bar{x}) = 0\}.$$

We call $J_0(\bar{x})$ the active inequality index set and $I_{00}(\bar{x})$ the bi-active index set at $\bar{x}$. Note, that locally around $\bar{x}$, for $m \in I_{0+}$ the function H_m behaves like an ordinary equality constraint ($H_m(x) = 0$). For $m \in I_{0-}$ or $m \in I_{+0}$ the

functions H_m and G_m behave locally like inequality constraints ($H_m(x) \geq 0$ or $G_m(x) \leq 0$, respectively).

Further, we recall the well-known Linear Independence Constraint Qualification (LICQ) for MPVC (e.g. [1]), which is said to hold at $\bar{x} \in M$ if the vectors

$$\begin{aligned} D^T h_i(\bar{x}), i \in I, D^T H_m(\bar{x}), m \in I_{0+}, \\ D^T g_j(\bar{x}), j \in J_0, D^T H_m(\bar{x}), m \in I_{0-}, D^T G_m(\bar{x}), m \in I_{+0}, \\ D^T H_m(\bar{x}), D^T G_m(\bar{x}), m \in I_{00} \end{aligned}$$

are linearly independent.

We introduce the notion of a T-stationary point which is crucial for the following.

Definition 4.2.1 (T-stationary point)

A point $\bar{x} \in M$ is called T-stationary for MPVC if there exist real numbers $\bar{\lambda}_i, i \in I, \bar{\alpha}_m, m \in I_{0+}, \bar{\mu}_j, j \in J_0, \bar{\beta}_m, m \in I_{0-}, \bar{\gamma}_m, m \in I_{+0}, \bar{\delta}_m^H, \bar{\delta}_m^G, m \in I_{00}$ (Lagrange multipliers), such that:

$$\begin{aligned} Df(\bar{x}) &= \sum_{i \in I} \bar{\lambda}_i Dh_i(\bar{x}) + \sum_{m \in I_{0+}} \bar{\alpha}_m DH_m(\bar{x}) + \\ &+ \sum_{j \in J_0} \bar{\mu}_j Dg_j(\bar{x}) + \sum_{m \in I_{0-}} \bar{\beta}_m DH_m(\bar{x}) + \sum_{m \in I_{+0}} \bar{\gamma}_m DG_m(\bar{x}) + \\ &+ \sum_{m \in I_{00}} (\bar{\delta}_m^H DH_m(\bar{x}) + \bar{\delta}_m^G DG_m(\bar{x})), \end{aligned} \quad (4.2)$$

$$\bar{\mu}_j \geq 0 \text{ for all } j \in J_0, \quad (4.3)$$

$$\bar{\beta}_m \geq 0 \text{ for all } m \in I_{0-}, \quad (4.4)$$

$$\bar{\gamma}_m \leq 0 \text{ for all } m \in I_{+0}, \quad (4.5)$$

$$\bar{\delta}_m^G \leq 0 \text{ and } \bar{\delta}_m^H \cdot \bar{\delta}_m^G \geq 0 \text{ for all } m \in I_{00}. \quad (4.6)$$

In the case where LICQ holds at $\bar{x} \in M$, the Lagrange multipliers in (4.2) are uniquely determined.

Given a T-stationary point $\bar{x} \in M$ for MPVC, we set:

$$\begin{aligned} M(\bar{x}) := \{x \in \mathbb{R}^n \mid & h_i(x) = 0, i \in I, H_m(x) = 0, m \in I_{0+}, g_j(x) = 0, j \in J_0, \\ & H_m(x) = 0, m \in I_{0-}, G_m(x) = 0, m \in I_{+0}, \\ & H_m(x) = 0, G_m(x) = 0, m \in I_{00}\}. \end{aligned}$$

Obviously, $M(\bar{x}) \subset M$ and, in the case where LICQ holds at $\bar{x}$, $M(\bar{x})$ is locally at $\bar{x}$ a C^2 -manifold.

Definition 4.2.2 (Nondegenerate T-stationary point)

A T -stationary point $\bar{x} \in M$ with Lagrange multipliers as in Definition 4.2.1 is called nondegenerate if the following conditions are satisfied:

ND1: LICQ holds at $\bar{x}$,

ND2: $\bar{\mu}_j > 0$ for all $j \in J_0$,
 $\bar{\beta}_m > 0$ for all $m \in I_{0-}$, $\bar{\gamma}_m < 0$ for all $m \in I_{+0}$,

ND3: $D^2L(\bar{x})|_{T_{\bar{x}}M(\bar{x})}$ is nonsingular,

ND4: $\bar{\delta}_m^H < 0$ and $\bar{\delta}_m^G < 0$ for all $m \in I_{00}$.

Here, the matrix D^2L stands for the Hessian of the Lagrange function L ,

$$\begin{aligned} L(x) := & f(x) - \sum_{i \in I} \bar{\lambda}_i h_i(x) - \sum_{m \in I_{0+}} \bar{\alpha}_m H_m(x) - \\ & - \sum_{j \in J_0} \bar{\mu}_j g_j(x) - \sum_{m \in I_{0-}} \bar{\beta}_m H_m(x) - \sum_{m \in I_{+0}} \bar{\gamma}_m G_m(x) - \\ & - \sum_{m \in I_{00}} (\bar{\delta}_m^H H_m(x) - \bar{\delta}_m^G G_m(x)) \end{aligned} \quad (4.7)$$

and $T_{\bar{x}}M(\bar{x})$ denotes the tangent space of $M(\bar{x})$ at $\bar{x}$,

$$\begin{aligned} T_{\bar{x}}M(\bar{x}) := \{ \xi \in \mathbb{R}^n \mid & Dh_i(\bar{x})\xi = 0, i \in I, \\ & DH_m(\bar{x})\xi = 0, m \in I_{0+}, \\ & Dg_j(\bar{x})\xi = 0, j \in J_0, \\ & DH_m(\bar{x})\xi = 0, m \in I_{0-}, \\ & DG_m(\bar{x})\xi = 0, m \in I_{+0}, \\ & DH_m(\bar{x})\xi = 0, DG_m(\bar{x})\xi = 0, m \in I_{00} \}. \end{aligned}$$

Condition ND3 means that the matrix $V^T D^2L(\bar{x})V$ is nonsingular, where V is some matrix whose columns form a basis for the tangent space $T_{\bar{x}}M(\bar{x})$.

Definition 4.2.3 (T-index)

Let $\bar{x} \in M$ be a nondegenerate T -stationary point with Lagrange multipliers as in Definition 4.2.2. The number of negative eigenvalues of $D^2L(\bar{x})|_{T_{\bar{x}}M(\bar{x})}$ in ND3 is called the quadratic index (QI) of $\bar{x}$. The number of negative pairs $(\bar{\delta}_m^H, \bar{\delta}_m^G)$, $m \in I_{00}$ in ND4 equals $|I_{00}|$ and is called the bi-active index (BI) of $\bar{x}$. The number $(QI + BI)$ is called the T -index of $\bar{x}$.

Note that in the absence of bi-active vanishing constraints, the T-index has only the QI-part and coincides with the well-known quadratic index of a nondegenerate Karush-Kuhn-Tucker-point in nonlinear programming or, equivalently, with the Morse index (cf. [61, 80, 91]). Also note, that the bi-active index BI is completely determined by the cardinality of I_{00} , in contrast to, e.g., the bi-active index for MPCCs as defined in [67].

The following proposition uses the T-index for the characterization of a local minimizer.

Proposition 4.2.4

- (i) *Assume that $\bar{x}$ is a local minimizer for MPVC and that LICQ holds at $\bar{x}$. Then, $\bar{x}$ is a T-stationary point for MPVC.*
- (ii) *Let $\bar{x}$ be a nondegenerate T-stationary point for MPVC. Then, $\bar{x}$ is a local minimizer for MPVC if and only if its T-index is equal to zero.*

Proof. (i) From [1] it is known that under LICQ a local minimizer $\bar{x}$ for MPVC is a strongly stationary point, i.e. (4.2)-(4.5) hold and

$$\bar{\delta}_m^G = 0 \text{ and } \bar{\delta}_m^H \geq 0 \text{ for all } m \in I_{00}. \quad (4.8)$$

Clearly, a strongly stationary point is T-stationary as well.

(ii) Let $\bar{x}$ be a nondegenerate T-stationary local minimizer for MPVC. As in (i), we claim that $\bar{x}$ is also strongly stationary. Comparing ND4 and (4.8) we see that $BI = |I_{00}| = 0$. Then, locally around $\bar{x}$ MPVC behaves like an ordinary nonlinear program, and using standard results on the quadratic index we obtain that $QI = 0$. The other direction is trivial. $\square$

The next genericity and stability results justify the LICQ assumption as well as the introduction of nondegeneracy for T-stationary points in MPVC.

Theorem 4.2.5 (Genericity and Stability)

- (i) *Let $\mathcal{F}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those (h, g, H, G) for which LICQ holds at all points $x \in M[h, g, H, G]$. Then, $\mathcal{F}$ is C_s^2 -open and -dense.*
- (ii) *Let $\mathcal{D}$ denote the subset of $C^2(\mathbb{R}^n, \mathbb{R}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|I|}) \times C^2(\mathbb{R}^n, \mathbb{R}^{|J|}) \times C^2(\mathbb{R}^n, \mathbb{R}^k) \times C^2(\mathbb{R}^n, \mathbb{R}^k)$ consisting of those problem data (f, h, g, H, G) for which each T-stationary point is nondegenerate. Then, $\mathcal{D}$ is C_s^2 -open and -dense.*

Proof. (i) We define the following set:

$$M_{\text{DISJ}} := \{x \in \mathbb{R}^n \mid \max\{H_m(x), G_m(x)\} \geq 0, m = 1, \dots, k, \\ h_j(x) = 0, i \in I, g_j(x) \geq 0, j \in J\}.$$

M_{DISJ} is the feasible set of a disjunctive optimization problem (cf. [69]). We obtain from the corresponding results on disjunctive optimization that the subset of problem data for which LICQ holds for all $x \in M_{\text{DISJ}}$ is C_s^2 -dense and C_s^2 -open (see [69], Lemmata 2.4 and 2.5.). Recalling, that the notions of LICQ for disjunctive optimization problems and MPVCs are the same, and that M is a subset of M_{DISJ} , the desired result follows immediately.

(ii) The proof is based on the application of the Jet Transversality Theorem, for details see e.g. [61]. For subsets $\tilde{J} \subseteq J$ and $\tilde{H}, \tilde{G} \subseteq \{1, \dots, k\}$, and sets $D_{\tilde{J}} \subseteq \tilde{J}$, $D_{\tilde{H}} \subseteq \tilde{H}$ and $D_{\tilde{G}} \subseteq \tilde{G}$, and $r \in \{0, \dots, \dim(T_{\bar{x}}M(\bar{x}))\}$ we consider the set Γ of x such that the following conditions are satisfied:

$$(m1) \quad g_j(x) = H_i(x) = G_l(x) = 0, \text{ for all } j \in \tilde{J}, i \in \tilde{H}, l \in \tilde{G}.$$

$$(m2) \quad Df(x) \in \text{span} \left\{ \begin{array}{l} Dg_j(x), j \in \tilde{J} \setminus D_{\tilde{J}}, \\ DH_i(x), i \in \tilde{H} \setminus D_{\tilde{H}}, \\ DG_l(x), l \in \tilde{G} \setminus D_{\tilde{G}} \end{array} \right\}.$$

$$(m3) \quad \text{The matrix } D^2L(x)|_{T_{\bar{x}}M(\bar{x})} \text{ has rank } r.$$

Now it suffices to show that Γ is generically empty, whenever one of the sets $D_{\tilde{J}}$, $D_{\tilde{H}}$ or $D_{\tilde{G}}$ is non-empty or the rank r of the matrix in (m3) is not full. It would mean, respectively, that a Lagrange multiplier in the equality (4.2) vanishes (cf. ND2, ND4) or the rank condition ND3 fails to hold.

In fact, the available degrees of freedom of the variables involved in Γ are n . The loss of freedom, caused by (m1) is at least $d := |\tilde{J}| + |\tilde{H}| + |\tilde{G}|$, and the loss of freedom caused by (m2) is at least (supposing that the gradients on the right-hand side are linearly independent (ND1) and the sets $D_{\tilde{J}}$, $D_{\tilde{H}}$, $D_{\tilde{G}}$ are empty) $n - d$. Hence, the total loss of freedom is n . We conclude that a further nondegeneracy would exceed the total available degree of freedom n . In virtue of the Jet Transversality Theorem, generically the set Γ must be empty.

For the openness result we can argue in a standard way (see e. g. [61]). Locally, T-stationarity can be rewritten via stable equations. Then, the Implicit Function Theorem for Banach-spaces can be applied to follow non-degenerate T-stationary points w.r.t. (local) C^2 -perturbations of defining functions. Then a standard globalization procedure exploiting the specific properties of the strong C^2 -topology can be used to construct a (global) C_s^2 -neighborhood of problem data for which the non-degeneracy property is stable. $\square$

Morse Lemma for MPVC

For the proof of the above mentioned results we locally describe the MPVC feasible set under the Linear Independence Constraint Qualification (see Lemma 4.2.7). Moreover, an equivariant Morse Lemma for MPVC is derived in order to obtain suitable normal forms for the objective function at nondegenerate T-stationary points (see Theorem 4.2.10).

Without loss of generality (w.l.o.g.), we assume that at the particular point of interest $\bar{x} \in M$ it holds:

$$J_0 = \{1, \dots, |J_0|\},$$

$$I_{0+} = \{1, \dots, |I_{0+}|\},$$

$$I_{0-} = \{|I_{0+}| + 1, \dots, |I_{0+}| + |I_{0-}|\},$$

$$I_{+0} = \{|I_{0+}| + |I_{0-}| + 1, \dots, |I_{0+}| + |I_{0-}| + |I_{+0}|\},$$

$$I_{00} = \{|I_{0+}| + |I_{0-}| + |I_{+0}| + 1, \dots, |I_{0+}| + |I_{0-}| + |I_{+0}| + |I_{00}|\}.$$

We put $s := |I| + |I_{0+}|$, $r := s + |J_0| + |I_{0-}|$, $q := r + |I_{+0}|$, $p := n - q - 2|I_{00}|$.

For the proof of Theorem 4.2.11 we need a local description of the MPVC feasible set under LICQ.

Definition 4.2.6 *The feasible set M admits a local C^r -coordinate system of $\mathbb{R}^n$ ($r \geq 1$) at $\bar{x}$ by means of a C^r -diffeomorphism $\Phi : U \longrightarrow V$ with open $\mathbb{R}^n$ -neighborhoods U and V of $\bar{x}$ and 0 , respectively, if it holds:*

$$(i) \quad \Phi(\bar{x}) = 0,$$

$$(ii) \quad \Phi(M \cap U) = \left(\{0_s\} \times \mathbb{H}^{|J_0|+|I_{0-}|} \times (-\mathbb{H})^{|I_{+0}|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p \right) \cap V.$$

Lemma 4.2.7 *Suppose that LICQ holds at $\bar{x} \in M$. Then M admits a local C^2 -coordinate system of $\mathbb{R}^n$ at $\bar{x}$.*

Proof. Choose vectors $\xi_l \in \mathbb{R}^n$, $l = 1, \dots, p$, which form - together with the vectors

$$\begin{aligned} & D^T h_i(\bar{x}), i \in I, D^T H_m(\bar{x}), m \in I_{0+}, \\ & D^T g_j(\bar{x}), j \in J_0, D^T H_m(\bar{x}), m \in I_{0-}, D^T G_m(\bar{x}), m \in I_{+0}, \\ & D^T H_m(\bar{x}), D^T G_m(\bar{x}), m \in I_{00} \end{aligned}$$

- a basis for $\mathbb{R}^n$. Next we put

$$\left. \begin{aligned}
y_i &:= h_i(x), i \in I \\
y_{|I|+m} &:= H_m(x), m \in I_{0+} \\
y_{|I|+|I_{0+}|+j} &:= g_j(x), j \in J_0 \\
y_{|I|+|J_0|+m} &:= H_m(x), m \in I_{0-} \\
y_{|I|+|J_0|+m} &:= G_m(x), m \in I_{+0} \\
y_{|I|+|J_0|+2m-1} &:= H_m(x), m \in I_{00} \\
y_{|I|+|J_0|+2m} &:= G_m(x), m \in I_{00} \\
y_{n-p+l} &:= \xi_l^T(x - \bar{x}), l = 1, \dots, p.
\end{aligned} \right\} \quad (4.9)$$

or, shortly,

$$y = \Phi(x). \quad (4.10)$$

Note that $\Phi \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $\Phi(\bar{x}) = 0$ and the Jacobian matrix $D\Phi(\bar{x})$ is nonsingular (in virtue of LICQ and the choice of ξ_l , $l = 1, \dots, p$). By means of the Implicit Function Theorem there exist open neighborhoods U of $\bar{x}$ and V of 0 such that $\Phi : U \rightarrow V$ is a C^2 -diffeomorphism. By shrinking U , if necessary, we can guarantee that $J_0(x) \subset J_0$, $I_{0-}(x) \subset I_{0-}$, $I_{+0}(x) \subset I_{+0}$ and $I_{00}(x) \subset I_{00}$ for all $x \in M \cap U$. Thus, the property (ii) in Definition 4.2.6 follows directly from the definition of Φ . $\square$

Definition 4.2.8 *We will refer to the C^2 -diffeomorphism Φ defined by (4.9), (4.10) as standard diffeomorphism.*

Remark 4.2.9 *It follows from the proof of Lemma 4.2.7 that the Lagrange multipliers at a nondegenerate T -stationary point are the corresponding partial derivatives of the objective function in new coordinates given by the standard diffeomorphism (cf. [63], Lemma 2.2.1). Moreover, the Hessian with respect to the last p coordinates corresponds to the restriction of the Lagrange function's Hessian on the respective tangent space (cf. [63], Lemma 2.2.10).*

We derive an equivariant Morse Lemma for MPVC in order to obtain suitable normal forms for the objective function at nondegenerate T -stationary points.

Theorem 4.2.10 (Morse Lemma for MPVC)

Suppose that $\bar{x}$ is a nondegenerate T -stationary point for MPVC with quadratic index QI , bi-active index BI and T -index $= QI + BI$. Then, there exists a local C^1 -coordinate system $\Psi : U \rightarrow V$ of $\mathbb{R}^n$ around $\bar{x}$ (according to Definition 4.2.6) such that:

$$f \circ \Psi^{-1}(0_s, y_{s+1}, \dots, y_n) =$$

$$f(\bar{x}) + \sum_{i=1}^{|J_0|+|I_{0-}|} y_{i+s} - \sum_{j=1}^{|I_{+0}|} y_{j+r} - \sum_{m=1}^{|I_{00}|} (y_{2j-1+q} + y_{2j+q}) + \sum_{k=1}^p \pm y_{k+n-p}^2, \quad (4.11)$$

where $y \in \{0_s\} \times \mathbb{H}^{|J_0|+|I_{0-}|} \times (-\mathbb{H})^{|I_{+0}|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p$. Moreover, in (4.11) there are exactly $BI = |I_{00}|$ negative linear pairs and QI negative squares.

Proof. W.l.o.g., we may assume $f(\bar{x}) = 0$. Let $\Phi : U \longrightarrow V$ be a standard diffeomorphism according to Definition 4.2.8. We put $\bar{f} := f \circ \Phi^{-1}$ on the set $\left(\{0_s\} \times \mathbb{H}^{|J_0|+|I_{0-}|} \times (-\mathbb{H})^{|I_{+0}|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p\right) \cap V$. We may assume $s = 0$ from now on. In view of Remark 4.2.9 we have at the origin:

- (i) $\frac{\partial \bar{f}}{\partial y_i} > 0, i = 1, \dots, |J_0| + |I_{0-}|,$
- (ii) $\frac{\partial \bar{f}}{\partial y_{j+r}} < 0, j = 1, \dots, |I_{+0}|,$
- (iii) $\frac{\partial \bar{f}}{\partial y_{2m-1+q}} < 0$ and $\frac{\partial \bar{f}}{\partial y_{2m+q}} < 0$ for exactly BI indices $m = 1, \dots, |I_{00}|,$
- (iv) $\frac{\partial \bar{f}}{\partial y_{k+n-p}} = 0, k = 1, \dots, p$ and $\left(\frac{\partial^2 \bar{f}}{\partial y_{k_1+n-p} \partial y_{k_2+n-p}}\right)_{1 \leq k_1, k_2 \leq p}$ is a non-singular matrix with QI negative eigenvalues.

We denote $\bar{f}$ by f . Under the following coordinate transformations the set $\mathbb{H}^{|J_0|+|I_{0-}|} \times (-\mathbb{H})^{|I_{+0}|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p$ will be transformed in itself (equivariance). As an abbreviation we put $y = (Y_{n-p}, Y^p)$, where $Y_{n-p} = (y_1, \dots, y_{n-p})$ and $Y^p = (y_{n-p+1}, \dots, y_n)$. We write

$$f(Y_{n-p}, Y^p) = f(0, Y^p) + \int_0^1 \frac{d}{dt} f(tY_{n-p}, Y^p) dt = f(0, Y^p) + \sum_{i=1}^{n-p} y_i d_i(y),$$

where $d_i \in C^1, i = 1, \dots, n-p$.

In view of (iv) we may apply the Morse Lemma on the C^2 -function $f(0, Y^p)$ (cf. [61], Theorem 2.8.2) without affecting the coordinates Y_{n-p} . The corresponding coordinate transformation is of class C^1 . Denoting the transformed functions f, d_j again by f, d_j , we obtain:

$$f(y) = \sum_{i=1}^{n-p} y_i d_i(y) + \sum_{k=1}^p \pm y_{k+n-p}^2.$$

Note that $d_i(0) = \frac{\partial f}{\partial y_i}(0)$, $i = 1, \dots, n - p$. Recalling (i)-(iii), we have

$$y_i |d_i(y)|, \quad i = 1, \dots, n - p, \quad y_j, \quad j = n - p + 1, \dots, n \quad (4.12)$$

as new local C^1 -coordinates. Denoting the transformed function f again by f and, recalling the signs in (i)-(iii), we obtain (4.11). Here, the coordinate transformation Ψ is understood as the composition of all previous ones. $\square$

Deformation and Cell-Attachment

We state and prove the main deformation and cell-attachment theorem for MPVC. Recall that for $a, b \in \mathbb{R}$, $a < b$ the sets M^a and M_a^b are defined as follows:

$$M^a := \{x \in M \mid f(x) \leq a\}$$

and

$$M_a^b := \{x \in M \mid a \leq f(x) \leq b\}.$$

Theorem 4.2.11 *Let M_a^b be compact and suppose that LICQ is satisfied at all points $x \in M_a^b$.*

- (a) **(Deformation Theorem)** *If M_a^b does not contain any T-stationary point for MPVC, then M^a is a strong deformation retract of M^b .*
- (b) **(Cell-attachment Theorem)** *If M_a^b contains exactly one (nondegenerate) T-stationary point for MPVC, say $\bar{x}$, and if $a < f(\bar{x}) < b$ and the T-index of $\bar{x}$ is equal to q , then M^b is homotopy-equivalent to M^a with a q -cell attached.*

Proof. (a) Let $\bar{x} \in M_a^b$. After a coordinate transformation with the standard diffeomorphism from Definition 4.2.6 and Remark 4.2.9 we may assume that $\bar{x} = 0$ and locally $M = \{0_s\} \times \mathbb{H}^{|J_0|+|I_{0-}|} \times (-\mathbb{H})^{|I_{+0}|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p$. Due to Remark 4.2.9 and the fact that $\bar{x}$ is not a T-stationary point (cf. Definition 4.2.1) one of the following cases holds:

- a) There exists $j \in \{1, \dots, p\}$ with $\frac{\partial f}{\partial y_{n-p+j}}(0) \neq 0$.
- b) There exists $j \in \{1, \dots, |J_0| + |I_{0-}| \}$ with $\frac{\partial f}{\partial y_{s+j}}(0) < 0$.

- c) There exists $j \in \{1, \dots, |I_{+0}|\}$ with $\frac{\partial f}{\partial y_{r+j}}(0) > 0$.
- d) There exists $m \in I_{00}$ with $\frac{\partial f}{\partial y_{q+2m}}(0) > 0$.
- e) There exists $m \in I_{00}$ with $\frac{\partial f}{\partial y_{q+2m-1}}(0) > 0$ and $\frac{\partial f}{\partial y_{q+2m}}(0) < 0$.

We set

$$D := \{x \in M_a^b \mid \text{one of the cases a)-d) holds}\} \quad \text{and} \quad L := M_a^b \setminus D.$$

The proof consists of the local argument and its globalization.

Local argument. We prove: For each $\bar{x} \in M_a^b$ there exists a $\mathbb{R}^n$ -neighborhood $\mathcal{U}_{\bar{x}}$ of $\bar{x}$, a $t_{\bar{x}} > 0$ and a flow $\Psi^{\bar{x}} : [0, t_{\bar{x}}] \times M^b \cap \mathcal{U}_{\bar{x}} \rightarrow M$, $(t, x) \mapsto \Psi^{\bar{x}}(t, x)$, with:

1. $\Psi^{\bar{x}}(0, x) = x$, for all $x \in M^b \cap \mathcal{U}_{\bar{x}}$.
2. $\Psi^{\bar{x}}(t_2, \Psi^{\bar{x}}(t_1, x)) = \Psi^{\bar{x}}(t_1 + t_2, x)$, for all $x \in M^b \cap \mathcal{U}_{\bar{x}}$ and $t_1, t_2 \geq 0$ with $t_1 + t_2 \in [0, t_{\bar{x}}]$.
3. $f(\Psi^{\bar{x}}(t, x)) \leq f(x) - t$, for all $x \in M^b \cap \mathcal{U}_{\bar{x}}$ and $t \in [0, t_{\bar{x}}]$.
4. If $\bar{x} \in D$, then $\Psi^{\bar{x}}$ is a C^2 -flow corresponding to a C^1 -vector field. If $\bar{x} \in L$, then $\Psi^{\bar{x}}$ is a Lipschitz flow.

We consider the constructions of the local flows in Cases a)-e).

Cases a)-c): We can use standard methods to construct a local flow induced by a C^1 -vector field. To see this, note that the behavior of partial derivatives in Cases a)-c) give us a descent direction which is - due to the structure of M in local coordinates - feasible for $t_{\bar{x}} > 0$. (This is a standard construction for generalized manifolds with boundary; see [61], Theorems 2.7.6 and 3.2.26 for details; cf. also with the proof of Theorem 3.2 in [67]).

If the violation of T-stationarity is exclusively due to the coordinates belonging to the set $\mathbb{V}^{|I_{00}|}$, i.e. one of the cases d) and e) holds, we have to construct a new flow.

Case d): Using a (additional) local coordinate transformation leaving M invariant - analogously as in the proof of Theorem 4.2.10 - we obtain:

$$f(y) = y_{q+2m} + f(y_1, \dots, y_{q+2m-1}, 0, y_{q+2m+1}, \dots, y_n).$$

We define a local vector field as $\tilde{F}^{\bar{x}}(y) := (0, \dots, 0, -1, 0, \dots, 0)^T$. After the inverse change of local coordinates, $\tilde{F}^{\bar{x}}$ induces the flow which fits the local argument.

Case e): Again, as in the proof of Theorem 4.2.10 we may assume that

$$f(y) = y_{q+2m-1} - y_{q+2m} + f(y_1, \dots, y_{q+2m-2}, 0, 0, y_{q+2m+1}, \dots, y_n).$$

We define a two-dimensional flow $\Phi(t, z)$ for $z = (z_1, z_2) \in \mathbb{V}$ as follows:

$$\Phi(t, z_1, z_2) := \begin{cases} \begin{pmatrix} \max \left\{ 0, \left(1 - \frac{t}{z_1 - z_2}\right) \cdot z_1 \right\} \\ \left[\left(1 - \frac{t}{z_1 - z_2}\right) \cdot z_2 \right]^- + [t - (z_1 - z_2)]^+ \end{pmatrix} & \text{for } z_2 < 0, \\ \begin{pmatrix} 0 \\ t - (z_1 - z_2) \end{pmatrix} & \text{for } z_2 \geq 0. \end{cases}$$

Here, $[\cdot]^-$ is the negative and $[\cdot]^+$ the positive part of a real number.

Note that the flow Φ is Lipschitz on $\mathbb{R} \times \mathbb{V}$. Moreover, due to the definition of Φ we get that the flow $\Psi^{\bar{x}}$ defined (again in new coordinates) by

$$\Psi_i(y) := \begin{cases} y_i & \text{for } i \in \{1, \dots, n\} \setminus \{q+2m-1, q+2m\} \\ \Phi_1(y_{q+2m-1}) & \text{for } i = q+2m-1 \\ \Phi_2(y_{q+2m}) & \text{for } i = q+2m \end{cases}.$$

fits the local argument. Here, Ψ_i and Φ_i stands for the i -th component of Ψ and Φ , respectively.

Globalization. Now we construct a global flow Ψ on M_a^b . Suppose for a moment that there exists a flow Ψ_L on a neighborhood $\mathcal{U}_L$ of L with the properties (i) to (iv). We choose a smaller neighborhood $\mathcal{W}_L$ of L such that the closure $\overline{\mathcal{W}_L}$ of $\mathcal{W}_L$ is contained in $\mathcal{U}_L$. Furthermore, we choose an arbitrary open covering $\{\mathcal{U}_x \mid x \in M_a^b \setminus \mathcal{U}_L\}$ of $M_a^b \setminus \mathcal{U}_L$ induced by the domains of the C^2 -flows corresponding to the cases a) to d). Since $M_a^b \setminus \mathcal{U}_L$ is compact we find a finite subcovering $\{\mathcal{U}_x \mid x \in \bar{D}\}$. Here $\bar{D}$ is a finite subset of D . W.l.o.g. we may assume that for all $x \in \bar{D}$ the closure $\overline{\mathcal{U}_x}$ of $\mathcal{U}_x$ is disjoint with $\overline{\mathcal{W}_L}$. By construction it holds that $\{\mathcal{U}_x \mid x \in \bar{D}\} \cup (\mathcal{U}_L \setminus \overline{\mathcal{W}_L})$ is a finite open covering of $M_a^b \setminus \mathcal{W}_L$. The crucial argument is now that outside the set L the flow Ψ_L is induced by a C^1 -vector field. (Note that Φ only has a singularity for $t = z_1 - z_2$.) Therefore, we can construct a flow on $M_a^b \setminus \overline{\mathcal{W}_L}$ by using a C^∞ -partition of unity subordinate to the open covering $\{\mathcal{U}_x \mid x \in \bar{D}\} \cup (\mathcal{U}_L \setminus \overline{\mathcal{W}_L})$. This enables us to construct a global C^1 -vector field. The flow Ψ_D obtained by integration fulfills the desired properties. (See [61] Theorem 3.3.14 for details on this procedure.) By construction Ψ_L and Ψ_D can be glued together to one flow Ψ on M_a^b .

We obtain for $x \in M_a^b$ a unique $t_a(x) > 0$ with $\Psi(t_a(x), x) \in M^a$ from the properties of Ψ (which are induced by local properties of the flows Ψ^x). It is

not hard (but technical) to realize that $t_a : x \mapsto t_a(x)$ is Lipschitz. Finally, we define $r : [0, 1] \times M^b \rightarrow M^b$ as follows:

$$r(\tau, x) := \begin{cases} x & \text{for } x \in M^a, \tau \in [0, 1] \\ \Psi(\tau \cdot t_a(x), x) & \text{for } x \in M_a^b, \tau \in [0, 1] \end{cases}.$$

The mapping r provides that M^a is a strong deformation retract of M^b .

It remains to construct the flow Ψ_L . Since this construction is highly technical we only present a short outline. The main idea is to construct the flow along strata inside L ; here the strata are induced by all possible subsets of active constraints $H_1, G_1, \dots, H_m, G_m$. Along a given stratum we find a differentiable family of standard coordinate systems (see Lemma 4.2.7). This enables us to define a flow along this stratum by just applying flows like Φ on fixed components which depend on the coordinate system. By introducing notions of a distance from a point in the embedding space to the strata we can construct homotopies (via Lipschitz continuous time-scaling) between the different branches of the stratification and the corresponding flows. (For details on such constructions by the aid of tube systems we refer to [28].)

(b) Due to the Deformation Theorem (Theorem 4.2.11(a)) we may assume that, w.l.o.g., a and b are small enough such that we can work in local coordinates. Therefore, we consider the normal form (2.19) from Theorem 4.2.10:

$$f(y) = \sum_{i=1}^{|J_0|+|I_0|-|} y_{s+i} - \sum_{j=1}^{|I_0|} y_{r+j} - \sum_{m=1}^{|I_{00}|} (y_{q+2m-1} + y_{q+2m}) + \sum_{l=1}^p \pm y_{n-p+l}^2,$$

with $y \in M := \{0_s\} \times \mathbb{H}^{|J_0|+|I_0|-|} \times (-\mathbb{H})^{|I_0|} \times \mathbb{V}^{|I_{00}|} \times \mathbb{R}^p$.

We set

$$M_{\text{MPCC}} := \{0_s\} \times \mathbb{H}^{|J_0|+|I_0|-|} \times (-\mathbb{H})^{|I_0|} \times (\partial\mathbb{H}^2)^{|I_{00}|} \times \mathbb{R}^p.$$

Note that M_{MPCC} differs from M by appearance of $(\partial\mathbb{H}^2)^{|I_{00}|}$ instead of $\mathbb{V}^{|I_{00}|}$.

For $c \in \mathbb{R}$ it holds: $M_{\text{MPCC}}^c := \{y \in M_{\text{MPCC}} \mid f(y) \leq c\}$ is a strong deformation retract of $M^c := \{y \in M \mid f(y) \leq c\}$. In fact, we define a mapping $g : M^c \rightarrow M_{\text{MPCC}}^c$ with

$$y_i \mapsto \begin{cases} 0 & i \in \{q+2m \mid m = 1, \dots, |I_{00}|\} \text{ and } y_i < 0 \\ y_i & \text{else} \end{cases}.$$

We see that there is a (convex combination) homotopy between g and the identity on M^c . If $(y_{q+2m-1}, y_{q+2m}) \in \mathbb{V}$ then $(y_{q+2m-1}, 0) \in \partial\mathbb{H}^2$ and, moreover, $f(g(y)) \leq f(y)$ for all $y \in M^c$, i.e. g , in fact, maps to M_{MPCC}^c . Hence, M_{MPCC}^c is a strong deformation retract of M^c .

According to Definitions 1.1 and 1.2 in [67] it holds that $\bar{y} = 0$ is a nondegenerate C-stationary point of the MPCC defined by f and the set M_{MPCC} . Since $\bar{y} = 0$ is the only C-stationary point, Theorem 3.2 b) from [67] implies that M_{MPCC}^b is homotopy equivalent to M_{MPCC}^a with a $\tilde{q}$ -cell attached. Note that $\tilde{q}$ is the so called C-index for the corresponding MPCC. Here, we have that the C-index $\tilde{q}$ w.r.t. MPCC coincides with the T-index q w.r.t. MPVC. Hence:

$$M_{\text{MPCC}}^b \simeq (M_{\text{MPCC}}^a \text{ with a } q\text{-cell attached}).$$

We know from the considerations above, that M^c is homotopy equivalent to M_{MPCC}^c for $c = a, b$. Furthermore, we note that the cell attachment on a homotopy-equivalent space is induced via the corresponding homotopy mapping. Finally, using the fact that homotopy equivalence is an equivalence relation we obtain that M^b is homotopy equivalent to M^a with a q -cell attached. $\square$

Different stationarity concepts

We briefly review well-known definitions of various stationarity concepts and connections between them (cf. [1, 43, 44, 45, 46, 57]).

Definition 4.2.12 *Let $\bar{x} \in M$.*

(i) *$\bar{x}$ is called weakly stationary if (4.2)-(4.5) hold and*

$$\bar{\delta}_m^G \leq 0 \text{ for all } m \in I_{00}.$$

(ii) *$\bar{x}$ is called M-stationary if (4.2)-(4.5) hold and*

$$\bar{\delta}_m^G \leq 0 \text{ and } \bar{\delta}_m^G \cdot \bar{\delta}_m^H = 0 \text{ for all } m \in I_{00}.$$

(iii) *$\bar{x}$ is called strongly stationary if (4.2)-(4.5) hold and*

$$\bar{\delta}_m^G = 0 \text{ and } \bar{\delta}_m^H \geq 0 \text{ for all } m \in I_{00}.$$

Note that a strongly stationary point is M-stationary, and the latter is T-stationary. We see that M- and strongly stationary points describe local minima tighter than T-stationary points. Moreover, strong stationarity is the tightest condition for a local minimizer under LICQ. It is worth to mention that M-stationarity exhibits a full calculus in the sense of Mordukhovich (cf. [92]). The scheme in Figure 13 illustrate the above stationarity concepts.

Figure 13: Stationarity concepts in MPVC

However, M- and strong stationarity exclude T-stationary points with $BI > 0$. These points are also crucial for the topological structure of MPVC (cf. Cell-attachment Theorem). For global optimization, points of T-index 1 play an important role (see the discussion in Section 1). We emphasize that among the points of T-index 1 from a topological point of view there is no substantial difference between the points with $BI = 1, QI = 0$ and $BI = 0, QI = 1$. It is worth to mention that a linear descent direction might exist in a nondegenerate T-stationary point with positive T-index. In particular, at points with $BI = 1, QI = 0$ there are exactly two directions of linear descent. Both of them are important from a global point of view. On the other hand, among weakly stationary points there are those with negative and positive Lagrange multipliers corresponding to the same bi-active vanishing constraint. Due to the Deformation Theorem such points are irrelevant for the topological structure of MPVC.

We mention that the nondegeneracy assumption (as in Definition 4.2.2, ND4) can not be stated for M- and strongly stationary points w.r.t bi-active vanishing constraints. This means that these points are singularities. Moreover, local minima for MPVC with bi-active vanishing constraints do not occur generically. We claim that their classification is sophisticated and might be established via singularity theory.

Links to MPCC

We point out that in [67] the analogous stationarity concept for MPCCs turned out to be C-stationarity. Indeed, the MPCC feasible set can be described by nonsmooth equality constraints of minimum-type. Moreover, generically the MPCC feasible set is a Lipschitz manifold of an appropriate dimension, that is, each nonsmooth equality constraint causes one loss of freedom (see [68]). This permits to use Clarke subdifferentials of these equality constraints to formulate the stationarity conditions, namely, the C-stationarity. As C-stationarity is the topologically relevant stationarity concept for MPCCs, we consider it as T-stationarity in the MPCC setting.

In contrast to the MPCC case, the MPVC feasible set (under LICQ) is not a Lipschitz manifold, but a set glued together from manifold pieces of **different dimensions** along their strata. Rather than by applying a general stationarity concept to MPVC, like C-stationarity for MPCCs, T-stationarity for MPVCs is motivated by understanding the geometrical properties of a typical MPVC feasible set $\mathbb{V}$ directly, where $\mathbb{V}$ represents the solution set of the basic vanishing constraint relations $x \geq 0, xy \geq 0$.

A further analogy between C-stationarity for MPCCs and T-stationarity for MPVCs is established via convergence theory of certain regularization methods. In fact, the MPCC regularization method from [109] yields C-stationary points as a limits of KKT points of the regularized problems ([109, Theorem 5.1]). The analogous limit points of an adaptation of this method to MPVCs from [47] are T-stationary, as expressed in [48].

Chapter 5

Bilevel optimization

5.1 Applications and examples

We consider bilevel optimization problems as hierarchical problems of two decision makers, the so-called leader and follower. The follower selects his decision knowing the choice of the leader, whereas the latter has to anticipate the followers response in his decision. Bilevel programming problems have been studied in the monographs [5] and [17]. We model the bilevel optimization problem in the so-called optimistic formulation. To this aim, assume that the follower solves the parametric optimization problem (lower level problem L)

$$L(x) : \min_y g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, j \in J \quad (5.1)$$

and that the leader's optimization problem (upper level problem U) is the following

$$U : \min_{(x,y)} f(x, y) \quad \text{s.t.} \quad y \in \text{Argmin } L(x). \quad (5.2)$$

Above we have $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and the real valued mappings $f, g, h_j, j \in J$ belong to $C^3(\mathbb{R}^n \times \mathbb{R}^m)$, the space of three times continuously differentiable mappings. $\text{Argmin } L(x)$ denotes the solution set of the optimization problem $L(x)$. For simplicity, additional (in)equality constraints in defining U are omitted.

In what follows, we present Stackelberg games as a classical application of bilevel programming. Further, optimistic and pessimistic versions of bilevel optimization are compared. It turns out that the main difficulty in studying both versions lies in the fact that the lower level contains a global constraint.

In fact, a point (x, y) is feasible if y solves a parametric optimization problem $L(x)$. It gives rise to study the structure of the bilevel feasible set

$$M := \{(x, y) \mid y \in \text{Argmin } L(x)\}.$$

Finally, we give some guiding examples on the possible structure of M in case of $\dim(x) = 1$.

We refer to [73] for details.

Stackelberg game

In a Stackelberg game there are two decision makers, the so-called leader and follower. The leader can adjust his variable $x \in \mathbb{R}^n$ at the upper level. This variable x influences as a parameter the follower's decision process $L(x)$ at the lower level. One might think of $L(x)$ being a minimization procedure w.r.t. the follower's variable $y \in \mathbb{R}^m$. Then, the follower chooses $y(x)$, a solution of $L(x)$ which is in general not unique. This solution $y(x)$ is anticipated by the leader at the upper level. After evaluating the leader's objective function $f(x, y(x))$, new adjustment of x is performed. The game circles till the leader obtains an optimal parameter x w.r.t. his objective function f . The scheme in Figure 14 describes a Stackelberg game.

Figure 14: Stackelberg game

The difficulties by modelling such a two-hierarchical Stackelberg game comes from its intrinsic dynamical behavior. In fact, the choice at the lower level is usually not unique. Hence, the feedback at the upper level can not be prescribed a-priori.

Pessimistic vs. Optimistic Versions

We model a Stackelberg game via bilevel optimization. For that, let the lower level be given by

$$L(x) : \min_y g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, j \in J.$$

Using the follower's objective function f , we define

$$\varphi_o(x) := \min_y \{f(x, y) \mid y \in \text{Argmin } L(x)\} \quad \text{and}$$

$$\varphi_p(x) := \max_y \{f(x, y) \mid y \in \text{Argmin } L(x)\}.$$

φ_o (resp. φ_p) is referred to an optimistic (resp. pessimistic) objective function value at the upper level. In the definition of φ_o a best reply $y \in \text{Argmin } L(x)$ from the leader's point of view is assumed. It happens in the case of cooperation between the leader and the follower. In opposite, if there is no cooperation between the players, the leader uses φ_p . In both situations φ_o or φ_p are in general nonsmooth functions. To obtain a solution of a bilevel optimization problem φ_o (resp. φ_p) are to be minimized w.r.t. x . The scheme in Figure 15 describes both optimistic and pessimistic perspective.

Figure 15: Optimistic vs. pessimistic perspective

The following example clarifies the difference between optimistic and pessimistic objective function value.

Example 5.1.1 Let $\dim(x) = 1$, $J = \emptyset$ and the graphs of $g(x, \cdot)$ be depicted in Figure 16 (for x close to $\bar{x}$).

Figure 16: Graphs of $g(x, \cdot)$ from Example 5.1.1

Clearly, $\text{Argmin } L(\bar{x}) = \{\bar{y}_1, \bar{y}_2\}$. For x close to $\bar{x}$ we obtain (cf. Figure 16):

$$\text{Argmin } L(x) = \begin{cases} \{y_1(x)\} & \text{if } x < \bar{x}, \\ \{y_2(x)\} & \text{if } x > \bar{x}. \end{cases}$$

Hence, we get with the leader's objective function f :

$$\varphi_o(x) = \begin{cases} f(x, y_1(x)), & \text{if } x < \bar{x} \\ \min \{f(\bar{x}, \bar{y}_1), f(\bar{x}, \bar{y}_2)\}, & \text{if } x = \bar{x} \\ f(x, y_2(x)), & \text{if } x > \bar{x} \end{cases}$$

$$\varphi_p(x) = \begin{cases} f(x, y_1(x)), & \text{if } x < \bar{x} \\ \max \{f(\bar{x}, \bar{y}_1), f(\bar{x}, \bar{y}_2)\}, & \text{if } x = \bar{x} \\ f(x, y_2(x)), & \text{if } x > \bar{x} \end{cases}$$

We see that the only difference between $\varphi_o(x)$ and $\varphi_p(x)$ is the value at $\bar{x}$.

Example 5.1.1 suggests that the main difficulty of bilevel programming lies in the structure of its feasible set M rather than in optimistic or pessimistic perspective. Thus, we concentrate on the optimistic formulation in the subsequent analysis.

Examples

We present several typical examples for the case $\dim(x) = 1$. They motivate our results on the structure of the bilevel feasible set M . In all examples the origin 0_{1+m} solves the bilevel problem U . Each example exhibits some kind of degeneracy in the lower level $L(x)$. Recall that $\dim(x) = 1$ throughout the paper.

Example 5.1.2

$$f(x, y) := -x + 2y_1 + \varphi(y_2, \dots, y_m) \text{ with } \varphi \in C^3(\mathbb{R}^{m-1}, \mathbb{R}),$$

$$g(x, y) := (x - y_1)^2 + \sum_{j=2}^m y_j^2, \quad J = \{1\} \text{ and } h_1(x, y) := y_1.$$

The degeneracy in the lower level $L(x)$ is the lack of strict complementarity at the origin 0_m .

The bilevel feasible set M becomes:

$$M = \{(x, \max(x, 0), 0, \dots, 0) \mid x \in \mathbb{R}\}.$$

This example refers to Type 2 in the classification of Section 5.2.

Example 5.1.3

$$f(x, y) := x + \sum_{j=1}^m y_j, \quad g(x, y) := -y_1,$$

$$J = \{1\}, \quad h_1(x, y) := x - \sum_{j=1}^m y_j^2.$$

The degeneracy in the lower level $L(x)$ is the violation of the so-called Mangasarian-Fromovitz Constraint Qualification (MFCQ) (see Section 5.2) at the origin 0_m . Moreover, the minimizer 0_m is a so-called Fritz-John point, but not a Karush-Kuhn-Tucker (KKT)-point.

The bilevel feasible set M is a (half-)parabola:

$$M = \{(x, \sqrt{x}, 0, \dots, 0) \mid x \geq 0\}.$$

This example refers to Type 4 in the classification of Section 5.2.

Example 5.1.4

$$f(x, y) := x + \sum_{j=1}^m y_j, \quad g(x, y) := \sum_{j=1}^m y_j, \quad J = \{1, \dots, m, m+1\},$$

$$h_j(x, y) := y_j, \quad j = 1, \dots, m, \quad h_{m+1}(x, y) := x - \sum_{j=1}^m y_j.$$

The degeneracy in $L(0)$ is again the violation of the MFCQ at the origin 0_m . However, in contrast to Example 5.1.3, the minimizer 0_m is a KKT-point now.

The bilevel feasible set M becomes:

$$M = \{(x, 0, \dots, 0) \mid x \geq 0\}.$$

This example refers to Type 5-1 in the classification of Section 5.2.

Example 5.1.5

$$f(x, y) := -x + 2 \sum_{j=1}^m y_j, \quad g(x, y) := \sum_{j=1}^m j y_j, \quad J = \{1, \dots, m, m+1\},$$

$$h_j(x, y) := y_j, \quad j = 1, \dots, m, \quad h_{m+1}(x, y) = -x + \sum_{j=1}^m y_j.$$

The degeneracy in $L(0)$ is the violation of the so-called Linear Independence Constraint Qualification (LICQ) at the origin 0_m , whereas MFCQ is satisfied.

The bilevel feasible set M becomes:

$$M = \{(x, \max(x, 0), 0, \dots, 0) \mid x \in \mathbb{R}\}.$$

This example refers to Type 5-2 in the classification of Section 5.2.

Note that the feasible set M exhibits a kink in Examples 5.1.2, 5.1.5, whereas it has a boundary in Examples 5.1.3, 5.1.4. Moreover, the minimizer 0_m in $L(0)$ is strongly stable (in the terminology of Kojima, [80]) in Examples 5.1.2, 5.1.5, but not in Examples 5.1.3, 5.1.4.

We note that, despite of degeneracies in the lower level, the structure of the bilevel feasible set M with its kinks and boundaries remains stable under small C_s^3 -perturbations of the defining functions.

5.2 Five types in parametric optimization

We consider the lower level problem $L(\cdot)$ in a one-dimensional parametric optimization setting, i.e. $\dim(x) = 1$:

$$L(x) : \min_y g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, j \in J.$$

We denote its feasible set by

$$M(x) := \{y \in \mathbb{R}^m \mid h_j(x, y) \geq 0, j \in J\}$$

and for $\bar{y} \in M(\bar{x})$ the active index set by

$$J_0(\bar{x}, \bar{y}) := \{j \in J \mid h_j(\bar{x}, \bar{y}) = 0\}.$$

Definition 5.2.1 (Generalized Critical Point)

A point $\bar{y} \in M(\bar{x})$ is called a generalized critical point (g.c. point) for $L(\bar{x})$ if the set of vectors

$$\{D_y g(\bar{x}, \bar{y}), D_y h_j(\bar{x}, \bar{y}), j \in J_0(\bar{x}, \bar{y})\} \quad (5.3)$$

is linearly dependent.

The critical set for $L(\cdot)$ is given by

$$\Sigma := \{(x, y) \in \mathbb{R}^{1+m} \mid y \text{ is g.c. point for } L(x)\}.$$

In [62] it is shown that generically each point of Σ is one of the Types 1-5. In what follows, we shortly recall the Types 1-5 and consider the structure of Σ locally around particular g.c. points being local minimizers for $L(\cdot)$. Here, we focus on such parts of Σ which correspond to (local) minimizers, i.e.

$$\Sigma_{min} := \{(x, y) \in \Sigma \mid y \text{ is a local minimizer for } L(x)\}$$

in a neighborhood of $(\bar{x}, \bar{y}) \in \Sigma_{min}$. We refer to [60] for the indication of the latter issue.

Points of Type 1

A point $(\bar{x}, \bar{y}) \in \Sigma$ is of Type 1 if $\bar{y}$ is a nondegenerate critical point for $L(\bar{x})$. It means that the following conditions ND1-ND3 hold.

ND1: Linear Independence Constraint Qualification (LICQ) is satisfied at $(\bar{x}, \bar{y})$, i.e. the set of vectors

$$\{D_y h_j(\bar{x}, \bar{y}), j \in J_0(\bar{x}, \bar{y})\} \quad (5.4)$$

is linearly independent.

From (5.3) and (5.4) we see that there exist (Lagrange multipliers) $\bar{\mu}_j$, $j \in J_0(\bar{x}, \bar{y})$, such that

$$D_y g(\bar{x}, \bar{y}) = \sum_{j \in J_0(\bar{x}, \bar{y})} \bar{\mu}_j D_y h_j(\bar{x}, \bar{y}). \quad (5.5)$$

ND2: $\bar{\mu}_j \neq 0$, $j \in J_0(\bar{x}, \bar{y})$,

ND3: $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$ is nonsingular.

Here, the matrix $D_{yy}^2 L(\bar{x}, \bar{y})$ stands for the Hessian w.r.t. y variables of the Lagrange function L ,

$$L(x, y) := g(x, y) - \sum_{j \in J_0(\bar{x}, \bar{y})} \bar{\mu}_j h_j(x, y). \quad (5.6)$$

and $T_{\bar{y}} M(\bar{x})$ denotes the tangent space of $M(\bar{x})$ at $\bar{y}$,

$$T_{\bar{y}} M(\bar{x}) := \{\xi \in \mathbb{R}^m \mid D_y h_j(\bar{x}, \bar{y}) \cdot \xi = 0, j \in J_0(\bar{x}, \bar{y})\}. \quad (5.7)$$

Condition ND3 means that the matrix $V^T D_{yy}^2 L(\bar{x}, \bar{y}) V$ is nonsingular, where V is some matrix whose columns form a basis for the tangent space $T_{\bar{y}} M(\bar{x})$.

The linear index LI, resp. linear coindex LCI, is defined to be the number of $\bar{\mu}_j$ in (5.5) which are negative, resp. positive. The quadratic index QI, resp. quadratic coindex QCI, is defined to be the number of negative, resp. positive eigenvalues of $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$.

Characteristic numbers: LI, LCI, QI, QCI

It is well-known that conditions ND1-ND3 allow us to apply the implicit function theorem and obtain unique C^2 -mappings $y(x)$, $\mu_j(x)$, $j \in J_0(\bar{x}, \bar{y})$ in an open neighborhood of $\bar{x}$. It holds: $y(\bar{x}) = \bar{y}$ and $\mu_j(\bar{x}) = \bar{\mu}_j$, $j \in J_0(\bar{x}, \bar{y})$, moreover, for x sufficiently close to $\bar{x}$ the point $y(x)$ is a nondegenerate critical point for $L(x)$ with Lagrange multipliers $\mu_j(x)$, $j \in J_0(\bar{x}, \bar{y})$ having the same indices LI, LCI, QI, QCI as $\bar{y}$. Hence, locally around $(\bar{x}, \bar{y})$ we can parametrize the set Σ by means of a unique C^2 -map $x \mapsto (x, y(x))$. If $\bar{y}$ is additionally a local minimizer for $L(\bar{x})$, i.e. LI=QI=0, then we get locally around $(\bar{x}, \bar{y})$:

$$\Sigma_{min} = \{(x, y(x)) \mid x \text{ sufficiently close to } \bar{x}\}.$$

Points of Type 2

A point $(\bar{x}, \bar{y}) \in \Sigma$ is of Type 2 if the following conditions A1-A6 hold:

A1: LICQ is satisfied at $(\bar{x}, \bar{y})$

A2: $J_0(\bar{x}, \bar{y}) \neq \emptyset$

After renumbering we may assume that $J_0(\bar{x}, \bar{y}) = \{1, \dots, p\}$, $p \geq 1$. Then, we have

$$D_y g(\bar{x}, \bar{y}) = \sum_{j=1}^p \bar{\mu}_j D_y h_j(\bar{x}, \bar{y}). \quad (5.8)$$

A3: In (5.8) exactly one of the Lagrange multipliers vanishes.

After renumbering we may assume that $\bar{\mu}_p = 0$ and $\bar{\mu}_j \neq 0$, $j = 1, \dots, p-1$.

Let L and $T_{\bar{y}} M(\bar{x})$ be defined as in (5.6) and (5.7), respectively.

A4: $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$ is nonsingular

We set

$$T_{\bar{y}}^+ M(\bar{x}) := \{\xi \in \mathbb{R}^m \mid D_y h_j(\bar{x}, \bar{y}) \cdot \xi = 0, j \in J_0(\bar{x}, \bar{y}) \setminus \{p\}\}.$$

A5: $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}}^+ M(\bar{x})}$ is nonsingular

Let W be a matrix with m rows, whose columns form a basis of the linear space $T_{\bar{y}}^+ M(\bar{x})$. Put $\Phi = (h_1, \dots, h_{p-1})^T$ and define the $m \times 1$ -vectors:

$$\alpha := - \left[(D_y \Phi \cdot D_y^T \Phi)^{-1} \cdot D_y \Phi \right]^T \cdot D_x \Phi,$$

$$\beta = -W \cdot (W^T \cdot D_{yy}^2 L \cdot W)^{-1} \cdot W^T [D_{yy}^2 L \cdot \alpha + D_x D_y^T L]$$

Note that all partial derivatives are evaluated at $(\bar{x}, \bar{y})$. Next, we put

$$\gamma := D_x h_p(\bar{x}, \bar{y}) + D_y h_p(\bar{x}, \bar{y})(\alpha + \beta).$$

A6: $\gamma \neq 0$

Let δ_1 and δ_2 denote the number of negative eigenvalues of $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}}^+ M(\bar{x})}$ and $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$, respectively, and put $\delta := \delta_1 - \delta_2$.

Characteristic numbers: $\text{sign}(\gamma)$, δ

We proceed with the local analysis of the set Σ in a neighborhood of $(\bar{x}, \bar{y})$.

a) We consider the following associated optimization problem (without the p -th constraint):

$$\tilde{L}(x) : \quad \underset{y \in \mathbb{R}^m}{\text{minimize}} \quad g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, j \in J \setminus \{p\}. \quad (5.9)$$

It is easy to see that $\bar{y}$ is a nondegenerate critical point for $\tilde{L}(\bar{x})$ due to A1, A3, A5. As in Type 1 we get a unique C^2 -map $x \mapsto (x, \tilde{y}(x))$. The latter curve belongs to Σ as far as $\psi(x)$ is nonnegative, where

$$\psi(x) := h_p(x, \tilde{y}(x)).$$

A few calculations show that

$$\frac{d\tilde{y}(\bar{x})}{dx} = \alpha + \beta \text{ and, hence, } \frac{d\psi(\bar{x})}{dx} = \gamma. \quad (5.10)$$

Consequently, if we walk along the curve $x \mapsto (x, \tilde{y}(x))$ as x increases, then at $x = \bar{x}$ we leave (enter) the feasible set $M(x)$ according to $\text{sign}(\gamma) = -1(+1)$ (cf. A6).

b) We consider the following associated optimization problem (with the p -th constraint as equality):

$$\hat{L}(x) : \quad \underset{y \in \mathbb{R}^m}{\text{minimize}} \quad g(x, y) \quad \text{s.t.} \quad h_j(x, y) \geq 0, \quad j \in J, \quad h_p(x, y) = 0. \quad (5.11)$$

It is easy to see that $\bar{y}$ is a nondegenerate critical point for $\hat{L}(\bar{x})$ due to A1, A3, A4. Using results for Type 1 we get a unique C^2 -map $x \mapsto (x, \hat{y}(x))$. Note that $h_p(x, \hat{y}(x)) \equiv 0$. Moreover, it can be calculated that

$$\text{sign}(\gamma) \cdot \text{sign} \left(\frac{d\mu_p(\bar{x})}{dx} \right) = -1 \text{ (resp. } +1) \text{ iff } \delta = 0 \text{ (resp. } \delta = 1). \quad (5.12)$$

Altogether, since the curve $x \mapsto (x, \tilde{y}(x))$ traverses the zero set " $h_p = 0$ " at $(\bar{x}, \bar{y})$ transversally (cf. A6), it follows that $x \mapsto (x, \tilde{y}(x))$ and $x \mapsto (x, \hat{y}(x))$ intersect at $(\bar{x}, \bar{y})$ under a nonvanishing angle. Obviously, in a neighborhood of $(\bar{x}, \bar{y})$ the set Σ consists of $x \mapsto (x, \hat{y}(x))$ and that part of $x \mapsto (x, \tilde{y}(x))$ on which h_p is nonnegative.

Let now additionally assume that $\bar{y}$ is a local minimizer for $L(\bar{x})$. Then, $\bar{\mu}_j > 0$, $j \in J_0(\bar{x}, \bar{y}) \setminus \{p\}$ in A3, and the matrix $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$ is positive definite in A4, hence, $\delta_2 = 0$.

We consider two cases for $\delta = 0$ or $\delta = 1$.

Case $\delta = 0$:

In this case $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}}^+ M(\bar{x})}$ is positive definite in A5. Hence, $\bar{y}$ is a strongly stable local minimizer for $L(\bar{x})$ (see [80] for details on the strong stability). Moreover, $\tilde{y}(x)$ is a local minimizer for $L(x)$ if $h_p(x, \tilde{y}(x)) > 0$. Otherwise, $\hat{y}(x)$ is a local minimizer for $L(x)$ since the corresponding Lagrange multiplier $\mu_p(x)$ becomes positive due to (5.12). Note that the sign of $h_p(x, \tilde{y}(x))$ is corresponding to $\text{sign}(\gamma)$ as obtained in (5.10).

Then, we get locally around $(\bar{x}, \bar{y})$:

$$\Sigma_{min} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} \tilde{y}(x), & x \leq \bar{x} \\ \hat{y}(x), & \bar{x} \leq x \end{cases} \right\} \text{ if } \text{sign}(\gamma) = -1$$

and

$$\Sigma_{min} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} \hat{y}(x), & x \leq \bar{x} \\ \tilde{y}(x), & \bar{x} \leq x \end{cases} \right\} \text{ if } \text{sign}(\gamma) = +1.$$

Case $\delta = 1$:

In this case $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}}^+ M(\bar{x})}$ has exactly one negative eigenvalue. Thus, we obtain that the optimal value of the following optimization problem is negative:

$$\underset{\xi \in \mathbb{R}^m}{\text{minimize}} \quad \xi^T \cdot D_{yy}^2 g(\bar{x}, \bar{y}) \cdot \xi \quad \text{s.t.} \quad \|\xi\| = 1, \xi \in T_{\bar{y}}^+ M(\bar{x}), D_y h_p(\bar{x}, \bar{y}) \cdot \xi \geq 0.$$

In view of that, at $(\bar{x}, \bar{y})$ we can find a quadratic descent direction ξ for $L(\bar{x})$. Thus, $\bar{y}$ is not a local minimizer for $L(\bar{x})$ which contradicts to the above assumption. We conclude that this case does not occur in Σ_{min} .

Points of Type 3

A point $(\bar{x}, \bar{y}) \in \Sigma$ is of Type 3 if the following conditions B1-B4 hold:

B1: LICQ is satisfied at $(\bar{x}, \bar{y})$

After renumbering we may assume in case $J_0(\bar{x}, \bar{y}) \neq \emptyset$ that $J_0(\bar{x}, \bar{y}) = \{1, \dots, p\}$, $p \geq 1$. Then, we have

$$D_y g(\bar{x}, \bar{y}) = \sum_{j=1}^p \bar{\mu}_j D_y h_j(\bar{x}, \bar{y}). \quad (5.13)$$

B2: In (5.13) we have $\bar{\mu}_j \neq 0$, $j = 1, \dots, p$.

Let L and $T_{\bar{y}} M(\bar{x})$ be defined as in (5.6) and (5.7), respectively.

B3: Exactly one eigenvalue of $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}} M(\bar{x})}$ vanishes.

Let V be a matrix, whose columns form a basis for the tangent space $T_{\bar{y}} M(\bar{x})$. According to B3, let w be a nonvanishing vector such that $V^T \cdot D_{yy}^2 L(\bar{x}, \bar{y}) \cdot V w = 0$, and put $v := V \cdot w$. Put $\Phi = (h_1, \dots, h_{p-1})^T$ and define

$$\beta_1 := v^T (D_{yyy}^3 L \cdot v) v - 3v^T D_{yy}^2 L \cdot \left((D_y \Phi \cdot D_y^T \Phi)^{-1} \cdot D_y \Phi \right) \cdot (v^T D_{yy}^2 \Phi v),$$

$$\beta_2 := D_x (D_y L \cdot v) - D_x^T \Phi \cdot \left((D_y \Phi \cdot D_y^T \Phi)^{-1} \cdot D_y \Phi \right) \cdot D_{yy}^2 L \cdot v.$$

Note that all partial derivatives are evaluated at $(\bar{x}, \bar{y})$. Next, we put

$$\beta := \beta_1 \cdot \beta_2.$$

B4: $\beta \neq 0$

Let α denote the number of negative eigenvalues of $D_{yy}^2 L(\bar{x}, \bar{y})|_{T_{\bar{y}}M(\bar{x})}$.

Characteristic numbers: $\text{sign}(\beta)$, α

It turns out that in a neighborhood of $(\bar{x}, \bar{y})$ the set Σ is a one-dimensional C^2 -manifold. Moreover, the parameter x , viewed as a function on Σ , has a (nondegenerate) local maximum, resp. local minimizer, at $(\bar{x}, \bar{y})$ according to $\text{sign}(\beta) = +1$, resp. $\text{sign}(\beta) = -1$. Consequently, the set Σ can be locally approximated by means of a parabola. In particular, if we approach the point $(\bar{x}, \bar{y})$ along Σ , the path of local minimizers (with $\text{QI} = \alpha = 0$) stops and the local minimizer switches into a saddlepoint (with $\text{QI} = \alpha + 1 = 1$). Moreover, note that at $(\bar{x}, \bar{y})$ there exists a unique (tangential) direction of cubic descent, hence, $\bar{y}$ can not be a local minimizer for $L(\bar{x})$. Hence, this case does not occur in $\Sigma_{\min}$.

Points of Type 4

A point $(\bar{x}, \bar{y}) \in \Sigma$ is of Type 4 if the following conditions C1-C6 hold:

C1: $J_0(\bar{x}, \bar{y}) \neq \emptyset$

After renumbering we may assume that $J_0(\bar{x}, \bar{y}) = \{1, \dots, p\}$, $p \geq 1$.

C2: $\dim \{ \text{span} \{ D_y h_j(\bar{x}, \bar{y}), j \in J_0(\bar{x}, \bar{y}) \} \} = p - 1$

C3: $p - 1 < m$

From C2 we see that there exist $\bar{\mu}_j$, $j \in J_0(\bar{x}, \bar{y})$, not all vanishing such that

$$\sum_{j=1}^p \bar{\mu}_j D_y h_j(\bar{x}, \bar{y}) = 0. \quad (5.14)$$

Note that the numbers $\bar{\mu}_j$, $j \in J_0(\bar{x}, \bar{y})$ are unique up to a common multiple.

C4: $\bar{\mu}_j \neq 0$, $j \in J_0(\bar{x}, \bar{y})$ and we normalize the $\bar{\mu}_j$'s by setting $\bar{\mu}_p = 1$

We define furthermore

$$L(x, y) := h_p(x, y) + \sum_{j=1}^{p-1} \bar{\mu}_j h_j(x, y) \text{ and}$$

$$T_{\bar{y}}M(\bar{x}) := \{ \xi \in \mathbb{R}^m \mid D_y h_j(\bar{x}, \bar{y}) \cdot \xi = 0, j \in J_0(\bar{x}, \bar{y}) \}$$

Let W be a matrix, whose columns form a basis for $T_{\bar{y}}M(\bar{x})$. Define

$$A := D_x L \cdot W^T \cdot D_{yy}^2 L \cdot W \text{ and } w := W^T \cdot D_y^T g,$$

all partial derivatives being evaluated at $(\bar{x}, \bar{y})$.

C5: A is nonsingular

Finally define

$$\alpha := w^T \cdot A^{-1} \cdot w.$$

C6: $\alpha \neq 0$

Let β denote the number of positive eigenvalues of A . Let γ be the number of negative $\bar{\mu}_j$, $j \in \{1, \dots, p-1\}$ and put $\delta := D_x L(\bar{x}, \bar{y})$.

Characteristic numbers: $\text{sign}(\alpha)$, $\text{sign}(\delta)$, γ , β

We proceed with the local analysis of the set Σ in a neighborhood of $(\bar{x}, \bar{y})$. Conditions C2, C4 and C5 imply that (locally around $(\bar{x}, \bar{y})$) at all points $(x, y) \in \Sigma$ - apart from $(\bar{x}, \bar{y})$ - LICQ holds. Moreover, the active set $J_0(\cdot)$ is locally constant ($= J_0(\bar{x}, \bar{y})$) on Σ . Having these facts in mind, we consider the following map $\Psi : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{p-1} \times \mathbb{R} \longrightarrow \mathbb{R}^m \times \mathbb{R}^p$:

$$\Psi(x, y, \mu, \lambda) := \begin{pmatrix} \lambda D_y g(x, y) + D_y h_p(x, y) + \sum_{j=1}^{p-1} \mu_j D_y h_j(x, y) \\ h_j(x, y) = 0, j = 1, \dots, p-1 \\ \lambda g(x, y) + h_p(x, y) + \sum_{j=1}^{p-1} \mu_j h_j(x, y) \end{pmatrix}.$$

Note that $\Psi(\bar{x}, \bar{y}, \bar{\mu}, 0) = 0$ and $D_{x,y,\mu} \Psi(\bar{x}, \bar{y}, \bar{\mu}, 0)$ is nonsingular due to C5 and C6. Hence, there exists the unique C^2 -mapping $\lambda \mapsto (x(\lambda), y(\lambda), \mu(\lambda))$ such that $\Psi(x(\lambda), y(\lambda), \mu(\lambda), \lambda) \equiv 0$ and $(x(0), y(0), \mu(0)) = (\bar{x}, \bar{y}, \bar{\mu})$. Further, it is not hard to see that locally around $(\bar{x}, \bar{y})$

$$\Sigma = \{(x(\lambda), y(\lambda)) \mid \lambda \text{ sufficiently close to } 0\}.$$

The Lagrange multipliers corresponding to $(x(\lambda), y(\lambda))$ are

$$\left(-\frac{\mu_j(\lambda)}{\lambda}, j = 1, \dots, p-1, -\frac{1}{\lambda} \right). \quad (5.15)$$

It turns out that in a neighborhood of $(\bar{x}, \bar{y})$ the set Σ is a one-dimensional C^2 -manifold. The parameter x , viewed as a function on Σ , has a (non-degenerate) local maximum, resp. local minimizer, at $(\bar{x}, \bar{y})$ according to $\text{sign}(\alpha) = +1$, resp. $\text{sign}(\alpha) = -1$. Consequently, the set Σ can be locally approximated by means of a parabola.

Let now additionally assume that $\bar{y}$ is a local minimizer for $L(\bar{x})$. Then, $\bar{\mu}_j > 0$, $j = 1, \dots, p-1$ in C4 and, hence,

$$\gamma = 0. \quad (5.16)$$

Moreover, the matrix $W^T \cdot D_{yy}^2 L \cdot W$ is negative definite. In particular, we get

$$\beta = \begin{cases} n - (p - 1) & \text{if } \text{sign}(\delta) = -1, \\ 0 & \text{if } \text{sign}(\delta) = 1 \end{cases} \quad (5.17)$$

We are interested in the local structure of Σ_{min} at $(\bar{x}, \bar{y})$. It is clear from (5.15) that λ must be nonpositive if following the branch of local minimizers.

We consider two cases with respect to $\text{sign}(\alpha)$ and $\text{sign}(\delta)$

Case 1: $\text{sign}(\alpha) = \text{sign}(\delta)$

A few calculations show that

$$D_\lambda g(x(\lambda), y(\lambda))_{\lambda=0} = -\alpha \cdot \delta.$$

Hence, $D_\lambda g(x(\lambda), y(\lambda))_{\lambda=0} < 0$ and $g(x(\cdot), y(\cdot))$ is strictly decreasing when passing $\lambda = 0$. Consequently, the possible branch of local minimizers corresponding to $\lambda \leq 0$ can not be one of global minimizers. We omit this case in view of our further interest in **global** minimizers in the context of bilevel programming problems.

Case 2: $\text{sign}(\alpha) \neq \text{sign}(\delta)$

In this case we get locally around $(\bar{x}, \bar{y})$:

$$\Sigma_{min} = \{(x(\lambda), y(\lambda)) \mid \lambda \leq 0\}.$$

In fact, for $\text{sign}(\alpha) = 1$ and $\text{sign}(\delta) = -1$ the linear and quadratic indices of $y(\lambda)$ for $L(x(\lambda))$, $\lambda < 0$ are

$$\text{LI} = \gamma = 0, \quad \text{QI} = n - p - \beta + 1 = n - p - (n - p + 1) + 1 = 0.$$

For $\text{sign}(\alpha) = -1$ and $\text{sign}(\delta) = 1$ the linear and quadratic indices of $y(\lambda)$ for $L(x(\lambda))$, $\lambda < 0$ are

$$\text{LI} = \gamma = 0, \quad \text{QI} = \beta = 0.$$

Confer (5.16) and (5.17) for the values of γ and β , respectively.

Points of Type 5

A point $(\bar{x}, \bar{y}) \in \Sigma$ is of Type 5 if the following conditions D1-D4 hold:

D1: $|J_0(\bar{x}, \bar{y})| = m + 1$

D2: The set of vectors

$$\{Dh_j(\bar{x}, \bar{y}), j \in J_0(\bar{x}, \bar{y})\}$$

is linearly independent (derivatives in $\mathbb{R}^{m+1}$)

After renumbering we may assume that $J_0(\bar{x}, \bar{y}) = \{1, \dots, p\}$, $p \geq 2$.

From D1, D2 we see that there exist μ_j , $j \in J_0(\bar{x}, \bar{y})$, not all vanishing such that

$$\sum_{j=1}^p \mu_j D_y h_j(\bar{x}, \bar{y}) = 0. \quad (5.18)$$

Note that the numbers μ_j , $j \in J_0(\bar{x}, \bar{y})$ are unique up to a common multiple.

D3: $\mu_j \neq 0$, $j \in J_0(\bar{x}, \bar{y})$

From D1, D2 it follows that there exist unique numbers β_j , $y \in J_0(\bar{x}, \bar{y})$ such that

$$Dg(\bar{x}, \bar{y}) = \sum_{j=1}^p \beta_j D h_j(\bar{x}, \bar{y}). \quad (5.19)$$

Put

$$\Delta_{ij} := \beta_i - \beta_j \cdot \frac{\mu_i}{\mu_j} \text{ for } i, j = 1, \dots, p$$

and let Δ be the $p \times p$ matrix with Δ_{ij} as its (i, j) -th element.

D4: All off-diagonal elements of Δ do not vanish

We set

$$L(\bar{x}, \bar{y}) = \sum_{j=1}^p \mu_j h_j(\bar{x}, \bar{y}).$$

From D2 we see that $D_x L(\bar{x}, \bar{y}) \neq 0$. We define:

$$\gamma_j := \text{sign}(\mu_j \cdot D_x L(\bar{x}, \bar{y})) \text{ for } j = 1, \dots, p.$$

By δ_j we denote the number of negative entries in the j -th column of Δ , $j = 1, \dots, p$.

Characteristic numbers: γ_j , δ_j , $j = 1, \dots, p$

We proceed with the local analysis of the set Σ in a neighborhood of $(\bar{x}, \bar{y})$. Conditions D1-D3 imply that locally around $(\bar{x}, \bar{y})$ at all points $(x, y) \in \Sigma \setminus \{(\bar{x}, \bar{y})\}$ LICQ holds. Combining (5.18) and (5.19) we obtain:

$$D_x g(\bar{x}, \bar{y}) = \sum_{j=1}^p \left(\beta_j - \beta_q \cdot \frac{\mu_j}{\mu_q} \right) D_x h_j(\bar{x}, \bar{y}), \quad q = 1, \dots, p. \quad (5.20)$$

These both facts imply that for all $(x, y) \in \Sigma \setminus \{(\bar{x}, \bar{y})\}$ in a neighborhood of $(\bar{x}, \bar{y})$:

$$\|J_0(x, y)\| = m \text{ and } J_0(x, y) = J_0(\bar{x}, \bar{y}) \setminus \{q\} \quad (5.21)$$

with some $q \in \{1, \dots, p\}$ (in general, depending on (x, y)).

We put

$$M_q := \{(x, y) \mid h_j(x, y) = 0, j \in J_0(\bar{x}, \bar{y}) \setminus \{q\}\} \text{ and}$$

$$M_q^+ := \{(x, y) \in M_q \mid h_q(x, y) \geq 0\}.$$

From (5.20) and (5.21) it is easy to see that locally around $(\bar{x}, \bar{y})$

$$\Sigma = \bigcup_{q=1}^p M_q^+.$$

The indices (LI, LCI, QI, QCI) along $M_q^+ \setminus \{(\bar{x}, \bar{y})\}$ are equal $(\delta_q, m - \delta_q, 0, 0)$. Let $q \in \{1, \dots, p\}$ be fixed. M_q is a one-dimensional C^3 -manifold due to D2. Since the set of vectors

$$\{D_y h_j(\bar{x}, \bar{y}), j \in J_0(\bar{x}, \bar{y}) \setminus \{q\}\}$$

is linearly independent, we can parametrize M_q by means of the unique C^3 -mapping $x \mapsto (x, y^q(x))$ with $y^q(\bar{x}) = \bar{y}$. A short calculation shows that

$$\text{sign} \left(\frac{dh_q(x, y^q(x))}{dx} \Big|_{x=\bar{x}} \right) = \gamma_q.$$

Hence, by increasing x , M_q^+ emanates from $(\bar{x}, \bar{y})$, resp. ends at $(\bar{x}, \bar{y})$ according to $\gamma_q = +1$, resp. $\gamma_q = -1$.

Let now additionally assume that $\bar{y}$ is a local minimizer for $L(\bar{x})$. For describing Σ_{min} we define the so-called Karush-Kuhn-Tucker subset

$$\Sigma_{KKT} := cl \{(x, y) \in \Sigma \mid (x, y) \text{ is of Type 1 with LI} = 0\}.$$

It is shown in [62, Theorem 4.1] that - generically - Σ_{KKT} is a one-dimensional (piecewise C^2 -) manifold with boundary. In particular, $(x, y) \in \Sigma_{KKT}$ is a boundary point iff at (x, y) we have: $J_0(x, y) \neq \emptyset$ and the Mangasarian-Fromovitz Constraint Qualification (MFCQ) fails to hold. We recall that MFCQ is said to be satisfied for (x, y) , $y \in M(x)$, if there exists a vector $\xi \in \mathbb{R}^m$ such that

$$D_y h_j(x, y) \cdot \xi > 0 \text{ for all } j \in J_0(x, y).$$

Now we consider two cases with respect to the signs of μ_j , $j \in J_0(\bar{x}, \bar{y})$:

Case 1: all μ_j , $j \in J_0(\bar{x}, \bar{y})$ have the same sign

Recalling (5.18) we obtain that MFCQ is not fulfilled at $(\bar{x}, \bar{y})$. Hence, $(\bar{x}, \bar{y})$ is a boundary point of Σ_{KKT} . Having in mind the formulas for the indices (LI= δ_q , LCI= $m - \delta_q$, QI=0, QCI=0) along $M_q^+ \setminus \{(\bar{x}, \bar{y})\}$ we obtain that $\delta_q = 0$ for some $q \in \{1, \dots, p\}$. Moreover, a simple calculation shows

$$\Delta_{ij} = -\frac{\mu_i}{\mu_j} \cdot \Delta_{ji}, \quad i, j = 1, \dots, p. \quad (5.22)$$

Since all μ_j , $j \in J_0(\bar{x}, \bar{y})$ have the same sign, we get from (5.22)

$$\text{sign}(\Delta_{ij}) = -\text{sign}(\Delta_{ji}), \quad i, j = 1, \dots, p.$$

Hence,

$$\delta_j > 0 \text{ for all } j \in \{1, \dots, p\} \setminus \{q\}.$$

Finally, in this case we get locally around $(\bar{x}, \bar{y})$:

$$\Sigma_{\min} = \{(x, y^q(x)) \mid x \geq \bar{x} \text{ (resp. } x \leq \bar{x}) \text{ if } \gamma_q = +1 \text{ (resp. } \gamma_q = -1), \delta_q = 0\}.$$

We refer to this case as Type 5-1.

Case 2: μ_j , $j \in J_0(\bar{x}, \bar{y})$ have different signs

The separation argument implies MFCQ to be satisfied at $(\bar{x}, \bar{y})$. Hence, a local minimizer $\bar{y}$ for $L(\bar{x})$ is also a KKT-point and $(\bar{x}, \bar{y}) \in \Sigma_{KKT}$. Due to MFCQ, $(\bar{x}, \bar{y})$ is not a boundary point of Σ_{KKT} . Thus, there exist $q, r \in \{1, \dots, p\}$, $q \neq r$ such that

$$\delta_q = 0, \gamma_q = -1 \text{ and } \delta_r = 0, \gamma_r = +1.$$

Moreover, such q, r are unique due to (5.22), D4 and definition of γ_j 's.

In this case we get locally around $(\bar{x}, \bar{y})$:

$$\Sigma_{\min} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} y^q(x), & x \leq \bar{x} \quad (\text{if } \delta_q = 0, \gamma_q = -1) \\ y^r(x), & x \geq \bar{x} \quad (\text{if } \delta_r = 0, \gamma_r = 1) \end{cases} \right\}.$$

We refer to the case as Type 5-2.

5.3 Structure of the feasible set: $\dim(x) = 1$

Our main goal is to describe the generic structure of the bilevel feasible set M , where

$$M := \{(x, y) \mid y \in \operatorname{Argmin} L(x)\}.$$

The special case with unconstrained one-dimensional lower level (i.e. $J = \emptyset$ and $m = 1$) is treated in [19]. In the latter paper the classification of 1-dimensional singularities was heavily used and for the higher dimensional case (i.e. $m > 1$) it is conjectured that a similar result will hold.

However, the situation becomes extremely difficult to describe if inequality constraints are present in the lower level (i.e. $J \neq \emptyset$). In particular, kinks and ridges will appear in the feasible set and such subsets might attract stable solutions of the bilevel problem. A simple example was presented in [19]. In this paper we restrict ourselves to the simplest case that the x -dimension is equal to one (i.e. $n = 1$), but no restrictions on the y -dimension. Then, the lower level $L(x)$ is a one-dimensional parametric optimization problem and we can exploit the well-known generic (five type) classification of so-called generalized critical points (cf. [62]) in order to describe the feasible set. Our main result (Theorems 5.3.2 and 5.3.3) states that - generically - the feasible set M is the union of C^2 curves with boundary points and kinks which can be parametrized by means of the variable x . The appearance of the boundary points and kinks is due to certain degeneracies of the corresponding local solutions in the lower level as well as the change from local to global solutions. Outside of the latter points, the feasible points $(x, y(x)) \in M$ correspond to nondegenerate minimizers of the lower level $L(x)$. Although $\dim(x) = 1$ might seem to be very restrictive, it should be noted that on typical curves in higher dimensional x -space the one-dimensional features as described in this paper will reappear on that curves.

Obtaining the generic and stable structure of the feasible set M we derive optimality criteria for the bilevel problem U . In order to guarantee the existence of solutions of the lower level we will assume an appropriate compactness condition (cf. (5.24)).

Simplicity of Bilevel Problems

First, we define simplicity of a bilevel programming problem at a feasible point. Recall again that $\dim(x) = 1$.

Definition 5.3.1 (Simplicity of Bilevel Problems)

A bilevel programming problem U (with $\dim(x) = 1$) is called simple at $(\bar{x}, \bar{y}) \in M$ if one of the following cases occurs:

Case I: $\text{Argmin } L(\bar{x}) = \{\bar{y}\}$ and $(\bar{x}, \bar{y})$ is of Type 1, 2, 4, 5-1 or 5-2,

Case II: $\text{Argmin } L(\bar{x}) = \{\bar{y}_1, \bar{y}_2\}$ and $(\bar{x}, \bar{y}_1), (\bar{x}, \bar{y}_2)$ are both of Type 1, additionally it holds:

$$\alpha := \text{sign} \left[\frac{d[g(x, y_2(x)) - g(x, y_1(x))]}{dx} \Big|_{x=\bar{x}} \right] \neq 0, \quad (5.23)$$

where $y_1(x), y_2(x)$ are unique local minimizers for $L(x)$ in a neighborhood of $\bar{x}$ with $y_1(\bar{x}) = \bar{y}$, $y_2(\bar{x}) = \bar{y}_2$ according to Type 1.

In order to avoid asymptotic effects, let $\mathcal{O}$ denote the set of $(g, h_j, j \in J) \in C^3(\mathbb{R}^{1+m}) \times [C^3(\mathbb{R}^{1+m})]^{|J|}$ such that

$$B_{g,h}(\bar{x}, c) \text{ is compact for all } (\bar{x}, c) \in \mathbb{R} \times \mathbb{R}, \quad (5.24)$$

where

$$B_{g,h}(\bar{x}, c) := \{(x, y) \mid \|x - \bar{x}\| \leq 1, g(x, y) \leq c, y \in M(x)\}.$$

Note that $\mathcal{O}$ is C_s^3 -open.

Now, we state our main result.

Theorem 5.3.2 (Simplicity is generic and stable)

Let $\mathcal{F}$ denote the set of defining functions $(f, g, h_j, j \in J) \in C^3(\mathbb{R}^{1+m}) \times \mathcal{O}$ such that the corresponding bilevel programming problem U is simple at all its feasible points $(\bar{x}, \bar{y}) \in M$. Then, $\mathcal{F}$ is C_s^3 -open and C_s^3 -dense in $C^3(\mathbb{R}^{1+m}) \times \mathcal{O}$.

Proof. It is well-known from the one-dimensional parametric optimization ([62]) that generically the points of Σ are of Types 1-5 as defined above. Moreover, for the points of $M \subset \Sigma$ only Types 1, 2, 4, 5-1 or 5-2 may occur generically (cf. Section 5.2). Further, the appearance of two different $y, z \in \text{Argmin } L(x)$ causes one loss of freedom to the equation

$$g(x, y) = g(x, z).$$

From the standard argument by counting the dimension and codimension of the corresponding manifold in multi-jet-space and by applying the Multi-Jet-Transversality Theorem (cf. [61]), we get generically:

$$|\text{Argmin } L(x)| \leq 2.$$

Now, $|\text{Argmin } L(x)| = 1$ corresponds to Case I in Definition 5.3.1. For the case $|\text{Argmin } L(x)| = 2$, we obtain the points of Type 1. It comes from the fact that the appearance of Types 2, 4, 5-1 or 5-2 would cause another loss of freedom due to their degeneracy. Analogously, (5.23) in Case II is generically valid.

The proof of the openness-part is standard (cf.[61]). $\square$

Reduced bilevel feasible set

Using the description of Σ_{min} from Section 5.2, a reducible bilevel programming problem U can be locally reduced as follows.

Theorem 5.3.3 (Bilevel feasible set and Reduced Problem)

Let the bilevel programming problem U (with $\dim(x) = 1$) be simple at $(\bar{x}, \bar{y}) \in M$. Then, locally around $(\bar{x}, \bar{y})$, U is equivalent to the following reduced optimization problem :

$$\text{Reduced-Problem: } \underset{(x,y) \in \mathbb{R}^1 \times \mathbb{R}^m}{\text{minimize}} f(x, y) \quad \text{s.t.} \quad (x, y) \in M_{loc}, \quad (5.25)$$

where M_{loc} is given according to the cases in Definition 5.3.1:

Case I, Type 1:

$$M_{loc} = \{(x, y(x)) \mid x \text{ sufficiently close to } \bar{x}\},$$

Case I, Type 2:

$$M_{loc} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} \tilde{y}(x), & x \leq \bar{x} \\ \hat{y}(x), & \bar{x} \leq x \end{cases} \right\} \text{ if } \text{sign}(\gamma) = -1$$

or

$$M_{loc} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} \hat{y}(x), & x \leq \bar{x} \\ \tilde{y}(x), & \bar{x} \leq x \end{cases} \right\} \text{ if } \text{sign}(\gamma) = +1,$$

Case I, Type 4:

$$M_{loc} = \{(x(\lambda), y(\lambda)) \mid \lambda \leq 0\},$$

Case I, Type 5-1:

$$M_{loc} = \{(x, y^q(x)) \mid x \geq \bar{x} \text{ (resp. } x \leq \bar{x}) \text{ if } \gamma_q = -1 \text{ (resp. } \gamma_q = +1), \delta_q = 0\},$$

Case I, Type 5-2:

$$M_{loc} = \left\{ (x, y(x)) \mid y(x) := \begin{cases} y^q(x), & x \leq \bar{x} \quad (\text{if } \delta_q = 0, \gamma_q = -1) \\ y^r(x), & x \geq \bar{x} \quad (\text{if } \delta_r = 0, \gamma_r = 1) \end{cases} \right\},$$

Case II:

$$M_{loc} = \{(x, y_1(x)) \mid x \geq \bar{x} \text{ (resp. } x \leq \bar{x}) \text{ if } \alpha = +1 \text{ (resp. } \alpha = -1)\}.$$

We refer to Section 5.2 for details on Types 1, 2, 4, 5-1 and 5-2.

In each case one of the possibilities for M_{loc} is depicted in Figure 17.

Figure 17: Bilevel feasible set M_{loc} from Theorem 5.3.3

Optimality criteria for bilevel problems

Theorem 5.3.3 allows to deduce optimality criteria for a reducible bilevel programming problem. In fact, the set M_{loc} from *Reduced-Problem* is the feasible set of either a standard nonlinear optimization problem - NLP - (Cases I, Type 1, 4, 5-1 and Case II) or a mathematical programming problem with complementarity constraints - MPCC - (Cases I, Type 2 and 5-2). Hence, we only need to use the corresponding optimality concepts of a Karush-Kuhn-Tucker point (for NLP) and of a S-stationary point (for MPCC), cf. [106] for the latter concept.

Theorem 5.3.4 (First-order optimality for simple bilevel problem)

Let a bilevel programming problem U (with $\dim(x) = 1$) be simple at its local minimizer $(\bar{x}, \bar{y}) \in M$. Then, according to the cases in Theorem 5.3.3 we obtain:

Case I, Type 1:

$$D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y(\bar{x}) = 0,$$

Case I, Type 2:

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x \tilde{y}(\bar{x})] \leq 0,$$

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x \hat{y}(\bar{x})] \leq 0,$$

$$\text{if } \text{sign}(\gamma) = -1$$

or

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x \hat{y}(\bar{x})] \leq 0,$$

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x \tilde{y}(\bar{x})] \leq 0,$$

$$\text{if } \text{sign}(\gamma) = +1,$$

Case I, Type 4:

$$D_x f(\bar{x}, \bar{y}) \cdot D_\lambda x(0) + D_y f(\bar{x}, \bar{y}) \cdot D_\lambda y(0) \leq 0,$$

Case I, Type 5-1:

$$D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y^q(\bar{x}) \geq 0, \text{ if } \gamma_q = -1, \delta_q = 0$$

or

$$D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y^q(\bar{x}) \leq 0, \text{ if } \gamma_q = +1, \delta_q = 0$$

Case I, Type 5-2:

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y^q(\bar{x})] \leq 0,$$

$$[D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y^r(\bar{x})] \leq 0,$$

Case II:

$$D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y(\bar{x}) \geq 0, \text{ if } \alpha = -1,$$

or

$$D_x f(\bar{x}, \bar{y}) + D_y f(\bar{x}, \bar{y}) \cdot D_x y(\bar{x}) \leq 0, \text{ if } \alpha = +1$$

Note that the derivatives of implicit functions above can be obtained from the defining equations as discussed in Section 5.2.

5.4 Towards the case $\dim(x) \geq 2$

In the higher dimensional case, i.e. $\dim(x) \geq 2$, there will appear more complicated singularities in the description of the feasible set. In particular, we will present stable examples when more than one Lagrange multiplier vanishes. This will be an extension of Type 2 (cf. Examples 5.4.2, 5.4.3). Here, combinatorial and bifurcation issues occur. On the other hand, we will not be able to describe all generic situations. This obstruction comes from classification in singularity theory. In fact, in one variable (y) there is already a countable infinite list of local minimizers: In the unconstrained case the functions y^{2k} , $k \geq 1$ and in the constrained case $y \geq 0$ the functions y^k , $k \geq 1$. However, a complete list of local minimizers for functions of two variables or more is even not known. Therefore, we have to bring the objective function of the bilevel problem into play as well. If we restrict ourselves to a neighborhood of a (local) solution of the bilevel problem, then the generic situation becomes easier. For example, the above mentioned singularities y^{2k} ($k \geq 2$) as well as the constrained singularities y^k ($k \geq 3$), $y \geq 0$, can generically be avoided at local solutions of the bilevel problem. The key idea is explained below and illustrated in Examples 5.4.5, 5.4.6.

Combinatorial and Bifurcation Issues

Remark 5.4.1 *We note that all singularities appearing for lower dimensional x may reappear at higher dimensional x in a kind of product structure. In fact, the lower dimensional singularity may appear as normal section in the corresponding normal-tangential stratification (cf. [30]). For example, let $x = (x_1, x_2, \dots, x_n)$ and let the lower level problem $L(x)$ be:*

$$L(x) : \min_y (y - x_1)^2 \text{ s.t. } y \geq 0$$

Then, the feasible set M becomes:

$$M = \{(x_1, x_2, \dots, x_n, \max\{x_1, 0\}) \mid x \in \mathbb{R}^n\}$$

and in this particular case we see that M is diffeomorphic to product

$$\{(x_1, \max\{x_1, 0\}) \mid x_1 \in \mathbb{R}\} \times \mathbb{R}^{n-1}.$$

At this point we come to typical examples with several vanishing Lagrange multipliers. Here, we assume that LICQ at the lower level is fulfilled, that the dimensions of the variables x and y coincide (i.e. $n = m$), that $J_0(\bar{x}, \bar{y}) = m$ and that $\bar{x} = \bar{y} = 0$. Taking the constraints h_j as new coordinates, we may assume that the lower level feasible set $M(0)$ is just the nonnegative orthant.

In this setting, the Lagrange multipliers of the lower level function g at the origin just become the partial derivatives with respect to the coordinates y_j , $j = 1, \dots, m$. Now we suppose that all these partial derivatives vanish (generalization of Type 2). Then, the Hessian $D_{yy}^2 g(0,0)$ comes into play and we assume that it is nonsingular. In order that the origin is a (local) minimizer for $L(0)$, a stable condition becomes that the positive cone of the Hessian $D_{yy}^2 g(0,0)$ contains the nonnegative orthant with deleted origin. This gives rise to several combinatorial possibilities, depending on the number of negative eigenvalues of $D_{yy}^2 g(0,0)$. In the next two examples, we restrict ourselves to two dimensions, i.e. $n = m = 2$.

Example 5.4.2 *In this example the Hessian $D_{yy}^2 g(0,0)$ has two (typically distinct) positive eigenvalues. In particular, $D_{yy}^2 g(0,0)$ is positive definite:*

$$f(x_1, x_2, y_1, y_2) = (-x_1 + 2y_1) + (-x_2 + 2y_2)$$

$$L(x_1, x_2) : \min_y g(x_1, x_2, y_1, y_2) := (y_1 - x_1)^2 + (y_1 - x_1) \cdot (y_2 - x_2) + (y_2 - x_2)^2$$

$$\text{s.t. } y_1 \geq 0, y_2 \geq 0.$$

In order to obtain the feasible set M , we have to consider critical points of $L(x_1, x_2)$ for the following four cases I-IV. These cases result from the natural stratification of the nonnegative orthant in y -space:

$$\begin{array}{ll} I : y_1 > 0, y_2 > 0 & II : y_1 = 0, y_2 > 0 \\ III : y_1 > 0, y_2 = 0 & IV : y_1 = 0, y_2 = 0. \end{array}$$

It turns out that the feasible set M is piecewise smooth two-dimensional manifold. Moreover, it can be parametrized via the x -variable by means of a subdivision of the x -space into four regions according to the above cases I-IV, see Figure 18.

Figure 18: Illustration of Example 5.4.2

On the regions I-IV the corresponding global minimizer $(y_1(\cdot), y_2(\cdot))$ is given by:

$$(y_1(x), y_2(x)) = \begin{cases} (x_1, x_2), & \text{if } (x_1, x_2) \in I, \\ (0, \frac{x_1}{2} + x_2), & \text{if } (x_1, x_2) \in II, \\ (\frac{x_2}{2} + x_1, 0), & \text{if } (x_1, x_2) \in III, \\ (0, 0), & \text{if } (x_1, x_2) \in IV. \end{cases} \quad (5.26)$$

In particular, we obtain $M = \{(x, y(x)) \mid y(x) \text{ as in (5.26)}\}$. A few calculations show that the origin $(0, 0)$ solves the corresponding bilevel problem U .

Example 5.4.3 In this example the Hessian $D_{yy}^2 g(0, 0)$ has one positive and one negative eigenvalue:

$$f(x_1, x_2, y_1, y_2) = -3x_1 + x_2 + 4y_1 + 5y_2$$

$$\begin{aligned} L(x_1, x_2) : \quad & \min_y g(x_1, x_2, y_1, y_2) := (y_1 - x_1)^2 + 4(y_1 - x_1) \cdot y_2 + 3(y_2 + \frac{1}{3}x_2)^2 \\ & \text{s.t.} \quad y_1 \geq 0, y_2 \geq 0. \end{aligned}$$

It is easy to see that $(y_1, y_2) = (0, 0)$ is the global minimizer for $L(0, 0)$. Analogously to Example 5.4.2 we subdivide the parameter space (x_1, x_2) into regions on which the global minimizer $(y_1(x), y_2(x))$ for $L(x)$ is a smooth function. Here, we obtain three regions II-IV, see Figure 19. Note that the region corresponding to the case I is empty.

Figure 19: Illustration of Example 5.4.3

In addition, for the parameters (x_1, x_2) lying on the half-line

$$G : x_1 = (2 + \sqrt{3})x_2, x_1 \geq 0$$

the problem $L(x)$ exhibits two different global minimizers. It is due to the fact that $(y_1, y_2) = (0, 0)$ is a saddlepoint of the objective function $g(0, y_1, y_2)$. Moreover, $(y_1, y_2) = (0, 0)$ is not strongly stable for $L(0, 0)$.

On the regions II-IV and on G the corresponding global minimizers $(y_1(\cdot), y_2(\cdot))$ are given by:

$$(y_1(x), y_2(x)) = \begin{cases} (0, \frac{2}{3}x_1 - \frac{1}{3}x_2), & \text{if } (x_1, x_2) \in II, \\ (x_1, 0), & \text{if } (x_1, x_2) \in III, \\ (0, 0), & \text{if } (x_1, x_2) \in IV, \\ \{(0, \frac{2}{3}x_1 - \frac{1}{3}x_2), (x_1, 0)\} & \text{if } (x_1, x_2) \in G. \end{cases} \quad (5.27)$$

Here, $M = \{(x, y(x)) \mid y(x) \text{ as in (5.27)}\}$. We point out that the bilevel feasible set M is now a two-dimensional nonsmooth Lipschitz manifold with boundary, but it cannot be parametrized by the variable x . Again, one calculates that origin $(0, 0)$ solves the corresponding bilevel problem U .

Remark 5.4.4 Let us consider in Examples 5.4.2, 5.4.3 a smooth curve around the origin which traverses the partition of x -space in a transversal way, for example a circle C . Then, restricted to C , the dimension of x reduces to one and we rediscover a simple bilevel problem.

Avoiding higher-order singularities

In order to avoid certain higher order singularities in the description of the feasible set M , we have to focus on a neighborhood of (local) solutions of the bilevel problem. The key idea is as follows. Suppose that the feasible set M contains a smooth curve, say C , through the point $(\bar{x}, \bar{y}) \in M$. Let the point $(\bar{x}, \bar{y})$ be a local solution of the bilevel problem U , i.e. $(\bar{x}, \bar{y})$ is a local minimizer for the objective function f on the set M . Then, $(\bar{x}, \bar{y})$ is also a local minimizer for f restricted to the curve C . If, in addition, $(\bar{x}, \bar{y})$ is a nondegenerate local minimizer for $f|_C$, then we may shift this local minimizer along C by means of a linear perturbation of f . After that perturbation with resulting $\tilde{f}$, the point $(\bar{x}, \bar{y})$ is not any more a local minimizer for $\tilde{f}|_C$ and, hence, it is not any more a local minimizer for $\tilde{f}|_M$. Now, if the singularities in M outside of the point $(\bar{x}, \bar{y})$ are of lower order, then in this way we are able to move away from the higher order singularity. This simple idea was used in particular in [19]. The key point however is to find a smooth curve through a given point of the feasible set M . An illustration will be presented in Examples 5.4.5 and 5.4.6. In contrast, note that in Examples 5.4.2 and 5.4.3 such a smooth curve through the origin $(0, 0)$ does not exist.

Example 5.4.5 Consider the one dimensional functions y^{2k} , $k = 1, 2, \dots$. The origin $y = 0$ is always the global minimizer. For $k = 1$ the latter is nondegenerate (Type 1), but for $k \geq 2$ it is degenerate. Let $k \geq 2$ and $x = (x_1, x_2, \dots, x_{2k-2})$. Then the function $g(x, y)$, with x as parameter,

$$g(x, y) = y^{2k} + x_{2k-2}y^{2k-2} + x_{2k-3}y^{2k-3} + \dots + x_1y$$

is a so-called universal unfolding of the singularity y^{2k} . Moreover, the singularities with respect to y have a lower codimension (i.e. lower order) outside the origin $x = 0$ (cf. [3, 13]). Consider the unconstrained lower level problem

$$L(x) : \min_y g(x, y)$$

with corresponding bilevel feasible set M . Let the smooth curve C in (x, y) -space be defined by the equations:

$$x_1 = x_2 = \dots = x_{2k-3} = 0, \quad ky^2 + (k-1)x_{2k-2} = 0.$$

It is not difficult to see that, indeed, C contains the origin and belongs to the bilevel feasible set M .

Example 5.4.6 Consider the one dimensional functions y^k , $k \geq 1$ under the constraint $y \geq 0$. The origin $y = 0$ is always the global minimizer. The case $k = 1$ is nondegenerate (Type 1), whereas the case $k = 2$ corresponds to Type 2. Let $k \geq 3$ and $x = (x_1, x_2, \dots, x_{k-1})$. Then, analogously to Example 5.4.5, the function $g(x, y)$,

$$g(x, y) = y^k + x_{k-1}y^{k-1} + x_{k-2}y^{k-2} + \dots + x_1y, \quad y \geq 0,$$

is the universal unfolding of the (constrained) singularity y^k , $y \geq 0$. Consider the constrained lower level problem

$$L(x) : \min_y g(x, y) \text{ s.t. } y \geq 0$$

with corresponding bilevel feasible set M .

In order to find a smooth curve C through the origin and belonging to M , we put

$$x_1 = x_2 = \dots = x_{k-3} = 0.$$

So, we are left with the reduced lower level problem function

$$\tilde{L}(x_{k-2}, x_{k-1}) : \min_y \tilde{g}(x_{k-2}, x_{k-1}, y) \text{ s. t. } y \geq 0$$

with reduced feasible set $\widetilde{M}$, where

$$\widetilde{g}(x_{k-2}, x_{k-1}, y) = y^k + x_{k-1}y^{k-1} + x_{k-2}y^{k-2}.$$

Firstly, let $x_{k-1} < 0$ and $x_{k-2} > 0$ and consider the curve defined by the equation

$$x_{k-2} - \frac{1}{4}x_{k-1}^2 = 0.$$

One calculates, that for points on this curve, the lower level $\widetilde{L}$ has two different global minimizers on the set $y \geq 0$ (with $\widetilde{g}$ -value zero), one of them being $y = 0$. Secondly, we note that the set $\{(x_{k-1}, x_{k-2}, 0) \mid x_{k-1} \geq 0, x_{k-2} \geq 0\}$ belongs to $\widetilde{M}$. Altogether, we obtain that the curve C defined by the equations

$$x_1 = x_2 = \dots = x_{k-3} = y = 0, \quad x_{k-2} - \frac{1}{4}x_{k-1}^2 = 0,$$

belongs to M .

We finally remark that a complete systematic generic description of the feasible bilevel set M in a neighborhood of local solutions of the bilevel problem U for higher x -dimensions is a very challenging issue for future research. Another interesting point for future research would be the discovery of a stable generic constraint qualification under which the whole feasible set M might be expected to be a Lipschitz manifold with boundary.

Chapter 6

Impacts on Nonsmooth Analysis

The crucial notion of analysis is that of regular/critical points for a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. It is well-known that in case of a smooth F the surjectivity of its Jacobian $DF(\bar{x})$ provides regularity at $\bar{x}$. In nonsmooth case we do have discrepancy between different suggested concepts. Moreover, it turns out that we have to take into account different classes of nonsmooth functions. Altogether, two main questions should be addressed by developing nonsmooth analysis:

- (1) What kind of nonsmooth functions do we study?
- (2) How can one define regular/critical points?

In the following scheme (see Figure 1) we illustrate what cases (regarding above questions) we deal with:

Nonsmooth functions	Regular/Critical points
Tame functions $\mathbb{R}^n \longrightarrow \mathbb{R}^k$ in o-minimal structures	Metric regularity
Lipschitz functions $\mathbb{R}^n \longrightarrow \mathbb{R}^1$ with Whitney-stratifiable graphs	via Clarke's subdifferential
Min-type functions $\mathbb{R}^n \longrightarrow \mathbb{R}^k$	via MFC

Figure 1: Nonsmooth analysis

We point out that the considerations of tame functions and metric regularity rely upon [51] and of Lipschitz functions $\mathbb{R}^n \longrightarrow \mathbb{R}^1$ and Clarke's subdifferentials upon [10].

Clearly, question (2) means that some desirable properties should hold at regular points, such as metric or topological properties. Section 6.2 will be devoted to this issue. Further, the set of critical values needs to be of Lebesgue-measure zero. The latter is a version of the classical Sard's Theorem and is the matter of Section 6.1.

6.1 Versions of Sard's Theorem

In this section we first recall the classical Sard's Theorem. Further, nonsmooth versions of Sard's Theorem are provided for

- (i) tame functions via metric regularity notion (cf. [51]),
- (ii) stratifiable functions via Clarke's subdifferentials (cf. [10]),
- (iii) min-type functions via Mangasarian-Fromovitz Constraint Qualification (cf. Section 2.2.1).

Smooth case

Definition 6.1.1 (Regular/critical points for smooth F , cf. e.g. [61])

Let $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$. A point $\bar{x} \in \mathbb{R}^n$ is called *critical* if the linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$ given by $\xi \mapsto DF(\bar{x})\xi$ is not surjective. In other words, $\bar{x}$ is *regular* (i.e. not critical) if and only if $DF(\bar{x})[\mathbb{R}^n] = \mathbb{R}^k$. A point $y \in \mathbb{R}^k$ is called a *regular, resp. critical value* for F if $F^{-1}(y)$ contains no critical points, resp. contains at least one critical point. $\square$

Remark 6.1.2 In the following cases the criticality of $\bar{x} \in \mathbb{R}^n$ is equivalent to:

- (a) $k=1$: $DF(\bar{x}) = 0$,
- (b) $k > n$: every $\bar{x} \in \mathbb{R}^n$ is critical,
- (c) $1 \leq k \leq n$: $D^T F_i(\bar{x})$, $i = 1, \dots, k$ are linearly independent. $\square$

The following theorem is well-known.

Theorem 6.1.3 (Sard's Theorem, cf. e.g. [61])

The set of critical values of $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ has Lebesgue-measure zero.

We point out that Sard's Theorem remains true for $F \in C^m(\mathbb{R}^n, \mathbb{R}^k)$, provided that $m > \max(n - k, 0)$ (cf. [113, 118]). Thus, a function $F \in C^1(\mathbb{R}^n, \mathbb{R}^k)$, $k > n$ can never be surjective. However, there exist surjective continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^k$, $k > n$ (cf. Peano space-filling curve from [108]).

Tame functions and metric regularity

We recall the notions of o-minimal structures and corresponding tame functions (cf. [16, 22, 51]).

Definition 6.1.4 (o-minimal Structure, cf. e.g. [16, 22])

A structure on $\mathbb{R}$ is a sequence $\mathcal{S} = (\mathcal{S}_n)$, $n \in \mathbb{N}$ such that:

- (D1) $\mathcal{S}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$, i.e. $\emptyset \in \mathcal{S}_n$ and $\mathcal{S}_n$ contains unions, intersections and complements of its elements,
- (D2) if $A \in \mathcal{S}_n$ then $A \times \mathbb{R}, \mathbb{R} \times A \in \mathcal{S}_{n+1}$,
- (D3) $\{(x_1, \dots, x_n) \mid x_i = x_j\} \in \mathcal{S}_n$ for all $1 \leq i < j \leq n$,
- (D4) if $A \in \mathcal{S}_{n+1}$ then $\pi(A) \in \mathcal{S}_n$, where $\pi : (x, x_{n+1}) \mapsto x$ is the projection onto $\mathbb{R}^n$.

A structure is called o-minimal if in addition:

- (D5) $\{(x, y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{S}_2$,
- (D6) the elements of $\mathcal{S}_1$ are finite unions of points and open intervals. $\square$

The elements of $\mathcal{S}_n$, $n \in \mathbb{N}$ are called definable in $\mathcal{S}$. $\square$

We give some examples of o-minimal structures.

Example 6.1.5 (Examples of o-minimal Structures, cf. e.g. [16, 22])

- (i) **Semialgebraic sets** are finite unions of the sets

$$\{x \in \mathbb{R}^n \mid p_i(x) < 0, i \in I, q_j(x) = 0, j \in J\},$$

where I and J are finite index sets and p_i , $i \in I$ and q_j , $j \in J$ are polynomials. Note that the validity of (D4) is due to the nontrivial Tarski-Seidenberg Theorem (cf. [8]).

- (ii) **Globally Subanalytic sets:** As above semianalytic sets can be constructed as finite unions of the sets

$$\{x \in \mathbb{R}^n \mid f_i(x) < 0, i \in I, g_j(x) = 0, j \in J\},$$

where I and J are finite index sets and f_i , $i \in I$ and g_j , $j \in J$ are real analytic functions. A set $A \subset \mathbb{R}^n$ is called subanalytic if for any $x \in A$ there is an open neighborhood U of x and a bounded semianalytic set $S \subset \mathbb{R}^{n+m}$ such that the projection of S onto $\mathbb{R}^n$ is $A \cap U$. Finally, a set $B \subset \mathbb{R}^n$ is called globally subanalytic if $G(B)$ is subanalytic, where G is a semialgebraic homeomorphism of $\mathbb{R}^n$ onto $(-1, 1)^n$. $\square$

Definition 6.1.6 (Tame sets and functions, cf. e.g. [16, 22])

Let $\mathcal{S}$ be an o-minimal structure on $\mathbb{R}$. A set A is called *tame* if its intersection with any bounded definable set is definable in $\mathcal{S}$. A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is definable (tame) if its graph is a definable (tame) set in $\mathcal{S}$. $\square$

Remark 6.1.7 Note that definable and tame sets are closed under interior and closure operations. Definable and tame functions are closed under taking infimum and supremum. $\square$

We recall the notion of metrically regular/critical points.

Definition 6.1.8 (Metrically regular/critical points, cf. [50])

A point $\bar{x}$ is called *metrically regular* for $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ if there exist $L > 0$ and neighborhoods U and V of $\bar{x}$ and $F(\bar{x})$, resp. such that:

$$\text{dist}(x, F^{-1}(y)) \leq L \text{dist}(y, F(x)) \text{ for all } x \in U, y \in V.$$

Otherwise, $\bar{x}$ is called *metrically critical*. A point $y \in \mathbb{R}^k$ is called a *metrically regular*, resp. *critical value* for F if $F^{-1}(y)$ contains no metrically critical points, resp. contains at least one metrically critical point. $\square$

Now we are ready to state the nonsmooth version of Sard's Theorem for tame functions.

Theorem 6.1.9 (Sard's Theorem for tame functions, [51])

The set of metrically critical values of a tame function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ has Lebesgue-measure zero.

The proof of Theorem 6.1.9 is mainly based on results from tame geometry, such as Monotonicity Theorem and Cell-Decomposition Theorem (cf. [51]).

Stratifiable functions and Clarke's subdifferentials

Here, we consider Lipschitz functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ whose graphs admit a C^∞ -Whitney stratification.

Definition 6.1.10 (Whitney stratification, cf. [89])

A C^∞ -stratification $\mathcal{X} = (X_i)$, $i \in \mathbb{N}$ of $X \subset \mathbb{R}^n$ is a locally finite partition of X into C^∞ manifolds $X_i \subset \mathbb{R}^n$ (called *strata* of X) such that:

$$\text{if } \overline{X_i} \cap X_j \neq \emptyset \text{ then } X_j \subset \overline{X_i} \setminus X_i.$$

A C^∞ -stratification $\mathcal{X} = (X_i)$, $i \in \mathbb{N}$ of $X \subset \mathbb{R}^n$ is called *Whitney stratification* if for each $x \in \overline{X_i} \cap X_j$, ($i \neq j$) and for each sequence $(x_l) \subset X_i$ it holds:

$$\text{if } x_l \longrightarrow x, T_{x_l}X_i \longrightarrow \mathcal{T} \text{ then } T_xX_j \subset \mathcal{T}.$$

Here, $T_{x_l}X_i$ denotes the tangent space of X_i at x_l . $\square$

Definition 6.1.11 (Whitney stratifiable functions)

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called C^∞ -Whitney stratifiable if its graph admits a C^∞ -Whitney stratification. $\square$

The critical point notion for Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ is based on Clarke's subdifferentials.

Definition 6.1.12 (Clarke regular/critical for $\mathbb{R}^n \longrightarrow \mathbb{R}$, cf. [15, 10])

A point $\bar{x}$ is called *Clarke critical* for a Lipschitz function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ if $0 \in \partial f(\bar{x})$, where $\partial f(\bar{x})$ is the Clarke's subdifferential of f at $\bar{x}$. Otherwise, $\bar{x}$ is called *Clarke regular*. A point $y \in \mathbb{R}^k$ is called a *Clarke regular*, resp. *critical value* for f if $f^{-1}(y)$ contains no Clarke critical points, resp. contains at least one Clarke critical point. $\square$

Remark 6.1.13 We refer to [64] for the similar treatment of continuous selections of smooth functions. $\square$

Theorem 6.1.14 (Sard's Theorem for $\mathbb{R}^n \longrightarrow \mathbb{R}$, [10])

The set of Clarke's critical values of a C^∞ -Whitney stratifiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has Lebesgue-measure zero.

The proof of Theorem 6.1.14 is based on projection formulas for Clarke's subdifferentials and involves the Whitney property of the corresponding stratification (cf. [10]).

We additionally refer to the recent work [52] for other Clarke-like notions of critical points for functions from $\mathbb{R}^n$ to $\mathbb{R}^k$ with stratifiable graphs.

Min-type functions and MFC

Let $F_1 := (F_{1,i}, i = 1, \dots, k)^T$, $F_2 := (F_{2,i}, i = 1, \dots, k)^T \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$. Setting $g_i := \min\{F_{1,i}, F_{2,i}\}$, $i = 1, \dots, k$ we define the min-type function

$$G : \begin{cases} \mathbb{R}^n & \longrightarrow & \mathbb{R}^k, \\ x & \mapsto & (g_1(x), \dots, g_k(x)). \end{cases} \quad (6.1)$$

For min-type functions we define the notion of topologically regular/critical points as follows.

Definition 6.1.15 (Topologically regular/critical for min-functions)

A point $\bar{x}$ is called *topologically regular* for a min-type function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ as in (6.1) if any k vectors $(w_1, \dots, w_k) \in \partial g_1(\bar{x}) \times \dots \times \partial g_k(\bar{x})$ are linearly independent, where $\partial g_i(\bar{x})$ is the Clarke's subdifferential of g_i at $\bar{x}$. Otherwise, $\bar{x}$ is called *topologically critical*. A point $y \in \mathbb{R}^k$ is called a *topologically regular, resp. critical value* for G if $G^{-1}(y)$ contains no topologically critical points, resp. contains at least one topologically critical point. $\square$

Remark 6.1.16 (Topologically regular for min-functions and MFC)

Note that $\bar{x}$ is a topologically regular point if and only if MFC holds at $\bar{x}$ (cf. Definition 2.2.2). Indeed,

$$\partial g_i(\bar{x}) = \partial \min\{F_{1,i}, F_{2,i}\}(\bar{x}) = \text{conv}\{\nabla F_{j,i}(\bar{x}) \mid F_{j,i}(\bar{x}) = g_i(x)\}.$$

In case $k = 1$ of a single min-function the notion of Clarke regular points according to Definition 6.1.12 coincides with that of topologically regular points according to Definition 6.1.15. $\square$

Theorem 6.1.17 (Sard's Theorem for min-functions)

The set of topologically critical values of a min-type function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ as in (6.1) has Lebesgue-measure zero.

Proof. Let $\mathcal{J}$ be a collection of all $\emptyset \neq J^i \subset \{1, 2\}$, $i = 1, \dots, k$. For each element $J = (J^i, i = 1, \dots, k)$ from this collection $\mathcal{J}$, we define a C^∞ -function:

$$F^J : \begin{cases} \mathbb{R}^n & \longrightarrow \mathbb{R}^{|J|}, \\ x & \mapsto (F_{j,i}, j \in J^i, i = 1, \dots, k), \end{cases}$$

where $|J| := \sum_{i=1}^k |J^i|$. Note that $k \leq |J| \leq 2k$.

Further, if $\bar{x}$ is a topologically critical point for G (cf. Definition 6.1.15) then $\bar{x}$ is critical for F^J (in the classical sense, cf. Definition 6.1.1) with

$$J := (J^i, i = 1, \dots, k), \quad J^i = \{j \in \{1, 2\} \mid F_{j,i}(x) = g_i(x)\} \quad \text{and} \\ F_{j,i}(x) = g_i(x) \quad \text{for all } j \in J^i. \quad (6.2)$$

Hence, applying the classical Sard's Theorem for F^J , $J \in \mathcal{J}$ we get the desired result. Note that the collection $\mathcal{J}$ is finite and critical points of G produce the critical values of F^J with the same components indexed by $j \in J^i$ due to (6.2). $\square$

6.2 Regularity and Implicit Functions

We discuss different notions of metric and topological regularity/criticality introduced in Section 6.1 for various nonsmooth functions. We show that these notions naturally generalize important consequences of the regularity property in the smooth setting, i.e.

$$DF(\bar{x})[\mathbb{R}^n] = \mathbb{R}^k \text{ for } F \in C^\infty(\mathbb{R}^n, \mathbb{R}^k).$$

Indeed, metric regularity corresponds to the well-known Lyusternik-Graves Theorem and topological regularity (at least for min-type functions) resembles Transversality and Implicit Function Theorem. We show that the concepts of metric and topological regularity do not coincide already for min-type functions. It gives rise to establish nonsmooth analysis along the lines of topological regularity based mainly on the application of Implicit Function Theorems (Near the very well developed nonsmooth analysis based on metric regularity, cf. [50, 78, 92, 103]). It is a very challenging issue to apply the ideas behind topological regularity for different kinds of nonsmooth functions and to get its analytical description. Note that the definition of topological regularity is given only for min-functions up to now (cf. Definition 6.1.15) and it is written in terms of Clarke's subdifferentials.

Consequences of regularity for smooth functions

We formulate Lyusternik-Graves Theorem as follows.

Theorem 6.2.1 (Lyusternik-Graves, cf. [31, 50, 87])

Let $F \in C^1(\mathbb{R}^n, \mathbb{R}^k)$ and $\bar{x} \in \mathbb{R}^n$ be regular (according to Definition 6.1.1). Then, there exists $K > 0$ such that:

- (a) $B(F(x), t) \subset F(B(x, Kt))$ for x close to $\bar{x}$ and small $t > 0$,
- (b) $\text{dist}(x, F^{-1}(y)) \leq K \text{dist}(F(x), y)$ for (x, y) close to $(\bar{x}, F(\bar{x}))$. $\square$

Remark 6.2.2 *It can be immediately seen that part (b) from Theorem 6.2.1 is exactly the definition of metric regularity (cf. Definition 6.1.8). Hence, in the smooth case metric regularity is a consequence of the property $DF(\bar{x})[\mathbb{R}^n] = \mathbb{R}^k$. It is worth to mention that the standard statement of Lyusternik is:*

$$T_{\bar{x}}F^{-1}(F(\bar{x})) = \{\xi \in \mathbb{R}^n \mid DF(\bar{x}) \cdot \xi = 0\},$$

where $T_{\bar{x}}F^{-1}(F(\bar{x}))$ is the tangent space of the level set $F^{-1}(F(\bar{x}))$ at $\bar{x}$. $\square$

Remark 6.2.2 gives rise to link regularity with results from transversality theory (cf. e.g. [61]).

Definition 6.2.3 (Transversality of manifolds)

Let M_1 and M_2 be manifolds in $\mathbb{R}^n$. We say that M_1 and M_2 intersect transversally if at every point $\bar{x} \in M_1 \cap M_2$ the following condition on the tangent spaces holds:

$$T_{\bar{x}}M_1 + T_{\bar{x}}M_2 = \mathbb{R}^n. \square$$

Definition 6.2.4 (Transversality of mappings)

Let M be a manifold in $\mathbb{R}^k$ and $F \in C^1(\mathbb{R}^n, \mathbb{R}^k)$. We say that F meets M transversally if the following two manifolds M_1, M_2 in $\mathbb{R}^n \times \mathbb{R}^k$ intersect transversally:

$$M_1 := \text{graph}(F), M_2 := \mathbb{R}^n \times M. \square$$

Note that F meets M transversally if and only if at every $x \in \mathbb{R}^n$ with $f(x) \in M$ the following holds:

$$DF(\bar{x})[\mathbb{R}^n] + T_{f(\bar{x})}M = \mathbb{R}^k.$$

Hence, $y \in \mathbb{R}^k$ is a regular value for F (according to Definition 6.1.1) if and only if F meets $\{y\}$ transversally.

Further, if F meets M (a manifold of codimension m) transversally then either $F^{-1}(M) = \emptyset$ or, otherwise, $F^{-1}(M)$ is a manifold in $\mathbb{R}^n$ of codimension m . Moreover, for $\bar{x} \in F^{-1}(M)$ we have:

$$T_{\bar{x}}F^{-1}(M) = DF(\bar{x})^{-1}T_{F(\bar{x})}M.$$

From the above considerations we get the following result.

Theorem 6.2.5 (Regular values and manifold)

Let $\bar{y}$ be a regular value of $F \in C^1(\mathbb{R}^n, \mathbb{R}^k)$. Then, $F^{-1}(\bar{y}) \neq \emptyset$ is a manifold in $\mathbb{R}^n$ of dimension $n - k$ and

$$T_{\bar{x}}F^{-1}(\bar{y}) = \{\xi \in \mathbb{R}^n \mid DF(\bar{x}) \cdot \xi = 0\} \text{ for all } \bar{x} \in F^{-1}(\bar{y}).$$

We point out that the proof of Theorem 6.2.5 is mainly based on the application of the standard Implicit Function Theorem. Indeed, we need to show that $F^{-1}(\bar{y})$ is locally diffeomorphic to $\mathbb{R}^{n-k}$ (i.e. is a manifold of dimension $n - k$). For that, we parametrize the solution set of $F(x) = \bar{y}$ using Implicit Function Theorem. The latter can be applied due to the fact that each $\bar{x} \in F^{-1}(\bar{y})$ is a regular point for F .

On metric and topological regularity for min-type functions

Theorems 6.2.1 and 6.2.5 give rise to concentrate either on metric or topological properties of $F^{-1}(\bar{y})$ for a possibly nonsmooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, $\bar{y} \in \mathbb{R}^k$. Indeed, taking min-type functions into consideration we obtain the following result (similar to that of Theorem 6.2.5 in the smooth setting).

Theorem 6.2.6 (Topological regularity and Lipschitz manifold)

Let $\bar{y}$ be a topologically regular value (cf. Definition 6.1.15) of a min-type function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ given as in (6.1). Further, assume the conjectured equivalence of MFC and SMFC (cf. Definition 2.2.2). Then, $F^{-1}(\bar{y})$ is a Lipschitz manifold in $\mathbb{R}^n$ of dimension $n - k$.

Proof. follows immediately from Corollary 2.2.30, Definition 6.1.15 and Remark 6.1.16. $\square$

It is worth to mention that metrically and topologically regular values do not coincide already in the setting of min-type functions.

Remark 6.2.7 (Metrically and topologically regular values)

Due to Propositions 2.2.12 and 2.2.15 each topologically regular value is also metrically regular, but not vice versa. In fact, let

$$G : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2, \\ x & \mapsto & (\min\{x, y\}, \min\{x + y - \sqrt{2}z, x + y + \sqrt{2}z\}). \end{cases}$$

From Example 2.2.14 we see that 0 is a metrically regular value for G , but not topologically regular. $\square$

Finally, we point out that topological considerations of $F^{-1}(\bar{y})$ for general nonsmooth functions $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ should involve conditions which guarantee that $F^{-1}(\bar{y})$ is a Lipschitz manifold of the right dimension. It requires an application of nonsmooth versions of Implicit Function Theorems (e.g. due to Clarke [15] or Kummer [78]). Certainly, analytical descriptions of the latter applicability depends heavily on the nonsmoothness type of F . For min-type functions we refer to Definition 6.1.15. Min-max functions might be handled using results on their Clarke's subdifferentials from Section 3.2.2. In general, topological properties of $\bar{y}$ -level sets of a nonsmooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a very challenging issue.

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Notation

Our notation is standard. The n -dimensional Euclidean space is denoted by $\mathbb{R}^n$ with the norm $\|\cdot\|$, its nonnegative orthant by $\mathbb{H}^n$ and its nonpositive orthant by $\mathbb{R}_-^n$. $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$. For $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^n$ the set $B_\varepsilon(\bar{x})$ (or $B(\bar{x}, \varepsilon)$) stands for the open Euclidean ball in $\mathbb{R}^n$ with radius ε and center $\bar{x}$. A closed ball with radius $\varepsilon > 0$ and center $\bar{x} \in \mathbb{R}^n$ is denoted by $\bar{B}(\bar{x}, \varepsilon)$.

Given an arbitrary set $K \subset \mathbb{R}^n$, $\bar{K}$, $\text{int}(K)$, ∂K denotes the topological closure, interior and boundary of K , respectively. $\text{span}(K)$, $\text{conv}(K)$ (or $\text{co}(K)$), $\text{cone}(K)$ denotes the set of all linear, convex, nonnegative combinations of elements of K , respectively. CK denotes the complement of $K \subset \mathbb{R}^n$. By $\text{span}\{a_1, \dots, a_t\}$ we denote the vector space over $\mathbb{R}$ generated by the finite number of vectors $a_1, \dots, a_t \in \mathbb{R}^n$ and $\dim\{\text{span}\{a_1, \dots, a_t\}\}$ stands for its dimension. The polar of K is defined by $K^\circ := \{v \in \mathbb{R}^n \mid v^T w \leq 0 \text{ for all } w \in K\}$. The distance from $x \in \mathbb{R}^n$ to $K \subset \mathbb{R}^n$ is denoted by $\text{dist}(x, K) = \inf_{y \in K} \|x - y\|$ with the convention $\text{dist}(x, \emptyset) = \infty$.

$T : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ denotes a multi-valued map defined on $\mathbb{R}^n$ with $T(x) \subset \mathbb{R}^k$, $x \in \mathbb{R}^n$. The graph of T is $\text{gph } T = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid y \in T(x)\}$ and the inverse of T is $T^{-1} : \mathbb{R}^k \rightrightarrows \mathbb{R}^n$, given by $T^{-1}(y) = \{x \in \mathbb{R}^n \mid y \in T(x)\}$.

Given a differentiable function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, DF denotes its $k \times n$ Jacobian matrix. Given a differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, Df denotes its gradient as a row vector and $D^T f$ (or ∇f) stands for the transposed vector. $C^l(\mathbb{R}^n, \mathbb{R}^k)$ denotes the space of l -times continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^k$. $C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ denotes the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^k$. $C^l(\mathbb{R}^n)$ stands for $C^l(\mathbb{R}^n, \mathbb{R})$ and $C^\infty(\mathbb{R}^n)$ for $C^\infty(\mathbb{R}^n, \mathbb{R})$.

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Summary

The main goal of our study is an attempt to understand and classify nonsmooth structures arising within the optimization setting:

$$P(f, F) : \min f(x) \text{ s.t. } x \in M[F],$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth real-valued objective function, $F : \mathbb{R}^n \rightarrow \mathbb{R}^l$ is a smooth vector-valued function and $M[F] \subset \mathbb{R}^n$ is a feasible set defined by F in some structured way. We focus rather on the underlying nonsmooth structures which fit the smooth function F to define the feasible set $M[F]$.

The basis of our study is the topological approach. It encompasses two objects:

- the feasible set $M[F]$ and
- the lower level sets $M[f, F]^a := \{x \in M[F] \mid f(x) \leq a\}$, $a \in \mathbb{R}$.

These objects are considered according to topological, optimization and stability issues. On the topology and stability level we deal with topological invariants of $M[F]$ and $M[f, F]^a$, $a \in \mathbb{R}$. Here the questionings mainly arise from. They lead to establishing of an adequate theory on the optimization level. For $M[F]$ Lipschitz manifold property and so-called topological stability are discussed. They naturally lead to constraint qualifications for $P(f, F)$. Topological changes of $M[f, F]^a$ (as $a \in \mathbb{R}$ varies) give rise to define stationary points and develop critical point theory for $P(f, F)$.

Each Chapter 2-5 is devoted to optimization problems with particular type of nonsmoothness:

- mathematical programming programs with complementarity constraints,
- general semi-infinite optimization problems,
- mathematical programming programs with vanishing constraints,
- bilevel optimization.

For these problems above topological and stability issues are elaborated and corresponding optimization concepts are introduced. It is worth to point out that the same topological questionings provide different (analytical) optimization concepts while applied to particular problems. The difference between these analytically described optimization concepts is a key point in understanding and comparing different kinds of nonsmoothness.

In Chapter 6 we enlighten the impacts of our topological approach on nonsmooth analysis theory. Topologically regular points of a min-type nonsmooth mappings $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ are introduced. The crucial property is that for topologically regular value $y \in \mathbb{R}^l$ of F the nonempty set $F^{-1}(y)$ is an $n - l$ dimensional Lipschitz manifold. Corresponding nonsmooth versions of Sard's Theorem are given.

Zusammenfassung

Man betrachtet folgendes allgemeines Optimierungsproblem:

$$P(f, F) : \quad \min f(x) \text{ s.t. } x \in M[F],$$

wobei $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ und $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ glatte Funktionen sind und $M[F] \subset \mathbb{R}^n$ die durch F definierte zulässige Menge bezeichnet. Die Nichtglatttheit wird dadurch gegeben, dass F die Menge $M[F]$ auf eine strukturierte Weise festlegt. Es werden nämlich vier Problemtypen untersucht:

- (i) Optimierungsprobleme mit Komplementaritätsnebenbedingungen (mathematical programming problems with complementarity constraints),
- (ii) Allgemeine Semi-Infinite Optimierungsprobleme (general semi-infinite optimization problems),
- (iii) Optimierungsprobleme mit verschwindenden Nebenbedingungen (mathematical programming programs with vanishing constraints),
- (iv) Zweistufige Optimierungsprobleme (bilevel optimization).

Das Hauptziel ist es, die nichtglatten Strukturen im Optimierungskontext topologisch zu untersuchen. Der topologische Zugang beinhaltet folgende Fragestellungen:

- (a) Unter welchen Bedingungen ist $M[F]$ eine Lipschitz Mannigfaltigkeit der passenden Dimension?
- (b) Unter welchen Bedingungen ist $M[F]$ stabil, d.h. $M[F]$ bleibt invariant bis auf Homöomorphismus im Bezug auf glatte Störungen von F ?
- (c) Wie ändert sich die Topologie der unteren Niveaumengen $M[f, F]^a := \{x \in M[F] \mid f(x) \leq a\}$, $a \in \mathbb{R}$, bis auf Homotopieäquivalenz?

Es wird gezeigt, dass die Fragestellungen (a) und (b) zu Constraint Qualifications führen. Über (c) gelangt man zur Stationarität und zur Kritische-Punkte-Theorie im Sinne von Morse. Man bekommt neue topologisch relevante Optimierungskonzepte in Termen von Ableitungen der definierenden Funktionen f und F . Es ist wichtig anzumerken: die selben Fragestellungen (a)-(c) liefern verschiedene analytische Optimierungskonzepte, wenn angewandt auf einzelne Problemtypen (i)-(iv). Genau der Unterschied zwischen diesen analytisch beschriebenen Optimierungskonzepten ist ein Schlüssel, die verschiedenen Typen der Nichtglattheit zu vergleichen und theoretisch zu verstehen.

Darüber hinaus werden die Auswirkungen von (a) und (b) auf die Theorie der nichtglatten Analysis dargestellt. Es werden topologisch reguläre Punkte für nichtglatte Abbildungen $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ vom Minimum-Typ eingeführt. Die ausschlaggebende Eigenschaft ist, dass für topologisch reguläre Werte $y \in \mathbb{R}^l$ von F die Menge $F^{-1}(y)$ eine $n - l$ dimensionale Lipschitz Mannigfaltigkeit ist. Hier ist die Anwendung nichtglatter Versionen des Satzes über implizite Funktionen von Bedeutung (von Clarke bzw. Kummer). Es wird herausgearbeitet, dass die Schwierigkeit deren Anwendung darin besteht, eine passende Aufspaltung des $\mathbb{R}^n$ zu finden. Dies führt zum besseren Verständnis der nichtglatten Geometrie und Topologie. Entsprechende nichtglatte Versionen des Satzes von Sard werden bewiesen.

Lebenslauf

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