A New Approach to Hybrid Censoring

Von der Fakultät für Mathematik, Informatik und Naturwissenschaften der RWTH Aachen University zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigte Dissertation

vorgelegt von

Julian Górny, M.Sc.
aus Meerbusch

Berichter: Universitätspfessor Dr. Erhard Cramer
Universitätspfessor Dr. Marco Burkschat
Narayanaswamy Balakrishnan, Distinguished University Professor, Ph.D.

Tag der mündlichen Prüfung: 25.04.2017

Diese Dissertation ist auf den Internetseiten der Hochschulbibliothek online verfügbar.
Acknowledgments

It took the support of many people for getting the opportunity to write a doctoral thesis as well as for being able to finish it. Hereby, I would like to express my gratitude to everyone of them: Thank you!

In particular, I am very grateful to Professor Erhard Cramer for giving me the possibility to work at the Institute of Statistics. Thank you for the opportunity of learning many new things and thank you for your permanent support.

Further, I would like to express my thankfulness to Professor Marco Burkschat for being the second referee. I sincerely appreciate the comments and ideas I received over several topic-related discussions.

For accepting to be the third referee, I want to thank Professor Narayanaswamy Balakrishnan, whom I had the pleasure of meeting at the conference of ordered statistical data in Piraeus.

Finally, I want to express my deepest thanks to my parents Heidi and Jan Górny, my aunt Jadwiga Górna, my uncle Jacenty Górny, my cousin Marcin Górny, his wife Katarzyna Górna, and my colleague Benjamin Laumen.
# Contents

## 1 Introduction

1.1 The Model Point of View ........................................ 4  
1.1.1 Type-I Sequential Hybrid Censoring ......................... 4  
1.1.2 Type-II Progressive Hybrid Censoring ....................... 7  
1.1.3 Generalized Type-II Progressive Hybrid Censoring ........ 8  
1.1.4 Further Hybrid Censoring Models ........................... 10  
1.2 The Distribution Point of View .................................. 11  
1.3 The Sample Situation Point of View ............................ 12  
1.4 Roadmap ........................................................... 13  
1.5 Further Aspects of Analysis and Possible Applications for Hybrid Censored Data ..... 21  

## 2 Preliminary Results

2.1 Order Statistics .................................................... 25  
2.2 Progressively Type-II Censored Order Statistics ............... 27  
2.3 Sequential Order Statistics ...................................... 31  

## 3 B-splines

3.1 Introduction and Basic Properties ................................ 38  
3.2 Geometrical Characterization and Volume Computation ....... 44  
3.3 Integration of B-spline Expressions ............................. 50  
3.3.1 B-spline Convolutions ....................................... 50  
3.3.2 Further Integration Formulas ................................. 53  

## 4 Type-I Hybrid Censoring

4.1 Basic Distributional Results ....................................... 57  
4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions .... 60  
4.2.1 Distribution Theory for a Known Location Parameter .......... 67  
4.2.2 Distribution Theory for an Unknown Location Parameter .......... 82  
4.3 Further Aspects Related to Type-I Hybrid Censoring from Exponential Distributions ...... 91  
4.3.1 From B-spline Representations to Gamma Representations .......... 91  
4.3.2 Limits for $T \to \infty$ ........................................ 93  
4.3.3 A Monotonicity Result w.r.t. $\psi$ .......................... 94  
4.4 Type-I Hybrid Censoring from Uniform Distributions ........ 96  
4.4.1 Fundamental Results ......................................... 97  
4.4.2 Distribution Theory for the MLEs ......................... 102  

## 5 Multi-Sample Type-I Sequential Hybrid Censoring

5.1 Model and Basic Distributional Results ......................... 110  
5.2 Distribution Theory for the MLE ............................... 112
Chapter 1
Introduction

A hybrid censoring model can be described as a censoring procedure where the censoring of a sample of size \( n \in \mathbb{N} \) depends on a previously fixed number of observations \( m \leq n \) and on a threshold time \( T \in (0, \infty) \). In the context of life tests, the integer \( m \) denotes the number of failures an experimenter intends to observe. By using a threshold time \( T \) on the other hand, one can control the test duration. Epstein (1954) was the first to present a censoring scheme of this type. He considered for the replacement as well as for the nonreplacement case a life test with exponential lifetimes, which is terminated at the stopping time \( \min\{X_{m:n}, T\} \). Hereby denotes \( X_{m:n} \) the \( m \)th order statistic. While Epstein (1954) called this censoring procedure ‘truncated life test’, this censoring model is usually denoted by the Type-I hybrid censoring scheme (Type-I HCS).

In the last two decades further hybrid censoring models have been proposed, e.g., Type-II hybrid censoring scheme by Childs et al. (2003), generalized Type-I and generalized Type-II hybrid censoring schemes by Chandrasekar et al. (2004) and four types of unified hybrid censoring schemes by Balakrishnan et al. (2008c), Huang and Yang (2010), and Park and Balakrishnan (2012). The model of unified Type-II hybrid censoring was introduced in Balakrishnan et al. (2008c) and firstly named as the unified hybrid censoring scheme. Then, Huang and Yang (2010) introduced a hybrid censoring scheme called combined hybrid censoring sampling. This model corresponds to the unified Type-I hybrid censoring scheme addressed later in Park and Balakrishnan (2012). The remaining two models (unified Type-III/IV hybrid censoring schemes) were presented in Park and Balakrishnan (2012) within the context of a newly introduced flexible hybrid censoring scheme. The above mentioned censoring procedures are different w.r.t. the set of integers and threshold times, used to specify the censoring properties of the model. For a comprehensive survey on hybrid censoring, we refer to Balakrishnan and Kundu (2013).

Apart from the unified Type-III/IV hybrid censoring schemes, the above mentioned models have been primarily discussed for a one-parameter exponential distribution. Further, the maximum likelihood estimators (MLEs) as well as the corresponding distribution have been derived, see, e.g., Bartholomew (1963), Chen and Bhattacharyya (1987), Childs et al. (2003), Chandrasekar et al. (2004) and Balakrishnan et al. (2008c). In all of these cases conditional moment generating function approach has been utilized. It should be noted that in the context of unified Type-I hybrid censoring the moment generating function of the MLE \( \hat{\vartheta} \) for the exponential parameter \( \vartheta \) has been derived (see Huang and Yang (2010)). However, an explicit expression for the distribution of \( \hat{\vartheta} \) has not been presented. The approach of the conditional moment generating function provides expressions in terms of gamma density functions. An alternative approach has been presented by Cramer and Balakrishnan (2013) within the context of Type-I progressive hybrid censoring. They considered the distribution of the spacings of the underlying hybrid censored order statistics and obtained a representation of the conditional density function \( f_{\hat{\vartheta}|D \geq 1} \) for the MLE \( \hat{\vartheta} \) in
terms of B-spline functions. For an increasing knot sequence \( \gamma_{d+1} < \cdots < \gamma_1, \ d \in \mathbb{N} \), the B-spline of degree \( d - 1 \) can be expressed as (cf. Curry and Schoenberg, 1947, 1966)

\[
B_{d-1}(s|\gamma_{d+1}, \ldots, \gamma_1) = \sum_{j=0}^{d} \frac{(\gamma_{d-j+1} - s)^{d-1}}{\prod_{i=0,i\neq j}^{d} (\gamma_{d-j+1} - \gamma_{d-i+1})}, \quad s \in \mathbb{R},
\]

(1.1)

where \( (s)_+ = \max\{0, s\} \) denotes the positive part of \( s \).

The results presented in Cramer and Balakrishnan (2013) illustrate that the representations for the density function \( \hat{\vartheta}_{|D\geq 1} \) of the MLE \( \hat{\vartheta} \) in terms of B-splines are simpler and more convenient to implement than those obtained by the moment generating function approach. Further, the B-splines have properties which allow a more detailed insight in the structure of the corresponding density function. Thus, it is desirable to obtain expressions in terms of B-splines.

A crucial step, which led Cramer and Balakrishnan (2013) to the B-spline representation of \( \hat{\vartheta}_{|D\geq 1} \), is the determination of the volume of a particular set. The fact that this set could be interpreted as the intersection of a simplex with a half-space, allowed the application of a volume formula established by Gerber (1981, p. 312). The resulting expression could then be rewritten as a scaled B-spline. Due to the results obtained by Cramer and Balakrishnan (2013), we intend to consider the spacing based approach in a more general scope. An essential issue is, to find a direct connection between the B-spline and the support of particular linear combinations of hybrid censored order statistics.

For convenience, we consider the following notations for the support of a real function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \),

\[
\text{supp}(f) = \{ x_d \in \mathbb{R}^d | f(x_d) > 0 \} \quad \text{and} \quad \overline{\text{supp}}(f) := \text{supp}(f),
\]

(1.2)

where \( \overline{A} \) denotes the closure of a set \( A \). Note that the usage of the term 'support' indicates, that it is not crucial whether we consider the closure of the support or not. If we want to distinguish between these two cases, then the respective notation (introduced in (1.2)) is used.

The aim of this thesis is to present a new approach which simplifies the analysis of hybrid censoring models. We focus on the two-parameter exponential and the two-parameter uniform distribution. While the exponential case has been extensively investigated in this area, it seems that there has been no effort made to consider hybrid censoring from uniform distributions.

The herein presented approach is based on the thorough analysis of the structure of an underlying hybrid censoring scheme. It provides a characterization of the model in terms of particular counter settings. In this work, we refer to expressions of the form \( D = d \) as to a counter setting, while \( D = \sum_{j=1}^{m} 1_{(-\infty, T]}(X_{j:n}) \) for \( d \in \{1, \ldots, m\} \). Hereby denotes \( 1_A \), \( A \subset \mathbb{R} \), the indicator function. The approach involves further a specification of the support w.r.t. the corresponding hybrid censored data. The support specification comprises the classification in terms of particular ordered cones \( \Sigma_F^d \) and \( \Sigma_{F,T}^d \), \( d \in \mathbb{N}, \ T \in (0, \infty) \), where

\[
\Sigma_F^d = \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_d < F^{-1}(1) \}
\]

(1.3)

and

\[
\Sigma_{F,T}^d = \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq \cdots \leq x_d \leq T \},
\]

(1.4)
respectively, and $F$ denotes a cumulative distribution function with quantile function $F^{-1}$ (see Definition C.1.5). Further, we use from now on the notation $\mathbf{x}_d = (x_1, \ldots, x_d)$ for a $d$-dimensional vector. In a next step, the distribution theory for the exponential and for the uniform distribution is considered. In this outline, we present two approaches leading to B-spline representations: the ‘volume approach’ (see Procedure 1.4.4) and the ‘expected value approach’ (see Procedure 1.4.6). Those two approaches only differ in the way of obtaining B-spline representations for the desired density functions. The volume approach can be viewed as a generalization of the spacing based approach introduced in Cramer and Balakrishnan (2013). The expected value approach on the other hand uses an identity for calculating the expected value of a continuously transformed random variable. A technique, similar to the expected value approach, has been recently utilized in Burkschat et al. (2016).

At this point, we restrict ourselves to the volume based approach, which proceeds as follows: after specifying the support and after applying suitable linear transformations, we consider a characterization of the transformed support in terms of simplices $S^{(s)}_{d-1,i^*}$ and intersections of simplices with half-spaces $\mathcal{H}^{(s)}_{d-1,i^*}$, $s \in \mathbb{R}$, $i^* \in \{1, \ldots, d\}$, where

$$
S^{(s)}_{d-1,i^*} = \left\{ (x_j)_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \in \mathbb{R}^{d-1} \bigg| x_j \geq 0, j \in \{1, \ldots, d\} \setminus \{i^*\}, \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} s_j x_j \leq s - s_{i^*} \right\} \quad (1.5)
$$

and

$$
\mathcal{H}^{(s)}_{d-1,i^*} = \left\{ (x_j)_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \in \mathbb{R}^{d-1} \bigg| \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} h_j x_j \geq s - h_{i^*} \right\}, \quad (1.6)
$$

with coefficients $s_j \in \{1, \ldots, d\}, h_j \in \{1, \ldots, d\} \in \mathbb{R}$. The derivation of the desired density functions, for instance of $f^0_{D \geq 1}$, corresponds to the problem of calculating the volumes

$$
\text{vol}_{d-1}(S^{(s)}_{d-1,i^*}) \quad \text{and} \quad \text{vol}_{d-1}(S^{(s)}_{d-1,i^*} \cap \mathcal{H}^{(s)}_{d-1,i^*}),
$$

where $\text{vol}_d(A)$ denotes the $d$-dimensional volume of a measurable set $A \subset \mathbb{R}^d$. By utilizing a new geometrical characterization for univariate B-splines, the complicated expression $\text{vol}_{d-1}(S^{(s)}_{d-1,i^*} \cap \mathcal{H}^{(s)}_{d-1,i^*})$ can be identified as a scaled B-spline.

The above approach enables us in the exponential case, to obtain explicit expressions for the distribution of the MLEs w.r.t. all hybrid censoring schemes which have been proposed so far. For the particular model of Type-I hybrid censoring the sequential order statistics are considered, as the underlying model of ordered random variables. For the other hybrid censoring models, we consider progressively Type-II censored order statistics. During these elaborations we obtain the following useful relation between the product of a B-spline with an exponential function and the gamma density function $f^{\alpha,\beta}_d$ (see Definition C.1.4), with parameters $\alpha, \beta > 0$:

$$
B_{d-1}(cs|\gamma_{d+1}, \ldots, \gamma_1) e^{-cs/\vartheta} = \frac{d!\vartheta^d}{c} \sum_{j=0}^{d} \frac{e^{-\gamma_{d-j+1}/\vartheta}}{\prod_{i=0, i \neq j} (\gamma_{d-i+1} - \gamma_{d-j+1})} \int \frac{f^{\alpha,\beta}_d(s - \frac{\gamma_{d-j+1}}{c})}{s^{d-1}}, \quad (1.7)
$$
where $0 \leq \gamma_{d+1} < \gamma_d < \cdots < \gamma_1$, $d \in \mathbb{N}$, and $c, \vartheta > 0$. This identity allows us, for instance, to obtain expressions for the distribution of the MLE $\hat{\vartheta}$ in terms of gamma density functions, from the B-spline based representations. For the two-parameter uniform distribution, we restrict ourselves to the setting of order statistics. We hereby find, that the distributions of the MLEs consist of degenerated parts.

The following three sections illustrate different ways of guiding the reader through this work. First, we consider in Section 1.1 a model point of view, where three particular censoring schemes are addressed in detail. Section 1.2 focuses on the readers interest in results w.r.t. a particular distribution. Following that, we address one-sample or multi-sample settings (see Section 1.3). Section 1.4 consists of a roadmap, which specifies in detail the way of approaching hybrid censoring models in this thesis. This chapter concludes with a brief overview of hybrid censoring related topics (see Section 1.5).

1.1 The Model Point of View

An intuitive approach to address a number of models is by considering them according to an increase of the models complexity. Hereby one may distinguish between the complexity w.r.t. the properties of a particular model and w.r.t. the complexity of the underlying model of ordered data. This section chooses the first option.

We give now a brief overview of the hybrid censoring models considered in this thesis. Hereby, we provide more detailed descriptions of three particular models. For these models some key results in the particular setting of ordinary order statistics for the one-parameter exponential distribution are presented in advance. Detailed descriptions of the remaining models are provided in the corresponding subsequent chapters.

1.1.1 Type-I Sequential Hybrid Censoring

Epstein’s model of Type-I hybrid censoring (Type-I HCS) (cf. Epstein, 1954) is based on order statistics (see Section 2.1 for an introduction). As an extension to sequential order statistics with proportional hazard rates (see Section 2.3 for an introduction), we introduce the Type-I sequential hybrid censoring scheme (Type-I SHCS). We apply the censoring mechanism for the Type-I hybrid censoring scheme described at the very beginning of this chapter to the sample of the first $m$ sequential order statistics, $X_{*1}, \ldots, X_{*m}$. (1.8)

The corresponding stopping time $T_1^*$ is given by

$$T_1^* = \min\{X_{*m}, T\}, \quad m \leq n, \quad T \in (0, \infty).$$

The above model can be seen as a Type-I censored sequential $(n - m + 1)$-out-of-$n$ system. For illustration, let us consider a system consisting of $n$ components, which is targeted for a mandatory maintenance at a prefixed threshold time $T$. Further, the system shuts down due to system breakdown, when the $m$th failure occurs. Then, the time of operation is limited by $\min\{X_{*m}, T\}$. For further reading on sequential $(n - m + 1)$-out-of-$n$ systems, we refer to Cramer and Kamps (1996, 1998b, 2001a) and Balakrishnan et al. (2008a).

For convenience, we define the *Type-I sequential hybrid censored order statistics* by

$$X_j^* := \min\{X_j^*, T\}, \quad 1 \leq j \leq m.$$ (1.9)
Further, we introduce a counter variable $D$, representing the number of observed failures till time $T$:

$$D = \sum_{j=1}^{m} 1_{(-\infty,T]}(X_j^*).$$  \hspace{1cm} (1.10)

In order to illustrate the censoring mechanism and the corresponding experimental outcomes, we use a graphical illustration inspired by Balakrishnan and Cramer (2014, pp. 141–142). Figure 1.1 depicts the possibly occurring sampling situations induced by $T^*_1$. Due to the proportional hazard rates assumption several Type-I hybrid censoring models are included in the model of Type-I sequential hybrid censoring. For special choices of the model parameters $\alpha_1, \ldots, \alpha_m > 0$ (see Remark 2.3.2, (i)), we obtain the model of Type-I hybrid censoring (cf. Epstein, 1954) or the model of Type-I progressive hybrid censoring, which was firstly introduced by Childs et al. (2008). The term progressive indicates the extension of a particular censoring scheme to progressively Type-II censored order statistics (see Section 2.2 for an introduction). Table 1.1 provides an overview of the most significant elaborations w.r.t. Type-I hybrid censoring from exponential distributions. We hereby restrict ourselves to these references, where the exact distribution of the MLEs is considered.

Further, we focus on the aspects as the parameter setting, the model of underlying order statistics and the utilized approach for deriving the desired distribution. It is obvious, that the most frequently pursued approach is the application of the conditional moment generating function (mgf). Firstly, Cramer and Balakrishnan (2013) presented an alternative approach for determining the distribution of the MLEs. This approach, denoted by the ‘spacing based approach’, provided representations in terms of B-splines. Burkschat et al.
(2016) used a method, similar to the herein presented expected value approach (see step \( \text{(II)} \) in Procedure 1.4.6), for Type-I censored sequential \( k \)-out-of-\( n \) systems. This approach leads also to a B-spline representation of the conditional density function \( f_{\hat{\theta}_{D \geq 1}} \). For details on their method, we refer to Remark 4.2.11, (iii).

For additional and early reference on estimating the scale parameter of the exponential distribution under Type-I hybrid censoring, we refer to Barlow et al. (1968), Fairbanks et al. (1982), Ebrahimi (1986) and Draper and Guttman (1987). Note that Barlow et al. (1968) referred to the Type-I hybrid censoring procedure as to the ‘burn-in’ process.

The desired distribution of the MLE \( \hat{\theta} \), for the setting of order statistics follows from the conditional distribution of the total time on test statistic \( S_D, D \in \{1, \ldots, m\} \), i.e., \( \hat{\theta} = \frac{S_D}{D} \). For \( D \in \{1, \ldots, m\} \), \( S_D \) is given by (see (4.18), with \( \gamma_j^* = n - j + 1, 1 \leq j \leq m, \gamma_{m+1}^* = 0 \), and \( W_j^* = (n - j + 1)(Z_{j:n} - Z_{j-1:n}), Z_{0:n} := \mu \),

\[
S_D = \sum_{j=1}^{D} \left( 1 - \frac{c(D)}{n - j + 1} \right)(n - j + 1)(Z_{j:n} - Z_{j-1:n}) + c^*(D) T, \quad \mu = 0,
\]

with

\[
c^*(D) = \begin{cases} 
   n - D, & D \in \{1, \ldots, m - 1\}, \\
   0, & D = m.
\end{cases}
\]

The random variables \( Z_{1:n}, \ldots, Z_{n:n} \) denote the order statistics based on \( Z_1, \ldots, Z_n \overset{iid}{\sim} \text{Exp}(0, \vartheta) \). By utilizing the volume approach (see Procedure 1.4.4), it can be shown that the conditional density function \( f_{S_D | D=d} \) is of the form (see equation (4.20) and Lemma 4.1.1),

\[
f_{S_D | D=d}(s) = \frac{(n - d + 1) e^{-s/\vartheta}}{\vartheta^d P(D = d)} \ vol_{d-1}(S_{d-1,d}^{(s)} \cap H_{d-1,d}^{(s)}), \quad s \geq 0,
\]

with

\[
S_{d-1,d}^{(s)} = \left\{ w_{d-1} \in \mathbb{R}^{d-1} \mid w_j \geq 0, 1 \leq j \leq d - 1, \sum_{j=1}^{d-1} \left( 1 - \frac{c^*(d)}{n - j + 1} \right) w_j \leq s - c^*(d) T \right\},
\]

\[
H_{d-1,d}^{(s)} = \left\{ w_{d-1} \in \mathbb{R}^{d-1} \mid \sum_{j=1}^{d-1} \left( 1 - \frac{n - d + 1}{n - j + 1} \right) w_j \geq s - (n - d + 1) T \right\}.
\]

Looking at the sets \( S_{d-1,d}^{(s)} \) and \( H_{d-1,d}^{(s)} \), we recognize the structures introduced in (1.5) and (1.6), respectively. The corresponding volume can be evaluated by using the geometrical characterization of the B-spline introduced in Theorem 3.2.1 (see also Corollary 3.2.3), such that

\[
f_{S_D | D=d}(s) = \binom{n}{d} \frac{e^{-s/\vartheta} T^d}{\vartheta^d P(D = d)} B_{d-1}(s|c^*(d) T, (n - d + 1) T, \ldots, n T), \quad s \geq 0,
\]

for \( d \in \{1, \ldots, m\} \).
1.1 The Model Point of View

1.1.2 Type-II Progressive Hybrid Censoring

The main drawback of Type-I hybrid censoring is that the experiment may be terminated without observing any failures. This case occurs if the lifespan of each object put on life test exceeds the threshold time \( T \). Childs et al. (2003) presented with the Type-II hybrid censoring scheme (Type-II HCS) a model, which guarantees the observation of at least \( m \) failures. Childs et al. (2008) proposed an extension on progressively Type-II censored order statistics, called Type-II progressive hybrid censoring scheme (Type-II PHCS). Due to the possibility that more than \( m \) failures can be observed, we consider the following extended sample of progressively Type-II censored order statistics

\[
X_{\tilde{m};m:n}, X_{\tilde{m}+1;m:n}, \ldots, X_{\tilde{m};m:n}, \quad \text{with} \quad \tilde{m} = m + R_m, \tag{1.11}
\]

and set \( X_{j;m:n} = X_{j;m:n}^{e}, 1 \leq j \leq m \) (see Remark 2.2.4). For brevity, we write \( \tilde{m} = m + R_m \) during this work. The random variables \( X_{j;m:n}^{e} \) and \( X_{j;m:n}^{e}^{e} \) denote the \( j \)th progressively Type-II censored order statistic w.r.t. the initially planned censoring plan \( R = (R_1, \ldots, R_m) \) and the corresponding (right) extended censoring plan \( \tilde{R}^{e} \), respectively. The extended censoring plan \( \tilde{R}^{e} \) (see Table 2.2) will be applied when the \( m \)th failure occurs before time \( T \). Now, the respective stopping time \( T_{II}^{e} \) can be written as

\[
T_{II}^{e} = \max \{X_{\tilde{m};m:n}, T\}.
\]

By analogy to (1.10), the corresponding counter \( D \) is given by

\[
D = \sum_{j=1}^{\tilde{m}} 1_{(-\infty,T]}(X_{j;m:n}^{e}).
\]

The censoring mechanism and the corresponding possible experimental outcomes are shown in Figure 1.2.

It has to be mentioned that some of the following results for Type-II progressive hybrid censoring from exponential distributions, have been partially presented in Cramer et al. (2016). An overview of the most significant work concerning the distribution of the MLEs for an exponential setup under Type-II hybrid censoring schemes, is given in Table 1.2. Similarly as in the situation of Type-I hybrid censoring, the conditional moment generating function approach is within the scope of Type-II hybrid censoring the most frequently applied approach. Only Cramer et al. (2016) chose with the spacings based approach a different technique.
The corresponding conditional density function $f_{S_d|D<m}(s)$ can be expressed as (cf. equations (6.4) and (6.7))

$$f_{S_d|D<m}(s) = \frac{e^{-s/\vartheta}}{\vartheta^m P(D < m)} \cdot \frac{\text{vol}_{d-1}(S_{m-1,m}^{(s)} \setminus (S_{m-1,m}^{(s)} \cap H_{m-1,m}^{(s)}))}{\text{vol}_{m-1}(S_{m-1,m}^{(s)}) - \text{vol}_{m-1}((S_{m-1,m}^{(s)} \cap H_{m-1,m}^{(s)}))},$$

with

$$S_{m-1,m}^{(s)} = \{w_{m-1} \in \mathbb{R}^{m-1} | w_j \geq 0, 1 \leq j \leq m-1, \sum_{j=1}^{m-1} w_j \leq s\}, \quad (1.12)$$

$$H_{m-1,m}^{(s)} = \{w_{m-1} \in \mathbb{R}^{m-1} | \sum_{j=1}^{m-1} \left(1 - \frac{n-m+1}{n-j+1}\right) w_j \geq s - (n-m+1)T\}.$$

While the volume of the simplex $S_{m-1,m}^{(s)}$ can be easily calculated (see (3.23)), the volume of $S_{m-1,m}^{(s)} \cap H_{m-1,m}^{(s)}$ follows with the same arguments as in Section 1.1.1. Hence, we obtain for $s \geq 0$,

$$f_{S_d|D<m}(s) = \frac{e^{-s/\vartheta}}{\vartheta^m P(D < m)} \left(\frac{s^{m-1}}{(m-1)!} - \binom{n}{m} T^m B_{m-1}(s|0, (n-m+1)T, \ldots, nT)\right).$$

### 1.1.3 Generalized Type-II Progressive Hybrid Censoring

Let $m \leq n$, and two previously fixed threshold-times $T_1, T_2 \in (0, \infty)$ with $T_1 < T_2$ be given. We call a censoring model a *generalized Type-II progressive hybrid censoring scheme (Type-II GPHCS)*, if it accomplishes the following scenarios:
– If the $m$th failure occurs before $T_1$, then the experiment stops at $T_1$;
– If the $m$th failure occurs in the interval $(T_1, T_2)$, then the censoring scheme terminates at $X^{(m)}_{m:n}$;
– Further, if less than $m$ failures have been observed till time $T_2$, the life test stops at $T_2$.

This model has been recently discussed in Górny and Cramer (2016).

The generalized Type-II progressive hybrid censoring scheme is an adaption of the generalized Type-II hybrid censoring scheme (Type-II GHCS) on progressively Type-II censored order statistics, introduced by Chandrasekar et al. (2004). One major drawback of the Type-II progressive hybrid censoring model is, that the experiment may last longer as desired. This happens when the $m$th failure occurs after the threshold time $T_1$. In order to avoid this, Chandrasekar et al. (2004) added a second threshold time $T_2 > T_1$, serving as an upper bound for the experiment duration. This, on the other hand, makes it possible that no failures may occur at all.

In the model of generalized Type-II progressive hybrid censoring, we consider the following extended sample of progressively Type-II censored order statistics (cf. Section 1.1.2),

$$X^{(m)}_{m:n}, \ldots, X^{(m)}_{m+1:n}, \ldots, X^{(m)}_{m:n},$$

with $X^{(m)}_{j:n} = X^{(m)}_{j:n}, 1 \leq j \leq m$, and $\mathcal{M}, \mathcal{P}$ as in Section 1.1.2. The stopping time is given by the random variable $T^*_\text{GH}$, with

$$T^*_\text{GH} = \min \left\{ \max\{T_1, X^{(m)}_{m:n}\}, T_2 \right\}.$$  \hspace{1cm} (1.13)

This expression has been introduced in Park and Balakrishnan (2012, p. 42) for the setting of order statistics. According to $T^*_\text{GH}$, the censoring mechanism can be described also as follows: if the $m$th failure occurs after $T_2$, then terminate the experiment at $T_2$. Otherwise, proceed as in the Type-II progressive hybrid censoring scheme with parameters $T_1$ and $m$.

For the two threshold times $T_1$ and $T_2$ we introduce the counters $D_1$ and $D_2$, where

$$D_i = \sum_{j=1}^{m} 1_{(-\infty, T_i]}(X^{(m)}_{j:n}), \quad i \in \{1, 2\}.$$  \hspace{1cm} (1.14)

Figure 1.3 illustrates the censoring mechanism of the generalized Type-II progressive hybrid censoring model. It should be mentioned that apart from the elaborations performed in this thesis as well as those partially presented in Górny and Cramer (2016), the model of generalized Type-II progressive hybrid censoring has been recently addressed in Lee et al. (2016). The therein presented expressions for the distribution of the MLE $\hat{\theta}$ were obtained by applying the conditional moment generating approach.

As depicted in Figure 1.3, the model of generalized Type-II progressive hybrid censoring consists of a new counter setting, i.e., $D_1 < m$, $D_2 = m$. In order to derive the density function of the MLE $\hat{\theta}$ w.r.t. order statistics, we proceed for $D_1 \in \{m, \ldots, n\}$ and $D_2 \in \{0, \ldots, m-1\}$, as in Section 1.1.1. For $D_1 < m$, $D_2 = m$, the total time on test statistic $S_m$ is given by (see (7.3), with $\gamma_j = n - j + 1, 1 \leq j \leq m$, and $W^\text{GH}_j = (n - j + 1)(Z_{j:n} - Z_{j-1:n}), Z_{0:n} := 0)$

$$S_m = \sum_{j=1}^{m} (n - j + 1)(Z_{j:n} - Z_{j-1:n}).$$
Figure 1.3: Experimental outcomes for the generalized Type-II progressive hybrid censoring scheme.

The corresponding conditional density function is given by (see equations (7.5) and (4.20))

\[
f_{S_{m} \mid D_1 < m, D_2 = m}(s) = e^{-s/\vartheta} \varphi^{-m} P(D_1 < m, D_2 = m) \cdot \text{vol}_{m-1} \left( S_{m-1,m} \cap H_{m-1,m;1}^{(s)} \right) - \text{vol}_{m-1} \left( S_{m-1,m} \cap H_{m-1,m;2}^{(s)} \right), \quad s \geq 0,
\]

with \( S_{m-1,m}^{(s)} \) as in (1.12) and

\[
H_{m-1,m;k}^{(s)} = \left\{ w_{m-1} \in \mathbb{R}^{m-1} \left| \sum_{j=1}^{m-1} \left( 1 - \frac{n - m + 1}{n - j + 1} \right) w_j \geq s - (n - m + 1)T_k \right\} \right., \quad k \in \{1, 2\}.
\]

We observe that the explicit representation of \( f_{S_{m} \mid D_1 < m, D_2 = m}(s) \) can be obtained by using the same arguments as in Section 1.1.1. Hence, we get for \( s \geq 0 \),

\[
f_{S_{m} \mid D_1 < m, D_2 = m}(s) = \binom{n}{m} e^{-s/\vartheta} \varphi^{-m} P(D_1 < m, D_2 = m) \cdot \left[ T_2^n B_{m-1}(s|0, (n - m + 1)T_2, \ldots, nT_2) - T_1^n B_{m-1}(s|0, (n - m + 1)T_1, \ldots, nT_1) \right].
\]

1.1.4 Further Hybrid Censoring Models

Together with the three previously introduced models, we consider in this work ten different hybrid censoring schemes (see Table 1.3 for an overview on these models and the corresponding model parameters). The herein introduced multi-sample Type-I sequential hybrid censoring scheme (Type-I MSHCS), is an extension of the Type-I sequential hybrid...
1.2 The Distribution Point of View

Motivated by the distributional setup established for the analysis of failure data in Davis (1952), Epstein considered in his work the setting of (one-parameter) exponentially distributed lifetimes (cf. Epstein, 1954). As mentioned at the beginning of this chapter, the subsequent elaborations extending the original model to more general hybrid censoring schemes also address the one-parameter exponential distribution (see, e.g., Childs et al., 2003; Chandrasekar et al., 2004; Balakrishnan et al., 2008c; Park and Balakrishnan, 2012). Although the Weibull distribution, for instance, has a much wider scope of applicability (see Section 1.5), the exponential distribution is favored. This basically is due to the simple structure, which allows the derivation of several exact distributional results. For an underlying Weibull distribution, the hybrid censoring related theory is much more sophisticated and therefore significantly less convenient to handle. Recently, the results for the one-parameter exponential distribution have been extended to the two-parameter case, see, e.g., Childs et al. (2012), Ganguly et al. (2012), Cramer and Balakrishnan (2013), Chan et al. (2015), Cramer et al. (2016) and Görny and Cramer (2016). For an comprehensive re-

censoring model to $k \geq 2$ independent samples. The generalized Type-I (progressive) hybrid censoring scheme (Type-I GPHCS) and the unified Type-II hybrid censoring scheme (Type-II UHCSs) have been discussed in the context of an underlying exponential distribution (cf. Chandrasekar et al. (2004), Cho et al. (2015b) and Balakrishnan et al. (2008c)). Huang and Yang (2010) introduced a model, which we call the unified Type-I hybrid censoring scheme (Type-I UHCS). The unified Type-III and unified Type-IV hybrid censoring schemes (Type-III/IV UHCSs) have been introduced for order statistics in Park and Balakrishnan (2012). They derived expressions for the corresponding Fisher information based on a continuous density function with parameter $\vartheta$. We present extensions of these models to progressively Type-II censored order statistics, so called Type-I/II/III/IV progressive hybrid censoring schemes (Type-I/II/III/IV UPHCSs). Finally, we present a new censoring model called general unified progressive hybrid censoring scheme (GUPHCS).

### Table 1.3: Model based guidance through this thesis.

<table>
<thead>
<tr>
<th>Censoring Model</th>
<th>Parameters</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type-I SHCS</td>
<td>$m$</td>
<td>$T$</td>
</tr>
<tr>
<td>Type-I MSHCS</td>
<td>$k, m_1, \ldots, m_k$</td>
<td>$T_1, \ldots, T_k$</td>
</tr>
<tr>
<td>Type-II PHCS</td>
<td>$m$</td>
<td>$T$</td>
</tr>
<tr>
<td>Type-I GPHCS</td>
<td>$k, m$</td>
<td>$T$</td>
</tr>
<tr>
<td>Type-II GPHCS</td>
<td>$m$</td>
<td>$T_1, T_2$</td>
</tr>
<tr>
<td>Type-I UPHCS</td>
<td>$k, m$</td>
<td>$T_1, T_2$</td>
</tr>
<tr>
<td>Type-II UPHCS</td>
<td>$k, m$</td>
<td>$T_1, T_2$</td>
</tr>
<tr>
<td>Type-III UPHCS</td>
<td>$k, m$</td>
<td>$T_1, T_2, T_3$</td>
</tr>
<tr>
<td>Type-IV UPHCS</td>
<td>$k, m, u$</td>
<td>$T_1, T_2$</td>
</tr>
<tr>
<td>GUPHCS</td>
<td>$k, m, u$</td>
<td>$T_1, T_2, T_3$</td>
</tr>
</tbody>
</table>
view on the exponential distribution as well as for a reference work for possible applications, we refer to Balakrishnan and Basu (1995).

Uniform distributions are often used for quantile transformations. E.g., results for progressively Type-II censored order statistics have been established in Balakrishnan and Dembińska (2008, 2009) and Balakrishnan and Cramer (2014, Chapter 2). Further, Kamps and Cramer (2001) considered with generalized order statistics a more general setting. Elandt-Johnson and Johnson (1980, Sections 3.10, 4.6 and 5.10) were one of the first to mention the uniform distribution in the context of lifetime experiments. Weissman and Cohen (1996) considered one-parameter uniform distributed failure times under random censoring. They established some basic model-specific results and considered likelihood inference. In the context of general progressive Type-II censoring from uniform distributions, Aggarwala and Balakrishnan (1998) presented some exact distributional results and considered the MLEs for the one-parameter as well as for the two-parameter case. Bayesian inferential results w.r.t. random censoring from the one-parameter uniform distribution have been established in Liang (2008) and Liang and Huang (2009). Hybrid censoring procedures have apparently not been considered for the uniform distribution so far. However, Ng et al. (2015) established formulas for the conditional moments of progressively Type-II censored order statistics under a time constraint. Among others, explicit formulas for the standard uniform distribution have been presented. Ng et al. (2015) stated further, that these formulas can be applied in the context of Type-I progressive hybrid censoring. Further distributional properties w.r.t. ordered data from uniform distributions have been established in Balakrishnan and Cramer (2014, Chapter 2), for ordinary and progressively Type-II censored order statistics, and in Bieniek (2007), for generalized order statistics.

Table 1.4 guides the reader through this thesis by focusing on the distribution of interest. For the exponential case, we consider sequential and progressively Type-II censored order statistics. For an underlying uniform distribution, we consider the setup of ordinary order statistics. Note that the denotation of a particular censoring scheme varies for the model of ordered data the censoring scheme is applied on. We assume further that the underlying random variables are independent and identically distributed (IID).

For the Type-I sequential, Type-II progressive and generalized Type-I/II progressive hybrid censoring schemes, we consider for an exponential assumption the distribution of the MLE $\hat{\vartheta}$ with location parameter $\mu$ known. Then, we determine the joint distribution of the MLEs $\hat{\mu}$ and $\hat{\vartheta}$. We proceed analogously for the uniform distribution w.r.t. the parameters $a$ and $b$. For the multi-sample Type-I sequential hybrid censoring model, we restrict ourselves to the case where the exponential distribution with $\mu$ known and $\vartheta$ unknown is assumed. In the context of the general unified progressive hybrid censoring model, we establish results for the MLEs $\hat{\vartheta}$ and $\hat{b}$ for an exponential and uniform assumption, respectively. Further, sufficient information is provided, which enables the derivation of the distributions from for the MLEs in the two-parameter case from the previously established results. We proceed similarly for the unified Type-I/II/III/IV progressive hybrid censoring models. There, the distribution for the MLE $\hat{\vartheta}$ is derived and information as well as references for calculating the distribution of the MLEs for the remaining setups are presented.

1.3 The Sample Situation Point of View

The most common situation within the scope of hybrid censoring, is the application of a censoring scheme on one particular sample at a time. In this thesis, we also consider
the application of a particular hybrid censoring procedure on \( k \) independent samples. In particular, we consider the model of performing independently \( k \)-times the Type-I sequential hybrid censoring scheme for the exponential case. This model, called multi-sample Type-I sequential hybrid censoring scheme, will be addressed in Chapter 5. The therein derived results provide the reader with sufficient tools to extend this multi-sample approach to the remaining hybrid censoring models for the exponential distribution. Multi-sample models have been so far among others considered for sequential order statistics in order to estimate the model parameters \( \alpha_1, \ldots, \alpha_n > 0 \) (see, e.g., Cramer and Kamps, 1996, 2001a) as well as in the context of step-stress models (see, e.g., Kateri et al., 2010; Bedbur et al., 2015). For more information and for additional references on multi-sample models in the context of ordered random variables, we refer to Chapter 5.

### 1.4 Roadmap

This work is structured as follows: in Chapter 2 we recall some basic results for particular models of ordered random variables. Chapter 3 introduces into the field of B-splines and focuses on the derivation of a new geometrical characterization of the univariate B-spline. In the subsequent Chapters 4 – 8 as well as in Appendix A, the hybrid censoring models enlisted in Table 1.3 are discussed. Following that, we provide some concluding remarks as well as an outlook. Note that in Appendix B a simulation study for all the herein discussed hybrid censoring models w.r.t. the scale parameter of the exponential distribution is conducted. Finally, Appendix C provides the definitions of the herein addressed distributions as well as some preliminary results.

<table>
<thead>
<tr>
<th>model</th>
<th>exponential distribution parameters</th>
<th>Section</th>
<th>uniform distribution parameters</th>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type-I SHCS</td>
<td>( \vartheta )</td>
<td>4.2.1</td>
<td>Type-I HCS</td>
<td>4.4.2</td>
</tr>
<tr>
<td></td>
<td>( \mu, \vartheta )</td>
<td>4.2.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type-I MSHCS</td>
<td>( \vartheta )</td>
<td>5.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type-II PHCS</td>
<td>( \vartheta )</td>
<td>6.2.1</td>
<td>Type-II HCS</td>
<td>6.3.2</td>
</tr>
<tr>
<td></td>
<td>( \mu, \vartheta )</td>
<td>6.2.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type-I GPHCS</td>
<td>( \vartheta )</td>
<td>7.1.2</td>
<td>Type-I GHCS</td>
<td>7.1.3</td>
</tr>
<tr>
<td></td>
<td>( \mu, \vartheta )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type-II GPHCS</td>
<td>( \vartheta )</td>
<td>7.2.2</td>
<td>Type-II GHCS</td>
<td>7.2.3</td>
</tr>
<tr>
<td></td>
<td>( \mu, \vartheta )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type-I UPHCS</td>
<td>( \vartheta )</td>
<td>A.1</td>
<td>Type-I UHCS</td>
<td>–</td>
</tr>
<tr>
<td>Type-II UPHCS</td>
<td>( \vartheta )</td>
<td>A.2</td>
<td>Type-II UHCS</td>
<td>–</td>
</tr>
<tr>
<td>Type-III UPHCS</td>
<td>( \vartheta )</td>
<td>A.3</td>
<td>Type-III UHCS</td>
<td>–</td>
</tr>
<tr>
<td>Type-IV UPHCS</td>
<td>( \vartheta )</td>
<td>A.4</td>
<td>Type-IV UHCS</td>
<td>–</td>
</tr>
<tr>
<td>GUPHCS</td>
<td>( \vartheta )</td>
<td>8.2</td>
<td>GUHCS</td>
<td>8.3</td>
</tr>
</tbody>
</table>

Table 1.4: Distribution based guidance through this thesis.
We proceed by providing two procedures (see Procedures 1.4.4 and 1.4.6), according to the volume and to the expected value approach, for deriving distributional results based on B-splines. During these elaborations a method is developed, with which hybrid censoring models can be structured by considering a particular way of classifying the experimental outcomes. However, first, we need to introduce some convenient notations.

**Notation 1.4.1 (Specification of a hybrid censoring model)** The following notations facilitate the structural as well as the distribution oriented analysis of an underlying hybrid censoring scheme.

- Let \( F \) be an absolutely continuous cumulative distribution function;
- The total amount of objects put on a life test is denoted by \( n \in \mathbb{N} \);
- Further, \( H \) denotes a particular hybrid censoring scheme, with threshold times \( 0 < T_1 < \cdots < T_l < \infty, l \in \mathbb{N} \), and integers \( 1 \leq m_1 \leq \cdots \leq m_h \leq n, h \in \mathbb{N} \);
- By \( X(1), \ldots, X(m_h) \), we denote a sample of the first \( m_h \leq n \) ordered random variables (e.g., sequential order statistics) based on \( F \);
- For \( d \in \{1, \ldots, m_h\} \), we denote by \( X^H_1, \ldots, X^H_d \) the sample of hybrid censored order statistics of length \( d \) associated to the hybrid censoring scheme \( H \) (e.g., Type-I sequential hybrid censored order statistics);
- The hybrid censored order statistics \( X^H_1, \ldots, X^H_d \) originate from the sample of ordered random variables \( X(1), \ldots, X(d), d \in \{1, \ldots, m_h\} \) as well as from the hybrid censoring procedure according to \( H \);
- We define the discrete random variables \( D_i, 1 \leq i \leq l \), by
  \[
  D_i = \sum_{j=1}^{m_h} 1_{(-\infty, T_i]}(X(j)),
  \]
  which specify the number of observed failures till time \( T_i \), respectively;
- For \( j \in \{1, \ldots, h\} \), we consider three particular sampling situations and its corresponding counter settings based on the parameters of the underlying censoring scheme \( H \):
  1. \( X(1), \ldots, X(D_i) \) \( D_i \in \{m_j-1, \ldots, m_j\}, m_0 := 1, i \in \{1, \ldots, l\} \),
  2. \( X(1), \ldots, X(m_j) \) \( D_i < m_j, i \in \{1, \ldots, l\} \), \hfill (1.15)
  3. \( X(1), \ldots, X(m_j) \) \( D_i < m_j, D_{i+1} = m_j, i \in \{1, \ldots, l-1\} \);
- Further, we denote by \( \mathcal{D} \) a counter setting of the form given in (1.15), i.e.,
  \[
  \mathcal{D} \in \left\{ [D_i = :], [D_i < :], [D_i < : D_{i+1} = :] | i \in \{1, \ldots, \tilde{l}\} \right\},
  \]
  where
  \[
  \tilde{l} := \begin{cases}
  l, & \mathcal{D} \in \{[D_i = :], [D_i < :]\}; \\
  l - 1, & \mathcal{D} = [D_i < :, D_{i+1} = :].
  \end{cases}
  \]
  Note that during this thesis the brackets \( [ \) and \( ] \) are only used for the specification of counter settings (as in (1.16)).
The set of valid integers w.r.t. the counter setting $\mathcal{D}$ is denoted by $\mathcal{I}(\mathcal{D})$. By valid integers, we hereby mean the set of integers for which the counter setting $\mathcal{D}$ is defined. The set of valid integers results from the definition of the hybrid censoring model $\mathcal{H}$;

- By the expression $\mathcal{D}(d)$, we imply the evaluation of the counter setting $\mathcal{D}$ w.r.t. a particular integer $d \in \mathcal{I}(\mathcal{D})$;

- The system of all counter settings, which are associated with a censoring model $\mathcal{H}$, is denoted by $\mathcal{S}_D(\mathcal{H})$. Note that $\mathcal{S}_D(\mathcal{H}) \subseteq \{ D \in \{ [D_i = \cdot], [D_i < \cdot], [D_i < \cdot, D_{i+1} = \cdot] | i \in \{1, \ldots, \tilde{l}\} \}$, with $\tilde{l}$ as in (1.17);

- By $\mathcal{I}(\mathcal{S}_D(\mathcal{H}))$, we denote the family of all sets of valid integers w.r.t. all counter settings for a censoring scheme $\mathcal{H}$, i.e.,

$$\mathcal{I}(\mathcal{S}_D(\mathcal{H})) := \{ \mathcal{I}(\mathcal{D}) | \mathcal{D} \in \mathcal{S}_D(\mathcal{H}) \}$$;

- The family of counter situations for a particular censoring scheme $\mathcal{H}$ can be specified by the pair

$$(\mathcal{S}_D(\mathcal{H}), \mathcal{I}(\mathcal{S}_D(\mathcal{H})))$$.

Let $k$ denote the number of elements of $\mathcal{S}_D(\mathcal{H})$. Then, it is obvious that $\mathcal{I}(\mathcal{S}_D(\mathcal{H}))$ consists also of $k$ elements. We state the following w.r.t. the pair $(\mathcal{S}_D(\mathcal{H}), \mathcal{I}(\mathcal{S}_D(\mathcal{H})))$: the $i$th element of $\mathcal{S}_D(\mathcal{H})$ corresponds to the $i$th element of $\mathcal{I}(\mathcal{S}_D(\mathcal{H}))$, $1 \leq i \leq k$. Further, once the sets $\mathcal{S}_D(\mathcal{H})$ and $\mathcal{I}(\mathcal{S}_D(\mathcal{H}))$ have been specified, the order of the elements contained therein is not to be changed.

The following example demonstrates the application of the above notations for the three hybrid censoring models introduced in Section 1.1.

**Example 1.4.2** We consider the models Type-I sequential hybrid censoring, Type-II progressive hybrid censoring and generalized Type-II progressive hybrid censoring introduced in Sections 1.1.1, 1.1.2 and 1.1.3, respectively.

(i) For the Type-I sequential hybrid censoring scheme, we have $l = h = 1$, $T_1 = T$ and $m_1 = m$. Hence, the sample of ordered data $X_{(1)}, \ldots, X_{(m)}$ is given by the first $m$ sequential order statistics (cf. (1.8)), i.e.,

$$X_1^*, \ldots, X_m^*.$$

The corresponding hybrid censored sample $X_1^H, \ldots, X_d^H$, $d \in \{1, \ldots, m\}$, is given by (cf. (1.9)),

$$X_1^*, \ldots, X_d^* \quad \text{with} \quad X_j^* := \min\{ X_j^*, T \}, \quad 1 \leq j \leq d.$$

Now, for the specification of the counter setting, we first find that the sampling and counter situations depicted in Figure 1.1 can be summarized as

$$X_1^*, \ldots, X_d^*, \quad D \in \{1, \ldots, m\}.$$
This corresponds to case 1 of (1.15). The respective counter setting \( \mathcal{D} \) is given by 
\( \mathcal{D} = [D = \cdot] \), while \( \mathcal{D}(d) = [D = \cdot] \), for \( d \in \mathcal{I}(\mathcal{D}) \), with \( \mathcal{I}(\mathcal{D}) = \{1, \ldots, m\} \). This leads us to

\[
(\mathcal{G}_D(\mathcal{I}^*), \mathcal{I}(\mathcal{G}_D(\mathcal{I}^*))) = \left( \{[D = \cdot]\}, \{1, \ldots, m\} \right)
\]

(ii) For the Type-II progressive hybrid censoring scheme, we find that \( l = 1, h = 2 \) with \( T_1 = T, m_1 = m \) and \( m_2 = \tilde{m} \). The underlying sample of ordered data (cf. (1.11)) as well as the corresponding hybrid censored sample of size \( d \in \{1, \ldots, \tilde{m}\} \) is given by

\[
X_{1;m_1:n}, \ldots, X_{\tilde{m};m_1:n}, X_{m_1+1;m_1:n}, \ldots, X_{\tilde{m};m_1:n},
\]

and \( X_1^{\text{HII}}, \ldots, X_d^{\text{HII}} \), with \( X_j^{\text{HII}} := X_{j;m_1:n}, 1 \leq j \leq d \), respectively. The following table illustrates the relation between the sampling and the counter situations from Figure 1.2 and the recently introduced notations:

<table>
<thead>
<tr>
<th>Sampling specifications</th>
<th>Corresponding notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{1;m_1:n}, \ldots, X_{\tilde{m};m_1:n} ), ( D \in {m, \ldots, \tilde{m}} )</td>
<td>( \mathcal{D} = [D = \cdot] ), ( \mathcal{I}(\mathcal{D}) = {m, \ldots, \tilde{m}} )</td>
</tr>
<tr>
<td>( X_{1;m_1:n}, \ldots, X_{\tilde{m};m_1:n} ), ( D &lt; m )</td>
<td>( \mathcal{D} = [D &lt; \cdot] ), ( \mathcal{I}(\mathcal{D}) = {m} )</td>
</tr>
</tbody>
</table>

Hence, the system of counter settings \( \mathcal{G}_D(\mathcal{I}^*) \) and the corresponding family of sets of valid integers \( \mathcal{I}(\mathcal{G}_D(\mathcal{I}^*)) \) is given by

\[
\mathcal{G}_D(\mathcal{I}^*) = \{[D = \cdot], [D < \cdot]\} \quad \text{and} \quad \mathcal{I}(\mathcal{G}_D(\mathcal{I}^{\text{HII}})) = \{\{m, \ldots, \tilde{m}\}, \{m\}\},
\]

respectively.

(iii) For the model of generalized Type-II progressive hybrid censoring, we have \( l = h = 2 \) with \( m_1 = m \), \( m_2 = \tilde{m} \), \( T_1 \) and \( T_2 \). The underlying sample of ordered random variables corresponds to the sample as given in the context of Type-II progressive hybrid censoring (cf. (1.11)). Further, the respective hybrid censored sample is given by

\[
X_1^{\text{GII}}, \ldots, X_d^{\text{GII}} \quad \text{with} \quad X_j^{\text{GII}} := \min \{X_{j;m_1:n}, T_2\}, 1 \leq j \leq d,
\]

with \( d \in \{1, \ldots, \tilde{m}\} \). The notations introduced above, read for the generalized Type-II progressive hybrid censoring scheme as follows (cf. Figure 1.3):

<table>
<thead>
<tr>
<th>Sample specifications</th>
<th>Corresponding notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{1;m_1:n}, \ldots, X_{D_1;m_1:n} ), ( D_1 \in {m, \ldots, \tilde{m}} )</td>
<td>( \mathcal{D} = [D_1 = \cdot] ), ( \mathcal{I}(\mathcal{D}) = {m, \ldots, \tilde{m}} )</td>
</tr>
<tr>
<td>( X_{1;m_1:n}, \ldots, X_{D_2;m_1:n} ), ( D_1 &lt; m, D_2 = m )</td>
<td>( \mathcal{D} = [D_1 &lt; \cdot, D_2 = \cdot] ), ( \mathcal{I}(\mathcal{D}) = {m} )</td>
</tr>
<tr>
<td>( X_{1;m_1:n}, \ldots, X_{D_2;m_1:n} ), ( D_2 \in {1, \ldots, m-1} )</td>
<td>( \mathcal{D} = [D_2 = \cdot] ), ( \mathcal{I}(\mathcal{D}) = {1, \ldots, m-1} )</td>
</tr>
</tbody>
</table>

Accordingly, we arrive for \( \mathcal{G}_D(\text{GII}) \) and \( \mathcal{I}(\mathcal{G}_D(\text{GII})) \) respective at

\[
\mathcal{G}_D(\text{GII}) = \{[D_1 = \cdot], [D_1 < \cdot, D_2 = \cdot], [D_2 = \cdot]\},
\]

and \( \mathcal{I}(\mathcal{G}_D(\text{GII})) = \{\{m, \ldots, \tilde{m}\}, \{m\}, \{1, \ldots, m-1\}\} \).
For two random variables $X$ and $Y$, we write $X \overset{d}{=} Y$ when $P(X \leq t) = P(Y \leq t)$, for all $t \in \mathbb{R}$. The following remark states a useful relation between the samples $X^H_1, \ldots, X^H_d$ and $X_{(1)}, \ldots, X_{(d)}$.

**Remark 1.4.3** Conditionally on a counter setting $\mathcal{D} \in \mathcal{S}_D(H)$, the hybrid censored sample $X^H_1, \ldots, X^H_d$ corresponds to the sample of the underlying ordered random variables $X_{(1)}, \ldots, X_{(d)}$, $d \in \mathcal{I}(\mathcal{D})$, i.e.,

$$X^H_1, \ldots, X^H_d | \mathcal{D}(d) \overset{d}{=} X_{(1)}, \ldots, X_{(d)} | \mathcal{D}(d), \quad d \in \mathcal{I}(\mathcal{D}), \quad \text{for all} \quad \mathcal{D} \in \mathcal{S}_D(H).$$

Hence, the distribution of the hybrid censored order statistics w.r.t. a particular censoring scheme $H$, is predetermined by the distributional setting of the underlying model of ordered random variables.

The above identity becomes more obvious during the elaborations in the subsequent chapters.

Except for the multi-sample Type-I sequential hybrid censoring scheme the subsequently addressed hybrid censoring models can be handled in the same manner. The following procedure serves as a general roadmap consisting of a volume based step by step technique for obtaining the distribution of a linear combination of hybrid censored order statistics.

**Procedure 1.4.4 (Volume approach)** Let $H$ denote the underlying censoring scheme with threshold times $0 < T_1 < \cdots < T_l < \infty$, $l \in \mathbb{N}$, and integers $1 \leq m_1 \leq \cdots \leq m_h \leq n_h$, $h \in \mathbb{N}$, where $n \in \mathbb{N}$ denotes the total amount of objects put on life test. The random variable $T^*_H$ denotes the stopping time of the censoring model $H$. Further, let $X_{(1)}, \ldots, X_{(m_h)}$ denote a sample of ordered random variables based on an absolutely continuous cumulative distribution function $F$, with $F^{-1}(0) = c_{lb}$ for a constant $c_{lb} \in \mathbb{R}$. A distribution oriented analysis of the hybrid censoring scheme $H$ by means of the volume approach can be realized by performing the following twelve steps:

1. Identify the utilized model of ordered data by specifying a sample of ordered random variables $X_{(1)}, \ldots, X_{(m_h)}$ based on $F$;

2. Define the hybrid censored order statistics $X^H_j$, $1 \leq j \leq m_h$, based on $X_{(1)}, \ldots, X_{(m_h)}$ and according to $T^*_H$;

3. Specify the system of counter settings $\mathcal{S}_D(H)$ associated to $H$, and determine the corresponding family of sets of valid integers $\mathcal{I}(\mathcal{S}_D(H))$;

4. Determine the probabilities of the counter settings, i.e.,

$$P(\mathcal{D}(d)), \quad d \in \mathcal{I}(\mathcal{D}), \quad \text{for all} \quad \mathcal{D} \in \mathcal{S}_D(H);$$

5. Calculate for each counter setting the joint probability of the hybrid censored random variables and the counter setting, i.e.,

$$P(X^H_j \leq x_j, 1 \leq j \leq d, \mathcal{D}(d)), \quad x_d \in \mathbb{R}^d, \quad d \in \mathcal{I}(\mathcal{D}), \quad \text{for all} \quad \mathcal{D} \in \mathcal{S}_D(H);$$
6) Determine the conditional joint density functions
\[ f_{X^H,1 \leq j \leq d | D(d)}^H, \quad d \in \mathcal{D}, \quad \text{for all} \quad \mathcal{D} \in \mathcal{G}_D(H), \]
and specify further the corresponding supports in terms of the cones \( \Sigma_F^d \) and \( \Sigma_{F,T}^d \),
d \in \mathcal{D}, \ T \in \{T_1, \ldots, T_l\} \}
where
\[ \Sigma_F^d = \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_d < F^{-1}(1) \} \]
and
\[ \Sigma_{F,T}^d = \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq \cdots \leq x_d \leq T \}, \ T \in \{T_1, \ldots, T_l\}, \]
respectively (cf. (1.3) and (1.4));

In order to determine the distribution of a linear statistic \( S \), where
\[
S := \begin{cases}
S_{d_1}, & d_1 \in \mathcal{D}(1), \\
\vdots & \vdots \\
S_{d_k}, & d_k \in \mathcal{D}(k),
\end{cases} = \begin{bmatrix}
\alpha_{1,0} + \sum_{j=1}^{d_1} \alpha_{1,j} X^H_j, & d_1 \in \mathcal{D}(1), \\
\vdots & \vdots \\
\alpha_{k,0} + \sum_{j=1}^{d_k} \alpha_{k,j} X^H_j, & d_k \in \mathcal{D}(k),
\end{bmatrix}, \quad (1.18)
\]
with \( k := |\mathcal{G}_D(H)| \), for coefficients \( \alpha_{i,0}, \ldots, \alpha_{i,d_i} \in \mathbb{R}, \ d_i \in \mathcal{D}(i), \ 1 \leq i \leq k \), and for \( \mathcal{D}_i \in \mathcal{G}_D(H) \) with \( \mathcal{D}_i \neq \mathcal{D}_j \), \( 1 \leq i, j \leq k \), \( i \neq j \), we proceed as follows:

7) Determine the conditional joint density functions,
\[ f_{X^H,1 \leq j \leq d | D(d)}^H, \quad d \in \mathcal{D}, \quad \text{for all} \quad \mathcal{D} \in \mathcal{G}_D(H), \]
by plugging in the underlying cumulative distribution function \( F \) and adjust the corresponding supports obtained in step 6);

8) Define for \( c^w_j > 0, \ 1 \leq j \leq m_h \), the spacings
\[ W^H_j = c^w_j (X^H_j - X^H_{j-1}), \quad 1 \leq j \leq m_h, \quad \text{with} \quad X^H_0 := c_b; \]

9) Calculate by adequate transformations the conditional joint density functions
\[ f_{W^H,1 \leq j \leq d | D(d)}^H, \quad d \in \mathcal{D}, \quad \text{for all} \quad \mathcal{D} \in \mathcal{G}_D(H), \]
and specify the corresponding supports;

10) Let \( i^* \in \{1, \ldots, d\} \) be arbitrarily chosen but fixed. Then, calculate by adequate transformations the conditional joint density functions
\[ f_{W^H,j \in \{1, \ldots, d\} \setminus \{i^*\}, D(d)}^H, \quad d \in \mathcal{D}, \quad \text{for all} \quad \mathcal{D} \in \mathcal{G}_D(H), \]
and specify the respective supports in terms of simplices $S_{d-1,i^*}^{(s)}$, and intersections of simplices with half-spaces $S_{d-1,i^*}^{(s)} \cap H_{d-1,i^*}^{(s)} =: M_{d-1}^{[i^*]}(s|)$, $s \in \mathbb{R}$, where (cf. (1.5) and (1.6))

$$S_{d-1,i^*}^{(s)} = \left\{ (x_j)_{j \in \{1, \ldots, d\}\backslash \{i^*\}} \in \mathbb{R}^{d-1} \left| x_j \geq 0, j \in \{1, \ldots, d\} \backslash \{i^*\}, \sum_{j \in \{1, \ldots, d\}\backslash \{i^*\}} s_j x_j \leq s - s_{i^*} \right. \right\},$$

and

$$H_{d-1,i^*}^{(s)} = \left\{ (x_j)_{j \in \{1, \ldots, d\}\backslash \{i^*\}} \in \mathbb{R}^{d-1} \left| \sum_{j \in \{1, \ldots, d\}\backslash \{i^*\}} h_j x_j \geq s - h_{i^*} \right. \right\}.$$

Note that $\cdot$ in $M_{d-1}^{[i^*]}(s|)$ is a placeholder for the parameter vectors $\beta \in \mathbb{R}^d$ and $t \in \mathbb{R}^{d+1}$ (see Theorem 3.2.1). These parameter vectors are not relevant here, such that they can be omitted;

1. Determine the conditional density functions

$$f^{S_d|\mathcal{D}(d)}, \quad d \in \mathcal{I}(\mathcal{D}), \quad \text{for all} \quad \mathcal{D} \in \mathcal{G}_\mathcal{D}(H),$$

by calculating the corresponding volumes $\text{vol}_{d-1}(S_{d-1,i^*}^{(s)})$ and $\text{vol}_{d-1}(M_{d-1}^{[i^*]}(s|))$, $s \in \mathbb{R}$;

2. Obtain the desired density function for the statistic $S$ by

$$f^S(s) = \sum_{\mathcal{D} \in \mathcal{G}_\mathcal{D}(H)} \sum_{d \in \mathcal{I}(\mathcal{D})} f^{S_d|\mathcal{D}(d)}(s) P(\mathcal{D}(d)), \quad s \in \mathbb{R}. \quad (1.19)$$

Note that the usage of $(\bar{i})$, $i \in \{1, \ldots, 11\}$, always refers to Procedure 1.4.4.

**Remark 1.4.5** (i) The foundation for a simplified analysis in terms of efficiency and reusability of an underlying hybrid censoring scheme $H$ is laid in the steps 3 – 6.

It should be noted that the specification of $\mathcal{G}_\mathcal{D}(H)$ is not a trivial task to accomplish, especially when the number of used threshold times is greater than one. In such a case the experimenter will be confronted with a high quantity of different experimental outcomes (see, e.g., general unified hybrid censoring in Chapter 8 and unified hybrid censoring in Appendix A). A significant subset of these experimental outcomes may due to their more complex structure not be at once identified with the situations given in (1.15). Hence, the experimenter should validate the underlying sampling situations for the possibility of combining particular cases to one singular sampling situation w.r.t. the counter settings given in (1.15). This process is strongly connected with the distribution based steps performed in 4 and 5, which also serve as a verification of the retrieved simplified representation for $H$, specified by the pair $(\mathcal{G}_\mathcal{D}(H), \mathcal{I}(\mathcal{G}_\mathcal{D}(H)))$.

Especially the effort made in 6, provides the desired specification of the supports for the respective joint density functions and as a consequence the basis for the geometry oriented approach.
(ii) Steps (7) – (12) address the calculation of the distribution of linear combinations of the underlying hybrid censored order statistics based on a particular cumulative distribution function $F$. A crucial step for the comfortable application of a particular volume formula, is the transfer of the cone structured supports to (d-dimensional) simplex structured supports. This can be realized by considering the spacings $W_j^H, 1 \leq j \leq m_h$ (cf. steps (8) and (9)). The approach proceeds by applying a suitable transformation in order to obtain an expression for the density function of the desired statistic $S$ out of the conditional joint distribution of the spacings. This induces the classification of the corresponding support in terms of simplices and half-spaces of dimension $(d-1)$, respectively (cf. step (10)). In this work we usually refer to the intersection of a simplex with a half-space $S_{d-1,1}^{(s)} \cap H_{d-1,1}^{(s)}$ as to the polytope $M_{d-1}^{(s)}(s \cdot)$ (see Lemma 3.2.4, (iii)). The desired density functions can be obtained by computing the volumes of $(d-1)$-dimensional simplices and of $(d-1)$-dimensional polytopes.

(iii) Should the resulting distribution of the statistic $S$ be degenerated, then we calculate the distribution of $S$ by

$$P(S \leq t) = \sum_{D \in \mathcal{D}_2(H)} \sum_{d \in \mathcal{I}(D)} P(S_d \leq t | \mathcal{D}(d)) P(\mathcal{D}(d)), \quad t \in \mathbb{R},$$

instead of performing step (12).

(iv) If the approach via the moment generating function is favored, one might perform the steps (1) – (7) and afterwards consider

$$M_S(s) = \sum_{D \in \mathcal{D}_2(H)} \sum_{d \in \mathcal{I}(D)} E(e^{s S_d | \mathcal{D}(d)}) P(\mathcal{D}(d)), \quad s \in \mathbb{R}.$$ 

As an alternative to Procedure 1.4.4, we present subsequently a second way for deriving the distribution of the statistic $S$ given in (1.18).

**Procedure 1.4.6 (Expected value approach)** Let $H$ denote the underlying censoring scheme with threshold times $0 < T_1 < \cdots < T_l < \infty$, $l \in \mathbb{N}$, and integers $1 \leq m_1 \leq \cdots \leq m_h \leq n$, $h \in \mathbb{N}$, where $n \in \mathbb{N}$ denotes the total amount of objects put on life test. The random variable $T^H_l$ denotes the stopping time of the censoring model $H$. Further, let $X_{(1)}, \ldots, X_{(m_h)}$ denote a sample of ordered random variables based on an absolutely continuous cumulative distribution function $F$, with $F^{-1}(0) = c_{lb}$ for a constant $c_{lb} \in \mathbb{R}$. A distribution oriented analysis of the hybrid censoring scheme $H$ by means of the expected value approach can be realized by performing the following steps:

1. Perform steps (1) – (7) given in Procedure 1.4.4;

2. Determine for all $D \in \mathcal{D}(H)$ the density function $f^{S_d | \mathcal{D}(d)}, d \in \mathcal{I}(D)$, with (cf. (1.18))

$$S_d = \alpha_0 + \sum_{j=1}^{d} \alpha_j X_j^H, \quad \alpha_0, \ldots, \alpha_d \in \mathbb{R},$$

by solving the integral equation

$$\int_{\mathbb{R}^d} g(\alpha_0 + \sum_{j=1}^{d} \alpha_j x_j) f^{X_j^H, 1 \leq j \leq d \mathcal{D}(d)}(x_d) dx_d = \int_{\mathbb{R}} f^{S_d | \mathcal{D}(s)}(s) g(s) ds.$$
for any continuous function \( g : \mathbb{R} \rightarrow \mathbb{R} \);

(iii) The desired density function \( f^S \), with \( S \) as given in (1.18), can be obtained by performing step (II) from Procedure 1.4.4.

By analogy with Procedure 1.4.4 we refer to Procedure 1.4.6 by using (I), (II) and (III).

**Remark 1.4.7**

(i) Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be an arbitrary continuous function. Then, step (II) is justified by

\[
E(g(S_d) | \mathcal{D}(d)) = \int_{\mathbb{R}} f^{S_d | \mathcal{D}(d)}(s) g(s) ds, \quad \text{for all } \mathcal{D} \in \mathcal{S}_D(H).
\]

(ii) We note that Procedure 1.4.4, the approach introduced in Remark 1.4.5, (iv) and Procedure 1.4.6 can be adapted to a multivariate setting, too.

(iii) The specification of an underlying sample \( X_1, \ldots, X_n \) in Notation 1.4.1, Procedure 1.4.4 and Procedure 1.4.6 have been deliberately omitted. It is not an essential requirement since the cumulative distribution function \( F \), the ordered sample \( X_{(1)}, \ldots, X_{(m_h)} \) and the total amount of objects \( n \in \mathbb{N} \) put on test are given.

The volume based approach and the expected value approach will be applied for the exponential case and for the uniform case, respectively. However, we provide some results, which are essential for conducting the expected value approach in the exponential case. Finally, we give two useful facts for an exponential setting.

**Remark 1.4.8**

Let \( P_F = \text{Exp}(\mu, \vartheta) \), \( \mu \geq 0 \) and \( \vartheta > 0 \).

(i) The density functions of the MLE \( \hat{\vartheta} \) as well as of the MLE \( (\hat{\mu}, \hat{\vartheta}) \) obtained in this thesis via Procedures 1.4.4 and 1.4.6 consist of B-splines.

(ii) By utilizing a particular identity (see Theorem 4.3.1 and equation (1.7)) the transfer from a B-spline representation to a representation in terms of gamma density functions can be easily accomplished.

1.5 Further Aspects of Analysis and Possible Applications for Hybrid Censored Data

The literature of the last two decades addressing the topic of hybrid censoring, shows a big variety of possibilities to investigate hybrid censoring models and to combine them with other models. Due to differentiation in content, we are not be able to address all of these topics in an adequate manner. Therefore, we provide the reader with basic information concerning a small selection of hybrid censoring related topics.

**Hybrid Censored Data from Weibull Distributions**

The density function of the Weibull distribution allows the emulation of many different types of shapes. Hence, it is considered to be one of the most frequently used distribution for lifetime experiments (see, e.g., Lawless (2003)). Being of high practical relevance, the Weibull distribution bears some difficulties in the context of hybrid censoring. In contrast
to the exponential case, the MLEs for an Weibull assumption cannot be obtained explicitly, such that the corresponding distribution can also not be determined (see, e.g., Kundu (2007) and Balakrishnan and Kundu (2013)). For an elaborate survey on the Weibull distribution, with some examples of practical application, we refer to Rinne (2009).

Kundu (2007) was the first to consider Weibull distributed lifetimes in the context of hybrid censoring. For the Type-I hybrid censoring a method was presented how the MLEs can be obtained by an iteratively resolution of a fixed-point formulation. An alternative approach for obtaining the values of the MLEs, among others applicable for Type-I censoring, was presented by Balakrishnan and Kateri (2008). Further effort has been made in the context of Type-II hybrid censoring by Banerjee and Kundu (2008), for Type-I progressive hybrid censoring by Lin et al. (2009), for Type-II progressive hybrid censoring by Mokhtari et al. (2011), and for adaptive Type-I progressive hybrid censoring by Lin et al. (2012). For the model of adaptive Type-I progressive hybrid censoring we refer to Lin and Huang (2012) (see also Remark 6.2.15). Notice that Lin et al. (2009) also considered an adaptive progressive scheme introduced by Ng et al. (2009). Recently, Cho et al. (2015a) considered Bayesian inference for the entropy of the Weibull distribution under generalized Type-I progressive hybrid censoring.

### Bayesian Inference Based on Hybrid Censored Data

Bayesian inference within the scope of Type-I hybrid censoring has been quite frequently addressed. Draper and Guttman (1987) established a two-sided credible interval using an inverted gamma prior. Subsequently, Gupta and Kundu (1998) investigated under an exponential assumption the Type-I hybrid censoring scheme from the Bayesian perspective and considered also the MLE of the respective scale parameter. Then, in the context of Type-I progressive hybrid censoring, Kundu and Joarder (2006) conducted Bayesian inference for the exponential scale parameter based on a gamma prior. In Kundu (2007) the Weibull distribution under Type-I hybrid censoring was considered and Bayesian estimators as well as credible intervals were established. Further results have been obtained for the Weibull distribution under the Type-II hybrid censoring scheme (see Banerjee and Kundu, 2008), for the two-parameter generalized exponential distribution under Type-I hybrid censoring (see Kundu and Pradhan, 2009) and for the Burr Type XII distribution under Type-II hybrid censoring (see Panahi and Asadi, 2011). For the two-parameter exponential distribution Kundu et al. (2013) obtained under Type-I hybrid censoring the Bayes estimators as well as symmetric credible intervals. According to Kundu et al. (2013), respective results for the Type-II hybrid censoring scheme and for the generalized Type-I/II hybrid censoring schemes can be obtained in analogy to the case where the Type-I hybrid censoring scheme was considered. Further, they provided the means to obtain the Bayesian estimators as well a symmetric credible intervals for the Type-II progressive hybrid censoring scheme. While Kundu et al. (2013) used a gamma prior for the scale parameter, Bayoud (2014) established Bayesian inferential results for the two-parameter exponential distribution under Type-II hybrid censoring based on an exponential prior for $\theta$. For the location parameter in both elaborations a uniform prior was chosen. The weighted exponential distribution has been considered for the Type-II hybrid censoring scheme in Kohansal et al. (2015). Recently Shafay (2016a) established Bayesian inferential results for the generalized Type-II hybrid censoring scheme from the one-parameter exponential distribution and from a two-parameter Pareto distribution. Further, Seo and Kim (2016) conducted robust Bayesian
inference for the two-parameter exponential distribution under generalized Type-I progressive hybrid censoring. The Bayes estimators for the entropy of a Weibull distribution under generalized Type-I progressive hybrid censoring have been deduced in the above mentioned work of Cho et al. (2015a).

Simple Step-Stress Models with Hybrid Censored Samples

As a form of accelerated life testing step-stress models have been proposed, see, e.g., DeGroot and Goel (1979), Nelson (1980) and Miller and Nelson (1983). Step-stress models can be used to simulate an increase of exposure for those objects put on test which exceed a prefixed time $\tau$, $\tau \in (0, \infty)$. Some effort has been made in the last decade to investigate simple step-stress models based on a hybrid censored sample from the one-parameter exponential distribution. For an elaborate review for inferential results concerning step-stress models in the exponential case, we refer to the review paper Balakrishnan (2009). Balakrishnan et al. (2009) considered a simple step-stress model with Type-I censored data, which can be described as follows: a change of stress is to be carried out at a prefixed time $\tau_1 \in (0, \infty)$, while the entire experiment should not exceed a prefixed threshold time $\tau_2 \in (0, \infty)$, with $\tau_2 > \tau_1$. Notice that Type-I censored data correspond to the sample which arises from the Type-I hybrid censoring scheme with $m = n$. For this setup Balakrishnan et al. (2009) determined the MLEs for the parameters $\vartheta_1$ and $\vartheta_2$. Further, the respective distributions have been obtained by means of the conditional moment generating function. Balakrishnan and Xie (2007a) considered with an underlying Type-I hybrid censored sample a more general setup. They derived expressions for the MLEs as well as the respective distributions for the one-parameter exponential distribution. Similar derivations have been conducted for a simple step-stress model with Type-II hybrid censored data (see Balakrishnan and Xie, 2007b). For a step-stress model, which can be obtained from sequential order statistics, we refer to Balakrishnan et al. (2012). Recently Shafay (2016b) considered the simple step-stress model for a generalized Type-I hybrid censored sample. The MLEs and the respective distributions for the one-parameter exponential case have been established. Further, approximate and exact confidence intervals have been determined as well as credible intervals have been presented.

It should be noted that the distributions of the MLEs, presented in the above mentioned literature, were in each case derived by the means of the conditional moment generating function. Balakrishnan and Cramer (2014, Section 23.1.1) used a different way for deriving the distribution of the MLEs in the one-parameter exponential case for a simple step-stress model based on progressively Type-II censored order statistics. By utilizing the arguments of the spacings based approach (cf. Cramer and Balakrishnan, 2013), they obtained a representation for the density function of $\hat{\vartheta}_1$ in terms of B-splines. This expression is more compact than the originally presented representation established in Xie et al. (2008).

Fisher Information

In the last years, the Fisher information in hybrid censoring schemes has been frequently addressed. Results for the Fisher information w.r.t. ordinary order statistics, progressively Type-II censored order statistics, and generalized order statistics can be found in Park (1996), Balakrishnan and Cramer (2014), and Burkschat and Cramer (2012), respectively. A review on Fisher information in the context of ordered data is given by Zheng et al. (2009).
Firstly in the context of hybrid censoring, Park et al. (2008) established formulas for the Fisher information under the Type-I/II hybrid censoring schemes, and applied them to the one-parameter exponential distribution and to the Weibull distribution. Subsequently, Park and Balakrishnan (2009) presented a more convenient way to calculate the Fisher information w.r.t. the Type-I/II hybrid censoring schemes. By utilizing these findings, they further provided formulas, such that the Fisher information for the generalized Type-I/II hybrid censoring schemes can be derived from the results obtained for the Type-I/II hybrid censoring models. Fisher information based on progressively hybrid censored data, has been firstly addressed by Park et al. (2011). They established formulas for the Type-I/II progressive hybrid censoring schemes, and applied them to the one-parameter exponential distribution. In the context of a newly introduced very flexible hybrid censoring scheme, Park and Balakrishnan (2012) established formulas for the Fisher information under unified Type-I/II/III/IV hybrid censoring schemes.
Chapter 2
Preliminary Results

This chapter gives a brief introduction to particular models of ordered data addressed in this work. We provide the respective basic distributional results and specify some probabilistic models, in order to facilitate the subsequent analysis of the different hybrid censoring schemes.

In Section 2.1, we consider order statistics. Following that, Section 2.2 addresses the distribution of progressively Type-II censored order statistics. Finally, the model of sequential order statistics under an proportional hazard rate assumption is considered in Section 2.3.

2.1 Order Statistics

If the random variables of an underlying sample $X_1, \ldots, X_n$ are sorted in ascending order, as $X_{1:n} \leq \cdots \leq X_{n:n}$, then they are called its order statistics (cf. Kamps, 1995a; Arnold et al., 2008). For elaborate surveys on order statistics and their applications, we refer to Arnold et al. (2008), David and Nagaraja (2003), Balakrishnan and Rao (1998b) and Balakrishnan and Rao (1998a).

First, we present a formal definition of order statistics, taken from Kamps (1995a, p. 21).

**Definition 2.1.1** Let the function $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by

$$T(x_n) = (x_{1:n}, \ldots, x_{n:n}), \quad x_n \in \mathbb{R}^n,$$

such that

$$x_{1:n} \leq \cdots \leq x_{n:n} \quad \text{and} \quad x_n P = (x_{1:n}, \ldots, x_{n:n}),$$

where $P$ is a $n \times n$ permutation matrix (which results from permuting the columns of an $n \times n$ identity matrix). Moreover, let the functions $T_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$T_i(x_n) = x_{i:n}, \quad 1 \leq i \leq n.$$

Then, with real valued random variables $X_1, \ldots, X_n$,

$$X_{i:n} = T_i(X_1, \ldots, X_n)$$

is called the $i$th order statistic, $1 \leq i \leq n$, and

$$(X_{1:n}, \ldots, X_{n:n}) = T(X_1, \ldots, X_n)$$

is the vector of order statistics (based on $X_1, \ldots, X_n$).
We assume in the following the random random variables \( X_1, \ldots, X_n \) to be independent and identically distributed (IID) with an absolutely continuous cumulative distribution function \( F \). For the proof of the following lemma, we refer to Arnold et al. (2008).

**Lemma 2.1.2** Let \( X_{1:n}, \ldots, X_{n:n} \) be the order statistics based on \( X_1, \ldots, X_n \overset{\text{iid}}{\sim} F \), with \( F \) absolutely continuous with density function \( f \).

(i) The joint density function of the order statistics \( X_{1:n}, \ldots, X_{r:n} \), \( 1 \leq r \leq n \), is given by
\[
f_{1:r:n}(x_r) = \frac{n!}{(n-r)!} \left( \prod_{i=1}^{r} f(x_i) \right) (1 - F(x_r))^{n-r}, \quad x_r \in \Sigma_r^c,
\]
where \( \Sigma_r^c \) denotes the trivial ordered cone of dimension \( r \) given in (1.3).

(ii) The density function of the \( r \)th order statistic \( X_{r:n} \), \( 1 \leq r \leq n \), is given by
\[
f_{r:n}(x) = r \binom{n}{r} F(x)^{r-1} (1 - F(x))^{n-r} f(x), \quad x \in \mathbb{R}.
\]

(iii) The joint density function of the order statistics \( X_{i:n} \) and \( X_{s:n} \), \( 1 \leq r < s \leq n \), is given by
\[
f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(x_r)^{r-1} (F(x_s) - F(x_r))^{s-r-1}
\times (1 - F(x_s))^{n-s} f(x_r)f(x_s), \quad x_r, x_s \in \mathbb{R}, \quad x_r < x_s.
\]

(iv) The cumulative distribution function of the \( r \)th order statistic \( X_{r:n} \), \( 1 \leq r \leq n \), is given by
\[
F_{r:n}(x) = \sum_{j=r}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}, \quad x \in \mathbb{R}.
\]

In order to distinguish between the different herein addressed models of ordered data, in future considerations the denotation *ordinary order statistics* for order statistics is used. We consider the following model.

**Model 2.1.3** (IID uniform model) In the IID uniform model, we assume the underlying sample \( U_1, \ldots, U_n \) to be independent and identically uniform distributed with parameters \( a \) and \( b \), with \( a < b \) (i.e., \( U_1, \ldots, U_n \overset{\text{iid}}{\sim} \text{U}(a,b) \), \( n \in \mathbb{N} \)). In the context of hybrid censoring, we assume further that \( a, b \in [0,\infty) \) and \( a < T < b \), for a threshold time \( T \in (0,\infty) \).

The density functions as well as the corresponding cumulative distribution functions for the distributions used in this thesis, are given in Appendix C.1.

**Remark 2.1.4** Let \( U_{1:n}, \ldots, U_{n:n}, \ n \in \mathbb{N} \), be ordinary order statistics based on \( U_1 \ldots, U_n \overset{\text{iid}}{\sim} \text{U}(a,b) \). Then, the support of the corresponding joint density function \( f_{1:n:n} \) (cf. Lemma 2.1.2, (i)) is given by
\[
\text{supp}(f_{1:n:n}) = \{u_n \in \mathbb{R}^n | a \leq u_1 \leq \cdots \leq u_n \leq b\} = \Sigma^c_n.
\]
2.2 Progressively Type-II Censored Order Statistics

In life tests the necessity for emulating situations, as the removal of a selection of objects after observing the failure of a component, occurs. Such a situation might be caused by the need of an earlier release of testing facilities. Further, an interception of unpredictable experiment behavior, as the accidental or premature failure of particular components, might be required. Such situations are not covered by the model of right censoring (Type-II censoring) an underlying sample of ordered data. Hence the model of progressively Type-II censored order statistics was introduced.

Progressive Type-II censoring has been among others presented in Herd (1956) and Cohen (1963, 1966). For an extensive discussion of the theory and the development of progressive censoring we refer to Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014), as well as to the review Balakrishnan (2007). Notice that recently Cramer and Navarro (2015) established some results concerning the field of application of progressively Type-II censored order statistics. They showed how the theory for particular coherent systems can be identified with progressive Type-II censoring and adaptive progressive Type-II censoring.

Basic Distribution Theory

Let a sample of size $n$ and a fixed censoring plan $\mathcal{R} = (R_1, \ldots, R_m) \in \mathbb{N}_0^m$, $m \leq n$ be given. Then, the observation of $m$ failures according to the model of progressive Type-II censoring can be described as follows: after observing the $i$th failure, a selection of $R_i$ randomly chosen components among the remaining working components is removed from the experiment, for $i \in \{1, \ldots, m\}$. The following procedure describes the generation process of progressively Type-II censored order statistics (see Balakrishnan and Cramer, 2014, p. 5).

Procedure 2.2.1 (Generation of progressively Type-II censored order statistics)

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X_1, \ldots, X_n$ be random variables on $(\Omega, \mathcal{A}, P)$. Let $\mathcal{R} = (R_1, \ldots, R_m)$ be a censoring plan.

For $\omega \in \Omega$, the progressively Type-II censored sample

$$X^{\mathcal{R}}_{1:m:n}(\omega), \ldots, X^{\mathcal{R}}_{m:m:n}(\omega),$$

based on $X_1(\omega), \ldots, X_n(\omega)$, is generated as follows:

1. Calculate the order statistics $X_{1:n}(\omega) \leq \ldots \leq X_{n:n}(\omega)$;
2. Let $\mathcal{N}_i = \{1, \ldots, n\}$, $i = 1$;
3. Let $k_i = \min \mathcal{N}_i$ and put $X^{\mathcal{R}}_{i:m:n}(\omega) = X_{k_i:n}(\omega)$;
4. Choose randomly a without-replacement sample $\mathcal{R}_i \subseteq \mathcal{N}_i \setminus \{k_i\}$ with $|\mathcal{R}_i| = R_i$;
5. If $i < m$, set $\mathcal{N}_{i+1} = \mathcal{N}_i \setminus (\{k_i\} \cup \mathcal{R}_i)$ and go to 3., or else stop.

Thus,

$$X^{\mathcal{R}}_{1:m:n}(\omega), \ldots, X^{\mathcal{R}}_{m:m:n}(\omega) = (X_{k_1:n}(\omega), \ldots, X_{k_m:n}(\omega)).$$
For $m, n \in \mathbb{N}$, $m \leq n$, an admissible censoring plan $\mathcal{R} = (R_1, \ldots, R_m)$ has to satisfy the conditions (cf. Balakrishnan and Cramer, 2014, p. 5)

$$(R_1, \ldots, R_m) \in \mathbb{N}_0^m \quad \text{and} \quad \sum_{j=1}^m R_j = n - m.$$  

The set of admissible censoring plans for $m, n \in \mathbb{N}$, $m \leq n$, is therefore given by

$$C_{m,n}^m = \left\{ (R_1, \ldots, R_m) \in \mathbb{N}_0^m \mid \sum_{j=1}^m R_j = n - m \right\}.$$  

All herein used censoring plans are assumed to be an element of $C_{m,n}^m$. Now, for a censoring plan $\mathcal{R} = (R_1, \ldots, R_m)$, we denote by

$$\gamma_j = \sum_{i=j}^m (R_i + 1), \quad 1 \leq j \leq m,$$  

the total number of objects remaining in the experiment before observing the $j$th failure, $1 \leq j \leq m$. For convenience, $\gamma_{m+1} = 0$. The proofs of the following results can be found in Balakrishnan and Cramer (2014, Chapter 2).

**Lemma 2.2.2** Let $X_{1,m,n}^{\mathcal{R}}, \ldots, X_{m,m,n}^{\mathcal{R}}$ be progressively Type-II censored order statistics from an absolutely continuous cumulative distribution function $F$ with density function $f$.

(i) The joint density function of the progressively Type-II censored order statistics $X_{1,m,n}^{\mathcal{R}}, \ldots, X_{r,m,n}^{\mathcal{R}}, 1 \leq r \leq m$, is given by

$$f_{1,r,m,n}^{\mathcal{R}}(x_r) = \left( \prod_{j=1}^r \gamma_j \right) \left( \prod_{j=1}^{r-1} f(x_j) (1 - F(x_j))^R_j \right) f(x_r) (1 - F(x_r))^\gamma_r - 1, \quad x_r \in \Sigma_{F}^r.$$  

(ii) The density function of the $r$th progressively Type-II censored order statistic $X_{r,m,n}^{\mathcal{R}}$, $1 \leq r \leq m$, is given by

$$f_{r,m,n}^{\mathcal{R}}(x) = f(x) \left( \prod_{j=1}^r \gamma_j \right) \sum_{j=1}^r a_{j,r} (1 - F(x))^{\gamma_j - 1}, \quad x \in \mathbb{R},$$  

with $a_{j,r} = \prod_{i=1, i \neq j}^{r} \frac{1}{\gamma_i - \gamma_j}, 1 \leq j \leq r$.

(iii) The cumulative distribution function of the $r$th progressively Type-II censored order statistic $X_{r,m,n}^{\mathcal{R}}$, $1 \leq r \leq m$, is given by

$$F_{r,m,n}^{\mathcal{R}}(x) = 1 - \left( \prod_{j=1}^r \gamma_j \right) \sum_{j=1}^r \frac{1}{\gamma_j} a_{j,r} (1 - F(x))^{\gamma_j}, \quad x \in \mathbb{R},$$  

where $a_{j,r}$ is given in (ii).
2.2 Progressively Type-II Censored Order Statistics

Table 2.1: Construction of the compressed censoring plan \( R_{\text{cor}} = (R_{1,\text{cor}}, \ldots, R_{r,\text{cor}}) \), \( r \in \{1, \ldots, m\} \), from the initially planned censoring plan \( A = (R_1, \ldots, R_m) \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>\ldots</th>
<th>( r-1 )</th>
<th>( r )</th>
<th>( r+1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{j,\text{cor}} )</td>
<td>( \sum_{i=1}^{m} (R_i + 1) )</td>
<td>\ldots</td>
<td>( \sum_{i=r-1}^{m} (R_i + 1) )</td>
<td>( \sum_{i=r}^{m} (R_i + 1) )</td>
<td>0</td>
</tr>
<tr>
<td>( R_{j,\text{cor}} )</td>
<td>( R_1 )</td>
<td>\ldots</td>
<td>( R_{r-1} )</td>
<td>( \sum_{i=r}^{m} R_i + (m-r) )</td>
<td>–</td>
</tr>
</tbody>
</table>

The following remark recalls some useful facts.

Remark 2.2.3

(i) For any setting \( A \in C_{m,n} \), the corresponding \( \gamma \)'s are strictly ordered, i.e.,

\[
0 < \gamma_m < \cdots < \gamma_2 < \gamma_1.
\]

(ii) The progressively Type-II censored order statistics correspond for \( m = n \) and \( A = (0^n) \) to ordinary order statistics presented in Section 2.1 (cf. Balakrishnan and Cramer, 2014, p. 7).

(iii) The censoring plan \( A = (0^{m-1}, n-m) \), \( m \leq n \), corresponds to the model of Type-II (right) censoring (cf. Balakrishnan and Cramer, 2014, p. 7).

Initially Planned and Modified Censoring Plans

In the context of hybrid censoring, we refer to \( A = (R_1, \ldots, R_m) \) as the initially planned censoring plan. We introduce two modified censoring plans \( A_{\text{cor}} \) and \( A_{\text{exp}} \), which are associated with \( A \). The compressed censoring plan \( A_{\text{cor}}, r \in \{1, \ldots, m\} \), is defined as (see also Table 2.1)

\[
A_{\text{cor}} = \left(R_1, \ldots, R_{r-1}, \sum_{i=r}^{m} (R_i + (m-r))\right), \quad r \in \{1, \ldots, m\}.
\] (2.3)

Note that \( A_{\text{cor},m} = A \) and \( \sum_{i=j}^{m} R_i + (m-j) = \gamma_j - 1, 1 \leq j \leq m \). Further, the (right) extended censoring plan \( A_{\text{exp}}, r \in \{m+1, \ldots, \tilde{m}\} \) (with \( \tilde{m} = m + R_m \), cf. (1.11)), is defined as (see Table 2.2)

\[
A_{\text{exp}} = \left(R_1, \ldots, R_{m-1}, 0^{(r-m)}, \tilde{m} - r\right), \quad r \in \{m+1, \ldots, \tilde{m}\}.
\] (2.4)

The notation \( 0^d \in \mathbb{R}^d, d \in \mathbb{N} \), is used for \( d \) successive zeros. The extended censoring plan \( A_{\text{exp}} \) has been among others used in Cramer et al. (2016), for \( r = \tilde{m} \). The coefficients representing the number of objects remaining in the experiment are for \( A_{\text{cor}}, r \in \{1, \ldots, m\} \) and \( A_{\text{exp}}, r \in \{m+1, \ldots, \tilde{m}\} \), given by

\[
\gamma_{j,\text{cor}} = \sum_{i=j}^{m} (R_i + 1), \quad 1 \leq j \leq r, \quad \text{with} \quad \gamma_{r+1,\text{cor}} = 0,
\] (2.5)
\[ \gamma_j = \sum_{i=1}^{m} (R_i + 1) - \sum_{i=m-1}^{m} (R_i + 1) + R_m + 1 - \bar{m} + r + 1 - \bar{m} + r \]

Table 2.2: Construction of the (right) extended censoring plan \( \mathcal{R}^r = (R_1^r, \ldots, R_r^r) \), \( r \in \{m + 1, \ldots, \bar{m}\} \) from the initially planned censoring plan \( \mathcal{R} = (R_1, \ldots, R_m) \).

\[ R_j^r = \begin{cases} \sum_{i=1}^{m} (R_i + 1), & 1 \leq j \leq m, \\ \bar{m} - j + 1, & j \in \{m + 1, \ldots, r\}, \end{cases} \quad \text{with } \gamma_r^r = 0, \quad (2.6) \]

respectively. Notice that \( \gamma_j^{\geq m} = \gamma_j, 1 \leq j \leq m + 1 \).

Remark 2.2.4 Notice that
\[ X_{j:r_1:n} = X^r_{j:r_2:n}, \quad 1 \leq j \leq r_1, \quad r_1 \in \{1, \ldots, m\}, \quad r_2 \in \{m + 1, \ldots, \bar{m}\}. \quad (2.7) \]

This can be seen from the cumulative distribution function given in Lemma 2.2.2, (ii). The relation given in (2.7) was firstly stated in Cramer et al. (2016) for \( r_1 = m \) and \( r_2 = \bar{m} \).

An important issue within the scope of progressive Type-II censoring is the design of optimal censoring plans. Therefore many different optimality criteria have been proposed. These criteria have been investigated for different distributions, such that accordingly optimal censoring plans have been established. For basic and further reference in this direction, we refer to Balakrishnan and Aggarwala (2000, Chapter 10), Ng et al. (2004), Burkschat et al. (2006), Burkschat et al. (2007), Balakrishnan (2007, Chapter 10), Balakrishnan et al. (2008b), Burkschat (2008), Cramer and Ensenbach (2011), Dahmen et al. (2011), and Bhattacharya et al. (2016). For elaborations on choosing optimal censoring parameters occurring in (progressive) hybrid censoring we refer to Ebrahimi (1988), Bhattacharya et al. (2014), and Bhattacharya and Pradhan (2015).

The following model provides the fundamental setup for the hybrid censoring related analysis based on progressively Type-II censored order statistics.

Model 2.2.5 (IID progressive model) In the IID progressive model, we assume the underlying sample \( X_1, \ldots, X_n \) to be independent and identically distributed w.r.t. an absolutely continuous cumulative distribution function \( F \). In the context of hybrid censoring, we consider distributions with \( F^{-1}(0) \geq 0 \). Furthermore, threshold times \( T \in (0, \infty) \), are supposed to satisfy the inequality \( F^{-1}(0) < T \).

For the exponential case we consider the following model.

Model 2.2.6 (IID progressive exponential model) In the IID progressive exponential model we consider the IID progressive model with \( P_F = \text{Exp}(\mu, \vartheta) \). The respective underlying sample is denoted by \( Z_1, \ldots, Z_n \). In the context of hybrid censoring, we assume \( \mu \geq 0 \) as well as \( \mu < T \), for a threshold \( T \in (0, \infty) \).
In realtime models the experimenter might like to adapt the number of progressively censored components during the experiment (see, e.g., Balakrishnan and Cramer, 2014, Chapter 6). This procedure is not covered by conventional Type-II progressive censoring, where the censoring plan $R$ is assumed to be prefixed. This issue was among others addressed by Ng et al. (2009). They presented a censoring model, called adaptive Type-II progressive censoring, which enables an adjustment of the censoring plan during the experiment. The adaptive progressive Type-II censoring scheme proceeds as follows: if the $m$th failure has not been observed till a prefixed time $T$, then the censoring plan will be accordingly modified, in order to achieve a soon termination of the experiment. If the $m$th failure occurs before $T$, then the initially planned censoring plan $R$ is applied. Generalizations of this model have been presented in Cramer and Iliopoulos (2010), Bairamov and Parsi (2011) and Kinaci (2013). For a review on adaptive progressive censoring, we refer to Balakrishnan and Cramer (2014, Chapter 6) as well as to Cramer and Iliopoulos (2015).

The above mentioned adaptive progressive censoring schemes guarantee the observation of exactly $m$ failures. The threshold times specified in these models serve solely for the modification and adaption of the underlying censoring plan. An alternative censoring model called adaptive Type-I progressive hybrid censoring scheme, was proposed in Lin and Huang (2012). In contrast to the previously addressed adaptive censoring schemes, this model underlies a time induced truncation, such that the quantity of observed failures is not predetermined. This model will be considered in more detail in the context of the Type-II progressive hybrid censoring scheme (see Remark 6.2.15).

### 2.3 Sequential Order Statistics

The concept of sequential order statistics was introduced by Kamps (1995a,b) in order to model the impact of failed components on the remaining ones in $(n-m+1)$-out-of-$n$ systems. As mentioned in Section 1.1.1, the systems failure corresponds to the failure of the $m$th component. We consider the following definition for sequential order statistics (see Cramer and Kamps, 2003, Definition 2.2).

**Definition 2.3.1** Let $F_1,\ldots,F_n$ be distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$, and let $V_1,\ldots,V_n$ be independent random variables with $V_r \sim \text{Beta}(n-r+1,1)$, $1 \leq r \leq n$. Then the random variables

$$X_r^* = F_r^{-1}(X_{[r]}) \quad \text{with} \quad X_{[r]} = 1 - V_r\left(1 - F_r(X_{r-1}^*)\right), \quad 1 \leq r \leq n, \quad X_0^* = -\infty,$$

are called sequential order statistics (based on $F_1,\ldots,F_n$).

The above construction of sequential order statistics serves as an alternative to the original one introduced in Kamps (1995a,b), where the construction via a triangular scheme of random variables was proposed. Given the coefficients $\alpha_j > 0$, $1 \leq j \leq n$, and the definition of an absolutely continuous cumulative distribution function $F$, we consider the following cumulative distribution functions $F_1,\ldots,F_n$:

$$F_j(x) = 1 - (1 - F(x))^{\alpha_j}, \quad 1 \leq j \leq n, \quad x \in \mathbb{R}. \quad (2.8)$$

We refer to (2.8) as the model of proportional hazard rates (see, e.g., Kamps, 1995a,b; Cramer and Kamps, 2003; Burkschat et al., 2016). Further, we consider the following coefficients which are related to the model of generalized order statistics:

$$\gamma_j^* = \alpha_j(n-j+1), \quad 1 \leq j \leq n \quad \text{and} \quad \gamma_{n+1}^* = 0. \quad (2.9)$$
The * implies that we refer to the context of proportional hazard rates within the model of sequential order statistics. Otherwise, the γ’s correspond to the model of progressive Type-II censoring, throughout this work.

**Remark 2.3.2** (i) According to Kamps (1995a,b) (see also Cramer and Kamps, 2001b, 2003) it is possible to obtain a number of models of ordered random variables out of the model of proportional hazard rates. In order to deduce a particular model, one has to choose a specific setting for the α’s. We therefore present an overview of a selection of possible setups (cf. Kamps, 1995a; Cramer and Kamps, 2001b):

<table>
<thead>
<tr>
<th>model</th>
<th>( \alpha_j ), ( 1 \leq j \leq n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ordinary order statistics</td>
<td>1</td>
</tr>
<tr>
<td>generalized order statistics</td>
<td>( \frac{\gamma_j}{n - j + 1} )</td>
</tr>
<tr>
<td>record values, minimal repair times</td>
<td>( \frac{1}{n - j + 1} )</td>
</tr>
<tr>
<td>kth record values</td>
<td>( k )</td>
</tr>
<tr>
<td>progressively Type-II censored order statistics</td>
<td>( \frac{N + 1 - j - \sum_{i=1}^{j-1} R_i}{n - j + 1} )</td>
</tr>
</tbody>
</table>

Hereby denotes \( N = \sum_{j=1}^{n} (R_j + 1) \) the number of objects targeted for a life test.

(ii) Notice that in contrast to the setting of progressively Type-II censored order statistics, the coefficients \( \gamma_j \), \( 1 \leq j \leq n \), need not to be pairwise distinct. E.g., we obtain for the setting of kth record values: \( \gamma_j = k \) for \( \alpha_j = \frac{k}{n-j+1} \), \( 1 \leq j \leq n \).

(iii) Due to \( \alpha_j > 0 \), \( 1 \leq j \leq n \), we also find that \( \gamma_j > 0 \), \( 1 \leq j \leq n \).

The following remark gives a brief introduction into the remaining four models addressed in Remark 2.3.2, (i).

**Remark 2.3.3** (i) The notion of generalized order statistics was introduced by Kamps (1995a,b). By covering a big variety of models of ordered random variables (among others those provided in Remark 2.3.2, (i)), the concept of generalized order statistics enables the treatment of several distributional models at once. For further reading on generalized order statistics and in particular on the respective marginal distributions, we refer to Kamps and Cramer (2001) and Cramer and Kamps (2003).

(ii) A strategy of restoring a system is called 'minimal repair' or 'bad-as-old' strategy, if the failed component is replaced by a working component which comprises the systems age (cf., e.g., Ascher, 1968; Beutner and Cramer, 2010; Burkschat et al., 2016). A system consisting of the components \( X_1^*, \ldots, X_n^* \), can hence be repaired \((n - 1)\)-times according to the minimal repair strategy. For further reading, we refer to Ascher and Feingold (1984) and Whitaker and Samaniego (1989).
(iii) Let \((X_j)_{j \in \mathbb{N}}\) be IID random variables. Motivated by modeling weather data, the (upper) record values, which denote the sequence of succeeding extremes (cf., e.g., Kamps, 1995a, p. 30), were introduced. Let \(L(1) = 1\) and \(L(n + 1) = \min\{j > L(n) \mid X_j > X_{L(n)}\}\), then the random variables
\[
(X_{L(n)})_{n \in \mathbb{N}},
\]
are called (upper) record values (cf. for instance Kamps (1995a, p. 31) or Nevzorov (2001, p. 57)). A generalization of the model of (upper) record values called kth record values, was proposed by Dziubdziela and Kopociński (1976). There the succeeding sequence of the kth maxima is considered (cf. Kamps (1995a, p. 33) and Nevzorov (2001, p. 82)). For further reading on record and kth record values, we refer to Chandler (1952), Nagaraja (1988), and Arnold et al. (1998).

For the distribution theory of sequential order statistics, we consider the following particular Meijer’s G-function (see Cramer and Kamps, 2003),
\[
G_{r,r}^{0,r}[s \mid \gamma_1^*, \ldots, \gamma_r^*] = \frac{1}{2\pi i} \int_L \frac{s^z}{\prod_{j=1}^r (\gamma_j^* - 1 - z)} \, dz, \quad z \in \mathbb{C}, \quad s \in (0, 1),
\]
where \(L\) denotes an appropriately chosen integration path. For brevity we write from now on
\[
G_{r,r}^{0,r}[s \mid \gamma_1^*, \ldots, \gamma_r^*] = G_{r,r}^{0,r} \left[ s \mid \gamma_1^* - 1, \ldots, \gamma_r^* - 1 \right].
\]
For an extensive survey on Meijer’s G-function, we refer to Mathai (1993). The following distributional results for sequential order statistics under the proportional hazard rates assumption (cf. (2.8)) are taken from Kamps (1995a,b), Cramer and Kamps (2003) and Burkschat and Lenz (2009).

**Lemma 2.3.4** Let \(F\) be an absolutely continuous cumulative distribution function with density function \(f\). Further, let \(\gamma_j^*, 1 \leq j \leq n\), be as in (2.9).

(i) The joint density function of the sequential order statistics \(X_1^*, \ldots, X_r^*, 1 \leq r \leq n\), for \(x_r \in \Sigma_{r,F}\), is given by
\[
f_{*,1\ldots r}(x_r) = \left( \prod_{j=1}^r \gamma_j^* \right) \left( \prod_{j=1}^{r-1} f(x_j) (1 - F(x_j))^{\gamma_j^* - \gamma_{j+1}^* - 1} \right) f(x_r) (1 - F(x_r))^{\gamma_r^* - 1}.
\]
(ii) The density function of the rth sequential order statistic \(X_r^*, 1 \leq r \leq n\), is given by
\[
f_{*,r}(x) = \left( \prod_{j=1}^r \gamma_j^* \right) G_{r,r}^{0,r}[1 - F(x) \mid \gamma_1^*, \ldots, \gamma_r^*] f(x), \quad x \in \mathbb{R}.
\]
(iii) The cumulative distribution function of the rth sequential order statistic \(X_r^*, 1 \leq r \leq n\), is given by
\[
F_{*,r}(x) = 1 - \left( \prod_{j=1}^r \gamma_j^* \right) \int_0^{1 - F(x)} G_{r,r}^{0,r}[s \mid \gamma_1^*, \ldots, \gamma_r^*] ds, \quad x \in \mathbb{R}.
\]
(iv) As an alternative to (iii), the cumulative distribution function $F_{s,r}, 1 \leq r \leq n$, can also be expressed as

$$F_{s,r}(x) = \left( \prod_{j=1}^{r} \gamma_j^* \right) G_{r+1,r+1}^{r+1,0} [1 - F(x) |\gamma_1^* + 1, \ldots, \gamma_r^* + 1, 1], \quad x \in \mathbb{R}.$$ 

**Proof.** Result (i) follows from the representation for the general case of sequential order statistics, firstly established in Kamps (1995a, p. 29) and Kamps (1995b, p. 4). For a proof, we refer to Cramer and Kamps (2003, p. 298). For the particular expression under the proportional hazard rate assumption, we refer to Kamps (1995a, p. 62). For results (ii) and (iii), we refer to Cramer and Kamps (2003). The representation given in (iv) is taken from Burkschat and Lenz (2009, Theorem 2.3).

Though, we express the density functions as well as the cumulative distribution functions of the sequential order statistics in terms of the $\gamma^*$'s, we recall that the $\gamma^*$'s (except for $\gamma_{n+1}^* = 0$) are predetermined by the model parameters $\alpha_1, \ldots, \alpha_n$ (see (2.9)). The following lemma recalls some basic properties of Meijer’s $G$-function (cf. Cramer et al., 2004; Burkschat and Lenz, 2009).

**Lemma 2.3.5** Let $r \geq 2, s \in (0, 1)$ and $\gamma_1^*, \ldots, \gamma_r^* > 0$.

(i) $G_{1,1}^{1,0}[s|\gamma_1^*] = s^{\gamma_1^* - 1}$.

(ii) $(\gamma_r^* - \gamma_{r+1}^*) G_{r,r}^{r,0}[s|\gamma_1^*, \ldots, \gamma_r^*] = G_{r-1,r-1}^{r-1,0}[s|\gamma_1^*, \ldots, \gamma_{r-1}^*] - G_{r-1,r-1}^{r-1,0}[s|\gamma_2^*, \ldots, \gamma_r^*]$.

(iii) $G_{r,r}^{r,0}[s|\gamma_1^*, \ldots, \gamma_r^*] = \frac{1}{s} G_{r-1,r-1}^{r-1,0}[s|\gamma_1^* + c, \ldots, \gamma_r^* + c], \quad c \in \mathbb{R}$.

(iv) If $\gamma_1^* = \ldots = \gamma_r^*$, then $G_{r,r}^{r,0}[s|\gamma_1^*, \ldots, \gamma_r^*] = \frac{1}{(r-1)!} s^{\gamma_1^* - 1} (- \ln(s))^{r-1}$.

**Proof.** For (i)–(iii), we refer to Cramer et al. (2004, Lemma 2.1). For property (iv), we refer to Cramer et al. (2004, Proof to Lemma 2.2).

Burkschat and Lenz (2009, Remark 2.2) presented a result concerning the distribution of counting processes.

**Lemma 2.3.6** Let $D = D(T), T \geq 0$, be a counting process based on non-negative sequential order statistics $X_{s(1)}^*, \ldots, X_{s(n)}^*$, with

$$D = \sum_{j=1}^{n} 1_{[0,T]}(X_j^*), \quad T > 0.$$ 

Then, the probability $P(D = d)$, for $d \in \{1, \ldots, n-1\}$, is given by

$$P(D = d) = \left( \prod_{j=1}^{d} \gamma_j^* \right) G_{d+1,d+1}^{d+1,0} [1 - F(T) |\gamma_1^* + 1, \ldots, \gamma_{d+1}^* + 1], \quad T > 0.$$ 

In the context of sequential order statistics, we consider throughout this thesis the following models.
2.3 Sequential Order Statistics

**Model 2.3.7 (Sequential model)** In the sequential model (under the proportional hazard rates assumption) we consider the sequential order statistics \(X^*_1, \ldots, X^*_n\) based on an absolutely continuous distribution function \(F\) and coefficients \(\gamma_1^*, \ldots, \gamma_{n+1}^*\) as specified in (2.9). In the context of hybrid censoring we further assume, that \(0 \leq F^{-1}(0) < T\) holds.

For the case of an underlying exponential distribution, we have:

**Model 2.3.8 (Sequential exponential model)** In the sequential exponential model we consider the sequential model with \(PF = Exp(\mu, \vartheta)\), for \(\mu \geq 0\) and \(T > \mu\). The corresponding sequential order statistics are denoted by \(Z^*_1, \ldots, Z^*_n\).

The following model considers the setting of \(k \in \mathbb{N}\) independent sequential \((n_i - m_i + 1)\)-out-of-\(n_i\) systems (cf. Cramer and Kamps, 1996, 2001b). In order to avoid ambiguous notations, we tag the threshold times and later on the counter variables as well as its realizations with the symbol \(\bigcirc\), such that \(T^\bigcirc_i, D^\bigcirc_i\) and \(d^\bigcirc_i\) are considered.

**Model 2.3.9 (Multi-sample sequential model)** In the multi-sample sequential model, we consider for a \(k \in \mathbb{N}\), the integers \(n_1, \ldots, n_k \in \mathbb{N}\) and \(m_1, \ldots, m_k \in \mathbb{N}\), with \(m_i \leq n_i, 1 \leq i \leq k\). Let further absolutely continuous cumulative distribution functions \(F_1, \ldots, F_k\) with respective density functions \(f_1, \ldots, f_k\) be given. Within the context of hybrid censoring the cumulative distribution functions are to be chosen such that for prefixed threshold times \(T^\bigcirc_1, \ldots, T^\bigcirc_k \in (0, \infty)\) the conditions

\[0 \leq F^{-1}_i(0) < T^\bigcirc_i, \quad 1 \leq i \leq k,\]

are satisfied. Then, the \(i\)th sample of sequential order statistics (under the proportional hazard rates assumption) based on \(F_i, 1 \leq i \leq k\), with its respective realization is given by

\[X^*_{i,1}, \ldots, X^*_{i,n_i}\] and \(x_{i,n_i} = (x_{i1}, \ldots, x_{im_i}), \quad 1 \leq i \leq k,\]

respectively. The total set of \(k\) samples and its realizations can be expressed as

\[(X^*_{i,j})_{1 \leq i \leq k, 1 \leq j \leq n_i}\] and \((x_{ij})_{1 \leq i \leq k, 1 \leq j \leq n_i} = (x_{1,n_1}, \ldots, x_{k,n_k}),\]

respectively.

For an underlying exponential distribution the above model can be formulated as follows.

**Model 2.3.10 (Multi-sample sequential exponential model)** In the multi-sample sequential exponential model, we consider the multi-sample sequential model where

\[PF_i = Exp(\mu, \vartheta_i), \quad \text{with} \quad \mu \geq 0, \quad T^\bigcirc_i > \mu \quad \text{and} \quad \vartheta_i > 0, \quad 1 \leq i \leq k.\]

The total amount of \(k\) samples and its respective realizations is further denoted by

\[(Z^*_{i,j})_{1 \leq i \leq k, 1 \leq j \leq n_i}\] and \((z_{ij})_{1 \leq i \leq k, 1 \leq j \leq n_i} = (z_{1,n_1}, \ldots, z_{k,n_k}),\]

respectively.

Based on Lemma 2.3.4, we introduce the following notations for the joint and marginal distributions in the multi-sample case.
Notation 2.3.11  

(i) The joint density function and the joint cumulative distribution function of the sequential order statistics $X_{i,1}^*, \ldots, X_{i,r}^*$, $1 \leq r \leq n_i$, $1 \leq i \leq k$, is denoted by

$$f_{s_i,1...r} \quad \text{and} \quad F_{s_i,1...r},$$

respectively.

(ii) The density function and the cumulative distribution function of the $r$th sequential order statistic w.r.t. the $i$th sample $X_{i,r}^*$, $1 \leq r \leq n_i$, $1 \leq i \leq k$, is denoted by

$$f_{s_{i,r}} \quad \text{and} \quad F_{s_{i,r}},$$

respectively.

We conclude this chapter with the trivial cone $\Sigma^n_{\bar{F}}$, for $\bar{P}_F = \text{Exp}(\mu, \vartheta)$.

Remark 2.3.12 Let $Z^*_1, \ldots, Z^*_n$, $n \in \mathbb{N}$, be sequential order statistics based on $F$ with $P_F = \text{Exp}(\mu, \vartheta)$. Then, the support of the corresponding joint density function $f_{s,1...n}$ (see Lemma 2.3.4, (i)) is given by

$$\text{supp}(f_{s,1...n}) = \{z_n \in \mathbb{R}^n | \mu \leq z_1 \leq \cdots \leq z_n < \infty\} = \Sigma^n_{\bar{F}}.$$
Chapter 3

B-splines

Schoenberg (1946) and Curry and Schoenberg (1947) were one of the first to address the B-spline. Since then a large number of theoretical results have been presented, see, e.g., Curry and Schoenberg (1966), de Boor (1972), Butterfield (1976), de Boor et al. (1976), de Boor (1976) and de Boor (1993a). In a comprehensive survey on B-splines, de Boor (1976) introduced the multivariate B-spline. Originating from the geometrical interpretation of the univariate B-spline established in Curry and Schoenberg (1966), the multivariate B-spline was defined as the volume of the intersection of a simplex with a hyperplane. Multivariate B-splines have been subsequently investigated by Micchelli (1979, 1980) and Dahmen (1980).

The definition of the multivariate B-spline, also referred to as simplicial spline, has been extended in de Boor and Höllig (1982) to more general geometrical objects. Instead of a simplex, a polyhedral convex set was considered. This allowed the deduction of other types of multivariate splines, like the box spline (cf., e.g., Höllig, 1985). For further reading on multivariate B-splines we refer to Dahmen and Micchelli (1983), Höllig (1985) and Goodman (1990). In a more recent work of Xu (2011) multivariate splines were considered within the scope of calculating the volume of polytopes.

Being frequently used in fields as data fitting, function approximation (see Schumaker, 2007) and geometrical modelling (cf. Piegl and Tiller, 1997), B-splines have been also intensively discussed from a probabilistic point of view. Goldman (1988b,a) considered B-splines in the context of urn models. Formulas for particular settings of the Dirichlet density function in terms of multivariate B-splines have been derived by Karlin et al. (1986) and Dahmen and Micchelli (1986). Further, Dahmen and Micchelli (1986) and Ignatov and Kaishev (1989) showed, that the density function of the linear combination of order statistics based on standard uniform random variables can be expressed as a B-spline. By using these results, they derived expressions for the density function of a the linear combination of standard uniform random variables. Further effort on this matter has been made by Agarwal et al. (2002) and Marichal and Kojadinovic (2008). For additional reading on applications of B-splines in statistics, we refer to Wegman and Wright (1983), Eilers and Marx (1996) and Alekseev (2010). For a basic reference on B-splines, we refer to de Boor (2001) and Schumaker (2007). For a handbook on developing tools for geometrical modeling based on B-splines and on non uniform rational B-splines (NURBS), we refer to Piegl and Tiller (1997).

This chapter gives an introduction to the field of univariate B-splines and provides essential tools for the volume approach as well as for the B-spline based distribution theory in hybrid censoring models. In Section 3.1, basic properties for B-splines and divided differences are recalled. Subsequently, we present a new geometric representation of the univariate B-spline and derive respective expressions for volume calculation. Finally, in Section 3.3 integration formulas for particular B-spline expressions are established.
3.1 Introduction and Basic Properties

Before we consider the definition of a B-spline, we provide some basic results on the theory of the divided differences. We start with the following definition (see de Boor, 2001, p. 6).

**Definition 3.1.1** Let \( t_1, \ldots, t_n \in \mathbb{R} \) be a sequence of knots not necessarily distinct and let \( p, g : \mathbb{R} \rightarrow \mathbb{R} \). We say that the function \( p \) agrees with the function \( g \) at \( (t_1, \ldots, t_n) \) provided that, for every knot \( \xi \) that occurs \( m \) times in the sequence \( t_1, \ldots, t_n \), \( p \) and \( g \) agree \( m \)-fold at \( \xi \), that is,

\[
p^{(i-1)}(\xi) = g^{(i-1)}(\xi), \quad \text{for} \quad i = 1, \ldots, m.
\]

The expression \( g^{(i)} \) denotes the \( i \)th derivative of \( g \). The divided difference of a function \( g \) is defined as follows (see de Boor, 2001, p. 3).

**Definition 3.1.2** Let \( g : \mathbb{R} \rightarrow \mathbb{R} \). The \( k \)th divided difference of the function \( g \) at the knots \( t_i, \ldots, t_{i+k} \) is the leading coefficient (that is, the coefficient of \( x^k \)) of the polynomial of order \( k+1 \) that agrees with \( g \) at the sequence \( t_i, \ldots, t_{i+k} \) (in the sense of Definition 3.1.1). It is denoted by

\[
[t_i, \ldots, t_{i+k}; g(\cdot)]. \tag{3.1}
\]

Throughout this work, we mostly make use of notation (3.1) for the divided difference of \( g \). In contrast to that, we consider within the context of B-spline convolutions (see Section 3.3.1) for convenience the notation,

\[
[t_0, \ldots, t_d]_x g(x) = [t_0, \ldots, t_d; g(\cdot)]. \tag{3.2}
\]

Note further, that the symbol \( \cdot \) in \([t_0, \ldots, t_d; g(\cdot)]\) suggests, where the knots are supposed to be plugged in when evaluating the expression w.r.t. the knot sequence \( t_0, \ldots, t_d \). For convenience, we often write \( g \) instead of \( g(\cdot) \). Let \( C^{(d)}(\mathbb{R}) \), \( d \in \mathbb{R} \), denote the set of the \( d \)-times continuously differentiable functions on \( \mathbb{R} \). We recall some basic properties of the divided differences.

**Lemma 3.1.3** Let \( t_0, \ldots, t_d \in \mathbb{R} \), \( d \in \mathbb{N} \), be a knot sequence, and let \( g \) be an arbitrary function. Then, the following properties for divided differences hold.

(i) Let \( t_0, \ldots, t_d \), \( d \in \mathbb{N} \), be pairwise distinct knots, then the divided difference of \( g \) can be expressed as

\[
[t_0, \ldots, t_d; g] = \sum_{j=0}^{d} \frac{g(t_j)}{\prod_{i=0, i \neq j}^{d} (t_j - t_i)}.
\]

(ii) For any distinct knots \( t_r, t_s \), \( 0 \leq r, s \leq d \), \( r \neq s \), in the knot sequence \( t_0, \ldots, t_d \), we have

\[
[t_0, \ldots, t_d; g] = \frac{[t_0, \ldots, t_{r-1}, t_{r+1}, \ldots, t_d; g] - [t_0, \ldots, t_{s-1}, t_{s+1}, \ldots, t_d; g]}{t_s - t_r}.
\]

If \( g \in C^{(d)}(\mathbb{R}) \), then, for \( t_0 = \cdots = t_d \) the respective divided difference of \( g \) is given by

\[
[t_0, \ldots, t_d; g] = \frac{g^{(d)}(t_0)}{d!}.
\]
(iii) Let \( g \) be a polynomial of degree less than \( d \). Then,
\[
[t_0, \ldots, t_d; g] = 0.
\]

(iv) The divided difference is linear in \( g \), such that for any functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) and coefficients \( \alpha, \beta \in \mathbb{R} \), we have
\[
[t_0, \ldots, t_d; \alpha f + \beta g] = \alpha [t_0, \ldots, t_d; f] + \beta [t_0, \ldots, t_d; g].
\]

(v) The divided difference \( [t_0, \ldots, t_d; g] \) is symmetric in the knots \( t_0, \ldots, t_d \), such that
\[
[t_0, \ldots, t_d; g] = [t_{\pi(0)}, \ldots, t_{\pi(d)}; g] \quad \text{for all} \quad \pi \in \Pi_{d+1},
\]
where
\[
\Pi_{d+1} = \{(i_0, \ldots, i_d) \in \{0, \ldots, d\}^{d+1} | i_j \neq i_k, j \neq k, 0 \leq j, k \leq d\}
\]
denotes the set of all permutations of \( (0, 1, \ldots, d) \).

(vi) Let \( \ell, d_1, \ldots, d_\ell, d \in \mathbb{N} \), with \( \ell \leq d \) and \( \sum_{j=1}^{\ell} d_j = d \). Further let \( t_1, \ldots, t_\ell \) be pairwise distinct knots and \( t_j^{d_j} \in \mathbb{R}^{d_j} \) denotes the \( d_j \)-times repetition of the knot \( t_j \), \( 1 \leq j \leq \ell \). Then, the divided differences of a sufficiently differentiable function \( g \), w.r.t. the knot sequence \( t_1^{d_1}, \ldots, t_\ell^{d_\ell} \), is given by
\[
[t_1^{d_1}, \ldots, t_\ell^{d_\ell}; g] = \sum_{j=1}^{\ell} \frac{(-1)^{d+d_j-1}}{(d-j)!} \frac{d^{d_j-1}}{dx_j^{d_j-1}} \left[ \frac{g(x_j)}{\prod_{i=1, i \neq j}^{\ell} (x_i - x_j)^{d_i}} \right],
\]
where \( \frac{d}{dx} g(x) \) denotes the derivative of a function \( g \) w.r.t. \( x \), and \( \prod_{i=1, i \neq j}^{\ell} (x_i - x_j)^{d_i} = 1 \), for \( \ell = 1 \).

Property (i) is taken from Schumaker (2007, p. 46). For (ii) – (v), we refer to de Boor (2001, pp. 4–6). The expression given in (vi) has been presented in Soltani and Roozegar (2012, Lemma 2.1). For further reading on divided differences, we refer to de Boor (2005). In the following, we consider a connection between the divided differences and the marginal distribution of ordered random variables.

Remark 3.1.4  
(i) According to Cramer (2009, p. 340), Meijer’s G-function introduced in (2.10) can be written as
\[
G_{r, i}^{\gamma_1, \ldots, \gamma_r; 0, 1}[x] = (-1)^{r-1} [\gamma_1 - 1, \ldots, \gamma_r - 1; x^{(r)}], \quad x \in (0, 1).
\]

(ii) If we consider the cumulative distribution function as well as the density function of the \( r \)th sequential order statistic \( X_r^* \), given in Lemma 2.3.4, (iv) and (ii), respectively, we find with the identity given in (3.3), that
\[
F_{r, r}(x) = (-1)^r \left( \prod_{j=1}^{r} \gamma_j^* \right) \left[ 0, \gamma_1^* - 1, \ldots, \gamma_r^* - 1; (1 - \Phi(x))^{(r)} \right], \quad x \in \mathbb{R},
\]
and
\[
f_{r, r}(x) = (-1)^{r-1} \left( \prod_{j=1}^{r} \gamma_j^* \right) f(x) \left[ \gamma_1 - 1, \ldots, \gamma_r^* - 1; (1 - \Phi(x))^{(r)} \right], \quad x \in \mathbb{R}.
\]
Cramer (2009) used the relation given in (3.3) to identify the survival function of the rth generalized order statistic with a particular Hermite interpolation polynomial evaluated at zero (see Cramer, 2009, Theorem 3.1).

Remark 3.1.4 and Lemma 3.1.3 can be utilized to calculate the distribution of \( X_r^* \) for any setting of \( \gamma_1^*, \ldots, \gamma_r^* > 0, 1 \leq r \leq n \).

**Example 3.1.5**  
(i) For the situation of kth record values we obtain with Lemma 3.1.3, (ii), equation (3.4) and equation (8) from Cramer (2009) the following expression for the density function \( f_{s,r} \) (cf., e.g., Dziubdziela and Kopociński, 1976, Theorem 1),

\[
f_{s,r}(x) = (-1)^{r-1} k^r f(x) \left( \frac{1}{(r-1)!} \frac{d^{r-1}(1-F(x))^{y}}{dy^{r-1}} \right)_{y=k-1} = \frac{k^r}{(r-1)!} f(x)(1-F(x))^{k-1} \left( -\ln(1-F(x)) \right)^{r-1}, \quad x \in \mathbb{R}.\]

(ii) Let \( n \geq m \) denote the number of objects put on life test. Further, let \( R_1, \ldots, R_m \), with \( R_m = n - m - \sum_{j=1}^{m-1} R_j \), denote the quantities of objects to be removed from the experiment after observing the jth failure, \( 1 \leq j \leq m \). By setting

\[
\alpha_j = \frac{(n+1-j-\sum_{i=j}^{m-1} R_i)}{(n-j+1)} = \frac{n-\sum_{i=j}^{m-1} (R_i+1)}{(n-j+1)} = \frac{\sum_{i=j}^{m} R_i+1}{(n-j+1)}, \quad 1 \leq j \leq m,
\]

in (2.9), we have \( \gamma_j^* = \gamma_j, 1 \leq j \leq m+1 \), and obtain the setting of progressively Type-II censored order statistics (see Remark 2.3.2, (i)). Then, we arrive with Lemma 3.1.3, (i), and (3.4) at the representation for the density function of the rth progressively Type-II censored order statistic \( X_{r:m:n}^* \), as given in Lemma 2.2.2, (ii).

We proceed by considering the definition of the B-spline.

**Definition 3.1.6** Let \( t_0, \ldots, t_d, d \in \mathbb{N} \), be a knot sequence. Curry-Schoenberg’s B-spline of degree \( d-1 \) is defined by

\[
B_{d-1}(s|t_0, \ldots, t_d) = d \left[ t_0, \ldots, t_d; (\cdot - s)^{d-1} \right], \quad s \in \mathbb{R}.
\]

The above definition follows from the definition given in de Boor (2001, p. 87) for the normalized B-spline \( N_{d-1}(|t_0, \ldots, t_d|) \), with (see de Boor, 2001, p. 88)

\[
N_{d-1}(s|t_0, \ldots, t_d) = \frac{t_d-t_0}{d} B_{d-1}(s|t_0, \ldots, t_d), \quad s \in \mathbb{R}, \tag{3.5}
\]

and by taking into account that for the non-normalized case the knot sequence can be chosen arbitrarily. Figures 3.1 (left) and 3.4 (non-solid lines) illustrate the plots of B-splines with equally spaced knots. Figure 3.1 (right) illustrates the plot of a B-spline with the variable knot sequence \( 0, a, \ldots, ma, a \in [0, 1.5] \) evaluated at \( s = 1 \).

**Remark 3.1.7**  
(i) The representation of the B-spline given in (1.1), with \( t_j = \gamma_{d-j+1}, \quad j \in \{0, \ldots, d\} \), follows from Definition 3.1.6 and Lemma 3.1.3, (i).

(ii) Schoenberg (1946) considered the B-spline for a sequence of equally spaced knots. Then, Curry and Schoenberg (1947) presented an expression for the B-spline w.r.t. an ascending knot sequence with distinct knots. A more general setup has been considered later by Curry and Schoenberg (1966). There the respective knot sequences consist of multiple knots.
We consider in the following examples for B-splines which occur in distributional setups related to the uniform distribution.

**Example 3.1.8**  
(i) According to Feller (1971, pp. 27–28) (see also Buonocore et al., 2009, Proposition 3.1) the density function of \( U_n^* = \sum_{i=1}^{n} U_i \), with \( U_i \sim U(0, a) \), \( a > 0 \), is given by

\[
f_{U_n^*}(u) = \frac{1}{(n-1)!a^n} \sum_{j=1}^{n} (-1)^j \binom{n}{j} (u - ja)_+^{n-1}, \quad u \in \mathbb{R}.
\]

Simple calculations verify that, for \( j \in \{0, \ldots, n\} \), \( n \in \mathbb{N} \),

\[
\prod_{i=0, i \neq j}^{n} (j - i) = j! (n - j)! (-1)^{(n-j)}, \quad (3.6)
\]

holds. Further, as we will see later (see Remark 4.3.2), the identity

\[
[0, a, \ldots, n a; (x - \cdot)_+^{n-1}] = (-1)^n [0, a, \ldots, n a; (\cdot - x)_+^{n-1}], \quad (3.7)
\]

holds. According to Lemma 3.1.3, (i) and together with equations (3.6) and (3.7), we get

\[
f_{U_n^*}(u) = \frac{n}{a^n} \sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{1}{j!(n-j)!} (u - ja)_+^{n-1}
\]

\[
= n(-1)^n \sum_{j=0}^{n} \frac{(u - ja)_+^{n-1}}{a^n \prod_{i=0, i \neq j}^{n} (j - i)}
\]

\[
= n(-1)^n [0, a, \ldots, na; (u - \cdot)_+^{n-1}]
\]

\[
= n [0, a, \ldots, na; (\cdot - u)_+^{n-1}]
\]

\[
= B_{n-1}(u|0, a, \ldots, na), \quad u \in \mathbb{R}.
\]

The above identity follows also from the calculations performed in Dahmen and Micchelli (1986, Section 3). The density function \( f_{U_n^*} \) for \( a = 0.5 \) and \( n \in \{1, 2, 3\} \), is depicted in Figure 3.1 (left).

(ii) Let \( U_1, \ldots, U_{m-1} \sim U(0, 1) \). Further, let the spacings \( Y_i \) be defined as

\[
Y_i = U_{i:m-1} - U_{i-1:m-1}, \quad i \in \{1, \ldots, m\}, \quad \text{with} \quad U_{0:m-1} = 0 \quad \text{and} \quad U_{m:m-1} = 1.
\]

Then, \( Y_{(m)} \) denotes the corresponding maximal spacing. According to David and Nagaraja (2003, p. 135) and Barlevy and Nagaraja (2015, pp. 137–138) the cumulative distribution function of \( Y_{(m)} \) is given by

\[
F_{Y_{(m)}}(a) = P(Y_{(m)} \leq a) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - ja)_+^{m-1}, \quad a \in [0, 1].
\]

Note that the domain of the parameter \( a \) can be extended to \( \mathbb{R} \). By utilizing the same arguments as in the previous example (i.e., Example 3.1.8, (i)), we find

\[
F_{Y_{(m)}}(a) = a^m m! (-1)^m [0, a, \ldots, ma; (1 - \cdot)_+^{m-1}]
\]

\[
= a^m (m - 1)! B_{m-1}(1|0, a, \ldots, ma), \quad a \in \mathbb{R}.
\]

Figure 3.1 (right) illustrates the plots of \( F_{Y_{(m)}} \) for three different \( m \)’s.
The following lemma recalls some basic B-spline properties.

**Lemma 3.1.9** Let $B_{d-1}(\cdot|t_0, \ldots, t_d)$ be a B-spline of degree $d-1$ w.r.t. the knot sequence $t_0, \ldots, t_d \in \mathbb{R}$, where not all knots are equal.

(i) For $t_0 \leq \cdots \leq t_d$, the following recurrence relation holds

$$
\frac{(d-1)(t_d - t_0)}{d} B_{d-1}(s|t_0, \ldots, t_d) = (s - t_0) B_{d-2}(s|t_0, \ldots, t_{d-1}) + (t_d - s) B_{d-2}(s|t_1, \ldots, t_d), \quad s \in \mathbb{R}.
$$

(ii) For the standard-d-simplex $S_d$, with

$$
S_d = \{ (x_0, x_d) \in \mathbb{R}^{d+1} | x_j \geq 0, 0 \leq j \leq d, \sum_{j=0}^{d} x_j = 1 \} \quad (3.8)
$$

and an arbitrary continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, the identity

$$
\frac{1}{d!} \int_{\mathbb{R}} B_{d-1}(s|t_0, \ldots, t_n) g(s)ds = \int_{S_d} g(t_0 x_0 + \cdots + t_d x_d) dx_d, \quad (3.9)
$$

holds. Note that $(x_0, x_d) = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$ and that $dx_d = dx_{d-1} \cdots dx_1$.

(iii) Let $g \in C^{(d-1)}\left( \bigoplus_{0 \leq j \leq d} [\min_{0 \leq j \leq d} t_j, \max_{0 \leq j \leq d} t_j] \right)$ and let $g^{(d-1)}$ be absolutely continuous on the interval $[\min_{0 \leq j \leq d} t_j, \max_{0 \leq j \leq d} t_j]$. Then, Peano’s formula

$$
[t_0, \ldots, t_d; g] = \frac{1}{d!} \int_{\mathbb{R}} B_{d-1}(x|t_0, \ldots, t_d) g^{(d)}(x)dx,
$$

holds.
(iv) Let \( t_0 < t_1 \). Then, the B-spline of degree zero \( B_0([t_0, t_1]) \) is given by

\[
B_0(s|t_0, t_1) = \begin{cases} \frac{1}{t_1-t_0}, & s \in [t_0, t_1), \\ 0, & s \notin [t_0, t_1). \end{cases}
\]

(v) Let \( d \in \mathbb{N}_{\geq 2} \) and \( t_0 \leq \cdots \leq t_d \). Further let \( a_0 = \max \{ j | t_0 = \cdots = t_{j-1} \} \) and \( b_d = \max \{ j | t_d = \cdots = t_{d-j+1} \} \). Then

\[
B_{d-1}(s|t_0, \ldots, t_d) > 0, \quad s \in (t_0, t_d) \quad \text{and} \quad B_{d-1}(s|t_0, \ldots, t_d) = 0, \quad s \notin [t_0, t_d].
\]

At the endpoints of the interval \([t_0, t_d]\) the B-spline is defined as follows

\[
(-1)^{k+d-a_0} \frac{d^k B_{d-1}(t_0, \ldots, t_d)}{ds^k} = \begin{cases} 0, & k \in \{0, \ldots, d - 1 - a_0\}, \\ > 0, & k \in \{d - a_0, \ldots, d - 1\}, \end{cases}
\]

and

\[
(-1)^{d-b_d} \frac{d^k B_{d-1}(t_0, \ldots, t_d)}{ds^k} = \begin{cases} 0, & k \in \{0, \ldots, d - 1 - b_d\}, \\ > 0, & k \in \{d - b_d, \ldots, d - 1\}, \end{cases}
\]

where

\[
\frac{d_+ f(s)}{ds} = \lim_{h \to 0^+} \frac{f(s + h) - f(s)}{h} \quad \text{and} \quad \frac{d_- f(s)}{ds} = \lim_{h \to 0^+} \frac{f(s) - f(s - h)}{h}.
\]

For result (i) of Lemma 3.1.9, we refer to de Boor (2001, p. 90), which was shown for the normalized B-spline given by \( N_{d-1}([t_0, \ldots, t_d]) \) (cf. (3.5)). Property (ii) can be found in Micchelli (1979, p. 213), Dahmen and Micchelli (1986, p. 21), and Forster and Massopust (2009, p. 326). The identity presented in (iii) can be found in Schumaker (2007, Theorem 4.23) and Strøm (1994, p. 5). We note that, for \( t_0 \leq \cdots \leq t_d \),

\[
\int_{\mathbb{R}} B_{d-1}(s|t_0, \ldots, t_d) g^{(d)}(s) ds = \int_{t_0}^{t_d} B_{d-1}(s|t_0, \ldots, t_d) g^{(d)}(s) ds.
\]

Result (iv) can be found in Schumaker (2007, p. 119). For (v) we refer to Schumaker (2007, pp. 13, 121–122). It should be noted that for property Lemma 3.1.9, (ii), the requirements for the function \( g \) vary between locally integrable (see, e.g., Micchelli, 1979) and continuous on \( \mathbb{R} \) (see, e.g., Forster and Massopust, 2009).

Let \( \Sigma^d \) denote the uniform cone

\[
\Sigma^d = \{ x_d \in [0, 1]^d | 0 \leq x_1 \leq \cdots \leq x_d \leq 1 \}.
\]

Finally, we present a result, which forms the basis of the expected value approach.

**Corollary 3.1.10** Let \( \Sigma^d \) be as in (3.11) and let further be \( g : \mathbb{R} \to \mathbb{R} \) an arbitrary continuous function. Then,

\[
\int_{\Sigma^d} g(t_0 + \Sigma_{j=1}^d t_j x_j) d\mathbf{x}_d = \frac{1}{d!} \int_{\mathbb{R}} B_{d-1}(x|t_0, t_0 + t_d, \ldots, t_0 + \sum_{j=1}^d t_{d-j+1}) g(x) dx.
\]

**Proof.** This result follows by applying the linear transformation \( \Phi : \mathcal{S}_d \to \Sigma^d \) with

\[
\Phi(\mathbf{x}_d) = (x_d, x_d + x_{d-1}, \ldots, \sum_{j=1}^d x_j),
\]

to the identity given in Lemma 3.1.9, (ii), with \( \mathcal{S}_d \) as in (3.8) (cf. Dahmen and Micchelli, 1986, Section 3). \( \square \)
3.2 Geometrical Characterization and Volume Computation

This section addresses B-splines from a geometric perspective. As mentioned at the very beginning of this chapter, a geometrical interpretation of the B-spline has been firstly presented in Curry and Schoenberg (1966, Section 2). The characterization as the volume of the intersection of a simplex and a hyperplane was then extended by de Boor (1976, p. 20), which led to the definition of the multivariate B-spline. For detailed information on this particular volume characterization as well as for supplementary illustrations we refer to Micchelli (1979, p. 214), Goodman (1990, pp. 348–349) and de Boor (1993b, Section 5).

We present an alternative geometry oriented characterization by specifying the B-spline as the volume of a particular polytope. This polytope can be further interpreted as the intersection of a simplex with a half-space. This new representation will be used to derive respective volume formulas.

**Theorem 3.2.1** Let \( t_0, \ldots, t_d, d \in \mathbb{N} \), be a knot sequence where not all knots are equal. Then, for each index \( i^* \in \{1, \ldots, d\} \), with \( t_0 < t_{i^*} \) and all \( \beta_1, \ldots, \beta_d > 0 \), the B-spline \( B_{d-1}(t_0, \ldots, t_d) \), can be expressed as the volume of a polytope \( M_{d-1}^{[i^*]}(s|\beta, t) \), \( s \in \mathbb{R} \), i.e.,

\[
B_{d-1}(s|t_0, \ldots, t_d) = \frac{d!}{(t_{i^*} - t_0) \prod_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \beta_j} \text{vol}_{d-1}(M_{d-1}^{[i^*]}(s|\beta, t)), \quad s \in \mathbb{R},
\]

where

\[
\beta = (\beta_1, \ldots, \beta_d), \quad t = (t_0, \ldots, t_d),
\]

and

\[
M_{d-1}^{[i^*]}(s|\beta, t) = \left\{ (x_j)_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \in \mathbb{R}^{d-1} \mid x_j \geq 0, j \in \{1, \ldots, d\} \setminus \{i^*\}, \quad \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \frac{t_j - t_0}{\beta_j} x_j \leq s - t_0, \right. \]

\[
\left. \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \frac{t_j - t_{i^*}}{\beta_j} x_j \geq s - t_{i^*} \right\}.
\]

For \( d = 1 \), we define

\[
\text{vol}_0(M_0^{[1]}(s|\beta, t)) := 1_{[t_0, t_{i^*})}(s).
\]

**Proof.** For \( d \in \mathbb{N} \) and an arbitrary continuous function \( g : \mathbb{R} \rightarrow \mathbb{R} \), we consider the integral

\[
\int_{S_d} g(t_0 x_0 + \cdots + t_d x_d) \, dx_d,
\]

where \( S_d \) denotes the standard-\( d \)-simplex given in (3.8). Notice that \( x_0 = 1 - \sum_{j=1}^d x_j \).

Then, we obtain for \( \beta_1, \ldots, \beta_d > 0 \)

\[
\int_{S_d} g(t_0 x_0 + \cdots + t_d x_d) \, dx_d = \int_{S_d} g \left( t_0 + \frac{(t_1 - t_0)}{\beta_1} \beta_1 x_1 + \cdots + \frac{(t_d - t_0)}{\beta_d} \beta_d x_d \right) \, dx_d.
\]
By applying the linear transformation \(\Phi_1 : S_d \rightarrow S_d^*\) with \(\Phi_1(x_d) = (\beta_1 x_1, \ldots, \beta_d x_d)\) and \(|\det(D\Phi_1)| = \prod_{j=1}^{d} \beta_j\) where

\[
S^*_d = \left\{ x_d \in \mathbb{R}^d \left| x_j \geq 0, 1 \leq j \leq d, \sum_{j=1}^{d} \frac{x_j}{\beta_j} \leq 1 \right. \right\},
\]  
(3.14)

we obtain

\[
\int_{S_d} g \left( t_0 + \frac{(t_1 - t_0)}{\beta_1} x_1 + \cdots + \frac{(t_d - t_0)}{\beta_d} x_d \right) dx_d = \frac{1}{\prod_{j=1}^{d} \beta_j} \int_{S^*_d} \frac{1}{(t_i^* - t_0)} \prod_{j \in I_{1,d}\backslash \{i^*\}} \beta_j \int_{\Phi_{2,i^*}(S^*_d)} g(s) d(s) ds.
\]

Hereby denotes \(D\Phi \in \mathbb{R}^{d \times d}\), the Jacobian matrix w.r.t. \(\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d\), and \(\det(A)\) denotes the determinant of a matrix \(A \in \mathbb{R}^{d \times d}\). For the sum \(t_0 + \frac{(t_1 - t_0)}{\beta_1} x_1 + \cdots + \frac{(t_d - t_0)}{\beta_d} x_d\), we consider for any index \(i^* \in \{1, \ldots, d\}\), with \(t_{i^*} > t_0\), the linear transformation \(\Phi_{2,i^*} : S_d^* \rightarrow \mathbb{R}^d\), with

\[
\Phi_{2,i^*}(x_d) = \left( x_1, \ldots, x_{i^*-1}, t_0 + \sum_{j=1}^{d} \frac{t_j - t_0}{\beta_j} x_j, x_{i^*+1}, \ldots, x_d \right), \quad |\det(D\Phi_{2,i^*})| = \frac{t_{i^*} - t_0}{\beta_{i^*}}.
\]

For brevity, we introduce the notation \(I_{m,n} = \{m, \ldots, n\}\), \(m < n\), \(m, n \in \mathbb{N}_0\) and \(n \in \mathbb{N}\). Then, we get by applying \(\Phi_{2,i^*}\),

\[
\frac{1}{\prod_{j=1}^{d} \beta_j} \int_{S^*_d} g \left( t_0 + \frac{(t_1 - t_0)}{\beta_1} x_0 + \cdots + \frac{(t_d - t_0)}{\beta_d} x_d \right) dx_d = \frac{1}{(t_i^* - t_0)} \prod_{j \in I_{1,d}\backslash \{i^*\}} \beta_j \int_{\Phi_{2,i^*}(S^*_d)} g(s) d(s) ds.
\]

Now, we determine the domain \(\Phi_{2,i^*}(S^*_d)\). By using the substitution \(s = t_0 + \sum_{j=1}^{d} \frac{t_j - t_0}{\beta_j} x_j\), we find

\[
s = t_0 + \sum_{j=1}^{d} \frac{t_j - t_0}{\beta_j} x_j \quad \iff \quad x_{i^*} = \frac{\beta_{i^*}}{t_{i^*} - t_0} \left( s - t_0 - \sum_{j \in I_{1,d}\backslash \{i^*\}} \frac{t_j - t_0}{\beta_j} x_j \right).
\]

(3.15)

In order to express the set \(\Phi_{2,i^*}(S^*_d)\) in terms of the variables \((x_j)_{j \in I_{1,d}\backslash \{i^*\}}\) and \(s\), we first find

\[
s - t_0 \geq \sum_{j \in I_{1,d}\backslash \{i^*\}} \frac{t_j - t_0}{\beta_j} x_j.
\]

By using (3.15) and by taking into account that \(t_{i^*} > t_0\), we rewrite \(\sum_{j=1}^{d} \frac{x_j}{\beta_j}\) as

\[
\sum_{j=1}^{d} \frac{x_j}{\beta_j} = \sum_{j \in I_{1,d}\backslash \{i^*\}} \frac{x_j}{\beta_j} + \frac{1}{t_{i^*} - t_0} \left( s - t_0 - \sum_{j \in I_{1,d}\backslash \{i^*\}} \frac{t_j - t_0}{\beta_j} x_j \right)
\]

\[
= \sum_{j \in I_{1,d}\backslash \{i^*\}} \left( \frac{1}{\beta_j} - \frac{t_j - t_0}{(t_{i^*} - t_0)\beta_j} \right) x_j + \frac{s - t_0}{t_{i^*} - t_0}.
\]
This yields for the constraint \( \sum_{j=1}^{d} \frac{x_j}{\beta_j} \leq 1 \) (cf. (3.14)) with \( t_r > t_0 \):

\[
\sum_{j \in \II_{1,d} \setminus \{i^*\}} \frac{t_r - t_j}{(t_r - t_0)\beta_j} x_j \leq 1 - \frac{s - t_0}{t_r - t_0} \iff \sum_{j \in \II_{1,d} \setminus \{i^*\}} \frac{t_r - t_j}{\beta_j} x_j \leq t_r - t_0 - (s - t_0) \\
\iff \sum_{j \in \II_{1,d} \setminus \{i^*\}} \frac{t_j - t_r}{\beta_j} x_j \geq s - t_r.
\]

Hence, we get: \( \Phi_{2,i^*}(S^*_d) = \mathcal{M}_{d-1}^{[i^*]}(s|\beta, t) \times \mathbb{R} \), with \( \mathcal{M}_{d-1}^{[i^*]}(s|\beta, t) \) as in (3.13). Now let \( d \geq 2 \). Then, due to

\[
\int_{S_d} g(t_0x_0 + \cdots + t_dx_d)dx_d = \frac{1}{(t_r - t_0) \prod_{j \in \II_{1,d} \setminus \{i^*\}} \beta_j} \int \Phi_{2,i^*}(S^*_d) g(s)d(x_j)_{j \in \II_{1,d} \setminus \{i^*\}} ds
\]

\[
= \frac{1}{d!} \frac{d!}{(t_r - t_0) \prod_{j \in \II_{1,d} \setminus \{i^*\}} \beta_j} \vol_{d-1}(\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)) g(s)ds
\]

and by utilizing Lemma 3.1.9, (ii), we obtain the identity

\[
B(s|t_0, \ldots, t_d) = \frac{d!}{(t_r - t_0) \prod_{j \in \II_{1,d} \setminus \{i^*\}} \beta_j} \vol_{d-1}(\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)), \quad s \in \mathbb{R}.
\]

For \( d = 1 \) the volume of \( \mathcal{M}_{d-1}^{[i^*]}(s|\beta, t) \) does not exist. Hence, we define

\[
\vol_0(\mathcal{M}_{0}^{[i^*]}(s|\beta, t)) := 1_{|t_0, t_r|}(s).
\]

This finishes the proof. \( \square \)

The following corollary addresses the case \( t_{j^*} < t_0 \), for a \( j^* \in \{1, \ldots, d\} \).

**Corollary 3.2.2** Let \( B_{d-1}(\cdot|t_0, \ldots, t_d) \), be a B-spline where not all knots are equal. Further, let \( j^* \in \{1, \ldots, d\} \) be an index such that \( t_0 > t_{j^*} \) and \( \beta_1, \ldots, \beta_d > 0 \). Then

\[
B_{d-1}(s|t_0, \ldots, t_d) = \frac{d!}{(t_r - t_{j^*}) \prod_{j \in \{1, \ldots, d\} \setminus \{j^*\}} \beta_j} \vol_{d-1}(\mathcal{M}_{d-1}^{[j^*]}(s|\beta, \tilde{t})), \quad s \in \mathbb{R},
\]

with \( \beta \) as in (3.12) and

\[
\tilde{t} = (t_{j^*}, t_1, \ldots, t_{j^*-1}, t_0, t_{j^*+1}, \ldots, t_d),
\]

and \( \mathcal{M}_{d-1}^{[j^*]}(s|\beta, \tilde{t}) \) as specified in (3.13).

**Proof.** According to Lemma 3.1.3, (v), the B-spline does not depend on the order of the knot sequence, that is,

\[
B_{d-1}(s|t_0, t_1, \ldots, t_{j^*-1}, t_{j^*}, t_{j^*+1}, \ldots, t_d) = B_{d-1}(s|t_{j^*}, t_1, \ldots, t_{j^*-1}, t_0, t_{j^*+1}, \ldots, t_d).
\]

The application of Theorem 3.2.1 to the right-hand side of the above equation proves the assertion. \( \square \)
We now present a volume formula for particular polytopes.

**Corollary 3.2.3** Let $s_1, \ldots, s_d, h_1, \ldots, h_d \in \mathbb{R}$, with
\[
  h_{i^*} > s_{i^*} \quad \text{and} \quad h_j < s_j, \quad j \in \{1, \ldots, d\} \setminus \{i^*\},
\]  
for an arbitrary but fixed $i^* \in \{1, \ldots, d\}$. Then, the polytope $\mathcal{M}^{[i^*]}_{d-1}(s|\beta, t)$, given in (3.13), can be expressed as
\[
\mathcal{M}^{[i^*]}_{d-1}(s|\beta, t) = \left\{ (x_j)_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \in \mathbb{R}^{d-1} \mid x_j \geq 0, j \in \{1, \ldots, d\} \setminus \{i^*\}, \right. \\
\left. \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} s_j x_j \leq s - s_{i^*}, \right. \\
\left. \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} h_j x_j \geq s - h_{i^*} \right\},
\]
with $\beta = (\beta_1, \ldots, \beta_d)$ and $t = (t_0, \ldots, t_d)$ where
\[
\beta_j = \frac{h_{i^*} - s_{i^*}}{s_j - h_j}, \quad j \in \{1, \ldots, d\} \setminus \{i^*\}, \quad \beta_j > 0,
\]
and $t_0 = s_{i^*}$, $t_{i^*} = h_{i^*}$, $t_j = \frac{s_j h_{i^*} - s_{i^*} h_j}{s_j - h_j}$.

Further, the corresponding volume can be calculated by
\[
\text{vol}_{d-1}(\mathcal{M}^{[i^*]}_{d-1}(s|\beta, t)) = \frac{(t_{i^*} - t_0) \prod_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \beta_j}{d!} B_{d-1}(s|t_0, \ldots, t_d), \quad s \in \mathbb{R}.
\]

**Proof.** The assertion can be directly deduced from Theorem 3.2.1 by comparing the parameters of $\mathcal{M}^{[i^*]}_{d-1}(s|\beta, t)$, as given in (3.13), with those of the polytope given in (3.17). Hence, we consider
\[
\sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \frac{t_j - t_0}{\beta_j} x_j \leq s - t_0 \quad \text{vs.} \quad \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} s_j x_j \leq s - s_{i^*} \]  
and 
\[
\sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \frac{t_j - t_{i^*}}{\beta_j} x_j \geq s - t_{i^*} \quad \text{vs.} \quad \sum_{j \in \{1, \ldots, d\} \setminus \{i^*\}} h_j x_j \geq s - h_{i^*}.
\]
By comparing the parameters from the constraints in (3.18), we get
\[
t_0 = s_{i^*} \quad \text{and} \quad \frac{t_j - s_{i^*}}{\beta_j} = s_j \iff t_j = s_j \beta_j + s_{i^*} \quad j \in \{1, \ldots, d\} \setminus \{i^*\}.
\]
Further, we find with the inequalities (3.19) and (3.20)
\[
t_{i^*} = h_{i^*} \quad \text{and} \quad \frac{t_j - h_{i^*}}{\beta_j} = h_j \iff s_j \beta_j + s_{i^*} - h_{i^*} = h_j \beta_j \iff \beta_j (s_j - h_j) = h_{i^*} - s_{i^*}.
\]
Due to the conditions $\beta_j > 0$ and $t_0 < t_i^*$ we get the constraints $h_{i^*} > s_{i^*}$ and $h_j < s_j$, $j \in \{1, \ldots, d\} \setminus \{i^*\}$. Notice that $\beta_{i^*} > 0$ can be chosen arbitrarily, since it does not occur in the expression for the B-spline $B_{d-1}(\cdot|t_0, \ldots, t_d)$ given in Theorem 3.2.1. By inserting (3.21) in (3.20), we further find

$$t_j = \frac{s_j (h_{i^*} - s_{i^*})}{s_j - h_j} + s_{i^*} = \frac{s_j h_{i^*} - s_{i^*} h_j}{s_j - h_j}, \quad j \in \{1, \ldots, d\} \setminus \{i^*\}.$$  

An application of Theorem 3.2.1 proves the result. 

In the following, we comment on some aspects of the B-spline characterization introduced in Theorem 3.2.1.

**Remark 3.2.4** (i) The purpose of applying transformation $\Phi_1$ in the proof of Theorem 3.2.1, is to add extra degrees of freedom to the structure of the polytope $\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)$, $i^* \in \{1, \ldots, d\}$. Further, this approach increases the number of sets for which Corollary 3.2.3 can be employed.

(ii) After applying Corollary 3.2.3 on a particular set, the knot sequence of the resulting B-spline may not be sorted in ascending order. However, this is irrelevant due to Lemma 3.1.3, (v).

(iii) The polytopes given in (3.13) and (3.17), can be written as the intersection of the simplex $S_{d-1,i^*}^{(s)}$ (cf. (1.5)) and the half-space $H_{d-1,i^*}^{(s)}$ (cf. (1.6)), that is

$$\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t) = S_{d-1,i^*}^{(s)} \cap H_{d-1,i^*}^{(s)}, \quad s \in \mathbb{R}.$$  

This relation has been also presented in Cramer and Balakrishnan (2013) for a particular setting for $\beta$ and $t$.

Note that the geometrical representation of the B-spline presented in Theorem 3.2.1 is less intuitive than the characterization established in Curry and Schoenberg (1966). Further, we find that the illustration of the alternative geometrical characterization is more sophisticated than it is in the original case. Hence, we consider the set $\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)$ in the two-dimensional case, i.e., $d = 3$. Figures 3.2 and 3.3 illustrate $\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)$ by the means of the simplex half-space characterization addressed in Remark 3.2.4, (iii). The light gray domains, correspond to the inequality inducing the half-space $H_{d-1,i^*}^{(s)}$. The areas, corresponding to the simplex $S_{2,3}^{(s)}$, are colored gray. Then, the desired set $\mathcal{M}_{d-1}^{[i^*]}(s|\beta, t)$ can be obtained (see dark gray domains) by considering the intersection of the sets corresponding to $H_{2,3}^{(s)}$ and to $S_{2,3}^{(s)}$.

The main purpose of the above introduced B-spline characterization, is to provide a convenient way to calculate the volume of particular polytopes. The necessity of such a volume formula in the context of hybrid censoring, arose with the results established Cramer and Balakrishnan (2013) and Cramer et al. (2016). They used a result presented in Gerber (1981, p. 312) in order to determine the volume. However, the application of Gerber’s general formula for the volume of the intersection of a simplex with a half-space, results in
Figure 3.2: Illustration of $M_{2}^{(s)}(s|\beta, t)$ (dark gray) as the intersection of the simplex $S_{2,3}^{(s)}$ (gray) and the half-space $H_{2,3}^{(s)}$ (light gray), with arbitrary $\beta$’s and $s \in [t_{0}, t_{d}]$. Left: $t_{0} < t_{1} < t_{2} < t_{3}$. Right: $t_{0} = t_{1} < t_{2} < t_{3}$.

complicated divided differences expressions. The effort to be made in order to transfer those expressions to more compact and most likely more informative B-spline expressions, is quite high. Hence, the derivation of a respective B-spline characterization seemed reasonable.

The problem of determining the volume of the intersection of a simplex with a half-space was apart from Gerber (1981) also addressed by Ali (1973), Varsi (1973), Cho and Cho (2001), Farber (2008) and Lasserre (2015). They pursued each a different approach for deriving the respective volume formulas and they considered further different settings for the simplex as well as for the half-space. Hence, it can be taken into consideration to use the above B-spline characterization in order to derive compact representations for related or more general settings for intersections of simplices with half-spaces.

Example 3.2.5 Let us consider the set

$$M_{d-1,i}^{[s]}(s|\beta, t)$$

$$= \left\{(x_{j})_{j \in \{1, \ldots, d\} \setminus \{i^{*}\}} \in \mathbb{R}^{d-1} \mid x_{j} \geq 0, j \in \{1, \ldots, d\} \setminus \{i^{*}\}, \sum_{j \in \{1, \ldots, d\} \setminus \{i^{*}\}} s_{j} x_{j} \leq s - s_{i^{*}}, \sum_{j \in \{1, \ldots, d\} \setminus \{i^{*}\}} h_{j} x_{j} \leq s - h_{i^{*}} \right\}, \quad (3.22)$$

with $\beta$ and $t$ as in Corollary 3.2.3. Note that the structure of $M_{d-1,i}^{[s]}(s|\beta, t)$ is very similar to $M_{d-1}^{[s]}(s|\beta, t)$, however the respective last inequalities are different. Now, we have

$$M_{d-1}^{[s]}(s|\beta, t) = S_{d-1,i}^{(s)} \cap H_{d-1,i}^{(s)},$$

where $H_{d-1,i}$ denotes a half-space

$$H_{d-1,i}^{(s)} = \left\{(x_{j})_{j \in \{1, \ldots, d\} \setminus \{i^{*}\}} \in \mathbb{R}^{d-1} \mid \sum_{j \in \{1, \ldots, d\} \setminus \{i^{*}\}} h_{j} x_{j} \leq s - h_{i^{*}} \right\}.$$
It is obvious, that $\mathcal{H}^{(s)}_{d-1,i^*} = \mathbb{R}^{d-1} \setminus \mathcal{H}^{(s)}_{d-1,i^*}$, with $\mathcal{H}^{(s)}_{d-1,i^*}$ as in (1.6). The volume of $\mathcal{M}^{[r^*]}_{d-1} (s|\beta, t)$ can be expressed as

$$\text{vol}_{d-1} (\mathcal{M}^{[r^*]}_{d-1} (s|\beta, t)) = \text{vol}_{d-1} (S^{(s)}_{d-1,i^*}) - \text{vol}_{d-1} (\mathcal{M}^{[r^*]}_{d-1} (s|\beta, t)), \quad s \in \mathbb{R}.$$ 

The simplex $S^{(s)}_{d-1,i^*}$ has vertices in $0^{(d-1)}$ and $\frac{(s-s_{i^*})}{s_j} e_j$, $j \in \{1, \ldots, d\} \setminus \{i^*\}$, where $e_1, \ldots, e_{i^*}, e_{i^*+1}, \ldots, e_d$ denote the standard basis of $\mathbb{R}^{d-1}$. The volume of $S^{(s)}_{d-1,i^*}$ is hence given by (cf., e.g., Gerber, 1981, p. 311)

$$\text{vol}_{d-1} (S^{(s)}_{d-1,i^*}) = \begin{cases} 
\frac{(s-s_{i^*})^{d-1}}{(d-1)! \prod_{j \in \{1, \ldots, d\} \setminus \{i^*\}} s_j}, & s \geq s_{i^*}, \\
0, & s < s_{i^*}.
\end{cases} \quad (3.23)$$

Then, the volume of $\mathcal{M}^{[r^*]}_{d-1} (s|\beta, t)$ can be determined by using Corollary 3.2.3 and (3.23), when the parameters $s_1, \ldots, s_d, h_1, \ldots, h_d$ satisfy the conditions given in (3.16) as well as $s_j \neq 0$, $j \in \{1, \ldots, d\} \setminus \{i^*\}$.

### 3.3 Integration of B-spline Expressions

In this section, we provide expressions for particular integrals which occur in the context of the B-spline based distribution theory for hybrid censored data. We start with the convolution of B-splines (see Section 3.3.1) and proceed in Section 3.3.2 with the calculation of additional B-spline related integral expressions.

#### 3.3.1 B-spline Convolutions

First, we define the convolution of two functions $f$ and $g$ (cf. Folland, 1999, p. 239).
Definition 3.3.1 Let $f$ and $g$ be measurable functions on $\mathbb{R}$. The convolution of $f$ and $g$ is the function $f \ast g$ defined by

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$ 

If $f$ and $g$ are the density functions of independent random variables $X$ and $Y$, respectively, then $f \ast g$ is the density function of the sum $X + Y$ (cf. Billingsley, 1995, pp. 266–267).

The objective is to determine an expression for the $k$-fold convolution of $k$ different B-splines. The corresponding setup is as follows: For integers $d_1, \ldots, d_k \in \mathbb{N}$, and knot sequences $t_{d_1}^{(1)}, \ldots, t_{d_k}^{(k)}$ where $t_{d_j}^{(j)} = (t_{d_j}^{(j)}_0, \ldots, t_{d_j}^{(j)}_j) \in \mathbb{R}^{d_j+1}$, $1 \leq j \leq k$, we consider the B-splines

$$B_{d_1-1}^{(1)}(\cdot | t_{d_1}^{(1)}), \ldots, B_{d_k-1}^{(k)}(\cdot | t_{d_k}^{(k)})$$

with degrees $d_1 - 1, \ldots, d_k - 1$ respectively.

A formula for the convolution of two B-splines has been presented in Strøm (1994, Proposition 4) (see also Lemma C.2.2). The crucial step is the application of the formula given in Lemma 3.3.9, (iii), on truncated power functions, i.e., functions of the form $a(x - b)^j$, $k \in \mathbb{N}$, $a, b \in \mathbb{R}$. The following lemma extends Strom’s formula to the case of $k$ B-splines.

Lemma 3.3.2 Let $t_{d_j}^{(j)}$, $1 \leq j \leq k$, be knot sequences, where the respective knots are not all equal. Then, the $k$-fold convolution of the B-splines $B_{d_j-1}^{(j)}(\cdot | t_{d_j}^{(j)})$, $1 \leq j \leq k$, $k \in \mathbb{N}_{\geq 2}$, can be calculated as iterated divided differences, i.e.,

$$
\left( B_{d_1-1}(\cdot | t_{d_1}^{(1)}) \ast \cdots \ast B_{d_k-1}(\cdot | t_{d_k}^{(k)}) \right)(s) = \left[ t_{d_1}^{(1)} \right]_{s_1} \left[ t_{d_2}^{(2)} \right]_{s_2} \cdots \left[ t_{d_k}^{(k)} \right]_{s_k} \left( \sum_{j=1}^{k} x_j - s \right)^{d_{s_k} - 1} \frac{\prod_{j=1}^{k} d_j!}{(d_{s_k} - 1)!}, \quad s \in \mathbb{R}.
$$

Proof. The assumption follows by induction. A proof is given in the appendix, see Lemma C.2.2.

Figure 3.4 illustrates the plots of B-spline convolutions (solid lines) for $k \in \{2, 3\}$. We consider in the following particular properties of the iterated divided differences, which can be derived from the properties of the common divided differences given in Lemma 3.1.3.

Lemma 3.3.3 Let $t_{d_j}^{(j)} = (t_{d_j}^{(j)}_0, \ldots, t_{d_j}^{(j)}_j) \in \mathbb{R}^{d_j+1}$, $1 \leq j \leq k$, $d_1, \ldots, d_k \in \mathbb{N}$, be knot sequences, where the respective knots are not all equal.

(i) Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Then, the iterated divided difference

$$\left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_k}^{(k)} \right]_{x_k} g(x_k),$$

is linear in $g$, i.e.,

$$\left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_k}^{(k)} \right]_{x_k} \left( \alpha f(x_k) + \beta g(x_k) \right)$$

$$= \alpha \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_k}^{(k)} \right]_{x_k} f(x_k) + \beta \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_k}^{(k)} \right]_{x_k} g(x_k).$$
(ii) Let $\ell_1, \ldots, \ell_k \in \mathbb{N}$, with $\ell_i \leq d_i$, $1 \leq i \leq k$. Then,
\[
\left[ t_{d_1}^{(1)} \right] x_1 \left[ t_{d_2}^{(2)} \right] x_2 \cdots \left[ t_{d_k}^{(k)} \right] x_k \left( \sum_{j=1}^{k} x_j - s \right)^{\ell_k - 1} = 0, \quad s \in \mathbb{R}.
\]

(iii) Further, we have
\[
\left[ t_{d_1}^{(1)} \right] x_1 \left[ t_{d_2}^{(2)} \right] x_2 \cdots \left[ t_{d_k}^{(k)} \right] x_k \left( \sum_{j=1}^{k} x_j - s \right)^{d_k - 1} = 0
\]
for
\[
s < \sum_{j=1}^{k} \min \{ t_0^{(j)}, \ldots, t_{d_j}^{(j)} \} \quad \text{and} \quad s > \sum_{j=1}^{k} \max \{ t_0^{(j)}, \ldots, t_{d_j}^{(j)} \}.
\]

**Proof.** (i) The result can be immediately verified by way of induction and by taking into account the linearity property for common divided differences (see Lemma 3.1.3, (iv)).

(ii) The desired identity can be also shown by induction. We provide the proof for $k = 2$. Then, the inductive step can be obtained by analogous calculations. The application of the binomial theorem to
\[
\left[ t_{d_1}^{(1)} \right] x_1 \left[ t_{d_2}^{(2)} \right] x_2 \left( x_1 + x_2 - s \right)^{\ell_1 + \ell_2 - 1}
\]
for $k = 2$, gives us together with the linearity of the iterated divided differences (see Lemma 3.3.3, (i))
\[
\left[ t_{d_1}^{(1)} \right] x_1 \left[ t_{d_2}^{(2)} \right] x_2 \sum_{j=0}^{\ell_1 + \ell_2 - 1} \binom{\ell_1 + \ell_2 - 1}{j} x_1^j (x_2 - s)^{\ell_1 + \ell_2 - 1 - j}
\]
3.3 Integration of B-spline Expressions

\[
= \sum_{j=0}^{\ell_1 + \ell_2 - 1} \left( \ell_1 + \ell_2 - 1 \right) \left[ \begin{array}{c} t_d^{(1)}_{r_1} x_1 \\ t_d^{(2)}_{r_2} (x_2 - s)^{\ell_1 + \ell_2 - 1 - j} = 0. \\
= 0, j \in \{0, \ldots, \ell_1 - 1\} \\
= 0, j \in \{\ell_1, \ldots, \ell_1 + \ell_2 - 1\}
\end{array} \right]
\]

The divided differences w.r.t. polynomials with degrees less than \(d_1\) and \(d_2\), respectively, vanish due to Lemma 3.1.3, (iii).

(iii) First, without loss of generality, let \(t_0^{(j)} \leq \cdots \leq t_d^{(j)}\) for \(1 \leq j \leq k\). The assertion for \(s > \sum_{j=1}^{k} t_d^{(j)}\) follows directly from the fact that

\[
\left( \sum_{j=1}^{k} x_j - s \right)^{d} = 0 \quad \text{for} \quad s > \sum_{j=1}^{k} t_d^{(j)} \quad \text{and each} \quad d \in \mathbb{N}.
\]

The second assumption follows with Lemma 3.3.3, (ii), and by taking into account that

\[
\left( \sum_{j=1}^{k} x_j - s \right)^{d_{\star k} - 1} = \left( \sum_{j=1}^{k} x_j - s \right)^{d_{\star k} - 1}, \quad s < \sum_{j=0}^{k} t_d^{(j)}. \quad \square
\]

Note that the result given in Lemma 3.3.3, (ii) can be most likely extended to arbitrary multivariate polynomials \(p : \mathbb{R}^k \to \mathbb{R}\) of degree less than \(d_{\star k}\), i.e.,

\[
p(x_k) = \sum_{i=1}^{n} c_i x_1^{d_{i,1}} \cdots x_k^{d_{i,k}},
\]

with prefixed \(c_1, \ldots, c_n \in \mathbb{R}\) and prefixed integers \(n \in \mathbb{N}, d_{i,1}, \ldots, d_{i,k} \in \mathbb{N}_0\), so that \(d_{i,\star k} < d_{\star k}, 1 \leq i \leq k\).

The expression for the iterated divided difference can be calculated for pairwise distinct knots (w.r.t. each knot sequence) by a \(k\)-times application of Lemma (3.1.3), (i). The corresponding implementation on a computer can be easily realized by using the recurrence relations given in Lemma (3.1.3), (ii). The evaluation for the setting of multiple knots becomes much more involved. Explicit formulas for \(k = 2\), which is also referred to as two-dimensional divided differences, have been presented in Bărbosu (2008), Pop and Bărbosu (2009) and Roozegar and Jafari (2013).

3.3.2 Further Integration Formulas

Formulas for calculating integrals of the form

\[
\int_a^b B_{d-1}(s|t_0, \ldots, t_d)ds, \quad a < b,
\]

have been established, for instance, in de Boor et al. (1976, Section 2) and Schumaker (2007, p. 201). These formulas, however, require an extension of the knot sequence \(t_0, \ldots, t_d\) and an evaluation of several B-splines of degree \(d\). Let \(t_0 \leq \cdots \leq t_d \leq t_{d+1} \leq \cdots \leq t_{2d}\) be a
knot sequence, where the knots \( t_0, \ldots, t_d \) are not all equal. Then, according to Schumaker (1988) the integral of the B-spline \( B_{d-1}(\cdot|t_0, \ldots, t_d) \) can be calculated by

\[
\int_{t_0}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds = \begin{cases} 
0, & t < t_0, \\
\sum_{i=0}^{d-1} N_d(t|i, \ldots, t_{d+i+1}), & t \in [t_0, t_d), \\
1, & t \geq t_d,
\end{cases}
\tag{3.25}
\]

where \( N_d(\cdot|t_i, \ldots, t_{d+i+1}) \) denotes the normalized B-spline of degree \( d \) w.r.t. the knots \( t_i, \ldots, t_{d+i} \) (cf. (3.5)), for \( 1 \leq i \leq d - 1 \). But, we would like to evaluate the integral in (3.24), without extending the knot sequence \( t_0, \ldots, t_d \). As far as we know, there has been no expression presented so far, which omits the evaluation of additional B-splines w.r.t. an extended knot sequence.

**Lemma 3.3.4** Let \( t_0, \ldots, t_d \in \mathbb{R}, \ d \in \mathbb{N}, \) be a knot sequence, where not all knots are equal. Then, the following identity holds,

\[
\int_{-\infty}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds = \begin{cases} 
0, & t < \min_{0 \leq j \leq d} t_j, \\
1 - [t_0, \ldots, t_d; (-t)_+^d], & t \in \left[ \min_{0 \leq j \leq d} t_j, \max_{0 \leq j \leq d} t_j \right), \\
1, & t \geq \max_{0 \leq j \leq d} t_j,
\end{cases}
\]

**Proof.** Without loss of generality, we assume \( t_0 \leq \cdots \leq t_d \) (cf. Lemma 3.1.3, (v)). According to de Boor (2001, p. 88), we have

\[
\int_{-\infty}^{\infty} B_{d-1}(s|t_0, \ldots, t_d)ds = \int_{t_0}^{t_d} B_{d-1}(s|t_0, \ldots, t_d)ds = 1. \tag{3.26}
\]

Further, we denote by \( g^{(-d)} \) the \( d \)-th antiderivative of a function \( g \). For \( g(x) = (x - t)_+^0 = \mathbb{I}_{(t,\infty)}(x) \) continuous on \( \mathbb{R} \setminus \{t\} \), and a fixed \( t \in [t_0, t_d) \), we have \( g^{(-d)}(x) = \frac{1}{d!} (x - t)_+^d \). It can be easily verified that \( g^{(-d)} \in C^{(d-1)}(\mathbb{R}) \), and, that \( g^{(-1)} = (-t)_+ = \mathbb{I}_{(t,\infty)}(\cdot) (-t) \) is Lipschitz continuous on \([t_0, t_d] \). Hence, \( g^{(-1)} \) is absolutely continuous on \([t_0, t_d] \), for \( t \in \mathbb{R} \) fixed (see Heuser, 1988, p. 117). Then, we find with equations (3.26) and (3.10), with Peano’s formula (see (iii) in Lemma 3.1.9), and with the linearity property of the divided differences (see result (iv) from Lemma 3.1.3), for \( t \in [t_0, t_d) \), that

\[
\int_{-\infty}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds = \int_{t_0}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds = 1 - \int_{t}^{t_d} B_{d-1}(s|t_0, \ldots, t_d)ds \\
= 1 - \int_{t_0}^{t_d} \mathbb{I}_{(t,\infty)}(s) B_{d-1}(s|t_0, \ldots, t_d)ds \\
= 1 - \int_{t_0}^{t_d} (s - t)_+^0 B_{d-1}(s|t_0, \ldots, t_d)ds \\
= 1 - \frac{d!}{d!} \left[ t_0, \ldots, t_d; \frac{(-t)_+^d}{d!} \right] = 1 - [t_0, \ldots, t_d; (-t)_+^d].
\]

The assumption for \( t < t_0 \) and \( t \geq t_d \) follows directly from (3.26). \qed
The consideration of \( g(x) = (x-t)^{0+} \) in the above proof, was motivated by the calculations performed by Marichal and Kojadinovic (2008, see proof to Theorem 4). They established a result similar to Lemma 3.3.4, in determining the cumulative distribution function of the linear combination of lattice polynomials by the means of the Hermite-Genocchi formula. Curry and Schoenberg (1947) referred to \( \int_{-\infty}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds \) as the spline distribution function. If the expression given in (3.25) is favored, in order to evaluate (3.24), then, the setting \( t_{d+1} = \cdots = t_{2d} \), for a knot \( t_{d+1} \) with \( t_{d+1} > t_d \), might be suitable.

**Remark 3.3.5** Let \( t_0, \ldots, t_d \) be a knot sequence, where not all knots coincide. Then, taking into account that \([t_0, \ldots, t_d; (-t)^d] = 1 \) and that \([t_0, \ldots, t_d; (-t)^d] = 0 \) for \( t \geq \max\{t_0, \ldots, t_d\} \), we obtain the following compact representation for \( \int_{-\infty}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds \), i.e.,

\[
\int_{-\infty}^{t} B_{d-1}(s|t_0, \ldots, t_d)ds = 1 - [t_0, \ldots, t_d; (-t)^d], \quad t \in \mathbb{R}.
\]

The identity \([t_0, \ldots, t_d; (-t)^d] = 1 \) used in Remark 3.3.5 follows directly with the Hermite-Genocchi formula (see, e.g., Strøm, 1994, p. 6). The following lemma presents two integral identities, which are useful in the context of the exponential distribution.

**Lemma 3.3.6**  
(i) Let \( \gamma_d, \ldots, \gamma_1 > 0 \) and \( \gamma_{d+1} \geq 0 \) be pairwise distinct. Then, for \( \vartheta, T > 0 \) and \( t \geq 0 \), we have

\[
\int_{0}^{dt} B_{d-1}(s|\gamma_{d+1}T, \ldots, \gamma_1T) e^{-s/\vartheta} ds = \sum_{j=0}^{d} \frac{(-1)^{j}d!d!}{\prod_{i=0}^{j}} e^{-\min\{\gamma_{d-j+1}T, \vartheta dt\}/\vartheta} \sum_{k=1}^{d-1} \frac{\vartheta^{k}d!(\vartheta)}{(d-k)!} \left( e^{-dt/\vartheta} (\gamma_{d-j+1}T - dt)^{d-k} - e^{-dt/\vartheta} (\gamma_{d-j+1}T - (dt)^{d-k} - e^{-dt/\vartheta} \gamma_{d-j+1}T - (dt)^{d-k} \right) \bigg|_{t=0}^{t}.
\]

(ii) Let \( \gamma_d, \ldots, \gamma_1 > 0 \) and \( \gamma_{d+1} \geq 0 \) be knots, which are not all equal. For \( \vartheta, T > 0 \) and \( d, k \in \mathbb{N} \), the following identity holds:

\[
\int_{\mathbb{R}} x^k B_{d-1}(x|\gamma_{d+1}T, \ldots, \gamma_1T) e^{-x/\vartheta} dx = d! k! (-\vartheta)^d \sum_{i=0}^{k} \sum_{i=2}^{k-i_1} \sum_{i=0}^{k-i_2} \cdots \sum_{i=0}^{k-i_{d-1}} \gamma_{d-i_{d}}^{k-i_{d}} e^{(k-i_{d})/\vartheta}.
\]

**Proof.**  
(i) The assumption follows by induction. We provide the proof for \( d = 1 \). The evaluation of the left-hand side gives

\[
\int_{0}^{t} B_{0}(s|\gamma_2T, \gamma_1T) e^{-s/\vartheta} ds = \int_{0}^{t} \frac{1}{T(\gamma_1 - \gamma_2)} e^{-s/\vartheta} ds = \frac{1}{T(\gamma_1 - \gamma_2)} \left( e^{-s/\vartheta} \right)_{s=\min\{t, \gamma_1T\}}^{s=\min\{t, \gamma_2T\}} = \frac{1}{T(\gamma_1 - \gamma_2)} \left( e^{-\min\{t, \gamma_1T\}/\vartheta} + e^{-\min\{t, \gamma_2T\}/\vartheta} \right).
\]

This is the assertion for \( d = 1 \). In order to perform the inductive step, one might integrate the left-hand side by parts using the representation from Lemma 3.1.3, (i).
for the divided differences. Then, after applying the formula for distinct knots, given in Lemma 3.1.3, (ii), the two resulting terms can be expressed as B-splines. Afterwards, we apply twice the induction hypothesis. Then, the resulting expressions can be combined by means of the first formula given in Lemma 3.1.3, (ii).

(ii) Let \( \gamma_d, \ldots, \gamma_1 > 0 \) and \( \gamma_{d+1} \geq 0 \) be knots, which are not all equal. For \( c \neq 0 \) and \( d, k \in \mathbb{N} \), we determine first the \( d \)th antiderivative of \( f(x) = x^k e^{cx} \). According to Lemma C.2.4, we get

\[
f^{(-d)}(x) = e^{cx} k! \sum_{i_1=0}^{k} \frac{(-1)^{i_1}}{c^{i_1+1}} \sum_{i_2=0}^{k-i_1} \frac{(-1)^{i_2}}{c^{i_2+1}} \cdots \sum_{i_d=0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c^{i_d+1}} x^{k-i_{d+1}} (k-i_{d+1})!, \quad x \in \mathbb{R}.
\]

Then, the application of Peano's formula (see Lemma 3.1.9, (iii)) with \( g = f^{(-d)} \) and \( g^{(d)} = f \) gives

\[
\int_{\mathbb{R}} x^k B_{d-1}(x|t_0, \ldots, t_d) e^{cx} dx = d! [t_0, \ldots, t_d] x^k e^{cx} k! \sum_{i_1=0}^{k} \frac{(-1)^{i_1}}{c^{i_1+1}} \sum_{i_2=0}^{k-i_1} \frac{(-1)^{i_2}}{c^{i_2+1}} \cdots \sum_{i_d=0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c^{i_d+1}} x^{k-i_{d+1}} (k-i_{d+1})!.
\]

For \( t_j = \gamma_{d-j+1} T, 0 \leq j \leq d, T > 0 \) and \( c = -\frac{1}{\theta} \), this yields the assumption. \( \Box \)

Note that the formula given in Lemma 3.3.6, (i), has recently been presented in Cramer et al. (2016, Section 4.1).
Chapter 4
Type-I Hybrid Censoring

This chapter addresses the model of Type-I hybrid censoring in detail. Since the censoring procedure according to the Type-I sequential hybrid censoring scheme has been already presented in Section 1.1.1, we proceed directly with the derivation of the basic distributional results (see Section 4.1). Hereby, we consider the general setting of an underlying absolutely continuous cumulative distribution function $F$. In Section 4.2, we consider the Type-I sequential hybrid censoring scheme for a two-parameter exponential distribution. Following, Section 4.3 addresses some additional aspects within the context of Type-I hybrid censoring for an exponential setup. Then, in Section 4.4, we discuss the Type-I hybrid censoring scheme for the two-parameter uniform distribution.

4.1 Basic Distributional Results

The analysis of the underlying Type-I sequential hybrid censoring scheme, proceeds as specified in the volume approach (see Procedure 1.4.4). Steps 1 and 2 involve the specification of the sample of ordered random variables as well as the specification of the sample of the respective hybrid censored order statistics. For that, we refer to Section 1.1.1 and Example 1.4.2, (i). Further, we recall the respective system of counter settings and its corresponding family of sets of valid integers, i.e. (see Example 1.4.2, (i)),

$$(\mathcal{S}_D(I^*), \mathcal{I}(\mathcal{S}_D(I^*))) = \left(\{D = \cdot\}, \{\{1, \ldots, m\}\} \right).$$

This completes step 3 of Procedure 1.4.4. Let us assume the sequential model presented in Model 2.3.7. Then, we proceed as in Cramer and Balakrishnan (2013) and calculate the distribution of $D = d$, $d \in \{0, \ldots, m\}$ (i.e., the distribution of $\mathcal{D}(d)$, $d \in \mathcal{I}(\mathcal{D})$ and $\mathcal{D} = [D = \cdot] \in \mathcal{S}_D(I)$, cf. step 4).

Lemma 4.1.1 Let $F$ be an absolutely continuous cumulative distribution function (cf. Model 2.3.7). Then, for $d = 0$, we have

$$P(D = 0) = (1 - F(T))^{\gamma^*_1}.$$  

For $d \in \{1, \ldots, m - 1\}$ and $f(T) > 0$, we get

$$P(D = d) = \frac{1 - F(T)}{\gamma^*_d f(T)} f_{s,d+1}(T).$$

Finally, we obtain for $d = m$,

$$P(D = m) = F_{s,m}(T).$$
Proof. First, we have for $D = 0$

$$P(D = 0) = P(X^*_1 > T) = 1 - F_{*,1}(T) = (1 - F(T))^{\gamma^*_1},$$

since $X^*_1$ is distributed as a minimum of IID random variables $X_1, \ldots, X_n$ which are distributed according to $1 - (1 - F)^{\gamma^*_1/n}$ (cf. Definition 2.3.1).

For $f(T) > 0$, $T \in (0, \infty)$, we obtain with Lemma 2.3.5, (iii) and with the representation of the density function $f_{*,r}$, $1 \leq r \leq n$ (see Lemma 2.3.4, (ii)),

$$G_{d+1,0}^{d+1,0}[1 - F(T)_{\gamma^*_1 + 1, \ldots, \gamma^*_d + 1} = (1 - F(T)) G_{d+1,0}^{d+1,0}[1 - F(T)_{\gamma^*_1, \ldots, \gamma^*_d + 1}$$

$$= (1 - F(T)) \frac{f_{*,d+1}(T)}{f(T) \prod_{j=1}^{d+1} \gamma^*_j}.$$

Then, Lemma 2.3.6 leads us for $d \in \{1, \ldots, m - 1\}$ to

$$P(D = d) = \left( \prod_{j=1}^{d} \gamma^*_j \right) (1 - F(T)) \frac{f_{*,d+1}(T)}{f(T) \prod_{j=1}^{d+1} \gamma^*_j} = \frac{1 - F(T)}{\gamma^*_d \prod_{j=1}^{d+1} \gamma^*_j} f_{*,d+1}(T).$$

Alternatively, due to $P(D = d) = F_{*,d}(T) - F_{*,d+1}(T)$, we can also use Corollary 2.6 from Cramer and Kamps (2003), in order to obtain the above expression for $P(D = d)$, $d \in \{1, \ldots, m - 1\}$.

For $D = m$, we find

$$P(D = m) = P(X^*_m \leq T) = F_{*,m}(T),$$

which concludes the proof. 

For $x_d \in \mathbb{R}^d$, we consider now the joint probability of $X^*_j \leq x_j$, $1 \leq j \leq d$ and $\mathfrak{D}(d)$, $d \in \mathcal{I}(\mathfrak{D})$, with $\mathfrak{D} = [D = 1]$ (see step 5). By recalling the definition of Type-I sequential hybrid censored order statistics (see (1.9) or Example 1.4.2, (iii)),

$$X^*_j := \min \{X^*_j, T\}, \quad 1 \leq j \leq m,$$

we obtain for $d \in \{1, \ldots, m - 1\}$,

$$P(X^*_j \leq x_j, 1 \leq j \leq d, D = d)$$

$$= P(X^*_j \leq x_j, 1 \leq j \leq d, D = d)$$

$$= P(\min \{X^*_j, T\} \leq x_j, 1 \leq j \leq d, X^*_d \leq T < X^*_d + 1)$$

$$= P(X^*_j \leq x_j, 1 \leq j \leq d - 1, X^*_d \leq \min \{x_d, T\}, X^*_d + 1 > T) \quad (4.1)$$

$$= P(X^*_j \leq x_j, 1 \leq j \leq d - 1, X^*_d \leq \min \{x_d, T\})$$

$$- P(X^*_j \leq x_j, 1 \leq j \leq d - 1, X^*_d \leq \min \{x_d, T\}, X^*_d + 1 \leq T)$$

$$= F_{*,1,2,d}(x_d, \min \{x_d, T\}) - F_{*,1,2,d+1}(x_d, \min \{x_d, T\}, T),$$

where the function $F_{*,1,2,d}$, $1 \leq d \leq m$, denotes the joint cumulative distribution function of the first $d$ sequential order statistics $X^*_1, \ldots, X^*_d$. We further state that $(x_d, y) = (x_1, \ldots, x_d, y) \in \mathbb{R}^{d+1}$, $y \in \mathbb{R}$. 

58 Chapter 4 Type-I Hybrid Censoring
For \( d = m \), we obtain for the joint probability of the event \( \{ X_j^r \leq x_j, 1 \leq j \leq m, D = m \} \),

\[
P(X_j^r \leq x_j, 1 \leq j \leq m, D = m) = P(\min\{X_j^r, T\} \leq x_j, 1 \leq j \leq m, X_m^* \leq T)
\]

\[
= P(X_j^r \leq x_j, 1 \leq j \leq m - 1, X_m^* \leq \min\{x_m, T\})
\]

\[
= F_{s,1...m}(x_{m-1}, \min\{x_m, T\}).
\]  \hspace{1cm} (4.2)

In the following, we denote by \( F_{s,1...d}(x_d | X_{d+1}^s = T) \) the joint cumulative distribution function of the sequential order statistics \( X_1^r, \ldots, X_d^r \), given \( X_{d+1}^s = T \), i.e.,

\[
F_{s,1...d}(x_d | X_{d+1}^s = T) = \int_{x_1}^{x_d} \cdots \int_{y_{d-1}}^{x_d} \frac{f_{s,1...d+1}(y_d, T)}{f_{s,d+1}(T)} dy_d.
\]  \hspace{1cm} (4.3)

We present the joint distribution of \( X_1^r, \ldots, X_d^r \) conditional on \( D = d \) (see step (5)).

Lemma 4.1.2 Let \( f(T) > 0 \). Then, for \( d \in \{1, \ldots, m - 1\} \), the conditional joint cumulative distribution function of \( X_1^r, \ldots, X_d^r \), is given by

\[
F_{X_j^r, 1 \leq j \leq d}^{X_d^r}(x_d) = F_{s,1...d}(x_d | X_{d+1}^s = T), \quad x_d \in \mathbb{R}^d.
\]

The corresponding conditional joint density function can be expressed as

\[
f_{X_j^r, 1 \leq j \leq d}^{X_d^r}(x_d) = 1_{\Sigma_{F,T}}(x_d) \frac{f_{s,1...d+1}(x_d, T)}{f_{s,d+1}(T)}.
\]

For \( d = m \), we have

\[
F_{X_j^r, 1 \leq j \leq m}^{X_m}(x_m) = \frac{F_{s,1...m}(x_{m-1}, \min\{x_m, T\})}{F_{s,m}(T)}, \quad x_m \in \mathbb{R}^m,
\]

and

\[
f_{X_j^r, 1 \leq j \leq m}^{X_m}(x_m) = 1_{\Sigma_{F,T}}(x_m) \frac{f_{s,1...m}(x_m)}{F_{s,m}(T)}.
\]

Proof. For \( d \in \{1, \ldots, m - 1\} \), we consider the auxiliary result

\[
\int_T^\infty f(x)(1 - F(x))^{\gamma_{d+1}-1} dx = -\int_0^{1-F(T)} y^{\gamma_{d+1}-1} dy = -\frac{y_0^{\gamma_{d+1}}}{\gamma_{d+1} - 1 - F(T)} = \frac{(1 - F(T))^{\gamma_{d+1}}}{\gamma_{d+1} - 1 - F(T)}.
\]

Let \( x_d \in \mathbb{R}^d \). Then, by using (4.1), Lemma (2.3.4), (i) as well as Lemma 4.1.1, we find that

\[
F_{X_j^r, 1 \leq j \leq d}^{X_d^r}(x_d)
\]

\[
= P(X_j^r \leq x_j, 1 \leq j \leq d, D = d)
\]

\[
= \frac{P(X_j^r \leq x_j, 1 \leq j \leq d, X_{d+1}^s > T)}{P(D = d)}
\]

\[
= \frac{\gamma_{d+1} f(T)}{f_{s,d+1}(T)(1 - F(T))} \int_{x_1}^{x_d} \cdots \int_{y_{d-1}}^{x_d} \int_T^\infty f_{s,1...d+1}(y_{d+1}) dy_{d+1}
\]

\[
\int_{x_1}^{x_d} \cdots \int_{y_{d-1}}^{x_d} \int_T^\infty f_{s,1...d+1}(y_{d+1}) dy_{d+1}
\]

\[
= \frac{\gamma_{d+1} f(T)}{f_{s,d+1}(T)(1 - F(T))} \int_{x_1}^{x_d} \cdots \int_{y_{d-1}}^{x_d} \int_T^\infty f_{s,1...d+1}(y_{d+1}) dy_{d+1}
\]
Then,
\[ \frac{\partial^d}{\partial x_{d} \ldots \partial x_{d}} F_{s,1 \ldots d+1}(x_d | X_{d+1}^*) = T) = \begin{cases} \frac{f_{s,1 \ldots d+1}(x_{d}, T)}{f_{s,d+1}(T)}, & x_d \in \Sigma_{F,T}, \\ 0, & x_d \notin \Sigma_{F,T}. \end{cases} \]
so that \( \text{supp}(f_X^*, 1 \leq j \leq |D = d|) = \Sigma_{F,T}^d. \)

For \( d = m \), we obtain with Lemma (4.1.1), equation (4.2) and \( x_m \in \mathbb{R}^m \),
\[ P(X^*_j, 1 \leq j \leq m | D = m)(x_m) = \frac{P(X^*_j \leq x_j, 1 \leq j \leq m, D = m)}{P(D = m)} = \frac{F_{s,1 \ldots m}(x_{m-1}, \min\{x_m, T\})}{F_{s,m}(T)}. \]
Then,
\[ \frac{\partial^m}{\partial x_{1} \ldots \partial x_{m}} \frac{F_{s,1 \ldots m}(x_{m-1}, \min\{x_m, T\})}{F_{s,m}(T)} = \begin{cases} \frac{f_{s,1 \ldots m}(x_m)}{F_{s,m}(T)}, & x_m \in \Sigma_{F,T}, \\ 0, & x_m \notin \Sigma_{F,T}. \end{cases} \]
This finishes the proof. \( \square \)

### 4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions

By assuming the sequential exponential model (see Model 2.3.8), we consider now the two-parameter exponential distribution. First, we determine the MLEs and derive some essential distributional results (see steps (6) – (8)). Then, we establish expressions for the exact distribution of the MLEs. In the following, we denote by \( Z_1^*, \ldots, Z_d^* \), \( d \in \{1, \ldots, m\} \) the Type-I sequential hybrid censored order statistics based on \( P_F = \text{Exp}(\mu, \theta) \).

**Notation 4.2.1** For \( D \in \{1, \ldots, m\} \), we define the total time on test statistic \( S_D \) and the modified total time on test statistic \( V_D \), by
\[ S_D = \sum_{j=1}^{D} (\gamma_j^* - \gamma_{j+1}^*)(Z_j^* - \mu) + \gamma_{D+1}^*(T - \mu), \quad (4.4) \]
and

\[ V_D = \sum_{j=2}^{D} (\gamma_j^* - \gamma_{j+1}^*)(Z_j^* - Z_1^*) + \gamma_{D+1}^*(T - Z_1^*), \tag{4.5} \]

respectively. It can be quickly verified that \( S_D = V_D + \gamma_1^*(Z_1^* - \mu) \), \( D \in \{1, \ldots, m\} \). The above representations of \( S_D \) and \( V_D \) are favorable w.r.t. the application of the expected value approach. The following spacings based representations facilitate the application of the volume approach, i.e.,

\[ S_D = \sum_{j=1}^{D} \left( 1 - \frac{\gamma_{D+1}^*}{\gamma_j^*} \right) \gamma_j^* (Z_j^* - Z_{j-1}^*) + \gamma_{D+1}^*(T - \mu) \tag{4.6} \]

\[ = \sum_{j=1}^{D} \gamma_j^* (Z_j^* - Z_{j-1}^*) + \gamma_{D+1}^*(T - Z_D^*), \quad Z_0^* := \mu, \]

and

\[ V_D = \sum_{j=2}^{D} \left( 1 - \frac{\gamma_{D+1}^*}{\gamma_j^*} \right) \gamma_j^* (Z_j^* - Z_{j-1}^*) + \gamma_{D+1}^*(T - Z_1^*) \tag{4.7} \]

\[ = \sum_{j=2}^{D} \gamma_j^* (Z_j^* - Z_{j-1}^*) + \gamma_{D+1}^*(T - Z_D^*) = S_D - \gamma_1^*(Z_1^* - \mu). \tag{4.8} \]

For the particular setting of progressively Type-II censored order statistics, representations (4.4) and (4.6) of \( S_D \) can be found in Childs et al. (2008) and Cramer and Balakrishnan (2013). For the present setting of sequential order statistics, we refer to the recent work of Burkschat et al. (2016). The modified total time on test statistic \( V_D \) has been addressed in Childs et al. (2012) and Cramer and Balakrishnan (2013), for the setting of ordinary order statistics and for the setting of progressively Type-II censored order statistics, respectively.

For an underlying exponential distribution, the truncated cone introduced in (1.4) is given by, \( d \in \mathbb{N} \),

\[ \Sigma_{d, T}^d = \{ z_d \in \mathbb{R}^d | \mu \leq z_1 \leq \cdots \leq z_d \leq T \}. \tag{4.9} \]

According to Lemma 4.1.2, we obtain the following conditional joint density functions for the Type-I sequential hybrid censored order statistics \( Z_1^*, \ldots, Z_d^* \), \( d \in \{1, \ldots, m\} \), from the exponential distribution.

**Lemma 4.2.2** Let \( z_0 := \mu \) and \( \Sigma_{d, T}^d \) as in (4.9), \( d \in \{1, \ldots, m\} \). Then, for \( d \in \{1, \ldots, m - 1\} \), the conditional joint density function is given by

\[ f_{Z_1^*, \ldots, Z_d^*} (z_d) I_{\Sigma_{d, T}^d} (z_d) \prod_{j=1}^{d} \frac{\gamma_j^*}{\frac{\gamma_j^*}{\gamma_j^*}} \exp \left\{ - \frac{1}{\partial} \left[ \sum_{j=1}^{d} \left( 1 - \frac{\gamma_{d+1}^*}{\gamma_j^*} \right) \gamma_j^* (z_j - z_{j-1}) + \frac{\gamma_{d+1}^*}{\gamma_j^*} (T - \mu) \right] \right\}. \]

For \( d = m \), we have

\[ f_{Z_m^*} (z_m) I_{\Sigma_{m, T}^m} (z_m) \prod_{j=1}^{m} \frac{\gamma_j^*}{\partial m} \exp \left\{ - \frac{1}{\partial} \sum_{j=1}^{m} \gamma_j^* (z_j - z_{j-1}) \right\}. \]
Proof. The assertion follows with Lemma 4.1.2, Lemma 2.3.4, (i), and by utilizing the spacings based representation of the total time on test statistic $S_d$ (see Notation 4.2.1). □

Now, we are able to determine the MLEs.

Lemma 4.2.3 The likelihood function $L(|z_d)$ for $\mu$ and $\vartheta$ is given by

$$L(\mu, \vartheta|z_d) = \begin{cases} \frac{1}{\gamma_{d+1} f(T)} f_{s,1:d+1}(z_d, T), & d \in \{1, \ldots, m - 1\}, \\
1, & d = m. \end{cases}$$

For a known location parameter $\mu$, the MLE $\hat{\vartheta}$ does not exist when $D = 0$. Hence, we have

$$\hat{\vartheta} = \begin{cases} \frac{1}{D} \sum_{j=1}^{D} \left( 1 - \frac{\gamma_{d+1}}{\gamma_j} \right) \gamma_j^*(Z_j^* - Z_{j-1}^*) + \gamma_{d+1}(T - \mu), & D \in \{1, \ldots, m - 1\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_j^*(Z_j^* - Z_{j-1}^*), & D = m. \end{cases}$$

We note that $\hat{\vartheta} = \frac{S_d}{D}$, $D \in \{1, \ldots, m\}$, holds.

If the parameter $\mu$ is unknown, then, the MLEs $\hat{\mu}$ and $\hat{\vartheta}$ are given by

$$\hat{\mu} = Z_1^*$$

and

$$\hat{\vartheta} = \begin{cases} \frac{1}{D} \sum_{j=2}^{D} \gamma_j^*(Z_j^* - Z_{j-1}^*) + \gamma_{d+1}(T - Z_D), & D \in \{1, \ldots, m - 1\}, \\
\frac{1}{m} \sum_{j=2}^{m} \gamma_j^*(Z_j^* - Z_{j-1}^*), & D = m. \end{cases}$$

Notice that $\hat{\vartheta} = \frac{V_d}{D}$, $D \in \{1, \ldots, m\}$.

Proof. For $d \in \{1, \ldots, m - 1\}$, we determine in a first step the joint density function of $Z_1^*, \ldots, Z_d^*$ and by utilizing the $d$-dimensional Lebesgue measure $\lambda^d$. According to Lemma 4.1.1, Lemma 4.1.2 and Lemma 4.2.2, we have for a realization $z_d \in \Sigma_d^{F,T}$,

$$f_{Z_1^*,1 \leq j \leq d, D=d}(z_d) = f_{Z_d^*,1 \leq j \leq d, D=d}(z_d) P(D = d)$$

$$= \frac{f_{s,1:d+1}(z_d, T)}{f_{s,d+1}(T)} \frac{1 - F(T)}{\gamma_{d+1} f(T)} f_{s,d+1}(z_d, T)$$

Then, the likelihood function $L(|z_d)$ is given by

$$L(\mu, \vartheta|z_d) = \frac{1 - F(T)}{\gamma_{d+1} f(T)} f_{s,1:d+1}(z_d, T) = \frac{\vartheta}{\gamma_{d+1}} f_{s,1:d+1}(z_d, T)$$

$$= \prod_{j=1}^{d} \gamma_j^* \frac{1}{\vartheta^d} \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^{d} (\gamma_j^* - \gamma_{j+1}^*)(z_j - \mu) + \gamma_{d+1}(T - \mu) \right\}$$
where \( z_0 := \mu \) and \( z_d \in \Sigma^d_{F,T} \). Notice that \( L(\cdot|z_d) \) is maximized w.r.t. \( \mu \) when \( \mu = z_1 \). Thus, \( \hat{\mu} = Z^*_1 \) and

\[
L(\mu, \vartheta|z_d) \leq L(z_1, \vartheta|z_d) = \left( \prod_{j=1}^{d} \gamma_j^* \right) \frac{1}{\vartheta^d} e^{-s_d/\vartheta}, \quad \vartheta > 0, \tag{4.11}
\]

where \( v_d = \sum_{j=2}^{d} (\gamma_j^* - \gamma_{j+1}^*)(z_j - z_1) + \gamma_{d+1}^*(T - z_1) \). Now, for a fixed \( \mu \) with \( \mu \leq z_1 \) the log-likelihood function \( \ell(\mu, \cdot|z_d) \), corresponding to \( L(\mu, \cdot|z_d) \) (see (4.10)), is given by

\[
\ell(\mu, \vartheta|z_d) = \ln(L(\mu, \vartheta|z_d)) = \ln \left( \prod_{j=1}^{d} \gamma_j^* \right) - d \ln(\vartheta) - \frac{s_d}{\vartheta}, \quad \vartheta > 0 \text{ and } z_d \in \Sigma^d_{F,T}.
\]

For the derivative of \( \ell(\mu, \cdot|z_d) \) w.r.t. \( \vartheta > 0 \), we find

\[
\frac{\partial}{\partial \vartheta} \ell(\mu, \vartheta|z_d) = -\frac{d}{\vartheta} + \frac{s_d}{\vartheta^2} \begin{cases} > 0 \quad \Longleftrightarrow \frac{s_d}{d} > \end{cases} \vartheta.
\tag{4.12}
\]

Due to the monotonicity behaviour of \( \ell(\mu, \cdot|z_d) \), we find that the MLE for \( \vartheta \) with \( \mu \leq z_1 \) fixed, is given by \( \hat{\vartheta} = \frac{S^*\mu}{D} \), \( D \in \{1, \ldots, m-1\} \), with \( S_D \) as in Notation 4.2.1. This gives the assumption for \( \mu \) known.

Notice that \( S_D = V_D \), when \( \mu = Z^*_1 \) (cf. (4.8)). Then, according to (4.11) and (4.12), the MLE \( (\hat{\mu}, \hat{\vartheta}) \) of \( (\mu, \vartheta) \) is given by \( (Z^*_1, \frac{V_D}{D}) \).

We proceed analogously for \( d = m \). \[ \square \]

According to Lemma 4.2.3, we find that the distribution of \( \hat{\vartheta} \) for a known location parameter \( \mu \) is degenerated when \( D = 0 \).

**Remark 4.2.4**  
(i) Let the parameter \( \mu \) be known. Then, we observe that for the setting \( \gamma_1^* = \cdots = \gamma_D^* = \gamma_D^*+1, \ D \in \{1, \ldots, m-1\} \), the MLE of \( \vartheta \) is given by

\[
\hat{\vartheta} = \frac{\gamma_D^*+1(T - \mu)}{D}, \quad D \in \{1, \ldots, m-1\}.
\]

Hence, for this particular setting, the distribution of \( \hat{\vartheta} \) is degenerated and a density function does not exist. The same holds for an unknown location parameter. There, we have for \( \gamma_2^* = \cdots = \gamma_m^* \), with \( D \in \{1, \ldots, m-1\} \),

\[
\hat{\vartheta} = \frac{\gamma_D^*+1(T - Z_D^*)}{D}, \quad D \in \{1, \ldots, m-1\}.
\]

The particular situation where all \( \gamma^* \)'s are equal, i.e., \( \gamma_1^* = \cdots = \gamma_m^* \), occurs, for instance, in the context of record values, minimal repair times and \( k \)th record values.
(ii) For \( d = 1 \), the bivariate MLE \((\hat{\mu}, \hat{\vartheta})\) is given by (see Lemma 4.2.3)
\[
(\hat{\mu}, \hat{\vartheta}) = (Z_1^*, \gamma_2^*(T - Z_1^*)).
\]
The respective conditional joint distribution is degenerated, too. Thus, in situations where the distribution of the bivariate MLE \((\hat{\mu}, \hat{\vartheta})\) is considered, we assume \( d \geq 2 \).

(iii) If not otherwise explicitly specified, we assume for a known parameter \( \mu \) that \( \gamma_1^* \neq \gamma_2^* \).
For an unknown parameter \( \mu \), we impose the restriction \( \gamma_2^* \neq \gamma_3^* \).

(iv) Let \( \mu \) be unknown and \( m = 1 \). Then, it can be quickly verified that the MLE \((\hat{\mu}, \hat{\vartheta})\) does not exist. Hence, we assume for an unknown \( \mu \) that \( m \geq 2 \).

(v) With Lemma 4.2.3 and the Neyman factorization criterion (see, e.g., Witting, 1985, Theorem 3.19 b)), we find that \((\hat{\vartheta}, D)\) is a sufficient statistic for \( \vartheta \), and that \((\hat{\mu}, \hat{\vartheta}, D)\) is a sufficient statistic for \((\mu, \vartheta)\).

For a brief description of a minimal repair procedure, the record as well as the \( k \)th record model (addressed in the end of Remark 4.2.4, (i)), we refer to Remark 2.3.3, (ii) and (iii). Note that MirMostafaei et al. (2017) conducted Bayesian prediction for minimal repair times w.r.t. a Type-I hybrid censored sequential system from the Rayleigh distribution. Further, Burkhschat et al. (2016) considered the record model as well as the setting of minimal repair times in the context of Type-I censored sequential \( k \)-out-of-\( n \) systems. We address the case \( \gamma_1^* = \cdots = \gamma_m^* \) later on in more detail (see pp. 80–82).

Note that the herein addressed Type-I sequential hybrid censoring scheme corresponds to the model of Type-I censored sequential \( k \)-out-of-\( n \) systems, addressed in Burkhschat et al. (2016). The underlying distribution theory is identical for the proportional hazard rate assumption. However, the model interpretation is slightly different. For further reading in that matter, we refer to the respective literature.

According to step (8), we define the normalized spacings based on Type-I sequential hybrid censored variables by
\[
W_j^* = \gamma_j^* (Z_j^* - Z_{j-1}^*), \quad 1 \leq j \leq m, \quad Z_0^* := \mu. \tag{4.13}
\]
Note that
\[
Z_j^* = \mu + \sum_{i=1}^{j} \frac{W_i^*}{\gamma_i^*}, \quad 1 \leq j \leq m. \tag{4.14}
\]

Further, we consider the linear transformation \( \Phi_W : \Sigma_{F,T}^d \to \mathcal{W}_d(T - \mu | \gamma_d^*) \), where
\[
\Phi_W(z_d) = (\gamma_1^*(z_1 - z_0), \ldots, \gamma_d^*(z_d - z_{d+1})) \quad \text{with} \quad |\text{det}(D\Phi_W)| = \prod_{j=1}^{d} \gamma_j^*. \tag{4.15}
\]

By utilizing (4.14), we see that the transformed set \( \mathcal{W}_d(T - \mu | \gamma_d^*) := \Phi_W(\Sigma_{F,T}^d) \), with \( \Sigma_{F,T}^d \) as in (4.9), is given by
\[
\mathcal{W}_d(T - \mu | \gamma_d^*) = \left\{ w_d \in \mathbb{R}^d \mid w_j \geq 0, 1 \leq j \leq d, \sum_{j=1}^{d} \frac{w_j}{\gamma_j^*} \leq T - \mu \right\}, \quad 1 \leq d \leq m, \tag{4.16}
\]
where \( \gamma_d^* = (\gamma_1^*, \ldots, \gamma_d^*) \). Together with Lemma 4.2.2, we obtain the following representation for the joint density function of the spacings \( W_1^*, \ldots, W_d^* \), conditional on \( D = d, d \in \{1, \ldots, m\} \) (cf. step (9)).
Lemma 4.2.5 For \( d \in \{1, \ldots, m-1\} \), we have

\[
f_{W_j^r,1 \leq j \leq |D|=d}(w_d) = 1_{W_d(T-\mu|\gamma_d^*)}(w_d) \frac{e^{-\gamma_{k+1}^*(T-\mu)/\theta}}{\theta f_{s,d+1}(T)} \prod_{j=1}^{d} \frac{1}{\theta} \exp \left\{ -\left(1 - \frac{\gamma_{k+1}^*}{\gamma_j^*} \right) \frac{w_j}{\theta} \right\},
\]

with \( W_d(T-\mu|\gamma_d^*) \) as in (4.16). For \( d = m \), the conditional joint density function of the spacings is given by

\[
f_{W_j^r,1 \leq j \leq |D|=m}(w_m) = 1_{W_m(T-\mu|\gamma_m^*)}(w_m) \frac{1}{F_{s,m}(T)} \prod_{j=1}^{m} \frac{1}{\theta} \exp \left\{ -\frac{w_j}{\theta} \right\}.
\]

If we consider the support \( W_d(T-\mu|\gamma_d^*) \), it becomes obvious that the spacings are not independent. In particular, the dependence structure of the random variables \( W_1^r, \ldots, W_d^r \), \( d \in \{1, \ldots, m\} \) is induced by the inequality \( \sum_{j=1}^{d} \frac{w_j}{\gamma_j^*} \leq T - \mu \). We proceed by considering two transformations of the spacings \( W_1^r, \ldots, W_d^r \), \( d \in \{1, \ldots, m\} \), for the particular case of ordinary order statistics.

Remark 4.2.6 Let \( Z \sim \text{Exp}(0, \vartheta) \). Then, we have for \( t \geq 0 \) and \( u \in [0,1] \)

\[
P(Z \leq u) = 1 - e^{-t/\vartheta} \iff P(F(Z) \leq u) = u \iff P(Z \leq F^{-1}(u)) = u,
\]

where \( F^{-1}(u) = -\vartheta \ln(1-u) \). This obviously leads us to \( Z \overset{d}{=} -\vartheta \ln(1-U) \) with \( U \sim \text{U}(0,1) \). Let \( Z_{1:n}, \ldots, Z_{d:n} \) denote the first \( d \) order statistics based on \( Z_1, \ldots, Z_n \overset{iid}{\sim} \text{Exp}(\vartheta) \), and let \( U_{1:n}, \ldots, U_{d:n} \) denote the first \( d \) order statistics based on \( U_1, \ldots, U_n \overset{iid}{\sim} \text{U}(0,1) \), for \( d \in \{1, \ldots, m\} \) with \( Z_{0:n} = U_{0:n} := 0 \). Then, for the setup of Type-I hybrid censoring, the spacings \( W_j^r, 1 \leq j \leq d \), can be written as

\[
W_j^r = (n-j+1)(Z_{j:n} - Z_{j-1:n}) \overset{d}{=} -\vartheta(n-j+1) \ln \left( \frac{1-U_{j:n}}{1-U_{j-1:n}} \right).
\]

Now, identity (4.17) can be used to transfer the density function \( f_{W_j^r,1 \leq j \leq |D|=d}(w_d) \), \( d \in \{1, \ldots, m\} \) (cf. Lemma 4.2.5) to expressions based on uniform and beta distributed random variables, respectively.

(i) Equation (4.17) gives us together with Malmquist (1950) (see also Arnold et al., 2008, Theorem 4.7.3)

\[
W_j = \frac{d}{j} \left( n-j+1 \right) \ln(U_j), \quad 1 \leq j \leq d,
\]

where \( U_1, \ldots, U_d \overset{iid}{\sim} \text{U}(0,1) \) denote uniformly distributed random variables. Now, by applying the (non-linear) transformation \( \Phi: U_d \rightarrow W_d(T|\gamma_d^*) \), with

\[
\gamma_d^* = \left( (n-d), (n-d+1), \ldots, n \right) \quad \text{and} \quad U_d = \left\{ u_d \in [0,1]^d \mid -\sum_{j=1}^{d} \frac{\ln(u_j)}{j} \leq \frac{T}{\vartheta} \right\},
\]
so that
\[ \Phi(u_d) = \left( -\vartheta n \ln(u_1), \ldots, -\vartheta \frac{n - d + 1}{d} \ln(u_d) \right) \]

with \(|\det(D\Phi)| = \frac{\vartheta^d}{n!} \prod_{j=1}^d \frac{n - j + 1}{u_j} = \vartheta^d (n_d) \prod_{j=1}^d v_j^{-1} \), to the density function given in Theorem 4.2.5, we obtain
\[ f_{V_j,1 \leq j \leq d|D=d}(u_d) = \frac{1_{V_d}(u_d)}{P(D=d)} \left( \frac{n}{d} \right) \exp \left\{ \sum_{j=1}^d \frac{d - j + 1}{j} \ln(u_j) \right\} \prod_{j=1}^d u_j^{-1} \]
\[ = \left( \frac{n}{d} \right) \frac{1_{V_d}(u_d)}{P(D=d)} \prod_{j=1}^d u_j^{-(d-j+1)/j}. \]

(ii) A more convenient expression can be obtained by considering a transformation based on beta distributed random variables. If we apply Theorem 4.7.2 from Arnold et al. (2008) to (4.17), we have
\[ W_j \overset{d}{=} -\vartheta (n - j + 1) \ln(V_j), \quad 1 \leq j \leq d, \]
where \( V_j \sim \text{Beta}(j, 1), \ 1 \leq j \leq d - 1, \ V_d \sim \text{Beta}(d, n - d + 1), d \in \{1, \ldots, m\} \). Now, by making adequate adjustments to the transformation used in (i), i.e.,
\[ \Phi(u_d) = \left( -\vartheta n \ln(u_1), \ldots, -\vartheta (n - d + 1) \ln(u_d) \right), \]
with \(|\det(D\Phi)| = \frac{\vartheta^d n_d}{(n-d)!} \prod_{j=1}^d u_j^{-1} \), so that \( \Phi : V_d \rightarrow W_d(T|\gamma_d^*) \), where
\[ V_d = \left\{ v_d \in [0,1]^d \mid - \sum_{j=1}^d \ln(v_j) \leq \frac{T}{\vartheta} \right\}, \]
we find
\[ f_{V_j,1 \leq j \leq d|D=d}(v_d) = 1_{V_d}(v_d) \frac{n!}{(n-d)!P(D=m)} \prod_{j=1}^d u_j \exp \left\{ \sum_{j=1}^d (d - j + 1) \ln(u_j) \right\} \]
\[ = 1_{V_d}(v_d) \frac{n!}{(n-d)!P(D=m)} \prod_{j=1}^d u_j^{d-j}. \]

Since \( 1_{[0,1]}(v) v^d = B_{d-1}(v0,1^d) \) (see, e.g., (4.30)), the conditional density function \( f_{V_j,1 \leq j \leq d|D=d} \) can also be written in terms of B-splines, that is,
\[ f_{V_j,1 \leq j \leq d|D=d}(v_d) = 1_{V_d}(v_d) \frac{n!}{(n-d)!P(D=m)} \prod_{j=1}^{d-1} B_{d-j-1}(v_j0,1^{*(d-j)}). \]

In the following, we consider a result by Burkschat et al. (2016).
Lemma 4.2.7  
(i) Let \( f_{*,d}^{(r)} \) denote the density function of the \( d \)th sequential order statistic \( Z_d^* \) w.r.t. the distribution \( P_F = \text{Exp}(0, \vartheta) \), and the model parameters \( \gamma_1^*, \ldots, \gamma_{r-1}^*, \gamma_{r+1}^*, \ldots, \gamma_{d+1}^* \), for \( 1 \leq r \leq d \leq m - 1 \). Then, for \( d \in \{1, \ldots, m-1\} \) and \( r \in \{1, \ldots, d\} \), the density function of the \( r \)th spacing \( W_r^* \), conditionally on \( D = d \), is given by

\[
f_{W_r^*|D=d}(w) = \frac{f_{*,d}^{(r)}(T - \frac{w}{\vartheta}) \exp(-w/\vartheta)}{\vartheta f_{*,d+1}(T)}, \quad 0 \leq w \leq \gamma_r^* T.
\]

The conditional cumulative distribution function is given by

\[
F_{W_r^*|D=d}(t) = 1 - \frac{f_{*,d+1}(T - \frac{t}{\vartheta}) \exp(-t/\vartheta)}{f_{*,d+1}(T)}, \quad 0 \leq t \leq \gamma_r^* T.
\]

(ii) Let \( F_{*,m-1}^{(r-1)} \), denote the cumulative distribution function of the \((m-1)\)th sequential order statistic \( Z_{m-1}^* \) w.r.t. the distribution \( P_F = \text{Exp}(0, \vartheta) \), and the model parameters \( \gamma_1^*, \ldots, \gamma_{r-1}^*, \gamma_{r+1}^*, \ldots, \gamma_{m}^* \), for \( 1 \leq r \leq m \). Then, for \( r \in \{1, \ldots, m\} \), the density function of the \( r \)th spacing \( W_r^* \), conditionally on \( D = m \), is given by

\[
f_{W_r^*|D=m}(w) = \frac{F_{*,m-1}^{(r)}(T - \frac{w}{\vartheta}) \exp(-w/\vartheta)}{\vartheta F_{*,m}(T)}, \quad 0 \leq w \leq \gamma_r^* T.
\]

The conditional cumulative distribution function is given by

\[
F_{W_r^*|D=m}(t) = 1 - \frac{F_{*,m}(T - \frac{t}{\vartheta}) \exp(-t/\vartheta)}{\vartheta F_{*,m}(T)}, \quad 0 \leq t \leq \gamma_r^* T.
\]

4.2.1 Distribution Theory for a Known Location Parameter

In order to determine the distribution of the MLE \( \widehat{\vartheta} \), we consider the total time on test statistic \( S_D \) (cf. Notation 4.2.1, in particular (4.6)) expressed in terms of the spacings \( W_1^*, \ldots, W_D^* \),

\[
S_D = \sum_{j=1}^{D} \left( 1 - \frac{\gamma_{D+1}^*}{\gamma_j^*} \right) W_j^* + \gamma_{D+1}^*(T - \mu) \quad D \in \{1, \ldots, m\}.
\]  

(4.18)

According to Remark 4.2.4, (iii) there exists at least one index \( i^* \in \{1, \ldots, d\} \) such that \( \gamma_{i^*}^* \neq \gamma_{d+1}^*, d \in \{1, \ldots, m\} \). For such an \( i^* \), we consider the linear transformation

\[
\Phi_{S,i^*} : W_d(T - \mu|\gamma_d^*) \rightarrow A_{d-1}^{[i^*]}(s) \times \mathbb{R}
\]

with

\[
\Phi_{S,i^*}(w_d) = \left( w_{i^* - 1}, \sum_{j=1}^{d} \left( 1 - \frac{\gamma_{d+1}^*}{\gamma_j^*} \right) w_j + \gamma_{d+1}^*(T - \mu), w_{i^* + 1}, \ldots, w_d \right)
\]  

(4.19)

and

\[
|\det(D\Phi_{S,i^*})| = \frac{|\gamma_{i^*}^* - \gamma_{d+1}^*|}{\gamma_{i^*}^*}.
\]
Instead of \( \{1, \ldots, d\} \setminus \{i^*\} \), we write for brevity from now on \( \mathcal{I}_{1,d}^{[i^*]} = \{1, \ldots, d\} \setminus \{i^*\} \).

The image of \( \Phi_{s,i^*} \), denoted by \( A_{d-1}^{[i^*]}(s) \times \mathbb{R} \), can be interpreted as follows: due to \( s = \sum_{j=1}^{d} (1 - \frac{\gamma_{d+1}}{\gamma_j}) w_j + \gamma_{d+1} (T - \mu) \), it is obvious that \( s \in \mathbb{R} \). Further, the \((d - 1)\)-dimensional set \( A_{d-1}^{[i^*]}(s) \) consists of the variables \( (w_j)_{j \in \mathcal{I}_{1,d}^{[i^*]}} \in \mathbb{R}^{d-1} \), which satisfy for a given \( s \in \mathbb{R} \) the conditions specified in the set \( \mathcal{W}_d(T - \mu | \gamma_d^*) \) (cf. (4.16)) as well as the condition imposed by \( \Phi_{s,i^*} \), that is \( s = \sum_{j=1}^{d} (1 - \frac{\gamma_{d+1}}{\gamma_j}) w_j + \gamma_{d+1} (T - \mu) \). In order to determine the set \( A_{d-1}^{[i^*]}(s) \) explicitly, we rewrite these conditions in terms of the variables \( (w_j)_{j \in \mathcal{I}_{1,d}^{[i^*]}} \) and \( s \).

The application of \( \Phi_{s,i^*} \) to the conditional joint density function \( f_{W_{[1]}^{[i^*]}}, 1 \leq j \leq d, D = d \) (cf. Lemma 4.2.5), yields for \( d \in \{1, \ldots, m - 1\} \),

\[
    f_{S_d}^{[d]}(s) = \frac{\gamma_{d+1} \gamma_d^* e^{-s/\theta}}{|\gamma_d^* - \gamma_{d+1} | \theta^{d+1} f_{s,d+1}(T)} \int_{A_{d-1}^{[i^*]}(s)} \prod_{j \in \mathcal{I}_{1,d}^{[i^*]}} \mathcal{W}_d(T - \mu | \gamma_j^*) \, dw_j.
\]

Now, due to

\[
    f_{S_d}^{[d]}(s) = \frac{\gamma_{d+1} \gamma_d^* e^{-s/\theta}}{|\gamma_d^* - \gamma_{d+1} | \theta^{d+1} f_{s,d+1}(T)} \prod_{j \in \mathcal{I}_{1,d}^{[i^*]}} \mathcal{W}_d(T - \mu | \gamma_j^*), \quad s \in \mathbb{R},
\]

the conditional density function of \( S_d \) can be obtained by calculating the volume of \( A_{d-1}^{[i^*]}(s) \), \( s \in \mathbb{R} \). For \( c > 0 \), let \( \gamma_{d+1} c = (\gamma_{d+1} c, \ldots, \gamma_d^* c) \in \mathbb{R}^{d+1} \). Note that in this work, bold print \( \gamma \)'s in combination with a factor, e.g., \( \gamma_d^* (T - \mu) \), are usually applied for specifying the knot lists of B-splines.

**Lemma 4.2.8** Let \( d \in \{1, \ldots, m\} \) with \( \gamma_{d+1} \neq \gamma_i^* \) for any \( i^* \in \{1, \ldots, d\} \). Then, for \( s \in \mathbb{R} \), the set \( A_{d-1}^{[i^*]}(s) \) is given by

\[
    A_{d-1}^{[i^*]}(s) = \begin{cases} 
        \mathcal{M}_{d-1}^{[i^*]}(s|\beta, \theta), & \gamma_{d+1} < \gamma_i^*, \\
        \mathcal{M}_{d-1}^{[i^*]}(s|\beta, \tilde{t}), & \gamma_{d+1} > \gamma_i^*,
    \end{cases}
\]

where \( \mathcal{M}_{d-1}^{[i^*]}(s|\beta, \theta) \) as specified in Theorem 3.2.1. The parameter vectors are given by \( \beta = (\beta_1, \ldots, \beta_d) \), \( t = (t_0, \ldots, t_d) \) and \( \tilde{t} = (t_i^*, t_{i^*}, \ldots, t_{i^*}, t_0, t_1^*, \ldots, t_d^*) \), respectively, with

\[
    \beta_j = \gamma_j^* (T - \mu), \quad j \in \{1, \ldots, d\} \setminus \{i^*\}, \quad \beta_{i^*} > 0 \quad \text{and} \quad t_j = \gamma_j^* (T - \mu), \quad 1 \leq j \leq d, \quad t_0 = \gamma_{d+1}^* (T - \mu).
\]

The volume of \( A_{d-1}^{[i^*]}(s) \) is given by

\[
    \text{vol}_{d-1}(A_{d-1}^{[i^*]}(s)) = \frac{(T - \mu)^d \gamma_d^* - \gamma_{d+1}^* \prod_{j \in \{1, \ldots, d\} \setminus \{i^*\}} \gamma_j^*}{d!} B_{d-1}(s | \gamma_{d+1}^* (T - \mu)), \quad s \in \mathbb{R}.
\]
Proof. For brevity, let $\mu = 0$ throughout this proof. In order to determine $A_{d-1}^{[*]}(s)$, $s \in \mathbb{R}$, for $d \in \{1, \ldots, m\}$, we first observe that

$$s = \sum_{j=1}^{d} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}}\right)w_{j} + \gamma_{d+1}^{*}T,$$

and

$$\sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}}\right)w_{j} \begin{cases} \leq s - \gamma_{d+1}^{*}T, & \gamma_{d+1}^{*} < \gamma_{i}^{*}, \\ \geq s - \gamma_{d+1}^{*}T, & \gamma_{d+1}^{*} > \gamma_{i}^{*}, \end{cases} \tag{4.21}$$

for $w_{j} \geq 0$, $1 \leq j \leq d$. Hence, we obtain for $\gamma_{d+1}^{*} \neq \gamma_{i}^{*}$,

$$w_{i}^{*} = \frac{\gamma_{i}^{*}}{\gamma_{i}^{*} - \gamma_{d+1}^{*}} \left(s - \gamma_{d+1}^{*}T - \sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}}\right)w_{j}\right).$$

Rewriting the sum $\sum_{j=1}^{d} \frac{w_{j}}{\gamma_{j}}$ in terms of the variables $(w_{j})_{j \in I_{1,d}^{[*]}}$ and $s$, yields

$$\sum_{j=1}^{d} \frac{w_{j}}{\gamma_{j}} = \sum_{j \in I_{1,d}^{[*]}} \frac{w_{j}}{\gamma_{j}} + \frac{1}{\gamma_{i}^{*} - \gamma_{d+1}^{*}} \left(s - \gamma_{d+1}^{*}T - \sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}}\right)w_{j}\right)$$

$$= \sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{j}^{*} - \gamma_{d+1}^{*}}{\gamma_{j}^{*}(\gamma_{i}^{*} - \gamma_{d+1}^{*})}\right)w_{j} + \frac{s - \gamma_{d+1}^{*}T}{\gamma_{i}^{*} - \gamma_{d+1}^{*}}.$$

Using $\sum_{j=1}^{d} \frac{w_{j}}{\gamma_{j}} \leq T$ (cf. $W_{d}(T|\gamma_{d}^{*})$ in (4.16)), we have for $\gamma_{d+1}^{*} < \gamma_{i}^{*}$,

$$\sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{j}^{*} - \gamma_{d+1}^{*}}{\gamma_{j}^{*}(\gamma_{i}^{*} - \gamma_{d+1}^{*})}\right)w_{j} + \frac{s - \gamma_{d+1}^{*}T}{\gamma_{i}^{*} - \gamma_{d+1}^{*}} \leq T$$

$$\iff \sum_{j \in I_{1,d}^{[*]}} \left(\frac{\gamma_{j}^{*} - \gamma_{d+1}^{*}}{\gamma_{j}^{*}(\gamma_{i}^{*} - \gamma_{d+1}^{*})}\right)w_{j} \leq T - s - \gamma_{d+1}^{*}T$$

$$\iff \sum_{j \in I_{1,d}^{[*]}} \left(\gamma_{j}^{*} - \gamma_{d+1}^{*}\right)w_{j} \geq (\gamma_{d+1}^{*} - \gamma_{i}^{*})T + s - \gamma_{d+1}^{*}T$$

$$\iff \sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}^{*}}\right)w_{j} \geq s - \gamma_{i}^{*}T. \tag{4.22}$$

It can be quickly verified that the application of the above calculations gives, for $\gamma_{d+1}^{*} > \gamma_{i}^{*}$,

$$\sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{j}^{*}}{\gamma_{j}^{*}}\right)w_{j} \leq s - \gamma_{i}^{*}T. \tag{4.23}$$

The inequality given in (4.21), for $\gamma_{d+1}^{*} < \gamma_{i}^{*}$, with $w_{j} \geq 0$, $j \in I_{1,d}^{[*]}$, leads us together with (4.22), to the following representation of $A_{d-1}^{[*]}(s)$,

$$A_{d-1}^{[*]}(s) = \left\{w_{j} \in I_{1,d}^{[*]} \in \mathbb{R}^{d-1} \mid w_{j} \geq 0, j \in I_{1,d}^{[*]}, \sum_{j \in I_{1,d}^{[*]}} \left(1 - \frac{\gamma_{d+1}^{*}}{\gamma_{j}^{*}}\right)w_{j} \leq s - \gamma_{d+1}^{*}T, \right\}.$$
\[
\sum_{j \in I^{[\ast]}_1} \left(1 - \frac{\gamma_j}{\gamma_j^s}\right)w_j \geq s - \gamma_i^* T \right\}.
\]

In order to determine the corresponding volume, let

\[
s_i^* = \gamma_{d+1} T, \quad s_j = \left(1 - \frac{\gamma_{d+1}}{\gamma_j^s}\right), \quad j \in I^{[\ast]}_1 \] and
\[
h_i^* = \gamma_i^* T, \quad h_j = \left(1 - \frac{\gamma_i}{\gamma_j^s}\right), \quad j \in I^{[\ast]}_1.
\]

Due to

\[
\begin{align*}
h_i^* > s_i^* & \iff \gamma_i^* > \gamma_{d+1}^* \quad \text{and} \quad h_j < s_j \iff \left(1 - \frac{\gamma_i^*}{\gamma_j^s}\right) < \left(1 - \frac{\gamma_{d+1}^*}{\gamma_j^s}\right) \\
& \iff \gamma_i^* - \gamma_j^s < \gamma_i^* - \gamma_{d+1}^* \\
& \iff \gamma_{d+1}^* < \gamma_i^*, \quad j \in I^{[\ast]}_1,
\end{align*}
\]

the conditions for applying Corollary 3.2.3 are satisfied for \(\gamma_{d+1}^* < \gamma_i^*\). Further, we have

\[
t_0 = \gamma_{d+1}^* T, \quad t_i^* = \gamma_i^* T, \quad t_j = \left(1 - \frac{\gamma_{d+1}}{\gamma_j^s}\right) \gamma_i^* T - \gamma_{d+1}^* T \left(1 - \frac{\gamma_i^*}{\gamma_j^s}\right) = \gamma_i^* T - \gamma_{d+1}^* T = \gamma_i^* T
\]

and

\[
\beta_j = \frac{\gamma_i^* T - \gamma_{d+1}^* T}{\left(1 - \frac{\gamma_{d+1}}{\gamma_j^s}\right) - \left(1 - \frac{\gamma_i^*}{\gamma_j^s}\right)} = \gamma_i^* T, \quad j \in \{1, \ldots, d\} \setminus \{i^*\}.
\]

According to Corollary 3.2.3, the volume of \(A^{[\ast]}_{d-1}(s)\) with \((t_i^* - t_0) = (\gamma_i^* - \gamma_{d+1}^*) T\) and \(\gamma_{d+1}^* < \gamma_i^*\), is given by

\[
\text{vol}_{d-1}(A^{[\ast]}_{d-1}(s)) = \frac{T^d \left(\gamma_i^* - \gamma_{d+1}^*\right) \prod_{j \in I^{[\ast]}_1} \gamma_j^s}{d!} B_{d-1}(s, \gamma_{d+1}^* T, \gamma_2^* T, \ldots, \gamma_1^* T), \quad s \in \mathbb{R}.
\]

Note that \(\beta_i^* > 0\) can be chosen arbitrarily, for \(\gamma_i^* \neq \gamma_{d+1}^*\). Now, for \(\gamma_i^* > \gamma_{d+1}^*\), we find together with equations (4.21) and (4.23) that

\[
A^{[\ast]}_{d-1}(s) = \left\{ w \in \mathbb{R}^{d-1} \mid w_j \geq 0, j \in I^{[\ast]}_1, \sum_{j \in I^{[\ast]}_1} \left(1 - \frac{\gamma_i^*}{\gamma_j^s}\right) w_j \leq s - \gamma_i^* T, \sum_{j \in I^{[\ast]}_1} \left(1 - \frac{\gamma_{d+1}^*}{\gamma_j^s}\right) w_j \geq s - \gamma_{d+1}^* T \right\}.
\]

By setting

\[
s_i^* = \gamma_i^* T, \quad s_j = \left(1 - \frac{\gamma_i}{\gamma_j^s}\right), \quad j \in I^{[\ast]}_1 \] and
\[
h_i^* = \gamma_{d+1}^* T, \quad h_j = \left(1 - \frac{\gamma_{d+1}}{\gamma_j^s}\right), \quad j \in I^{[\ast]}_1,
\]

we can again apply Corollary 3.2.3 and obtain the same values for \(\beta_1, \ldots, \beta_{i^* - 1}, \beta_{i^* + 1}, \ldots, \beta_d\) and \(t_1, \ldots, t_{i^* - 1}, t_{i^* + 1}, \ldots, t_d\) as for the setting where \(\gamma_{d+1}^* < \gamma_i^*\), while \(t_0 = \gamma_i^* T\) and
\( t_\ast = \gamma_{d+1}^\ast T \). Hence, the volumes for \( \gamma_{d+1}^\ast < \gamma_i^\ast \) and \( \gamma_{d+1}^\ast > \gamma_i^\ast \) coincide. Then, the assertion follows by taking into account that

\[
|\gamma_i^\ast - \gamma_{d+1}^\ast| = \begin{cases} 
\gamma_i^\ast - \gamma_{d+1}^\ast, & \gamma_i^\ast > \gamma_{d+1}^\ast, \\
\gamma_{d+1}^\ast - \gamma_i^\ast, & \gamma_i^\ast < \gamma_{d+1}^\ast.
\end{cases}
\]

Now, we are able to present expressions for the conditional density function of the total time on test statistic \( S_d, d \in \{1, \ldots, m\} \).

**Theorem 4.2.9** Let \( \gamma_1^\ast, \ldots, \gamma_m^\ast > 0 \) and \( \gamma_{m+1}^\ast = 0 \). Then, for \( d \in \{1, \ldots, m-1\} \) and \( \gamma_1^\ast, \ldots, \gamma_{d+1}^\ast \) not all equal, the conditional density function of \( S_d \) is given by

\[
f^{S_d \mid D = d}(s) = \frac{(T-\mu)^d \prod_{j=1}^{d+1} \gamma_j^\ast}{d! \hat{\vartheta}_{d+1} f_{s,d+1}(T)} B_{d-1}(s|\gamma_{d+1}^\ast(T-\mu)) e^{-s/\vartheta}, \quad s \in \mathbb{R}.
\]

For \( d = m \), we have

\[
f^{S_m \mid D = m}(s) = \frac{(T-\mu)^m \prod_{j=1}^{m} \gamma_j^\ast}{d! \hat{\vartheta}_m f_{s,m}(T)} B_{m-1}(s|\gamma_m^\ast(T-\mu)) e^{-s/\vartheta}, \quad s \in \mathbb{R}.
\]

**Proof.** For \( d \in \{1, \ldots, m-1\} \), we insert the volume representation from Lemma 4.2.8 in (4.20). For \( d = m \), we recall \( \gamma_{m+1}^\ast = 0 \). By applying transformation \( \Phi_{S,m} \) (cf. (4.19)) to the conditional joint density function \( f^{W_j^\ast, 1 \leq j \leq m \mid D = m} \) (cf. Lemma 4.2.5) and by taking into account \( \gamma_m^\ast > 0 \), we get with Lemma 4.2.8 and \( i^\ast = m \),

\[
f^{S_m \mid D = m}(s) = \int_{A_{m-1}^{[m]}(s)} f^{W_j^\ast, 1 \leq j < m, S_m \mid D = m}(w_{m-1}, s) dw_{m-1} = e^{-s/\vartheta} \frac{e^{-s/\vartheta}}{d! \hat{\vartheta}_m f_{s,m}(T)} \text{vol}_{m-1}(A_{m-1}^{[m]}(s)) = T^m \frac{e^{-s/\vartheta}}{d! \hat{\vartheta}_m f_{s,m}(T)} B_{m-1}(s|0, \gamma_m^\ast T, \ldots, \gamma_1 T) e^{-s/\vartheta}.
\]

The fact that the index \( i^\ast \) can be chosen arbitrarily as long as the required condition is satisfied, enables us to weaken the constraints imposed on the \( \gamma^\ast \)'s given in Theorem 4.2.9. Notice further that Lemma 4.2.8, the calculations leading to Lemma 4.2.8 and Theorem 4.2.9, correspond to the steps \( \mathbb{I} \) and \( \mathbb{II} \) of the volume approach (see Procedure 1.4.4). In the following, we perform step \( \mathbb{II} \) from the expected value approach (see Procedure 1.4.6), providing a way to obtain the distribution of \( S_d \).

**Theorem 4.2.10** Let the total time on test statistic \( S_d \) be given as in (4.4) and let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function. Then, the conditional expectation of \( g(S_d) \) given \( D = d \), with \( d \in \{1, \ldots, m-1\} \), is given by

\[
E(g(S_d) \mid D = d) = \int_{\mathbb{R}} g(s) \frac{(T-\mu)^d \prod_{j=1}^{d+1} \gamma_j^\ast}{d! \hat{\vartheta}_{d+1} f_{s,d+1}(T)} B_{d-1}(s|\gamma_{d+1}^\ast(T-\mu)) e^{-s/\vartheta} ds.
\]
For \( d = m \), we have
\[
E(g(S_m)|D = m) = \int \limits_\mathbb{R} g(s) \frac{(T - \mu)^m}{m! \mu^m} \prod_{j=1}^m \gamma_j^s \ B_{m-1}(s|0, \gamma_m^s(T - \mu)) e^{-s/\vartheta} ds.
\]

**Proof.** Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function and let \( \tilde{g} : \mathbb{R} \rightarrow \mathbb{R} \), as \( \tilde{g}(x) = g(x) e^{-x/\vartheta} \), \( \vartheta > 0 \). Further, we consider the linear transformation \( \Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d \), where
\[
\Phi(y_1, \ldots, y_d) = \left( \frac{y_1 - \mu}{T - \mu}, \ldots, \frac{y_d - \mu}{T - \mu} \right), \quad \text{with} \quad \det(\Phi) = (T - \mu)^{-d}. \tag{4.26}
\]

In order to derive the distribution of the total time on test statistic \( S_d \), for \( d \in \{1, \ldots, m-1\} \), we calculate the conditional expectation of \( g(S_d) \). By using representation (4.4) of \( S_d \) and by recalling the definitions of the sets \( \Sigma_{F, T}^d \) and \( \Sigma^d \) (cf. (1.4) and (3.11), respectively), we find
\[
E(g(S_d)|D = d) = \int \limits_{\mathbb{R}^d} g\left( \sum_{j=1}^d (\gamma_j^s - \gamma_j^{s+1})(z_j - \mu) + \gamma_{d+1}^s(T - \mu) \right) f_{\Sigma_{F, T}^d} d \Sigma_{F, T}^d d \Sigma^d
\]
\[
= \prod_{j=1}^{d+1} \gamma_j^s \int \limits_{\Sigma^d} \tilde{g}\left( \sum_{j=1}^d (\gamma_j^s - \gamma_j^{s+1})(z_j - \mu) + \gamma_{d+1}^s(T - \mu) \right) d \Sigma^d.
\]

Now, by applying transformation (4.26) and by using Corollary 3.1.10 with \( \tilde{g} \), we arrive at
\[
E(g(S_d)|D = d) = \frac{(T - \mu)^d}{\vartheta^{d+1} f_{\Sigma_{F, T}^d}(T)} \int \limits_{\Sigma^d} \tilde{g}\left( \sum_{j=1}^d (\gamma_j^s - \gamma_j^{s+1})(T - \mu) y_j + \gamma_{d+1}^s(T - \mu) \right) d \Sigma^d
\]
\[
= \frac{(T - \mu)^d}{\vartheta^{d+1} f_{\Sigma_{F, T}^d}(T)} \int \limits_\mathbb{R} \tilde{g}(s) B_{d-1}(s|\gamma_{d+1}^s(T - \mu), \ldots, \gamma_1^s(T - \mu)) ds
\]
\[
= \frac{(T - \mu)^d}{\vartheta^{d+1} f_{\Sigma_{F, T}^d}(T)} \int \limits_\mathbb{R} g(s) B_{d-1}(s|\gamma_{d+1}^s(T - \mu), \ldots, \gamma_1^s(T - \mu)) e^{-s/\vartheta} ds.
\]

For \( D = m \), we proceed analogously. \( \Box \)

Thus, it is possible to derive a representation for \( f_{S_d}^{D=d} \) in terms of B-splines without considering the distribution of the spacings and without calculating particular volumes. Hence, the expected value approach needs less effort than the volume approach in order to obtain an expression for \( f_{S_d}^{D=d} \). Further, using the expected value approach, we do not have to distinguish between the two cases \( \gamma_{d+1}^s < \gamma_1^s \) and \( \gamma_{d+1}^s > \gamma_1^s \). We only need the coefficients \( \gamma_1^s, \ldots, \gamma_d^s \) not all to be equal. However, the expected value approach may seem less intuitive than the volume approach. Further, the expected value approach requires in contrast to the volume approach a quite profound knowledge of B-spline theory.

**Remark 4.2.11** (i) Cramer and Balakrishnan (2013) calculated the volume of \( A_{d-1}^d(s) \), with \( d \in \{2, \ldots, m\} \), for the particular setting of progressively Type-II censored order statistics. They used a result presented in Gerber (1981, p. 312). The calculations to be conducted in order to obtain an expression for the volume as an scaled B-spline, are more complicated and more extensive, than those performed in the proof of Lemma 4.2.8.
(ii) The volume of $A_{d-1}^{[s]}(s)$, $i^* \in \{1, \ldots, d\}$, for $d \in \{2, \ldots, m\}$, can be also calculated by using the results established in Varsi (1973), Ali (1973), and Cho and Cho (2001). The effort to be made for obtaining the desired volume formulas is very similar to the case where Gerber’s result is employed.

(iii) A technique, quite similar to the expected value approach, has been recently presented in Burkschat et al. (2016, Theorem 3.3) in the context of Type-I sequential k-out-of-n-systems. There, an indirect approach was chosen to obtain the B-spline. In particular, a relation between the B-spline and the distribution of a particular linear combination of order statistics based on $U_1, \ldots, U_m \overset{\text{iid}}{\sim} U(0,1)$ was used (see, e.g., Adell and Sangüesa, 2005, Proposition 2.1).

(iv) Using $g_w$ with $g_w(t) = e^{wt}$, $t \in \mathbb{R}$, for a fixed $w \in \mathbb{R}$, in Theorem 4.2.10, we arrive at the conditional moment generating function of $S_d$.

We present now the conditional density function of the MLE $\hat{\Theta}$ (see step 12 from Procedure 1.4.4 or step 111 from Procedure 1.4.6).

**Theorem 4.2.12** Let $\gamma_1^*, \ldots, \gamma_m^* > 0$ with $\gamma_1^* \neq \gamma_2^*$ and $\gamma_{m+1}^* = 0$. Then, the conditional density function $f_{\hat{\Theta}|D \geq 1}(s)$ of the MLE $\hat{\Theta}$ is given by

$$f_{\hat{\Theta}|D \geq 1}(s) = \frac{1}{1 - e^{-\gamma_1^*(T-\mu)/\vartheta}} \sum_{d=1}^{m} \frac{(T - \mu)^d \prod_{j=1}^{d} \gamma_j^*}{(d - 1)! \vartheta^d} B_{d-1}(ds|\gamma_{d+1}^*(T-\mu)) e^{-ds/\vartheta}, \quad s \geq 0.$$ 

**Proof.** We recall that $S_D = D\hat{\Theta}$, $D \in \{1, \ldots, m\}$. Further,

$$f_{\hat{\Theta}|D \geq 1}(s) = \frac{1}{P(D \geq 1)} \sum_{d=1}^{m} d f_{S_D=d|(D=ds)} P(D=d), \quad s \geq 0.$$ 

Then, the assertion follows immediately from Lemma 4.1.1, Theorem 4.2.9 and by using

$$P(D \geq 1) = 1 - P(D = 0) = 1 - (1 - F(T))^{\gamma_1^*} = 1 - e^{-\gamma_1^*(T-\mu)/\vartheta}. \quad \square$$

Let $V_1, \ldots, V_n \overset{\text{iid}}{\sim} \text{Exp}(0,1)$. Then, according to Burkschat et al. (2016, eq. (4.4)) the identity

$$Z_j^* = \vartheta \sum_{i=j}^{m} \frac{V_i}{\gamma_i^*}, \quad 1 \leq j \leq m,$$

holds, for the sequential order statistics $Z_1, \ldots, Z_n^*$ (w.r.t. the sequential exponential model, cf. Model 2.3.8). Based on the simulated exponential data $v_1, \ldots, v_{10}$ given in Table 4.1, we calculate the samples of sequential order statistics (see Table 4.2) for different settings of the $\gamma^*$’s. Scenarios I and II from Table 4.1 correspond to a setting of an alternating sequence and to a setup of a decreasing sequence of $\gamma^*$’s, respectively. Both sequences consist of multiple occurring values. Figure 4.1 illustrates the corresponding plots with different settings for the threshold time $T$. While the solid line denotes the plot of $f_{\hat{\Theta}|D \geq 1}$ for $\vartheta = 1$, the dashed line corresponds to the density function based on the respective observed data, i.e., $\vartheta = \hat{\vartheta}_{obs}$. Further, we state that the variable $D$ denotes in the respective scenario the number of failures observed till time $T$. 

Corollary 4.2.13 Let \( \gamma^*_1, \ldots, \gamma^*_m > 0 \), with \( \gamma^*_1 \neq \gamma^*_2 \) and \( \gamma^*_m+1 = 0 \). Then, the conditional kth moment of \( \hat{\vartheta} \) is given by

\[
E(\hat{\vartheta}^k|D \geq 1) = \frac{1}{1 - e^{-\gamma^*_1(T-\mu)/\vartheta}} \left[ \sum_{d=1}^m \frac{k!(T-\mu)^d(-1)^d d! \prod_{j=1}^d \gamma^*_j}{d^k} \times \left[ \gamma^*_{d+1}(T-\mu) \right] \sum_{i_1=0}^{k-i_1} \sum_{i_2=0}^{k-i_2} \cdots \sum_{i_d=0}^{k-i_d} \vartheta^{i_d} \left( \frac{x^k}{k - i_d!} \right) \right].
\]

Proof. From the representation given in Theorem 4.2.12, we find

\[
E(\hat{\vartheta}^k|D \geq 1) = \frac{1}{1 - e^{-\gamma^*_1(T-\mu)/\vartheta}} \sum_{d=1}^m \frac{(T-\mu)^d}{(d-1)!} \prod_{j=1}^d \gamma^*_j \int_{R} x^k B_{d-1}(x|\gamma^*_{d+1}(T-\mu)) e^{-x/\vartheta} dx
\]

\[
= \frac{1}{1 - e^{-\gamma^*_1(T-\mu)/\vartheta}} \sum_{d=1}^m \frac{(T-\mu)^d}{d^k d! \vartheta^d} \int_{R} x^k B_{d-1}(x|\gamma^*_{d+1}(T-\mu)) e^{-x/\vartheta} dx.
\]

Then, the application of Lemma 3.3.6, (ii) gives the assumption. \(\square\)

The following theorem provides expressions for the conditional cumulative distribution function of \( \hat{\vartheta} \) for different settings of the \( \gamma^* \)s.

Theorem 4.2.14 Let \( \gamma^*_1, \ldots, \gamma^*_m > 0 \) and \( \gamma^*_m+1 = 0 \).

(i) Let \( \gamma^*_1 = \cdots = \gamma^*_{\ell+1} \neq \gamma^*_\ell+2 \), for a \( \ell \in \{0, \ldots, m-1\} \). Then, the cumulative distribution function \( F_{\hat{\vartheta}^k|D \geq 1} \) is given by, for \( t \geq 0 \),

\[ F_{\hat{\vartheta}^k|D \geq 1}(t) \]
4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions

Figure 4.1: Plots of $f_{D \geq 1}$ (solid line) and $f_{\hat{D} \geq 1}$ (dashed line) with $m = n = 10$. Left: $\gamma^*$'s as in I (cf. Table 4.2), $T = 2.3$, $D = 4$ and $\hat{\theta}_{obs} = 1.320845$. Right: $\gamma^*$'s as in II, $T = 7$, $D = 5$ and $\hat{\theta}_{obs} = 1.10276$.

\[
\begin{align*}
F_{\hat{D}}(t) & = \frac{1}{1 - e^{-\gamma(T-\mu)/\theta}} \left[ \sum_{d=1}^{\ell} \prod_{j=1}^{d} \gamma_j \right] \\
& \quad + \sum_{d=\ell+1}^{m} (T - \mu)^d \prod_{j=1}^{d} \gamma_j \int_0^t B_{d-1}(s) \gamma_{d+1}(T - \mu) e^{-s/\theta} ds.
\end{align*}
\]

(ii) Let $\gamma_1^*, \ldots, \gamma_m^*$ be pairwise distinct. Then, the conditional cumulative distribution function for $\hat{\theta}$, with $t \geq 0$, is given by

\[
F_{\hat{\theta}|D \geq 1}(t) = \frac{1}{1 - e^{-\gamma(T-\mu)/\theta}} \left[ \sum_{d=1}^{m} \prod_{j=1}^{d-1} \gamma_j \right] \\
\times \sum_{j=0}^{d} (-1)^d e^{-\min(\gamma_{d-j+1}(T-\mu))/\theta} \left[ \frac{\gamma_{d+1}(T - \mu)^d}{\prod_{i=0, i \neq j} \gamma_{d-j+1} - \gamma_{d-i+1}} \right].
\]

Proof. (i) According to Lemma 2.3.6 and Lemma 2.3.5, (iv), we find for $d \in \{1, \ldots, \ell\}$ with $\gamma_1^* = \cdots = \gamma_{d+1}^*$ and $\ell \in \{0, \ldots, m-1\}$, that

\[
P(D = d) = \left( \prod_{j=1}^{d} \gamma_j \right) G_{d+1, d+1}^{+1, 0}[e^{-(T-\mu)/\theta}| \gamma_1^* + 1, \ldots, \gamma_{d+1}^* + 1] = (\gamma_1^*)^d \frac{1}{d!} e^{-\gamma(T-\mu)/\theta} \frac{(T - \mu)^d}{\theta^d}.
\]

Further, due to Lemma 4.2.3, we get

\[
P(\hat{\theta} \leq t|D = d) = P\left( \frac{\gamma_{d+1}^*(T - \mu)}{d} \leq t | D = d \right)
\]
Then, the result follows together with Theorem 4.2.12 and

\[
\int_0^t B_{d-1}(ds|\gamma_{d+1}(T-\mu))e^{-ds/\vartheta} ds = \frac{1}{d} \int_0^d t B_{d-1}(s|\gamma_{d+1}(T-\mu))e^{-s/\vartheta} ds. \tag{4.27}
\]

(ii) The application of (4.27) and Lemma 3.3.6, (i) to \( \int_0^t f_{\bar{\vartheta}|D \geq 1}(s) ds \) yields the assertion.

Figure 4.2 shows the plots for the cumulative distribution function \( F_{\bar{\vartheta}|D \geq 1} \) as well as for the density function \( f_{\bar{\vartheta}|D \geq 1} \) for \( \vartheta \in \{1, \hat{\vartheta}_{obs}\} \) and w.r.t. the scenarios III and IV given in Table 4.2. Plots for the setups of ordinary and of progressively Type-II censored order statistics are shown in Figure 4.3 (i.e., Type-I hybrid censoring and Type-I progressive hybrid censoring, respectively).

We proceed, by deriving a result useful for the construction of exact confidence intervals for \( \vartheta \).

**Theorem 4.2.15** Let \( \gamma_1^*, \ldots, \gamma_m^* \) be pairwise distinct and \( \gamma_{m+1}^* = 0 \). Then, the following limits hold

\[
\lim_{\vartheta \to 0} F_{\bar{\vartheta}|D \geq 1}(t) = 1, \quad t \geq 0,
\]

\[
\lim_{\vartheta \to \infty} F_{\bar{\vartheta}|D \geq 1}(t) = \begin{cases} 
0, & t \in \left[0, \gamma_2^*(T-\mu)\right], \\
\frac{t - \gamma_2^*(T-\mu)}{\gamma_1^*(T-\mu)}, & t \in \left(\gamma_2^*(T-\mu), \gamma_1^*(T-\mu)\right), \quad \gamma_1^* > \gamma_2^*, \\
1, & t \in \left[\gamma_1^*(T-\mu), \infty\right),
\end{cases}
\]

and

\[
\lim_{\vartheta \to \infty} F_{\bar{\vartheta}|D \geq 1}(t) = \begin{cases} 
0, & t \in \left[0, \gamma_1^*(T-\mu)\right], \\
\frac{t - \gamma_1^*(T-\mu)}{\gamma_2^*(T-\mu)}, & t \in \left(\gamma_1^*(T-\mu), \gamma_2^*(T-\mu)\right), \quad \gamma_1^* < \gamma_2^*, \\
1, & t \in \left[\gamma_2^*(T-\mu), \infty\right),
\end{cases}
\]

**Proof.** Without loss of generality, we assume \( \mu = 0 \) throughout this proof. We consider the expression for \( F_{\bar{\vartheta}|D \geq 1} \) presented in Theorem 4.2.14, (ii), so that

\[
F_{\bar{\vartheta}|D \geq 1}(t) = \frac{1}{1 - e^{-\gamma_1^*(T-\mu)/\vartheta}} \left[ \sum_{d=1}^{m} \left( \prod_{j=1}^{d} \gamma_j^* \right) \right] \times \sum_{j=0}^{d} (-1)^d e^{-\min(\gamma_{d+j+1}^*(T, dt)/\vartheta)} + \sum_{k=1}^{d-1} \frac{\vartheta k - (d-k)}{(d-k)!} e^{-dt/\vartheta} \left( \gamma_{d+j+1}^* T - dt \right)^{d-k} \]
4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions

\[
m = n = 10, \gamma^* \text{'s as in III}, T = 3, D = 3 \text{ and } \hat{\vartheta}_{\text{obs}} = 1.663238.\]

\[
m = n = 10, \gamma^* \text{'s as in IV}, T = 7, D = 5 \text{ and } \hat{\vartheta}_{\text{obs}} = 1.13696.\]

**Figure 4.2:** Plots of the density function \( f_{\hat{\vartheta}|D \geq 1} \) (left) and of the cumulative distribution function \( F_{\hat{\vartheta}|D \geq 1} \) (right) for \( \vartheta = 1 \) (solid line) and for \( \hat{\vartheta}_{\text{obs}} \) (dashed line).

First, we find that

\[
\lim_{\vartheta \to 0} \frac{1}{1 - e^{-\gamma^*_T/\vartheta}} = 1
\]

and

\[
\lim_{\vartheta \to 0} \sum_{j=0}^{d} (-1)^{d-k} e^{-\min\{\gamma_{d-j+1}^* T, dt\}/\vartheta} + \sum_{k=1}^{d-1} \frac{d-k}{(d-k)!} (-1)^k e^{-d-\vartheta (\gamma_{d-j+1}^* T - dt)} (\gamma_{d-j+1}^* - \gamma_{d-i+1}^*)^{d-k} \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-1}^*) = \begin{cases} 0, & d \in \{1, \ldots, m - 1\}, \\ \frac{(-1)^m}{\prod_{i=1}^{m} (-\gamma_{m-i+1})^d}, & d = m, \end{cases}
\]
Setting of Type-I hybrid censoring with \( n = 10, m = 5 \) and \( T = 1 \), such that \((\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*, \gamma_5^*) = (10, 9, 8, 7, 6)\).

Setting of Type-I progressive hybrid censoring with \( n = 10, m = 5 \) and \( T = 1 \), such that \((\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*, \gamma_5^*) = (10, 8, 6, 4, 2)\).

**Figure 4.3:** Plots of \( \hat{F}_{\vartheta|D \geq t} \) and \( \hat{f}_{\vartheta|D \geq t} \) (solid lines) for \( \vartheta = 1 \) w.r.t. Type-I hybrid censoring (top) and Type-I progressive hybrid censoring (bottom). The dashed lines represent the cumulative distribution function and the density function of \( \hat{\vartheta} \) under (progressive) Type-II censoring.

for any \( t \geq 0 \). Thus, we obtain

\[
\lim_{\vartheta \to 0} F_{\vartheta}^{[D \geq t]}(t) = 1 \cdot \left( \frac{(-1)^m \prod_{j=1}^{m} \gamma_j^*}{\prod_{i=1}^{m} (-\gamma_{m-i+1}^*)} \right) = \frac{\prod_{j=1}^{m} \gamma_j^*}{\prod_{j=1}^{m} \gamma_j^*} = 1,
\]

which yields the assertion for \( \vartheta \to 0 \).

In order to determine \( \lim_{\vartheta \to \infty} F_{\vartheta}^{[D \geq t]}(t) \), we consider for \( d \in \{1, \ldots, m\} \) the limit

\[
\lim_{\vartheta \to \infty} \sum_{j=0}^{d} \frac{(-1)^d e^{-\min(\gamma_{d-j+1}^* T, dt)/\vartheta}}{(d-k)!} \frac{e^{-dt/\vartheta} (\gamma_{d-j+1}^* T - dt)^{d-k}}{\left(1 - e^{-\gamma_{d-j+1}^* T/\vartheta}\right) \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-i+1}^*)} = \sum_{j=0}^{d} \lim_{\vartheta \to \infty} \frac{(-1)^d e^{-\min(\gamma_{d-j+1}^* T, dt)/\vartheta}}{(d-k)!} \frac{e^{-dt/\vartheta} (\gamma_{d-j+1}^* T - dt)^{d-k}}{\left(1 - e^{-\gamma_{d-j+1}^* T/\vartheta}\right) \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-i+1}^*)},
\]

(4.28)
By applying l’Hospital’s rule, we first find for \( d = 1 \) that
\[
\lim_{\vartheta \to \infty} \frac{1}{\sum_{j=0}^{\vartheta - 1}} \frac{(-1)e^{-\min(\gamma_{d-j+1}^*, T, t)/\vartheta}}{(1 - e^{-\gamma_{d-j+1}^*/\vartheta}) \prod_{i=0, i \neq j}^{\vartheta - 1} (\gamma_{d-j+1}^* - \gamma_{d-i}^*)} = \lim_{\vartheta \to \infty} \frac{(-\vartheta)^{-2} \min(\gamma_{d,j+1}^*, T, t) e^{-\min(\gamma_{d,j+1}^*, T, t)/\vartheta}}{(-\vartheta)^{-2} \gamma_{1}^* T e^{-\gamma_{1}^*/\vartheta}(\gamma_{1}^* - \gamma_{2}^*)} = \frac{\min\{\gamma_{1}^*, T, t\} - \min\{\gamma_{2}^*, T, t\}}{(\gamma_{1}^* - \gamma_{2}^*) \gamma_{1}^*/T}, \quad \gamma_{1}^* \neq \gamma_{2}^*.
\]

For \( d \in \{2, \ldots, m\} \), we calculate the derivative of the numerator in (4.28),
\[
\frac{d}{d\vartheta} \left[ (-1)^d e^{-\min(\gamma_{d-j+1}^*, T, dt)/\vartheta} + \sum_{k=1}^{d-1} \frac{\vartheta^{k-d}(\gamma_{d-j+1}^* T - dt)^{d-k}}{(d-k)!} e^{-dt/\vartheta} (\gamma_{d-j+1}^* T - dt)^{d-k} \right] = (-1)^d (\vartheta)^{-2} \min(\gamma_{d-j+1}^*, T, dt) e^{-\min(\gamma_{d-j+1}^*, T, dt)/\vartheta}
\]
\[
+ \sum_{k=1}^{d-1} \frac{(-1)^k}{(d-k)!} (\gamma_{d-j+1}^* T - dt)^{d-k} \left[ (k-d)\vartheta^{k-d-1} e^{-dt/\vartheta} + dt \vartheta^{-2} e^{-dt/\vartheta} \vartheta^{k-d} \right] = \frac{1}{\vartheta^d} \left[ (-1)^d \min(\gamma_{d,j+1}^*, T, dt) e^{-\min(\gamma_{d,j+1}^*, T, dt)/\vartheta}
\]
\[
+ \sum_{k=1}^{d-1} \frac{(-1)^k}{(d-k)!} (\gamma_{d-j+1}^* T - dt)^{d-k} \left[ (k-d)\vartheta^{k-d-1} e^{-dt/\vartheta} + dt \vartheta^{-2} e^{-dt/\vartheta} \vartheta^{k-d} \right].
\]

Further, by taking into account that
\[
\lim_{\vartheta \to \infty} \frac{(k-d)}{\vartheta^{d-k-1}} e^{-dt/\vartheta} = \begin{cases} 0, & k \in \{2, \ldots, d-2\}, \\ -1, & k = d - 1, \end{cases}
\]
and
\[
\lim_{\vartheta \to \infty} \frac{dt}{\vartheta^{d-k}} e^{-dt/\vartheta} = 0, \quad d \in \{2, \ldots, k-1\},
\]
we arrive at
\[
\sum_{j=0}^{d} \lim_{\vartheta \to \infty} \frac{(-1)^d e^{-\min(\gamma_{d-j+1}^*, T, dt)/\vartheta} + \sum_{k=1}^{d-1} \frac{\vartheta^{k-d}(\gamma_{d-j+1}^* T - dt)^{d-k}}{(d-k)!} e^{-dt/\vartheta} (\gamma_{d-j+1}^* T - dt)^{d-k}}{(1 - e^{-\gamma_{d-j+1}^*/\vartheta}) \prod_{i=0, i \neq j}^{\vartheta - 1} (\gamma_{d-j+1}^* - \gamma_{d-i}^*)} = \sum_{j=0}^{d} \frac{(-1)^d \min(\gamma_{d-j+1}^*, T, dt) + (-1)^d \max(\gamma_{d-j+1}^*, T - dt, 0)}{\gamma_{1}^* T \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-i}^*)}
\]
\[
= \sum_{j=0}^{d} \frac{(-1)^d \gamma_{d-j+1}^* T \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-i}^*)}{\gamma_{1}^* \prod_{i=0, i \neq j}^{d} (\gamma_{d-j+1}^* - \gamma_{d-i}^*)} = \frac{(-1)^d}{\gamma_{1}^*} \left[ \gamma_{d+1}^*, \ldots, \gamma_{1}^*, (\cdot) \right] = 0.
\]
The last equation follows due to the properties for divided differences (see Lemma 3.1.3, (iii), see also Remark 2.8 in Balakrishnan et al. (2001a)). The above results lead us finally to

\[
\lim_{\vartheta \to \infty} F_\vartheta^{\hat{\vartheta}|D \geq 1}(t) = \gamma_1^* \min\{\gamma_1^* T, t\} - \min\{\gamma_2^* T, t\} = \min\{\gamma_1^* T, t\} - \min\{\gamma_2^* T, t\}.
\]

The evaluation of the above expression for different intervals for \( t \geq 0 \) w.r.t. the cases \( \gamma_1^* > \gamma_2^* \) and \( \gamma_1^* < \gamma_2^* \), proves the assumption for \( \vartheta \to \infty \).

It should be noted that the way of proving Theorem 4.2.15 has been used in the context of generalized progressive hybrid censoring (cf. Górný and Cramer, 2016). The limits for \( F_\vartheta^{\hat{\vartheta}|D \geq 1} \) are necessary for the derivation of exact confidence intervals for \( \vartheta \). As shown in Balakrishnan et al. (2014), the condition that the equations

\[
F_\vartheta^{\hat{\vartheta}|D \geq 1}(t) = 1 - \alpha_1 \quad \text{and} \quad F_\vartheta^{\hat{\vartheta}|D \geq 1}(t) = 1 - \alpha_2
\]

for \( \alpha_1, \alpha_2 > 0 \), with \( \alpha_1 + \alpha_2 = \alpha \in (0, 1) \), have to be solvable in order to employ the pivoting method (see, e.g., Casella and Berger, 2002, Theorem 9.2.12), are not always satisfied. The limits of \( F_\vartheta^{\hat{\vartheta}|D \geq 1} \) for \( \vartheta \to 0 \) and \( \vartheta \to \infty \) are crucial in order to determine whether the equations given in (4.29) are solvable or not. Figure 4.4 illustrates the plots of \( F_\vartheta^{\hat{\vartheta}|D \geq 1} \) for an increasing \( \vartheta \) w.r.t. \( \gamma_1^* > \gamma_2^* \) (left) and \( \gamma_1^* < \gamma_2^* \) (right).

We conclude our investigations for the distribution of the MLE \( \hat{\vartheta} \) by considering the particular case where \( \gamma_1^* = \cdots = \gamma_m^* \). According to Remark 4.2.4, (i), the distribution is degenerated for \( d \in \{1, \ldots, m - 1\} \). However, the density function \( f^{\hat{\vartheta}|D=m} \) exists.

**Corollary 4.2.16** Let \( \gamma_1^* = \cdots = \gamma_m^* > 0 \) and \( \gamma_{m+1}^* = 0 \). Then, the conditional cumulative distribution function of \( \hat{\vartheta} \) for \( t \geq 0 \) is given by

\[
F^{\hat{\vartheta}|D \geq 1}(t) = \frac{1}{1 - e^{-\gamma_1^*(T-\mu)/\vartheta}}.
\]

**Figure 4.4:** Plots of \( F_\vartheta^{\hat{\vartheta}|D \geq 1} \) for \( T = 1 \) and \( m = n = 4 \) w.r.t. \( \vartheta = 5 \) (dotted line), \( \vartheta = 25 \) (dashed line) and \( \vartheta = 5^6 = 15625 \) (solid line). Left: \( (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*) = (10, 6, 2, 4) \). Right: \( (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*) = (4, 6, 2, 10) \).
4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions

![Plots of \( F_{\vartheta|D \geq 1}^{\vartheta} \) for \( n = m = 5 \) and \( \vartheta = 1 \). Left: \( \gamma_j^* = 2, 1 \leq j \leq 5 \) and \( T = 1 \). Right: \( \gamma_j^* = 3, 1 \leq j \leq 5 \) and \( T = 2 \).](image)

**Figure 4.5:** Plots of \( F_{\vartheta|D \geq 1}^{\vartheta} \) for \( n = m = 5 \) and \( \vartheta = 1 \). Left: \( \gamma_j^* = 2, 1 \leq j \leq 5 \) and \( T = 1 \). Right: \( \gamma_j^* = 3, 1 \leq j \leq 5 \) and \( T = 2 \).

\[
\times \left[ \sum_{d=1}^{m-1} 1_{[\gamma_1^*(T-\mu), \infty)}(dt) \vartheta \frac{(\gamma_1^*(T-\mu))^d}{\vartheta^d d!} e^{-\gamma_1^*(T-\mu)/\vartheta} + \int_0^{mt} 1_{[0, \gamma_1^*(T-\mu)]}(s) \frac{s^{m-1}}{(m-1)! \vartheta^m} e^{-s/\vartheta} ds \right].
\]

**Proof.** Let \( t_0 < t_1 \) and \( n \in \mathbb{N} \). Then, we note that

\[
B_{n-1}(x|t_0, t_1^n) = 1_{[t_0, t_1]}(x) \frac{(x - t_0)^{n-1}}{(t_1 - t_0)^n}, \quad x \in \mathbb{R},
\]

holds. This identity can be easily verified using Lemma 3.1.9, (i). If we apply (4.30) to the representation of \( F_{\vartheta|D \geq 1}^{\vartheta} \) given in Theorem 4.2.14, (i), we arrive at

\[
F_{\vartheta|D \geq 1}^{\vartheta}(t) = \frac{1}{1 - e^{\gamma_1^*(T-\mu)/\vartheta}} \left[ \sum_{d=1}^{m-1} 1_{[\gamma_1^*(T-\mu), \infty)}(dt) \frac{(\gamma_1^*(T-\mu))^d}{\vartheta^d d!} e^{-\gamma_1^*(T-\mu)/\vartheta} \right. \\
+ \left. \int_0^{mt} 1_{[0, \gamma_1^*(T-\mu)]}(s) \frac{s^{m-1}}{(m-1)! \vartheta^m} e^{-s/\vartheta} ds \right].
\]

The assertion follows by using the gamma density function \( f_{\vartheta, d+1}^{\vartheta} \) (see Definition C.1.4) and by using the notation \( F_{\vartheta, m}^{\vartheta} \) for the cumulative distribution function based on \( f_{\vartheta, m}^{\vartheta} \). \( \square \)

Notice that the representations of \( F_{\vartheta|D \geq 1}^{\vartheta} \) given in Theorem 4.2.14, (i) and in Corollary (4.2.16) have been already presented in Burkschat et al. (2016).

**Remark 4.2.17** According to Corollary 4.2.16 the conditional cumulative distribution function is (due to \( F_{\vartheta, m}^{\vartheta} \)) continuous on the interval \([0, \gamma_1^*(T-\mu)/m]\). It further has discontinuities at \( \gamma_1^*(T-\mu)/d, d \in \{1, \ldots, m-1\} \), resulting from the indicator functions \( 1_{[\gamma_1^*(T-\mu)/d, \infty)} \).

Examples for the shapes of \( F_{\vartheta|D \geq 1}^{\vartheta} \) are depicted in Figure 4.5. The therein considered setups correspond to Type-I censored second record values and to Type-I censored third record values.
Table 4.3: Values based on $N = 10^6$ simulated samples.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\vartheta$</th>
<th>$N$</th>
<th>$N^*$</th>
<th>$T$</th>
<th>$\bar{\vartheta}$</th>
<th>s.e.</th>
<th>s.d.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>$10^6$</td>
<td>999121</td>
<td>7</td>
<td>1.198332</td>
<td>0.7322245</td>
<td>0.7586094</td>
</tr>
</tbody>
</table>

Figure 4.6: Histogram and the corresponding cumulative distribution function for Type-I censored (upper) record values, with class width 0.01, $N = 10^6$, $N^* = 999121$, $n = m = 10$, $\vartheta = 1$, $\mu = 0$, $\gamma_j^* = 1$, $1 \leq j \leq 10$ and $T = 7$.

In the following, let $\vartheta = 1$, $\mu = 0$, $n = m = 10$, $T = 7$ and $\gamma_j^* = 1$, $1 \leq j \leq 10$. For this setup of record values (minimal repair times), the corresponding MLE is given by (see Remark 4.2.4, (i))

$$
\hat{\vartheta} = \begin{cases} 
\frac{7}{D}, & D \in \{1, \ldots, 9\}, \\
\frac{Z_{10}^*}{10}, & D = 10.
\end{cases}
$$

We conducted a simulation based on $N = 10^6$ samples. In Table 4.3 the model parameters as well as the mean $\bar{\vartheta}$, the standard error (s.e.) and the standard deviation (s.d.) from the true value $\vartheta = 1$, that is

$$
\bar{\vartheta} = \frac{1}{N^*} \sum_{i=1}^{N^*} \hat{\vartheta}_i, \quad \text{(s.e.): } \sqrt{\frac{1}{N^* - 1} \sum_{i=1}^{N^*} (\hat{\vartheta}_i - \bar{\vartheta})^2}, \quad \text{(s.d.): } \sqrt{\frac{1}{N^* - 1} \sum_{i=1}^{N^*} (\hat{\vartheta}_i - \vartheta)^2}; \quad (4.31)
$$

are listed. We denote by the counter $N^* \leq N$ the number of samples where at least one failure has been observed. Table 4.3 provides the values arising from the simulation and Figure 4.6 shows the histogram for a class width of 0.01 as well as the respective cumulative distribution function.

### 4.2.2 Distribution Theory for an Unknown Location Parameter

In this section, we assume the location parameter $\mu$ to be unknown. Hence, we consider the modified total time on test statistic $V_D$ of the form given in (4.7). By taking into account
that

\[ T - Z_i^* = T - \mu - \frac{\gamma_i^* (Z_i^* - \mu)}{\gamma_i^*}, \]

we obtain the following expression for \( V_D \) in terms of the normalized spacings \( W_i^* \), \( 1 \leq j \leq m \), (cf. (4.13)),

\[ V_D = \sum_{j=2}^{D} \left( 1 - \frac{\gamma_{D+1}^*}{\gamma_j^*} \right) W_j^* + \gamma_{D+1}^* \left( T - \mu - \frac{W_D^*}{\gamma_1^*} \right), \quad D \in \{2, \ldots, m\}. \]

In order to determine the conditional joint density function of \( W_i^* \) and \( V_d \), for \( d \in \{1, \ldots, m\} \), we define the linear transformation \( \Phi_{V,i^*} : W_d(T - \mu | \gamma_d^*) \rightarrow [0, \infty) \times B_{d-2}^{[i^*]}(v) \times \mathbb{R} \), with

\[
\Phi_{V,i^*}(w_d) = \left( w_{i^*-1}, \sum_{j=2}^{d} \left( 1 - \frac{\gamma_{d+1}^*}{\gamma_j^*} \right) w_j + \gamma_{d+1}^* \left( T - \mu - \frac{w_1}{\gamma_1^*} \right), w_{i^*}, \ldots, w_d \right),
\]

and

\[
|\det(D\Phi_{V,i^*})| = \frac{\gamma_{i^*}^* - \gamma_{d+1}^*}{\gamma_{i^*}^*}, \quad (4.32)
\]

for \( i^* \in \{2, \ldots, d\} \). The arguments leading to the specification of the set \( B_{d-2}^{[i^*]}(v) \) are very similar to those used for the set \( A_{d-1}^{[\star]}(s) \) (see Section 4.2.1).

Let \( T_{d-1}^{[\star]} = \{2, \ldots, d\} \). Then, the variables \( (w_j)_{j \in T_{d-1}^{[\star]}} \), belonging to the set \( B_{d-2}^{[i^*]}(v) \), have to satisfy the conditions specified in \( W_d(T - \mu | \gamma_d^*) \). They have further to satisfy the condition imposed by \( \Phi_{V,i^*} \) for \( v \in \mathbb{R} \), i.e.,

\[
v = \sum_{j=2}^{d} \left( 1 - \frac{\gamma_{d+1}^*}{\gamma_j^*} \right) w_j + \gamma_{d+1}^* \left( T - \mu - \frac{w_1}{\gamma_1^*} \right).
\]

Further, it is clear that \( w_1 \in [0, \infty) \).

Applying \( \Phi_{V,i^*} \) to \( f_{W_j^*, 1 \leq j \leq d | D=d, v} \), \( d \in \{1, \ldots, m-1\} \), yields

\[
f_{W_j^*, 1 \leq j \leq d-1, V_d | D=d}(w_{d-1}, v) = 1_{[0, \infty) \times B_{d-2}^{[i^*]}(v) \times \mathbb{R}} \left( w_1, (w_j)_{j \in T_{d-1}^{[\star]}}, v \right)
\times \frac{\gamma_{d+1}^* \gamma_{i^*}^*}{|\gamma_{i^*}^* - \gamma_{d+1}^*|^d f_{d+1} \{ \gamma_{d+1}^* \}^{(d+1) \mathbb{R}}} e^{-(w_1 + v)/\theta}.
\]

Integrating w.r.t. the variables \( (w_j)_{j \in T_{d-1}^{[\star]}} \) gives for \( (w, v) \in [0, \infty) \times \mathbb{R} \)

\[
f_{W_i^*, V_d | D=d}(w, v) = \frac{\gamma_{d+1}^* \gamma_{i^*}^* e^{-(w_1 + v)/\theta}}{|\gamma_{i^*}^* - \gamma_{d+1}^*|^d f_{d+1} \{ \gamma_{d+1}^* \}^{(d+1) \mathbb{R}}} \text{vol}_{d-2}(B_{d-2}^{[i^*]}(v)) \quad (4.33)
\]

The volume of \( B_{d-2}^{[i^*]}(v) \) can be determined in Section 4.2.1. For \( c \in (0, \infty) \) let \( \gamma_{2d+1}c = (\gamma_{d+1}c, \ldots, \gamma_{2c}) \in \mathbb{R}^d \). 

Lemma 4.2.18 Let \( d \in \{2, \ldots, m\} \) with \( \gamma_{d+1}^* \neq \gamma_i^* \) for any \( i^* \in \{1, \ldots, d\} \). Then, for \( v \in \mathbb{R} \),
the set \( B_{d-2}^{[i^*]}(v) \) is given by

\[
B_{d-2}^{[i^*]}(v) = \begin{cases} 
\mathcal{M}_{\ell^*-1}(v|\beta, t), & \gamma_{d+1}^* < \gamma_i^*, \\
\mathcal{M}_{\ell^*-1}(v|\beta, \tilde{t}), & \gamma_{d+1}^* > \gamma_i^*, 
\end{cases}
\]

where \( j^* = i^* - 1, \ell^* = d - 1 \) and \( \mathcal{M}_{\ell^*-1}(v|\beta, t) \) as given in Theorem 3.2.1. The corresponding parameter vectors are given by \( \beta = (\beta_1, \ldots, \beta_{\ell^*}), t = (t_0, \ldots, t_{\ell^*}) \) and \( \tilde{t} = (t_{j^*}, t_1, \ldots, t_{j^*-1}, t_0, t_{j^*+1}, \ldots, t_{\ell^*}) \), with

\[
\beta_j = \gamma_{j+1}^* \left( T - \mu - \frac{w}{\gamma_1^*} \right), \quad j \in \{1, \ldots, d-1\} \setminus \{i^*-1\}, \quad \beta_{i^*-1} > 0
\]

and

\[
t_j = \gamma_{j+1}^* \left( T - \mu - \frac{w}{\gamma_1^*} \right), \quad 1 \leq j \leq d - 1, \quad t_0 = \gamma_{d+1}^* \left( T - \mu - \frac{w}{\gamma_1^*} \right).
\]

The volume of \( B_{d-2}^{[i^*]}(s) \), is for \( (w, v) \in [0, \gamma_1^*(T - \mu)] \times \mathbb{R} \) given by

\[
\text{vol}_{d-2}(B_{d-2}^{[i^*]}(s)) = \frac{(d-1)!}{(T - \mu - \frac{w}{\gamma_1^*})} \left| \gamma_{d+1}^* - \gamma_i^* \right| B_{d-2}(v|\gamma_{2,d+1}^*(T - \mu - \frac{w}{\gamma_1^*})).
\]

**Proof.** In order to determine \( B_{d-2}^{[i^*]}(v), v \in \mathbb{R}, \) for \( d \in \{2, \ldots, m\} \), we observe that

\[
v = \sum_{j=2}^{d} \left( 1 - \frac{\gamma_{d+1}}{\gamma_j^*} \right) w_j + \gamma_d^* \left( T - \mu - \frac{w}{\gamma_1^*} \right),
\]

and

\[
\sum_{j \in I_{2,d}^{[i^*]}} \left( 1 - \frac{\gamma_{d+1}}{\gamma_j^*} \right) w_j \begin{cases} 
\leq v - \gamma_{d+1}^*(T - \mu - \frac{w}{\gamma_1^*}), & \gamma_{d+1}^* < \gamma_i^*, \\
\geq v - \gamma_{d+1}^*(T - \mu - \frac{w}{\gamma_1^*}), & \gamma_{d+1}^* > \gamma_i^*, 
\end{cases}
\]

for \( w_j \geq 0, 2 \leq j \leq d \). Then, we obtain for \( \gamma_{d+1}^* \neq \gamma_i^* \),

\[
w_{i^*} = \frac{\gamma_{i^*} \gamma_d^*}{\gamma_i^* - \gamma_{d+1}^*} \left( v - \gamma_{d+1}^*(T - \mu - \frac{w}{\gamma_1^*}) \right) - \sum_{j \in I_{2,d}^{[i^*]}} \left( 1 - \frac{\gamma_{d+1}}{\gamma_j^*} \right) w_j.
\]

Rewriting the sum \( \sum_{j=2}^{d} \frac{w_j}{\gamma_j^*} \) in terms of the variables \( w_j \) \( j \in I_{2,d}^{[i^*]} \) and \( v \) yields

\[
\sum_{j=2}^{d} \frac{w_j}{\gamma_j^*} = \sum_{j \in I_{2,d}^{[i^*]}} \frac{w_j}{\gamma_j^*} + \frac{1}{\gamma_i^* - \gamma_{d+1}^*} \left( v - \gamma_{d+1}^*(T - \mu - \frac{w}{\gamma_1^*}) \right) - \sum_{j \in I_{2,d}^{[i^*]}} \left( 1 - \frac{\gamma_{d+1}}{\gamma_j^*} \right) w_j
\]

\[
= \sum_{j \in I_{2,d}^{[i^*]}} \left( \frac{1}{\gamma_j^*} - \frac{\gamma_i^* - \gamma_{d+1}^*}{\gamma_j^* (\gamma_i^* - \gamma_{d+1}^*)} \right) w_j + \frac{v - \gamma_{d+1}^*(T - \mu - \frac{w}{\gamma_1^*})}{\gamma_i^* - \gamma_{d+1}^*}.
\]
If we take into account that \( \sum_{j=1}^{d} \frac{w_j}{\gamma_j} \leq (T - \mu) \iff \sum_{j=2}^{d} \frac{w_j}{\gamma_j} \leq (T - \mu - \frac{w_1}{\gamma_1}) \) (cf. \( \mathcal{W}_d(T - \mu|\gamma^*_d) \) in (4.16)), we have for \( \gamma^*_{d+1} < \gamma^*_i \),

\[
\sum_{j \in \mathcal{I}^*_d} \left( \frac{1}{\gamma_j} - \frac{\gamma^*_j - \gamma^*_{d+1}}{\gamma_j (\gamma^*_i - \gamma^*_{d+1})} \right) w_j + \frac{v - \gamma^*_{d+1} \left( T - \mu - \frac{w}{\gamma_1} \right)}{\gamma^*_i - \gamma^*_{d+1}} \leq (T - \mu - \frac{w}{\gamma_1}) \]

\[
\iff \sum_{j \in \mathcal{I}^*_d} \left( \frac{\gamma^*_j - \gamma^*_d}{\gamma_j} \right) w_j \leq (T - \mu - \frac{w}{\gamma_1}) - \frac{v - \gamma^*_{d+1} \left( T - \mu - \frac{w}{\gamma_1} \right)}{\gamma^*_i - \gamma^*_{d+1}}
\]

\[
\iff \sum_{j \in \mathcal{I}^*_d} \left( \frac{\gamma^*_j - \gamma^*_d}{\gamma_j} \right) w_j \geq \left( \gamma^*_{d+1} - \gamma^*_i \right) (T - \mu - \frac{w}{\gamma_1}) + v - \gamma^*_{d+1} \left( T - \mu - \frac{w}{\gamma_1} \right)
\]

\[
\iff \sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \geq v - \gamma^*_i \left( T - \mu - \frac{w}{\gamma_1} \right). \tag{4.35}
\]

By analogy, we find for \( \gamma^*_{d+1} > \gamma^*_i \) that

\[
\sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \leq v - \gamma^*_i \left( T - \mu - \frac{w}{\gamma_1} \right). \tag{4.36}
\]

The first inequality in (4.34) yields together with (4.35) and \( \gamma^*_{d+1} < \gamma^*_i \),

\[
B^{|i^{*}|}_{d-2}(v) = \left\{ w_{j \in \mathcal{I}^*_d} \in \mathbb{R}^{d-2} \middle| w_j \geq 0, j \in \mathcal{I}^*_d, \sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \leq v - \gamma^*_{d+1} \left( T - \mu - \frac{w}{\gamma_1} \right), \sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \geq v - \gamma^*_i \left( T - \mu - \frac{w}{\gamma_1} \right) \right\}.
\]

For \( \gamma^*_{d+1} > \gamma^*_i \), we obtain with the second inequality from (4.34) and with (4.36)

\[
B^{|i^{*}|}_{d-2}(v) = \left\{ w_{j \in \mathcal{I}^*_d} \in \mathbb{R}^{d-2} \middle| w_j \geq 0, j \in \mathcal{I}^*_d, \sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \leq v - \gamma^*_i \left( T - \mu - \frac{w}{\gamma_1} \right), \sum_{j \in \mathcal{I}^*_d} \left( 1 - \frac{\gamma^*_j}{\gamma^*_d} \right) w_j \geq v - \gamma^*_{d+1} \left( T - \mu - \frac{w}{\gamma_1} \right) \right\}.
\]

If we apply Corollary 3.2.3 for \( j^* = i^* - 1 \) and \( \ell^* = d - 1 \), we find by performing exactly the same steps as in the proof to Lemma 4.2.8 that

\[
\beta_j = \gamma^*_{j+1} \left( T - \mu - \frac{w}{\gamma_1} \right), \quad j \in \{1, \ldots, d-1\} \setminus \{i^* - 1\}
\]
and \( t_j = \gamma^*_j + 1(T - \mu - \frac{w}{\gamma^*_1}), \quad 1 \leq j \leq d - 1, \quad t_0 = \gamma^*_{d+1}(T - \mu - \frac{w}{\gamma^*_1}), \)

for \( \gamma^*_{d+1} < \gamma^*_1 \). For \( \gamma^*_{d+1} > \gamma^*_1 \), we obtain the same values for the \( \beta \)'s and the \( t \)'s, except for \( t_0 = \gamma^*_0(T - \mu - \frac{w}{\gamma^*_1}) \) and \( t_{i+1} = \gamma^*_{d+1}(T - \mu - \frac{w}{\gamma^*_1}) \). According to \( \beta_j > 0, \) \( j \in \{1, \ldots, d-1\} \), we find that \( w < \gamma^*_1(T - \mu) \). This finishes the proof. \( \square \)

Lemma 4.2.18 allows us to derive the joint density function of the first spacing \( W^*_1 \) and the modified total time on test statistic \( V_d, \) \( d \in \{2, \ldots, m\} \).

**Theorem 4.2.19** Let \( \gamma^*_1, \gamma^*_2, \ldots, \gamma^*_m > 0, \) \( \gamma^*_{m+1} = 0 \) and \( \gamma^*_2 \neq \gamma^*_3 \). Then, the conditional joint density function of \( W^*_1 \) and \( V_d \) is for \( d \in \{2, \ldots, m-1\} \) and \( (w, v) \in [0, \gamma^*_1(T - \mu)] \times [0, \infty) \), given by

\[
 f_{W^*_1, V_d|D=d}(w, v) = \frac{(T - \mu - \frac{w}{\gamma^*_1})^{d-1}}{(d-1)!} \prod_{j=2}^{d+1} \gamma^*_j B_{d-2}(v|\gamma^*_2, d+1(T)) \gamma^*_i e^{-(w+v)/\mu}. 
\]

For \( d = m \) and \( (w, v) \in [0, \gamma^*_1(T - \mu)] \times [0, \infty) \), we have

\[
 f_{W^*_1, V_m|D=m}(w, v) = \frac{(T - \mu - \frac{w}{\gamma^*_1})^{m-1} \prod_{j=2}^{m} \gamma^*_j}{(m-1)!} \int_{\omega_{d-2}(B_{d-2}(v))} B_{m-2}(v|0, \gamma^*_2, m(T - \mu - \frac{w}{\gamma^*_1})) e^{-(w+v)/\mu}. 
\]

**Proof.** The assertion for \( d \in \{2, \ldots, m-1\} \) results from (4.33) and Lemma 4.2.18.

For \( d = m \), we apply transformation \( \Phi_{V,m} \) on \( f_{W^*_1, 1\leq j \leq m|D=m} \) given in (4.2.5). Then, we find together with Lemma 4.2.18 and \( \hat{v}^* = m \) that

\[
 f_{W^*_1, 1\leq j \leq m|D=m}(w, v) = \int_{\omega_{d-2}(B_{d-2}(v))} \frac{e^{-(w+v)/\mu}}{\omega_{d-2}(B_{d-2}(v))} dw_{m-1} \ldots dw_2 
\]

\[
 = \frac{(T - \mu - \frac{w}{\gamma^*_1})^{m-1} \prod_{j=2}^{m} \gamma^*_j}{(m-1)!} \int_{\omega_{d-2}(B_{d-2}(v))} B_{m-2}(v|0, \gamma^*_2, m(T - \mu - \frac{w}{\gamma^*_1})) e^{-(w+v)/\mu}, 
\]

for \( 0 \leq w < \gamma^*_1(T - \mu) \). Due to \( B_{m-2}(v|0, \gamma^*_2, m(T - \mu - \frac{w}{\gamma^*_1})) = 0 \) for \( v < 0 \), we restrict the domain of \( f_{W^*_1, 1\leq j \leq m|D=d} \) to \( [0, \infty) \), \( d \in \{2, \ldots, m\} \).

The above result leads us with the substitution \( w = \gamma^*_1(s - \mu) \) to the bivariate density function of the MLEs \( \hat{\mu} \) and \( \hat{\theta} \).

**Theorem 4.2.20** Let \( \gamma^*_1, \gamma^*_2, \ldots, \gamma^*_m > 0 \) with \( \gamma^*_2 \neq \gamma^*_3 \) and \( \gamma^*_{m+1} = 0 \). Then, the conditional density function \( f_{\hat{\mu}, \hat{\theta}|D \geq 2} \) of the MLEs \( \hat{\mu} \) and \( \hat{\theta} \) is for \( m \geq 2 \) given by

\[
 f_{\hat{\mu}, \hat{\theta}|D \geq 2}(s, t) = \frac{\gamma^*_1}{P(D \geq 2)} \sum_{d=2}^{m} \frac{d(T - s)^{d-1} \prod_{j=2}^{d} \gamma^*_j}{(d-1)!} \int_{0}^{\infty} B_{d-2}(dt|\gamma^*_2, d+1(T - s)) e^{-(\gamma^*_1(s - \mu) + dt)/\mu}. 
\]
4.2 Type-I Sequential Hybrid Censoring from Exponential Distributions

with \((s, t) \in [\mu, T) \times [0, \infty)\) and

\[
P(D \geq 2) = \begin{cases} 
1 - \left[ e^{-\gamma_1(T-\mu)/\vartheta} + \frac{\gamma_1}{\gamma_2 - \gamma_1} \left( e^{-\gamma_1(T-\mu)/\vartheta} - e^{-\gamma_2(T-\mu)/\vartheta} \right) \right], & \gamma_1 \neq \gamma_2, \\
1 - \left[ e^{-\gamma_1(T-\mu)/\vartheta} + \frac{\gamma_1(T-\mu)}{\gamma_2} e^{-\gamma_1(T-\mu)/\vartheta} \right], & \gamma_1 = \gamma_2.
\end{cases}
\] (4.38)

**Proof.** For the probability \(P(D \geq 2)\) (cf. Remark 4.2.4, (ii)), we get according to Lemma 4.1.1

\[
P(D \geq 2) = 1 - \left[ (1 - F(T))^{\gamma_1} + P(D = 1) \right].
\]

Lemma 2.3.6 gives

\[
P(D = 1) = \gamma_1^* G_{2,2}^2 \left[ e^{-(T-\mu)/\vartheta} | \gamma_1^* + 1, \gamma_2^* + 1 \right].
\]

The function \(G_{2,2}^2 \left[ e^{-(T-\mu)/\vartheta} | \gamma_1^* + 1, \gamma_2^* + 1 \right]\) can be calculated for \(\gamma_1^* \neq \gamma_2^*\) as well as for \(\gamma_1^* = \gamma_2^*\), by utilizing Lemma 2.3.5, (i), (ii) and (iv). The assumption then follows by using Theorem 4.2.19 and by proceeding as in the proof to Theorem 4.2.12. \(\square\)

Figure 4.7 depicts two plots of \(f^{\hat{\mu}, \hat{\vartheta}} D \geq 2\), for setups corresponding to Type-I hybrid censoring and Type-I progressive hybrid censoring, respectively.

**Remark 4.2.21** By utilizing the calculations conducted in the proof to Lemma 4.1.2, we find

\[f^{\hat{\mu}, D \geq 2}(s) = \frac{\gamma_1^*}{\vartheta} e^{-\gamma_1^*(s-\mu)/\vartheta} \left( 1 - e^{-\gamma_2^*(T-s)/\vartheta} \right), \quad s \in [\mu, T).\]

Now, the density function of \(\hat{\vartheta}\) conditional on \(\hat{\mu} = s\), for a \(s \in [\mu, T)\), and \(D \geq 2\) is given by

\[f^{\hat{\vartheta}|\hat{\mu} = s, D \geq 2}(t) = \frac{f^{\hat{\vartheta}, D \geq 2}(s, t)}{f^{\hat{\mu}, D \geq 2}(s)}, \quad t \geq 0.
\]

This leads us together with Theorem 4.2.20 to

\[f^{\hat{\vartheta}|\hat{\mu} = s, D \geq 2}(t) = \frac{1}{1 - e^{-\gamma_2^*(T-s)/\vartheta}} \sum_{d=2}^{m} d(T-s)^{d-1} \prod_{j=2}^{d} \gamma_j^* \frac{B_{d-2}(dt|\gamma_{2d+1}^*, T-s)}{(d-1)! \vartheta^{d-1}} e^{-dt/\vartheta},\]

with \((s, t) \in [\mu, T) \times [0, \infty)\). We observe that \(f^{\hat{\vartheta}|\hat{\mu} = s, D \geq 2}\) has the same structure as the density function \(f^{\hat{\vartheta}, D \geq 1}\) presented in Theorem 4.2.12.

In the following, we derive the conditional density function of the bivariate MLE \((\hat{\mu}, \hat{\vartheta})\) according to step (1) of the expected value approach.

**Theorem 4.2.22** Let \(g : \mathbb{R}^2 \rightarrow \mathbb{R}\) be a continuous function. Then, for \(d \in \{2, \ldots, m-1\}\) the conditional expectation of \(g(Z_1^*, V_d)\), with \(V_d\) as in Notation 4.2.1 (see in particular (4.5)), is given by

\[E(g(Z_1^*, V_d)|D = d)\]
Then, censoring, i.e., hybrid censoring, i.e., define for any $D$

$$
\gamma = \left\{ (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (10, 9, 8, 7, 6) \right\}.
$$

**Figure 4.7:** Plots of $f^D_{\gamma | D \geq 2}$ for $\gamma = 1$, $\mu = 0.1$, $n = 10$, $m = 5$ and $T = 1$. Left: setup for Type-I hybrid censoring, i.e., $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (10, 9, 8, 7, 6)$. Right: setup for Type-I progressive hybrid censoring, i.e., $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (10, 8, 6, 4, 2)$.

$$\begin{align*}
E &= (T - z)^{d-1} \prod_{j=1}^{d+1} \gamma_j^* \\
&= \frac{(d-1)!^d}{(m-1)!^m} \int_{[\mu, T]} \int_{[\mu, T]} g(z, s) B_{d-2}(s | \gamma_2^*(T - z)) e^{-[\gamma_1^*(z-\mu)+s]/\vartheta} ds dz.
\end{align*}$$

The expectation of $g(Z_1^*, S_m)$ conditional on $D = m$ is given by

$$
E(g(Z_1^*, V_m) | D = m)
= \frac{(T - z)^{d-1} \prod_{j=1}^{m} \gamma_j^*}{(m-1)!^m} \int_{[\mu, T]} \int_{[\mu, T]} g(z, s) B_{m-2}(s | \gamma_2^*(T - z)) e^{-[\gamma_1^*(z-\mu)+s]/\vartheta} ds dz.
$$

**Proof.** Due to (4.4) and (4.8), we have

$$S_D = V_D + \gamma_1^*(Z_1^* - \mu),$$

for $D \in \{2, \ldots, m\}$. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an arbitrary continuous function. Further, we define for any $z_1 \in \mathbb{R}$ the function $\tilde{g}_{z_1} : \mathbb{R} \rightarrow \mathbb{R}$, with

$$\tilde{g}_{z_1}(x) = g(z_1, x) e^{-[\gamma_1^*(z_1-\mu)+x]/\vartheta}, \quad \vartheta > 0.$$ 

Then, $g_{z_1}$ is continuous on $\mathbb{R}$ for any $z_1 \in \mathbb{R}$, and we obtain for the conditional expectation of $g(Z_1^*, V_d)$, with $f_{Z_1^*, 1 \leq j \leq d | D = d}$ as in Lemma 4.2.2, for $d \in \{2, \ldots, m - 1\},$

$$E(g(Z_1^*, V_d) | D = d)
= \int_{\mathbb{R}^d} g(z_1, \sum_{j=2}^{d} (\gamma_j^* - \gamma_{j+1})(z_j - z_1) + \gamma_d^*(T - z_1)) f_{Z_1^*, 1 \leq j \leq d | D = d}(z_d) dz_d.$$
We proceed by considering the (non-linear) transformation \( \Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d \), with
\[
\Phi(y_1, \ldots, y_d) = \left( y_1, \frac{y_2 - y_1}{T - y_1}, \ldots, \frac{y_d - y_1}{T - y_1} \right)
\]
and \( \det(D\Phi) = (T - y_1)^{(d-1)} \).

Then, we find together with (4.39) and \( y_1 = z_1 \) that
\[
\Phi(\Sigma_{F,T}^d) = \left\{ (z_1, y_2, \ldots, y_d) \in \mathbb{R}^d | \mu \leq z_1 \leq y_2 \leq \cdots \leq y_d \leq 1 \right\}.
\]

This leads us together with Corollary 3.1.10 to
\[
E(g(Z_1^d, V_d) | D = d) = \frac{(T - z_1)^{d-1}\prod_{j=1}^{d+1} \gamma_j^*}{(d-1)!d^{d+1} f_{s,d+1}(T)} \int_{[\mu,T]} \int_{\{(y_2, \ldots, y_d) \in \mathbb{R}^{d-1} | 0 \leq y_2 \cdots \leq y_d \leq 1\}} \times \tilde{g}_z_1 \left( \sum_{j=2}^{d} \left( \gamma_j^* - \gamma_{j+1}^* \right) (T - z_1)y_j + \gamma_{d+1}^* (T - z_1) \right) dy_2 \cdots dy_d dz_1
\]
\[
= \frac{(T - z_1)^{d-1}\prod_{j=1}^{d+1} \gamma_j^*}{(d-1)!d^{d+1} f_{s,d+1}(T)} \times \int_{[\mu,T]} \int_{\mathbb{R}} \tilde{g}_z_1(s) B_{d-2}(s|\gamma_{d+1}^*(T - z_1), \ldots, \gamma_2^*(T - z_1)) ds dz_1
\]
\[
= \frac{(T - z_1)^{d-1}\prod_{j=1}^{d+1} \gamma_j^*}{(d-1)!d^{d+1} f_{s,d+1}(T)} \times \int_{[\mu,T]} \int_{\mathbb{R}} g(z_1, s) B_{d-2}(s|\gamma_2^*(T - z_1)) e^{-|\gamma_1^*(z_1 - \mu) + s|/\vartheta} ds dz_1.
\]

For \( d = m \) we can proceed analogously. \( \square \)

The calculations conducted in the proof of Theorem 4.2.22 show that the adaption of the expected value approach on the bivariate situation is more sophisticated than it is in the context of the volume approach.

We denote in the following by \( F_{s,1:2} \) the joint cumulative distribution function of the sequential order statistics \( Z_1^* \) and \( Z_2^* \).

**Theorem 4.2.23** Let \( \gamma_1^*, \ldots, \gamma_m^* > 0, \gamma_{m+1}^* = 0 \) and \( \gamma_2^* \neq \gamma_3^* \). The distribution of the bivariate MLE \((\hat{\mu}, \hat{\vartheta})\) conditionally on \( D \geq 1 \), is for \( s, t \geq 0 \) and with \( m \geq 2 \), given by
\[
F_{\hat{\mu},\hat{\vartheta},(s,t)|D \geq 1}(s,t)
\]
= \frac{\gamma_1^s}{1 - e^{-\gamma_1^s(T-t)/\vartheta}} \left[ \mathbb{1}_{((s,t)|T-t/\gamma_2^s \leq \min\{s,T\})}(s,t) \right.
\times \left( F_{s,1}(\min\{s,T\}) + F_{s,1:2}\left(T - \frac{t}{\gamma_2^s}, T\right) - F_{s,1}(T - \frac{t}{\gamma_2^s}) - F_{s,1:2}(\min\{s,T\}, T) \right)
\times \sum_{d=2}^{m} \prod_{j=2}^{d} \gamma_j^s \int_{0}^{\mu} \int_{0}^{\mu} (T - x)^{d-1} B_{d-2}(y|\gamma_j^s, (T - x)) e^{-\gamma_1^s(x-t)/\vartheta} dy dx \right].

\textbf{Proof.} The representation for } d \in \{2, \ldots, m\} \text{ follows directly from Theorem 4.2.20. For } D = 1, \text{ we obtain with Remark 4.2.4, (ii) and Lemma 4.1.1}

\begin{align*}
P(\hat{\mu} \leq s, \hat{\vartheta} \leq t|D = 1) &= \frac{1}{P(D = 1)} \left[ \mathbb{1}_{((s,t)|T-t/\gamma_2^s \leq \min\{s,T\})}(s,t) \right.
\times \left( F_{s,1}(\min\{s,T\}) - F_{s,1}(T - \frac{t}{\gamma_2^s}, T) - F_{s,1:2}(\min\{s,T\}, T) \right)
\times \sum_{d=2}^{m} \prod_{j=2}^{d} \gamma_j^s \int_{0}^{\mu} \int_{0}^{\mu} (T - x)^{d-1} B_{d-2}(y|\gamma_j^s, (T - x)) e^{-\gamma_1^s(x-t)/\vartheta} dy dx \right].
\end{align*}

This finishes the proof. \hfill \Box

The above formula holds true for } \gamma_1^s = \gamma_2^s \text{ as well as for } \gamma_1^s \neq \gamma_2^s. \text{ In order to derive an explicit expression for } G_{2,2}^{0,0}[\cdot|\gamma_1^s + 1, \gamma_2^s + 1] \text{ one might make use of the properties of the Meijer’s G-function (see Lemma 2.3.5) as in the proof to Theorem 4.2.20.}

\textbf{Remark 4.2.24} The conditional marginal distribution for } \hat{\vartheta} \text{ is given by

\begin{align*}
F^{\hat{\vartheta}|D \geq 1}(t) &= \int_{\mu}^{T} F^{\hat{\mu},\hat{\vartheta}|D \geq 1}(s,t) ds.
\end{align*}

For } \gamma_1^s, \ldots, \gamma_m^s > 0 \text{ pairwise distinct and by taking into account

\begin{align*}
\int_{\mu}^{T} (T-s)^{d-1} B_{d-2}(t|\gamma_2^s, (T-s)) e^{-\gamma_1^s(s-t)/\vartheta} ds
&= \int_{\mu}^{T} \sum_{j=0}^{d-1} \prod_{i=0, i \neq j}^{d-1} (\gamma_{d-j+1}^s - \gamma_{d-i+1}^s) e^{-\gamma_1^s(s-t)/\vartheta} ds,
\end{align*}

For } \gamma_1^s, \ldots, \gamma_m^s > 0 \text{ pairwise distinct and by taking into account

\begin{align*}
\int_{\mu}^{T} (T-s)^{d-1} B_{d-2}(t|\gamma_2^s, (T-s)) e^{-\gamma_1^s(s-t)/\vartheta} ds
&= \int_{\mu}^{T} \sum_{j=0}^{d-1} \prod_{i=0, i \neq j}^{d-1} (\gamma_{d-j+1}^s - \gamma_{d-i+1}^s) e^{-\gamma_1^s(s-t)/\vartheta} ds,
\end{align*}
the conditional cumulative distribution function $F_{\hat{\theta}|D \geq 1}$ can be obtained by integrating partially the expression on the right-hand side. In fact, one can employ the same arguments as in the proof to Lemma 3.3.6, (i).

For the particular setting of Type-I progressive hybrid censoring, the marginal distribution for $\hat{\theta}$ can be also derived by the means of the moment generating function approach as done in Chan et al. (2015, Theorem 3). They obtained an expression of the respective density function in terms of gamma density functions.

We conclude this section with the following result.

**Theorem 4.2.25** The distribution of $\hat{\mu}$ conditional on $D \geq 1$ is given by

$$F_{\hat{\mu}|D \geq 1}(s) = \mathbb{I}_{[\mu, \infty)}(s) \frac{1 - e^{\gamma_1 (\min(s,T)-\mu)/\vartheta}}{1 - e^{\gamma_1 (T-\mu)/\vartheta}}.$$

**Proof.** The assumption follows as in Cramer and Balakrishnan (2013, Theorem 5.1), where Type-I progressive hybrid censoring was considered. □

### 4.3 Further Aspects Related to Type-I Hybrid Censoring from Exponential Distributions

In the following (Section 4.3.1), we present a result, which allows a comfortable transfer from B-spline based representations for the density function of the MLE $\hat{\theta}$, to representations in terms of gamma density functions. In Section 4.3.2, we calculate the limit of $f_{\hat{\theta}|D \geq 1}$ for $T \to \infty$ w.r.t. Type-I progressive hybrid censoring. Further, the monotonicity of the conditional distribution for the total time on test statistic $S_d$ in $\hat{\theta}$ under Type-I progressive hybrid censoring is considered.

#### 4.3.1 From B-spline Representations to Gamma Representations

The approaches leading to expressions in terms of B-splines and the moment generating function approach both have advantages, such that one representation seems more suitable than the other w.r.t. a particular aspect. The conditional moment generating function approach allows a convenient derivation of the moments as well as the calculation of marginal distributions. Further, the resulting expressions for the density functions can be written as linear combinations of gamma density functions. The B-spline representation on the other hand, as we have seen, is very compact and can be implemented very easy and efficiently. This holds also true for related expressions, as for formulas for the moments (see Corollary 4.2.13) and for the cumulative distribution function (see Theorem 4.2.14, (ii)). In order to exploit the advantages of both representations and to make use of the theoretical tools both representations have to offer, it is desirable to have access to both representations. It would further be preferable to achieve this goal without paying the price of conducting both very different and elaborate strategies. We hence present a result which transfers the product of a B-spline with the exponential function to a linear combination of gamma density functions. Let $f_{\vartheta, \beta}$ denote the density function of the gamma distribution $\Gamma(\vartheta, \beta)$, with parameters $\vartheta, \beta > 0$ (see Definition C.1.4).
Theorem 4.3.1 Let \( t_0, \ldots, t_d \geq 0 \), be a knot sequence where not all knots are equal, \( d \in \mathbb{N} \). Then, for \( c, \vartheta > 0 \), the following identity holds

\[
B_{d-1}(cx|t_1, \ldots, t_{d+1}) e^{-cx/\vartheta} = \frac{d! \vartheta^d}{c} \left[ t_0, \ldots, t_d; f^\vartheta_{\vartheta, d}(x - \frac{c}{\vartheta}) e^{-c/(d\vartheta)} \right], \quad x \in \mathbb{R}.
\]

Proof. By analogy with the positive part of \( x \), denoted by \( (x)_+ \), we introduce the negative part of \( x \) as \( (x)_- = \min\{0, x\} \). Thus, we consider the following two identities (cf. Cramer and Balakrishnan (2013) and Cramer et al. (2016), respectively)

\[
(x)^d_+ = (-1)^d(-x)^d_- \quad \text{and} \quad (x)^d = x^d - (x)^d_+.
\] (4.40)

By exploiting (4.40) and by taking into account the linearity property and the vanishing property of the divided differences (see Lemma 3.1.3, (iv) and (iii)), we arrive at

\[
B_{d-1}(cx|t_0, \ldots, t_d) e^{-cx/\vartheta} = \frac{d! \vartheta^d}{c} \left[ t_0, \ldots, t_d; f^\vartheta_{\vartheta, d}(x - \frac{c}{\vartheta}) e^{-c/(d\vartheta)} \right] = \frac{d! \vartheta^d}{c} \left[ t_0, \ldots, t_d; f^\vartheta_{\vartheta, d}(x - \frac{c}{\vartheta}) e^{-c/(d\vartheta)} \right].
\]

This proves the assumption. \( \square \)

Remark 4.3.2 According to the steps performed in the above proof, we see that

\[
[t_0, \ldots, t_d; (x - \gamma^d_+)] = (-1)^d [t_0, \ldots, t_d; (x - \gamma^d_-)], \quad d \in \mathbb{N},
\]

holds, for any knots \( t_0, \ldots, t_d \), which are not all equal.

The above theorem allows us to establish an alternative representation for \( f^{\vartheta}_{\vartheta, D} \) based on the expression given in Theorem 4.2.12.

Theorem 4.3.3 Let \( \gamma_1, \ldots, \gamma_m > 0 \) with \( \gamma_1 \neq \gamma_2 \) and \( \gamma_{m+1} = 0 \). Then, the conditional density function of the MLE \( \hat{\vartheta} \) in terms of gamma density functions is given by,

\[
f^{\vartheta}_{\vartheta, D}(s) = \frac{1}{1 - e^{-\gamma_1 T/\vartheta}} \sum_{d=1}^{m} \left( \prod_{j=1}^{d} (-\gamma^*_j T) \right) \left[ \gamma^*_{d+1} T, \ldots, \gamma^*_d T; f^\vartheta_{\vartheta, d}(x - \frac{c}{d}) e^{-c/(d\vartheta)} \right], \quad s \geq 0.
\]

Proof. The assumption follows by applying Theorem 4.3.1 with \( t_j = \gamma^*_j T, 0 \leq j \leq d \), \( d \in \{1, \ldots, m\} \), to \( f^{\vartheta}_{\vartheta, D} \) as given in Theorem 4.2.12. \( \square \)
4.3 Further Aspects Related to Type-I Hybrid Censoring from Exponential Distributions

Due to the implicit usage of the divided differences the above expression is of limited practical relevance. Therefore, we provide an explicit expression for $f^{\bar{\theta}[D] \geq 1}$ in terms of gamma density functions w.r.t. different settings for the $\gamma$'s.

**Corollary 4.3.4** Let $\mu = 0$.

(i) Let $\gamma_1^*, \ldots, \gamma_m^* > 0$ be pairwise distinct and $\gamma_{m+1}^* = 0$. Then, the density function given in Theorem 4.3.3, can be written as, for $s \in \mathbb{R}$,

$$f^{\bar{\theta}[D \geq 1]}(s) = \frac{1}{1-e^{-\gamma/T}} \sum_{d=1}^{n} \left( \prod_{j=1}^{d} \gamma_j^* \right) \sum_{j=0}^{d} \frac{e^{-\gamma_{d-j+1}^* T/\vartheta}}{\prod_{i=0, i \neq j}^{d} \left( \gamma_{d-i+1}^* - \gamma_{d-j+1}^* \right) } f_{\vartheta,d}^{\gamma} \left( x - \frac{\gamma_{d-j+1}^* T}{d} \right).$$

(ii) Let $\gamma_1^*, \ldots, \gamma_m^* > 0$ with $\gamma_1^* \neq \gamma_2^*$ and $\gamma_{m+1}^* = 0$. Further, let $\beta_1, \ldots, \beta_\ell > 0$, $\ell \leq m + 1$, be distinct knots and $n_1, \ldots, n_\ell$ be integers with $\sum_{j=1}^{\ell} n_j = m + 1$ and $n_1 = n_\ell = 1$, with

$$\beta_1 = \gamma_1^* T, \quad \{\beta_2^{n_2}, \ldots, \beta_\ell^{n_\ell} \} = \{\gamma_2^* T, \ldots, \gamma_m^* T \} \quad \text{and} \quad \beta_\ell = \gamma_{m+1}^* T = 0.$$

Then, the density function given in Theorem 4.3.3 can be written as, for $s \in \mathbb{R}$,

$$f^{\bar{\theta}[D \geq 1]}(s) = \frac{1}{1-e^{-\gamma/T}} \sum_{d=1}^{m} \left( \prod_{j=1}^{d} (-\gamma_j^* T) \right) \sum_{j=1}^{\ell} \frac{(-1)^{m+n_j} d^{n_j-1}}{(n_j - 1)!} \frac{d^\beta_{\gamma_j} - x \beta_{\gamma_j} d^\beta_{\gamma_j}}{\prod_{i=1, i \neq j}^{\ell} (\beta_i - \beta_j)^{d^\beta_{\gamma_j}}}. $$

**Proof.** (i) The assumption follows immediately from Theorem 4.3.3 together with the representations for divided differences for pairwise distinct knots (see Lemma 3.1.3, (i)).

(ii) The result follows from Theorem 4.3.3, the formula for the divided differences for multiple knots (see Lemma 3.1.3, (vi)) and together with the construction of the $\beta$'s.

□

**Remark 4.3.5** For $\gamma_j^* = \gamma_j$, $1 \leq j \leq m+1$ (see also Example 3.1.5, (ii)), the density function given in Corollary 4.3.4, (i), corresponds to the density function of the MLE $\hat{\theta}$ under Type-I progressive hybrid censoring. It can be easily verified, that this representation corresponds to the expression derived in Childs et al. (2008, Theorem 23.2.2).

### 4.3.2 Limits for $T \to \infty$

The following limits for $T \to \infty$ can be used to establish relations among the distributions of the MLEs for different censoring models. Lemma 4.3.6 has been shown in Cramer and Balakrishnan (2013, Remark 4.4).
Lemma 4.3.6 Let $f_{S_i|D=d}^{m}$, $d \in \{1, \ldots, m\}$, denote the conditional density function as given in Theorem 4.2.9 for the particular setting of progressively Type-II censored order statistics (cf. Remark 2.3.2, (i)). Then,

$$
\lim_{T \to \infty} f_{S_i|D=d}(s) = \begin{cases} 
0, & d \in \{0, \ldots, m - 1\}, \\
\int_{R} f_{\hat{\vartheta},m}(s), & d = m,
\end{cases} \quad s \geq 0.
$$

The above result yields the following remark.

Remark 4.3.7 Let $f_{S_i|D\geq 1}^{\hat{\vartheta}}$ denote the conditional density function of $\hat{\vartheta}$ given in Theorem 4.2.12 for Type-I progressive hybrid censoring. Then, we find with Lemma 4.3.6 as well as with $m f_{\hat{\vartheta},m}(ms) = f_{\hat{\vartheta},m}(s)$, $s \geq 0$, that

$$
\lim_{T \to \infty} f_{S_i|D\geq 1}(s) = f_{\hat{\vartheta},m}(s), \quad s \geq 0.
$$

This gives us immediately the following $L^1$-convergence result of $f_{S_i|D\geq 1}^{\hat{\vartheta}}$.

Theorem 4.3.8 Let $f_{S_i|D\geq 1}^{\hat{\vartheta}}$ denote the density function of the MLE $\hat{\vartheta}$ under Type-I progressive hybrid censoring. Then

$$
\lim_{T \to \infty} \int_{R} \left| f_{S_i|D\geq 1}^{\hat{\vartheta}}(s) - f_{\hat{\vartheta},m}(s) \right| ds = 0.
$$

Proof. By taking into account $\int_{R} f_{S_i|D\geq 1}^{\hat{\vartheta}}(s)ds = \int_{R} f_{\hat{\vartheta},m}(s)ds = 1 < \infty$, the result follows with Remark 4.3.7 and with Scheffé’s theorem (cf. Billingsley, 1995, Theorem 16.12). □

It is obvious that $F_{S_i|D\geq 1}^{\hat{\vartheta}}$ converges weakly to $F_{\hat{\vartheta},m}$, for $T \to \infty$.

4.3.3 A Monotonicity Result w.r.t. $\vartheta$

We present a monotonicity result for the distribution of the total time on test statistic $S_D$ conditional on $D = d$, $d \in \{1, \ldots, m\}$. The following lemma is motivated by the results obtained in Otten (2014).

Lemma 4.3.9 Let $S_D$, $D \in \{1, \ldots, m\}$, be the total time on test statistics for the setting of progressively Type-II censored order statistics with initially planned censoring plan $R$, i.e. (cf. (4.6)),

$$
S_D = \sum_{j=1}^{D} \left(1 - \frac{\gamma_{D+1}}{\gamma_j}\right) \gamma_j (Z_{j;m:n}^{R} - Z_{j-1;m:n}^{R}) + \gamma_{D+1}(T - \mu), \quad D \in \{1, \ldots, m\}.
$$

Then, the survival function

$$
P_{\vartheta}(S_D > t|D = d)
$$

is increasing in $\vartheta > 0$ for all $t \in R$. In particular,

$$
P_{\vartheta_1}(S_D > t|D = d) = P_{\vartheta_2}(S_D > t|D = d), \quad t \notin (\gamma_{d+1}(T - \mu), \gamma_1(T - \mu)),
$$

and

$$
P_{\vartheta_1}(S_D > t|D = d) < P_{\vartheta_2}(S_D > t|D = d), \quad t \in (\gamma_{d+1}(T - \mu), \gamma_1(T - \mu)),
$$

for all $0 < \vartheta_1 < \vartheta_2$. 
4.3 Further Aspects Related to Type-I Hybrid Censoring from Exponential Distributions

**Proof.** Let \( f_{\tilde{\theta}|D=d} \) (see Theorem 4.2.9) denote the conditional density function of \( S_d \) (as in (4.41)) for \( \tilde{\theta} > 0 \). Then, according to Theorem 4.2.9 (with \( \mu = 0 \)), we find for \( 0 < \tilde{\theta}_1 < \tilde{\theta}_2 \)

\[
f_{\tilde{\theta}_2|D=d}(s) = \frac{T^d \prod_{j=1}^{d+1} \gamma_j}{d!} f_{\tilde{\theta}_2,d+1:m:n}(T) \frac{B_{d-1}(s|\gamma_{d+1}T)}{\exp \left\{ -s \left( \frac{1}{\tilde{\theta}_2} - \frac{1}{\tilde{\theta}_1} \right) \right\} e^{-s/\tilde{\theta}_1}} = \frac{\tilde{\theta}_2^{d+1} f_{\tilde{\theta}_2,d+1:m:n}(T)}{\tilde{\theta}_2^{d+1} f_{\tilde{\theta}_2,d+1:m:n}(T)} \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} f_{\tilde{\theta}_1|D=d}(s).
\]

Notice that \( f_{\tilde{\theta}_1,d+1:m:n} \) denotes the density function of \( Z_{d+1:m:n}^\tilde{\theta} \) based on \( P_F = \exp(\mu, \tilde{\theta}_1), i \in \{1, 2\} \). The above identity leads us to

\[
\int_{\gamma_{d+1}T}^{\gamma_T} f_{\tilde{\theta}_2|D=d}(s) ds = 1 \iff \int_{\gamma_{d+1}T}^{\gamma_T} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds = \frac{\tilde{\theta}_2^{d+1} f_{\tilde{\theta}_2,d+1:m:n}(T)}{\tilde{\theta}_2^{d+1} f_{\tilde{\theta}_2,d+1:m:n}(T)}.
\]

Therefore, we get for any \( t > \gamma_{d+1}T \)

\[
P_{\tilde{\theta}_2}(S_D > t|D = d) = \int_t^{\gamma_T} f_{\tilde{\theta}_2|D=d}(s) ds = \frac{\int_{\gamma_{d+1}T}^{\gamma_T} f_{\tilde{\theta}_2|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds}{\int_{\gamma_{d+1}T}^{\gamma_T} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds}.
\]

Due to \( f_{\tilde{\theta}_1|D=d} \geq 0 \), \( \int_{\tilde{\theta}_1} f_{\tilde{\theta}_1|D=d}(s) ds = 1 < \infty \) and the continuity of \( \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} \), we can apply the generalized mean value theorem, and find that for any \( t \in (\gamma_{d+1}T, \gamma_1T) \) there exists an \( \xi_1 \in [\gamma_{d+1}T, t] \), such that

\[
\int_{\gamma_{d+1}T}^{t} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds = \exp \left\{ \xi_1 \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} \int_{\gamma_{d+1}T}^{t} f_{\tilde{\theta}_1|D=d}(s) ds.
\]

By utilizing the same arguments, we get, for \( \xi_2 \in [t, \gamma_1T] \),

\[
\int_{t}^{\gamma_1T} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds = \exp \left\{ \xi_2 \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} \int_{t}^{\gamma_1T} f_{\tilde{\theta}_1|D=d}(s) ds.
\]

Now, by taking into account \( f_{\tilde{\theta}_1|D=d}(s) > 0 \), for \( s \in (\gamma_{d+1}T, \gamma_1T) \) and that \( \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} \) is strictly increasing, we obtain the following inequality

\[
\int_{t}^{\gamma_1T} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ s \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds > \int_{t}^{\gamma_1T} f_{\tilde{\theta}_1|D=d}(s) \exp \left\{ t \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} ds \iff \exp \left\{ t \left( \frac{\tilde{\theta}_2 - \tilde{\theta}_1}{\tilde{\theta}_1 \tilde{\theta}_2} \right) \right\} \int_{t}^{\gamma_1T} f_{\tilde{\theta}_1|D=d}(s) ds.
\]

According to equations (4.42) and (4.43) as well as according to inequality (4.44), we have \( \xi_1 \leq t < \xi_2 \). Together with the preceding derivations, we finally find for \( t \in (\gamma_{d+1}T, \gamma_1T) \)

\[
P_{\tilde{\theta}_2}(S_D > t|D = d)
\]
\[ \exp \left\{ \xi_2 \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 \vartheta_2} \right\} \int_t^{T'} f_\vartheta_1 f_{S_d|D=d}(s) ds \\
\exp \left\{ \xi_1 \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 \vartheta_2} \right\} \int_t^{T'} f_\vartheta_1 f_{S_d|D=d}(s) ds + \exp \left\{ \xi_2 \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 \vartheta_2} \right\} \int_t^{T'} f_\vartheta_1 f_{S_d|D=d}(s) ds \]
\[ \exp \left\{ \xi_2 \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 \vartheta_2} \right\} \int_t^{T'} f_\vartheta_1 f_{S_d|D=d}(s) ds \]
\[ = P_{\vartheta_1}(S_D > t|D = d). \]

For \( t \notin [\gamma_{d+1}, T] \), we have obviously
\[ P_{\vartheta_2}(S_D > t|D = d) = P_{\vartheta_1}(S_D > t|D = d). \]

The assumption for \( d = m \) follows analogously. This concludes the proof.

The above lemma proves the second condition of the Three Monotonicities Lemma introduced in Balakrishnan and Iliopoulos (2009, Lemma 1). The Three Monotonicities Lemma is used to show that the MLE \( \vartheta \) is stochastically increasing in \( \vartheta \). Whereas the monotonicity of the MLE \( \vartheta \) is necessary in order to construct exact confidence intervals for \( \vartheta \) (cf. Casella and Berger, 2002, Theorem 9.2.12).

Balakrishnan and Iliopoulos (2009) showed that, for the Type-I hybrid censoring scheme as well as for the Type-II hybrid censoring scheme the respective MLEs are stochastically increasing in \( \vartheta \). The following result is taken from Burkschat et al. (2016, Theorem 4.6).

**Theorem 4.3.10** Let \( \gamma_1^* \geq \cdots \geq \gamma_m^* > 0 \) and \( \gamma_{m+1}^* = 0 \). Further, let \( \hat{\vartheta} \) denote the MLE for \( \vartheta \) under Type-I sequential hybrid censoring (cf. Lemma 4.2.3) with \( \mu = 0 \). Then, the survival function
\[ P_{\vartheta}(\hat{\vartheta} > t|D \geq 1) \]
is increasing in \( \vartheta > 0 \), for every \( t \in (0, \gamma_1^*) \).

Although we assumed the \( \gamma^* \)'s to be pairwise distinct, the limits for \( F_{\vartheta|D \geq 1}^{(\vartheta)} \) w.r.t \( \vartheta \) are similar whether the \( \gamma^* \)'s are ascending or descending or alternating (see Theorem 4.2.15). Thus, it seems to be reasonable for future considerations, to investigate whether and how the monotonicity result for \( P_{\vartheta}(\hat{\vartheta} > t|D \geq 1) \) can be extended to a more general setting of the \( \gamma^* \)'s.

### 4.4 Type-I Hybrid Censoring from Uniform Distributions

In the following, we assume the IID uniform model (see Model 2.1.3). The results established in Section 4.1 for an arbitrary absolutely continuous distribution \( F \), can be adapted to \( P_F = U(a, b) \). For a sample of uniform Type-II censored order statistics
\[ U_{1:m}, \ldots, U_{m:n}, \quad m \leq n, \]
we define the **uniform Type-I hybrid censored order statistics** by
\[ U_j := \min\{U_{j:n}, T\}, \quad T \in (a, b), \quad 1 \leq j \leq m. \]

Further, let \( D = \sum_{j=1}^m I_{(-\infty,T]}(U_j;n) \), denote the number of failures observed till time \( T \).

First, we establish some basic distributional results for uniform Type-I hybrid censored order statistics including the derivation of the respective MLEs. Then, we consider in Section 4.4.2 the distribution of the MLEs for the one-parameter as well as for the two-parameter case.
4.4.1 Fundamental Results

As specified in Model 2.1.3, we assume \( a < T < b \). The corresponding truncated cone \( \Sigma_{F,T}^d \), \( d \in \{1, \ldots, m\} \), is given by (see 1.4)

\[
\Sigma_{F,T}^d = \{ u_d \in \mathbb{R}^d | a \leq u_1 \leq \cdots \leq u_d \leq T \}. \tag{4.45}
\]

It is obvious that for \( T > b \) the trivial cone \( \Sigma_{F}^d \) and the truncated cone \( \Sigma_{F,T}^d \) coincide, and that \( U_{j} = U_{j,n}, 1 \leq j \leq m \). In that case, the Type-I hybrid censoring scheme corresponds to the model of Type-II censoring.

According to Lemma 6.1.2 the conditional joint density function of \( U_{1}, \ldots, U_{d}, \) \( d \in \{1, \ldots, m\} \) is given as follows.

**Lemma 4.4.1** For \( d \in \{1, \ldots, m-1\} \), the conditional joint density function \( f_{U_{1}, \ldots, U_{d}|D=d}, 1 \leq j \leq d \), is given by

\[
f_{U_{1}, \ldots, U_{d}|D=d}(u_d) = \mathbb{1}_{\Sigma_{F,T}^d}(u_d) \frac{n!}{(n-d)! (b-a)^{d+1} f_{d+1:n}(T)} \left( 1 - \frac{T-a}{b-a} \right)^{n-d-1}. \tag{4.46}
\]

For \( d = m \), we find

\[
f_{U_{1}, \ldots, U_{m}|D=m}(u_m) = \mathbb{1}_{\Sigma_{F,T}^m}(u_m) \frac{n!}{(n-m)! (b-a)^{m} F_{m:n}(T)} \left( 1 - \frac{u_m-a}{b-a} \right)^{n-m}. \tag{4.47}
\]

**Proof.** According to Lemma 4.1.2, we have for \( u_d \in \Sigma_{F,T}^d \)

\[
f_{U_{1}, \ldots, U_{d}|D=d}(u_d) = \frac{f_{1,\ldots,d+1:n}(u_d,T)}{f_{d+1:n}(T)}, \quad d \in \{1, \ldots, m-1\},
\]

\[
f_{U_{1}, \ldots, U_{m}|D=m}(u_m) = \frac{f_{1,\ldots,m:n}(u_m)}{F_{m:n}(T)}, \quad d = m. \tag{4.46}
\]

The result follows with Lemma 2.1.2, (i) and from the uniform distribution (see Definition C.1.1). \( \square \)

Let \( d \in \mathbb{N} \). Then, elementary calculations show that

\[
\text{vol}_d(\Sigma_{F,T}^d) = \text{vol}_d(\{ u_d \in \mathbb{R}^d | a \leq u_1 \leq \cdots \leq u_d \leq T \}) = \frac{(T-a)^d}{d!}, \quad 0 < a < T, \tag{4.47}
\]

with \( \Sigma_{F,T}^d \) as in (4.45).

**Remark 4.4.2** We observe that according to Lemma 4.4.1 and together with (4.47), the density function \( f_{d+1:n}(T) \) can be written as, for \( d \in \{1, \ldots, m-1\} \),

\[
f_{d+1:n}(T) = \frac{n!}{(n-d-1)! (b-a)^{d+1} (1 - \frac{T-a}{b-a})^{n-d-1}} \text{vol}_d(\Sigma_{F,T}^d)
\]

\[
= n \binom{n-1}{d} \frac{(T-a)^d}{(b-a)^{d+1} (1 - \frac{T-a}{b-a})^{n-d-1}}, \quad T \in (a,b).
\]
If we plug in $f_{d+1:n}(T)$ in $f_{U_d^*,1 \leq j \leq |D|=d}$, we obtain

\[ f_{U_d^*,1 \leq j \leq |D|=d}(u_d) = \frac{d!}{(T-a)^d} I_{\Sigma_{F,T}^d}(u_d). \tag{4.48} \]

Further, we find for $m = n$

\[ F_{n:n}(T) = \frac{n!}{(b-a)^n} \text{vol}_n(\Sigma_{F,T}^n) = \left( \frac{T-a}{b-a} \right)^n, \quad T \in (a, b). \]

This leads us to the following simplified expression for $f_{U_d^*,1 \leq j \leq n|D|=n}$,

\[ f_{U_d^*,1 \leq j \leq n|D|=n}(u_n) = \frac{n!}{(T-a)^n} I_{\Sigma_{F,T}^n}(u_n). \]

The findings from above allow us to derive compact expressions for the conditional distribution of the statistic $Y_d$, $d \in \{1, \ldots, m\}$, with

\[ Y_d = \alpha_0 + \sum_{j=1}^{d} \alpha_j U^j_d, \quad \alpha_0 \in \mathbb{R}, \quad (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \setminus 0^d. \tag{4.49} \]

**Theorem 4.4.3** Let the statistic $Y_d$ be given as in (4.49) and let further

\[ \alpha_0^* := \alpha_0 + a \sum_{j=1}^{d} \alpha_j, \quad d \in \{1, \ldots, m\}. \tag{4.50} \]

(i) Then, the conditional density function of $Y_d$ with $d \in \{1, \ldots, m-1\}$ is given by

\[ f_{Y_d|D=d}(x) = B_{d-1}(x|\alpha_0^*, \alpha_0^* + (T-a)\alpha_d, \ldots, \alpha_0^* + (T-a)\sum_{j=1}^{d-1} \alpha_{d-j+1}), \quad x \in \mathbb{R}. \]

The corresponding conditional cumulative distribution function is given by, $y \in \mathbb{R}$, \[ F_{Y_d|D=d}(y) = 1 - \left[ \alpha_0^*, \alpha_0^* + (T-a)\alpha_d, \ldots, \alpha_0^* + (T-a)\sum_{j=1}^{d-1} \alpha_{d-j+1}; (\cdot - y)^d_+ \right]. \]

(ii) For $d = m = n$, the density function of $Y_d$ conditionally on $D = n$ is given by

\[ f_{Y_n|D=n}(x) = B_{n-1}(x|\alpha_0^*, \alpha_0^* + (T-a)\alpha_n, \ldots, \alpha_0^* + (T-a)\sum_{j=1}^{n-1} \alpha_{n-j+1}), \quad x \in \mathbb{R}. \]

Further, the corresponding conditional cumulative distribution function is given by, $y \in \mathbb{R}$, \[ F_{Y_n|D=n}(y) = 1 - \left[ \alpha_0^*, \alpha_0^* + (T-a)\alpha_n, \ldots, \alpha_0^* + (T-a)\sum_{j=1}^{n-1} \alpha_{n-j+1}; (\cdot - y)^n_+ \right]. \]

**Proof.** We restrict ourselves to the case where $d \in \{1, \ldots, m-1\}$. The results for $m = n = d$ can be obtained by analogous calculations. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary continuous function. Then, we find with (4.48) and the linear transformation $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

\[ \Phi(v_d) = \left( \frac{v_1 - a}{T - a}, \ldots, \frac{v_d - a}{T - a} \right)' \]

with \( \det(D\Phi) = (T - a)^{-d} \).
that
\[
E(g(Y_d)|D = d) = \frac{d!}{(T - a)^d} \int_{\Sigma_{F,T}^d} g(\alpha_0 + \sum_{j=1}^{d} \alpha_j u_j) d\mathbf{u}_d \\
= d! \int_{\Sigma_{F,T}^d} g(\alpha_0 + \sum_{j=1}^{d} \alpha_j ((T - a)v_j + a)) d\mathbf{v}_d \\
= d! \int_{\Sigma_{F,T}^d} g(\alpha_0^* + \sum_{j=1}^{d} \alpha_j (T - a)v_j) d\mathbf{v}_d \\
= \int_{\mathbb{R}} g(x) B_{d-1}(x|\alpha_0^*, \alpha_0^* + (T - a)\alpha_d, \ldots, \alpha_0^* + (T - a)\sum_{j=1}^{d} \alpha_{d-j+1}) dx.
\]

The last equation follows due to the application of Corollary 3.1.10, where \( \Sigma^d \) is given as in (3.11) and \( \alpha_0^* \) is specified as in (4.50). The above calculations lead us to
\[
f_{Y_d|D=d}(x) = B_{d-1}(x|\alpha_0^*, \alpha_0^* + (T - a)\alpha_d, \ldots, \alpha_0^* + (T - a)\sum_{j=1}^{d} \alpha_{d-j+1}), \quad x \in \mathbb{R}.
\]
The corresponding cumulative distribution function follows immediately from Lemma 3.3.4 and Remark 3.3.5. This finishes the proof. \( \square \)

Note that the above proof is similar to the calculations performed in Dahmen and Michelli (1986). We proceed by determining the MLEs for \( a \) and \( b \).

**Lemma 4.4.4** The likelihood function \( L(\cdot|\mathbf{u}_d) \) for \( a \) and \( b \) is given by
\[
L(a, b|\mathbf{u}_d) = \begin{cases} 
1_{\Sigma_{F,T}^d}^d(\mathbf{u}_d) \frac{1 - F(T)}{(n - d)f(T)} f_{1\ldots d+1:n}(\mathbf{u}_d, T), & d \in \{1, \ldots, m - 1\}, \\
1_{\Sigma_{F,T}^m}^m(\mathbf{u}_m) f_{1\ldots m:n}(\mathbf{u}_m), & d = m.
\end{cases}
\]

Note that \( \hat{b} \) does not exist when \( D = 0 \). Hence, we restrict the domain for \( D \) to \( \{1, \ldots, m\} \).

By assuming the parameter \( a \) to be known, the MLE for \( b \) is given by
\[
\hat{b} = \begin{cases} 
\frac{nT - (n - D)a}{D}, & D \in \{1, \ldots, m - 1\}, \\
\frac{nU_{m:n} - (n - m)a}{m}, & D = m.
\end{cases}
\]

For an unknown parameter \( a \), the MLEs \( \hat{a} \) and \( \hat{b} \) are given by
\[
\hat{a} = U_{1:n} \quad \text{and} \quad \hat{b} = \begin{cases} 
\frac{nT - (n - D)U_{1:n}}{D}, & D \in \{1, \ldots, m - 1\}, \\
\frac{nU_{m:n} - (n - m)U_{1:n}}{m}, & D = m.
\end{cases}
\]

**Proof.** Due to Lemma 4.1.1, the distribution of \( D = d \) is for the setup of ordinary order statistics given by
\[
P(D = d) = \frac{1 - F(T)}{(n - d)f(T)} f_{d+1:n}(T), \quad d \in \{1, \ldots, m - 1\},
\]
\[
P(D = m) = F_{m:n}(T), \quad d = m.
\]

(4.51)
Due to the monotonicity of \( U \), differentiating \( \ell(a, | u_d) \) for a fixed parameter \( a \) with \( a \leq u_1 \), which is given by

\[
\ell(a, b| u_d) = \ln(\Lambda(a, b| u_d)) = \ln\left( \frac{n!}{(n-d)!} \right) - d \ln(b-a) + (n-d) \ln\left( \frac{b-T}{b-a} \right).
\]

Differentiating \( \ell(a, | u_d) \) w.r.t. \( b > T \) yields

\[
\frac{\partial}{\partial b} \ell(a, b| u_d) = -\frac{n}{b-a} + \frac{n-d}{b-T}
\]

\[
\iff
\frac{n-d}{b-a} > 0 \iff \frac{n}{b-T} > b
\]

\[
\iff nb - (n-d)a > nb - nT
\]

\[
\iff \frac{nT - (n-d)a}{d} > b.
\]

Due to the monotonicity of \( \ell(a, | u_d) \), we have \( \hat{b} = \frac{nT - (n-D)a}{D} \) for \( a \leq u_1 \) fixed. This yields the result for a known parameter \( a \). For an unknown parameter \( a \), the result follows with the choice \( a = u_1 \). Hence, we obtain \( \hat{b} = \frac{nT - (n-D)u_1}{D} \).

The assertion for \( d = m \) follows by analogous calculations.}

**Remark 4.4.5**  
(i) For a known parameter \( a \), we find that \( \hat{b} \) is discrete for \( d \in \{1, \ldots, m-1\} \). Thus, the respective distribution is degenerated and the conditional density function of \( \hat{b} \) does not exist for \( d \in \{1, \ldots, m-1\} \).
Figure 4.8: Plots of $F_{\tilde{a}, \tilde{b}}|D \geq 1$ for $m = 1$, $a = 1$, $b = 2$ and $T = 1.5$. Left: $n = 2$. Right: $n = 20$.

(ii) For an unknown parameter $a$ and $m = 1$, we find that the bivariate MLE $(\hat{a}, \hat{b})$ is given by

$$(\hat{a}, \hat{b}) = (U_{1:n}, U_{1:n}).$$

The corresponding distribution is obviously degenerated, i.e.,

$$F_{\tilde{a}, \tilde{b}}|D \geq 1(y_1, y_2) = \frac{1}{P(D \geq 1)} P(U_{1:n} \leq y_1, U_{1:n} \leq y_2| U_{1:n} \leq T) P(U_{1:n} \leq T)$$

$$= \frac{1}{1 - P(D = 0)} P(U_{1:n} \leq y_1, U_{1:n} \leq y_2, U_{1:n} \leq T)$$

$$= \frac{1}{1 - P(U_{1:n} > T)} P(U_{1:n} \leq \min\{y_1, y_2, T\})$$

$$= \frac{F_{1:n}(\min\{y_1, y_2, T\})}{F_{1:n}(T)}, \quad y_1, y_2 \geq a.$$

Figure 4.8 illustrates the plots of $F_{\tilde{a}, \tilde{b}}|D \geq 1$ for two $n$'s. The shapes of the conditional cumulative distribution function suggest that $F_{\tilde{a}, \tilde{b}}|D \geq 1$ converges to $\varepsilon_{(p_1, p_2)}$. Hereby denotes $\varepsilon_{(p_1, p_2)}$ the one-point distribution in $(p_1, p_2) \in \mathbb{R}^2$ with cumulative distribution function $1_{[p_1, \infty) \times [p_2, \infty)}$. We can easily verify that

$$\lim_{n \to \infty} F_{\tilde{a}, \tilde{b}}|D \geq 1(y_1, y_2) = 1_{[a, \infty)^2}(y_1, y_2),$$

for

$$(y_1, y_2) \in \mathbb{R}^2 \setminus \{(y_1, y_2) \in \mathbb{R}^2|y_1 = a, y_2 \geq a\} \cup \{(y_1, y_2) \in \mathbb{R}^2|y_1 \geq a, y_2 = a\}.$$
(iii) Let the parameter $a$ be unknown and let $m \geq 2$. Then, we find that the MLE $\hat{b}$ established in Lemma 4.4.4 can be also expressed as

$$\hat{b} = \begin{cases} U_{1:n} + \frac{n}{D}(T - U_{1:n}), & D \in \{1, \ldots, m - 1\}, \\ U_{1:n} + \frac{n}{m}(U_{m:n} - U_{1:n}), & D = m. \end{cases} \tag{4.52}$$

This representation is admittedly less compact than the expression proposed in Lemma 4.4.4. However, it reveals immediately that $\hat{b}$ can be interpreted as a shift of $\hat{a}$, while the respective shift parameter can be readily obtained from the expression given in (4.52).

(iv) By taking into account Neyman’s factorization criterion (see, e.g., Witting, 1985, Theorem 3.19 b)), the calculations conducted in the proof to Lemma 4.4.4 reveal that $(\hat{b}, D)$ is a sufficient statistic for $b$, and that $(\hat{a}, \hat{b}, D)$ is a sufficient statistic for $(a, b)$.

### 4.4.2 Distribution Theory for the MLEs

We start by considering the conditional density function of $U^1_m$ in order to determine the distribution of the MLE $\hat{b}$ for a known parameter $a$ in the case $D = m$.

**Theorem 4.4.6** The density function of $U^1_m$, conditionally on $D = m$, is given by

$$f_{U^1_m|D=m}(x) = 1_{[a,T]}(x) \frac{f_{m:n}(x)}{F_{m:n}(T)}, \tag{4.53}$$

**Proof.** Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary continuous function. Further, the function $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$\tilde{g}(x) = g(x) \left(1 - \frac{x - a}{b - a}\right)^{n-m}, \quad x \in \mathbb{R}, \quad a < b, \tag{4.54}$$

is also continuous. Let $c_m = \frac{m!}{(n-m)!(b-a)^m F_{m:n}(T)}$. Then, we obtain with the representation for $f_{U^1_m}^{1 \leq j \leq m|D=m}$ given in Lemma 4.4.1, with Corollary 3.11.10 for $\tilde{g}$ as in (4.54) and with the linear transformation $\Phi$ used in the proof to Theorem 4.4.3:

$$E(g(U^1_m)|D = m) = c_m \int_{\Sigma_{F,T}^m} g(u_m) \left(1 - \frac{u_m - a}{b - a}\right)^{n-m} du_m$$

$$= c_m \int_{\Sigma_{F,T}^m} g((T - a) \frac{u_m - a}{T - a} + a) \left(1 - \frac{(T - a) \frac{u_m - a}{T - a} + a}{b - a}\right)^{n-m} du_m$$

$$= c_m (T - a)^m \int_{\Sigma_{F,T}^m} g((T - a)v_m + a) \left(1 - \frac{(T - a)v_m + a}{b - a}\right)^{n-m} dv_m$$

$$= c_m (T - a)^m \int_{\mathbb{R}} g(x) \left(1 - \frac{x - a}{b - a}\right)^{n-m} B_{m-1}(x|a, T^*) dx.$$ 

We recall that $T^*$ denotes the $n$-times repetition of $T$ and that $\Sigma^m$ denotes the bounded uniform cone introduced in (3.11). This yields together with equation (4.30)

$$f_{U^1_m|D=m}(x) = \frac{n!}{(n-m)!(b-a)^m F_{m:n}(T)} \frac{(T - a)^m}{m!} \left(1 - \frac{x - a}{b - a}\right)^{n-m} B_{m-1}(x|a, T^*)$$
Theorem 4.4.7 Let \( a < T < b \). Then, the conditional distribution of the MLE \( \hat{b} \) under Type-I hybrid censoring is given by

\[
F_{\hat{b}|D \geq 1}(y) = \frac{1}{1 - \left( \frac{b-T}{b-a} \right)^n} \times \left[ \sum_{d=1}^{m-1} 1_{[a,T-(d-n)a,\infty)}(dy) \frac{(b-T)f_{d+1:n}(T)}{(n-d)} + F_{m:n} \left( \min \left\{ \frac{my + (n-m)a}{n}, T \right\} \right) \right],
\]

for \( y \geq 0 \). Further,

\[
\lim_{T \to b^-} F_{\hat{b}|D \geq 1}(y) = F_{m:n} \left( \frac{my + (n-m)a}{n} \right), \quad y \geq 0.
\]

**Proof.** The distribution of \( \hat{b} \) can be calculated from (cf. proof to Theorem 4.2.12 or Remark 4.4.5, (ii))

\[
P(\hat{b} \leq y|D \geq 1) = \frac{1}{P(D \geq 1)} \sum_{j=1}^{m} P(\hat{b} \leq y|D = d) P(D = d), \quad y \geq 0.
\]

For \( d \in \{1, \ldots, m-1\} \), we have

\[
P(\hat{b} \leq y|D = d) = P \left( \frac{nT - (n-d)a}{d} \leq y|D = d \right) = 1_{[a,T-(d-n)a,\infty)}(dy).
\]

According to Theorem 4.4.6 and with

\[
U_{m:n} = \frac{m\hat{b} + (n-m)a}{n},
\]

we obtain for \( d = m \)

\[
F_{\hat{b}|D=m}(y) = F^{U_{m:n}|D=m} \left( \frac{my + (n-m)a}{n} \right) = \frac{1}{F_{m:n}(T)} F_{m:n} \left( \min \left\{ \frac{my + (n-m)a}{n}, T \right\} \right).
\]

With (4.51), we arrive at

\[
P(D = d) = \frac{1 - F(T)}{(n-d)f(T)} f_{d+1:n}(T) = 1_{[a,b]}(T) \frac{b-T}{n-d} f_{d+1:n}(T), \quad d \in \{1, \ldots, m-1\}.
\]

For \( d = m \), we have \( P(D = m) = F_{m:n}(T) \). Further, we obtain according to Lemma 4.1.1

\[
P(D \geq 1) = 1 - P(D = 0) = 1 - \left( 1 - F(T) \right)^n = 1 - \left( \frac{b-T}{b-a} \right)^n. \quad (4.55)
\]
Figure 4.9: Plots of the conditional cumulative distribution function $\hat{F}^{\hat{D}\geq 1}$, for $n = 10$, $a = 1$, $b = 2$ and $T = 1.5$. Left: $m = 10$. Right: $m = 5$.

With the representation of $\hat{F}^{\hat{D}\geq 1}$, we arrive for the limit at

$$\lim_{T \to b} \hat{F}^{\hat{D}\geq 1}(y) = F_{m:n}\left(\min\left\{\frac{my + (n - m)a}{n}, b\right\}\right), \quad y \geq 0.$$  

The assertion follows by taking into account that $F_{m:n}$ is the cumulative distribution function of $U_{m:n}$ based on $U_1, \ldots, U_n \overset{\text{iid}}{\sim} U(a, b)$. This concludes the proof. □

Remark 4.4.8  
(i) In Theorem 4.4.7, we avoided to use the explicit expression of the density functions $f_{2:n}, \ldots, f_{m:n}$ as well as of the cumulative distribution function $F_{m:n}$, in order to keep things simple and structured. For a convenient implementation of the marginal distributions of uniformly distributed order statistics one might use the formulas provided in Lemma 2.1.2, (ii) and (iv). Note further that the evaluation of the density functions $f_{2:n}, \ldots, f_{m:n}$ at $T$, has to be conducted only once per experiment setting. Only the indicator functions as well as the cumulative distribution function $F_{m:n}$ have to be evaluated repeatedly.

(ii) Theorem 4.4.7 reveals further that $\hat{F}^{\hat{D}\geq 1}$ is continuous on the interval $[a, \frac{nT - (n - m)a}{m}]$ and that it has discontinuities at $\frac{nT - (n - d)a}{d}$, $d \in \{1, \ldots, m - 1\}$. Figure 4.9 depicts the plots of $\hat{F}^{\hat{D}\geq 1}$ for the setup of Type-I censoring with $n = 10$ (left) and for the setup of Type-I hybrid censoring with $n = 10$ and $m = 5$.

(iii) If we recall the distribution of the MLE $\hat{\vartheta}$ given for the setup of minimal repair times and $k$th record values (see Corollary 4.2.16 and Remark 4.2.17) we observe that the conditional cumulative distribution functions $\hat{F}^{\hat{D}\geq 1}$ and $\hat{F}^{\hat{D} \geq 1}$ exhibit many structural similarities.

(iv) It should be noted that in the Type-I hybrid censoring model, according to Theorem 4.4.7, the distribution of the MLE converges for $T \to b$ to the distribution of the MLE in the Type-II censoring scenario.

We proceed with the distribution theory for the MLE $(\hat{a}, \hat{b})$. 
Theorem 4.4.9 The joint density function of $U_1^l$ and $U_m^l$ conditionally on $D = m$, is given by
\[
 f_{U_1^l, U_m^l | D = m}(u, x) = 1_{(u, x) \in [a, T]^2} |u < x| \frac{f_{1, m; n}(u, x)}{F_{m; n}(T)}.
\]

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary continuous function and let the constant $c_m$ be chosen as in the proof of Theorem 4.4.6. We define, for $u \in \mathbb{R}$ the continuous function $\tilde{g}_u : \mathbb{R} \to \mathbb{R}$ by
\[
 \tilde{g}_u(x) = g(u, x) \left(1 - \frac{x - a}{b - a}\right)^{n-m}, \quad x \in \mathbb{R}.
\]
Further, we consider the (non-linear) transformation $\Phi : \mathbb{R}^m \to \mathbb{R}^m$, with
\[
 \Phi(v_1, \ldots, v_m) = \left(\frac{v_1 - v_1}{T - v_1}, \ldots, \frac{v_m - v_1}{T - v_1}\right), \quad \text{and} \quad |\det(D\Phi)| = (T - v_1)^-(m-1).
\]
As in the proof of Theorem 4.2.22, we have
\[
 \Phi(\Sigma_{F; T}^n) = \{(u_1, v_2, \ldots, v_m) \in \mathbb{R}^m | a \leq u_1 \leq T, 0 \leq v_2 \leq \cdots \leq v_m \leq 1\},
\]
while $v_1 = u_1$. Then, the conditional expectation of $g(U_1^l, U_m^l)$ can be calculated by
\[
 E(g(U_1^l, U_m^l) | D = m)
 = c_m \int_{\Sigma_{F; T}^n} g(u_1, u_m) \left(1 - \frac{u_m - a}{b - a}\right)^{n-m} \, du_m
 = c_m \int_{\Sigma_{F; T}^n} \tilde{g}_u(u_m) \, du_m
 = c_m (T - u_1)^{m-1} \int_{[a, T]} \int_{\{v_2, \ldots, v_m \in \mathbb{R}^{m-1} | 0 \leq v_2 \leq \cdots \leq v_m \leq 1\}} \tilde{g}_u \left((T - u_1)v_m + u_1\right) dv_m \cdots dv_2 \, du_1
 = c_m (T - u_1)^{m-1} \frac{1}{(m-1)!} \int_{[a, T]} \int_{\mathbb{R}} \tilde{g}_u(x) B_{m-2}(x|u_1, T)^{(m-1)} dx \, du_1
 = \int_{[a, T]} \int_{\mathbb{R}} g(u_1, x) \frac{c_m (T - u_1)^{m-1}}{(m-1)!} \left(1 - \frac{x - a}{b - a}\right)^{n-m} \mathbbm{1}_{[u_1, T]}(x) (m-1) \frac{(x - u_1)^{m-2}}{(T - u_1)^{m-1}} \, dx \, du_1.
\]
The last identity follows from equation (4.30). By taking into account the formula for the joint density function of two order statistics (see Lemma 2.1.2, (iii)), with $r = 1$ and $s = m$, we arrive at
\[
 f_{U_1^l, U_m^l | D = m}(u, x) = \mathbbm{1}_{[a, T]}(u) \mathbbm{1}_{[u, T]}(x) m(m-1) \frac{(b - x)^{n-m}(x - u)^{m-2}}{(b - a)^n F_{m; n}(T)}
 = \mathbbm{1}_{[a, T]}(u) \mathbbm{1}_{[u, T]}(x) \frac{f_{1, m; n}(u, x)}{F_{m; n}(T)}, \quad (u, x) \in \mathbb{R}^2,
\]
which yields the desired result. □
Figure 4.10: Plots of $f_{U_1^m, U_m^m | D = m}$ for $n = 10$, $a = 1$, $b = 2$ and $T = 1.6$. Left: $m = 6$. Right: $m = 4$.

Figure 4.10 illustrates the plots of $f_{U_1^m, U_m^m | D = m}$ for two different $m$'s. We proceed by providing an expression for the conditional joint distribution of $\hat{a}$ and $\hat{b}$.

**Theorem 4.4.10** Let $m \geq 2$. Then, the conditional cumulative distribution function $F_{\hat{a}, \hat{b} | D \geq 1}$ is given by

$$F_{\hat{a}, \hat{b} | D \geq 1}(v, y) = \frac{1}{1 - \left(\frac{b - T}{b - a}\right)^n} \times$$

$$\left[1\{v, y) \leq (nT - dy)/(n - d) \leq \min\{v, T\}\right](v, y) \times$$

$$\times \left[F_{1:n}(\min\{v, T\}) - F_{1:n}(\frac{nT - dy}{n - d}) + F_{1,2:n}(\frac{nT - dy}{n - d}, T) - F_{1,2:n}(\min\{v, T\}, T)\right]$$

$$+ \sum_{d=2}^{m-1} 1\{(v, y) \leq (nT - dy)/(n - d) \leq v\} \times$$

$$\times \left[F_{1,d:n}(v, T) - F_{1,d:d}(\frac{nT - dy}{n - d}, T) + F_{1,d+1:n}(\frac{nT - dy}{n - d}, T) - F_{1,d+1:n}(v, T)\right]$$

$$+ \frac{m}{n} \int_a^v \int_a^y f_{1,m:n}(u, \frac{mx + (n - m)u}{n}) dx \ du, \quad v, y \geq 0.$$

**Proof.** For $d = 1$, we have

$$P\left(\frac{nT - dy}{n - d} \leq U_{1:n} \leq v, U_{d:n} \leq T < U_{d+1:n}\right)$$

$$= P\left(\frac{nT - dy}{n - d} \leq U_{1:n} \leq \min\{v, T\}\right) - P\left(\frac{nT - dy}{n - d} \leq U_{1:n} \leq \min\{v, T\}, U_{2:n} \leq T\right)$$

$$= 1\{(v, y) \leq (nT - dy)/(n - d) \leq \min\{v, T\}\}(v, y) \times$$

$$\times \left[F_{1:n}(\min\{v, T\}) - F_{1:n}(\frac{nT - dy}{n - d}) + F_{1,2:n}(\frac{nT - dy}{n - d}, T) - F_{1,2:n}(\min\{v, T\}, T)\right].$$
For \( d \in \{2, \ldots, m-1\} \), we obtain further

\[
P\left(\frac{nT - dy}{n-d} \leq U_{1:n} \leq v, U_{d:n} \leq T < U_{d+1:n}\right)
= P\left(\frac{nT - dy}{n-d} \leq U_{1:n} \leq v, U_{d:n} \leq T\right) - P\left(\frac{nT - dy}{n-d} \leq U_{1:n} \leq v, U_{d+1:n} \leq T\right)
= \mathbb{1}_{\{(v,y)\mid (nT-dy)/(n-d) \leq v\}}(v,y)
\times \left[F_{1,d:n}(v,T) - F_{1,d:d}(\frac{nT - dy}{n-d}, T) + F_{1,d+1:n}(\frac{nT - dy}{n-d}, T) - F_{1,d+1:n}(v,T)\right].
\]

Finally, the expression for \( d = m \) follows with Theorem 4.4.9 and with

\[
U_{m:n} = \frac{\hat{m}b + (n-m)U_{1:n}}{n},
\]

which holds according to Lemma 4.4.4. This proves the assumption.

The conditional distribution of \( \hat{a} \) is given as follows.

**Theorem 4.4.11** The cumulative distribution function of \( \hat{a} \), conditionally on \( D \geq 1 \), is given by

\[
F_{\hat{a}\mid D \geq 1}(v) = 1 - \left(\frac{b - \min\{T,v\}}{b-a}\right)^n, \quad v \geq a.
\]

**Proof.** The assumption follows by analogy with Cramer and Balakrishnan (2013, Theorem 5.1) (see also Theorem 4.2.25).

In the subsequent models, we focus for an unknown parameter \( a \) on establishing expressions for the conditional joint density functions in the non-degenerated case. Then, the distribution of the MLE \((\hat{a}, \hat{b})\) can be determined by performing steps analogous to those conducted in the proof of Theorem 4.4.10.
Chapter 5
Multi-Sample Type-I Sequential Hybrid Censoring

For sequential order statistics, the scenario of \( k \in \mathbb{N} \) independent \( m \)-out-of-\( n \)-systems is often considered. Much work on estimating the model parameters \( \alpha_1, \ldots, \alpha_m \) (cf. Cramer and Kamps, 1996, 1998a, 2001a) as well as on estimating the distribution parameters of the exponential and the Weibull distribution (cf. Cramer and Kamps, 1998b,a, 2001b,a) has been done. Then, Balakrishnan et al. (2001b) determined for an underlying exponential distribution the MLEs for both \( k \) independent progressive Type-II censored and for \( k \) independent general progressive Type-II censored experiments. Further, Burkschat et al. (2015) considered \( k \) independent sequential \( m \)-out-of-\( n \) systems under an ordered hazard rate assumption. In these articles, different ways of estimating the corresponding model parameters have been proposed. For non-parametric inferential results obtained in the context of multi-sample (progressive) Type-II censored experiments, we refer to Volterman and Balakrishnan (2010), Volterman et al. (2012), Balakrishnan et al. (2013) and Volterman et al. (2014).

It should be noted that also step-stress models (cf. Section 1.5) have been considered in a multi-sample situation. Kateri et al. (2009) considered \( k \) independent Type-II censored (simple) step-stress experiments. Following that, Kateri et al. (2010) presented a multi-sample approach to simple step-stress experiments under Type-I censoring. In both cases the one-parameter exponential distribution was considered and the density function of the MLE \( \hat{\vartheta} \) has been derived. In each of the two setups, the moment generating function approach was applied. A more general approach, applicable for any absolutely continuous cumulative distribution function, has been presented by Bedbur et al. (2015). They considered \( k \) independent step-stress experiments which each consist of \( m \) stress levels, and which underlie a repeated Type-II censoring procedure. As an underlying model of ordered data, sequential order statistics under a proportional hazard rate assumption were assumed. As a key result the MLEs of the model parameters have been determined.

In the following, we present a multi-sample model which considers \( k \) independent Type-I sequential hybrid censored experiments under an exponential assumption. This model is useful when several experiments are performed with the same assumption for the parameter \( \vartheta \), and the experimenter wants to include all data for the estimation of \( \vartheta \). Further, it includes among others the scenario where several experiments took place at different locations or at different times (cf. Kateri et al., 2009, p. 2907). This multi-sample situation covers a strategy which subdivides one experiment in several sub-experiments. For instance, instead of conducting one Type-I sequential hybrid censoring experiment with parameters \( n, m \) and \( T \), one might perform \( k \) independent experiments with parameters \( n_1, \ldots, n_k, m_1, \ldots, m_k \) and \( T_1^\diamond, \ldots, T_k^\diamond \), where \( n = \sum_{i=1}^k n_i \) and \( m = \sum_{i=1}^k m_i \).

In Section 5.1, we derive some preliminary results and establish the MLEs for two different
assumptions for the scale parameter \( \vartheta \). Following that, we derive the density functions of the previously determined MLEs (see Section 5.2).

### 5.1 Model and Basic Distributional Results

Let the multi-sample sequential exponential model be given as specified in Model 2.3.10. Then, the multi-sample Type-I sequential hybrid censoring scheme (Type-I MSHCS) consists of \( k \geq 2 \) independently conducted Type-I sequential hybrid censoring experiments with parameters \( n_i, m_i \leq n_i \) and \( T_i^\diamond > 0, 1 \leq i \leq k \) (cf. Model 2.3.10). The termination of each censoring experiment is also accomplished independently from the other \((k-1)\) experiments.

In the following, we refer to this particular independence situation as to the 'independence property' of the multi-sample Type-I sequential hybrid censoring model. Further, we note that, for \( k = 1 \) the multi-sample Type-I sequential hybrid censoring scheme corresponds to the common Type-I sequential hybrid censoring scheme with parameters \( n, m \leq n \) and \( T > 0 \). Notice that the multi-sample Type-I sequential hybrid censoring scheme is for the particular setting of ordinary order statistics and of progressively Type-II censored order statistics denoted by multi-sample Type-I hybrid censoring scheme (Type-I MHCS) and by multi-sample Type-I progressive hybrid censoring scheme (Type-I MPHCS), respectively.

The stopping time \( T_i^* \) for the \( i \)th sample is given by

\[
T_i^* = \min \{ Z_{i,m_i}^*, T_i^\diamond \}, \quad m_i \leq n_i, \quad T_i^\diamond \in (0, \infty), \quad 1 \leq i \leq k.
\]

The counter variables \( D_i^\diamond \), which denote the number of observed failures till time \( T_i^\diamond \), are for the respective \( i \)th sample given by

\[
D_i^\diamond = \sum_{j=1}^{m_i} \mathbf{1}_{(-\infty, T_i^\diamond]}(Z_{i,j}^*), \quad 1 \leq i \leq k.
\]

Further, we define the Type-I sequential hybrid censored order statistics for the \( i \)th sample, \( 1 \leq i \leq k \), by

\[
Z_{i,j}^I := \min \{ Z_{i,j}^*, T_i^\diamond \}, \quad 1 \leq j \leq m_i.
\]

The multi-sample Type-I sequential hybrid censored order statistics are then specified by the random matrix \( Z_k^I \), with

\[
Z_k^I = (Z_{i,j}^I)_{1 \leq i \leq k, 1 \leq j \leq d_i^\diamond}, \quad (5.1)
\]

where the integers \( d_1^\diamond, \ldots, d_k^\diamond \) denote the realizations of the counters \( D_1^\diamond, \ldots, D_k^\diamond \). Note that the random matrix \( Z_k^I \) may consist of rows with different lengths. This convention is tolerable, since we do not intend to apply matrix operations on \( Z_k^I \).

First, we determine the probabilities \( P(D_i^\diamond = d_i^\diamond, 1 \leq i \leq k), d_i^\diamond \in \{0, \ldots, m_i\}, 1 \leq i \leq k \). They can be obtained with Lemma 4.1.1 and by taking into account the 'independence property' of the multi-sample Type-I sequential hybrid censoring scheme.

**Lemma 5.1.1** For \( d_1^\diamond = \cdots = d_k^\diamond = 0 \), we find

\[
P(D_i^\diamond = 0, 1 \leq i \leq k) = \prod_{i=1}^k P(D_i^\diamond = 0) = \prod_{i=1}^k (1 - F_i(T_i^\diamond))^{\gamma_i^\diamond},
\]

where the integers \( \gamma_1^\diamond, \ldots, \gamma_k^\diamond \) denote the realizations of the counters \( D_1^\diamond, \ldots, D_k^\diamond \). Note that the random matrix \( Z_k^I \) may consist of rows with different lengths. This convention is tolerable, since we do not intend to apply matrix operations on \( Z_k^I \).
For \( d_i \in \{1, \ldots, m_i\}, 1 \leq i \leq k \), the joint distribution of \( D_i = d_i \), \( 1 \leq i \leq k \), is given by

\[
P(D_i = d_i, 1 \leq i \leq k) = \prod_{i=1}^{k} P(D_i = d_i),
\]

with

\[
P(D_i = d_i) = \begin{cases} 
1 - F_i(T_i) \gamma_{i,d_i+1} f_i(T_i) \gamma_{i,d_i+1} f_i(T_i) \gamma_{i,d_i+1} f_i(T_i) \gamma_{i,d_i+1} f_i(T_i), & d_i \in \{1, \ldots, m_i - 1\}, \\
F_i,d_i(T_i) & d_i = m_i.
\end{cases}
\]

For the exponential distribution, the above probabilities read

\[
\frac{1}{P(D_i = d_i)} = \begin{cases} 
\frac{\gamma_{i,d_i+1}}{\partial_{y_i} f_i(T_i)} & d_i \in \{1, \ldots, m_i - 1\}, \\
1 & d_i = m_i,
\end{cases} \tag{5.2}
\]

We recall that \( z_i,d_i = (z_{i,1}, \ldots, z_{i,d_i}) \), \( 1 \leq i \leq k \) (cf. Model 2.3.10).

**Lemma 5.1.2** Let \( z_{i0} := \mu \), \( 1 \leq i \leq k \). Then, for \( d_i \in \{1, \ldots, m_i\}, 1 \leq i \leq k \), the conditional joint density function of the multi-sample Type-I sequential hybrid censored order statistics, specified by the random matrix \( Z_i^* \) (see (5.1)), is given by

\[
f_{Z_i^* \mid D_i = d_i, 1 \leq i \leq k} = \prod_{i=1}^{k} \frac{\prod_{j=1}^{d_i} \gamma_{ij}^*}{\prod_{j=1}^{d_i} P(D_i = d_i) y_{ij}^*} \\
\times \left[ \prod_{i=1}^{k} \left( z_{i,d_i} \right) \exp \left\{ - \frac{1}{\partial_{y_i}} \left( \sum_{j=1}^{d_i} \left( \gamma_{ij}^* - \gamma_{ij+1}^* \right) (z_{ij} - \mu) + \gamma_{ij+1}^* (T_i - \mu) \right) \right\} \right].
\]

where \( \prod_{i=1}^{k} P(D_i = d_i)^{-1} \) according to (5.2).

**Proof.** Due to the ‘independence property’ of the multi-sample Type-I sequential hybrid censoring scheme, we obtain with Lemma 4.2.2

\[
f_{Z_i^* \mid D_i = d_i, 1 \leq i \leq k} = \prod_{i=1}^{k} f_{Z_i^* \mid D_i = d_i} \left( z_{i,d_i} \right),
\]

where

\[
f_{Z_i^* \mid D_i = d_i} = \begin{cases} 
\prod_{i=1}^{m_i} \frac{f_{i,m_i}(z_{i,m_i})}{F_{i,m_i}(T_i)}, & d_i \in \{1, \ldots, m_i - 1\}, \\
\prod_{i=1}^{m_i} (z_{i,d_i}) f_{i,m_i}(z_{i,m_i}) \frac{f_{i,m_i}(z_{i,m_i})}{F_{i,m_i}(T_i)}, & d_i = m_i,
\end{cases}
\]

for \( 1 \leq i \leq k \). Then, the assumption follows together with (5.2). \( \square \)

The MLEs can be obtained by utilizing the same arguments as for the Type-I sequential hybrid censoring scheme. Therefore, we refer to the proof of Lemma 4.2.3.
Lemma 5.1.3  Let the location parameter $\mu$ be known and let $P_{F_{i}} = \text{Exp}(\mu, \vartheta_{i}), 1 \leq i \leq k$. Then, the MLE for $(\vartheta_{1}, \ldots, \vartheta_{k})$, is given by

$$\hat{\vartheta}_{k} = (\hat{\vartheta}_{1}, \ldots, \hat{\vartheta}_{k}),$$

with

$$\hat{\vartheta}_{i} = \frac{1}{D_{i}} \sum_{i,j=1}^{D_{i}^\circ} (\gamma_{i,j}^{\ast} - \gamma_{i,j+1}^{\ast})(Z_{i,j}^{\ast} - \mu) + \gamma_{i,D_{i}^\circ+1}^{\ast}(T_{i}^{\circ} - \mu), \quad D_{i}^\circ \in \{1, \ldots, m_{i}\}, \quad 1 \leq i \leq k.$$

For $P_{F_{i}} = \text{Exp}(\mu, \vartheta)$, $1 \leq i \leq k$, the MLE for $\vartheta$ is further given by

$$\hat{\vartheta} = \frac{1}{D_{k}^\circ} \sum_{i=1}^{k} \sum_{i,j=1}^{D_{i}^\circ} (\gamma_{i,j}^{\ast} - \gamma_{i,j+1}^{\ast})(Z_{i,j}^{\ast} - \mu) + \gamma_{i,D_{i}^\circ+1}^{\ast}(T_{i}^{\circ} - \mu),$$

with $D_{i}^\circ \in \{1, \ldots, m_{i}\}, 1 \leq i \leq k$.

5.2 Distribution Theory for the MLE

For $1 \leq i \leq k$, we define the total time on test statistic $S_{D_{i}^\circ}$ (cf. (4.4)) for the $i$th sample by

$$S_{D_{i}^\circ} = \sum_{j=1}^{D_{i}^\circ} (\gamma_{i,j}^{\ast} - \gamma_{i,j+1}^{\ast})(Z_{i,j}^{\ast} - \mu) + \gamma_{i,D_{i}^\circ+1}^{\ast}(T_{i}^{\circ} - \mu), \quad D_{i}^\circ \in \{1, \ldots, m_{i}\}. \quad (5.3)$$

Remark 5.2.1  For $P_{F_{i}} = \text{Exp}(\mu, \vartheta_{i}), 1 \leq i \leq k$, we have

$$\hat{\vartheta}_{k} = \left( \frac{S_{D_{1}^\circ}}{D_{1}^\circ}, \ldots, \frac{S_{D_{k}^\circ}}{D_{k}^\circ} \right), \quad D_{i}^\circ \in \{1, \ldots, m_{i}\}, \quad 1 \leq i \leq k.$$

Further, we can easily verify that, for $P_{F_{i}} = \text{Exp}(\mu, \vartheta)$, $1 \leq i \leq k$,

$$\hat{\vartheta} = \frac{\sum_{i=1}^{k} S_{D_{i}^\circ}}{\sum_{i=1}^{k} D_{i}^\circ} = \frac{S_{\bullet k}}{D_{\bullet k}}, \quad D_{i}^\circ \in \{1, \ldots, m_{i}\}, \quad 1 \leq i \leq k.$$

The desired joint density function $f_{S_{D_{i}^\circ}; 1 \leq i \leq k|D_{i}^\circ=d_{i}^\circ}$ follows directly from Theorem 4.2.9, that is

$$f_{S_{D_{i}^\circ}; 1 \leq i \leq k|D_{i}^\circ=d_{i}^\circ}(s_{k})$$

$$= \prod_{i=1}^{k} f_{S_{D_{i}^\circ}|D_{i}^\circ=d_{i}^\circ}(s_{i})$$

$$= \frac{1}{\prod_{i=1}^{k} P(D_{i}^\circ = d_{i}^\circ)} \left[ \prod_{i=1}^{k} \frac{(T_{i}^{\circ} - \mu)^{d_{i}^\circ} \prod_{j=1}^{d_{i}^\circ} \gamma_{i,j}^{\ast}}{\vartheta_{i,j}^{\ast} d_{i}^{\circ}!} B_{d_{i}^{\circ}-1}(s_{i} | \gamma_{i,d_{i}^{\circ}+1}^{\ast}(T_{i}^{\circ} - \mu)) e^{-s_{i}/\vartheta_{i}} \right],$$

with $\left( \prod_{i=1}^{k} P(D_{i}^\circ = d_{i}^\circ) \right)^{-1}$ as in (5.2). This result leads us for $P_{F_{i}} = \text{Exp}(\mu, \vartheta_{i}), 1 \leq i \leq k$, directly to the conditional density function of the MLE $\hat{\vartheta}_{k}$. 


Figure 5.1: Plots of $f_{\hat{\theta}_k|D_k^2 \geq 1, 1 \leq i \leq k}$, with $k = 2$, $\mu = 0$, $n_1 = n_2 = 6$, $m_1 = m_2 = 3$, $T_1^\circ = T_2^\circ = 1$ and $(\gamma_{1,1}, \gamma_{1,2}, \gamma_{1,3}) = (\gamma_{2,1}, \gamma_{2,2}, \gamma_{2,3}) = (6, 5, 4)$. Left: $\vartheta_1 = 1$ and $\vartheta_2 = 1.1$. Right: $\vartheta_1 = 2$ and $\vartheta_2 = 2.1$.

Theorem 5.2.2 Let $\gamma_{i,1}, \ldots, \gamma_{i,m_i} > 0$ with $\gamma_{i,1} \neq \gamma_{i,2}$ and $\gamma_{i,m_i+1} = 0$, $1 \leq i \leq k$. Then, the conditional density function $f_{\hat{\theta}_k|D_k^2 \geq 1, 1 \leq i \leq k}$ for $s_k \in [0, \infty)^k$ given by

$$f_{\hat{\theta}_k|D_k^2 \geq 1, 1 \leq i \leq k}(s_k) = \frac{1}{\prod_{i=1}^{k} \left(1 - e^{-\gamma_{i,1}(T_i^\circ - \mu)/\vartheta_i}\right)} \times \prod_{i=1}^{k} \left[ \sum_{d_i^k=1}^{m_i} \frac{(T_i^\circ - \mu)^{d_i^k}}{(d_i^k - 1)!} \vartheta_i^{d_i^k} B_{d_i^k-1}(d_i^k s_i | \gamma_{i,d_i^k+1}(T_i^\circ - \mu)) e^{-d_i^k s_i / \vartheta_i} \right].$$

Proof. Due to (5.4), we find that

$$f_{\hat{\theta}_k|D_k^2 \geq 1, 1 \leq i \leq k}(s_k) = \prod_{i=1}^{k} f_{\hat{\theta}_i|D^2_i \geq 1}(s_i), \quad s_k \in [0, \infty)^k.$$ 

Then, the assumption follows directly with Theorem 4.2.12. \hfill \Box

Figure 5.1 shows two plots of $f_{\hat{\theta}_k|D_k^2 \geq 1, 1 \leq i \leq k}$ for $k = 2$. We proceed by considering $\vartheta_i = \vartheta$, $1 \leq i \leq k$, and recall that $S_{\bullet k} = D_{\bullet k}^\circ$ holds (cf. Remark 5.2.1). For the density function $f_{S_{\vartheta}^\circ|1 \leq i \leq k|D_k^2 = d_i^k}$ given in (5.4), we define the linear transformation $\Phi_{S_k} : \mathbb{R}^k \rightarrow \mathbb{R}$, with

$$\Phi_{S_k}(s_k) = (s_{k-1}, \sum_{i=1}^{k} s_i) \quad \text{and} \quad \det(D\Phi_{S_k}) = 1.$$ 

The application of $\Phi_{S_k}$ on (5.4) gives us for $d_i^k \in \{1, \ldots, m_i\}$, $1 \leq i \leq k$,

$$f_{S_{\vartheta}^\circ|1 \leq i \leq k-1, S_{\bullet k}|D_k^2 = d_i^k, 1 \leq i \leq k}(s_{k-1}, s_k) \quad \text{(5.5)}$$
\[ e^{-s/\varphi} \left[ \prod_{i=1}^{k} P(D_i^o = d_i^o) \prod_{i=1}^{k} d_i^{o+1} \right] \prod_{j=1}^{k} \left( T_i^o - \mu \right)^{d_j^o} \gamma_{i,j}^o \]
\times \left[ \prod_{i=1}^{k-1} B(d_i^o-1)(s_i|\gamma_{i,d_i^o+1}^o(T_i^o - \mu)) \right] B(d_k^o-1)(s - \sum_{i=1}^{k-1} s_i|\gamma_{i,d_i^o+1}^o(T_k^o - \mu)).

Let \( g : \mathbb{R}^k \rightarrow \mathbb{R} \), with
\[ g(s_{k-1}, s) = \left( \prod_{i=1}^{k-1} B(d_i^o-1)(s_i|\gamma_{i,d_i^o+1}^o(T_i^o - \mu)) \right) B(d_k^o-1)(s - \sum_{i=1}^{k-1} s_i|\gamma_{i,d_i^o+1}^o(T_k^o - \mu)). \]

Then, the integration of \( g \) w.r.t. the variables \( s_1, \ldots, s_{k-1} \) results in determining the \((k-1)\)-fold B-spline convolution addressed in Section 3.3.1. For brevity, let \( \mu = 0 \). Hence, we obtain with equation (C.1) and Lemma 3.3.2
\[ \int_{\mathbb{R}^{k-1}} g(s_{k-1}, s) ds_{k-1} \]
\[ = \int_{\mathbb{R}} B(d_1^o-1)(s_1|\gamma_{1,d_1^o+1}^o(T_1^o)) \]
\[ \cdots \int_{\mathbb{R}} B(d_k^o-1)(s_{k-1}|\gamma_{k-1,d_k^o+1}^o(T_{k-1}^o)) B(d_k^o-1)(s - \sum_{i=1}^{k-1} s_i|\gamma_{i,d_i^o+1}^o(T_k^o)) ds_{k-1} \cdots ds_1 \]
\[ = [\gamma_{1,d_1^o+1}^o T_1^o]_{x_1} [\gamma_{2,d_2^o+1}^o T_2^o]_{x_2} \cdots [\gamma_{k,d_k^o+1}^o T_k^o]_{x_k} \left( \sum_{i=1}^{k} x_i - s \right)^{d_k^o-1} + \prod_{i=1}^{k} \frac{d_i^o}{(d^o_{*k} - 1)}! \]

**Remark 5.2.3** The expression
\[ [\gamma_{1,d_1^o+1}^o T_1^o]_{x_1} [\gamma_{2,d_2^o+1}^o T_2^o]_{x_2} \cdots [\gamma_{k,d_k^o+1}^o T_k^o]_{x_k} \left( \sum_{i=1}^{k} x_i - s \right)^{d_k^o-1} \]
means that the operator of the divided differences has to be applied \( k \) times to the term \( \left( \sum_{i=1}^{k} x_i - s \right)^{d_k^o-1} \). I.e., in the \( i \)th step the term \( \left( \sum_{i=1}^{k} x_i - s \right)^{d_k^o-1} \) will be evaluated w.r.t. the variable \( x_i \) by applying the divided differences operator w.r.t. the knot sequence \( \gamma_{i,d_i^o+1}^o T_i^o = (\gamma_{i,d_i^o+1}^o T_i^o, \gamma_{i,d_i^o+1}^o T_i^o, \ldots, \gamma_{i,d_i^o+1}^o T_i^o) \), \( 1 \leq i \leq k \).

For example, let \( k = 2 \) and \( T_1^o = T_2^o = 1 \). Further, we assume the \( \gamma_{1,}\)'s as well as the \( \gamma_{2,}\)'s to be pairwise distinct. Then, for \( d_1^o, d_2^o \geq 1 \), the expression
\[ [\gamma_{1,d_1^o+1}^o]_{x_1} [\gamma_{2,d_2^o+1}^o]_{x_2} (x_1 + x_2 - s)^{d_1^o+d_2^o-1} \]
can be together with Lemma 3.3.1, (i) calculated by
\[ [\gamma_{1,d_1^o+1}^o]_{x_1} [\gamma_{2,d_2^o+1}^o]_{x_2} (x_1 + x_2 - s)^{d_1^o+d_2^o-1} \]
\[ = \left[ \gamma_{1,d_1^o+1}^o \right]_{x_1} \sum_{j=0}^{d_2^o} \frac{(x_1 + \gamma_{2,d_2^o+1-j_2}^o - s)^{d_1^o+d_2^o-1}}{\prod_{j=0}^{d_2^o} \gamma_{2,d_2^o+1-j_2}^o - \gamma_{2,d_2^o+1-i_2}^o} \]
\[ = \sum_{j_1=0}^{d_1^o} \left( \prod_{i_1=0}^{d_1^o} \left( \gamma_{1,d_1^o+1-j_1}^o - \gamma_{1,d_1^o+1-i_1}^o \right)^{-1} \right) \sum_{j_2=0}^{d_2^o} \frac{(\gamma_{1,d_1^o+1-j_1}^o + \gamma_{2,d_2^o+1-j_2}^o - s)^{d_1^o+d_2^o-1}}{\prod_{j_2=0}^{d_2^o} \gamma_{2,d_2^o+1-j_2}^o - \gamma_{2,d_2^o+1-i_2}^o} \]
for \( s \geq 0 \).
The above derivations lead us to the following expression for the conditional density function of $S_k$.

**Theorem 5.2.4** Let $\gamma_{i,1}, \ldots, \gamma_{i,m_i} > 0$ with $\gamma_{i,1} \neq \gamma_{i,2}$ and $\gamma_{i,m_i+1} = 0$, $1 \leq i \leq k$. Then, the conditional density function $f_{S_k | D_i^o = d_i^o, 1 \leq i \leq k}$ for $d_i^o \in \{1, \ldots, m_i\}$, $1 \leq i \leq k$, and $s \geq 0$ is given by

$$f_{S_k | D_i^o = d_i^o, 1 \leq i \leq k}(s) = \frac{e^{-s/\theta} \left[ \prod_{i=1}^{k} \left[ \prod_{j=1}^{d_i^o} (T_j^o - \mu) \gamma_{i,j}^* \right] \right]}{(d_{i,k}^o - 1)! \prod_{i=1}^{k} P(D_i^o = d_i^o)} \times \left[ \gamma_{1,d_1^o+1}(T_1^o - \mu) \right] x_1 \cdots \left[ \gamma_{k,d_k^o+1}(T_k^o - \mu) \right] x_k \left( \sum_{i=1}^{k} x_i - s \right)^{d_{i,k}^o-1}.$$ 

This yields the distribution of $\hat{\theta}$.

**Theorem 5.2.5** Let $\gamma_{i,1}, \ldots, \gamma_{i,m_i} > 0$ with $\gamma_{i,1} \neq \gamma_{i,2}$ and $\gamma_{i,m_i+1} = 0$, $1 \leq i \leq k$. Then, the density function $f_{\hat{\theta} | D_i^o \geq 1, 1 \leq i \leq k}$ is given by

$$f_{\hat{\theta} | D_i^o \geq 1, 1 \leq i \leq k}(s) = \frac{1}{\prod_{i=1}^{k} (1 - e^{-\gamma_{i,1}(T_i^o - \mu) / \theta})} \sum_{d_1^o=1}^{m_1} \cdots \sum_{d_k^o=1}^{m_k} \frac{e^{-\sum_{i=1}^{k} d_i^o s / \theta} \left[ \prod_{i=1}^{k} \left[ \prod_{j=1}^{d_i^o} (T_j^o - \mu) \gamma_{i,j}^* \right] \right]}{(d_{i,k}^o - 1)! \prod_{i=1}^{k} P(D_i^o = d_i^o)} \times \left[ \gamma_{1,d_1^o+1}(T_1^o - \mu) \right] x_1 \cdots \left[ \gamma_{k,d_k^o+1}(T_k^o - \mu) \right] x_k \left( \sum_{i=1}^{k} x_i - d_{i,k}^o s \right)^{d_{i,k}^o-1},$$

with $\text{supp}(f_{\hat{\theta} | D_i^o \geq 1, 1 \leq i \leq k}) \subseteq [0, M]$, where

$$M = \frac{1}{k} \sum_{i=1}^{k} \max \left\{ \gamma_{i,1}(T_i^o - \mu), \ldots, \gamma_{i,m_i}(T_i^o - \mu) \right\}.$$ 

**Proof.** The conditional density function follows from

$$f_{\hat{\theta} | D_i^o \geq 1, 1 \leq i \leq k}(s) = \frac{1}{\prod_{i=1}^{k} (1 - P(D_i^o = 0))} \sum_{d_1^o=1}^{m_1} \cdots \sum_{d_k^o=1}^{m_k} d_i^o s \left[ \prod_{i=1}^{k} f_{S_k | D_i^o = d_i^o, 1 \leq i \leq k}(d_{i,k}^o s) \prod_{i=1}^{k} P(D_i^o = d_i^o) \right],$$

with $f_{S_k | D_i^o = d_i^o}$ as in Theorem 4.2.9, $d_i^o \in \{1, \ldots, m_i\}$, $1 \leq i \leq k$. The assertion for the support follows directly from Lemma 3.3.3, (iii), i.e.,

$$\text{supp}(f_{\hat{\theta} | D_i^o \geq 1, 1 \leq i \leq k}) = \bigcup_{d_i^o \in \times_{i=1}^{k} \{1, \ldots, m_i\}} \left[ 0, \frac{1}{d_{i,k}^o} \sum_{i=1}^{k} \max \left\{ \gamma_{i,1}(T_i^o - \mu), \ldots, \gamma_{i,d_i^o+1}(T_i^o - \mu) \right\} \right]$$

$$\subseteq \left[ 0, \frac{1}{k} \sum_{i=1}^{k} \max \left\{ \gamma_{i,1}(T_i^o - \mu), \ldots, \gamma_{i,m_i}(T_i^o - \mu) \right\} \right],$$

where $d_{i,k}^o = (d_1^o, \ldots, d_k^o)$. \qed
In $I$, $n_1 = 3$, $m_1 = 2$, $T_1^{\diamond} = 1$; $n_2 = 4$, $m_2 = 2$, $T_2^{\diamond} = 1$; $n_3 = 5$, $m_3 = 3$, $T_3^{\diamond} = 1$

In $II$, $n_1 = 3$, $m_1 = 2$, $T_1^{\diamond} = 1$; $n_2 = 4$, $m_2 = 2$, $T_2^{\diamond} = 1$; $n_3 = 5$, $m_3 = 3$, $T_3^{\diamond} = 1$

Table 5.1: Parameter settings for the subdivision of one Type-I hybrid censoring experiment into three Type-I hybrid censoring sub-experiments (case I) as well as for the subdivision of one Type-I progressive hybrid censoring experiment into three Type-I progressive hybrid censoring sub-experiments (case II).

Figure 5.2: Plots of $\hat{\vartheta}_{\left| D^{\diamond}_i \geq 1 \right|, 1 \leq i \leq k}$ for $\vartheta = 1$, $\mu = 0$, and $k = 3$ (solid line), and of $\hat{\vartheta}^{D^{\diamond} \geq 1}$ (as in Theorem 4.2.12) for $n = 12$, $m = 7$, $\vartheta = T = 1$ (dashed line). Left: setting of ordinary order statistics, see scenario I from Table 5.1. Right: setting of progressively Type-II censored order statistics, see scenario II. The dotted lines represent the density function of $\hat{\vartheta}$ under (progressive) Type-II censoring.

Figure 5.2 depicts the plots of $f^{\hat{\vartheta}|D^{\diamond}_i \geq 1, 1 \leq i \leq k}$ for $k = 3$. The shapes are compared to the density functions for Type-I (progressive) hybrid censoring (dashed lines) and to the density functions w.r.t. (progressive) Type-II censoring in each case. On the left-hand side we consider the subdivision of a Type-I hybrid censored experiment (with parameters $n = 12$, $m = 7$, $T = 1$) into three Type-I hybrid censored sub-experiments (with parameters as specified in scenario I from Table 5.1). On the right-hand side a Type-I progressive hybrid censoring experiment (with $n = 12$, $m = 7$, $T = 1$ and censoring scheme $R = (1, 1, 1, 1, 0, 0)$), with the subdivision given in II from Table 5.1, is considered. Note that the $\gamma^*$'s given in II from Table 5.1 correspond to the censoring schemes $R_1 = (1, 0)$, $R_2 = (1, 1)$ and $R_3 = (1, 1, 0)$.

We further observe that the density function for the subdivision of the original experiment into sub-experiments, seem to approximate the density function of the MLE $\hat{\vartheta}$ under (progressive) Type-II censoring better, than the density functions of the MLE $\hat{\vartheta}$ w.r.t. the original experiment of Type-I (progressive) hybrid censoring.
can be extended to the setting

\[\gamma_i, j = \begin{cases} 0.5 & i = 1 \text{ or } j = 1, 2 \text{ or } 3, \\ 1.5 & j = 1 \text{ for } i = 2, 3. \end{cases}\]

In the following, we consider for \(k = 2 \text{ and } k = 3\), respectively, three different scenarios for the \(\gamma^*\)’s (see Table 5.2). Figure 5.3 depicts the density functions of the MLEs for \(k\) different experiments (dashed, dotted, and dashed dotted lines) as well as the density function of the MLE for the multi-sample case (solid lines). In scenario I from Table 5.2, we choose an increasing, a decreasing and an alternating sequence for the \(\gamma^*\)’s. Scenarios II and III correspond for \(k = 3\), except for the threshold times \(T^*_i, T^*_2\) and \(T^*_3\), to the scenarios I and II from Table 5.1.

Table 5.2: Three different settings for the \(\gamma^*\)’s for \(k = 2\) and \(k = 3\) respectively. The setups provided in scenario II correspond to Type-I hybrid censoring experiments. The settings given in scenario III correspond to the Type-I progressive hybrid censoring scheme.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(T^*_1)</th>
<th>(T^*_2)</th>
<th>(T^*_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(n_1 = 4, m_1 = 2, T^*_1 = 1)</td>
<td>(n_2 = 6, m_2 = 2, T^*_2 = 1.25)</td>
<td>(n_3 = 5, m_3 = 3, T^*_3 = 1.1)</td>
</tr>
<tr>
<td>2</td>
<td>(j, \gamma_{1,j} = 0.5, 1)</td>
<td>(j, \gamma_{2,j} = 2, 1.5)</td>
<td>(j, \gamma_{3,j} = 1, 0.333, 0.666)</td>
</tr>
<tr>
<td>II</td>
<td>(n_1 = 4, m_1 = 2, T^*_1 = 1)</td>
<td>(n_2 = 6, m_2 = 2, T^*_2 = 1.25)</td>
<td>(n_3 = 5, m_3 = 3, T^*_3 = 1.1)</td>
</tr>
<tr>
<td>3</td>
<td>(j, \gamma_{1,j} = 1, 2)</td>
<td>(j, \gamma_{2,j} = 2, 3)</td>
<td>(j, \gamma_{3,j} = 4, 3)</td>
</tr>
<tr>
<td>III</td>
<td>(n_1 = 4, m_1 = 2, T^*_1 = 1)</td>
<td>(n_2 = 6, m_2 = 2, T^*_2 = 1.25)</td>
<td>(n_3 = 5, m_3 = 3, T^*_3 = 1.1)</td>
</tr>
<tr>
<td>3</td>
<td>(j, \gamma_{1,j} = 1, 2)</td>
<td>(j, \gamma_{2,j} = 2, 3)</td>
<td>(j, \gamma_{3,j} = 4, 3)</td>
</tr>
</tbody>
</table>

Remark 5.2.6: We consider the scenario of subdividing one experiment into \(k\) sub-experiments. Then, it is obvious that the probability for satisfying the condition of observing at least one failure per experiment decreases with an increase of the number of sub-experiments \(k\). If we consider for example the simulation study conducted in Appendix B, we see that for \(k = 2\), the condition has been satisfied in 998793 out of \(10^6\) samples (cf. Table B.1). For \(k = 3\), however, the condition has been satisfied only in 979882 out of \(10^6\) cases.

One possibility to overcome this drawback, is to condition on the event that at least one failure among the \(k\) experiments has been observed. The MLE \(\hat{\theta}\) remains the same as in Lemma 5.1.3. But instead of conditioning on \(D^*_i \geq 1\), \(1 \leq i \leq k\), we condition on the event \(D^*_{\cdot,k} \geq 1\). First, we calculate together with Lemma 4.1.1,

\[
P(D^*_{\cdot,k} \geq 1) = 1 - P(D^*_i = 0, 1 \leq i \leq k) = 1 - \prod_{i=1}^{k} P(D^*_i = 0) = 1 - \prod_{i=1}^{k} e^{-\gamma_i^* (T^*_i - \mu) / \hat{\theta}}
\]
Plots w.r.t. scenario I from Table 5.2.

Plots w.r.t. the model of Type-I hybrid censoring (see scenario II).

Plots w.r.t. the model of Type-I progressive hybrid censoring (see scenario III).

Figure 5.3: Plots of $f_{\hat{\theta}\mid D^i \geq 1, 1 \leq i \leq k}$ (solid lines) with $\theta = 1$ and $\mu = 0$, for $k = 2$ (left) and $k = 3$ (right). The dashed lines correspond to the density function $f_{\hat{\theta}\mid D \geq 1}$ (see Theorem 4.2.12) w.r.t. the first parameter setting provided in Table 5.2 (left part of the table). The dotted and the dashed dotted lines correspond to the density function $f_{\hat{\theta}\mid D \geq 1}$ of the second and the third parameter setting, respectively (middle and right part of the table).
By taking into account the representation provided in Theorem 5.2.5, we are able to derive an expression for the density function of the MLE \( \hat{\theta} \) (as given in Lemma 5.1.3) conditional on \( D_{\bullet k}^{\circ} \geq 1 \),

\[
f_{\hat{\theta}|D_{\bullet k}^{\circ} \geq 1}(s) = \frac{1}{1 - \exp \left\{ - \frac{1}{d} \sum_{i=1}^{k} \gamma_i^*(T_i^\circ - \mu) \right\}} \left[ \sum_{d_{\bullet k} \in \mathcal{M}_k} e^{-d_{\bullet k} s/d} d_{\bullet k} \left[ \prod_{i=1}^{k} \left[ \prod_{j=1}^{d_{\bullet k}} (T_i^\circ - \mu)^{d_{\bullet k}^*} \gamma_i^{*j} \right] \left( \sum_{x_i = 1}^{x} x_i - d_{\bullet k}^* s \right) \right] \right],
\]

where

\[
\mathcal{M}_k = \left\{ d_{\bullet k}^* \in \{1, \ldots, m_i\} \mid d_{\bullet k}^* \geq 1 \right\}.
\]

For the evaluation of the above expression, we require that

\[
\left[ t_0^{(1)} x_1 \right] \cdots \left[ t_0^{(k)} x_k \right] \left[ t_0^{(k+1)} x_{k+1} \right] \cdots \left[ t_0^{(k+s)} x_s \right] g \left( \sum_{x=1}^{k} x_i \right)
\]

for \( \ell \in \{1, \ldots, k-1\} \) and \( d_{\bullet +1}^*, \ldots, d_k^* \geq 1 \). Then, for this new setting, the number of relevant terms to be evaluated in order to calculate the density function of \( \hat{\theta} \) increases significantly. In particular, the expression provided in Theorem 5.2.5 consists of \( \prod_{i=1}^{k} m_i \) terms. The representation given in (5.6), however, requires the evaluation of \( \left( \prod_{i=1}^{k} (m_i+1) \right) - 1 \) relevant terms.

In order to get an impression of how the two herein addressed conditions affect the structure and the shape of the density function of \( \hat{\theta} \), we consider the following example.

**Example 5.2.7** Let \( k = 2, m_1 = 2, m_2 = 3 \) and \( \mu = 0 \). The remaining parameters can be chosen arbitrarily. Then, according to (5.6), the density function \( f_{\hat{\theta}|D_{\bullet k}^{\circ} \geq 1}(s) \) is given by

\[
f_{\hat{\theta}|D_{\bullet k}^{\circ} \geq 1}(s) = \frac{1}{1 - e^{-\gamma_1^* T_1^\circ / \theta}} \left( 1 - e^{-\gamma_2^* T_2^\circ / \theta} \right) f_{\hat{\theta}|D_{\bullet k}^{\circ} \geq 1,D_{\bullet k}^{\circ} \geq 1}(s)
\]

\[
+ e^{-s/\theta} \gamma_1^* T_1^\circ B_0(s - \gamma_1^* T_1^\circ, \gamma_2^* T_2^\circ, \gamma_2^* T_2^\circ)
\]

\[
+ e^{-2s/\theta} \gamma_2^* \gamma_2^* (T_2^\circ)^2 B_1(2s - \gamma_1^* T_1^\circ, \gamma_2^* T_2^\circ, \gamma_2^* T_2^\circ, \gamma_2^* T_2^\circ)
\]

\[
+ e^{-3s/\theta} \frac{3}{2} \gamma_1^* \gamma_2^* \gamma_2^* (T_2^\circ)^3 B_2(3s - \gamma_1^* T_1^\circ, \gamma_2^* T_2^\circ, \gamma_2^* T_2^\circ, \gamma_2^* T_2^\circ)
\]

\[
+ e^{-2s/\theta} \gamma_1^* \gamma_1^* \gamma_1^* (T_1^\circ)^2 B_1(2s - \gamma_1^* T_1^\circ, \gamma_1^* T_1^\circ, \gamma_1^* T_1^\circ)
\]

\[
+ e^{-s/\theta} \gamma_1^* T_1^\circ B_0(s - \gamma_2^* T_1^\circ, \gamma_1^* T_1^\circ, \gamma_1^* T_1^\circ), \quad s \geq 0.
\]

(5.9)
contains four different scenarios for the two-sample case. While the scenarios 5.4 is easier to handle than the 5.2.5 5.2.6 5.2.6 5.2.5.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( m_1 = 1, m_2 = 2, T_1^\gamma = 1.5, T_2^\gamma = 1 )</td>
<td>( m_1 = 1, m_2 = 2, T_1^\gamma = 1.5, T_2^\gamma = 1 )</td>
</tr>
<tr>
<td></td>
<td>( j ) 1 2 3</td>
<td>( j ) 1 2 3</td>
</tr>
<tr>
<td></td>
<td>( \gamma_{1,j} ) 1 0 –</td>
<td>( \gamma_{1,j} ) 1 0 –</td>
</tr>
<tr>
<td></td>
<td>( \gamma_{2,j} ) 0.5 1 0</td>
<td>( \gamma_{2,j} ) 2 1 0</td>
</tr>
<tr>
<td>II</td>
<td>( m_1 = 2, m_2 = 3, T_1^\gamma = 1.5, T_2^\gamma = 1 )</td>
<td>( m_1 = 2, m_2 = 3, T_1^\gamma = 1.5, T_2^\gamma = 1 )</td>
</tr>
<tr>
<td></td>
<td>( j ) 1 2 3 4</td>
<td>( j ) 1 2 3 4</td>
</tr>
<tr>
<td></td>
<td>( \gamma_{1,j} ) 0.5 1 0 –</td>
<td>( \gamma_{1,j} ) 2 1 0</td>
</tr>
<tr>
<td></td>
<td>( \gamma_{2,j} ) 1.5 0.5 1 0</td>
<td>( \gamma_{2,j} ) 3 2 1 0</td>
</tr>
</tbody>
</table>

Table 5.3: Four different settings (I.(a), I.(b), II.(a) and II.(b)) for the multi-sample Type-I sequential hybrid censoring model with \( k = 2 \).

It is clear that the density function w.r.t. the condition \( D_{k}^{\circ} \geq 1 \) contains all the terms which occur w.r.t. the condition \( D_{i}^{\circ} \geq 1 \), \( 1 \leq i \leq k \). Further, by incorporating the factor \( d_{i+1}^{\circ} \), we can express for the present case of \( k = 2 \) the resulting (one-dimensional) divided differences (cf. (5.8)) as B-splines of degree \( d_{i+1}^{\circ} – 1 \).

Table 5.3 contains four different scenarios for the two-sample case. While the scenarios I.(a) and II.(a) consist of alternating sequences for the \( \gamma^* \)'s, strictly decreasing sequences are considered in the remaining two scenarios. Figure 5.4 shows the plots of \( f_{\hat{\theta}^\circ D_{i}^{\circ} \geq 1, D_{i}^{\circ} \geq 1} \) (solid line) in comparison to \( f_{\hat{\theta}^\circ D_{i}^{\circ} \geq 1} \) (dotted lines) w.r.t. the four scenarios given in Table 5.3. As suggested by the formula given in (5.9), we observe that the shapes of the different density functions are similar to each other on the interval \([0, c] \), for \( c \in \mathbb{R} \). However, the support of \( f_{\hat{\theta}^\circ D_{i}^{\circ} \geq 1} \) is in most cases larger than the support of the density function \( f_{\hat{\theta}^\circ D_{i}^{\circ} \geq 1} \).

Remark 5.2.8

(i) We realize that w.r.t. practical applicability, it makes more sense to condition on \( D_{k}^{\circ} \geq 1 \), instead of assuming \( D_{k}^{\circ} \geq 1 \) for \( 1 \leq i \leq k \). However, due to its structure the expression given in Theorem 5.2.5 is easier to handle than the representation introduced in Remark 5.2.6. By conditioning on \( D_{k}^{\circ} \geq 1 \), \( 1 \leq i \leq k \), we are confronted with iterated divided differences of length \( k \). Whereas, for \( D_{k}^{\circ} \geq 1 \), we have to consider iterated divided differences of lengths 1 to \( k \).

(ii) In this work the function given in Remark 5.2.6 was directly deduced from the representation given in Theorem 5.2.5, by making obvious adjustments. For a more formal derivation one might consider the following decomposition of the set \( \mathcal{M}_{k} \), i.e.,

\[
\mathcal{M}_{k} = \left\{ d_{k}^{\circ} \in \prod_{i=1}^{k} \{1, \ldots, m_i\} \right\} \cup \mathcal{M}_{k}^{*}, \quad \text{with} \quad \mathcal{M}_{k}^{*} := \mathcal{M}_{k} \setminus \left\{ d_{k}^{\circ} \in \prod_{i=1}^{k} \{1, \ldots, m_i\} \right\},
\]

where \( \mathcal{M}_{k} \) as in (5.7). Then, the distribution of \( \hat{\theta}^\circ \) for \( d_{k}^{\circ} \in \prod_{i=1}^{k} \{1, \ldots, m_i\} \) can be obtained by using the approach of convolving the B-splines as performed in this work. Let \( d_{k}^{\circ}\ell, 1 \leq \ell \leq k – 1 \) denote the vector \( d_{k}^{\circ} \) with \( \ell \) zero-entries. Then, for \( d_{k}^{\circ}\ell, \in \mathcal{M}_{k}^{*} \), the joint distribution of the quantity of \( k \) samples is degenerated. This holds true due to the fact that in the case \( k \) among the samples no observation has been made. Hence, the distribution w.r.t. \( d_{k}^{\circ}\ell, \) follows from the distribution in the case where for exactly \( k – \ell \) among the \( k \) samples, at least one observation has been made. Elementary calculations lead to the representation given in Remark 5.2.6.
5.2 Distribution Theory for the MLE

We conclude our investigations for the multi-sample situation by providing an expression for the conditional density function of the MLE $\hat{\vartheta}$ in terms of gamma density functions. Hereby, we restrict ourselves to the condition $D_1^0 \geq 1, 1 \leq i \leq k$. First, we need a generalization of Remark 4.3.2 to iterated divided differences.

**Lemma 5.2.9** Let $t_j^{(i)} = (t_0^{(i)}, \ldots, t_{d_j}^{(i)}) \in \mathbb{R}^{d_j+1}$, $1 \leq j \leq k$, $d_1, \ldots, d_k \in \mathbb{N}$, be knot sequences, where the respective knots are not all equal. Then, we have for $c > 0$,

\[
\left[ \frac{t_1}{d_1} x_1 \right]_{x_2} \ldots \left[ t_{d_k}^{(k)} \right]_{x_k} \left( \sum_{j=1}^k x_j - cs \right)^{d_k - 1} + \\
= - \left( -c \right)^{d_k - 1} \left[ t_1^{(1)} \right]_{x_1} \left[ t_2^{(2)} \right]_{x_2} \ldots \left[ t_{d_k}^{(k)} \right]_{x_k} \left( s - \frac{\sum_{j=1}^k x_j}{c} \right)^{d_k - 1}, \quad s \in \mathbb{R}.
\]

**Proof.** The assertion follows nearly in analogy to the proof of Theorem 4.3.1. In particular, we need the identities for truncated power functions provided in (4.40), the linearity for iterated divided differences (see Lemma 3.3.3, (i)), and property (ii) from Lemma 3.3.3. □
Lemma 5.2.9 leads us immediately to the desired representation in terms of gamma density functions.

**Theorem 5.2.10** Let \( \gamma_{i,1}, \ldots, \gamma_{i,m_i} > 0 \) with \( \gamma_{i,1} \neq \gamma_{i,2} \) and \( \gamma_{i,m_i+1} = 0, \ 1 \leq i \leq k \). Then, an expression for \( f_\theta[D_i^\circ \geq 1, i \leq k] \) in terms of gamma density functions is given by

\[
f_\theta[D_i^\circ \geq 1, i \leq k](s) = \frac{1}{\prod_{i=1}^{k} (1 - e^{-\gamma_{i,1}(T_i^\circ - \mu)/\vartheta})} \left[ \sum_{d_1=1}^{m_1} \cdots \sum_{d_k=1}^{m_k} (-1)^{d_k} \prod_{i=1}^{k} \left( \prod_{j=1}^{d_i} (T_i^\circ - \mu) \gamma_{i,j}^s \right) \right] \times \left[ \gamma_{1,d_1+1}^s (T_1^\circ - \mu) \right] \cdots \left[ \gamma_{k,d_k+1}^s (T_k^\circ - \mu) \right] \frac{d_i^s}{x_i} \left( s - \sum_{i=1}^{k} x_i/s \right) \exp \left\{ - \frac{\sum_{i=1}^{k} x_i}{\vartheta} \right\},
\]

for \( s \geq 0 \).

**Proof.** For brevity, let \( \mu = 0 \). Then, we find together with Lemma 5.2.9

\[
\frac{e^{-d_k^s/\vartheta} d_k^s}{(d_k^s - 1)!} \theta^{d_k^s} \left[ \gamma_{1,d_1+1}^s + \cdots + \left( \sum_{i=1}^{k} x_i - d_k^s s \right) + \left( \gamma_{k,d_k+1}^s + T_k^\circ \right) \frac{d_k^s - 1}{x_k} \right] = \frac{(-1)^{d_k^s}}{(d_k^s - 1)!} \theta^{d_k^s} \left[ \gamma_{1,d_1+1}^s + \cdots + \left( \sum_{i=1}^{k} x_i - d_k^s s \right) + \left( \gamma_{k,d_k+1}^s + T_k^\circ \right) \frac{d_k^s - 1}{x_k} \right] = (-1)^{d_k^s} \left[ \gamma_{1,d_1+1}^s + \cdots + \left( \sum_{i=1}^{k} x_i - d_k^s s \right) + \left( \gamma_{k,d_k+1}^s + T_k^\circ \right) \frac{d_k^s - 1}{x_k} \right] \exp \left\{ - \frac{\sum_{i=1}^{k} x_i}{\vartheta} \right\}.
\]

The assertion follows with Theorem 5.2.5 and by plugging in the right-hand side of the above identity.

**Remark 5.2.11** (i) A representation for \( f_\theta[D_i^\circ \geq 1] \) in terms of gamma density functions, can be obtained in analogy to Theorem 5.2.10.

(ii) Explicit expressions in terms of gamma density functions for the density function of the MLE \( \theta \) under multi-sample Type-I hybrid and multi-sample Type-I progressive hybrid censoring, can be obtained by employing Lemma 3.1.3, (i) in Theorem 5.2.10 (cf., e.g., Remark 5.2.3).
Chapter 6

Type-II Hybrid Censoring

In this chapter, we consider the model of Type-II progressive hybrid censoring introduced in Section 1.1.2. While in Section 6.1 basic distributional results are derived, we consider in Section 6.2, the Type-II progressive hybrid censoring scheme for an underlying exponential distribution. Following that, we investigate the model of Type-II hybrid censoring for the uniform distribution (see Section 6.3).

Notice that recently Alma and Belaghi (2016) considered the Type-II progressive hybrid censoring model for the extreme value as well as for the normal distribution. By using the Newton-Raphson method as well as the EM algorithm, they determined for both distributional setups the respective estimators.

6.1 Basic Distributional Results

We assume the IID progressive model specified in Model 2.2.5. According to Example 1.4.2, the system of counter settings as well as the respective sets of valid integers, are given by

\( ( \mathcal{S}_D(\|), \mathcal{I}(\mathcal{S}_D(\|)) ) \).

The distributions of \( D(d) \), for \( d \in \mathcal{I}(\mathcal{D}) \) and with \( \mathcal{D} \in \mathcal{S}_D(\|) \), are given in Lemma 6.1.1. Let \( \mathcal{R} \) denote the initially planned censoring plan. Then, we use the notations for the cumulative distribution functions as well as for the density functions of progressively Type-II censored order statistics as specified in Section 2.2.

Lemma 6.1.1 For \( d < m \), we have,

\[ P(D < m) = 1 - F_{m:m:n}(T). \]

If \( d \in \{ m, \ldots, \tilde{m} - 1 \} \), then we have with \( f(T) > 0 \)

\[ P(D = d) = \frac{1 - F(T)}{\frac{d^m}{d + 1} f(T)} f_{d+1:m:n}(T). \]

Finally, we get for \( d = \tilde{m} \)

\[ P(D = \tilde{m}) = F_{\tilde{m}:m:n}(T). \]

Proof. For \( d < m \), we find that

\[ P(D < m) = P(X_{m:m:n} > T) = 1 - F_{m:m:n}(T). \]

The remaining two cases can be derived from the results established in the context of Type-I sequential hybrid censoring (see Lemma 4.1.1). \( \square \)
We recall that the Type-II progressive hybrid censored order statistics are defined by (see Example 1.4.2, (ii))

\[ X_j^{\Pi} := X_{j;\tilde{m};n}, \quad 1 \leq j \leq \tilde{m}. \]

Let \( x_m \in \mathbb{R}^m \) and \( d < m \). Then,

\[
P(X_j^{\Pi} \leq x_j, 1 \leq j \leq m, D < m) = P(X_{j;\tilde{m};n} \leq x_j, 1 \leq j \leq m, X_{\tilde{m};m;n} > T) = P(X_{j;\tilde{m};n} \leq x_j, 1 \leq j \leq m, X_{\tilde{m};m;n} \leq T) = F_{1;\tilde{m};m;n}(x_m) - F_{1;\tilde{m};m;n}(x_{m-1}, \min\{x_m, T\}). \tag{6.1}
\]

Notice, that \( F_{1;\tilde{m};m;n} \) denotes the joint cumulative distribution function of the progressively Type-II censored order statistics \( X_{1;\tilde{m};n}, \ldots, X_{\tilde{m};m;n} \). Further, we find for \( d \in \{m, \ldots, \tilde{m} - 1\} \),

\[
P(X_j^{\Pi} \leq x_j, 1 \leq j \leq d, D = d) = P(X_{j;\tilde{m};n} \leq x_j, 1 \leq j \leq d, X_{\tilde{m};m;n} \leq T) = P(X_{j;\tilde{m};n} \leq x_j, 1 \leq j \leq d - 1, X_{\tilde{m};m;n} \leq \min\{x_d, T\}, X_{\tilde{m};m;n} > T) = F_{1;\tilde{m};m;n}(x_{d-1}, \min\{x_d, T\}) - F_{1;\tilde{m};m;n}(x_{d-1}, \min\{x_d, T\}, T).
\]

For \( D = \tilde{m} \), we finally obtain

\[
P(X_j^{\Pi} \leq x_j, 1 \leq \tilde{m}, D = \tilde{m}) = P(X_{j;\tilde{m};n} \leq x_j, 1 \leq \tilde{m}, X_{\tilde{m};m;n} \leq T) = F_{1;\tilde{m};m;n}(x_{\tilde{m}-1}, \min\{x_{\tilde{m}}, T\}).
\]

The above results lead us to the conditional joint distribution of the Type-II progressively hybrid censored order statistics \( X_1^{\Pi}, \ldots, X_d^{\Pi}, d \in \{1, \ldots, \tilde{m}\} \).

**Lemma 6.1.2** Let \( f(T) > 0 \). Then, the joint cumulative distribution function of \( X_1^{\Pi}, \ldots, X_d^{\Pi}, d \in \{m, \ldots, \tilde{m} - 1\} \), conditional on \( D = d \), is given by

\[
F_{X_1^{\Pi}, \ldots, X_d^{\Pi}|D=d}(x_d) = F_{1;\tilde{m};m;n}(x_d|X_{d+1;\tilde{m};n} = T), \quad x_d \in \mathbb{R}^d,
\]

where (cf. (4.3))

\[
F_{1;\tilde{m};m;n}(x_d|X_{d+1;\tilde{m};n} = T) = \int_{\mathbb{R}^{d-1}} \cdots \int_{\mathbb{R}^{d-1}} f_{1;\tilde{m};m;n}(y_d, T) \frac{1}{f_{d+1;\tilde{m};n}(T)} dy_d \cdots dy_1.
\]

The respective conditional joint density function is given by

\[
f_{X_1^{\Pi}, \ldots, X_d^{\Pi}|D=d}(x_d) = \mathbb{1}_{\mathcal{Y}_d,T}(x_d) \frac{f_{1;\tilde{m};m;n}(x_d, T)}{f_{d+1;\tilde{m};n}(T)}.
\]

For \( d = \tilde{m} \), we have

\[
F_{X_1^{\Pi}, \ldots, X_{\tilde{m}}^{\Pi}|D=\tilde{m}}(x_{\tilde{m}}) = \frac{F_{1;\tilde{m};m;n}(x_{\tilde{m}-1}, \min\{x_{\tilde{m}}, T\})}{F_{\tilde{m};m;n}(T)}, \quad x_{\tilde{m}} \in \mathbb{R}^{\tilde{m}},
\]
and
\[ f_{X_j,1 \leq j \leq m \mid D=m}(x_m) = 1_{\Sigma_{F,T}^m}(x_m) \frac{f_{X_1 \ldots m \mid m}(x_m)}{F_{m:m:n}(T)}. \]

Further, the conditional joint cumulative distribution function for \( d < m \) is given by
\[ F_{X_j,1 \leq j \leq m \mid D=m}(x_m) = \frac{F_{X_1 \ldots m \mid m}(x_m) - F_{X_1 \ldots m \mid m}(x_{m-1}, \min\{x_m, T\})}{1 - F_{m:m:n}(T)}, \quad x_m \in \mathbb{R}^m. \]

The corresponding conditional joint density function is given by
\[ f_{X_j,1 \leq j \leq m \mid D=m}(x_m) = 1_{\Sigma_{F,T}^m \setminus \Sigma_{F,T}^m}(x_m) \frac{f_{X_1 \ldots m \mid m}(x_m)}{1 - F_{m:m:n}(T)}. \]

**Proof.** The assertion for \( d \in \{m, \ldots, \bar{m} - 1\} \) and for \( d = \bar{m} \) follows immediately from Lemma 4.1.2. The representation for the cumulative distribution function conditional on \( D < m \) follows with Lemma 6.1.1 and equation (6.1). Then, partial differentiation of \( F_{X_j,1 \leq j \leq m \mid D=m} \) gives
\[ \frac{\partial^n}{\partial x_1 \ldots \partial x_m} \frac{F_{X_1 \ldots m \mid m}(x_m) - F_{X_1 \ldots m \mid m}(x_{m-1}, \min\{x_m, T\})}{1 - F_{m:m:n}(T)} = 1_{\Sigma_{F,T}^m}(x_m) f_{X_1 \ldots m \mid m}(x_m) - 1_{\Sigma_{F,T}^m}(x_m) f_{X_1 \ldots m \mid m}(x_m). \]

Since \( \Sigma_{F,T}^m \subset \Sigma_{F,T}^m \), for any \( T > 0 \), we have for the support of \( f_{X_j,1 \leq j \leq m \mid D=m} \),
\[ \text{supp}(f_{X_j,1 \leq j \leq m \mid D=m}) = \Sigma_{F,T}^m \setminus \Sigma_{F,T}^m. \]

This concludes the proof. \( \square \)

### 6.2 Type-II Progressive Hybrid Censoring from Exponential Distributions

We consider now the IID progressive exponential censoring model (see Model 2.2.6). In this setup, the respective Type-II progressive hybrid censored order statistics are denoted by \( Z_{d1}^\mu, \ldots, Z_{d\bar{m}}^\mu \), \( d \in \{1, \ldots, \bar{m}\} \). Lemma 6.1.2 gives with \( P_F = \text{Exp}(\mu, \vartheta) \) the following result.

**Lemma 6.2.1** Let \( z_0 := \mu \). For \( d \in \{m, \ldots, \bar{m} - 1\} \), the conditional joint density function \( f_{Z_j,1 \leq j \leq d \mid D=d} \) is given by
\[ f_{Z_j,1 \leq j \leq d \mid D=d}(z_d) = 1_{\Sigma_{F,T}^d}(z_d) \frac{\prod_{j=1}^{d+1} \gamma_j^{\bar{m}}}{\partial^{d+1} f_{X_1 \ldots d+1 \mid m}(T)} \times \exp \left\{ -\frac{1}{\vartheta} \sum_{j=1}^{d} \left( 1 - \frac{\bar{m}}{\gamma_j^{\bar{m}}} \right) \gamma_j^{\bar{m}} (z_j - z_{j-1}) + \gamma_j^{\bar{m}} (T - \mu) \right\}. \]
For $d = \bar{m}$, we have

$$f_{j \leq j \leq \bar{m} | D = \bar{m}}(z_{\bar{m}}) = \mathbf{1}_{\Sigma_{j}}(z_{\bar{m}}) \prod_{j=1}^{\bar{m}} \gamma_{j}(z_{j} - z_{j-1}) \exp\left\{ -1 \frac{1}{\vartheta} \sum_{j=1}^{\bar{m}} \gamma_{j}(z_{j} - z_{j-1}) \right\}.$$ 

Finally, we obtain for $d < \bar{m}$,

$$f_{j \leq j \leq \bar{m} | D < \bar{m}}(z_{m}) = \mathbf{1}_{\Sigma_{j} \setminus \Sigma_{\bar{m}}}(z_{m}) \prod_{j=1}^{\bar{m}} \gamma_{j}(z_{j} - z_{j-1}) \exp\left\{ -1 \frac{1}{\vartheta} \sum_{j=1}^{\bar{m}} \gamma_{j}(z_{j} - z_{j-1}) \right\}.$$ 

As for Type-I sequential hybrid censoring, we obtain with Lemmas 6.2.1 and 6.1.1, the following expression of the likelihood function as well as of the respective MLEs.

**Lemma 6.2.2** The likelihood function $L(\cdot | z_{d})$ for $\mu$ and $\vartheta$ is given by

$$L(\mu, \vartheta | z_{d}) = \begin{cases} 
\mathbf{1}_{\Sigma_{j}}(z_{d}) \frac{1 - F(T)}{\gamma_{d+1}} f_{1, \ldots, d+1 : \bar{m}}(z_{d}, T), & d \in \{m, \ldots, \bar{m} - 1\}, \\
\mathbf{1}_{\Sigma_{j}}(z_{m}) f_{1, \ldots, \bar{m}}(z_{m}), & d = \bar{m}, \\
\mathbf{1}_{\Sigma_{j} \setminus \Sigma_{\bar{m}}}(z_{m}) f_{1, \ldots, \bar{m}}(z_{m}), & d < \bar{m}.
\end{cases}$$

Let $\mu$ be known. Then, the MLE of $\vartheta$ is given by

$$\hat{\vartheta} = \begin{cases} 
\frac{1}{D} \sum_{j=1}^{D} \left( 1 - \frac{\gamma_{D+1}}{\gamma_{j}} \right) \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}) + \gamma_{D+1}^{\bar{m}}(T - \mu), & D \in \{m, \ldots, \bar{m} - 1\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}), & D = \bar{m}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}), & D < \bar{m}.
\end{cases}$$

Let $\mu$ be unknown. Then, the MLEs $\hat{\mu}$ and $\hat{\vartheta}$ are given by

$$\hat{\mu} = Z_{1, \bar{m}}$$

and

$$\hat{\vartheta} = \begin{cases} 
\frac{1}{D} \sum_{j=2}^{D} \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}) + \gamma_{D+1}^{\bar{m}}(T - Z_{D}^{\bar{m}}), & D \in \{m, \ldots, \bar{m} - 1\}, \\
\frac{1}{m} \sum_{j=2}^{m} \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}), & D = \bar{m}, \\
\frac{1}{m} \sum_{j=2}^{m} \gamma_{j}^{\bar{m}} (Z_{j}^{\bar{m}} - Z_{j-1}^{\bar{m}}), & D < \bar{m}.
\end{cases}$$

Note that the MLE $(\hat{\mu}, \hat{\vartheta})$ does not exist for an unknown parameter $\mu$ and $m = 1$ (cf. Remark 4.2.4, (iv)).
We proceed by defining the normalized spacings based on Type-II progressive hybrid censored variables as

\[
W_j^{\|} = \begin{cases} 
\gamma_j(Z_j^{\|} - Z_{j-1}^{\|}), & j \in \{1, \ldots, m\}, \\
\gamma_j^{\|}(Z_j^{\|} - Z_{j-1}^{\|}), & j \in \{m + 1, \ldots, \tilde{m}\}, 
\end{cases}
\]

while \(Z_0^{\|} := \mu\), so that

\[
Z_j^{\|} = \begin{cases} 
\mu + \sum_{i=1}^{j} \frac{W_i^{\|}}{\gamma_i}, & j \in \{1, \ldots, m\}, \\\n\mu + \sum_{i=1}^{j} \frac{W_i^{\|}}{\gamma_i^{\|}}, & j \in \{m + 1, \ldots, \tilde{m}\}. 
\end{cases}
\]

The conditional joint density functions of the spacings \(W_1^{\|}, \ldots, W_m^{\|}\), with \(d \in \{1, \ldots, \tilde{m}\}\), can be directly derived from the results presented for Type-I sequential hybrid censoring (cf. Section 4.2). Hence we consider the case \(d < m\) only.

**Lemma 6.2.3** The joint density function of the spacings \(W_1^{\|}, \ldots, W_m^{\|}\), conditionally on \(D < m\), is given by

\[
f_{W_1^{\|} \leq \ldots \leq W_m^{\|}, D < m}(w_m) = 1_{[0, \infty)^m \setminus W_m(T - \mu|\gamma_m)}(w_m) \frac{1}{1 - F_{m,m,m}(T)} \prod_{j=1}^{m} \frac{1}{\vartheta} \exp \left\{ - \frac{w_j}{\vartheta} \right\}.
\]

**Proof.** The assertion follows due to the application of the linear transformation \(\Phi_W\) (cf. (4.15)). The corresponding support results by imposing \(\Phi_W\) on the cones \(\Sigma_F^m\) and \(\Sigma_{F,T}^m\), respectively. This is legitimate due to the following calculation. Let \(A, B, C \subset \mathbb{R}^m\) be three measurable sets. Then

\[
(A \setminus B^c) \cap C = A \cap B \cap C = (A \cap C) \setminus B^c = (A \cap C) \setminus (B^c \cap C).
\]

This concludes the proof. \(\square\)

**6.2.1 Distribution Theory for a Known Location Parameter**

The total time on test statistic \(S_D\) is given by

\[
S_D = \begin{cases} 
\sum_{j=1}^{D} \left( 1 - \frac{\gamma_j^{\|}}{\gamma_j^{\|}} \right) W_j^{\|} + \gamma_D^{\|} (T - \mu), & D \in \{m, \ldots, \tilde{m}\}, \\
\sum_{j=1}^{D} W_j^{\|} & D < m.
\end{cases}
\]

(6.2)

Then, \(S_D = \max \{D, m\} \cdot \tilde{\vartheta}\), \(D \in \{0, \ldots, \tilde{m}\}\). In the following we assume \(\mu\) to be known.

Applying the linear transformation \(\Phi_{S,m}\) (cf. (4.19)) on \(f_{W_1^{\|} \leq \ldots \leq W_m^{\|}, D < m}\) (given in Lemma 6.2.3), we find

\[
f_{W_1^{\|} \leq \ldots \leq W_m^{\|}, S_m | D < m}(w_{m-1}, s) = 1_{A_{m-1}(s) \times \mathbb{R}^m}(w_{m-1}, s) \frac{1}{\vartheta^m(1 - F_{m,m,m}(T))} \exp \left\{ - \frac{s}{\vartheta} \right\}.
\]

(6.3)
The support $A_{m-1}^{[m]}(s) \times \mathbb{R}$ can be determined by applying (as in the proof to Lemma 6.2.3) $\Phi_{S,m}$ on the sets $[0, \infty)^m$ and $\mathcal{W}_m(T - \mu|\gamma_m)$, respectively. By integrating w.r.t. the set $A_{m-1}^{[m]}(s)$ in (6.3), we obtain

$$f^{S_m|D<m}(s) = \frac{e^{-s/\theta}}{\theta^m(1 - F_{m;m,n}(T))} \text{vol}_{m-1}(A_{m-1}^{[m]}(s)), \ s \in \mathbb{R}. \quad (6.4)$$

**Lemma 6.2.4** Let $\gamma_1, \ldots, \gamma_m > 0$ and $\gamma_{m+1} = 0$. Then, the set $A_{m-1}^{[m]}(s)$ is given by

$$A_{m-1}^{[m]}(s) = S_{m-1}^{(s)} \setminus M_{m-1}^{[m]}(s|\beta, t),$$

where

$$S_{m-1}^{(s)} = \left\{ w_{m-1} \in \mathbb{R}^{m-1} \mid w_j \geq 0, 1 \leq j \leq m-1, \sum_{j=1}^{m-1} w_j \leq s \right\}, \ s \geq 0, \quad (6.5)$$

and $M_{m-1}^{[m]}(s|\beta, t)$ as in Theorem 3.2.1 with parameters $\beta = (\beta_1, \ldots, \beta_m)$ and $t = (t_0, \ldots, t_m)$, such that

$$\beta_j = \gamma_j(T - \mu), \ 1 \leq j \leq m-1, \ \beta_m > 0 \text{ and } t_j = \gamma_j(T - \mu), \ 1 \leq j \leq m, \ t_0 = 0. \quad (6.6)$$

For $s \geq 0$, the corresponding volume can be calculated by

$$\text{vol}_{m-1}(A_{m-1}^{[m]}(s)) = \frac{s^{m-1}}{(m-1)!} - \frac{(T - \mu)^{m} \prod_{j=1}^{m} \gamma_j}{m!} B_{m-1}(s|0, \gamma_m(T - \mu)).$$

**Proof.** Due to $w_m \geq 0$, the application of $\Phi_{S,m}$ yields

$$s = \sum_{j=1}^{m} w_j \text{ and } s \geq \sum_{j=1}^{m-1} w_j.$$ 

By taking into account the above inequality, the set $[0, \infty)^m$ changes to $S_{m-1}^{(s)} \times \mathbb{R}$, $s \geq 0$, with $S_{m-1}^{(s)}$ as in (6.5). The corresponding volume can be calculated from equation (3.23).

However, by taking into account the inequality $s \geq \sum_{j=1}^{m-1} w_j$, the set $\mathcal{W}_m(T - \mu|\gamma_m)$ changes to $M_{m-1}^{[m]}(s|\beta, t) \times \mathbb{R}$. This can be verified by conducting the same calculations as in the proof to Lemma 4.2.8. The volume of $M_{m-1}^{[m]}(s|\beta, t)$ follows also by performing the same calculations as in the proof to Lemma 4.2.8. Then, the assertion follows due to

$$\text{vol}_{m-1}(A_{m-1}^{[m]}(s)) = \text{vol}_{m-1}(S_{m-1}^{(s)}) - \text{vol}_{m-1}(M_{m-1}^{[m]}(s|\beta, t)), \ s \geq 0. \quad (6.7)$$

The following result follows with Theorem 4.2.9, for $D \in \{m, \ldots, \tilde{m}\}$, and by applying Lemma 6.2.4 on (6.4), for $D < m$. 


Theorem 6.2.5 For $d \in \{m, \ldots, \tilde{m} - 1\}$, the density function $f^{S_d|D=d}$ is given by

$$f^{S_d|D=d}(s) = \frac{(T - \mu)^d \prod_{j=1}^{d+1} \gamma_j^{\beta}}{d! \gamma_d^{\beta+\tilde{m}} f^{S_{\tilde{d}+1},m,m}(T)} \cdot B_{d-1}(s|\gamma_d^{\beta+\tilde{m}}(T - \mu)) e^{-s/\theta}, \quad s \geq 0.$$  

For $d = \tilde{m}$, we have

$$f^{S_{\tilde{m}}|D=\tilde{m}}(s) = \frac{(T - \mu)\tilde{m} \prod_{j=1}^{\tilde{m}} \gamma_j^{\beta}}{m! \gamma_d^{\beta+\tilde{m}} f^{S_{\tilde{d}+1},m,m}(T)} \cdot B_{m-1}(s|0, \gamma_m(T - \mu)) e^{-s/\theta}, \quad s \geq 0.$$

Finally, for $d < m$ and $s \geq 0$, we arrive at

$$f^{S_{m}|D<m}(s) = \frac{f^{\mathcal{F}_{m,m}(s)}}{(1 - F^{\mathcal{F}_{m,m}}(T))} - \frac{(T - \mu)^m \prod_{j=1}^{m} \gamma_j}{m! \gamma_m(1 - F^{\mathcal{F}_{m,m}}(T))} \cdot B_{m-1}(s|0, \gamma_m(T - \mu)) e^{-s/\theta}. \quad \text{(6.8)}$$

Remark 6.2.6 Recalling the structure of the set $\mathcal{M}^{[s]|\beta,t}_{d-1}(s|\beta,t)$ introduced in Example 3.2.5, we find that

$$\mathcal{M}^{[m]|\beta,t}_{m-1}(s|\beta,t) = S^{(s)}_{m-1} \setminus \mathcal{M}^{[m]|\beta,t}_{m-1}(s|\beta,t), \quad s \geq 0,$$

with $S^{(s)}_{m-1}$ and $\beta, t$ as in (6.5) and (6.6), respectively. Cramer et al. (2016) obtained the above expression for $f^{S_{m}|D<m}$ by calculating the volume of $\mathcal{M}^{[m]|\beta,t}_{m-1}(s|\beta,t)$ as given in (3.22) using Gerber’s formula (cf. Gerber, 1981, p. 312). This approach requires more effort than choosing the way pursued in the proof to Lemma 6.2.4, where already established formulas for $\text{vol}_{m-1}(S^{(s)}_{m-1})$ and $\text{vol}_{m-1}(\mathcal{M}^{[m]|\beta,t}_{m-1}(s|\beta,t))$ have been used.

The following theorem provides the basis for deriving the density function of $S_m$ conditionally on $D < m$, with the expected value approach.

Theorem 6.2.7 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary continuous function. Further, let $S_m = \sum_{j=1}^{m} (R_j + 1) Z_{j:m}^\mathcal{F}$ (cf. (6.2)). Then, the expectation of $g(S_m)$ conditionally on $D < m$, is given by

$$E(g(S_m)|D < m) = \int_{\mathbb{R}} \frac{f^{\mathcal{F}_{m,m}(s)}}{1 - F^{\mathcal{F}_{m,m}}(T)} - \frac{(T - \mu)^m \prod_{j=1}^{m} \gamma_j}{m! \gamma_m(1 - F^{\mathcal{F}_{m,m}}(T))} \cdot B_{m-1}(s|0, \gamma_m(T - \mu)) e^{-s/\theta} ds.$$  

Proof. First, $\tilde{S}_m \sim \Gamma(\tilde{\theta}, m)$ (cf. Balakrishnan and Cramer, 2014, Theorem 12.1.1.) and hence $S_m \sim \Gamma(\tilde{\theta}, m)$. Let $\tilde{g}$ be defined as in the proof to Theorem 4.2.10. Then, we find together with the result for $D = m$ given in Theorem 4.2.10, that

$$E(g(S_m)|D < m) = \int_{\mathbb{R}^d} g(\sum_{j=1}^{m} (R_j + 1) z_j) f^{Z^\mathcal{F},1\leq j\leq m|D<m}(z_m) d z_m$$

$$= \int_{\sum_{j=1}^{m} \gamma_j} \tilde{g}(\sum_{j=1}^{m} (R_j + 1) z_j) d z_m.$$
6.2.2

Now, we present two different representations for the density function of \( \hat{\vartheta} \).

**Theorem 6.2.8** Let \( \hat{\vartheta} \) be the MLE for \( \vartheta \) under the Type-II progressive hybrid censoring scheme (see Lemma 6.2.2).

(i) The density function of \( \hat{\vartheta} \), is given by

\[
\begin{align*}
  f_{\hat{\vartheta}}(s) &= f_{\hat{\vartheta} m}^\vartheta(s) - \frac{(T - \mu)^m}{(m - 1)! \hat{\vartheta} m} B_{m-1}(ms|0, \vartheta(T - \mu)) e^{-ms/\vartheta} \\
  &\quad + \sum_{d=m}^{d=m} \frac{T^d \prod_{j=1}^{d} \gamma_j^{\hat{\vartheta} m}}{(d-1)! \hat{\vartheta} d} B_{d-1}(ds|\gamma_{d-1}^{\hat{\vartheta} m}(T - \mu)) e^{-ds/\vartheta}, \quad s \geq 0.
\end{align*}
\]

(ii) An expression for \( f_{\hat{\vartheta}}(s) \), which consists solely of gamma density functions is given by, for \( s \geq 0 \),

\[
\begin{align*}
  f_{\hat{\vartheta}}(s) &= \sum_{d=m}^{d=m} \left( \prod_{j=1}^{d} \gamma_j^{\hat{\vartheta} m} \right) \sum_{j=0}^{d} \frac{\epsilon_j^{\vartheta d-\vartheta+1}(T-\mu)/\vartheta}{\prod_{i=0, i \neq j}^{d} (\gamma_i^{\hat{\vartheta} m} - \gamma_j^{\hat{\vartheta} m})} f_{\vartheta m}^\vartheta \left( s - \frac{\gamma_j^{\hat{\vartheta} m} (T - \mu)}{m} \right) \\
  &\quad - \left( \prod_{j=1}^{m} \gamma_j \right) \sum_{j=1}^{m} \frac{\epsilon_j^{\vartheta m-\vartheta+1}(T-\mu)/\vartheta}{\prod_{i=0, i \neq j}^{m} (\vartheta m - \vartheta i - \vartheta m - j + 1)} f_{\vartheta m}^\vartheta \left( s - \frac{\gamma_j^{\hat{\vartheta} m} (T - \mu)}{m} \right).
\end{align*}
\]

**Proof.**

(i) The assumption follows by applying Lemma 6.1.1 and Theorem 6.2.5 to the expression given in equation (1.19).

(ii) A representation in terms of gamma density functions can be obtained, by applying Theorem 4.3.1 to the expression of \( f_{\hat{\vartheta}}(s) \) in terms of B-splines, given in (i). By using the divided differences representation for pairwise distinct knots (see Lemma 3.1.3, (i)), we obtain

\[
\begin{align*}
  f_{\hat{\vartheta}}(s) &= f_{\vartheta m}^\vartheta(s) \\
  &\quad - \left( \prod_{j=1}^{m} \gamma_j \right) \sum_{j=0}^{m} \frac{\epsilon_j^{\vartheta m-\vartheta+1}(T-\mu)/\vartheta}{\prod_{i=0, i \neq j}^{m} (\vartheta m - \vartheta i - \vartheta m + 1)} f_{\vartheta m}^\vartheta \left( s - \frac{\vartheta m - \vartheta j + 1 (T - \mu)}{m} \right) \\
  &\quad + \sum_{d=m}^{d=m} \left( \prod_{j=1}^{d} \gamma_j^{\hat{\vartheta} m} \right) \sum_{j=0}^{d} \frac{\epsilon_j^{\vartheta d-\vartheta+1}(T-\mu)/\vartheta}{\prod_{i=0, i \neq j}^{d} (\vartheta d - \vartheta i - \vartheta d + 1)} f_{\vartheta d}^\vartheta \left( s - \frac{\vartheta d - \vartheta j + 1 (T - \mu)}{d} \right).
\end{align*}
\]
Then, the assertion follows with the identity, for $j = 0$,
\[
\left(\prod_{q=1}^{m} \gamma_j \right) \prod_{i=0, i\neq j}^{m} \left( \gamma_{m-i+1} - \gamma_{m-j+1} \right) f_{m,m}^\theta \left( s - \frac{\gamma_{m-j+1}(T-\mu)}{m} \right) = f_{m,m}^\theta (s). \]

Figure 6.1 depicts the plots of $f^\theta$ for two different censoring plans $\mathcal{R}$.

**Remark 6.2.9**

(i) It is obvious that the representation of $f^\theta$ in terms of B-splines (see Theorem 6.2.8, (i)) is more compact than the expression provided in terms of gamma density functions, provided in Theorem 6.2.8, (ii). This enables a more convenient implementation of $f^\theta$. By recalling that the support of the B-spline is determined by its knots we realize that the deduction of support related properties for the density function $f^\theta$ is much more convenient when dealing with the B-spline representation.

(ii) Childs et al. (2008) were the first to derive an expression for $f^\theta$ (with $\mu = 0$). They used the method of the conditional moment generating function. Notice that the notation used therein, is significantly different from the notation used in this work. Adapting the notations, the expression of $f^\theta$ given in Childs et al. (2008, Theorem 23.3.2) reads
\[
f^\theta (s) = \sum_{d=0}^{m-1} \left( \prod_{j=1}^{d} \gamma_j \right) \sum_{i=0}^{d} \prod_{i=0, i\neq j}^{m} \left( \gamma_{m-i+1} - \gamma_{m-j+1} \right) f_{m,m}^\theta \left( s - \frac{\gamma_{d-i+1}(T)}{m} \right) \\
+ \sum_{d=m}^{\bar{m}} \left( \prod_{j=1}^{d} \gamma_{\bar{m}}^j \right) \sum_{i=0}^{d} \prod_{i=0, i\neq j}^{d} \left( \gamma_{d-i+1} - \gamma_{d-j+1} \right) f_{d,d}^\theta \left( s - \frac{\gamma_{d-i+1}(T)}{d} \right),
\]
with $s \geq 0$. By comparing the above formula with the representation given in Theorem 6.2.8, (ii), we find that the number of relevant terms has been reduced by $\frac{m(m+1)}{2}$ in our expression.
(iii) Consider the particular case of Type-II hybrid censoring (i.e., \( R_1 = \ldots = R_{m-1} = 0 \) and \( R_m = n - m \)). Then, a similar observation as in (ii) is present. According to Childs et al. (2003, Theorem 3.2), the density function of \( \hat{\vartheta} \) under Type-II hybrid censoring is given by

\[
\hat{f}(s) = e^{-\eta T/\vartheta} f_{\frac{m}{m},m}(s - \frac{nT}{m}) + \sum_{d=1}^{m-1} \sum_{j=0}^{d} (-1)^j \binom{n}{d} \binom{d}{j} e^{-(n-d+j)T/\vartheta} f_{\frac{m}{m},m}(s - \frac{(n-d+j)T}{m}) + \sum_{d=m}^{n} \sum_{j=0}^{d} (-1)^j \binom{n}{d} \binom{d}{j} e^{-(n-d+j)T/\vartheta} f_{\frac{m}{m},m}(s - \frac{(n-d+j)T}{m}), \quad s \geq 0.
\]

By employing the initially planned censoring plan \( \mathcal{R} = (0^{(m-1)}, n - m) \) for the representation established in Theorem 6.2.8, (ii), we arrive together with performing some elementary calculations at the following simplified formula for the density function of \( \hat{\vartheta} \) under Type-II hybrid censoring,

\[
\hat{f}(s) = \sum_{d=m}^{n} \sum_{j=0}^{d} (-1)^j \binom{n}{d} \binom{d}{j} e^{-(n-d+j)T/\vartheta} f_{\frac{m}{m},m}(s - \frac{(n-d+j)T}{m}),
\]

with \( s \geq 0 \). Again, the number of relevant terms has been reduced by \( \frac{m(m+1)}{2} \).

The next two results provide expressions for particular limits of \( \hat{f}(s) \) and \( \hat{F}(s) \).

**Lemma 6.2.10** Let \( \hat{f}(s) \) denote the density function as given in Theorem 6.2.8, (i). Then,

\[
\lim_{T \to \infty} \hat{f}(s) = f_{\frac{m}{m},m}(s) \quad \text{and} \quad \lim_{T \to \mu+} \hat{f}(s) = f_{\frac{m}{m},m}(s), \quad s \geq 0.
\]

**Proof.** For \( T \to \mu+ \), the assumption follows from

\[
\lim_{T \to \mu+} B_{d-1}(ds|\gamma_{d+1}^\vartheta m(T - \mu)) = B_{d-1}(ds|0^{d+1}) = \lim_{T \to \mu+} B_{m-1}(ds|0, \gamma_m(T - \mu)) = B_{m-1}(ds|0^{m+1}) = 0.
\]

The result for \( T \to \infty \) follows directly from Lemma 4.3.6. \( \square \)

It should be mentioned that the result for \( T \to 0^+ \) (i.e., \( \mu = 0 \)), has been provided for Type-II hybrid censoring in Childs et al. (2003, Remark 3.1).

**Theorem 6.2.11** Let \( \hat{F}(s) \) denote the cumulative distribution function of the MLE \( \hat{\vartheta} \) for \( \vartheta > 0 \). Then, the following limits hold

\[
\lim_{\vartheta \to 0} \hat{F}(t) = 1 \quad \text{and} \quad \lim_{\vartheta \to \infty} \hat{F}(t) = 0, \quad t \geq 0.
\]
6.2 Type-II Progressive Hybrid Censoring from Exponential Distributions

**Proof.** We consider the following implicit expression of $F^\vartheta\varphi$:

$$F^\vartheta\varphi(t) = \int_0^{mt/\vartheta} f_{1,m}^{\varphi}(x) dx - \frac{(T - \mu)^m}{m! \vartheta^m} \int_0^{mt} B_{m-1}(x|0, \gamma_m(T - \mu)) e^{-x/\vartheta} dx \quad (6.9)$$

Then, the first integral converges to zero for $\vartheta \to \infty$. For the remaining terms we find that

$$0 \leq \frac{1}{d!} \int_0^{dt} B_{d-1}(x|t_0, \ldots, t_d) e^{-x/\vartheta} dx \leq \frac{1}{d!} \int_0^{dt} B_{d-1}(x|t_0, \ldots, t_d) dx \to 0 \quad \text{for} \quad \vartheta \to \infty.$$ 

with accordingly chosen knots $t_0, \ldots, t_d$, $d \in \{m, \tilde{m}\}$. This proves the assertion for $\vartheta \to \infty$.

For $\vartheta \to 0$, we first note that $\int_0^{mt/\vartheta} f_{1,m}^{\varphi}(x) dx \to 1$. Further, we obtain by arguing as in the proof to Theorem 4.2.15, that

$$\lim_{\vartheta \to 0} \frac{(T - \mu)^m}{m! \vartheta^m} \int_0^{mt} B_{m-1}(x|0, \gamma_m(T - \mu)) e^{-x/\vartheta} dx = \lim_{\vartheta \to 0} \sum_{d=m}^{\tilde{m}} \frac{(T - \mu)^d}{d! \vartheta^d} \int_0^{dt} B_{d-1}(x|\gamma_{d+1}^m(T - \mu)) e^{-x/\vartheta} dx = 1.$$ 

This concludes the proof. \hfill \Box

Notice that the way of proving the above theorem has been also used in the context of generalized progressive hybrid censoring (cf. Górný and Cramer, 2016).

### 6.2.2 Distribution Theory for an Unknown Location Parameter

Assume the parameter $\mu$ is unknown. Hence, we consider the modified total time on test statistic $V_D$ with

$$V_D = \begin{cases} 
\sum_{j=2}^{m} W_j^D, & D < m, \\
\sum_{j=2}^{D} \left(1 - \frac{\gamma_{D+1}^m}{\vartheta_j}\right) W_j^D + \gamma_{D+1}^m \left(T - \mu - \frac{W^D_D}{m}\right), & D \in \{m, \tilde{m}\},
\end{cases}$$

so that $\tilde{\vartheta} = \frac{V_D}{D}$, $D \in \{2, \ldots, \tilde{m}\}$ (cf. Lemma 6.2.2). Applying the transformation $\Phi_{V,m}$ (cf. (4.32)) to the conditional joint density function $f_{W_j^D, D < m, D < m, V_j|D < m}$, gives

$$f_{W_j^D, D < m, V_j|D < m}(w_{m-1}, v) = \frac{1}{(1 - F_{m,m,m}(T)) \vartheta^m} e^{-(w_1 + v)/\vartheta}. \quad (6.10)$$

Let $w = w_1$, and let $B_{m-2}(v)$ denote the support of $f_{W_j^D, D < m, V_j|D < m}$ w.r.t. the variables $w_2, \ldots, w_{m-1}, w$ and $v$, so that

$$f_{W_j^D, V_j|D < m}(w, v) = \frac{1}{(1 - F_{m,m,m}(T)) \vartheta^m} e^{-(w_1 + v)/\vartheta} \text{vol}_{m-2}(B_{m-2}(v)). \quad (6.11)$$
Lemma 6.2.12 For $\gamma_2, \ldots, \gamma_m > 0$ and $\gamma_{m+1} = 0$, the set $B_{m-2}^{[m]}(v)$ is given by
\[
B_{m-2}^{[m]}(v) = S_{m-2}^{(v)} \setminus M_{m-2}^{[m]}(s|\beta, t),
\]
where
\[
S_{m-2}^{(v)} = \left\{(w_2, \ldots, w_{m-1}) \in \mathbb{R}^{m-2} \mid w_j \geq 0, \ 2 \leq j \leq m - 1, \sum_{j=2}^{m-1} w_j \leq v\right\}, \quad v \geq 0, \quad (6.12)
\]
and $M_{m-2}^{[m]}(v|\beta, t)$ is as in Theorem 3.2.1 with parameters $\beta = (\beta_1, \ldots, \beta_{m-1})$ and $t = (t_0, \ldots, t_{m-1})$, with
\[
\beta_j = \gamma_{j+1} \left( T - \mu - \frac{w}{n} \right), \quad 1 \leq j \leq m - 2, \quad \beta_{m-1} > 0
\]
and $t_j = \gamma_{j+1} \left( T - \mu - \frac{w}{n} \right), \quad 1 \leq j \leq m - 1, \quad t_0 = 0. \quad (6.13)
Then, for $v \geq 0$, the volume $\text{vol}_{m-2}(B_{m-2}^{[m]}(v))$ is given by
\[
\text{vol}_{m-2}(B_{m-2}^{[m]}(v)) = \frac{v^{m-2}}{(m-2)!} - \frac{(T - \mu - \frac{w}{n})^{m-1} \prod_{j=2}^{m} \gamma_j}{(m-1)!} B_{m-2}(v|0, \gamma_{2,m}(T - \mu - \frac{w}{n})).
\]
\[\textbf{Proof.}\] We proceed as in the previous subsection. Due to $\Phi_{V,m}$ we have
\[
v = \sum_{j=2}^{m} w_j \implies v \geq \sum_{j=2}^{m-1} w_j. \quad (6.14)
\]
The above inequality yields together with $[0, \infty)^{m-1}$ the set $\mathcal{S}_{m-2}^{(v)} \times \mathbb{R}$, with $\mathcal{S}_{m-2}^{(v)}$ as in (6.12), and further $\text{vol}_{m-2}(\mathcal{S}_{m-2}^{(v)}) = \frac{v^{m-2}}{(m-2)!}$, for $v \geq 0$ (cf. (3.23)). Then, the assumption follows readily with the calculations performed in the proofs to Lemma 6.2.4 and Lemma 4.2.18.

This yields the following theorem.

Theorem 6.2.13 For $d \in \{m, \ldots, \tilde{m} - 1\}$ and $0 \leq w < n(T - \mu)$, $v \geq 0$, the density function $f_{W_{m}, W_{d}}|D=d$ is given by
\[
f_{W_{m}, W_{d}}|D=d(w, v) = \frac{(T - \mu - \frac{w}{n})^{d-1} \prod_{j=2}^{d+1} \gamma_j}{(d-1)!} \prod_{j=2}^{d+1} \gamma_j B_{d-2}(v|\gamma_{2,d+1}(T - \mu - \frac{w}{n})) e^{-(w+v)/\vartheta}.
\]

For $d = \tilde{m}$, we obtain with $0 \leq w < n(T - \mu)$ and $v \geq 0$,
\[
f_{W_{m}, W_{\tilde{m}}}|D=\tilde{m}(w, v) = \frac{(T - \mu - \frac{w}{n})^{\tilde{m}-1} \prod_{j=2}^{\tilde{m}} \gamma_j}{(\tilde{m} - 1)!} \prod_{j=2}^{\tilde{m}} \gamma_j B_{\tilde{m}-2}(v|0, \gamma_{2,\tilde{m}}(T - \mu - \frac{w}{n})) e^{-(w+v)/\vartheta}.
\]

Further, for $d < m$ and $v \geq 0$, we get
\[
f_{W_{m}, W_{d}}|D<m(w, v) = \begin{cases} 
1_{[0, \infty)}(w) & f_{\theta, m-1}(v) \frac{e^{-w/\vartheta}}{\vartheta(1 - F_{m,m,n}(T))} \\
-1_{[0,n(T-\mu))]}(w) & (T - \mu - \frac{w}{n})^{m-1} \prod_{j=2}^{m} \gamma_j \frac{1}{(m-1)!} \prod_{j=2}^{m} \gamma_j B_{m-2}(v|0, \gamma_{2,m}(T - \mu - \frac{w}{n})) e^{-(w+v)/\vartheta}.
\end{cases}
\]

(6.15)
6.2 Type-II Progressive Hybrid Censoring from Exponential Distributions

Figure 6.2: Plots of \( f_{\hat{\mu}, \hat{\vartheta}} \) for \( n = 10, m = 5, T = 1, \vartheta = 1 \) and \( \mu = 0.1 \). Left: \( \mathcal{R} = (0^{*4}, 5) \). Right: \( \mathcal{R} = (1^{*5}) \).

**Proof.** For \( D < m \), the result follows with (6.11) and by applying Lemma 6.2.12. The remaining two cases are derived from Theorem 4.2.19. \( \square \)

Finally, Theorem 6.2.13 yields an expression for the density function of the bivariate MLE \((\hat{\mu}, \hat{\vartheta})\).

**Theorem 6.2.14** Let \( m \geq 2 \). The density function \( f_{\hat{\mu}, \hat{\vartheta}} \) of the MLEs \( \hat{\mu} \) and \( \hat{\vartheta} \) for \( t \geq 0 \) is given by

\[
f_{\hat{\mu}, \hat{\vartheta}}(s, t) = \mathbb{1}_{[\mu, \infty)}(s) \frac{mn}{\vartheta} f_{\vartheta, m-1}(mt) e^{-n(s-\mu)/\vartheta} \\
- \mathbb{1}_{[\mu, T]}(s) \frac{m(T-s)^{m-1}}{(m-1)!} \prod_{j=1}^{m} \gamma_j B_{m-2}(mt|0, \gamma_2, m(T-s)) e^{-[n(s-\mu)+mt]/\vartheta} \\
+ \mathbb{1}_{[\mu, T]}(s) \sum_{d=m}^{\tilde{m}} \frac{d(T-s)^{d-1}}{(d-1)!} \prod_{j=1}^{d} \gamma_j^{\tilde{m}} B_{d-2}(dt|\gamma_2, d+1(T-s)) e^{-[n(s-\mu)+dt]/\vartheta}.
\]

**Proof.** The result follows with Theorem 6.2.13, the substitutions \( w = n(s-\mu), v = dt \), and by taking into account that \( \gamma_1^{\tilde{m}} = \gamma_1 = n \). \( \square \)

Figure 6.2 depicts the plots of \( f_{\hat{\mu}, \hat{\vartheta}} \) for two initially planned censoring plans \( \mathcal{R} \). The results established in the context of Type-I sequential hybrid censoring as well as the notations used in this chapter allow us to present expressions for the MLE for \( \vartheta \) under adaptive Type-I progressive hybrid censoring.

**Remark 6.2.15 (Adaptive Type-I progressive hybrid censoring)** In order to overcome the drawback of long experiment duration of the Type-II progressive hybrid censoring scheme, Lin and Huang (2012) introduced the adaptive Type-I progressive hybrid censoring scheme. It proceeds as follows: if the \( m \)th failure occurs after a prefixed threshold time \( T \), the
experiment stops at $T$. Otherwise, i.e., the $m$th failure occurs before $T$, the experiment proceeds as in the model of Type-II progressive hybrid censoring. Notice that it does not matter, whether for the $(m-1)$ first failures the initially planned $R$ or the extended censoring plan $R^{\hat{m}}$ is applied. Then, after observing the $m$th failure the extended censoring plan $R^{\hat{m}}$ is used. It is obvious that for $R = (0^{m-1}, n-m)$ this model corresponds to the well-known model of Type-I censoring. Hence, the adaptive Type-I progressive hybrid censoring scheme can be viewed as an adaption of Type-I censoring to the progressively Type-II censored order statistics $X_{1:m:n}, \ldots, X_{\hat{m}:m:n}$.

Lin and Huang (2012) derived, among others, distributional results for the MLE for the one-parameter exponential distribution by using the conditional moment generating function approach. In the following we provide results for the MLEs as well as for the respective conditional density functions with $\mu$ known and $\mu$ unknown. The results can be directly obtained by using the expressions established in the context of Type-I sequential hybrid censoring w.r.t. setting $\gamma_j = \gamma^0 \hat{m}$, $1 \leq j \leq \hat{m} + 1$.

The likelihood function $L(\cdot | z_d)$ for $\mu$ and $\vartheta$ in the model of adaptive Type-I progressive hybrid censoring, is given by

$$L(\mu, \vartheta | z_d) = \begin{cases} 
1_{\Sigma_{F,T}}(z_d) \frac{1 - F(T)}{\gamma^0 \hat{m}} \prod_{j=1}^{\hat{m}} f_{1: \cdots: d+1:m:n}(z_d, T), & d \in \{1, \ldots, \hat{m} - 1\}, \\
1_{\Sigma_{F,T}}(z_d) f_{1: \cdots: d+1:m:n}(z_d), & d = \hat{m}.
\end{cases}$$

Let $\mu$ be known. Then, the MLE $\hat{\vartheta}$ is given by (cf. Lin and Huang, 2012)

$$\hat{\vartheta} = \frac{1}{D} \sum_{j=1}^{D} \left( 1 - \frac{\gamma^0 \hat{m}}{\gamma_j^p} \right) \gamma_j^p (Z_{1:m:n} - Z_{j-1:m:n}) + \gamma^0 \hat{m} (T - \mu), \quad D \in \{1, \ldots, \hat{m}\}.$$ 

Note that $\hat{\vartheta}$ does not exist for $d = 0$. For an unknown parameter $\mu$, the MLEs for $\mu$ and $\vartheta$ are given by

$$\hat{\mu} = Z_{1:m:n}^\hat{m},$$

and

$$\hat{\vartheta} = \frac{1}{D} \sum_{j=2}^{D} \gamma_j^p (Z_{j:m:n} - Z_{j-1:m:n}) + \gamma^0 \hat{m} (T - Z_{d:m:n}), \quad D \in \{1, \ldots, \hat{m}\}.$$ 

As for the Type-I sequential hybrid censoring scheme, we have to condition on $D \geq 2$ in order to obtain the joint density function of $\hat{\mu}$ and $\hat{\vartheta}$.

According to Theorem 4.2.12, the density function of $\hat{\vartheta}$ under adaptive Type-I progressive hybrid censoring can be expressed as

$$f_{\hat{\vartheta}|D \geq 1}(s) = \frac{1}{1 - e^{-n(T-\mu)/\vartheta}} \sum_{d=1}^{\hat{m}} \frac{(T - \mu)^d \prod_{j=1}^{d} \gamma_j^p}{(d-1)!\vartheta^d} B_{d-1}(ds|\gamma^0 \hat{m} (T - \mu)) e^{-ds/\vartheta}, \quad s \geq 0.$$ 

The density function $f_{\hat{\mu}, \hat{\vartheta}|D \geq 2}$ follows directly from Theorem 4.2.20, so that

$$f_{\hat{\mu}, \hat{\vartheta}|D \geq 2}(s, t) = \frac{1}{P(D \geq 2)} \sum_{d=2}^{m} \frac{d(T - s)^d \prod_{j=1}^{d} \gamma_j^*}{(d-1)!\vartheta^d} B_{d-2}(dt|\gamma^*\hat{m}^* (T - s)) e^{-(\gamma^* (s-\mu) + dt)/\vartheta},$$
6.3 Type-II Hybrid Censoring from Uniform Distributions

Throughout this section, we assume the IID uniform model (see Model 2.1.3). Let us consider the complete set of \( n \) ordinary order statistics, i.e., \( U_{1:n}, \ldots, U_{m:n}, U_{m+1:n}, \ldots, U_{n:n} \). Then, for \( m \leq n \), and \( T \in (0, \infty) \) the corresponding stopping time is given by \( T_{II}^* = \max\{U_{m:n}, T\} \). Further, we define the uniform Type-II hybrid censored order statistics \( U_{j}^{II} \) by

\[
U_{j}^{II} := U_{j:n}, \quad 1 \leq j \leq n.
\]

The counter variable \( D \), denoting the number of failures till time \( T \), is given by \( D = \sum_{j=1}^{n} \mathbf{1}_{(-\infty, T]}(U_{j:n}) \).

6.3.1 Fundamental Results

We start with the conditional joint distribution of the first \( d \) uniform Type-II hybrid censored order statistics \( U_{1}^{II}, \ldots, U_{d}^{II}, d \in \{1, \ldots, n\} \). The next lemma follows directly from Lemma 6.1.2, with \( P_F = \mathbb{U}(a, b) \).
Lemma 6.3.1 For $d = n$, the conditional joint density function is given by

$$f_{U_j^1, 1 \leq j \leq n|D=n}(u_n) = I_{\Sigma_{F,T}}^n(u_n) \frac{n!}{(b-a)^n F_{m,n}(T)}.$$ 

Let $d \in \{m, \ldots, n-1\}$. Then, we have

$$f_{U_j^1, 1 \leq j \leq d|D=d}(u_d) = I_{\Sigma_{F,T}}^n(u_d) \frac{n!}{(n-d-1)!(b-a)^{d+1} f_{d+1:n}(T)} \left(1 - \frac{T-a}{b-a}\right)^{n-d-1}.$$ 

For $d < m$, we find

$$f_{U_j^1, 1 \leq j \leq m|D=m}(u_m) = I_{\Sigma_{F,T}}^m \times \Sigma_{F,T}^m(u_m) \frac{n!}{(n-m)!(b-a)^m (1 - F_{m,n}(T))} \left(1 - \frac{u_m-a}{b-a}\right)^{n-m}.$$ 

This leads us to the following lemma.

Lemma 6.3.2 The likelihood function $L(\cdot|u_d)$ for $a$ and $b$ is given by

$$L(a, b|u_d) = \begin{cases} 
I_{\Sigma_{F,T}}^n(u_n) f_{1\ldots n:n}(u_n), & d = n \\
I_{\Sigma_{F,T}}^n(u_d) \frac{1 - F(T)}{(n-d)f(T)} f_{1\ldots d+1:n}(u_d, T), & d \in \{m, \ldots, n-1\}, \\
I_{\Sigma_{F,T}}^m \times \Sigma_{F,T}^m(u_m) f_{1\ldots m:n}(u_m), & d < m.
\end{cases}$$

For a known parameter $a$, the MLE $\hat{b}$ is given by

$$\hat{b} = \left\{ \begin{array}{ll} 
\frac{nU_{m:n} - (n-m)a}{m}, & D < m, \\
\frac{nT - (n-D)a}{D}, & D \in \{m, \ldots, n-1\}, \\
U_{n:n}, & D = n.
\end{array} \right.$$ 

Let the parameter $a$ be unknown. Then, the MLEs $\hat{a}$ and $\hat{b}$ are given by

$$\hat{a} = U_{1:n} \quad \text{and} \quad \hat{b} = \left\{ \begin{array}{ll} 
\frac{nU_{m:n} - (n-m)U_{1:n}}{m}, & D < m, \\
\frac{nT - (n-D)U_{1:n}}{D}, & D \in \{m, \ldots, n-1\}, \\
U_{n:n}, & D = n.
\end{array} \right.$$ 

Proof. The likelihood function $L(\cdot|u_d)$ follows from Lemma 6.3.1 and with the distribution of $D = d$, given by (cf. Lemma 6.1.1)

$$P(D < m) = 1 - F_{m:n}(T), \quad d < m,$$

$$P(D = d) = \frac{1 - F(T)}{(n-d)f(T)} f_{d+1:n}(T), \quad d \in \{m, \ldots, n-1\},$$

$$P(D = n) = F_{n:n}(T), \quad d = n.$$ 

By proceeding as in the model of Type-I hybrid censoring (cf. Lemma 4.4.4), we obtain the desired expressions for the MLEs. \qed
Remark 4.4.5, (iii) leads us for an unknown parameter $a$ to the following alternative representation of the MLE $\hat{b}$

$$
\hat{b} = \begin{cases} 
U_{1:n} + \frac{n}{m} (U_{m:n} - U_{1:n}), & D < m, \\
U_{1:n} + \frac{n}{D} (T - U_{1:n}), & D \in \{m, \ldots, n-1\}, \\
U_{1:n} + (U_{n:n} - U_{1:n}), & D = n.
\end{cases}
$$

### 6.3.2 Distribution Theory for the MLE

The structure of the MLE $\hat{b}$ (see Lemma 6.3.2) reveals, that the distribution of $\hat{b}$ for $D < m,$ can be derived from the distribution of $U_{m:n}$ conditionally on $D < m.$

**Theorem 6.3.3** The density function of the $m$th uniform Type-II hybrid censored order statistic $U_{m:n}^{\text{ll}}$ conditional on $D < m,$ is given by

$$
f_{U_{m:n}^{\text{ll}}|D<m}(x) = \mathbb{1}_{[T,b]}(x) \frac{f_{m:n}(x)}{1 - F_{m:n}(T)}. 
$$

**Proof.** Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function. Then, we calculate for the conditional expectation of $g(U_{m:n}^{\text{ll}}),$

$$
E(g(U_{m:n}^{\text{ll}})|D < m) = \int_{\Sigma_m} \frac{n! g(u_m)}{(n-m)!(b-a)^m(1 - F_{m:n}(T))} \left(1 - \frac{u_m - a}{b-a}\right)^{n-m} du_m
$$

$$
= \int_{\Sigma_m} \frac{n! g(u_m)}{(n-m)!(b-a)^m(1 - F_{m:n}(T))} \left(1 - \frac{u_m - a}{b-a}\right)^{n-m} du_m
$$

$$
- \int_{\Sigma_m} \frac{n! g(u_m)}{(n-m)!(b-a)^m(1 - F_{m:n}(T))} \left(1 - \frac{u_m - a}{b-a}\right)^{n-m} du_m
$$

$$
= \frac{1}{1 - F_{m:n}(T)} E(g(U_{m:n})) - \frac{F_{m:n}(T)}{1 - F_{m:n}(T)} E(g(U_{m:n})|D = m)
$$

$$
= \frac{1}{1 - F_{m:n}(T)} \left[ \int_{\mathbb{R}} g(x) f_{m:n}(x) dx - \int_{\mathbb{R}} g(x) \mathbb{1}_{[a,T]}(x) f_{m:n}(x) dx \right].
$$

The last equality follows from Theorem 4.4.6 and by taking into account

$$
E(g(U_{m:n})|D = m) = E(g(U_{m:n}^{\text{ll}})|D = m).
$$

This concludes the proof. \qed

We immediately obtain the distribution of $\hat{b}.$

**Theorem 6.3.4** Let $a < T < b.$ Then, the cumulative distribution function of the MLE $\hat{b}$ under Type-II hybrid censoring is given by

$$
F_{\hat{b}}(y) = F_{m:n} \left( \frac{my + (n-m)a}{n} \right) - F_{m:n} \left( \min \left\{ \frac{my + (n-m)a}{n}, T \right\} \right)
$$

$$
+ \sum_{d=m}^{n-1} \mathbb{1}_{[nT-(n-d)a,\infty)}(dy) \frac{(b - T)f_{d+1:n}(T)}{(n-d)} + F_{n:n}(\min\{y,T\}), \quad y \geq 0.
$$
Proof. The assumption follows with Theorem 6.3.3 and with the steps performed in the proof to Theorem 4.4.7.

Figure 6.4 illustrates plots of the cumulative distribution function $\hat{F}$ for two settings.

Remark 6.3.5 It can be quickly verified, that the limits

$$\lim_{T \to a^+} F_{\hat{b}D}^\hat{b}(y) = F_{m:n}(\frac{my + (n - m)a}{n}) \quad \text{and} \quad \lim_{T \to b^-} F_{\hat{b}D}^\hat{b}(y) = F_{n:n}(y), \quad y \geq 0,$$

hold.

We proceed by calculating the density function $f_{U_{1}^{\text{II}},U_{m}^{\text{II}}|D<m}$.

Theorem 6.3.6 Conditionally on $D < m$, the joint density function of the uniform Type-II hybrid censored order statistics $U_{1}^{\text{II}}$ and $U_{m}^{\text{II}}$, is given by

$$f_{U_{1}^{\text{II}},U_{m}^{\text{II}}|D<m}(u, x) = 1_{(u,x) \in [T,b]^2} \frac{f_{1,m:n}(u, x)}{1 - F_{m:n}(T)}.$$

(6.18)

Proof. The result can be immediately obtained by proceeding as in the proof to Theorem 6.3.3 and by making use of the result presented in Theorem 4.4.9.

By proceeding as in Theorem 4.4.10, the above theorem can be used to establish an expression for the joint cumulative distribution function $F_{\hat{a},\hat{b}}$. Examples for the shape of $f_{U_{1}^{\text{II}},U_{m}^{\text{II}}|D<m}$ are depicted in Figure 6.5.
Figure 6.5: Plots of $f^{U_n, U_n|D < m}$, for $n = 10$, $a = 1$, $b = 2$ and $T = 1.5$. Left: $m = 7$. Right: $m = 3$. 
Chapter 7

Generalized Hybrid Censoring Schemes

The notion of generalized Type-I/II hybrid censoring has been introduced in Chandrasekar et al. (2004) in order to overcome the drawbacks of the Type-I/II hybrid censoring models addressed in Chapters 4 and 6. While in the context of Type-I hybrid censoring no failures might be observed, the generalized Type-I hybrid censoring scheme guarantees the observation of at least \( k < m \) failures. When, on the other hand, it comes down to Type-II hybrid censoring, the experimenter has no control on the duration of the experiment. The generalized Type-II hybrid censoring model fixes this drawback by guaranteeing that the experiment duration does not exceed a prefixed threshold time \( T_2 \), where \( T_1, T_2 \in (0, \infty) \) with \( T_1 < T_2 \).

This chapter addresses the adaptations of the models introduced in Chandrasekar et al. (2004) on progressively Type-II censored order statistics. For both censoring schemes basic distributional results as well as likelihood inference is conducted. Further, distributional results for the MLEs w.r.t. the exponential and the uniform distribution are established. It should be noted that the herein presented results have been partially presented in Górny and Cramer (2016).

We note that once the volumes given in Lemmas 4.2.8 and 6.2.4 have been calculated, it is very convenient to derive expressions for the MLEs for the generalized Type-I/II progressive hybrid censoring models. The desired distributional results can be immediately derived from those established in the context of Type-I sequential hybrid censoring and Type-II progressive hybrid censoring. Hence, it is for those two models not necessary to derive the distribution of the spacings, in order to determine the distribution of the MLEs.

7.1 Generalized Type-I Hybrid Censoring

Chandrasekar et al. (2004) considered the one-parameter exponential case and derived an expression for the density function of the MLE \( \hat{\vartheta} \) in terms of gamma density functions. An extension to progressively Type-II censored order statistics has been presented in Cho et al. (2015b) and Górny and Cramer (2016). Chandrasekar et al. (2004) as well as Cho et al. (2015b) employed the method of the conditional moment generating function in order to obtain the distribution of the MLE \( \hat{\vartheta} \).

7.1.1 Model and Basic Distributional Results

Let \( k, m, \) with \( k < m \leq n \), be previously fixed integers and \( T \in (0, \infty) \) be a previously fixed threshold time. The generalized Type-I progressive hybrid censoring scheme (Type-I GPHCS) aims to stop the life test after observing at least \( k \) failures. In particular, when the \( k \)th failure occurs after time \( T \), the experiment terminates after observing the \( k \)th failure.
If the $k$th failure is observed before reaching $T$, then the life test will be continued till $\min\{X^\mathcal{F}_{m:n}, T\}$.

We consider the sample of $m$ progressively Type-II censored order statistics, given by

$$X^\mathcal{F}_{1:m:n}, \ldots, X^\mathcal{F}_{m:m:n}.$$  

The stopping time $T^*_\mathcal{G}_{\text{I}}$, which accomplishes the intended behavior, is given by

$$T^*_\mathcal{G}_{\text{I}} = \max\{X^\mathcal{F}_{k:m:n}, \min\{X^\mathcal{F}_{m:n}, T\}\}. \tag{7.1}$$

It should be noted, that the above expression was originally introduced for the setting of ordinary order statistics by Park and Balakrishnan (2012, p. 42). Figure 7.1 illustrates the censoring mechanism by providing the possible experimental outcomes.

**Figure 7.1:** Experimental outcomes for the generalized Type-I progressive hybrid censoring scheme.

...
7.1 Generalized Type-I Hybrid Censoring

Generalized Type-I progressive hybrid censored order statistics are defined by

$$X_{j}^{Gl} := \min \{ \max \{ X_{j,k,m:n}^{\sigma}, T \}, X_{j,m:n}^{\sigma} \}, \quad 1 \leq j \leq m.$$ 

In a next step, we calculate the probability of the event \( \{ X_{j}^{Gl} \leq x_j, 1 \leq j \leq d, \mathcal{D}(d) \} \), where \( d \in \mathcal{I}(\mathcal{D}), \ x_d \in \mathbb{R}^d \), for each \( \mathcal{D} \in \mathcal{G}_{\mathcal{D}}(Gl) \). First for \( D < k \), we obtain

$$P(\ X_{j}^{Gl} \leq x_j, 1 \leq j \leq k, D < k)$$

$$= P(\ X_{j}^{Gl} \leq x_j, 1 \leq j \leq k, X_{k,m:n}^{\sigma} > T)$$

$$= P(\ \min \{ \max \{ X_{k,m:n}^{\sigma}, T \}, X_{j,m:n}^{\sigma} \} \leq x_j, 1 \leq j \leq k, X_{k,m:n}^{\sigma} > T)$$

$$= P(\ \min \{ X_{k,m:n}^{\sigma}, X_{j,m:n}^{\sigma} \} \leq x_j, 1 \leq j \leq k, X_{k,m:n}^{\sigma} > T)$$

For \( d \in \{ k, \ldots, m - 1 \} \), we find

$$P(\ X_{j}^{Gl} \leq x_j, 1 \leq j \leq d, D = d)$$

$$= P(\ \min \{ \max \{ X_{k,m:n}^{\sigma}, T \}, X_{j,m:n}^{\sigma} \} \leq x_j, 1 \leq j \leq d, D = d)$$

$$= P(\ \min \{ T, X_{j,m:n}^{\sigma} \} \leq x_j, 1 \leq j \leq d, X_{d,m:n}^{\sigma} \leq T \ < X_{d+1,m:n}^{\sigma} \)$$

Finally, for \( d = m \), we arrive at

$$P(\ X_{j}^{Gl} \leq x_j, 1 \leq j \leq m, D = m)$$

This yields together with Lemma 6.1.2 the following result.

**Lemma 7.1.2** For \( d \in \{ k, \ldots, m - 1 \} \) the conditional joint cumulative distribution function of \( X_{1}^{Gl}, \ldots, X_{d}^{Gl} \) is given by

$$P(\ X_{j}^{Gl}, 1 \leq j \leq d, D = d \mid \mathcal{D}(d) = x_d = F_{1,d+1,m:n}(x_d) = F_{d+1,m:n}(x_d, T) \),$$

For the respective conditional joint density function, we have

$$f_{X_{j}^{Gl}, 1 \leq j \leq d, D = d} \mid \mathcal{D}(d) = x_d = f_{1,d+1,m:n}(x_d, T) = f_{d+1,m:n}(T).$$
Let \( d = m \). Then, we obtain for the cumulative distribution function \( F^{X_j,1 \leq j \leq m|D=m} \),
\[
F^{X_j,1 \leq j \leq m|D=m}(x_m) = \frac{F_{1 \cdots m:m:n}^\varrho(x_m)}{F_{m:m:n}(T)}, \quad x_m \in \mathbb{R}^m.
\]

Further, the corresponding density function is given by
\[
f^{X_j,1 \leq j \leq m|D=m}(x_m) = 1_{\Sigma_m^m}(x_m) \frac{f_{1 \cdots m:m:n}^\varrho(x_m)}{F_{m:m:n}(T)}.
\]

Finally, for \( d < k \), we obtain
\[
F^{X_j,1 \leq j \leq k|D=k}(x_k) = \frac{F_{1 \cdots k:k:n}^\varrho(x_k)}{1 - F_{k:k:n}(T)}, \quad x_k \in \mathbb{R}^k,
\]

and
\[
f^{X_j,1 \leq j \leq k|D=k}(x_k) = 1_{\Sigma_k^k \setminus \Sigma_k^k,T}(x_k) \frac{f_{1 \cdots k:k:n}^\varrho(x_k)}{1 - F_{k:k:n}(T)}.
\]

The preceding results show that the structure of the underlying model is very similar to the model of progressive Type-II hybrid censoring. For both, the exponential and the uniform case, the distributional results can be derived by analogy with Chapter 6.

### 7.1.2 Generalized Type-I Progressive Hybrid Censoring from Exponential Distributions

Assume the IID progressive exponential model (see Model 2.2.6). Then, we denote by \( Z_1^{GI}, \ldots, Z_d^{GI}, \ d \in \{1, \ldots, m\} \), the generalized Type-I progressive hybrid censored order statistics from \( \text{Exp}(\mu, \vartheta) \). The conditional joint density of \( Z_1^{GI}, \ldots, Z_d^{GI}, \ d \in \{1, \ldots, m\} \), follows with Lemma 6.2.1.

**Lemma 7.1.3** Let \( z_0 = \mu \). Then, for \( d < k \) the conditional joint density function is given by
\[
f^{Z_j,1 \leq j \leq k|D=k}(z_k) = 1_{\Sigma_k^k \setminus \Sigma_k^k,T}(z_k) \frac{1}{\vartheta k} \prod_{j=1}^k \gamma_j^{\varrho \vartheta k} \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^k \gamma_j^{\varrho \vartheta k} (z_j - z_{j-1}) \right\}.
\]

For \( d \in \{k, \ldots, m-1\} \), we have
\[
f^{Z_j,1 \leq j \leq d|D=d}(z_d) = 1_{\Sigma_d^d \setminus \Sigma_d^d,T}(z_d) \frac{1}{\vartheta d+1} \prod_{j=1}^{d+1} \gamma_j^{\varrho \vartheta} \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^d \left( 1 - \frac{\gamma_{d+1}^{\varrho \vartheta}}{\gamma_j^{\varrho \vartheta}} \right) \gamma_j (z_j - z_{j-1}) + \gamma_{d+1}^{\varrho \vartheta} T \right\}.
\]

Further, for \( d = m \), we find
\[
f^{Z_j,1 \leq j \leq m|D=m}(z_m) = 1_{\Sigma_m^m \setminus \Sigma_m^m,T}(z_m) \frac{1}{\vartheta m} \prod_{j=1}^m \gamma_j^{\varrho \vartheta} \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^m \gamma_j (z_j - z_{j-1}) \right\}.
\]
The MLEs under generalized Type-I progressive hybrid censoring are given as in Lemma 7.1.4.

**Lemma 7.1.4** The likelihood function $L(\cdot | z_d)$ for $\mu$ and $\vartheta$ is given by

$$L(\mu, \vartheta | z_d) = \begin{cases} \prod_{j=1}^{k} \gamma_j(Z_{j:k:n}^{\vartheta \mu} - Z_{j-1:k:n}^{\vartheta \mu}), & D < k, \\ \frac{1}{D} \sum_{j=1}^{D} \left(1 - \frac{\gamma_{d+1}}{\gamma_j}\right) \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}) + \gamma_{D+1}(T - \mu), & D \in \{k, \ldots, m-1\}, \\ \frac{1}{m} \sum_{j=1}^{m} \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}), & D = m. \end{cases}$$

Let the location parameter $\mu$ be known. Then, the MLE for $\vartheta$ is given by

$$\hat{\vartheta} = \begin{cases} \frac{1}{k} \sum_{j=1}^{k} \gamma_j(Z_{j:k:n}^{\vartheta s} - Z_{j-1:k:n}^{\vartheta s}), & D < k, \\ \frac{1}{D} \sum_{j=1}^{D} \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}) + \gamma_{D+1}(T - \mu), & D \in \{k, \ldots, m-1\}, \\ \frac{1}{m} \sum_{j=1}^{m} \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}), & D = m. \end{cases}$$

For an unknown location parameter $\mu$, the MLEs $\hat{\mu}$ and $\hat{\vartheta}$ are given by

$$\hat{\mu} = Z_{1:m:n}^{\vartheta}$$

and

$$\hat{\vartheta} = \begin{cases} \frac{1}{k} \sum_{j=2}^{k} \gamma_j(Z_{j:k:n}^{\vartheta s} - Z_{j-1:k:n}^{\vartheta s}), & D < k, \\ \frac{1}{D} \sum_{j=2}^{D} \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}) + \gamma_{D+1}(T - Z_{D:m:n}^{\vartheta}), & D \in \{k, \ldots, m-1\}, \\ \frac{1}{m} \sum_{j=2}^{m} \gamma_j(Z_{j:m:n}^{\vartheta} - Z_{j-1:m:n}^{\vartheta}), & D = m. \end{cases}$$

Let $\mu$ be known. Then, the conditional distribution of the total time on test statistic $S_D$ follows directly from Theorem 6.2.5.

**Theorem 7.1.5** For $D < k$, the density function of $S_k$ is given by

$$f_{S_k | D < k}(s) = \frac{f_{S_k}(s)}{1 - F_{k:k:n}(T)} = \frac{(T - \mu)^k \prod_{j=1}^{k} \gamma_j^{\vartheta s}}{k! \vartheta (1 - F_{k:k:n}(T)) B_{k-1}(s | 0, \gamma_k^{\vartheta s} (T - \mu)) e^{-s/\vartheta}}, \quad s \geq 0.$$
For $d \in \{k, \ldots, m-1\}$, we find
\[
f_{S_d|D=d}(s) = \frac{(T-\mu)^d}{d!} \prod_{j=1}^{d+1} \gamma_j B_{d+1}(s|\gamma_{d+1}(T-\mu)) e^{-s/\theta}, \quad s \geq 0.
\]
Finally, for $d = m$, we obtain
\[
f_{S_m|D=m}(s) = \frac{(T-\mu)^m}{m!} \prod_{j=1}^{m} \gamma_j B_m(s|0, \gamma_m(T-\mu)) e^{-s/\theta}, \quad s \geq 0.
\]

The above lemma leads us to the following representation for the density function of $\hat{\theta}$.

**Theorem 7.1.6** For the MLE $\hat{\theta}$ under generalized Type-I progressive hybrid censoring, as specified in Lemma 7.1.4, the following representations for $f_{\hat{\theta}}$ hold.

(i) The density function $f_{\hat{\theta}}$ in terms of B-splines is given by
\[
f_{\hat{\theta}}(s) = f_{\bar{\theta},k}(s) - \frac{(T-\mu)^k}{(k-1)!} \sum_{j=0}^{k} \prod_{i=0, i \neq j}^{d} \frac{\gamma_{d-i+1}(T-\mu)/\theta}{\gamma_{d-i+1} - \gamma_{d-j+1}} f_{\bar{\theta}}(s - \frac{\gamma_{d-j+1}(T-\mu)}{d})
\]
\[
- \sum_{j=1}^{k} \prod_{i=0, i \neq j}^{d} \frac{\gamma_{d-j+1}(T-\mu)/\theta}{\gamma_{d-j+1} - \gamma_{k-j+1}} f_{\bar{\theta}}(s - \frac{\gamma_{k-j+1}(T-\mu)}{k}).
\]

(ii) A gamma representation for the density function of $\hat{\theta}$ is for $s \geq 0$ given by
\[
f_{\hat{\theta}}(s) = \sum_{d=k}^{m} \left( \prod_{j=1}^{d} \gamma_j \right) \frac{e^{\gamma_{d-j+1}(T-\mu)/\theta}}{\gamma_{d-i+1} - \gamma_{d-j+1}} f_{\bar{\theta}}(s - \frac{\gamma_{d-j+1}(T-\mu)}{d})
\]
\[
- \left( \prod_{j=1}^{k} \gamma_j \right) \frac{e^{\gamma_{k-j+1}(T-\mu)/\theta}}{\gamma_{d-j+1} - \gamma_{k-j+1}} f_{\bar{\theta}}(s - \frac{\gamma_{k-j+1}(T-\mu)}{k}).
\]

**Proof.** (i) The result follows directly with Theorem 7.1.5.

(ii) An expression for $f_{\hat{\theta}}$ in terms of gamma density functions based on the B-spline representation given in (i), can be obtained by using Theorem 4.3.1. Then, the assumption follows by proceeding as in the proof to Theorem 6.2.8. (ii) \qed

Figure 7.2 depicts plots of $f_{\hat{\theta}}$ in comparison with the density function of the MLE $\hat{\theta}$ in the (progressive) Type-II censoring case.

Let $\mu$ be unknown. Hence, we consider the following modified total time on test statistic
\[
V_D = \begin{cases} 
\sum_{j=2}^{k} \gamma_j (Z_{j;\tilde{c}k} - Z_{j-1;\tilde{c}k}), & D < k, \\
D \sum_{j=2}^{D} \gamma_j (Z_{j;\tilde{c}m} - Z_{j-1;\tilde{c}m}) + \gamma_{D+1} (T - \mu - \frac{W_{\tilde{c}1}}{n}), & D \in \{k, \ldots, m\},
\end{cases}
\]
where $W_{\tilde{c}1} = n(Z_{\tilde{c}1} - \mu)$, denotes the first spacing based on the generalized Type-I progressive hybrid censored order statistics $Z_{\tilde{c}1}, \ldots, Z_{\tilde{c}m}$. By using Theorem 6.2.13 we can calculate the conditional joint density functions of $W_{\tilde{c}1}$ and $V_D$. 

Theorem 7.1.7 For \( d < k \), we have with \( v \geq 0 \)
\[
f_{W_{v|D<k}}(w, v) = 1_{[0, \infty)}(w) f_{\theta_d, k-1}(v) \frac{e^{-w/\theta}}{\theta(1 - F_{k:k:n}^{\text{progressive}}(T))} \]
\[
- 1_{[0, n(T-\mu))}(w) \frac{(T - \mu - \frac{w}{n})^{k-1} \prod_{j=2}^{k} \gamma_j^{\text{jack}}}{(k-1)! \theta^k} B_{k-2}\left( v, 0, \gamma_{2,k}^{\text{jack}}(T - \mu - \frac{w}{n}) \right) e^{-(w+v)/\theta}.
\]
Considering \( d \in \{k, \ldots, m-1\} \) and \( 0 \leq w < n(T-\mu) \), \( v \geq 0 \), we find
\[
f_{W_{v|D=d}}(w, v) = \frac{(T - \mu - \frac{w}{n})^{d-1} \prod_{j=2}^{d+1} \gamma_j}{(d-1)! \theta^{d+1} f_{d+1:m:n}(T)} B_{d-2}\left( v, 0, \gamma_{2,d+1}(T - \mu - \frac{w}{n}) \right) e^{-(w+v)/\theta}.
\]
For \( d = m \), we finally obtain with \( 0 \leq w < n(T-\mu) \) and \( v \geq 0 \),
\[
f_{W_{v|m|D=m}}(w, v) = \frac{(T - \mu - \frac{w}{n})^{m-1} \prod_{j=2}^{m} \gamma_j}{(m-1)! \theta^m F_{m:m:n}(T)} B_{m-2}\left( v, 0, \gamma_{2,m}(T - \mu - \frac{w}{n}) \right) e^{-(w+v)/\theta}.
\]
The above theorem leads us to the following expression for \( f_{\hat{\mu}, \hat{\theta}} \).

Theorem 7.1.8 Let \( k, m \geq 2 \). Then, the joint density function of the MLEs \( \hat{\mu} \) and \( \hat{\theta} \) under generalized Type-I progressive hybrid censoring is given by, for \( t \geq 0 \),
\[
f_{\hat{\mu}, \hat{\theta}}(s, t) = 1_{[\mu, \infty)}(s) \frac{kn}{\theta} f_{\theta_d, k-1}(kt) e^{-n(s-\mu)/\theta} \]
\[
- 1_{[\mu, T)}(s) \frac{k(T-s)^{k-1} \prod_{j=2}^{k} \gamma_j^{\text{jack}}}{(k-1)! \theta^k} B_{k-2}\left( kt, 0, \gamma_{2,k}(T-s) \right) e^{-[n(s-\mu)+kt]/\theta} \]
\[
+ 1_{[\mu, T)}(s) \sum_{d=k}^{m} \frac{d(T-s)^{d-1} \prod_{j=1}^{d} \gamma_j}{(d-1)! \theta^d} B_{d-2}(dt, \gamma_{2,d+1}(T-s)) e^{-[n(s-\mu)+dt]/\theta}.
\]

Figure 7.3 shows the plots of \( f_{\hat{\mu}, \hat{\theta}} \) for two censoring plans \( \mathcal{R} \).
7.1.3 Generalized Type-I Hybrid Censoring from Uniform Distributions

In this section, we suppose the IID uniform model (see Model 2.1.3). The underlying sample is given by the first \( m \leq n \) uniform order statistics \( U_{1:n}, \ldots, U_{m:n} \). The stopping time \( T_{Gl}^* \) is given by

\[
T_{Gl}^* = \max\{U_{k:n}, \min\{U_{m:n}, T\}\}.
\]

The corresponding uniform generalized Type-I hybrid censored order statistics are defined by

\[
U_{Gl}^j := \min\{\max\{T, U_{k:n}\}, U_{j:n}\}, \quad 1 \leq j \leq m.
\]

Further we introduce the counter \( D = \sum_{j=1}^{m} 1_{(\infty, T]}(U_{j:n}) \). The conditional joint density functions of \( U_{Gl}^1, \ldots, U_{Gl}^d, \quad d \in \{1, \ldots, m\} \), can be directly deduced from Lemma 6.3.1.

**Lemma 7.1.9** For \( d < k \), the joint density function of the uniform generalized Type-I hybrid censored order statistics is given by

\[
f_{U_{Gl}^j, 1 \leq j \leq k | D < k}(u_k) = 1_{\Sigma_{k}^{\infty} \backslash \Sigma_{F,T}(u_k)} \frac{n!}{(n-k)!(b-a)^k(1-F_{k:n}(T))} \left(1 - \frac{u_k - a}{b-a}\right)^{n-k}.
\]

For \( d \in \{k, \ldots, m-1\} \), we have

\[
f_{U_{Gl}^j, 1 \leq j \leq d | D = d}(u_d) = 1_{\Sigma_{d}^{\infty} \backslash \Sigma_{F,T}(u_d)} \frac{n!}{(n-d-1)!(b-a)^{d+1}f_{d+1:n}(T)} \left(1 - \frac{T - a}{b-a}\right)^{n-d-1}.
\]

For \( d = m \), we find

\[
f_{U_{Gl}^j, 1 \leq j \leq m | D = m}(u_m) = 1_{\Sigma_{m}^{\infty} \backslash \Sigma_{F,T}(u_m)} \frac{n!}{(n-m)!(b-a)^mF_{m:n}(T)} \left(1 - \frac{u_m - a}{b-a}\right)^{n-m}.
\]

By utilizing the same arguments as in the previous models, we obtain the likelihood function \( L(\cdot | u_d) \) (cf. Lemma 6.3.2).
Lemma 7.1.10 The likelihood function $L(\cdot|u_d)$ for $a$ and $b$ is given by

$$L(a,b|u_d) = \begin{cases}
 1_{\sum_k \not= \sum_{F,T}}(u_k) f_{1..k:n}(u_k), & d < k, \\
 1_{\sum_{F,T}}(u_d) \frac{1-F(T)}{(n-d)f(T)} f_{1..d+1:n}(u_d,T), & d \in \{k, \ldots, m-1\}, \\
 1_{\sum_{m}}(u_m) f_{1..m:n}(u_m), & d = m.
\end{cases}$$

Let the parameter $a$ be known. Then, the MLE for $b$ is given by

$$\hat{b} = \begin{cases}
 \frac{nU_{k:n} - (n-k)a}{k}, & D < k, \\
 \frac{nT - (n-D)a}{D}, & D \in \{k, \ldots, m-1\}, \\
 \frac{nU_{m:n} - (n-m)a}{m}, & D = m.
\end{cases}$$

For an unknown parameter $a$, the MLEs for $a$ and $b$ are further given by

$$\hat{a} = U_{1:n} \quad \text{and} \quad \hat{b} = \begin{cases}
 \frac{nU_{k:n} - (n-k)U_{1:n}}{k}, & D < k, \\
 \frac{nT - (n-D)U_{1:n}}{D}, & D \in \{k, \ldots, m-1\}, \\
 \frac{nU_{m:n} - (n-m)U_{1:n}}{m}, & D = m.
\end{cases}$$

We proceed by determining the distribution of $\hat{b}$ for a known parameter $a$. First we need the distribution of $U_{k:n}^{\hat{a}}$ conditional on $D < k$. By utilizing the same arguments as in the proof to Theorem 6.3.3 we obtain the following result.

Theorem 7.1.11 The density function $f^{U_{k:n}^{\hat{a}}|D<k}$ is given by

$$f^{U_{k:n}^{\hat{a}}|D<k}(x) = 1_{[T,b]}(x) \frac{f_{k:n}(x)}{1-F_{k:n}(T)}, \quad x \in \mathbb{R}.$$  

The above theorem leads us to the following expression for the cumulative distribution function $F^{\hat{b}}$.

Theorem 7.1.12 Let $a < T < b$. Then, the cumulative distribution function of the MLE $\hat{b}$ under generalized Type-I hybrid censoring is given by, for $y \geq 0$,

$$F^{\hat{b}}(y) = F_{k:n}\left(\frac{ky + (n-k)a}{n}\right) - F_{k:n}\left(\min\left\{\frac{ky + (n-k)a}{n}, T\right\}\right) + \sum_{d=k}^{m-1} 1_{[nT-(n-d)a,\infty)}(dy) \frac{(b-T)f_{d+1:n}(T)}{(n-d)} + F_{m:n}\left(\min\left\{\frac{my + (n-m)a}{n}, T\right\}\right).$$

Figure 7.4 shows the plots of $F^{\hat{b}}$ for two setups. Notice that the conditional joint density function $f^{U_{k:n}^{\hat{a}},U_{k:n}^{\hat{b}}|D<k}$ can be obtained directly from Theorem 6.3.6. Then, an expression for the joint distribution of the MLEs $\hat{a}$ and $\hat{b}$ can be derived.
7.2 Generalized Type-II Hybrid censoring

For a detailed description of the herein addressed generalized Type-II progressive hybrid censoring model, we refer to Section 1.1.3. We proceed directly with the derivation of some basic distributional results (see Section 7.2.1). In Section 7.2.2, the generalized Type-II progressive hybrid censoring scheme under an exponential assumption is discussed. We conclude this chapter by considering the generalized Type-II hybrid censoring scheme from uniform distributions (see Section 7.2.3).

7.2.1 Model and Basic Distributional Results

First, we recall the underlying system of counter settings and its corresponding family of sets of valid integers, i.e. (see Example 1.4.2, (iii)),

\[ S_D(GII) = \{[D_1 = \cdot], [D_1 < \cdot, D_2 = \cdot], [D_2 = \cdot]\}, \]

and \( \mathcal{I}(S_D(GII)) = \{\{m, \ldots, \tilde{m}\}, \{m\}, \{1, \ldots, m - 1\}\}, \)

respectively. Further, we assume the IID progressive model (see Model 2.2.5). The following lemma presents the distribution of \( D_1 = d_1 \) can be calculated by

\[ P(D_1 = d_1) = 1 - F(T_1) \prod_{d_1 + 1}^{\tilde{m}} f_{d_1 + 1; m:n}(T_1). \]

For \( d_1 = \tilde{m} \), we have

\[ P(D_1 = \tilde{m}) = F_{\tilde{m}; m:n}(T_1). \]

For \( d_1 < m \) and \( d_2 = m \), we find further

\[ P(D_1 < m, D_2 = m) = F_{m; m:n}(T_2) - F_{m; m:n}(T_1). \]
Finally, for \( d_2 \in \{0, \ldots, m-1\} \), we get
\[
P(D_2 = d_2) = \frac{1 - F(T_2)}{\gamma_{d_2+1} f(T_2)} f_{d_2+1:m:n}(T_2).
\]

**Proof.** Let \( d_1 \in \{0, \ldots, m-1\} \) and \( d_2 = m \). Then, by realizing that \( D_1 \) cannot exceed \( D_2 \), we obtain
\[
P(D_1 \in \{0, \ldots, m-1\}, D_2 = m) = P(D_1 < m, D_2 = m)
\]
\[
= P(D_2 = m) - P(D_1 \geq m, D_2 = m)
\]
\[
= P(D_2 = m) - P(D_1 = m, D_2 = m)
\]
\[
= P(D_2 = m) - P(D_1 = m)
\]
\[
= P(X_{<j:m:n} \leq T_2) - P(X_{<j:m:n} \leq T_1)
\]
\[
= F_{m:m:n}(T_2) - F_{m:m:n}(T_1).
\]

The identity \( P(D_1 = m, D_2 = m) = P(D_1 = m) \) used in the above calculations, follows from the ordering \( T_1 < T_2 \), i.e.,
\[
P(D_1 = m, D_2 = m) = P(X_{<j:m:n} \leq T_1, X_{<j:m:n} \leq T_2) = P(X_{<j:m:n} \leq T_1) = P(D_1 = m).
\]

The remaining cases can be deduced from Lemma 6.1.1. \( \square \)

We proceed by calculating the probabilities \( P(X_{<j}^{GII} \leq x_j, 1 \leq j \leq d, D(d)), x_d \in \mathbb{R}^d, d \in \mathcal{I}(D) \), for each \( D \in \mathcal{G}_2(GII) \). According to Example 1.4.2, (iii), the *generalized Type-II progressive hybrid censored order statistics* are defined by
\[
X_{<j}^{GII} := \min\{X_{<j:m:n}, T_2\}, \quad 1 \leq j \leq \tilde{m}.
\]
Then, for \( d_1 \in \{m, \ldots, \tilde{m} - 1\} \), we get
\[
P(X_{<j}^{GII} \leq x_j, 1 \leq j \leq d_1, D_1 = d_1)
\]
\[
= P(\min\{X_{<j}^{m:n}, T_2\} \leq x_j, 1 \leq j \leq d_1, X_{<j}^{m:n} \leq T_1 < X_{d_1+1:m:n})
\]
\[
= P(X_{<j}^{m:n} \leq x_j, 1 \leq j \leq d_1 - 1, X_{d_1+1:m:n} \leq \min\{x_{d_1}, T_1\}, X_{d_1+1:m:n} > T_1)
\]
\[
= F_{1\ldots,d_1:m:n}(x_{d_1-1}, \min\{T_1, x_{d_1}\}) - F_{1\ldots,d_1+1:m:n}(x_{d_1-1}, \min\{T_1, x_{d_1}\}, T_1).
\]

For \( d_1 = \tilde{m} \), we find
\[
P(X_{<j}^{GII} \leq x_j, 1 \leq j \leq \tilde{m}, D_1 = \tilde{m})
\]
\[
= P(X_{<j}^{m:n} \leq x_j, 1 \leq \tilde{m} - 1, X_{\tilde{m}:m:n} \leq \min\{x_{\tilde{m}}, T_1\})
\]
\[
= F_{1\ldots,\tilde{m}:m:n}(x_{\tilde{m}-1}, \min\{T_1, x_{\tilde{m}}\}).
\]

For \( d_1 < m, d_2 = m \), we obtain
\[
P(X_{<j}^{GII} \leq x_j, 1 \leq j \leq m, D_1 < m, D_2 = m)
\]
\[
= P(X_{<j}^{m:m:n} \leq x_j, 1 \leq j \leq m, D_1 < m, D_2 = m)
\]
\[
= P(X_{<j}^{m} \leq x_j, 1 \leq j \leq m, D_2 = m)
\]
For the corresponding density function, we find

\[
P(X_j^\text{GII} \leq x_j, 1 \leq j \leq m, D_1 = m, D_2 = m)
\]
\[
= P(X_j^\text{GII} \leq x_j, 1 \leq j \leq m - 1, X_{m:m:n}^\text{GII} \leq \min\{T_2, x_m\})
\]
\[
- P(X_j^\text{GII} \leq x_j, 1 \leq j \leq m, D_1 = m)
\]
\[
= F_{1...m:m:n}(x_{m-1}, \min\{T_2, x_m\}) - F_{1...m:m:n}(x_{m-1}, \min\{T_1, x_m\}).
\]

Finally, we find for \(d_2 \in \{1, \ldots, m-1\}\)

\[
P(X_j^\text{GII} \leq x_j, 1 \leq j \leq d_2, D_2 = d_2)
\]
\[
= P(\min\{X_j^\text{GII}, T_2\} \leq x_j, 1 \leq j \leq d_2, X_{d_2+1:m:n}^\text{GII} \leq T_2 < X_{d_2+1:m:n}^\text{GII})
\]
\[
= P(X_j^\text{GII} \leq x_j, 1 \leq j \leq d_2 - 1, X_{d_2+1:m:n}^\text{GII} \leq \min\{T_2, x_{d_2}\}, X_{d_2+1:m:n}^\text{GII} > T_2)
\]
\[
= F_{1...d_2:m:n}(x_{d_2-1}, \min\{T_2, x_{d_2}\}) - F_{1...d_2+1:m:n}(x_{d_2-1}, \min\{T_2, x_{d_2}\}, T_2).
\]

Now, we are able to derive the conditional joint distribution of \(X_1^\text{GII}, \ldots, X_d^\text{GII}, d \in \{1, \ldots, \tilde{m}\}\).

**Lemma 7.2.2** Let \(f(T_i) > 0, i \in \{1, 2\}\). For \(d_1 \in \{m, \ldots, \tilde{m} - 1\}\) the conditional joint cumulative distribution function is given by

\[
F_{X_j^\text{GII},1 \leq j \leq d_1|D=d_1}(x_{d_1}) = F_{1...d_1+1:m:n}^\text{GII}(x_{d_1})|X_{d_1+1:m:n}^\text{GII} = T_1), \ x_{d_1} \in \mathbb{R}^{d_1}.
\]

For the corresponding density function, we find

\[
f_{X_j^\text{GII},1 \leq j \leq d_1|D=d_1}(x_{d_1}) = \mathbf{1}_{\Sigma_{F,T_1}^\text{GII}}(x_{d_1}) \frac{f_{\tilde{m}}(x_{d_1}, T_1)}{f_{d_1+1:m:n}(T_1)}.
\]

For \(d_1 = \tilde{m}\), we have

\[
F_{X_j^\text{GII},1 \leq j \leq \tilde{m}|D=\tilde{m}}(x_{\tilde{m}}) = \frac{F_{\tilde{m}}(x_{\tilde{m}-1}, \min\{x_{\tilde{m}}, T_1\})}{F_{\tilde{m}:m:n}(T_1)}, \ x_{\tilde{m}} \in \mathbb{R}^{\tilde{m}},
\]

and

\[
f_{X_j^\text{GII},1 \leq j \leq \tilde{m}|D=\tilde{m}}(x_{\tilde{m}}) = \mathbf{1}_{\Sigma_{F,T_1}^\text{GII}}(x_{\tilde{m}}) \frac{f_{\tilde{m}}(x_{\tilde{m}})}{F_{\tilde{m}:m:n}(T_1)}.
\]

For \(d_1 < m, d_2 = m\), the conditional joint distribution is given by

\[
F_{X_j^\text{GII},1 \leq j \leq m|D=m}^\text{GII}(x_{m})
\]
\[
= F_{1...m:m:n}(x_{m-1}, \min\{x_{m}, T_2\}) - F_{1...m:m:n}(x_{m-1}, \min\{x_{m}, T_1\}), \ x_{m} \in \mathbb{R}^{m}.
\]

Accordingly, the respective density function is given by

\[
f_{X_j^\text{GII},1 \leq j \leq m|D=m}^\text{GII}(x_{m}) = \mathbf{1}_{\Sigma_{F,T_2}^\text{GII} \setminus \Sigma_{F,T_2}^\text{GII}}(x_{m}) \frac{f_{\tilde{m}}(x_{m})}{F_{\tilde{m}:m:n}(T_2) - F_{\tilde{m}:m:n}(T_1)}.
\]

Finally, we get for \(d_2 \in \{1, \ldots, m-1\}\),

\[
F_{X_j^\text{GII},1 \leq j \leq d_2|D=d_2}(x_{d_2}) = F_{1...d_2+1:m:n}^\text{GII}(x_{d_2})|X_{d_2+1:m:n}^\text{GII} = T_2), \ x_{d_2} \in \mathbb{R}^{d_2},
\]

and

\[
f_{X_j^\text{GII},1 \leq j \leq d_2|D=d_2}(x_{d_2}) = \mathbf{1}_{\Sigma_{F,T_2}^\text{GII}}(x_{d_2}) \frac{f_{\tilde{m}}(x_{d_2}, T_2)}{f_{d_2+1:m:n}(T_2)}.
\]
Proof. The joint cumulative distribution function conditional on $D_1 < m$, $D_2 = m$ follows from (7.2). The corresponding density function is obtained from

\[
\frac{\partial^m}{\partial x_1 \ldots \partial x_m} F_{1,\ldots,m;m:n}(x_{m-1}, \min\{x_m, T_2\}) - F_{1,\ldots,m;m:n}(x_{m-1}, \min\{x_m, T_1\})
\]

\[
= \frac{1_{\sum_m T_2}(x_m) f_{1,\ldots,m;m:n}(x_m) - 1_{\sum_m T_1}(x_m) f_{1,\ldots,m;m:n}(x_m)}{F_{m;m:m:n}(T_2) - F_{m;m:m:n}(T_1)},
\]

so that \( \supp(f_{GII,1 \leq i \leq m|D_1 < m, D_2 = m}) = \sum_{T_2} \setminus \sum_{T_1} \). The results for the remaining cases can be taken from Lemma 6.1.2.

Remark 7.2.3 Looking at the calculations leading to (7.2), we realize that the density function for $D_1 < m$, $D_2 = m$ can be rewritten as

\[
f_{\tilde{X}_{GII,1 \leq i \leq m|D_1 < m, D_2 = m}}(x_m) = c(T_2) f_{\tilde{X}_{GII,1 \leq i \leq m|D_1 = m}}(x_m) - c(T_1) f_{\tilde{X}_{GII,1 \leq i \leq m|D_1 = m}}(x_m),
\]

with

\[
f_{\tilde{X}_{GII,1 \leq i \leq m|D_1 = m}}(x_m) = 1_{\sum_m T_2}(x_m) \frac{f_{1,\ldots,m;m:n}(x_m)}{F_{m;m:m:n}(T_1)} \cdot c(T_1) = \frac{F_{m;m:m:n}(T_1)}{F_{m;m:m:n}(T_2) - F_{m;m:m:n}(T_1)},
\]

for $i \in \{1, 2\}$. Thus, the conditional density function $f_{\tilde{X}_{GII,1 \leq i \leq m|D_2 < m, D_2 = m}}$ can be written as a linear combination of already known conditional density functions. This simplifies the derivation of related distributional results considerably.

7.2.2 Generalized Type-II Progressive Hybrid Censoring from Exponential Distributions

Consider the IID progressive exponential model (see Model 2.2.6). Further, we denote by $Z_{GII}^1, \ldots, Z_{GII}^d$, $d \in \{1, \ldots, \tilde{m}\}$, the generalized Type-II progressive hybrid censored order statistics from the two-parameter exponential distribution. The next result follows immediately with Lemma 7.2.2.

Lemma 7.2.4 Let $z_0 := \mu$. Then, for $d_1 \in \{m, \ldots, \tilde{m} - 1\}$, we have

\[
f_{Z_{GII,1 \leq j \leq D = d_1}|D = d_1}(z_{d_1}) = 1_{\sum_{T_1}}(z_{d_1}) \frac{\prod_{j=1}^{d_1+1} \gamma_j^{\tilde{m}}} {\partial_{d_1+1} \sum_{T_1}} \cdot \exp \left\{ - \frac{1}{\vartheta} \left[ \sum_{j=1}^{d_1} \left( 1 - \frac{\gamma_j^{\tilde{m}}}{\gamma_j^{\tilde{m}}} \right) \gamma_j^{\tilde{m}} (z_j - z_{j-1}) + \gamma_j^{\tilde{m}} T_1 \right] \right\}.
\]

Further, for $d_1 = \tilde{m}$, we find

\[
f_{Z_{GII,1 \leq j \leq \tilde{m}|D = \tilde{m}}}(z_{\tilde{m}}) = 1_{\sum_{T_1}}(z_{\tilde{m}}) \frac{\prod_{j=1}^{\tilde{m}} \gamma_j^{\tilde{m}}} {\partial_{\tilde{m}} \sum_{T_1}} \cdot \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^{\tilde{m}} \gamma_j^{\tilde{m}} (z_j - z_{j-1}) \right\}.
\]

For $d_1 < m$, $d_2 = m$, the conditional joint density function is given by

\[
f_{Z_{GII,1 \leq j \leq m|D_1 < m, D_2 = m}}(z_m) = 1_{\sum_{T_2} \setminus \sum_{T_1}}(z_m) \frac{\prod_{j=1}^{m} \gamma_j^{\tilde{m}}}{\partial_m \sum_{T_2} \setminus \sum_{T_1}} \cdot \exp \left\{ - \frac{1}{\vartheta} \sum_{j=1}^{m} \gamma_j^{\tilde{m}} (z_j - z_{j-1}) \right\}.
\]
Finally, we have for \( d_2 \in \{1, \ldots, m-1\}, \)
\[
f_{Z_j^{GII}, 1 \leq j \leq d_2 | D_2 = d_2}(z_{d_2}) = \mathbf{1}_{z_{d_2}}(z_{d_2}) \prod_{j=1}^{d_2+1} \gamma_j \left( \frac{1}{\theta} \sum_{j=1}^{\min(m, d_2+1)} \gamma_j (z_j - z_{j-1}) + \gamma_{d_2+1} \right) \times \exp \left\{ - \frac{1}{\theta} \sum_{j=1}^{\min(m, d_2+1)} \gamma_j (z_j - z_{j-1}) \right\}.
\]

The above expressions yield the MLEs for \( \mu \) and \( \vartheta \).

**Lemma 7.2.5** The likelihood function \( \mathcal{L}(\cdot | z_d) \) for \( \mu \) and \( \vartheta \) is given by
\[
\mathcal{L}(\mu, \vartheta | z_d) = \begin{cases} 
\mathbf{1}_{z_{d_1}}(z_{d_1}) f_{1 \ldots m}^z(z_{d_1}), & d_1 = \tilde{m}, \\
\mathbf{1}_{z_{d_2}}(z_{d_2}) \frac{1 - F(T_1)}{\gamma_1 + \sum_{j=1}^{d_2} \gamma_j} f_1^{z_{d_2}}(z_{d_2}, T_1), & d_1 \in \{m, \ldots, \tilde{m} - 1\}, \\
\mathbf{1}_{z_{d_2}}(z_{d_2}) \frac{1 - F(T_2)}{\gamma_{d_2+1}} f_1 z_{d_2}(z_{d_2}, T_2), & d_2 \in \{1, \ldots, m - 1\}.
\end{cases}
\]

Let \( \mu \) be known. Then, the MLE \( \hat{\vartheta} \) is given by
\[
\hat{\vartheta} = \begin{cases} 
\frac{1}{D_1} \sum_{j=1}^{d_1} \left( 1 - \frac{\gamma_j}{\gamma_{j+1}} \right) \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}) + \gamma_{D_1+1} (T_1 - \mu), & D_1 \in \{m, \ldots, \tilde{m}\}, \\
\frac{1}{m} \sum_{j=1}^{d_1} \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}), & D_1 < m, D_2 = m, \\
\frac{1}{D_2} \sum_{j=1}^{d_2} \left( 1 - \frac{\gamma_j}{\gamma_{j+1}} \right) \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}) + \gamma_{D_2+1} (T_2 - \mu), & D_2 \in \{1, \ldots, m - 1\}.
\end{cases}
\]

The MLE \( \hat{\vartheta} \) does not exist for \( D_2 = 0 \).

For an unknown location parameter \( \mu \), the MLEs \( \hat{\mu} \) and \( \hat{\vartheta} \) are given by
\[
\hat{\mu} = Z_1^{GII}
\]
and
\[
\hat{\vartheta} = \begin{cases} 
\frac{1}{D_1} \sum_{j=1}^{d_1} \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}) + \gamma_{D_1+1} (T_1 - Z_{D_1}^{GII}), & D_1 \in \{m, \ldots, \tilde{m}\}, \\
\frac{1}{m} \sum_{j=2}^{d_2} \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}), & D_1 < m, D_2 = m, \\
\frac{1}{D_2} \sum_{j=2}^{d_2} \gamma_j (Z_j^{GII} - Z_{j-1}^{GII}) + \gamma_{D_2+1} (T_2 - Z_{D_2}^{GII}), & D_2 \in \{1, \ldots, m - 1\},
\end{cases}
\]
respectively. It should be noted that the MLE \( \hat{\vartheta} \) equals \( \gamma_2 (T_2 - Z_1^{GII}) \), for \( D_2 = 1 \). In this particular case the joint distribution of \( \hat{\mu} \) and \( \hat{\vartheta} \) is degenerated (see also Remark 4.2.4, (ii)). Therefore, we impose the condition by \( D_2 \geq 2 \).
Though it is not necessary to determine the distribution of the spacings in order to derive the density function of \( \hat{\vartheta} \), we provide the respective density function for the new counter setting \( D_1 < m, D_2 = m \). The normalized spacings based on generalized Type-II progressive hybrid censored random variables are defined by

\[
W_j^{\text{GII}} = \begin{cases} 
\gamma_j \left( Z_j^{\text{GII}} - Z_{j-1}^{\text{GII}} \right), & j \in \{1, \ldots, m\}, \\
\gamma_j \left( Z_j^{\text{GII}} - Z_{j-1}^{\text{GII}} \right), & j \in \{m + 1, \ldots, \tilde{m}\},
\end{cases}
\]

where \( Z_0^{\text{GII}} := \mu \). The next lemma follows by taking into account the representation given in Remark 7.2.3 and by using Lemma 4.2.5.

**Lemma 7.2.6** For \( d_1 < m, d_2 = m \), the conditional joint density function of the spacings \( W_1^{\text{GII}}, \ldots, W_m^{\text{GII}} \) is given by

\[
f_{W_1^{\text{GII}}, \ldots, W_m^{\text{GII}}} = 1_{W_m(T_2 - \mu|\gamma_m), W_m(T_1 - \mu|\gamma_m)}(w_m) \times \frac{1}{F_{m:m:n}(T_2) - F_{m:m:n}(T_1)} \prod_{j=1}^m \frac{1}{\vartheta} \exp \left\{ -\frac{w_j}{\vartheta} \right\}.
\]

Let the parameter \( \mu \) be known. Then, the conditional distribution of the total time on test statistic

\[
S_D = \begin{cases} 
\sum_{j=1}^{m} \left( 1 - \frac{\gamma_j \gamma_{D_1 + 1}}{\gamma_j} \right) \gamma_j (Z_j^{\text{GII}} - Z_{j-1}^{\text{GII}}) + \gamma_{D_1 + 1} (T_1 - \mu), & D_1 \in \{m, \ldots, \tilde{m}\}, \\
\sum_{j=1}^{D_2} \left( 1 - \frac{\gamma_j \gamma_{D_1 + 1}}{\gamma_j} \right) \gamma_j (Z_j^{\text{GII}} - Z_{j-1}^{\text{GII}}) + \gamma_{D_2 + 1} (T_2 - \mu), & D_2 \in \{1, \ldots, m - 1\},
\end{cases}
\]

where obviously \( S_D = D \hat{\vartheta} \), for \( D \in \{D_1, m, D_2\} \), follows immediately.

**Theorem 7.2.7** For \( d \in \{m, \ldots, m - 1\} \), the density function \( f_{S_D | D_1 = d_1} \) is given by

\[
f_{S_D | D_1 = d_1}(s) = \frac{(T_1 - \mu)^{d_1} \prod_{j=1}^{d_1 + 1} \gamma_j \gamma_{D_1 + 1} \gamma_{D_2 + 1}}{d_1! \vartheta^{d_1 + 1} f_{\tilde{m}:m:n}(T) B_{d_1 - 1}(s|\gamma \gamma_{D_1 + 1} (T_1 - \mu))} e^{-s/\vartheta}, \quad s \geq 0.
\]

Further, for \( d_1 = \tilde{m} \), we have

\[
f_{S_{\tilde{m}} | D_1 = \tilde{m}}(s) = \frac{(T_1 - \mu)^{\tilde{m}} \prod_{j=1}^{\tilde{m}} \gamma_j \gamma_{D_1 + 1}}{\tilde{m}! \vartheta^{\tilde{m}} f_{\tilde{m}:m:n}(T_1)} B_{\tilde{m} - 1}(s|0, \gamma \gamma_{D_1 + 1} (T_1 - \mu)) e^{-s/\vartheta}, \quad s \geq 0.
\]

For \( d_1 < m, d_2 = m \), we get

\[
f_{S_m | D_1 < m, D_2 = m}(s) = \prod_{j=1}^m \gamma_j \frac{m! \vartheta^m (F_{m:m:n}(T_2) - F_{m:m:n}(T_1))}{e^{-s/\vartheta}}
\]

\[
\times \left[ (T_2 - \mu)^m B_{m-1}(s|0, \gamma_{D_2 + 1} (T_2 - \mu)) - (T_1 - \mu)^m B_{m-1}(s|0, \gamma_{D_1 + 1} (T_1 - \mu)) \right], \quad s \geq 0.
\]
Finally, we get for $d_2 \in \{1, \ldots, m-1\}$,

$$f_{S_{d_2} \mid D_2 = d_2}(s) = \frac{(T_2 - \mu)^{d_2}}{d_2!} \prod_{j=1}^{d_2+1} \gamma_j \frac{(d_2+1)!}{(d_2+1)!} f_{d_2+1:1:m:n}(T_1) B_{d_2+1}(s) \gamma_{d_2+1}(T_2 - \mu) e^{-s/\vartheta}, \quad s \geq 0.$$  

**Proof.** Due to Remark 7.2.3, we find that

$$f_{S_m \mid D_1 < m, D_2 = m}(s) = c(T_2) f_{S_m \mid D_2 = m}(s) - c(T_1) f_{S_m \mid D_1 = m}(s), \quad s \geq 0,$$

where $S_m = \sum_{j=1}^{m} \gamma_j(Z_j^{\vartheta} - Z_{j-1:m:n}^{\vartheta})$. Then, the assertion follows with the expression for $f_{S_m \mid D_1 = m}$, $i \in \{1, 2\}$, as provided in Theorem 7.1.5. The remaining expressions can be obtained with Theorem 4.2.9.

These findings lead us to the distribution of $\hat{\vartheta}$. While the first expression for $f_{\hat{\vartheta} \mid D_2 \geq 1}$ (see (i)) follows from Theorem 7.2.7, the second representation given in (ii) follows with Theorem 4.3.1.

**Theorem 7.2.8** Let $\hat{\vartheta}$ be the MLE for $\vartheta$ under generalized Type-II progressive hybrid censoring.

(i) The conditional density function $f_{\hat{\vartheta} \mid D_2 \geq 1}$ is given by

$$f_{\hat{\vartheta} \mid D_2 \geq 1}(s) = \frac{1}{1 - e^{-m(T_2 - \mu)/\vartheta}} \times 
\left[ \sum_{d_1=m}^{m} \frac{(T_1 - \mu)^{d_1}}{(d_1-1)!} \frac{\prod_{j=1}^{d_1} \gamma_j^{\vartheta/m}}{d_1!} B_{d_1}(d_1s) \gamma_{d_1+1}(T_1 - \mu) e^{-d_1 s/\vartheta} \right] 
+ \frac{e^{-ms/\vartheta}}{(m-1)!} \frac{\prod_{j=1}^{m} \gamma_j}{\vartheta^m} \times 
\left[ (T_2 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_2 - \mu)) - (T_1 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_2 - \mu)) \right] 
+ \sum_{d_2=1}^{m-1} \frac{(T_2 - \mu)^{d_2}}{(d_2-1)!} \frac{\prod_{j=1}^{d_2} \gamma_j}{d_2!} B_{d_2-1}(d_2s) \gamma_{d_2+1}(T_2 - \mu) e^{-d_2 s/\vartheta}, \quad s \geq 0.$$  

(ii) Further, an expression for $f_{\hat{\vartheta} \mid D_2 \geq 1}$ in terms of gamma density functions is for $s \geq 0$ given by

$$f_{\hat{\vartheta} \mid D_2 \geq 1}(s) = \frac{1}{1 - e^{-m(T_2 - \mu)/\vartheta}} \times 
\left[ \sum_{d_1=m}^{m} \left( \prod_{j=1}^{d_1} \gamma_j^{\vartheta/m} \right) \sum_{j=0}^{d_1} \frac{e^{\gamma_{d_1-j+1}(T_1 - \mu)/\vartheta}}{(d_1-j)!} \frac{\prod_{i=0, i \neq j}^{d_1} \gamma_{d_1-i+1} - \gamma_{d_1-j+1}}{\vartheta^{d_1}} f_{\hat{\vartheta}} \frac{\vartheta}{d_1} \bigg( s - \frac{\gamma_{d_1-j+1}(T_1 - \mu)}{d_1} \bigg) \right] 
+ \left( \prod_{j=1}^{m} \gamma_j \right) \left( \sum_{j=0}^{m} \frac{e^{\gamma_{m-j+1}(T_2 - \mu)/\vartheta}}{(m-j)!} \frac{\prod_{i=0, i \neq j}^{m} \gamma_{m-i+1} - \gamma_{m-j+1}}{m} f_{\hat{\vartheta}} \frac{\vartheta}{m} \bigg( s - \frac{\gamma_{m-j+1}(T_2 - \mu)}{m} \bigg) \right)$$  

where $\gamma_{d_1-j+1}(T_1 - \mu)$ is the gamma density function. The remaining expressions can be obtained with Theorem 4.2.9.
Further, for generalised type-II hybrid censoring.

Theorem 7.2.9 For \( d_1 \in \{ m, \ldots, \tilde{m} - 1 \} \) and \( 0 \leq w < n(T_1 - \mu) \), \( v \geq 0 \), the conditional joint density function of \( W_1^{\text{GII}} \) and \( V_d \) is given by

\[
    f_{W_1^{\text{GII}} \mid V_d = d_1}(w, v) = \frac{(T_1 - \mu - \frac{w}{n})^{d_1 - 1} \prod_{j=2}^{d_1 + 1} \gamma_j^{\bar{p}_j \bar{m}}}{(d_1 - 1)! \theta^{d_1 + 1} f_{d_1 + 1; m; n}(T_1)} \times B_{d_1-2} \left( v \left| \frac{\bar{p}_2 \bar{m}}{d_1+1} (T_1 - \mu - \frac{w}{n}) \right. \right) e^{-(w+v)/\theta}.
\]

Further, for \( d_1 = \tilde{m} \) and \( 0 \leq w < n(T_1 - \mu) \), \( v \geq 0 \),

\[
    f_{W_1^{\text{GII}} \mid V_d \mid d_1 = \tilde{m}}(w, v) = \frac{(T_1 - \mu - \frac{w}{n})^{\tilde{m} - 1} \prod_{j=2}^{\tilde{m}} \gamma_j^{\bar{p}_j \bar{m}}}{(\tilde{m} - 1)! \theta^{\tilde{m}} F_{m; \tilde{m}; n}(T_1)}.
\]
For \( d_1 < m, d_2 = m \), we get
\[
f_{W_1^{G_n}, W_m | D_1 < m, D_2 = m}(w, v) = e^{-(w+v)/\vartheta} \prod_{j=2}^{m} \gamma_j \frac{1}{(m-1)! \vartheta^m (F_{m:m:n}^{\#}(T_2) - F_{m:m:n}^{\#}(T_1))} \times \left[ 1_{[0, n(T_2 - \mu))}(w) (T_2 - \mu - \frac{w}{n})^{m-1} B_{m-2}(v | 0, \gamma_{2,m}, (T_2 - \mu - \frac{w}{n})) \right. \\
- \left. 1_{[0, n(T_1 - \mu))}(w) (T_1 - \mu - \frac{w}{n})^{m-1} B_{m-2}(v | 0, \gamma_{2,m}, (T_1 - \mu - \frac{w}{n})) \right], \quad v \geq 0.
\]

Finally, we obtain for \( d_2 \in \{2, \ldots, m-1\} \) and with \( 0 \leq w < n(T_2 - \mu), v \geq 0 \),
\[
f_{W_1^{G_n}, W_{d_2} | D_1 = d_2, D_2 = d_2}(w, v) = \frac{(T_2 - \mu - \frac{w}{n})^{d_2-1} \prod_{j=2}^{d_2} \gamma_j}{(d_2-1)! \vartheta^{d_2+1} f_{d_2+1:m:n}^\#(T_2)} \times B_{d_2-2}(v | \gamma_{2,d_2+1}, (T_2 - \mu - \frac{w}{n})) e^{-(w+v)/\vartheta}.
\]

The above result yields the density function \( f^{\hat{\mu}, \hat{\vartheta}} | D_2 \geq 2 \).

**Theorem 7.2.10** Let \( m \geq 2 \). The conditional joint density function for the MLEs \( \hat{\mu} \) and \( \hat{\vartheta} \), for \( t \geq 0 \), is given by
\[
f_{\hat{\mu}, \hat{\vartheta} | D_2 \geq 2}(z, t) = \frac{n}{P(D_2 \geq 2)} \times \\
\left[ 1_{[\mu, T_1]}(z) \sum_{d_1 = m}^{m} \frac{d_1(T_1 - z)^{d_1-1} \prod_{j=2}^{d_1} \gamma_j \vartheta^{d_1}}{(d_1-1)! \vartheta^{d_1}} B_{d_1-2}(d_1 | \gamma_{2,d_1+1}, (T_1 - z)) e^{-n(z - \mu)/d_1 \vartheta} \right. \\
+ \frac{e^{-n(z - \mu)/d_1 \vartheta} m \prod_{j=2}^{m} \gamma_j}{(m-1)! \vartheta^m} \left. \times \left[ 1_{[0, T_2]}(z)(T_2 - z)^m B_{m-2}(mt | 0, \gamma_{2,m}, (T_2 - z)) - 1_{[0, T_1]}(z)(T_1 - z)^m B_{m-2}(mt | 0, \gamma_{2,m}, (T_1 - z)) \right] \right. \\
+ \left. 1_{[\mu, T_2]}(z) \sum_{d_2 = 2}^{m-1} \frac{d_2(T_2 - z)^{d_2-1} \prod_{j=2}^{d_2} \gamma_j}{(d_2-1)! \vartheta^{d_1}} B_{d_2-2}(d_2 | \gamma_{2,d_2+1}, (T_2 - z)) e^{-n(z - \mu)/d_2 \vartheta} \right],
\]

with
\[
P(D_2 \geq 2) = 1 - \left( e^{-n(T - \mu)/\vartheta} + \frac{n}{\gamma_2 - n} \left( e^{-n(T - \mu)/\vartheta} - e^{-\gamma_2(T - \mu)/\vartheta} \right) \right).
\]

Figure 7.6 depicts the shapes of the density function given in Theorem 7.2.10 for two censoring plans.

**Remark 7.2.11** It should be noted that the expressions provided in Theorems 7.1.6, (ii) and 7.2.8, (ii), are more compact than those established in Cho et al. (2015b, Theorem 2) and Lee et al. (2016, Theorem 2), respectively.
7.2 Generalized Type-II Hybrid censoring

Figure 7.6: Plots of $f_{\hat{\mu}, \hat{\theta} | D_2 \geq 2}$ for $n = 10$, $m = 5$, $T_1 = 1$, $T_2 = 1.5$ $\vartheta = 1$ and $\mu = 0.1$. Left: $\mathcal{R} = (0^1, 5)$. Right: $\mathcal{R} = (1^{15})$.

7.2.3 Generalized Type-II Hybrid Censoring from Uniform Distributions

For the IID uniform model (see Model 2.1.3), we consider the complete set of order statistics $U_{1:n}, \ldots, U_{m:n}, U_{m+1:n}, \ldots, U_{n:n}, m \leq n$. For the threshold times $T_1$ and $T_2$, with $T_1, T_2 \in (a, b)$, $T_1 < T_2$, we consider the stopping time $T_{\text{GII}}^2 = \min\{\max\{T_1, U_{m:n}\}, T_2\}$. The corresponding uniform generalized Type-II hybrid censored order statistics are then defined by

$$U_{\text{GII}}^j := \min\{U_{j:n}, T_2\}, \quad 1 \leq j \leq n.$$ 

The counter variables $D_i, i \in \{1, 2\}$, denoting the number of failures observed till time $T_i$, are given by $D_i = \sum_{j=1}^{n} 1_{[-\infty, T_i]}(U_{j:n}), i \in \{1, 2\}$.

According to Theorem 7.2.2 the conditional joint density functions of $U_{\text{GII}}^1, \ldots, U_{\text{GII}}^d$, $d \in \{1, \ldots, n\}$, are given as follows.

**Lemma 7.2.12** For $d_1 \in \{m, \ldots, n - 1\}$, we find

$$f_{U_{\text{GII}}^1, 1 \leq j \leq d_1 | D_1 = d_1}(u_{d_1}) = 1_{\Sigma_{F,T_1}^{d_1}}(u_{d_1}) \times \frac{n!}{(n - d_1 - 1)!(b - a)^{d_1+1}} f_{d_1+1:n}(T_1) \left(1 - \frac{T_1 - a}{b - a}\right)^{n-d_1-1}.$$ 

Further, for $d_1 = n$, we have

$$f_{U_{\text{GII}}^1, 1 \leq j \leq n | D_1 = n}(u_n) = 1_{\Sigma_{F,T_1}^{n}}(u_n) \frac{n!}{(b - a)^n} F_{m:n}(T_1).$$

For $d_1 < m$, $d_2 = m$, we get

$$f_{U_{\text{GII}}^1, 1 \leq j \leq m | D_1 < m, D_2 = m}(u_m) = 1_{\Sigma_{F,T_2}^{m} \setminus \Sigma_{F,T_1}^{m}}(u_m) \frac{n!}{(n - m)!((b - a)^m F_{m:n}(T_2) - F_{m:n}(T_1))}.$$
Finally, we obtain for \(d_2 \in \{1, \ldots, m - 1\},
\)

\[
f^{U_{ij}}_{d_2}(u_{d_2}) = \begin{cases} 
1 - \frac{u_{m} - a}{b - a} & b = 1, \\
1 - \frac{u_{m} - a}{b - a} - \frac{u_{m} - a}{b - a} & b = 2, \\
1 - \frac{u_{m} - a}{b - a} - \frac{u_{m} - a}{b - a} & b = 3,
\end{cases}
\]

\[
\times \left(1 - \frac{u_{m} - a}{b - a}\right)^{n-m}. 
\]

We proceed by presenting the corresponding likelihood function as well as the MLEs.

**Lemma 7.2.13** The likelihood function \(L(\cdot; u_d)\) for \(a\) and \(b\) is given by

\[
L(a, b|u_d) = \begin{cases} 
\frac{1 - F(T_1)}{(n - d_1)f(T_1)} f_{1_d_1+1:n}(u_{d_1}, T_1), & d_1 \in \{m, \ldots, n - 1\}, \\
\frac{1 - F(T_1)}{(n - d_n)f(T_1)} f_{1:m:n}(u_n), & d_n = n, \\
\frac{1 - F(T_2)}{(n - d_2)f(T_2)} f_{1_d_2+1:n}(u_{d_2}, T_2), & d_2 \in \{1, \ldots, m - 1\},
\end{cases}
\]

Let the parameter \(a\) be known. Then, the MLE for \(b\) is given by

\[
\hat{b} = \begin{cases} 
\frac{nt_1 - (n - D_1)a}{D_1}, & D_1 \in \{m, \ldots, n - 1\}, \\
U_{n:n}, & D_1 = n, \\
\frac{nt_2 - (n - D_2)a}{D_2}, & D_2 \in \{1, \ldots, m - 1\},
\end{cases}
\]

Note that \(\hat{b}\) does not exist for \(D_2 = 0\). For an unknown parameter \(a\), the MLEs for \(a\) and \(b\) are given by

\[
\hat{a} = U_{1:n} \quad \text{and} \quad \hat{b} = \begin{cases} 
\frac{nt_1 - (n - D_1)u_{1:n}}{D_1}, & D_1 \in \{m, \ldots, n - 1\}, \\
U_{n:n}, & D_1 = n, \\
\frac{nt_2 - (n - D_2)u_{1:n}}{D_2}, & D_2 \in \{1, \ldots, m - 1\},
\end{cases}
\]

Likelihood inference under generalized Type-II hybrid censoring can be entirely developed from the results presented for Type-II hybrid censoring (cf. Section 6.3).

**Theorem 7.2.14** The density function of the \(m\)th uniform generalized Type-II hybrid censored order statistic \(U_{m}^{GII}\) conditionally on \(D_1 < m\), \(D_2 = m\) is given by

\[
f^{U_{m}}_{D_1 < m, D_2 = m}(x) = \begin{cases} 
1(T_1, T_2)^2(x) \frac{f_{m:n}(x)}{F_{m:n}(T_2) - F_{m:n}(T_1)}, & x \in (\tau_{m,n}(T_2)^{-1}, \tau_{m,n}(T_1)^{-1}), \\
0, & x \notin (\tau_{m,n}(T_2)^{-1}, \tau_{m,n}(T_1)^{-1}).
\end{cases}
\]
The conditional cumulative distribution function of $\hat{b}$, under the assumption that the parameter $a$ is known, is given by the following Theorem.

**Theorem 7.2.15** Let $a < T < b$. Then, the conditional cumulative distribution function of the MLE $\hat{b}$ under generalized Type-II hybrid censoring is given by

$$F_{\hat{b}|D_{2} \geq 1}(y) = 1 - \frac{1}{1 - \left(\frac{b - T_2}{b - a}\right)^n} \times$$

$$\left[\sum_{d_1=m}^{n-1} 1_{[nT_1-(n-d_1)a,\infty]}(d_1 y) \frac{(b - T_1)f_{d_1+1:n}(T_1)}{(n - d_1)} + F_{m:n}(\min\{y, T_1\}) + F_{m:n}\left(\min\left\{\frac{my + (n - m)a}{n}, T_2\right\}\right) - F_{m:n}\left(\min\left\{\frac{my + (n - m)a}{n}, T_1\right\}\right)\right]$$

$$+ \sum_{d_2=1}^{m-1} 1_{[nT_2-(n-d_2)a,\infty]}(d_2 y) \frac{(b - T_2)f_{d_2+1:n}(T_2)}{(n - d_2)}, \quad y \geq 0.$$

Figure 7.7 depicts two plots of $F_{\hat{b}|D_{2} \geq 1}$.

**Theorem 7.2.16** The conditional joint density function $f_{U_1^{\text{GII}},U_2^{\text{GII}}|D_1 < m,D_2 < m}$ is given by

$$f_{U_1^{\text{GII}},U_2^{\text{GII}}|D_1 < m,D_2 < m}(u, x) = 1_{\{(u,x)\in(T_1,T_2)^2 \mid u < x\}}(u, x) \frac{f_{1:m:n}(u, x)}{F_{m:n}(T_2) - F_{m:n}(T_1)}.$$

For illustrations of the shape of $f_{U_1^{\text{GII}},U_2^{\text{GII}}|D_1 < m,D_2 < m}$, we refer to Figure 7.8 (see p. 164).
Figure 7.8: Plots of $f_{U_1,U_2}^{GII|D_1=m,D_2=m}$ for $n = 10$, $a = 1$, $b = 2$, $T_1 = 1.5$ and $T_2 = 1.75$. Left: $m = 7$. Right: $m = 3$. 
Chapter 8

Application: General Unified Progressive Hybrid Censoring

Park and Balakrishnan (2012) introduced a censoring model called 'a very flexible hybrid censoring scheme', which enables the experimenter to choose in addition to a value for the desired experiment duration also a lower as well as an upper bound. They showed how already known hybrid censoring models as the Type-I/II hybrid censoring schemes, the generalized Type-I/II hybrid censoring schemes and the unified Type-I/II hybrid censoring schemes, can be constructed using the flexible hybrid censoring methodology. Further, two new hybrid censoring models called unified Type-III hybrid censoring scheme and unified Type-IV hybrid censoring scheme (Type-III/IV UHCSs) were introduced. These two hybrid censoring models are very complex due to the large number of parameters used to specify the censoring procedure.

In order to emphasize the benefit of characterizing the structure of an underlying censoring model as a combination of three types of counter settings (cf. step 3 from Procedure 1.4.4 and Remark 1.4.5, (i)), we consider a more general and more complex new hybrid censoring model. This model is a generalization of a significant set of models addressed in this thesis. Further, it provides the experimenter with more options to overcome drawbacks such as long experiment duration and the possibility of not observing any failures.

This chapter serves as an example and as a roadmap for simplifying the structure of an underlying hybrid censoring model. This strategy can be applied to the four unified progressive hybrid censoring schemes (see Appendix A) which have been briefly addressed at the end of Section 1.1.4 (see p. 10). Notice that once the structure of the underlying hybrid censoring model has been simplified, the desired distributional results can be directly obtained by using already known results. In particular, the distribution theory for both the exponential and the uniform distribution follows with the results established within the scope of Type-I sequential hybrid censoring (cf. Chapter 4), Type-II progressive hybrid censoring (cf. Chapter 6) and generalized Type-II progressive hybrid censoring (cf. Section 7.2) (see also Sections 1.1.1 – 1.1.3).

We start by presenting the model of general unified progressive hybrid censoring and therefore perform the steps 1 – 6 from Procedure 1.4.4. Then, in Section 8.2, we address the general unified progressive hybrid censoring scheme for the exponential distribution, where the MLE for \( \vartheta \) as well as an expression for the corresponding density function is presented. Following that (see Section 8.3), we derive for the general unified hybrid censoring scheme with an underlying uniform distribution the distribution of the MLE \( \hat{\vartheta} \).
8.1 Model and Model-Simplification

Let $\mathcal{R}$ be the initially planned censoring plan. Then, we consider the following extended sample of progressively Type-II censored order statistics

$$X_{1;m,n}^{\varphi_k}, \ldots, X_{k;m,n}^{\varphi_k}, X_{m+1;m,n}^{\varphi_k}, \ldots, X_{m+m;n}^{\varphi_k}. \quad (8.1)$$

Further, let the integers $k, m, u \in \mathbb{N}$, with $1 \leq k < m < u \leq \tilde{m} \leq n$, as well as the threshold times $T_1, T_2, T_3 \in (0, \infty)$, with $T_1 < T_2 < T_3$, be given. Then, the stopping time $T^*_\text{GU}$, with

$$T^*_\text{GU} = \max \left\{ \max \{X_{k;m,n}^{\varphi_k}, \min \{T_3, X_{m;m,n}^{\varphi_k}\}\}, \min \{T_2, \max \{X_{u;m,n}^{\varphi_k}, T_1\}\} \right\},$$

defines the general unified progressive hybrid censoring scheme (GUPHCS). By using the approach of Park and Balakrishnan (2012), for designing hybrid censoring schemes, the general unified progressive hybrid censoring scheme can also be specified by using the following values for the lower bound, the censoring time and for the upper bound of the experiment:

- lower bound: $\max \{X_{k;m,n}^{\varphi_k}, \min \{T_3, X_{m;m,n}^{\varphi_k}\}\}$,
- censoring time: $T_2$,
- upper bound: $\max \{X_{u;m,n}^{\varphi_k}, T_1\}$.

The possible experimental outcomes for the general unified progressive hybrid censoring scheme are listed in Table 8.1. They can be obtained by verifying all possibilities of censoring the sample given in (8.1) according to the stopping time $T^*_\text{GU}$. There are 17 possibilities for terminating the experiment. However, consider, for instance, the cases 2, 5 and 10 from Table 8.1. Then, the sample $X_{1;n,n}, \ldots, X_{u;n,n}$ corresponds to three different counter settings. Now, the key idea of simplifying the experimental structure is to combine those three cases by considering the union of the domains of the respective counter variables.

Based on the stopping time $T^*_\text{GU}$ as well as on the experimental outcomes provided in Table 8.1, we define the general unified progressive hybrid censored order statistics as

$$X^*_j := \min \{ \max \{X^{\varphi_k}_{k;m,n}, T_3\}, X^{\varphi_k}_{j;m,n} \}, \quad 1 \leq j \leq \tilde{m}. \quad (8.2)$$

Assume the IID progressive model (see Model 2.2.5). Then, we determine the joint probabilities of the general unified progressive hybrid censored order statistics and the respective counter variables by considering the experimental outcomes listed in Table 8.1. This process involves the simplification of the experimental structure.

By starting with $d_1 \in \{u, \ldots, \tilde{m} - 1\}$ (see the first case from Table 8.1), we have

$$P(X^*_j \leq x_j, 1 \leq j \leq d_1, D_1 = d_1)$$

$$= P(\min \{ \max \{X^{\varphi_k}_{k;m,n}, T_3\}, X^{\varphi_k}_{j;m,n} \} \leq x_j, 1 \leq j \leq d_1, D_1 = d_1)$$

$$= P(\min \{T_3, X^{\varphi_k}_{j;m,n}\} \leq x_j, 1 \leq j \leq d_1, D_1 = d_1)$$

$$= P(X^{\varphi_k}_{j;m,n} \leq x_j, 1 \leq j \leq d_1, D_1 = d_1)$$

$$= F^\varphi_{1;\ldots;d_1;\tilde{m};n}(x_{d_1 - 1}; \min \{T_1, T_1\}) - F^\varphi_{1;\ldots;d_1+1;\tilde{m};n}(x_{d_1 - 1}; \min \{T_1, T_1\} ; T_1).$$

Further, for $d_1 = \tilde{m}$,

$$P(X^*_j \leq x_j, 1 \leq j \leq \tilde{m}, D_1 = \tilde{m}) = P(X^{\varphi_k}_{j;m,n} \leq x_j, 1 \leq j \leq \tilde{m}, D_1 = \tilde{m})$$
<table>
<thead>
<tr>
<th>case</th>
<th>ordered values</th>
<th>sample</th>
<th>relevant counters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td>2.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 = u$</td>
</tr>
<tr>
<td>3.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_1 \leq D_2$</td>
</tr>
<tr>
<td>4.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_1 \leq D_2$</td>
</tr>
<tr>
<td>5.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 = u$</td>
</tr>
<tr>
<td>6.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td>7.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td>8.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {k, \ldots, m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_3 = m, D_1 \leq D_2$</td>
</tr>
<tr>
<td>9.</td>
<td>$X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {k, \ldots, m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_1 \leq D_2 \leq D_3$</td>
</tr>
<tr>
<td>10.</td>
<td>$T_1 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 = u$</td>
</tr>
<tr>
<td>11.</td>
<td>$T_1 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td>12.</td>
<td>$T_1 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {m, \ldots, u-1}$</td>
</tr>
<tr>
<td>13.</td>
<td>$T_1 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {k, \ldots, m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_3 = m$</td>
</tr>
<tr>
<td>14.</td>
<td>$T_1 &lt; X_{k,m:n}^{---} &lt; T_2 &lt; T_3 &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {k, \ldots, m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \leq D_3$</td>
</tr>
<tr>
<td>15.</td>
<td>$T_1 &lt; T_2 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {0, \ldots, k-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_3 \in {k, \ldots, m-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_1 \leq D_2$</td>
</tr>
<tr>
<td>16.</td>
<td>$T_1 &lt; T_2 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_2 \in {0, \ldots, k-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$D_1 \leq D_2, D_3 = m$</td>
</tr>
<tr>
<td>17.</td>
<td>$T_1 &lt; T_2 &lt; T_3 &lt; X_{k,m:n}^{---} &lt; X_{m:n}^{---} &lt; X_{u,m:n}^{---}$</td>
<td>$T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{1:m}^{---}, \ldots, X_{D_1:m}^{---}$</td>
</tr>
</tbody>
</table>

**Table 8.1:** Experimental outcomes for the general unified progressive hybrid censoring scheme.
Now, consider the counter settings for the cases 2, 5 and 10 given in Table 8.1 as mentioned above, we get

\[
P(D_1 \in \{m, \ldots, u - 1\}, D_2 = u) + P(D_1 \in \{k, \ldots, m - 1\}, D_2 = u)
\]

This yields

\[
P(X_j^{GU} \leq x_j, 1 \leq j \leq u, D_1 \in \{m, \ldots, u - 1\}, D_2 = u)
\]

We state that by using the term 'combine' in the context of the experimental simplification process, we refer to the union of the domains of the respective counter variables. The counter settings for the cases 3, 4, 6, 7, 11 and 12, can be combined as follows

\[
P(D_1 \in \{m, \ldots, u - 1\}, D_2 \in \{m, \ldots, u - 1\}, D_1 \leq D_2)
\]

Notice that due to \(T_1 < T_2 < T_3\), the identities \(P(D_i \leq D_{i+1}) = 1, i \in \{1, 2\}\), and \(P(D_1 \leq D_2 \leq D_3) = 1\) hold. For \(d_2 \in \{m, \ldots, u - 1\}\), the above findings let us arrive at

\[
P(X_j^{GU} \leq x_j, 1 \leq j \leq d_2, D_1 \in \{m, \ldots, u - 1\}, D_2 = d_2, D_1 \leq D_2)
\]

We proceed by considering the cases 8, 13 and 16, so that

\[
P(D_1, D_2 \in \{k, \ldots, m - 1\}, D_3 = m, D_1 \leq D_2)
\]
Hence, we find by analogy with the case $D_1 < u$, $D_2 = u$, that

$$P(X_j^{GU} \leq x_j, 1 \leq j \leq m, D_1, D_2 \in \{k, \ldots, m-1\}, D_3 = m, D_1 \leq D_2)$$

$$+ P(X_j^{GU} \leq x_j, 1 \leq j \leq m, D_1 \in \{0, \ldots, k-1\}, D_2 \in \{k, \ldots, m-1\}, D_3 = m)$$

$$= P(X_j^{GU} \leq x_j, 1 \leq j \leq m, D_1, D_2 \in \{0, \ldots, k-1\}, D_3 = m, D_1 \leq D_2)$$

$$= P(D_2 < m, D_3 = m, D_1 \leq D_2)$$

$$= P(D_2 < m, D_3 = m).$$

The combination of the counter settings for the cases 9, 14 and 15, gives us

$$P(D_1, D_2, D_3 \in \{k, \ldots, m-1\}, D_1 \leq D_2 \leq D_3)$$

$$+ P(D_1 \in \{0, \ldots, k-1\}, D_2, D_3 \in \{k, \ldots, m-1\}, D_2 \leq D_3)$$

$$+ P(D_1, D_2 \in \{0, \ldots, k-1\}, D_3 \in \{k, \ldots, m-1\}, D_1 \leq D_3)$$

$$= P(D_1, D_2 \in \{0, \ldots, m-1\}, D_3 \in \{k, \ldots, m-1\}, D_1 \leq D_2 \leq D_3)$$

$$= P(D_3 \in \{k, \ldots, m-1\}).$$

For $d_3 \in \{k, \ldots, m-1\}$, the above calculations let us arrive at

$$P(X_j^{GU} \leq x_j, 1 \leq j \leq d_3, D_1, D_2, \in \{k, \ldots, m-1\}, D_3 = d_3, D_1 \leq D_2 \leq D_3)$$

$$+ P(X_j^{GU} \leq x_j, 1 \leq j \leq d_3, D_1 < k, D_2 \in \{k, \ldots, m-1\}, D_3 = d_3, D_2 \leq D_3)$$

$$+ P(X_j^{GU} \leq x_j, 1 \leq j \leq d_3, D_1, D_2 < k, D_3 = d_3, D_1 \leq D_3)$$

$$= P(X_j^{GU} \leq x_j, 1 \leq j \leq d_3, D_3 = d_3)$$

$$= P(X_{j,m:n}^{d} \leq x_j, 1 \leq j \leq d_3, D_3 = d_3)$$

$$= F_{1...m:n}(x_{d_3-1}, \min\{T_3, x_{d_3}\}) - F_{1...m:n+1}(x_{d_3-1}, \min\{T_3, x_{d_3}\}, T_3).$$

Finally, we find for the last case from Table 8.1 (i.e., $d_3 < k$), that

$$P(X_j^{GU} \leq x_j, 1 \leq j \leq k, D_3 < k) = P(X_j^{GU} \leq x_j, 1 \leq j \leq k, D_3 < k)$$

$$= P(\min\{X_{k,m:n}^{d}, X_{j,m:n}^{d}\} \leq x_j, 1 \leq j \leq k, D_3 < k)$$

$$= P(X_{j,m:n}^{d} \leq x_j, 1 \leq j \leq k, D_3 < k)$$

$$= P(X_{j,k:n}^{d} \leq x_j, 1 \leq j \leq k, D_3 < k)$$

$$= F_{1...k:n}(x_k) - F_{1...k:n+1}(x_{k-1}, \min\{x_k, T_3\}).$$

Note that the cases $d_1 \in \{u, \ldots, \tilde{m} - 1\}$ and $d_1 = \tilde{m}$ will be treated as one case. Figure
8.1 illustrates the censoring mechanism of the general unified progressive hybrid censoring model in the simplified version. The system of counter settings $\mathcal{S}_D(GU)$ and the corresponding family of sets of valid integers $\mathcal{I}(\mathcal{S}_D(GU))$ can now be specified as

$$\mathcal{S}_D(GU) = \{[D_1 = \cdot], [D_1 \leq \cdot, D_2 = \cdot], [D_2 < \cdot, D_3 = \cdot], [D_2 \leq \cdot, D_3 = \cdot], [D_3 < \cdot]\},$$

and

$$\mathcal{I}(\mathcal{S}_D(GU)) = \{\{u, \ldots, \tilde{m}\}, \{u\}, \{m, \ldots, u - 1\}, \{m\}, \{k, \ldots, m - 1\}, \{k\}\},$$

respectively. The occurring counter settings are all of the types specified in (1.15) (see p. 14). Hence, the entire distribution theory for the general unified progressive hybrid censoring model can be derived by using the results established in the previously addressed hybrid censoring models. We start with the distribution of the counter variables.

**Lemma 8.1.1** Let $f(T_i) > 0$, $i \in \{1, 2, 3\}$. For $d_1 = \tilde{m}$, we have

$$P(D_1 = \tilde{m}) = F_{\tilde{m};\tilde{m};\tilde{m}}(T_1).$$

Further for $d_1 \in \{u, \ldots, \tilde{m} - 1\}$, the probability of $D_1 = d_1$, is given by

$$P(D_1 = d_1) = \frac{1 - F(T_1)}{\gamma_{d_1+1}} f_{d_1+1;\tilde{m};\tilde{m}}(T_1).$$

---

**Figure 8.1:** Simplified structure of the general unified progressive hybrid censoring scheme.
Let \( d_1 < u, d_2 = u \), then
\[
P(D_1 < u, D_2 = u) = F_{u;u;u}^{p_u}(T_2) - F_{u;u;u}^{p_u}(T_1).
\]
For \( d_2 \in \{m, \ldots, u - 1\} \), we obtain
\[
P(D_2 = d_2) = \frac{1 - F(T_2)}{\sum_{d_2+1}^{u} f(T_2)} f_{d_2+1;u;u}^{p_u}(T_2).
\]
For \( d_2 < m, d_3 = m \), we have
\[
P(D_2 < m, D_3 = m) = F_{m;m;m}^{\mathfrak{R}}(T_3) - F_{m;m;m}^{\mathfrak{R}}(T_2).
\]
For \( d_3 \in \{k, \ldots, m - 1\} \), we get
\[
P(D_3 = d_3) = \frac{1 - F(T_3)}{\sum_{d_3+1}^{k} f(T_3)} f_{d_3+1;k;n}^{p_{\mathfrak{R}e}}(T_3).
\]
Finally, we have for \( d_3 < k \),
\[
P(D_3 < k) = 1 - F_{k;k;n}^{p_{\mathfrak{R}e}}(T_3).
\]
The above result leads us immediately to the conditional joint distribution of the general unified progressive hybrid censored order statistics \( X_1^{GU}, \ldots, X_d^{GU}, d \in \{1, \ldots, \tilde{m}\} \).

**Lemma 8.1.2** Let \( f(T_i) > 0, i \in \{1,2,3\} \). Then, we have for \( d_1 = \tilde{m} \)
\[
f_{X_1^{GU},1 \leq j \leq \tilde{m} | D_1 = \tilde{m}}^{m}(x_{\tilde{m}}) = \frac{1_{\Sigma^{m}_F{T_2}}(x_{\tilde{m}})}{F_{m;m;n}(T_1)}.
\]
For \( d_1 \in \{u, \ldots, \tilde{m} - 1\} \), the conditional joint density function is given by
\[
f_{X_1^{GU},1 \leq j \leq d_1 | D_1 = d_1}^{m}(x_{d_1}) = \frac{1_{\Sigma^{d_1}_F{T_2}}(x_{d_1})}{f_{d_1;m;n}(T_1)}.
\]
For \( d_1 < u, d_2 = u \), we obtain
\[
f_{X_1^{GU},1 \leq j \leq u | D_1 = u, D_2 = u}^{m}(x_u) = \frac{1_{\Sigma^{u}_F{T_2} \setminus \Sigma^{u}_F{T_1}}(x_u)}{f_{u;u;u}^{p_u}(T_2) - f_{u;u;u}^{p_u}(T_1)}.
\]
Further, we have for \( d_2 \in \{m, \ldots, u - 1\} \),
\[
f_{X_1^{GU},1 \leq j \leq d_2 | D_2 = d_2}^{m}(x_{d_2}) = \frac{1_{\Sigma^{d_2}_F{T_2}}(x_{d_2})}{f_{d_2;u;u}^{p_u}(T_2)}.
\]
Let \( d_2 < m, d_3 = m \). Then, we arrive at
\[
f_{X_1^{GU},1 \leq j \leq m | D_2 < m, D_3 = m}^{m}(x_{m}) = \frac{1_{\Sigma^{m}_F{T_2} \setminus \Sigma^{m}_F{T_1}}(x_{m})}{f_{m;m;m}^{p_{\mathfrak{R}e}}(T_3) - F_{m;m;m}^{p_{\mathfrak{R}e}}(T_2)}.
\]
For \( d_3 \in \{k, \ldots, m - 1\} \), we get
\[
f_{X_1^{GU},1 \leq j \leq d_3 | D_3 = d_3}^{m}(x_{d_3}) = \frac{1_{\Sigma^{d_3}_F{T_3}}(x_{d_3})}{f_{d_3;m;m}^{p_{\mathfrak{R}e}}(T_3)}.
\]
Finally, we have for \( d_3 < k \),
\[
f_{X_1^{GU},1 \leq j \leq k | D_3 < k}^{m}(x_{k}) = \frac{1_{\Sigma^{k}_F{T_3}}(x_{k})}{1 - F_{k;k;n}^{p_{\mathfrak{R}e}}(T_3)}.
\]
Chapter 8 Application: General Unified Progressive Hybrid Censoring

Table 8.2: Sampling situations for the general unified progressive hybrid censoring scheme after the simplification process.

<table>
<thead>
<tr>
<th>Case</th>
<th>Ordered Values</th>
<th>Sample</th>
<th>Relevant Counters</th>
<th>Dist. References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $X_{1,m:n}^{\text{GU}} &lt; X_{m,m:n} &lt; X_{u,m:n} &lt; T_1$</td>
<td>$X_{1,m:n}^{\text{GU}}, \ldots, X_{D_{1,m:n}}^{\text{GU}}$</td>
<td>$D_1 \in {u, \ldots, \tilde{m}}$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(4.25) $\hat{b}$ (4.53)</td>
</tr>
<tr>
<td>2. $X_{m,m:n} &lt; T_1 &lt; X_{u,m:n} &lt; T_2$</td>
<td>$X_{1,u:m:n}^{\text{GU}}, \ldots, X_{D_{2,u:m:n}}^{\text{GU}}$</td>
<td>$D_1 &lt; u, D_2 = u$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(7.4) $\hat{b}$ (7.7)</td>
</tr>
<tr>
<td>3. $X_{m,m:n} &lt; T_1 &lt; T_2 &lt; X_{u,m:n} &lt; T_3$</td>
<td>$X_{1,u:m:n}^{\text{GU}}, \ldots, X_{D_{2,u:m:n}}^{\text{GU}}$</td>
<td>$D_2 \in {m, \ldots, u - 1}$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(4.25) $\hat{b}$ (4.53)</td>
</tr>
<tr>
<td>4. $X_{m,m:n} &lt; T_1 &lt; T_2 &lt; X_{u,m:n} &lt; T_3$</td>
<td>$X_{1,u:m:n}^{\text{GU}}, \ldots, X_{D_{2,u:m:n}}^{\text{GU}}$</td>
<td>$D_2 &lt; m, D_3 = m$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(7.4) $\hat{b}$ (7.7)</td>
</tr>
<tr>
<td>5. $X_{m,m:n} &lt; T_1 &lt; T_2 &lt; T_3 &lt; X_{u,m:n}$</td>
<td>$X_{1,u:m:n}^{\text{GU}}, \ldots, X_{D_{2,u:m:n}}^{\text{GU}}$</td>
<td>$D_3 \in {k, \ldots, m - 1}$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(4.25) $\hat{b}$ (4.53)</td>
</tr>
<tr>
<td>6. $T_3 &lt; X_{1,m:n}^{\text{GU}} &lt; X_{u,m:n} &lt; X_{m,m:n}^{\text{GU}}$</td>
<td>$X_{1,k:m:n}^{\text{GU}}, \ldots, X_{k,k:n}^{\text{GU}}$</td>
<td>$D_3 &lt; k$</td>
<td>$\hat{\mu}, \hat{\vartheta}$</td>
<td>(6.8) $\hat{b}$ (6.17)</td>
</tr>
</tbody>
</table>

Table 8.2 lists the possible experimental outcomes for the general unified progressive hybrid censoring model after the simplification process. The last two columns provide us with references to results established in this work, which allow the derivation of the distribution of the MLEs for the exponential and for the uniform distribution under general unified (progressive) hybrid censoring.

8.2 Distribution Theory for the Exponential Distribution

Let us assume the IID progressive exponential model (see Model 2.2.6). Further, let $Z_1^{\text{GU}}, \ldots, Z_d^{\text{GU}}, d \in \{1, \ldots, \tilde{m}\}$, denote the general unified progressive hybrid censored order statistics from the exponential distribution. Then, the likelihood function as well as the MLE for $\vartheta$ under general unified progressive hybrid censoring are given as follows.

**Lemma 8.2.1** The likelihood function $L(\mu, \vartheta | z_d)$ for $\mu$ and $\vartheta$ is given by

\[
L(\mu, \vartheta | z_d) = \begin{cases} 
  \frac{1 - F(T_1)}{\gamma \tilde{m} + 1} f_{1, \ldots, \tilde{m} + 1}^{\text{GU}}(z_{\tilde{m}}, 1, \ldots, T_1) \mathbf{1}_{\Sigma_{T_1}^{\mu}}(z_{\tilde{m}}), & d_1 = \tilde{m}, \\
  \frac{1 - F(T_1)}{\gamma \tilde{m} + 1} f_{1, \ldots, \tilde{m} + 1}^{\text{GU}}(z_{d_1}, 1, T_1) \mathbf{1}_{\Sigma^{\vartheta}_{T_1}}(z_{d_1}), & d_1 \in \{u, \ldots, \tilde{m} - 1\}, \\
  \frac{1 - F(T_2)}{\gamma u + 1} f_{1, \ldots, u + 1}^{\text{GU}}(z_u, 1, \ldots, T_2) \mathbf{1}_{\Sigma^{\vartheta}_{T_2}}(z_u), & d_1 < u, d_2 = u, \\
  \frac{1 - F(T_2)}{\gamma u + 1} f_{1, \ldots, u + 1}^{\text{GU}}(z_{d_2}, 1, T_2) \mathbf{1}_{\Sigma^{\mu}_{T_2}}(z_{d_2}), & d_2 \in \{m, \ldots, u - 1\}, \\
  \frac{1 - F(T_3)}{\gamma \tilde{m} + 1} f_{1, \ldots, \tilde{m} + 1}^{\text{GU}}(z_{\tilde{m}}, 1, \ldots, T_3) \mathbf{1}_{\Sigma^{\mu}_{T_3}}(z_{\tilde{m}}), & d_2 < m, d_3 = m, \\
  \frac{1 - F(T_3)}{\gamma \tilde{m} + 1} f_{1, \ldots, \tilde{m} + 1}^{\text{GU}}(z_{d_3}, 1, T_3) \mathbf{1}_{\Sigma^{\vartheta}_{T_3}}(z_{d_3}), & d_3 \in \{k, \ldots, m - 1\}, \\
  \frac{1 - F(T_3)}{\gamma d_3 + 1} f_{1, \ldots, d_3 + 1 + \tilde{m}}^{\text{GU}}(z_{d_3}, 1, T_3) \mathbf{1}_{\Sigma^{\vartheta}_{T_3}}(z_{d_3}), & d_3 < k.
\end{cases}
\]
For a known location parameter \( \mu \), the MLE for \( \vartheta \) is given by

\[
\hat{\vartheta} = \left\{ \begin{array}{l}
\frac{1}{D_1} \sum_{j=1}^{D_1} \left( 1 - \frac{\gamma_j^{m \hat{D}_1+1}}{\gamma_j^{m \hat{D}_1}} \right) \gamma_j^{m \hat{D}_1} (Z_j^{GU} - Z_{j-1}^{GU}) + \gamma_j^{m \hat{D}_1+1}(T_1 - \mu), \quad D_1 \in \{u, \ldots, \hat{m}\}, \\
\frac{1}{m} \sum_{j=1}^{u} \gamma_j^{u} (Z_j^{GU} - Z_{j-1}^{GU}), \quad D_1 < u, D_2 = u, \\
\frac{1}{D_2} \sum_{j=1}^{D_2} \left( 1 - \frac{\gamma_j^{m \hat{D}_1+1}}{\gamma_j^{m \hat{D}_1}} \right) \gamma_j^{m \hat{D}_1} (Z_j^{GU} - Z_{j-1}^{GU}) + \gamma_j^{m \hat{D}_1+1}(T_2 - \mu), \quad D_2 \in \{m, \ldots, u - 1\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_j (Z_j^{GU} - Z_{j-1}^{GU}), \quad D_2 < m, D_3 = m, \\
\frac{1}{D_3} \sum_{j=1}^{D_3} \left( 1 - \frac{\gamma_j^{m \hat{D}_1+1}}{\gamma_j^{m \hat{D}_1}} \right) \gamma_j^{m \hat{D}_1} (Z_j^{GU} - Z_{j-1}^{GU}) + \gamma_j^{m \hat{D}_1+1}(T_3 - \mu), \quad D_3 \in \{k, \ldots, m - 1\}, \\
\frac{1}{k} \sum_{j=1}^{k} \gamma_j^{m \hat{D}_1} (Z_j^{GU} - Z_{j-1}^{GU}), \quad D_3 < k.
\end{array} \right.
\]

The information provided in Table 8.2 enables us to establish an expression for the density function of \( \hat{\vartheta} \) given in Lemma 8.2.1.

**Theorem 8.2.2** Let the parameter \( \mu \) be known. Then, the density function \( f_{\hat{\vartheta}}(s) \) of the MLE \( \hat{\vartheta} \) under general unified progressive hybrid censoring for \( s \geq 0 \), is given by

\[
f_{\hat{\vartheta}}(s) = \sum_{d_1 = u}^{\hat{m}} \frac{(T_1 - \mu)^{d_1} \prod_{j=1}^{d_1} \gamma_j^{m \hat{D}_1+1}}{(d_1 - 1)! \hat{d}_1} B_{d_1-1}(d_1 s | \gamma_j^{m \hat{D}_1+1}(T_1 - \mu)) e^{-d_1 s / \vartheta} \\
+ \frac{e^{-u / \vartheta} \prod_{j=1}^{u} \gamma_j^{m \hat{D}_1+1}}{(u - 1)! \hat{u}} \times \left[ (T_2 - \mu)^u B_{u-1}(us | 0, \gamma_j^{m \hat{D}_1+1}(T_2 - \mu)) - (T_1 - \mu)^u B_{u-1}(us | 0, \gamma_j^{m \hat{D}_1+1}(T_1 - \mu)) \right] \\
+ \frac{e^{-m s / \vartheta} \prod_{j=1}^{m} \gamma_j}{(m - 1)! \hat{m}} \times \left[ (T_3 - \mu)^m B_{m-1}(ms | 0, \gamma_j(T_3 - \mu)) - (T_2 - \mu)^m B_{m-1}(ms | 0, \gamma_j(T_2 - \mu)) \right] \\
+ \prod_{d_2 = m}^{m-1} \frac{(T_3 - \mu)^{d_2} \prod_{j=1}^{d_2} \gamma_j}{(d_2 - 1)! \hat{d}_2} B_{d_2-1}(d_2 s | \gamma_j^{m \hat{D}_1+1}(T_2 - \mu)) e^{-d_2 s / \vartheta} \\
+ \frac{f_{\hat{\vartheta}}(s) - (T_3 - \mu)^k \prod_{j=1}^{k} \gamma_j^{m \hat{D}_1+1}}{\vartheta(k - 1)!} B_{k-1}(ks | 0, \gamma_j^{m \hat{D}_1+1}(T_2 - \mu)) e^{-k s / \vartheta}.
\]

Figure 8.2 illustrates the shapes of the density function \( f_{\hat{\vartheta}} \) for four settings. Remark 8.2.3 considers some particular limits of \( f_{\hat{\vartheta}} \), which relates the density function of the MLE under general unified progressive hybrid censoring to the density functions in other (hybrid) censoring models.
Figure 8.2: Plots of $f^\hat{\vartheta}$ (solid line) for $n = 10$, $k = 3$, $m = 5$, $u = 6$, $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$ and $\mu = 0$. Left: $R = (0^4, 5)$. Right: $R = (1^5, 5)$. The dashed lines represent the density function of $\hat{\vartheta}$ under (progressive) Type-II censoring.

Remark 8.2.3 Let $f^\hat{\vartheta}$ denote the density function given in Theorem 8.2.2. Then, the following limits hold.

(i) Since $T_1 < T_2 < T_3$, $T_1 \to \infty$ implies that $T_2 \to \infty$ as well as $T_3 \to \infty$. Hence, we have

$$\lim_{T_1 \to \infty} f^\hat{\vartheta}(s) = f_{\frac{\vartheta}{\tilde{m}}}^\vartheta(s), \quad s \geq 0.$$ 

(ii) Let $T_2 \to \infty$ (and $T_3 \to \infty$). Then, we arrive at

$$\lim_{T_2 \to \infty} f^\hat{\vartheta}(s) = \sum_{d_1 = u}^{\tilde{m}} \frac{(T_1 - \mu)^{d_1} \prod_{j=1}^{d_1} \gamma_{\tilde{m}}^{\rho_{d_1}}}{(d_1 - 1)! d_1!} B_{d_1-1}(d_1 s | \gamma_{\tilde{m}}^{\rho_{d_1}} (T_1 - \mu)) e^{-d_1 s / \vartheta}$$

$$+ f_{\frac{\vartheta}{u}}^u(s) \frac{(T_1 - \mu)^{u} \prod_{j=1}^{u} \gamma_{u}^{\rho_{u}}}{(u - 1)! u!} B_{u-1}(us | 0, \gamma_{u}^{\rho_{u}} (T_1 - \mu)) e^{-us / \vartheta}, \quad s \geq 0.$$
It is obvious that the above limit corresponds to the density function of the MLE under Type-II progressive hybrid censoring (see Theorem 6.2.8, (i)) with \( m = u, \ T = T_1 \) and \( \mathcal{R} = \mathcal{R}^\circ u \).

(iii) For \( T_3 \to \infty \),
\[
\lim_{T_3 \to \infty} f^{\hat{\theta}}(s) = \sum_{d_1 = u}^{\bar{m}} \frac{(T_1 - \mu)^{d_1} \prod_{j=1}^{d_1} \gamma_j^{\mu_d} B_{d_1-1}(d_1 s|\gamma_{d_1+1}^{\mu}(T_1 - \mu))}{(d_1 - 1)!\theta^{d_1}} e^{-d_1 s/\theta} \\
+ \frac{e^{-ut/\theta} \prod_{j=1}^{u} \gamma_j^{\mu}}{(u - 1)!\theta^u} \left[ (T_2 - \mu)^u B_{u-1}(us|0, \gamma_u^{\mu}(T_2 - \mu)) - (T_1 - \mu)^u B_{u-1}(us|0, \gamma_u^{\mu}(T_1 - \mu)) \right] \\
+ \sum_{d_2 = m}^{u-1} \frac{(T_2 - \mu)^{d_2} \prod_{j=1}^{d_2} \gamma_j^{\mu}}{(d_2 - 1)!\theta^{d_2}} B_{d_2-1}(d_2 s|\gamma_{d_2+1}^{\mu}(T_2 - \mu)) e^{-d_2 s/\theta} \\
+ f^{\hat{\theta}}_{\frac{m}{m}}(s) - \frac{(T_2 - \mu)^m \prod_{j=1}^{m} \gamma_j^{\mu}}{(m - 1)!\theta^m} B_{m-1}(ms|0, \gamma_u^{\mu}(T_2 - \mu)) e^{-ms/\theta}, \quad s \geq 0,
\]
so that the density function under unified Type-II progressive hybrid censoring with parameters \( k = m, \ m = u \) and \( T_1 = T_i, \ i \in \{1, 2\} \) (see Theorem A.2.3) results.

(iv) For the limit w.r.t. \( T_2 \to \mu^+ \), we find for \( s \geq 0 \)
\[
\lim_{T_2 \to \mu^+} f^{\hat{\theta}}(s) = \sum_{d_3 = k}^{m} \frac{(T_3 - \mu)^{d_3} \prod_{j=1}^{d_3} \gamma_j^{\mu}}{(d_3 - 1)!\theta^{d_3}} B_{d_3-1}(d_3 s|\gamma_{d_3+1}^{\mu}(T_3 - \mu)) e^{-d_3 s/\theta} + f^{\hat{\theta}}_{\frac{m}{m}}(s) \\
- \frac{(T_3 - \mu)^k \prod_{j=1}^{k} \gamma_j^{\mu}}{\theta^k(k - 1)!} B_{k-1}(ks|0, \gamma_k^{\mu}(T_3 - \mu)) e^{-ks/\theta}.
\]
The right-hand side of the above equation corresponds to the density function of the MLE \( \hat{\theta} \) under generalized Type-I progressive hybrid censoring with \( T = T_3 \) (cf. Theorem 7.1.6, (i)).

(v) For \( T \to \mu^+ \), we obtain the density function of the unified Type-IV progressive hybrid censoring scheme, with \( T_i = T_{i+1}, \ i \in \{1, 2\} \), i.e. (see Theorem A.4.2),
\[
\lim_{T_3 \to \mu^+} f^{\hat{\theta}}(s) = \sum_{d_2 = m}^{u} \frac{(T_2 - \mu)^{d_2} \prod_{j=1}^{d_2} \gamma_j^{\mu}}{(d_2 - 1)!\theta^{d_2}} B_{d_2-1}(d_2 s|\gamma_{d_2+1}^{\mu}(T_2 - \mu)) e^{-d_2 s/\theta} \\
+ \frac{e^{-ms/\theta} \prod_{j=1}^{m} \gamma_j^{\mu}}{(m - 1)!\theta^m} \left[ (T_3 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_3 - \mu)) \\
- (T_2 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_2 - \mu)) \right] \\
+ \sum_{d_3 = k}^{m-1} \frac{(T_3 - \mu)^{d_3} \prod_{j=1}^{d_3} \gamma_j^{\mu}}{(d_3 - 1)!\theta^{d_3}} B_{d_3-1}(d_3 s|\gamma_{d_3+1}(T_3 - \mu)) e^{-d_3 s/\theta}.
\]
unified progressive hybrid censoring model is denoted by the
The MLE for
Notice that for the setting of ordinary order statistics as the underlying sample, the general
consider the total set of
Assume the IID uniform model (see Model
8.3 Distribution Theory for the Uniform Distribution
(vii) The expressions obtained in (i) – (vi) have in common that the resulting density functions correspond to censoring models where the observation of a prefixed quantity is guaranteed.

8.3 Distribution Theory for the Uniform Distribution
Assume the IID uniform model (see Model 2.1.3), so that \(T_1, T_2, T_3 \in (a, b)\). Hence, we consider the total set of \(n\) ordinary order statistics, i.e., \(U_{1:n}, \ldots, U_{m:n}, U_{m+1:n}, \ldots, U_{n:n}\). Notice that for the setting of ordinary order statistics as the underlying sample, the general unified progressive hybrid censoring model is denoted by the general unified censoring scheme (GUHC). Further, according to (8.2) the uniform general unified hybrid censored order statistics are defined as

\[ U_{j}^{GU} := \min \{ \max\{U_{k:n}, T_3\}, U_{j:n}\}, \quad 1 \leq j \leq n. \]

The MLE for \(b\) under the general unified hybrid censoring model is given as follows.

**Lemma 8.3.1** The likelihood function \(L(\cdot | u_d)\) for \(a\) and \(b\) is given by

\[
L(a, b | u_d) = \begin{cases} 
1_{\Sigma F, T_1}(u_{d_1}) \frac{1 - F(T_1)}{(n - d_1)f(T_1)} f_{1\ldots d_1+1:n}(u_{d_1}, T_1), & d_1 \in \{u, \ldots, n - 1\}, \\
1_{\Sigma F, T_1}(u_n) f_{1\ldots n:n}(u_n), & d_1 = n, \\
1_{\Sigma F, T_2 \setminus \Sigma F, T_1}(u_u) f_{1\ldots u:n}(u_u), & d_1 < u, d_2 = u, \\
1_{\Sigma F, T_3 \setminus \Sigma F, T_1}(u_m) f_{1\ldots m:n}(u_m), & d_2 < m, d_3 = m, \\
1_{\Sigma F, T_3 \setminus \Sigma F, T_2}(u_d) \frac{1 - F(T_3)}{(n - d_3)f(T_3)} f_{1\ldots d_3+1:n}(u_{d_3}, T_3), & d_3 \in \{k, \ldots, m - 1\}, \\
1_{\Sigma F, T_3 \setminus \Sigma F, T_2}(u_k) f_{1\ldots k:n}(u_k), & d_3 < k.
\end{cases}
\]
For a known parameter $a$, the MLE $\hat{b}$ is given by

$$
\hat{b} =\begin{cases}
\frac{nT_1 - (n-D_1)a}{D_1}, & D_1 \in \{u, \ldots, n - 1\}, \\
U_{n:n}, & D_1 = n, \\
\frac{nU_{u:n} - (n-u)a}{u}, & D_1 < u, D_2 = u, \\
\frac{nT_2 - (n-D_2)a}{D_2}, & D_2 \in \{m, \ldots, u - 1\}, \\
\frac{nU_{m:n} - (n-m)a}{m}, & D_2 < m, D_3 = m, \\
\frac{nT_3 - (n-D_3)a}{D_3}, & D_3 \in \{k, \ldots, m - 1\}, \\
\frac{nU_{k:n} - (n-k)a}{k}, & D_3 < k.
\end{cases}
$$

Then, the distribution of $\hat{b}$ follows together with Table 8.2.

**Theorem 8.3.2** Let $a < T_1 < T_2 < T_3 < b$. Then, the cumulative distribution function of the MLE $\hat{b}$ under general unified hybrid censoring is given by

$$
F_{\hat{b}}(y) = \sum_{d_1=u}^{n-1} 1_{[nT_1 - (n-d_1)a, \infty)}(d_1 y) \frac{(b - T_1) f_{d_1 + 1:n}(T_1)}{(n - d_1)} + F_{n:n}(\min\{y, T_1\})
$$

$$
+ \sum_{d_2=m}^{u-1} 1_{[nT_2 - (n-d_2)a, \infty)}(d_2 y) \frac{(b - T_2) f_{d_2 + 1:n}(T_2)}{(n - d_2)} + F_{u:n}(\min\{\frac{uy + (n-u)a}{n}, T_2\}) - F_{u:n}(\min\{\frac{uy + (n-u)a}{n}, T_1\})
$$

$$
+ \sum_{d_3=k}^{m-1} 1_{[nT_3 - (n-d_3)a, \infty)}(d_3 y) \frac{(b - T_3) f_{d_3 + 1:n}(T_3)}{(n - d_3)} + F_{m:n}(\min\{\frac{my + (n-m)a}{n}, T_3\}) - F_{m:n}(\min\{\frac{my + (n-m)a}{n}, T_2\})
$$

$$
+ \sum_{d_4=k}^{n-1} 1_{[nT_4 - (n-d_4)a, \infty)}(d_4 y) \frac{(b - T_4) f_{d_4 + 1:n}(T_4)}{(n - d_4)} + F_{k:n}(\min\{\frac{ky + (n-k)a}{n}, T_3\}) - F_{k:n}(\min\{\frac{ky + (n-k)a}{n}, T_2\})
$$

$$
, y \geq 0.
$$

Figure 8.3 depicts the plots of $F_{\hat{b}}$ for two scenarios (see p. 178).
Figure 8.3: Plots of the cumulative distribution function $F^{\hat{b}}$, for $n = 10$, $a = 1$, $b = 2$, $T_1 = 1.25$, $T_2 = 1.5$ and $T_3 = 1.75$. Left: $k = 4$, $m = 7$ and $u = 9$. Right: $k = 2$, $m = 5$ and $u = 7$. 
Conclusion and Outlook

In this thesis, the distribution theory of various hybrid censoring models from exponential and uniform distributions has been considered. The key tool for deriving the desired distribution of the MLEs was the application of one of the two herein proposed techniques leading to B-spline based representations.

The process of deriving B-spline based representations for particular density functions entailed first of all a new way of approaching hybrid censoring models. This new approach does not only simplify the analysis of new hybrid censoring models, it can also be used to derive simplified distributional results for well known and established censoring models.

The huge amount of theoretical results on B-splines facilitates the extension of already introduced models, the specification of new models as well as the access to new distributional setups. In particular: the effort of applying B-spline based approaches does not depend on the underlying structure of ordered random variables. Hence, without undertaking additional effort, the models can be extended to sequential order statistics or to progressively Type-II censored order statistics. Further, well known results for B-spline convolutions can be extended in order to access the distribution theory of multi-sample models. Finally, the connections between B-splines and the uniform distribution (see, e.g., Dahmen and Micchelli, 1986; Ignatov and Kaishev, 1989) encourage to consider the notion of hybrid censoring from uniform distributions.

Although, much effort has been done in this work to exploit the advantages of the B-splines as well as of the new approach of attacking hybrid censoring models, there remains much more to do. For instance, a more profound knowledge on B-splines, in particular on B-spline spaces (see, e.g., Strøm, 1994), may allow to derive more elegant expressions for the density functions for the MLE in the multi-sample exponential case. A next step, which could facilitate the analysis of hybrid censoring methodology, is the verification whether it is possible to determine the distribution of a class of hybrid censoring models, instead of deriving distributional results for each hybrid censoring scheme separately. In particular, one might consider for a set of threshold times and integers a class of hybrid censoring models, which can be characterized by a particular subset of the corresponding and admissible lattice polynomials. It seems further reasonable to verify whether the herein addressed classification into counter settings can be extended to other models, such as progressive Type-I censoring or step-stress related models.
Appendix A
Unified Progressive Hybrid Censoring Schemes

In Chapter 8, we have seen how a sophisticated hybrid censoring model can be simplified using the herein presented approach. In the following, we apply this approach to the unified Type-I/II/III/IV hybrid censoring models proposed in the literature (see Balakrishnan et al., 2008c; Huang and Yang, 2010; Park and Balakrishnan, 2012). Further, we consider extensions of these hybrid censoring schemes on progressively Type-II censored order statistics. For each censoring model the necessary information for simplifying the structure of the underlying censoring procedure is provided. Further, the MLEs as well as the respective density functions for the one parameter exponential case are derived. In addition, sufficient information is provided (as in Table 8.2) allowing the deduction of distributional results for the remaining distributional setups.

For information on the matter how particular hybrid censoring models can be obtained out of the models addressed in the following sections, we refer to Balakrishnan et al. (2008c) and Park and Balakrishnan (2012). Further, relations among the respective density functions can be established in the same manner as for general unified progressive hybrid censoring (cf. Remark 8.2.3).

Notice that in this chapter, the IID progressive model (see Model 2.2.5) as well as the IID progressive exponential model (see Model 2.2.6) are assumed.

A.1 Unified Type-I Progressive Hybrid Censoring

The unified Type-I hybrid censoring scheme (Type-I UHCS) was originally introduced by Huang and Yang (2010) who called the scheme combined hybrid censoring sampling. Then, the unified Type-I hybrid censoring scheme was addressed in Park and Balakrishnan (2012). While Huang and Yang (2010, pp. 358–360) derived the MLE as well as the corresponding moment generating function for an underlying one-parameter exponential distribution, Park and Balakrishnan (2012, Theorem 3.1) established a formula for the Fisher information w.r.t. a continuous density function with positive support.

We present with the unified Type-I progressive hybrid censoring scheme (Type-I UPHCS) an extension to progressively Type-II censored order statistics. The corresponding stopping time $T_{UI}^*$ can be expressed as (cf. Park and Balakrishnan, 2012, Lemma 2.1)

$$T_{UI}^* = \max \left\{ \min \{X_{k;m:n}, T_2\}, \min \{X_{m;m:n}, T_1\} \right\}, \quad 1 \leq k < m \leq n, \quad 0 < T_1 < T_2.$$ 

The possible experimental outcomes are listed in Table A.1. Further, we define the unified Type-I progressive hybrid censored order statistics by

$$X_j^{UI} := \min \{X_{j;m:n}, T_2\}, \quad 1 \leq j \leq m.$$
A.1 shows the occurring sampling situations after the simplification process. The integers for the unified Type-I progressive hybrid censoring scheme is given by the exponential and the uniform distribution.

Figure A.1 depicts the unified Type-I progressive hybrid censoring procedure based on the simplified case ordered values sample relevant counters distr. references

<table>
<thead>
<tr>
<th>case</th>
<th>ordered values sample relevant counters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$X_{k;m:n} &lt; X_{m;m:n} &lt; T_1 &lt; T_2$ $X_{k;m:n}, \ldots, X_{m;m:n}$ $D_1 = m$ $\hat{\mu}, \hat{\theta}$ (4.25) $\hat{a}, \hat{b}$ (4.53)</td>
</tr>
<tr>
<td>2.</td>
<td>$X_{k;m:n} &lt; T_1 &lt; X_{m;m:n} &lt; T_2$ $X_{1:m:n}, \ldots, X_{D_1;m:n}$ $D_1 \in {k, \ldots, m - 1}$ $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.56)</td>
</tr>
<tr>
<td>3.</td>
<td>$X_{k;m:n} &lt; T_1 &lt; T_2 &lt; X_{m;m:n}$ $X_{1:m:n}, \ldots, X_{D_2;m:n}$ $D_1 \in {k, \ldots, m - 1}$ $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.56)</td>
</tr>
<tr>
<td>4.</td>
<td>$T_1 &lt; X_{k;m:n} &lt; X_{m;m:n} &lt; T_2$ $X_{k;k:n}$, $X_{k;k:n}$ $D_1 \in {0, \ldots, k - 1}$ $\hat{\mu}, \hat{\theta}$ (4.53) $\hat{a}, \hat{b}$ (4.56)</td>
</tr>
<tr>
<td>5.</td>
<td>$T_1 &lt; X_{k;m:n} &lt; T_2 &lt; X_{m;m:n}$ $X_{k;k:n}$, $X_{k;k:n}$ $D_1 \in {0, \ldots, k - 1}$ $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.56)</td>
</tr>
<tr>
<td>6.</td>
<td>$T_1 &lt; T_2 &lt; X_{k;m:n} &lt; X_{m;m:n}$ $X_{k;k:n}$, $X_{k;k:n}$ $D_2 \in {0, \ldots, k - 1}$ $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.56)</td>
</tr>
</tbody>
</table>

We realize that the cases 2 and 3 as well as the cases 4 and 5 can be combined. Figure A.1 depicts the unified Type-I progressive hybrid censoring scheme based on the simplified sampling situations.

Table A.2 shows the occurring sampling situations after the simplification process. It provides additional information to derive the distributional results for the MLEs for both the exponential and the uniform distribution.

Further, the system of relevant counter settings as well as the family of sets of valid integers for the unified Type-I progressive hybrid censoring scheme is given by

$$\mathcal{S}_2(\text{UI}) = \{ [D_1 = \cdot], [D_1 < \cdot, D_2 = \cdot], [D_2 = \cdot] \},$$

and

$$\mathcal{I}(\mathcal{S}_2(\text{UI})) = \{ \{k, \ldots, m\}, \{k\}, \{1, \ldots, k - 1\} \},$$

respectively. Let $Z_{1}^{\text{UI}}, \ldots, Z_{d}^{\text{UI}}, d \in \{1, \ldots, m\}$, denote the unified Type-I progressive hybrid censored order statistics from the exponential distribution.
A.1 Unified Type-I Progressive Hybrid Censoring

Let the parameter $\vartheta$. Then, the MLE for $\vartheta$ is given by

$\hat{\vartheta} = \begin{cases} 
\frac{1}{D_1} \sum_{j=1}^{D_1} \left( 1 - \frac{\gamma_{D_1+1,j}}{\gamma_{D_1+1}} \right) \gamma_j (Z_{j-1}^U - Z_j^U) + \gamma_{D_1+1} (T_1 - \mu), & D_1 \in \{k, \ldots, m\}, \\
\frac{1}{k} \sum_{j=1}^{k} \frac{p_{D_1+1,j}}{p_{D_1+1}} (Z_{j-1}^U - Z_j^U), & D_1 < k, D_2 = k, \\
\frac{1}{D_2} \sum_{j=1}^{D_2} \left( 1 - \frac{p_{D_2+1,j}}{p_{D_2+1}} \right) \gamma_{D_2+1} (Z_{j-1}^U - Z_j^U) + \gamma_{D_2+1} (T_2 - \mu), & D_2 \in \{1, \ldots, k-1\}. 
\end{cases}$

Notice that the MLE $\hat{\vartheta}$ does not exist for $D_2 = 0$.

**Theorem A.1.2** The conditional density function $f_{\hat{\vartheta}}|D_2 \geq 1$ of the MLE $\hat{\vartheta}$ under unified Type-I progressive hybrid censoring for $s \geq 0$, is given by

$f_{\hat{\vartheta}|D_2 \geq 1} (s) = \frac{1}{1 - e^{-n(T_2 - \mu)/\vartheta}}$
Appendix A Unified Progressive Hybrid Censoring Schemes

Figure A.2: Plots of $f^{\hat{\theta}|D_2 \geq 1}$ (solid line) for $n = 10, m = 5, T_1 = 0.5, T_2 = 1 \ \vartheta = 1$ and $\mu = 0$. Left: $\mathcal{R} = (0^4, 5)$. Right: $\mathcal{R} = (1^5)$. The dashed lines represent the density function of $\hat{\theta}$ under (progressive) Type-II censoring.

$$\begin{align*}
&\sum_{d_1=k}^{m} (T_1 - \mu)^{d_1} \frac{\prod_{j=1}^{d_1} \gamma_j^\vartheta}{(d_1 - 1)! \gamma_1^\vartheta} B_{d_1 - 1}(d_1 s | \gamma_{d_1 + 1}(T_1 - \mu)) e^{-d_1 s/\vartheta} \\
&\quad + \sum_{d_2=1}^{k-1} (T_2 - \mu)^{d_2} \frac{\prod_{j=1}^{d_2} \gamma_j^{\vartheta \delta_k}}{(d_2 - 1)! \gamma_1^{\vartheta \delta_k}} B_{d_2 - 1}(d_2 s | \gamma_{d_2 + 1}(T_2 - \mu)) e^{-d_2 s/\vartheta} \\
&\quad + \frac{e^{-ks/\vartheta} \prod_{j=1}^{k} \gamma_j^{\vartheta \delta_k}}{(k - 1)! \gamma_1^{\vartheta \delta_k}} \\
&\quad \times \left[(T_2 - \mu)^k B_{k-1}(ks | \gamma_k^{\vartheta \delta_k}(T_2 - \mu)) - (T_1 - \mu)^k B_{k-1}(ks | \gamma_k^{\vartheta \delta_k}(T_1 - \mu))\right].
\end{align*}$$

Figure A.2 illustrates the plots of $f^{\hat{\theta}|D_2 \geq 1}$ for two censoring plans.

A.2 Unified Type-II Progressive Hybrid Censoring

The unified Type-II hybrid censoring scheme (Type-II UHCS) was originally introduced by Balakrishnan et al. (2008c), who called it unified hybrid censoring scheme. For further elaborations on unified Type-II hybrid censoring, we refer to Rad and Izanlo (2011) and Panahi and Sayyareh (2015).

By extending the unified Type-II hybrid censoring scheme to progressively Type-II censored order statistics, we introduce the unified Type-II progressive hybrid censoring scheme (Type-II UPHCS). The respective stopping time $T^*_U$ can be formulated as (cf. Park and Balakrishnan, 2012, Lemma 2.1),

$$T^*_U = \min \left\{ \max\{X_{\vartheta m}^{\vartheta \delta_k}, T_2\}, \max\{X_{\vartheta m}^{\vartheta \delta_k}, T_1\} \right\}, \quad 1 \leq k < m \leq n, \quad 0 < T_1 < T_2 < \infty.$$  

For the corresponding experimental outcomes, we refer to Table A.3. The unified Type-II progressive hybrid censored order statistics are then defined by

$$X^*_j = \min \left\{ \max\{X_{\vartheta m}^{\vartheta \delta_k}, T_2\}, X_{\vartheta j}^{\vartheta \delta_k} \right\}, \quad 1 \leq j \leq \tilde{m}.$$
A.2 Unified Type-II Progressive Hybrid Censoring

Table A.3: Experimental outcomes for the unified Type-II progressive hybrid censoring scheme.

<table>
<thead>
<tr>
<th>case</th>
<th>ordered values</th>
<th>sample</th>
<th>relevant counters</th>
<th>distr. references</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(X^{\text{m,n}}<em>{k:m:n} &lt; X^{\text{m,n}}</em>{k:m:n} &lt; T_1)</td>
<td>(X^{\text{m,n}}<em>{1:m:n}, \ldots, X^{\text{m,n}}</em>{D_1:m:n})</td>
<td>(D_1 \in {m, \ldots, \tilde{m}})</td>
<td>exponential (4.25) (\hat{b}) (4.53)</td>
</tr>
<tr>
<td>2.</td>
<td>(X^{\text{m,n}}<em>{k:m:n} &lt; T_1 &lt; X^{\text{m,n}}</em>{m:m:n} &lt; T_2)</td>
<td>(X^{\text{m,n}}<em>{1:m:n}, \ldots, X^{\text{m,n}}</em>{m:m:n})</td>
<td>(D_1 \in {k, \ldots, m - 1})</td>
<td>uniform (\hat{\mu}, \hat{\sigma}) (4.37) (\hat{a}, \hat{b}) (4.66)</td>
</tr>
<tr>
<td>3.</td>
<td>(T_1 &lt; X^{\text{m,n}}<em>{k:m:n} &lt; X^{\text{m,n}}</em>{m:m:n} &lt; T_2)</td>
<td>(X^{\text{m,n}}<em>{1:m:n}, \ldots, X^{\text{m,n}}</em>{m:m:n})</td>
<td>(D_1 \in {0, \ldots, k - 1})</td>
<td>exponential (\hat{\mu}, \hat{\sigma}) (4.75) (\hat{a}, \hat{b}) (4.56)</td>
</tr>
<tr>
<td>4.</td>
<td>(X^{\text{m,n}}<em>{k:m:n} &lt; T_1 &lt; T_2 &lt; X^{\text{m,n}}</em>{m:m:n})</td>
<td>(X^{\text{m,n}}<em>{1:m:n}, \ldots, X^{\text{m,n}}</em>{2:m:n})</td>
<td>(D_1, D_2 \in {k, \ldots, m - 1})</td>
<td>exponential (\hat{\mu}, \hat{\sigma}) (4.75) (\hat{a}, \hat{b}) (4.53)</td>
</tr>
<tr>
<td>5.</td>
<td>(T_1 &lt; X^{\text{m,n}}<em>{k:m:n} &lt; T_2 &lt; X^{\text{m,n}}</em>{m:m:n})</td>
<td>(X^{\text{m,n}}<em>{1:m:n}, \ldots, X^{\text{m,n}}</em>{2:m:n})</td>
<td>(D_1 \in {0, \ldots, k - 1})</td>
<td>exponential (\hat{\mu}, \hat{\sigma}) (4.75) (\hat{a}, \hat{b}) (4.56)</td>
</tr>
<tr>
<td>6.</td>
<td>(T_1 &lt; T_2 &lt; X^{\text{m,n}}<em>{k:m:n} &lt; X^{\text{m,n}}</em>{m:m:n})</td>
<td>(X^{\text{m,n}}<em>{1:k:n}, \ldots, X^{\text{m,n}}</em>{k:k:n})</td>
<td>(D_2 \in {0, \ldots, k - 1})</td>
<td>exponential (\hat{\mu}, \hat{\sigma}) (4.75) (\hat{a}, \hat{b}) (4.67)</td>
</tr>
</tbody>
</table>

Table A.4: Sampling situations for the unified Type-II progressive hybrid censoring scheme after the simplification process.

By combining the cases 2 and 3 as well as the cases 4 and 5 given in Table A.3, we arrive at

\[ \mathcal{G}_D(\text{UII}) = \{ \| D_1 = \cdot \|, \| D_1 < \cdot \|, \| D_2 = \cdot \|, \| D_2 < \cdot \| \}, \]

and

\[ \mathcal{I}(\mathcal{G}_D(\text{UII})) = \{ \{m, \ldots, \tilde{m}\}, \{m\}, \{k, \ldots, m - 1\}, \{k\} \}. \]

Figure A.3 shows the simplified structure of the unified Type-II progressive hybrid censoring scheme. Let \(Z^{\text{m,n}}_{1:k,n}, \ldots, Z^{\text{m,n}}_{d:k,n}, d \in \{1, \ldots, \tilde{m}\}\), denote the unified Type-II progressive hybrid censored order statistics from the exponential distribution.
Remark A.2.2 According to the sampling situations presented in Tables A.3 and A.4 as well as in Figure A.3, we find that after terminating the life test at least \( k \) failures are observed. In contrast to Balakrishnan et al. (2008c), conditioning on \( D_2 \geq 1 \), is not necessary.
A.3 Unified Type-III Progressive Hybrid Censoring

By extending the unified Type-III hybrid censoring scheme, introduced in Park and Balakrishnan (2012), to progressively Type-II censored order statistics, we define the stopping time $T_{\text{UIII}}^*$ by (cf. Park and Balakrishnan, 2012, Lemma 2.2),

$$T_{\text{UIII}}^* = \max \left\{ \min \left\{ X_{k,m;n}, T_3 \right\}, \min \left\{ \max \left\{ X_{m:n}, T_1 \right\}, T_2 \right\} \right\},$$

for $1 \leq k < m \leq n$, and $0 \leq T_1 < T_2 < T_3 < \infty$. The resulting experimental outcomes for the unified Type-III progressive hybrid censoring scheme (Type-III UPHCS) are listed in Table A.5. The unified Type-III progressive hybrid censored order statistics are defined
The resulting simplified structure of the unified Type-III progressive hybrid censoring scheme is illustrated in Figure A.5.

By combining the cases 2 with 5, 3 with 4, 6 with 7 and 8 with 9 from Table A.5, we find

$$\mathcal{S}_D(\text{UIII}) = \{ [D_1 = \cdot], [D_1 < \cdot, D_2 = \cdot], [D_2 < \cdot, D_3 = \cdot], [D_3 = \cdot] \},$$

and

$$\mathcal{I}(\mathcal{S}_D(\text{UIII})) = \{ \{m, \ldots, \tilde{m}\}, \{m\}, \{k, \ldots, m - 1\}, \{k\}, \{1, \ldots, k - 1\} \}.$$

The resulting simplified structure of the unified Type-III progressive hybrid censoring scheme is illustrated in Figure A.5.

We denote in the following by $Z_1^{\text{UIII}}, \ldots, Z_d^{\text{UIII}}$, $d \in \{1, \ldots, \tilde{m}\}$, the unified Type-III progressive hybrid censored order statistics from an exponential distribution.

<table>
<thead>
<tr>
<th>case</th>
<th>ordered values</th>
<th>sample</th>
<th>relevant counters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$X_{k_1;\tilde{m}<em>1} &lt; X</em>{m_2;\tilde{m}_2} &lt; T_1 &lt; T_2 &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {m, \ldots, \tilde{m}}$</td>
</tr>
<tr>
<td>2.</td>
<td>$X_{k_1;\tilde{m}<em>1} &lt; T_1 &lt; X</em>{m_2;\tilde{m}_2} &lt; T_2 &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {k, \ldots, m - 1}$</td>
</tr>
<tr>
<td>3.</td>
<td>$X_{k_1;\tilde{m}<em>1} &lt; T_1 &lt; T_2 &lt; X</em>{m_2;\tilde{m}_2} &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {k, \ldots, m - 1}$</td>
</tr>
<tr>
<td>4.</td>
<td>$X_{k_1;\tilde{m}<em>1} &lt; T_1 &lt; T_2 &lt; T_3 &lt; X</em>{m_2;\tilde{m}_2}$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {k, \ldots, m - 1}$</td>
</tr>
<tr>
<td>5.</td>
<td>$T_1 &lt; X_{k_1;\tilde{m}<em>1} &lt; X</em>{m_2;\tilde{m}_2} &lt; T_2 &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
<tr>
<td>6.</td>
<td>$T_1 &lt; X_{k_1;\tilde{m}<em>1} &lt; T_2 &lt; X</em>{m_2;\tilde{m}_2} &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
<tr>
<td>7.</td>
<td>$T_1 &lt; X_{k_1;\tilde{m}<em>1} &lt; T_2 &lt; T_3 &lt; X</em>{m_2;\tilde{m}_2}$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
<tr>
<td>8.</td>
<td>$T_1 &lt; T_2 &lt; X_{k_1;\tilde{m}<em>1} &lt; X</em>{m_2;\tilde{m}_2} &lt; T_3$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
<tr>
<td>9.</td>
<td>$T_1 &lt; T_2 &lt; X_{k_1;\tilde{m}<em>1} &lt; T_3 &lt; X</em>{m_2;\tilde{m}_2}$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
<tr>
<td>10.</td>
<td>$T_1 &lt; T_2 &lt; T_3 &lt; X_{k_1;\tilde{m}<em>1} &lt; X</em>{m_2;\tilde{m}_2}$</td>
<td>$X_{k_1;\tilde{m}<em>1}, \ldots, X</em>{D_2;\tilde{m}_2}$</td>
<td>$D_1 \in {0, \ldots, k - 1}$</td>
</tr>
</tbody>
</table>

Table A.5: Experimental outcomes for the unified Type-III progressive hybrid censoring scheme.
Figure A.5: Simplified structure of the unified Type-III progressive hybrid censoring scheme.

<table>
<thead>
<tr>
<th>case</th>
<th>ordered values</th>
<th>sample</th>
<th>relevant counters</th>
<th>distr. references</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$X_{m;m:n}^{\text{d}} &lt; X_{m;m:n}^{\text{d}} &lt; T_1$</td>
<td>$X_{1:n}^{\text{d}}, \ldots, X_{D_1:n}^{\text{d}}$</td>
<td>$D_1 \in {m, \ldots, \hat{m}}$</td>
<td>$\hat{\theta}$ (4.25) $\hat{\mu}$, $\hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.53)</td>
</tr>
<tr>
<td>2.</td>
<td>$T_1 &lt; X_{m;n:n} &lt; T_2$</td>
<td>$X_{1:n}^{\text{d}}, \ldots, X_{D_2:n}^{\text{d}}$</td>
<td>$D_1 &lt; m$, $D_2 = m$</td>
<td>$\hat{\theta}$ (7.4) $\hat{\mu}, \hat{\theta}$ (7.6) $\hat{a}, \hat{b}$ (7.8)</td>
</tr>
<tr>
<td>3.</td>
<td>$X_{k;m:n}^{\text{d}} &lt; T_2 &lt; X_{m;m:n}^{\text{d}}$</td>
<td>$X_{1:n}^{\text{d}}, \ldots, X_{D_2:n}^{\text{d}}$</td>
<td>$D_2 \in {k, \ldots, m-1}$</td>
<td>$\hat{\theta}$ (4.25) $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.53)</td>
</tr>
<tr>
<td>4.</td>
<td>$T_2 &lt; X_{k;m:n}^{\text{d}} &lt; T_3$</td>
<td>$X_{1:k}^{\text{d}}, \ldots, X_{D_3:k}^{\text{d}}$</td>
<td>$D_2 &lt; k$, $D_3 = k$</td>
<td>$\hat{\theta}$ (7.4) $\hat{\mu}, \hat{\theta}$ (7.6) $\hat{a}, \hat{b}$ (7.8)</td>
</tr>
<tr>
<td>5.</td>
<td>$T_3 &lt; X_{k;m:n}^{\text{d}}$</td>
<td>$X_{1:k}^{\text{d}}, \ldots, X_{D_3:k}^{\text{d}}$</td>
<td>$D_3 \in {0, \ldots, k-1}$</td>
<td>$\hat{\theta}$ (4.25) $\hat{\mu}, \hat{\theta}$ (4.37) $\hat{a}, \hat{b}$ (4.53)</td>
</tr>
</tbody>
</table>

Table A.6: Sampling situations for the unified Type-III progressive hybrid censoring scheme after the simplification process.
Lemma A.3.1 The likelihood function $L(|z_d)$ for $\mu$ and $\vartheta$ is given by

\[
L(\mu, \vartheta|z_d) = \begin{cases}
\mathbb{I}_{\tilde{z}^m_{F,T_1}}(z_d) \frac{F^{-\tilde{m}}}{1, \ldots, \tilde{m}; m}(z_m), & d_1 = \tilde{m}, \\
\mathbb{I}_{z_{F,T_1}}(z_d) \frac{1 - F(T_1)}{\gamma_{d_1}^{\tilde{m}} + 1} f(T_1)^{\tilde{m}}_{1, \ldots, d_1 + 1: \tilde{m}; j_1}(z_{d_1}, T_1), & d_1 \in \{m, \ldots, \tilde{m} - 1\}, \\
\mathbb{I}_{z_{F,T_1}}(z_m) \frac{F^{\tilde{m}}}{1, \ldots, m; m}(z_m), & d_1 < m, d_2 = m, \\
\mathbb{I}_{z_{F,T_2}}(z_{d_2}) \frac{1 - F(T_2)}{\gamma_{d_2+1}^{\tilde{m}} + 1} f(T_2)^{\tilde{m}}_{1, \ldots, d_2 + 1: \tilde{m}; j_2}(z_{d_2}, T_2), & d_2 \in \{k, \ldots, m - 1\}, \\
\mathbb{I}_{z_{F,T_2}}(z_k) \frac{F^{\tilde{m}}}{1, \ldots, k; k}(z_k), & d_2 < k, d_3 = k, \\
\mathbb{I}_{z_{F,T_3}}(z_{d_3}) \frac{1 - F(T_3)}{\gamma_{d_3+1}^{\tilde{m}} + 1} f(T_3)^{\tilde{m}}_{1, \ldots, d_3 + 1: \tilde{m}; j_3}(z_{d_3}, T_3), & d_3 \in \{1, \ldots, k - 1\}.
\end{cases}
\]

For a known location parameter $\mu$, the MLE $\hat{\vartheta}$ is given by

\[
\hat{\vartheta} = \begin{cases}
\frac{1}{D_1} \sum_{j=1}^{D_1} \left( 1 - \frac{\gamma_{D_1+1}^{\tilde{m}}}{\gamma_j} \right) \gamma_j^{\tilde{m}} (Z_j^{\text{III}} - Z_{j-1}^{\text{III}}) + \gamma_{D_1+1}^{\tilde{m}} (T_1 - \mu), & D_1 \in \{m, \ldots, \tilde{m}\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_j (Z_j^{\text{III}} - Z_{j-1}^{\text{III}}), & D_1 < m, D_2 = m, \\
\frac{1}{D_2} \sum_{j=1}^{D_2} \left( 1 - \frac{\gamma_{D_2+1}^{\tilde{m}}}{\gamma_j} \right) \gamma_j (Z_j^{\text{III}} - Z_{j-1}^{\text{III}}) + \gamma_{D_2+1}^{\tilde{m}} (T_2 - \mu), & D_2 \in \{k, \ldots, m - 1\}, \\
\frac{1}{k} \sum_{j=1}^{k} \gamma_j^{\text{III}} (Z_j^{\text{III}} - Z_{j-1}^{\text{III}}), & D_2 < k, D_3 = k, \\
\frac{1}{D_3} \sum_{j=1}^{D_3} \left( 1 - \frac{\gamma_{D_3+1}^{\tilde{m}}}{\gamma_j} \right) \gamma_j^{\text{III}} (Z_j^{\text{III}} - Z_{j-1}^{\text{III}}) + \gamma_{D_3+1}^{\tilde{m}} (T_3 - \mu), & D_3 \in \{1, \ldots, k - 1\}.
\end{cases}
\]

It should be noted that the MLE $\hat{\vartheta}$ does not exist for $D_3 = 0$.

Theorem A.3.2 The conditional density function $f_{\hat{\vartheta}|D_3 \geq 1}(s)$ of the MLE $\hat{\vartheta}$ under unified Type-III progressive hybrid censoring for $s \geq 0$ is given by

\[
f_{\hat{\vartheta}|D_3 \geq 1}(s) = \frac{1}{1 - e^{-n(T_3 - \mu)/\vartheta}} \times \left[ \left( T_1 - \mu \right)^{d_1} \prod_{j=1}^{d_1} \frac{\gamma_j^{\tilde{m}}}{(d_1 - 1)! \vartheta^{d_1}} B_{d_1-1}(d_1 s | \gamma_{D_1+1}^{\tilde{m}} (T_1 - \mu)) e^{-d_1 s/\vartheta} \right. \\
+ \frac{e^{-ms/\vartheta}}{(m - 1)! \vartheta^{m}} \times \left[ (T_2 - \mu)^m B_{m-1}(ms | 0, \gamma_m (T_2 - \mu)) - (T_1 - \mu)^m B_{m-1}(ms | 0, \gamma_m (T_1 - \mu)) \right] \\
+ \sum_{d_2=k}^{m-1} \frac{(T_2 - \mu)^{d_2} \prod_{j=1}^{d_2} \gamma_j}{(d_2 - 1)! \vartheta^{d_2}} B_{d_2-1}(d_2 s | \gamma_{d_2+1}^{\tilde{m}} (T_2 - \mu)) e^{-d_2 s/\vartheta} \\
\left. \right) \right]
\]
Figure A.6: Plots of \( f_{\hat{\varphi}|D_3 \geq 1} \) (solid line) for \( n = 10, k = 2, m = 5, T_1 = 0.5, T_2 = 1.0, T_3 = 1.4 \) \( \varphi = 1 \) and \( \mu = 0 \). Left: \( \mathcal{R} = (0^4,5) \). Right: \( \mathcal{R} = (1^5) \). The dashed lines represent the density function of \( \hat{\varphi} \) under (progressive) Type-II censoring.

\[
\begin{align*}
&\text{Figure A.6 shows the plots of } f_{\hat{\varphi}|D_3 \geq 1} \text{ for two censoring plans.}

\text{A.4 Unified Type-IV Progressive Hybrid Censoring}

The unified Type-IV progressive hybrid censoring scheme (Type-IV UPHCS) is specified by the following stopping time (cf. Park and Balakrishnan, 2012, Lemma 2.2),

\[
T_{UIV}^* = \max \left\{ \min \{ X_{[\hat{\varphi},\varphi;n], T_1}, \min \{ \max \{ X_{[\hat{\varphi},\varphi;n], T_2}, X_{[\hat{\varphi},\varphi;n]}, X_{[\varphi,\varphi;n], T_3} \} \} \right\},
\]

with \( 1 \leq m < u \leq \bar{m} \leq n \), and \( 0 < T_1 < T_2 < \infty \). The resulting experimental outcomes are listed in Table A.7. We define the unified Type-IV progressive hybrid censored order statistics by

\[
X_{UIV}^j := \min \{ \max \{ X_{[\hat{\varphi},\varphi;n], T_2}, X_{[\varphi,\varphi;n], T_3} \}, 1 \leq j \leq u.
\]

Now, by combining the cases 2 with 5, 3 with 4 with 6 with 7, and 8 with 9, we find

\[
\mathcal{D}(UIV) = \{ [D_1 = \cdot], [D_1 < \cdot], D_2 = \cdot], [D_2 = \cdot], [D_2 < \cdot] \},
\]

and

\[
\mathcal{I}(\mathcal{D}(UIV)) = \{ \{m, \ldots, u\}, \{m\}, \{k, \ldots, m - 1\}, \{k\} \}.
\]
<table>
<thead>
<tr>
<th>Case</th>
<th>Ordered Values</th>
<th>Sample</th>
<th>Relevant Counters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( T_1 &lt; T_2 &lt; X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u )</td>
<td>( X_{1:k:n}^{\text{occ}}, \ldots, X_{k:k:n}^{\text{occ}} )</td>
<td>( D_2 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>2.</td>
<td>( T_1 &lt; X_{k,u:n}^u &lt; T_2 &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_2 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>3.</td>
<td>( T_1 &lt; X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; T_2 &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_2 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>4.</td>
<td>( T_1 &lt; X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u &lt; T_2 )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_2 = m )</td>
</tr>
<tr>
<td>5.</td>
<td>( X_{k,u:n}^u &lt; T_1 &lt; T_2 &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>6.</td>
<td>( X_{k,u:n}^u &lt; T_1 &lt; T_2 &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>7.</td>
<td>( X_{k,u:n}^u &lt; T_1 &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u &lt; T_2 )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>8.</td>
<td>( X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u &lt; T_1 &lt; T_2 )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>9.</td>
<td>( X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u &lt; T_1 &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
<tr>
<td>10.</td>
<td>( X_{k,u:n}^u &lt; X_{m,u:n}^u &lt; X_{n,u:n}^u &lt; X_{n,u:n}^u )</td>
<td>( X_{1:m:n}^{\text{occ}}, \ldots, X_{n:m:n}^{\text{occ}} )</td>
<td>( D_1 \in {0, \ldots, k - 1} )</td>
</tr>
</tbody>
</table>

Table A.7: Experimental outcomes for the unified Type-IV progressive hybrid censoring scheme.

**Figure A.7:** Simplified structure of the unified Type-IV progressive hybrid censoring scheme.
The resulting simplified structure of the unified Type-IV progressive hybrid censoring scheme is given by

\[ \theta \hat{\theta} = \begin{cases} 
\frac{1}{D_1} \sum_{j=1}^{D_1} \left( 1 - \frac{\gamma_j^{D_1+1}}{\gamma_j^{D_1}} \right) \gamma_j^{D_1} (Z_j^{UV} - Z_j^{UV}) + \gamma_j^{D_1+1} (T_1 - \mu), & D_1 \in \{m, \ldots, u\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_j (Z_j^{UV} - Z_j^{UV}), & D_1 < m, D_2 = m, \\
\frac{1}{D_2} \sum_{j=1}^{D_2} \left( 1 - \frac{\gamma_j^{D_2+1}}{\gamma_j} \right) \gamma_j (Z_j^{UV} - Z_j^{UV}) + \gamma_j^{D_2+1} (T_2 - \mu), & D_2 \in \{k, \ldots, m - 1\}, \\
\frac{1}{k} \sum_{j=1}^{k} \gamma_j^{D_2} (Z_j^{UV} - Z_j^{UV}), & D_2 < k. 
\end{cases} \]

Let the location parameter \( \mu \) be known. Then, the MLE for \( \theta \) is given by

\[ \hat{\theta} = \begin{cases} 
\frac{1}{D_1} \sum_{j=1}^{D_1} \left( 1 - \frac{\gamma_j^{D_1+1}}{\gamma_j^{D_1}} \right) \gamma_j^{D_1} (Z_j^{UV} - Z_j^{UV}) + \gamma_j^{D_1+1} (T_1 - \mu), & D_1 \in \{m, \ldots, u\}, \\
\frac{1}{m} \sum_{j=1}^{m} \gamma_j (Z_j^{UV} - Z_j^{UV}), & D_1 < m, D_2 = m, \\
\frac{1}{D_2} \sum_{j=1}^{D_2} \left( 1 - \frac{\gamma_j^{D_2+1}}{\gamma_j} \right) \gamma_j (Z_j^{UV} - Z_j^{UV}) + \gamma_j^{D_2+1} (T_2 - \mu), & D_2 \in \{k, \ldots, m - 1\}, \\
\frac{1}{k} \sum_{j=1}^{k} \gamma_j^{D_2} (Z_j^{UV} - Z_j^{UV}), & D_2 < k. 
\end{cases} \]

**Theorem A.4.2** The density function \( f_{\hat{\theta}}(s) \) of the MLE \( \hat{\theta} \) under unified Type-IV progressive hybrid censoring is given by

\[ f_{\hat{\theta}}(s) = \sum_{d_1=m}^{u} \left( \frac{(T_1 - \mu)}{(d_1 - 1)! \gamma_{d_1}^{D_1}} B_{d_1-1} \left( d_1 s | \gamma_{d_1+1}, (T_1 - \mu) \right) \right) e^{-d_1 s/\hat{\theta}}. \]
Figure A.8: Plots of $f^{\hat{\vartheta}}$ (solid line) for $n = 10$, $k = 3$, $m = 5$, $u = 6$, $T_1 = 0.5$, $T_2 = 1.0$, $\vartheta = 1$ and $\mu = 0$. Left: $\mathcal{R} = (0^4, 5)$. Right: $\mathcal{R} = (1^5)$. The dashed lines represent the density function of $\hat{\vartheta}$ under (progressive) Type-II censoring.

$$+rac{e^{-ms/\vartheta} \prod_{j=1}^{m} \gamma_j}{(m-1)! \vartheta^m} \left[ (T_2 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_2 - \mu)) - (T_1 - \mu)^m B_{m-1}(ms|0, \gamma_m(T_1 - \mu)) \right]$$

$$+ \sum_{d_2=k}^{m-1} \frac{(T_2 - \mu)^{d_2} \prod_{j=1}^{d_2} \gamma_j}{(d_2-1)! \vartheta^{d_2}} B_{d_2-1}(d_2s|\gamma_{d_2+1}(T_2 - \mu)) e^{-d_2s/\vartheta}$$

$$+ f^{T}_{k,k}(s) - \frac{(T_2 - \mu)^k \prod_{j=1}^{k} \gamma_{j}^{\text{ck}}}{(k-1)! \vartheta^k} B_{k-1}(ks|0, \gamma_{k}^{\text{ck}}(T_2 - \mu)) e^{-ks/\vartheta}, \quad s \geq 0.$$
Appendix B

Simulation Study

In order to get an idea of the differences among the herein addressed hybrid censoring models, we consider in this chapter a brief simulation study for the exponential distribution. Let $\mu = 0$ and $\vartheta = 1$. Then, we simulate $N = 10^6$ exponential samples with length $n = 15$. For each progressive censoring model (i.e., Type-I/II progressive hybrid censoring schemes, generalized Type-I/II progressive hybrid censoring schemes, unified Type-I/II/III/IV progressive censoring schemes and general unified progressive hybrid censoring scheme) addressed in this thesis, we consider the initially planned censoring plan $R = (0^6, 8)$, so that $m = 7$. The particular parameter settings for the hybrid censoring models are presented in Table B.1. We further include in our simulation study two hybrid censoring models arising from the multi-sample Type-I hybrid censoring scheme. In particular, we consider the setups

\begin{equation}
\begin{aligned}
n_1 &= 7, \; m_1 = 3, \; T^o_1 = 1, \; (\gamma^*_1, 1, \gamma^*_1, 2, \gamma^*_1, 3) = (7, 6, 5), \\
n_2 &= 8, \; m_2 = 4, \; T^o_2 = 1, \; (\gamma^*_2, 1, \gamma^*_2, 2, \gamma^*_2, 3, \gamma^*_2, 4) = (8, 7, 6, 5),
\end{aligned}
\tag{B.1}
\end{equation}

for $k = 2$, and

\begin{equation}
\begin{aligned}
n_1 &= 5, \; m_1 = 3, \; T^o_1 = 1, \; (\gamma^*_1, 1, \gamma^*_1, 2, \gamma^*_1, 3) = (5, 4, 3), \\
n_2 &= 5, \; m_2 = 2, \; T^o_2 = 1, \; (\gamma^*_2, 1, \gamma^*_2, 2) = (5, 4), \\
n_3 &= 5, \; m_3 = 2, \; T^o_3 = 1, \; (\gamma^*_3, 1, \gamma^*_3, 2) = (5, 4),
\end{aligned}
\tag{B.2}
\end{equation}

for $k = 3$. We consider in total twelve hybrid censoring models applied to the simulated data $Z_{N,n} := (z_{i,j})_{1 \leq i \leq N, 1 \leq j \leq n}$ with $n = 15$ and $N = 10^6$. Apart from the multi-sample Type-I hybrid censoring scheme (Type-I MHCS), we apply the hybrid censoring procedures to the respective ordered simulated data $Z_{N,n:n}$, where

\begin{equation}
Z_{N,n:n} := (z_{i,j:n})_{1 \leq i \leq N, 1 \leq j \leq n} = \begin{pmatrix}
z_{1,1:n} & \cdots & z_{1,n:n} \\
\vdots & \ddots & \vdots \\
z_{N,1:n} & \cdots & z_{N,n:n}
\end{pmatrix}.
\end{equation}

For the multi-sample Type-I hybrid censoring scheme with $k \in \{2, 3\}$, we proceed as follows: we subdivide the data matrix $Z_{N,n}$ into $k$ data submatrices $Z^{(1)}_{N,n_{i}}, \ldots, Z^{(k)}_{N,n_{k}}$, so that

\begin{equation}
Z_{N,n} = \left( Z^{(1)}_{N,n_{1}}, \ldots, Z^{(k)}_{N,n_{k}} \right),
\end{equation}

where obviously $n = \sum_{i=1}^{k} n_i$. Then, we apply the Type-I hybrid censoring scheme to each of the data submatrices $Z^{(i)}_{N,n_{i}}, 1 \leq i \leq k$. The resulting data is then used to compute the MLE $\hat{\vartheta}$ given in Lemma 5.1.3.
and where at least one failure has been observed. For the particular case of multi-sample Type-I experiments at least one failure has been observed.

For the multi-sample Type-I hybrid censoring scheme, we have

\[ k = 2, \text{ see (B.1)} \]

and

\[ k = 3, \text{ see (B.2)} \]

The simulation results are given in Table B.1.

**Table B.1:** Parameter settings of the considered hybrid censoring schemes and simulation results w.r.t. the exponential distribution with parameters \( \mu = 0 \) and \( \vartheta = 1 \). Further, \( n = 15 \) and \( N = 10^6 \).

For the multi-sample Type-I hybrid censoring scheme, we have \( N^* = 998793 \) and \( N^* = 979882 \) for \( k = 2 \) and \( k = 3 \), respectively. In the remaining models, we have \( N^* = N \).

For each hybrid censoring model we provide the mean \( \bar{\vartheta} \), the standard deviation (s.d.), the mean experiment duration \( T^* \) and the mean of observed failures \( D^* \). For all models \( \bar{\vartheta} \), s.e. and s.d. are computed as in (4.31). The mean values \( T^* \) and \( D^* \) are defined by

\[
T^* = \begin{cases} 
\frac{1}{N} \sum_{i=1}^{N} \max_{q \in \{1, \ldots, k\}} T_{i;1,q}^* \text{ Type-I MHCS,} \\
\frac{1}{N} \sum_{i=1}^{N} T_{i;H}^* \text{ else,}
\end{cases}
\]

and

\[
D^* = \begin{cases} 
\frac{1}{N} \sum_{i=1}^{N} \sum_{q=1}^{k} D_{i;1,q}^* \text{ Type-I MHCS,} \\
\frac{1}{N} \sum_{i=1}^{N} D_{i;H}^* \text{ else,}
\end{cases}
\]

respectively. Here, we denote by \( T_{i;1,1}^*, \ldots, T_{i;1,k}^* \) the stopping times and by \( D_{i;1}^*, \ldots, D_{i;k}^* \) the numbers of observed failures for the the multi-sample Type-I hybrid censoring scheme w.r.t. the ith sample. Further, by \( T_{i;H}^* \) and \( D_{i;H}^* \) we denote the stopping time as well as the number of observed failures, respectively, for a particular hybrid censoring scheme \( H \) among the remaining hybrid censoring schemes w.r.t. the ith sample. Finally, the quantity \( N^* \) (cf. p. 82) denotes for an underlying hybrid censoring scheme the number of experiments where at least one failure has been observed. For the particular case of multi-sample Type-I hybrid censoring, the value \( N^* \) denotes the number of times where in each of the \( k \) sub experiments at least one failure has been observed.

The simulation results are given in Table B.1. They yield the following observations:

- The Type-I hybrid censoring scheme has a relatively short experimental duration, whereas for Type-II hybrid censoring relatively small values for s.e. and s.d. are obtained;
- The values of \( \bar{\vartheta}, \text{s.e.}, \text{and s.d. for multi-sample Type-I hybrid censoring improve for an increasing } k \in \{2, 3\};\)
– The general unified hybrid censoring scheme allows the experimenter to cover different desired features of the experiment by choosing the parameters adequately, e.g., short experimental duration or the observation of a relatively large number of failures.

Figures B.1 and B.2 (see pp. 198–199) depict the histograms together with the plots of the density function of the underlying MLE for the models addressed in Table B.1.
Figure B.1: Histograms with class width 0.01 and plots of the density function of the respective MLEs for the first six hybrid censoring schemes listed in Table B.1.
Figure B.2: Histograms with class width 0.01 and plots of the density function of the respective MLEs for the last six hybrid censoring schemes listed in Table B.1.

GUHCS $\left(T_1 = 0.5, T_2 = 0.75, T_3 = 1\right)$  
GUHCS $\left(T_1 = 0.5, T_2 = 1, T_3 = 1.5\right)$
Appendix C
Distributions and Preliminary Results

This chapter provides the definitions of the distributions addressed in this work. Further, some preliminary results are presented.

C.1 Definitions of Distributions

The following distributions are considered (see Balakrishnan and Cramer, 2014, pp. 571–572).

**Definition C.1.1 (Uniform distribution)** The uniform distribution $U(a, b)$ with parameters $a, b \in \mathbb{R}, a < b$, is defined by the density function

$$f(x) = \frac{1}{b - a} \mathbb{1}_{[a, b]}(x), \quad x \in \mathbb{R}.$$  

The corresponding cumulative distribution function is given by (cf. Johnson et al., 1995, p. 276)

$$F(x) = \begin{cases} \frac{x - a}{b - a} \mathbb{1}_{[a, b]}(x), & x \leq b, \\ 1, & x > b, \end{cases} \quad x \in \mathbb{R}.$$  

**Definition C.1.2 (Beta distribution)** The beta distribution $\text{Beta}(\alpha, \beta)$ with parameters $\alpha, \beta > 0$, is defined by the density function

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha - 1}(1 - x)^{\beta - 1} \mathbb{1}_{(0,1)}(x), \quad x \in \mathbb{R},$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}$ denotes the complete beta function and $\Gamma(\cdot)$ denotes the gamma function.

**Definition C.1.3 (Exponential distribution)** The two-parameter exponential distribution $\text{Exp}(\mu, \vartheta)$ with parameters $\mu \in \mathbb{R}$ and $\vartheta > 0$ is defined by the density function

$$f(x) = \frac{1}{\vartheta} e^{-(x - \mu)/\vartheta} \mathbb{1}_{(\mu, \infty)}(x), \quad x \in \mathbb{R}.$$  

The corresponding cumulative distribution function is given by (cf. Johnson et al., 1994, p. 496)

$$F(x) = (1 - e^{-(x - \mu)/\vartheta}) \mathbb{1}_{(\mu, \infty)}(x), \quad x \in \mathbb{R}.$$
Definition C.1.4 (Gamma distribution) The gamma distribution $\Gamma(\vartheta, \beta)$ with parameters $\vartheta, \beta > 0$, is defined by the density function

$$f_{\vartheta, \beta}(x) = \frac{1}{\Gamma(\beta)\vartheta^\beta} x^{\beta-1}e^{-x/\vartheta} 1_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$ 

Since this thesis addresses the field of life tests, we usually restrict the domain of $\mu$ to $[0, \infty)$. The quantile function $F^{-1}$ (see Balakrishnan and Cramer, 2014, p. 574) is defined as follows.

Definition C.1.5 Let $F$ be a cumulative distribution function. Then, the quantile function $F^{-1} : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$F^{-1}(y) = \inf\{x \in \mathbb{R}| F(x) \geq y\}, \quad 0 < y < 1.$$ 

The values at zero and one are defined by the one-sided limits $F^{-1}(0) = F^{-1}(0+)$ and $F^{-1}(1) = F^{-1}(1-)$, respectively.

C.2 Preliminary Results

In this section, some results, established for the integration of B-spline related expressions (cf. Section 3.3), are presented. We start by recalling a result for the convolution of two B-splines, established in Strøm (1994, Proposition 4).

Proposition C.2.1 Let $t^{(1)}_{d_1} = (t^{(1)}_{0}, \ldots, t^{(1)}_{d_1})$, $t^{(1)}_{0} \leq t^{(1)}_{d_1}$, and $t^{(2)}_{d_2} = (t^{(2)}_{0}, \ldots, t^{(2)}_{d_2})$, $t^{(2)}_{0} \leq t^{(2)}_{d_2}$, be two knot sequences. Then, the convolution of the two B-splines $B_{d_1-1}(|t^{(1)}_{d_1}|)$ and $B_{d_2-1}(|t^{(2)}_{d_2}|)$ can be, for $s \in \mathbb{R}$, calculated by

$$\left( B_{d_1-1}(|t^{(1)}_{d_1}|) \ast B_{d_2-1}(|t^{(2)}_{d_2}|) \right)(s) = \frac{d_1!d_2!}{(d_{*2}-1)!} [t^{(1)}_{d_1}]_{x_1} [t^{(2)}_{d_2}]_{x_2} (x_1 + x_2 - s)^{d_{*2}-1}.$$ 

We proceed with the $k$-fold convolution of B-splines.

Lemma C.2.2 Let the setup described in Section 3.3.1, be given. Then, the $k$-fold convolution of the B-splines $B_{d_{j-1}}(|t^{(j)}_{d_j}|)$, $1 \leq j \leq k$, $k \in \mathbb{N}_{\geq 2}$, can be calculated by

$$\left( B_{d_{1-1}}(|t^{(1)}_{d_1}|) \ast \cdots \ast B_{d_{k-1}}(|t^{(k)}_{d_k}|) \right)(s) = [t^{(1)}_{d_1}]_{x_1} [t^{(2)}_{d_2}]_{x_2} \cdots [t^{(k)}_{d_k}]_{x_k} \left( \sum_{j=1}^{k} x_j - s \right)^{d_{*k}-1} \prod_{j=1}^{k} d_j! \frac{1}{(d_{*k}-1)!}, \quad s \in \mathbb{R}.$$ 

Proof. The repeated application of the convolution operator $\ast$ (see Definition 3.3.1) corresponds to the following expression of the $k$-fold convolution of the B-splines $B_{d_{j-1}}(|t^{(j)}_{d_j}|)$, $1 \leq j \leq k$,

$$\left( B_{d_{1-1}}(|t^{(1)}_{d_1}|) \ast \cdots \ast B_{d_{k-1}}(|t^{(k)}_{d_k}|) \right)(s) = \int_{\mathbb{R}} B_{d_{1-1}}(x_1|t^{(1)}_{d_1}|) \cdots \int_{\mathbb{R}} B_{d_{k-1}-1}(x_{k-1}|t^{(k-1)}_{d_{k-1}}|) B_{d_k-1}(s - \sum_{i=1}^{k-1} x_i|t^{(k)}_{d_k}|) dx_{k-1} \cdots dx_1.$$(C.1)

Proof. The repeated application of the convolution operator $\ast$ (see Definition 3.3.1) corresponds to the following expression of the $k$-fold convolution of the B-splines $B_{d_{j-1}}(|t^{(j)}_{d_j}|)$, $1 \leq j \leq k$,

$$\left( B_{d_{1-1}}(|t^{(1)}_{d_1}|) \ast \cdots \ast B_{d_{k-1}}(|t^{(k)}_{d_k}|) \right)(s) = \int_{\mathbb{R}} B_{d_{1-1}}(x_1|t^{(1)}_{d_1}|) \cdots \int_{\mathbb{R}} B_{d_{k-1}-1}(x_{k-1}|t^{(k-1)}_{d_{k-1}}|) B_{d_k-1}(s - \sum_{i=1}^{k-1} x_i|t^{(k)}_{d_k}|) dx_{k-1} \cdots dx_1.$$
For $k = 2$ the assumption follows from Proposition C.2.1. Suppose the assertion holds true for an arbitrarily chosen but fixed $k \in \mathbb{N}_{\geq 2}$ (i.h.). Then, we find together with the commutativity of the convolution operator (cf., e.g., Folland, 1999, p. 240)

$$
\left( B_{d_{k+1}}(\cdot | t_{d_{k+1}}) * \cdots * B_{d_{k}}(\cdot | t_{d_{k}}) * B_{d_{k-1}}(\cdot | t_{d_{k-1}}) \right) (s)
$$

$$
= \int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}) \int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}) \cdots
$$

$$
= \int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}) B_{d_{k+1}}((s - x_{k+1}) - \sum_{i=1}^{k-1} x_i \cdot t_{d_{i}}) \, dx_{k+1} \, dx_{k-1} \cdots \, dx_1
$$

i.h.

$$
= \int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}) \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - (s - x_{k+1}) \right)_+ \frac{1}{d_{k-1}!} dx_{k+1}
$$

$$
'= \int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}) \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - s \right)_+ \frac{1}{d_{k-1}!} dx_{k+1}.
$$

(C.2)

Now, by setting

$$
f(x_{k+1}) = \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - s \right)_+ \frac{1}{d_{k-1}!} \prod_{j=1}^{k} \frac{j!}{d_{k+1-j}!},
$$

we find that the $d_{k+1}$th antiderivative of $f$, denoted by $f^{(-d_{k+1})}$, is given by

$$
f^{(-d_{k+1})}(x_{k+1})
$$

$$
= \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - s \right)_+ \frac{1}{d_{k-1}!} \prod_{j=1}^{k} \frac{j!}{d_{k+1-j}!},
$$

w.r.t. the knots $t_{d_{k+1}}^{(k+1)}$ yields the following expression for (C.2):

$$
\int_{\mathbb{R}} B_{d_{k+1}}(x_{k+1} | t_{d_{k+1}}^{(k+1)}) \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - s \right)_+ \frac{1}{d_{k-1}!} \prod_{j=1}^{k} \frac{j!}{d_{k+1-j}!} \, dx_{k+1}
$$

$$
= d_{k+1}! \left[ t_{d_{k+1}}^{(k+1)} \right]_{x_{k+1}} f^{(-d_{k+1})}(x_{k+1})
$$

$$
= \left[ t_{d_1}^{(1)} \right]_{x_1} \left[ t_{d_2}^{(2)} \right]_{x_2} \cdots \left[ t_{d_{k}}^{(k)} \right]_{x_k} \left( \sum_{j=1}^{k+1} x_j - s \right)_+ \frac{1}{d_{k+1-j}!} \prod_{j=1}^{k+1} \frac{j!}{d_{k+1-j}!}.
$$

This proves the assumption for $k + 1$.

The following results address the antiderivative of $x^k e^{cx}$, $x > 0$, $k \in \mathbb{N}$.
Lemma C.2.3 For \( c \neq 0 \) and \( k \in \mathbb{N} \), the following identity holds:

\[
\int_{\mathbb{R}} x^k e^{cx} \, dx = e^{cx} \sum_{i=0}^{k} \frac{(-1)^i}{(k-i)!} \frac{k!}{c^{i+1}} x^{k-i}, \quad x > 0.
\]

**Proof.** We prove the assertion via induction. Partial integration gives

\[
\int x e^{cx} \, dx = \frac{e^{cx} x}{c} - \frac{e^{cx}}{c^2},
\]

which corresponds to the assumption for \( k = 1 \). Suppose the assumption holds for an arbitrarily chosen but fixed \( k \in \mathbb{N} \) (i.h.). Then, we get by partial integration

\[
\int x^{k+1} e^{cx} \, dx = x^{k+1} \frac{1}{c} e^{cx} - \frac{k+1}{c} \int x^k e^{cx} \, dx
\]

\[
\equiv \text{i.h.} \quad x^{k+1} \frac{1}{c} e^{cx} - \frac{k+1}{c} e^{cx} \sum_{i=0}^{k} \frac{(-1)^i}{(k-i)!} \frac{k!}{c^{i+1}} x^{k-i}
\]

\[
= x^{k+1} \frac{1}{c} e^{cx} + e^{cx} \sum_{i=0}^{k} \frac{(-1)^i+1}{(k-i)!} \frac{(k+1)!}{c^{i+2}} x^{k-i}
\]

\[
= x^{k+1} \frac{1}{c} e^{cx} + e^{cx} \sum_{i=1}^{k+1} (-1)^i \frac{(k+1)!}{((k+1)-i)!} \frac{x^{(k+1)-i}}{c^{i+1}}
\]

\[
= e^{cx} \sum_{i=0}^{k+1} (-1)^i \frac{(k+1)!}{((k+1)-i)!} \frac{x^{(k+1)-i}}{c^{i+1}}.
\]

This is the assumption for \( k + 1 \), which proves the result. \( \square \)

We consider now the \( d \)th antiderivative of \( x^k e^{cx} \), \( x > 0 \), with \( d \geq 1 \).

**Lemma C.2.4** For \( d, k \in \mathbb{N} \) and \( c \neq 0 \), the \( d \)th antiderivative of \( f(x) = x^k e^{cx} \), \( x > 0 \), is given by

\[
f^{(-d)}(x) = e^{cx} \sum_{i_1=0}^{k} \frac{(-1)^{i_1}}{c_1^{i_1+1}} \sum_{i_2=0}^{k-i_1} \frac{(-1)^{i_2}}{c_2^{i_2+1}} \cdots \sum_{i_d=0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c_d^{i_d+1}} \frac{x^{k-i_d}}{(k-i_d)!}.
\]

**Proof.** We proceed via induction. The result for \( d = 1 \) follows from Lemma C.2.3. Suppose the assumption holds for an arbitrarily chosen but fixed \( d \in \mathbb{N} \) (i.h.). Then, we find with Lemma C.2.3

\[
f^{(-n+1)}(x)
\]

\[
= \int f^{(-n)}(x) \, dx
\]

\[
\equiv \text{i.h.} \quad \sum_{i_1=0}^{k} \frac{(-1)^{i_1}}{c_1^{i_1+1}} \sum_{i_2=0}^{k-i_1} \frac{(-1)^{i_2}}{c_2^{i_2+1}} \cdots \sum_{i_d=0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c_d^{i_d+1}} \frac{1}{(k-i_d)!} \int x^{k-i_d} e^{cx} \, dx
\]
\[= k! \sum_{i_1 = 0}^{k} \frac{(-1)^{i_1}}{c^{i_1+1}} \sum_{i_2 = 0}^{k-i_1} \frac{(-1)^{i_2}}{c^{i_2+1}} \cdots \sum_{i_d = 0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c^{i_d+1}} \frac{1}{(k - i_d)!} \]

\[e^{cx} \sum_{i_{d+1} = 0}^{k-i_d} \frac{(-1)^{i_{d+1}}}{(k - i_d - i_{d+1})!} \frac{x^{k-i_d-i_{d+1}}}{c^{i_{d+1}+1}} \]

\[= e^{cx} k! \sum_{i_1 = 0}^{k} \frac{(-1)^{i_1}}{c^{i_1+1}} \sum_{i_2 = 0}^{k-i_1} \frac{(-1)^{i_2}}{c^{i_2+1}} \cdots \sum_{i_d = 0}^{k-i_{d-1}} \frac{(-1)^{i_d}}{c^{i_d+1}} \sum_{i_{d+1} = 0}^{k-i_{d-1}} \frac{(-1)^{i_{d+1}}}{c^{i_{d+1}+1}} \frac{x^{k-i_{d+1}}}{(k - i_{d+1})!}.\]

This is the assertion for \(d + 1\), which proves the result. \(\square\)
### Notation

The following table provides an overview on the notations used in this thesis. In particular cases, the page number for the respective definition or explanation is provided.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{j:n}$</td>
<td>$j$th order statistic (based on IID random variables $X_1, \ldots, X_n$)</td>
<td>25</td>
</tr>
<tr>
<td>$U_{j:n}$</td>
<td>$j$th uniform order statistic</td>
<td></td>
</tr>
<tr>
<td>$f_{1:r:n}, F_{1:r:n}$</td>
<td>Joint density function/cumulative distribution function of $X_{1:n}, \ldots, X_{r:n}$, $1 \leq r \leq n$</td>
<td>26</td>
</tr>
<tr>
<td>$f_{r:n}, F_{r:n}$</td>
<td>Density function/cumulative distribution function of $X_{r:n}$, $1 \leq r \leq n$</td>
<td>26</td>
</tr>
<tr>
<td>$f_{r:s:n}, F_{r:s:n}$</td>
<td>Joint density function/cumulative distribution function of $X_{r:n}$ and $X_{s:n}$, $1 \leq r &lt; s \leq n$</td>
<td>26</td>
</tr>
</tbody>
</table>

**Progressively Type-II censored order statistics**

- $\mathcal{R}$: Censoring plan, denotes usually the initially planned censoring plan $(R_1, \ldots, R_m)$
- $\mathcal{E}^m_{m,n}$: Set of admissible censoring plans, $m \leq n$ | 28 |
- $X^\mathcal{R}_{j:m:n}$: $j$th progressively Type-II censored order statistic w.r.t. a censoring plan $\mathcal{R}$ | 27 |
- $Z^\mathcal{R}_{j:m:n}$: $j$th exponential progressively Type-II censored order statistic w.r.t. a censoring plan $\mathcal{R}$ | |
- $\tilde{m}$: $m + R_m$ | 28 |
- $\gamma_j^\mathcal{R}$: $\sum_{i=j}^{m}(R_i + 1), 1 \leq j \leq m, \gamma_{m+1} = 0$ | 28 |
- $\mathcal{R}^{\mathcal{R} \supset} (R_1, \ldots, R_{r-1}, \sum_{i=r}^{m} R_i + (m-r))$, compressed censoring plan w.r.t. the initially planned censoring plan $\mathcal{R}$, $r \in \{1, \ldots, m\}$ | 29 |
- $\gamma_j^{\mathcal{R}, r}$: $\sum_{i=j}^{m}(R_i + 1), 1 \leq j \leq r, \gamma_{r+1} = 0$, for $r \in \{1, \ldots, m\}$ | 28 |
- $\mathcal{R}^{\mathcal{R} \shoveleft} (R_1, \ldots, R_{m-1}, 0^{(r-m)}, m + R_m - r)$, (right) extended censoring plan w.r.t. the initially planned censoring plan $\mathcal{R}$, $r \in \{m + 1, \ldots, m + R_m\}$ | 29 |
- $\gamma_j^{\mathcal{R}, r}$: $\sum_{i=j}^{m}(R_i + 1), 1 \leq j \leq m$ and $\tilde{m} - j + 1, m + 1 \leq j \leq r$, $\gamma_{r+1} = 0$, for $r \in \{m + 1, \ldots, \tilde{m}\}$ | (continued) |
Notation Explanation

\( f_{1\ldots r;m:n}, F_{1\ldots r;m:n} \) Joint density function/cumulative distribution function of 
\( X_{1\ldots r;m:n}, 1 \leq r \leq m \), w.r.t. a censoring plan 
\( \mathcal{R} \in \mathcal{C}_{m,n}, m \leq n \)

\( f_{r;m:n}, F_{r;m:n} \) Density function/cumulative distribution function of 
\( X_{r;m:n}, 1 \leq r \leq m \), w.r.t. a censoring plan 
\( \mathcal{R} \in \mathcal{C}_{m,n}, m \leq n \)

**Sequential order statistics**

\( X_j^* \) \( j \)th sequential order statistic based on \( F \)

\( Z_j^* \) \( j \)th exponential sequential order statistic (based on \( \text{Exp}(\mu, \vartheta) \))

\( \alpha_j \) Parameter w.r.t. the model of proportional hazard rates

\( \gamma_j^* \) \( \alpha_j(n - j + 1), 1 \leq j \leq n, \gamma_n^* = 0 \)

\( f_{s,1\ldots r}, F_{s,1\ldots r} \) Joint density function/cumulative distribution function of 
\( X_1^*, \ldots, X_r^*, 1 \leq r \leq n \)

\( f_{s,r}, F_{s,r} \) Density function/cumulative distribution function of \( X_r^* \), 
\( 1 \leq r \leq n \)

\( F_{s,1\ldots d}(|X_{d+1}^* = T) \) Joint distribution of \( X_1^*, \ldots, X_{d+1}^* \), conditional on \( X_{d+1}^* = T, d \in \mathbb{N} \)

\( X_{i,j}^* \) \( j \)th sequential order statistic w.r.t. the \( i \)th sample (based on \( F_i \)), \( 1 \leq i \leq k \)

\( Z_{i,j}^* \) \( j \)th exponential sequential order statistic w.r.t. the \( i \)th sample (based on \( \text{Exp}(\mu_i, \vartheta_i) \)), \( 1 \leq i \leq k \)

\( f_{si,1\ldots r}, F_{si,1\ldots r} \) Joint density function/cumulative distribution function of 
\( X_{i,1}^*, \ldots, X_{i,r}^*, 1 \leq r \leq n_i \)

\( f_{si,r}, F_{si,r} \) Density function/cumulative distribution function of \( X_{i,r}^* \), 
\( 1 \leq r \leq n_i \)

**Hybrid censored order statistics**

\( X_j^H \) \( j \)th hybrid censored order statistic, w.r.t. a hybrid censoring scheme \( H \) and an underlying set of ordered random variables
\( X_{(1)}, \ldots, X_{(m_h)}, m_h \leq n \)

\( U_j^H \) \( j \)th uniform hybrid censored order statistic, w.r.t. a hybrid censoring scheme \( H \) and an underlying set of ordinary order statistics
\( U_{1,n}, \ldots, U_{m_h,n}, m_h \leq n \)

\( Z_j^H \) \( j \)th exponential hybrid censored order statistic, w.r.t. a hybrid censoring scheme \( H \) and an underlying set of sequential order statistics
\( Z_{1,m_h}, \ldots, Z_{m_h}, m_h \leq n \)

\( f_{X_{i,j}^H, 1 \leq d \mid D(d)}, F_{X_{i,j}^H, 1 \leq d \mid D(d)} \) Joint density function/cumulative distribution function of
\( X_1^H, \ldots, X_d^H \) w.r.t. the counter setting \( D \in \mathcal{S}_{D}(H), d \in \mathcal{I}(D) \)

(continued)
### Notation Explanation Page

<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{U^H,1 \leq j \leq d}(d) )</td>
<td>Joint density function/cumulative distribution function of the uniform hybrid censored order statistics ( U^H_1, \ldots, U^H_d ) w.r.t. the counter setting ( D \in \mathcal{S}(H), d \in \mathcal{I}(D) )</td>
<td></td>
</tr>
<tr>
<td>( F_{U^H,1 \leq j \leq d}(d) )</td>
<td>Joint density function/cumulative distribution function of the exponential hybrid censored order statistics ( Z^H_1, \ldots, Z^H_d ) w.r.t. the counter setting ( D \in \mathcal{S}(H), d \in \mathcal{I}(D) )</td>
<td></td>
</tr>
</tbody>
</table>

### Sets

- **A**
  \[ \{ x \in \mathbb{R}^d | x \notin A \}, A \subseteq \mathbb{R}^d \]

- **A \setminus B**
  \[ \{ x \in A | x \notin B \} = A \cap B^c, A, B \subseteq \mathbb{R}^d, B \subseteq A \]

- **C^{(d)}(A)**
  \[ \{ g : A \rightarrow \mathbb{R} | g \text{ is } d\text{-times continuously differentiable on } A \}, A \subseteq \mathbb{R}, d \in \mathbb{N} \]

- **I_{m,n}**
  \[ \{ m, \ldots, n \} \]

- **I_{k,d}^{[*]}**
  \[ \{ k, \ldots, d \} \setminus \{ i^* \}, i^* \in \{ k, \ldots, d \}, k < d \]

- **N**
  \[ \{ 1, 2, \ldots \} \]

- **N_0**
  \[ \{ 0 \} \cup N \]

- **N_{\geq n}**
  \[ \{ n, n+1, \ldots \}, n \in \mathbb{N} \]

- **\( \mathcal{H}_{d-1,i^*}^{(s)} \)**
  \[ \text{Half-space of dimension } (d-1), d \in \mathbb{N} \text{ w.r.t. an index } i^* \in \{ 1, \ldots, d \} \text{ and a parameter } s \in \mathbb{R} \]

- **\( \Pi_{d+1} \)**
  \[ \text{Set of all permutations of } (0, 1, \ldots, d) \]

- **\( \mathbb{R} \)**
  \[ \mathbb{R} \setminus \{ -\infty, \infty \} \]

- **\( \mathbb{R}^d \)**
  \[ \text{d-fold Cartesian product of } \mathbb{R} \]

- **\( \mathcal{M}_{d-1,i^*}^{[r]}(s|\beta,t) \)**
  \[ \text{Particular polytope} \]

- **\( \mathcal{S}_d \)**
  \[ \text{Standard } d\text{-simplex} \]

- **\( \mathcal{S}^{(s)}_{d-1,i^*} \)**
  \[ \text{Simplex of dimension } (d-1), d \in \mathbb{N} \text{ w.r.t. an index } i^* \in \{ 1, \ldots, d \} \text{ and a parameter } s \in \mathbb{R} \]

### Ordered cones

- **\( \Sigma_d^{[r]} \)**
  \[ \text{Trivial cone: } \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq \ldots \leq x_d < F^{-1}(1) \} \]

- **\( \Sigma_{d,F,T} \)**
  \[ \text{Truncated cone: } \{ x_d \in \mathbb{R}^d | F^{-1}(0) < x_1 \leq \ldots \leq x_d \leq T \}, T \in (F^{-1}(0), \infty) \]

- **\( \Sigma^d \)**
  \[ \text{Uniform cone: } \{ x_d \in [0,1]^d | 0 \leq x_1 \leq \ldots \leq x_d \leq 1 \} \]

### Symbols

- **\( D \)**
  \[ \sum_{j=1}^m \mathbb{1}_{(-\infty,T_j]}(X_j^*), T \in (0, \infty) \]

- **\( D_i \)**
  \[ \sum_{j=1}^m \mathbb{1}_{(-\infty,T_i]}(X_j^*), T_i \in (0, \infty) \]

- **\( \mathcal{D} \)**
  \[ \text{Counter setting w.r.t. a hybrid censoring model} \]

(continued)
<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d \mathbf{x}_d )</td>
<td>( d \mathbf{x}_d \cdots d \mathbf{x}_1 )</td>
</tr>
<tr>
<td>( \gamma_{d+1,c} )</td>
<td>((\gamma_{d+1,c}, \ldots, \gamma_{1,c}) \in \mathbb{R}^{d+1}, d \in \mathbb{N}, c \in \mathbb{R})</td>
</tr>
<tr>
<td>( \gamma_{2,d+1,c} )</td>
<td>((\gamma_{d+1,c}, \ldots, \gamma_{2,c}) \in \mathbb{R}^d, d \in \mathbb{N}, c \in \mathbb{R})</td>
</tr>
<tr>
<td>( I(\mathcal{D}) )</td>
<td>Set of valid integers w.r.t. the counter setting ( \mathcal{D} )</td>
</tr>
<tr>
<td>( P_F = F )</td>
<td>Implies that the distribution with the cumulative distribution function ( F ) according to the distribution ( F ) is considered. E.g., for ( F = \text{Exp}(\mu, \vartheta) ), we have ( F = 1 - e^{-(\cdot - \mu)/\vartheta}, \mu \in \mathbb{R}, \vartheta &gt; 0 )</td>
</tr>
<tr>
<td>( \mathfrak{S}_\mathcal{D}(H) )</td>
<td>System of counter settings w.r.t. a hybrid censoring model ( H )</td>
</tr>
<tr>
<td>( \mathbf{x}_d )</td>
<td>( (x_1, \ldots, x_d), d \in \mathbb{N} )</td>
</tr>
<tr>
<td>( \mathbf{x}_{i,d} )</td>
<td>( (x_{i1}, \ldots, x_{id}), d, i \in \mathbb{N} )</td>
</tr>
<tr>
<td>( X \sim F )</td>
<td>( X ) is distributed according to a cumulative distribution function ( F )</td>
</tr>
<tr>
<td>( \overset{\text{iid}}{\sim} )</td>
<td>Independent and identically distributed</td>
</tr>
<tr>
<td>( \overset{\text{d}}{=} )</td>
<td>Identical distribution</td>
</tr>
<tr>
<td>( x \to c^- )</td>
<td>Limit at ( c ) from the left, ( c \in \mathbb{R} )</td>
</tr>
<tr>
<td>( x \to c^+ )</td>
<td>Limit at ( c ) from the right, ( c \in \mathbb{R} )</td>
</tr>
</tbody>
</table>

**Operations**

\( \times_{j=1}^k A_i \) | \( k \)-fold Cartesian product of the sets \( A_1, \ldots, A_k \) |
| \( d_{\bullet_k} \) | \( \sum_{j=1}^k d_j \) for \( d_k \in \mathbb{R}^k \) |
| \( D_{\bullet_k} \) | \( \sum_{j=1}^k D_j \) for discrete random variables \( D_1, \ldots, D_k \) |
| \( \prod_{j=1}^0 c_j \) | equals one |
| \( \sum_{j=1}^0 c_j \) | equals zero |

**Distributions**

\( \text{Beta}(\alpha, \beta) \) | Beta distribution |
| \( \text{U}(a, b) \) | Uniform distribution |
| \( \text{Exp}(\mu, \vartheta) \) | Exponential distribution |
| \( \Gamma(\alpha, \beta) \) | Gamma distribution |

**Special functions**

\( n! \) | \( \prod_{j=1}^n j \), \( n \)-factorial |
<p>| ( \begin{pmatrix} n \end{pmatrix}<em>k ) | ( \frac{n!}{k!(n-k)!} ), binomial coefficient |
| ( B</em>{d-1}(\cdot|t_0, \ldots, t_d) ) | (Curry-Schoenberg) B-spline of degree ( d - 1 ) with knots in ( t_0, \ldots, t_d ) |
| ( \det(A) ) | Determinant of ( A \in \mathbb{R}^{d \times d} ) |</p>
<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{d-1}([t_0, \ldots, t_d])$</td>
<td>(Normalized) B-spline of degree $d-1$ with knots in $t_0, \ldots, t_d$</td>
<td>40</td>
</tr>
<tr>
<td>$[t_0, \ldots, t_d; g]$</td>
<td>Divided differences of $g$ w.r.t. the knot sequence $t_0, \ldots, t_d$</td>
<td>38</td>
</tr>
<tr>
<td>$[t_0, \ldots, t_d]xg(x)$</td>
<td>Divided differences of $g$ w.r.t. the knot sequence $t_0, \ldots, t_d$</td>
<td>38</td>
</tr>
<tr>
<td>$[t_d^{(1)}]<em>{x_1} \ldots [t_d^{(k)}]</em>{x_k}g$</td>
<td>$k$th iterated divided differences of $g$ w.r.t. the knot sequences $t^{(1)}_d, \ldots, t^{(k)}_d$</td>
<td>51</td>
</tr>
<tr>
<td>$f^{(n)}$, $\frac{d^n}{dx^n}f$</td>
<td>$n$th derivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial}{\partial x}f$</td>
<td>Partial derivative of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, w.r.t. $x_i$, $1 \leq i \leq d$</td>
<td></td>
</tr>
<tr>
<td>$D\Phi$</td>
<td>Jacobian matrix w.r.t. $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$</td>
<td></td>
</tr>
<tr>
<td>$f^{(-n)}$</td>
<td>$n$th antiderivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$</td>
<td></td>
</tr>
<tr>
<td>$f^\Gamma_{\vartheta, \beta}$</td>
<td>Density function of the gamma distribution $\Gamma(\vartheta, \beta)$, with parameters $\vartheta, \beta &gt; 0$</td>
<td>202</td>
</tr>
<tr>
<td>$\exp{x}$, $e^x$</td>
<td>Exponential function</td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>Cumulative distribution function</td>
<td></td>
</tr>
<tr>
<td>$F^{-1}$</td>
<td>Quantile function of a cumulative distribution function $F$</td>
<td>202</td>
</tr>
<tr>
<td>$1_A$</td>
<td>Indicator function on the set $A \subset \mathbb{R}$, i.e., $1_A(x) = \begin{cases} 1, &amp; x \in A, \ 0, &amp; x \notin A \end{cases}$</td>
<td></td>
</tr>
<tr>
<td>$L(\cdot</td>
<td>x_d)$</td>
<td>Likelihood function w.r.t. the data $x_d \in \mathbb{R}^d$</td>
</tr>
<tr>
<td>$\ell(\cdot</td>
<td>x_d)$</td>
<td>Log-likelihood function w.r.t. the data $x_d \in \mathbb{R}^d$</td>
</tr>
<tr>
<td>$\ln(x)$</td>
<td>Natural logarithm</td>
<td></td>
</tr>
<tr>
<td>$\Gamma(\alpha)$</td>
<td>Gamma function</td>
<td></td>
</tr>
<tr>
<td>$G_{r,r}^{0</td>
<td>\gamma^<em>_1, \ldots, \gamma^</em>_r}$</td>
<td>Meijer’s G-function</td>
</tr>
<tr>
<td>$\text{vol}_d(A)$</td>
<td>$d$-dimensional volume of a measurable set $A \subset \mathbb{R}^d$</td>
<td></td>
</tr>
<tr>
<td>$(x)^+$</td>
<td>$\max{0, x}$, the positive part of $x$</td>
<td></td>
</tr>
<tr>
<td>$(x)^-$</td>
<td>$\min{0, x}$, the negative part of $x$</td>
<td></td>
</tr>
</tbody>
</table>

**Abbreviations**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>IID</td>
<td>Independent and identically distributed</td>
</tr>
<tr>
<td>cf.</td>
<td>Compare (confer)</td>
</tr>
<tr>
<td>e.g.</td>
<td>For example (exempli gratia)</td>
</tr>
<tr>
<td>i.e.</td>
<td>That is (id est)</td>
</tr>
<tr>
<td>MLE</td>
<td>Maximum likelihood estimator</td>
</tr>
<tr>
<td>Type-I (S/P)HCS</td>
<td>Type-I (sequential/progressive) hybrid censoring scheme</td>
</tr>
<tr>
<td>Type-I M(S)HCS</td>
<td>Multi-sample Type-I (sequential) hybrid censoring scheme</td>
</tr>
<tr>
<td>Type-II (P)HCS</td>
<td>Type-II (progressive) hybrid censoring scheme</td>
</tr>
<tr>
<td>Type-I G(P)HCS</td>
<td>Generalized Type-I (progressive) hybrid censoring scheme</td>
</tr>
<tr>
<td>Type-II G(P)HCS</td>
<td>Generalized Type-II (progressive) hybrid censoring scheme</td>
</tr>
</tbody>
</table>
(continued)

<table>
<thead>
<tr>
<th>Notation</th>
<th>Explanation</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type-i U(P)HCS</td>
<td>Unified Type-i (progressive) hybrid censoring scheme, $i \in {I, II, III, IV}$</td>
<td>181</td>
</tr>
<tr>
<td>GU(P)HCS</td>
<td>General unified (progressive) hybrid censoring scheme</td>
<td>165</td>
</tr>
<tr>
<td>w.r.t.</td>
<td>With respect to</td>
<td></td>
</tr>
</tbody>
</table>
Bibliography


Index

A
approach
expected value, 3, 6, 20, 43, 71–73, 87, 89, 129
moment generating function, 1, 5, 7, 20, 23, 91, 109, 131, 136, 143
volume, 6, 17, 37, 57, 71, 72, 89

B
B-spline, 2–6, 21, 23, 37–56, 72, 81, 91, 120, 179
convolution, see convolution
Curry-Schoenberg, 40
geometric characterization, 6, 44, 68, 84, 128, 134
integration, 51–55
knots, see knot sequence
normalized, 40, 43, 54
properties, 42
bad-as-old, 32
binomial theorem, 52
burn-in process, 6

C
censoring plan, 27
(right) extended, 7, 9, 29, 136, 166
admissible, 28
compressed, 29
initially planned, 7, 29, 31, 132, 135, 136, 166, 195
optimal, 30
combined hybrid censoring sampling, 1, 181
conditional expectation, see expected value
cone
trivial, 2, 26, 36
truncated, 2, 61, 97
uniform, 43, 102
convergence
$L^1$, 94

weak, 94, 101
convolution, 51
B-spline, 51, 113–114, 120, 202
counter setting, 2, 14–19, 152
counting process, 14, 34, 57
cumulative distribution function
conditional
for the exponential distribution, 74, 76, 80, 91
for the uniform distribution, 103, 163
conditional joint
for generalized Type-I progressive hybrid censored order statistics, 145
for generalized Type-II progressive hybrid censored order statistics, 154
for the exponential distribution, 89
for the uniform distribution, 106
for Type-I sequential hybrid censored order statistics, 59
for Type-II progressive hybrid censored order statistics, 124
for the uniform distribution, 139, 151, 177, 201
marginal
for order statistics, 26
for progressively Type-II censored order statistics, 28
for sequential order statistics, 33, 34

D
density function
conditional joint
for general unified progressive hybrid censored order statistics, 171
for generalized Type-I progressive
hybrid censored order statistics, 145
for generalized Type-II progressive hybrid censored order statistics, 154
for the exponential distribution, 61, 65, 111, 125, 127, 146, 155
for the uniform distribution, 61, 138, 150, 161
for Type-I sequential hybrid censored order statistics, 59
for Type-II progressive hybrid censored order statistics, 124
gamma, 3, 92
joint
for order statistics, 26
for progressively Type-II censored order statistics, 28
for sequential order statistics, 33
marginal
for order statistics, 26
for progressively Type-II censored order statistics, 28
for sequential order statistics, 33
distribution
beta, 31, 66, 201
degenerated, 20, 63, 80, 100, 120, 156
exponential, 4–10, 21, 30, 35, 112, 201
gamma, 202
one-point, 101
spline, 55
uniform, 26, 41, 65, 96, 201
Weibull, 21
distribution function, see cumulative distribution function
divided differences, 38–40, 92
iterated, 51–53, 114–119, 202
knots, see knot sequence
properties, 38
two-dimensional, 53, 202
E
EM algorithm, 123
estimation
Bayes, 22
maximum likelihood, 62, 99, 112, 126, 138, 147, 151, 156, 162, 173, 177, 183, 186, 190, 193
expected value, 129, 139
conditional, 21, 71, 74, 87, 102, 105, 129, 139
expected value approach, see approach
F
Fisher information, 23
function
B-spline, see B-spline
complete beta, 201
density, see density function
gamma, 201
likelihood, see maximum likelihood estimation
log-likelihood, 63, 100
Meijer’s G-function, 33–34, 39, 90
moment generating, 20, 73, 181
G
gamma coefficients, see values for γ∗’s
general unified (progressive) hybrid censoring, see hybrid censoring scheme
generalized (progressive) hybrid censoring, see hybrid censoring scheme
generalized mean value theorem, 95
H
half-space, 3, 19, 20, 44–50
hybrid censoring scheme
adaptive Type-I progressive, 22, 31, 135–137
general unified, 176–178, 195–197
general unified progressive, 11, 12, 165–176
generalized Type-I, 1, 11, 22, 23, 143, 150–151, 195–197
generalized Type-I progressive, 11, 12, 23, 143–149
generalized Type-II, 1, 9, 22, 161–164, 195–197
generalized Type-II progressive, 8–10, 12, 16, 152–160, 165
multi-sample Type-I, 110, 195–197
multi-sample Type-I progressive, 110
multi-sample Type-I sequential, 10, 12, 13, 109–121
Type-I, 1, 3–7, 22–24, 57, 64, 76, 87, 96–107, 116, 143, 195–197
Type-I progressive, 1, 5, 12, 22, 76, 87, 91, 93, 116
Type-I sequential, 4–6, 12, 15, 57–91, 96, 110, 111, 136, 143, 165
Type-II, 1, 7, 22, 96, 123, 132, 137–141, 143, 195–197
Type-II progressive, 7–9, 16, 22, 123–137, 143, 165
unified Type-I, 1, 11, 181, 195–197
unified Type-I progressive, 181–184
unified Type-II, 1, 11, 184, 195–197
unified Type-II progressive, 184–187
unified Type-III, 1, 11, 187, 195–197
unified Type-III progressive, 187–191
unified Type-IV, 1, 11, 191, 195–197
unified Type-IV progressive, 191–194
hyperplane, 44

K
knot sequence
for the B-spline
with knots not all equal, 54, 55, 92
with multiple knots, 81
in divided differences
with knots all equal, 38
with knots not all equal, 51, 92
with multiple knots, 39, 93
with pairwise distinct knots, 38, 53, 93, 130
knots, see knot sequence

M
mean experiment duration, 196
mean of observed failures, 196
minimal repair, 32, 63–64
model
hybrid censoring, see hybrid censoring scheme
probability, see probability model
model-simplification, 165–172
modified total time on test statistic, 60
moment generating function, see function generating function approach, see approach
moments, 74
monotonicity
for the maximum likelihood estimator, 96
for the total time on test statistic, 94
multi-sample, see hybrid censoring scheme

N
negative part, 92
Newton-Raphson method, 123
NURBS, 37

O
order statistics, 25–32
generalized, 32
progressively Type-II censored, 27–30, 32, 40
sequential, 31–36

P
Peano’s formula, 42, 54, 56, 203
polytope, 20, 44, 47
positive part, 2, 92
probability model
IID progressive exponential model, 30
IID progressive model, 30
IID uniform model, 26
multi-sample sequential exponential model, 35
multi-sample sequential model, 35
sequential exponential model, 35
sequential model, 35
progressive censoring
adaptive progressive Type-II censoring, 31
general progressive Type-II censoring, 12, 109
progressive hybrid censoring, see hybrid censoring scheme
progressive Type-I censoring, 179
progressive Type-II censoring, 23, 27–30, 109, 116, 148
proportional hazard rate, 31

Q
quantile function, 3, 202

R
record values, 32, 63–64, 82
kth, 32, 40, 63, 81
recurrence relation for B-splines, 42
representation
B-spline
for adaptive Type-I progressive hybrid censoring, 136
for general unified progressive hybrid censoring, 173
for generalized Type-I progressive hybrid censoring, 148, 149
for generalized Type-II progressive hybrid censoring, 158, 160
for Type-I sequential hybrid censoring, 73, 86
for Type-II progressive hybrid censoring, 130, 135
for unified Type-I progressive hybrid censoring, 183
for unified Type-II progressive hybrid censoring, 187
for unified Type-III progressive hybrid censoring, 190
for unified Type-IV progressive hybrid censoring, 193

gamma
for generalized Type-I progressive hybrid censoring, 148
for generalized Type-II progressive hybrid censoring, 158
for Type-I progressive hybrid censoring, 93
for Type-I sequential hybrid censoring, 92–93
for Type-II hybrid censoring, 132
for Type-II progressive hybrid censoring, 130, 131

total time on test statistic, 6, 8, 9, 60
transformation
linear, 43, 45, 102, 113
non-linear, 65, 89, 105
trunched life test, 1
trunched power
negative, see negative part
positive, see positive part

Type-I censored sequential k-out-of-n systems, 6, 73
Type-I censoring, 23, 104, 109, 136
Type-II censoring, 27, 29, 97, 104, 109

unified (progressive) hybrid censoring, see hybrid censoring scheme
unified hybrid censoring scheme, 1, 184

values for γ∗’s
definition, 31
in multi-sample Type-I sequential hybrid censoring
alternating, 117, 120
decreasing, 117, 120
increasing, 117
pairwise distinct, 114
restriction, 113, 115
in Type-I sequential hybrid censoring
alternating, 73
decreasing, 73
pairwise distinct, 75, 76, 90, 93, 96
restriction, 64
with multiple occurrences, 63, 73, 74, 80
volume approach, see approach
volume formula
for a simplex, 50
for the intersection of a simplex with a half-space, 47, 50