

The XFEM for Hydraulic Fracture Mechanics

Die XFEM für die hydraulische Bruchmechanik

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Nikolai Weber

Berichter: Univ. Prof. Marek Behr, Ph.D.
Univ. Prof. Dr.-Ing. Thomas-Peter Fries

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Abstract

The aim of this thesis is the numerical treatment of the hydraulic fracturing process based on the extended finite element method (XFEM). Three main physical phenomena are considered: a viscous fluid flows into an existing crack, the crack propagation due to the fluid pressurization, and solid deformation due to internal and external loads. The XFEM accurately accounts for the displacement jumps across the crack surface which is numerically interpreted as a discontinuity. This is achieved by locally enriching the approximation space with discontinuous functions such as the sign function. The singular behavior at the crack tip is captured by adding four tip-enrichment functions. Thus, the crack is not required to align with the element edges of the finite element mesh and no remeshing is necessary for crack propagation. A twofold crack description is utilized to properly incorporate the enrichments and to perform the crack update. First, a level-set method with three level-set functions is used to describe the crack implicitly. Thereby, the enriched location can be determined and the level-set functions are directly used to evaluate the enrichments. Moreover, a local coordinate system at the crack tip can be formulated based on the implicit description. Second, an explicit crack description is used on the basis of a triangular mesh. The crack update is then performed by adding additional elements to the existing mesh at those crack front locations where propagation takes place. Since the crack front represents a moving boundary, its location is easily tracked as the exact crack front position is given throughout the whole simulation. Since both descriptions are easily transformed into each other, they complement one another extremely well. This combines the advantages of the evaluation of the enrichment functions with the implicit and the crack update with the explicit description.

The fluid flow equations on the crack surface are described using a finite element method on triangular meshes. Additionally, the mesh is locally refined in the crack front region in order to accurately capture the steep gradient in the pressure field. The finite element formulation is extended to the case of arbitrarily curved cracks. A local coordinate system on the surface is introduced in order to properly transform and integrate the fluid flow equations. This approach is implemented and verified for transport equations on curved surfaces. The governing equations are coupled iteratively and, therefore, maintain the flexibility of treating each model part independently. At each propagation step, the coupled equations are solved such that the propagation condition is met and the crack is updated. When the fluid front does not reach the crack front, both fronts are tracked separately. In this case, the fluid front moves forward until the propagation condition is satisfied and then a new propagation step can be performed. This thesis presents a fully coupled modeling approach that allows arbitrary crack growth in three-dimensional space. Another focus is on the treatment of transport equations

on manifolds and the associated fluid-induced crack propagation.

The coupled problem is verified against simplified analytical solutions and against experimental results that were obtained from fracturing rock samples of the size $30 \times 30 \times 45 \text{ cm}^3$. An initial penny-shaped crack is assumed and the verification focuses on the pressure response for the pressure buildup and propagation phase. Since no satisfying agreement in the pressure curve close to the breakdown pressure is achieved, the fluid flow model is extended to capture the non-linear effects of the system due to the compressibility. The simulation results obtained with this approach agree well with the experimental results.

Zusammenfassung

Das Ziel dieser Arbeit ist die numerische Behandlung des hydraulischen Bruchprozesses auf der Grundlage der erweiterten Finite Elemente Methode (XFEM). Drei wesentliche physikalische Phänomene werden hier betrachtet: eine viskose Flüssigkeit fließt in einen bestehenden Riss, der Riss breitet sich aufgrund des Fluiddrucks aus und es erfolgt eine Verformung des Körpers durch interne und externe Lasten. Die XFEM berücksichtigt die Sprünge im Verschiebungsfeld über die Rissoberfläche, welche numerisch als Diskontinuität interpretiert wird. Dies wird durch eine lokale Anreicherung des Approximationsraumes mit diskontinuierlichen (vorzeichenbehafteten) Funktionen erreicht. Das singuläre Verhalten an der Risspitze wird durch Einbeziehen von vier Rissspitzen-Anreicherungsfunktionen erfasst. Somit ist eine Ausrichtung des Risses mit den Elementkanten des Finite Elemente Netzes nicht erforderlich und eine Neuvernetzung aufgrund der Rissausbreitung daher nicht notwendig. In dieser Arbeit wird eine hybride Rissbeschreibung verwendet, um die Anreicherungen korrekt abzubilden und um das Risswachstum durchzuführen. Zunächst wird ein Level-Set Verfahren mit drei Level-Set Funktionen verwendet, um den Riss implizit zu beschreiben. Dadurch kann der zu anreichende Bereich lokalisiert werden und die Level-Set Funktionen werden unmittelbar verwendet, um die Anreicherungen zu auswerten. Darüber hinaus kann ein lokales Koordinatensystem an der Risspitze auf der Grundlage der impliziten Beschreibung formuliert werden. Des Weiteren wird eine explizite Beschreibung des Risses auf der Basis eines Dreiecksnetzes verwendet. Die Fortschreibung des Risses wird dann durch das Hinzufügen zusätzlicher Elemente an diejenigen Stellen des bestehenden Gitters realisiert, wo Risswachstum tatsächlich stattfindet. Obwohl die Rissfront eine sich bewegende Grenze darstellt, kann die genaue Lage der Rissfront während der gesamten Simulation leicht verfolgt werden. Beide Rissbeschreibungen können leicht ineinander umgewandelt werden und ergänzen sich daher sehr gut. Die Vorteile beider Beschreibungen, die Auswertung der Anreicherungsfunktionen mit dem impliziten und die Rissaktualisierung mit dem expliziten Ansatz, können gleichzeitig verwendet werden.

Die dem Fluid zugrundeliegende partielle Differentialgleichung, welche auf der Rissfläche definiert ist, wird mit einer Finite Elemente Methode auf Dreieckselementen beschrieben. Zusätzlich wird das Netz lokal um die Rissfront verfeinert, um die an dieser Stelle vorherrschenden steilen Gradienten in dem Druckfeld zu erfassen. Die Finite Elemente Formulierung ist für den Fall von beliebig gekrümmten Rissen erweitert. Ein lokales Koordinatensystem auf der Oberfläche des Risses wird eingeführt, um die Fluidströmungsgleichungen angemessen zu transformieren und integrieren. Dieser Ansatz ist für Transportgleichungen auf gekrümmten Oberflächen umgesetzt und verifiziert. Die Grundgleichungen sind iterativ gekoppelt und gewährleisten die Flexibilität, dass jeder Modellteil unabhängig voneinander behandelt werden

kann. Die gekoppelten Gleichungen werden an jedem Rissausbreitungsschritt gelöst, so dass der Riss bei Erfüllung des Rissfortschrittskriteriums aktualisiert wird. Wenn die Fluidfront die Rissfront nicht erreicht und sich eine Kavität an der Rissfront bildet, werden beide Fronten getrennt berücksichtigt. In diesem Fall bewegt sich die Flüssigkeitsfront solange vorwärts bis das Rissfortschrittskriterium erfüllt ist. Dann erst wird ein neuer Risswachstumsschritt durchgeführt. Diese Arbeit stellt einen voll gekoppelten Modellierungsansatz vor, der beliebiges Risswachstum im dreidimensionalen Raum ermöglicht. Ein weiterer Fokus liegt auf der Behandlung von Transportgleichungen auf Mannigfaltigkeiten und der damit einhergehenden fluid-induzierten Rissausbreitung.

Das gekoppelte Problem wird gegen vereinfachte analytische Lösungen und gegen experimentelle Ergebnisse, die durchs Spalten von Gesteinsproben der Größe $30 \times 30 \times 45 \text{ cm}^3$ gewonnen wurden, verifiziert. Unter der Annahme eines initialen münzförmigen Risses konzentriert sich die Überprüfung auf die Druckkurve für den Bereich des Druckaufbaus und der Ausbreitungsphase. Da keine zufriedenstellende Übereinstimmung in der Druckkurve nahe des Druckmaximums erreicht wird, wurde das Modell um einen zusätzlichen Term erweitert, der die nichtlinearen Effekte des Systems aufgrund der Kompressibilität erfasst. Die mit diesem Ansatz gewonnenen Simulationsergebnisse stimmen sehr gut mit den experimentellen Ergebnissen überein.

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1 Introduction

Hydraulic fracturing is a widely known technique in a variety of industrial applications. One of the most commonly used reservoir stimulation techniques is the fracturing of oil and gas reservoirs using a mixture of viscous hydraulic fluids and proppant [6]. Applications of hydraulic fracturing also include the disposal of wastes in shale [58, 165], volcano geodesy [67], goafing [183] and fault reactivation in mining [26], the measurement of in situ stresses [50, 52], and heat production from geothermal reservoirs [130, 139].

Hydraulic fracturing is typically modeled as a multi-field problem with the three main physical processes (1) solid deformation, (2) viscous fluid flow, and (3) crack propagation [6]. Each process is described separately and coupled to governing equations that lead to a non-linear, non-local, and time-dependent behavior. The solid deformation is described with the theory of linear elasticity while the theory of linear elastic fracture mechanics (LEFM) is used to describe the crack growth. Finally, the viscous fluid flow is modeled within the lubrication theory as a stationary, laminar flow between two parallel plates. Moreover, the moving boundaries of the fluid front and the crack front significantly contribute to the complexity of the problem and, therefore, have to be properly taken into consideration. A focus of this work is on the numerical treatment of the hydraulic fracturing process with a moving boundary based on the extended finite element method (XFEM) [18, 119]. A combination of an implicit and explicit crack description is used to capture the singular behavior at the crack tip and to enable a simple crack update. Additionally, an approach is presented to solve for transport equations on arbitrarily curved cracks.

1.1 The physics of hydraulic fracturing

A common link between these applications is the access to the geologic formations through drilled wells. Fluids are injected into a selected section of the wellbore at a pressure that is high enough to induce a fracturing of the formation. Since the fluid flow generates a pressure differential between the wellbore pressure and the original reservoir pressure, it also creates stress in the formation. By artificially increasing the stress, a point will eventually be reached where the stress becomes greater than the maximum stress that can be sustained by the formation. Consequently, the rock physically splits apart [63]. By this procedure, existing natural joints can be reopened and new cracks may be created.

This early fracture growth exposes new formation area to the injected fluid, and thus the rate of fluid leaking off into the surrounding medium starts to increase. The crack continues to grow and enlarge the formation flow area as long as the pumping rate is maintained at a rate

higher than the fluid-loss rate [157]. But once the pumping stops and the injection pressure that is used to generate the fracture has been relieved, the injected fluid passes off, the fracture closes and the new formation area may not be available for production [157].

After the injection pressure that is used to generate the fracture has been relieved, the rock's in-situ stress forces the fracture to close. In order to prevent the fracture from closing, granulated materials are pumped into the created fracture as a slurry, see Figure 1.1. These proppants range from natural sands to high-strength, granular substitute and hold the created fracture open. A fracture that is filled with proppant maintains a narrow but very conductive flow path towards the wellbore [62]. The flow path has ideally a very large permeability to maximize the production increase obtained from fracturing.

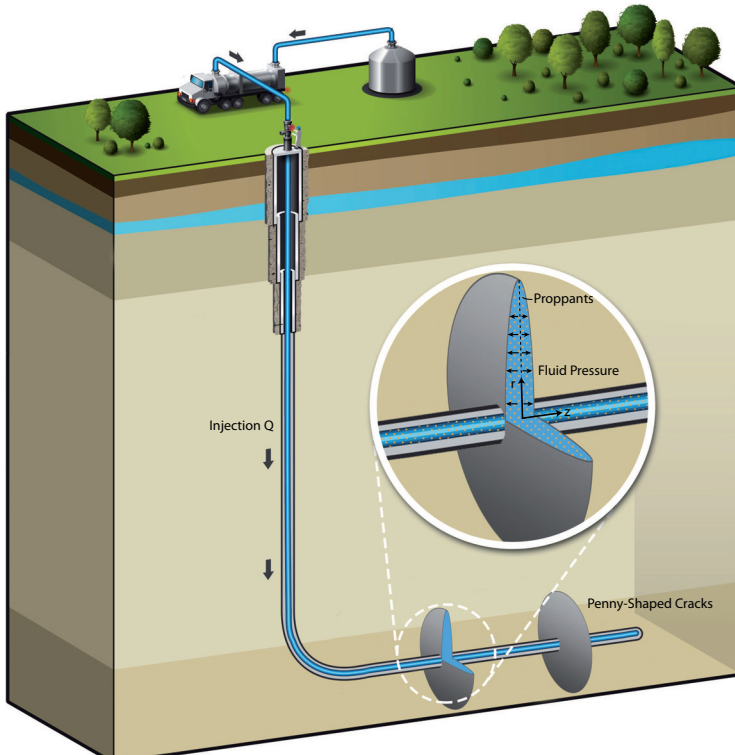


Figure 1.1: A schematic representation of a hydraulic fracture treatment according to [155]. Two penny-shaped fractures propagate from a horizontal wellbore due to fluid pressurization.

1.2 Scope of work

The aim of this work is to provide a strong and flexible model for hydraulic fracture simulation on the basis of the extended finite element method (XFEM). With XFEM, the crack propagation can be modeled very effectively. A twofold crack description allows to combine the advantages of an implicit crack description for the enrichment scheme and an explicit description for the crack update. Furthermore, a focus in this study is solving transport equations on arbitrarily curved cracks. The iterative coupling of the involved models to a flexible tool for assessing hydraulic fracture applications.

This paper is organized as follows: In Section 2, the hydraulic fracturing process is reviewed. The most important models that were development within the last decades are discussed and their main underlying assumption are presented. Moreover, a general modeling approach to the physical phenomena of hydraulic fracturing is described and an overview is given over the numerical techniques to solve these models. In Section 3, the extended finite element method (XFEM) is presented which is used here to solve for the coupled hydraulic fracture problem. The XFEM-description consists of an explanation of the explicit-implicit crack description, the XFEM-formulation for crack problems, the enrichment scheme, and the numerical integration. Section 4 discusses how PDEs are solved on manifolds. A basic background is given on manifolds in general and how differential operators are formulated on curves and surfaces. The Section closes with a discussion on implementation aspects and some numerical examples. In Section 5, the numerical modeling of a coupled hydraulic fracture problem is presented. Three main areas are discretized, the fluid flow on a manifold (*i.e.*, the curved crack surface), the solid deformation, and the fracture propagation. Finally, the iterative coupling procedure is discussed. The final Section 6 presents some numerical results that are verified against 2D and 3D examples. Moreover, a comparison was made of the simulation results with experiments that were conducted in laboratory scale.

2 Review of hydraulic fracture modeling

The simulation of the physical phenomena contributes to the understanding of hydraulic fracturing processes. Incorporating this knowledge into hydraulic fracture operations is the key factor for a successful and safe application. Due to the wide range of specification parameters for hydraulic fracture treatments, simulating the fracturing process is quite challenging. Such parameters specify the reservoir scale, soil conditions, the complex geologic formations, and in particular, the area of application.

The coupled processes that are present in hydraulic fracturing include stress/deformation in the rocks, rock failure and fracture propagation, fluid flow within the fracture and the surrounding medium, heat transfer and chemical interactions [94, 157, 166, 175]. According to the field of application, some of these processes are given a greater significance than others, and that strongly affects the development of fracture simulators. Since the first models from the 1950s, no fracture simulator covers the fracturing process in its entire complexity for the whole range of applications. Therefore, fracture models often describe the physical phenomena in a simplified manner and are only useful for a certain type of application [63]. Usually, one of the most important features that is needed in designing fracturing operations is the ability to predict the geometry and the propagation characteristics of the hydraulically induced fractures [180].

2.1 Classical hydraulic fracture models

The foundation for hydraulic fracture modeling was developed between the late 1950s and early 1970s. During this time, a number of simplified 2D models were proposed which make different assumptions concerning the importance and influence of the involved physical phenomena [33, 79, 101, 126, 138] (summarized in [116, 157]). These basic models for the propagation of a hydraulically induced fracture combine elasticity, fluid flow, material balance, and a propagation criterion. They aim to predict the fracture length and the average width with a fixed fracture height. The focus is on specific dominant conditions which might be the fluid flow or the fracture mechanics.

2.1.1 2D models

The two most important “first generation” models are commonly known as the Khristianovic-Geertsma-de Klerk (KGD) and Perkins-Kern-Nordgren (PKN) model. They are named after the authors that made key contributions to their development. The main assumptions for both

are that the fracture is planar and that it propagates in a particular direction which is the direction of the minimum stress. The fluid flow is one-dimensional along the fracture length, whereas the fracture height is fixed or completely confined in one layer. The leak-off process is governed by a simple expression derived from filtration theory [33] and the rock is considered as a continuous, homogeneous, linear elastic solid. However, there are some important differences between these two models.

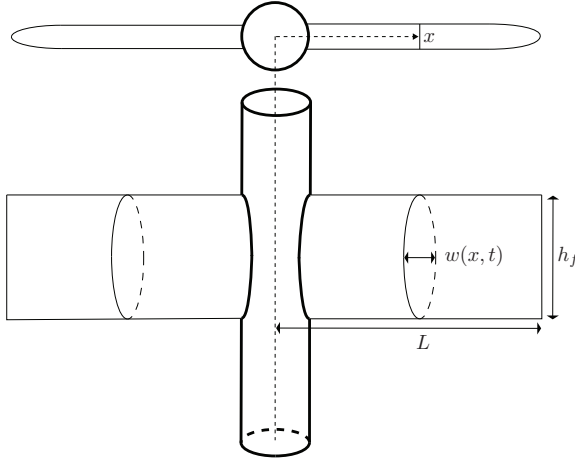


Figure 2.1: Geometry of a PKN model with L being the fracture length, h_f the fracture height, and $w(x, t)$ the fracture width. The vertical cross section has an elliptical shape [61].

The PKN model [126, 138] assumes a plane-strain behavior in the vertical direction, the fracture length being much larger than the height. Furthermore, it is assumed that each vertical cross section, which has an elliptical form, acts independently. A schematic representation of the PKN fracture is shown in Figure 2.1 with a view into the wellbore on the upper illustration and a sideview of the wellbore with two identical fractures propagating in opposite direction on the lower illustration. The mechanics at the crack tip is not taken into account. Instead, the focus is on fluid flow effects and the corresponding pressure gradients. Hence, the governing equation for the flow rate per unit fracture height q of a Newtonian fluid in an elliptical cross section is

$$q = -\frac{Q}{h_f} = -\frac{\pi}{64\mu}w^3\frac{dp}{dx}, \quad (2.1)$$

where Q is the total injection rate, h_f the fixed fracture height, μ the dynamic viscosity, p the fluid pressure, and x the distance along the fracture. The maximum fracture width w for an

elliptical cross section is at its center [80] and is defined as

$$w = \frac{1 - \nu}{G} h_f p_{net}. \quad (2.2)$$

Here, the shear modulus G and Poisson's ratio ν are the rock's elastic properties, and ($p_{net} = p_f - \sigma_H$) is the net pressure defined as the difference between the fluid pressure and the in-situ stress acting perpendicular to the fracture faces. The justification, that in this model fracture mechanics effects could be neglected, was shown through an energy analysis [157]. Under typical hydraulic fracture conditions, a significantly higher amount of energy was required to allow fluid flow along the fracture than to propagate a fracture.

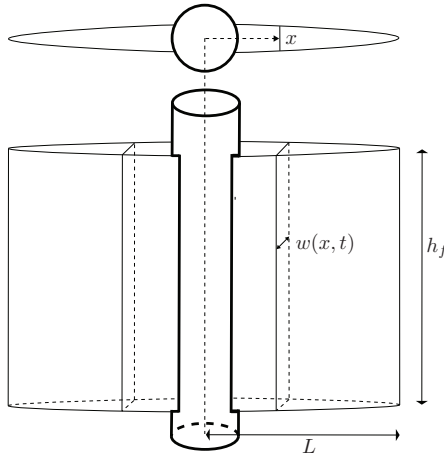


Figure 2.2: Geometry of a KGD model with L being the fracture length, h_f the fracture height, and $w(x, t)$ the fracture width. The vertical cross section has a rectangular shape [61].

In contrast, the KGD model [79, 101] is based on a plane-strain assumption in the horizontal direction with the horizontal cross sections being independent. This is true for a fracture whose height is much larger than its length. Again, a view into the wellbore and a sideview on the wellbore with two identical fractures is shown in Figure 2.2. Since the fluid pressure gradient is highly sensitive to crack width, the crack-tip processes have a much greater importance and dominate propagation. The fluid flow resistance for a narrow rectangular slit determines the pressure gradient in propagation direction [80]:

$$q = \frac{Q}{h_f} = -\frac{w^3}{12\mu} \frac{dp}{dx}. \quad (2.3)$$

The crack tip region is identified as the area with the largest fluid flow resistance where the pressure gradient is maximal indicated by a sharp drop in the fluid pressure. Since the crack opening has an elliptical shape in the horizontal plane, the maximum fracture width for this model is at the wellbore

$$w(0, t) = \frac{2(1 - \nu)}{G} L p_{net}. \quad (2.4)$$

In general, fracture widths predicted with the KGD model are larger than the fracture widths from the PKN model. Moreover, the pressure behavior with increasing crack length obtained from the KGD model distinguishes from the pressure predicted by the PKN model. According to the KGD model, the pressure decreases with time (and thus with crack length) whereas it increases in the PKN model [116].

Finally, the radial model considers a penny-shaped fracture that develops either horizontally from vertical wells or vertically from horizontal wells [62]. Due to the radial symmetry, the problem can be considered as two dimensional where the pressure field depends on the radial coordinate r (Figure 2.3).

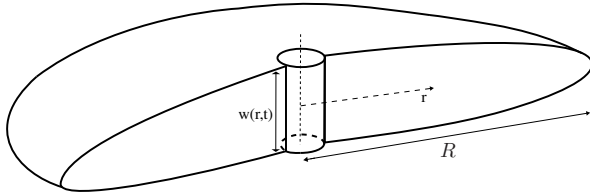


Figure 2.3: Geometry of a radial model with R being the fracture radius, and $w(r, t)$ the fracture width.

Thus, the flow rate per unit length can be formulated in terms of r

$$q = \frac{Q}{2\pi R_0} = -\frac{w^3}{12\mu} \frac{dp}{dr}, \quad (2.5)$$

where the injection rate Q is divided through the length of the wellbore's circumference $2\pi R_0$ with R_0 being the wellbore radius. Also here, the fracture width at the borehole for a KGD-type radial model can be stated as

$$w(R_0, t) = 2.15 \left[\frac{(1 - \nu)\mu q_0 R}{G} \right]^{1/4}. \quad (2.6)$$

2.1.2 Pseudo 3D models

The limitations of a constant height model were overcome through the development of three dimensional models [40, 41, 131, 152]. The simplest models which account for height growth are

known as pseudo 3D (P3D) models and attempt to capture the significant behavior of planar models at low computational costs [157]. The two main types of P3D models are lumped and cell-based. In the lumped models as depicted in Figure 2.4a, an elliptical geometry in the fracture length direction is assumed whereas the vertical profile is assumed to consist of two half ellipses joined at the center. At each time step, the vertical extension at the borehole and at the crack tip away from the borehole are calculated. The predefined shape of the half-ellipses is matched to these fracture endpoints. Furthermore, the fluid is assumed to flow along streamlines from the fracture inlet to the tip [80, 157, 182]. In contrast, cell-based P3D models (Figure 2.4b) divide the fracture in a series of connected cells. It is assumed that for each cell the plane strain condition holds and that all cells act independently.

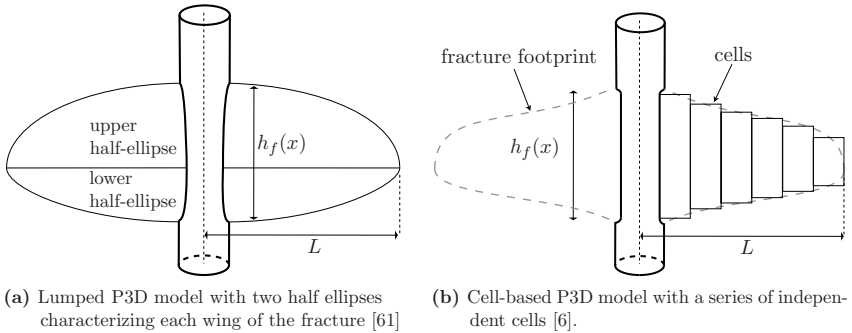


Figure 2.4: Conceptual representation of the P3D geometry.

2.1.3 Planar 3D models

The main restriction in planar 3D models is that the fracture is aligned and propagates perpendicular to the far-field minimum principal stress. Thus, its behavior is always planar. The fluid flow is not restricted to one dimensional streamlines as in P3D models, but is rather a two dimensional flow within the fracture which is controlled by the fracture geometry and the injection rate [2, 3, 182]. Since this is a coupled problem of the entire 2D fluid-flow field in the fracture and the 3D elastic response of the rock, the computational demand associated with the simulation is much larger than in P3D models. A two-dimensional mesh, either fixed [8, 43, 123, 168] or moving [14, 154], is used to discretize the fracture footprint and to solve for the fluid flow equations. This is directly related to interface capturing or tracking, respectively. Both representations are schematically depicted in Figure 2.5a and 2.5b.

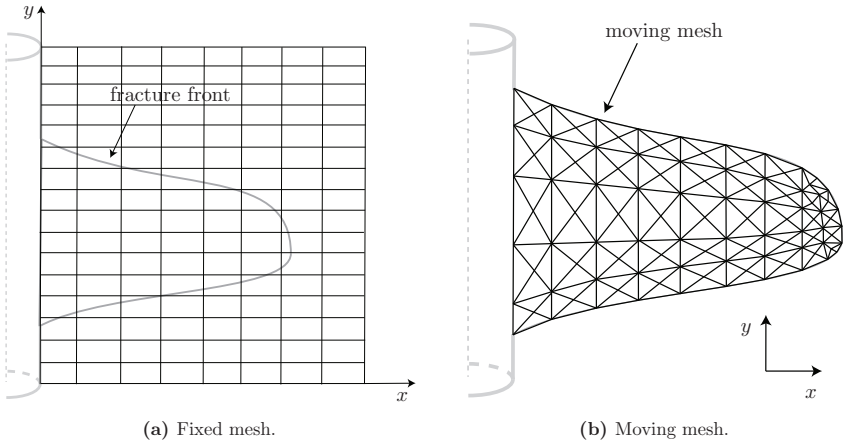


Figure 2.5: PL3D geometry based on different meshing strategies [6].

2.1.4 General 3D models

The fracture geometry of general 3D models is not restricted in its orientation and propagation to any particular plane. In fact, the fracture propagates arbitrarily within the 3D domain according to the principles of fracture mechanics [32]. The propagation direction depends on the actual fracture geometry and the stress state of the rock and can be determined by a number of different propagation criteria [9, 105]. Recent developments in high performance computing make the immense computational costs of general 3D models more tractable. Thus, 3D simulations become more feasible.

In this thesis, the focus is on simulating arbitrarily evolving fractures in three-dimensional domains in the context of hydraulic fracturing. The topology of the fracture is not restricted in any direction; rather, a fully coupled simulator is proposed which solves the fluid flow equations on the curved crack surface.

2.2 New insight into classical hydraulic fracture models

In recent years, a return to fundamental research was carried out with the objective to gain insight into the properties of the classical hydraulic fracture models. This is in opposition to developing entirely new models. A considerable effort has been made to understand the different regimes of propagation in order to deal with new complex and challenging environments in hydraulic fracturing [6]. First, the process in the vicinity of the crack tip was excessively analyzed based on the asymptotic theory. The results of the research significantly extended the

pioneering work and included not only the asymptotic behavior of the crack for zero-toughness or fluid lag, but also the intermediate asymptotic regimes [51, 52, 54, 56, 76, 110]. Incorporating the knowledge of the near-tip behavior into the numerical models increased the credibility and efficiency of computations [178]. Moreover, the analysis of the dominant dimensionless groups that control the fracturing process has revealed that hydraulic fractures can be categorized within a parametric space, confined by four limiting regimes. Within this parametric space, whose limits are controlled by the dominant physical processes of leak-off, storage, toughness, or viscosity, a fluid-filled fracture evolves with time. The main variables which specify the dominant regime are the injection rate, the properties of the fluid (mainly the viscosity), and of the rock such as elastic modulus, fracture toughness and leak-off coefficient [6]. Based on this framework, semi-analytical and numerical solutions [4, 5, 31, 71, 72, 77, 117, 118, 147, 159] have been constructed for different regimes and simplified geometries such as the penny-shaped crack and the plane-strain fracture that is assumed in the KGD model. They now serve as important benchmark solutions in the verification of newly developed numerical models.

Furthermore, the newly developed scaling law methodology also helps to interpret and extrapolate properly the experimental data from laboratory scale to field scale where fractures can extend to hundreds of meters. The hydraulic fracture growth, in turn, can be modeled for the desired scale by identifying the appropriate range of parameters. Unlike treatments in the laboratory scale, where toughness is the dominant factor for fracture propagation, in most field-scale treatments fracture growth is controlled by fluid viscosity, and toughness has only a minor role [6]. Consequently, the use of experimental results from the laboratory scale requires special care for the application on the field scale.

2.2.1 Propagation regimes

In order to create and propagate a fracture to appropriate dimensions, energy has to be transferred to the formation. This is accomplished by increasing the pressure (and stress) which is defined as stored energy per unit volume. Energy is also consumed by overcoming the highly viscous frac fluid's resistance while pumping the fluid into the formation [63]. Thus, hydraulic fracturing is governed by the two competing energy dissipating processes: the fracture toughness and fluid viscosity. A specific dominant propagation regime is related to each of these dissipating processes. In a fracture that is propagating in the viscosity-dominated regime, the energy dissipation due to the fluid viscosity dominates over the dissipated energy from the rock when splitting it apart [103]. In contrast, if the energy loss related to the rock's toughness prevails over the dissipated energy due to moving the viscous fluid, the propagation regime is toughness dominated. The dominant dissipative mechanism affects, to a great extent, the near-tip behavior of a fracture and consequently the whole crack growth process.

Parametric space

Based on scaling considerations, the dominant processes that govern the energy balance for

a specific hydraulic fracture treatment can be identified. Moreover, solutions of hydraulic fractures for simple geometries can be organized and classified within a parametric space [53]. Detournay et al. [57] formed a concept for the representation of the parametric space in a phase diagram whose corner points describe the governing asymptotic regimes. Such a phase diagram for the radial fracture is shown in Figure 2.6.

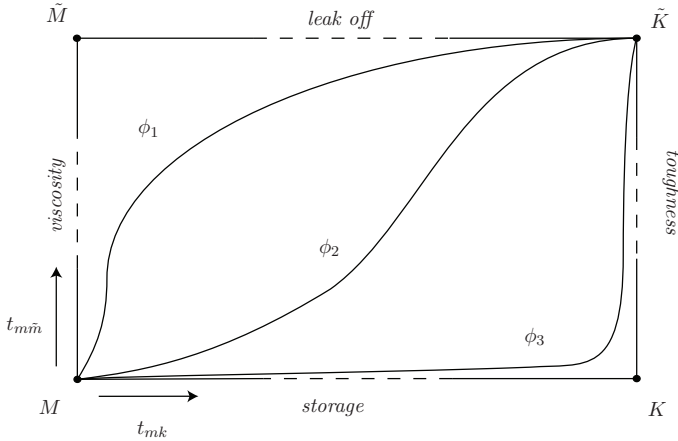


Figure 2.6: Parametric space for a radial fracture [57]. The fracture evolves with time within the parametric space and follows a trajectory characterized by a constant value of ϕ . Three solution trajectories are outlined here as an example.

Here, four primary asymptotic regimes can be distinguished: the two dissipative processes of viscosity and toughness, and the two competing fluid balance mechanisms of leak off and fluid storage in the fracture. Thus, the vertices of the phase diagram represent the solution for a propagating radial fracture dominated by storage-viscosity (M), storage-toughness (K), leak-off-viscosity ($\tilde{M}$), and leak-off-toughness ($\tilde{K}$).

Since fracture propagation is a transient process, two time scales can be identified which are related to the transition from the viscosity- to a toughness-dominated regime (t_{mk}) and from the storage- to the leak-off-dominated regime ($t_{m\tilde{m}}$) [57]. In the phase diagram, these transitions take place from one edge to the opposite as indicated in Figure 2.6. The two time scales are defined as

$$t_{mk} = \left(\frac{\mu'^5 E'^5 Q_0^3}{K'^{18}} \right)^{\frac{1}{2}}, \quad \text{and} \quad t_{m\tilde{m}} = \left(\frac{\mu'^4 Q_0^6}{E'^4 C'^{18}} \right)^{\frac{1}{2}}. \quad (2.7)$$

Here, the parameters are given in condensed form with $\mu' = 12\mu$ denoting the dynamic viscosity, $E' = \frac{E}{1-\nu^2}$ the plane strain modulus (Young's modulus and Poisson's ratio), $K' = \left(\frac{32}{\pi}\right)^{(1/2)} K_{IC}$

the fracture toughness, and $C' = 2C_l$ the leak-off coefficient. Q_0 is the constant fluid injection rate. Starting from the short-term solution at the M -vertex (viscosity- and storage dominated), the solution follows a trajectory characterized by a constant ratio ϕ of the two time scales with $\phi = \left(\frac{t_{mk}}{t_{min}}\right)^{\frac{14}{9}}$ and eventually ends at the K -vertex (dominated by toughness and leak off). Bungier and Detournay [30] confirmed experimentally the existence of the solution for the complete transition along the storage-edge (MK) from the viscosity- (M) to the toughness-dominated (K) regime.

2.2.2 Scaling of fundamental solutions

The main variables describing a radial fracture propagating in an impermeable material are the crack width w , the pressure p , and the crack radius R . They build the fundamental solutions which are the vertices of the phase diagram denoted with $M, \bar{M}, K, \bar{K}$ (Figure 2.6) and obtained from the scaled governing equations. In general, these similarity solutions have been constructed for limiting propagation regimes at the vertices of the parametric space and for some cases also for the transition along one edge of the space for simple fracture geometries (PKN, KGD, and radial fracture) [4, 5, 71–74, 76, 77, 147]. They serve as important benchmark solutions in the development of new hydraulic fracture simulators.

The solution of the problem for a radial fracture is given by means of a general scaling [53, 147]:

$$w(r, t) = \varepsilon(t) L(t) \Omega(\rho, \mathcal{P}(t)) \quad (2.8)$$

$$p(r, t) = \varepsilon(t) E' \Pi(\rho, \mathcal{P}(t)) \quad (2.9)$$

$$R(t) = L(t) \gamma(\mathcal{P}(t)) \quad (2.10)$$

Here, the dimensionless crack opening Ω , the pressure Π , and the fracture radius γ are scaled with a small number ε and a length scale L , which is of the same order as the radius ($O(L) \sim O(R)$). This form of scaling is motivated by elementary elasticity considerations with the aim that the order of the fracture width $w(r, t)$ scaled by the radius $R(t)$ should be of the same order as the product of net pressure and elastic modulus ($O(wR) \sim O(pE)$) [147]. The main dimensionless unknowns (Ω , Π , and γ) depend on the dimensionless radius $\rho = \frac{r}{R(t)}$ ($0 \leq \rho \leq 1$) and the evolution parameter $\mathcal{P}(t)$. The explicit forms of the parameters $\varepsilon(t)$, $L(t)$, and $\mathcal{P}(t)$ depend on the choice of the scaling which can be either the viscosity scaling or the toughness scaling. In the viscosity scaling, the evolution parameter becomes

$$\mathcal{K} = K' \left(\frac{t^2}{\mu^{1/5} Q_0^3 E^{1/3}} \right)^{\frac{1}{18}} \quad (2.11)$$

and is referred to as dimensionless toughness. In contrast, the dimensionless viscosity can be

identified as evolution parameter in the toughness scaling:

$$\mathcal{M} = \mu' \left(\frac{Q_0^3 E'^{13}}{t^2 K'^{18}} \right)^{\frac{1}{5}}. \quad (2.12)$$

Nonetheless, it can be shown that for this problem only one dimensionless parameter determines the global solution behavior, since both parameters are directly related to each other through

$$\mathcal{M} = \mathcal{K}^{-\frac{18}{5}}. \quad (2.13)$$

Thus, the evolution from the viscosity dominated regime, which is identified as the initial solution, to the toughness-dominated regime can be expressed in terms of only one single evolution parameter, either the dimensionless viscosity $\mathcal{M}$ or toughness $\mathcal{K}$ [147].

Given the set of parameters describing the fluid flow and material properties, the dimensionless evolution parameter determines the dominating fundamental solution which governs the propagation process. The point of transition from one dominating process to the other, and thus, the region in which a limiting solution is valid, has been discussed, e.g., in [31, 77, 147].

2.3 Governing equations of hydraulic fracturing

The main physical processes of hydraulic fracturing are the viscous fluid flow in the fracture associated with fluid leakoff in the formation, the deformation of solid media under preexisting compressive stresses and internal pressurization by the injected fluid, the crack growth, and, for some applications, the proppant transport. The following assumptions are usually made for a hydraulic fracture model [6]: i) the fluid flow is governed by the lubrication theory, ii) solid deformation is modeled using the theory of linear elasticity, and iii) the propagation criterion is given by the conventional energy-release-rate approach of linear elastic fracture mechanics (LEFM). Even under ideal assumptions of an isotropic homogenous linear elastic body and a laminar 1D flow, simulating the propagation of a fluid-induced fracture remains a challenge [55]. One reason lies in the non-linear, non-local, and time-dependent behavior of the governing equations. Another reason is the moving boundary nature of the problem which leads to additional complexities. In particular, this moving boundary has been a subject of debate in recent years [55].

2.3.1 Fluid flow within the fracture

The non-linear partial differential equations governing the flow of a viscous fluid in a deformable permeable channel can be described by means of the lubrication theory [133]. Consequently, the injection rate is assumed to be small enough that the flow in the fracture is laminar. A laminar flow subjected to friction between two parallel infinitely long plates is known as Poiseuille flow and can be derived from the Navier-Stokes equations and the conservation of

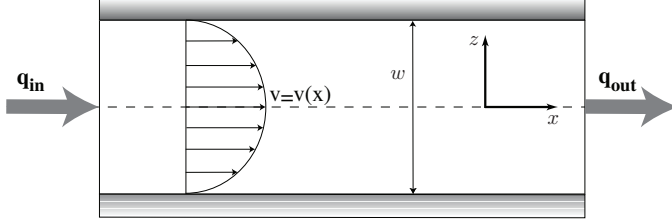


Figure 2.7: Plane Poiseuille flow with the volume rates q_{in} and q_{out} .

mass [66, 111]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \mathbf{v}, \quad (2.14)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.15)$$

where ρ is the fluid density, μ the dynamic viscosity, p the fluid pressure, and $\mathbf{v}$ the fluid velocity. The body force is neglected due to the high pressure required to propagate a fracture. The flow is assumed to be incompressible and at steady state [168]. Furthermore, the fluid velocity component in normal direction to the channel is assumed negligible. The consequence of this assumption is that the pressure is uniform for each channel cross section. Applying these assumptions to the Navier-Stokes equations for Newtonian fluids leads to

$$\frac{\partial^2 v_x}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}, \quad \frac{\partial^2 v_y}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial y} \quad (2.16)$$

Here, v_x and v_y are the fluid flow velocity components in x- and y-direction whereas the z-direction is normal to the crack faces, μ is the dynamic viscosity, and p is the fluid pressure. Together with the no-slip boundary conditions at the walls, $\mathbf{v}(z = \pm \frac{w}{2}) = 0$, equations (2.16) yield the velocity profile

$$v_x = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) \left[z^2 - \left(\frac{w}{2} \right)^2 \right], \quad v_y = \frac{1}{2\mu} \left(\frac{\partial p}{\partial y} \right) \left[z^2 - \left(\frac{w}{2} \right)^2 \right]. \quad (2.17)$$

The fluid flux Q is a function of the channel's cross sectional shape. For a fracture with its height h_f much larger than the length, a rectangular cross section can be assumed (see Figure 2.2). For a one-dimensional problem, the area is given by $A(x, t) = h_f \cdot w(x, t)$, with a

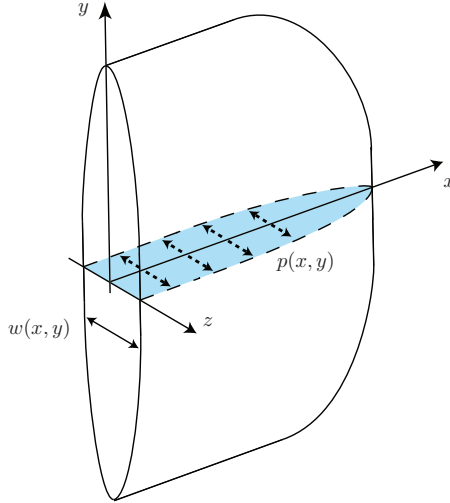


Figure 2.8: Schematic representation of a three-dimensional hydraulic fracture.

constant cross-sectional height h_f and a varying width $w(x, t)$. The associated fluid flux is

$$Q = \int_A \mathbf{v} \cdot d\mathbf{A} = \int_{-\frac{w}{2}}^{+\frac{w}{2}} \mathbf{v} \cdot h_f \, dz = -h_f \cdot \frac{w^3}{12\mu} \left(\frac{\partial p}{\partial x} \right). \quad (2.18)$$

Choosing an elliptical cross section (Figure 2.1) $A(x, t) = \pi h_f \cdot w_{max}$ with w_{max} being the maximum (width in the middle of the ellipse) leads to

$$Q = -\frac{\pi h_f w^3}{64\mu} \left(\frac{\partial p}{\partial x} \right). \quad (2.19)$$

Since the plan-strain condition is assumed, the model described herein considers a rectangular cross-section.

The total volume rate of flow of the fracture per unit length in the third dimension is obtained by integrating (2.17) across the fracture width w which is equivalent to using (2.18)

$$\mathbf{q} = \frac{Q}{h_f}. \quad (2.20)$$

Consequently, the flow rate in x- and y-direction is given by [42, 180]

$$q_x = -\frac{w^3}{12\mu} \left(\frac{\partial p}{\partial x} \right), \quad q_y = -\frac{w^3}{12\mu} \left(\frac{\partial p}{\partial y} \right). \quad (2.21)$$

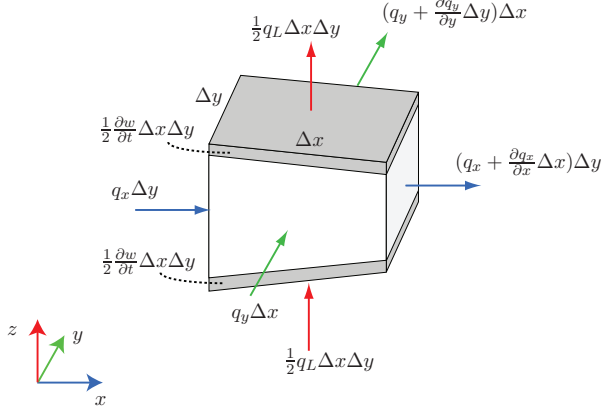


Figure 2.9: The local continuity condition on a control volume $\Delta x \Delta y$ and a varying height z . The grey area indicates the growing control volume which is the increase in fracture width w over time t [180].

Applying the local continuity equation to a 3D control volume (Figure 2.9) of sides Δx and Δy [180]

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} + q_L = 0 \quad (2.22)$$

and using the definition of the flow rate (2.21) yields the formulation of the nonlinear fluid flow equation, which is known as the Reynolds equation [15, 102, 136]

$$\frac{\partial w}{\partial t} - \nabla \cdot \left(\frac{w^3}{12\mu} \nabla p \right) + q_L = 0. \quad (2.23)$$

A source injection rate $Q(t)$ is incorporated into the formulation as an inflow boundary condition. q_L is the leak-off term and represents the fluid losses from the fracture into the surrounding formation.

Tip conditions

In general, a hydraulic fracture is characterized by two distinct moving fronts: the fluid front

and the fracture front [55]. They both advance in time with different velocities. Whether a lag is negligible depends mostly on the far-field stress [76]. The boundary conditions for a fluid-filled fracture with a non-negligible lag are

$$p = p_0 \quad \text{in } \Omega_\lambda \text{ and} \quad (2.24)$$

$$\mathbf{q} = \mathbf{q}_f = \mathbf{v}_f \cdot w_f \quad \text{at the fluid front } L_f. \quad (2.25)$$

In the lag region Ω_λ , the fluid pressure p is equal to the vapor pressure p_0 which is assumed to be zero. The latter condition expresses that the flux $\mathbf{q}_f$ is equal to the product of the average velocity $\mathbf{v}_f$ and the crack width w_f at the fluid front. The subscript f denotes quantities related to the fluid and is used here in order to distinguish between the fluid filled part of the crack and the lag zone.

When the fluid front reaches the crack front, the lag becomes negligibly small and the two previously distinct fronts effectively merge into one single front. This situation leads to a degenerated fluid flow equation at the tip [55]:

$$\mathbf{q} = 0, \quad (2.26)$$

since w is equal to zero and the velocity $\mathbf{v}$ at the tip is finite. Consequently, the solution from the fluid flow equation is no longer finite at the tip but yields a negative singularity in the pressure [76].

Global volume balance

Finally, the set of governing equations is completed with the mass conservation law for the fluid in the fracture [89]

$$\int_0^t Q(\tau) d\tau - \int_0^t \int_\Omega q_L d\tau d\Omega = \int_\Omega w d\Omega. \quad (2.27)$$

Here, the first term denotes the injected fluid into the fracture, and the second is the fluid leaked into the surrounding formation. The term on the right-hand side is the fracture volume obtained by integrating the fracture width.

The mass conservation law is formulated in terms of a fluid volume balance equation. This is valid for the case when the change in fluid density, and thus, fluid compressibility in the fracture is neglected [75, 111]. Nonetheless, for the specific case of laboratory experiments as conducted in [111], compressibility effects are accounted for in the global system containing the pumping system and the complete testing facility.

2.3.2 Fluid leak off

In hydraulic fracture treatments, where fractures evolve in porous and permeable materials, the rate of fluid leak-off is a critical factor in determining the fracture's geometry. Since the volume lost to the formation during the treatment lowers the overall efficiency, an accurate estimation of fluid loss has to be provided through a hydraulic fracture model [80]. The simplest model for taking into account fluid leaking into the surrounding formation is the one-dimensional formulation of Carter [33]. The fluid leak-off is described as a diffusion process through the lateral walls of the crack [63]. The outward flow is assumed to be normal to the propagation direction of the crack and the difference between the fluid and pore pressure is assumed to be small. Hence, the leak-off sink term in equation (2.23) is defined as

$$q_L = -\frac{2C_L}{\sqrt{t - t_0(\mathbf{x})}} \quad (2.28)$$

with $t_0(\mathbf{x})$ being the initiation time for leak-off at position $\mathbf{x}$. C_L is the Carter leak-off coefficient and is an empirical parameter that is related to the permeability and porosity of the rock [32]. The factor 2 in the numerator is due to the fact that the fluid is leaking through both faces of the crack.

Three separate leak-off processes are covered by Carter's model, namely the displacement and compressibility of reservoir fluid, the invasion of the formation by filtrate or fracturing fluid, and the buildup of an external filter cake [151, 157]. Instead of considering three different physical processes, Williams [174] considered three time periods of leak off. The initial period comes along with a high leak-off rate which is controlled by the resistance of the formation to fluid flow [157]. A filter cake has not formed at that time. In the second period, the external filter cake builds up. Finally, in the last period (dynamic leak off) the high velocity of the fracturing fluid prevents polymer deposition and the cake building stops.

2.3.3 Solid deformation

The solid deformation is modeled within the theory of linear elasticity [114]. The equation of motion for a body in static equilibrium can be written as

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega. \quad (2.29)$$

The indices $i, j = 1, 2, 3$ represent the three dimensions with respect to a rectangular coordinate system and f_i are the body forces defined in the domain Ω (see figure (2.10)). The Einstein summation convention of summing on repeated indices is used above.

The Cauchy stress tensor $\sigma_{ij} = \sigma_{ji}$ for an isotropic media is defined as

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2G \epsilon_{ij}, \quad (2.30)$$

where δ_{ij} is the Kronecker's delta and ϵ_{ij} the linear strain tensor associated with the displace-

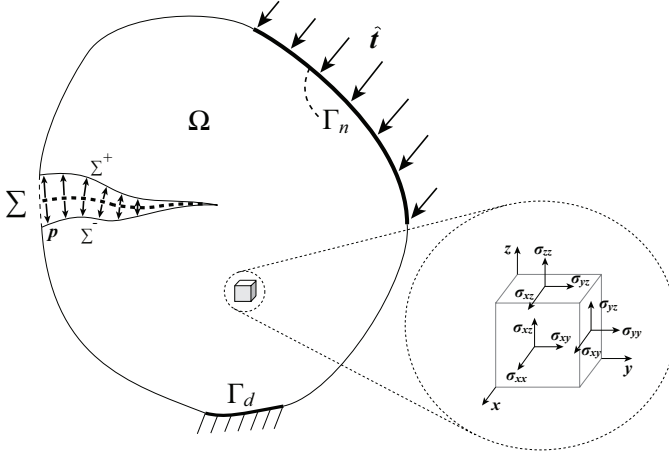


Figure 2.10: The domain contains a crack and is subjected to a load at the outer boundary and the crack face. An arbitrary point at equilibrium is highlighted.

ment u ,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (2.31)$$

There are two independent elastic moduli, the Lamé (λ) and shear (G) modulus, which read

$$\lambda = \frac{E \cdot \nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad G = \frac{E}{2(1 + \nu)}. \quad (2.32)$$

Here, the Young's modulus E and the Poisson's ratio ν are material parameters. Finally, the relationship between the stresses σ and strains ϵ can also be written in the generalized Hooke's Law

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (2.33)$$

with C_{ijkl} being the fourth order elasticity tensor

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (2.34)$$

Additionally, essential and natural boundary conditions are defined at the boundary Γ_d and

Γ_n , respectively, as

$$u_i = u_i^d \quad \text{on } \Gamma_d \quad (2.35)$$

$$\sigma_{ij}n_j = \bar{t}_i \quad \text{on } \Gamma_n \quad (2.36)$$

where u_i^d are the prescribed displacements and $\bar{t}_i$ the stresses at the outer boundary Γ_n .

Due to the presence of a crack which represent a discontinuity within the domain Ω , the stress and displacement field contain a jump along the crack interface Σ . The separation $\mathbf{w}$ between the two surfaces of the crack is defined by means of the displacement jump in normal direction $\mathbf{n}$

$$[[\mathbf{u}]]_\Sigma \cdot \mathbf{n} = (\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{n} = \mathbf{w}. \quad (2.37)$$

The superscripts $+$ and $-$ denote the two crack faces. In case of a pressurized crack, additional boundary conditions arise on the interface Σ

$$\sigma \mathbf{n}^+ = -\sigma \mathbf{n}^- = p \mathbf{n}^+ = -p \mathbf{n}^- \quad \text{on } \Sigma. \quad (2.38)$$

Here, the pressure p is a scalar field which is acting against the crack surfaces.

Singular integral formulation

It is noteworthy, that in hydraulic fracture modeling an integrational form of the elasticity equation between the fracture aperture w and the fluid pressure p_f is widely used [6, 42, 134].

$$p_f(x, y, t) - \sigma_c(x, y) = \int_{\mathcal{A}(t)} \mathcal{C}(x, y; \xi, \eta) \, d\xi d\eta. \quad (2.39)$$

Here, $\mathcal{C}$ is a non-local kernel function which contains the information about the elastic medium, σ_c is the prescribed *in situ* (or confining) stress field, and $\mathcal{A}(t)$ the fracture footprint. The non-local integral equation of elasticity for a penny-shaped crack in an infinite medium is [135]

$$p_f(x, y, t) - \sigma_c(x, y) = -\frac{E}{8\pi(1-\nu^2)} \int_{\mathcal{A}(t)} \frac{w(\xi, \eta, t)}{[(\xi-x)^2 + (\eta-y)^2]^{3/2}} \, d\xi d\eta. \quad (2.40)$$

The formulation yields a direct relationship between the pressure and the crack width. Furthermore, in direct conjunction with the displacement discontinuity method [44], the computational effort is less compared to a discretization of the 3D partial differential equations. However, employing a finite element scheme as is done in this work, a discretization of the whole domain is necessary, and thus the singular integral equation is not used here.

2.3.4 Fracture propagation

The theory of linear elastic fracture mechanics (LEFM) is used to model fracture propagation. According to the definition of fracture mechanics an explicit requirement is the existence of an identifiable crack [98, 105]. Hence, a main assumption here is that crack growth always starts from an initial crack with a given length l . A crack extension occurs when the applied loading overcomes the material's resistance against failure. In LEFM, this resistance arises due to the fact that surface energy is needed for the newly created crack surfaces. The material parameter which is associated to the fracture's resistance is known as fracture toughness K_{IC} and is, in LEFM, directly related to the energy dissipated during the fracturing process [181]. Since no energy is absorbed by non-linear or non-elastic effects such as plastic deformation, energy in the form of stress is directly transferred to create new fractures in the material [61].

Crack stability analysis

According to Griffith's theory [85], the global energy balance of the entire system containing a crack is

$$\dot{W}_{ext} = \dot{E}_{int} + \dot{E}_{kin} + \dot{E}_{\Gamma} \quad (2.41)$$

and states that the work performed per unit time by the applied forces $\dot{W}_{ext}$ is equal to the sum of the rate of change of internal energy $\dot{E}_{int}$, the rate of change of kinetic energy $\dot{E}_{kin}$, and the energy per unit time spent in increasing the crack area $\dot{E}_{\Gamma}$. For a slowly propagating crack and applied loads that are time independent, $\dot{E}_{kin}$ can be neglected.

Since all changes with respect to time are caused by changes in crack size, the energy balance during crack growth can be formulated as [78]

$$\frac{\partial W_{ext}}{\partial A} = \left(\frac{\partial U^e}{\partial A} + \frac{\partial U^p}{\partial A} \right) + \frac{\partial E_{\Gamma}}{\partial A}. \quad (2.42)$$

The internal energy E_{int} includes the elastic energy U^e and plastic work U^p . For brittle materials the rate of energy dissipated in plastic deformation is zero, and thus, the internal energy is equal to the elastic energy. Rearranging the balance equation, one obtains the energy available for crack growth which is referred to as the crack driving force [22, 23]

$$\frac{\partial}{\partial A}(W_{ext} - U^e) = \frac{\partial E_{\Gamma}}{\partial A} = 2\gamma. \quad (2.43)$$

The term on the left hand side is the rate of energy dissipated in crack growth and is defined as the energy release rate G with

$$G := -\frac{\partial \Pi}{\partial A} = -\frac{\partial}{\partial A}(U^e - W_{ext}). \quad (2.44)$$

Here, it is implicitly assumed through the equality in equation (2.43) that G meets the critical condition for crack propagation, which is the energy γ required to form 2 new crack surfaces. Thus, the rate of potential energy decrease during crack growth is a measure of the energy available for an increment of crack extension [9]. If $G < 2\gamma$, the crack is static and no extension occurs.

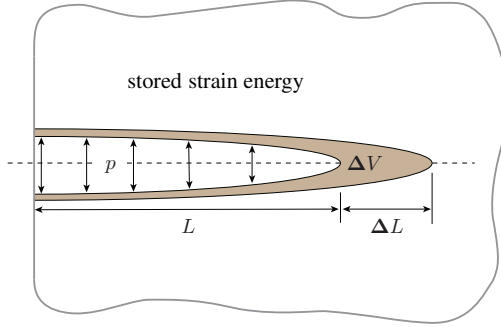


Figure 2.11: Half-crack in an infinite medium opened by a constant pressure p [167].

For a fluid-driven crack growth, the boundary conditions to the fluid flow analysis control whether a fracture propagates or not; however the propagation direction remains to be determined [168]. A basic assumption is that a fracture evolves in the direction of least resistance which is perpendicular to the direction of the minimum principal stress [137].

Propagation criterion

A number of different propagation criteria can be utilized to model the growth of a fracture. Local approaches such as the maximum circumferential stress criterion [65] and the maximum strain criterion [113] are based on the local fields at the crack tip. In turn, global approaches consider the energy distribution along the crack where the fracture extends, e.g., in the direction of the maximal strain energy release rate [93]. A commonly used propagation condition in hydraulic fracturing is based on the stress intensity factor, which is a measure for the strength of the inverse square-root stress singularity at the tip. According to this conventional LEFM criterion, the fracture propagates if the stress intensity factor for mode I, K_I , is in balance with the fracture toughness K_{IC} of the rock:

$$K_I = K_{IC}. \quad (2.45)$$

It was reported that for 3D simulations in XFEM, the determination of the stress intensity factor bears some difficulties and uncertainties, see [140, 179]. In this thesis, the propagation direction is obtained through the maximum circumferential stress criterion which was

proven to have good accuracy and efficiency in the context of XFEM [16, 69]. Expressing the circumferential stress $\sigma_{\theta\theta}$ in terms of the stress intensity factors for mode I and II:

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \left[\frac{K_I}{4} \left(3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{4} \left(-3 \sin \frac{\theta}{2} - 3 \sin \frac{3\theta}{2} \right) \right] \quad (2.46)$$

Here, r and θ are the coordinates of a polar coordinate system with its origin in the crack tip. For the propagation angle θ_c the relation

$$\left. \frac{\partial \sigma_{\theta\theta}}{\partial \theta} \right|_{\theta=\theta_c} = 0 \quad (2.47)$$

holds. This criterion can also be used in 3D simulations for situations where mode III loading which refers to out-of-plane shear can be neglected.

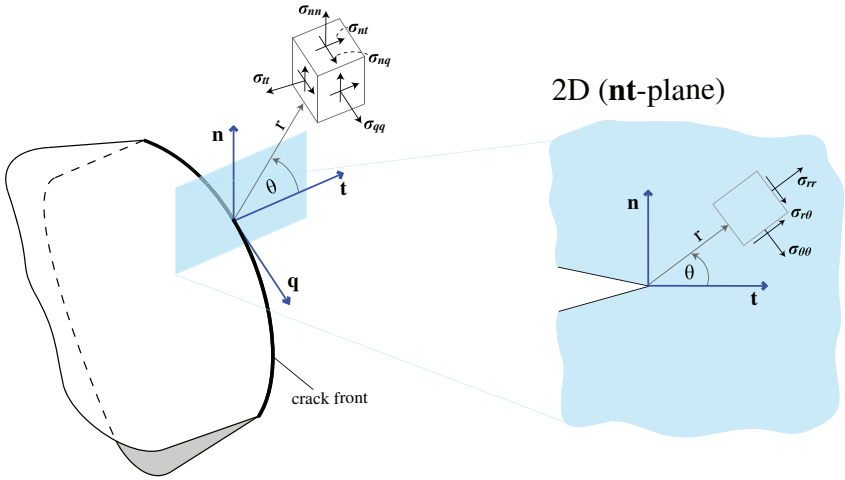


Figure 2.12: Definition of a local coordinate system $(\mathbf{n}, \mathbf{t}, \mathbf{q})$ and the stress state at the crack front.

Alternatively, the hoop stress can also be computed by means of the local stress field at the crack front. Based on a local coordinate system $(\mathbf{n}, \mathbf{t}, \mathbf{q})$ at the crack front (see figure (2.12)), the hoop stress in the direction of propagation is given by [16]

$$\sigma_{\theta\theta} = \sigma_{nn} \sin^2(\theta_c) + \sigma_{tt} \cos^2(\theta_c) - \sigma_{nt} \sin^2(2\theta_c) \quad (2.48)$$

Since the stress in the direction $\mathbf{q}$ tangent to the crack front is neglected, this formulation is valid when the out-of-plane shearing mode is vanishingly small ($K_{III} \approx 0$).

2.4 Numerical methods for hydraulic fracturing

This section gives a brief overview of the numerical methods used to solve a typical hydraulic fracture problem which couples fluid flow, solid deformation, and fracture propagation.

Elasticity equations

Two main approaches might be used to discretize the elasticity equation. In the first approach, the discretization of partial differential equations can be performed by means of a finite difference (FDM) or finite element method (FEM) [7, 27, 34, 92]; however, the computational requirements rapidly increase with the extension of the problem to 3D since the mesh has to be adapted according to the propagating crack [168]. Furthermore, a large reservoir volume in the vicinity of the fracture has to be properly discretized in order to obtain accurate results for the crack opening [6]. Another accepted approach is the boundary element method (BEM) [44]. In particular, the displacement discontinuity (DD) method, which belongs to the family of indirect boundary-integral methods [168], employs an integral equation formulation as given in (2.39). Though this method is limited to cases where fundamental solutions are available, it is well suited for hydraulic fracture problems since a direct relation between the pressure and the crack width is obtained. Moreover, multi-layered reservoirs have been approximated in different ways in the context of boundary integral schemes. A discussion can be found in Adachi et al. [6].

Fluid flow equations

A finite element, finite difference, or finite volume [171] scheme is usually used to solve for the pressure field in the fluid flow equations. For this purpose, the discretization of the fluid flow equations in classical hydraulic fracture models is performed over the fracture footprint. For a non-planar fracture that is arbitrarily formed within the 3D domain, the fluid pressure has to be obtained by integrating the flow equations over a manifold. This special approach in the context of the finite element method is presented in this thesis.

Interface meshing

For hydraulic fracture propagation, two meshing strategies are pursued: the Lagrangian description with a moving mesh and the Eulerian description with a fixed mesh. In the case of a Lagrangian mesh, the fracture footprint is typically described by triangular elements. When the crack propagates, new surface area is created and the fracture footprint becomes larger. These advancing fracture sections are accounted for by adding new elements at the crack front of the mesh. Additionally, the entire fracture footprint is periodically re-meshed in order to avoid a loss in accuracy [59]. A drawback of this approach is that the state variables such as crack opening, pressure, and leak-off data have to be mapped to the new mesh after each

re-meshing step, invariably leading to additional interpolation errors. Further difficulties arise for layered materials where the fracture elements partially cross the interface of layers and cause less accurate results [6]. In the Eulerian description with a fixed mesh, no interpolation issues as known from the Lagrangian description are present anymore. Nonetheless, due to the steep pressure gradients near the fracture front a finer mesh is needed to capture this solution behavior properly. A sub-meshing strategy near the fracture front when the crack is advancing is only achieved at larger computational costs. Even re-meshing of a fixed mesh can become necessary in order to achieve a reasonable mesh density for late-time solutions. Within the framework of the Eulerian description, interface capturing methods are used to locate the fracture front; the most common are the level set [129, 150] and volume of fluid (VOF) [88]. Using the level set method, the interface is captured by the zero level set of a scalar function. Then, there are no restrictions on the topology of the crack front; however, this method can become computationally expensive since multiple iterations of a coupled system of equations within one growth step have to be performed to obtain an accurate fracture footprint [6].

2.5 Literature overview for XFEM in hydraulic fracturing

In order to solve for the coupled problem of fluid flow, solid deformation, and fracture propagation, an XFEM approach is employed in this thesis with a twofold description of the crack: an implicit description realized through the level set concept and an explicit description based on a meshing of the fracture footprint. Since this numerical approach is attributed great significance in this work, a brief overview is given in this section about simulating hydraulic fracturing processes with XFEM.

Since the first papers published on XFEM in 1999 [18, 119], only a small number of researchers applied XFEM on hydraulic fracture problems. Although XFEM was used in coupled hydro-mechanical problems [17, 95, 96, 100, 120, 173] and fluid flow in fractured porous media [142, 143], the first works on XFEM related to hydraulic fracturing were published in 2009 by [45, 90, 108, 141] which were all 2D applications.

Lecampion [108] presented special tip-enrichment functions for the viscosity and the toughness dominated regime by generalizing the four standard tip-enrichment functions used in classical linear elastic fracture mechanics (LEFM) problems. It was shown, that the enrichment functions for the toughness dominated regime conform exactly to the known LEFM enrichment functions. The enrichment functions, in turn, for the viscosity dominated regime are obtained by only changing the exponent defining the asymptotic behavior. Therefore, the fracture front is tracked by using the proper tip asymptotics; however, no coupling and fracture growth were considered in this work.

Ren et al. [141] modeled hydraulic fracturing in concrete with XFEM with the standard enrichment functions from LEFM. It is noteworthy that no fluid flow model was used, rather a prescribed pressure was imposed on the crack interface.

Dahi-Taleghani [45] considered in his dissertation the interaction of induced and natural frac-

tures for the stimulation of tight gas areas. The hydraulic fracturing process was fully coupled by an iterative coupling scheme. The crack growth was modeled by the XFEM approach based on the theory of linear elastic fracture mechanics (LEFM). Dahi-Taleghani [45] also published a number of papers on XFEM and hydraulic fracturing. For example, he studied fracture re-initiation as a possible branching mechanism [46] and simultaneous growth of multi-branch hydraulic fractures and their interaction with natural fractures [47, 48].

Finally, Huang et al. [90] simulated hydrocarbon reservoirs containing subsurface features with XFEM. He also proposed a new enrichment scheme to model fractures and conduits in porous media flow problems in 3D [91]; however, a coupling to the geomechanics was not provided.

Gordeliy and Peirce [81] investigated two hydrodynamic coupling schemes in 2D-XFEM which can cope with a representative variety of tip behaviors and propagation regimes typically encountered in hydraulic fracture modeling. For zero-lag problems, with a tip singularity in the pressure field, a novel mixed algorithm was proposed that incorporates the tip width asymptotic solution into the XFEM framework. Additionally, they presented in [82] a level set strategy that also exploits the asymptotic solution to provide a robust procedure to locate the free boundary. Two enrichment strategies were discussed that were used to represent the fracture tips: a scheme with full tip enrichment and a simpler more efficient, scheme in which the tip asymptotic behavior is only imposed in a weak sense.

Keshavarzi and Mohammadi [99] investigated 2D hydraulic fracturing propagation with XFEM in naturally fractured reservoirs and studied the dominant factors governing the propagation direction of hydraulic fractures in the presence of natural fractures.

Chen et al. [35] presented an ABAQUS implementation in 2D of the XFEM for hydraulic fracture problems. He used a fully coupled monolithic approach for the discretization of the pressure and the displacement field. Although, his fully coupled approach was not verified against fully coupled solutions, rather against decoupled solutions for the two limiting cases of a toughness-dominated and viscosity-dominated plane-strain hydraulic fracture. Furthermore, the author notes that the fluid pressure predictions of his model were unsatisfactory.

A 2D fracture mapping approach combined with the XFEM was presented by Lamb et al. [107] to simulate the coupled deformation and fluid flow in fractured porous media. The fracture mapping approach combines the dual continuum concept of Barenblatt [13] and the discrete fracture flow model approach. The flow interaction between the porous matrix and the existing fractures was established via a transfer function.

In [121, 122] a fully coupled numerical model was developed for modeling 2D hydraulic fracture propagation in porous media using XFEM in conjunction with a cohesive zone model. Here, the XFEM and the generalized Newmark scheme [124, 184] were used for the spatial and time domain discretizations, respectively.

Finally, non-planar fracture propagation in three dimensions modeled with XFEM was presented by Gupta and Duarte [87]. The focus was on propagating fractures with complex geometries for hydraulic fracture applications. Since the crack was subjected to a uniform

time-dependent fluid pressure, no coupling to the fluid flow was conducted. A fully coupled three dimensional hydraulic fracture simulator, which accounts for the solid deformation and crack propagation as well as for the fluid flow, is not presented in the context of XFEM so far.

3 The extended Finite Element Method (XFEM)

A propagating fracture that arises due to a hydraulic stimulation manifests itself in the form of a discontinuity in the displacement and stress fields. Moreover, according to LEFM, the stresses become singular at the crack front. In the classical finite element method (FEM), non-smooth characteristics such as discontinuities and singularities cause approximation errors because the FEM-approximation is built upon polynomial shape functions which require sufficiently smooth solution behavior. The finite element mesh has to be refined in those areas. In the case of a propagating crack, the element edges of the FE-mesh have to be aligned with the crack geometry. Furthermore, the domain has to be constantly remeshed while the crack is growing. These issues can be overcome within the framework of the extended finite element method (XFEM).

The key feature of the XFEM is its ability to solve problems with a discontinuous and singular solution behavior [18, 119]. In particular, the XFEM has shown to be advantageous for crack problems. The crack which represents a discontinuity in the displacement field can be arbitrarily located within the domain and no alignment with the element edges is required. This is achieved by means of the partition of unity method (PUM) [11, 115] which uses *a priori* knowledge about the differential equation being solved and includes it into the finite element approximation space. The *a priori* knowledge of a discontinuous displacement field at the location of the crack can be used in order to enrich the ansatz space by additional functions which are able to reproduce such non-smooth behavior. In contrast to the early work on PUM, which applied the enrichments globally, the XFEM differs in that only a local area around the discontinuous field is enriched. In conjunction with the XFEM, a finite element mesh could be used which is independent of the crack geometry and with no or minimal need for remeshing. Nowadays, XFEM has been applied to a number of different problems such as solidification [37], dislocations [169], holes and inclusions [163], particulated flows [172], two-phase flows [36], boundary layer problems [156], and cracks as in this work.

It is noteworthy that the generalized finite element method (GFEM) [160, 161] also follows the idea of PUM and XFEM and was developed parallel to the XFEM. An overview and a discussion of the concepts of the methods can be found in [70].

3.1 Crack description

In this section, an overview is given of the two crack descriptions used in this work: explicit and implicit. For a more detailed discussion on these crack descriptions, the reader is referred to [69]. As mentioned in the introduction of this section, non-smooth features in the partial differential equations can be properly addressed by the XFEM. The location of these discontinuities and singularities is often defined implicitly by means of the level-set method [12, 127, 128]. Moreover, using the XFEM together with the level-set method has become a quasi-standard, since the level-set field is ideal for determining the enriched region and evaluating the enrichment functions. Nevertheless, complications arise when using a purely implicit approach in the context of fracture propagation problems. While the crack grows, propagation steps have to be performed in order to keep the crack geometry up to date. With a purely implicit description, the level-set field also has to be updated, which is a non-trivial task (see e.g. [84, 129, 164]). In contrast, a representation of the crack geometry by means of a finite element mesh provides the benefit of a relatively simple crack update. This crack description is called *explicit*. Nonetheless, a fully satisfying coupling of a purely explicit formulation with the XFEM is not readily achieved [70, 162]. Therefore, a mixed approach is used in this work to take advantage of both descriptions for the coupling with the XFEM and the crack update.

3.1.1 Explicit crack description

The crack geometry is given explicitly in terms of a polygonal surface. In 2D, a set of connected (straight) line segments which build a polygon is used to discretize the crack path. Additionally, the crack tips are defined by their position and their inclination angle. In 3D, the crack represents not a line anymore, but an arbitrarily curved surface. Flat triangles are connected here to a closed surface mesh which represents the crack geometry. Figure 3.2 illustrates the crack description in two and three dimensions. A local coordinate system can be defined at the crack tips (crack front) and new elements can be simply added to the given crack in order to extend it. $\vec{n}$ is the normal vector perpendicular to the crack surface. $\vec{q}$ is the vector tangential to the crack front, and $\vec{t}$ is the extension vector orthogonal to the crack front. Since these three vectors build a local coordinate system at the nodes of a discretized crack front, they are not necessarily orthogonal but linearly independent. They are determined by averaging the vectors that are orthogonal to the facets of the crack mesh.

For a triangular facet k with the vertices A , B , and C the unit normal vector $\vec{n}_k^e$ is the cross product of the distance between these vertices

$$\vec{n}_k^e = \frac{(\vec{r}_B - \vec{r}_A) \times (\vec{r}_C - \vec{r}_A)}{\|(\vec{r}_B - \vec{r}_A) \times (\vec{r}_C - \vec{r}_A)\|}, \quad (3.1)$$

with $\vec{r}$ being the corresponding position vector. By averaging the normal vectors with the area

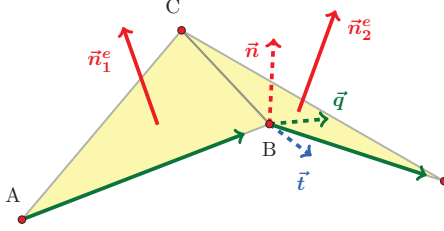


Figure 3.1: The computation of the normal and tangential vectors $\vec{n}$, $\vec{t}$, and $\vec{q}$ at the element node is illustrated on two neighboring facets. For each facet k , a unit normal and tangential vector $\vec{n}_k^e$ and $\vec{q}_k^e$ is determined. The tangential vectors $\vec{q}_k^e$ at the element edges (which are assumed to be part of the crack front) are indicated by the green arrows; however they are not normalized here for illustration purposes. The nodal vectors are finally obtained by averaging the element vectors.

A_k of the neighboring elements one obtains the nodal normal vector

$$\vec{n} = \frac{\vec{n}'}{\|\vec{n}'\|} \quad \text{with} \quad \vec{n}' = \frac{\sum_k A_k \cdot \vec{n}_k^e}{\sum_k A_k}. \quad (3.2)$$

The tangential vector at the crack front $\vec{q}^e$ is calculated from the difference of the nodal position vectors of that element edge which belongs to the crack front. This situation is illustrated in Figure 3.1. The unit tangential vector for each element is

$$\vec{q}_k^e = \frac{(\vec{r}_B - \vec{r}_A)}{\|(\vec{r}_B - \vec{r}_A)\|} \quad (3.3)$$

and leads to the nodal unit tangential vector

$$\vec{q} = \frac{\vec{q}'}{\|\vec{q}'\|} \quad \text{with} \quad \vec{q}' = \frac{\sum_k l_k \cdot \vec{q}_k^e}{\sum_k l_k}, \quad (3.4)$$

and with l_k being the length of the element edge. Finally, the nodal tangential extension vector $\vec{t}$ is perpendicular to the plane spanned with $\vec{n}$ and $\vec{q}$:

$$\vec{t} = \vec{q} \times \vec{n}. \quad (3.5)$$

The direction of crack growth is determined by means of the local stress field as discussed in Section 2.3.4. The extension procedure is highlighted for the simple case of a tangential extension in Figure 3.3. The extension vectors are scaled according to the local maximum

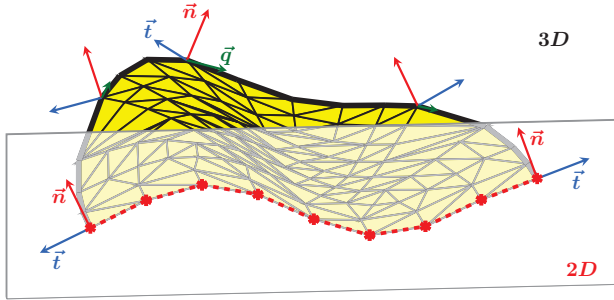


Figure 3.2: An explicitly described crack is illustrated here in 2D and in 3D. The crack in three dimensions is an arbitrarily curved surface approximated by flat triangles. The thick black line represents the crack front. The two dimensional representation is illustrated in the cross section where the crack is described as a curved line and composed of straight line segments (red dashed line). Additionally, a local coordinate system is given at the crack tips for the 2D $(\vec{n}, \vec{t})$ and at some nodes at the crack front for the 3D $(\vec{n}, \vec{t}, \vec{q})$ cases, respectively.

circumferential stress which is described in more detail in [69]. New triangles are built by means of the extension vectors (denoted as $\vec{t}$) and added to the existing crack geometry. As can be seen in Figure 3.3, the propagation step can be performed equivalently in 2D and in 3D and, therefore, allows a straightforward extension from two to three dimensions.

Extending the crack in the tangential direction, as depicted in Figure 3.3, and scaling the extension vectors such that the new elements reach the domain limits leads to the definition of the *extended* crack path (surface). The definition of the extended crack is crucial since it divides the whole domain into two parts. These two domain parts can be assigned with a positive (+) and a negative (−) sign by applying the level-set function ϕ_3 .

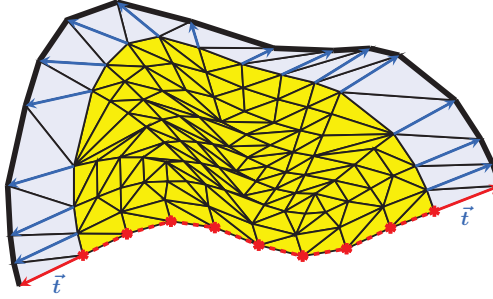


Figure 3.3: The propagation step is performed by creating new flat triangles by means of the extension vectors $\vec{t}$. The new elements are added to the crack front of the existing crack geometry.

3.1.2 Implicit crack description

The implicit crack description is realized by means of the level-set approach. In general, an implicit representation of an n -dimensional hypersurface Σ is defined as

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \phi(\mathbf{x}) = 0\}, \quad (3.6)$$

with $\mathbf{x}$ being a point on the hypersurface implicitly defined by the function $\phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ [97]. The hypersurface is the zero-level (or isocontour) of the implicit function and is one dimension lower than the region where it is embedded in [97].

A set of three level-set functions ϕ_1 , ϕ_2 , ϕ_3 is used in this work to describe the location of the crack path in 2D and the crack surface in 3D, respectively [69]. Moreover, in conjunction with the XFEM, the level-set method is particularly well suited. First, the local regions to be enriched are determined directly from the level-set functions. Second, the definition and evaluation of the enrichment functions are typically based on the level-set functions as can be directly seen from the definition of the step-enrichment Ψ_{step} (3.15) which will be discussed later in this chapter. Finally, the numerical integration over the elements through which the crack passes can be applied by easily decomposing these cut elements. The decomposition is based on the information given by the level-set functions [70].

The three level-set functions are defined as follows [69]:

1. $\phi_1(\mathbf{x})$ is the (un-signed) distance function to the crack path (2D) or surface (3D). That is, the level-set value at position $\mathbf{x}$ is the shortest distance to the crack path/surface.
2. $\phi_2(\mathbf{x})$ is the (un-signed) distance function to the crack tip (2D) or front (3D). That is, the level-set value at position $\mathbf{x}$ is the shortest distance to the crack tip/front.

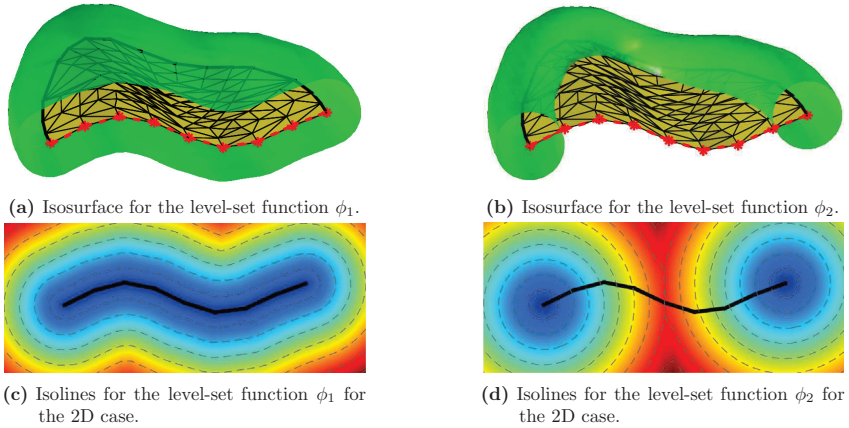


Figure 3.4: The level-set functions ϕ_1 and ϕ_2 are visualized here in 3D (a,b) and in 2D (c,d). Isosurfaces (in 3D) and isolines (in 2D) represent a fixed distance to the crack surface and crack front, respectively. The colors in the 2D plots represent the specific level-set values with a small distance in the blue and a bigger distance in the red area.

3. $\phi_3(\mathbf{x})$ is a signed distance function to the *extended* crack path (2D) or surface (3D). Accordingly, the domain is assigned as positive (+) or as negative (−) with the interface being represented by the extended crack. That region, in which the normal vectors on the crack surface are pointing is defined as positive.

By means of this set of level-set functions a local coordinate system can be defined at the crack front which, in turn, is extremely useful in realizing the crack propagation step. The coordinate system implied by the level-set functions can be either formulated in terms of a polar coordinate system (r, θ)

$$r(\mathbf{x}) = \phi_2(\mathbf{x}) \quad \text{and} \quad \theta^*(\mathbf{x}) = \sin^{-1} \frac{\phi_3(\mathbf{x})}{\phi_2(\mathbf{x})}, \quad (3.7)$$

or a local cartesian coordinate system (a, b) with its origin at the crack tip/front

$$a^*(\mathbf{x}) = \sqrt{[\phi_2(\mathbf{x})]^2 - [\phi_3(\mathbf{x})]^2} \quad \text{and} \quad b(\mathbf{x}) = \phi_3(\mathbf{x}). \quad (3.8)$$

θ and a are determined from θ^* and a^* by decomposing the area at the crack front into different regions. This is discussed in detail in Fries and Baydoun [69] and will not be discussed in this section.

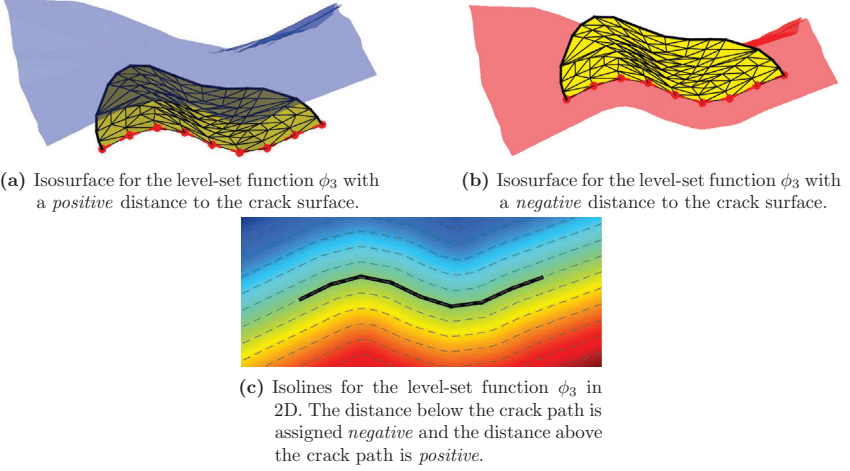


Figure 3.5: The level-set function ϕ_3 is visualized here in 3D (a,b) and in 2D (c). Isosurfaces (in 3D) and isolines (in 2D) represent a fixed distance to the *extended* crack surface/path. The colored contours in the 2D case represent the distance to the extended crack path with positive level-set values in the blue and negative values in the red area.

As a result, the following relationship holds

$$a(\mathbf{x}) = r(\mathbf{x}) \cdot \cos \theta(\mathbf{x}), \quad (3.9)$$

$$b(\mathbf{x}) = r(\mathbf{x}) \cdot \sin \theta(\mathbf{x}). \quad (3.10)$$

3.2 The XFEM formulation

In general, the XFEM approximation of an arbitrary scalar quantity f is formulated as

$$f^h = f_{FEM}^h + f_{Enriched}^h. \quad (3.11)$$

Here, the approximated variable f^h is composed by a classical finite element approximation f_{FEM}^h and an additional (enriched) approximation $f_{Enriched}^h$ which accounts for the non-smooth solution behavior. The standard XFEM approximation of the scalar function $f^h(\mathbf{x})$ in a d -dimensional domain $\Omega \in \mathbb{R}^d$ with multiple enriched terms is given by

$$f^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) f_i + \sum_{j=1}^m \sum_{i \in I_j^*} N_i^*(\mathbf{x}) \cdot \Psi_j(\mathbf{x}) \alpha_i^j, \quad (3.12)$$

with $N_i(\mathbf{x})$ being the standard FE shape functions defined at each node i within the subset I , and f_i the nodal unknowns of the standard FE approximation. The functions $N_i^*(\mathbf{x})$ are also standard FE shape functions defined within the subset I^* . They build a partition of unity in the enriched elements [70] which is crucial for the XFEM. A typical choice for the shape functions is $N_i^*(\mathbf{x}) = N_i(\mathbf{x})$ (which will be the case in this work); however, $N_i^*(\mathbf{x})$ can be chosen independently from $N_i(\mathbf{x})$. $\Psi(\mathbf{x})$ is the global enrichment function which incorporates the special knowledge about the solution behavior (e.g. jumps, kinks, singularities, etc.) into the approximation space [70] and the coefficients a_i^j are the corresponding nodal unknowns associated with the enrichments. Through the multiplication of the global enrichment function with the FE shape function, $N_i^*(\mathbf{x}) \cdot \Psi(\mathbf{x})$, the *local* enrichment function is obtained [1].

The XFEM-approximation (3.12) has two main limitations. First, the imposition of the Dirichlet boundary conditions is not straightforward anymore since additional degrees of freedom from the enrichment terms have to be considered at the nodes. Second, the final solution at the nodes cannot be interpreted solely on the basis of the FEM-unknowns anymore. Ideally, the equality $f^h(\mathbf{x}_i) = f_i$ should hold within the whole computational domain. The Kronecker- δ property of the shape functions, $N_i(\mathbf{x}_j) = \delta_{ij}$, guarantees this equality for vanishing enrichment terms at the nodes (since the standard finite element approximation is recovered then). Of course, if the enrichment terms are non-zero at the enriched nodes $i \in I_j^*$ it follows that $f^h(\mathbf{x}_i) \neq f_i$. A shifted version [19] of the XFEM-approximation (3.12) is therefore introduced in order to maintain the desired properties:

$$f^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) f_i + \sum_{j=1}^m \sum_{i \in I_j^*} N_i^*(\mathbf{x}) \cdot [\Psi_j(\mathbf{x}) - \Psi_j(\mathbf{x}_i)] a_i^j. \quad (3.13)$$

Due to the *shifting* of the global enrichment function, the enrichment terms vanish at the nodes and recover the equality $f^h(\mathbf{x}_i) = f_i$.

3.3 Enrichments schemes for crack problems

The crack problem yields a twofold non-smooth solution behavior. First, along the crack path (or in 3D crack area), the deformation and stress fields have a strong discontinuity—a jump within these fields can be observed across the fracture. This comes from the fact that the crack faces move in opposite directions when the crack opens. Second, according to the theory of linear elasticity, the stress field at the crack tip is singular. This situation is illustrated in Figure 3.6 where a crack path is shown in a two-dimensional domain Ω . Two nodal subsets, I^{step} and I^{tip} , with different enrichments are present in the domain. Enriched nodes in I^{step} are located along the crack path where elements are completely cut by the interface. Enriched nodes in I^{tip} can be either chosen within a prescribed radius, r , around the crack tip or by choosing only the crack-tip element nodes to be enriched. The corresponding *step*-enriched and *tip*-enriched elements are indicated by the colored regions in Figure 3.6. Elements that are only partly enriched or located between the two enrichment zones are called blending elements

and are colored grey. The enrichment function cannot be reproduced exactly in the blending elements since $N_i^*(\mathbf{x})$ does not build a partition of unity there. Issues with blending elements are widely discussed (*e.g.* [38, 68, 83, 170]). If all nodes belonging to a single element are enriched, this element is called a reproducing element and the enrichment function can be reproduced exactly in it.

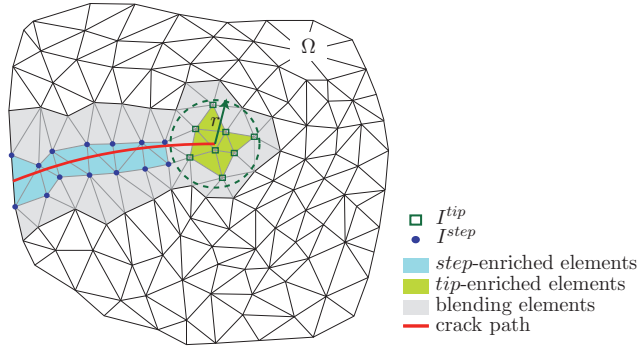


Figure 3.6: Domain with a crack illustrating the subsets of enriched nodes I^{step} and I^{tip} and the corresponding enriched and blending elements.

Figure 3.7 illustrates the jump in the displacement field u and the crack-tip singularity in the stress σ which is present LEFM.

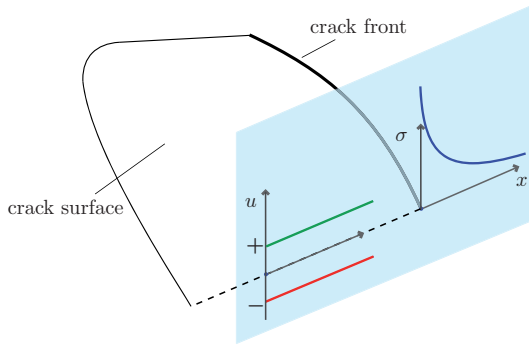


Figure 3.7: Illustration of the discontinuous displacement u across the crack surface and the singular stress σ at the crack front.

$$u^h(\mathbf{x}) = \underbrace{\sum_{i \in I} N_i(\mathbf{x}) u_i}_{\text{continuous}} + \underbrace{\sum_{j \in I^{step}} N_j^*(\mathbf{x}) \cdot \Psi_{step}(\mathbf{x}) a_j + \sum_{k \in I^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(r, \theta) b_k^m \right)}_{\text{discontinuous}} \quad (3.14)$$

In order to consider these discontinuities and singularities in crack problems, two sets of enrichment functions are incorporated into the approximation space. The first term on the right-hand side describes the classical FEM approximation with continuous shape functions $N_i(\mathbf{x})$ and nodal unknowns u_i . The second term accounts for the discontinuity in the displacement field across the crack path by incorporating a step-function Ψ_{step} with additional nodal unknowns a_i into the enrichment space. The tip region is enriched with a set of enrichment functions, $\Psi_{tip}^m(r, \theta)$, $m = 1, 2, 3 \text{ \& } 4$, that account for the singularity in the stress field (see Figure 3.7). The tip enrichment functions are evaluated at (r, θ) which describe a polar coordinate system at the crack front as shown in Figure 2.12. As discussed in Section 3.1.2, this coordinate system can be directly extracted from the level-set field.

A typical choices for the global enrichment function $\Psi_{step}(\mathbf{x})$ in the presence of strong discontinuities are the sign-function (3.15) and the Heaviside function (3.16) of the level-set function

$$\Psi_{step}(\mathbf{x}) = \text{sign}(\phi(\mathbf{x})) = \begin{cases} -1, & \phi(\mathbf{x}) < 0 \\ 0, & \phi(\mathbf{x}) = 0, \\ 1, & \phi(\mathbf{x}) > 0 \end{cases} \quad (3.15)$$

$$\Psi_{step}(\mathbf{x}) = H(\phi(\mathbf{x})) = \begin{cases} 0, & \phi(\mathbf{x}) \leq 0 \\ 1, & \phi(\mathbf{x}) > 0 \end{cases}. \quad (3.16)$$

Here, the level-set function $\phi(\mathbf{x})$ is the signed distance to the crack surface and is equal to $\phi_3(\mathbf{x})$ as was introduced in Section 3.1.2. For the reason of simplification, $\phi(\mathbf{x}) = \phi_3(\mathbf{x})$ in this section. It is noteworthy, that no problems arise in blending elements with the *step*-enrichment in conjunction with the shifted version of the XFEM (3.13) since the piecewise-constant enrichment function vanishes outside the reproducing elements.

$$\psi_{tip}^{1-4}(r, \theta) = \begin{Bmatrix} r^\lambda \cos(\lambda\theta) \\ r^\lambda \sin(\lambda\theta) \\ r^\lambda \sin(\theta) \sin(\lambda\theta) \\ r^\lambda \sin(\theta) \cos(\lambda\theta) \end{Bmatrix} \quad (3.17)$$

with λ being the exponent that describes the asymptotic behavior. The choice of λ is discussed

in view of the limiting regimes which are present in hydraulic fracturing.

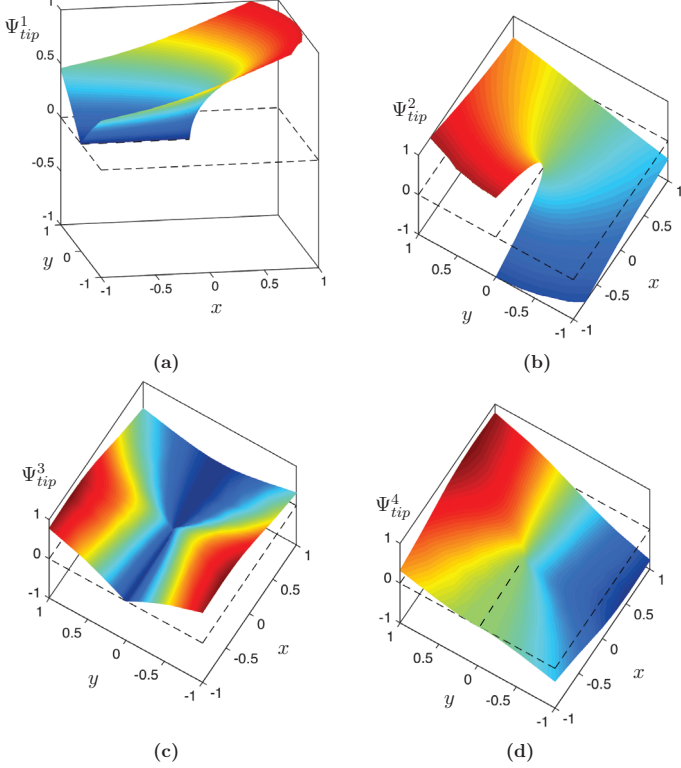


Figure 3.8: An arbitrary domain $[-1,1] \times [-1,1]$ and a crack with $(x,y) = (-1,0)$ to $(0,0)$ is considered. The enrichment functions at the crack tip/front Ψ_{tip}^m with $m = 1, 2, 3$ and 4 are evaluated with $\lambda = \frac{1}{2}$ which represents the inverse square root asymptote given in LEFM.

Implications on hydraulic fracturing: choosing λ

The hydraulic fracture problem leads to a strong fluid-solid coupling that is mainly confined to a small region near the crack tip where rapid variation in the fluid pressure is taking place. Analyzing the physical process at the tip leads to a better understanding of the propaga-

tion regime of a fluid-driven fracture. The work done by the fluid in extending a fracture is compared with the energy required to create new crack surfaces. Two limiting regimes are particularly important: toughness- and viscosity-dominated [51] (see also Section 2.2.1). In the toughness-dominated regime, the inverse square root singularity of LEFM ($\frac{1}{\sqrt{x}}$, with x being the distance from the crack tip/front) captures the effect of the crack-tip process on the total fracture. In contrast, the viscosity-dominated regime is characterized by a singularity that is weaker than the singularity predicted by LEFM [76]. Lecampion [108] suggested a modification of the tip-enrichment functions for the different limiting regimes. When propagating in the toughness-dominated regime, enrichment functions compatible with the classical square root singularity of LEFM are used to enrich the region near the crack tip by choosing $\lambda = \frac{1}{2}$. In the viscosity-dominated regime, the weaker singularity is taken into account with $\lambda = \frac{2}{3}$.

The four tip-enrichment functions with the square root singularity of LEFM ($\lambda = \frac{1}{2}$) are visualized in Figure 3.8. With these functions, the asymptotic behavior at the crack tip can be reproduced for classical crack problems.

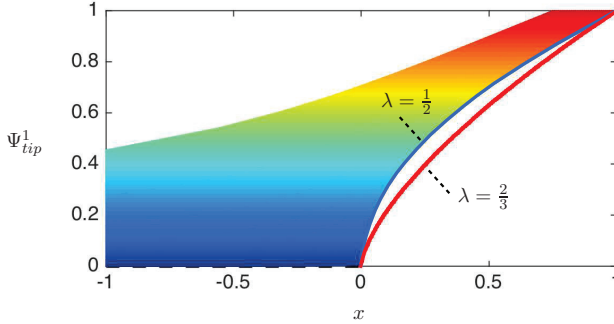


Figure 3.9: The enrichment functions at the crack tip/front Ψ_{tip}^1 evaluated at $y = 0$ with $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{3}$.

A modification of the enrichment function might be appropriate in this case because the asymptotic behavior of fluid-induced crack propagation can change as is shown for the viscosity dominated problem. Figure 3.9 shows the enrichment function Ψ_{tip}^1 evaluated along the line at $y = 0$ for both $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{3}$. The comparison of both asymptotes in this enrichment function clearly shows the different asymptotic behaviors. The blue line at the edge of the colored graph corresponds to $\lambda = \frac{1}{2}$ and represents $f(x) = x^{\frac{1}{2}}$ whereas the red line is the corresponding solution for $\lambda = \frac{2}{3}$ and $f(x) = x^{\frac{2}{3}}$. It is assumed that the crack tip is located at $x = 0$, making x the distance to the crack tip. The red curve shows the weaker singularity compared to the blue curve as is expected in hydraulic fracture problems where viscosity dominates the propagation process.

3.4 Numerical integration

In XFEM, some aspects have to be considered when performing the numerical integration in elements which contain the discontinuity. In order to avoid inaccuracy in the numerical integration due to the presence of discontinuities, elements are decomposed along the interface and numerical integration is performed on each subcell separately (*e.g.*, [18, 70]). There is no unique way of decomposing elements into parts along a given interface that passes through the element. The procedure used here for the element decomposition for hexahedra and tetrahedra is illustrated in Figure 3.10 and in Figure 3.11, respectively. A crack passes through the hexahedral element; however, the element is not completely cut by the crack. The crack front is partly present within this element. Regarding the hexahedral element, it is decomposed into six tetrahedra. Each tetrahedron is then subdivided according to the location of the crack interface. Two important aspects should be mentioned at this point. The decomposition of the elements is based on the *implicit* crack description, that is, the level-set information is used in order to determine the crack location. The crack intersection points for the subdivision are then easily calculated through interpolation. Furthermore, as indicated in Figure 3.10, the decomposition is based on the *extended* crack surface; consequently the position of the crack front within the element is basically irrelevant. As a result, only the level-set function ϕ_3 is used for the decomposition procedure.

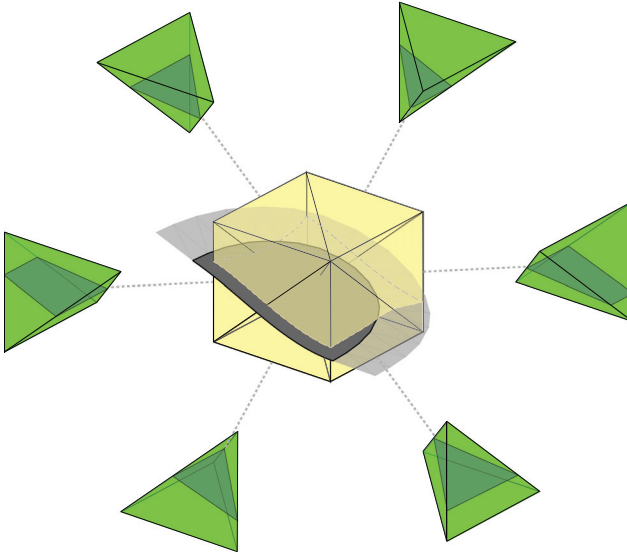


Figure 3.10: Element decomposition of a hexahedron.

The placement of the integration points is carried out for each individual tetrahedron and is then projected into the hexahedron. More precisely, integration points are positioned in a reference tetrahedron and projected into the real tetrahedron which represents one part of the whole hexahedron. Figure 3.11 illustrates the situation for setting integration points in a cut tetrahedron for volume and for surface integration.

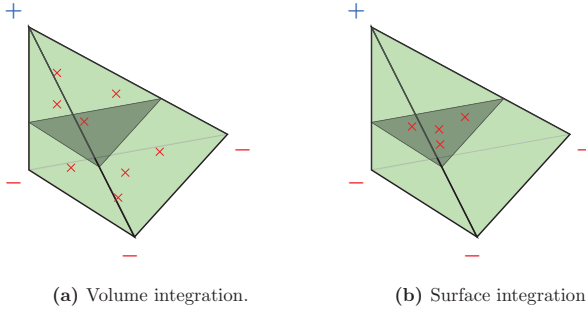


Figure 3.11: Element decomposition and placement of integration points.

It is noteworthy that the explicitly described crack surface and the implicitly determined surface do not perfectly coincide. This might lead to inaccuracies in the numerical integration. Therefore, in order to keep the whole framework consistent, the explicit crack description should only be used for crack propagation purposes whereas all other calculations should be based on the level-set field.

4 Solving PDEs on manifolds

In general, a crack that is evolving in the three-dimensional space can be considered narrow—the topology in one dimension is less pronounced than in the other two dimensions. Thus, modeling the crack geometry as a curved surface in 3D by neglecting the third dimension is a valid assumption. An arbitrary curve with a lower dimension with respect to the embedded physical space is called a manifold. Consequently, a one-dimensional manifold is called a curve and a two-dimensional manifold in a 3D-space is a surface [20].

In order to allow calculations on manifolds and especially solving partial differential equations (PDEs), manifolds must have certain properties. An important property is the differentiability so that a tangent space can be defined on each point on the manifold.

4.1 Basis vectors on manifolds

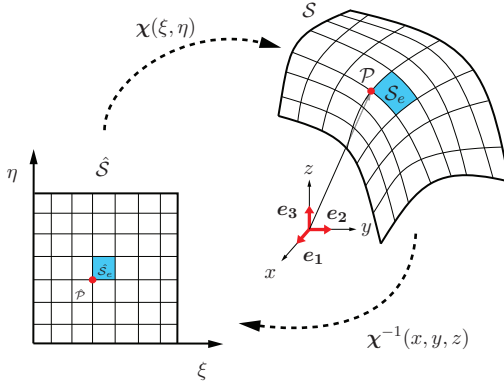


Figure 4.1: An element $\hat{S}_e$ in parameter space is mapped to an element S_e in physical space. [49]

As denoted in Figure 4.1 a point $\mathcal{P}$ on a manifold $S \in \mathbb{R}^2$ in a three-dimensional Euclidean space is given in terms of a Cartesian coordinate system $x^i(\xi^1, \xi^2)$, spanned by an orthonormal set of vectors e_i [24]. The position vector is a function of a natural coordinate system ξ^i defined

on the manifold and takes the form

$$\mathbf{x} = \sum_{i=1}^3 x^i(\xi^1, \xi^2) \mathbf{e}_i = x^1(\xi^1, \xi^2) \mathbf{e}_1 + x^2(\xi^1, \xi^2) \mathbf{e}_2 + x^3(\xi^1, \xi^2) \mathbf{e}_3. \quad (4.1)$$

Using the Einstein summation convention [64] the position vector can be written in its compact form

$$\mathbf{x} = x^i \mathbf{e}_i. \quad (4.2)$$

According to differential geometry, upper indices are related to a contravariant and lower indices to a covariant object [125]. The covariant and contravariant basis vectors are defined as

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i} \quad \text{and} \quad \mathbf{g}^i = \frac{\partial \xi^i}{\partial \mathbf{x}} \quad (4.3a,b)$$

and describe the transformation between the coordinates systems. Thus, an arbitrary vector can be represented by a linear combination of covariant basis vectors and contravariant components or by contravariant basis vectors and covariant components:

$$\mathbf{x} = x^i \mathbf{g}_i = x_i \mathbf{g}^i \quad (4.4)$$

Covariant basis vectors are always tangent to the gridlines on the manifold and scaled in proportion to the scaling of the grid lines [29]. In contrast, a contravariant basis does not move with the grid lines but remains perpendicular to the material plane spanned by two covariant basis vectors at any point as illustrated in Figure 4.2. A cross-pairing of co- and contravariant basis vectors and components ensures that the vector is invariant according to a coordinate transformation [29], *i.e.*, the sum of components times their base vectors is invariant. It is noteworthy, that for an orthonormal basis the co- and contravariant components are exactly the same. By means of the co- and contravariant basis, a local coordinate system at an arbitrary point on the manifold $\mathcal{S}$ can be formulated.

A direct link can be established between the Jacobian matrix $\mathbf{J}$ and the basis vectors $\mathbf{g}$ for the transformation between the Cartesian and a curvilinear coordinate system. $\mathbf{J}$ is given in a general form with $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \cdots & \frac{\partial x^1}{\partial \xi^n} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \cdots & \frac{\partial x^2}{\partial \xi^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial \xi^1} & \frac{\partial x^n}{\partial \xi^2} & \cdots & \frac{\partial x^n}{\partial \xi^n} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_n \end{bmatrix} \quad (4.5)$$

Here, the covariant basis vectors $\mathbf{g}_i$ are easily identified as the column vectors of the Jacobian matrix, whereas the contravariant basis vectors $\mathbf{g}^i$ represent the row vectors of the inverse

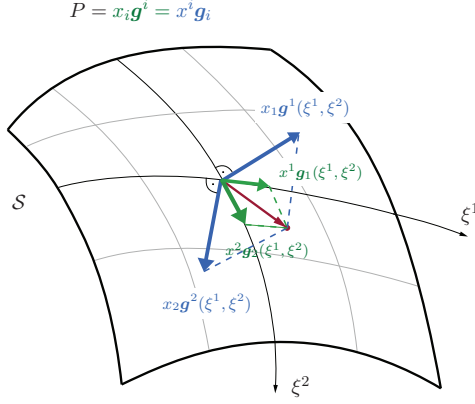


Figure 4.2: A co- and contravariant basis is illustrated. A vector at an arbitrary node (ξ^1, ξ^2) that points to a point $\mathcal{P}$ can be described with a contravariant and a contravariant (or tangent) basis.

Jacobian matrix

$$\mathbf{J}^{-1} = \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right)^{-1} = \begin{bmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \cdots & \frac{\partial \xi^1}{\partial x^n} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \cdots & \frac{\partial \xi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi^n}{\partial x^1} & \frac{\partial \xi^n}{\partial x^2} & \cdots & \frac{\partial \xi^n}{\partial x^n} \end{bmatrix} = \left[\mathbf{g}^1, \mathbf{g}^2, \dots, \mathbf{g}^n \right]^T. \quad (4.6)$$

Since the transformation can be interpreted as solving a system of linear equations, the determinant of the Jacobian matrix has to be nonzero ($\det(\mathbf{J}) \neq 0$).

For a manifold $\mathcal{S} \in \mathbb{R}^2$ in a three-dimensional Euclidean space the Jacobian matrix is not square anymore but has the form

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} \end{bmatrix} = \left[\mathbf{g}_1, \mathbf{g}_2 \right], \quad (4.7)$$

with $\mathbf{g}_1$ and $\mathbf{g}_2$ being the local tangent vectors on the manifold.

Moreover, the scalar product between the basis vectors in its covariant and contravariant representation define the components of a metric tensor, known as metric coefficients [125]

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j. \quad (4.8a,b)$$

The metric tensor can also be formulated in terms of the Jacobian matrix

$$\mathbf{g} = \mathbf{J}^T \mathbf{J} = \begin{bmatrix} \mathbf{g}_1, & \mathbf{g}_2, & \cdots, & \mathbf{g}_n \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{g}_1, & \mathbf{g}_2, & \cdots, & \mathbf{g}_n \end{bmatrix} \quad (4.9)$$

with the Gram determinant [60]

$$g = \det(\mathbf{g}) = \det(\mathbf{J}^T \mathbf{J}). \quad (4.10)$$

With the definition of the co- and contravariant basis vectors and the Jacobian matrix differential operators on manifolds such as curves and surfaces can now be formulated.

4.2 Differential operators on curves and surfaces

A vector formulated in terms of a natural basis as in (4.4) is differentiated by means of the product rule

$$\frac{\partial \mathbf{x}}{\partial \xi^k} = \frac{\partial(x^i \mathbf{g}_i)}{\partial \xi^k} = \frac{\partial x^i}{\partial \xi^k} \mathbf{g}_i + x^i \frac{\partial \mathbf{g}_i}{\partial \xi^k}. \quad (4.11)$$

The derivative in the last term is simply a linear combination of all basis vectors

$$\frac{\partial \mathbf{g}_i}{\partial \xi^k} = \sum_{j=1}^n \Gamma_{ki}^j \mathbf{g}_j \quad (4.12)$$

with the coefficients Γ_{ki}^j known as Christoffel symbols in the context of differential geometry [86, 104].

The gradient on the manifold expressed in coordinates of the physical space can be written in quantities of the parametric space which describes the manifold

$$\nabla \tilde{f}(\mathbf{x}) = [\mathbf{J}(\mathbf{J}^T \mathbf{J})^{-1} \nabla_{\xi} \hat{f}(\xi)] \circ \mathbf{x}^{-1}, \quad (4.13)$$

where $\nabla_{\xi} \hat{f}(\xi)$ denotes the gradient in parameter space. Here, the geometrical mapping from the parameter space on the manifold to the physical space and vice versa is formally expressed through

$$\hat{f}(\xi) = f(\mathbf{x}(\xi)) \quad \text{and} \quad f(\mathbf{x}) = \hat{f}(\xi) \circ \mathbf{x}^{-1}. \quad (4.14a,b)$$

This expression is essentially used in context of the finite element method on manifolds to transform the gradient of shape functions from the natural coordinates $\nabla_{\xi} \mathbf{N}(\xi)$ into the physical space $\nabla \mathbf{N}(\mathbf{x})$ which will be seen in Section 5.1. The tangential gradient of a differentiable function $f(\mathbf{x})$ is [49]

$$\nabla_{\Sigma} f(\mathbf{x}) = \mathbf{P}(\mathbf{x}) \nabla \tilde{f}(\mathbf{x}), \quad (4.15)$$

where the tangential projector operator is defined as

$$\mathbf{P}(\mathbf{x}) = [\mathbf{I} - \mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x})] = [\mathbf{I} - \mathbf{n} \cdot \mathbf{n}^T]. \quad (4.16)$$

Here, $\mathbf{I}$ is the identity matrix and $\mathbf{n}$ a unit normal vector on the manifold given in the physical space.

4.3 Implementation aspects

In this section, some aspects on numerical implementation are pointed out when mapping Gaussian points from a standard reference element in natural coordinates in $\mathbb{R}^n$ into a domain $\Omega \in \mathbb{R}^{n+1}$. Moreover, partial derivatives for shape and enrichment functions on manifolds are determined.

The isoparametric concept [28, 176, 177] is utilized here to construct the shape and enrichment functions on a reference element and to transfer them to the global elements as illustrated in Figure 4.3. Here, a transformation from the domain $\Omega_{Ref} \subset \mathbb{R}^2$ to $\Omega_{Real} \subset \mathbb{R}^3$ is considered.

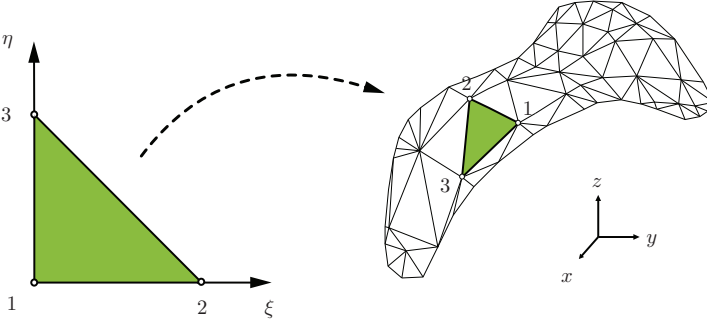


Figure 4.3: Illustration of a mapping from a reference triangular element in natural coordinates (ξ, η) to an element on a discretized manifold in the physical space (x, y, z) .

The global coordinate system (x, y, z) can be expressed in terms of the natural coordinates

ξ and η as

$$x(\xi, \eta) = \sum_{i=1}^{n_e} N_i(\xi, \eta) x_i \quad (4.17a)$$

$$y(\xi, \eta) = \sum_{i=1}^{n_e} N_i(\xi, \eta) y_i \quad (4.17b)$$

$$z(\xi, \eta) = \sum_{i=1}^{n_e} N_i(\xi, \eta) z_i \quad (4.17c)$$

where n_e is the number of nodes for one element and $N_i(\xi, \eta)$ are the shape functions defined in the local coordinate system. Since the derivatives are sought in terms of the global coordinate system (x, y, z) , the shape functions have to be transformed by means of the relationship

$$\begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \\ \frac{\partial N}{\partial z} \end{bmatrix} \quad (4.18)$$

where the transposed Jacobian matrix on a manifold is defined according to [176]

$$\mathbf{J}^T = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}. \quad (4.19)$$

By rearranging equation (4.18) and solving the resulting linear system of equations for $\frac{\partial N}{\partial \mathbf{x}}$ one arrives at

$$\begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \\ \frac{\partial N}{\partial z} \end{bmatrix} = \mathbf{J}(\mathbf{J}^T \mathbf{J})^{-1} \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix}. \quad (4.20)$$

The evaluation of the enrichment functions M_i is now based on the determination of the shape functions N_i and the global enrichment $\Psi(x, y, z)$, both in the global coordinate system

$$M_i(x, y, z) = N_i(x, y, z) \cdot \Psi(x, y, z). \quad (4.21)$$

Equivalently, the derivatives of the enrichment functions are obtained by applying the product rule through

$$\begin{bmatrix} \frac{\partial}{\partial x} M(\mathbf{x}) \\ \frac{\partial}{\partial y} M(\mathbf{x}) \\ \frac{\partial}{\partial z} M(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} N(\mathbf{x}) \\ \frac{\partial}{\partial y} N(\mathbf{x}) \\ \frac{\partial}{\partial z} N(\mathbf{x}) \end{bmatrix} \cdot \Psi(\mathbf{x}) + N(\mathbf{x}) \cdot \begin{bmatrix} \frac{\partial}{\partial x} \Psi(\mathbf{x}) \\ \frac{\partial}{\partial y} \Psi(\mathbf{x}) \\ \frac{\partial}{\partial z} \Psi(\mathbf{x}) \end{bmatrix}. \quad (4.22)$$

As can be seen from this equation, the enrichment $\Psi(x, y, z)$ is directly evaluated in the global

coordinates and no transformation from the natural coordinates system is required. Thus, only the shape function N_i are mapped from the local into the global domain.

Quadrature

For the purpose of numerical integration the transformation of the area in the reference element A_{Ref} to the real element A_{Real} is in general performed according to [106]

$$dA_{Real} = |\det(\mathbf{J})| dA_{Ref} \quad (4.23)$$

with the determinant of the Jacobian matrix describing the relationship between the differential areas in the reference and real domain. Since a 2D reference element is projected into a plane element embedded in a manifold in 3D, the Jacobian matrix is not square but rectangular. Thus, the ratio of the differential areas is formulated as

$$dA_{Real} = \sqrt{\det(\mathbf{J}^T \mathbf{J})} dA_{Ref} \quad (4.24)$$

which is equivalent to equation (4.23) with $\mathbf{J}$ being square.

4.4 Numerical examples

Some numerical examples for solving PDEs on manifolds are presented in this section. The Laplace-Beltrami problem is solved in 2D on a circular manifold and in 3D on a quarter-cylindrical shape. The strong form of the Laplace-Beltrami problem on a manifold $\Sigma \in \mathbb{R}^n$ is equivalent to the Poisson problem in a domain $\Omega \in \mathbb{R}^n$ and is written as [97]

$$-\Delta_\Sigma u = f \quad \text{on} \quad \Sigma, \quad (4.25a)$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \quad (4.25b)$$

$$\mathbf{n}_\Gamma \nabla_\Sigma u = 0 \quad \text{on} \quad \Gamma_N. \quad (4.25c)$$

Here, f is the source term, Γ_D is the Dirichlet boundary, Γ_N is the Neumann boundary, and $\mathbf{n}_\Gamma$ the unit normal vector on the boundary. For a closed manifold with no outer boundaries, the problem is ill-posed [49, 97]. Thus, the zero-mean constraint

$$\int_\Sigma u \, dx = 0 \quad (4.25d)$$

is introduced in order guarantee a unique solution u .

4.4.1 Laplace-Beltrami problem in 2D

First, the problem is solved on a closed circular curve which is embedded in the 2D space. Figure 4.4 shows the circle with the radius $r = 1$ as a black curve and the numerical approximation of that circle indicated by the red curve. The solution in polar coordinates (r, ϕ) is chosen as in [97] to

$$u(r, \phi) = 12 \sin 3 \cdot \phi. \quad (4.26)$$

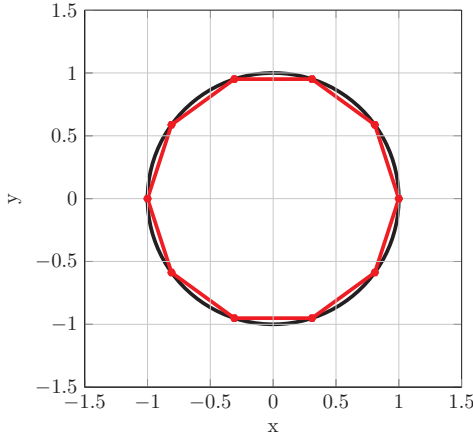


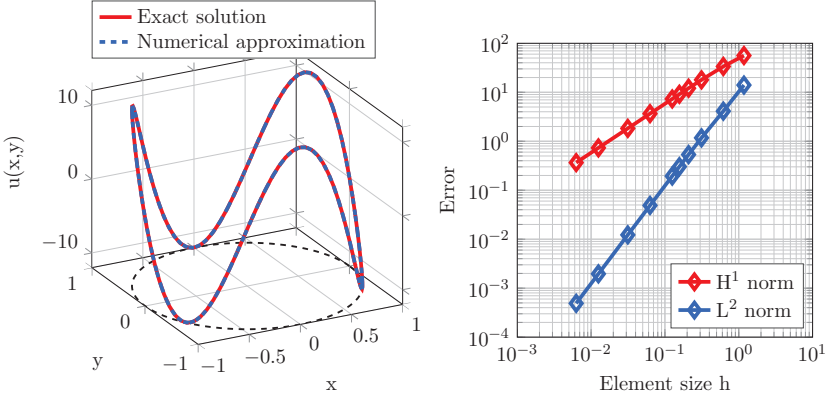
Figure 4.4: The Laplace-Beltrami problem is defined on a manifold in the form of a unit circle (black curve) embedded in the domain with the coordinates (x, y) . The approximation of this circle is given by the red mesh. For illustration purposes, 10 elements are used for the discretization.

By applying the Laplace-Beltrami operator on the exact solution yields the source function f with

$$f(r, \phi) = 108 \sin 3 \cdot \phi, \quad (4.27)$$

which, in turn, is used in the numerical approach as the source term on the right hand side of the linear system of equations.

The numerical solution is presented in Figure 4.5a. As can be seen from the figure, it fits quite well with the analytical solution. Figure 4.5b shows the error subjected to the element size h of the mesh in logarithmic scale. The convergence analysis in the L^2 -norm and H^1 -norm confirm the theoretical order of convergence for linear elements from a priori error estimates (of $O(h^2)$ for the L^2 -norm) [49, 97].



(a) Comparison of the numerical with exact solution for a curved manifold. (b) Error analysis for the 2D Laplace-Problem.

Figure 4.5: The numerical solution of the 2D-Laplace-Beltrami problem on the unit circle is compared with the exact solution.

4.4.2 Laplace-Beltrami problem in 3D

Second, the Laplace-Beltrami problem is solved on the surface of a quarter cylinder with the radius $r = 1$ and the height $L = 4$. The surface mesh is illustrated in Figure 4.6 in a domain with the Cartesian coordinates (x, y, z) .

Defining the parameters and functions [49, 97]

$$\alpha = 3, \quad \beta = \frac{1}{3/2 - \sqrt{2}}, \quad (4.28)$$

$$g_{\phi,1}(\phi) = (1 - \cos \phi)(1 - \sin \phi), \quad (4.29)$$

$$g_{\phi,2}(\phi) = (\cos \phi + \sin \phi - 4 \sin \phi \cos \phi), \quad \text{and} \quad (4.30)$$

$$g_z(z) = \sin \frac{\alpha \pi z}{L}, \quad (4.31)$$

the exact solution is chosen in cylindrical coordinates (r, ϕ, z) as

$$u(r, \phi, z) = \beta g_{\phi,1}(\phi) g_z(z). \quad (4.32)$$

Again, by applying the Laplace-Beltrami operator on the exact solution with the second deriva-

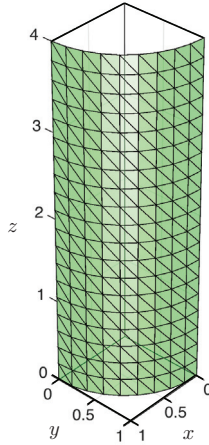


Figure 4.6: The Laplace-Beltrami problem is solved on a surface mesh of a quarter of a cylinder.

tives given

$$\frac{\partial^2 u}{\partial \phi^2} = \beta g_z(z) [\cos(\phi)(1 - \sin(\phi)) + \sin(\phi)(1 - \cos(\phi)) - 2 \sin(\phi) \cos(\phi)] \quad \text{and} \quad (4.33a)$$

$$\frac{\partial^2 u}{\partial z^2} = -\beta g_{\phi,1}(\phi) \left(\frac{\alpha\pi}{L} \right)^2 \sin\left(\frac{\alpha\pi}{L} z \right) \quad (4.33b)$$

the source function f is obtained as

$$\begin{aligned} f(r, \phi, z) &= -\left(\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ &= \beta g_z(z) \left(\left(\frac{\alpha\pi}{L} \right)^2 g_{\phi,1}(\phi) - g_{\phi,2}(\phi) \right). \end{aligned} \quad (4.34)$$

It is mentioned that the solution does not depend on the radius r , since the radius is constant for this problem.

The comparison of the exact and the numerical solution is shown in Figure 4.7. The colored region indicates the solution values on the surface. As can be seen from the results, the exact solution can be reproduced with optimal accuracy for linear elements in this numerical approach. A convergence analysis that is also performed for this case confirms the optimal theoretical rates of convergence, as well.

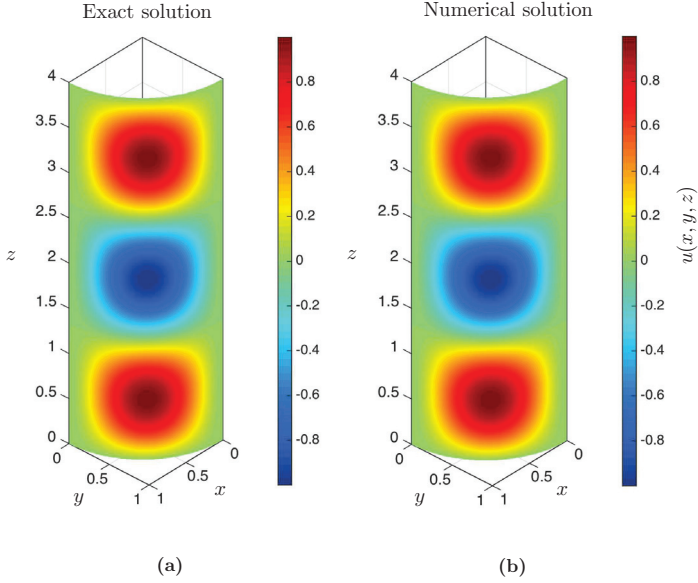


Figure 4.7: Comparison of the numerical with the exact solution of the Laplace-Beltrami problem on a curved manifold in the 3D space.

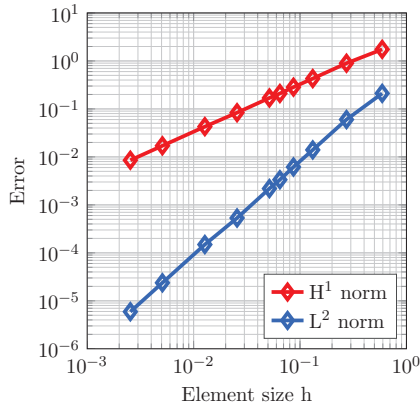


Figure 4.8: Error analysis for the 3D Laplace-Problem.

4.4.3 Discontinuity on a manifold

Finally, a discontinuity Σ_I on the manifold is considered and solved by means of the XFEM. The situation of a manifold containing a discontinuity is illustrated in Figure 4.9. The manifold is divided into two regions along the prescribed interface Γ_I . $\mathbf{n}_\Gamma$ is the normal vector on the interface. We aim to solve the elliptic equation

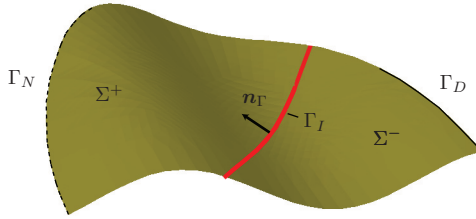


Figure 4.9: A discontinuous manifold is separated into two regions, Σ^- and Σ^+ . A normal vector $\mathbf{n}_\Gamma$ is defined at the interface Σ_I .

$$-\nabla \cdot (k \nabla_\Sigma u) = b \quad \text{on} \quad \Sigma = \Sigma^- \cup \Sigma^+, \quad (4.35a)$$

$$u = u_D \quad \text{on} \quad \Gamma_D, \quad (4.35b)$$

$$\mathbf{n}_{\Gamma_N} \nabla_\Sigma u = f \quad \text{on} \quad \Gamma_N, \quad (4.35c)$$

with the material coefficient k being smooth in the manifold regions Σ^- and Σ^+ and discontinuous at the manifold interface Γ_I between Σ^- and Σ^+

$$k(x) = \begin{cases} k^-, & \forall x \in \Sigma^- \\ k^+, & \forall x \in \Sigma^+ \end{cases}. \quad (4.36)$$

u is the unknown variable and can be interpreted as a displacement with the body force b , $\mathbf{n}_{\Gamma_N}$ is the normal vector, and f the applied force at the Neumann boundary Γ_N . Moreover, the interface condition is implied here by the underlying conservation law [112]

$$[[k \nabla_\Sigma u \cdot \mathbf{n}_\Gamma]] = 0 \quad \text{on} \quad \Gamma_I. \quad (4.37)$$

In 2D, equation (4.35a) is solved on a curve with the discontinuity being a point in the middle of that curve as indicated in Figure 4.10. The curve is taken as part of a circle with the transformation from the local coordinate system (s) to the Cartesian coordinate system (x, y)

$$x = \cos(s) \quad \text{and} \quad y = \sin(s). \quad (4.38a,b)$$

The endpoints of the curve are defined within the local coordinate system at $s = 0$ being and $s = 1$.

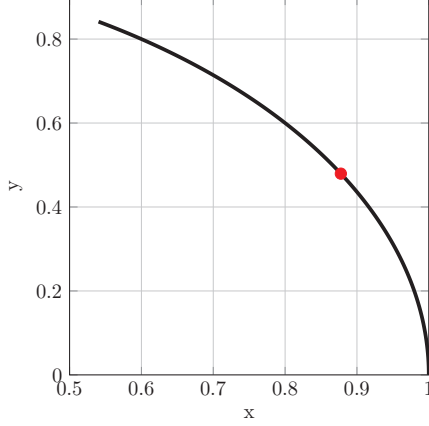


Figure 4.10: Curved manifold in the domain (x, y) with a discontinuity defined as a (red) point on the curve.

In the first example, the boundary conditions are being chosen to $u(0) = 0$, $b = 0$, and $\mathbf{n}_{\Gamma_N} \nabla_{\Sigma} u(1) = f = 10$, and the material coefficient is

$$k(s) = \begin{cases} 1, & \forall s \in \Sigma^- \\ 10, & \forall s \in \Sigma^+ \end{cases} \quad (4.39)$$

This problem yields the exact solution u^{ex} which is piecewise linear [25]

$$u^{ex}(s) = \begin{cases} \frac{f}{k^-} s, & \forall s \in \Sigma^- \\ \frac{f}{k^+} s^* + \frac{f}{k^+} (s - s^*), & \forall s \in \Sigma^+ \end{cases} \quad (4.40)$$

It is compared to the solutions obtained with the classical FEM and the XFEM in Figure 4.11. The discontinuity is intended to be inside of the element of the discretized manifold in order to demonstrate the ability of XFEM to be able to handle such problems. The dashed curve in the xy -plane represents the manifold with the black dot being the location of the discontinuity.

Figure 4.12 shows the convergence of both, the FEM- and the XFEM-solution, in the L^2 norm. The convergence rate for FEM with linear elements is $= 1$ and for the XFEM $= 2$ for the errors in the L^2 norm.

A second example is considered with pure Dirichlet boundary conditions $u(0) = 0$, $u(1) = 0$, and with a non-constant source term $b(s) = -4e^{-2s}$. The exact solution has a weak disconti-

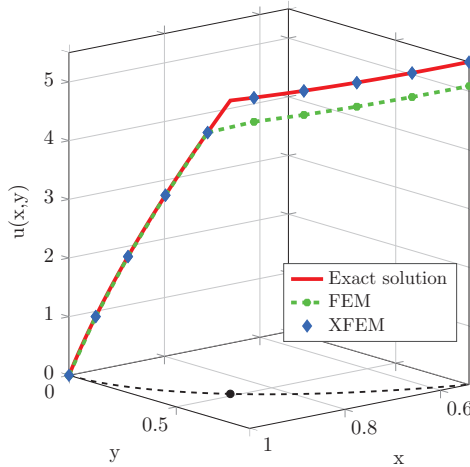


Figure 4.11: Comparison of the numerical with the piecewise linear exact solution for a curved manifold.

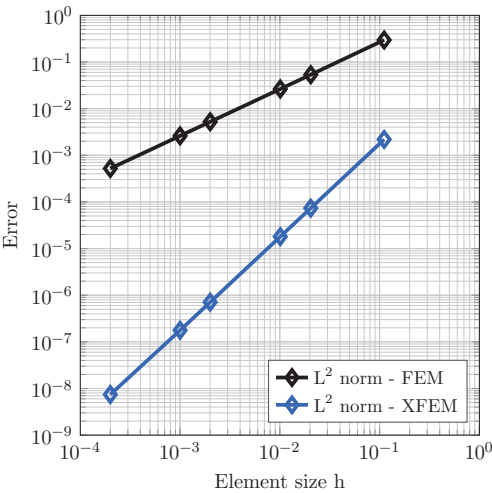


Figure 4.12: Convergence analysis

nuity and is non-linear in each region [112]

$$u^{ex}(s) = \begin{cases} 1 + \frac{1}{k^-}(e^{-2s} - 1) + \frac{2k^+\alpha}{k^-(k^-+k^+)}s, & \forall s \in \Sigma^- \\ \frac{1}{k^+}(e^{-2s} - e^{-2}) + \frac{2\alpha}{(k^-+k^+)}(s - 1), & \forall s \in \Sigma^+ \end{cases} \quad (4.41)$$

with

$$\alpha = e^{-1} \left(\frac{k^-}{k^+} - 1 \right) - \frac{k^-}{k^+} e^{-2} + 1 - k^-.$$

As in the previous example, the FEM- and XFEM-solution is compared against the analytical solution and is shown in Figure 4.13.

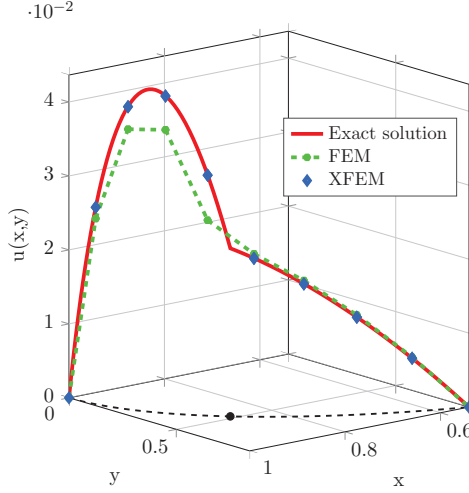


Figure 4.13: Comparison of the numerical with the non-linear exact solution for a curved manifold.

Again, Figure 4.14 shows the log-log plot of the L^2 norm of the error against the mesh size h for the FEM- and the XFEM-solution. The convergence rate for FEM with linear elements is ≈ 1 and for the XFEM ≈ 2 .

In both examples, the Heaviside enrichment (3.16) was used in the XFEM. Due to the mathematical nature of the Heaviside function which uncouples the differential equation, the continuity in the primary unknown u is enforced by means of a Lagrange multiplier

$$\lambda = [u^-(s^*) - u^+(s^*)].$$

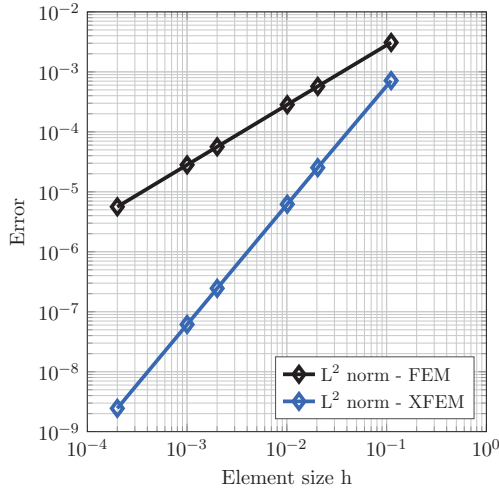


Figure 4.14: Convergence analysis

4.4.4 Laplace-Beltrami problem in 3D with a discontinuity

In addition to the numerical example given in Section 4.4.2 where the Laplace-Beltrami problem was solved on a quarter cylinder, a discontinuity is placed on the manifold at a constant height ($z = 2$) as illustrated in Figure 4.15. The region on the lower part of the manifold ($z < 2$) is named Σ^- and the region above is Σ^+ . The simulation is performed by means of the XFEM. The shape of the manifold is discretized using triangular elements. Cut (or fully enriched) elements are colored light blue, and blending elements are grey in the Figure.

The exact solution is chosen to be

$$u^{ex}(\phi, z) = \begin{cases} -\beta g_{\phi,1}(\phi) g_z(z), & \forall (\phi, z) \in \Sigma^- \\ +\beta g_{\phi,1}(\phi) g_z(z), & \forall (\phi, z) \in \Sigma^+ \end{cases} \quad (4.42)$$

with the source function

$$f(\phi, z) = \begin{cases} -\beta g_z(z) \left(\left(\frac{\alpha\pi}{L} \right)^2 g_{\phi,1}(\phi) - g_{\phi,2}(\phi) \right), & \forall (\phi, z) \in \Sigma^- \\ +\beta g_z(z) \left(\left(\frac{\alpha\pi}{L} \right)^2 g_{\phi,1}(\phi) - g_{\phi,2}(\phi) \right), & \forall (\phi, z) \in \Sigma^+ \end{cases} \quad (4.43)$$

and homogeneous Dirichlet boundary conditions at the whole manifold boundary. The parameters α , β and the functions $g_{\phi,1}$, $g_{\phi,2}$, and $g_z(z)$ are defined in (4.28) to (4.31). The solution in Figure 4.16 is presented in terms of a displacement relative to the shape of the quarter

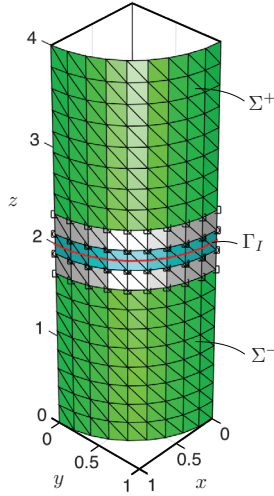


Figure 4.15: A *discontinuous* Laplace-Beltrami problem is solved on a surface mesh of a quarter of a cylinder. The discontinuity Γ_I (red line) is located at a constant height of $z = 2$.

cylinder in order to point out the solution behavior close to the interface. The XFEM is able to reproduce such solution behavior and converges towards the exact solution; however, with a lower convergence rate of ≈ 0.51 . This can be traced back to the awkward influence of blending elements. A number of approaches exist to fix this problem [68].

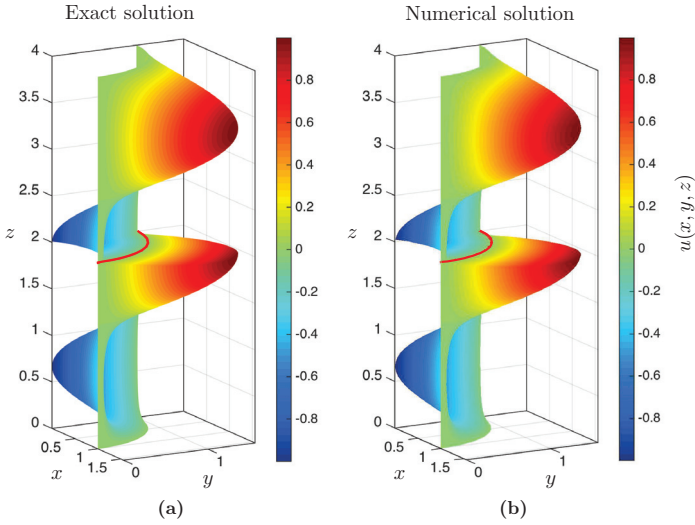


Figure 4.16: Comparison of the numerical with the exact solution of the discontinuous Laplace-Beltrami problem on a curved manifold in the 3D space.

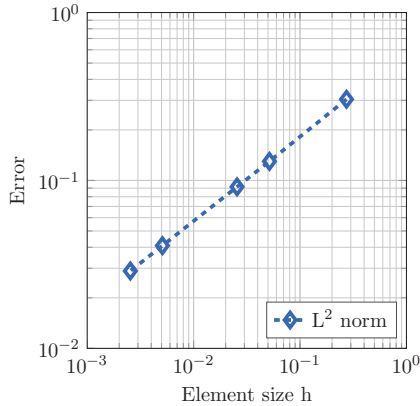


Figure 4.17: Error analysis for the 3D Laplace-Problem.

5 Numerical Modeling

This chapter discusses the numerical modeling of the fluid flow on a curved crack, the solid deformation and the crack propagation. Finally, an iterative procedure is presented that couples these three processes.

5.1 Fluid flow on manifolds

The governing equations for the fluid flow on a curved crack are formulated within the framework of the finite element method.

5.1.1 Variational formulation

The weak form of the Reynolds equation is obtained by multiplying Equation (2.23) with a test function δp and integrating over the surface domain Σ :

$$\int_{\Sigma} \delta p^T \left(\frac{\partial w}{\partial t} - \nabla \cdot \left(\frac{w^3}{12\mu} \nabla p \right) + q_L \right) d\Sigma = 0. \quad (5.1)$$

After integrating by parts and introducing the fluid flux as an essential boundary condition, one obtains

$$\int_{\Sigma} \nabla^T(\delta p) \left(\frac{w^3}{12\mu} \nabla p \right) d\Sigma = - \int_{\Sigma} \delta p^T \frac{\partial w}{\partial t} d\Sigma - \int_{\Sigma} \delta p^T q_L d\Sigma - \delta p^T \mathbf{q}|_{\partial\Sigma}. \quad (5.2)$$

Here, $\partial\Sigma$ is the boundary of the fluid filled region of the crack. When the fluid front reaches the crack front, $\partial\Sigma$ corresponds exactly to the crack front.

5.1.2 Finite element representation

The Galerkin finite element method [21] is used to discretize the variational formulation of the fluid flow problem. Hereby, the function space $\mathcal{V}$ is substituted by a finite dimensional subspace $\mathcal{V}^h \subset \mathcal{V}$. Using a set of basis shape functions $N(\mathbf{x})$, the pressure and its derivative are approximated by

$$p^h(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) p_i, \quad \frac{\partial p^h(\mathbf{x})}{\partial \mathbf{x}} = \sum_{i \in I} \frac{\partial N_i(\mathbf{x})}{\partial \mathbf{x}} p_i. \quad (5.3)$$

The pressures at the nodes of the discrete surface, p_i , are subsequently solved for.

The finite element representation, proceeding from equations (5.2) and (5.3), reads

$$\frac{1}{12\mu} \int_{\Sigma} w^3 \nabla \mathbf{N} \nabla \mathbf{N}^T d\Sigma \cdot \mathbf{p} = \left[\mathbf{N} \frac{w^3}{12\mu} \nabla \mathbf{p} \right]_{\partial\Sigma} - \int_{\Sigma} \mathbf{N} q_L d\Sigma - \int_{\Sigma} \mathbf{N} \frac{\partial w}{\partial t} d\Sigma \quad (5.4)$$

The term on the left is the system matrix for the unknown pressure $\mathbf{p}$. The time derivative $\frac{\partial w}{\partial t}$ in (5.1) can be approximated by a backward finite difference method at time step n

$$\left. \frac{\partial w}{\partial t} \right|_n = \frac{w^n - w^{n-1}}{t^n - t^{n-1}} = \frac{\Delta w}{\Delta t} \quad (5.5)$$

where w^n and w^{n-1} are the known fracture widths at the time step t^n and t^{n-1} , respectively.

For a constant injection rate Q and a negligible fluid leak-off the equation (2.27) can be formulated in terms of the crack volume V

$$Q \cdot t = \int_{\Omega} w d\Omega = V. \quad (5.6)$$

Accordingly, the time increment can be calculated from

$$\Delta t = \frac{\int_{\Omega^n} w d\Omega - \int_{\Omega^{n-1}} w d\Omega}{Q} = \frac{V^n - V^{n-1}}{Q}. \quad (5.7)$$

5.2 Discretization of solid deformation

The displacement field of the solid is obtained by solving the elasticity equation (2.29) in conjunction with the boundary conditions (2.35), (2.36) and (2.38). The XFEM approach, presented in Section 3, is used here to discretize the governing equations.

5.2.1 Variational formulation

The weak form of the equilibrium equation is given by the weighted integral formulation in terms of the stresses

$$\int_{\Omega} (\boldsymbol{\sigma} + \mathbf{f}) \cdot \delta \mathbf{u} d\Omega = 0. \quad (5.8)$$

Here, the weighting function vector is defined as $\delta \mathbf{u} = [\delta u_1, \delta u_2, \delta u_3]^T$ with the index values $(= 1, 2, 3)$ denoting the three directions of the Cartesian coordinate system. Since an elasticity problem in its weak form is equivalent to the principle of virtual work [21, 185], $\delta \mathbf{u}$ is termed a virtual displacement. By applying the Green-Gauss theorem and introducing the definition

of the virtual strain

$$\delta \boldsymbol{\varepsilon} = \mathbf{S} \delta \mathbf{u}, \quad (5.9)$$

the weak form of the elasticity equation, with the displacement being the primary variable, can finally be written as

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} \, d\Omega - \int_{\Gamma} \delta \mathbf{u}^T \bar{\mathbf{t}} \, d\Gamma - \int_{\Sigma^+} \delta \mathbf{u}_{\Sigma^+}^T \mathbf{p}_{\Sigma^+} \, d\Sigma - \int_{\Sigma^-} \delta \mathbf{u}_{\Sigma^-}^T \mathbf{p}_{\Sigma^-} \, d\Sigma = 0. \quad (5.10)$$

The elasticity tensor $\boldsymbol{\varepsilon}$ and the strain operator $\mathbf{S}$ are defined as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix}. \quad (5.11)$$

With the definition of the pressure acting normal to the crack face (2.38) and with

$$\mathbf{p}_{\Sigma} = \mathbf{p}_{\Sigma^+} = -\mathbf{p}_{\Sigma^-} \quad (5.12)$$

$$\mathbf{p}_{\Sigma^+} = p \mathbf{n}^+ \quad (5.13)$$

$$\mathbf{p}_{\Sigma^-} = -p \mathbf{n}^- \quad (5.14)$$

the integral over the crack surface in Equation (5.10) can be rewritten as

$$\int_{\Sigma^+} \delta \mathbf{u}_{\Sigma^+}^T \mathbf{p}_{\Sigma^+} \, d\Sigma + \int_{\Sigma^-} \delta \mathbf{u}_{\Sigma^-}^T \mathbf{p}_{\Sigma^-} \, d\Sigma = \int_{\Sigma} (\delta \mathbf{u}_{\Sigma^+}^T - \delta \mathbf{u}_{\Sigma^-}^T) \mathbf{p}_{\Sigma} \, d\Sigma. \quad (5.15)$$

5.2.2 Extended finite element representation

Recalling the XFEM formulation in the context of crack problems, the displacement approximation in vector notation becomes

$$\begin{aligned}
 \mathbf{u}^h(\mathbf{x}) &= \begin{bmatrix} u_1^h(\mathbf{x}) \\ u_2^h(\mathbf{x}) \\ u_3^h(\mathbf{x}) \end{bmatrix} \\
 &= \begin{bmatrix} \sum_{i \in I_{el}} N_i(\mathbf{x}) u_{1i} + \sum_{j \in I_{el}^{step}} N_j^*(\mathbf{x}) \Psi_{step}(\mathbf{x}) a_{1j} + \sum_{k \in I_{el}^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(\mathbf{x}) b_{1k}^m \right) \\ \sum_{i \in I_{el}} N_i(\mathbf{x}) u_{2i} + \sum_{j \in I_{el}^{step}} N_j^*(\mathbf{x}) \Psi_{step}(\mathbf{x}) a_{2j} + \sum_{k \in I_{el}^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(\mathbf{x}) b_{2k}^m \right) \\ \sum_{i \in I_{el}} N_i(\mathbf{x}) u_{3i} + \sum_{j \in I_{el}^{step}} N_j^*(\mathbf{x}) \Psi_{step}(\mathbf{x}) a_{3j} + \sum_{k \in I_{el}^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(\mathbf{x}) b_{3k}^m \right) \end{bmatrix} \\
 &= \underbrace{\begin{bmatrix} \mathbf{N}_{std}^T & \emptyset & \emptyset & \mathbf{M}_{step}^T & \emptyset & \emptyset & \mathbf{M}_{tip}^T & \emptyset & \emptyset \\ \emptyset & \mathbf{N}_{std}^T & \emptyset & \emptyset & \mathbf{M}_{step}^T & \emptyset & \emptyset & \mathbf{M}_{tip}^T & \emptyset \\ \emptyset & \emptyset & \mathbf{N}_{std}^T & \emptyset & \emptyset & \mathbf{M}_{step}^T & \emptyset & \emptyset & \mathbf{M}_{tip}^T \end{bmatrix}}_{\mathbf{N}} \cdot \mathbf{u} \\
 &= \mathbf{N} \cdot \mathbf{u}. \tag{5.16}
 \end{aligned}$$

This formulation consists of a classical FEM approximation which accounts for the continuous solution behavior and is composed of standard finite element shape functions $\mathbf{N}_{std} = [N_1(\mathbf{x}) \dots N_{n_{el}}(\mathbf{x})]^T$ and nodal unknowns u_i . The discontinuous solution behavior is captured by two additional terms. The first term is based on the enrichment function $\Psi_{step}(\mathbf{x})$ and takes the discontinuity in the displacement field across the crack path/surface into account. The second term, based on the four enrichment functions $\Psi_{tip}^i(\mathbf{x})$, accounts for the singular stresses and strains at the crack tip/front. The enrichments are applied to the nodes in I^{step} and I^{tip} . At these nodes, additional nodal degrees of freedom a_j and b_k^m are introduced into the approximation. The compact form of the unknowns $\mathbf{u}$ contains the nodal unknowns of the classical FEM approximation (u_1, u_2, u_3), the nodal unknowns of the step-enriched terms (a_1, a_2, a_3), and, finally, the nodal unknowns of the four tip-enrichments ($b_1^{1-4}, b_2^{1-4}, b_3^{1-4}$):

$$\mathbf{u} = \left[u_1, u_2, u_3, a_1, a_2, a_3, b_1^1, \dots, b_1^4, b_2^1, \dots, b_2^4, b_3^1, \dots, b_3^4 \right]^T \tag{5.17}$$

The enrichment vectors $\mathbf{M}_{step}$ and $\mathbf{M}_{tip}$ are defined as

$$\mathbf{M}_{step} = \left[N_1^*(\mathbf{x})\Psi_{step}(\mathbf{x}), \dots, N_{n_{el}}^*(\mathbf{x})\Psi_{step}(\mathbf{x}) \right]^T \quad (5.18)$$

$$\mathbf{M}_{tip} = \left[\begin{array}{c} \left[N_1^*(\mathbf{x})\Psi_{tip}^1(\mathbf{x}), \dots, N_{n_{el}}^*(\mathbf{x})\Psi_{tip}^1(\mathbf{x}) \right]^T \\ \left[N_1^*(\mathbf{x})\Psi_{tip}^2(\mathbf{x}), \dots, N_{n_{el}}^*(\mathbf{x})\Psi_{tip}^2(\mathbf{x}) \right]^T \\ \left[N_1^*(\mathbf{x})\Psi_{tip}^3(\mathbf{x}), \dots, N_{n_{el}}^*(\mathbf{x})\Psi_{tip}^3(\mathbf{x}) \right]^T \\ \left[N_1^*(\mathbf{x})\Psi_{tip}^4(\mathbf{x}), \dots, N_{n_{el}}^*(\mathbf{x})\Psi_{tip}^4(\mathbf{x}) \right]^T \end{array} \right] \quad (5.19)$$

Thus, approximating the strain tensor by means of the shape functions leads to

$$\boldsymbol{\varepsilon}^h = \mathbf{S}\mathbf{u}^h = \mathbf{S}\mathbf{N} \cdot \mathbf{u} = \mathbf{B} \cdot \mathbf{u}. \quad (5.20)$$

Equivalently, the discrete stress tensor becomes

$$\boldsymbol{\sigma}^h = \mathbf{C} \cdot \boldsymbol{\varepsilon}^h = \mathbf{C}\mathbf{S}\mathbf{N} \cdot \mathbf{u} = \mathbf{C}\mathbf{B} \cdot \mathbf{u} \quad (5.21)$$

with the discretized gradient operator

$$\mathbf{B} = \left[\begin{array}{cccccccc} \frac{\partial}{\partial x_1} \mathbf{N}_{std}^T & \emptyset & \emptyset & \frac{\partial}{\partial x_1} \mathbf{M}_{step}^T & \emptyset & \emptyset & \frac{\partial}{\partial x_1} \mathbf{M}_{tip}^T & \emptyset & \emptyset \\ \emptyset & \frac{\partial}{\partial x_2} \mathbf{N}_{std}^{*T} & \emptyset & \emptyset & \frac{\partial}{\partial x_2} \mathbf{M}_{step}^{*T} & \emptyset & \emptyset & \frac{\partial}{\partial x_2} \mathbf{M}_{tip}^{*T} & \emptyset \\ \emptyset & \emptyset & \frac{\partial}{\partial x_3} \mathbf{N}_{std}^{*T} & \emptyset & \emptyset & \frac{\partial}{\partial x_3} \mathbf{M}_{step}^{*T} & \emptyset & \emptyset & \frac{\partial}{\partial x_3} \mathbf{M}_{tip}^{*T} \\ \frac{\partial}{\partial x_2} \mathbf{N}_{std}^T & \frac{\partial}{\partial x_1} \mathbf{N}_{std}^T & \emptyset & \frac{\partial}{\partial x_2} \mathbf{M}_{step}^T & \frac{\partial}{\partial x_1} \mathbf{M}_{step}^T & \emptyset & \frac{\partial}{\partial x_2} \mathbf{M}_{tip}^T & \frac{\partial}{\partial x_1} \mathbf{M}_{tip}^T & \emptyset \\ \emptyset & \frac{\partial}{\partial x_3} \mathbf{N}_{std}^T & \frac{\partial}{\partial x_2} \mathbf{N}_{std}^T & \emptyset & \frac{\partial}{\partial x_3} \mathbf{M}_{step}^T & \frac{\partial}{\partial x_2} \mathbf{M}_{step}^T & \emptyset & \frac{\partial}{\partial x_3} \mathbf{M}_{tip}^T & \frac{\partial}{\partial x_2} \mathbf{M}_{tip}^T \\ \frac{\partial}{\partial x_3} \mathbf{N}_{std}^T & \emptyset & \frac{\partial}{\partial x_1} \mathbf{N}_{std}^T & \frac{\partial}{\partial x_3} \mathbf{M}_{step}^T & \emptyset & \frac{\partial}{\partial x_1} \mathbf{M}_{step}^T & \frac{\partial}{\partial x_3} \mathbf{M}_{tip}^T & \emptyset & \frac{\partial}{\partial x_1} \mathbf{M}_{tip}^T \end{array} \right] \quad (5.22)$$

The element stiffness matrix can now be written in its well known form [185]

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega. \quad (5.23)$$

Note that $\mathbf{C}$ is given in equation (2.33).

The equivalent load vector due to the applied tractions at the outer boundary is

$$\mathbf{b}_{\Gamma} = \int_{\Gamma} \mathbf{N}^T \bar{\mathbf{t}} \, d\Gamma \quad (5.24)$$

5.2.3 Incorporation of interface boundary conditions

In contrast to the classical FEM approach, the crack surface is not aligned along the element faces in the XFEM. Rather, the surface passes through a set of elements which are defined as cut elements. Since the crack surface is subjected to a pressure load obtained from the fluid flow equations, a distributed load is incorporated into the weak formulation as a natural boundary condition on the interface Σ . Since the fluid flow is bounded by the upper and lower crack surfaces, there are physically no degrees of freedom for the fluid pressure at the nodes of the cut elements. In order to establish the right-hand-side vector of the linear system of equations, the pressure has to be transferred into equivalent forces at the element nodes.

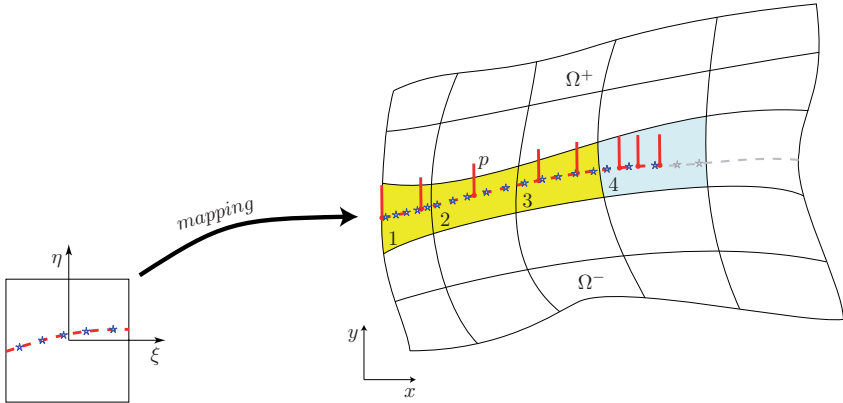


Figure 5.1: The integration points (blue stars) are positioned along the interface (red, dotted line $\dots\dots$) and mapped from the reference into the real element. The red lines ($\color{red}{\longrightarrow}$) represent pressure values at the nodes of the crack which are used to determine the pressure values at the integration points.

Applying the XFEM approximation to the formulation in (5.15) leads to

$$\mathbf{b}_\Sigma = \int_{\Sigma} (\mathbf{N}_{\Sigma^+}^T - \mathbf{N}_{\Sigma^-}^T) \mathbf{p}_\Sigma d\Sigma \quad (5.25)$$

with $\mathbf{N}_{\Sigma^+}$ and $\mathbf{N}_{\Sigma^-}$ being the element shape functions (5.16) evaluated at the interface integration points. Since the interface represents a discontinuity, the evaluation of the enrichment functions is not performed directly on the interface. Rather, the enrichment functions for the positive and negative crack surfaces (Σ^+ and Σ^-) are calculated by asserting that the integration points are either on the positive or negative part of the region. The positive and negative

step-enrichment functions are defined in the shifted version [70] as

$$\mathbf{M}_{step,\Sigma+}^T = \mathbf{N}^*(\mathbf{x})[\Psi_{step}(+\mathbf{x}) - \Psi_{step}(\mathbf{x}_i)] = \mathbf{N}^*(\mathbf{x})[-1 - \Psi_{step}(\mathbf{x}_i)], \quad (5.26)$$

and

$$\mathbf{M}_{step,\Sigma-}^T = \mathbf{N}^*(\mathbf{x})[\Psi_{step}(-\mathbf{x}) - \Psi_{step}(\mathbf{x}_i)] = \mathbf{N}^*(\mathbf{x})[-1 - \Psi_{step}(\mathbf{x}_i)]. \quad (5.27)$$

The global enrichment function $\Psi_{step}(\mathbf{x})$ is chosen as the sign-function of the level-set function defined in (3.15).

Accordingly, the four step-enrichment functions $\Psi_{tip}^{1-4}(r, \theta)$ are evaluated by setting the angle θ at the crack front to $\pm\pi$ as demonstrated in Figure 5.9.

$$\mathbf{M}_{tip,\Sigma+}^T = \mathbf{N}^*(\mathbf{x}) \sum_{m=1}^4 [\Psi_{tip}^m(r, +\pi) - \Psi_{tip}^m(r_i, \pi_i)], \quad (5.28)$$

and

$$\mathbf{M}_{tip,\Sigma-}^T = \mathbf{N}^*(\mathbf{x}) \sum_{m=1}^4 [\Psi_{tip}^m(r, -\pi) - \Psi_{tip}^m(r_i, \pi_i)]. \quad (5.29)$$

Here, the sign $\pm\mathbf{x}$ has the symbolical meaning that the enrichment function are evaluated for the positive and negative region, although, the position $\mathbf{x}$ remains unchanged. The definition of $\Psi_{tip}^{1-4}(r, \theta)$ can be found in (3.17).

Figure 5.1 depicts schematically how the integration points are mapped from the reference into the real element. The integration points are set along the implicitly defined interface based on the level-set functions. It is noted, that the implicit and explicit crack description do not necessarily coincide. This is because the zero-isoline which represent the crack interface has to be interpolated from the level-set values at the element. Integration points in the tip element (denoted with number 4) which are not located at the real crack path are not taken into account in the integration routine. The integration weights are corrected accordingly.

The full procedure of incorporating the interface boundary condition is described in Algorithm 1. All necessary steps are performed within the element loop.

Test cases

In order to prove the validity of this implementation two simple tests were conducted. In the first test case (see Figure 5.2) the opening of a half crack under various pressure profiles is calculated with the XFEM and compared to a equivalent standard finite element approach. Both, the XFEM- and FEM-mesh have the same order of refinement. In Figure 5.2a a linear pressure profile $p(x) = mx$ and in Figure 5.2b a pressure profile with $p(x) = \sin(x)$ is applied on the crack surfaces.

Algorithm 1 Interface boundary conditions

- 1: **Loop over cut elements**
 - ▷ elements that contain the real crack
- 2: **Set integration points in the reference element along the interface**
 - ▷ integration points are set on the basis of the implicit crack description (and the level set functions)
- 3: **Project integration points from reference to real element**
- 4: **Evaluate enrichment functions**
 - ▷ enrichment functions have to be evaluated at the positive and negative crack surface
- 5: **Determine pressure values at the integration points**
 - ▷ through interpolation
- 6: **Add element contribution to right hand side**
- 7: **End Loop**

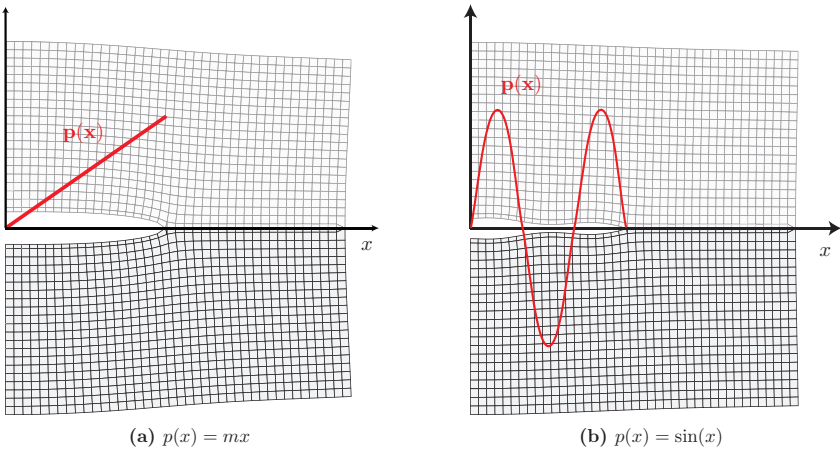


Figure 5.2: A half crack opens under a given pressure profile $p(x)$.

Figures 5.3 show the stress field calculations normal to the crack location for the XFEM (left figures) and the FEM (right figures). These calculations were conducted on a coarse grid with 40×40 (Figures 5.3a and 5.3b) and a fine grid with 160×160 (Figures 5.3c and 5.3d) elements. The stress field in the XFEM-solution yields extremely high values along the interface whereas no stress peaks are present in the FEM-solution. A good agreement between the XFEM- and FEM-results is also achieved for the deformation field on the coarse grid as is shown in Figures 5.3.

Analogously for the pressure profile given in Figure 5.2b, the stress state (Figures 5.5) and the deformation field in y -direction (Figures 5.6) show the same good agreement with the

FEM-calculations.

In order to analyze the influence of the mesh on the interface integration method, the problem of a horizontal crack under a uniform pressure with a zero-displacement condition at the outer boundary is considered. The domain is chosen big enough, in order to avoid any influence of the domain geometry on the crack opening. The underlying mesh is rotated by an increment of 10° (see Figure 5.7).

The calculated crack opening w is compared with the equivalent FEM-solution and the analytical solution of a Griffith crack [158]:

$$w(x) = 4pl \frac{1 - \nu^2}{E} \sqrt{1 - \left(\frac{x}{l}\right)^2}, \quad (5.30)$$

where p is the uniform pressure, l the half crack length, and ν, E are material parameters. Each single crack opening solution obtained from the XFEM approach is plotted in Figure 5.8 together with an equivalent FEM solution and the analytical solution. The XFEM solutions for different mesh configurations are all almost identical. The only visible difference is in the FEM approach close to the crack tips. This is due to the relatively coarse mesh used here for the calculations.

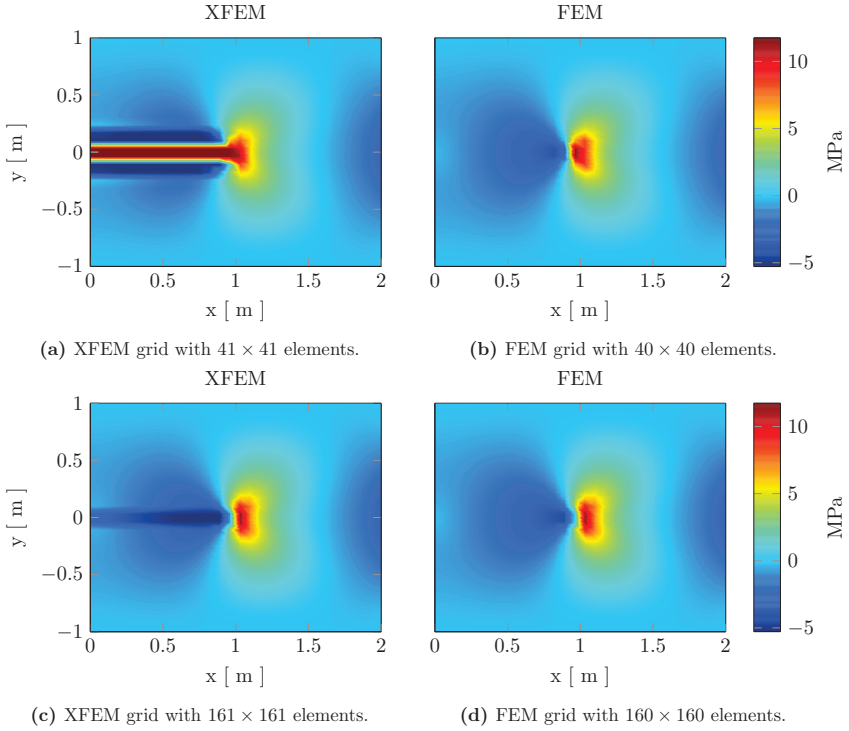


Figure 5.3: Comparison of the stress field σ_{yy} for the XFEM and FEM.

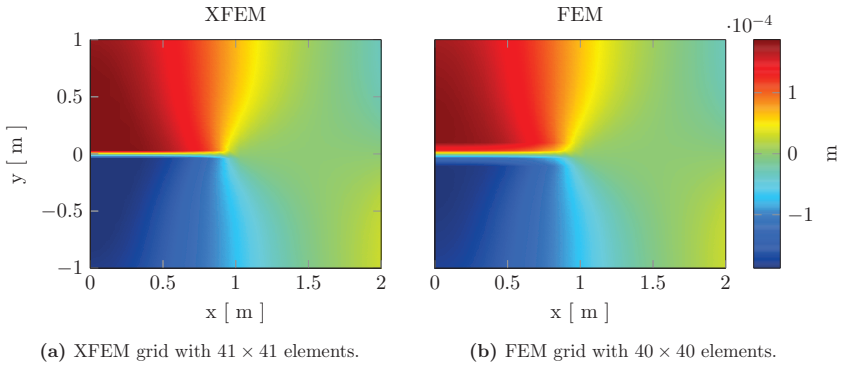


Figure 5.4: Comparison of the deformation field in y -direction for the XFEM and FEM.

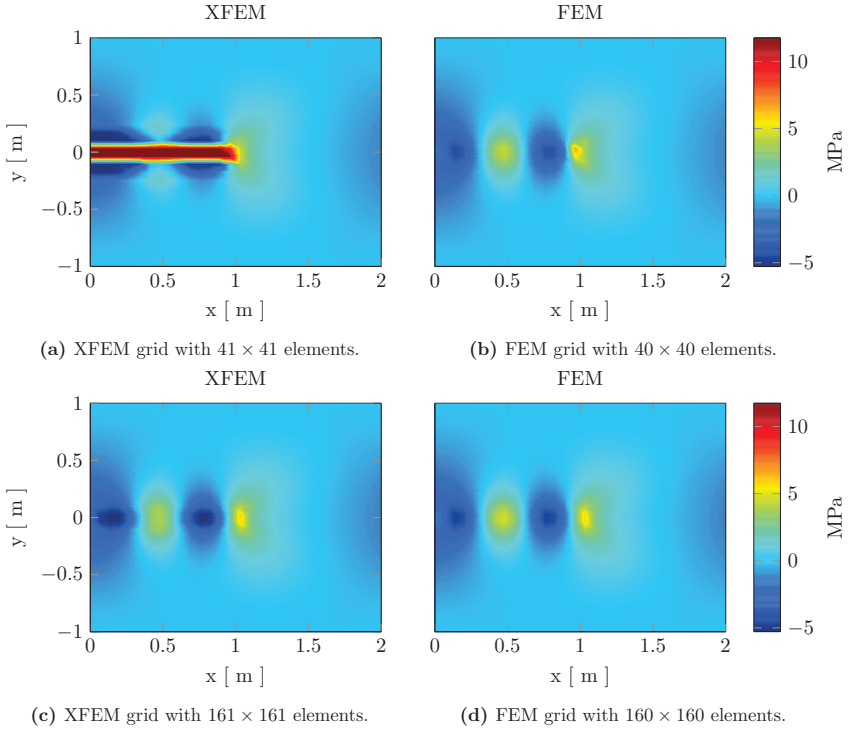


Figure 5.5: Comparison of the stress field σ_{yy} for the XFEM and FEM.

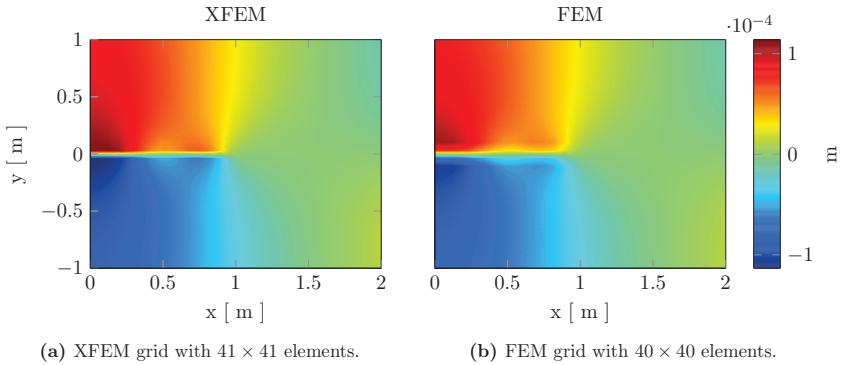


Figure 5.6: Comparison of the deformation field in y -direction for the XFEM and FEM.

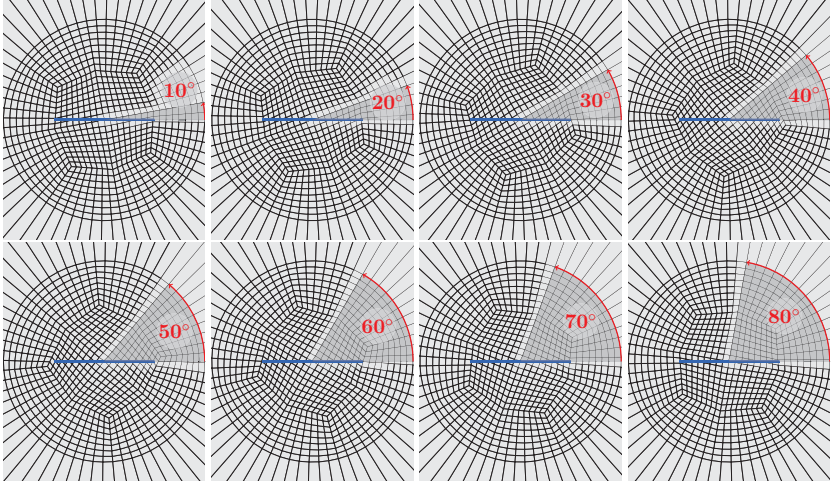


Figure 5.7: Mesh configuration. The interface is represented by the blue line (—). The underlying mesh is rotated with an angle of 10° , 20° , 30° , 40° , 50° , 60° , 70° , 80° .

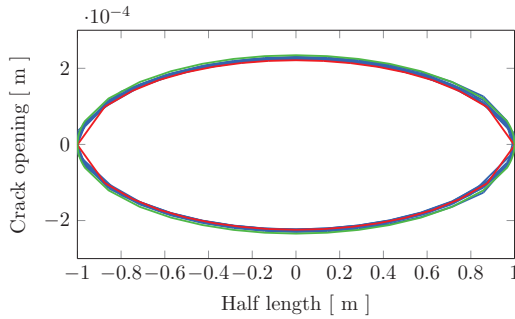


Figure 5.8: Crack opening calculation. The blue line (—) corresponds to the XFEM solution, the red line (—) is the equivalent FEM solution, and the light green line (—) is the analytical solution given by [158].

5.2.4 Crack opening

The displacement field obtained with the FEM or XFEM only delivers a solution at the nodes of each element; however, the solution behavior inside elements is easily determined by means of interpolation. In order to obtain the crack width, the displacement field has to be interpolated at the crack path/surface. The width is then defined as

$$w(\mathbf{x}) = u^+(\mathbf{x}) - u^-(\mathbf{x}), \quad (5.31)$$

where $u^+(\mathbf{x})$ and $u^-(\mathbf{x})$ are the displacements at the crack path in positive and negative (normal) direction, respectively. The sign is hereby defined through the level-set field ϕ_3 .

Since the crack itself represents a discontinuity, interpolation is not straightforward. One possible solution might be interpolating slightly above and below the point of interest. Another way is to *assign* the correct values to the corresponding enrichment functions as is shown in Figure 5.9.

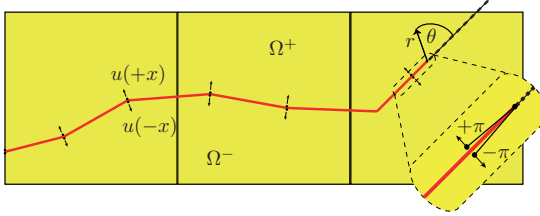


Figure 5.9: The positive and negative crack surface are considered for the evaluation of the enrichment functions according to this picture. The red line (—) represents the crack.

Evaluating the step-enrichment function Ψ_{step} from Equation (3.15) and the tip-enrichment function Ψ_{tip} from Equation (3.17) we arrive at

$$u^+(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) u_i + \sum_{j \in I^{step}} N_j^*(\mathbf{x}) a_j + \sum_{k \in I^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(r, +\pi) b_k^m \right) \quad (5.32)$$

$$u^-(\mathbf{x}) = \sum_{i \in I} N_i(\mathbf{x}) u_i - \sum_{j \in I^{step}} N_j^*(\mathbf{x}) a_j + \sum_{k \in I^{tip}} N_k^*(\mathbf{x}) \cdot \left(\sum_{m=1}^4 \Psi_{tip}^m(r, -\pi) b_k^m \right) \quad (5.33)$$

which can be interpreted as interpolating the displacement at the positive and negative crack face, respectively.

In the case when the crack itself is moving due to internal and external loads as depicted in Figure 5.10a, the individual parts that contribute to the displacement of the crack have to be determined. In a deformed configuration, the crack path is defined to be in the middle of the crack opening as is indicated by the dashed red line in Figure 5.10b, so that the crack opening in positive and in negative direction has the same magnitude but opposite sign. The

crack is simply the distance between the crack position in the deformed and the undeformed configuration.

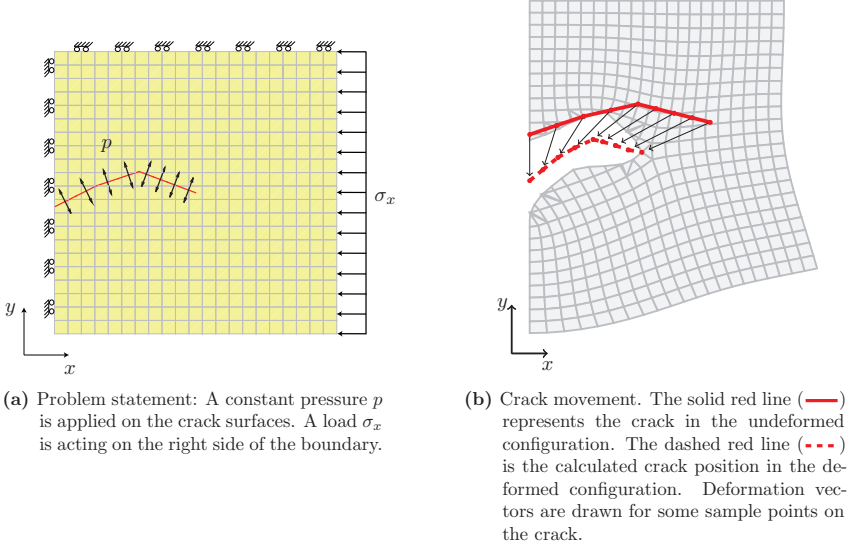


Figure 5.10: Crack opening decomposition.

Finally, the contribution of the crack opening in both, positive and negative direction is then obtained by subtracting the crack-path motion from the actual deformation

$$\bar{u}_x(+x) = u_x(+x) - dx_c \quad (5.34a)$$

$$\bar{u}_x(-x) = u_x(-x) - dx_c \quad (5.34b)$$

$$\bar{u}_y(+x) = u_y(+x) - dy_c \quad (5.34c)$$

$$\bar{u}_y(-x) = u_y(-x) - dy_c. \quad (5.34d)$$

Here, $\bar{u}_x(\pm x)$ is the crack opening of the positive and negative face in x -direction, $\bar{u}_y(\pm y)$ is the crack opening of the positive and negative face in y -direction, $u_x(\pm x)$ and $u_y(\pm x)$ are the deformations at the crack position x , and dx_c, dy_c are the contributions of the crack-path movement in x - and y -direction, respectively.

5.3 Fracture propagation

In this section, the numerical realization of the crack growth process is described. Once the condition for crack growth is satisfied the crack growth direction is calculated for every single node at the crack front on the basis of the local stresses [69]. Given the local coordinate system at the crack front as depicted in Figure 2.12, stresses are evaluated within the $(\vec{n}, \vec{t})$ -plane at a limited number of discrete points. These points are placed on a circle with a user-specified radius r_c ; however, since the crack propagation angle is limited to an interval of $\pm 70.529^\circ$ for pure crack mode II [69], all points are placed only within the range of $\pm 75^\circ$. The crack is then extended along the vectors starting at the front nodes and pointing towards the *local* maximum circumferential stress, which is the maximum stress for all points in one circle (see Figure 5.11). The length of this extension vector, and thus the crack increment da , is defined by the user. Since the *local* maximum stresses will not necessarily be the same for all front nodes, the crack increment is applied only to the maximum of all *local* maximum stresses. For all other front nodes, the crack extension length is scaled according to their maximum local values.

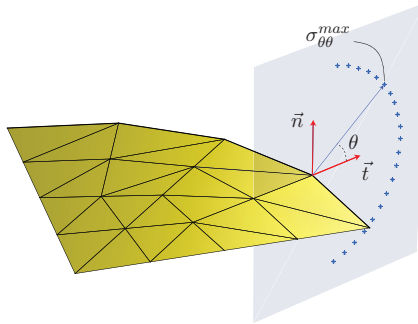


Figure 5.11: An illustration on placing points around a virtual circle in a plane spanned by the vectors $\vec{n}$ and $\vec{t}$.

It is noteworthy, that for C_0 -continuous approximations the stress field is discontinuous over the element boundaries [69]. In order to avoid stress jumps that might be present on the circle profiles, a smooth stress field is obtained from the XFEM-solution in a post-processing step. The actual hoop stress is then calculated on the basis of the smoothed stress field.

5.4 Coupling

The overall model can be separated into the sub-modules *Fluid* and *Structure* which are coupled through an iterative process. The crack propagation criterion is integrated into the coupling and is used for determining whether the crack grows or not. The advantage of an iterative

coupling is the flexibility in developing the fluid and the structure model separately. Moreover, different fluid models can be tested in conjunction with the structure model or vice versa.

5.4.1 General procedure

The entire coupling is shown in the flowchart 5.12. The fluid equations are solved here by utilizing the finite element method on the fracture surface and the structure equations are solved by means of the extended finite element method.

The time loop starts with an initial solution at the time step t_k . The initialization includes the routines to determine all relevant information for the fluid and the structural solver, such as the crack description, the level-set field, and the XFEM enrichments. The time is calculated on the basis of the global volume balance condition as shown in Equation (6.13).

In the convergence loop, the fluid pressure and the deformation (and crack width) are coupled iteratively. In the beginning, the fluid equation is solved for an initial crack width w_1^k . In each iteration step n , the structure module receives the fluid pressure as a load boundary condition on the crack surface and delivers a new crack width $\tilde{w}_n^k$, from which eventually the actual crack width w_n^k is determined by relaxation. These steps are repeated until the relative error between the crack widths w_n^k and $\tilde{w}_n^k$ is below a given tolerance.

Finally, the a check is done by means of the crack propagation criterion whether the crack grows or not. If the stress intensity factor K_I matches the fracture toughness K_{IC} the current time step is completed and new crack increments are added to the current crack. In the flowchart, this step is referred to as *Crack growth step* and covers all processing steps that are necessary to provide a new crack configuration for the next time step. The whole procedure is repeated for any further time step.

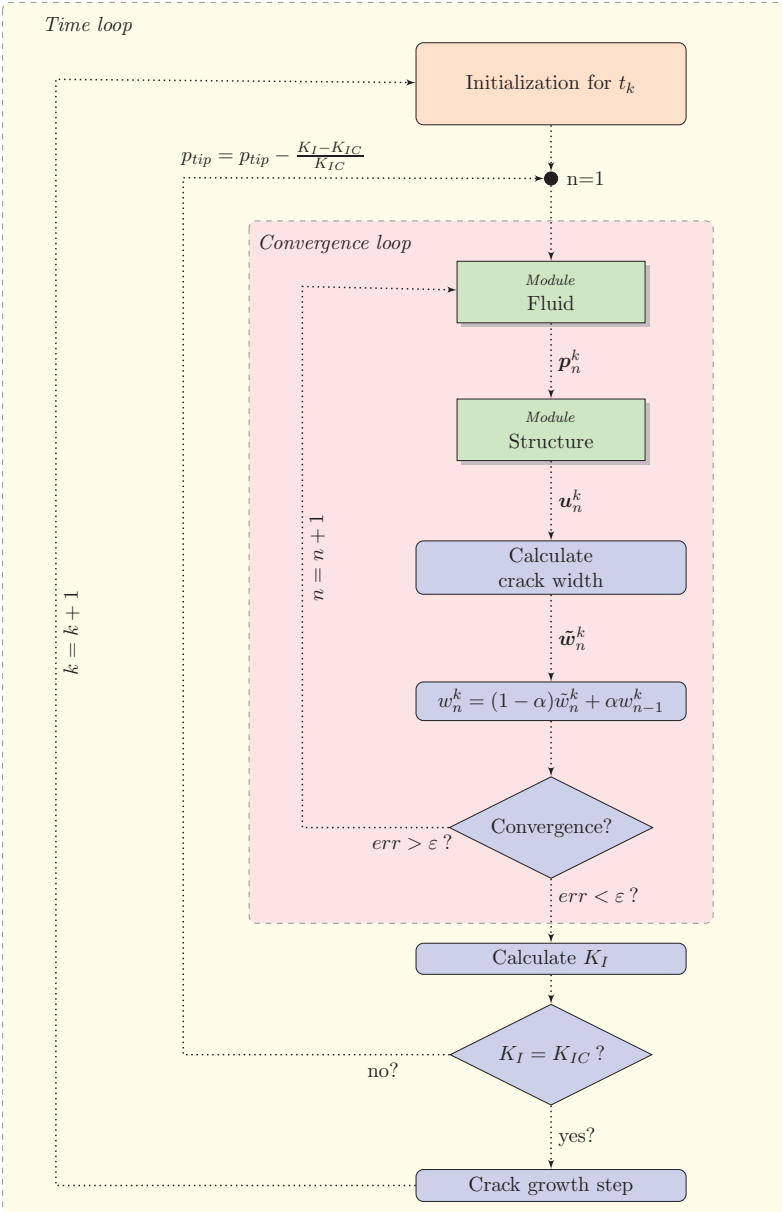


Figure 5.12: The flow chart illustrated the iterative coupling procedure between the fluid and solid modules.

5.4.2 Fluid-Lag coupling

The laboratory fracturing tests with a high viscous fluid and a block sample of $30 \times 30 \times 45 \text{ cm}^3$ suggested that a period occurs right after the breakdown pressure where a significant fluid lag is present [39]. Therefore, the coupling procedure for a fluid-induced crack propagation with an existing fluid lag is discussed in this Section.

The fluid-lag coupling is considered for the case of a crack that is initially completely filled with fluid and the fluid pressure is sufficiently high to initiate crack propagation (i.e. the propagation condition is met). The algorithm now starts with adding a crack increment to the existing crack; however, the fluid is not moved, yet. An increased fluid lag will lead to an instantaneous pressure drop. The overall pressure might not be sufficient anymore to fulfill the growth condition and the fluid has to be moved forward such that crack propagation is possible again. The pressure in the lag region is set to the pore pressure. Typically, the pore pressure is of little importance and, therefore, set to zero.

The size of the lag becomes a part of the solution process and is not known *a priori*. Furthermore, it is precluded that the fluid lag vanishes completely; however, the fluid-lag size can become vanishingly small. Since the fluid-lag size can vary within a complex crack geometry, the level-set approach is used here to track the fluid front position. Here, the fluid-front position can be interpreted as a moving interface and described with the zero level-set. The fluid-filled part of the crack is assigned with a positive level-set value and the lag region is assigned with a negative level-set value. In the fluid module, all nodes within the lag region are then prescribed with the predefined pore pressure and enter the system of equations as a boundary condition.

The algorithm is illustrated in flowchart 5.13

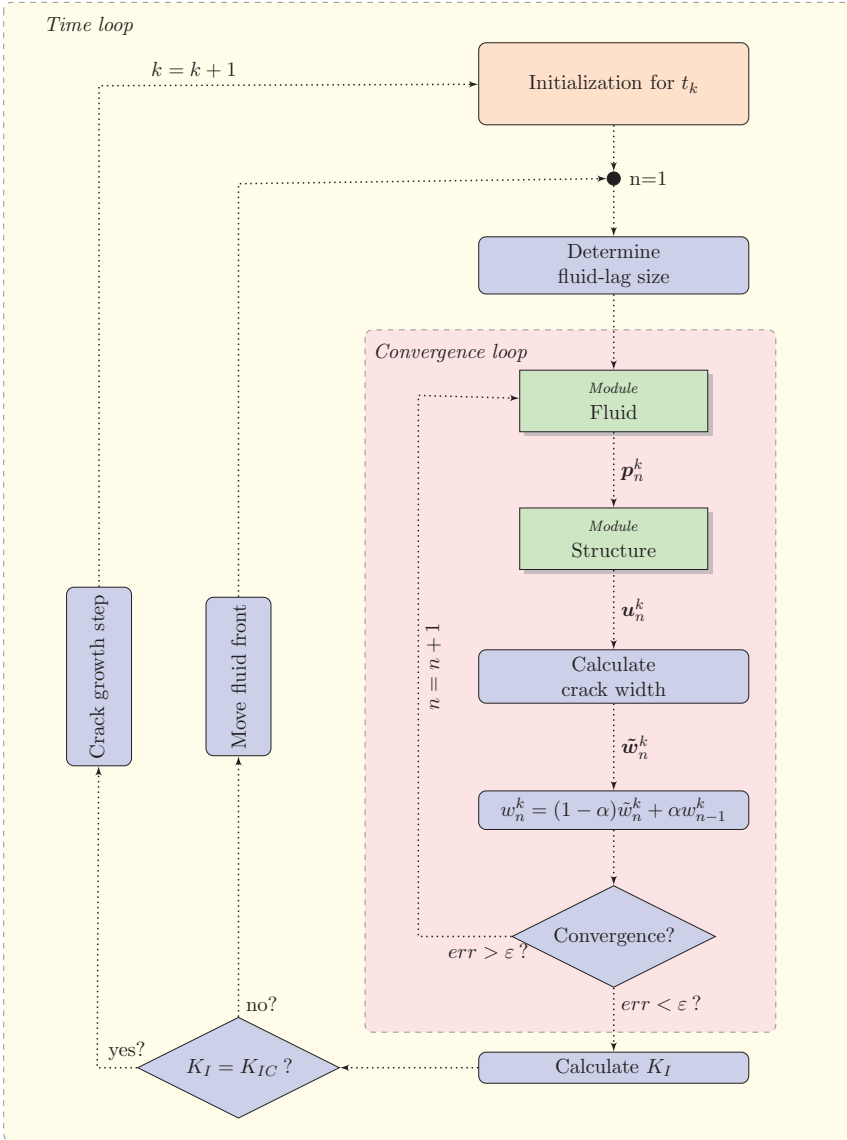


Figure 5.13: The flow chart illustrated the iterative coupling procedure between the fluid and solid modules for a problem where a fluid lag is present.

6 Modeling results and verification

This chapter discusses some results in 2D and 3D for the presented model. First, the solutions of the two limiting regimes - the toughness and the viscosity dominated regime - are used to verify the model. Second, in view of the experiments conducted on block samples in the laboratory scale the model is also tested for a non-planar crack propagation. Hereby, the crack is not aligned with the maximum stresses but initially inclined with an angle α_0 . The crack propagation direction is studied in the beginning of crack growth.

Moreover, the model is compared with laboratory experiments with a penny-shaped crack. A rock sample is fractured through injection of a high viscous fluid and the crack growth is monitored by means of acoustic emissions.

6.1 Numerical examples in 2D

The 2D model is implemented in MATLAB and verified against a KGD-type model [45] and against the toughness solution with fluid lag presented in Garagash [73], Hunsweck et al. [92].

KGD model

As discussed in Section 2.1.1, the KGD model is one of the first models in hydraulic fracturing which accounts for the fracture toughness. Analytical solutions are given for the main parameters of fracture length $L(t)$, fracture width at the crack inlet $w(0, t)$, and the net pressure level at the crack inlet $\Delta p_{net}(0, t)$.

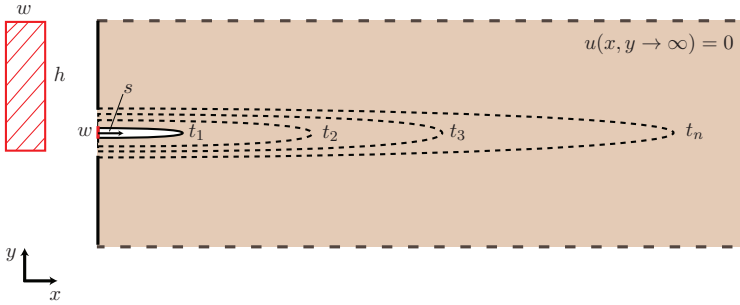
$$L(t) = 2 \left(\frac{16}{21\pi^3} \frac{E' q_0^3}{\mu} \right)^{\frac{1}{6}} \cdot t^{\frac{2}{3}}, \quad (6.1)$$

$$w(0, t) = \left(\frac{5376}{\pi^3} \frac{\mu q_0^3}{E'} \right)^{\frac{1}{6}} \cdot t^{\frac{1}{3}}, \text{ and} \quad (6.2)$$

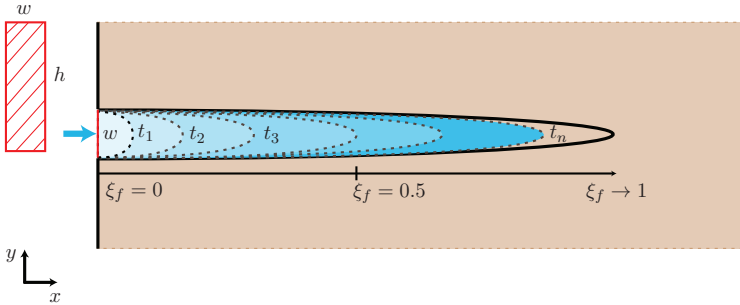
$$\Delta p_{net}(0, t) = \left(\frac{21}{16} \mu E'^2 \right)^{\frac{1}{3}} \cdot t^{-\frac{1}{3}} \quad (6.3)$$

E' is the plane strain modulus as already used in (2.7), and $q_0 = \frac{Q_0}{h_f}$ is the fluid flux at the crack inlet (see (2.3)).

Figure 6.1a shows a sketch of the problem statement. Since this problem is symmetric, *i.e.*, fluid is injected at the center of the crack such that the fracture develops in two opposite



(a) KGD model: A half-wing fracture with a rectangular cross section.



(b) Fluid-lag model: A crack is gradually filled with fluid.

directions, a semi-infinite domain is considered with $u(x, y \rightarrow \infty) = 0$. The cross sectional area of the crack is according to the definition of the KGD-model rectangular with $A = w \cdot h$. The solution (6.2) and (6.3) corresponds to the opening and pressure at the crack inlet at $s = 0$ with s being the local coordinate system along the crack. The solution is calculated for each time step t_1 to t_n .

For the two dimensional domain, a mesh with bilinear elements is used. The crack is initially discretized with 10 linear elements. Since the pressure distribution along the crack is almost uniform and shows close to the crack tip a sharp decrease, additional elements are added close to the crack tip region. However, refining the one-dimensional mesh of the crack does not lead to any significant shortcomings in the calculation performance. The time consuming part is the integration procedure in the 2D domain of the enriched region around the crack tip.

The simulation results shown in Figure 6.2 for the coupled 2D problem fit very well with the fracture length, fracture opening, and pressure of the KGD model. In Figure 6.3a and 6.3b,

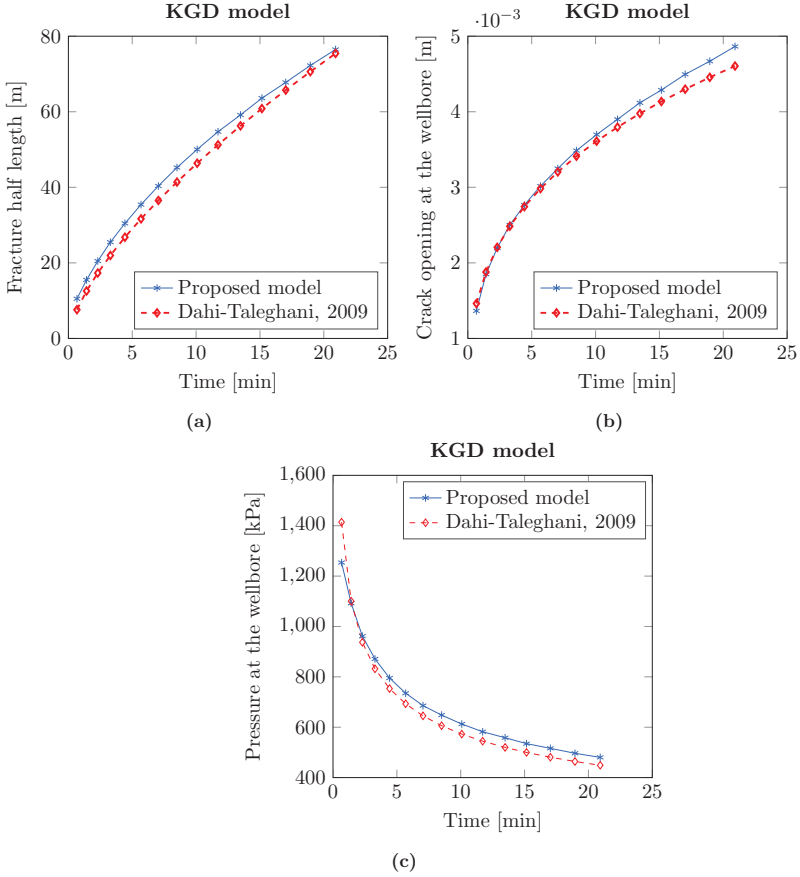


Figure 6.2: The fracture half length (a), the maximum crack width (b), and the pressure (c) over time for a symmetrical crack propagating in two opposite directions is shown here. The simulation results are compared with the KGD model given in [45].

the pressure and crack opening profiles are plotted for each iteration step.

Fluid-lag model

The second test case considers the problem of a toughness dominated crack propagation with

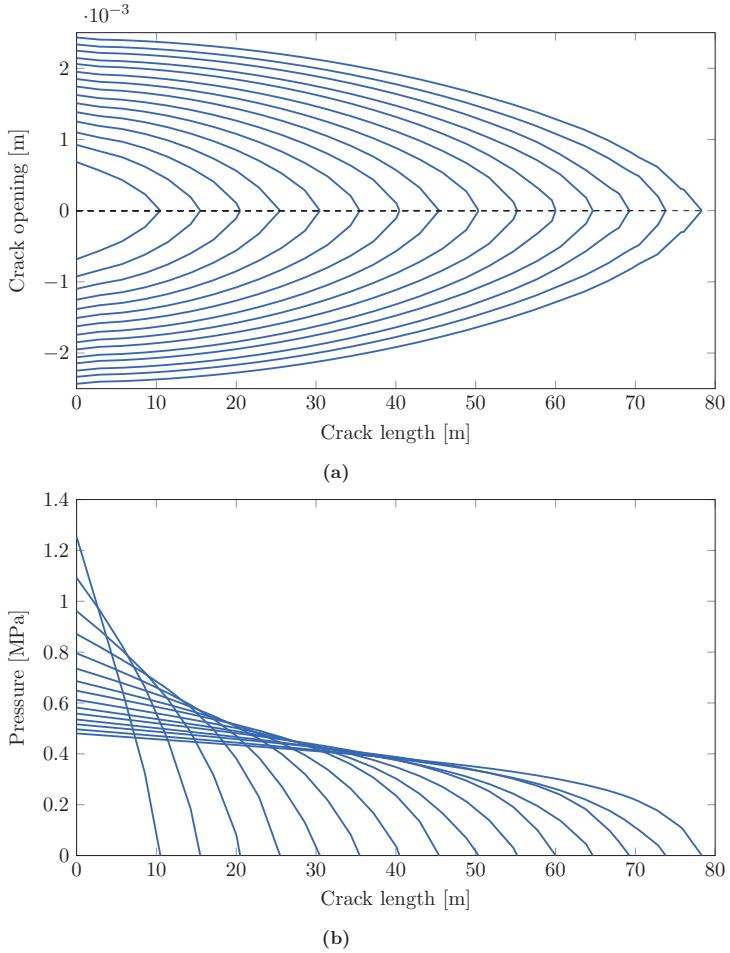


Figure 6.3: Simulation results for the crack opening (a) and pressure (b) profiles for a symmetrical crack propagating in two opposite directions. Only one half is shown here.

fluid lag. The fluid front position ξ_f assumes values between 0 and 1. Here $\xi_f = 0$ means that the crack does not contain any liquid and therefore pore pressure prevails in the entire crack. Accordingly, for $\xi_f = 0.5$ the crack is half filled with liquid. Finally, for $\xi_f \rightarrow 1$ the

cavity vanishes, so that the fluid front and crack front coalesce. This situation is illustrated in Figure 6.1b.

The solution in the literature is not given in a closed analytical form but is calculated numerically (see [73, 92]). The simulation results are in a very good agreement with the solution given in the literature. The crack length, the crack width at the wellbore and the wellbore pressure are shown in scaled (dimensionless) quantities.

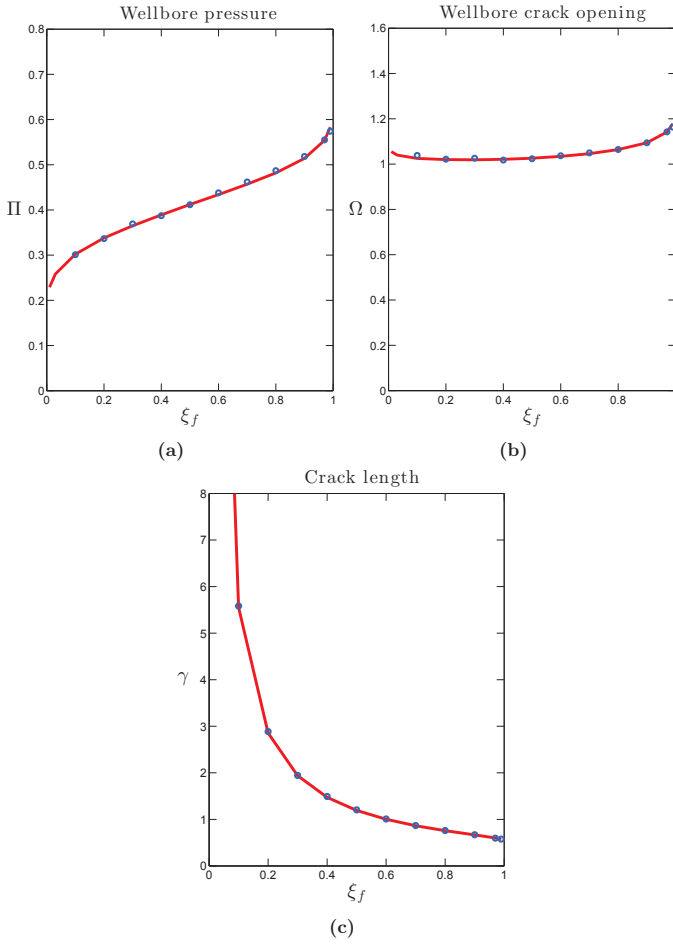


Figure 6.4: A scaled pressure Π (a) and a scaled crack width Ω (b) at the injection point, and a scaled crack half length γ (c) are shown here. The blue dots correspond to the simulation results whereas the red line shows the solution given in the literature.

6.2 Numerical examples in 3D

In the first step, the verification of the fluid and structure model is performed separately. Benchmark solutions are provided for the penny-shaped model [31, 147] and are compared with the simulation results for the impermeable case of viscosity-dominated, toughness-dominated, and large-toughness problem. It is assumed that the penny-shaped crack propagates in an infinite medium. A fluid is pressed into the crack with a constant injection rate Q_0 . The fluid model considers for the fluid injection in the form of a point source in the center of the penny-shaped crack. As a consequence, the pressure profile shows not only a singular behavior at the crack front but also towards the crack center.

The model of the penny-shaped crack is one of the most important benchmarks in hydraulic fracturing. In recent years, a number of solutions have been developed that describe the behavior of the penny-shaped model in limiting cases. This limiting behavior is characterized by the dominant physical processes of viscosity and toughness which are responsible for crack growth. The dominant regime during the growth phase was elaborated in detail in the literature and depends on the material parameters of the fluid and the structure. Accordingly, the crack growth starts in the viscosity-dominated regime (M -vertex), and shifts into the toughness-dominated regime (K -vertex). In between, a mixed regime is present wherein both the viscosity and the toughness are responsible for crack growth and is referred to as K -large. These three regimes are used for a comparison with the simulation. The dimensionless fracture toughness $\mathcal{K}$ is used as a parameter to determine the dominant regime. The range for $\mathcal{K}$ is given as

$$\mathcal{K} < 1 \quad \text{viscosity dominated regime (} M\text{-vertex)} \quad (6.4a)$$

$$1 < \mathcal{K} < 3.5 \quad \text{mixed regime (} K\text{-large)} \quad (6.4b)$$

$$3.5 < \mathcal{K} \quad \text{toughness dominated regime (} K\text{-vertex)} \quad (6.4c)$$

The simulations were conducted with the following parameters:

K_{IC}	1.66	$MPa\sqrt{m}$	fracture toughness
E	37800	MPa	elastic modulus
ν	0.25	-	Poisson's ratio
μ	$1.48 \cdot 10^{-6}$	-	dynamical viscosity
Q_0	0.1	$\frac{mL}{min}$	fluid injection rate

Table 6.1: Parameters for penny-shaped crack growth.

The dimensional toughness $\mathcal{K}$ can be set by adjusting the crack radius and holding all other

parameters constant.

$$R = 0.001 \text{ m} \quad \rightarrow \mathcal{K} \approx 0.87 \quad M - \text{vertex} \quad (6.5a)$$

$$R = 0.05 \text{ m} \quad \rightarrow \mathcal{K} \approx 2.43 \quad K - \text{large} \quad (6.5b)$$

$$R = 0.5 \text{ m} \quad \rightarrow \mathcal{K} \approx 4.57 \quad K - \text{vertex} \quad (6.5c)$$

6.2.1 Uncoupled penny-shaped crack

In the decoupled test case, either the exact crack opening or the pressure field is given to the appropriate solver. That is, the fluid model receives the exact crack opening as an input variable and gives the pressure profile as an output. In contrast, the exact pressure profile is given to the structure model which yields the crack opening as an output.

Fluid

The Reynolds equation (2.23) is solved for the pressure profile by means of the classical finite element method (see Section 5.1). Solutions for the viscosity-dominated, toughness-dominated, and mixed regime are taken from the literature [109, 147]. The fluid solver returns as a result a pressure profile over the crack. The level of this pressure profile is determined by the pressure boundary condition at the crack front. The choice of the pressure boundary condition is determined on the basis of the fracture toughness of the rock. For the simulation, this means that the fluid module is first solved for an arbitrary pressure tip condition, say $p_{tip} = 0$. Subsequently, the entire pressure level is adjusted so that the crack propagation criterion $K_I = K_{IC}$ (2.45) is met. Here, the stress intensity factor K_I must be calculated. This requires the solution of a singular integral, which is given explicitly for the penny-shaped model as [145]

$$K_{IC} = K_I = \frac{2}{\sqrt{\pi R}} \int_0^R \frac{p_{net}(r, t)}{\sqrt{R^2 - r^2}} r dr. \quad (6.6)$$

Here, R denotes the crack radius, p_{net} is the net pressure, and r is the radial coordinate of a penny-shaped crack (see Figure 2.3).

In view of the dominating viscosity over the fracture toughness in the M -vertex solution, this approach of coupling the stress intensity factor with the crack front pressure condition might be not well suited. Nevertheless, good results are obtained for the fluid model, in particular for the viscosity-dominated problem.

Figure 6.5 shows the mesh of the penny-shaped crack, whereas the crack radius is scaled to 1 for a more general view. The crack is discretized with triangular (linear) elements of size $h \approx 0.3 \times r$. Additionally, a refined region at the crack front accounts for the steep pressure gradient. The region size to be refined which is the distance to the crack front is chosen to be $\approx 0.03 \times r$. Since this is a symmetrical problem, the solution is evaluated along the x -axis with $x > 0$ and $y = 0$ as indicated in Figure 6.6.

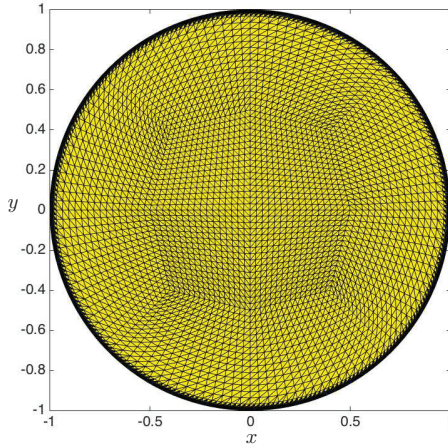


Figure 6.5: Mesh of the penny-shaped crack.

Solutions for the three considered test cases are given in Figures 6.7, 6.8, and 6.9. The M -vertex solution (Figure 6.7) shows a good agreement in the pressure profile. Though, due to numerical errors in calculating the stress intensity factor from (6.6), the pressure level is slightly overestimated here. The K -large solution (Figure 6.8) is also in a good agreement with the solution given in the literature [146, 147]. This solution has no closed analytical form but is numerically evaluated. Finally, the K -vertex solution shows the greatest deviations from the exact solution. The reason for that lies in the character of the K -vertex solution. Since the assumption holds, that fracture toughness is predominant and the viscosity does not play a role in propagating the fracture, the solution becomes uncoupled with a uniform pressure on the crack surface and an elliptical crack opening. Nevertheless, the fluid flow model aims for solving an equation with a non-vanishing viscosity and a constant injection rate incorporated as a point-source into the model. This leads to the already discussed singular behavior at the center of the crack. Furthermore, the singularity at the crack front still prevails (under the assumption of a negligible fluid lag) and is limited to a very small region near the tip. Thus, the pressure profile obtained from the fluid model can not be uniform over the whole crack area; however, it is still used here in this section in order to show the results obtained for a zero-viscosity solution.

Structure

The structure module is based on the extended finite element method (XFEM) and calcu-

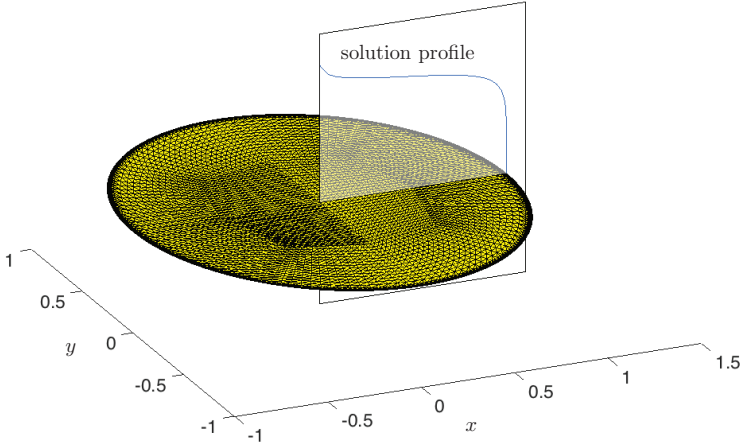


Figure 6.6: The solution is evaluated at the plane $x > 0, y = 0$.

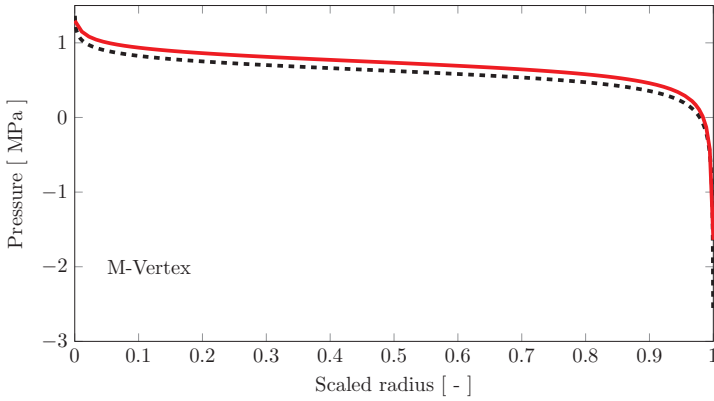


Figure 6.7: The pressure profile for a circular crack for the viscosity-dominated case (M -vertex) is shown here. The black dotted line (.....) corresponds to the exact solution and the red solid line (—) to the simulation.

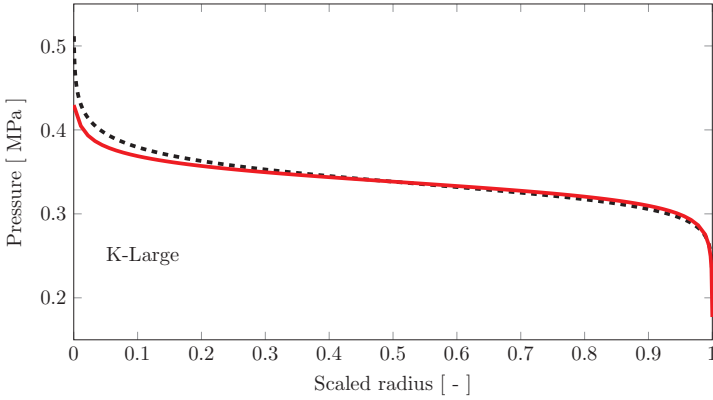


Figure 6.8: The pressure profile for a circular crack for the large-toughness case (K -large) is shown here. The black dotted line (.....) corresponds to the exact solution and the red solid line (—) to the simulation.

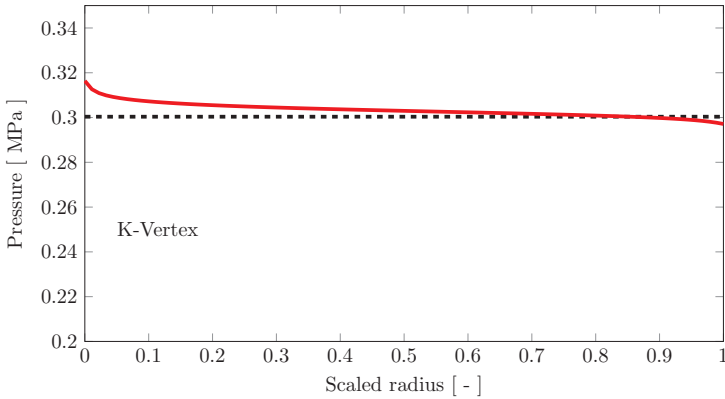


Figure 6.9: The pressure profile for a circular crack for the toughness-dominated case (K -vertex) is shown here. The black dotted line (.....) corresponds to the exact solution and the red solid line (—) to the simulation.

lates the displacements of a body for a given stress state. In the context of fracture mechanics, the focus is on the displacement along the crack surface rather the displacements in the whole body. The exact pressure profile enters the structure module as an interface boundary condition. For all the test cases one obtains results that are in a good agreement with the reference

solutions. One limiting factor is the mesh refinement of the whole body and the refinement of the crack interface itself. In order to ensure an equal treatment for the uncoupled and the coupled problem, the exact pressure profile is not applied directly on the integration points when integrating the crack surface. Instead, the exact pressure values are stored on the interface mesh which represents the crack and transferred as a boundary condition to the structure module. This approach does not give the exact pressure on every single integration point because of interpolation errors, but it ensures an equal treatment of the uncoupled and coupled problem.

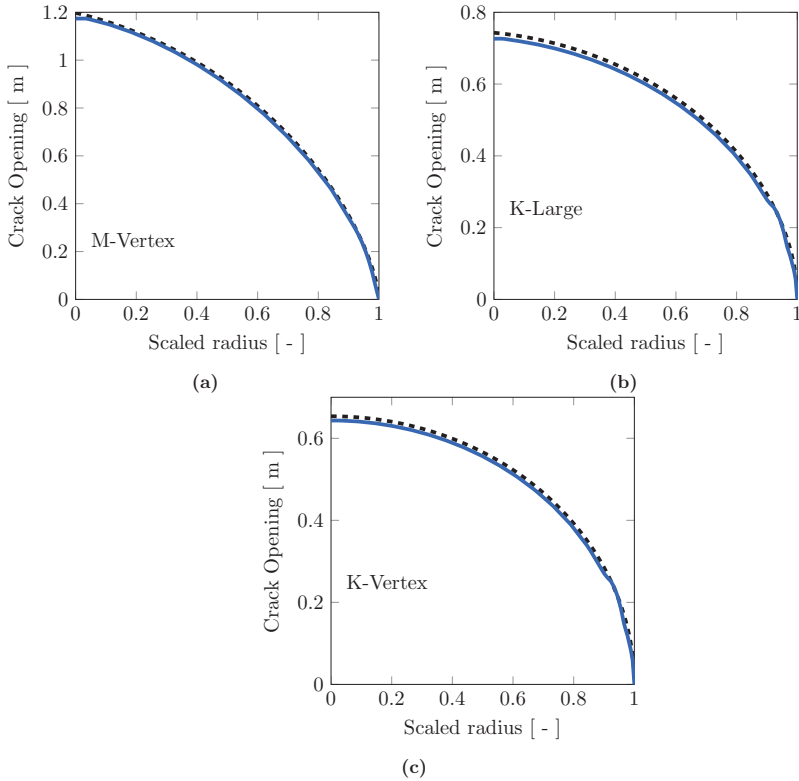


Figure 6.10: The pressure profile for a circular crack for all three test cases (*M*-vertex, *K*-vertex, and *K*-large) is shown here. The black dotted line (.....) corresponds to the exact solution and the red solid line (—) to the simulation.

6.2.2 Inclined penny-shaped crack

In this study, an initially penny-shaped crack inclined at an angle α_0 propagates through a rock sample that is subjected to an external load. The rock's material properties are the Young's modulus $E = 37800$ MPa and Poisson's ratio $\nu = 0.25$. The block dimensions with $1 \times 1 \times 1$ m³ are chosen such that the crack does not reach the edges of the geometry. The external loads applied on that body are $\sigma_x = 1$, $\sigma_y = 5$, $\sigma_z = 1$ MPa. The initial crack radius is 0.2 m and the crack is rotated around the x-axis with an angle $\alpha = 30^\circ$, so that it is not aligned with the maximum outer stress (σ_y) anymore.

The fluid is injected into the crack as a point source (the borehole radius is neglected) at a injection rate $Q_0 = 0.05$ mL/s and has a dynamic viscosity of $\mu = 1.48$ Pa·s. The material properties of the solid and the fluid are chosen according to the laboratory setup discussed in Section 6.3.

This study reveals that crack reorientation occurs almost instantly. The crack turns into the direction of the maximum outer stress as soon as propagation starts. This is also true for other tested inclination angles of $\alpha_0 = 60^\circ$ and $\alpha = 75^\circ$ (see Figure 6.13). Figure 6.11 shows the crack surface after 20 propagation steps. It clearly shows a dominant propagation direction towards the maximum outer stress σ_y .

As indicated in Figure 6.12, where only one half of the crack is shown for illustration purposes, the crack reaches a length of $dy = 0.33$ m and $dz = 0.014$ m after these first propagation steps. It is pointed out, that the propagation direction in the crack initiation phase is of interest here. At that point, the crack is almost fully horizontally aligned and no substantial changes in the propagation angle are expected anymore.

Figure 6.13 compares the crack surfaces of three different simulations. In each simulation, the initial inclination angle has been modified from $\alpha_0 = 30^\circ$ (red, transparent surface) to $\alpha_0 = 60^\circ$ (yellow surface) and $\alpha_0 = 75^\circ$ (blue, transparent surface). After 20 propagation steps, all three cracks are basically aligned to the maximum horizontal stress and crack propagation continues in the direction of σ_y .

Finally, the crack opening profile is illustrated in Figure 6.14 for one propagation step. As can be seen from the figure, the crack opening within the initially existing crack part is almost uniform with a maximum crack opening of $2.5 \cdot 10^{-6}$ m.

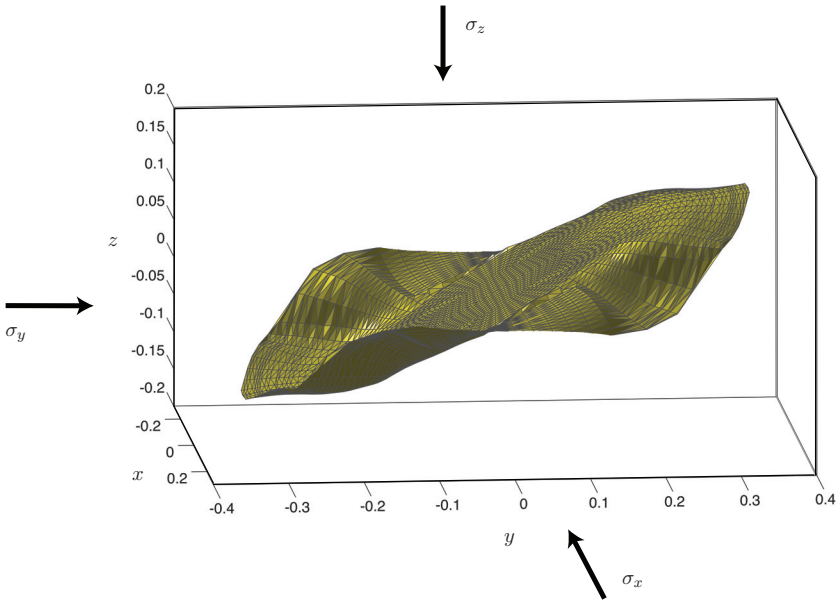


Figure 6.11: Inclined penny-shaped crack with an initial radius $R_0 = 0.2\text{m}$ and an inclination angle $\alpha_0 = 30^\circ$.

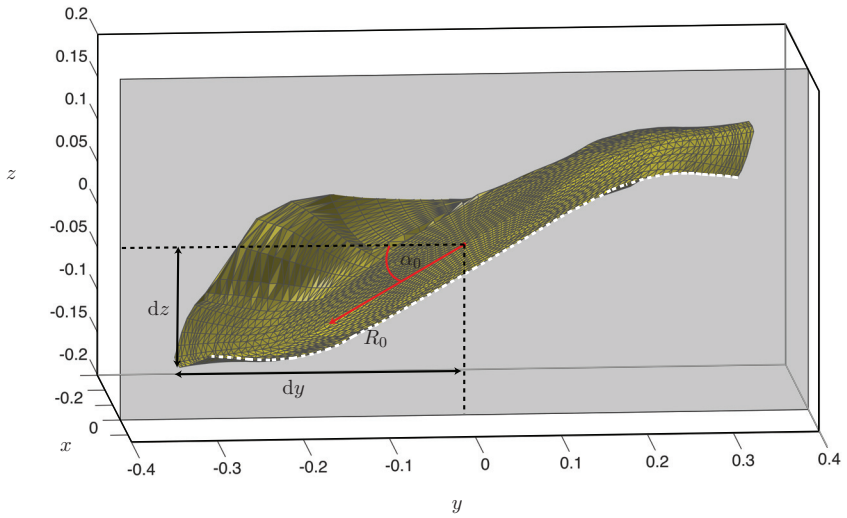


Figure 6.12: Inclined penny-shaped crack with an initial radius $R_0 = 0.2\text{m}$ and an inclination angle $\alpha_0 = 30^\circ$.

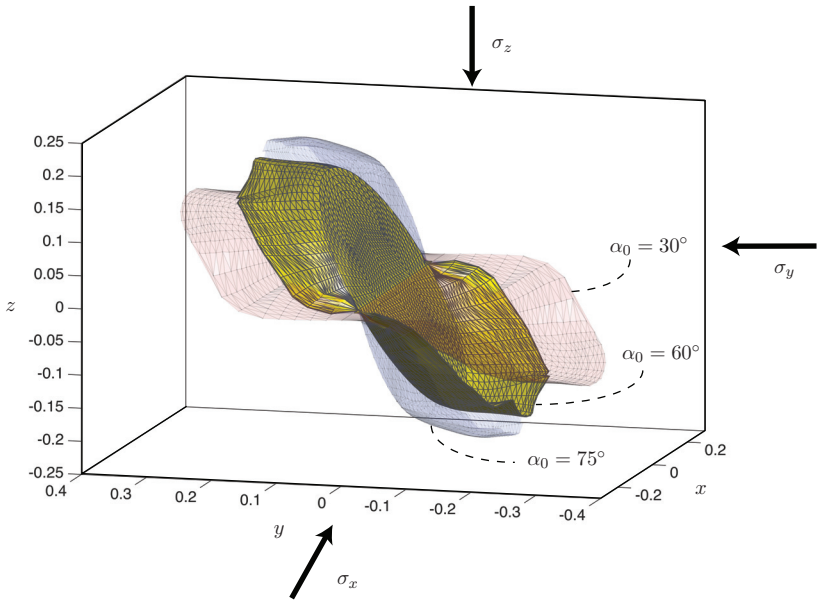


Figure 6.13: Inclined penny-shaped crack with an initial radius $R_0 = 0.2\text{m}$ and an different inclination angles $\alpha_0 = 30^\circ$, $\alpha_0 = 60^\circ$, $\alpha_0 = 75^\circ$.

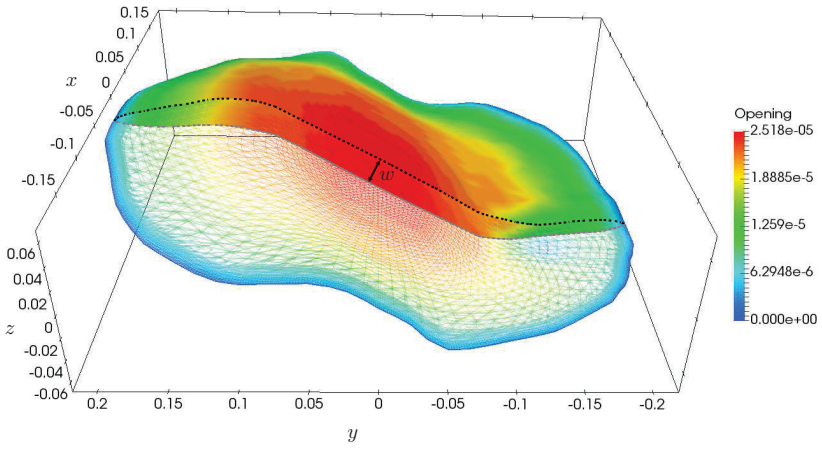


Figure 6.14: Crack opening of an inclined penny-shaped crack with an initial radius $R_0 = 0.2\text{m}$.

6.3 Experiments

In this section a brief overview is given over the laboratory setup of the testing facility and the conducted experiments. Subsequently, some results that are obtained from the experiments are discussed and compared with simulation results. The realization of the experiments is documented in [10, 144, 148, 149] and summarized in [39]. It is noted that all experiments and experimental results are not part of the author's work but mentioned in order to give a clear picture about the comparison with the simulations.

6.3.1 Experimental setup

The basic design of the testing device with all of its components is outlined in Figure 6.15. A rectangular block specimen with a borehole is stressed with a predefined external load. Through the injection system, a hydraulic pressure is generated within a small area in the borehole which eventually leads to a initiation of a crack. This small area is well-defined by means of borehole packers that are placed in the borehole. In order to prevent the crack to initiate arbitrarily the borehole is perforated in order to weaken the material and to force the initial crack growth into a specific direction. The whole experiment is monitored with acoustic emissions technology where sensors are placed all around the block to measure the crack movement. Moreover, the injection pressure is stored together with the fluid injection rate for further analysis and comparison with numerical simulations.

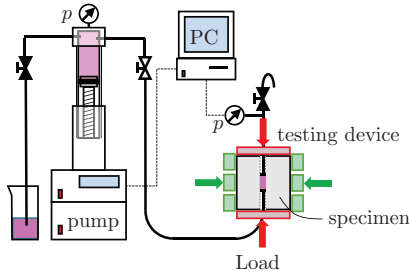


Figure 6.15: A schematic representation of the testing facility and its components according to [39, 148].

The testing facility is able to apply a load of up to 30 MPa vertically and 15 MPa horizontally on the specimen with the dimensions of $30 \times 30 \times 45 \text{ cm}^3$. The specimen is surrounded by three steel rings for the horizontal abutment and two steel plates for the vertical abutment. The whole device is hold together by four tension rods. It has the dimensions of approximately 1 m in width and height as shown in Figure 6.16. Acoustic events of fracturing experiments are recorded during fracture creation and propagation. The crack is observed through passive monitoring of the acoustic emissions that are released when changes in the stress field arise

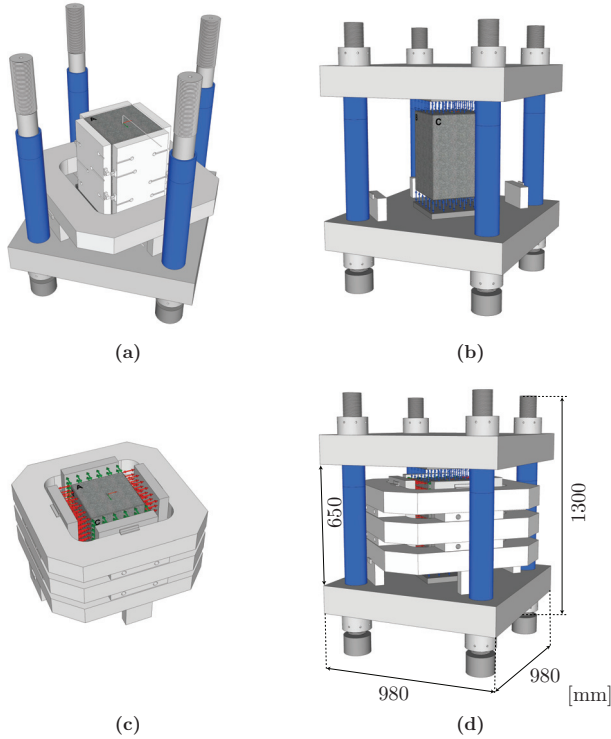


Figure 6.16: These sketches show the testing device as presented in [39]. (a) Loading plates cover the specimen and transmit a uniform load on it. The slots on the plates are used to place sensors for acoustic emission monitoring. (b) The vertical abutment is realized with two steel plates. (c) The horizontal abutment is realized with three steel rings.

within the block. Sensors have to be positioned in such a way that they are able to locate each event unambiguously. Therefore, 6 sensors are placed on each vertical, and 4 sensors on each horizontal loading plane around the rectangular block. The slots on the plates that cover the specimen on Figure 6.16a indicate the sensor positions. A more detailed discussion on acoustic monitoring is given in [39].

Figure 6.17 illustrates the ideal pressure response during hydraulic fracturing for a constant fluid injection rate Q_0 . When fluid is injected into the borehole the pressure begins to rise. Due to some residual air in the system, the pressure rise is low at the very beginning. In the ideal case, the pressure rate becomes linear until the characteristic point p_i which is denoted

as the *initiation pressure*. Newly developed microcracks and flaws weaken the material at the borehole wall and the pressure increase slows down. Eventually, these microcracks coalesce to one or more cracks which are big enough to be filled with fluid. When the pressure maximum is reached (known as the *breakdown pressure* p_b) material failure occurs. At this time, all fluid injected by the pump can enter the crack. Since the fluid is further pumped continuously into the crack, the crack volume grows and fracture propagation takes place. During this propagation phase the pressure rate decreases; however, the curve progression highly depends on boundary conditions and the experimental setup and can significantly differ from the ideal case. When the fluid injection is stopped (at the *shut-in pressure* p_{si}), the pressure falls instantaneously due to balance effects in the system and the fracture and due to poroelastic effects. The fluid pressure is released at the point p_{pr} which leads to a sudden pressure drop equal the atmospheric pressure.

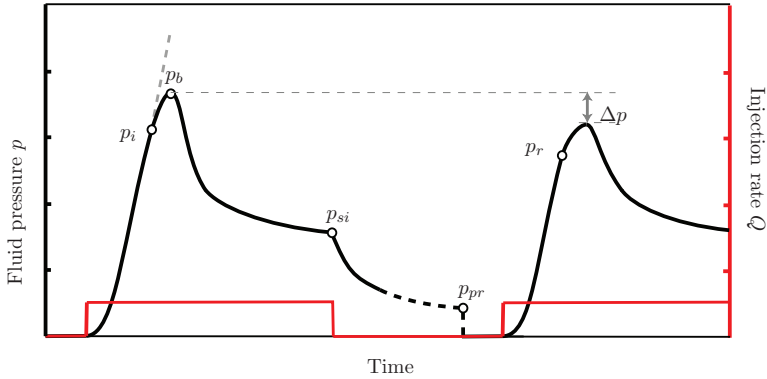


Figure 6.17: Idealized pressure curve over time for hydraulic fracture propagation.

Two injection cycles with a constant fluid rate are shown in Figure 6.17. After creating a small initial crack in the first cycle, the fracture is reopened again in the second cycle (that point is denoted as *reopening pressure* p_r). The pressure curve in both cycles is qualitatively similar; however, less energy is needed in the second cycle to reopen a fracture and initiate crack propagation. Thus, the maximum pressure of the second cycle is lower than of the first one.

Some fracking results obtained from the experiments are shown in Figure 6.18. The rectangular specimen is perforated with a circular notch and fluid is pumped into the borehole. Although the borehole penetrates the block completely, fluid pressure is only built up within a small area around the notch which is achieved through the usage of packers [39]. The figures show a vertical and a horizontal cross section of the block. The colored points are the acoustic events measured during fluid injection. Different colors are assigned to the individual points in

time. The acoustic emissions monitoring indicates a plane crack that propagates perpendicular to the borehole; however, the results in the horizontal cross section do not show a clear fracture footprint. Additionally, the block is sliced into two parts perpendicular the borehole position and it is found that the fracture footprint is nearly circular.

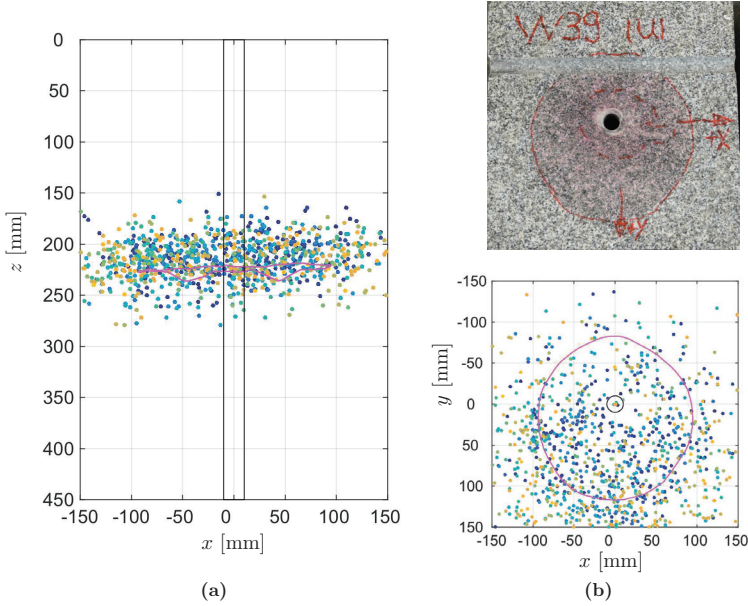


Figure 6.18: Acoustic emission monitoring and crack surface detection for experiment VV39.

The comparison of the numerical results with the experiments will focus on the pressure-over-time curve measured during the fracturing treatments. The reason for referring to the pressure is that this is the most reliable output obtained from the experiments. Since the XFEM requires an existing initial crack, only the second cycle of the fluid injection (for details see [39]) is considered for comparison. Nonetheless, comparisons of the fracture footprint, its area and the fracture radius at the end of the simulation are also part of this study and will be discussed in the next Section.

6.3.2 System compressibility effects

As indicated in Figure 6.17 two injection cycles are driven during the experimental treatment. In the ideal case, each cycle can be subdivided into two phases: the pressure buildup phase

until the pressure maximum is reached, and the propagation phase which is characterized by a pressure drop. For the numerical modeling a penny-shaped and a transverse crack (see Figure 6.19) are assumed, respectively. In order to capture the elastic response of the testing device together with the injection system, and the compressibility of the fluid, a heuristic term named system compressibility is introduced according to [111]. Since minimal or no information on those effects for such a complex system are available, they are derived from the pressure response measured at the injection pump.

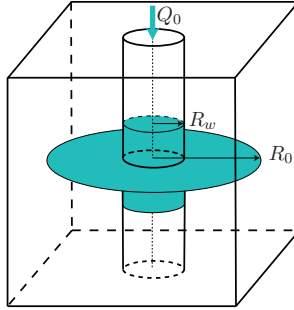


Figure 6.19: Transverse crack according to [111].

Determining the initial crack radius R_0

Following the assumption that the propagation condition $K_I = K_{IC}$ is just fulfilled at the pressure maximum and that no propagation takes place in the pressure buildup phase, the initial crack radius R_0 can be estimated on the basis of the transverse crack model described in [111]. The condition reads

$$K_I = f\left(\frac{R_0 - R_w}{R_w}\right) \cdot 2p_{net} \sqrt{\frac{R_0}{\pi}} \stackrel{!}{=} K_{IC}. \quad (6.7)$$

As can be seen from this equation, this is the propagation condition for a penny-shaped crack for a uniform net pressure p_{net} multiplied with a correction term f that is defined as

$$f(a) = \frac{2(1+a)^4(2a+\pi) + 0.3(4a-\pi a+2\pi-4)}{4(1+a)^5} \sqrt{\frac{a}{1+a}}. \quad (6.8)$$

Given a borehole radius $R_w = 0.01$ m, a maximum net pressure $p_{net} \approx 15$ MPa and a fracture toughness of $K_{IC} = 1.66 \text{ MPa}\sqrt{\text{m}}$, the initial crack radius becomes $R_0 \approx 0.015$ m.

Volume balance

The volume balance due to system compressibility effects can be formulated as

$$dV_{frac} = dV_Q - dV_{comp} = Q_0 dt - c_{sys} V_0 dp_{net}, \quad (6.9)$$

where dV_{frac} is the volume of the fracture created during the time interval dt , $dV_Q = Q_0 dt$ is the volume pumped into the system, $dV_{comp} = c_{sys} V_0 dp_{net}$ is the compressed volume, and V_0 is the initial fluid volume in the system before pressure buildup. The system compressibility c_{sys} [1/MPa] is chosen as constant and is obtained from the pressure curve measured in the experiments [39]. The volume change of a transverse fracture during pressure buildup is defined as

$$dV_{frac} = g\left(\frac{R_0 - R_w}{R_w}\right) \int_0^R (dw) r dr = g\left(\frac{R_0 - R_w}{R_w}\right) \frac{16}{3E'} R_0^3 dp_{net} \quad (6.10)$$

with $g(a)$ being the volume ratio of the transverse fracture to the penny-shaped fracture [111], and w the crack opening. Since the fluid injection into the fracture is

$$Q_{frac} = \frac{dV_{frac}}{dt}, \quad (6.11)$$

it can be expressed in terms of Q_0 with

$$Q_{frac} = Q_0 \cdot \frac{dV_{frac}}{dV_{frac} + dV_{comp}}. \quad (6.12)$$

Here, dt is obtained from Equation (6.9)

$$dt = \frac{dV_{frac} + dV_{comp}}{Q_0}. \quad (6.13)$$

Moreover, the pressure term can be canceled out and one obtains the constant relation

$$\frac{Q_{frac}}{Q_0} = \frac{g\left(\frac{R_0 - R_w}{R_w}\right) R_0^3}{g\left(\frac{R_0 - R_w}{R_w}\right) R_0^3 + \frac{3E'}{16} c_{sys} V_0} = const. \quad (6.14)$$

It is noteworthy that this expression holds for a constant crack radius R_0 (no crack propagation) as is the case for the pressure-buildup phase.

6.3.3 Effective fracture toughness

It was observed in hydraulic fracture operations that the net-pressures in the field are much higher than the pressures calculated by conventional hydraulic fracturing models [132]. The inconsistency can be explained with high friction losses within a fracture and effects due to micro-cracking or softening behavior of rocks [132]. The latter effects refer to an effective

fracture toughness that can be significantly higher than the fracture toughness measured in the laboratory. [153] showed that typical toughness values from field data are up to two orders of magnitude higher than values obtained from laboratory tests.

Fracture toughness values obtained in experiments in [39] are within the range of $K_{IC} = 1.61 \pm 0.03 \text{ MPa}\sqrt{m}$. However, the pressure level predicted with $K_{IC} = 1.6 \text{ MPa}\sqrt{m}$ is considerably lower than the values in the experimental results. Thus, a higher fracture toughness value is assumed due to plastic yielding near the tip of the propagating fracture.

6.3.4 Comparison with the experiments

The comparison with the experiments are carried out here first for the pressure buildup phase. During this first phase, a uniform pressure in a preexisting fracture is assumed. The pressure level is slowly increased until the stress intensity factor matches the fracture toughness K_{IC} . The corresponding point in time is then calculated from Equation (6.13).

The material parameters given in Table 6.2 are used as mean values obtained from the experiments. From the experiments one yields a maximum fluid pressure of 20 MPa. The

R_w	0.01	m	borehole radius
R_0	0.015	m	initial fracture radius
K_{IC}	1.66	$\text{MPa}\sqrt{m}$	fracture toughness
E	37800	MPa	elastic modulus
ν	0.25	-	Poisson's ratio
μ	$1.48 \cdot 10^{-6}$	-	dynamical viscosity
Q_0	0.05	$\frac{mL}{min}$	fluid injection rate
V_0	10	mL	initial fluid volume in the system
c_{sys}	0.004	$\frac{1}{MPa}$	system compressibility

Table 6.2: Parameters, that were used for the comparison with the experimental results.

outer stresses acting on the block sample are $\sigma_x = 15 \text{ MPa}$, $\sigma_y = 15 \text{ MPa}$, and $\sigma_z = 5 \text{ MPa}$ which finally leads to a maximum net pressure of 15 MPa. With this net pressure and the assumption of a circular crack geometry, the initial fracture radius is estimated as described above.

Uniform pressure in the buildup phase

The pressure buildup phase is basically characterized through a linear increase of the fluid pressure in time. The fracture toughness has not reached the critical value, yet. For a circular crack, this can be simply modeled with a uniform pressure assumption, *i.e.* the pressure loads

are equal everywhere on the crack surface. The pressure level is successively raised until the stress intensity factor K_I meets the fracture toughness K_{IC} . It is noted, that the nonlinear behavior of the pressure curve very close to the (maximum) breakdown or reopening pressure cannot be taken into account with this modeling approach. However, the discrepancy from this nonlinear behavior is assessed as very low and is not crucial for the simulation.

As can be seen from Figure 6.20, the time difference for the pressure from an almost closed crack at 5 MPa to the maximum pressure is ≈ 767 seconds. Furthermore, due to the constant fluid injection rate, the crack volume change in time becomes also linear as long as compressibility effects are neglected. The maximum crack opening (at the center of the crack) is only a function of the stress intensity factor

$$w(r) = \frac{K'}{E'} \sqrt{\frac{R}{2}} \cdot \sqrt{1 - \frac{r^2}{R^2}}, \quad (6.15)$$

with K' and E' being the effective value of the critical stress intensity factor and Young's modulus, R is the fracture radius and r is the local coordinate from the crack center to the crack front. The maximum aperture w is therefore at $r = 0$ and grows linearly with increasing stress intensity factor. Both, the linear dependency of the volume and the crack opening can be seen in Figure 6.24 and Figure 6.25.

Uniform pressure in the propagation phase

The pressure response in the propagation phase is modeled with two different approaches. First, the uniform pressure assumption is also applied in the propagation phase and is characterized by a rapid pressure decline right after the maximum pressure level is reached. And second, a coupling between the fluid and the structure model is used in order to obtain the pressure curve in time.

The comparison with the experiments are carried out at first on the basis of the decoupled model. Here, a constant pressure level is assumed on the crack surface. The pressure level in each time step t_k is chosen such that $K_I = K_{IC}$ is valid. Under the uniform-pressure assumption and a penny-shaped crack, the calculation of the stress intensity factor K_I can be simplified to [111]

$$K_I = 2(p_f - \sigma_n) \sqrt{\frac{R}{\pi}} \quad (6.16)$$

where p_f is the fluid pressure in the crack, σ_n is the outer stress acting normal to the crack surface and R is the fracture radius.

As mentioned earlier, the simulation needs a preexisting crack (with radius R). The initial crack radius is estimated as described in the previous section. Since little information about the initial radius is given from the experiments, it is estimated for the time of the maximum

(breakdown) pressure. The initial radius in combination with the stress intensity factor is chosen such, that the maximum pressure at breakdown in the simulation fits the experimental data.

As can be seen in Figure 6.20, the simulation results fit the experimental data very well. This is also supported by the findings from fracking experiments in the laboratory setup [111]. It is stated that the hydraulic fracturing process in the laboratory scale is mainly driven by the toughness dominated process. And the toughness dominated process is characterized by decoupling the fluid and structural behavior. The pressure becomes uniform within the crack. Figure 6.20 shows four pressure curves obtained from the experiments, *VV37* to *VV39*, to emphasize that four repeatable laboratory runs were conducted with the same results.

However, the behavior near the breakdown pressure (before and after breakdown) cannot be reproduced properly. It is assumed, that other than the described phenomena are dominant in this phase such as a strong nonlinear crack-tip behavior.

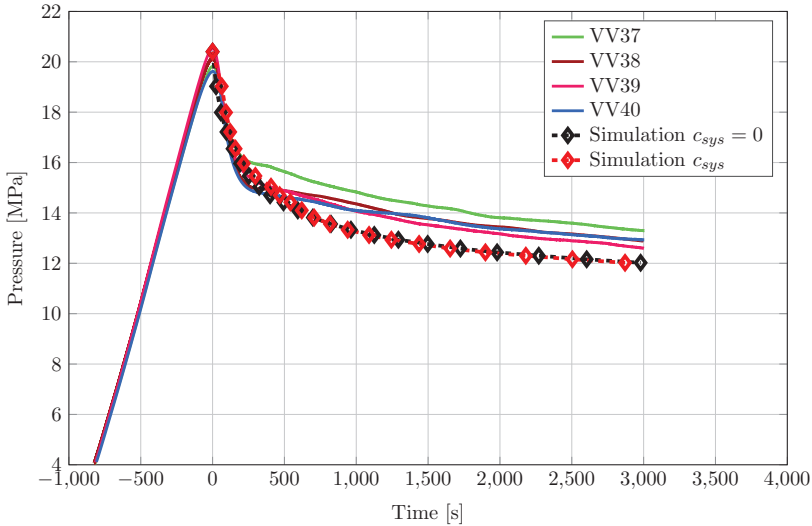


Figure 6.20: Propagation phase: The pressure curve over time of the experiments *VV37* to *VV39* is shown here and the comparison with the simulation. The simulation is based on the decoupled model with a uniform pressure over the whole crack and a pressure level that fulfills $K_I = K_{IC}$.

Moreover, it has also been shown that a parameter defined as a system compressibility c_{sys} and discussed in the previous section is suitable to reproduce the pressure response near the breakdown. Figures 6.20, 6.21, and 6.22 all show two curves obtained from simulations. The

simulation results labeled with $c_{sys} = 0$ does not contain any effects referred to the compressibility. However, no major difference in the pressure profile with and without compressibility is visible in Figure 6.20 which is due to the coarse time steps after the maximum pressure. Indeed, as is pointed out in Figure 6.21, the effect of the system compressibility is only visible shortly after breakdown within the first 200 seconds. After breakdown, the pressure decreases very slowly in the beginning and that effect can be captured by means of the system compressibility. The time interval has to be refined in this area by choosing smaller propagation steps. The long-term pressure behavior is not affected significantly by the system compressibility, since the pressure change over time is small for $t > 200$ seconds.

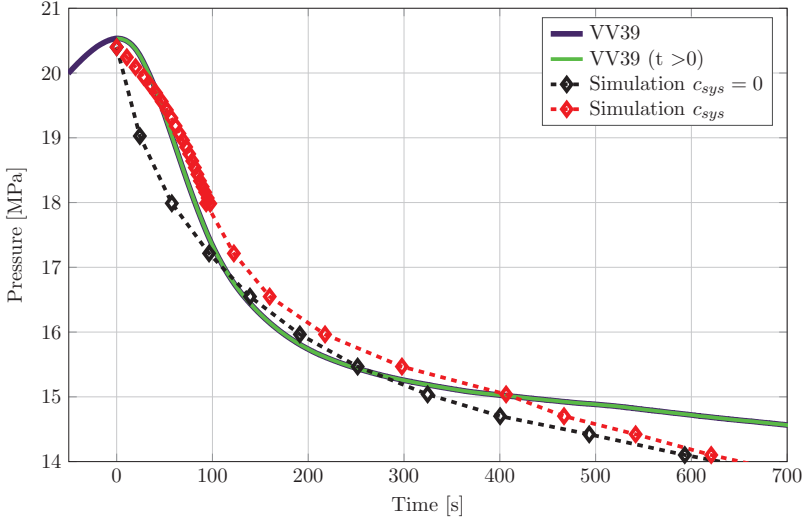
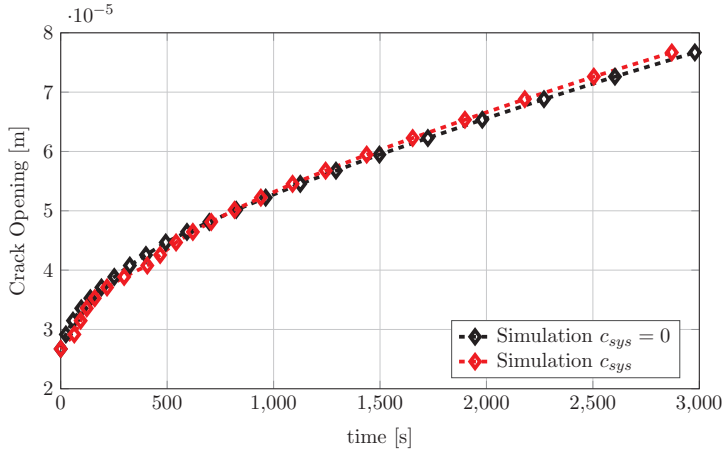


Figure 6.21: The pressure curve over time for the propagation phase of the experiment VV39 is shown here and compared with the simulations with and without compressibility effects.

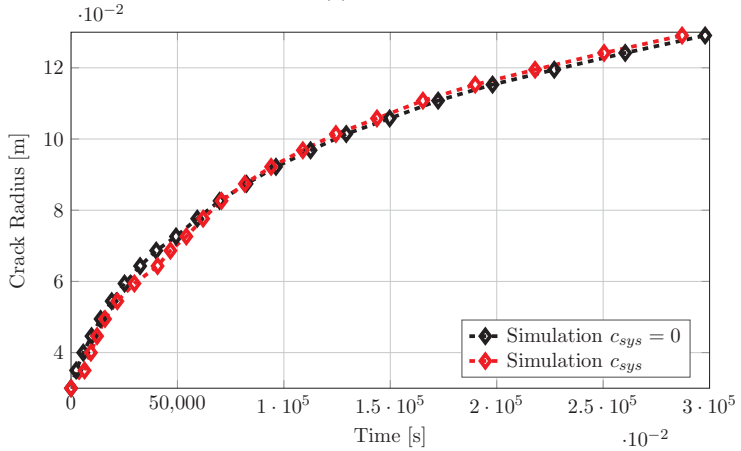
The fracture radius reaches ≈ 13 cm at a time $t = 2980$ seconds. The corresponding fracture area is 502 cm^2 . The fracture volume with compressibility effects is with $V = 2.39 \text{ cm}^3$ slightly lower than without compressibility ($V = 2.48 \text{ cm}^3$).

Cubic law for the fluid model in the propagation phase

Next, the Reynolds equation (cubic law) is chosen for the fluid model and the coupled problem is solved for the penny-shaped crack. The full simulation starts in the pressure buildup phase with a uniform pressure model. When reaching the breakdown pressure, the algorithm switches



(a)



(b)

Figure 6.22: Simulation results are shown for the crack opening (a) and the fracture radius (b) over time.

to the fully coupled hydraulic fracture model with the Reynolds equation for the fluid model.

Moreover, a fluid-lag approach is incorporated into the solution process. The fluid front and

the crack front are tracked independently, such that a cavity develops in the crack front region. The numerical results presented in Figure 6.23 show a good agreement with the experimental data. The coupled solution slightly overestimates the pressure after breakdown. The pressure decline is not as steep as measured during the experiments. This could be interpreted as a lower crack propagation velocity in the simulation. A faster crack growth after breakdown would lead to a steeper pressure decline and, consequently, the fracture radius would be longer (compare with Figure 6.26).

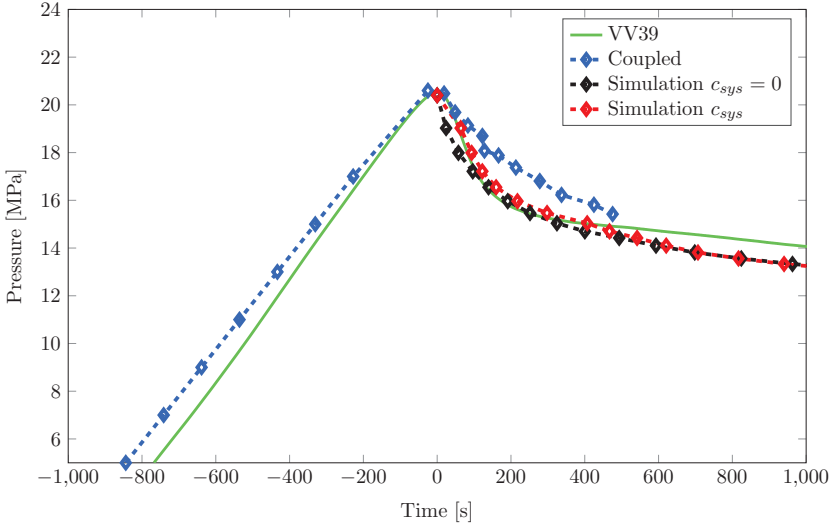


Figure 6.23: The pressure curve over time for the buildup and propagation phase is shown here. The solution of the coupled problem is compared to the experimental results *VV39* and to the simulations with and without compressibility effects (and a uniform pressure).

Figure 6.24 presents the crack volume calculated from the crack opening and compares it with the crack volume derived from the experiments. Since this is the volume only of the crack itself and not of the whole laboratory system, the volume has to be estimated. The simplest method is taking the fluid pumped into the whole system and subtracting the fluid-filled part of the borehole and the piping system. If the system compressibility is taken into account, the compressible volume part which depends mainly on the pressure has to be subtracted, as well. The crack volume curve is derived from the experimental pressure data according to

$$V_{frac} = \int_t Q_{frac}(t) dt \quad (6.17)$$

with $Q_{frac}(t)$ being the rate of the crack volume

$$Q_{frac}(t) = Q_0 - c_{sys}(t_i) \cdot \frac{dp}{dt} \quad (6.18)$$

$c_{sys}(t_i)$ is the volumetric system compressibility at initiation time t_i .

The volume of the coupled problem agrees very well with the experimental data. The time $t < 0$ seconds corresponds to the pressure buildup phase and is characterized through an almost linear increase in the volume. The non-linearity is owed to the system compressibility and other effects that cannot easily be measured. At time $t = 0$ seconds which mark the end of the pressure buildup phase and the start of the propagation phase the volume starts to increase rapidly. This is due the fact, that the crack starts to grow and, therefore, more fluid enters the crack region. Moreover, compressed fluid instantly decompresses and fills the crack, especially shortly after the breakdown time. This effect can not be observed with the uniform-pressure model; however, the coupled model accounts for that effect quite well. The reason may lie within the assumption of an existing fluid lag.

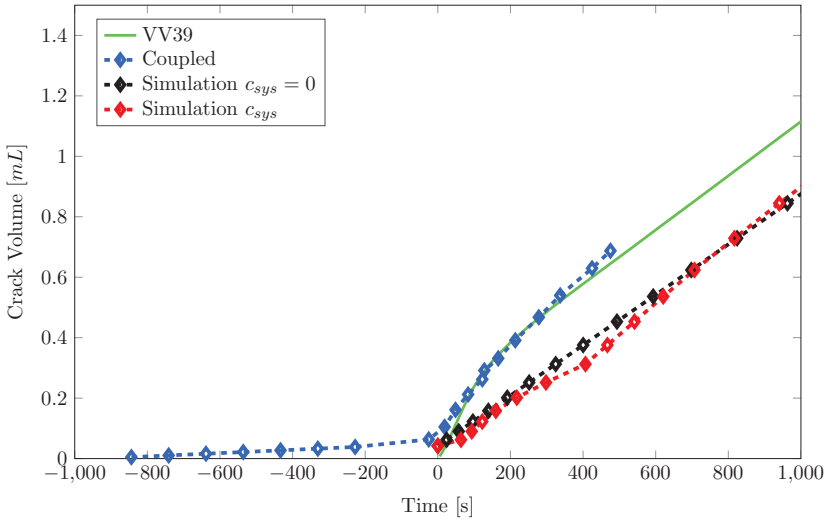


Figure 6.24: The fracture volume over time obtained from the coupled problem and the simulations with uniform pressure is compared to the volume derived from the experiments.

No reliable results on the crack opening are available from the conducted experiments. Therefore, the crack opening is compared in Figure 6.25 with the two different model assumptions used to simulate the fracture propagation. Here again, the pressure buildup phase corresponds

to the time $t < 0$ seconds and the propagation phase is for $t > 0$ s. Although the crack opening curve in the pressure buildup phase is only associated with the coupled problem, it is an equal approach with equal results for all simulation runs. The pressure is always assumed uniform before the actual propagation starts.

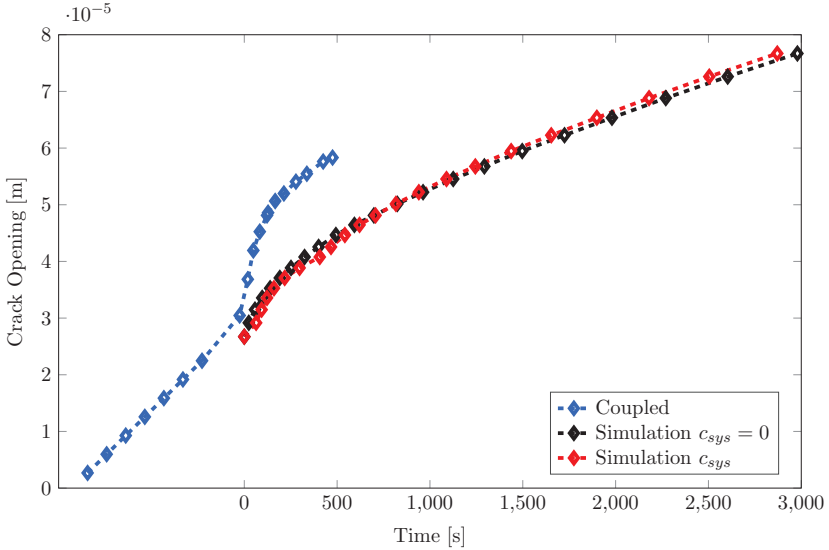


Figure 6.25: The crack opening is presented here for the coupled problem and the simulation with a uniform pressure (with and without compressibility effects).

As can be seen in Figure 6.25, the coupled problem leads to a higher crack opening in the propagation phase. This can be linked to the lower crack propagation velocity. Since the crack is not growing fast enough, the fluid that penetrates the crack leads to higher pressure values and consequently bigger crack widths. In all cases, the crack opening is in the order of 10^{-5} m which is a reasonable value for the laboratory experiments.

The fracture radius is presented in Figure 6.26. The green squares are data extracted from the experiments by averaging the acoustic emission data. The simulation results with uniform pressure overestimate the radius at the end of the simulation time. The experiments reveal a fracture radius around 10 cm at $t = 3000$ s. The simulation obtain a fracture radius of around 13 cm; however, the fracture shaped in the experiments is not perfectly circular anymore. Unlike in the experiments the circular fracture shape is preserved until the fracture almost reaches the boundary of the block sample.

Finally, Figure 6.27 shows the pressure distribution within a cross section of the penny-

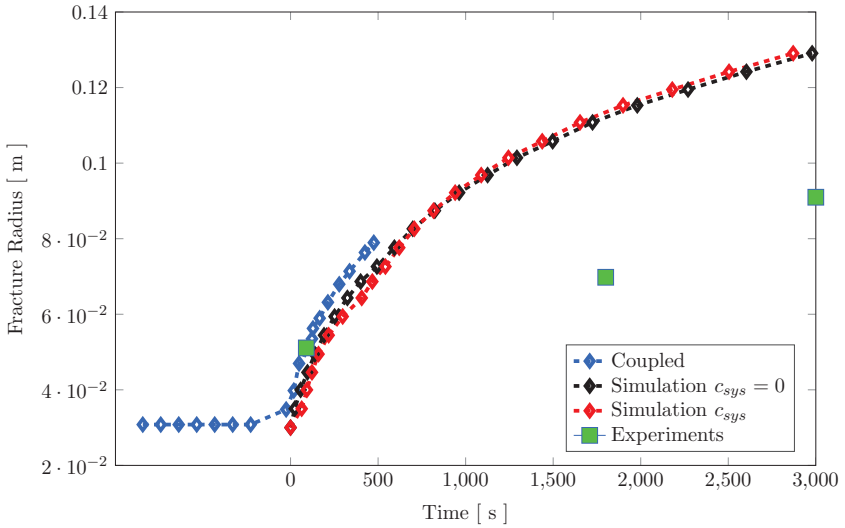


Figure 6.26: The fracture radius over time is shown here for the coupled problem and compared to the simulations with a uniform pressure assumption (c_{sys}).

shaped crack. Here, r is the local coordinate starting at the center of the crack. The blue curves are the pressure distributions from $-R$ to $+R$ as the crack propagates where R is the fracture radius. Consequently, the position of each blue curve at the zero-pressure level indicates the crack length. Since, the borehole is also considered (with a borehole radius $R_w = 0.01$ m), the pressure is uniform for $r < |\pm 0.01|$ m.

In summary, the numerical model has been successfully verified against simplified analytical models in 2D and 3D. Moreover, a good agreement could be achieved with the experimental data for a penny-shaped crack.

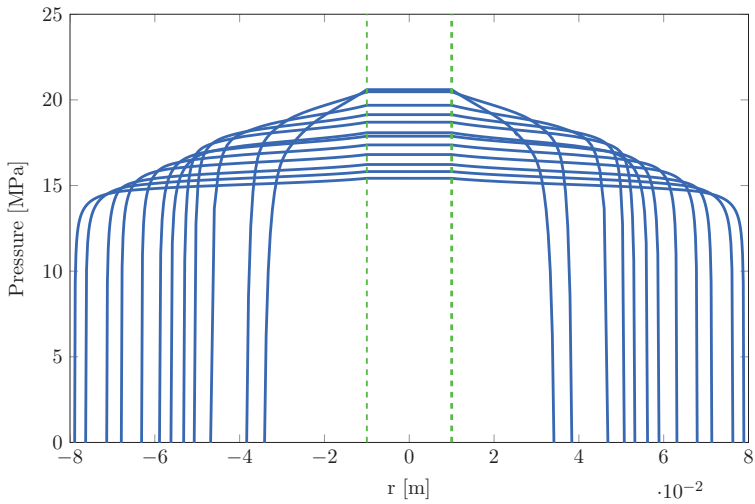


Figure 6.27: The pressure profile within a cross section of a penny shaped crack is shown here. The green vertical lines represent the borehole boundary. Consequently, the pressure within ± 0.01 m is the pressure in the borehole and, thus, constant.

7 Summary and conclusion

The focus of this work is on numerical simulation of hydraulic fracturing based on the XFEM. The hydraulic fracturing process is a multi-field problem where at least three dominating physical phenomena are present: a deformable solid that is subjected to an external load, an existing crack within that solid which propagates, and a viscous fluid taking place on the crack surface. Each of these physical processes are modeled separately and these models may differ largely in terms of simplification and abstraction. In the recent decades, numerous modeling approaches have been discussed in the literature and some popular models in 2D and 3D have been established. The main goal of these models was to provide an understanding of the relevant processes and support the technical realization of fracking by drilling companies. The most well-known models are the PKN- and KGD-model in 2D and the penny-shaped model in 3D. Recent research activities lead to a new approach in assessing hydraulic fracture models through a dimensionless analysis and identifying the dominant processes during a hydraulic fracture treatment.

The modeling approach in this work considers a linear elastic medium and a fracture propagating according to the linear elastic fracture mechanics theory. The viscous fluid is described with a cubic's law which is known as the Reynolds equation in hydraulic fracture applications. The Reynolds equation is basically a 1D-equation and assumes typically a rectangular crack cross section. An iterative coupling of all three processes is used to provide maximal flexibility and an independent consideration of all fields.

The solid and the crack are treated numerically by means of the extended finite element method (XFEM). The XFEM is extremely suitable for problems with a discontinuous or singular behavior in the solution field. It has been successfully applied to numerous fracture mechanics problems. Hereby, the crack defines the discontinuity which passes through the finite elements. Furthermore, the discontinuity does not need to align with the edges of the finite elements. It is described by means of a level-set field that determines the distance at each node of the mesh to the discontinuity.

One major part in this work, is the extension of the XFEM approach for problems on manifolds. The Reynolds equation has to be solved on the crack surface; however, the crack surface can be arbitrarily curved. In order to solve a transport equation on an arbitrarily curved surface, a local coordinate system on that surface has to be introduced, the transport equation has to be properly transformed and integrated. This approach also handles discontinuous transport equations on arbitrarily curved manifolds.

The model presented here is verified in 2D against the KGD-type model and in 3D against the penny-shaped model. Moreover, a verification has been conducted against laboratory

experiments on rock samples of the size $30 \times 30 \times 45 \text{ cm}^3$. The verification focused on the pressure buildup phase where it is assumed that an initial crack already exists and on the propagation phase. The propagation phase starts at the maximum pressure and is characterized by a continuous pressure decline. It is noted, that the physical processes shortly before and after breakdown are not fully investigated and a discrepancy in the pressure between the simulations and the experiments remains. In particular, the maximal pressure gradient in the simulation is found close to the breakdown point whereas the experimental pressure values showed a transition from a low gradient shortly after breakdown to high gradients in between and again low values towards the end. This transition in the pressure gradient could be reproduced by introducing a term that accounts for the nonlinear compressibility effects in the system.

In conclusion, the presented modeling approach for hydraulic fracture propagation has proved to be suitable. The XFEM builds a robust and excellent basis for modeling crack propagation and the iterative coupling allows the flexibility of using different approaches for different physical problems. However, issues with the crack propagation criterion have to be addressed more properly. The stress intensity factor approach used in this work to decide whether a crack grows or not, could not be implemented to complete satisfaction. Especially, numerical inaccuracies in the stress intensity factor have a significant effect on the coupled solution. Moreover, non-linear or plasticity behavior at the crack tip has to be accounted for with proper model extensions. A promising model might be the cohesive zone approach which was already successfully applied to hydraulic fracture problems. Non-linear effects that were already observed in rock could be effectively modeled.

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