Lifting properties of blocks

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1 Introduction

Representations of finite groups over a field K are often studied in the form of modules over the group algebra KG. Representation theory can therefore be understood as the study of the module category of the group algebra KG. If K is a field of characteristic 0, the algebra KG is semisimple. This implies that every KG-module is a direct sum of simple modules and thus the module category of KG can be studied by focusing on its finitely many simple modules.

Modular representation theory studies the situation where the characteristic p of the field K divides the group order. In this case KG is not semisimple and there are usually infinitely many non-isomorphic indecomposable modules. Nevertheless, the algebra KG still has only finitely many simple modules.

A bridge between these two situations is given by the group algebra RG, where R is a complete discrete valuation ring with maximal ideal m whose field of fractions $K = \mathrm{Quot}(R)$ has characteristic zero and whose residue field $F = R/m$ has characteristic p. The algebra RG can be considered as a subalgebra of the semisimple algebra KG, and FG is the quotient algebra RG/mG. Every simple FG-module is a simple RG-module. On the other hand, we can use the Wedderburn decomposition of the semisimple algebra KG to embed RG into an algebra of the form $\bigoplus_{i\in I} D_i^{n_i\times n_i}$ for K-division algebras D_i. Consequently RG can be explicitly described by matrices.

The algebra RG admits a unique decomposition $RG = \bigoplus B_j$ as a finite direct sum of indecomposable algebras B_j, which are called *blocks*. Each block is of the form $B_j = RG\varepsilon_j$ for some central primitive idempotent ε_j of RG. By Hensel's Lemma, every central idempotent of FG lifts to a central idempotent of RG. The induced decomposition $FG = \bigoplus F \otimes_R B_j$ is therefore also the decomposition of FG into indecomposable algebras, which we call *blocks* as well. Furthermore, for every indecomposable RG-module M there is at most one block B with $M.B \neq 0$. Therefore, it makes sense to study the group algebras one block at a time. To a block B one associates its *defect group* D_B, see Definition 2.2.19, and we call the p-valuation of the order of D the *defect* of the block.

Since we are only interested in the module category of the blocks up to equivalence of categories, we consider the blocks up to Morita-equivalence. The lowest-dimensional algebra in a Morita-equivalence class of algebras, called the *basic algebra*, is unique up to algebra isomorphisms. Note that the basic algebra $\overline{\Lambda}$ of a block over FG can be obtained by tensoring the basic algebra Λ of the corresponding block in RG with F, see Lemma 2.3.5. Consequently, the algebra Λ is a *lift* of $\overline{\Lambda}$, i.e. an R-algebra Γ such that $F \otimes \Gamma \cong \overline{\Lambda}$.

One goal of this work is to give explicit descriptions of the basic algebra Λ of a block in RG as a subalgebra in the Wedderburn decomposition $KG \cong \bigoplus D_i^{n_i\times n_i}$ of

KG. Although the details of the procedure depend on Λ, the strategy is roughly the following: First describe $\overline{\Lambda}$ as a quiver algebra. In our examples a description is either readily available in the literature or we found one by hand.

As the second step, we construct an algebra $\Gamma_0 \subset \bigoplus_{i \in I} R_i^{n_i \times n_i}$, where R_i is the integral closure of R in D_i, such that every algebra Γ for which

1. Γ is a lift of $\overline{\Lambda}$,
2. $K \otimes \Gamma$ is semisimple with the same center as $K \otimes \Lambda$,
3. Γ has the same decomposition matrix as Λ,
4. Γ is self-dual with respect to the same form as Λ,

is isomorphic to Γ_0. We call an R-algebra Γ fulfilling these conditions a Λ-lift of $\overline{\Lambda}$. Checking whether certain algebras have a unique Λ-lift is the second goal of this work.

The method outlined above can be considered as an inverse process to the construction of the basic algebra of $\overline{\Lambda}$ as the factor algebra $F \otimes_R \Lambda$. This idea is based on work of Wilhelm Plesken [Ple83], which uses the p-modular representation theory of a group G to obtain information about the integral p-adic group ring $\mathbb{Z}_p G$. This strategy was further developed by Gabriele Nebe [Neb99] to obtain the full ring-theoretic structure of suitable lifts Λ of group algebras and their blocks over fields of positive characteristic. Florian Eisele [Eis12] then studied the question for which algebras such lifts are unique, developed methods to transfer lifts along derived equivalences of algebras and applied them to several infinite series of examples.

Blocks with semidihedral defect

The first types of blocks we study are those with a semidihedral defect group. Their corresponding blocks over the residue field of characteristic 2 are a special kind of tame algebras, which have been classified by Karin Erdmann [Erd90b]. It is not completely known which tame algebras are Morita-equivalent to blocks of group algebras. However, it is known what their decomposition matrix and their representations in characteristic 0 would look like if they were blocks. We will use this information to show the theorem below.

Theorem. *The algebras of type $SD(2\mathcal{B})_1^0$ and $SD(2\mathcal{A})_2^0$ are not Morita-equivalent to blocks of group algebras.*

Proof. See Theorem 3.2.11 and Theorem 3.2.14. □

We will further show that if A is either $SD(2\mathcal{B})_1^1$ or $SD(2\mathcal{A})_2^1$ and Λ is the basic algebra of a block of a group algebra such that A has a Λ-lift, then A has infinitely many Λ-lifts, see Theorem 3.2.11 and Theorem 3.2.14. For a basic algebra Λ of a block with a semidihedral defect group such that $F \otimes_R \Lambda \cong SD(3\mathcal{B})_1$, we will construct a Λ-lift of $F \otimes_R \Lambda$ and show that it is the unique lift. Then, we will apply methods developed by Florian Eisele [Eis12] to show the following result.

Theorem. *Let Λ be the basic algebra of a block with a semidihedral defect group. Then every tame algebra of semidihedral type with three simple modules has at most one Λ-lift for every given center.*

Proof. See Theorem 3.2.21. □

These methods use derived equivalences between those blocks, which have been determined by Thorsten Holm [Hol01].

Defect 3 blocks of symmetric groups

The second type of block we investigate is the principal block B_0 of the group algebra $\mathbb{Z}_pS_p \wr S_3$ for $p > 3$, that is the block such that $TB_0 \neq 0$ for the trivial $\mathbb{Z}_pS_p \wr S_3$-module T. We will use results by Joseph Chuang and Kai Meng Tang [CT03] about wreath products of algebras to determine its decomposition matrix and the quiver of the corresponding block over $\mathbb{F}_p$. Then we will construct a lift of its basic algebra $\overline{\Lambda}$ and obtain:

Theorem. *Let Λ be the basic algebra of the principal block $B_0(\mathbb{Z}_pS_p \wr S_3)$. Then the algebra $\overline{\Lambda}$ has a unique Λ-lift.*

Proof. See Theorem 4.4.35. □

This block is of particular interest since it is Morita-equivalent to a defect 3 block of a symmetric group and it has been shown that all defect 3 blocks of symmetric groups in characteristic $p > 3$ are derived equivalent. Our result therefore lays the ground work for further investigation of those blocks.

Outline

In Chapter 2 we will recall the theory required to achieve our results. We will start with general properties of finite-dimensional algebras, continue to introduce properties of algebras over discrete valuation rings R and their connection to the corresponding algebras over the field of fractions and the residue field of R and finish by considering semisimple algebras. The second section will discuss special properties of group algebras and define defect groups. In the third section, we introduce two notions of equivalence of rings, Morita-equivalence and derived equivalence, and some of their properties. We continue by introducing the theory of graduated orders and conclude the chapter by introducing specialized methods to calculate lifts and to transfer unique lifting results along derived equivalences.

In Chapter 3 we consider tame blocks. We will start by recalling the classification of tame blocks by Erdmann [Erd90b] and other general properties of blocks of semidihedral defect. The main results of this chapter are the construction of infinitely many lifts for the algebra $SD(2\mathcal{B})_1^c$ for $c = 1$ and the non-existence of a lift if $c = 0$, see Theorem 3.2.11, the construction and uniqueness of a lift of $SD(3\mathcal{B})_1$, see Theorem 3.2.18,

and the transfer of the uniqueness and non-existence of the lifts to derived equivalent algebras, see Theorem 3.2.14 and Theorem 3.2.21.

In Chapter 4 we determine a unique lift of the basic algebra of the principal block $B_0(\mathbb{F}_p S_p \wr S_3)$, see Theorem 4.4.35. We will start the chapter by recalling the results by Chuang and Tang about wreath products and the structure of the principal block $B_0(\mathbb{F}_p S_p)$. We continue by applying those theories to find an explicit description of the basic algebra Λ of the block $B_0(\mathbb{Z}_p S_p \wr S_3)$. This lifting will be done in roughly three steps. At first we determine the subalgebras $e\Lambda e$ for primitive idempotents e of Λ. Next we determine the *exponent matrices*, see Definition 2.4.4, and finally we will give explicit descriptions of generators of the algebra.

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2 Representation theory

In this chapter we give a short introduction to the methods of representation theory used in this work. All topics are widely covered in literature and we will therefore give references for most proofs instead of repeating them. For a more detailed introduction to representation theory the reader is referred to [PD77] or [NT89].

All modules we consider will be finitely generated right modules unless stated otherwise. All R-algebras over a ring R will be R-free, and all modules over an R-algebra will be R-lattices, which means they are R-free. We will use "." to denote the multiplication of a ring on its modules and "$\cdot$" to denote multiplication inside the ring to avoid confusion. If confusion is unlikely we will leave out the operator for any multiplication.

2.1 Finite-dimensional algebras

In this section we will introduce important properties of finite-dimensional algebras.

2.1.1 General properties

Let R be a commutative ring and A a finite-dimensional R-algebra.

Definition 2.1.1. *Let $B \leq A$ be a subalgebra of A, V a B-module and W an A-module.*

Then $\mathrm{Ind}_B^A V := V \otimes_B A$ is an A-module with multiplication $(v \otimes a_1).a_2 := v \otimes a_1 a_2$ for $v \in V, a_1, a_2 \in A$, the induced *module of V to A.*

The A-module W becomes a B-module by restricting the multiplication of A on W to B. We denote this restricted module by $\mathrm{Res}_B^A W$.

Lemma 2.1.2. *Let $B \leq A$ be a subalgebra of A, V a B-module and W an A-module. The map*

$$\begin{aligned} f : \mathrm{Hom}_B(V, \mathrm{Res}_B^A W) &\longrightarrow \mathrm{Hom}_A(\mathrm{Ind}_B^A V, W) \\ \varphi &\longmapsto (v \otimes a \mapsto \varphi(v).a) \end{aligned}$$

is an isomorphism of R-modules.

Proof. See [NT89, Theorem 11.3 (i)] □

Definition 2.1.3. *1. A system $e_1, \ldots, e_n$ of primitive orthogonal idempotents is called* complete *if*

$$1_A = e_1 + \ldots + e_n.$$

2. *A system $P_1, \dots, P_n$ of projective indecomposable A-modules is called* complete *if*

$$A = P_1 \oplus \dots \oplus P_n.$$

Remark 2.1.4. *If $e_1, \dots, e_n$ is a complete system of primitive orthogonal idempotents, then $e_1A, \dots, e_nA$ is a complete system of projective indecomposable A-modules.*

Conversely, every decomposition of A into indecomposable A-modules yields a decomposition of 1_A into primitive orthogonal idempotents.

Lemma 2.1.5. *Assume that R is a field. Let $\{P_1, \dots, P_n\}$ be a complete system of projective indecomposable A-modules.*

1. *Every (finitely generated) projective indecomposable A-module is isomorphic to P_i for some $i \in \{1, \dots, n\}$.*

2. *The head $S_i := P_i/\operatorname{rad}(P_i)$ of P_i is simple for every $i \in \{1, \dots, n\}$.*

3. *Every simple A-module is isomorphic to S_i for some $i \in \{1, \dots, n\}$.*

Proof. The first part follows since every projective module is a direct summand of a free module and if such a direct summand is indecomposable it has to be a direct summand of the free A-module A. For the other parts see [NT89, Theorem 8.10]. □

In particular, the above lemma shows that there are only finitely many isomorphism classes of simple A-modules and that they all occur as heads of direct summands of A. In the semisimple case the projective indecomposable modules are themselves simple A-modules.

Definition 2.1.6. 1. *We say that two primitive orthogonal idempotents $e, f \in A$ are* isomorphic *($e \cong f$) if $eA \cong fA$.*

2. *A set $\{P_1, \dots, P_l\}$ of non-isomorphic projective indecomposable A-modules is called* complete *if for every projective indecomposable module P there is some $i \in \{1, \dots, l\}$ such that $P \cong P_i$.*

3. *A set $\{e_1, \dots, e_l\}$ of non-isomorphic primitive orthogonal idempotents is called* complete *if $\{e_1A, \dots, e_lA\}$ is complete.*

4. *A set $\{S_1, \dots, S_l\}$ of non-isomorphic simple A-modules is called* complete *if for every simple module S there is some $i \in \{1, \dots, l\}$ such that $S \cong S_i$.*

Lemma 2.1.7. *Assume that R is a field and let P, Q be two projective indecomposable A-modules. Then*

$$P/\operatorname{rad}(P) \cong Q/\operatorname{rad}(Q) \Leftrightarrow P \cong Q.$$

Proof. See [PD77, Theorem 1.8]. □

Corollary 2.1.8. *Assume that R is a field and let $\{P_1, \dots, P_l\}$ be a complete set of non-isomorphic finitely generated projective indecomposable A-modules. Then the set $\{P_1/\operatorname{rad}(P_1), \dots, P_l/\operatorname{rad}(P_l)\}$ is a complete set of non-isomorphic simple A-modules.*

Lemma 2.1.9. *Let e and f be two idempotents of A. Then the map*

$$\begin{aligned} eAf &\longrightarrow \operatorname{Hom}_A(fA, eA) \\ a &\longmapsto (x \mapsto a \cdot x) \end{aligned}$$

is an isomorphism of R-modules. If $e = f$ then the map is a ring-isomorphism. We therefore call the ring eAe an endomorphism ring.

Proof. See [NT89, Theorem 4.3]. □

Lemma 2.1.10 (Schur)**.** *Assume that R is a field and let V be a simple A-module. Then $\operatorname{End}_A(V)$ is a division algebra.*

Proof. Since both the kernel and the image of an endomorphism of V are submodules, both have to be either zero or V. Therefore, the only non-bijective endomorphism of V is the zero-homomorphism. □

Definition 2.1.11. *Assume that $R = K$ is a field. K is called a splitting field for A if $\operatorname{End}_A(V) = K$ for every simple A-module V.*

2.1.2 Algebras over discrete valuation rings

Definition 2.1.12. *Let p be a prime number. A* p-modular system *is a triple (F, R, K) where R is a complete discrete valuation ring with maximal ideal m, the residue field $F = R/m$ is a field of characteristic p and the field of fractions $K = \operatorname{Quot}(R)$ of R has characteristic 0.*

For the rest of the section, let p be a prime number, (F, R, K) a p-modular system and Λ a finite-dimensional R-algebra. Assume that F is a splitting field for $F \otimes_R \Lambda$ and that the K-Algebra $K \otimes_R \Lambda$ is semisimple. Let $\phi : \Lambda \to \overline{\Lambda} := F \otimes_R \Lambda$ denote the natural epimorphism and denote the generator of the maximal ideal of R by π.

Lemma 2.1.13. *The radical of Λ is* pro-nilpotent*, i.e. $\pi R \subseteq \operatorname{rad}(\Lambda)$ and there is some $n \in \mathbb{Z}_{\geq 0}$ with $\operatorname{rad}(\Lambda)^n \subseteq \pi R$.*

Proof. See [NT89, Theorem 14.1]. □

Lemma 2.1.14 (Hensel)**.** *Let $\overline{e_1}, \dots, \overline{e_l}$ be orthogonal primitive idempotents of $\overline{\Lambda}$. Then there are orthogonal idempotents $e_1, \dots, e_l$ such that $\phi(e_i) = \overline{e_i}$ for $i \in \{1, \dots, l\}$.*

Proof. See [NT89, Theorem 14.2]. □

Lemma 2.1.15. *Let e, f be primitive idempotents of Λ. Then*

$$\phi(e)\overline{\Lambda} \cong \phi(f)\overline{\Lambda} \Leftrightarrow e\Lambda \cong f\Lambda$$

Proof. See [NT89, Theorem 14.2]. □

Hensel's Lemma and Lemma 2.1.15 give us a very strong connection between the representation theory of Λ and that of $\overline{\Lambda}$ via the relationship between idempotents, projective indecomposable modules and simple modules. In the following corollary we summarise these connections.

Corollary 2.1.16. *Let $\{e_1, \dots, e_l\}$ be a complete set of non-isomorphic primitive orthogonal idempotents of Λ. Then*

1. *$\{P_1 := e_1\Lambda, \dots, P_n := e_n\Lambda\}$ is a complete set of non-isomorphic projective indecomposable Λ-modules.*
2. *We have $S_i := P_i/\operatorname{rad}(P_i) = \overline{S_i} := \phi(P_i)/\operatorname{rad}(\phi(P_i))$ for $i \in \{1, \dots, l\}$ and hence $\{S_1, \dots, S_n\}$ is both a complete set of non-isomorphic simple Λ-modules and non-isomorphic simple $\overline{\Lambda}$-modules.*
3. *$\{\overline{e_1} := \phi(e_1), \dots, \overline{e_n} := \phi(e_n)\}$ is a complete set of non-isomorphic primitive orthogonal idempotents of $\overline{\Lambda}$.*
4. *$\{\overline{P_1} := \overline{e_1}\overline{\Lambda} = \phi(P_1), \dots, \overline{P_n} := \overline{e_n}\overline{\Lambda} = \phi(P_n)\}$ is a complete set of non-isomorphic projective indecomposable $\overline{\Lambda}$-modules.*

Proof. The equality of the simple Λ- and $\overline{\Lambda}$-module follows from Lemma 2.1.13. The rest follows by combining Hensel's Lemma and Lemma 2.1.15 with Lemma 2.1.8. □

Definition 2.1.17. *Let S be a ring, $K_S = \operatorname{Quot}(S)$ its field of fractions and Λ an S-algebra which is finitely generated as an S-module. Let V be a $K_S \otimes_S \Lambda$-module. Then a Λ-lattice L is called an S-form of V if $K_S \otimes_S L \cong V$ as $K_S \otimes_S \Lambda$-modules.*

Lemma 2.1.18. *Let S be a principal ideal domain, $K_S = \operatorname{Quot}(S)$ its field of fractions and Λ an S-algebra. Then every $K_S \otimes_S \Lambda$-module has an S-form.*

Proof. See [NT89, Theorem II.1.6] □

We fix the following notation. Let $\{V_1, \dots, V_k\}$ be a complete set of non-isomorphic simple $K \otimes_R \Lambda$-modules. Let L_i be a Λ-lattice with $K \otimes_R L_i = V_i$ for $i \in \{1, \dots, k\}$.

Let $\{e_1, \dots, e_l\}$ be a complete set of non-isomorphic primitive orthogonal idempotents of Λ. Define for $j \in \{1, \dots, l\}$

$$\begin{aligned} \overline{e_j} &= \phi(e_j) \\ P_j &:= e_j\Lambda \\ \overline{P_j} := F \otimes_R P_j &= \overline{e_j}\overline{\Lambda} \\ S_j &:= P_j/\operatorname{rad} P_j \\ \overline{S_j} := F \otimes_R S_j &= \overline{P_j}/\operatorname{rad} \overline{P_j}. \end{aligned}$$

Then $\overline{S_1}, \dots, \overline{S_k}$ is a system of representatives of the isomorphism classes of simple $\overline{\Lambda}$-modules.

Notation 2.1.19. *Let B be a ring for which the Krull-Schmidt theorem holds and let M and N be two B-modules. Then we denote the multiplicity of N in a composition series of M by $[M : N]$.*

Definition 2.1.20. *Let B be a ring and $\mathcal{A}$ be a subcategory of Mod_A. Let G be the free abelian group generated by $\{[M] \mid M \in \mathrm{Obj}(A)\}$ where $[M]$ denotes the isomorphism class of M. Let*

$$H := \langle \{[M_1] + [M_3] - [M_2] \mid 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is a short exact sequence}\} \rangle \leq G.$$

Then we define the Grothendieck group *of $\mathcal{A}$ as*

$$K_0(\mathcal{A}) = G/H.$$

Definition 2.1.21.

1. *For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ the* decomposition number

$$d_{ij} := [F \otimes_R P_j : \overline{V_i}]$$

is the multiplicity of the simple module $\overline{V_i}$ as a direct summand of $K \otimes_R P_j$.

2. *The matrix $D := (d_{ij})_{i,j}$ is called the* decomposition matrix *of Λ.*

3. *The map*

$$\begin{aligned} \theta^{\Lambda} : K_0(\mathrm{proj}_{\Lambda}) &\longrightarrow K_0(\mathrm{mod}_{K \otimes_R \Lambda}) \\ P &\longmapsto K \otimes P \end{aligned}$$

is called the decomposition map *of Λ.*

Note that the decomposition matrix is the matrix of the decomposition map with respect to the isomorphism classes of the projective indecomposable modules as a basis of proj_{Λ} and the isomorphism classes of simple modules as a basis of $\mathrm{mod}_{K \otimes_R \Lambda}$.

Definition 2.1.22. *For $i, j \in \{1, \ldots, l\}$ we denote by $c_{ij} := [\overline{P}_i : \overline{S}_j]$ the multiplicity of the simple $\overline{\Lambda}$-module $\overline{S}_j$ in the projective indecomposable module $\overline{P}_i$. The matrix $C := (c_{ij})_{i,j}$ is called the* Cartan matrix *of Λ.*

Lemma 2.1.23. *For $i, j \in \{1, \ldots, l\}$ we have $c_{ij} = \dim_F(\mathrm{Hom}_F(\overline{P}_i, \overline{P}_j))$.*

Proof. See [Ben98, Lemma 1.7.6]. □

Lemma 2.1.24 (Brauer reciprocity)**.** *Assume that K is a splitting field for $K \otimes_R \Lambda$. Then the following equalities hold.*

1. $d_{ij} = [F \otimes_R L_i : S_j]$

2. $C = D^T \cdot D$.

Proof. See [Ben98, Proposition 1.9.6]. □

Definition 2.1.25. *For $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,l\}$ define*

$$d_i := \{t \in \{1,\dots,l\} \mid d_{it} > 0\}$$
$$c_j := \{s \in \{1,\dots,k\} \mid d_{sj} > 0\}.$$

Analogously to the decomposition of modules into indecomposable modules, we decompose algebras into their *blocks*.

Definition 2.1.26.
1. *Let $1_\Lambda = \varepsilon_1 + \ldots + \varepsilon_n$ be a decomposition into centrally primitive idempotents. Then $\Lambda = \varepsilon_1\Lambda \oplus \ldots \oplus \varepsilon_n\Lambda$ is a decomposition of Λ into indecomposable algebras. We call the algebras $\varepsilon_i\Lambda$ the* blocks *of Λ.*
2. *For every indecomposable Λ-lattice L there is exactly one $i \in \{1,\dots,n\}$ such that $L\varepsilon_i \neq 0$. In that case we say that L* lies in *the block $\varepsilon_i\Lambda$.*
3. *Let V be a simple $K \otimes_R \Lambda$-module. Then there is exactly one $i \in \{1,\dots,n\}$ such that $V\varepsilon_i \neq 0$. We say that V* lies in *the block $\varepsilon_i\Lambda$.*
4. *Let T be the trivial Λ-module and assume that T lies in the block $\varepsilon_i\Lambda$. Then we call $\varepsilon_i\Lambda$ the* principal block *of Λ and denote it by $B_0(\Lambda)$.*

2.1.3 Semisimple algebras

Lemma 2.1.27. *Let K be a field, A a semisimple K-algebra and let*

$$A = \bigoplus_{i=1}^{k} \bigoplus_{\lambda=1}^{n_i} e_{i\lambda}A$$

be an A-module decomposition of a such that $e_{i\lambda}$ is primitive for every i and λ and $e_{i\lambda} \cong e_{j\mu}$ if and only if $i = j$. Then we have the following.

1. *The module $U_i := \bigoplus_{i=1}^{n_i} e_{i\lambda}A$ is a two-sided ideal in A and therefore a block.*
2. *Let ε_i be the central primitive idempotent with $U_i = \varepsilon_i A$ and let V be a simple A-module. Then $V \cong e_{i1}A$ if and only if $V\varepsilon_i \neq 0$.*

Proof. First note that since A is semisimple every module $e_{i\lambda}A$ is simple.

1. Left-multiplication by an element of A is a right A-module homomorphism and can thus only send simple modules to isomorphic modules or 0.
2. This follows since by assumptions $e_{i\lambda}A \ncong e_{j\lambda}A$ for $i \neq j$.

□

Lemma 2.1.28 (Wedderburn)**.** *Let K be a field, A a semisimple K-algebra and $\{S_1,\dots,S_l\}$ a complete set of non-isomorphic simple A-modules. Then $D_i := \mathrm{End}_A(S_i)$ is a division algebra for every $i \in \{1,\dots l\}$ and there is an isomorphism of K-algebras*

$$\Phi : A \longrightarrow \bigoplus_{i=1}^{l} D_i^{n_i \times n_i},$$

where $n_i = \dim(S_i)$ *and with* $\Phi(\varepsilon_i) = I_{D_i^{n_i \times n_i}}$ *for* $i \in \{1, \dots, l\}$. *We call such an isomorphism* Φ *a* Wedderburn decomposition *of* A.

Recall that if K *is a splitting field for* A *then* $D_i = K$ *for every* i.

Proof. See [Ben98, Theorem 1.3.5]. □

Definition 2.1.29. *Let* K *be a field,* A *a semisimple* K*-algebra and* $\{S_1, \dots, S_l\}$ *a complete set of non-isomorphic simple* A*-modules. Assume further that there is a seperable field extension* L/K *such that* L *is splitting field for* $L \otimes_K A$. *Then the* reduced trace $\mathrm{Tr}_{red}(a)$ *of an element* $a \in A$ *is defined as follows. Let*

$$\Psi : L \otimes_K A \longrightarrow \bigoplus_{i=1}^{l} L^{m_i \times m_i}$$

be a Wedderburn decomposition of $L \otimes_K A$, $\iota : A \hookrightarrow L \otimes_K A, a \mapsto 1 \otimes a$ *and* $(a_i)_{i=1}^{l} := \Psi(\iota(a))$. *Then* $\mathrm{Tr}_{red}(a) := \sum_{i=1}^{l} \mathrm{Tr}(a_i)$.

Lemma 2.1.30. *Let* K *and* A *be as above. The reduced trace* Tr_{red} *does not depend on the choice of the splitting field or the Wedderburn decomposition and* $\mathrm{Tr}_{red}(a) \in K$ *for all* $a \in A$.

Definition 2.1.31. *Let* K *be a field,* A *a semisimple* K*-algebra and* $u \in Z(A)^\times$ *a central unit in* A. *Then the bilinear form* T_u *is defined as follows:*

$$\begin{aligned} \mathrm{T}_u : A \times A &\longrightarrow K \\ (a, b) &\longmapsto \mathrm{Tr}_{red}(u \cdot a \cdot b). \end{aligned}$$

Lemma 2.1.32. *Let* R *be a complete discrete valuation ring with field of fractions* $K = \mathrm{Quot}(R)$. *Let* Λ *be an* R*-algebra such that* $K \otimes_R \Lambda$ *is semisimple. Let* $\{S_1, \dots, S_l\}$ *be a complete set of non-isomorphic simple* $K \otimes_R \Lambda$*-modules,* $D_i := \mathrm{End}_A(S_i)$, $R_i \subseteq D_i$ *be the integral closure of* R *in* D_i *and let* $e_1, \dots, e_n$ *be a complete system of primitive orthogonal idempotents of* Λ. *Then there is a Wedderburn decomposition* Φ *of* $K \otimes_R \Lambda$ *such that both*

$$\Phi(\Lambda) \subseteq \bigoplus_{i=1}^{l} R_i^{m_i \times m_i}$$

and the idempotents $\Phi(e_1), \dots, \Phi(e_n)$ *are diagonal.*

Proof. Since D_i is a finite extension of K, R_i is a discrete valuation ring and thus in particular a principal ideal domain for every i. Let Φ be a Wedderburn decomposition. We are going to construct an algebra automorphism Ψ of $\bigoplus_{i=1}^{l} D_i^{m_i \times m_i}$ such that $\Psi \circ \Phi$ is the required Wedderburn decomposition, that is if $\iota_\Lambda : \Lambda \to K \otimes \Lambda, \lambda \mapsto 1 \otimes \lambda$ then $\Psi \circ \Phi \circ \iota_\Lambda$ factors through $\bigoplus_{i=1}^{l} R_i^{m_i \times m_i}$ as depicted by the following diagram:

$$\begin{array}{ccc} K \otimes_R \Lambda & \xrightarrow{\Psi \circ \Phi} & \bigoplus_{i=1}^{l} D_i^{m_i \times m_i} \\ {\scriptstyle \iota_\Lambda}\uparrow & & \uparrow{\scriptstyle \iota_W} \\ \Lambda & \dashrightarrow & \bigoplus_{i=1}^{l} R_i^{m_i \times m_i} \end{array}$$

where ι_W is the obvious inclusion. The automorphism Ψ will also be chosen in such a way that the idempotents $\Psi(\Phi(e_1)), \dots, \Psi(\Phi(e_n))$ are diagonal. We construct Ψ component wise by considering each direct summand $D_i^{m_i \times m_i}$ separately. Note that if ε_i is a centrally primitive idempotent of $K \otimes_R \Lambda$ then $\Phi(\varepsilon_i K \otimes_R \Lambda) = D_i^{m_i \times m_i}$, $\varepsilon_i \Lambda$ is an R-algebra with $K \otimes_R \varepsilon_i \Lambda = \varepsilon_i K \otimes_R \Lambda$ and $\{\varepsilon_i e_j \mid j \in \{1, \dots, n\}\}$ is a set of orthogonal idempotents.

Let ε_i be a centrally primitive idempotent of $K \otimes_R \Lambda$ with $V_i \varepsilon_i \neq 0$. Then we can identify $\Phi(\varepsilon_i K \otimes_R \Lambda) = D_i^{m_i \times m_i}$ with $\mathrm{End}_{D_i}(D_i^{m_i})$. With this identification $X_i := D_i^{m_i}$ becomes an $\varepsilon_i K \otimes_R \Lambda$-module and thus by Lemma 2.1.18, we can find an R_i-form $L \subseteq X_i$. Let $B = (b_1, \dots, b_{m_i})$ be an R_i-basis of L compatible with the decomposition $L = \bigoplus_{j=1}^{n} L \varepsilon_i e_j$. Then B is a D_i-basis of X_i. Let $\Psi_i : D_i^{m_i \times m_i} \to D_i^{m_i \times m_i}$ be the conjugation with the base change matrix from the standard D_i-basis of X_i to B. Then $M = \Psi_i(\Phi(a)) \in \Psi_i(\Phi(K \otimes_R \Lambda))$ is the matrix describing the multiplication of a on X_i with respect to the basis B. If $a \in \Lambda$ then $b_j.a \in L = \langle b_1, \dots, b_{m_i} \rangle_{R_i}$ for every $j \in \{1, \dots, m_i\}$ and thus the entries of M all lie in R_i. Now assume that $a = \varepsilon_i e_j$, let $k \in \{1, \dots, m_i\}$ and let $m \in \{1, \dots, n\}$ such that $b_k \in L \varepsilon_i e_m$. Then $b_k \varepsilon_i e_j = b_k e_j \varepsilon_i = b_k \delta_{mj}$ and thus M is a diagonal matrix.

By combining these base changes as $\Psi := \bigoplus_{i=1}^{l} \Psi_i$ we get an algebra automorphism of $\bigoplus_{i=1}^{l} D_i^{m_i \times m_i}$ induced by conjugation such that $\Psi \circ \Phi$ is a Wedderburn decomposition as in the statement of the lemma. □

Definition 2.1.33. *Let K be a field, R a subring of K and A a K-algebra. We say that an R-subalgebra Λ of A is an R-order in A if Λ generates A as a K-vector space.*

Definition 2.1.34. *Let K be a field, R a subring of K and A a K-algebra. Let further $T : A \times A \longrightarrow K$ be a non-degenerate bilinear form on A.*

For an R-order Λ in A we define its dual $\Lambda^\#$ to be

$$\Lambda^\# := \{a \in A \mid T(a, \Lambda) \subseteq R\}.$$

We call Λ symmetric *or* self-dual *in A with respect to T if $\Lambda = \Lambda^\#$.*

If $K = \mathrm{Quot}(R)$ and $A = K \otimes \Lambda$ we say Λ is self-dual *omitting the reference to A.*

This notion is strongly related to the usual notion of the dual space:

Lemma 2.1.35. *For every R-order Λ and every non-degenerate bilinear form on $K \otimes \Lambda$ we have $\Lambda^\# \cong \mathrm{Hom}_R(\Lambda, R) = \Lambda^*$.*

Proof. As T is non-degenerate, it induces an isomorphism between A and A^*. The order Λ^* can be embedded in A^* as the set of all K-homomorphisms ϕ from A to

K for which $\phi(\Lambda) \subseteq R$. Then $\Lambda^{\#}$ is the preimage of this set under the isomorphism induced by T. □

Lemma 2.1.36. *Let Λ as above be self-dual with respect to T and e, f be idempotents in Λ. Then $(e\Lambda f)^{\sharp} = f\Lambda e$. In particular $e\Lambda e$ is self-dual with respect to T.*

Proof. See [Thé95, Proposition 6.4]. □

2.2 Group algebras

In this section, we will discuss properties of group algebras. For the whole section, let G be a finite group and R be a field or a complete discrete valuation ring.

Definition 2.2.1. *We define the group algebra RG of G over R as the free R-module with basis G, where the multiplication is given by the distributive extension of the group operation.*

Definition 2.2.2. *A* representation *of G over R of rank n is a group homomorphism $X : G \longrightarrow \mathrm{Gl}_n(R)$.*

Lemma 2.2.3. *Let X be a representation of G over R of rank n. Then R^n becomes an RG-module via*

$$\begin{aligned} V \times RG &\longrightarrow R \\ (v, g) &\longmapsto vX(g). \end{aligned}$$

Denote this module by V_X.

Conversely, let V be an RG-module with $n = \dim V < \infty$. Then the following map is a group homomorphism.

$$\begin{aligned} X_V : G &\longrightarrow \mathrm{End}_R(V)^* \cong \mathrm{Gl}_n(R) \\ g &\longmapsto (v \mapsto v.g) \end{aligned}$$

For two group homomorphisms $X, Y : G \longrightarrow \mathrm{Gl}_n(R)$ we have $V_X \cong V_Y$ if and only if X and Y are conjugate in $\mathrm{Gl}_n(R)$. Further, we have X_{V_X} is conjugate to X and V_{X_V} is isomorphic to V.

Proof. This follows from straightforward calculations □

Definition 2.2.4. *If $R = K$ is a field we call a representation X* irreducible *if the KG-module V_X is simple.*

The above lemma shows that the study of RG-modules is the same as the study of representations of the group G over R. The following Lemma of Maschke implies that the representation theory of a group over a field of characteristic 0 is generally much simpler than that over a field whose characteristic divides the group order.

Lemma 2.2.5 (Maschke)**.** *Let F be a field of characteristic $p \geq 0$. Then FG is semisimple if and only if p does not divide the order of G.*

Proof. See [NT89, Theorem 1.22] □

Definition 2.2.6. *1. We denote the set of conjugacy classes of G by $\mathrm{Cl}(G)$.*

2. For $g \in G$ we denote the conjugacy class containing g by g^G.

3. We define the class sum *of a conjugacy class $C \in \mathrm{Cl}(G)$ as $\hat{C} := \sum_{g \in C} g \in KG$.*

Over a field of characteristic 0 a lot of information about the representation theory of a group can be gathered from the traces of the irreducible representations.

Definition 2.2.7. *Let $R = K$ be a field of characteristic 0 , V be a KG-module and X_V the corresponding representation.*

1. We define the character χ_V *of V as*

$$\begin{aligned} \chi_V : \mathrm{Cl}(G) &\longrightarrow K \\ g^G &\longmapsto \mathrm{Tr}(X_V(g)). \end{aligned}$$

2. The character χ_V is called irreducible *if V is simple. Both V and χ_V are called* absolutely irreducible *if $L \otimes V$ is a simple LG-module for every algebraic field extension L/K.*

3. Let $\overline{K}$ be the algebraic closure of K. For every character χ of $\overline{K}G$, we define

$$K(\chi) := K(\{\chi(g^G) \mid g \in G\})$$

to be the character field of χ over K.

Lemma 2.2.8. *Let K be a field of characteristic zero and V, W be KG-modules. Then*

1. $\chi_V(1) = \dim_K(V)$ and

2. $\chi_V = \chi_W \Leftrightarrow V \cong W$.

Proof. See [CR62, Theorem 30.12]. □

Definition 2.2.9. *Given a character χ, we define the module V_χ* corresponding to *χ to be a module such that $\chi_{V_\chi} = \chi$. Lemma 2.2.8 implies that V_χ is unique up to isomorphism.*

Remark 2.2.10. *With Lemma 2.2.8 it also makes sense to enumerate the rows of the decomposition matrix of a group algebra RG by the irreducible characters of KG.*

Lemma 2.2.11. *Let V be a simple KG-module and let $\overline{K} \otimes V = \bigoplus_{i \in I} V_i$ with simple $\overline{K}G$-modules V_i. Then*

$$\mathrm{Z}(\mathrm{End}_{KG}(V)) \cong K(\chi_{V_i})$$

for every $i \in I$.

Proof. See [NT89, Theorem 6.2] □

Lemma 2.2.12. *Let L/K be two fields of characteristic zero, V an absolutely irreducible LG-module and assume that $\chi_V(g^G) \in K$ for all $g \in G$. Then there is an $m \in \mathbb{Z}_{>0}$ and a K-module W such that $\chi_W = m \cdot \chi_V$.*

Proof. See [CR62, Lemma 70.12]. □

Lemma 2.2.13. *Let $H \leq G$ be a subgroup, T be a right transversal of H in G, and M be an $RH-$module. Then $\mathrm{Ind}_{RH}^{RG} M = M \otimes_{RH} RG$ is generated by $\{m \otimes t \mid m \in M, t \in T\}$ as an R-module.*

Proof. It is clear that $\{m \otimes_{RH} g \mid m \in M, g \in G\}$ generates $\mathrm{Ind}_{RH}^{RG} M$. Now by the choice of T there is for every $g \in G$ an $h \in H$ and a $t \in T$ with $g = h \cdot t$ and thus $m \otimes g = m.h \otimes t$ for every $m \in M$. □

One important property to determine a group algebra over a discrete valuation ring is its self-duality.

Lemma 2.2.14. *Let (K, R, F) be a p-modular system such that KG is semisimple. We define $u = (u_i)_i \in \mathrm{Z}(KG) = \bigoplus_{i=1}^{k} \mathrm{End}_{KG}(V_i)$ via $u_i := \frac{\dim V_i}{|G|}$. Then RG is self-dual with respect to the form T_u as in Definition 2.1.31.*

Proof. See [CR62, Remark 2]. □

Lemma 2.2.15. *Let $x = \sum_{g \in G} a_g g \in KG$. Then*

$$x \in \mathrm{Z}(G) \Leftrightarrow a_g = a_{h^{-1}gh} \quad \forall g, h \in G.$$

Thus $\mathrm{Z}(KG)$ is generated by the class sums $\{\hat{C} \mid C \in \mathrm{Cl}(G)\}$.

Proof. Since G is a basis for KG, we know that

$$x \in \mathrm{Z}(G) \leftrightarrow h^{-1}xh = x.$$

The assertion follows by comparing the coefficients. □

We know that every indecomposable representation of an algebra lies in a unique block, see Definition 2.1.26. Each block B is obtained by multiplying the algebra with a centrally primitive idempotent, which we will denote by ε_B. In the case of group algebras one can also associate to each block a subgroup of G, its *defect group*, which we will define below.

For the rest of the section let (K, R, F) be a p-modular system.

Theorem 2.2.16. *Let P be a finitely generated projective FG-module. Then $p^{\nu_p(|G|)} \mid \dim(P)$.*

Proof. See [NT89, Theorem 1.26]. □

Definition 2.2.17. *Let k be the number of simple KG-modules and l the number of simple RG-modules. The* Brauer graph *of Λ is the graph $\Gamma = (V, E)$ with $V = \{1, \dots, k\}$ and $E = \{(i, j) \mid \exists x \in \{1, \dots, l\}, d_{ix}d_{jx} \neq 0\}$.*

Lemma 2.2.18. *1. If V is a simple KG-module, P is a projective RG-module, and V and P lie in different blocks of RG, then $[P : V] = 0$.*

2. If V and W are simple KG-module lying in the same block of RG, then the corresponding vertices in the Brauer graph are connected.

3. The Brauer graph has exactly one connected component for each block of RG.

Proof. See [PD77, Section 4.2 (d)]. □

Definition 2.2.19. *1. For two subsets $X, Y \subseteq G$ of G, we write $X \subseteq_G Y$ if there is a $g \in G$ such that $g^{-1}Xg \subseteq Y$.*

2. Let $C = g^G \in \mathrm{Cl}(G)$. A p-subgroup D of G is called a defect group *of C if D is a p-Sylow subgroup in $C_G(g)$. Note that all defect groups of a conjugacy class C are conjugate in G.*

3. For a p-subgroup D of G, we define

$$I_D := \sum_{C : D_C \subseteq_G D} F\hat{C} \subseteq FG$$

where D_C is a defect group of C.

Lemma 2.2.20. *For every p-subgroup D of G, the set I_D is a two-sided ideal in $\mathrm{Z}(FG)$.*

Proof. See [PD77, Lemma 4.3A]. □

Theorem 2.2.21. *Let B be a block of FG. Then there exists a subgroup D of G such that*

1. $\varepsilon_B \in I_D$ and

2. if $D' \leq G$ is a p-subgroup with $\varepsilon_B \in I_{D'}$ then $D \leq_G D'$.

This group is unique up to conjugation in G.

Proof. [PD77, Lemma 4.3A] □

Definition 2.2.22. *Let B be a block of FG.*

1. *A group D as in Theorem 2.2.21 is called a* defect group *of B and is denoted by D_B.*

2. *We call $d_B := \nu_p(|D_B|)$ the* defect *of B.*

Lemma + Definition 2.2.23. *Let B be a block of FG and let $a = \nu_p(|G|)$. Then*

$$p^{a-d_B} \mid \dim_K V$$

for every simple KG-module V and there is one simple KG module V_0 such that

$$p^{a-d_B+1} \nmid \dim_K V_0.$$

We define the height *of an irreducible module V to be*

$$\mathrm{ht}(V) := d_B - a - \nu_p(\dim_K V)$$

We say that $\mathrm{ht}(\chi_V) = \mathrm{ht}(V)$ is the height *of the corresponding character.*

Proof. See [PD77, Theorem 4.5A]. □

Example 2.2.24. *Assume that $d_B = 0$ and thus $|D_B| = 1$. Then there is exactly one simple KG-module V which lies in B. Furthermore, for an R-form L in V the module $S = F \otimes_R L$ is simple and the only FG-module lying in B. Every block containing an irreducible KG-module V with $\nu_p(\dim V) = \nu_p(|G|)$ is a defect zero block. For a proof see [PD77, Theorem 4.6A, 4.5B].*

Next we consider blocks with defect 1. From now on, let $|G| = p^a q$ where $p \nmid q$. Let $L_1 = \mathbb{Q}_p(\zeta_{p^m q})$ and $L_2 = \mathbb{Q}_p(\zeta_q)$ where $\zeta_{p^m q}$ is a primitive $p^m q$-th and ζ_q a primitive q-th root of unity and (L_1, R, F) be the corresponding p-modular system.

Lemma 2.2.25. 1. *The field L_1 is a splitting field for G.*

2. *For $\sigma \in \mathrm{Gal}(L_1/L_2)$ and an irreducible character χ of L_1G, χ^σ is also an irreducible character of L_1G. We say that χ and χ^σ are* p-conjugate *and write $\chi \sim_p \chi^\sigma$.*

3. *Let χ and χ^σ be as above with corresponding simple L_1G-modules V and V_σ and R-forms M and M_σ. Then $F \otimes_R M$ and $F \otimes_R M_\sigma$ have the same irreducible constituents. In particular, the row of the decomposition matrix of G corresponding to χ is equal to that corresponding to χ^σ.*

Proof. See [PD77]. □

Definition 2.2.26. *Let $\chi_1, \ldots, \chi_u$ be a complete set of representatives of p-conjugacy classes of characters of L_1G and assume that FG has exactly l non-isomorphic simple modules. We define the* reduced decomposition matrix *of L_1G as $D^0 = (d^0_{ij})$, where $d^0_{ij} = d_{\chi_i j}$ is the j-th entry of the row of the decomposition matrix corresponding to χ_i.*

Theorem 2.2.27. *Let B be a block with $d_B = 1$, $a := \nu_p(|G|)$, $\chi_1, \dots, \chi_k$ a complete set of irreducible characters of L_1G and assume (by reordering) that $\chi_1, \dots, \chi_u$ is a complete set of representatives of the p-conjugacy classes of L_1G-characters. Let further l be the number of isomorphism classes of simple $F \otimes_R B$-modules, $D = (d_{ij})$ the decomposition matrix and $D^0 = (d^0_{ij})$ the reduced decomposition matrix of B. Furthermore define*

$$\begin{aligned} t_i &:= |\{j \mid \chi_j \sim_p \chi_i\}|, \\ S &:= \{\chi_i \mid t_i\chi_i(1) \equiv t_1\chi_1(1) \mod 2^a\}, \\ T &:= \{\chi_i \mid t_i\chi_i(1) \equiv -t_1\chi_1(1) \mod 2^a\}. \end{aligned}$$

Then the characters and the decomposition matrix have the following properties.

1. $S \cup T = \{\chi_1, \dots, \chi_k\}$.
2. *If $\chi_i \not\sim_p \chi_j$ and we have either $\chi_i, \chi_j \in S$ or $\chi_i, \chi_j \in T$, then $d_{ia}d_{ja} = 0$ for every $a \in \{1, \dots, l\}$.*
3. *For every $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$, we have $d_{ij} \in \{0, 1\}$.*
4. *For $j \in \{1, \dots, l\}$, there are exactly two characters χ_{i_1} and χ_{i_2} with $i_1, i_2 \in \{1, \dots, u\}$ such that $d_{i_1 j}, d_{i_2 j} \neq 0$. One of those lies in S and the other in T.*

Proof. These are results by Brauer, see [Bra41]. □

The theory of block with defect 1 by Brauer has been generalized to blocks with a cyclic defect group by Dade [Dad66].

2.3 Morita- and derived equivalence

2.3.1 Morita-equivalence

We are interested in the module category of algebras. We therefore introduce the notion of *Morita-equivalence* and formalize what it means for two rings to have the same module category.

Definition 2.3.1. 1. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are* equivalent *if there are two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to C$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to Id_D and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to Id_C.*

2. *Two rings R and S are called* Morita-equivalent *if the categories* Mod_R *and* Mod_S *are equivalent.*

Theorem 2.3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. Then $\mathcal{F}$ is an equivalence of categories if and only if $\mathcal{F}$ induces bijections on the homomorphism sets and for every object D in $\mathcal{D}$ there is an object C in $\mathcal{C}$ such that $D \cong \mathcal{F}(C)$.*

Proof. See [Zim14, Proposition 3.1.28] □

This theorem yields the following alternative characterisation of Morita-equivalence.

Corollary 2.3.3. *Two rings R and S are Morita-equivalent if and only if there is a functor $\mathcal{F} : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S$ such that*

1. *For any two R-modules M, N the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(\mathcal{F}(M), \mathcal{F}(N))$ induced by $\mathcal{F}$ is bijective.*
2. *For every S-module V there is some R-module M such that $V \cong \mathcal{F}(M)$, i.e. $\mathcal{F}$ induces a bijection between the isomorphism classes of R-modules and the isomorphism classes of S-modules.*

Definition 2.3.4. *Let R be a field or complete discrete valuation ring, A a finite-dimensional R-algebra and $e_1, \ldots, e_n$ a system of primitive orthogonal idempotents such that $1 = \sum_{i=1}^{n} e_i$. Then A is called* basic *if $e_i A \ncong e_j A$, for all $i \neq j$.*

Lemma 2.3.5. *If R is a complete discrete valuation ring with residue field F and Λ is a basic R-algebra, then $F \otimes \Lambda$ is a basic F-algebra.*

Proof. This follows from the definition and Corollary 2.1.16. □

Lemma 2.3.6. *Let K be a field, A be a basic K-algebra and assume that K is a splitting field for A. Then every simple A-module is one-dimensional.*

Proof. See [ASS06, Proposition I.6.2]. □

Lemma 2.3.7. *Let A be finite-dimensional algebra.*

1. *There is a a basic algebra B such that A is Morita-equivalent to B. This algebra is unique up to isomorphism and it is called the* basic algebra of A.
2. *There is an idempotent $e \in A$ such that eAe is basic.*

Proof. See [Zim14, Proposition 4.3.5] □

2.3.2 Quiver algebras

As we are going to consider algebras up to Morita-equivalence, we are looking for a way to describe the Morita-equivalence class of an algebra A. From Lemma 2.3.3 we can see that this is equivalent to describing the isomorphism classes of A-modules together with the homomorphisms between them. This leads to the notion of the *quiver* of an algebra.

Definition 2.3.8. *A* quiver $Q = (Q_0, Q_1, s, t)$ *is given by*

- *a set Q_0 of* vertices,
- *a set Q_1 of* arrows *and*
- *two maps $s, t : Q_1 \to Q_0$ associating to each arrow a its* source $s(a)$ *and* target $t(a)$.

This means a quiver is a directed graph, where there can be any number of edges between two vertices.

Definition 2.3.9. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver*

1. *Let $a, b \in Q_0$. Then a* path *of length $l \geq 1$ from a to b is a sequence*

$$(a|\alpha_1, \ldots, \alpha_l|b) \tag{2.1}$$

with $\alpha_i \in Q_1$ for $i \in \{1, \ldots l\}$, $s(\alpha_1) = a$, $t(\alpha_l) = b$ and $t(\alpha_i) = s(\alpha_{i+1})$ for $i \in \{1, \ldots, l-1\}$.

We also define a path of length 0 for each $a \in Q_0$ and denote it by $\varepsilon_a = (a||a)$

2. *The* path algebra *KQ of Q over a field K is the K-algebra having the set of all paths in Q as a basis and the product between to paths of length $l \geq 0$ is defined as*

$$(a|\alpha_1, \ldots \alpha_l|b)(c|\beta_1, \ldots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k|d).$$

We extend this product distributively to KQ.

Definition 2.3.10. *Let K be a field and A a finite-dimensional K-algebra. We define the* quiver *Q_A of A to be the following quiver.*

- *The vertices of Q_A are a complete set of non-isomorphic primitive orthogonal idempotents $\{e_1, \ldots, e_l\}$ of A.*
- *For any pair e_i, e_j of vertices we have $\dim((e_i \operatorname{rad}(A) e_j)/(e_i \operatorname{rad}^2(A) e_j))$ arrows from e_i to e_j.*

The quiver Q_A does not depend on the choice of a complete set of non-isomorphic primitive orthogonal idempotents of A [ASS06, Lemma 3.2].

Lemma 2.3.11. *Let R be a commutative ring, A a finite-dimensional R-algebra and $A' \subseteq A$ a subalgebra of A with $A' + \operatorname{rad}^2(A) = A$. Then $A' = A$.*

Proof. See [Ben98, Proposition 1.2.8]. □

Lemma 2.3.12. *Assume that K is an algebraically closed field and A a basic and indecomposable K-algebra. Let further*

- *$\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A*
- *For $i, j \in \{1, \ldots, n\}$ let $B_{ij} \subseteq e_i \operatorname{rad} A e_j$ be such that*

$$\overline{B_{ij}} := \{b + e_i \operatorname{rad}^2 A e_j \mid b \in B_{ij}\}$$

is a basis of $(e_i \operatorname{rad} A e_j)/(e_i \operatorname{rad}^2 A e_j)$.

Then A is generated by

$$\{e_1, \dots, e_n\} \cup \bigcup_{i,j=1}^{n} B_{ij}.$$

Proof. See [Zim14, Proposition 4.5.1]. □

Corollary 2.3.13. *Assume that K is algebraically closed, A is basic and indecomposable and let Q_A be its quiver. Then the algebra A is isomorphic to a quotient of the path algebra of Q_A.*

2.3.3 Derived equivalence

We will give a short introduction to derived equivalences. For more details the reader is referred to [Zim14].

Definition 2.3.14. *1. Let A be a ring. A chain complex C over A consists of a sequence of A-modules $(C^i)_{i\in\mathbb{Z}}$ together with a sequence of A-homomorphisms $(d^i : C^i \longrightarrow C^{i+1})_{i\in\mathbb{Z}}$ such that $d^{i+1} \circ d^i = 0$.*

2. A homomorphism between two chain complexes (C_1, d_1) and (C_2, d_2) is defined to be a sequence $(\varphi^i : C_1^i \longrightarrow C_2^i)$ of A-module homomorphisms such that $\varphi^{i+1} \circ d_1^i = d_2^{i+1} \circ \varphi^i$ for every i. For the last condition we also write $\varphi \circ d_1 = d_2 \circ \varphi$.

3. Composition of two homomorphisms is defined component wise and we denote the resulting category of all chain complexes of A-modules by $\mathcal{C}(A)$. More generally, for every subcategory $\mathcal{A}$ of Mod_A we denote the category of chain complexes with objects in $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$.

4. For every chain complex (C, d) we define the shifted *chain complex $(C[i], d[i])$ by $C[i]^j := C^{i+j}$ and $d[i]^j := (-1)^i d^{i+j}$.*

5. We let $\mathcal{C}^+(A)$, $\mathcal{C}^-(A)$ and $\mathcal{C}^b(A)$ denote the category of left-bounded ($C^i = 0$ for $i << 0$), right-bounded ($C^i = 0$ for $i >> 0$) and bounded complexes (left- and right-bounded) respectively.

Chain complexes are defined in such a way that $\mathrm{Im}(d^i) \subseteq \mathrm{Ker}(d^{i+1})$ and thus the homology $H^i(C) = \mathrm{Ker}(d^{i+1})/\mathrm{Im}(d^i)$ is well-defined. Additionaly homomorphisms induce homomorphisms on the homology. The following definition introduces a class of homomorphisms which are zero on homology.

Definition 2.3.15. *Let (C_1, d_1) and (C_2, d_2) be chain complexes over a ring A. A sequence of A-homomorphisms $(\varphi^i : C_1^i \longrightarrow C_2^i)_{i\in\mathbb{Z}}$ is called* zero-homotopic *if there is a sequence $(h^i : C_1^i \longrightarrow C_2^{i-1})_{i\in\mathbb{Z}}$ of A-homomorphisms such that*

$$\varphi = d_2 \circ h + h \circ d_1.$$

Lemma 2.3.16. *Zero-homotopic maps are homomorphisms of chain complexes. The set of all zero-homotopic maps is closed under addition and the composition of two homomorphisms of chain complexes is zero-homotopic whenever one of the homomorphisms is zero-homotopic.*

Proof. This follows from straightforward calculations. □

This lemma assures that the following definition is well-defined.

Definition 2.3.17. *Let $\mathcal{A}$ be a subcategory of* Mod_A*. Then we define the* homotopy category $\mathcal{K}(\mathcal{A})$ *to be the category where the objects are chain complexes and the homomorphisms are homomorphisms of chain complexes modulo the zero-homotopic maps.*
We also define $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ analogously to Definition 2.3.14.

Any A-module M can be considered as an object in $\mathcal{K}(A)$ as the complex $0 \to M \to 0$. In the derived category this complex is isomorphic to any projective or injective resolution of M. We define the category in such a way that any exact complex becomes the zero complex.

Definition 2.3.18. *Let $\mathcal{A}$ be a subcategory of* Mod_A *for some ring A. We define the* derived category *as*

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{K}(\mathcal{A})/\mathcal{N}$$

where $\mathcal{N}$ are the exact chain complexes in $\mathcal{K}(\mathcal{A})$. We again define $\mathcal{D}(\mathcal{A})^-, \mathcal{D}(\mathcal{A})^+$ and $\mathcal{D}(\mathcal{A})^b$ analogously to Definition 2.3.14. For more details about the construction of categories as a quotient by a null-system, see [Eis12, Remark 2.65].

Lemma 2.3.19. *Let φ be a homomorphism of chain complexes such that its induced homomorphism on homology is an isomorphism. Then φ is an isomorphism in the derived category.*

Proof. See [Zim14, Remark 3.5.38] □

Remark 2.3.20. *Both $\mathcal{D}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ can be promoted to triangulated categories. As neither the definition of a triangulated category nor the way those categories become triangulated categories is essential for our discussions we will skip it and refer the reader to [Zim14] for details.*

Lemma 2.3.21. *Let A be an algebra over a commutative ring R. When we restrict the quotient functor*

$$\mathcal{Q} : \mathcal{K}^-(A) \longrightarrow \mathcal{D}^-(A)$$

to complexes containing only projective modules

$$\mathcal{Q}_{res} : \mathcal{K}^-(\mathrm{proj}_A) \longrightarrow \mathcal{D}^-(A)$$

we obtain an equivalence of triangulated categories.

Proof. [Zim14, Proposition 3.5.43] □

Definition 2.3.22. *Let A be a ring and $T \in \mathcal{K}^b(\mathrm{proj}_A)$ be a chain complex. We say that T is a* tilting complex *if*

$$\mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}_A)}(T[i], T) = 0 \text{ for all } i \in \mathbb{Z} \setminus \{0\} \text{ and}$$
$$\mathrm{add}(T) = \mathcal{K}^b(\mathrm{proj}_A),$$

where $\mathrm{add}(T)$ *is the smallest triangulated subcategory of* $\mathcal{K}^b(\mathrm{proj}_A)$ *containing* T *and being closed under taking direct summands and direct sums.*

Theorem 2.3.23 (Rickard)**.** *Let A and B be algebras over a commutative ring R which are projective as R-modules. Then the following are equivalent.*

1. *The bounded derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*
2. *The right-bounded homotopy categories $\mathcal{K}^-(\mathrm{proj}_A)$ and $\mathcal{K}^-(\mathrm{proj}_B)$ are equivalent as triangulated categories.*
3. *The bounded homotopy categories $\mathcal{K}^b(\mathrm{proj}_A)$ and $\mathcal{K}^b(\mathrm{proj}_B)$ are equivalent as triangulated categories.*
4. *There is a tilting complex $T \in \mathcal{K}^b(\mathrm{proj}_A)$ with $B = \mathrm{End}_{\mathcal{K}^b(A)}(T)$.*

In this case we say that A and B are derived equivalent.

Proof. See [Zim14, Theorem 6.5.1] □

Lemma 2.3.24. *Let A be as above and $T \in \mathcal{K}^b(\mathrm{proj}_A)$ be a tilting complex. Then there is an equivalence of triangulated categories*

$$\mathcal{G}_T : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(\mathrm{End}_{\mathcal{D}^b(A)}(T))$$

with $\mathcal{G}_T(T) = [0 \longrightarrow \mathrm{End}_{\mathcal{D}^b(A)}(T) \longrightarrow 0]$.

Proof. See [Zim14, Theorem 6.5.1] □

Definition 2.3.25. *Let A and B be two rings and let $(C_1, d_1) \in \mathcal{C}(\mathrm{Mod}_B)$ and $(C_2, d_2) \in \mathcal{C}({}_B\mathrm{Mod}_A)$ be two chain complexes. Then we define the tensor product $C_1 \otimes_B C_2$ to be the complex where*

$$(C_1 \otimes_B C_2)^i = \sum_{j+k=i} C_1^j \otimes_B C_2^k$$
$$(d_1 \otimes_B d_2)^i = \sum_{j+k=i} d_1^j \otimes id_{C_2^k} + (-1)^j id_{C_1^j} \otimes d_2^k.$$

It is straight forward to check that $(d_1 \otimes_B d_2)^{i+1} \circ (d_1 \otimes_B d_2)^i = 0$ and therefore we have $(C_1 \otimes_B C_2, d_1 \otimes_B d_2) \in \mathcal{C}(\mathrm{Mod}_A)$.

Lemma 2.3.26. *The tensor product descends to the homotopy category $\mathcal{K}(\mathcal{A})$.*

Definition 2.3.27. *Let A and B be two algebras over a commutative ring R.*

1. *Let $\mathcal{F} : \mathcal{K}^-(A) \longrightarrow \mathcal{K}^-(B)$ be an exact functor. Then we call the functor*

$$\mathrm{L}\mathcal{F} = \mathcal{Q} \circ \mathcal{F} \circ (\mathcal{Q}_{res})^{-1} : \mathcal{D}^-(A) \longrightarrow \mathcal{D}^-(B)$$

   the left derived functor of $\mathcal{F}$.

2. *We define the left derived tensor product as follows:*

$$\begin{aligned}(-\otimes_A^{\mathbb{L}} =) : \mathcal{D}^-(A) \times \mathcal{D}^-(A^{op} \otimes_R B) &\longrightarrow \mathcal{D}^-(B)\\ (C_1, C_2) &\mapsto \mathcal{Q}((\mathcal{Q}_{res})^{-1}(C_1) \otimes_A (\mathcal{Q}_{res})^{-1}(C_2)).\end{aligned}$$

   Then for a complex $Y \in \mathcal{K}^-(A^{op} \otimes_R B)$ the functor

$$- \otimes_A^{\mathbb{L}} \mathcal{Q}(Y) : \mathcal{D}^-(A) \longrightarrow \mathcal{D}^-(B)$$

   is the left derived functor of

$$- \otimes_A Y : \mathcal{K}^-(A) \longrightarrow \mathcal{K}^-(B).$$

Now we are ready to define two-sided tilting complexes, which give us a different way to describe derived equivalences.

Definition 2.3.28. *Let A and B be R-algebras. We call $X \in \mathcal{D}^b(A^{op} \otimes_R B)$ an* invertible object *if there is a complex $Y \in \mathcal{D}^b(B^{op} \otimes_R A)$ such that*

$$\begin{aligned}X \otimes_B^{\mathbb{L}} Y &\cong [0 \to {}_AA_A \to 0] \text{ and}\\ Y \otimes_A^{\mathbb{L}} X &\cong [0 \to {}_BB_B \to 0].\end{aligned}$$

Theorem 2.3.29. *Let A and B be two R-algebras which are projective as R-modules.*

1. *The algebras A and B are derived equivalent if and only if there exists an invertible object $X \in \mathcal{D}^b(A^{op} \otimes_R B)$. Such an object is called a* two-sided tilting complex *and*

$$- \otimes_A^{\mathbb{L}} X : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$$

   is an equivalence.

2. *Every equivalence of triangulated categories between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ is induced by a two-sided tilting complex.*

Proof. See [Zim14, Proposition 6.5.5] □

There is a strong connection between one- and two-sided tilting complexes.

Theorem 2.3.30. *Let A and B be as above.*

1. *Let $X \in \mathcal{D}^b(A^{op} \otimes_R B)$ be a two-sided tilting complex. Then $\mathcal{Q}^{-1}(X)$ considered as an element of $\mathcal{K}^-(B)$ by restriction is isomorphic to a one-sided tilting complex in $\mathcal{D}^-(B)$.*

2. *For every one-sided tilting complex $T \in \mathcal{K}^b(B)$ there is a two-sided tilting complex $X \in \mathcal{D}^b(A^{op} \otimes_R B)$ which is isomorphic to T in $\mathcal{D}^b(B)$.*

Proof. See [Zim14, Corollary 6.1.6] □

2.4 Graduated orders

In this section we introduce the notion of *graduated orders*. A helpful way to understand group algebras is by embedding them into such graduated orders. This method was introduced by Plesken [Ple83].

Let R be a complete discrete valuation ring with maximal ideal πR and field of fractions K.

Definition 2.4.1. *Let Γ be an R-order such that $K \otimes \Gamma$ is semisimple and $\{e_1, \dots, e_l\}$ be a complete system of primitive orthogonal idempotents. Then Γ is called* graduated *if $e_i \Gamma e_i$ is a maximal order in $e_i(K \otimes \Gamma)e_i$ for all i.*

Definition 2.4.2. *For $\underline{n} = (n_1, \dots, n_t) \in \mathbb{Z}_{>0}^{1\times t}$, $n := \sum_{i=1}^t$ and $M = (m_{ij})_{i,j\in\{1,\dots,t\}} \in \mathbb{Z}^{t\times t}$ we define the algebra*

$$\Lambda(n, M) = \{(a_{ij})_{i,j=1}^t \in K^{n\times n} \mid a_{ij} \in (\pi^{m_{ij}})^{n_i \times n_j}, 1 \le i, j \le t\}$$

Lemma 2.4.3. *Let Γ be a graduated order in $K^{n\times n}$. Then there are $t \in \mathbb{N}$, $\underline{n} = (n_1, \dots, n_t) \in \mathbb{Z}_{>0}^{1\times t}$ and $M = (m_{ij})_{i,j\in\{1,\dots,t\}} \in \mathbb{Z}_{\ge 0}^{t\times t}$ such that Γ is isomorphic to $\Lambda(n, M)$ and*

$$\begin{aligned} \sum_{i=1}^t n_i &= n \\ m_{ij} + m_{jk} &\ge m_{ik} \\ m_{ii} &= 0 \\ i \ne j \Rightarrow m_{ij} + m_{ji} &> 0 \end{aligned}$$

for all $i, j \in \{1, \dots t\}$.

Proof. See [Ple83, II.3]. □

Definition 2.4.4. *We call a matrix M such that $\Gamma \cong \Lambda(n, M)$ an* exponent matrix *of Γ. If Γ is already of the form $\Lambda(n, M)$ we say M is* the exponent matrix *of Γ.*

2.5 Lifting

Let (K, R, F) be a p-modular system where πR is the maximal ideal of R and let $\nu_\pi : K \to \mathbb{Z}$ denote the valuation of R.

Definition 2.5.1. *Let $\overline{\Lambda}$ be a finite-dimensional F-algebra.*

1. *We call an R-order Λ with $F \otimes_R \Lambda \cong \overline{\Lambda}$ a* lift *of $\overline{\Lambda}$.*
2. *We call an element $\hat{\alpha} \in \Lambda$ a lift of $\alpha \in \overline{\Lambda}$ if $F \otimes \hat{\alpha}$ is mapped to α by the isomorphism $F \otimes_R \Lambda \to \overline{\Lambda}$.*

Definition 2.5.2. *Let $\overline{\Lambda}$ be a finite dimensional F-algebra and Λ a lift of $\overline{\Lambda}$. For $i \in \{1, \dots, k\}$ let K_i/K be a field extension, $u_i \in K_i$ and let $d_{ij} \in \mathbb{Z}_{\geq 0}$ for every $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$. Then we say that Λ fulfills the* rational conditions

$\mathrm{Z}(A)$	u	1	$\dots$	l
K_1	u_1	d_{11}	$\dots$	d_{1l}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_k	u_k	d_{k1}	$\dots$	d_{kl}

if

- *$A := K \otimes \Lambda$ is semisimple,*
- *there is an isomorphism*

$$\mathrm{Z}(A) \cong \bigoplus_{i=1}^{k} K_i,$$

- *Λ is self-dual with respect to T_u for $u := (u_i)_{i=1}^k \in \mathrm{Z}(A)$ and*
- *the decomposition matrix of Λ is $D = (d_{ij})_{i,j}$ where the i-th row corresponds to the i-th summand of the center.*

Our goal is to classify all lifts fulfilling certain rational conditions of certain F-algebras. This question is motivated by the following special case.

Definition 2.5.3. *Let B be an R-order such that $K \otimes B$ is semisimple and B is self-dual with respect to T_u for $u \in \mathrm{Z}(K \otimes B)$. We say that an R-order Λ is a B-lift of $\overline{B}$ if Λ is a lift of $\overline{B}$ and Λ and B fulfill the same rational conditions.*

Thus, given an algebra B of interest, for example the basic algebra of a block of a group algebra, our goal will be to classify all B-lifts of $\overline{B}$ and thereby in particular gaining information about B. We will sometimes consider slight variations of this question, but in every case we want to derive as much information as possible about an R-order Λ given its residue algebra $F \otimes \Lambda$ and its rational conditions. In this section we will introduce some methods to achieve this goal.

2.5.1 General methods

For this section let Λ be an R-algebra such that $K \otimes_R \Lambda$ is semisimple. Let $\{e_i \mid i \in I_p\}$ be a complete set of non-isomorphic orthogonal idempotents of Λ, $\{V_i \mid i \in I_0\}$ a complete set of non-isomorphic simple $K \otimes \Lambda$-module, $D_i := \mathrm{End}_A(S_i)$ and $R_i \subseteq D_i$ the integral closure of R in D_i.

Then, by Lemma 2.1.32, there are $m_i \in \mathbb{Z}_{>0}$ such that Λ can be embedded in

$$\Gamma := \bigoplus_{i \in I_0} R_i^{m_i \times m_i}.$$

We will identify Λ with the image of this embedding and for the rest of the section assume that $\Lambda \subseteq \Gamma$. We also assume that all decomposition numbers of Λ are 0 or 1.

Notation 2.5.4. *By $E_{ll} \in K^{n \times n}$ we denote the matrix such that $(E_{ll})_{ll} = 1$ and $(E_{ll})_{km} = 0$ if $(k,m) \neq (l,l)$.*

Lemma 2.5.5. *Let $u \in \mathrm{Z}(K \otimes \Gamma)$. Then the dual of Γ with respect to the form T_u is as follows:*

$$\Gamma^\sharp = \bigoplus_{i \in I_0} \pi^{-\nu_\pi(u_i)} R_i^{m_i \times m_i}.$$

Proof. Let $\alpha = (\alpha_i)_i \in \bigoplus_{i \in I_0} \pi^{-\nu_\pi(u_i)} R_i^{m_i \times m_i}$ and $\beta = (\beta_i)_i \in \Gamma$. Then

$$\nu_\pi(\mathrm{T}_u(\beta, \alpha)) = \nu_\pi(\sum_{i \in I_0} \mathrm{Tr}(\beta_i \alpha_i) u_i) \geq \min\{\nu_\pi(\mathrm{Tr}(\alpha_i \beta_i) u_i) | i \in I_0\}$$

By assumption we have $\nu_\pi(\mathrm{Tr}(\alpha_i \beta_i) u_i) = \nu_\pi(\mathrm{Tr}(\alpha_i \beta_i)) + \nu_\pi(u_i) \geq -\nu_\pi(u_i) + \nu_\pi(u_i) = 0$ and thus also $\nu_\pi(\mathrm{T}_u(\alpha\beta)) \geq 0$.

Conversely let $\alpha = (\alpha_i)_i \in K \otimes \Gamma$, let $j \in I_0$ with $\nu_\pi((\alpha_j)_{kl}) < -\nu_\pi(u_j)$ and let $\beta = (\beta_i)_i \in \Gamma$ with $\beta_j = E_{ll} \in R_i^{m_i \times m_i}$ and $\beta_i = 0$ for $i \neq j$. Then

$$\begin{aligned} \nu_\pi(\mathrm{T}_u(\alpha\beta)) &= \nu_\pi(\mathrm{Tr}(\alpha\beta) u_j) \\ &= \nu_\pi((\alpha_j)_{kl} u_j) \\ &= \nu_\pi((\alpha_j)_{kl}) + \nu_\pi(u_j) < 0 \end{aligned}$$

and therefore $\alpha \notin \Gamma$. □

Corollary 2.5.6. *Assume that $u \in \mathrm{Z}(K \otimes \Gamma)$ such that Λ is self-dual with respect to T_u. Then the following algebra is a subalgebra of Λ:*

$$\bigoplus_{i \in I_0} \pi^{-\nu_\pi(u_i)} R_i^{m_i \times m_i} \subseteq \Lambda.$$

Proof. By assumption it is $\Lambda \subseteq \Gamma$ and therefore $\Gamma^\sharp \subseteq \Lambda^\sharp = \Lambda$. □

Lemma 2.5.7. *Let e be a primitive idempotent of Λ, ε a central primitive idempotent of $K \otimes \Lambda$ and V a simple $K \otimes \Lambda$-module with $V\varepsilon \neq 0$.*

- *Then εe is either zero or a primitive idempotent of $K \otimes_R \Lambda$.*
- *If f is a primitive idempotent orthogonal to e and both εe and εf are primitive idempotents then $\varepsilon eK \otimes_R \Lambda \cong \varepsilon fK \otimes_R \Lambda \cong V$ and we obtain a K-vector space isomorphism*

$$\begin{aligned} \varepsilon e \Lambda f &\longrightarrow \mathrm{End}_{K\otimes\Lambda}(V) \\ \lambda &\longmapsto (x \mapsto x.\lambda) \end{aligned}$$

Proof. It is easy to see that εe is an idempotent of $K \otimes \Lambda$. From Lemma 2.1.27 it follows that every simple summand of $\varepsilon K \otimes_R e\Lambda$ is isomorphic to V. On the other hand we obtain a decomposition $K \otimes_R e\Lambda = \bigoplus_{k\in I_0} K \otimes_R \varepsilon e\Lambda$. Thus, since all the decomposition numbers of Λ are 0 or 1, $\varepsilon eK \otimes \Lambda$ is either 0 or isomorphic to V_k, so εe is either 0 or primitive. The isomorphism is the one given by Lemma 2.1.9. □

Lemma 2.5.8. *Let e, f be two primitive orthogonal idempotents of Λ which are diagonal in Γ and ε_i the central primitive idempotent of $K \otimes \Lambda$ with $\varepsilon_i V_i \neq 0$. If both $\varepsilon_i e$ and $\varepsilon_i f$ are non-zero then there are $j_1, j_2 \in \{1, \dots, m_i\}$ with $\varepsilon_i e = E_{j_1 j_1}$ and $\varepsilon_i f = E_{j_2 j_2} \in K_i^{m_i \times m_i}$. Thus we can describe $\varepsilon_i f K \otimes_R \Lambda e$ as follows.*

$$\varepsilon_i f K \otimes_R \Lambda e = \{(a_{mn})_{m,n} \in K_i^{m_i \times m_i} \mid (m,n) \neq (j_2, j_2) \Rightarrow a_{mn} = 0\}$$

Proof. From the previous lemma and the assumptions we know that both $\varepsilon_i e$ and $\varepsilon_i f$ are primitive diagonal idempotents in in $\varepsilon_i K \otimes \Gamma$, so they have to be of the form E_{jj}. The rest follows by applying matrix multiplication. □

Definition 2.5.9. *Let $j, k \in I_p$ and $\alpha \in e_k \Lambda e_j$. From Lemma 2.5.8 we conclude that $\varepsilon_i \alpha$ has at most one non-zero entry. We call this entry α_i and define*

$$\begin{array}{rccc} F_{kj}: & e_k \Lambda e_j & \longrightarrow & \bigoplus_{i\in I_0} R_i \\ & \alpha & \longmapsto & (\alpha_i)_{i\in I_0}. \end{array}$$

The maps above are multiplicative in the following sense.

Lemma 2.5.10. *If $j, k, l \in I_p$, $\alpha \in e_k \Lambda e_j$ and $\beta \in e_l \Lambda e_k$ then*

$$F_{lk}(\beta) F_{kj}(\alpha) = F_{lj}(\beta\alpha)$$

In particular if $l = j$ we obtain

$$F_{jj}(\beta\alpha) = F_{jj}(\alpha\beta)$$

Proof. This follows directly by matrix multiplication. □

Corollary 2.5.11. *Let $j_1, \dots, j_n \in I_p$, $\alpha_i \in e_{j_{i+1}} \Lambda e_{j_i}$ and $\beta_i \in e_{j_i} \Lambda e_{j_{i+1}}$. Then the we obtain the following equalities:*

$$\begin{aligned} F_{j_1 j_1}(\beta_1 \alpha_1) &= F_{j_2 j_2}(\alpha_1 \beta_1) \\ F_{j_1 j_1}(\beta_1 \dots \beta_{n-1} \alpha_{n-1} \dots \alpha_1) &= F_{j_1 j_1}(\beta_1 \alpha_1) \dots F_{j_{n-1} j_{n-1}}(\beta_{n-1} \alpha_{n-1}). \end{aligned}$$

Lemma 2.5.12. *Let $j \in I_p$ and $\alpha \in \operatorname{rad}\Lambda$. Then*

$$F_{jj}(\alpha) \in \bigoplus_{i \in I_0} \pi \cdot R_i.$$

In particular if $k \in I_p$, $\beta \in e_k\Lambda e_j$ and $\gamma \in e_j\Lambda e_k$. Then

$$F_{jj}(\beta\gamma) \in \bigoplus_{i \in I_0} \pi \cdot R_i$$

Proof. By Lemma 2.1.13, we know that some power of α lies in $\pi \cdot \Lambda$. Since $F_{jj}((\alpha)^n) = (F_{jj}(\alpha))^n$ this is only possible if $F_{jj}(\alpha) \in \bigoplus_{i \in I_0} \pi \cdot R_i$. □

Remark 2.5.13. *For $j \in I_p$ the algebra $e_j\Lambda e_j$ is isomorphic to its image under F_{jj}. We can even omit every component where $d_{ij} = 0$ since e_j is zero in these components and consider the image of the following homomorphism.*

$$pr \circ F_{jj} : e_j\Lambda e_j \longrightarrow \bigoplus_{i \in c_j} R_i \tag{2.2}$$

Lemma 2.5.14. *Let $j \in I_p$ such that $K_i = K$ for all $i \in c_j$. Consider $e_j\Lambda e_j$ as an R-subalgebra of $R^{|c_j|}$ as in Remark 2.5.13. Then there is a martix A of the form*

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \pi^{a_2} & \cdots & \xi_{2k}\pi^{a_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{a_k} \end{pmatrix}$$

such that $e_j\Lambda e_j$ is the row space of A. For any such matrix we have

$$\sum_{i=2}^{k} a_k = (\sum_{i=1}^{k} -u_k)/2.$$

Proof. We cann assume the row-reduced form since R is a principal ideal domain. The rest follows from the self-duality of Λ. □

Lemma 2.5.15. *For every $i \in I_0$ such that $D_i = K$ the order $\varepsilon_i\Lambda$ is a graduated order.*

Proof. As the decomposition numbers are zero or one and the order Λ is basic, we know that $\dim(\varepsilon_i e_j \Lambda e_j) \leq 1$ for every $i \in I_0$ and $j \in I_p$. Furthermore, if $\dim_R(\varepsilon_i e_j \Lambda e_j) = 1$ then, as $\varepsilon e_j \in \varepsilon e_j \Lambda e_j$ is an idempotent, $\varepsilon e_j \Lambda e_j = R$. Therefore $\varepsilon_i\Lambda \subseteq K^{n_i \times n_i}$ is maximal. □

Definition 2.5.16. *For $i \in I_0$ such that $D_i = K$ we denote the exponent matrix of $\varepsilon_i\Lambda$ by $M_{i,\Lambda}$ or, if Λ is clear from the context, by M_i.*

Conjugation

Our strategy to determine the isomorphism type of an R-order will often be to show that it can be embedded into the Wedderburn decomposition of $K \otimes \Lambda$ in a certain way. Those proofs will often include assumptions we can obtain by base changes in the matrix algebras $D_i^{m_i \times m_i}$. The following lemmas show that certain properties can be achieved this way.

Lemma 2.5.17. *Let $A = (a_{ij})_{i,j=1}^n$ be a matrix in $K^{n\times n}$. Let*

$$I(l,x) = diag(1,\dots,1,x,1,\dots,1)$$

be the diagonal matrix with ones in all diagonal entries except the $l-th$ where it has entry $x \in K$. Then the matrix $B = (b_{ij})_{i,j=1}^n := I_j(x)A(I_j(x))^{-1}$ has the following form

- $b_{ij} = a_{ij}$ *if* $i \neq l$ *and* $j \neq l$ *or* $(i,j) = (l,l)$
- $b_{lj} = x \cdot a_{lj}$ *if* $l \neq j$
- $b_{il} = x^{-1} \cdot a_{il}$ *if* $l \neq i$

Proof. This follows from a straightforward calculation. □

In words this means that we can by base change manipulate a particular entry of the matrix while leaving everything but one row and one column unchanged.

Lemma 2.5.18. *Let Λ be a finite-dimensional R-order such that $K\otimes_R\Lambda$ is semisimple, assume that the simple Λ-modules are indexed over I_p and the simple $K\otimes_R\Lambda$-modules are indexed over I_0. Assume that I_p is ordered by "$<$", let $i \in I_0$ and let $d_i = \{j_1,\dots,j_k\}$ with $j_h < j_{h+1}$ for $h < k$.*

1. *Let $d_i' := d_i \setminus \{j_1\}$ and $\varphi : d_i' \to d_i$ with $\varphi(j) < j$ for all j. Then there is an $a \in K \otimes \Lambda$ such that for the exponent matrix $M_i = M_{i,a^{-1}\Lambda a}$ we get*

   $$(M_i)_{j\varphi(j)} = 0 \ \forall j \in d_i'$$

   and all other exponent matrices of $a^{-1}\Lambda a$ are the same as in Λ.

   For every $j \in d_i'$ let $\alpha_j \in e_j a^{-1}\Lambda a e_{\varphi(j)}$ with $\nu_\pi((\alpha_j)_i) = 0$. Then there is a $b \in K\otimes\Lambda$ such that

   $$(b^{-1}\alpha_j b)_i = 1$$

   and the exponent matrices of $b^{-1}a^{-1}\Lambda ab$ are the same as those of $a^{-1}\Lambda a$.

2. *Let $d_i' := d_i \setminus \{j_k\}$ and $\varphi \in d_i^{d_i'}$ with $\varphi(j) < j$ for all j. Then there is an $a \in K \otimes \Lambda$ such that for $M_i = M_{i,a^{-1}\Lambda a}$ we get*

   $$(M_i)_{\varphi(j)j} = 0 \ \forall j \in d_i'$$

and all other exponent matrices of $a^{-1}\Lambda a$ are the same as in Λ

For every $j \in d_i'$ let $\alpha_j \in e_\varphi(j)a^{-1}\Lambda a e_j$ with $\nu_\pi((\alpha_j)_i) = 0$. Then there is a $b \in K \otimes \Lambda$ such that

$$((b)^{-1}\alpha_j b)_i = 1$$

and the exponent matrices of $b^{-1}a^{-1}\Lambda ab$ are the same as those of $a^{-1}\Lambda a$.

Proof. 1. Define $N_g \in \mathbb{Z}_{\geq 0}^{m_i \times m_i}$ and $A_g \in K \otimes \Lambda$ inductively for $j \in d_i'$ such that

$$\begin{aligned} N_1 &= M_{i,\Lambda} \\ g > 1 &: \\ \varepsilon_{i'} A_g &= I_{m_{i'}} \ \forall i' \in I_0 \setminus \{i\} \\ (\varepsilon_i A_g)_{xy} &= (I_{m_i})_{x,y} \text{ if } (x,y) \neq (j_g, j_g) \\ (\varepsilon_i A_g)_{j_g j_g} &= (p^{N_{j_{g-1}}})_{j_g \varphi(j_g)} \\ a_g &= A_2 \cdot \ldots \cdot A_g \\ N_g &= M_{i,(a_g)^{-1}\Lambda a_g} \end{aligned}$$

Then $a := a_k$ fulfils the assumptions above:

First of all we can see that no exponent matrices except that of $\varepsilon_i\Lambda$ are changed as a is just the identity in those components.

Further note that by lemma 2.5.17 we see that A_g leaves all entries of the exponent matrix except those in row and column j_g invariant. Therefore if we assume by induction that $(N_{g-1})_{j_h \varphi(j_h)} = 0$ for $h < g$ then the same is true for N_g, as $\varphi(j_h) < h < g$. From lemma 2.5.17 it also follows that

$$(N_g)_{j_g \varphi(j_g)} = (N_{g-1})_{j_g \varphi(j_g)} - (N_{g-1})_{j_g \varphi(j_g)} = 0$$

With a similar approach we can achieve the second part of 1. We define B_g, N_g and b_g similar to A_g, N_g and a_g above only changing

$$(\varepsilon_i B_g)_{j_g j_g} = b_{g-1}^{-1}(\alpha_{j_g})_i b_{g-1}$$

As $(\alpha_{j_g})_i$ is always a unit we can inductively see that $b_{g-1}^{-1}(\alpha_{j_g})_i b_{g-1}$ is a unit and thus the exponent matrices will never change.

2. In this case we work inductively from k to 1.

$$\begin{aligned} N_k &= M_{i,\Lambda} \\ g < k &: \\ \varepsilon_{i'} A_g &= I_{m_{i'}} \ \forall i' \in I_0 \setminus \{i\} \\ (\varepsilon_i A_g)_{xy} &= (I_{m_i})_{x,y} \text{ if } (x,y) \neq (j_g, j_g) \\ (\varepsilon_i A_g)_{j_g j_g} &= (p^{N_{j_{g+1}}})_{j_g \varphi(j_g)} \\ a_g &= A_{k-1} \cdot \ldots \cdot A_g \\ N_g &= M_{i,(a_g)^{-1}\Lambda a_g} \end{aligned}$$

and B_g, N_g and b_g as above. This implies the assertion with the same arguments as in 1.

□

Remark 2.5.19. *The first part of the lemma above assures that we can choose for each but the first row of a matrix one entry and manipulate those entries independently by base change. The second part assures the same thing if we choose one entry in each column.*

2.5.2 Lifting and derived equivalence

The following results due to Eisele [Eis12] show that the uniqueness of lifts can be translated along a derived equivalence. In this section, we will state the results from [Eis12].

We let (K, R, F) denote a p-modular system.

Definition 2.5.20. *Let $\overline{\Lambda}$ be a finite-dimensional F-algebra. We define the* set of lifts *of $\overline{\Lambda}$ as follows.*

$$\widehat{\mathfrak{L}}(\overline{\Lambda}) := \left\{(\Lambda, \varphi) \mid \Lambda \text{ is an } R\text{-order and } \varphi : F \otimes \Lambda \xrightarrow{\sim} \overline{\Lambda} \text{ is an isomorphism}\right\} \Big/ \sim$$

where $(\Lambda, \varphi) \sim (\Lambda', \varphi')$ if and only if

1. *There is an isomorphism $\alpha : \Lambda \longrightarrow \Lambda'$ of R-orders*
2. *There is an automorphism $\beta \in \mathrm{Aut}_F(\overline{\Lambda})$ such that the functor $- \otimes^{\mathbb{L}}_{\overline{\Lambda}} {}_{\beta}\overline{\Lambda}_{id}$ fixes all the isomorphism classes of tilting complexes in $\mathcal{K}^b(proj_{\overline{\Lambda}})$ and*
3. *$\varphi = \beta \circ \beta' \circ (id_k \otimes \alpha)$.*

We define both

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) := \{(\Lambda, \varphi) \in \widehat{\mathfrak{L}}(\overline{\Lambda}) \mid K \otimes \Lambda \text{ is semisimple}\} \text{ and}$$
$$\mathfrak{L}(\overline{\Lambda}) := \{[\Lambda] \mid \Lambda \text{ lift of } \overline{\Lambda}\},$$

where $[\Lambda]$ is the isomorphism class of Λ and

$$\Pi : \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \mathfrak{L}(\overline{\Lambda}) : (\Lambda, \varphi) \longmapsto \Lambda.$$

Let $\overline{\Lambda}$ and $\overline{\Gamma}$ be two derived equivalent finite-dimensional F-algebras. For every two-sided tilting complex $X \in \mathcal{D}^b(\overline{\Lambda}^{op} \otimes_F \overline{\Gamma})$. Eisele defined a map [Eis12, Definition 3.5]

$$\Phi_X : \widehat{\mathfrak{L}}(\overline{\Lambda}) \longrightarrow \widehat{\mathfrak{L}}(\overline{\Gamma}).$$

We introduce the abbreviation $\Phi = \Pi \circ \Phi_X$.

In the following theorem we cite the most important properties of the map Φ_X.

Theorem 2.5.21. *Let* $(\Lambda, \varphi), (\Lambda', \varphi') \in \widehat{\mathfrak{L}}(\overline{\Lambda})$.

1. *For the inverse* X^{-1} *of* X *the map* $\Phi_{X^{-1}}$ *is the inverse of* Φ_X. *In particular* Φ_X *is bijective.*

2. *The map* Φ_X *restricts to a bijection*

$$\widehat{\mathfrak{L}}_s(\overline{\Lambda}) \longleftrightarrow \widehat{\mathfrak{L}}_s(\overline{\Gamma}).$$

3. *There is an isomorphism* $\eta_\Lambda : \mathrm{Z}(\Lambda) \longrightarrow \mathrm{Z}(\Phi(\Lambda, \varphi))$.

4. *Every isomorphism*

$$\gamma : \mathrm{Z}(\Lambda) \longrightarrow \mathrm{Z}(\Lambda')$$

gives rise to the isomorphism

$$\Phi(\gamma) = \eta_{\Lambda'} \circ \gamma \circ (\eta_\Lambda)^{-1} : \mathrm{Z}(\Phi(\Lambda, \varphi)) \longrightarrow \mathrm{Z}(\Phi(\Lambda', \varphi')).$$

5. *The map* $K \otimes \eta_\Lambda : \mathrm{Z}(K \otimes \Lambda) \longrightarrow \mathrm{Z}(K \otimes \Phi(\Lambda, \varphi))$ *is an isomorphism.*

6. *Every isomorphism*

$$\gamma : \mathrm{Z}(K \otimes \Lambda) \longrightarrow \mathrm{Z}(K \otimes \Lambda')$$

gives rise to the isomorphism

$$\Phi(\gamma) = (K \otimes \eta_{\Lambda'}) \circ \gamma \circ (K \otimes \eta_\Lambda)^{-1} : \mathrm{Z}(K \otimes \Phi(\Lambda, \varphi)) \longrightarrow \mathrm{Z}(K \otimes \Phi(\Lambda', \varphi')).$$

7. *If*

$$\gamma : \mathrm{Z}(K \otimes \Lambda) \longrightarrow \mathrm{Z}(K \otimes \Lambda')$$

is an isomorphism such that $D^\Lambda = D^{\Lambda'}$ *when identifying the rows via* γ *up to permutation of columns, then* $D^{\Phi(\Lambda,\varphi)} = D^{\Phi(\Lambda',\varphi')}$ *when identifying the rows via* $\Phi(\gamma)$ *up to permutation of columns.*

Proof. Part 1 is proven in [Eis12, Proposition 3.6], the rest follows from [Eis12, Theorem 3.20]. □

Remark 2.5.22. *From the proof of [Eis12, Theorem 3.20] we obtain an algorithm to calculate the decomposition matrix of* $\Phi(\Lambda, \varphi)$ *given the decomposition matrix of* Λ. *To do this let* $\overline{T}$ *be a one-sided tilting complex with* $\mathrm{End}_{\mathcal{K}(\overline{\Lambda})}(\overline{T}) = \overline{\Gamma}$. *Then by Corollary 2.3.30 there is a two-sided tilting complex* $Y \in \mathcal{D}^b(\overline{\Gamma^{op}} \otimes_F \overline{\Lambda})$ *such that* Y *restricted to* $\overline{\Lambda}$ *is isomorphic to* $\overline{T}$. *Let* $X = Y^{-1}$ *and let* $\Phi = \Pi \circ \Phi_X$.

Decompose $\overline{T}$ *as*

$$\overline{T} = \bigoplus_{j \in J} \overline{T_j}$$

into indecomposable complexes. We obtain a decomposition corresponds via $\mathcal{G}_{\overline{T}}$*, see Lemma 2.3.24, to a decomposition of* $\overline{\Gamma}$ *as*

$$[0 \longrightarrow \overline{\Gamma} \longrightarrow 0] = \bigoplus_{j \in J} \mathcal{G}_{\overline{T}}(\overline{T}_j).$$

Denote by $\overline{P_j}$ *the projective indecomposable* $\overline{\Gamma}$*-module with* $[0 \longrightarrow \overline{P_j} \longrightarrow 0] = \mathcal{G}_{\overline{T}}(\overline{T}_j)$*. Let* P_j *be a projective indecomposable* Γ*-module with* $F \otimes P_j \cong \overline{P_j}$*. The isomorphism between centers from Part 5 of the preceeding theorem induces a correspondence between the simple modules of* Λ *and those of* $\Phi(\Lambda, \varphi)$ *and thus an isomorphism* $\delta : K_0(\Lambda) \xrightarrow{\sim} K_0(\Phi(\Lambda, \varphi))$*. Then we obtain the following equality.*

$$\theta^{\Gamma}([P_j]) = \delta(\theta^{\Lambda}(\sum_k (-1)^k [T_j^k])) \in K_0(K \otimes \Phi(\Lambda, \varphi)).$$

The following theorem shows how self-duality of algebras is translated along the map Φ_X.

Theorem 2.5.23. *Let* $\overline{\Lambda}$ *and* $\overline{\Gamma}$ *be two derived equivalent finite-dimensional* F*-algebras and* $X \in \mathcal{D}^b(\overline{\Lambda}^{op} \otimes_F \overline{\Gamma})$ *be a two-sided tilting complex. Let* $(\Lambda, \varphi) \in \widehat{\mathfrak{Q}}(\overline{\Lambda})$ *with* $A := K \otimes \Lambda$ *semisimple with simple modules* $\{V_i \mid i \in I\}$*. Then* $\mathrm{Z}(A) \cong \bigoplus_{i \in I} \mathrm{Z}(\mathrm{End}_A(V_i))$*. Assume that* Λ *is self-dual with respect to* T_u *for* $u = (u_i)_{i \in I} \in \mathrm{Z}(A)$*. Let* $\Gamma = \Phi(\Lambda, \varphi)$ *and* $B := K \otimes \Gamma$*. There is an isomorphism* $\gamma : \mathrm{Z}(A) \longrightarrow \mathrm{Z}(B)$*. In particular, there is a bijection between the simple* A*-modules and the simple* B*-modules, say* $\{W_i \mid i \in I\}$*, and we can assume that* γ *restrict to isomorphisms* $\gamma_i : \mathrm{End}_A(V_i) \longrightarrow \mathrm{End}_B(W_i)$*.*

Then there are signs $\xi_i \in \{-1, 1\}$ *for* $i \in I$ *such that* Γ *is self-dual with respect to* $\mathrm{T}_{\tilde{u}}$ *with* $\tilde{u} = (\xi_i \gamma_i(u_i))$ *and* $\xi_i = \xi_j$ *if* V_i *and* V_j *have the same decomposition numbers.*

Proof. See [Eis12, Theorem 3.19]. □

3 Tame blocks

In this chapter we will investigate blocks of group algebras of semidihedral defect, that is blocks whose defect group is a semidihedral group over an algebraically closed field of characteristic 2. Those are a special case of tame algebras, which have been classified up to Morita-equivalence by Erdmann [Erd90b]. Amongst the tame algebras Erdmann singled out the ones of semidihedral type. All blocks with a semidihedral defect group are algebras of semidihedral type and if an algebra of semidihedral type is a block of a group algebra, then its defect group is a semidihedral group. However, not all algebras of semidihedral type are Morita equivalent to blocks of group algebras. Erdmann's classification contains partial answers to the question which algebras occur as blocks. This classification uses results by Olsson [Ols75] about character values and heights of characters of blocks with a semidihedral defect group.

From this discussion two question arise. One is whether we can determine for more algebras whether they can occur as blocks of group algebras. The second is if the corresponding blocks over a discrete valuation ring are uniquely determined by the algebras given in Erdmann's classification.

Eisele [Eis12] completely answered the second question for tame algebras of dihedral type which could occur as a block by Erdmann's classification. He showed that those blocks have unique lifts, if one assumes certain rational conditions a block of a group algebra would fulfill. On top of that, he was able to show that for certain algebras no such lifts exist and thus these algebras cannot be Morita-equivalent to blocks of group algebras.

He additionally showed that tame algebras of quaternion type with three simple modules lift uniquely in the same sense as above [Eis16].

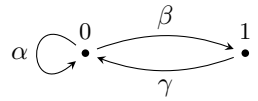
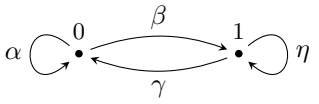
We show that the algebras of type $SD(2\mathcal{B})_1^0$ and $SD(2\mathcal{A})_2^0$ are not Morita-equivalent to blocks of group algebras. We further show that the algebras $SD(2\mathcal{B})_1^1$ and $SD(2\mathcal{A})_2^1$ have infinitely many lifts if they have one lift.

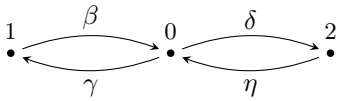
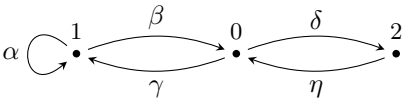
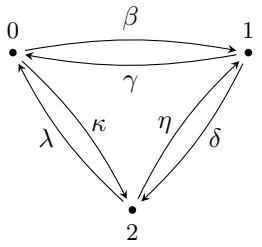
Additionally, we show that all blocks of semidihedral type with three simple modules lift uniquely.

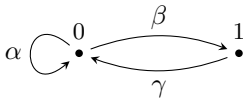
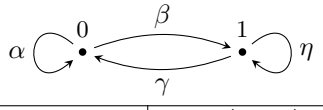
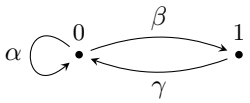
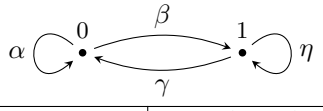
3.1 Classification

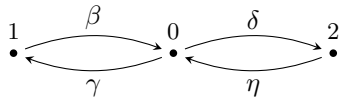
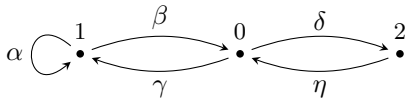
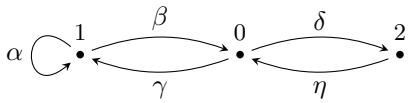
The following tables give the complete classification of tame algebras with two or three simple modules that can occur as blocks by Erdmann's classification [Erd90b]. The list is taken from [Hol01] with the decomposition matrices added from [Erd90b].

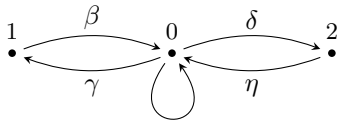
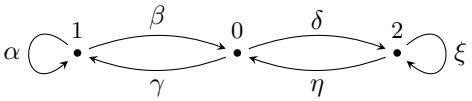
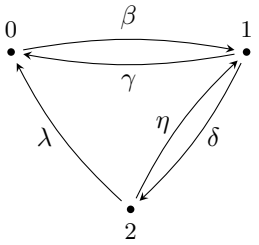
For $k = 2^{n-2}$ the blocks with the structure given below will have a defect group of size 2^n.

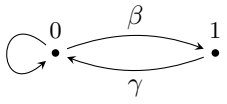
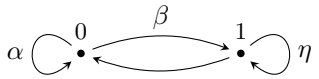
Algebras of Dihedral Type with Two Simple Modules				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$D(2\mathcal{A})^c$, $c \in \{0,1\}$			$\gamma\beta = 0, \alpha^2 = c(\alpha\beta\gamma)^k$, $(\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k$	
	$\begin{pmatrix} 4k & 2k \\ 2k & k+1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}_*$	$B_0(H)$, $L_2(q) < H$, $q \equiv 1 \mod 4$	$c = 0$: unique lift $c = 1$: no lift [Eis12]
$D(2\mathcal{B})^c$, $c \in \{0,1\}$			$\beta\eta = \eta\gamma = \gamma\beta = 0$, $\alpha^2 = c(\alpha\beta\gamma), \gamma\alpha\beta = \eta^k$ $\alpha\beta\gamma = \beta\gamma\alpha$	
	$\begin{pmatrix} 4 & 2 \\ 2 & k+1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}_*$	$B_0(H)$, $L_2(q) < H$, $q \equiv 3 \mod 4$	$c = 0$: unique lift $c = 1$: no lift [Eis12]

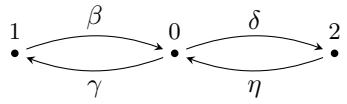
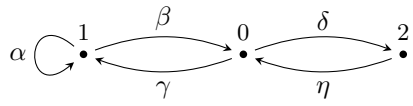
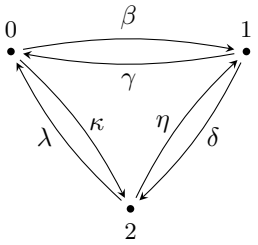
Algebras of Dihedral Type with Three Simple Modules				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$D(3\mathcal{A})$			$\beta\gamma = \eta\delta = 0$ $(\gamma\beta\delta\eta)^k = (\delta\eta\gamma\beta)^k$	
	$\begin{pmatrix} 4k & 2k & 2k \\ 2k & k+1 & k \\ 2k & k & k+1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}_*$	$B_0(PSL(q))$, $q \equiv 1 \mod 4$	unique [Eis12]
$D(3\mathcal{B})_1$			$\alpha\beta = \gamma\alpha = \beta\gamma = \eta\delta = 0$ $\alpha^k = \beta\delta\eta\gamma, \gamma\beta\delta\eta = \delta\eta\gamma\beta$	
	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & k+1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}_*$	$n = 3 : B_0(A_7)$, no blocks for $n > 3$	unique [Eis12]
$D(3\mathcal{K})$			$\beta\delta = \delta\lambda = \lambda\beta = \gamma\kappa = 0$ $\kappa\eta = \eta\gamma = 0, (\beta\gamma)^k = \kappa\lambda$ $\lambda\kappa = \eta\delta, \delta\eta = (\gamma\beta)^k$	
	$\begin{pmatrix} k+1 & k & 1 \\ k & k+1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}_*$	$B_0(PSL_2(q))$ $q \equiv 3 \mod 4$	unique [Eis12]

Algebras of Semidihedral Type with Two Simple Modules				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$SD(2\mathcal{A})_1^c$, $c \in \{0,1\}$			$\alpha^2 = c(\alpha\beta\gamma)^k$ $\beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta$ $\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$ $(\alpha\beta\gamma)^k\alpha = 0$	
	$\begin{pmatrix} 4k & 2k \\ 2k & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}_*$	$B_1(U_3(q))$, $q \equiv 1 \mod 4$	?
$SD(2\mathcal{B})_2^c$, $c \in \{0,1\}$			$\beta\eta = \alpha\beta\gamma\alpha\beta, \eta\gamma = \gamma\alpha\beta\gamma\alpha,$ $\gamma\beta = \eta^{k-1}, \alpha^2 = c(\alpha\beta\gamma)^2$ $\beta\eta^2 = \eta^2\gamma = 0$	
	$\begin{pmatrix} 8 & 4 \\ 4 & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}_*$	$B_1(PSL_3(q))$ $q \equiv 3 \mod 4$	?
$SD(2\mathcal{A})_2^c$, $c \in \{0,1\}$			$\gamma\beta = 0, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k$ $\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\alpha\beta\gamma)^k$	
	$\begin{pmatrix} 4k & 2k \\ 2k & k+1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}_*$	$B_0(PSL_2(q)^*)$, $q = r^2$ odd	$c = 0$: no lift $c = 1$: not unique, Thm 3.2.14
$SD(2\mathcal{B})_1^c$, $c \in \{0,1\}$			$\gamma\beta = \eta\gamma = \beta\eta = 0,$ $\alpha^2 = \beta\gamma + c(\alpha\beta\gamma)$ $\eta^k = \gamma\alpha\beta, \alpha\beta\gamma = \beta\gamma\alpha$	
	$\begin{pmatrix} 4 & 2 \\ 2 & k+1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}_*$	$B_1(3 \cdot M_{10})$ no blocks for $n > 4$	$c = 0$: no lift $c = 1$: not unique, Thm 3.2.14

Algebras of Semidihedral Type with Three Simple Modules I				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$SD(3\mathcal{A})_1$			$\beta\gamma = 0, \delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$ $\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$	
	$\begin{pmatrix} 4k & 2k & 2k \\ 2k & k+1 & k \\ 2k & k & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}_*$	$B_0(U_3(q))$, $q \equiv 1 \mod 4$	unique, see Thm 3.2.21
$SD(3\mathcal{B})_1$			$\alpha\beta = \gamma\alpha = \beta\gamma = 0$ $\alpha^k = \beta\delta\eta\gamma, \eta\delta\eta = \eta\gamma\beta$ $\delta\eta\delta = \gamma\beta\delta$	
	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & k+1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_*$	$n = 4 : B_0(M_{11})$ $n > 4$?	unique, see Thm 3.2.21
$SD(3\mathcal{B})_2$			$\eta\delta = 0, \gamma\alpha = \delta\eta\gamma\beta\delta\eta\gamma$ $\beta\gamma = \alpha^{k-1}, \alpha\beta = \beta\delta\eta\gamma\beta\delta\eta$	
	$\begin{pmatrix} 8 & 4 & 4 \\ 4 & k+2 & 2 \\ 4 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}_*$	?	unique, see Thm 3.2.21

Algebras of Semidihedral Type with Three Simple Modules II				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$SD(3\mathcal{C})^{a,b}$ $\{a,b\} = \{2, 2^{n-2}\}$			$\beta\rho = \rho\delta = \eta\rho = \rho\gamma = 0$ $\gamma\beta = \delta\eta, (\gamma\beta)^a = \rho^b$ $(\beta\gamma)^{a-1}\beta\delta = 0, (\eta\delta)^{a-1}\eta\gamma = 0$	
	$\begin{pmatrix} a+b & a & a \\ a & a+1 & a-1 \\ a & a-1 & a+1 \end{pmatrix}$	$\begin{pmatrix} 1&1&0 \\ 0&1&0 \\ 1&0&1 \\ 0&0&1 \\ 1&1&1 \\ 1&0&0 \end{pmatrix}_*$ and $\begin{pmatrix} 0&1&0 \\ 1&1&0 \\ 0&0&1 \\ 1&0&1 \\ 1&0&0 \\ 1&1&1 \end{pmatrix}_*$	?	unique, see Thm 3.2.21
$SD(3\mathcal{D})$			$\delta\xi = \xi\eta = \eta\delta = 0$ $\beta\gamma = \alpha^{k-1}, \gamma\alpha = \delta\eta\gamma$ $\alpha\beta = \beta\delta\eta, \xi^2 = \eta\gamma\beta\delta$	
	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & k+1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 1&1&0 \\ 1&0&1 \\ 1&1&1 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}_*$	$B_0(PSL_3(q))$, $q \equiv 3 \mod 4$	unique, see Thm 3.2.21
$SD(3\mathcal{H})$			$\delta\lambda = \gamma\beta\gamma, \lambda\beta = (\eta\delta)^{k-1}\eta$ $\beta\delta\eta = \gamma\beta\delta = \eta\gamma = 0$	
	$\begin{pmatrix} 3 & 2 & 1 \\ 2 & k+2 & k \\ 1 & k & k+1 \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \\ 1&1&1 \\ 1&1&0 \\ 0&1&1 \end{pmatrix}_*$	?	unique, see Thm 3.2.21

Algebras of Quaternion Type with Two Simple Modules				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$Q(2\mathcal{A})^c$, $c \in K$			$\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$, $\beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta$, $\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$, $\alpha^2\beta = 0$	
	$\begin{pmatrix} 4k & 2k \\ 2k & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}_*$	$B_0(G), SL_2(q) < G$ $q \equiv 1 \mod 4$	?
$Q(2\mathcal{B})_1^{c,a}$, $a \in K^*$, $c \in K$			$\gamma\beta = \eta^{k-1}, \beta\eta = \alpha\beta\gamma\alpha\beta$, $\eta\gamma = \gamma\alpha\beta\gamma\alpha, \alpha^2\beta = \gamma\alpha^2 = 0$ $\alpha^2 = (\alpha\beta\gamma)^2 + c(\beta\gamma\alpha)$	
	$\begin{pmatrix} 8 & 4 \\ 4 & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}_*$	$B_0(G), SL_2(q) < G$ $q \equiv 3 \mod 4$	?

Algebras of Quaternion Type with Three Simple Modules I				
Name	Quiver		Relations	
	Cartan matrix	Decomposition matrix	Examples	Lifting properties
$Q(3\mathcal{A})_2$			$\beta\gamma\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$, $\gamma\beta\gamma = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$ $\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$ $\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$ $\beta\gamma\beta\delta = \eta\delta\eta\gamma = 0$	
	$\begin{pmatrix} 4k & 2k & 2k \\ 2k & k+2 & k \\ 2k & k & k+2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}_*$	$B_0(SL_2(q))$, $q \equiv 1 \mod 4$	unique [Eis16]
$Q(3\mathcal{B})$			$\alpha\beta = \beta\delta\eta, \gamma\alpha = \delta\eta\gamma$ $\eta\delta\eta = \eta\gamma\beta, \delta\eta\delta = \gamma\beta\delta$ $\beta\gamma = \alpha^{k-1}, \alpha^2\beta = \beta\delta\eta\delta = 0$	
	$\begin{pmatrix} 8 & 4 & 4 \\ 4 & k+2 & 2 \\ 4 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_*$	$n = 4 : B_0(2A_7)$, $n > 4$?	unique [Eis16]
$Q(3\mathcal{K})$			$\beta\delta = (\kappa\lambda)^{k-1}\kappa, \eta\gamma = (\lambda\kappa)^{k-1}\lambda$ $\delta\lambda = \gamma\beta\gamma, \kappa\eta = \beta\gamma\beta$ $\lambda\beta = \eta\delta\eta, \gamma\kappa = \delta\eta\delta$ $\gamma\beta\delta = \delta\eta\gamma = \lambda\kappa\eta = 0$	
	$\begin{pmatrix} k+2 & k & 2 \\ k & k+2 & 2 \\ 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_*$	$B_0(SL_2(q))$, $q \equiv 1 \mod 4$	unique [Eis16]

3.2 Blocks of semidihedral type

Our goal in this section is to determine the basic algebra of a block of a group algebra with semidihedral defect group over a discrete valuation ring using the classification of the blocks over a splitting field of positive characteristic and additional information about the decomposition matrix, character fields and character heights.

3.2.1 General properties

Let K be the 2-adic completion of the maximal unramified extension of $\mathbb{Q}_2$. Let (K, R, F) be the corresponding 2-modular system. Then F is algebraically closed, see for example [Neu92, p.162].

Let G be a finite group and let Λ be a block of RG with semidihedral defect group SD_{2^n} for some $n \geq 3$. Define $A := K \otimes \Lambda$ and $\overline{\Lambda} := F \otimes \Lambda$. First we recall some basic results from number theory.

Lemma 3.2.1. *1. Let ζ_i be a primitive 2^i-th root of unity in $\overline{K}$ and let $K_i := K(\zeta_i + \zeta_i^{-1})$ and $K_i' := K(\zeta_i - \zeta_i^{-1})$. Then the Galois groups of both K_i and K_i' over K are cyclic. The unique subfield of index 2 in both K_i and K_i' is K_{i-1}. Thus, the subfields of K_i' form a chain $K = K_2 \subset K_3 \subset \cdots \subset K_{i-1} \subset K_i'$. We denote by R_i and R_i' the integral closure of R in K_i and K_i', respectively.*

2. The field extensions K_i'/K and K_i/K are totally ramified and the 2-valuations of their discriminants are equal to $(i-1) \cdot 2^{i-2} - 1$.

Lemma 3.2.2. *For any finite group H all the division algebras occurring in the Wedderburn decomposition of KH are commutative. This implies that for any irreducible KH-module V and any completely irreducible constituent χ of its character*

$$\mathrm{End}_{KH}(V) = \mathrm{Z}(\mathrm{End}_{KH}(V)) \cong K(\chi)$$

Proof. See [Eis12, Lemma 4.1 (iii)]. □

Lemma 3.2.3. *Let L/K be a splitting field for A.*

1. *Then $L \otimes A$ has $2^{n-2} + 3$ or $2^{n-2} + 4$ simple modules. Of those $2^{n-2} - 1$ are of height 1, 4 of height 0 and the remaining simple module, if it exists, has height $n-2$.*

2. *All character values of irreducible characters of $L \otimes A$ lie in K_{n-1}'. The characters of height 0 and $n-2$ take values in K.*

   The characters whose values do not lie in K are distributed into $n-3$ families $F_1, \ldots, F_{n-3}$, where each family is an orbit under the Galois group with $|F_i| = 2^i$, where $i \in \{1 \ldots n-3\}$. Thus, using Lemma 3.2.1 and Galois theory, we can see that the character field of characters in F_i is K_{i+2} for $i < n-3$ and for the characters in F_{n-3} it is K_{n-1}'.

Note that $\sum_{i=1}^{n-3} 2^i = 2^{n-2} - 2$ *and thus we have exactly one character of height 1 with values in* $K = K_2$*. We will define the family* F_0 *to be the set containing only this character.*

3. *Some multiple of the sum over the characters in one family* F_j *for* $j \in \{0, \dots, n-3\}$ *is a character of* A*.*

Proof. The first part follows from [Ols75, Theorem 3.13–3.16], the second part from [Ols75, Proposition 4.1, 4.2, 4.5] and the third part from the second part together with Lemma 2.2.12. □

Lemma 3.2.4. *Let*

$$\Gamma \subseteq R \oplus R \oplus \bigoplus_{r=0}^{n-4} R_{r+2} \oplus R'_{n-1}$$

be a local R*-order such that* $F \otimes \Gamma$ *is generated by a single nilpotent element* η*. Furthermore, assume that* Γ *is self-dual with respect to* T_u*, where* $u = (u_1, \dots, u_n) \in K \oplus K \oplus \bigoplus_{r=0}^{n-4} K_{r+2} \oplus K'_{n-1}$ *with* $\nu_2(u_1) = \nu_2(u_2) = -n$ *and* $\nu_2(u_i) = -n+1$ *for all* $i > 2$*. Then for some* $x \in F^*$ *there exists a preimage* $\hat{\eta}$ *of* $x \cdot \eta$ *under the residue map* $\Gamma \mapsto F \otimes \Gamma$ *in* Γ *of the form*

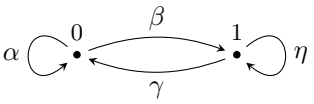
$$(0, 4, \pi_0, \dots, \pi_{n-3}),$$

where the π_r *are prime elements in the ring* R_{r+2} *for* $0 \le r < n-3$ *and in* R'_{n-1} *for* $r = n-3$*.*

Proof. See [Eis12, Lemma 4.7]. □

3.2.2 Blocks with two simple modules

Let $n \ge 4$ be fixed and for $c \in \{0, 1\}$ let $\overline{\Lambda}_c$ be a basic algebra of type $SD(2\mathcal{B})_1^c$ given by the quiver



with relations as follows.

$$\gamma\beta = 0 = \eta\gamma = \beta\eta, \quad \alpha^2 = \beta\gamma + c \cdot (\beta\gamma\alpha), \quad \eta^{2^{n-2}} = \gamma\alpha\beta, \quad \alpha\beta\gamma = \beta\gamma\alpha$$

We will assume the following rational structure on the lifts:

$\mathrm{Z}(A)$	u	P_0	P_1	
K	u_1	1	0	
K	u_1	1	0	
K	u_2	1	1	
K	u_2	1	1	
K_{r+2}	u_3	0	1	for $r \in \{0, \dots, n-4\}$
K'_{n-1}	u_3	0	1	

(3.1)

where $u_1, u_2, u_3 \in K$ with $\nu_2(u_1) = \nu_2(u_2) = -n$ and $\nu_2(u_3) = -n+1$.

Remark 3.2.5. *If a lift is a block of a group algebra it will fulfill the rational conditions (3.1) by the classification of tame algebras and Lemma 3.2.3.*

Definition 3.2.6. *For $\xi \in R^*$, $\pi_i \in K_{i+2}$ prime for $i \in \{0,\dots,n-4\}$ and $\pi_{n-3} \in K'_{i-1}$ prime we define $\Lambda_\xi(\pi_0,\dots,\pi_{n-3})$ to be the subalgebra of*

$$R \oplus R \oplus R^{2\times 2} \oplus R^{2\times 2} \oplus \bigoplus_{d=0}^{n-4} R_{d+2} \oplus R'_{n-1}$$

generated by the elements

$$\begin{aligned}
e_0 &= (1,1,\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},0,\dots,0)\\
e_1 &= (0,0,\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},1,\dots,1)\\
\widetilde{\eta} &= (0,0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&4\end{pmatrix},\pi_0,\dots,\pi_{n-3})\\
\widetilde{\alpha} &= (0,2,\begin{pmatrix}2^{n-2}\xi&0\\0&0\end{pmatrix},\begin{pmatrix}2-2^{n-2}\xi&0\\0&0\end{pmatrix},0,\dots,0)\\
\widetilde{\beta} &= (0,0,\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},0,\dots,0)\\
\widetilde{\gamma} &= (0,0,\begin{pmatrix}0&0\\2^{n-1}&0\end{pmatrix},\begin{pmatrix}0&0\\-2^{n-1}&0\end{pmatrix},0,\dots,0).
\end{aligned}$$

Theorem 3.2.7. *Let Λ be a lift of $\overline{\Lambda}_c$ satisfying the rational conditions (3.1). Then there are $\xi \in R^*$, $\pi_i \in K_{i+2}$ prime for $i \in \{0,\dots,n-4\}$ and $\pi_{n-3} \in K'_{i-1}$ prime such that $\Lambda \cong \Lambda_\xi(\pi_0,\dots,\pi_{n-3})$. Furthermore $\widetilde{\alpha}$ is a lift of α, $\widetilde{\beta}$ is a lift of β, $\xi\widetilde{\gamma}$ is a lift of γ and there is some unit $\mu \in R^*$ such that $\mu\widetilde{\eta}$ is a lift of η. The existence of such a lift implies that $c = 1$.*

Proof. By Lemma 2.1.32 we find an embedding

$$\Lambda \subseteq \Gamma := R \oplus R \oplus R^{2\times 2} \oplus R^{2\times 2} \oplus \bigoplus_{d=0}^{n-4} R_{d+2} \oplus R'_{n-1}$$

where the primitive idempotents are mapped to

$$\begin{aligned}
e_0 &= (1,1,\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},0,\dots,0) \text{ and}\\
e_1 &= (0,0,\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},1,\dots,1).
\end{aligned}$$

We want to apply Lemma 3.2.4 to determine a lift of η. Therefore we show that $F \otimes e_1\Lambda e_1$ is generated by η. Indeed we have

$$\gamma\beta = 0, \quad \gamma\alpha\beta = \eta^{2^{n-2}}, \quad \gamma\alpha^l\beta = \gamma\alpha^{l-2}(\beta\gamma + c\cdot(\alpha\beta\gamma))\beta = 0, \quad \forall l \geq 2.$$

Then Lemma 3.2.4 implies that $e_1\Lambda e_1$ is generated by an element $\hat{\eta}$ of the form

$$\hat{\eta} = (0, 4, \pi_0, \ldots, \pi_{n-3})$$

which is a lift of a scalar multiple of η and where π_r is prime in R_{r+2} for $r \in \{0, \ldots, n-4\}$ and π_{n-3} is prime in R'_{n-1}.

Considering $e_0\Lambda e_0$ let us first observe some properties of the elements of $F \otimes e_0\Lambda e_0$. If $s > 0$ is some natural number then $\beta\eta^s\gamma = 0$. Thus we see that $F \otimes e_0\Lambda e_0$ is generated by α and $\beta\gamma$ as an algebra. Furthermore, using the defining relations as follows, one can see that $F \otimes e_0\Lambda e_0$ is generated by e_0, α, $\beta\gamma$ and $\alpha\beta\gamma$ as a vector space:

$$\begin{aligned} \alpha^2 = \beta\gamma + c \cdot (\alpha\gamma\beta), \alpha^2\beta\gamma = (\beta\gamma)^2 + c \cdot (\alpha(\gamma\beta)^2) = 0, \\ \beta\gamma\alpha = \alpha\gamma\beta, (\beta\gamma)^2 = 0, \beta\gamma\alpha\beta\gamma = \beta\gamma\beta\gamma\alpha = 0, \\ \alpha\beta\gamma\alpha = \alpha^2\beta\gamma = 0, \alpha(\beta\gamma)^2 = 0, (\alpha\beta\gamma)^2 = \alpha(\beta\gamma)^2\alpha = 0. \end{aligned}$$

Since, by the Cartan matrix $C = DD^T$, the dimension of $F \otimes e_0\Lambda e_0$ is 4, these elements form a basis.

By Nakayama's Lemma lifts of these elements also form a basis of $e_0\Lambda e_0$. Since $2e_0 = (2, 2, 2, 2) \in 2e_0\Lambda e_0$ and by Lemma 2.5.12 there are $a \in \mathbb{Z}_{>0}$ and $x, y \in R$ such that for $\hat{\alpha} = (0, 2^a, x, y)$ there is a unit $\rho \in R^*$ such that $\rho\hat{\alpha}$ is a lift of α. There are lifts $\hat{\beta}$ and $\hat{\gamma}$ of β and γ such that there are $b \in \mathbb{Z}_{>0}$, $\zeta \in R^*$ and $z \in R$ with $\hat{\beta}\hat{\gamma} = (0, 0, \zeta 2^b, z)$. Then

$$\hat{\alpha}\hat{\beta}\hat{\gamma} - x\hat{\beta}\hat{\gamma} \in \langle (0, 0, 0, 2^d) \rangle_R$$

for some $d \geq 0$ and since $\nu_2(u_2) = n$ we know that $d = n$. We define $z' = \zeta^{-1}z$. Then, since $x \in 2R$, $e_0\Lambda e_0$ is the row space of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^a & x & y \\ 0 & 0 & 2^b & z' \\ 0 & 0 & 0 & 2^n \end{pmatrix}.$$

Since $\gamma\beta = 0$, the following product lies in $2e_1\Lambda e_1$.

$$\hat{\gamma}\hat{\beta}\hat{\eta} = (0, \zeta \cdot 2^{2+b}, 0, \ldots, 0) \in 2 \cdot e_1\Lambda e_1.$$

Since $\nu_2(u_2) = n$ it follows that $2 + b \geq n + 1$ and thus $b \geq n - 1$. Since, by Lemma 2.5.14, we have $a + b + n = \frac{4n}{2}$ and since $a \geq 1$, we know that $b = n - 1$ and therefore $a = 1$. We use the fact that Λ is self-dual to obtain some information on our open parameters:

$$\begin{aligned} \mathrm{T}_u(e_0, e_0) = 2 \cdot u_1 + 2 \cdot u_2 = u_1 \cdot (2 + 2 \cdot \frac{u_2}{u_1}) \in R &\Rightarrow 2^n | 2 + 2 \cdot \frac{u_2}{u_1} \\ &\Rightarrow \frac{u_1}{u_2} \equiv -1 \bmod 2^{n-1} \\ \mathrm{T}_u(e_0, \hat{\alpha}) = 2 \cdot u_1 + (x + y) \cdot u_2 = u_1 \cdot (2 + (x + y) \cdot \frac{u_2}{u_1}) \in R &\Rightarrow x + y \equiv -2 \cdot \frac{u_1}{u_2} \bmod 2^n \\ &\Rightarrow y \equiv 2 - x \bmod 2^n. \end{aligned}$$

Since $\gamma\beta = 0$ we obtain the following result for $\hat{\beta}\hat{\gamma}$:

$$\begin{aligned} \mathrm{T}_u(e_1, \frac{1}{2}\hat{\gamma}\hat{\beta}) = u_2 \cdot (2^{n-2}\zeta + z) \in R &\Rightarrow z \equiv -2^{n-2}\zeta \bmod 2^n \\ &\Rightarrow z \equiv -2^{n-1}\zeta \bmod 2^{n+1} \\ &\Rightarrow \hat{\beta}\hat{\gamma} \equiv (0, 0, 2^{n-1}\zeta, -2^{n-1}\zeta) \bmod 2\Lambda. \end{aligned}$$

Next we use the fact that $e_0\Lambda e_0$ is multiplicatively closed to obtain more information on the parameter x. Since $(0, 0, 0, 2^{n+1}) \in 2\Lambda$ we can ignore the difference $y - (2 - x) \in 2^n R$ in the following calculation:

$$\begin{aligned} \hat{\alpha}^2 - 2\hat{\alpha} \equiv (0, 0, x^2 - 2x, x^2 - 2x) \bmod 2\Lambda &\Rightarrow 2^{n-1} \mid x^2 - 2x \\ &\Rightarrow 2^{n-2} \mid x \text{ or } 2^{n-2} \mid x - 2. \end{aligned}$$

If $2^{n-2} \mid x - 2$ then $2^{n-2} \mid y \equiv 2 - x \bmod 2^n$. Therefore we can get from the second case to the first by interchanging the third and fourth Wedderburn components of Λ. We can do this without changing the form of any of the elements we already determined modulo 2Λ:

$$\hat{\eta} = (0, 4, \pi_0, \ldots, \pi_{n-3}) \equiv (4, 0, 4 - \pi_0, \ldots, 4 - \pi_{n-3}) \bmod 2\Lambda$$

and if π_i is prime then so is $4 - \pi_i$. When interchanging the third and fourth Wedderburn components $\hat{\beta}\hat{\gamma}$ gets multiplied by -1 . Therefore we can assume that $2^{n-2} \mid x$ and let $\xi \in R$ with $x = 2^{n-2}\xi$. We conclude that $e_0\Lambda e_0$ is the row space of the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2^{n-2}\xi & 2 - 2^{n-2}\xi \\ 0 & 0 & 2^{n-1} & -2^{n-1} \\ 0 & 0 & 0 & 2^n \end{pmatrix}$$

Next consider the remaining relations. Recall that $\hat{\alpha}$ was defined such that $\rho\hat{\alpha}$ is a lift of α. Therefore we obtain the following relations:

$$\begin{aligned} (0, 0, 0, -c \cdot 2^n \rho\zeta) &\equiv c \cdot \rho\hat{\alpha}\hat{\beta}\hat{\gamma} \\ &\equiv \rho^2\hat{\alpha}^2 - \hat{\beta}\hat{\gamma} \\ &\equiv \rho^2\hat{\alpha}^2 - 2\rho^2\hat{\alpha} - \hat{\beta}\hat{\gamma} \\ &\equiv (0, 0, -2^{n-1}\xi\rho^2 + 2^{2n-4}\xi^2 - 2^{n-1}\zeta, -2^{n-1}\xi\rho^2 + 2^{2n-4}\xi^2 + 2^{n-1}\zeta) \\ &\equiv (0, 0, -2^{n-1}\xi\rho^2 - 2^{n-1}\zeta, -2^{n-1}\xi\rho^2 + 2^{n-1}\zeta) \bmod 2\Lambda \end{aligned}$$

First of all this implies that $2 \mid \xi\rho^2 + \zeta$, so in particular $\xi \in R^*$ and we obtain

$$\begin{aligned} \hat{\beta}\hat{\gamma} &\equiv (0, 0, 2^{n-1}\xi\rho^2, -2^{n-1}\xi\rho^2) \bmod 2\Lambda \text{ and} \\ \rho^2\hat{\alpha}^2 - \hat{\beta}\hat{\gamma} \equiv (0, 0, 0, 2^n\xi\rho^2) &\equiv (0, 0, 0, -c \cdot 2^n \rho\xi\rho^2) \bmod 2\Lambda. \end{aligned}$$

This implies that $2 \mid c\rho^3\xi + \xi\rho^2$. Since ξ and ρ are units, we see that $c \neq 0$. Then, since $c = 1$, we know that $2 \mid \rho^2\xi(\rho+1)$, so $\rho \equiv -1 \equiv 1 \bmod 2$. It follows that

$$\rho\hat{\alpha} \equiv \hat{\alpha} \bmod 2\Lambda \text{ and}$$
$$\hat{\beta}\hat{\gamma} \equiv (0,0,2^{n-1}\xi,-2^{n-1}\xi) \bmod 2\Lambda$$

Now $\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{\eta}$ are lifts of scalar multiples of α,β,γ and η respectively and thus by Nakayama's Lemma form together with e_0 and e_1 a generating system for Λ. We know that there is a factor $\vartheta \in R^*$ such that

$$\hat{\alpha} + \vartheta\hat{\alpha}\hat{\beta}\hat{\gamma} \equiv (0,2,2^{n-2}\xi,2-2^{n-2}\xi) \bmod 2\Lambda$$

so in particular $\hat{\alpha} \equiv \widetilde{\alpha} \bmod \mathrm{rad}^2(\Lambda)$. Furthermore $\hat{\alpha}$ fulfills the same relations as $\widetilde{\alpha}$ modulo 2Λ and we can therefore assume that $\widetilde{\alpha}$ is a lift of α. By Lemma 2.5.18 there is an element $d \in K \otimes \Gamma$ such that

$$\hat{\beta}^d = \widetilde{\beta} = (0,0,\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},0,\dots,0)$$
$$\hat{\gamma}^d = (0,0,\begin{pmatrix} 0 & 0 \\ 2^{n-1}\xi & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ -2^{n-1}\xi & 0 \end{pmatrix},0,\dots,0)$$

Thus we know that $\{e_0,e_1,\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\eta}\}$ is a generating system for Λ^d and Λ is isomorphic to $\Lambda_\xi(\pi_0,\dots,\pi_{n-3})$.

□

Theorem 3.2.8. *Let $\pi_i \in K_{i+2}$ prime for $i \in \{0,\dots,n-4\}$, $\pi_{n-3} \in K'_{i-1}$ prime and $\xi,\xi' \in R^*$. If $\Lambda_\xi := \Lambda_\xi(\pi_0,\dots,\pi_{n-3})$ is a lift of $\overline{\Lambda}_1$ fulfilling the rational conditions (3.1) then so is $\Lambda_{\xi'} := \Lambda_{\xi'}(\pi_0,\dots,\pi_{n-3})$.*

Proof. We denote the generators of $\Lambda_{\xi'}$ by $e'_0,e'_1,\widetilde{\alpha}',\widetilde{\beta}',\widetilde{\gamma}'$ and $\widetilde{\eta}'$ to distinguish them from the generators of Λ_ξ. It is clear that $\Lambda_{\xi'}$ fulfills the rational conditions (3.1). It remains to show that $\Lambda_{\xi'}$ is a lift of $\overline{\Lambda}_1$.

First note that since $\widetilde{\eta}$ generates $e_1\Lambda_\xi e_1$ and $\widetilde{\eta}' = \widetilde{\eta}$ generates $e_1\Lambda_{\xi'}e_1$ we know that $e_1\Lambda_\xi e_1 = e'_1\Lambda_{\xi'}e'_1$. Let $\mu \in R^*$ such that $\mu\widetilde{\eta}$ is a lift of η. Then

$$\widetilde{\alpha}\widetilde{\beta}\xi\widetilde{\gamma} \equiv (0,0,0,2^n\xi) \equiv \mu^{2^{n-2}}\eta^{2^{n-2}} \bmod 2e_1\Lambda e_1.$$

Since F is algebraically closed, there exists a $\mu' \in R^*$ such that $\frac{\mu'^{2^{n-2}}}{\mu^{2^{n-2}}} \equiv \frac{\xi'}{\xi} \bmod 2$ and thus

$$\begin{aligned}(\mu'\hat{\eta}')^{2^{n-2}} &\equiv (0,0,0,2^n\xi') \\ &\equiv \widetilde{\alpha}'\widetilde{\beta}'\xi'\widetilde{\gamma}' \bmod 2\Lambda.\end{aligned}$$

Then it is easy to check that $\alpha' := F \otimes \widetilde{\alpha}'$, $\beta' := F \otimes \widetilde{\beta}'$, $\gamma' := \xi' := F \otimes \widetilde{\gamma}'$ and $\eta' := \mu'\widetilde{\eta}$ fulfill the defining relations of $\overline{\Lambda}_1$. □

Next we want to show that the different parameters in the previous theorem yield non-isomorphic algebras. We do this by applying the following lemma.

Lemma 3.2.9. *Let Λ, Λ' be two R-orders with decomposition numbers 0 and 1. Let $\varphi : \Lambda \longrightarrow \Lambda'$ be an R-algebra isomorphism and $\Phi : K \otimes_R \Lambda \longrightarrow K \otimes_R \Lambda'$ be the induced K-algebra isomorphism. Let further e be a primitive idempotent of Λ such that $\mathrm{Z}(\varepsilon K \otimes \Lambda) = K$ for all centrally primitive idempotents ε of $K \otimes_R \Lambda$ with $\varepsilon e \neq 0$. Then $\Phi \restriction_{\varepsilon K \otimes_R e\Lambda e}$ is uniquely determined by the images of ε and e.*

Proof. Since the decomposition numbers are 1 or 0, we know that $\varepsilon_a e\Lambda e \in \{0, R\varepsilon_a e\}$. □

Theorem 3.2.10. *Let $\pi_i \in K_{i+2}$ prime for $i \in \{0, \dots, n-4\}$, $\pi_{n-3} \in K'_{i-1}$ prime and $\xi, \xi' \in R^*$. Then*

$$\Lambda_\xi := \Lambda_\xi(\pi_0, \dots, \pi_{n-3}) \cong \Lambda_{\xi'} := \Lambda_{\xi'}(\pi_0, \dots, \pi_{n-3}) \Leftrightarrow \xi \equiv \xi' \bmod 2.$$

Proof. Let $\varphi : \Lambda_\xi \longrightarrow \Lambda_{\xi'}$ be an isomorphism with induced isomorphism

$$\Phi : K \otimes_R \Lambda_\xi \longrightarrow K \otimes_R \Lambda_{\xi'}.$$

Denote the primitive central idempotents of $K \otimes \Lambda_\xi$ by $\varepsilon_1, \dots, \varepsilon_{n+2}$ and those of $K \otimes \Lambda_{\xi'}$ by $\varepsilon'_1, \dots, \varepsilon'_{n+2}$. We further denote the primitve idempotents of $\Lambda_{\xi'}$ by e'_0, e'_1. Then we know that $\Phi(\{\varepsilon_1, \dots \varepsilon_4\}) \subseteq \{\varepsilon'_1, \dots, \varepsilon'_4\}$ since $\varphi(e_0) = e'_0$, $e_0\varepsilon_i \neq 0 \Leftrightarrow i \in \{1, \dots, 4\}$ and $e'_0\varepsilon'_i \neq 0 \Leftrightarrow i \in \{1, \dots, 4\}$. We can further see from the definition of the generators that for $i \in \{1, \dots, 4\}$ we have

$$\varepsilon_i e_0 \Lambda e_0 - \varepsilon_j e_0 \Lambda e_0 \subseteq 2^{n-2}\Gamma \Leftrightarrow (i, j) \in \{(1, 3), (2, 4)\}$$

and the same holds for ε'_i, e'_0. Since $e_1\varepsilon_i \neq 0$ if and only if $i \geq 3$ and $e'_1\varepsilon'_i \neq 0$ if and only if $i \geq 3$ this implies that $\Phi(\varepsilon_i) = \varepsilon'_i$ for $i \in \{1, \dots, 4\}$. With Lemma 3.2.9 we see that, using the notation from Remark 2.5.13, for all $\gamma \in e_0\Lambda e_0$ we have $\gamma_i = \phi(\gamma)_i$ for all $i \in \{1, \dots, 4\}$. Therefore the row space of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2^{n-2}\xi & 2 - 2^{n-2}\xi \\ 0 & 0 & 2^{n-1} & -2^{n-1} \\ 0 & 0 & 0 & 2^n \end{pmatrix}$$

is the same subspace of R^4 as the row space of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2^{n-2}\xi' & 2 - 2^{n-2}\xi' \\ 0 & 0 & 2^{n-1} & -2^{n-1} \\ 0 & 0 & 0 & 2^n \end{pmatrix},$$

so in particular

$$(0,0,2^{n-2}(\xi-\xi'),-2^{n-2}(\xi-\xi')) \in \langle (0,0,2^{n-1},-2^{n-1}),(0,0,0,2^n)\rangle_R$$

This is only the case if $\xi \equiv \xi' \bmod 2$. In that case the algebras Λ_ξ and $\Lambda_{\xi'}$ are the same subalgebra of Γ and thus of course isomorphic. □

Theorem 3.2.11. *Let B be a block with basic algebra Λ such that $F \otimes \Lambda \cong SD(2\mathcal{B})_1^c$. Then $c = 1$ and $F \otimes \Lambda$ has infinitely many non-isomorphic Λ-lifts.*

Proof. The rational conditions (3.1) were chosen in such a way that Λ-lifts are exactly the ones fulfilling them. Thus, this theorem summarises Theorems 3.2.8, 3.2.10 and 3.2.7. □

Lemma 3.2.12. *There is a tilting complex $\overline{T}$ of $SD(2\mathcal{A})_2^c$ of the form*

$$[0 \longrightarrow P_1 \oplus P_1 \longrightarrow P_0 \longrightarrow 0] \oplus [0 \longrightarrow P_1 \longrightarrow 0]$$

such that $\mathrm{End}(Q) \cong (SD(2\mathcal{B})_1^c)^{op}$.

Proof. See [Hol01, Proposition 1.3.2]. □

Lemma 3.2.13. *Let $\overline{T}$ be the tilting complex in Lemma 3.2.12 and X be a two-sided tilting complex whose inverse restricts to $\overline{T}$. Then Φ_X sends a lift satisfying the rational conditions (3.1) to a lift satisfying the rational conditions*

$\mathbb{Z}(A)$	u	P_0	P_1	
K	u_1	1	0	
K	u_1	1	0	
K	u_2	1	1	.
K	u_2	1	1	
K_{r+2}	u_3	2	1	*for* $r \in \{0,\dots,n-4\}$
K'_{n-1}	u_3	2	1	

(3.2)

Proof. This follows from Theorem 2.5.21, Remark 2.5.22 and the transfer of the self-duality by Theorem 2.5.23. □

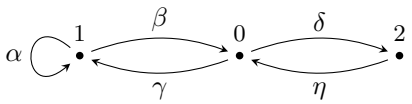
Theorem 3.2.14. *Let B be a block with basic algebra Λ such that $F \otimes \Lambda \cong SD(2\mathcal{A})_2^c$. Then $c = 1$ and $F \otimes \Lambda$ has infinitely many non-isomorphic Λ-lifts.*

Proof. With the previous lemma this follows directly from Theorem 3.2.11. □

Remark 3.2.15. *The centers of the algebras $\Lambda_\xi(\pi_0,\dots,\pi_{n-3})$ and $\Lambda_{\xi'}(\pi_0,\dots,\pi_{n-3})$ coincide and it is therefore not possible to identify the algebra isomorphic to a given block by its center.*

3.2.3 Blocks with three simple modules

Let $n \geq 3$ be fixed and let $\overline{\Lambda}$ be a basic algebra of type $SD(3\mathcal{B})_1$ given by the quiver



with the following relations:

$$\alpha\beta = \gamma\delta = \beta\gamma = 0, \quad \alpha^{2^{n-2}} = \beta\delta\eta\gamma, \quad \eta\delta\eta = \eta\gamma\beta, \quad \delta\eta\delta = \gamma\beta\delta.$$

We will only consider lifts fulfilling the following rational conditions.

$\mathrm{Z}(A)$	u	P_0	P_1	P_2	
K	u_1	1	0	0	
K	u_2	1	1	0	
K	u_3	1	0	1	
K	u_4	1	1	1	
K	u_5	0	0	1	
K_{r+2}	v_r	0	1	0	for $r \in \{1, \ldots, n-4\}$
K'_{n-1}	v_{n-3}	0	1	0	

(3.3)

where $u_1, \ldots, u_5, v_r \in \mathbb{Q}$, $\nu_2(u_i) = -n$ for $i \in \{1, \ldots, 4\}$, $\nu_2(u_5) = 2$ and $\nu_2(v_r) = -n+1$ for $r \in \{1, \ldots, n-3\}$.

Theorem 3.2.16. *Let Λ be a lift of $\overline{\Lambda}$ satisfying the rational conditions (3.3). Then Λ is isomorphic to the subalgebra of*

$$\Gamma := R \oplus R^{2\times 2} \oplus R^{2\times 2} \oplus R^{3\times 3} \oplus R \oplus \bigoplus_{d=0}^{n-4} R_{d+2} \oplus R'_{n-1}$$

generated by the elements

$$e_0 = (1, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

$$e_1 = (0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, 0, 1, \dots, 1),$$

$$e_2 = (0, 0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, 1, 0, \dots, 0),$$

$$\widetilde{\eta} = (0, 0, \begin{pmatrix}0&0\\2&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\2^{n-1}+2&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

$$\widetilde{\delta} = (0, 0, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

$$\widetilde{\gamma} = (0, \begin{pmatrix}0&2^{n-1}\\0&0\end{pmatrix}, 0, \begin{pmatrix}0&2^{n-1}&0\\0&0&0\\0&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

$$\widetilde{\beta} = (0, \begin{pmatrix}0&0\\1&0\end{pmatrix}, 0, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

where π_i is prime in K_{i+2} for $i \in \{0, \dots, n-4\}$, π_{n-3} is prime in K_{n-1}, $\widetilde{\alpha} \in e_1 \mathrm{Z}(\Lambda) e_1$ and $\widetilde{\alpha}$ generates $e_1 \Lambda e_1$.

Proof. By Lemma 2.1.32 we find an embedding $\Lambda \subseteq \Gamma$ where the primitive idempotents are embedded as

$$e_0 = (1, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, 0, 0, \dots, 0),$$

$$e_1 = (0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 0, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, 0, 1, \dots, 1) \text{ and}$$

$$e_2 = (0, 0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, 1, 0, \dots, 0).$$

First we consider $e_1 \Lambda e_1$. Since $\beta\gamma = 0$ and $\beta\delta\eta\gamma = \alpha^{2^{n-2}}$ we can see that α generates $F \otimes e_1 \Lambda e_1$. Thus the assumptions of Lemma 3.2.4 are fulfilled and there exists a lift $\hat{\alpha}$ of a scalar multiple of α in $e_1 \Lambda e_1$ of the form $(0, 4, \pi_0, \dots, \pi_{n-3})$, where π_r is prime in R_{r+2} for $i \in \{0, \dots, n-4\}$ and π_{n-1} is prime in R'_{n-1}.

Let $e_2\Lambda e_2$ be the row space of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2^a & 2^b\xi & 0 \\ 0 & 2^d & 0 \end{pmatrix}$$

with $\xi \in R^*$ and such that $(2^a, 2^b\xi, 0) = \hat{\eta}\hat{\delta}$ where $\hat{\eta}$ and $\hat{\delta}$ are lifts of scalar multiples of η and δ. We have $d = n$ since $\nu(u_4) = -n$. Furthermore $a + n = \frac{1}{2}(n + n + 2) = n + 1$ holds and thus $a = 1$. By the self-duality of Λ we obtain

$$\mathrm{T}_u(e_2, \hat{\eta}\hat{\delta}) = u_3 \cdot 2 + u_4 \cdot 2^b\xi \in R \Rightarrow 2^n \mid 2 + \frac{u_4}{u_3} \cdot 2^b\xi$$

We can further deduce that $b = 1$, as otherwise $\nu_2(2 + \frac{u_4}{u_3} \cdot 2^b\xi) \leq 1 < n$. This implies $-2 \equiv \frac{u_4}{u_3} \cdot 2\xi \bmod 2^n$ and thus $e_2\Lambda e_2$ is the row space of the matrix

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2\frac{u_3}{u_4} & 0 \\ 0 & 2^n & 0 \end{pmatrix}.$$

Note that

$$(\hat{\eta}\hat{\delta})^2 - 2\hat{\eta}\hat{\delta} = (0, 4(\frac{u_3^2}{u_4^2} + \frac{u_3}{u_4}), 0) \in e_2\Lambda e_2$$

and therefore $\frac{u_3}{u_4} \equiv -1 \bmod 2^{n-2}$. We define $\widetilde{u_i} = \frac{u_i}{2^n}$ for $i \in \{1, \ldots, 4\}$. Then we know that

$$\mathrm{T}_u(e_2, e_2) = u_3 + u_4 + u_5 = \frac{1}{2^n}(\widetilde{u_3} + \widetilde{u_4}) + \frac{1}{4}\widetilde{u_5} \in R$$

so $\nu_2(\widetilde{u_3} + \widetilde{u_4}) = n - 2$.
Since u_3 and u_4 are rational numbers, we know that either $\frac{u_3}{u_4} \equiv -1 \bmod 2^{n-1}$ or $\frac{u_3}{u_4} \equiv -2^{n-2} - 1 \bmod 2^{n-1}$. The case $\frac{u_3}{u_4} \equiv -1 \mod 2^{n-1}$ is impossible since $\nu_2(\widetilde{u_3} + \widetilde{u_4}) = n - 2$ and thus $2^{n-1} \nmid \widetilde{u_3} + \widetilde{u_4}$. Therefore we know that $\frac{u_3}{u_4} \equiv -2^{n-2} - 1 \bmod 2^{n-1}$ and $e_2\Lambda e_2$ is the row space of

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2^{n-1} + 2 & 0 \\ 0 & 2^n & 0 \end{pmatrix}.$$

Now let $e_0\Lambda e_0$ be the row space of the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^a & 0 & 2^b\xi \\ 0 & 0 & 2^c & 2^d\vartheta \\ 0 & 0 & 0 & 2^n \end{pmatrix}$$

where $\xi, \vartheta \in R^*$, $(0,0,2^c,2^d\vartheta) = \hat{\delta}\hat{\eta}$ and $(0,2^a,0,2^b\xi) = \hat{\gamma}\hat{\beta}$ where $\hat{\gamma}$ and $\hat{\beta}$ are unit scalar multiples of lifts of γ and β respectively. Then by our previous calculation and Lemma 2.5.11 we have $c = d = 1$. By Lemma 2.5.14 we have $a = \frac{1}{2}(4n) - n - 1 = n - 1$, and as before we can show that $e_0\Lambda e_0$ is the row space of the matrix

$$\tilde{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^{n-1} & 0 & 2^b\xi \\ 0 & 0 & 2 & 2^{n-1}+2 \\ 0 & 0 & 0 & 2^n \end{pmatrix}.$$

By self-duality and since u_4 and u_2 have the same 2-valuation, we have $2^n | 2^{n-1} + \frac{u_4}{u_2} \cdot 2^b\xi$ and thus $b = n - 1$. Therefore $e_0\Lambda e_0$ is the row space of the matrix

$$\hat{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^{n-1} & 0 & 2^{n-1}\xi \\ 0 & 0 & 2 & 2^{n-1}+2 \\ 0 & 0 & 0 & 2^n \end{pmatrix}.$$

Since $2^n | 2^{n-1} + 2^{n-1}\xi\frac{u_4}{u_3}$ we can without loss assume that $\xi \in R^* \cap \mathbb{Q}$ therefore $\xi \equiv 1 \mod 2$. Thus we can without loss assume that

$$\hat{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2^{n-1} & 0 & 2^{n-1} \\ 0 & 0 & 2 & 2^{n-1}+2 \\ 0 & 0 & 0 & 2^n \end{pmatrix}$$

There is an element $c \in K \otimes \Gamma$ such that

$$\hat{\eta}^c = (0,0,\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\vartheta & 0 & 0 \end{pmatrix}, 0, 0, \ldots, 0)$$

$$\hat{\delta}^c = (0,0,\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0, \ldots, 0)$$

$$\hat{\gamma}^c = (0,\begin{pmatrix} 0 & 2^{n-1} \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 2^{n-1}\xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0, \ldots, 0)$$

$$\hat{\beta}^c = (0,\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, 0, \ldots, 0)$$

for some units ξ and ϑ. Therefore we will from now on assume that $\hat{\eta}, \hat{\delta}, \hat{\gamma}$ and $\hat{\beta}$ are as above. We have seen that by the self-duality $2\vartheta \equiv 2^{n-1} + 2 \mod 2^n$ and $2^{n-1}\xi \equiv 2^{n-1}$

mod 2^n. Furthermore we have

$$\hat{\eta}\hat{\gamma}\hat{\beta} = (0,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2^n\xi\vartheta & 0 & 0 \end{pmatrix},0,0,\ldots,0),$$

$$\hat{\delta}\hat{\eta}\hat{\gamma} = (0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},0,\begin{pmatrix} 0 & 2^n\xi\vartheta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},0,0,\ldots,0) \in \mathrm{rad}(\Lambda)^2,$$

Since adding elements of $\mathrm{rad}(\Lambda)^2$ does not change the property of being a generating system we can conclude that Λ is generated by $\{e_0, e_1, e_2, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}, \widetilde{\eta}\}$.

All that remains to show is that $\hat{\alpha} \in e_1\,\mathrm{Z}(\Lambda)e_1$. This is true since the element

$$(0,0,\begin{pmatrix} -4\frac{u_4}{u_3} & 0 \\ 0 & -4\frac{u_4}{u_3} \end{pmatrix},\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},0,\pi_0,\ldots,pi_{n-3})$$

lies in the center of Λ. □

Definition 3.2.17. *We say that two lifts Λ and Λ' are* centrally equal *if there is an isomorphism $\mathrm{Z}(\Lambda) \cong \mathrm{Z}(\Lambda')$ such that the induced isomorphism between $\mathrm{Z}(K \otimes_R \Lambda)$ and $\mathrm{Z}(K \otimes_R \Lambda')$ only relates summands of the center corresponding to equal rows in the decomposition matrix.*

Theorem 3.2.18. *Let Λ and Λ' be centrally equal lifts of $\overline{\Lambda}$ satisfying the rational conditions (3.3). Then $\Lambda \cong \Lambda'$.*

Proof. Both Λ and Λ' are isomorphic to algebras as in Theorem 3.2.16. Since they are centrally equal we can assume that the elements π_i are the same in both cases. □

Next we will again use derived equivalences to extend our result to all blocks of semidihedral type with three simple modules. The tilting complexes inducing those derived equivalences in the following theorem were determined by Holm [Hol01].

Theorem 3.2.19. *1. There is a tilting complex of the form*

$$T := [0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow 0] + [0 \longrightarrow P_1 \oplus P_1 \longrightarrow P_0 \longrightarrow 0]$$
$$+[0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{A})_1)$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{B})_1$.

2. There is a tilting complex of the form

$$T := [0 \longrightarrow 0 \longrightarrow P_2 \longrightarrow 0] + [0 \longrightarrow P_2 \oplus P_2 \longrightarrow P_0 \longrightarrow 0]$$
$$+[0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{A})_1)$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{B})_2$.

3. *There is a tilting complex of the form*

$$T := [0 \longrightarrow 0 \longrightarrow P_2 \longrightarrow 0] + [0 \longrightarrow 0 \longrightarrow P_0 \longrightarrow 0] \\ +[0 \longrightarrow P_0 \oplus P_2 \longrightarrow P_1 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{H})_1)$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{D})_2$.

4. *There is a tilting complex of the form*

$$T := [0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow 0] + [0 \longrightarrow 0 \longrightarrow P_0 \longrightarrow 0] \\ +[0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{H}))$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{C})_{2,I}$.

5. *There is a tilting complex of the form*

$$T := [0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow 0] + [0 \longrightarrow 0 \longrightarrow P_0 \longrightarrow 0] \\ +[0 \longrightarrow P_0 \longrightarrow P_2 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{C}_{2,II}))$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{H})$.

6. *There is a tilting complex of the form*

$$T := [0 \longrightarrow 0 \longrightarrow P_2 \longrightarrow 0] + [0 \longrightarrow P_2 \oplus P_2 \longrightarrow P_0 \longrightarrow 0] \\ +[0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0] \in \mathcal{K}^b(SD(3\mathcal{B}_2))$$

with $\mathrm{End}(T)^{op} \cong SD(3\mathcal{D})$.

Proof. See [Hol01, Proposition 1.3.3–1.3.8]. □

For the following discussion we fix the following notation. We say that an order Λ satisfies the rational conditions

$\mathrm{Z}(A)$	$\nu_2(u)$	1	…	l
K_1	n_1	d_{11}	…	d_{1l}
⋮	⋮	⋮	⋮	⋮
K_k	n_k	d_{k1}	…	d_{kl}

if there is an element

$$u = (u_i)_{i \in \{1,\dots,k\}} \in \bigoplus_{i=1}^{k} \mathbb{Q} \subseteq \mathrm{Z}(K \otimes \Lambda) = \bigoplus_{i=1}^{k} K_i$$

with $\nu_2(u_i) = n_i$ such that Λ satisfies the rational conditions

$\mathrm{Z}(A)$	u	1	…	l
K_1	u_1	d_{11}	…	d_{1l}
⋮	⋮	⋮	⋮	⋮
K_k	u_k	d_{k1}	…	d_{kl}

.

Theorem 3.2.20. *1. Let Λ and Λ' be centrally equal lifts of $SD(3A)_1$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.*

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	1	0	0	
K	$-n$	1	1	1	
K	$-n$	1	0	1	
K	$-n$	1	1	0	
K	2	0	0	1	
K_{r+2}	$-n+1$	2	1	1	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	2	1	1	

2. Let Λ and Λ' be centrally equal lifts of $SD(3B)_2$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	1	0	0	
K	$-n$	1	1	0	
K	$-n$	1	1	1	
K	$-n$	1	0	1	
K	2	2	1	1	
K_{r+2}	$-n+1$	0	1	0	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	0	1	0	

3. Let Λ and Λ' be centrally equal lifts of $SD(3C)_I$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	1	1	0	
K	$-n$	0	1	0	
K	$-n$	0	0	1	
K	$-n$	1	0	1	
K	2	1	1	1	
K_{r+2}	$-n+1$	1	0	0	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	1	0	0	

4. Let Λ and Λ' be centrally equal lifts of $SD(3C)_{II}$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	1	0	1	
K	$-n$	0	1	0	
K	$-n$	0	0	1	
K	$-n$	1	1	0	
K	2	1	0	0	
K_{r+2}	$-n+1$	1	1	1	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	1	1	1	

5. *Let Λ and Λ' be centrally equal lifts of $SD(3D)$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.*

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	1	0	0	
K	$-n$	1	1	0	
K	$-n$	1	0	1	
K	$-n$	1	1	1	
K	2	0	0	1	
K_{r+2}	$-n+1$	0	1	0	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	0	1	0	

6. *Let Λ and Λ' be centrally equal lifts of $SD(3H)$ satisfying the rational conditions below. Then $\Lambda \cong \Lambda'$.*

$\mathrm{Z}(A)$	$\nu_2(u)$	P_0	P_1	P_2	
K	$-n$	0	1	0	
K	$-n$	0	0	1	
K	$-n$	1	0	0	
K	$-n$	1	1	1	
K	2	1	1	0	
K_{r+2}	$-n+1$	0	1	1	*for* $r \in \{1, \ldots, n-4\}$
K'_{n-1}	$-n+1$	0	1	1	

Proof. The decomposition matrices in our table at the start of this chapter are given in such a way that the first four rows always correspond to the height zero characters, the fifth row to the character of height $n-2$ and the last row to the height 1 characters [Erd90a, Lemmas (11.4),(11.6),(11.11),(11.9)].

Let T be a tilting complex of the form as in Theorem 3.2.19 in $\mathcal{K}^b(A)$ with $B \cong \mathrm{End}(T)$ and X a two-sided tilting complex whose inverse restricts to T. Then straightforward calculations using the instructions from Remark 2.5.22 show that Φ_X will send a lift fulfilling the rational conditions given in the assertion for A to a lift fulfilling the rational conditions given for B. Therefore, if there is only one lift fulfilling the conditions for A there can also only be one for B and the other way round. Together with Theorem 3.2.18 the assertion follows. □

We summarise our result by the following theorem.

Theorem 3.2.21. *Let Λ be the basic algebra of a block of a group algebra over R with a semidihedral defect group and three simple modules. If Λ_1 and Λ_2 are two centrally equal Λ-lifts of $\overline{\Lambda}$, then $\Lambda_1 \cong \Lambda_2$.*

4 Defect 3 blocks of symmetric groups

Blocks, and more generally the representation theory, of symmetric groups, have been a popular subject of research. Scopes showed that Donovan's famous conjecture is true for blocks of symmetric groups [Sco91]. The conjecture states that for every isomorphism type of defect groups there are only finitely many Morita-equivalence classes of blocks with that defect group.

A special case are p-blocks with a defect $d < p$. All of those block have isomorphic defect groups [JK81, Theorem 6.2.45]. Chuang and Rouquier further proved the following theorem [CR08, Theorem 7.2].

Theorem 4.0.1. *Let R be a field of characteristic p or $\mathbb{Z}_p$. Let A and B be two blocks of symmetric groups over R with isomorphic defect groups. Then, A and B are derived equivalent.*

So we see that all p-blocks of defect $d < p$ of symmetric groups are derived equivalent. It is therefore natural to study blocks with a small defect, especially a defect $d < p$. Blocks of defect 0 and 1 have been covered by the more general research on blocks with cyclic defect. In particular, the structure of basic algebras of blocks with a cyclic defect group over discrete valuation rings has been described by Plesken [Ple83]. The basic algebras of defect 2 blocks over complete discrete valuation rings have been described by Eisele [Eis12], generalising a result by Nebe [Neb02] where the basic order of the principal block of $\mathbb{Z}_p S_{2p}$ was determined.

We want to consider blocks with defect 3 for primes $p > 3$. We will not give a description of all such blocks and in fact not study any block of a symmetric group directly, but instead use the following result [CK02, Theorem 2].

Theorem 4.0.2. *Given any $d < p$ there exist blocks of defect d of symmetric groups which are Morita equivalent to the principal block of $R(S_p \wr S_d)$.*

Therefore, we will use results about wreath products of algebras developed by Chuang and Tan [CT03] to investigate the basic algebra of the principal block B_0 of $R(S_p \wr S_3)$. We will first repeat important results from [CT03], then use them to determine the decomposition numbers and the quiver of $\mathbb{F}_p \otimes B_0$ and subsequently use similar methods as in Chapter 3 to lift the basic algebra of $\mathbb{F}_p \otimes B_0$.

4.1 The representation theory of wreath products

We recall the most important results and notation from [CT03].

In this section let R be either a field or a discrete valuation ring and let w be a positive integer such that $w!$ is invertible in R. Let A be a finite-dimensional R-algebra.

Definition 4.1.1. *Let G be a group and $H \leq S_n$ a subgroup of the symmetric group. Then the* wreath product *of G and H is defined as*

$$G \wr H := G^n \times H$$

with multiplication

$$(g_1, \ldots, g_n; \sigma)(h_1, \ldots, h_n; \mu) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}; \sigma\mu)$$

We have an inclusion of H into $G \wr H$ as follows.

$$\begin{aligned} \iota : H &\longrightarrow G \wr H \\ \sigma &\mapsto \hat{\sigma} := (1, \ldots, 1; \sigma) \end{aligned}$$

Definition 4.1.2. *1. Let n be a positive integer. We call a sequence of integers $\underline{n} = (n_1, n_2, \ldots)$ a* composition *of n, written $\underline{n} \vDash n$, if $\sum_{i \in \mathbb{Z}_{>0}} n_i = n$. We also write $\underline{n} = (n_1, \ldots, n_l)$ if $n_i = 0$ for $i > l$. We call a composition a* partition *if it is non-increasing and write $\underline{n} \vdash n$. Given a composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we define $|\lambda| := \sum_{i=1}^{l} \lambda_i$.*

2. Let B be a ring, V a B-module and n a positive integer. Then we let $T^n(V)$ denote the n-fold tensor product $\bigotimes^n V$ of V.

3. We define the algebra $A(w)$ as $T^w(A) \otimes RS_w$ with the following multiplication.

$$(a_1 \otimes \ldots \otimes a_w \otimes \sigma)(b_1 \otimes \ldots \otimes b_w \otimes \rho) = a_1 b_{\sigma^{-1}(1)} \otimes \ldots \otimes a_w b_{\sigma^{-1}(w)} \otimes \sigma\rho$$

4. Let $\underline{w} = (w_1, \ldots, w_l) \vDash w$. We define

$$S_{\underline{w}} = S_{w_1} \times \ldots \times S_{w_l}$$

and consider it as a subgroup of S_w by letting the factor S_{w_r} act on

$$\left\{ \sum_{i=1}^{r-1} w_i + 1, \ldots, \sum_{i=1}^{r} w_i \right\} \subseteq \{1, \ldots, w\}$$

Then we define the algebra $A(\underline{w})$ as the subalgebra $T^w(A) \otimes RS_{\underline{w}}$ of $A(w)$. Then there is an isomorphism

$$A(\underline{w}) \cong A(w_1) \otimes \ldots \otimes A(w_l).$$

If V is an $A(\underline{w})$-module and W an $A(w)$-module we use the following short hand notation for induction and restriction.

$$\begin{aligned} \mathrm{Ind}_{\underline{w}}^{w}(V) &:= \mathrm{Ind}_{A(\underline{w})}^{A(w)}(V) \\ \mathrm{Res}_{\underline{w}}^{w}(W) &:= \mathrm{Res}_{A(\underline{w})}^{A(w)}(W) \end{aligned}$$

For every field K and partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n one can define a KS_n-module S^λ, the so called *Specht module*. We will skip the definition and refer the reader to [JK81]. If K is a field of characteristic 0 the Specht module S^λ is simple for every $\lambda \vdash n$. If $\mathrm{char}(K) = p > 0$, then S^λ is simple if and only if λ is *p-regular*, i.e. no number $k < n$ appears p or more times in λ.

Example 4.1.3. *1. For $n = 2$ there are two Specht modules, the trivial module $S^{(2)}$ and the signum representation $S^{(1,1)}$.*

2. For $n = 3$ there are three Specht modules, the trivial module $S^{(3)}$, the signum representation $S^{(1,1,1)}$ and the standard representation $S^{(2,1)}$.

Remark 4.1.4. *If A is the group algebra of a group G then $A(w)$ is the group algebra of $G \wr S_w$.*

Lemma 4.1.5. *1. If V is an A-module then $T^w(V)$ becomes an $A(w)$-module by letting $T^w(A)$ act component wise and letting RS_w permute the components as follows.*

$$v_1 \otimes \ldots \otimes v_n.(1 \otimes \sigma) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \quad \forall v_1, \ldots, v_n \in A, \sigma \in S_w$$

We denote this module by $T^{(w)}(V)$.

2. If V is an $A(w)$-module and S is an RS_w-module then we can consider $V \otimes S$ as an $A(w)$-module with the following action.

$$(v \otimes s).(a \otimes \sigma) = v.(a \otimes \sigma) \otimes s.\sigma \quad \forall v \in V, s \in S, a \in T^w(A), \sigma \in S_w$$

We will denote this module by $V \oslash S$

3. Let now $\underline{w} \vDash w$. Similar to 2, for any $A(\underline{w})$-module V and any $RS_{\underline{w}}$-module S, $V \otimes S$ becomes an $A(\underline{w})$-module which we will also denote by $V \oslash S$.

Proof. This can be proven by straightforward calculations. □

Definition 4.1.6. *Let $\lambda \vdash w$ be a partition and S^λ be the corresponding Specht module over R. Then we define the $A(w)$-module $T^\lambda(V) := T^{(w)}(M) \oslash S^\lambda$.*

Definition 4.1.7. *For every index set I we define*

$$\Lambda_w^I := \{\underline{\lambda} = (\lambda_i)_{i \in I} \mid \lambda_i \text{ partition}, \sum_{i \in I} |\lambda_i| = w\}.$$

Definition 4.1.8. *Let $\{V(i) \mid i \in I\}$ be a set of A-modules and $\underline{\lambda} = (\lambda_i)_{i \in I} \in \Lambda_w^I$. Then we obtain an $A(w)$-module by the following construction:*

$$V(\underline{\lambda}) := \mathrm{Ind}^w_{(|\lambda_i|)_{i \in I}} \left(\bigotimes_{i \in I} T_i^\lambda(V(i)) \right)$$

Note that this definition depends on the set $\{V(i) \mid i \in I\}$.

The following theorems show that both the simple and the projective indecomposable modules of $A(w)$ can be constructed in the way above.

Theorem 4.1.9. *Suppose that R is a splitting field for A and let $\{V(i) \mid i \in I\}$ be a complete set of representatives of isomorphism classes of simple A-modules. Then R is a splitting field for $A(w)$ and $\{V(\underline{\lambda}) \mid \underline{\lambda} \in \Lambda_w^I\}$ is a complete set of representatives of isomorphism classes of simple $A(w)$-modules.*

Proof. See [CT03, Lemma 3.8]. □

Theorem 4.1.10. *Let $\{P(i) \mid i \in I\}$ be a set of projective A-modules and let $\underline{\lambda} \in \Lambda_w^I$. Then $P(\underline{\lambda})$ is projective.*

Proof. See [CT03, Corollary 3.9]. □

Lemma 4.1.11. *Suppose that R is a splitting field for A and let $\{M(i) \mid i \in I\}$, $\{N(i) \mid i \in I\}$ be sets of A-modules such that each $M(i)$ has simple head $N(i)$ and $N(i) \ncong N(j)$ for $i \neq j$. Then $M(\underline{\lambda})$ has simple head $N(\underline{\lambda})$ for any $\underline{\lambda} \in \Lambda_w^I$.*

Proof. See [CT03, Lemma 4.5]. □

Lemma 4.1.12. *Let $\underline{w} = (w_1, \dots, w_r) \vDash w$.*

1. *For $i \in \{1, \dots, r\}$ let V_i be an $A(w_i)$-module and X_i an RS_w-module. Then*

$$(V_1 \oslash X_1) \otimes \dots \otimes (V_r \oslash X_r) \cong (V_1 \otimes \dots \otimes V_r) \oslash (X_1 \otimes \dots \otimes X_r)$$

2. *Let V be an $A(w)$-module and Y an $RS_{\underline{w}}$-module. Then*

$$V \oslash (\mathrm{Ind}_{\underline{w}}^{w} Y) \cong \mathrm{Ind}_{A(\underline{w})}^{A(w)}((\mathrm{Res}_{A(\underline{w})}^{A(w)} V) \oslash Y).$$

3. *Let W be an $A(\underline{w})$-module and X an RS_w-module. Then*

$$(\mathrm{Ind}_{A(\underline{w})}^{A(w)} W) \oslash X \cong \mathrm{Ind}_{A(\underline{w})}^{A(w)}(W \oslash \mathrm{Res}_{\underline{w}}^{w} X)).$$

Proof. See [CT03, Lemma 3.2]. □

Lemma 4.1.13. *Suppose that R is a splitting field for A.*

1. *We have*

$$\mathrm{rad}(T^w(A)) = \sum_{i=0}^{w-1} T^i(A) \otimes \mathrm{rad}(A) \otimes T^{w-i-1}(A).$$

2. *Let V be an $A(w)$-module and $n \in \mathbb{Z}^{>0}$. Then*

$$\mathrm{rad}^n(V) = \mathrm{rad}^n(\mathrm{Res}_{T^w(A)}^{A(w)}(V))$$

3. Let V be an $A(w)$-module, X an RS_w-module and $n \in \mathbb{Z}^{>0}$. Then

$$\mathrm{rad}^n(V \oslash X) = \mathrm{rad}^n(V) \oslash X$$

Proof. See [CT03, Lemma 3.4, 3.5]. □

Definition 4.1.14. *Let $n \in \mathbb{Z}_{>0}$ and $\lambda \vdash n$. Let further $\underline{\mu} = (\mu_i)_{i\in I}$ be a sequence of partitions with $\sum_{i\in I} |\mu_i| = n$ and $\underline{n} := (|\mu_i|)_{i\in I}$. Then we define*

$$c(\lambda; \underline{\mu}) := [\mathrm{Res}^n_{\underline{n}}(S^\lambda) : \bigotimes_{i\in I} S^{\mu_i}].$$

Further we define $c((0); ((0), \ldots, (0))) = 1$ and $c(\lambda; \underline{\mu'}) = 0$ if $\sum_{i\in I} |\mu_i| \neq n$.

Definition 4.1.15. *Let $\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots)$ be two partitions. We define the distance between λ and μ as*

$$d(\lambda, \mu) = \sum_{i\in\mathbb{Z}} |\lambda_i - \mu_i|.$$

The following theorem determines the quiver of $A(w)$.

Theorem 4.1.16. *Let $R = k$ be a splitting field for a, let $\{S(i) \mid i \in I\}$ be the simple A-modules and $\{P(i) \mid i \in I\}$ their projective covers.*

1. *For every $\underline{\lambda} \in \Lambda^I_w$ the module $P(\underline{\lambda})$ is the projective cover of $L(\underline{\lambda})$.*
2. *Let $\underline{\lambda} = (\lambda_i)_{i\in I} \neq \underline{\mu} = (\mu_i)_{i\in I} \in \Lambda^I_w$.*
   a) We have

$$\dim_k \mathrm{Hom}_{A(w)}(P(\underline{\lambda}), \mathrm{rad}(P(\underline{\lambda}))/\mathrm{rad}^2(P(\underline{\lambda}))) = \sum_{i\in I} p(\lambda_i) \dim_k \mathrm{Hom}_A(P(i), \mathrm{rad}(P(i))/\mathrm{rad}^2(P(i)))$$

   where $p(\lambda_i)$ is the number of distinct parts of λ_i.

   b) We have

$$\mathrm{Hom}_{A(w)}(P(\underline{\lambda}), \mathrm{rad}(P(\underline{\mu}))/\mathrm{rad}^2(P(\underline{\mu}))) = 0$$

   unless either

   i. there exists $j \in I$ such that $\lambda_i = \mu_i$ for all $i \in I, i \neq j$,

   ii. we have $d(\lambda_j, \mu_j) = 2$ and

   iii. $\mathrm{Hom}_A(P(j), \mathrm{rad}(P(j))/\mathrm{rad}^2(P(j))) \neq 0$,

   in which case

$$\dim_k \mathrm{Hom}_{A(w)}(P(\underline{\lambda}), \mathrm{rad}(P(\underline{\mu}))/\mathrm{rad}^2(P(\underline{\mu}))) = \dim_k \mathrm{Hom}_A(P(j), \mathrm{rad}(P(j))/\mathrm{rad}^2(P(j)))$$

   or

i. *there exist* $j, j' \in I, j \neq j'$ *such that* $\lambda_i = \mu_i$ *for all* $i \in I, i \neq j, i \neq j'$,

ii. $|\mu_j| > |\lambda_j|$ *and* $d(\mu_j, \lambda_j) = 1$

iii. $|\mu_{j'}| > |\lambda_{j'}|$ *and* $d(\mu_{j'}, \lambda_{j'}) = 1$

in which case

$$\dim_k \mathrm{Hom}_{A(w)}(P(\underline{\lambda}), \mathrm{rad}(P(\underline{\mu}))/\mathrm{rad}^2(P(\underline{\mu}))) = \dim_k \mathrm{Hom}_A(P(j), \mathrm{rad}(P(j'))/\mathrm{rad}^2(P(j')))$$

Proof. See [CT03, Proposition 4.6]. □

4.2 The principal block of $\mathbb{Z}_p S_p$

Recall that the simple modules of the symmetric group over any field are enumerated by partitions. The defect one characters of the symmetric group S_p correspond to the following partitions [JK81, 6.3.9]:

$$(p), (p-1, 1), (p-2, 1^2), \ldots, (2, 1^{p-2}), (1^p).$$

They are all in the same block, the principal block, of S_p. Furthermore, all but the last partition are p-regular and thus correspond to irreducible modular representations. The Brauer graph of the principal block has the following form:

Note that the form of this graph as a single path follows from Theorem 2.2.27 since all characters of S_p have values in $\mathbb{Q}$ and therefore every p-conjugacy class contains just one element. It also follows from the theorem that all decomposition numbers are zero or one.

Notation 4.2.1. *We will denote the simple* $\mathbb{Q}_p S_p$ *module corresponding to the partition* $(p-i, 1^i)$ *by* $V(i)$ *and the simple and projective indecomposable, module of* $\mathbb{Z}_p S_p$ *corresponding to* $(p-i, 1^i)$ *by* $S(i)$ *and* $P(i)$ *respectively. Furthermore, we define* $\overline{M} := \mathbb{F}_p \otimes M$ *for any* $\mathbb{Z}_p$*-module* M, $\overline{S}(i) := \mathbb{F}_p \otimes S(i)$ *and* $\overline{P}(i) := \mathbb{F}_p \otimes P(i)$.

Lemma 4.2.2. *With the notation above, we obtain the following decompositions:*

$$\mathbb{Q}_p \otimes P(k) \cong V(k) \oplus V(k+1) \tag{4.1}$$

$$\mathrm{rad}(\overline{P}(k))/\mathrm{rad}^2(\overline{P}(k)) \cong \overline{S}(k+1) \oplus \overline{S}(k-1) \text{ if } 1 < k < p-1 \tag{4.2}$$

$$\mathrm{rad}(\overline{P}(1))/\mathrm{rad}^2(\overline{P}(1)) \cong \overline{S}(2) \tag{4.3}$$

$$\mathrm{rad}(\overline{P}(p-1))/\mathrm{rad}^2(\overline{P}(1)) \cong \overline{S}(p-2) \tag{4.4}$$

$$\mathrm{rad}^2(\overline{P}(k)) \cong \overline{S}(k) \tag{4.5}$$

The ext-quiver of $\mathbb{F}_p S_p$ *is*

with relations

$$\beta_i \circ \beta_{i+1} = 0, \alpha_{i+1} \circ \alpha_i = 0, \beta_{i+1} \circ \alpha_i \circ \beta_{i+1} = 0, \alpha_i \circ \beta_{i+1} \circ \alpha_i = 0, \alpha_{i-1} \circ \beta_i = \beta_{i+1} \circ \alpha_i$$

Proof. See [EM94, Section 4.1, Lemma 4.5]). □

4.3 The principal block of $\mathbb{F}_p(S_p \wr S_3)$

The fields $\mathbb{Q}_p$ and $\mathbb{F}_p$ are splitting fields for every symmetric group and wreath product $S_m \wr S_n$ [JK81] and we will therefore fix the p-modular system $(K, R, F) = (\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$.

Remark 4.3.1. *With the notation from Section 4.1 we have that* $B_0(RS_p(3)) = B_0(RS_p)(3)$.

Proof. We consider $B_0(RS_p)(3)$ as a subalgebra of $RS_p(3)$. If S is the trivial RS_p-module, then the trivial $RS_p(3)$-module is the threefold tensor product $T^{(3)}(S)$ of S, so in particular it lies in $B_0(RS_p)(3)$. In Corollary 4.3.15 we will further see that the Brauer graph of $B_0(RS_p)(3)$ is a connected component of the Brauer graph of $RS_p(3)$ and therefore $B_0(RS_p)(3)$ is a block by Lemma 2.2.18. □

4.3.1 Simple and projective indecomposable modules

Let A an R-algebra. We introduce new notation for $A(3)$-modules.

Definition 4.3.2. *Let* I *be an index set and* $x = (i, j, k) \in I^3$.

1. *We define the* type *of* x *to be* $\mathrm{type}(x) = 4 - |\{i, j, k\}|$, *i.e.* $\mathrm{type}(x) = n \in \{1, 2, 3\}$ *if there are* n *occurrences of the same index in* (i, j, k) *and no index occurs more than* n *times.*
2. *Let* $\lambda \vdash \mathrm{type}((i, j, k))$. *Then we define* $\mathrm{type}((i, j, k; \lambda)) = \mathrm{type}((i, j, k))$.

Definition 4.3.3. *Let* $\{M(i) \mid i \in I\}$ *be a set of* A*-modules,* $i, j, k \in I$, $|\{i, j, k\}| = 3$, $\mu \vdash 2$ *and* $\lambda \vdash 3$.

$$M(i,j,k) := M(i,j,k;(1)) := \mathrm{Ind}_{(1,1,1)}^{(3)}(M(i) \otimes M(j) \otimes M(k))$$
$$M(i,i,j;\mu) := M(i,j,i;\mu) := M(j,i,i;\mu) := \mathrm{Ind}_{(2,1)}^{(3)}((T^{(2)}(M(i)) \oslash S^{\mu}) \otimes M(j))$$
$$M(i,i,i;\lambda) := T^{(3)}(M(i)) \oslash S^{\lambda}$$

For any $i, j, k \in I$ *and* $\lambda \vdash \mathrm{type}((i, j, k))$ *we define*

$$M_{(i,j,k;\lambda)} := M(i, j, k; \lambda).$$

Lemma 4.3.4. *Let $\{M(i) \mid i \in I\}$ be a set of A-modules, $i,j,k \in I$ and let λ be a partition with $|\lambda| = \mathrm{type}((i,j,k))$. Then the dimension of $M(i,j,k;\lambda)$ can be calculated as follows:*

$$C(\lambda) = \begin{cases} 6 & \text{if } \lambda = (1) \\ 3 & \text{if } \lambda \vdash 2 \\ 2 & \text{if } \lambda = (2,1) \\ 1 & \text{otherwise} \end{cases}$$

$$\dim(M(i,j,k;\lambda)) = \dim(M(i))\dim(M(j))\dim(M(k)) \cdot \dim(S^\lambda) \cdot C(\lambda).$$

Proof. We can see that the factor $C(\lambda)$ is the index of the subalgebra from which the module $M(i,j,k;\lambda)$ is induced. Therefore the formula follows directly from the definition. □

Definition 4.3.5. *We define two index sets:*

$$I_0 = \{(i,j,k;\lambda) \mid i,j,k \in \{1,\dots,p\}, i \le j \le k, \lambda \vdash \mathrm{type}((i,j,k))\}$$
$$I_p = \{(i,j,k;\lambda) \mid i,j,k \in \{1,\dots,p-1\}, i \le j \le k, \lambda \vdash \mathrm{type}((i,j,k))\}.$$

We will also use the short hand notation $ijk\lambda$ for $(i,j,k;\lambda)$.

Theorem 4.3.6. *With the notation from 4.2.1 the simple $B_0(LS_p)(3)$ modules are as follows.*

1. *$L = K$*

$$\{V_a \mid a \in I_0\}$$

2. *$L = F$*

$$\{\overline{S}_x \mid x \in I_p\}$$

Proof. This follows directly from Theorem 4.1.9. □

Remark 4.3.7. *With Theorem 4.1.10 and Lemma 4.1.11 we can see that for $x \in I_p$ the projective cover of $\overline{S}_x$ is $\overline{P}_x$.*

Lemma 4.3.8. *The constructions from Section 4.1 are all compatible with a change of scalars, e.g. $\overline{S}(i,j,k) = k \otimes S(i,j,k)$. Thus it follows that the simple/ projective $\Lambda := B_0(\mathbb{Z}_p S_p)(3)$ modules are the following.*

1. *The simple Λ-modules are*

$$\{S_x \mid x \in I_p\}$$

2. *and the projective indecomposable modules of Λ are*

$$\{P_x \mid x \in I_p\}.$$

Definition 4.3.9. *Let $a = (i, j, k; \lambda), b = (i', j', k'; \lambda') \in I_0$.*

1. *We define the length of a as*

$$l(a) := i + j + k$$

2. *We define a partial order $\leq$ on I_0 as follows.*

$$a \leq b \Leftrightarrow l(a) \leq l(b)$$

4.3.2 Decomposition numbers

Recall that $\mathbb{Q} \otimes P(i) = V(i) \oplus V(i+1)$ for every $i \in \{1 \dots p-1\}$. Thus we obtain the following decompositions for the tensor products.

$$K \otimes P(i) \otimes P(j) \otimes P(k) = \bigoplus_{\substack{i' \in \{i, i+1\} \\ j' \in \{j, j+1\} \\ k' \in \{k, k+1\}}} V(i') \otimes V(j') \otimes V(k')$$

Remark 4.3.10. *If $w \vDash 3$ is a composition, T is a right transversal of $S_{\underline{w}}$ in S_3 and A an R-algebra where R is a discrete valuation ring or a field, then for a $A(\underline{w})$-module M every element of $\mathrm{Ind}_{\underline{w}}^{(3)} M$ is of the form $m \otimes \hat{t}$ with $m \in M$ and $t \in T$, see Lemma 2.2.13. In the following discussions we will always write the elements of induced modules in that form and choose $T = \{id, (123), (132)\}$ if $\underline{w} = (2, 1)$.*

To calculate the decomposition numbers, we will first make a few observations.

Remark 4.3.11. 1. *There is an isomorphism*

$$\mathrm{Ind}_{(1,1)}^{(2)}(V(i) \otimes V(i)) \cong V(i, i; (2)) \oplus V(i, i; (1, 1)).$$

2. *Define an $RS_p \wr S_2$ module $\widehat{V(i,j)}$ as follows:*

$$\mathrm{Res}_{(1,1)}^{(2)}(\widehat{V(i,j)}) = (V(i) \otimes V(j)) \oplus (V(j) \otimes V(i))$$
$$(a_i \otimes a_j, b_j \otimes b_i).\hat{\tau} = (b_i \otimes b_j, a_j \otimes a_i) \quad \forall \tau \in S_3.$$

Then the following map is an $RS_p \wr S_2$-module isomorphism.

$$\varphi : \widehat{V(i,j)} \longrightarrow V(i,j) = \mathrm{Ind}_{(1,1)}^{(2)}(V(i) \otimes V(j))$$
$$(a_i \otimes a_j, b_j \otimes b_i) \longmapsto a_i \otimes a_j \otimes id + (b_i \otimes b_j) \otimes \hat{\tau}$$

3. *There is a decomposition*

$$K \otimes T^{(2)}(P(i)) = T^{(2)}(V(i)) \oplus V(i, i+1) \oplus T^{(2)}(V(i+1)). \tag{4.6}$$

4. Define an $RS_p \wr S_3$ module $V\widehat{(i,i,j)}$ as follows:

$$\begin{aligned}
&\mathrm{Res}^{(3)}_{(1,1,1)}(V(\widehat{i,i,j};(2))) = \\
&(V(i)\otimes V(i)\otimes V(j)) \oplus (V(i)\otimes V(j)\otimes V(i)) \oplus (V(j)\otimes V(i)\otimes V(i)) \\
&(a_{i1}\otimes a_{i2}\otimes a_j, b_{i1}\otimes b_j\otimes b_{i2}, c_j\otimes c_{i1}\otimes c_{i2}).\hat{\tau} \\
&= (a_{i2}\otimes a_{i1}\otimes a_j, c_{i1}\otimes c_j\otimes c_{i2}, b_j\otimes b_{i1}\otimes b_{i2}) \\
&(a_{i1}\otimes a_{i2}\otimes a_j, b_{i1}\otimes b_j\otimes b_{i2}, c_j\otimes c_{i1}\otimes c_{i2}).\hat{\sigma} \\
&= (c_{i1}\otimes c_{i2}\otimes c_j, a_{i2}\otimes a_j\otimes a_{i1}, b_j\otimes b_{i2}\otimes b_{i1}).
\end{aligned}$$

Then the following map is an $RS_p \wr S_3$-module isomorphism.

$$\begin{aligned}
\varphi : V\widehat{(i,i,j)} &\longrightarrow V(i,i,j;(2)) = \mathrm{Ind}^{(3)}_{(1,1,1)}(V(i)\otimes V(i)\otimes V(j)) \\
&(a_{i1}\otimes a_{i2}\otimes a_j, b_{i1}\otimes b_j\otimes b_{i2}, c_j\otimes c_{i1}\otimes c_{i2}) \longmapsto \\
&a_{i1}\otimes a_{i2}\otimes a_j + b_{i2}\otimes b_{i1}\otimes b_j\hat{\sigma} + c_{i1}\otimes c_{i2}\otimes c_j\hat{\sigma}^2
\end{aligned}$$

Proof. Part 1:

$$\begin{aligned}
&\mathrm{Ind}^{(2)}_{(1,1)}(V(i)\otimes V(i)) \\
&= \mathrm{Ind}^{(2)}_{(1,1)}((V(i)\otimes V(i)) \oslash (S^{(1)}\otimes S^{(1)})) \\
&= \mathrm{Ind}^{(2)}_{(1,1)}(\mathrm{Res}^{(2)}_{(1,1)}(T^{(2)}(V(i))) \oslash (S^{(1)}\otimes S^{(1)})) \\
&\overset{4.1.12.2}{=} T^{(2)}(V(i)) \oslash \mathrm{Ind}^{(2)}_{(1,1)}(S^{(1)}\otimes S^{(1)}) \\
&= T^{(2)}(V(i)) \oslash (S^{(2)} \oplus S^{(1,1)}) \\
&= V(i,i;(2)) \oplus V(i,i;(1,1))
\end{aligned}$$

Part 2 and 4 are straightforward calculations.
Part 3: As a vector space $K\otimes T^{(2)}(P(i))$ decomposes as follows:

$$\begin{aligned}
&K\otimes T^{(2)}(P(i)) = \\
&\qquad V(i)\otimes V(i) \oplus V(i)\otimes V(i+1) \oplus V(i+1)\otimes V(i) \oplus V(i+1)\otimes V(i+1)
\end{aligned}$$

Here $V(i)\otimes V(i)$ and $V(i+1)\oplus V(i+1)$ are also invariant under $S_p \wr S_2$. The remaining two summands form another $RS_p \wr S_2$ submodule.

$$V(i)\otimes V(i+1) \oplus V(i+1)\otimes V(i) \cong V\widehat{(i,i+1)} \cong \mathrm{Ind}^{(2)}_{(1,1)}(V(i)\otimes V(i+1))$$

□

Definition 4.3.12. *Let $n \le 3$ and $\lambda \vdash n$ be a partition of n.*

Then we define a set of partitions $dc(\lambda)$ as follows.

$$\begin{aligned} dc((1)) &= \{\lambda' \mid \lambda' \vdash m, m \leq 3\} \\ dc((2,1)) &= \{\lambda' \mid \lambda' \vdash m, m \leq 2\} \cup \{(2,1)\} \\ dc((1,1,1)) &= \{(1),(1,1),(1,1,1)\} \\ dc((3)) &= \{(1),(2),(3)\} \\ dc((1,1)) &= \{(1),(1,1),(1,1,1),(2,1)\} \\ dc((2)) &= \{(1),(2),(3),(2,1)\} \end{aligned}$$

Remark 4.3.13. *Let $\lambda \vdash 3$ and $\mu \vdash 2$. Then $dc(\lambda)$ and $dc(\mu)$ are defined such that the following holds.*

$$\lambda \in dc(\mu) \Leftrightarrow \mu \in dc(\lambda) \Leftrightarrow c(\lambda;((1),\mu)) > 0$$

Theorem 4.3.14. *Let $a = (i',j',k';\lambda') \in I_0$ and $b = (i,j,k;\lambda) \in I_p$. Then*

$$\begin{aligned} d_{ab} \neq 0 \Leftrightarrow i' \in \{i,i+1\}, j' \in \{j,j+1\}, k' \in \{k,k+1\}, \text{ and } \lambda' \in dc(\lambda) \text{ or} \\ a = (i,i+1,i+1;\mu), b = (i,i,i+1;\mu'), \mu,\mu' \in \{(2),(1,1)\}. \end{aligned}$$

If $d_{ab} \neq 0$ then $d_{ab} = 1$.

Proof. We calculate the decomposition of the projective indecomposable modules by considering several cases.

- $i < j < k$

$$\begin{aligned} K \otimes P(i,j,k) &= \mathrm{Ind}_{(1,1,1)}^{(3)}(P(i) \otimes P(j) \otimes P(k)) \\ &= \bigoplus_{\substack{i' \in \{i,i+1\} \\ j' \in \{j,j+1\} \\ k' \in \{k,k+1\}}} \mathrm{Ind}_{(1,1,1)}^{(3)}(V(i') \otimes V(j') \otimes V(k')) \end{aligned}$$

  If $|i-j| > 1$ and $|j-k| > 1$ then this is already the decomposition into irreducible modules. Now assume that $j = i+1$. Then for $k' \in \{k, k+1\}$ we obtain a further decomposition as follows.

$$\begin{aligned} &\mathrm{Ind}_{(1,1,1)}^{(3)}(V(i+1) \otimes V(i+1) \otimes V(k')) \\ &= \mathrm{Ind}_{(2,1)}^{(3)}(\mathrm{Ind}_{(1,1,1)}^{(2.1)}(V(i+1) \otimes V(i+1) \otimes V(k'))) \\ &= \mathrm{Ind}_{(2,1)}^{(3)}(\mathrm{Ind}_{(1,1)}^{(2)}(V(i+1) \otimes V(i+1)) \otimes V(k')) \\ &\overset{4.3.11.1}{=} \mathrm{Ind}_{(2,1)}^{(3)}((T^{(2)}(V(i+1)) \oplus T^{(1,1)}(V(i+1))) \otimes V(k')) \\ &= \mathrm{Ind}_{(2,1)}^{(3)}(T^{(2)}(V(i+1)) \otimes V(k')) \oplus \mathrm{Ind}_{(2,1)}^{(3)}(T^{(1,1)}(V(i+1)) \otimes V(k')) \\ &= V(i+1,i+1,k';(2)) \oplus V(i+1,i+1,k';(1,1)) \end{aligned}$$

Similarly, we obtain the following decomposition if $k = j + 1$:

$$\begin{aligned} &\operatorname{Ind}_{(1,1,1)}^{(3)}(V(i') \otimes V(j+1) \otimes V(j+1)) \\ &= V(i', j+1, j+1; (2)) \oplus V(i', j+1, j+1; (1,1)) \end{aligned}$$

- $i = j, j < k$

  Using Remark 4.3.11 Part 3 we obtain the following decomposition for $P(i, i, k; \lambda)$

$$\begin{aligned} &K \otimes P(i,i,k;\lambda) \\ &\cong K \otimes \operatorname{Ind}_{(2,1)}^{(3)}(T^{(2)}(P(i)) \oslash S^{\lambda} \otimes P(k)) \\ &\overset{4.3.11.3}{\cong} \bigoplus_{k' \in \{k, k+1\}} \operatorname{Ind}_{(2,1)}^{(3)}(T^{(2)}(V(i)) \oplus V(i, i+1) \oplus T^{(2)}(V(i+1))) \oslash S^{\lambda} \otimes V(k')) \\ &\cong \bigoplus_{k' \in \{k, k+1\}} \operatorname{Ind}_{(2,1)}^{(3)}(T^{\lambda}(V(i)) \otimes V(k')) \oplus \operatorname{Ind}_{(2,1)}^{(3)}(T^{\lambda}(V(i+1)) \otimes V(k')) \oplus \\ &\qquad \bigoplus_{k' \in \{k, k+1\}} \operatorname{Ind}_{(2,1)}^{(3)}((V(i, i+1) \oslash S^{\lambda}) \otimes V(k')) \end{aligned}$$

  Note that $V(i, i+1) \oslash S^{\lambda} \cong V(i, i+1)$ for $\lambda \in \{(2), (1,1)\}$. Thus, if $k' \neq i+1$, then the last summand is $V(i, i+1, k')$ and we already have a decomposition into simple modules.

$$\begin{aligned} &K \otimes P(i,i,k;\lambda) \\ &\cong \bigoplus_{k' \in \{k, k+1\}} V(i, i, k'; \lambda) \oplus V(i+1, i+1, k'; \lambda) \oplus V(i, i+1, k') \end{aligned}$$

  Now assume that $k = k' = i + 1$. Then we obtain the following additional decompositions

$$\begin{aligned} &\operatorname{Ind}_{(2,1)}^{(3)}(T^{\lambda}(V(i+1)) \otimes V(i+1)) \\ &= \operatorname{Ind}_{(2,1)}^{(3)}(T^{(2)}(V(i+1)) \oslash S^{\lambda} \otimes V(i+1) \oslash S^{(1)}) \\ &\overset{4.1.12.1}{=} \operatorname{Ind}_{(2,1)}^{(3)}((T^{(2)}(V(i+1)) \otimes V(i+1)) \oslash (S^{\lambda} \otimes S^{(1)})) \\ &= \operatorname{Ind}_{(2,1)}^{(3)}(\operatorname{Res}_{(2,1)}^{(3)}(T^{(3)}(V(i+1))) \oslash (S^{\lambda} \otimes S^{(1)})) \\ &\overset{4.1.12.2}{=} T^{(3)}(V(i+1)) \oslash \operatorname{Ind}_{(2,1)}^{(3)}(S^{\lambda} \otimes S^{(1)}) \\ &= T^{(3)}(V(i+1)) \oslash (S^{(2,1)} \oplus S^{\lambda'}) \\ &= V(i+1, i+1, i+1; (2,1)) + V(i+1, i+1, i+1; \lambda') \end{aligned}$$

  where $(\lambda, \lambda') \in \{((2), (3)), ((1,1), (1,1,1))\}$.

$$
\begin{aligned}
&\operatorname{Ind}_{(2,1)}^{(3)}(\operatorname{Ind}_{(1,1)}^{(2)}(V(i)\otimes V(i+1))\oslash S^{\lambda}\otimes V(i+1))\\
&=\operatorname{Ind}_{(2,1)}^{(3)}((\operatorname{Ind}_{(1,1)}^{(2)}(V(i)\otimes V(i+1))\otimes V(i+1))\oslash (S^{\lambda}\otimes S^{(1)}))\\
&=\operatorname{Ind}_{(2,1)}^{(3)}(\operatorname{Ind}_{(1,1,1)}^{(2.1)}((V(i)\otimes V(i+1))\otimes V(i+1))\oslash S^{\lambda}\otimes S^{(1)})\\
&\overset{4.1.1,2.3}{=}\operatorname{Ind}_{(2,1)}^{(3)}(\operatorname{Ind}_{(1,1,1)}^{(2.1)}((V(i)\otimes V(i+1)\otimes V(i+1))\oslash (\operatorname{Res}_{(1,1,1)}^{(3)}(S^{\lambda}\otimes S^{(1)}))))\\
&=\operatorname{Ind}_{(1,1,1)}^{(3)}((V(i)\otimes V(i+1)\otimes V(i+1))\oslash (S^{(1)}\otimes S^{(1)}\otimes S^{(1)}))\\
&=\operatorname{Ind}_{(1,2)}^{(3)}(\operatorname{Ind}_{(1,1,1)}^{(1,2)}(V(i)\otimes V(i+1)\otimes V(i+1)))\\
&=\operatorname{Ind}_{(1,2)}^{(3)}(V(i)\otimes \operatorname{Ind}_{(1,1)}^{(2)}(V(i+1)\otimes V(i+1)))\\
&=\operatorname{Ind}_{(1,2)}^{(3)}(V(i)\otimes (T^{(2)}(V(i+1))\oplus T^{(1,1)}(V(i+1))))\\
&=V(i,i+1,i+1;(2))\oplus V(i,i+1,i+1;(1,1))
\end{aligned}
$$

- $i<j, j=k$

  With similar arguments as above, we obtain the decomposition

$$
\begin{aligned}
&K\otimes P(i,j,j;\lambda)\\
&\cong \bigoplus_{i'\in\{i,i+1\}} V(i',j,j;\lambda)\oplus V(i',j+1,j+1;\lambda)\oplus V(i',j,j+1)
\end{aligned}
$$

  if $j>i+1$ and

$$
\begin{aligned}
&K\otimes P(j-1,j,j;\lambda)\\
&\cong V(j-1,j,j;\lambda)\oplus V(j,j,j;(2,1))\oplus V(j,j,j;\lambda')\\
&\oplus V(j-1,j+1,j+1;\lambda)\oplus V(j,j+1,j+1;\lambda)\\
&\oplus V(j-1,j,j+1)\oplus V(j,j,j+1;(2))\oplus V(j,j,j+1;(1,1))
\end{aligned}
$$

  if $j=i+1$.

- $i=j=k$

  Similarly to the decomposition of $K\otimes T^{(2)}(P(i))$ we obtain the following decomposition.

$$
\begin{aligned}
&K\otimes T^{(3)}(P(i))\cong\\
&T^{(3)}(V(i))\oplus T^{(3)}(V(i+1))\oplus V(i,i,i+1;(2))\oplus V(i,i+1,i+1;(2))
\end{aligned}
$$

For the general case, we observe the following:

$$\begin{aligned}
&V(i,i,i+1;(2)) \oslash S^\lambda \\
&\cong \operatorname{Ind}_{(2,1)}^{(3)}(T^{(2)}(V(i)) \otimes V(i+1)) \oslash S^\lambda \\
&\cong \operatorname{Ind}_{(2,1)}^{(3)}((T^{(2)}(V(i)) \otimes V(i+1)) \oslash \operatorname{Res}_{(2,1)}^{(3)} S^\lambda) \\
&\cong \begin{cases} V(i,i,i+1;(2)) & \text{if } \lambda = (3) \\ V(i,i,i+1;(2)) & \text{if } \lambda = (1,1,1) \\ V(i,i,i+1;(1,1)) \oplus V(i,i,i+1;(2)) & \text{if } \lambda = (2,1) \end{cases}.
\end{aligned}$$

Using this and the analogous result for $V(i,i+1,i+1;(2))$ we calculate the decomposition of $K \otimes P(i,i,i;\lambda)$:

$$\begin{aligned}
&K \otimes P(i,i,i;\lambda) \\
&= K \otimes T^{(3)}(P(i)) \oslash S^\lambda \\
&\cong (T^{(3)}(V(i)) \oplus T^{(3)}(V(i+1)) \oplus V(i,i,i+1;(2)) \oplus V(i,i+1,i+1;(1,1))) \oslash S^\lambda \\
&\cong V(i,i,i;\lambda) \oplus V(i+1,i+1,i+1;\lambda) \oplus \\
&\qquad \begin{cases} V(i,i,i+1;(2)) \oplus V(i,i+1,i+1;(2)) & \text{if } \lambda = (3) \\ V(i,i,i+1;(1,1)) \oplus V(i,i+1,i+1;(1,1)) & \text{if } \lambda = (1,1,1) \\ V(i,i,i+1;(1,1)) \oplus V(i,i+1,i+1;(1,1)) \oplus \\ \quad V(i,i,i+1;(2)) \oplus V(i,i+1,i+1;(2)) & \text{if } \lambda = (2,1) \end{cases}.
\end{aligned}$$

□

Corollary 4.3.15. *The Brauer graph of $B_0(RS_p)(3)$ is connected*

Proof. For ease of notations we will label the vertices of the Brauer graph by I_0. For $x = (i,j,k;\lambda)$ and $y = (i',j',k';\lambda)$ let $\operatorname{dist}(x,y) = |i-i'| + |j-j'| + |k-k'|$. We show that the distance of the vertex $x_0 = (1,2,3)$ to every other vertex y in the Brauer graph is at most $\operatorname{dist}(x_0,y)$. We use induction on $\operatorname{dist}(x_0,y)$. The induction base for $\operatorname{dist}(x_0,y)$ is given since there is only one $\lambda \vdash 1$. Now let $y = (i,j,k;\lambda) \in I_0$ with $\operatorname{dist}(x_0,y) = l > 0$. Assume first that $i \neq 1$. Then for every $z_{\lambda'} = (i-1,j,k;\lambda')$ with $\lambda' \vdash \operatorname{type}((i-1,j,k))$ it is $\operatorname{dist}(x_0,z_{\lambda'}) < \operatorname{dist}(x_0,y)$. From the decomposition numbers it is not hard to see that there is at least one λ' such that $V(y)$ and $V(z_{\lambda'})$ share a composition factor. Thus, by the induction hypothesis, the distance from x_0 to y in the quiver is at most $\operatorname{dist}(x_0,z)+1 = \operatorname{dist}(x_0,y)$. The cases where $j \neq 2$ or $k \neq 3$ work completely analogous, with the small difference that, if $j < 2$ or $k < 3$ we have to add 1 to the differing component instead of substracting 1. □

4.3.3 Quiver

Our next goal is to determine the quiver of $\overline{\Lambda} \coloneqq B_0(\mathbb{F}_p S_p)(3)$. Let $\overline{S}(1),\dots,\overline{S}(p-1)$ and $\overline{P}(1),\dots,\overline{P}(p-1)$ be the simple and projective indecomposable $B_0(\mathbb{F}_p S_p)$-modules

respectively. Recall the radical series of $\overline{P}(i)$:

$$\begin{aligned}\overline{P}(i)/\operatorname{rad}(\overline{P}(i)) &\cong \overline{S}(i)\\ \operatorname{rad}(\overline{P}(i))/\operatorname{rad}^2(\overline{P}(i)) &\cong \overline{S}(i-1)\oplus\overline{S}(i+1)\\ \operatorname{rad}^2(\overline{P}(i)) &\cong \overline{S}(i).\end{aligned}$$

Theorem 4.3.16. *Let $x=(i_1,i_2,i_3;\lambda), y=(j_1,j_2,j_3;\mu)\in I_p$. Then*

$$\operatorname{Hom}_{\overline{\Lambda}}(\overline{P}(x),\operatorname{rad}(\overline{P}(y))/\operatorname{rad}^2(\overline{P}(y)))$$

is non-trivial if and only if $i_l\in\{j_l-1,j_l+1\}$ for exactly one $l\in\{1,2,3\}$ and $i_k=j_k$ for $l\neq k\in\{1,2,3\}$ and we have one of the following cases:

- *$\mu\in dc(\lambda)$ or*
- *$i_1=i_2=i_3-1$ and $j_1=i_1, j_2=i_2+1=i_3=j_3$ or*
- *$i_1+1=i_2=i_3$ and $j_1=i_1=j_2=i_2-1, j_3=i_3$.*

Then $\dim_F \operatorname{Hom}_{\overline{\Lambda}}(\overline{P}(x),\operatorname{rad}(\overline{P}(y))/\operatorname{rad}^2(\overline{P}(y)))=1$.

Proof. We apply Theorem 4.1.16. Recall that

$$\operatorname{Hom}_{B_0(\mathbb{F}_pS_p)}(\overline{P}(i),\operatorname{rad}(\overline{P}(j))/\operatorname{rad}^2(\overline{P}(j)))\neq 0\Leftrightarrow j\in\{i+1,i-1\}.$$

In particular we have for all i:

$$\operatorname{Hom}_{B_0(\mathbb{F}_pS_p)}(\overline{P}(i),\operatorname{rad}(\overline{P}(i))/\operatorname{rad}^2(\overline{P}(i)))=0.$$

for all i. This means only the second case of Theorem 4.1.16.2.b applies. The rest is just a translation of the theorem to our notation. □

Next we are going to determine some relations on the quiver of $\overline{\Lambda}$. Those are not all the relations, but they suffice to show that the algebra lifts uniquely. Note that the existence of a lift is not in question since we know that the algebra is Morita-equivalent to a block.

We will give explicit generators for $\operatorname{Hom}_{\overline{\Lambda}}(\overline{P}(x),\operatorname{rad}(\overline{P}(y))/\operatorname{rad}^2(\overline{P}(y)))$, $x,y\in I_p$ to determine the relations. To do this, we will use different presentations of the projective indecomposable modules. In this context we will have to differentiate between the subalgebras of $A(3)$ corresponding to the subgroup $S_{\{\sigma(1),\sigma(2)\}}\times S_{\sigma(3)}$ of S_3 for permutations $\sigma\in S_3$ for an algebra A. For this purpose we introduce the following notation: For $\sigma\in S_3$ define

$$\begin{aligned}A(\{\sigma(1),\sigma(2)\},\sigma(3)) &:= \{a\otimes\rho\mid\rho\in S_{\sigma(1),\sigma(2)}\times S_{\sigma(3)}\}\leq A(3),\\ \operatorname{Ind}^{(3)}_{(\{\sigma(1),\sigma(2)\},\sigma(3))}(V) &:= \operatorname{Ind}^{A(3)}_{A(\{\sigma(1),\sigma(2)\},\sigma(3))}(V),\\ \operatorname{Res}^{(3)}_{(\{\sigma(1),\sigma(2)\},\sigma(3))}(V) &:= \operatorname{Res}^{A(3)}_{A(\{\sigma(1),\sigma(2)\},\sigma(3))}(W),\end{aligned}$$

where V is an $A(\{\sigma(1),\sigma(2)\},\sigma(3))$-module and W is an $A(3)$-module.

Definition 4.3.17. *Let $A := B_0(\mathbb{F}_p S_p)$, $x = (i_1, i_2, i_3; \lambda) \in I_p$ and $\sigma \in S_3$. Then we define the $A(3)$-module $\overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)})$ as follows.*

If $\lambda = (2,1)$ we define $\overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) = \overline{P}(x)$. Otherwise let B be the following subalgebra of $A(3)$.

- *If $i_1 = i_2 = i_3$, then $B = A(3)$.*
- *If $i_j = i_k \neq i_l$, then $B = A(\{\sigma^{-1}(j), \sigma^{-1}(k)\}, \sigma^{-1}(l))$.*
- *If $|\{i_1, i_2, i_3\}| = 3$, then $B = A((1,1,1))$*

Then we define a B-module M_λ where $M_\lambda =_{F_p} \overline{P}(i_{\sigma(1)}) \otimes \overline{P}(i_{\sigma(2)}) \otimes \overline{P}(i_{\sigma(3)})$ and the multiplication of B on M_λ is as follows.

$$(v_1 \otimes v_2 \otimes v_3).(a_1 \otimes a_2 \otimes a_3 \otimes \rho)$$
$$= \begin{cases} v_{\rho(1)}a_{\rho(1)} \otimes v_{\rho(2)}a_{\rho(2)} \otimes v_{\rho(3)}a_{\rho(3)} & \text{if } \lambda \in \{(1),(2),(3)\} \\ sgn(\rho) \cdot v_{\rho(1)}a_{\rho(1)} \otimes v_{\rho(2)}a_{\rho(2)} \otimes v_{\rho(3)}a_{\rho(3)} & \text{if } \lambda \in \{(1,1),(1,1,1)\} \end{cases}$$

for $v_1 \otimes v_2 \otimes v_3 \in M$ and $a_1 \otimes a_2 \otimes a_3 \otimes \rho \in B$. Note that B is chosen such that the multiplication is well-defined.

We define $\overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) := \mathrm{Ind}_B^{A(3)} M_\lambda$.

Lemma 4.3.18. *Let $A := B_0(\mathbb{F}_p S_p)$, $x = (i_1, i_2, i_3; \lambda) \in I_p$ and $\sigma \in S_3$.*

1. *If $|\{i_1, i_2, i_3\}| = 3$, then the following map is an isomorphism of $A(3)$-modules:*

$$\Psi_\sigma : \overline{P}(x) \longrightarrow \overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}\lambda)$$
$$(v_1 \otimes v_2 \otimes v_3) \otimes \hat{\rho} \longmapsto (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1}\hat{\rho}.$$

2. *Assume that $i_1 = i_2 \neq i_3$. Then the following map is an isomorphism:*

$$\Psi_\sigma : \overline{P}(x) \longrightarrow \overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}; \lambda)$$
$$((v_1 \otimes v_2 \oslash s) \otimes v_3) \otimes \hat{\rho} \longmapsto (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}\hat{\rho}.$$

If $i_1 \neq i_2 = i_3$, then the following is an isomorphism:

$$\Psi_\sigma : \overline{P}(x) = \mathrm{Ind}_{(2,1)}^{(3)} T^2(\overline{P}(i_2)) \oslash S^\lambda \otimes \overline{P}(i_1) \longrightarrow \overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}; \lambda)$$
$$v_2 \otimes v_3 \otimes v_1 \oslash s \otimes \hat{\rho} \longmapsto (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}\hat{\rho}.$$

3. *Assume that $i_1 = i_2 = i_3$ and let $\lambda \neq (2,1)$. Then the following is an isomorphism.*

$$\Psi : \overline{P}(x) \longrightarrow \overline{P}'(i_1, i_1, i_1; \lambda)$$
$$v_1 \otimes v_2 \otimes v_3 \otimes s \longmapsto v_1 \otimes v_2 \otimes v_3.$$

If $\lambda = (2,1)$ then $\overline{P}(x) = \overline{P}'(i_1, i_1, i_1; \lambda)$.

Proof. For $\rho \in S_3$ define $\hat{\rho} := 1 \otimes 1 \otimes 1 \otimes \rho \in A(3)$.

1. First note that Ψ_σ is the image of the homomorphism

$$\Phi : \overline{P}(i_1) \otimes \overline{P}(i_2) \otimes \overline{P}(i_3) \longrightarrow \mathrm{Res}^{A(3)}_{A(1,1,1)} \mathrm{Ind}^{A(3)}_{A((1,1,1))} \overline{P}(i_{\sigma(1)}) \otimes \overline{P}(i_{\sigma(2)}) \otimes \overline{P}(i_{\sigma(3)})$$
$$v_1 \otimes v_2 \otimes v_3 \longmapsto (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1}$$

under the isomorphism from Lemma 2.1.2. Therefore we only have to check the homomorphism property for Φ. Let $a_1, a_2, a_3 \in A$. Then, by definition, we have the following relation in the algebra $A(3)$:

$$\begin{aligned}(\hat{\sigma}^{-1}) \cdot (a_1 \otimes a_2 \otimes a_3 \otimes 1) \\ = (1 \otimes 1 \otimes 1 \otimes \sigma^{-1}) \cdot (a_1 \otimes a_2 \otimes a_3 \otimes 1) \\ = (a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)} \otimes 1) \cdot (1 \otimes 1 \otimes 1 \otimes \sigma^{-1}).\end{aligned}$$

Now let $v_1 \in \overline{P}(i_1), v_2 \in \overline{P}(i_2)$ and $v_3 \in \overline{P}(i_3)$ and $a = a_1 \otimes a_2 \otimes a_3 \in A \otimes A \otimes A$. We get

$$\begin{aligned}\Phi(v_1 \otimes v_2 \otimes v_3).(a \otimes 1) \\ = ((v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1}).(a \otimes 1) \\ = (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes (\hat{\sigma}^{-1} \cdot (a \otimes 1)) \\ = (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes (((a.\sigma) \otimes 1) \cdot \hat{\sigma}^{-1}) \\ = (v_{\sigma(1)} a_{\sigma(1)} \otimes v_{\sigma(2)} a_{\sigma(2)} \otimes v_{\sigma(3)} a_{\sigma(3)}) \otimes \hat{\sigma}^{-1} \\ = \Phi((v_1 \otimes v_2 \otimes v_3).(a \otimes 1)).\end{aligned}$$

The map Ψ_σ is clearly bijective.

2. We show that

$$\Phi : T^2(\overline{P}(i_1)) \oslash S^\lambda \otimes \overline{P}(i_3) \longrightarrow$$
$$\mathrm{Res}^{A(3)}_{A(1,1,1)} \mathrm{Ind}^{A(3)}_{(\{\sigma^{-1}(1),\sigma^{-1}(2)\},\sigma^{-1}(3))} \overline{P}(i_{\sigma(1)}) \otimes \overline{P}(i_{\sigma(2)}) \otimes \overline{P}(i_{\sigma(3)})$$
$$(v_1 \otimes v_2 \oslash s) \otimes v_3 \longmapsto (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1}$$

is a homomorphism and obtain the homomorphism property of Ψ_σ again by Lemma 2.1.2. The homomorphism property for elements in $A((1,1,1))$ can be proven exactly as above, as $A((1,1,1))$ acts neutrally on the S^λ component. It remains to check that the map Φ commutes with multiplication with the transposition $\eta = (12)$. Let $v_1 \in \overline{P}(i_1), v_2 \in \overline{P}(i_2)$ and $v_3 \in \overline{P}(i_3)$. Now one computes

$$\begin{aligned}\Phi(v_1 \otimes v_2 \otimes v_3).\hat{\eta} \\ = ((v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1}).\hat{\eta} \\ = (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1} \cdot \hat{\eta} \cdot \hat{\sigma} \cdot \hat{\sigma}^{-1}.\end{aligned}$$

Note that $\sigma^{-1}\eta\sigma = (\sigma^{-1}(1)\sigma^{-1}(2))$ and thus

$$\hat{\sigma}^{-1}\hat{\eta}\hat{\sigma} \in A(\{\sigma^{-1}(1), \sigma^{-1}(2)\}, \sigma^{-1}(3)).$$

Let $\varepsilon_{(2)} = 1$ and $\varepsilon_{(1,1)} = -1$. Then we get the following.

$$\begin{aligned}
&(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \otimes \hat{\sigma}^{-1} \cdot \hat{\eta} \cdot \hat{\sigma} \cdot \hat{\sigma}^{-1} \\
&= \varepsilon_\lambda \cdot (v_{\sigma(\sigma^{-1}(\eta(\sigma(1))))} \otimes v_{\sigma(\sigma^{-1}(\eta(\sigma(2))))} \otimes v_{\sigma(\sigma^{-1}(\eta(\sigma(3))))}) \otimes \hat{\sigma}^{-1} \\
&= \varepsilon_\lambda \cdot (v_{\eta(\sigma(1))} \otimes v_{\eta(\sigma(2))} \otimes v_{\eta(\sigma(3))}) \otimes \hat{\sigma}^{-1} \\
&= \Phi(\varepsilon_\lambda \cdot v_{\eta(1)} \otimes v_{\eta(2)} \oslash s \otimes v_{\eta(3)}) \\
&= \Phi(v_{\eta(1)} \otimes v_{\eta(2)} \oslash \varepsilon_\lambda \cdot s \otimes v_3) \\
&= \Phi(v_{\eta(1)} \otimes v_{\eta(2)} \oslash (s.\eta) \otimes v_3) \\
&= \Phi(v_1 \otimes v_2 \oslash s \otimes v_3.\hat{\eta}).
\end{aligned}$$

3. Let $v_1, v_2, v_3 \in \overline{P}(i_1)$, $a_1, a_2, a_3 \in A$, $\rho \in S_3$, $s \in S^\lambda$ and define $\varepsilon_{(3)} = 1$ and $\varepsilon_{(1,1,1)} = sgn(\rho)$.

$$\begin{aligned}
&\Psi(v_1 \otimes v_2 \otimes v_3 \otimes s).(a_1 \otimes a_2 \otimes a_3 \otimes \rho) \\
&= (v_1 \otimes v_2 \otimes v_3).(a_1 \otimes a_2 \otimes a_3 \otimes \rho) \\
&= \varepsilon_\lambda \cdot (a_{\rho(1)}v_{\rho(1)} \otimes a_{\rho(2)}v_{\rho(2)} \otimes a_{\rho(3)}v_{\rho(3)}) \\
&= \Psi(\varepsilon_\lambda a_{\rho(1)}v_{\rho(1)} \otimes a_{\rho(2)}v_{\rho(2)} \otimes a_{\rho(3)}v_{\rho(3)} \oslash s) \\
&= \Psi(a_{\rho(1)}v_{\rho(1)} \otimes a_{\rho(2)}v_{\rho(2)} \otimes a_{\rho(3)}v_{\rho(3)} \oslash \varepsilon_\lambda s) \\
&= \Psi((v_1 \otimes v_2 \otimes v_3 \otimes s).(a_1 \otimes a_2 \otimes a_3 \otimes \rho))
\end{aligned}$$

□

Corollary 4.3.19. *Let $x = (i_1, i_2, i_3; \lambda) \in I_p$ with $\lambda \neq (2,1)$ and $\sigma \in S_3$.*

Then $\overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}; \lambda)$ has the following properties. There is a subalgebra B of $A(3)$ and a B-module M such that

1. $\overline{P}(i_1, i_2, i_3; \lambda) \cong_{A(3)} \overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}; \lambda)$
2. $A((1,1,1)) \subseteq B$,
3. $a_1 \otimes a_2 \otimes a_3 \otimes \hat{\rho} \in B \Leftrightarrow \rho(\sigma^{-1}(j)) = \sigma^{-1}(j)$ *for every* $j \in \{1,2,3\}$, $\rho \in S_3$ *and all* $a_1, a_2, a_3 \in A$,
4. $\overline{P}'(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}; \lambda) = \mathrm{Ind}_B^{A(3)} M$ *and*
5. $M =_{\mathbb{F}_p} \overline{P}(i_{\sigma(1)}) \otimes \overline{P}(i_{\sigma(2)}) \otimes \overline{P}(i_{\sigma(3)})$.

The above Corollary assure that in all cases where $S^{(2,1)}$ does not occur, we only need to define homomorphisms between modules of the form $\overline{P}'(i,j,k;\lambda)$ and $\overline{P}'(i',j,k;\lambda')$ where $i' \in \{i+1, i-1\}$. The homomorphisms between the standard presentations of the projective indecomposable modules will be given by composition with the isomorphisms above, see Definition 4.3.22.

Remark 4.3.20. *The standard representation $S^{(2,1)}$ of S_3 will be described as follows.*

Let (b_1, b_2, b_3) be a basis of the vector space F^3, then F^3 becomes a right S_3-module via $b_i.\sigma = b_{\sigma^{-1}(i)}$ for $\sigma \in S_3$. Let further $x_1 = b_1 - b_2$ and $x_2 = b_2 - b_3$. Then $\langle x_1, x_2 \rangle_F \cong S^{(2,1)}$.

Lemma 4.3.21. *Let $x, y \in I_p$ $i,j,k \in \{1,\dots,p-1\}$, $|\{i,j,k\}| = 3$, $i' \in \{i+1, i-1\}$ and*

$$\mathrm{Hom}_{B_0(\mathbb{F}_p S_p)}(\overline{P}(i), \mathrm{rad}(\overline{P}(i'))/\mathrm{rad}^2(\overline{P}(i'))) = \langle\gamma\rangle.$$

Then $\beta'_{(y,x)}$ defined as below is a generator of

$$\mathrm{Hom}_{\overline{\Lambda}}(\overline{P}(x), \mathrm{rad}(\overline{P}(y))/\mathrm{rad}^2(\overline{P}(y)))$$

1. $x = (i,i,i;(2,1)), y = (i',i,i;(2))$

$$a \otimes b \otimes c \otimes id \otimes x_1 \mapsto \gamma(a) \otimes b \otimes c \otimes id - \gamma(b) \otimes c \otimes a \otimes (132) \tag{4.7}$$
$$a \otimes b \otimes c \otimes id \otimes x_2 \mapsto \gamma(b) \otimes c \otimes a \otimes (132) - \gamma(c) \otimes a \otimes b \otimes (123) \tag{4.8}$$

2. $x = (i,i,i;(2,1)), y = (i',i,i,(1,1))$

$$a \otimes b \otimes c \otimes id \otimes x_1 \mapsto 2\gamma(c) \otimes a \otimes b \otimes (123) - \gamma(a) \otimes b \otimes c \otimes id - \gamma(b) \otimes c \otimes a \otimes (132) \tag{4.9}$$
$$a \otimes b \otimes c \otimes id \otimes x_2 \mapsto 2\gamma(a) \otimes b \otimes c \otimes id - \gamma(b) \otimes c \otimes a \otimes (132) - \gamma(c) \otimes a \otimes b \otimes (123) \tag{4.10}$$

3. $x = (i,i,i;\lambda), y = (i',i,i;\mu)$, $(\lambda,\mu) \in \{((3),(2)),((1,1,1),(1,1))\}$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id + \gamma(b) \otimes c \otimes a \otimes (132) + \gamma(c) \otimes a \otimes b \otimes (123) \tag{4.11}$$

4. $x = (i,i',i';(2)), y(i',i',i';(2,1))$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id \otimes (2x_1 + x_2) \tag{4.12}$$

5. $x = (i,i',i';(1,1)), y = (i',i',i';(2,1))$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id \otimes x_2 \tag{4.13}$$

6. $x = (i,j,j;\mu), y = (i',j,j;\lambda)$, $\lambda \neq (2,1)$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id \tag{4.14}$$

7. $x = (i,i,j;(2)), y(i',i,j;\lambda)$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id + \gamma(b) \otimes a \otimes c \otimes (12) \tag{4.15}$$

8. $x = (i,i,j;(1,1)), y = (i',i,j;\lambda)$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id - \gamma(b) \otimes a \otimes c \otimes (12) \tag{4.16}$$

9. $x = (i,j,k;(2)), y = (i',j,k;\lambda)$

$$a \otimes b \otimes c \otimes id \mapsto \gamma(a) \otimes b \otimes c \otimes id \tag{4.17}$$

Proof. Let $A = B_0(\mathbb{F}_p S_p)(3)$. Straightforward calculations very similar to the ones in Lemma 4.3.18 show that the above maps are indeed homomorphisms.

All that remains is to show is that if $f : \overline{P}_1 \longrightarrow \overline{P}_2$ is one of the homomorphisms above, then $\mathrm{Im}(f) \nsubseteq \mathrm{rad}^2(\overline{P}_2)$. This follows from Lemma 4.1.13 as follows. First assume that $\overline{P}_2 = \overline{P}'(i_1,i_2,i_3;\lambda)$ with $i_1,i_2,i_3 \in \{1,\dots,p-1\}$.

$$\begin{aligned}
\mathrm{rad}(\overline{P}'(i_1,i_2,i_3;\lambda)) &= \mathrm{rad}(\mathrm{Ind}_B^{A(3)}(\overline{P}(i_1) \otimes \overline{P}(j) \otimes \overline{P}(k))) \\
&= \mathrm{rad}(\mathrm{Res}^{A(3)}_{T^3(A)}(\mathrm{Ind}_B^{A(3)}(\overline{P}(i_1) \otimes \overline{P}(j) \otimes \overline{P}(k)))) \\
&= \mathrm{Res}^{A(3)}_{T^3(A)}(\mathrm{Ind}_B^{A(3)}(\overline{P}(i_1) \otimes \overline{P}(j) \otimes \overline{P}(k))).(\sum_{i=0}^{2} T^i(A) \otimes \mathrm{rad}(A) \otimes T^{2-i}(A)) \\
&= \mathrm{Ind}_B^{A(3)}\big[(\mathrm{rad}(\overline{P}(i_1)) \otimes \overline{P}(j) \otimes \overline{P}(k)) \\
&\qquad + (\overline{P}(i_1) \otimes \mathrm{rad}(\overline{P}(j)) \otimes \overline{P}(k)) + (\overline{P}(i_1) \otimes \overline{P}(j) \otimes \mathrm{rad}(\overline{P}(k)))\big]
\end{aligned}$$

Applying the same argument again, we can see that an element $a_1 \otimes a_2 \otimes a_3 \otimes \rho \in \mathrm{Ind}_B^A(\overline{P}(i_1) \otimes \overline{P}(j) \otimes \overline{P}(k))$ lies in $\mathrm{rad}^2(\overline{P}'(i_1,j,k;\lambda))$ if and only if

$$\begin{aligned}
&\exists j \in \{1,2,3\}\ , a_{i_j} \in \mathrm{rad}^2(\overline{P}(i_j)) \text{ or} \\
\exists j,k \in \{1,2,3\}, j \neq k,\ &a_{i_j} \in \mathrm{rad}(\overline{P}(i_j)), a_{i_k} \in \mathrm{rad}(\overline{P}(i_k)).
\end{aligned}$$

Thus if we choose a such that $\gamma(a) \notin \mathrm{rad}^2(\overline{P}(i_1'))$ and $b \notin \mathrm{rad}(\overline{P}(i_2)), c \notin \mathrm{rad}(\overline{P}(i_3))$ we have $f(a \otimes b \otimes c \otimes id) \notin \mathrm{rad}^2(\overline{P}_2)$.

The same argument works for cases including $S^{(2,1)}$ as we know from Lemma 4.1.13 that

$$\mathrm{rad}^2(T^3(\overline{P}(i)) \oslash S^{(2,1)}) = \mathrm{rad}^2(T^3(\overline{P}(i))) \oslash S^{(2,1)}.$$

□

Theorem 4.3.22. *Let $x = (i_1, i_2, i_3; \lambda), y = (i_1', i_2', i_3'; \lambda') \in I_p$. Let $j \in \{1,2,3\}$ with $i_j' \in \{i_j + 1, i_j - 1\}$ and let $i_k = i_k'$ for $k \neq j$. Let further λ, λ' be such that*

$$\mathrm{Hom}_{\overline{\Lambda}}(\overline{P}(x), \mathrm{rad}(\overline{P}(y))/\mathrm{rad}^2(\overline{P}(y))) \neq 0.$$

Let $\Phi_1 : \overline{P}(x) \longrightarrow \overline{P}'(i_j, i_k, i_l; \lambda)$ and $\Phi_2 : \overline{P}(y) \longrightarrow \overline{P}'(i_j', i_k, i_l; \lambda')$ and be the isomorphisms from Lemma 4.3.18 where $i_k \leq i_l$ and $\{j,k,l\} = \{1,2,3\}$. Then

$$\beta_{(y,x)} := \Phi_2^{-1} \circ \beta'_{(i_j' i_k i_l \lambda', i_j i_k i_l \lambda)} \circ \Phi_1$$

is a generator of $\mathrm{Hom}_{\overline{\Lambda}}(\overline{P}(x), \mathrm{rad}(\overline{P}(y))/\mathrm{rad}^2(\overline{P}(y)))$.

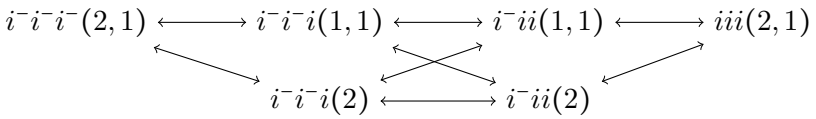
Proof. This is a direct consequence of Lemma 4.3.21. □

Remark 4.3.23. *The following diagrams depict parts of the quiver, for each index $a = (i,j,k;\lambda) \in I_0$ the quiver restricted to d_a. To improve readability, we write i^- for $i-1$ and i^+ for $i+1$.*

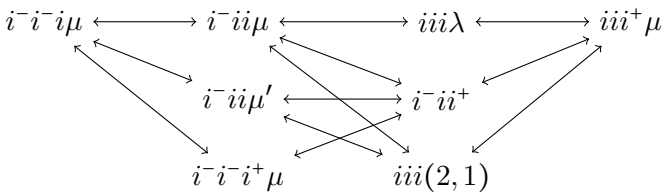
- $i = j = k, (\lambda, \mu) \in \{((1,1,1),(1,1)), ((3),(2))\}$

  $$i^-i^-i^-\lambda \longleftrightarrow i^-i^-i\mu \longleftrightarrow i^-ii\mu \longleftrightarrow iii\lambda$$

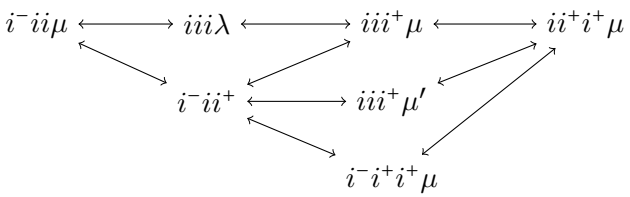
- $i = j = k, \lambda = (2,1)$

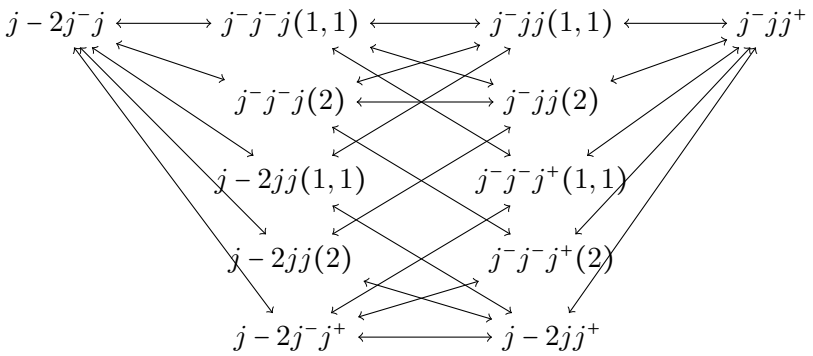
- $i = j = k - 1, \{\mu, \mu'\} = \{(1,1),(2)\}$

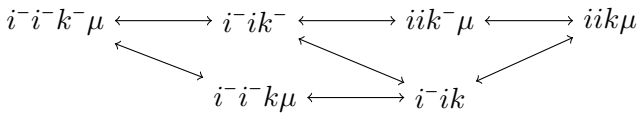
- $j = k = i + 1, (\lambda, \mu, \mu') \in \{((1,1,1),(1,1),(2)), ((3),(2),(1,1))\}$

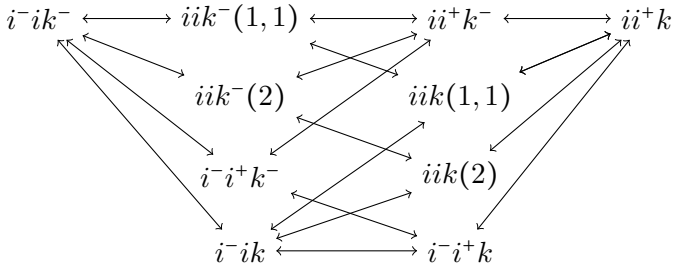
- $i+1=j=k-1$



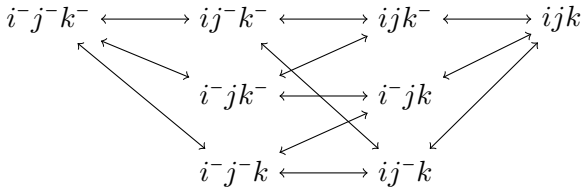
- $i=j, k-j>1, \mu\in\{(1,1),(2)\}$



- $j=i+1, k-j>1$



- $j-i>1, k-j>1$



Definition 4.3.24. *Let x and y be two vertices in the quiver $Q_{\overline{\Lambda}}$ of $\overline{\Lambda}$. We call an arrow from x to y in $Q_{\overline{\Lambda}}$* ascending *(*descending*) if $l(x)<l(y)$ ($l(x)>l(y)$). Note that every arrow is either ascending or descending as $|l(x)-l(y)|=1$ whenever an arrow between x and y exists. We call a path in the quiver* ascending *(*descending*) if all the arrows on the path are ascending (descending).*

Lemma 4.3.25. *Let $x, y, y', z \in I_p$ with $x > y, x > y', y > z, y' > z$ or $x < y, x < y', y < z, y' < z$ and assume that there is an arrow in $Q_{\overline{\Lambda}}$ for the pairs of vertices $(x, y), (x, y'), (y, z), (y', z)$ and assume that $d_x \cap d_y \cap d_z \neq \emptyset$ and $c_x \cap c_{y'} \cap c_z \neq \emptyset$.*

$$\beta_{(z,y)}\beta_{(y,x)} = F_{zyxy'}\beta_{(z,y')}\beta_{(y',x)}$$

Where $F_{zyxy'}$ is as in Table 4.1. Note that we list every combination $\{x, y, y', z\}$ just once since $F_{zyxy'} = \frac{1}{F_{zy'xy}}$. The annotations $(', *)$ can be replaced both by $(+, -)$ and $(-, +)$ to cover the ascending and descending cases at the same time. Note that the components of the indices below are not assumed to be ordered to allow us to consider ascending and descending paths simultaneously.

Proof. We will prove that Table 4.1 covers all possible cases. If we don't assume any order in the components, every descending/ascending path of length 2 such that all vertices share some composition factor will be between two vertices of the form

$$x = (i, j, k; \rho), z = (i', j', k; \rho').$$

with the middle vertices being of the form

$$y \in \{(i', j, k; \eta), (i, j', k; \eta')\}.$$

Now we consider several cases. We define $\mu_{(3)} = (2)$ and $\mu_{(1,1,1)} = (1, 1)$.

1. $i = j = k$:
   a) If $x = (i, i, i; (2, 1))$ then
   $$y \in \{(i', i, i; (2)), (i', i, i; (1, 1))\}, z \in \{(i', i', i; (2)), (i', i', i; (1, 1))\}$$
   and all combinations are possible.
   b) If $x = (i, i, i; \lambda)$, $\lambda \neq (2, 1)$ then $y = y' = (i', i, i; \mu_\lambda)$ and $z = (i', i', i; \mu_\lambda)$. So there is just one path and we do not have to consider it.
2. $i = j \neq k$:
   a) If $i' = j' = k$, i.e. $x = (k^*, k^*, k; \rho)$, then $y = (k, k^*, k; \mu)$ for some $\mu \vdash 2$ and $z = (k, k, k; \rho')$ for some $\rho' \vdash 3$. If $\rho' = (2, 1)$ then both options for μ are possible. Otherwise $\mu = \mu_{\rho'}$.
   b) If $i' \neq k$ then there is just one possibility $y = (i', i, k)$ for the middle vertex and thus a unique path which we do not have to consider.
3. $i = k \neq j$ (yields the same possibilities as $j = k \neq i$):
   a) If $i' = k' = j$, i.e. $x = (i, i^*, i; \rho)$, then $y \in \{(i', i^*, i), (i, i, i; \eta)\}$ and $z = (i', i, i; \rho')$. The partition $\eta = (2, 1)$ is possible independent from ρ' but only one of $\eta \in \{(3), (1, 1, 1)\}$ is possible dependent on ρ' and if $\rho \neq \rho'$ that only $\eta = (2, 1)$ is possible.

x	y	y'	z	F_{zyx}	
$(i,i,i;(2,1))$	$(i',i,i;(2))$	$(i',i,i;(1,1))$	$(i',i',i;(2))$	-3	
$(i,i,i;(2,1))$	$(i',i,i;(2))$	$(i',i,i;(1,1))$	$(i',i',i;(1,1))$	1	
$(i,i^*,i;(2))$	(i',i,i^*)	$(i,i,i;(3))$	$(i',i,i;(2))$	1	
$(i,i^*,i;(2))$	(i',i,i^*)	$(i,i,i;(2,1))$	$(i',i,i;(2))$	-1	
$(i,i^*,i;(2))$	$(i,i,i;(2,1))$	$(i,i,i;(3))$	$(i',i,i;(2))$	-1	
$(i,i^*,i;(2))$	(i',i,i^*)	$(i,i,i;(2,1))$	$(i',i,i;(1,1))$	3	
$(i,i^*,i;(1,1))$	(i',i,i^*)	$(i,i,i;(1,1,1))$	$(i',i,i;(1,1))$	-1	
$(i,i^*,i;(1,1))$	(i',i,i^*)	$(i,i,i;(2,1))$	$(i',i,i;(1,1))$	1	
$(i,i^*,i;(1,1))$	$(i,i,i;(2,1))$	$(i,i,i;(1,1,1))$	$(i',i,i;(1,1))$	-1	
$(i,i^*,i;(1,1))$	(i',i,i^*)	$(i,i,i;(2,1))$	$(i',i,i;(2))$	1	
$(i,i^*,i^*;(2))$	$(i',i^*,i^*;(2))$	$(i,i,i^*;(2))$	(i',i,i^*)	1	
$(i,i^*,i^*;(2))$	$(i',i^*,i^*;(2))$	$(i,i,i^*;(1,1))$	(i',i,i^*)	-1	
$(i,i^*,i^*;(2))$	$(i,i,i^*;(2))$	$(i,i,i^*;(1,1))$	(i',i,i^*)	-1	
$(i^*,i^*,i;(2))$	$(i,i^*,i;(2))$	$(i,i^*,i;(1,1))$	$(i,i,i(2,1))$	1	
$(j,i,i;(2))$	$(j',i,i;(2))$	(j,i',i)	(j',i',i)	1	$\lvert j-i\rvert>1$
$(i,i^*,i^*;(1,1))$	$(i',i^*,i^*;(1,1))$	$(i,i,i^*;(1,1))$	(i',i,i^*)	-1	
$(i,i^*,i^*;(1,1))$	$(i',i^*,i^*;(1,1))$	$(i,i,i^*;(2))$	(i',i,i^*)	1	
$(i,i^*,i^*;(1,1))$	$(i,i,i^*;(1,1))$	$(i,i,i^*;(2))$	(i',i,i^*)	-1	
$(i^*,i^*,i;(1,1))$	$(i,i^*,i;(1,1))$	$(i,i^*,i;(2))$	$(i,i,i(2,1))$	-3	
$(j,i,i;(1,1))$	$(j',i,i;(1,1))$	(j,i',i)	(j',i',i)	1	$\lvert j-i\rvert>1$
(i,i^*,i')	$(i',i^*,i'(2))$	$(i,i,i'(2))$	$(i',i,i'(2))$	1	
(i,i^*,i')	$(i',i^*,i'(2))$	$(i,i,i'(1,1))$	$(i',i,i'(2))$	-1	
(i,i^*,i')	$(i,i,i'(2))$	$(i,i,i'(1,1))$	$(i',i,i'(2))$	-1	
(i,i^*,i')	$(i',i^*,i'(1,1))$	$(i,i,i'(1,1))$	$(i',i,i'(1,1))$	-1	
(i,i^*,i')	$(i',i^*,i'(1,1))$	$(i,i,i'(2))$	$(i',i,i'(1,1))$	1	
(i,i^*,i')	$(i,i,i'(1,1))$	$(i,i,i'(2))$	$(i',i,i'(1,1))$	-1	
(j,i^*,i)	(j',i^*,i)	$(j,i,i;(2))$	$(j',i,i;(2))$	1	$j\notin\{i,i^*\}$, $j'\notin\{i,i^*\}$
(j,i^*,i)	(j',i^*,i)	$(j,i,i;(1,1))$	$(j',i,i;(1,1))$	1	$j\notin\{i,i^*\}$, $j'\notin\{i,i^*\}$
(i,i^*,j)	(i',i^*,j)	$(i,i,j;(2))$	(i',i,j)	1	$j\notin\{i',i,i^*\}$
(i,i^*,j)	(i',i^*,j)	$(i,i,j;(1,1))$	(i',i,j)	1	$j\notin\{i',i,i^*\}$
(i,i^*,j)	$(i,i,j;(2))$	$(i,i,j;(1,1))$	(i',i,j)	1	$j\notin\{i',i,i^*\}$
(i,j,k)	(i',j,k)	(i,j',k)	(i',j',k)	1	$\lvert\{i,j,k\}\rvert=3$, $\lvert j-i\rvert>1$

Table 4.1: Factors

b) If $i' \neq j$, i.e. $x = (i, j, i; \rho)$, then $y \in \{(i, j', i; \rho), (i', j, i)\}$ and $z = (i', j', i)$.

4. $i \neq j \neq k$ First note that $i' = j \Rightarrow j' \neq i$, $j' = i \Rightarrow i' \neq j$, $i' = k \Rightarrow j' \neq k$ and $j' = k \Rightarrow i' \neq k$.
   a) If $i' = j$ and $j' = k$ ($(i, j, k) = (j^*, j, j')$) we have a permutation of case 4c) below.
   b) If $i' = j$ and $j' \neq k$ ($(i, j, k) = (j^*, j, k)$) we have a permutation of case 4e) below.
   c) If $i' = k$ and $j' = i$ ($(i, j, k) = (i, i^*, i')$) then both y and z have type 2. If $y = (i', i^*, i'; \eta)$ then $\eta = \rho'$. Otherwise, both choices are possible.
   d) If $i' = k$ and $j' \neq k$ ($(i, j, k) = (k^*, j, k)$) then we have a permutation of case 4f) below.
   e) If $j' = i$ and $i' \neq k$ ($(i, j, k) = (i, i^*, k)$) then y and z have type 1 and y' has type 2. Either partition of type 2 is possible.
   f) If $j' = k$ and $i' \neq j$ ($(i, j, k) = (i, k^*, k)$) then then y has type 1 and y' and z have type 2 and it is $\lambda' = \lambda''$.
   g) If $j', i' \notin \{i, j, k\}$ then all indices are of type 1 and all partitions are 1.

That the homomorphisms are scalar multiples of one another with the scalars above follows from straightforward if somewhat tedious calculations. Details can be found in the appendix. □

Notation 4.3.26. *Since there is at most one arrow between two vertices in the quiver $Q_{\overline{\Lambda}}$ of $\overline{\Lambda}$ we will sometimes describe paths as a sequence of vertices. For a path $\rho = (x_1, \ldots, x_n)$ in $Q_{\overline{\Lambda}}$ we define the reverse path $\rho^{-1} = (x_n, \ldots, x_1)$. This will always describe a path in $Q_{\overline{\Lambda}}$ since there is an arrow from a vertex x to another vertex y if and only if there is an arrow from y to x.*

Lemma 4.3.27. *Let $a = (i, j, k; \lambda) \in I_0$ such that $a^- := (i-1, j-1, k-1; \lambda) \in I_0$. Then there is a descending path ρ of length 3 from a to a^- in the quiver $Q_{\overline{\Lambda}}$ of $\overline{\Lambda}$ such that $\rho\rho^{-1}$ corresponds to a non-zero endomorphism of $\overline{P}(a^-)$.*

Proof. This follows again from straight forward calculations, which can be found in the appendix. □

4.4 The principal block of $\mathbb{Z}_p(S_p \wr S_3)$

Recall that we fixed the p-modular system $(K, R, F) = (\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$. Throughout this section we let $\overline{\Lambda}$ be the basic algebra of the principal block of $\mathbb{F}_p(S_p \wr S_3)$ and Λ_0 be the basic algebra of the principal block of $\mathbb{Z}_p(S_p \wr S_3)$. Let I_0 and I_p be as in Definition 4.3.5. We use the notation from 4.2.1 for the S_p-modules. Let Λ be a Λ_0-lift of $\overline{\Lambda}$, i.e.

1. $\Lambda/p\Lambda \cong \overline{\Lambda}$

2. $K \otimes \Lambda$ is semisimple with center $\mathrm{Z}(K \otimes \Lambda) = \bigoplus_{a \in I_0} K$.

3. The decomposition matrix of Λ is the same as that of Λ_0.

4. Λ is self dual in $K \otimes \Lambda$ with respect to the form T_u with $u = (u_a)_{a \in I_0}$ and $u_a = \frac{\dim(V_a)}{|S_p \wr S_3|}$

Because $\overline{\Lambda}$ is basic and all decomposition numbers are one or zero, we know that $m_a := \dim V_a = |\{x \in I_p \mid d_{ax} \neq 0\}|$ for every $a \in I_0$. By Lemma 2.1.32 we can assume that

$$\Lambda \subseteq \bigoplus_{a \in I_0} R^{m_a \times m_a}.$$

We write $(x, y) \in Q_{\overline{\Lambda}}$ if $x, y \in I_p$ and there is an arrow between x and y in the quiver $Q_{\overline{\Lambda}}$ of $\overline{\Lambda}$. Recall that this is a symmetric property in this particular case.

Lemma 4.4.1. *For every $a \in I_0$ we obtain*

$$\nu_p(u_a) = \nu_p(\frac{\dim(V_a)}{|S_p \wr S_3|}) = -3$$

Proof. We know that every simple KS_p-module whose dimension is divisible by p lies in a defect zero block by Example 2.2.24. Therefore every simple $B_0(KS_p)$-module has a dimension prime to p. With the notation in Lemma 4.3.4 for $a = (i, j, k; \lambda) \in I_0$, the dimension of V_a is

$$\dim(V_a) = \dim(V(i)) \cdot \dim(V(j)) \cdot \dim(V(k)) \cdot C(\lambda) \cdot \dim(S^\lambda).$$

and as by assumption $p > 3$ we know that both $C(\lambda)$ and $\dim(S^\lambda)$ are prime to p. □

Notation 4.4.2. *We denote the primitive idempotents of Λ by e_x for $x \in I_p$ and the centrally primitive idempotents of $K \otimes \Lambda$ by ε_a for $a \in I_0$. For $x, y \in I_p$ with $(x, y) \in Q_{\overline{\Lambda}}$ we let $\beta_{(y,x)}$ be as in Definition 4.3.22. Let further $\hat{\beta}_{(y,x)}$ denote a lift of $\beta_{(y,x)}$ such that $\hat{\beta}_{(y,x)}\varepsilon_a = 0$ for every $a \in I_p \setminus c_x \cap c_y$. We further define*

$$\Gamma := \bigoplus_{a \in I_0} \varepsilon_a \Lambda = \bigoplus_{a \in I_0} R^{m_a \times m_a}.$$

With the notation above we know that

$$\mathrm{Gen}(\Lambda) := \{e_x \mid x \in I_p\} \cup \{\hat{\beta}_{(y,x)} \mid x, y \in I_p \text{ and } (x, y) \in Q_{\overline{\Lambda}}\}$$

is a generating system of Λ. Our goal is to determine as much information as we can about those generators to define a corresponding generating system of Λ that we can determine completely.

4.4.1 Endomorphism rings

Definition 4.4.3. *To every path $p = (x_1, \dots, x_n)$ in the quiver of Λ, we associate the homomorphism*

$$\Phi(p) = \beta_{(x_n, x_{n-1})} \circ \dots \circ \beta_{(x_2, x_1)}.$$

Lemma 4.4.4. *Let $a = (i, j, k; \lambda) \in I_p$ such that $a^- = (i-1, j-1, k-1; \lambda) \in I_p$.*

1. *For any two ascending (descending) paths p_1 and p_2 of length two between two vertices in the quiver there is a unit $F \in R^*$ such that $\Phi(p_1) = F\Phi(p_2)$.*

2. *For any two ascending (descending) paths p_1 and p_2 of length 3 between a and a^- there is a unit $F \in R^*$ such that $\Phi(p_1) = F\Phi(p_2)$.*

3. *For every ascending (descending) path p of length 3 between a^- and a we have $\Phi(pp^{-1}) \neq 0$.*

Proof. 1. This follows directly from Lemma 4.3.25.

2. Let $b = (i', j', k'; \lambda')$ be some vertex on an ascending path of length 3 between a^- and a. Then we have either $(a^-, b) \in Q_{\overline{\Lambda}}$ or $(a, b) \in Q_{\overline{\Lambda}}$, so $\lambda' \in c(\lambda)$. Also note that $i' \in \{i, i-1\}, j' \in \{j, j-1\}$ and $k' \in \{k, k-1\}$ and therefore $d_{ab} \neq 0$. Considering the quiver restricted to a as depicted in Remark 4.3.23 it is easy to see that any ascending path of length 3 from a^- to a can be transformed to any other such path by repeatedly exchanging ascending paths of length two. By Part 1 these actions only change the homomorphism by multiplication with units. The same argument works for the descending case.

3. This part follows from Part 2 together with Lemma 4.3.27.

□

Corollary 4.4.5. *Let $a = (i, j, k; \lambda) \in I_p$ such that $a^- = (i-1, j-1, k-1; \lambda) \in I_p$. Let further $x, y \in I_p$ with $a \in c_x \cap c_y$ and assume that $(x, y) \in Q_{\overline{\Lambda}}$. Then $(\hat{\beta}_{(y,x)}\hat{\beta}_{(x,y)})_a$ has p-valuation 1.*

Proof. As $(\hat{\beta}_{(y,x)}\hat{\beta}_{(x,y)})_a = (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})_a$ we can without loss assume that $x < y$. Then the arrow (x, y) lies on an ascending path $p = (a^-, a_1, a_2, a)$ from a^- to a, so $x, y \in \{a^-, a_1, a_2, a\}$. Then

$$\gamma := \beta_{(a,a_2)}\beta_{(a_2,a_1)}\beta_{(a_1,a^-)}\beta_{(a^-,a_1)}\beta_{(a_1,a_2)}\beta_{(a_2,a)}$$

is non-zero and therefore for the following lift of γ we get

$$\hat{\gamma} := \hat{\beta}_{(a,a_2)}\hat{\beta}_{(a_2,a_1)}\hat{\beta}_{(a_1,a^-)}\hat{\beta}_{(a^-,a_1)}\hat{\beta}_{(a_1,a_2)}\hat{\beta}_{(a_2,a)} \notin p\Lambda.$$

In particular Lemma 2.5.6 implies that $\hat{\gamma} \notin p^4\Gamma \subseteq p\Lambda$. Note that $c_a \cap c_{a^-} = \{a\}$ and thus the product $\hat{\gamma}$ is zero in every component except a. Therefore the entry $\hat{\gamma}_a$ has p-valuation at most 3. Lemma 2.5.11 implies that

$$F_{a^-a^-}(\hat{\beta}_{(a^-,a_1)}\hat{\beta}_{(a_1,a_2)}\hat{\beta}_{(a_2,a)}\hat{\beta}_{(a,a_2)}\hat{\beta}_{(a_2,a_1)}\hat{\beta}_{(a_1,a^-)}) =$$
$$F_{a^-a^-}(\hat{\beta}_{(a^-,a_1)}\hat{\beta}_{(a_1,a^-)})F_{a_1a_1}(\hat{\beta}_{(a_1,a_2)}\hat{\beta}_{(a_2,a_1)})F_{a_2a_2}(\hat{\beta}_{(a_2,a)}\hat{\beta}_{(a,a_2)}).$$

On the other hand we know that $F_{zz}(\hat{\beta}_{(z,t)}\hat{\beta}_{(t,z)}) \in pR^{|I_0|}$ for any $z,t \in I_p$. So $(\hat{\beta}_{(z,t)}\hat{\beta}_{(t,z)})_a$ has p-valuation 1 for $(z,t) \in \{(a,a_2),(a_2,a_1),(a_1,a^-)\}$, in particular for $(z,t) = (x,y)$. □

Definition 4.4.6. *1. For $i \in \{1,\dots,p\}$ or $i \in I_0$ let $r_i = \dim(V(i))$.*

2. For $i \in \{1,\dots,p-1\}$ define $\xi_i := \frac{r_{i+1}}{r_i}$ and $\eta_i := \frac{\xi_i+1}{p}$, so $\xi_i = -1 + \eta_i p$.

Lemma 4.4.7. *1. Using the notation above, we have*

$$\xi_i \equiv -1 \mod p \text{ and } \eta_i \in \mathbb{Z}.$$

2. Let $x \in I_p$. Then if an element of $e_x\Lambda e_x$ has exactly one non-zero entry, this entry needs to have p-valuation at least 3.

Proof. 1. We know that $\dim(P(i)) = \dim(V(i)) + \dim(V(i+1))$ and therefore by Theorem 2.2.16

$$r_i + r_{i+1} \equiv 0 \mod p \Rightarrow r_i \equiv -r_{i+1} \mod p$$
$$\Rightarrow \xi_i \equiv -1 \mod p \text{ and } \eta_i \in \mathbb{Z}.$$

2. This follows directly from the self-duality of $e_x\Lambda e_x$.

□

Notation 4.4.8. *1. For $a \in I_0$ denote the components of a via $a = (a_1,a_2,a_3;\lambda_a)$ and define $C_a := C(\lambda_a)$.*

2. For $x \in I_p$ and $a \in c_x$ define

$$n_a^x := |\{x_1,x_2,x_3\} \setminus \{a_1,a_2,a_3\}|,$$
$$\sigma_a^x := \frac{r_{x_1}r_{x_2}r_{x_3}}{r_{a_1}r_{a_2}r_{a_3}}$$
$$\rho_a^x := \frac{\sigma_a^x - (-1)^{n_a^x}}{p}.$$

We will often have a context where x is fixed. We will then omit the index x and write n_a, σ_a and ρ_a.

Lemma 4.4.9. *Let $x \in I_p$ and $a \in c_x$. Then we have*

$$\frac{r_{x_i}}{r_{a_i}} \equiv \begin{cases} 1 \mod p \text{ if } a_i = x_i \\ \xi_{x_i} \mod p \text{ otherwise} \end{cases}.$$

In particular

$$\sigma_a^x \equiv (-1)^{n_a^x} \mod p \text{ and } \rho_a^x \in \mathbb{Z}.$$

Proof. If $a_i \neq x_i$ then $a_i = x_{i+1}$ since $a \in c_x$ and the assertion follows by Lemma 4.4.7. □

Lemma 4.4.10. *Let $x \in I_p$ and $a \in c_x$. Then we obtain the congruence*

$$\frac{r_a}{r_{x_1} r_{x_2} r_{x_3}} \equiv C_a (-1)^{n_a^x} \mod p.$$

Proof. By Lemma 4.3.4 we know that $r_a = C_a r_{a_1} r_{a_2} r_{a_3}$. □

Lemma 4.4.11. *Let $x \in I_p$ and $\gamma \in e_x \Lambda e_x$. Then we obtain the congruences*

$$\sum_{a \in I_0} r_a^x \gamma_a \equiv 0 \mod p^3 \tag{4.18}$$

$$0 \equiv \sum_{a \in I_0} C_a (-1)^{n_a^x} \gamma_a \mod p^2. \tag{4.19}$$

Proof. Self-duality of Λ implies

$$\begin{aligned} \mathrm{T}_u(\gamma, e_x) = \frac{1}{|S_p \wr S_3|} \sum_{a \in I_0} r_a \gamma_a \in R \\ \Rightarrow p^3 \mid \sum_{a \in I_0} r_a \gamma_a \\ \Rightarrow 0 \equiv \sum_{a \in I_0} C_a \frac{r_a}{r_{x_1} r_{x_2} r_{x_3}} \gamma_a \equiv \sum_{a \in I_0} C_a (-1)^{n_a} \gamma_a \mod p^2 \end{aligned}$$

where the last equivalence follows since $\nu_p(\gamma_a) \geq 1$ and thus

$$\frac{r_a}{r_{x_1} r_{x_2} r_{x_3}} \gamma_a - C_a (-1)^{n_a} \gamma_a \equiv 0 \mod p^2.$$

□

Lemma 4.4.12. *Let xI_p, $\gamma \in e_x \Lambda e_x$ and assume that there is a unit $\beta_a \in R^*$ such that $\gamma_a = \beta_a p$ whenever $\gamma_a \neq 0$. Define $\beta_a = 0$ otherwise. Then we obtain the congruences*

$$\sum_{a \in I_0} r_a^x \beta_a \equiv 0 \mod p^2 \tag{4.20}$$

$$0 \equiv \sum_{a \in I_0} C_a (-1)^{n_a^x} \beta_a \mod p \tag{4.21}$$

$$0 \equiv \sum_{a \in I_0} C_a (-1)^{n_a^x} \beta_a^2 \mod p. \tag{4.22}$$

Proof. The first two congruences follow from Lemma 4.4.11:

$$\begin{aligned} 4.18: \sum_{a\in I_0} r_a\gamma_a &\equiv 0 \mod p^3 \\ \Rightarrow \sum_{a\in I_0} r_a\beta_a &\equiv 0 \mod p^2 \\ 4.19: \sum_{a\in I_0} C_a(-1)^{n_a}\gamma_a &\equiv 0 \mod p^2 \\ \Rightarrow \sum_{a\in I_0} C_a(-1)^{n_a}\beta_a &\mod p \end{aligned}$$

The third congruence follows from the same arguments as in Lemma 4.4.11 by the considering the trace of γ with itself.

$$\begin{aligned} \mathrm{T}_u(\gamma,\gamma) &= \frac{1}{|S_p \wr S_3|}\sum_{a\in I_0} r_a\gamma_a^2 \in R \\ \Rightarrow p^3 | \sum_{a\in I_0} r_a\gamma_a^2 &= \sum_{a\in I_0} r_a\beta_a^2 p^2 \\ \Rightarrow 0 \equiv \sum_{a\in I_0} C_a(-1)^{n_a}\beta_a^2 &\mod p \end{aligned}$$

□

Lemma 4.4.13. *Let $x \in I_p$, $\gamma \in e_x\Lambda e_x$, $\gamma' \in e_x\Gamma e_x$ such that $\gamma_a \equiv \gamma'_a \mod p^2$ for every $a \in I_0$ and assume that $\mathrm{T}_u(\gamma', 1) \in R$. Then $\gamma - \gamma' \in \Lambda \cap p^2\Gamma$. In particular, $\gamma' \in \Lambda$.*

Proof. It is clear that $\gamma - \gamma' \in p^2\Gamma$. So all that remains is to show that $\gamma' \in \Lambda$. By assumption we know that $\mathrm{T}_u(\gamma, e_x) \in R$. Now let $\delta \in e_x\Lambda e_x \setminus e_xR \subseteq p\Gamma$. Then

$$\begin{aligned} p^3(\mathrm{T}_u(\gamma',\delta) - \mathrm{T}_u(\gamma,\delta)) &= p^3\,\mathrm{T}_u(\gamma'-\gamma,\delta) \\ &\in p^3\,\mathrm{T}_u(p^2\Gamma, p\Gamma) \subseteq p^3R. \end{aligned}$$

Thus we obtain $\mathrm{T}_u(\gamma', \delta) \in R$. Therefore $\gamma' \in (e_x\Lambda e_x)^\sharp = e_x\Lambda e_x$ which concludes the proof. □

Lemma 4.4.14. *Let $x, y \in I_p$, $\gamma = \gamma^1 \cdot \gamma^2 \in e_x\Lambda e_y e_y\Lambda e_x$ and γ' as above. Define $\delta := \gamma^2\gamma^1 \in e_y\Lambda e_y$ and $\delta' \in e_y\Lambda e_y$ with $\delta'_a = \gamma'_a$ for every $a \in I_0$. Then*

$$\delta - \delta' \in \Lambda \cap p^2\Gamma.$$

Proof. By Lemma 2.5.11 we know that $\gamma_a = \delta_a$ for every $a \in I_0$. Furthermore $\mathrm{T}_u(\delta', 1) = \mathrm{T}_u(\gamma', 1)$ and therefore the assumptions of Lemma 4.4.13 are fullfilled for δ, δ'. □

Lemma 4.4.15. *Let $\gamma \in e_x\Gamma e_x$, $a, b \in c_x$ such that $\gamma_c = 0$ if $c \notin \{a, b\}$ and $\gamma_a = \beta_a p^2, \gamma_b = \beta_b p^2$ with $\beta_a, \beta_b \in R^*$. Then*

$$\gamma \in \Lambda \Leftrightarrow (-1)^{n_a}C_a\beta_a \equiv -(-1)^{n_b}C_b\beta_b \mod p.$$

Proof. First consider the trace of γ with e_x:

$$\begin{aligned}\frac{|S_p \wr S_3|\,\mathrm{T}_u(\gamma, e_x)}{r_{x_1} r_{x_2} r_{x_3}} &\equiv (-1)^{n_a} C_a \gamma_a + (-1)^{n_b} C_b \gamma_b \equiv 0 \mod p^3 \\ &\Rightarrow (-1)^{n_a} C_a \beta_a + (-1)^{n_b} C_b \beta_b \equiv 0 \mod p,\end{aligned}$$

which implies the direction from left to right. For the converse recall that $e_x \Lambda e_x \smallsetminus e_x R^* \subseteq p\Gamma$. By assumption we know that $\mathrm{T}_u(\gamma, e_x) \in R$ and $\gamma e_x (p\Gamma) e_x \subseteq p^3\Gamma$, which implies that $\mathrm{T}_u(\gamma\beta) \in R$ for any $\beta \in e_x \Lambda e_x \smallsetminus e_x R^*$. Thus we conclude that $\gamma \in (e_x \Lambda e_x)^{\#} = e_x \Lambda e_x$. □

Lemma 4.4.16. *Let $x \in I_p$, $a \neq b \in I_0$ and $\gamma, \delta \in e_x \Lambda e_x \smallsetminus e_x R^*$ with $(\gamma\delta)_c \neq 0$ if and only if $c \in \{a, b\}$.*

1. *If $\nu_p((\gamma\delta)_a) = 2$, then $\nu_p(\gamma_b) = \nu_p(\delta_b) = 1$.*
2. *Assume that $(\gamma\delta)_a \equiv \alpha p^2 \mod p^3$ and $\delta_b = \beta p \mod p^2$ with $\alpha, \beta \in R^*$. Then*

$$\gamma_p \equiv \frac{-(-1)^{n_a} C_a \alpha p}{(-1)^{n_b} C_b \beta} \mod p^2.$$

Proof. By assumption $\nu_p(\gamma_b), \nu_p(\delta_b) \geq 1$. If one of them had p-valuation larger than zero, then for $\vartheta \in e_x \Lambda e_x$ with $\vartheta_b = \gamma_b \delta_b$ and $\vartheta_c = 0$ for all $c \neq 0$ we would have $\vartheta \in e_x \Lambda e_x$ since $\nu_p(\gamma_b \delta_b) \geq 3$. Then $\gamma\delta - \vartheta$ would be an element contradicting Lemma 4.4.7 Part 2.

For the second part, first note that we already know that $p \mid \gamma_b$. Then consider the trace of $\gamma\delta$ with e_x. We have

$$\begin{aligned}\frac{|S_p \wr S_3|}{r_{x_1} r_{x_2} r_{x_3}} \mathrm{T}_u(\gamma\delta, e_x) &= \sum_{c \in I_0} \frac{r_c}{r_{x_1} r_{x_2} r_{x_3}} \gamma_c \delta_c \\ &= \frac{r_a}{r_{x_1} r_{x_2} r_{x_3}} \gamma_a \delta_a + \frac{r_b}{r_{x_1} r_{x_2} r_{x_3}} \gamma_b \delta_b \\ &\equiv C_a \cdot (-1)^{n_a} \alpha p^2 + C_b \cdot (-1)^{n_b} \frac{\delta_b}{p} \cdot p \cdot \beta p \mod p^3 \\ \Rightarrow \gamma_b &\equiv \frac{-(-1)^{n_a} C_a \alpha p}{(-1)^{n_b} C_b \beta} \mod p^2\end{aligned}$$

□

Lemma 4.4.17. *Let $x \in I_p$, $\gamma \in e_x \Lambda e_x$, $a \neq b \in I_0$ and assume that $\gamma_c \neq 0 \Leftrightarrow c \in \{a, b\}$ and assume further that $\gamma_a = \beta_a p, \gamma_b = \beta_b p$ with $\beta_a, \beta_b \in R^*$. Then $C_a(-1)^{n_a} \equiv -C_b(-1)^{n_b} \mod p$.*

Proof. By congruences 4.21 and 4.22 we obtain that

$$\begin{aligned} 0 &\equiv C_a(-1)^{n_a}\beta_a + C_b(-1)^{n_b}\beta_b \mod p \\ 0 &\equiv C_a(-1)^{n_a}\beta_a^2 + C_b(-1)^{n_b}\beta_b^2 \mod p \\ \Rightarrow \beta_a &\equiv \frac{C_b(-1)^{n_b}\beta_b}{C_a(-1)^{n_a}} \mod p \\ \Rightarrow 0 &\equiv C_b(-1)^{n_b}\beta_b^2\left(\frac{C_b(-1)^{n_b}}{C_a(-1)^{n_a}} + 1\right) \\ \Rightarrow C_a(-1)^{n_a} &\equiv -C_b(-1)^{n_b} \end{aligned}$$

□

Notation 4.4.18. *For two elements $\alpha, \beta \in \Lambda$ we write $\alpha \sim \beta$ if there is a unit $\xi \in R^*$ such that $\alpha \equiv \xi\beta \mod \Lambda \cap p^2\Gamma$.*

Definition 4.4.19. *Let $x, y \in I_p$ be two indices such that $(x, y) \in Q_{\overline{\Lambda}}$.*

1. *We call $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)} \in e_x\Lambda e_x$ a* standard endomorphism.
2. *We call the elements of $c_x \cap c_y$ the* relevant positions *of $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$. If $a \in I_0$ is a relevant positions, we say $(\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})_a$ is a* relevant entry *of $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$.*

We are now going to determine the standard endomorphisms up to multiplication with a unit and modulo $p^2\Gamma \cap \Lambda$. For these calculations we will identify the endomorphism rings $e_x\Lambda e_x$ for $x \in I_p$ with the direct sum $\bigoplus_{i \in c_x} K$ as in Remark 2.5.13. We sometimes write i^+ for $i+1$ and i^- for $i-1$.

Theorem 4.4.20. *The endomorphism rings are the row spaces of the following matrices. The standard endomorphisms are given up to multiplication with a unit and modulo $\Lambda \cap p^2\Gamma$. If γ is a standard endomorphism we denote the endomorphism defined as in the matrix below by γ'.*

- *$e_{ijk}\Lambda e_{ijk}$ for $|i-j| > 1$, $|j-k| > 1$*

	ijk	i^+jk	ij^+k	ijk^+	ij^+k^+	i^+jk^+	i^+j^+k	$i^+j^+k^+$
	1	1	1	1	1	1	1	1
(i^+,j,k)	0	p	0	0	0	p	p	p
(i,j^+,k)	0	0	p	0	p	0	p	p
(i,j,k^+)	0	0	0	p	p	p	0	p
	0	0	0	0	p^2	0	0	p^2
	0	0	0	0	0	p^2	0	p^2
	0	0	0	0	0	0	p^2	p^2
	0	0	0	0	0	0	0	p^3

The standard endomorphisms written as row vectors are as follows.

$$\begin{array}{c} \\ (i^-,j,k) \\ (i^+,j,k) \\ (i,j^-,k) \\ (i,j^+,k) \\ (i,j,k^-) \\ (i,j,k^+) \end{array} \begin{array}{c} \begin{array}{cccccccc} ijk & i^+jk & ij^+k & ijk^+ & ij^+k^+ & i^+jk^+ & i^+j^+k & i^+j^+k^+ \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p & 0 & p & p & p & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & p & p & p \\ p & p & 0 & p & 0 & p & 0 & 0 \\ 0 & 0 & p & 0 & p & 0 & p & p \\ p & p & p & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & p & p & p & 0 & p \end{pmatrix} \end{array}$$

- $e_{iii\lambda}\Lambda e_{iii\lambda}$, $\mu \in c_\lambda$

$$\begin{array}{c} \\ iii^+\mu \end{array} \begin{array}{c} \begin{array}{cccc} iii\lambda & iii^+\mu & ii^+i^+\mu & i^+i^+i^+\lambda \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & p & 2p & 3p \\ 0 & 0 & p^2 & 3p^2 \\ 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

$$\begin{array}{c} \\ i^-ii\mu \\ iii^+\mu \end{array} \begin{array}{c} \begin{array}{cccc} iii\lambda & iii^+\mu & ii^+i^+\mu & i^+i^+i^+\lambda \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3p & 2p & p & 0 \\ 0 & p & 2p & 3p \end{pmatrix} \end{array}$$

- $e_{iii(2,1)}\Lambda e_{iii(2,1)}$

$$\begin{array}{c} \\ iii^+\mu \\ iii^+\mu' \end{array} \begin{array}{c} \begin{array}{cccccc} iii(2,1) & iii^+\mu & iii^+\mu' & ii^+i^+\mu & ii^+i^+\mu' & i^+i^+i^+(2,1) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2p & 0 & p & 3p & 3p \\ 0 & 0 & 2p & 3p & p & 3p \\ 0 & 0 & 0 & 2p^2 & 0 & 3p^2 \\ 0 & 0 & 0 & 0 & 2p^2 & 3p^2 \\ 0 & 0 & 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

	$iii(2,1)$	$iii^+\mu$	$iii^+\mu'$	$ii^+i^+\mu$	$ii^+i^+\mu'$	$i^+i^+i^+(2,1)$
	1	1	1	1	1	1
$i^-ii\mu$	$3p$	p	$3p$	$2p$	0	0
$i^-ii\mu'$	$3p$	$3p$	p	0	$2p$	0
$iii^+\mu$	0	$2p$	0	p	$3p$	$3p$
$iii^+\mu'$	0	0	$2p$	$3p$	p	$3p$

- $e_{iii^+\mu}\Lambda e_{iii^+\mu}$

	$iii^+\mu$	$ii^+i^+\mu'$	$ii^+i^+\mu$	$ii(i+2)\mu$	$i^+i^+i^+\lambda$	$i^+i^+i^+(2,1)$	$ii^+(i+2)$	$i^+i^+(i+2)\mu$
	1	1	1	1	1	1	1	1
$ii^+i^+\mu'$	0	$2p$	0	0	0	$3p$	p	$2p$
$ii^+i^+\mu$	0	0	$2p$	0	$4p$	p	p	$2p$
$ii(i+2)\mu$	0	0	0	p	0	0	p	p
	0	0	0	0	$3p^2$	0	0	p^2
	0	0	0	0	0	$3p^2$	0	$2p^2$
	0	0	0	0	0	0	p^2	$2p^2$
	0	0	0	0	0	0	0	p^3

Standard endomorphisms:

	$iii^+\mu$	$ii^+i^+\mu'$	$ii^+i^+\mu$	$ii(i+2)\mu$	$i^+i^+i^+\lambda$	$i^+i^+i^+(2,1)$	$ii^+(i+2)$	$i^+i^+(i+2)\mu$
	1	1	1	1	1	1	1	1
$iii\lambda$	p	0	$2p$	0	$3p$	0	0	0
$iii(2,1)$	$2p$	$3p$	p	0	0	$3p$	0	0
$ii(i+2)\mu$	0	0	0	p	0	0	p	p
i^-ii^+	$2p$	p	p	$2p$	0	0	p	0
$ii^+i^+\mu'$	0	$2p$	0	0	0	$3p$	p	$2p$
$ii^+i^+\mu$	0	0	$2p$	0	$4p$	p	p	$2p$

- $e_{i^-ii\mu}\Lambda e_{i^-ii\mu}$

$$\begin{array}{c} \\ iii\lambda \\ iii(2,1) \\ i^-ii^+ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccccc} i^-ii\mu & iii\lambda & iii(2,1) & i^-ii^+ & iii^+\mu' & iii^+\mu & i^-i^+i^+\mu & ii^+i^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3p & 0 & 0 & 0 & 2p & 0 & p \\ 0 & 0 & 3p & 0 & 3p & p & 0 & 2p \\ 0 & 0 & 0 & p & p & p & 2p & 2p \\ 0 & 0 & 0 & 0 & p^2 & 0 & 0 & p^2 \\ 0 & 0 & 0 & 0 & 0 & p^2 & 0 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

$$\begin{array}{c} \\ i^-i^-i\mu' \\ i^-i^-i\mu \\ i\text{–}2ii\mu \\ iii\lambda \\ iii(2,1) \\ i^-ii^+ \\ \end{array}\begin{array}{c} \begin{array}{cccccccc} i^-ii\mu & iii\lambda & iii(2,1) & i^-ii^+ & iii^+\mu' & iii^+\mu & i^-i^+i^+\mu & ii^+i^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2p & 0 & 3p & p & 2p & 0 & 0 & 0 \\ 2p & 4p & p & p & 0 & 2p & 0 & 0 \\ p & 0 & 0 & p & 0 & 0 & p & 0 \\ 0 & 3p & 0 & 0 & 0 & 2p & 0 & p \\ 0 & 0 & 3p & 0 & 3p & p & 0 & 2p \\ 0 & 0 & 0 & p & p & p & 2p & 2p \end{pmatrix} \end{array}$$

- $e_{iij\mu}\Lambda e_{iij\mu}$

$$\begin{array}{c} \\ ii^+j \\ iij^+\mu \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} iij\mu & ii^+j & iij^+\mu & i^+i^+j\mu & ii^+j^+ & i^+i^+j^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & p & 0 & 2p & p & 2p \\ 0 & 0 & p & 0 & p & p \\ 0 & 0 & 0 & p^2 & 0 & p^2 \\ 0 & 0 & 0 & 0 & p^2 & 2p^2 \\ 0 & 0 & 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

$$\begin{array}{c} \\ iij^-\mu \\ i^-ij \\ ii^+j \\ iij^+\mu \end{array}\begin{array}{c} \begin{array}{cccccc} iij\mu & ii^+j & iij^+\mu & i^+i^+j\mu & ii^+j^+ & i^+i^+j^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ p & p & 0 & p & 0 & 0 \\ 2p & p & 2p & 0 & p & 0 \\ 0 & p & 0 & 2p & p & 2p \\ 0 & 0 & p & 0 & p & p \end{pmatrix} \end{array}$$

- $e_{jii\mu}\Lambda e_{jii\mu}$

$$\begin{array}{c} \\ j^+ii\mu \\ jii^+ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} jii\mu & j^+ii\mu & jii^+ & j^+ii^+ & ji^+i^+\mu & j^+i^+i^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & p & 0 & p & 0 & p \\ 0 & 0 & p & p & 2p & 2p \\ 0 & 0 & 0 & p^2 & 0 & 2p^2 \\ 0 & 0 & 0 & 0 & p^2 & p^2 \\ 0 & 0 & 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

$$\begin{array}{c} \\ ji^-i \\ j^-ii\mu \\ j^+ii\mu \\ jii^+ \end{array}\begin{array}{c} \begin{array}{cccccc} jii\mu & j^+ii\mu & jii^+ & j^+ii^+ & ji^+i^+\mu & j^+i^+i^+\mu \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2p & 2p & p & p & 0 & 0 \\ p & 0 & p & 0 & p & 0 \\ 0 & p & 0 & p & 0 & p \\ 0 & 0 & p & p & 2p & 2p \end{pmatrix} \end{array}$$

- $e_{jii^+\mu}\Lambda e_{jii^+\mu}$, $i_2 := i+2$

$$\begin{array}{c} \\ j^+ii^+ \\ ji^+i^+(1{,}1) \\ ji^+i^+(2) \\ jii_2 \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccccccc} jii^+ & j^+ii^+ & ji^+i^+(1{,}1) & ji^+i^+(2) & jii_2 & j^+i^+i^+(1{,}1) & j^+i^+i^+(2) & j^+ii_2 & ji^+i_2 & j^+i^+i_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & p & 0 & 0 & 0 & p & p & p & 0 & p \\ 0 & 0 & 2p & 0 & 0 & 2p & 0 & 0 & p & p \\ 0 & 0 & 0 & 2p & 0 & 0 & 2p & 0 & p & p \\ 0 & 0 & 0 & 0 & p & 0 & 0 & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 2p^2 & 0 & 0 & 0 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2p^2 & 0 & 0 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & p^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^3 \end{pmatrix} \end{array}$$

Standard endomorphisms:

	jii^+	j^+ii^+	$ji^+i^+(1,1)$	$ji^+i^+(2)$	jii_2	$j^+i^+i^+(1,1)$	$j^+i^+i^+(2)$	j^+ii_2	ji^+i_2	$j^+i^+i_2$
	1	1	1	1	1	1	1	1	1	1
$jii(1,1)$	p	p	$2p$	0	0	$2p$	0	0	0	0
$jii(2)$	p	p	0	$2p$	0	0	$2p$	0	0	0
ji^-i^+	p	p	0	0	p	0	0	p	0	0
j^-ii^+	p	0	p	p	p	0	0	0	p	0
j^+ii^+	0	p	0	0	0	p	p	p	0	p
$ji^+i^+(1,1)$	0	0	$2p$	0	0	$2p$	0	0	p	p
$ji^+i^+(2)$	0	0	0	$2p$	0	0	$2p$	0	p	p
jii_2	0	0	0	0	p	0	0	p	p	p

- $e_{i^-ij\mu}\Lambda e_{i^-ij\mu}$

	i^-ij	$iij(1,1)$	$iij(2)$	i^-i^+j	i^-ij^+	ii^+j	$iij^+(1,1)$	$iij^+(2)$	$i^-i^+j^+$	ii^+j^+
	1	1	1	1	1	1	1	1	1	1
$iij(1,1)$	0	$2p$	0	0	0	p	$2p$	0	0	p
$iij(2)$	0	0	$2p$	0	0	p	0	$2p$	0	p
i^-i^+j	0	0	0	p	0	p	0	0	p	p
i^-ij^+	0	0	0	0	p	0	p	p	p	p
	0	0	0	0	0	p^2	0	0	0	p^2
	0	0	0	0	0	0	$2p^2$	0	0	p^2
	0	0	0	0	0	0	0	$2p^2$	0	p^2
	0	0	0	0	0	0	0	0	p^2	p^2
	0	0	0	0	0	0	0	0	0	p^3

Standard endomorphisms:

	i^-ij	$iij(1,1)$	$iij(2)$	i^-i^+j	i^-ij^+	ii^+j	$iij^+(1,1)$	$iij^+(2)$	$i^-i^+j^+$	ii^+j^+
	1	1	1	1	1	1	1	1	1	1
i^-ij^-	p	p	p	p	0	p	0	0	0	0
$i^-i^-j(1,1)$	p	$2p$	0	0	p	0	$2p$	0	0	0
$i^-i^-j(2)$	p	0	$2p$	0	p	0	0	$2p$	0	0
$i–2ij$	p	0	0	p	p	0	0	0	p	0
$iij(1,1)$	0	$2p$	0	0	0	p	$2p$	0	0	p
$iij(2)$	0	0	$2p$	0	0	p	0	$2p$	0	p
i^-i^+j	0	0	0	p	0	p	0	0	p	p
i^-ij^+	0	0	0	0	p	0	p	p	p	p

- $e_{i^-ii^+}\Lambda e_{i^-ii^+}$

	i^-ii^+	$iii^+(1,1)$	$iii^+(2)$	$i^-i^+i^+(1,1)$	$i^-i^+i^+(2)$	$i^-i(i{+}2)$	$ii^+i^+(1,1)$	$ii^+i^+(2)$	$ii(i{+}2)(1,1)$	$ii(i{+}2)(2)$	$i^-i^+(i{+}2)$	$ii^+(i{+}2)$
	1	1	1	1	1	1	1	1	1	1	1	1
$iii^+(1,1)$	0	$2p$	0	0	0	0	p	p	$2p$	0	0	p
$iii^+(2)$	0	0	$2p$	0	0	0	p	p	0	$2p$	0	p
$i^-i^+i^+(1,1)$	0	0	0	$2p$	0	0	$2p$	0	0	0	p	p
$i^-i^+i^+(2)$	0	0	0	0	$2p$	0	0	$2p$	0	0	p	p
$i^-i(i{+}2)$	0	0	0	0	0	p	0	0	p	p	p	p
	0	0	0	0	0	0	$2p^2$	0	0	0	0	p^2
	0	0	0	0	0	0	0	$2p^2$	0	0	0	p^2
	0	0	0	0	0	0	0	0	$2p^2$	0	0	p^2
	0	0	0	0	0	0	0	0	0	$2p^2$	0	p^2
	0	0	0	0	0	0	0	0	0	0	p^2	p^2
	0	0	0	0	0	0	0	0	0	0	0	p^3

Standard endomorphisms:

	i^-ii^+	$iii^+(1,1)$	$iii^+(2)$	$i^-i^+i^+(1,1)$	$i^-i^+i^+(2)$	$i^-i(i{+}2)$	$ii^+i^+(1,1)$	$ii^+i^+(2)$	$ii(i{+}2)(1,1)$	$ii(i{+}2)(2)$	$i^-i^+(i{+}2)$	$ii^+(i{+}2)$
	1	1	1	1	1	1	1	1	1	1	1	1
$i^-ii(1,1)$	p	p	p	$2p$	0	0	$2p$	0	0	0	0	0
$i^-ii(2)$	p	p	p	0	$2p$	0	0	$2p$	0	0	0	0
$i^-i^-i^+(1,1)$	p	$2p$	0	0	0	p	0	0	$2p$	0	0	0
$i^-i^-i^+(2)$	p	0	$2p$	0	0	p	0	0	0	$2p$	0	0
$i{-}2ii^+$	p	0	0	p	p	p	0	0	0	0	p	0
$iii^+(1,1)$	0	$2p$	0	0	0	0	p	p	$2p$	0	0	p
$iii^+(2)$	0	0	$2p$	0	0	0	p	p	0	$2p$	0	p
$i^-i^+i^+(1,1)$	0	0	0	$2p$	0	0	$2p$	0	0	0	p	p
$i^-i^+i^+(2)$	0	0	0	0	$2p$	0	0	$2p$	0	0	p	p
$i^-i(i{+}2)$	0	0	0	0	0	p	0	0	p	p	p	p

Proof. We will first determine the standard endomorphisms and then conclude that the endomorphism spaces are as in the assertion. To follow the proof, the reader is strongly adviced to follow the arguments using the matrices above.

We will do the calculations in two steps. First we will determine all entries modulo p^2. Then we will show that the assumptions of Lemma 4.4.13 are fulfilled for the endomorphisms defined.

1. $x = (i, i, i; \lambda)$, $(\lambda, \mu) \in \{((3), (2)), ((1,1,1), (1,1))\}$.

   First we want to show that the entries at all relevant positions have p-valuation one for both $\hat{\beta}_{(iii\lambda, iii^+\mu)}\hat{\beta}_{(iii^+\mu, iii\lambda)}$ and $\hat{\beta}_{(iii\lambda, iii^-\mu)}\hat{\beta}_{(iii^-\mu, iii\lambda)}$.

   This condition is fulfilled whenever $p-1 > i > 1$ by Corollary 4.4.5. The arguments for the two edge cases are very similar so we we will only consider the case $i = 1$. Then only $\gamma = \hat{\beta}_{(iii\lambda, iii^+\mu)}\hat{\beta}_{(iii^+\mu, iii\lambda)}$ exists.

   By Corollary 4.4.5 the entry $\gamma_{i^+i^+i^+\lambda}$ has p-valuation one. Assume that both other entries have a p-valuation greater that one. By considering $p\gamma$ we can see that this would contradict Remark 4.4.7 Part 2, since $p^3\Gamma \subseteq \Lambda$.

   The case where γ has exactly one entry with p-valuation one besides $\gamma_{i^+i^+i^+\lambda}$ is impossible by Lemma 4.4.17 since $C_{i^+i^+i^+\lambda} = -1$, $C_{iii^+\mu} = -3$ and $C_{ii^+i^+\mu} = 3$.

   We can conclude that for any i all relevant entries have p-valuation 1, so there are units $a, b, c, d \in R$ fulfilling the following.

$$\begin{array}{rcccc} & iii\lambda & iii^+\mu & ii^+i^+\mu & i^+i^+i^+\lambda \\ e_{iii\lambda} = & (1, & 1, & 1, & 1) \\ \hat{\beta}_{(iii\lambda, iii^-\mu)}\hat{\beta}_{(iii^-\mu, iii\lambda)} \sim & (ap, & bp, & p, & 0) \\ \hat{\beta}_{(iii\lambda, iii^+\mu)}\hat{\beta}_{(iii^+\mu, iii\lambda)} \sim & (0, & p, & cp, & dp) \end{array}$$

   First, consider $\gamma := (\hat{\beta}_{(iii\lambda, iii^-\mu)}\hat{\beta}_{(iii^-\mu, iii\lambda)})' = (ap, bp, p, 0)$. From 4.21 and 4.22 we get the following two congruences.

$$\begin{aligned} a - 3b + 3 &\equiv 0 \mod p \\ a^2 - 3b^2 + 3 &\equiv 0 \mod p \end{aligned}$$

   These congruences have two solutions, $a \equiv 0, b \equiv 1 \mod p$ and $a \equiv 3, b \equiv 2 \mod p$. The first solution is impossible as we already know that $\nu_p(ap) = 1$. Using the same arguments for $\hat{\beta}_{(iii\lambda, iii^+\mu)}\hat{\beta}_{(iii^+\mu, iii\lambda)}$ we get that $c \equiv 2 \mod p$ and $d \equiv 3 \mod p$.

2. $x = (i, i, i+1; \mu)$, $\mu, \mu' \vdash 2$, $\mu \neq \mu'$, $\lambda \vdash 3$, $\lambda \in c_\mu$, $1 \leq i < p-1$

   By Lemma 2.5.11 we have already determined the entries of a unit multiple of $\hat{\beta}_{(x, iii\lambda)}\hat{\beta}_{(iii\lambda, x)}$ modulo p^2.

   First we will use Lemma 4.4.16 Part 1 to show that the standard endomorphisms have p-valuation 1 at some positions not covered by Lemma 4.4.5. The following

table lists the combinations of endomorphisms and indices to which we apply the lemma. Here y,z are indices such that we apply the lemma to $\gamma = \hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$ and $\delta = \hat{\beta}_{(x,z)}\hat{\beta}_{(z,x)}$.

y	z	a	b	i
$(i,i,i;(2,1))$	$(i,i^+,i^+;\mu)$	$(i^+,i^+,i^+;(2,1))$	$(i,i^+,i^+;\mu)$	1
$(i,i,i;(2,1))$	$(i,i^+,i^+;\mu')$	$(i^+,i^+,i^+;(2,1))$	$(i,i^+,i^+;\mu')$	1
$(i,i,i;(2,1))$	$(i,i,i;\lambda)$	$(i,i^+,i^+;\mu)$	$(i,i,i^+;\mu)$	1
$(i,i,i+2;\mu)$	$(i,i^+,i^+;\mu))$	$(i^+,i^+,i+2;\mu)$	$(i,i^+,i+2)$	1
$(i,i,i+2;\mu)$	$(i,i^+,i^+;\mu'))$	$(i^+,i^+,i+2;\mu')$	$(i,i^+,i+2)$	1
(i^-,i,i^+)	$(i,i^+,i^+;\mu)$	$(i,i,i^+;\mu)$	$(i,i^+,i+2)$	$p-2$
(i^-,i,i^+)	$(i,i^+,i^+;\mu')$	$(i,i,i^+;\mu')$	$(i,i^+,i+2)$	$p-2$

This covers all entries except the following.

$$
\begin{aligned}
i = 1 &: \\
(\hat{\beta}_{(x,iii+2\mu)}\hat{\beta}_{(iii+2\mu,x)})_{iii+2\mu} & \\
i = p-2 &: \\
(\hat{\beta}_{(x,i^-ii^+)}\hat{\beta}_{(i^-ii^+,x)})_{iii+2\mu}&, \\
(\hat{\beta}_{(x,ii^+i^+\mu)}\hat{\beta}_{(ii^+i^+\mu,x)})_{i^+i^+i+2\mu}&, \\
(\hat{\beta}_{(x,ii^+i^+\mu')}\hat{\beta}_{(ii^+i^+\mu',x)})_{i^+i^+i+2\mu} &
\end{aligned}
$$

From Lemma 4.4.17 it follows that $(\hat{\beta}_{(x,113\mu)}\hat{\beta}_{(113\mu,x)})_{112\mu}$ has p-valuation one. We will not use the p-valuation of the remaining entries.

For every standard endomorphism γ in $e_x\Lambda e_x$ we pick one position $a \in I_0$ for which we know that $\nu_p(\gamma_a) = 1$ and multiply γ with a unit to obtain another endomorphism γ^* for which γ^*_a is a certain multiple of p:

$$
\begin{aligned}
\gamma^* &:= n\left(\frac{\gamma_a}{p}\right)^{-1}\gamma \\
&\Rightarrow \gamma^*_a = np
\end{aligned}
$$

In cases where the endomorphism only exists for certain i we add a restriction on i.

γ	a	γ_a	i
$\hat{\beta}_{(x,iii(2,1))}\hat{\beta}_{(iii(2,1),x)}$	$(i,i+1,i+1;\mu)$	p	–
$\hat{\beta}_{(x,ii(i+2)\mu)}\hat{\beta}_{(ii(i+2)\mu,x)}$	$(i,i,i+2;\mu)$	p	$i < p-2$
$\hat{\beta}_{(x,i^-ii^+)}\hat{\beta}_{(i^-ii^+,x)}$	$(i,i+1,i+1;\mu)$	p	–
$\hat{\beta}_{(x,ii^+i^+\mu')}\hat{\beta}_{(ii^+i^+\mu',x)}$	$(i,i+1,i+1;\mu')$	$2p$	–
$\hat{\beta}_{(x,ii^+i^+\mu)}\hat{\beta}_{(ii^+i^+\mu,x)}$	$(i,i+1,i+1;\mu)$	$2p$	$i > 1$

We use the congruences 4.19, 4.21 and 4.22 and Lemma 4.4.16 to determine the remaining entries modulo p^2. We will again only list the endomorphisms and indices for which we apply the lemmas and the results. A row in the following

table means: We apply the lemma/ congruence from the first column to $\gamma = (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})^*$, and $\delta = (\hat{\beta}_{(x,z)}\hat{\beta}_{(z,x)})^*$ when we apply Lemma 4.4.16, where y, z are given as in the second column. From this we deduce the entry γ_a modulo p^2 at position a as in the fourth and fifth column. The last column gives restrictions to i as above.

Lemma/ Congruence	y	z	a	γ_a	i
4.21,4.22	$(i,i,i+2;\mu)$	-	$(i,i^+,i+2)$	p	$i < p-2$
			$(i^+,i^+,i+2;\mu)$	p	
4.4.16	$(i,i,i;(2,1))$	$(i,i,i;\lambda)$	$(i,i,i^+;\mu)$	$2p$	–
4.21,4.22	$(i,i,i;(2,1))$	-	$(i,i^+,i^+;\mu')$	$3p$	–
			$(i^+,i^+,i^+;(2,1))$	$3p$	
4.4.16	(i^-,i,i^+)	$(i,i,i;\lambda)$	$(i,i,i^+;\mu)$	$2p$	
4.4.16	$(i,i^+,i^+;\mu')$	$(i,i,i;(2,1))$	$(i^+,i^+,i^+;(2,1))$	$3p$	–
4.4.16	$(i,i^+,i^+;\mu)$	$(i,i,i;\lambda)$	$(i^+,i^+,i^+;\lambda)$	$4p$	–
4.4.16	$(i,i^+,i^+;\mu)$	$(i,i,i;(2,1))$	$(i^+,i^+,i^+;(2,1))$	p	–

Now consider $\gamma = \hat{\beta}_{(x,i^-ii^+)}\hat{\beta}_{(i^-ii^+,x)}$ once more.

$$\begin{aligned} 0 &\equiv |S_p \wr S_3|\,\mathrm{T}_u(\gamma, \hat{\beta}_{(x,iii(2,1))}\hat{\beta}_{(iii(2,1),x)}) \\ &\equiv 3\cdot 2p\cdot 2p - 3\cdot 3p\cdot \gamma_{ii^+i^+\mu'} - 3\cdot p\cdot p \\ &\equiv 9p^2 - 9\gamma_{ii^+i^+\mu'}p \mod p^3 \\ &\Rightarrow \gamma_{ii^+i^+\mu'} \equiv p \mod p^2 \end{aligned}$$

Next we consider the trace of γ and γ^2 with e_x.

$$\begin{aligned} 0 &\equiv 6p - 3p - 3p - 3\gamma_{ii(i+2)\mu} + 6\gamma_{ii^+(i+2)} \mod p^2 \\ &\Rightarrow 3\gamma_{ii(i+2)\mu} \equiv 6\gamma_{ii^+(i+2)} \mod p^2 \\ 0 &\equiv 12p^2 - 3p^2 - 3p^2 - 3\gamma^2_{ii(i+2)\mu} + 6\gamma^2_{ii^+(i+2)} \\ &\equiv 6p^2 - 12\gamma^2_{ii^+(i+2)} + 6\gamma^2_{ii^+(i+2)} \mod p^3 \\ &\Rightarrow p^2 \equiv \gamma^2_{ii^+(i+2)} \mod p^3 \\ \Rightarrow \gamma_{ii^+(i+2)} &\equiv p \mod p^2 \text{ and } \gamma_{ii(i+2)\mu} \equiv 2p \mod p^2 \end{aligned}$$

Note that we did not need any information on the p-valuation of $\gamma_{ii(i+2)\mu}$ so the argument also works for $i = p-2$. By using our knowledge of $\hat{\beta}_{(x,i^-ii+)}\hat{\beta}_{(i^-ii+,x)}$ we can make further deductions if $i > 1$.

Lemma/ Congruence	y	z	a	γ_a	i
4.4.16	$(i,i^+,i^+;\mu')$	(i^-,i,i^+)	$(i,i^+,i+2)$	p	$i>1$
4.21	$(i,i^+,i^+;\mu')$		$(i^+,i^+,i+2;\mu)$	p	$i>1$
4.4.16	$(i,i^+,i^+;\mu)$	(i^-,i,i^+)	$(i,i^+,i+2)$	p	$i>1$
4.21	$(i,i^+,i^+;\mu)$		$(i^+,i^+,i+2;\mu)$	p	$i>1$

Thus we have determined all endomorphisms except $\hat{\beta}_{(112\mu,122\mu)}\hat{\beta}_{(122\mu,112\mu)}$ and $\hat{\beta}_{(112\mu',122\mu')}\hat{\beta}_{(122\mu',112\mu')}$. We will determine those endomorphisms in the next part of the proof.

In the following discussions we will use the same table notations as in this one without further explanation.

3. $x = (i^-, i, i; \mu)$, $i > 1$

We will give a bijection between $c_{ii^+i^+\mu}$ and $c_{(p-i-1)(p-i)(p-i)\mu}$ for $1 \le i < p-1$ and thus $p-1 \ge p-i > 1$ along which we can transfer all the calculations from the last part of the proof to this case..

$$\begin{aligned}
\zeta : c_{iii^+\mu} &\longrightarrow c_{(p-i-1)(p-i)(p-i)\mu} \\
(i, i, i+1; \mu) &\longmapsto (p-i, p-i+1, p-i+1; \mu) \\
(i, i+1, i+1; \mu') &\longmapsto (p-i, p-i, p-i+1; \mu') \\
(i, i+1, i+1; \mu) &\longmapsto (p-i, p-i, p-i+1; \mu) \\
(i, i, i+2; \mu) &\longmapsto (p-i-1, p-i+1, p-i+1; \mu) \\
(i+1, i+1, i+1; \lambda) &\longmapsto (p-i, p-i, p-i; \lambda) \\
(i+1, i+1, i+1; (2,1)) &\longmapsto (p-i, p-i, p-i; (2,1)) \\
(i, i+1, i+2) &\longmapsto (p-i-1, p-i, p-i+1) \\
(i+1, i+1, i+2; \mu) &\longmapsto (p-i-1, p-i, p-i; \mu)
\end{aligned}$$

This bijection is chosen such that $C_a = C_{\zeta(a)}$ and $(-1)^{n_a} = -(-1)^{n_{\zeta(a)}}$ for every $a \in c_{iii^+\mu}$. Further it is $a \notin I_p \Leftrightarrow \zeta(a)^- \notin I_p$ and $a^- \notin I_p \Leftrightarrow \zeta(a) \notin I_p$. For $x \in I_p$ let

$$E_x = \{z \in I_p \mid (x, z) \in Q_{\overline{\Lambda}}\}.$$

We define a bijection between $E_{iii^+\mu}$ and $E_{(p-i-1)(p-i)(p-i)\mu}$.

$$\begin{aligned}
\omega : E_{iii^+\mu} &\longrightarrow E_{(p-i-1)(p-i)(p-i)\mu} \\
(i, i+1, i+1; \mu') &\longmapsto (p-i-1, p-i-1, p-i; \mu') \\
(i, i+1, i+1; \mu') &\longmapsto (p-i-1, p-i-1, p-i; \mu) \\
(i, i, i+2; \mu) &\longmapsto (p-i-2, p-i, p-i; \mu) \\
(i, i, i; \lambda) &\longmapsto (p-i, p-i, p-i; \lambda) \\
(i, i, i; (2,1)) &\longmapsto (p-i, p-i, p-i; (2,1)) \\
(i-1, i, i+1) &\longmapsto (p-i-1, p-i, p-i+1)
\end{aligned}$$

Then

$$a \in c_{iii^+\mu} \cap c_z \Leftrightarrow \zeta(a) \in c_{(p-i-1)(p-i)(p-i)\mu} \cap c_{\omega(z)}.$$

Therefore we can transfer all the arguments made above via the bijections to deduce that if we transfer the choices we made by multiplication with a unit we obtain

$$(\hat{\beta}_{(iii^+\mu,x)}\hat{\beta}_{(x,iii^+\mu)})_a^* \equiv (\hat{\beta}_{(i^-ii\mu,\omega(x))}\hat{\beta}_{(\omega(x),i^-ii\mu)})_{\zeta(a)}^* \mod p^2$$

for every x such that $(iii^+\mu, x) \in Q_{\overline{\Lambda}}$ and $a \in I_0$. This yields the endomorphisms in the assertion.

Note that this means that the endomorphisms which cannot be completely determined by these arguments are $\hat{\beta}_{((p-2)p^-p^-\mu,(p-2)(p-2)p^-\mu)}\hat{\beta}_{((p-2)(p-2)p^-\mu,(p-2)p^-p^-\mu)}$ and $\hat{\beta}_{((p-2)p^-p^-\mu,(p-2)(p-2)p^-\mu)}\hat{\beta}_{((p-2)(p-2)p^-\mu,(p-2)p^-p^-\mu)}$. However these endomorphism were already determined in the discussion of $x = (i, i, i+1; \mu)$ by Lemma 4.4.14. Additionally, the standard endomorphisms which were not determined in the last part, $\hat{\beta}_{(112\mu,122\mu)}\hat{\beta}_{(122\mu,112\mu)}$ and $\hat{\beta}_{(112\mu,122\mu')}\hat{\beta}_{(122\mu',112\mu)}$, are covered by this discussion.

4. $x = (i, i, i; (2, 1))$

   This case has already been completely determined by the other discussions.

5. $x = (i-1, i, i+1)$, $1 < i < p-1$

   For the following $y \in I_p$ the entries of $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$ are already determined modulo p^2.

$$(i-1,i,i;(1,1)),(i-1,i,i;(2)),(i,i,i+1;(1,1)),(i,i,i+1;(2))$$

   We use Lemma 4.4.16 Part 1 to show that the standard endomorphisms all have p-valuation 1 at every relevant position.

y	z	a	b	i
$(i^-,i^-,i^+;(1,1))$	$(i^-,i,i;(1,1))$	$(i,i,i^+;(1,1))$	(i^-,i,i^+)	2
$(i^-,i^-,i^+;(2))$	$(i^-,i,i;(2))$	$(i,i,i^+;(2))$	(i^-,i,i^+)	2
$(i^-,i^-,i^+;(1,1))$	$(i^-,i^-,i^+;(2))$	(i^-,i,i^+)	$(i^-,i,i+2))$	2
$(i^-,i^-,i^+;(1,1))$	$(i^-,i,i+2)$	$(i,i,i+2;(1,1)))$	$(i^-,i,i+2))$	2
$(i^-,i^+,i^+;(1,1))$	$(i^-,i,i;(1,1))$	$(i,i^+,i^+;(1,1))$	$(i^-,i^+,i^+(1,1))$	2
$(i^-,i^+,i^+;(1,1))$	$(i^-,i^+,i^+;(2))$	$(i,i^+,i+2)$	$(i^-,i^+,i+2)$	2
$(i^-,i^+,i^+;(2))$	$(i^-,i,i;(2))$	$(i,i^+,i^+;(2))$	$(i^-,i^+,i^+(2))$	2
$(i^-,i^+,i^+;(1,1))$	$(i^-,i,i+2)$	$(i,i^+,i+2)$	$(i^-,i^+,i+2))$	2
$(i^-,i^-,i^+;(1,1))$	$(i^-,i^-,i^+;(2))$	$(i^-,i,i^+))$	$(i^-,i,i+2)$	$p-2$
$(i^-,i^-,i^+;(1,1))$	$(i-2,i,i^+)$	(i^-,i,i^+)	$(i^-,i,i+2)$	$p-2$
$(i^-,i^-,i^+;(1,1))$	$(i,i,i^+;(1,1))$	$(i,i,i^+;(1,1)))$	$(i,i,i+2;(1,1)))$	$p-2$
$(i^-,i^-,i^+;(2))$	$(i,i,i^+;(2))$	$(i,i,i^+;(2))$	$(i,i,i+2;(2))$	$p-2$
$(i-2,i,i^+)$	$(i^-,i^+,i^+;(1,1))$	$(i^-,i^+,i^+;(1,1))$	$(i^-,i^+,i+2)$	$p-2$
$(i-2,i,i^+)$	$(i^-,i^+,i^+;(2))$	$(i^-,i^+,i^+;(2))$	$(i^-,i^+,i+2)$	$p-2$
$(i^-,i^+,i^+;(1,1))$	$(i^-,i^+,i^+;(2))$	$(i^-,i^+,i+2)$	$(i,i^+,i+2))$	$p-2$

   Next we choose one entry for each endomorphism.

γ	a	γ_a	i
$\hat\beta_{(x,i^-i^-i^+(1,1))}\hat\beta_{(i^-i^-i^+(1,1),x)}$	(i^-,i,i^+)	p	–
$\hat\beta_{(x,i^-i^-i^+(2))}\hat\beta_{(i^-i^-i^+(2),x)}$	(i^-,i,i^+)	p	–
$\hat\beta_{(x,(i-2)ii^+)}\hat\beta_{((i-2)ii^+,x)}$	(i^-,i,i^+)	p	$i>2$
$\hat\beta_{(x,i^-i^+i^+(1,1))}\hat\beta_{(i^-i^+i^+(1,1),x)}$	$(i^-,i^+,i^+;(1,1))$	$2p$	–
$\hat\beta_{(x,i^-i^+i^+(2))}\hat\beta_{(i^-i^+i^+(2),x)}$	$(i^-,i^+,i^+;(2))$	$2p$	–
$\hat\beta_{(x,i^-i(i+2))}\hat\beta_{(i^-i(i+2),x)}$	$(i^-,i,i+2)$	p	$i<p-2$

Finally we deduce the remaining entries.

Lemma/ Congruence	y	z	a	γ_a	i
4.4.16	$(i^-,i^-,i^+;(1,1))$	$(i^-,i,i;(1,1))$	$(i,i,i^+;(1,1))$	$2p$	–
4.4.16	$(i^-,i^-,i^+;(1,1))$	$(i,i,i^+;(1,1))$	$(i,i,i+2;(1,1))$	$2p$	–
4.21	$(i^-,i^-,i^+;(1,1))$	–	$(i^-,i,i+2)$	p	–
4.4.16	$(i^-,i^-,i^+;(2))$	$(i^-,i,i;(2))$	$(i,i,i^+;(2))$	$2p$	–
4.4.16	$(i^-,i^-,i^+;(2))$	$(i,i,i^+;(2))$	$(i,i,i+2;(2))$	$2p$	–
4.21	$(i^-,i^-,i^+;(2))$	–	$(i^-,i,i+2)$	p	–
4.4.16	$(i-2,i,i^+)$	$(i^-,i,i;(1,1))$	$(i^-,i^+,i^+;(1,1))$	p	$i>2$
4.4.16	$(i-2,i,i^+)$	$(i^-,i,i;(2))$	$(i^-,i^+,i^+;(2))$	p	$i>2$
4.4.16	$(i-2,i,i^+)$	$(i^-,i^-,i^+;(1,1))$	$(i^-,i,i+2)$	p	$i>2$
4.21	$(i-2,i,i^+)$	–	$(i^-,i^+,i+2)$	p	$i>2$
4.4.16	$(i^-,i^+,i^+;(1,1))$	$(i^-,i,i;(1,1))$	$(i,i^+,i^+;(1,1))$	$2p$	–
4.4.16	$(i^-,i^+,i^+;(1,1))$	$(i,i,i^+;(1,1))$	$(i,i^+,i+2)$	p	–
4.21	$(i^-,i^+,i^+;(1,1))$	–	$(i^-,i^+,i+2)$	p	–
4.4.16	$(i^-,i^+,i^+;(2))$	$(i^-,i,i;(2))$	$(i,i^+,i^+;(2))$	$2p$	–
4.4.16	$(i^-,i^+,i^+;(2))$	$(i,i,i^+;(2))$	$(i,i^+,i+2)$	p	–
4.21	$(i^-,i^+,i^+;(2))$	–	$(i^-,i^+,i+2)$	p	–
4.4.16	$(i^-,i,i+2)$	$(i^-,i^-,i^+;(1,1))$	$(i,i,i+2;(1,1))$	p	$i<p-2$
4.4.16	$(i^-,i,i+2)$	$(i^-,i^-,i^+;(2))$	$(i,i,i+2;(2))$	p	$i<p-2$
4.4.16	$(i^-,i,i+2)$	$(i,i,i^+;(1,1))$	$(i,i^+,i+2)$	p	$i<p-2$
4.21	$(i^-,i,i+2)$	–	$(i^-,i^+,i+2)$	p	$i<p-2$

6. $x=(i,i,j;\mu)$ and $x=(i-1,i,j)$, $j>i$.

   We consider to cases simultaneously and use induction on $j-i$. If $j-i=1$ then both cases have already been covered ($x=(i,i,i+1;\mu), x=(i-1,i,i+1)$). We will apply the induction hypothesis only to endomorphisms where the asserted forms agree for $j-i=1$ and $j-i>1$.

   Now assume that every entry of a standard endomorphism in $e_{i'i'j'\mu}\Lambda e_{i'i'j'\mu}$ or $e_{i'-i'j'}\Lambda e_{i'-i'j'}$ with $j'-i'<j-i$ is already determined to be as in the assertion of the lemma mudolo p^2.

   First consider $x=(i,i,j;\mu)$. Here the induction hypothesis implies that that $\hat\beta_{(x,iij^-\lambda)}\hat\beta_{(iij^-\lambda,x)}$ and $\hat\beta_{(x,ii^+j)}\hat\beta_{(ii^+j,x)}$ are already determined. Those endomorphisms have the same asserted form both if $j^-=i^+$ and if $j^->i^+$.

We show that some additional entries have p-valuation 1.

y	z	a	b	i/j
(i,i,j^+)	$(i,i^+,j;\mu)$	$(i^+,i^+,j^+;\mu)$	(i,i^+,j^+)	$i=1, j<p-1$
(i^-,i,j)	$(i,i^+,j;\mu)$	(i,i^+,j)	(i,i^+,j^+)	$i>1, j=p-1$

The entry $(\hat{\beta}_{(x,iij^+)}\hat{\beta}_{(iij^+,x)})_{iij^+\mu}$ has p-valuation 1 for $i=1$ by Lemma 4.4.17. We will not use the fact that $\nu_p(\hat{\beta}_{(x,i^-ij)}\hat{\beta}_{(i^-ij,x)})_{iij^+\mu}=1$ to determine the entry modulo p^2.

We choose the following entries by multiplication with a unit.

γ	a	γ_a	i,j
$\hat{\beta}_{(x,i^-ij)}\hat{\beta}_{(i^-ij,x)}$	$(i,i,j;\mu)$	$2p$	$i>1$
$\hat{\beta}_{(x,iij^+)}\hat{\beta}_{(iij^+,x)}$	$(i,i+1,j+1)$	p	$j<p-1$

Now we determine the rest of the entries.

Lemma/ Congruence	y	z	a	γ_a	i/j
4.4.16	(i^-,i,j)	$(i,i,j^-;\mu)$	(i,i^+,j)	p	$i>1$
4.4.16	(i^-,i,j)	(i,i^+,j)	(i,i^+,j^+)	p	$i>1$
4.19	(i^-,i,j)	-	$(i,i,j^+;\mu)$	$2p$	$i>1$
4.4.16	(i,i,j^+,μ)	(i,i^+,j)	$(i^+,i^+,j^+;\mu)$	p	$i<p-1$
4.21	(i,i,j^+,μ)	-	$(i,i,j^+;\mu)$	p	$i<p-1$

Next we consider $x=(i-1,i,j)$. For $x\in I_p$ we have to additionally assume $i>1$. By the induction hypothesis we assume that $\hat{\beta}_{(x,i^-ij^-)}\hat{\beta}_{(i^-ij^-,x)}$ is already determined. This endomorphism has the same asserted form for $j-1=i+1$ as for $j-1>i+1$. We have also already determined $\hat{\beta}_{(x,iij\mu)}\hat{\beta}_{(iij\mu,x)}$ for $\mu\in\{(1,1),(2)\}$.

Most relevant entries have p-valuation 1.

y	z	a	b	i/j
$(i^-,i^-,j;(1,1))$	(i^-,i,j^-)	$(i,i,j;(1,1))$	(i^-,i,j)	$i=2$
$(i^-,i^-,j;(2))$	(i^-,i,j^-)	$(i,i,j;(2))$	(i^-,i,j)	$i=2$
$(i^-,i^-,j;(1,1))$	$(i^-,i^-,j;(2))$	(i^-,i,j)	(i^-,i,j^+)	$i=2$
(i^-,i^+,j)	(i^-,i,j^-)	(i,i^+,j)	(i^-,i^+,j)	$i=2$
(i^-,i,j^+)	$(i^-,i^-,j;(1,1))$	$(i,i,j^+;(1,1))$	(i^-,i,j^+)	$i=2, j<p-1$
(i^-,i,j^+)	(i^-,i^+,j)	(i,i^+,j^+)	(i^-,i^+,j^+)	$i=2, j<p-1$
$(i^-,i^-,j;(1,1))$	$(i^-,i^-,j;(2))$	(i^-,i,j)	(i^-,i,j^+)	$j=p-1$
$(i^-,i^-,j;(1,1))$	$(i-2,i,j)$	(i^-,i,j)	(i^-,i,j^+)	$j=p-1, i>2$
$(i^-,i^-,j;(1,1))$	$(i,i,j;(1,1))$	$(i,i,j;(1,1))$	$(i,i,j^+;(1,1))$	$j=p-1$
$(i^-,i^-,j;(2))$	$(i,i,j;(2))$	$(i,i,j;(2))$	$(i,i,j^+;(2))$	$j=p-1$
$(i-2,i,j)$	(i^-,i^+,j)	(i^-,i^+,j)	(i^-,i^+,j^+)	$j=p-1, i>2$
$(i-2,i,j)$	$(i^-,i^-,j;(1,1))$	(i^-,i,j)	(i^-,i,j^+)	$j=p-1, i>2$

The only entry missing is $(\hat{\beta}_{(x,12(p-1))}\hat{\beta}_{(12(p-1),x)})_{13p}$.

We choose the following entries by multiplication with a unit.

γ	a	γ_a	i,j
$\hat{\beta}_{(x,i^-i^-j(1,1))}\hat{\beta}_{(i^-i^-j(1,1),x)}$	$(i-1,i,j)$	p	-
$\hat{\beta}_{(x,i^-i^-j(2))}\hat{\beta}_{(i^-i^-j(2),x)}$	$(i-1,i,j)$	p	-
$\hat{\beta}_{(x,i-2ij)}\hat{\beta}_{(i-2ij,x)}$	$(i-1,i,j)$	p	$i>2$
$\hat{\beta}_{(x,i^-i^+j)}\hat{\beta}_{(i^-i^+j,x)}$	$(i-1,i+1,j)$	p	-
$\hat{\beta}_{(x,i^-ij^+)}\hat{\beta}_{(i^-ij^+,x)}$	$(i-1,i,j+1)$	p	$j<p-1$

Now we determine the rest of the entries.

Lemma/ Congruence	y	z	a	γ_a	i/j
4.4.16	$(i^-,i^-,j;(1,1))$	(i^-,i,j^-)	$(i,i,j;(1,1))$	$2p$	-
4.4.16	$(i^-,i^-,j;(1,1))$	$(i,i,j(1,1))$	$(i,i,j^+;(1,1))$	$2p$	-
4.21	$(i^-,i^-,j;(1,1))$	-	(i^-,i,j^+)	p	-
4.4.16	$(i^-,i^-,j;(2))$	(i^-,i,j^-)	$(i,i,j;(2))$	$2p$	-
4.4.16	$(i^-,i^-,j;(2))$	$(i,i,j(2))$	$(i,i,j^+;(2))$	$2p$	-
4.21	$(i^-,i^-,j;(2))$	-	(i^-,i,j^+)	p	-
4.4.16	$(i-2,i,j)$	(i^-,i,j^-)	(i^-,i^+,j)	p	$i>2$
4.4.16	$(i-2,i,j)$	$(i^-,i^-,j(1,1))$	(i^-,i,j^+)	p	$i>2$
4.21	$(i-2,i,j)$	-	(i^-,i^+,j^+)	p	$i>2$
4.4.16	(i^-,i^+,j)	(i^-,i,j^-)	(i,i^+,j)	p	-
4.4.16	(i^-,i^+,j)	$(i,i,j(1,1))$	(i,i^+,j^+)	p	-
4.19	(i^-,i^+,j)	-	(i^-,i^+,j^+)	p	-
4.4.16	(i^-,i,j^+)	$(i^-,i^-,j;(1,1))$	$(i,i,j^+;(1,1))$	p	$j<p-1$
4.4.16	(i^-,i,j^+)	$(i^-,i^-,j;(2))$	$(i,i,j^+;(2))$	p	$j<p-1$
4.4.16	(i^-,i,j^+)	$(i,i,j(1,1))$	(i,i^+,j^+)	p	$j<p-1$
4.21	(i^-,i,j^+)	-	(i^-,i^+,j^+)	p	$j<p-1$

7. $x=(j,i,i;\mu)$ and $x=(j,i,i+1)$, $j<i$

   We will handle this case similar to the last one by induction over $i-j$. The induction base for $i-j=1$ is covered by previous discussions.

   We assume that any entry of a standard endomorphism in $e_{j'i'i'\mu}\Lambda e_{j'i'i'\mu}$ or $e_{j'i'i'^+}\Lambda e_{j'i'i'^+}$ with $i'-j'<i-j$ is already determined to be as in the assertion of the lemma mudolo p^2.

   First consider $x=(j,i,i;\mu)$. The induction hypothesis implies that that both $\hat{\beta}_{(x,j^+ii\mu)}\hat{\beta}_{(j^+ii\mu,x)}$ and $\hat{\beta}_{(x,ji^-i)}\hat{\beta}_{(ji^-i,x)}$ are already determined. Those endomorphisms have the same asserted form both in the case where $i^-=j^+$ and where $i^->j^+$.

   The following entries have p-valuation 1.

y	z	a	b	i/j
$(j^-,i,i;\mu)$	(j,i^-,i)	$(j,i,i;\mu)$	(j,i,i^+)	$i=p-1,j>1$
(j,i,i^+)	(j,i^-,i)	(j^+,i,i^+)	(j,i,i^+)	$j=1,i<p-1$

   Then $\nu_p((\hat{\beta}_{(x,j^-ii\mu)}\hat{\beta}_{(j^-ii\mu,x)})_{ji^+i^+\mu})=1$ by Lemma 4.4.17. We will not use the

fact that $\nu_p((\hat{\beta}_{(x,jii^+)}\hat{\beta}_{(jii^+,x)})_{ji^+i^+\mu}) = 1$ to determine it modulo p^2.

We make the following choices.

γ	a	γ_a	i,j
$\hat{\beta}_{(x,j^-ii\mu)}\hat{\beta}_{(j^-ii\mu,x)}$	$(j,i,i;\mu)$	p	$j>1$
$\hat{\beta}_{(x,jii^+)}\hat{\beta}_{(jii^+,x)}$	$(j,i,i+1)$	p	$i<p-1$

Now we determine the rest of the entries.

Lemma/ Congruence	y	z	a	γ_a	i/j
4.4.16	$(j^-,i,i;\mu)$	(j,i^-,i)	(j,i,i^+)	p	$j>1$
4.21	$(j^-,i,i;\mu)$		$(j,i^+,i^+;\mu)$	p	$j>1$
4.4.16	(j,i,i^+)	(j,i^-,i)	(j^+,i,i^+)	p	$i<p-1$
4.4.16	(j,i,i^+)	$(j^+,i,i;\mu)$	$(j^+,i^+,i^+;\mu)$	$2p$	$i<p-1$
4.19	(j,i,i^+)		$(j,i^+,i^+;\mu)$	$2p$	$j>1$

Next we consider $x=(j,i,i^+)$. For $x\in I_p$ we have to additionally assume $i<p-1$. By the induction hypothesis we assume that $\hat{\beta}_{(x,j^+ii^+)}\hat{\beta}_{(j^+ii^+,x)}$ is already determined. This endomorphism has the same asserted form for $j+1=i-1$ as for $j+1>i-1$. We have also already determined $\hat{\beta}_{(x,jii\mu)}\hat{\beta}_{(jii\mu,x)}$ for $\mu\in\{(1,1),(2)\}$.

The following relevant entries have p-valuation 1.

y	z	a	b	i/j
(j,i^-,i^+)	$(j,i,i;(1,1))$	(j^+,i,i^+)	(j,i,i^+)	$j=1$
(j,i^-,i^+)	$(j,i,i+2)$	$(j^+,i,i+2)$	$(j,i,i+2)$	$j=1,i<p-2$
$(j,i^+,i^+;(1,1))$	$(j,i^+,i^+;(2))$	$(j^+,i^+,i+2)$	$(j,i^+,i+2)$	$j=1$
$(j,i^+,i^+;(1,1))$	$(j,i,i+2)$	$(j^+,i^+,i+2)$	$(j,i^+,i+2)$	$j=1,i<p-2$
(j^-,i,i^+)	(j^+,i,i^+)	(j^+,i,i^+)	$(j^+,i,i+2)$	$i=p-2$
(j^-,i,i^+)	$(j,i^+,i^+;(1,1))$	$(j,i^+,i^+;(1,1))$	$(j,i^+,i+2)$	$i=p-2$
(j^-,i,i^+)	$(j,i^+,i^+;(2))$	$(j,i^+,i^+;(2))$	$(j,i^+,i+2)$	$i=p-2$
$(j,i^+,i^+;(1,1))$	$(j,i^+,i^+;(2))$	$(j,i^+,i+2)$	$(j^+,i^+,i+2)$	$i=p-2$

The missing entries are $(\hat{\beta}_{(x,ji^-i^+)}\hat{\beta}_{(ji^-i^+,x)})_{ji(i+2)}$ and $(\hat{\beta}_{(x,j^-ii^+)}\hat{\beta}_{(j^-ii^+,x)})_{ji(i+2)}$. We choose the following entries by multiplication with a unit.

γ	a	γ_a	i,j
$\hat{\beta}_{(x,ji^-i^+)}\hat{\beta}_{(ji^-i^+,x)}$	$(j,i,i+1)$	p	–
$\hat{\beta}_{(x,j^-ii^+)}\hat{\beta}_{(j^-ii^+,x)}$	$(j,i,i+1)$	p	–
$\hat{\beta}_{(x,ji^+i^+;(1,1))}\hat{\beta}_{(ji^+i^+;(1,1),x)}$	$(j,i+1,i+1;(1,1))$	$2p$	–
$\hat{\beta}_{(x,ji^+i^+;(2))}\hat{\beta}_{(ji^+i^+;(2),x)}$	$(j,i+1,i+1;(2))$	$2p$	–
$\hat{\beta}_{(x,ji(i+2))}\hat{\beta}_{(ji(i+2),x)}$	$(j,i,i+2)$	p	–

Now we determine the rest of the entries.

Lemma/ Congruence	y	z	a	γ_a	i/j
4.4.16	(j,i^-,i^+)	$(j,i,i;(1,1))$	(j^+,i,i^+)	p	-
4.4.16	(j,i^-,i^+)	(j^+,i,i^+)	$(j^+,i,i+2)$	p	-
4.19	(j,i^-,i^+)	-	$(j,i,i+2)$	p	-
4.4.16	(j^-,i,i^+)	$(j,i,i;(1,1))$	$(j,i^+,i^+(1,1))$	p	$j>1$
4.4.16	(j^-,i,i^+)	$(j,i,i;(2))$	$(j,i^+,i^+(2))$	p	$j>1$
4.4.16	(j^-,i,i^+)	(j,i^-,i^+)	$(j,i,i+2)$	p	$j>1$
4.21	(j^-,i,i^+)	-	$(j,i^+,i+2)$	p	$j>1$
4.4.16	$(j,i^+,i^+;(1,1))$	$(j,i,i;(1,1))$	$(j^+,i^+,i^+;(1,1))$	$2p$	-
4.4.16	$(j,i^+,i^+;(1,1))$	(j^+,i,i^+)	$(j^+,i^+,i+2)$	p	-
4.21	$(j,i^+,i^+;(1,1))$	-	$(j,i^+,i+2)$	p	-
4.4.16	$(j,i^+,i^+;(2))$	$(j,i,i;(2))$	$(j^+,i^+,i^+;(2))$	$2p$	-
4.4.16	$(j,i^+,i^+;(2))$	(j^+,i,i^+)	$(j^+,i^+,i+2)$	p	-
4.21	$(j,i^+,i^+;(2))$	-	$(j,i^+,i+2)$	p	-
4.4.16	$(j,i,i+2)$	(j,i^-,i^+)	$(j^+,i,i+2)$	p	$i<p-2$
4.4.16	$(j,i,i+2)$	(j^+,i,i^+)	$(j^+,i^+,i+2)$	p	$i<p-2$
4.21	$(j,i,i+2)$	-	$(j,i^+,i+2)$	p	$i<p-2$

8. $x=(i,j,k)$, $i<j<k$

   We will make an induction over the pairs $(j-i,k-j)$ ordering them lexicographically. We claim that any endomorphism $\hat{\beta}_{(ijk,i'j'k')}\hat{\beta}_{(i'j'k',ijk)}$ where $|\{i,j,k\}|=|\{i',j',k'\}|=3$ has entry p modulo p^2 at every relevant position. The induction base is given since we already know $e_{i^-ii^+}\Lambda e_{i^-ii^+}$ and the entries there fulfill this assertion. We can also assume that $j-i>1$ and $k-j>1$ since we have already determined the cases where $j-i=1$ or $k-j=1$ and here the assertion is fulfilled as well.

   So the induction hypothesis implies that $\hat{\beta}_{(x,i^+jk)}\hat{\beta}_{(i^+jk,x)}$, $\hat{\beta}_{(x,ij^-k)}\hat{\beta}_{(ij^-k,x)}$ and $\hat{\beta}_{(x,ijk^-)}\hat{\beta}_{(ijk^-,x)}$ are already of the asserted form.

   The following relevant entries have p-valuation 1.

y	z	a	b	i/k
(i,j^+,k)	(i,j,k^-)	(i^+,j^+,k)	(i,j^+,k)	$i=1$
(i,j,k^+)	(i,j^-,k)	(i^+,j,k^+)	(i,j,k^+)	$i=1,k<p-1$
(i^-,j,k)	(i,j^-,k)	(i,j,k)	(i,j,k^+)	$k=p-1,i>1$
(i,j^+,k)	(i^+,j,k)	(i^+,j^+,k)	(i^+,j^+,k^+)	$k=p-1$

   We will deduce the entries of the endomorphisms at position (i,j^+,k^+) without using knowledge of their valuation.

   We make the following choices.

γ	a	γ_a	i
$\hat{\beta}_{(x,i^-jk)}\hat{\beta}_{(i^-jk,x)}$	(i,j,k)	p	$i>1$
$\hat{\beta}_{(x,ij^+k)}\hat{\beta}_{(ij^+k,x)}$	(i,j^+,k)	p	-
$\hat{\beta}_{(x,ijk^+)}\hat{\beta}_{(ijk^+,x)}$	(i,j,k^+)	p	$k<p-1$

We determine the remaining entries.

Lemma/ Congruence	y	z	a	γ_a	i
4.4.16	(i^-,j,k)	(i,j,k^-)	(i,j^+,k)	p	$i>1$
4.4.16	(i^-,j,k)	(i,j^-,k)	(i,j,k^+)	p	$i>1$
4.19	(i^-,j,k)		(i,j^+,k^+)	p	$i>1$
4.4.16	(i,j^+,k)	(i,j,k^-)	(i^+,j^+,k)	p	-
4.4.16	(i,j^+,k)	(i^+,j,k)	(i^+,j^+,k^+)	p	-
4.19	(i,j^+,k)		(i,j^+,k^+)	p	-
4.4.16	(i,j,k^+)	(i,j^-,k)	(i^+,j,k^+)	p	$k<p-1$
4.4.16	(i,j,k^+)	(i^+,j,k)	(i^+,j^+,k^+)	p	$k<p-1$
4.19	(i,j,k^+)		(i,j^+,k^+)	p	$k<p-1$

Now we have determined all entries modulo p^2. Our next step will be to show that the asserted endomorphisms fulfill the assumptions of Lemma 4.4.13 and thus are equivalent to the standard endomorphisms modulo $\Lambda \cap p^2\Gamma$.

Let $x,y \in I_p$ with $(x,y) \in Q_{\overline{\Lambda}}$ and $\gamma' \in e_x\Gamma e_x$ be the corresponding element defined as in the assertion of the lemma.

Note that $r_{x_1}r_{x_2}r_{x_3} \in \mathbb{Z}_p^*$ and thus $\mathrm{T}_u(\gamma',e_x) \in \mathbb{Z}_p$ if and only if $\frac{\mathrm{T}_u(\gamma',e_x)}{r_{x_1}r_{x_2}r_{x_3}} \in \mathbb{Z}_p$. We get

$$\begin{aligned}|S_p \wr S_3|\frac{\mathrm{T}_u(\gamma',e_x)}{r_{x_1}r_{x_2}r_{x_3}} &= \sum_{a\in I_0}\frac{r_a}{r_{x_1}r_{x_2}r_{x_3}}\gamma'_a \\ &= \sum_{a\in I_0} C_a((-1)^{n_a}\gamma'_a + (p\rho_a)\gamma'_a).\end{aligned}$$

Our construction was done in such a way that we have always chosen the last entry such that $\sum_{a\in I_0} C_a(-1)^{n_a}\gamma'_a = 0$. So all that remains to show is that $\sum_{a\in I_0} C_a p\rho_a\gamma'_a \in p^3\mathbb{Z}_p$. We calculate $p\rho_a$:

$$\begin{aligned}(-1)^{n_a} + p\rho_a &= \frac{r_{x_1}r_{x_2}r_{x_3}}{r_{a_1}r_{a_2}r_{a_3}} \\ &= \prod_{i\in\{1,2,3\}}\frac{r_{a_i}}{r_{x_i}} \\ &= \prod_{i\in\{1,2,3|a_i>x_i\}}\frac{r_{x_i+1}}{r_{x_i}} \\ &= \prod_{i\in\{1,2,3|a_i>x_i\}}\xi_{x_i} \\ &= \prod_{i\in\{1,2,3|a_i>x_i\}}(-1+\eta_i p) \\ &\equiv (-1)^{n_a} - \sum_{i\in\{1,2,3|a_i>x_i\}}\eta_{x_i}p \mod p^2.\end{aligned}$$

Since $p \mid \gamma'_a$, it suffices to consider $p\rho_a$ modulo p^2. So it suffices to show that

$$\sum_{a\in I_0} C_a\Big(\sum_{i\in\{1,2,3|a_i>x_i\}} \eta_{x_i} p\Big)\gamma'_a = 0$$

This can be easily checked for all the endomorphisms. Here we show it in two examples for clarification. For $\gamma = \hat{\beta}_{(iii\lambda,iii^+\mu)}\hat{\beta}_{(iii^+\mu,iii\lambda)}$ we obtain

$$\begin{aligned}
\gamma'_{iii^+\mu} &= p\\
\gamma'_{ii^+i^+\mu} &= 2p\\
\gamma'_{i^+i^+i^+\lambda} &= 3p\\
\sum_{a\in I_0} C_a\Big(\sum_{l\in\{1,2,3|a_m>x_m\}} \eta_{x_m} p\Big)\gamma'_a&\\
= -3(\eta_i p)p + 3(\eta_i p + \eta_i p)2p - 1(\eta_i p + \eta_i p + \eta_i p)3p &= 0.
\end{aligned}$$

In the case of $\gamma = \hat{\beta}_{(ijk,i^-jk)}\hat{\beta}_{(i^-jk,ijk)}$ for $i < j < k$ we obtain

$$\begin{aligned}
\gamma'_{ijk} = \gamma_{ij^+k} = \gamma_{ijk^+} = \gamma_{ij^+k^+} &= p\\
\sum_{a\in I_0} C_a\Big(\sum_{i\in\{1,2,3|a_m>x_m\}} \eta_{x_m} p\Big)\gamma'_a &=\\
-6(\eta_j)p - 6(\eta_k)p + 6(\eta_j + \eta_k)p &= 0.
\end{aligned}$$

This completes the proof that the standard endomorphisms are as asserted up to multiplication by a unit and modulo $p^2\Gamma\cap\Lambda$. It remains to show that the endomorphism spaces are the row spaces of the matrices in the assertion. Let e be a primitive idempotent of Λ. First we see that the row space of the matrix is contained in $e\Lambda e$. The first row of the matrix corresponds to e, the rows with entries with p-valuation 1 are in $e\Lambda e$ by Lemma 4.4.13, the rows with p-valuation 2 are contained by Lemma 4.4.15 and the last row is always contained, as $p^3\Gamma \subseteq \Lambda$.

For the other direction note that if n is the dimension of $e\Lambda e$ then the sum of the valuations of the diagonal entries in the corresponding matrix above is always

$$0 + \frac{n-1}{2} + \frac{n-1}{2}\cdot 2 + 3 = \frac{3n}{2}.$$

This concludes the proof by Lemma 2.5.14. □

We conclude the section by one more observation on the endomorphism rings.

Remark 4.4.21. *Let $x \in I_p$, $a, b \in c_x$ and $\gamma \in e_x\Gamma e_x$ with $\nu_p(\gamma_a) = \nu_p(\gamma_b) = 1$ and $\gamma_c = 0$ for $c \notin \{a, b\}$. Then $\gamma \notin e_x\Lambda e_x$.*

Proof. The matrices are all of the following form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \xi_1 p & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \xi_{\frac{n-1}{2}} p \end{pmatrix}, C \in p^2 R^{\frac{n}{2} \times \frac{n}{2}}$$

with $\xi_i \in R^*$. We denote the rows $2 \dots \frac{n}{2}$ by $\alpha_1 \dots \alpha_{\frac{n-1}{2}}$. Then there is an element $\gamma' \in \langle p \cdot e_x, \alpha_1 \dots \alpha_{\frac{n-1}{2}} \rangle$ such that $\gamma \equiv \gamma' \mod p^2\Gamma$. Furthermore $\gamma' \in p\Gamma$, so $\delta := \frac{\gamma'}{p} \in \Gamma$. Then $\delta \in \langle e_x, \frac{\alpha_1}{p}, \dots, \frac{\alpha_{n-1}}{p} \rangle$ with exactly two unit entries.

Now we make a case distinction depending on where those two unit entries lie. First note that by the row reduced form of the matrices at least one of the unit entries lies in the left half of the matrix. First assume that just one of those entries lies in the left half. As the second to $\frac{n-1}{2}$-th rows are strictly reduced, this is only possible if either $\gamma' = \alpha_i$ for some i or $\gamma' = p \cdot e_x + \frac{1}{\xi_1}\alpha_1 + \dots + \frac{1}{\xi_{\frac{n-1}{2}}}\alpha_{\frac{n-1}{2}}$. It is straight forward to consider all elements of this form for all endomorphism spaces and see that neither is of this form.

Now assume that both unit entries of δ lie in the left half of the matrix. This would imply that the second half of the vector has only entries divisible by p. Let $B' = B \cdot \mathrm{diag}(1, \frac{1}{p}, \dots \frac{1}{p})$, i.e. we divide all rows but the first by p. Now the existence of a $\delta \in \langle e, \frac{\alpha_1}{p}, \dots, \frac{\alpha_{n-1}}{p} \rangle$ where the only unit entries are in the left half of the vector implies that B' is singular modulo p. Straightforward calculations show that the determinant of B' is not divisible by p in all the cases above. Thus, the second case cannot occur either. □

4.4.2 Exponent matrices

The next step is to determine the generators $\hat{\beta}_{(x,y)}$. We are going to show that there is an automorphism of Λ such that after applying this automorphism the elements $\hat{\beta}_{(x,y)}$ with $x < y$, which correspond to descending arrows in the quiver of $\overline{\Lambda}$, will have entry 1 at some positions and the other positions can be determined by our knowledge of $\overline{\Lambda}$.

First, we will determine the exponent matrices of Λ.

Lemma 4.4.22. *Let M be the exponent matrix of $\varepsilon_a\Lambda$. The rows and columns of the matrix are assumed to be labeled by $d_a = \{x \in I_p \mid d_{ax} \neq 0\}$. Then $M = (m_{xy})_{x,y \in d_a}$ has the following properties.*

1. *$m_{xy} \leq m_{xz} + m_{zy}$ $\forall x, y, z \in d_a$.*
2. *Let $x, y \in d_a$ such that $(x, y) \in Q_{\overline{\Lambda}}$. Then $m_{xy} + m_{yx} = 1$.*
3. *Let $x, y \in d_a$ and assume that there is an index $z \in I_p$ such that (x, z), (z, y) are descending arrows in $Q_{\overline{\Lambda}}$. Then $m_{xy} + m_{yx} = 2$.*
4. *Let $x, y, y', z \in d_a$ such that $(x, y), (y, z), (x, y'), (y', z)$ are descending arrows in $Q_{\overline{\Lambda}}$. Assume further that $m_{zy} = m_{yx} = 0$. Then $m_{zy'} + m_{y'x} = 0$.*

Proof. 1. This is a general property of exponent matrices, see Lemma 2.4.3.

2. We have already determined the endomorphism ring $e_x\Lambda e_x$ and can see that $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$ has p-valuation 1 at position a. On the other hand the only elements of $e_x\Lambda e_x$ with valuation 0 at position a are unit multiples of e_x, so there is no element of $x\Lambda yy\Lambda x$ with p-valuation less than 1 at position a. This implies that $m_{xy} + m_{yx} \geq 1$.

3. Again by considering the endomorphism rings we can see that

$$(\hat{\beta}_{(x,z)}\hat{\beta}_{(z,y)}\hat{\beta}_{(y,z)}\hat{\beta}_{(z,x)})_a = (\hat{\beta}_{(x,z)}\hat{\beta}_{(z,x)}\hat{\beta}_{(z,y)}\hat{\beta}_{(y,z)})_a$$

has p-valuation 2. On the other hand x any y will always have exactly 2 composition factors in common. Thus by Remark 4.4.21 we can see that no element of $x\Lambda yy\Lambda x$ can have p-valuation less than 2 at any position.

4. From the other parts of this lemma we can make the following deductions.

$$\begin{aligned} m_{zy} = 0 &\overset{2.}{\Rightarrow} m_{yz} = 1 \\ m_{yx} = 0 &\overset{2.}{\Rightarrow} m_{xy} = 1 \\ &\overset{1.}{\Rightarrow} m_{zx} \leq 0, m_{xz} \leq 2 \\ &\overset{3.}{\Rightarrow} m_{zx} = 0, m_{xz} = 2 \end{aligned}$$

Now since $\nu_p((\hat{\beta}_{(z,y)}\hat{\beta}_{(y,z)})_a) = 1$ and $1 = m_{yz} \leq \nu_p((\hat{\beta}_{(y,z)})_a)$ it follows that $\nu_p((\hat{\beta}_{(y,z)})_a) = 1$ and $\nu_p((\hat{\beta}_{(z,y)})_a) = 0$. Similarly it is $\nu_p((\hat{\beta}_{(x,y)})_a) = 1$ and $\nu_p((\hat{\beta}_{(y,x)})_a) = 0$ and in particular $\nu_p((\hat{\beta}_{(z,y)}\hat{\beta}_{(y,x)})_a) = 0$.

By Lemma 4.3.25 there is a unit $F \in R^*$ such that

$$\hat{\beta}_{(z,y)}\hat{\beta}_{(y,x)} \equiv F\hat{\beta}_{(z,y')}\hat{\beta}_{(y',z)} \mod p\Lambda.$$

Since $m_{zx} = 0$ it follows that $\nu_p((\hat{\beta}_{(z,y')}\hat{\beta}_{(y',x)})_a) = 0$, so $m_{zy'} + m_{y'x} \leq 0$. On the other hand $0 = m_{zx} \leq m_{zy'} + m_{y'x}$, which proves the assertion.

□

Definition 4.4.23. *We define an equivalence relation on I_p:*

$$x \sim y \Leftrightarrow l(x) = l(y)$$

For $a \in I_0$ and $i \in \{0,1,2,3\}$ we denote the equivalence classes in d_a with respect to $\sim$ by

$$\mathrm{cl}_i(a) = \{x \in d_a \mid l(x) = l(a) - i\}.$$

Then $\mathrm{cl}_0(a) = \{a\}$ and $\mathrm{cl}_3(a) = \{a^-\}$ and there are only arrows between indices in $\mathrm{cl}_i(a)$ and $\mathrm{cl}_{i+1}(a)$ where $i \in \{0,1,2\}$.

Now for the determination of the exponent matrices first note that it suffices to determine the entries corresponding to our generators $\hat{\beta}_{(x,y)}$. Further, using Part 2 of Lemma 4.4.22, we can see that it even suffices to determine the entries lying below the diagonal.

Lemma 4.4.24. *There is an element $c \in K \otimes \Lambda$ such that the exponent matrices of Λ^c have the following property: Let $a \in I_0$ and $M_a = (m_{xy})_{x,y \in d_a}$ be the exponent matrix of $\varepsilon_a \Lambda^c$. Then $m_{xy} = 0$ for all $x < y \in d_a$.*

Proof. First assume that the $\mathrm{cl}_i(a) \neq \varnothing$ for all i, so in particular $a \in I_p$. We can pick one entry below the diagonal in each row and there is a $c \in K \otimes \Lambda$ such that the exponent matrix of $\varepsilon_a \Lambda^c$ has entry 0 at every entry we picked by Lemma 2.5.18. This also implies that we can choose m_{xa} to be zero for every $x \in \mathrm{cl}_1(a)$. Further we pick for each $x \in \mathrm{cl}_1(a)$ a $y \in \mathrm{cl}_2(a)$ and choose m_{yx} to be 0.

Now let $y \neq y' \in \mathrm{cl}_2(a)$. Then we can use Part 4 of Lemma 4.4.22 to deduce that $m_{xy'} = 0$ since we know that $m_{xy} = m_{ya} = m_{y'a} = 0$. Thus, we are finished for every generator corresponding to an arrow between $\mathrm{cl}_1(a)$ and $\mathrm{cl}_2(a)$.

For the arrows between $\mathrm{cl}_2(a)$ and $\mathrm{cl}_3(a)$ we can choose one entry to be 0 and with the same argument as above, again using Part 4 of Lemma 4.4.22, we can see that all the entries corresponding to such arrows are zero.

If $a \in I_p$ and $\mathrm{cl}_i(a) = \varnothing$ for some i then $\mathrm{cl}_j = \varnothing$ for all $j > i$. Therefore the arguments above work for every $a \in I_p$.

The only case that remains is the one where $a \in I_0 \setminus I_p$. In that case we choose one entry corresponding to a generator in each column to be zero. This means $m_{a^- x} = 0$ for every $x \in \mathrm{cl}_2$. Now we use the same argument as above to show that all entries corresponding to arrows between cl_2 and cl_1 are zero. Since we assumed $a \notin I_p$ the class cl_0 is empty. □

Corollary 4.4.25. *If $a \in I_0$ then there is a $c \in K \otimes \Lambda$ such that the exponent matrices of Λ^c look as follows. In edge cases the matrices look the same with the rows/columns corresponding to indices not in I_p removed. The annotated + signs will be explained later on.*

$$
\begin{array}{l}
M_{iii\lambda} \\
iii\lambda \\
i^- ii\mu \\
i^- i^- i\mu \\
i^- i^- i^- \mu
\end{array}
\begin{array}{c}
\begin{array}{cccc}
iii\lambda & i^- ii\mu & i^- i^- i\mu & i^- i^- i^- \lambda
\end{array} \\
\begin{pmatrix}
. & 1 & 2 & 3 \\
0^+ & . & 1 & 2 \\
0 & 0^+ & . & 1 \\
0 & 0 & 0^+ & .
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{l|cccccc}
M_{iii(2,1)} & iii(2,1) & i^-ii(2) & i^-ii(1,1) & i^-i^-i(2) & i^-i^-i(1,1) & i^-i^-i^-(2,1) \\
iii(2,1) & . & 1 & 1 & 2 & 2 & 3 \\
i^-ii(2) & 0^+ & . & 1 & 1 & 1 & 2 \\
i^-ii(1,1) & 0^+ & 1 & . & 1 & 1 & 2 \\
i^-i^-i(2) & 0 & 0^+ & 0 & . & 1 & 1 \\
i^-i^-i(1,1) & 0 & 0^+ & 0 & 1 & . & 1 \\
i^-i^-i^-(2,1) & 0 & 0 & 0 & 0^+ & 0 & .
\end{array}
$$

$$
\begin{array}{l|cccccccc}
M_{iii^+\mu} & iii^+\mu & iii(2,1) & iii\lambda & i^-ii^+ & i^-ii\mu & i^-ii\mu' & i^-i^-i^+\mu & i^-i^-i\mu \\
iii^+\mu & . & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
iii(2,1) & 0^+ & . & 1 & 1 & 1 & 1 & 2 & 2 \\
iii\lambda & 0^+ & 1 & . & 1 & 1 & 2 & 2 & 2 \\
i^-ii^+ & 0^+ & 1 & 1 & . & 1 & 1 & 1 & 2 \\
i^-ii\mu & 0 & 0 & 0 & 0^+ & . & 1 & 1 & 1 \\
i^-ii\mu' & 0 & 0 & 1 & 0^+ & 1 & . & 1 & 1 \\
i^-i^-i^+\mu & 0 & 1 & 1 & 0^+ & 1 & 1 & . & 1 \\
i^-i^-i\mu & 0 & 0 & 0 & 0 & 0 & 0^+ & 0 & .
\end{array}
$$

$$
\begin{array}{l|cccccccc}
M_{ii^+i^+\mu} & ii^+i^+\mu & iii^+\mu & iii^+\mu' & i^-i^+i^+\mu & iii(2,1) & iii\lambda & i^-ii^+ & i^-ii\mu \\
ii^+i^+\mu & . & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
iii^+\mu & 0^+ & . & 1 & 1 & 1 & 1 & 1 & 2 \\
iii^+\mu' & 0^+ & 1 & . & 1 & 1 & 2 & 1 & 2 \\
i^-i^+i^+\mu & 0^+ & 1 & 1 & . & 2 & 2 & 1 & 2 \\
iii(2,1) & 0 & 0 & 0^+ & 1 & . & 1 & 1 & 1 \\
iii\lambda & 0 & 0^+ & 1 & 1 & 1 & . & 1 & 1 \\
i^-ii^+ & 0 & 0 & 0 & 0^+ & 1 & 1 & . & 1 \\
i^-ii\mu & 0 & 0 & 0 & 0 & 0 & 0^+ & 0 & .
\end{array}
$$

$$
\begin{array}{l|cccccc}
M_{iij\mu} & iij\mu & iij^-\mu & i^-ij & i^-ij^- & i^-i^-j\mu & i^-i^-j^-\mu \\
iij\mu & . & 1 & 1 & 2 & 2 & 3 \\
iij^-\mu & 0^+ & . & 1 & 1 & 2 & 2 \\
i^-ij & 0^+ & 1 & . & 1 & 1 & 2 \\
i^-ij^- & 0 & 0 & 0^+ & . & 1 & 1 \\
i^-i^-j\mu & 0 & 1 & 0^+ & 1 & . & 1 \\
i^-i^-j^-\mu & 0 & 0 & 0 & 0^+ & 0 & .
\end{array}
$$

$$
M_{jiiμ}\begin{array}{c} \\ jiiμ \\ ji^-i \\ j^-iiμ \\ ji^-i^-μ \\ j^-i^-j \\ j^-i^-i^-μ \end{array}
\begin{array}{c}
\begin{array}{cccccc} jiiμ & ji^-i & j^-iiμ & ji^-i^-μ & j^-i^-i & j^-i^-i^-(2) \end{array} \\
\begin{pmatrix}
. & 1 & 1 & 2 & 2 & 3 \\
0^+ & . & 1 & 1 & 2 & 2 \\
0^+ & 1 & . & 1 & 1 & 2 \\
0 & 0^+ & 1 & . & 1 & 1 \\
0 & 0 & 0^+ & 1 & . & 1 \\
0 & 0 & 0 & 0^+ & 0 & .
\end{pmatrix}
\end{array}
$$

$$
M_{ijk}\begin{array}{c} \\ ijk \\ ijk^- \\ ij^-k \\ i^-jk \\ ij^-k^- \\ i^-jk^- \\ i^-j^-k \\ i^-j^-k^- \end{array}
\begin{array}{c}
\begin{array}{cccccccc} ijk & ijk^- & ij^-k & i^-jk & ij^-k^- & i^-jk^- & i^-j^-k & i^-j^-k^- \end{array} \\
\begin{pmatrix}
. & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
0^+ & . & 1 & 1 & 1 & 1 & 2 & 2 \\
0^+ & 1 & . & 1 & 1 & 2 & 1 & 2 \\
0^+ & 1 & 1 & . & 2 & 1 & 1 & 2 \\
0 & 0 & 0^+ & 1 & . & 1 & 1 & 1 \\
0 & 0 & 1 & 0^+ & 1 & . & 1 & 1 \\
0 & 1 & 0 & 0^+ & 1 & 1 & . & 1 \\
0 & 0 & 0 & 0 & 0^+ & 0 & 0 & .
\end{pmatrix}
\end{array}
$$

$$
M_{ii^+j}\begin{array}{c} \\ ii^+j \\ ii^+j^- \\ iij(2) \\ iij(1,1) \\ i^-i^+j \\ iij^-(2) \\ iij^-(1,1) \\ i^-i^+j^- \\ i^-ij \\ i^-ij^- \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} ii^+j & ii^+j^- & iij(2) & iij(1,1) & i^-i^+j & iij^-(2) & iij^-(1,1) & i^-i^+j^- & i^-ij & i^-ij^- \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\
0^+ & . & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
0^+ & 1 & . & 1 & 1 & 1 & 2 & 2 & 1 & 2 \\
0^+ & 1 & 1 & . & 1 & 2 & 1 & 2 & 1 & 2 \\
0^+ & 1 & 1 & 1 & . & 2 & 2 & 1 & 1 & 2 \\
0 & 0 & 0^+ & 1 & 1 & . & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0^+ & 1 & 1 & . & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0^+ & 1 & 1 & . & 1 & 1 \\
0 & 1 & 0 & 0 & 0^+ & 1 & 1 & 1 & . & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0^+ & 0 & 0 & .
\end{pmatrix}
\end{array}
$$

M_{jii^+}	jii^+	$jii(2)$	$jii(1,1)$	ji^-i^+	j^-ii^+	ji^-i	$j^-ii(2)$	$j^-ii(1,1)$	$j^-i^-i^+$	j^-i^-i
jii^+	0	1	1	1	1	2	2	2	2	3
$jii(2)$	0^+	.	1	1	1	1	1	2	2	1
$jii(1,1)$	0^+	1	.	1	1	1	2	1	2	2
ji^-i^+	0^+	1	1	.	1	1	2	2	1	2
j^-ii^+	0^+	1	1	1	.	2	1	1	1	2
ji^-i	0	0	0	0^+	1	.	1	1	1	1
$j^-ii(2)$	0	0	1	1	0^+	1	.	1	1	1
$j^-ii(1,1)$	0	1	0	1	0^+	1	1	.	1	1
$j^-i^-i^+$	0	1	1	0	0^+	1	1	1	.	2
j^-i^-i	0	0	0	0	0	0^+	0	0	0	.

$M_{i^-ii^+}$	i^-ii^+	$i^-ii(2)$	$i^-ii(1,1)$	$i^-i^-i^+(2)$	$i^-i^-i^+(1,1)$	$i\text{–}2ii^+$	$i^-i^-i(2)$	$i^-i^-i(1,1)$	$i\text{–}2ii(2)$	$i\text{–}2ii(1,1)$	$i\text{–}2i^-i^+$	$i\text{–}2i^-i$
i^-ii^+	.	1	1	1	1	1	2	2	2	2	2	3
$i^-ii(2)$	0^+	.	1	1	1	1	1	1	1	2	2	2
$i^-ii(1,1)$	0^+	1	.	1	1	1	1	1	2	1	2	2
$i^-i^-i^+(2)$	0^+	1	1	.	1	1	1	2	2	2	1	2
$i^-i^-i^+(1,1)$	0^+	1	1	1	.	1	2	1	2	2	1	2
$i\text{–}2ii^+$	0^+	1	1	1	1	.	2	2	1	1	1	2
$i^-i^-i(2)$	0	0	0	0^+	1	1	.	1	1	1	1	1
$i^-i^-i(1,1)$	0	0	0	1	0^+	1	1	.	1	1	1	1
$i\text{–}2ii(2)$	0	0	1	1	1	0^+	1	1	.	1	1	1
$i\text{–}2ii(1,1)$	0	1	0	1	1	0^+	1	1	1	.	1	1
$i\text{–}2i^-i^+$	0	1	1	0	0	0^+	1	1	1	1	.	1
$i\text{–}2i^-i$	0	0	0	0	0	0	0	0^+	0	0	0	.

Proof. Lemma 4.4.24 implies that there is a $c \in \Lambda$ such that the entries in the exponent matrices of Λ^c below the diagonal corresponding to generators are all zero. Therefore, by Lemma 4.4.22 Part 2, all the entries above the diagonal corresponding to generators are 1. Now, as we have calculated the entries for all the generators, we can inductively determine the other entries via

$$m_{ij} = \min\{m_{ik} + m_{kj} | m_{ik} \text{ and } m_{kj} \text{ are already determined}\}.$$

□

4.4.3 Conclusion

Our next step will be to determine the entries of all generators $\hat{\beta}_{(x,y)}$ where $x, y \in I_p$ with $(x, y) \in Q_{\overline{\Lambda}}$ and $x < y$ modulo p. We assume that the exponent matrices of Λ are as in Corollary 4.4.25.

Definition 4.4.26. *Let $a = (i, j, k, \lambda), b = (i', j', k', \lambda') \in I_0$ where $i \leq j \leq k, i' \leq j' \leq k'$. We extend our partial order on I_0 to a total order as follows.*

$$\begin{aligned} a < b \Leftrightarrow \\ l(a) < l(b) \text{ or} \\ l(a) = l(b) \text{ and } (i, j, k) <_{lex} (i', j', k') \text{ or} \\ (i, j, k) = (i', j', k') \text{ and} \\ (\lambda = (2, 1) \text{ or } (\lambda = (2) \text{ and } \lambda' = (1, 1)) \text{ or } (\lambda = (3) \text{ and } \lambda' = (1, 1, 1))) \end{aligned}$$

Lemma 4.4.27. *1. Let $a \in I_p$, $y \in d_a$ and*

$$x = \max\{z \in d_a \mid z > y, (z, y) \in Q_{\overline{\Lambda}}\}.$$

Then we can by conjugation without loss assume that $(\hat{\beta}_{(y,x)})_a = 1$.

2. Let $a \in I_0 \setminus I_p$ and assume that $a^- \in I_p$. Let further $x \in d_a$ and

$$y = \max\{z \in d_a \mid z < x, (y, z) \in Q_{\overline{\Lambda}}\}.$$

Then we can by conjugation without loss assume that $(\hat{\beta}_{(y,x)})_a = 1$.

3. Let $a \in I_0 \setminus I_p$ and $a^- \notin I_p$. Then for every $x \in d_a$ there exists at most one $y \in d_a$ such that $x < y$ and $(x, y) \in Q_{\overline{\Lambda}}$. We can assume that $(\hat{\beta}_{(y,x)})_a = 1$

Proof. The first two parts can be achieved by Lemma 2.5.18, since in the first case we conjugate one entry per row and in the second case one entry per column. The

exponent matrices for the third case look as follows.

$$i=1, j=p:$$

$$\begin{array}{cc} M_{iij\mu} & {\scriptstyle iij^-\mu} \\ {\scriptstyle iij^-\mu} & \begin{pmatrix} . \end{pmatrix} \end{array}$$

$$j=1, i=p:$$

$$\begin{array}{cc} M_{jii\mu} & {\scriptstyle ji^-i^-\mu} \\ {\scriptstyle ji^-i^-\mu} & \begin{pmatrix} . \end{pmatrix} \end{array}$$

$$i=1, k=p, j-i>1, k-j>1:$$

$$\begin{array}{ccc} M_{ijk} & {\scriptstyle ijk^-} & {\scriptstyle ij^-k^-} \\ {\scriptstyle ijk^-} & . & 1 \\ {\scriptstyle ij^-k^-} & 0 & . \end{array}$$

$$i=1, j=p:$$

$$\begin{array}{ccccc} M_{ii^+j} & {\scriptstyle ii^+j^-} & {\scriptstyle iij^-(2)} & {\scriptstyle iij^-(1,1)} \\ {\scriptstyle ii^+j^-} & . & 1 & 1 \\ {\scriptstyle iij^-(2)} & 0 & . & 1 \\ {\scriptstyle iij^-(1,1)} & 0 & 1 & . \end{array}$$

$$j=1, i=p-1:$$

$$\begin{array}{ccccc} M_{jii^+} & {\scriptstyle jii(2)} & {\scriptstyle jii(1,1)} & {\scriptstyle ji^-i} \\ {\scriptstyle jii(2)} & . & 1 & 1 \\ {\scriptstyle jii(1,1)} & 1 & . & 1 \\ {\scriptstyle ji^-i} & 0 & 0 & . \end{array}$$

□

From now on we will assume that Λ is as in Lemma 4.4.27.

Definition 4.4.28. *If $a \in I_0$ and $x, y \in d_a$ such that $(\hat{\beta}_{(y,x)})_a$ is assumed to be 1 by Lemma 4.4.27 we say $\hat{\beta}_{(y,x)}$ is* normalized *at position a.*

Remark 4.4.29. *In the exponent matrices above the entries m_{yx} of M_a such that $\hat{\beta}_{(y,x)}$ is normalized at position a are marked with $a+$ for the cases where $a \in I_p$ and for $\mu = (2)$.*

Lemma 4.4.30. *1. If $x, y \in I_p$ with $(x, y) \in Q_{\overline{\Lambda}}$ and $y < x$ then $(\hat{\beta}_{(y,x)})_x = 1$.*

2. If $a \in I_0 \setminus I_p$ with $a^- \in I_p$ and $x \in d_a$ with $x > a^-$ and $(a^-, x) \in Q_{\overline{\Lambda}}$ then $(\hat{\beta}_{(a^-,x)})_a = 1$.

Proof. 1. Since $(y, x) \in Q_{\overline{\Lambda}}$ and $y < x$ we know that $y \in d_x$ and $l(y) = l(x) - 1$. Thus for every element $t \in \{z \in d_x \mid z > y, (z, y) \in Q_{\overline{\Lambda}}\}$ we know that $l(t) = l(x)$ and $x \in d_x$, so $t = x$. In particular, x is the maximum of this set.

2. With similar arguments as above we see that

$$\{z \in d_{a^-} \mid z < x, (x,z) \in Q_{\overline{\Lambda}}\} = \{a^-\}.$$

□

Lemma 4.4.31. *Let $x, y \in I_p$ with $(x,y) \in Q_{\overline{\Lambda}}$ and $x > y$. Let further $a \in c_x \cap c_y$ such that $\gamma := \hat{\beta}_{(y,x)}$ is not normalized at position a. Note that this implies $a \neq x$.*

1. *Let $a \in I_p$ and $x' \in I_p$ such that $\hat{\beta}_{(y,x')}$ is normalized at a. Then we have $\gamma_a \equiv F_{yxa} \mod p$ where $F_{yxa} \in \mathbb{Z}_p$ such that*

$$\beta_{(y,x)}\beta_{(x,a)} = \overline{F_{yxa}} \cdot \beta_{(y,x')}\beta_{(x',a)}$$

2. *Let $a \notin I_p$ but $a^- \in I_p$ and $y' \in I_p$ such that $(\hat{\beta}_{(y',x)})_a = 1$. Then we have $\gamma_a \equiv F_{a^-yx} \mod p$ where $F_{a^-yx} \in \mathbb{Z}_p$ such that*

$$\beta_{(a^-,y)}\beta_{(y,x)} = \overline{F_{a^-yx}} \cdot \beta_{(a^-,y')}\beta_{(y',x)}$$

3. *If $a \notin I_p$ and $a^- \notin I_p$ then every generator is normalized at position a.*

Proof. 1. We have chosen the generators such that $\overline{\hat{\beta}_{(s,t)}} = \beta_{(s,t)}$ for all $s, t \in I_p$, and therefore we have

$$\hat{\beta}_{(y,x)}\hat{\beta}_{(x,a)} \equiv F_{xya} \cdot \hat{\beta}_{(y,x')}\hat{\beta}_{(x',a)} \mod p\Lambda.$$

As $\Lambda \subseteq \Gamma$ this implies that

$$(\hat{\beta}_{(y,x)})_a(\hat{\beta}_{(x,a)})_a \equiv F_{xya} \cdot (\hat{\beta}_{(y,x')})_a(\hat{\beta}_{(x',a)})_a \mod p.$$

From the assumptions and Remark 4.4.30 we know that $(\hat{\beta}_{(x,a)})_a = (\hat{\beta}_{(y,x')})_a = (\hat{\beta}_{(x',a)})_a = 1$ and therefore $\gamma_x \equiv F_{xya} \mod p$.

2. In this case we know that $(\hat{\beta}_{(a^-,y)})_a = (\hat{\beta}_{(a^-,y')})_a = (\hat{\beta}_{(y',x)})_a = 1$ and therefore the same arguments as in Part 1 prove the assertion.

3. This follows directly from Lemma 4.4.27.

□

Lemma 4.4.32. *Let $x > y \in I_p$ with $(x,y) \in Q_{\overline{\Lambda}}$. Then for every $z, t \in c_x \cap c_y$ there is an element $\delta \in e_y \operatorname{rad}^2(\Lambda)e_x + pe_y\Lambda e_x$ with $\nu_p(\delta_t) = \nu_p(\delta_z) = 1$ and $\delta_w = 0$ for $t \notin \{t, z\}$*

Proof. First note that $\hat{\beta}_{(y,x)}$ has unit entries at all relevant positions by Lemma 4.4.31. Therefore $\hat{\beta}_{(x,y)}$ has p-valuation 1 at all relevant positions. From the exponent matrix and knowing that $\gamma := \hat{\beta}_{(x,y)}$ is the only element of the generating system $\operatorname{Gen}(\Lambda)$ in $e_x\Lambda e_y$, we can see that

$$e_x\Lambda e_y = R\gamma + p^2\Gamma \cap e_x\Lambda e_y.$$

By Lemma 2.1.36, we know that $e_y\Lambda e_x = (e_x\Lambda e_y)^\sharp$. Let $t, z \in c_x \cap c_y$ and let $\delta \in e_x\Gamma e_y$ such that $\delta_z = p, \delta_t = -\frac{r_z\gamma_z}{r_t\gamma_t}p$, $\delta_w = 0$ for $w \notin \{z,t\}$. Then

$$T_u(\delta, r\gamma) = r \cdot \frac{1}{|S_p \wr S_3|} \cdot (r_z \cdot p \cdot \gamma_z + r_t \cdot (-\frac{r_z\gamma_z}{r_t\gamma_t}p) \cdot \gamma_t) = 0 \in R$$

and $T_u(\delta, \vartheta) \in R$ for all $\vartheta \in p^2\Gamma$ as $\delta\vartheta \in p^3\Gamma$. It follows that $\delta \in (e_x\Lambda e_y)^\sharp = e_y\Lambda e_x$.

Of the generators in $\mathrm{Gen}(\Lambda)$ only $\hat{\beta}_{(y,x)}$ lies in $e_y\Lambda e_x$. Furthermore $\hat{\beta}_{(s,t)} \in \mathrm{rad}(\Lambda)$ for all s, t. Therefore $e_y\Lambda e_x/(e_y \operatorname{rad}^2(\Lambda)e_x)$ is generated by products of $\hat{\beta}_{(y,x)}$ and idempotents. But as $\hat{\beta}_{(y,x)}$ multiplied with any idempotent is either $\hat{\beta}_{(y,x)}$ or zero we can conclude that $\hat{\beta}_{(y,x)}$ generates $e_y\Lambda e_x/(e_y \operatorname{rad}^2(\Lambda)e_x)$ as a $\mathbb{Z}_p$-module.

On the other hand, since the exponent matrices have no entry 0 above the diagonal, no element of $e_y \operatorname{rad}^2(\Lambda)e_x$ can have unit entries. Since $\hat{\beta}_{(y,x)}$ has unit entries this implies that any element of $e_y\Lambda e_x \setminus (e_y \operatorname{rad}^2(\Lambda)e_x + p\Lambda)$ will have unit entries as well. In particular it follows that $\delta \in e_y \operatorname{rad}^2(\Lambda)e_x + p\Lambda$. □

Lemma 4.4.33. *There is an element $\alpha \in \Gamma^*$ and for every pair of indices $x, y \in I_p$ with $(x,y) \in Q_{\overline{\Lambda}}$ and $x > y$ there is an element $\tilde{\beta}_{(y,x)} \in e_y\Lambda e_x$ with the following properties:*

- $((\tilde{\beta}_{(y,x)})^\alpha)_a = 1$ *if the $\hat{\beta}_{(y,x)}$ is normalized at a,*
- $((\tilde{\beta}_{(y,x)})^\alpha)_a = F_{yxa}$ *if $\hat{\beta}_{(y,x)}$ is not normalized at a and $a \in I_p$,*
- $((\tilde{\beta}_{(y,x)})^\alpha)_a = F_{a^-yx}$ *if $\hat{\beta}_{(y,x)}$ is not normalized at a and $a \in I_0 \setminus I_p$,*
- $\{e_x \mid x \in I_p\} \cup \{(\tilde{\beta}_{(y,x)})^\alpha \mid (y,x) \in Q_{\overline{\Lambda}}, y < x\} \cup \{(\hat{\beta}_{(y,x)})^\alpha \mid (y,x) \in Q_{\overline{\Lambda}}, y > x\}$ *generate Λ^α as a $\mathbb{Z}_p$-algebra.*

Proof. We will describe how to construct the new generators from the old ones.

First note that the entries of the generators $\hat{\beta}_{(y,x)}\hat{\beta}_{(x,y)}$ already fulfill the assumptions modulo p. We use the fact that any set generating $\Lambda/\operatorname{rad}^2(\Lambda)$ also generates Λ, see Lemma 2.3.11, and thus changing the generators by elements of $\operatorname{rad}^2(\Lambda)$ does not change the fact that they are a generating system for Λ. By Nakayama's lemma the same is true for changes modulo $p\Lambda$.

We will construct the generators inductively starting with the largest generators with respect to the order $\hat{\beta}_{(y,x)} < \hat{\beta}_{(y',x')} \Leftrightarrow x < x'$. By Lemma 4.4.32 we can find a generator $\tilde{\beta}^0_{(y,x)}$ such that $\tilde{\beta}^0_{(y,x)} \equiv \hat{\beta}_{(y,x)} \mod \operatorname{rad}^2(\Lambda) + p\Lambda$ and $\tilde{\beta}^0_{(y,x)}$ fulfills the requirements above at all positions except x. We can now choose $\alpha_{yx} \in \Gamma^*$ such that $(\varepsilon_x\alpha_{yx})_{yy} = (\tilde{\beta}_{(y,x)})_x$, $(\varepsilon_x\alpha_{yx})_{zz} = 1$ if $z \neq y$ and $\varepsilon_a\alpha_{yx}$ is the identity matrix if $a \neq x$.

Assume that $z, t \in I_p$ with $(z,t) \in Q_{\overline{\Lambda}}$ and $z < t$ and for $\gamma = \hat{\beta}_{(z,t)}$ it is $\gamma^{\alpha_{yx}} \neq \gamma$. Then $t = y$ and thus $z < x$ and in particular $\gamma < \hat{\beta}_{(y,x)}$. Also note that $(\tilde{\beta}_{(y,x)})_x \equiv 1 \mod p$ so we do not change the value of γ modulo p at any position.

Therefore if we work from largest to smallest generator, this method assures the following two points.

- Conjugation with α_{yx} will never change any of the $\tilde{\beta}_{(y',x')}$ we considered before.
- We never change any entry modulo p.

Therefore we can inductively choose α_{yx} and $\tilde{\beta}_{(y,x)}$ for every generator $\hat{\beta}_{(y,x)}$ such that for $\alpha := \prod_{(y,x)\in Q(\Lambda),y<x} \alpha_{yx}$ the conditions above are fulfilled. □

Now we have determined all the generators corresponding to descending arrows.

Lemma 4.4.34. *Let $x, y \in I_p$ with $x > y$ and $(x, y) \in Q(\Lambda)$. Then there is an element $\tilde{\beta}_{(x,y)} \in e_x\Lambda e_y$ such that*

1. *$\tilde{\beta}_{(x,y)} \equiv \xi\hat{\beta}_{(x,y)} \mod \mathrm{rad}^2(\Lambda) + p\Lambda$ for some $\xi \in R^*$.*
2. *$\tilde{\beta}_{(x,y)}\tilde{\beta}_{(y,x)} = (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})'$ as in Lemma 4.4.20.*

Note that the second point means that $\tilde{\beta}_{(x,y)}$ is uniquely determined by $\tilde{\beta}_{(y,x)}$.

Proof. From the exponent matrices and since the standard endomorphisms have p-valuation 1 at every relevant position we can deduce the p-valuations of $\hat{\beta}_{(x,y)}$:

$$\nu_p((\hat{\beta}_{(x,y)})_a) = \begin{cases} 1 & \text{if } a \in d_x \cap d_y \\ -\infty & \text{otherwise} \end{cases}.$$

Therefore $\gamma := \hat{\beta}_{(x,y)}\tilde{\beta}_{(y,x)}$ is an element of $e_x\Lambda e_x$ such that

$$\nu_p(\gamma_a) = \begin{cases} 1 & \text{if } a \in d_x \cap d_y \\ -\infty & \text{otherwise} \end{cases}.$$

Following the proof of Theorem 4.4.20 we can see that this is already enough to deduce that $\xi\gamma \equiv (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})' \mod p^2\Gamma \cap \Lambda$ for some $\xi \in R^*$ as this is all the information we used about $\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)}$. We define the entries of $\tilde{\beta}_{(x,y)}$ as follows

$$(\tilde{\beta}_{(x,y)})_a := \begin{cases} \frac{(\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})'_a}{(\tilde{\beta}_{(y,x)})_a} & \text{if } a \in c_x \cap c_y \\ 0 \text{ otherwise.} \end{cases}$$

Now let $\delta \in p^2\Gamma \cap \Lambda$ with $\xi\gamma = (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})' + \delta$ and define $\delta' \in p^2 e_x\Gamma e_y$ as

$$\delta'_a := \begin{cases} \frac{\delta_a}{(\tilde{\beta}_{(y,x)})_a} & \text{if } a \in c_x \cap c_y \\ 0 \text{ otherwise} \end{cases}.$$

Then $\xi\hat{\beta}_{(x,y)} = \tilde{\beta}_{(x,y)} + \delta'$. Furthermore $\mathrm{T}_u(\delta', \tilde{\beta}_{(y,x)}) = \mathrm{T}_u(\delta, 1) \in \mathbb{Z}_p$ and since $\delta' \in p^2\Gamma$ this suffices to show that $\delta' \in e_y\Lambda e_x^\sharp = e_x\Lambda e_y$ and therefore $\xi\hat{\beta}_{(x,y)} \equiv \tilde{\beta}_{(x,y)} \mod p^2\Gamma\cap\Lambda$.

Since γ is the only generator in $e_x\Lambda e_y$, the same arguments as in the proof of Lemma 4.4.32 show that $p^2\Gamma \cap e_x\Lambda e_y \subseteq \mathrm{rad}^2(\Lambda) + p\Lambda$ which proves the assertion. □

All the calculations of this chapter are summarised in the following theorem.

Theorem 4.4.35. *Let $\overline{\Lambda}$ be the basic algebra of the principal block of $\mathbb{F}_p(S_p \wr S_3)$ and Λ_0 be the basic algebra of the principal block of $\mathbb{Z}_p(S_p \wr S_3)$. Let Λ be a Λ_0-lift, i.e.*

1. $\Lambda/p\Lambda \cong \overline{\Lambda}$
2. $K \otimes \Lambda$ *is semisimple with center* $\mathrm{Z}(K \otimes \Lambda) = \bigoplus_{a \in I_0} K$.
3. *The decomposition matrix of* Λ *is the same as that of* Λ_0
4. Λ *is self dual in* $K \otimes \Lambda$ *with respect to the form* T_u *with* $u = (u_a)_{a \in I_0}$ *and* $u_a = \frac{\dim(V_a)}{|S_p \wr S_3|}$

Then Λ is isomorphic to the subalgebra of

$$\Lambda \subseteq \bigoplus_{a \in I_0} R^{m_a \times m_a}$$

generated by

$$\{e_x \mid x \in I_p\} \cup \{\tilde{\beta}_{(y,x)} \mid x, y \in I_p \text{ and } (x,y) \in Q_{\overline{\Lambda}}\}$$

where for $x, y \in I_p$ with $(x,y) \in Q_{\overline{\Lambda}}$ the element $\tilde{\beta}_{(y,x)}$ is defined as in Lemma 4.4.34 if $y > x$ and Lemma 4.4.33 if $y < x$.

Proof. All the assumptions we made can be achieved by conjugation leaving the diagonal entries invariant. Therefore the conjugation will not change the product $\tilde{\beta}_{(x,y)}\tilde{\beta}_{(y,x)} = (\hat{\beta}_{(x,y)}\hat{\beta}_{(y,x)})'$ in Lemma 4.4.34. □

5 Appendix

The following calculations show that any two ascending / descending paths of length two in the quiver correspond to linearly dependent homomorphisms in $B_0(\mathbb{F}_p S_p \wr S_3)$. Observe that it suffices to consider compositions of the homomorphisms $\beta'_{(x,y)}$ where the three first components of x and y are not necessarily ordered as the isomorphisms Φ is the definition of $\beta_{(x,y)}$ will cancel out or be the same at the start and end of the composition for all cases we compare. We can also consider ascending and descending paths at the same time. To do this, we write i'/i^* instead of i^-/i^+. Those are to be understood in the following way: If you consider descending paths, replace i' by i^- and i^* by i^+, so $(i^*)' = i$.

The calculations were done by a C++ program which can be found on github (https://github.com/CorinnaL/pimhoms/). To ease the automatisation all the homomorphisms of projective $\mathbb{F}_p S_p$-modules are denoted by γ. It is always clear from the context between which projective modules γ is defined. To see that the results are indeed scalar multiples of one another recall the relations $\beta_i \circ \beta_{i+1} = 0, \alpha_{i+1} \circ \alpha_i = 0$ from the quiver of $\mathbb{F}_p S_p$. So whenever γ is applied twice to one component in the calculations below, the result is zero.

Homomorphisms starting in $\overline{P'}(i,i,i;(2,1))$ need to be considered twice, once for each of the basis elements of $S^{(2,1)}$.

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(2)) \cong \overline{P'}(i,i',i;(2)) \to \overline{P'}(i',i',i;(2)) \\
(a\otimes b\otimes c)\otimes id\otimes x_1 &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (\gamma(b)\otimes c\otimes a)\otimes (132) \\
&\mapsto -(c\otimes a\otimes \gamma(b))\otimes (123) + (b\otimes c\otimes \gamma(a))\otimes (132) \\
&\mapsto (\gamma(a)\otimes b\otimes \gamma(c))\otimes id - (\gamma(c)\otimes a\otimes \gamma(b))\otimes (123) \\
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(1,1)) \cong \overline{P'}(i,i',i;(1,1)) \to \overline{P'}(i',i',i;(2)) \\
(a\otimes b\otimes c)\otimes id\otimes x_1 &\mapsto -(\gamma(a)\otimes b\otimes c)\otimes id + 2(\gamma(c)\otimes a\otimes b)\otimes (123) \\
&\qquad - (\gamma(b)\otimes c\otimes a)\otimes (132) \\
&\mapsto -2(a\otimes b\otimes \gamma(c))\otimes id + (c\otimes a\otimes \gamma(b))\otimes (123) \\
&\qquad + (b\otimes c\otimes \gamma(a))\otimes (132) \\
&\mapsto -3(\gamma(a)\otimes b\otimes \gamma(c))\otimes id + 3(\gamma(c)\otimes a\otimes \gamma(b))\otimes (123)
\end{aligned}
$$

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(2)) \cong \overline{P'}(i,i',i;(2)) \to \overline{P'}(i',i',i;(2)) \\
(a\otimes b\otimes c)\otimes id\otimes x_2 &\mapsto -(\gamma(c)\otimes a\otimes b)\otimes(123)+(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id+(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132) \\
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(1,1)) \cong \overline{P'}(i,i',i;(1,1)) \to \overline{P'}(i',i',i;(2)) \\
(a\otimes b\otimes c)\otimes id\otimes x_2 &\mapsto 2(\gamma(a)\otimes b\otimes c)\otimes id-(\gamma(c)\otimes a\otimes b)\otimes(123) \\
&\qquad -(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto (a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad -2(b\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto 3(\gamma(a)\otimes b\otimes\gamma(c))\otimes id-3(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132)
\end{aligned}
$$

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(2)) \cong \overline{P'}(i,i',i;(2)) \to \overline{P'}(i',i',i;(1,1)) \\
(a\otimes b\otimes c)\otimes id\otimes x_1 &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id-(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -(c\otimes a\otimes\gamma(b))\otimes(123)+(b\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id-(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad +2(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132) \\
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(1,1)) \cong \overline{P'}(i,i',i;(1,1)) \to \overline{P'}(i',i',i;(1,1)) \\
(a\otimes b\otimes c)\otimes id\otimes x_1 &\mapsto -(\gamma(a)\otimes b\otimes c)\otimes id+2(\gamma(c)\otimes a\otimes b)\otimes(123) \\
&\qquad -(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -2(a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad +(b\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id-(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad +2(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132)
\end{aligned}
$$

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i',i,i;(2)) \cong \overline{P'}(i,i',i;(2)) \to \overline{P'}(i',i',i;(1,1)) \\
(a\otimes b\otimes c)\otimes id\otimes x_2 &\mapsto -(\gamma(c)\otimes a\otimes b)\otimes(123)+(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id+2(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad -(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132)
\end{aligned}
$$

$$\begin{aligned}
\overline{P'}(i,i,i;(2,1)) \to \overline{P'}(i',i,i;(1,1)) &\cong \overline{P'}(i,i',i;(1,1)) \to \overline{P'}(i',i',i;(1,1)) \\
(a\otimes b\otimes c)\otimes id\otimes x_2 &\mapsto 2(\gamma(a)\otimes b\otimes c)\otimes id-(\gamma(c)\otimes a\otimes b)\otimes(123) \\
&\quad -(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto (a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\quad -2(b\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id+2(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\quad -(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132)
\end{aligned}$$

$$\begin{aligned}
\overline{P'}(i,i^*,i;(2)) \to \overline{P'}(i',i^*,i) &\cong \overline{P'}(i^*,i',i) \to \overline{P'}(i,i',i;(2)) \cong \overline{P'}(i',i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id+(\gamma(c)\otimes b\otimes a)\otimes(13) \\
&\mapsto (b\otimes\gamma(c)\otimes a)\otimes(132)+(b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123)+(\gamma(b)\otimes\gamma(c)\otimes a)\otimes(132) \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
\overline{P'}(i,i^*,i;(2)) \cong \overline{P'}(i^*,i,i;(2)) &\to \overline{P'}(i,i,i;(3)) \cong \overline{P'}(i,i,i;(3)) \to \overline{P'}(i',i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes c\otimes a)\otimes(132) \\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\quad +(\gamma(\gamma(b))\otimes c\otimes a)\otimes(132) \\
\overline{P'}(i,i^*,i;(2)) \cong \overline{P'}(i^*,i,i;(2)) &\to \overline{P'}(i,i,i;(2,1)) \cong \overline{P'}(i,i,i;(2,1)) \to \overline{P'}(i',i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\otimes x_1+(a\otimes\gamma(b)\otimes c)\otimes id\otimes x_2 \\
&\mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\quad +2(\gamma(\gamma(b))\otimes c\otimes a)\otimes(132)
\end{aligned}$$

$$\begin{aligned}
\overline{P'}(i,i^*,i;(2)) \to \overline{P'}(i',i^*,i) &\cong \overline{P'}(i^*,i',i) \to \overline{P'}(i,i',i;(1,1)) \cong \overline{P'}(i',i,i;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id+(\gamma(c)\otimes b\otimes a)\otimes(13) \\
&\mapsto (b\otimes\gamma(c)\otimes a)\otimes(132)+(b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto -(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123)+(\gamma(b)\otimes\gamma(c)\otimes a)\otimes(132) \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
\overline{P'}(i,i^*,i;(2)) \cong \overline{P'}(i^*,i,i;(2)) &\to \overline{P'}(i,i,i;(2,1)) \cong \overline{P'}(i,i,i;(2,1)) \to \overline{P'}(i',i,i;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\otimes x_1+(a\otimes\gamma(b)\otimes c)\otimes id\otimes x_2 \\
&\mapsto 3(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-3(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)
\end{aligned}$$

$$\overline{P'}(i,i^*,i;(1,1)) \to \overline{P'}(i',i^*,i) \cong \overline{P'}(i^*,i',i) \to \overline{P'}(i,i',i;(1,1)) \cong \overline{P'}(i',i,i;(1,1))$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (\gamma(c)\otimes b\otimes a)\otimes(13)\\
&\mapsto -(b\otimes\gamma(c)\otimes a)\otimes(132) + (b\otimes\gamma(a)\otimes c)\otimes(12)\\
&\mapsto -(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) - (\gamma(b)\otimes\gamma(c)\otimes a)\otimes(132)\\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id + (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)
\end{aligned}$$
$$\overline{P'}(i,i^*,i;(1,1)) \cong \overline{P'}(i^*,i,i;(1,1)) \to \overline{P'}(i,i,i;(1,1,1)) \to \overline{P'}(i',i,i;(1,1))$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto -(b\otimes c\otimes a)\otimes(132)\\
&\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\\
&\mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id - (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)\\
&\qquad - (\gamma(\gamma(b))\otimes c\otimes a)\otimes(132)
\end{aligned}$$
$$\overline{P'}(i,i^*,i;(1,1)) \cong \overline{P'}(i^*,i,i;(1,1)) \to \overline{P'}(i,i,i;(2,1)) \to \overline{P'}(i',i,i;(1,1))$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto -(b\otimes c\otimes a)\otimes(132)\\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id\otimes x_1 + (a\otimes\gamma(b)\otimes c)\otimes id\otimes x_2\\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id + (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)\\
&\qquad - 2(\gamma(\gamma(b))\otimes c\otimes a)\otimes(132)
\end{aligned}$$

$$\overline{P'}(i,i^*,i;(1,1)) \to \overline{P'}(i',i^*,i) \cong \overline{P'}(i^*,i',i) \to \overline{P'}(i,i',i;(2)) \cong \overline{P'}(i',i,i;(2))$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (\gamma(c)\otimes b\otimes a)\otimes(13)\\
&\mapsto -(b\otimes\gamma(c)\otimes a)\otimes(132) + (b\otimes\gamma(a)\otimes c)\otimes(12)\\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) - (\gamma(b)\otimes\gamma(c)\otimes a)\otimes(132)\\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id - (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)
\end{aligned}$$
$$\overline{P'}(i,i^*,i;(1,1)) \cong \overline{P'}(i^*,i,i;(1,1)) \to \overline{P'}(i,i,i;(2,1)) \to \overline{P'}(i',i,i;(2))$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto -(b\otimes c\otimes a)\otimes(132)\\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id\otimes x_1 + (a\otimes\gamma(b)\otimes c)\otimes id\otimes x_2\\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id - (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)
\end{aligned}$$

$$\overline{P'}(i,i^*,i^*;(2)) \to \overline{P'}(i',i^*,i^*;(2)) \cong \overline{P'}(i^*,i',i^*;(2)) \to \overline{P'}(i,i',i^*) \cong \overline{P'}(i',i,i^*)$$
$$\begin{aligned}
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\
&\mapsto (c\otimes\gamma(a)\otimes b)\otimes(123)\\
&\mapsto (\gamma(c)\otimes\gamma(a)\otimes b)\otimes(123) + (\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12)\\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id + (\gamma(a)\otimes\gamma(c)\otimes b)\otimes(23)
\end{aligned}$$

$$
\begin{aligned}
&\overline{P'}(i,i^*,i^*;(2)) \cong \overline{P'}(i^*,i,i^*;(2)) \to \overline{P'}(i,i,i^*;(2)) \to \overline{P'}(i',i,i^*)\\
&(a\otimes b\otimes c)\otimes id \mapsto (c\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto (a\otimes \gamma(b)\otimes c)\otimes id + (\gamma(c)\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes \gamma(b)\otimes c)\otimes id + (\gamma(\gamma(c))\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad\qquad + (\gamma(\gamma(b))\otimes a\otimes c)\otimes (12) + (\gamma(a)\otimes \gamma(c)\otimes b)\otimes (23)\\
&\overline{P'}(i,i^*,i^*;(2)) \cong \overline{P'}(i^*,i,i^*;(2)) \to \overline{P'}(i,i,i^*;(1,1)) \to \overline{P'}(i',i,i^*)\\
&(a\otimes b\otimes c)\otimes id \mapsto (c\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto -(a\otimes \gamma(b)\otimes c)\otimes id + (\gamma(c)\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto -(\gamma(a)\otimes \gamma(b)\otimes c)\otimes id + (\gamma(\gamma(c))\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad\qquad + (\gamma(\gamma(b))\otimes a\otimes c)\otimes (12) - (\gamma(a)\otimes \gamma(c)\otimes b)\otimes (23)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P'}(i^*,i^*,i;(2)) \to \overline{P'}(i,i^*,i;(2)) \cong \overline{P'}(i^*,i,i;(2)) \to \overline{P'}(i,i,i;(2,1))\\
&(a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id + (c\otimes a\otimes \gamma(b))\otimes (123)\\
&\qquad\qquad \mapsto (a\otimes \gamma(b)\otimes c)\otimes id + (b\otimes c\otimes \gamma(a))\otimes (132)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes \gamma(b)\otimes c)\otimes id\otimes x_1 + 2(\gamma(a)\otimes \gamma(b)\otimes c)\otimes id\otimes x_2\\
&\overline{P'}(i^*,i^*,i;(2)) \to \overline{P'}(i,i^*,i;(1,1)) \cong \overline{P'}(i^*,i,i;(1,1)) \to \overline{P'}(i,i,i;(2,1))\\
&(a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (c\otimes a\otimes \gamma(b))\otimes (123)\\
&\qquad\qquad \mapsto (a\otimes \gamma(b)\otimes c)\otimes id - (b\otimes c\otimes \gamma(a))\otimes (132)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes \gamma(b)\otimes c)\otimes id\otimes x_1 + 2(\gamma(a)\otimes \gamma(b)\otimes c)\otimes id\otimes x_2
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P'}(j,i,i;(2)) \to \overline{P'}(j',i,i;(2)) \cong \overline{P'}(i,j',i;(2)) \to \overline{P'}(i',j',i) \cong \overline{P'}(j',i',i)\\
&(a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\
&\qquad\qquad \mapsto (c\otimes \gamma(a)\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto (\gamma(c)\otimes \gamma(a)\otimes b)\otimes (123) + (\gamma(b)\otimes \gamma(a)\otimes c)\otimes (12)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes \gamma(b)\otimes c)\otimes id + (\gamma(a)\otimes \gamma(c)\otimes b)\otimes (23)\\
&\overline{P'}(j,i,i;(2)) \cong \overline{P'}(i,j,i;(2)) \to \overline{P'}(i',j,i) \cong \overline{P'}(j,i',i) \to \overline{P'}(j',i',i)\\
&(a\otimes b\otimes c)\otimes id \mapsto (c\otimes a\otimes b)\otimes (123)\\
&\qquad\qquad \mapsto (\gamma(c)\otimes a\otimes b)\otimes (123) + (\gamma(b)\otimes a\otimes c)\otimes (12)\\
&\qquad\qquad \mapsto (a\otimes \gamma(b)\otimes c)\otimes id + (a\otimes \gamma(c)\otimes b)\otimes (23)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes \gamma(b)\otimes c)\otimes id + (\gamma(a)\otimes \gamma(c)\otimes b)\otimes (23)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P'}(i,i^*,i^*;(1,1)) \to \overline{P'}(i',i^*,i^*;(1,1)) \cong \overline{P'}(i^*,i',i^*;(1,1)) \to \overline{P'}(i,i',i^*) \cong \overline{P'}(i',i,i^*)\\
&\quad (a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\
&\qquad\qquad \mapsto -(c\otimes\gamma(a)\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto -(\gamma(c)\otimes\gamma(a)\otimes b)\otimes(123)+(\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(\gamma(a)\otimes\gamma(c)\otimes b)\otimes(23)\\
&\overline{P'}(i,i^*,i^*;(1,1)) \cong \overline{P'}(i^*,i,i^*;(1,1)) \to \overline{P'}(i,i,i^*;(1,1)) \to \overline{P'}(i',i,i^*)\\
&\quad (a\otimes b\otimes c)\otimes id \mapsto -(c\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto -(a\otimes\gamma(b)\otimes c)\otimes id-(\gamma(c)\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(\gamma(\gamma(c))\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad\qquad +(\gamma(\gamma(b))\otimes a\otimes c)\otimes(12)+(\gamma(a)\otimes\gamma(c)\otimes b)\otimes(23)\\
&\overline{P'}(i,i^*,i^*;(1,1)) \cong \overline{P'}(i^*,i,i^*;(1,1)) \to \overline{P'}(i,i,i^*;(2)) \to \overline{P'}(i',i,i^*)\\
&\quad (a\otimes b\otimes c)\otimes id \mapsto -(c\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto (a\otimes\gamma(b)\otimes c)\otimes id-(\gamma(c)\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(\gamma(\gamma(c))\otimes a\otimes b)\otimes(123)\\
&\qquad\qquad\qquad +(\gamma(\gamma(b))\otimes a\otimes c)\otimes(12)-(\gamma(a)\otimes\gamma(c)\otimes b)\otimes(23)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P'}(i^*,i^*,i;(1,1)) \to \overline{P'}(i,i^*,i;(1,1)) \cong \overline{P'}(i^*,i,i;(1,1)) \to \overline{P'}(i,i,i;(2,1))\\
&\quad (a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123)\\
&\qquad\qquad \mapsto -(a\otimes\gamma(b)\otimes c)\otimes id-(b\otimes c\otimes\gamma(a))\otimes(132)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\otimes x_1\\
&\overline{P'}(i^*,i^*,i;(1,1)) \to \overline{P'}(i,i^*,i;(2)) \cong \overline{P'}(i^*,i,i;(2)) \to \overline{P'}(i,i,i;(2,1))\\
&\quad (a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id-(c\otimes a\otimes\gamma(b))\otimes(123)\\
&\qquad\qquad \mapsto -(a\otimes\gamma(b)\otimes c)\otimes id+(b\otimes c\otimes\gamma(a))\otimes(132)\\
&\qquad\qquad \mapsto -3(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\otimes x_1
\end{aligned}
$$

$$
\begin{aligned}
&\overline{P'}(j,i,i;(2)) \to \overline{P'}(j',i,i;(2)) \cong \overline{P'}(i,j',i;(2)) \to \overline{P'}(i',j',i) \cong \overline{P'}(j',i',i)\\
&(a\otimes b\otimes c)\otimes id \mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\
&\qquad\qquad \mapsto (c\otimes\gamma(a)\otimes b)\otimes(123)\\
&\qquad\qquad \mapsto (\gamma(c)\otimes\gamma(a)\otimes b)\otimes(123)+(\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12)\\
&\qquad\qquad \mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(\gamma(a)\otimes\gamma(c)\otimes b)\otimes(23)
\end{aligned}
$$

$$\overline{P'}(j,i,i;(2)) \cong \overline{P'}(i,j,i;(2)) \to \overline{P'}(i',j,i) \cong \overline{P'}(j,i',i) \to \overline{P'}(j',i',i)$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (c\otimes a\otimes b)\otimes (123)\\ &\mapsto (\gamma(c)\otimes a\otimes b)\otimes (123)+(\gamma(b)\otimes a\otimes c)\otimes (12)\\ &\mapsto (a\otimes\gamma(b)\otimes c)\otimes id+(a\otimes\gamma(c)\otimes b)\otimes (23)\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(\gamma(a)\otimes\gamma(c)\otimes b)\otimes (23)\end{aligned}$$

$$\overline{P'}(i,i^*,i') \to \overline{P'}(i',i^*,i';(2)) \cong \overline{P'}(i^*,i',i';(2)) \to \overline{P'}(i,i',i';(2)) \cong \overline{P'}(i',i,i';(2))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\ &\mapsto (b\otimes c\otimes\gamma(a))\otimes (132)\\ &\mapsto (\gamma(b)\otimes c\otimes\gamma(a))\otimes (132)\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\end{aligned}$$
$$\overline{P'}(i,i^*,i') \cong \overline{P'}(i^*,i,i') \to \overline{P'}(i,i,i';(2)) \cong \overline{P'}(i,i,i';(2)) \to \overline{P'}(i',i,i';(2))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto (a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(c\otimes a\otimes\gamma(\gamma(b)))\otimes (123)\end{aligned}$$
$$\overline{P'}(i,i^*,i') \cong \overline{P'}(i^*,i,i') \to \overline{P'}(i,i,i';(1,1)) \cong \overline{P'}(i,i,i';(1,1)) \to \overline{P'}(i',i,i';(2))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id+(c\otimes a\otimes\gamma(\gamma(b)))\otimes (123)\end{aligned}$$

$$\overline{P'}(i,i^*,i') \to \overline{P'}(i',i^*,i';(1,1)) \cong \overline{P'}(i^*,i',i';(1,1)) \to \overline{P'}(i,i',i';(1,1)) \cong \overline{P'}(i',i,i';(1,1))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\ &\mapsto -(b\otimes c\otimes\gamma(a))\otimes (132)\\ &\mapsto -(\gamma(b)\otimes c\otimes\gamma(a))\otimes (132)\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\end{aligned}$$
$$\overline{P'}(i,i^*,i') \cong \overline{P'}(i^*,i,i') \to \overline{P'}(i,i,i';(1,1)) \cong \overline{P'}(i,i,i';(1,1)) \to \overline{P'}(i',i,i';(1,1))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(c\otimes a\otimes\gamma(\gamma(b)))\otimes (123)\end{aligned}$$
$$\overline{P'}(i,i^*,i') \cong \overline{P'}(i^*,i,i') \to \overline{P'}(i,i,i';(2)) \cong \overline{P'}(i,i,i';(2)) \to \overline{P'}(i',i,i';(1,1))$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto (a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id-(c\otimes a\otimes\gamma(\gamma(b)))\otimes (123)\end{aligned}$$

$$\begin{aligned}
\overline{P'}(j,i^*,i) \to \overline{P'}(j',i^*,i) &\cong \overline{P'}(i^*,j',i) \to \overline{P'}(i,j',i;(2)) \cong \overline{P'}(j',i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto (b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id \\
\overline{P'}(j,i^*,i) \cong \overline{P'}(i^*,j,i) &\to \overline{P'}(i,j,i;(2)) \cong \overline{P'}(j,i,i;(2)) \to \overline{P'}(j',i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes(12) \\
&\mapsto (c\otimes a\otimes\gamma(b))\otimes(123) \\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id
\end{aligned}$$

$$\begin{aligned}
\overline{P'}(j,i^*,i) \to \overline{P'}(j',i^*,i) &\cong \overline{P'}(i^*,j',i) \to \overline{P'}(i,j',i;(1,1)) \cong \overline{P'}(j',i,i;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto (b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto -(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id \\
\overline{P'}(j,i^*,i) \cong \overline{P'}(i^*,j,i) &\to \overline{P'}(i,j,i;(1,1)) \cong \overline{P'}(j,i,i;(1,1)) \to \overline{P'}(j',i,i;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes(12) \\
&\mapsto -(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id
\end{aligned}$$

$$\begin{aligned}
\overline{P'}(i,i^*,j) \to \overline{P'}(i',i^*,j) &\cong \overline{P'}(i^*,i',j) \to \overline{P'}(i,i',j) \cong \overline{P'}(i',i,j) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto (b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id \\
\overline{P'}(i,i^*,j) \cong \overline{P'}(i^*,i,j) &\to \overline{P'}(i,i,j;(2)) \cong \overline{P'}(i,i,j;(2)) \to \overline{P'}(i',i,j) \\
(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes(12) \\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id + (\gamma(\gamma(b))\otimes a\otimes c)\otimes(12)
\end{aligned}$$

$$\overline{P'}(i,i^*,j) \cong \overline{P'}(i^*,i,j) \to \overline{P'}(i,i,j;(1,1)) \cong \overline{P'}(i,i,j;(1,1)) \to \overline{P'}(i',i,j)$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto -(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id + (\gamma(\gamma(b))\otimes a\otimes c)\otimes (12)\end{aligned}$$

$$\overline{P'}(i,j,k) \to \overline{P'}(i',j,k) \cong \overline{P'}(j,i',k) \to \overline{P'}(j',i',k) \cong \overline{P'}(i',j',k)$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id\\ &\mapsto (b\otimes\gamma(a)\otimes c)\otimes (12)\\ &\mapsto (\gamma(b)\otimes\gamma(a)\otimes c)\otimes (12)\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\end{aligned}$$
$$\overline{P'}(i,j,k) \cong \overline{P'}(j,i,k) \to \overline{P'}(j',i,k) \cong \overline{P'}(i,j',k) \to \overline{P'}(i',j',k)$$
$$\begin{aligned}(a\otimes b\otimes c)\otimes id &\mapsto (b\otimes a\otimes c)\otimes (12)\\ &\mapsto (\gamma(b)\otimes a\otimes c)\otimes (12)\\ &\mapsto (a\otimes\gamma(b)\otimes c)\otimes id\\ &\mapsto (\gamma(a)\otimes\gamma(b)\otimes c)\otimes id\end{aligned}$$

The following calculations prove Lemma 4.3.27.

$$\begin{aligned}
\overline{P'}(i,i,i;(3)) &\to \overline{P'}(i^-,i,i;(2)) \cong \overline{P'}(i,i^-,i;(2)) \to \overline{P'}(i^-,i^-,i;(2)) \cong \overline{P'}(i,i^-,i^-;(2)) \\
&\to \overline{P'}(i^-,i^-,i^-;(3)) \to \overline{P'}(i,i^-,i^-;(2)) \cong \overline{P'}(i^-,i^-,i;(2)) \\
&\to \overline{P'}(i,i^-,i;(2)) \cong \overline{P'}(i^-,i,i;(2)) \to \overline{P'}(i,i,i;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id + (\gamma(c)\otimes a\otimes b)\otimes(123) + (\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto (a\otimes b\otimes \gamma(c))\otimes id + (c\otimes a\otimes \gamma(b))\otimes(123) + (b\otimes c\otimes \gamma(a))\otimes(132) \\
&\mapsto 2(\gamma(a)\otimes b\otimes\gamma(c))\otimes id \\
&\qquad + 2(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) + 2(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto 2(a\otimes\gamma(b)\otimes\gamma(c))\otimes id + 2(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\qquad + 2(b\otimes\gamma(c)\otimes\gamma(a))\otimes(132) \\
&\mapsto 6(\gamma(a)\otimes\gamma(b)\otimes\gamma(c))\otimes id \\
&\mapsto 6(\delta(\gamma(a))\otimes\gamma(b)\otimes\gamma(c))\otimes id + 6(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\qquad + 6(\delta(\gamma(b))\otimes\gamma(c)\otimes\gamma(a))\otimes(132) \\
&\mapsto 6(\gamma(a)\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id + 6(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123) \\
&\qquad + 6(\gamma(b)\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132) \\
&\mapsto 12(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id \\
&\qquad + 12(\delta(\gamma(c))\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123) \\
&\qquad + 12(\delta(\gamma(b))\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132) \\
&\mapsto 12(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id \\
&\qquad + 12(\gamma(c)\otimes\delta(\gamma(a))\otimes\delta(\gamma(b)))\otimes(123) \\
&\qquad + 12(\gamma(b)\otimes\delta(\gamma(c))\otimes\delta(\gamma(a)))\otimes(132) \\
&\mapsto 36(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}$$

$$\begin{aligned}
\overline{P'}(i,i,i;(1,1,1)) &\to \overline{P'}(i^-,i,i;(1,1)) \cong \overline{P'}(i,i^-,i;(1,1)) \\
&\to \overline{P'}(i^-,i^-,i;(1,1)) \cong \overline{P'}(i,i^-,i^-;(1,1)) \to \overline{P'}(i^-,i^-,i^-;(1,1,1)) \\
&\to \overline{P'}(i,i^-,i^-;(1,1)) \cong \overline{P'}(i^-,i^-,i;(1,1)) \\
&\to \overline{P'}(i,i^-,i;(1,1)) \cong \overline{P'}(i^-,i,i;(1,1)) \to \overline{P'}(i,i,i;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id + (\gamma(c)\otimes a\otimes b)\otimes(123) + (\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes b\otimes\gamma(c))\otimes id - (c\otimes a\otimes\gamma(b))\otimes(123) - (b\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto -2(\gamma(a)\otimes b\otimes\gamma(c))\otimes id - 2(\gamma(c)\otimes a\otimes\gamma(b))\otimes(123) \\
&\qquad - 2(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto 2(a\otimes\gamma(b)\otimes\gamma(c))\otimes id + 2(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\qquad + 2(b\otimes\gamma(c)\otimes\gamma(a))\otimes(132) \\
&\mapsto 6(\gamma(a)\otimes\gamma(b)\otimes\gamma(c))\otimes id
\end{aligned}$$

$$
\begin{aligned}
&\mapsto 6(\delta(\gamma(a))\otimes\gamma(b)\otimes\gamma(c))\otimes id + 6(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123)\\
&\qquad + 6(\delta(\gamma(b))\otimes\gamma(c)\otimes\gamma(a))\otimes(132)\\
&\mapsto -6(\gamma(a)\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id - 6(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123)\\
&\qquad - 6(\gamma(b)\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132)\\
&\mapsto -12(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id - 12(\delta(\gamma(c))\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123)\\
&\qquad - 12(\delta(\gamma(b))\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132)\\
&\mapsto 12(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id + 12(\gamma(c)\otimes\delta(\gamma(a))\otimes\delta(\gamma(b)))\otimes(123)\\
&\qquad + 12(\gamma(b)\otimes\delta(\gamma(c))\otimes\delta(\gamma(a)))\otimes(132)\\
&\mapsto 36(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i^-,i,i;(2)) \cong \overline{P'}(i,i^-,i;(2))\\
&\to \overline{P'}(i^-,i^-,i;(2)) \cong \overline{P'}(i,i^-,i^-;(2)) \to \overline{P'}(i^-,i^-,i^-;(2,1))\\
&\to \overline{P'}(i,i^-,i^-;(2)) \cong \overline{P'}(i^-,i^-,i;(2))\\
&\to \overline{P'}(i,i^-,i;(2)) \cong \overline{P'}(i^-,i,i;(2)) \to \overline{P'}(i,i,i;(2))\\
(a\otimes b\otimes c)\otimes id\otimes x_1 &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (\gamma(b)\otimes c\otimes a)\otimes(132)\\
&\mapsto -(c\otimes a\otimes\gamma(b))\otimes(123) + (b\otimes c\otimes\gamma(a))\otimes(132)\\
&\mapsto (\gamma(a)\otimes b\otimes\gamma(c))\otimes id - (\gamma(c)\otimes a\otimes\gamma(b))\otimes(123)\\
&\mapsto -(a\otimes\gamma(b)\otimes\gamma(c))\otimes id + (b\otimes\gamma(c)\otimes\gamma(a))\otimes(132)\\
&\mapsto -3(\gamma(a)\otimes\gamma(b)\otimes\gamma(c))\otimes id\otimes x_1\\
&\mapsto -3(\delta(\gamma(a))\otimes\gamma(b)\otimes\gamma(c))\otimes id + 3(\delta(\gamma(b))\otimes\gamma(c)\otimes\gamma(a))\otimes(132)\\
&\mapsto 3(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123) - 3(\gamma(b)\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132)\\
&\mapsto -3(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id + 3(\delta(\gamma(c))\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123)\\
&\mapsto 3(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id - 3(\gamma(b)\otimes\delta(\gamma(c))\otimes\delta(\gamma(a)))\otimes(132)\\
&\mapsto 9(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id\otimes x_1
\end{aligned}
$$

$$
\begin{aligned}
\overline{P'}(i,i,i;(2,1)) &\to \overline{P'}(i^-,i,i;(2)) \cong \overline{P'}(i,i^-,i;(2)) \\
&\to \overline{P'}(i^-,i^-,i;(2)) \cong \overline{P'}(i,i^-,i^-;(2)) \to \overline{P'}(i^-,i^-,i^-;(2,1)) \\
&\to \overline{P'}(i,i^-,i^-;(2)) \cong \overline{P'}(i^-,i^-,i;(2)) \\
&\to \overline{P'}(i,i^-,i;(2)) \cong \overline{P'}(i^-,i,i;(2)) \to \overline{P'}(i,i,i;(2)) \\
(a\otimes b\otimes c)\otimes id\otimes x_2 &\mapsto -(\gamma(c)\otimes a\otimes b)\otimes(123)+(\gamma(b)\otimes c\otimes a)\otimes(132) \\
&\mapsto -(a\otimes b\otimes\gamma(c))\otimes id+(c\otimes a\otimes\gamma(b))\otimes(123) \\
&\mapsto -(\gamma(a)\otimes b\otimes\gamma(c))\otimes id+(\gamma(b)\otimes c\otimes\gamma(a))\otimes(132) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123)-(b\otimes\gamma(c)\otimes\gamma(a))\otimes(132) \\
&\mapsto -3(\gamma(a)\otimes\gamma(b)\otimes\gamma(c))\otimes id\otimes x_2 \\
&\mapsto 3(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123)-3(\delta(\gamma(b))\otimes\gamma(c)\otimes\gamma(a))\otimes(132) \\
&\mapsto 3(\gamma(a)\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id-3(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123) \\
&\mapsto 3(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id \\
&\qquad -3(\delta(\gamma(b))\otimes\gamma(c)\otimes\delta(\gamma(a)))\otimes(132) \\
&\mapsto -3(\gamma(c)\otimes\delta(\gamma(a))\otimes\delta(\gamma(b)))\otimes(123) \\
&\qquad +3(\gamma(b)\otimes\delta(\gamma(c))\otimes\delta(\gamma(a)))\otimes(132) \\
&\mapsto 9(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id\otimes x_2
\end{aligned}
$$

$$
\begin{aligned}
& i<j \\
\overline{P'}(i,i,j;(2)) &\to \overline{P'}(i^-,i,j) \cong \overline{P'}(i,i^-,j) \\
&\to \overline{P'}(i^-,i^-,j;(2)) \cong \overline{P'}(j,i^-,i^-;(2)) \to \overline{P'}(j^-,i^-,i^-;(2)) \\
&\to \overline{P'}(j,i^-,i^-;(2)) \cong \overline{P'}(i^-,i^-,j;(2)) \\
&\to \overline{P'}(i,i^-,j) \cong \overline{P'}(i^-,i,j) \to \overline{P'}(i,i,j;(2)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id+(\gamma(b)\otimes a\otimes c)\otimes(12) \\
&\mapsto (a\otimes\gamma(b)\otimes c)\otimes id+(b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto 2(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto 2(c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto 2(\gamma(c)\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto 2(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto 2(\gamma(a)\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id \\
&\mapsto 2(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id+2(\delta(\gamma(b))\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes(12) \\
&\mapsto 2(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id+2(\gamma(b)\otimes\delta(\gamma(a))\otimes\delta(\gamma(c)))\otimes(12) \\
&\mapsto 4(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

$$
\begin{aligned}
& & & i<j \\
\overline{P'}(i,i,j;(1,1)) &\to \overline{P'}(i^-,i,j) \cong \overline{P'}(i,i^-,j) \\
&\to \overline{P'}(i^-,i^-,j;(1,1)) \cong \overline{P'}(j,i^-,i^-;(1,1)) \to \overline{P'}(j^-,i^-,i^-;(1,1)) \\
&\to \overline{P'}(j,i^-,i^-;(1,1)) \cong \overline{P'}(i^-,i^-,j;(1,1)) \\
&\to \overline{P'}(i,i^-,j) \cong \overline{P'}(i^-,i,j) \to \overline{P'}(i,i,j;(1,1)) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id - (\gamma(b)\otimes a\otimes c)\otimes (12) \\
&\mapsto -(a\otimes\gamma(b)\otimes c)\otimes id + (b\otimes\gamma(a)\otimes c)\otimes (12) \\
&\mapsto -2(\gamma(a)\otimes\gamma(b)\otimes c)\otimes id \\
&\mapsto 2(c\otimes\gamma(a)\otimes\gamma(b))\otimes (123) \\
&\mapsto 2(\gamma(c)\otimes\gamma(a)\otimes\gamma(b))\otimes (123) \\
&\mapsto 2(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes (123) \\
&\mapsto -2(\gamma(a)\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id \\
&\mapsto -2(\delta(\gamma(a))\otimes\gamma(b)\otimes\delta(\gamma(c)))\otimes id \\
&\qquad + 2(\delta(\gamma(b))\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes (12) \\
&\mapsto 2(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id - 2(\gamma(b)\otimes\delta(\gamma(a))\otimes\delta(\gamma(c)))\otimes (12) \\
&\mapsto 4(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

$$
\begin{aligned}
& & & j<i \\
\overline{P'}(j,i,i;(2)) &\to \overline{P'}(j^-,i,i;(2)) \cong \overline{P'}(i,j^-,i;(2)) \\
&\to \overline{P'}(i^-,j^-,i) \cong \overline{P'}(i,j^-,i^-) \to \overline{P'}(i^-,j^-,i^-;(2)) \\
&\to \overline{P'}(i,j^-,i^-) \cong \overline{P'}(i^-,j^-,i) \\
&\to \overline{P'}(i,j^-,i;(2)) \cong \overline{P'}(j^-,i,i;(2)) \to \overline{P'}(j,i,i) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto (c\otimes\gamma(a)\otimes b)\otimes (123) \\
&\mapsto (\gamma(c)\otimes\gamma(a)\otimes b)\otimes (123) + (\gamma(b)\otimes\gamma(a)\otimes c)\otimes (12) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes (123) + (b\otimes\gamma(a)\otimes\gamma(c))\otimes (12) \\
&\mapsto 2(\gamma(c)\otimes\gamma(a)\otimes\gamma(b))\otimes (123) \\
&\mapsto 2(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes (123) + 2(\delta(\gamma(b))\otimes\gamma(a)\otimes\gamma(c))\otimes (12) \\
&\mapsto 2(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes (123) + 2(\gamma(b)\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes (12) \\
&\mapsto 4(\delta(\gamma(c))\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes (123) \\
&\mapsto 4(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id \\
&\mapsto 4(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

$$
\begin{aligned}
& & & j < i \\
\overline{P'}(j,i,i;(1,1)) &\to \overline{P'}(j^-,i,i;(1,1)) \cong \overline{P'}(i,j^-,i;(1,1)) \\
&\to \overline{P'}(i^-,j^-,i) \cong \overline{P'}(i,j^-,i^-) \to \overline{P'}(i^-,j^-,i^-;(1,1)) \\
&\to \overline{P'}(i,j^-,i^-) \cong \overline{P'}(i^-,j^-,i) \\
&\to \overline{P'}(i,j^-,i;(1,1)) \cong \overline{P'}(j^-,i,i;(1,1)) \to \overline{P'}(j,i,i) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto -(c\otimes\gamma(a)\otimes b)\otimes(123) \\
&\mapsto -(\gamma(c)\otimes\gamma(a)\otimes b)\otimes(123)+(\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123)-(b\otimes\gamma(a)\otimes\gamma(c))\otimes(12) \\
&\mapsto 2(\gamma(c)\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto 2(\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123)-2(\delta(\gamma(b))\otimes\gamma(a)\otimes\gamma(c))\otimes(12) \\
&\mapsto -2(\gamma(c)\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123)+2(\gamma(b)\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes(12) \\
&\mapsto -4(\delta(\gamma(c))\otimes\gamma(a)\otimes\delta(\gamma(b)))\otimes(123) \\
&\mapsto 4(\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id \\
&\mapsto 4(\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

$$
\begin{aligned}
& & & i < j < k \\
\overline{P'}(i,j,k) &\to \overline{P'}(i^-,j,k) \cong \overline{P'}(j,i^-,k) \to \overline{P'}(j^-,i^-,k) \cong \overline{P'}(k,i^-,j^-) \to \overline{P'}(k^-,i^-,j^-) \\
&\to \overline{P'}(k,i^-,j^-) \cong \overline{P'}(j^-,i^-,k) \to \overline{P'}(j,i^-,k) \cong \overline{P'}(i^-,j,k) \to \overline{P'}(i,j,k) \\
(a\otimes b\otimes c)\otimes id &\mapsto (\gamma(a)\otimes b\otimes c)\otimes id \\
&\mapsto (b\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (\gamma(b)\otimes\gamma(a)\otimes c)\otimes(12) \\
&\mapsto (c\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto (\gamma(c)\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto (\delta(\gamma(c))\otimes\gamma(a)\otimes\gamma(b))\otimes(123) \\
&\mapsto (\gamma(b)\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes(12) \\
&\mapsto (\delta(\gamma(b))\otimes\gamma(a)\otimes\delta(\gamma(c)))\otimes(12) \\
&\mapsto (\gamma(a)\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id \\
&\mapsto (\delta(\gamma(a))\otimes\delta(\gamma(b))\otimes\delta(\gamma(c)))\otimes id
\end{aligned}
$$

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