Shear Strength Models for Reinforced and Prestressed Concrete Members

A thesis submitted in fulfillment of the requirements for the degree of
Doktor der Ingenieurwissenschaften
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submitted by

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This thesis is available online on the website of the university library.
Abstract

In this thesis, shear strength models for slender concrete members without and with shear reinforcement are developed. Although well over 100 years of research have been carried out in this field, the basic failure mode of flexural shear failure is still under discussion. Until today, empirical models for shear design of members without shear reinforcement have been the common case for design practice as it was not possible to derive simple design equations from more mechanically oriented theories. On the other hand, a mechanically sound model for the design of beams with shear reinforcement was available from the very beginning with the 45°-truss analogy. In the following, more refined models with a concrete contribution or lower strut inclinations were introduced to decrease the necessary amount of stirrups. However, existing prestressed concrete bridges were often designed based on the principal tensile stress criterion and therefore built with very little shear reinforcement which was less than the currently required minimum shear reinforcement. Since the shear design check of these structures can often not be performed with current code models based on truss models with a variable strut inclination, refined models are therefore necessary for the assessment of these structures.

In this thesis, a mechanical flexural shear model for beams without shear reinforcement is derived that accounts for the shear transfer actions from direct strut action, compression zone, crack processing zone, aggregate interlock and dowel action. Based on the mechanical model, a simplified closed form Critical Crack Width Model is derived. By linking the flexural shear capacity with flexural crack widths, the influence of axial forces can be accounted for consistently within this model. The comparison of the model with shear tests on RC beams, PC beams and RC beams in tension shows a very good agreement and it can be concluded that all relevant influence parameters are considered correctly.

Moreover, the shear capacity of beams with shear reinforcement was investigated. The behavior of beams with very little shear reinforcement can be considered similar to the behavior of beams without shear reinforcement, but with an additional stirrup contribution. For higher shear reinforcement ratios, the beams behave in agreement with an equilibrium based truss model with a variable strut inclination. To distinguish these failure modes in a consistent manner, a criterion based on the mechanical shear reinforcement ratio of the beam was derived. On this basis, shear design procedures for the design of new structures as well as for the economic assessment of existing structures are presented. The partial safety factors for the proposed models are determined by probabilistic evaluations according to EN 1990. This thesis thus presents a comprehensive procedure for design and assessment of structures under shear loading. Judging from test evaluations it can be expected that the presented approaches will be especially beneficiary for the assessment of existing structures like bridges.
Preface

This dissertation on shear strength of reinforced and prestressed concrete members was written during my time as a researcher at the Institute of Structural Concrete at RWTH Aachen University.

Several research projects funded by the Federal Highway Research Institute of Germany (BASt) have been my main motivation to work on the problem of shear strength of beams and the development of guidelines for the recalculation of existing bridges. I therefore thank the BASt for funding and supporting this work. Also, my work as an assistant to the convenor Josef Hegger in the Task Group on Shear, Punching Shear and Torsion of CEN/TC250/SC2/WG1 for the next generation of EC2 was a great help for me to gain knowledge in this field, particularly by the insightful presentations and discussions by the Professors A. Cladera, L.C. Hoang, A. Mari and A. Muttoni.

I especially want to thank the supervisor of my doctoral thesis, Prof. Josef Hegger, head of the Institute of Structural Concrete at RWTH Aachen University, for his continued support and mentoring over the years. This work would not have possible without the possibilities that I was given by Prof. Hegger. I also want to thank my second examiners Prof. Viet Tue Nguyen and Prof. Karl-Heinz Reineck for their thorough and critical review of my thesis.

The success of this thesis is also a product of the teamwork among the colleagues of our Institute, in particular with my friends and colleagues from Test Hall C. Without our creative exchange, mutual motivation and long discussions during and after work, this project would have been incomparably more difficult to finish. I also want to thank my student assistants and bachelor and master students who helped me to manage the daily workload and experimental investigations.

Last, but most importantly I want to thank my family for their support: my parents for raising me and providing me with an education and my two sisters. I want to thank my wife He-Jung for always supporting me in my work and for taking care of our daughter Min-Ah.

Aachen, November 2016

Martin Herbrand
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Notations

In the following, the most important units, abbreviations and symbols are listed. Unlisted symbols are explained within the thesis.

Units

Strain: %, ‰
Force: N, kN, MN
Stress: N/mm², MN/m², MPa
Distance: mm, cm, m

Abbreviations

CCCM Compression Chord Capacity Model
CCWM Critical Crack Width Model
cf. confer (“see also”)
CFT Compression Field Theory
COV Coefficient of Variation
CSCT Critical Shear Crack Theory
CSDT Critical Shear Displacement Theory
MCFT Modified Compression Field Theory
MV Mean Value
RC Reinforced Concrete
PC Prestressed Concrete
SCCWM Simplified Critical Crack Width Model

Capital Latin Letters

$A$ area, equation parameter
$A_s$ reinforcement area
$A_{sw}$ transverse reinforcement area
$A_{sw, min}$ minimum transverse reinforcement area
### Notations

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<tr>
<td>$A_{sw,req}$</td>
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<td>$B$</td>
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<td>$V_{cc}$</td>
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<tr>
<td>$V_{Rd,ct}$</td>
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<td>$V_{Rd,c,max}$</td>
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<tr>
<td>$V_{Rd,c,min}$</td>
<td>minimum design shear capacity of beams without shear reinforcement</td>
</tr>
</tbody>
</table>
Notations

- $V_{Rd,\text{cp},\text{min}}$: minimum design shear capacity of prestressed concrete beams without shear reinforcement
- $V_{Rd,\text{ct},\text{min}}$: minimum design shear capacity of beams in tension without shear reinforcement
- $V_{Rm,c}$: mean shear capacity of beams without shear reinforcement
- $V_{sy}$: stirrup contribution
- $V_u$: ultimate shear capacity
- $W$: section modulus
- $W_v$: internal work

Lowercase Latin Letters

- $a$: length of the shear span
- $b$: width
- $b_v$: shear width
- $b_{v,\text{eff}}$: effective shear width
- $b_w$: web width
- $d$: effective depth
- $d_{ag}$: maximum aggregate size
- $d_0$: size effect constant
- $d_{b,\text{crit}}$: critical crack band width
- $e$: eccentricity
- $f_c$: concrete compressive strength
- $f_{ck}$: characteristic concrete compressive strength
- $f_{cm}$: mean concrete compressive strength
- $f_{ct}$: concrete tensile strength
- $f_{ctm}$: mean concrete tensile strength
- $f_p$: prestressing steel strength
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<td>$f_y$</td>
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<td>$v$</td>
<td>dimensionless shear capacity</td>
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### Notations

- \( v_c \): dimensionless shear capacity of beams without shear reinforcement
- \( w \): crack width
- \( w_1 \): crack width for which concrete tensile stresses are zero
- \( w_0 \): critical crack width above which aggregate interlock stresses cannot be transmitted
- \( w_k \): crack width at reinforcement level
- \( x \): depth of the compression zone
- \( x_0 \): depth of the compression zone
- \( x_1 \): distance between neutral axis and maximum tensile stresses in the concrete tension zone
- \( x_2 \): depth of the remaining post-peak concrete tension zone
- \( x_{dw} \): point from which dowel action is activated
- \( y_s \): distance of mild reinforcement from the center line of gravity
- \( y_p \): distance of tendon from the center line of gravity
- \( z \): inner lever arm
- \( z_{cog} \): distance of center of gravity from cross-section bottom

### Greek Letters and Symbols

- \( \varnothing \): diameter
- \( \alpha_e \): ratio of the Young’s moduli of reinforcement and prestressing steel
- \( \alpha_p \): coefficient for shear capacity in compression
- \( \alpha_t \): coefficient for shear capacity in tension
- \( \beta_r \): shear crack angle
- \( \gamma_c \): partial safety factor for concrete for uncertainty in material properties
- \( \gamma_C \): partial safety factor for flexural shear failure for uncertainties in material properties (aleatoric) and model (epistemic)
- \( \gamma_m \): partial safety factor for uncertainty in material properties
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<td>$\nu$</td>
<td>effectiveness factor for concrete compressive strength</td>
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1 Introduction

1.1 Motivation

Although well over 100 years of research have been carried out on structural concrete, one of its basic failure modes, shear failure of beams without shear reinforcement, is still under discussion. While flexural failures can be predicted with great precision using a sound theoretical framework, one of the problems in shear research seems to be that a commonly accepted conceptual framework of its basic mechanisms could not be established at an early age. Early design methods from the beginning of the 20th century can be regarded as incomplete or flawed for lack of better knowledge. As research in shear got more attention in the 1950s and 60s, more basic research on fundamental shear transfer actions like dowel action or aggregate interlock was carried out, allowing for a mechanical analysis of the internal stress state in RC beams. However, empirical models for shear design of members without shear reinforcement have been and are still the common case for design practice as it was not possible to derive simple design equations from more sophisticated mechanical theories. Since empirical equations are often based on regressive analysis of test data, extending their scope of use on structures which exhibit parameters beyond the verified range may pose a threat. Reasonably simple design equations based on models that allow for their general use within the scope of the model are therefore necessary. This is also a necessity since it is not possible to verify all combinations of possible model parameters for flexural shear failure with tests.

In contrast, a mechanically sound model for the design of beams with shear reinforcement was available from the end of the 19th century with the 45°-truss analogy. In the following, more refined models with a concrete contribution or lower strut inclinations were introduced to decrease the necessary amount of stirrups. The invention of prestressed concrete even allowed for the construction of T-beam or box girder bridges without requiring shear reinforcement. These bridges were often built with very little shear reinforcement which was less than the currently required minimum shear reinforcement. In view of increasing traffic loads, the shear design check of these structures can often not be performed with current code models. Therefore, refined models have to be deployed for the assessment of these structures.

1.2 Goals of this thesis

The aim of this thesis is to provide shear strength models on a rational basis for RC and PC beams with and without shear reinforcement. The models should fulfill the following requirements to be of use for practical design work:
1.2 Goals of this thesis

- **Ease of use**: The models should be easy to use by experienced designers in design practice. Models that require iterations and are thus elaborate for calculations by hand should be avoided. The models and the rules should be comprehensive.

- **Safe design of large structures**: The scope of the model should include regular beams and slabs in buildings as well as large structures such as bridge superstructures and large footings. For this, the model must account for the influence of the member size on the nominal shear capacity in a justified way.

- **Influence of statical system and load condition**: Most shear tests are performed on point loaded single span beams. However, regular structures are often built as continuous systems and are loaded predominantly by uniformly distributed loads. The model should therefore be able to account for different load conditions as it might otherwise result in conservative or unsafe design.

- **Influence of axial loads**: Many shear models consider the influence of axial loads resulting from prestressing in a simplified manner. For example, databank evaluations indicate that the model of EC2 is conservative for prestressed concrete. On the other hand, the EC2 model overestimates the effect of axial tensile forces which becomes a problem in structures like circular silos. The model should therefore realistically account for axial loads to allow for economic design of these structures.

- **Influence of the cross-section geometry**: The current shear model of EC2 only accounts for rectangular cross-sections. It is well known that the top and bottom chord of T- and I-beams also contribute to the shear capacity / Gör04/. With regard to the design and assessment of existing structures like bridge girders, a shear model should be able to consider the beneficiary effects of different types of cross-section geometries appropriately.

- **Influence of the reinforcement material**: The existing shear models for regular RC and PC structures are only applicable for regular steel reinforcement bars and tendons. As other reinforcement materials like Carbon Fiber Reinforced Polymers (CFRP) are becoming increasingly important, the model should be able to account for different mechanical properties of the reinforcement material.

- **Influence of various concrete properties**: In addition to normal strength concrete, there exist variations in aggregate types and sizes or concrete strengths types like HPC and UHPC. Although this thesis is focused on normal strength concrete, the effect of these important additional parameters should be accounted for.
2 State of the Art on Shear Strength Models

2.1 General

In this chapter, an overview of the development and the current state of art on shear strength models for beams without and with shear reinforcement will be given. Chapter 2.2 summarizes the generally accepted shear transfer actions. In chapter 0 the different modes of shear failure are described. Chapter 2.4 gives an overview of shear strength models for beams without shear reinforcement. Due to the large amount of available literature on this topic, the descriptions are kept brief. In chapter 2.5, the shear strength models for beams with shear reinforcement are summarized.

2.2 Shear Transfer Actions

In beams without or with shear reinforcement, the imposed shear forces must be equal to the internal shear forces of the beam. In general, several different shear transfer actions can be identified (Figure 2-1). Whereas the quantification of these actions is still subject of discussion, their existence is generally agreed upon:

- \(V_{cc}\): additional arch or direct strut action in non-slender or prestressed beams
- \(V_{sy}\): tensile forces in stirrups
- \(V_{cs}\): shear stresses in the uncracked compression zone
- \(V_{CPZ}\): tensile stresses in the crack processing zone
- \(V_{ai}\): shear stresses transferred via crack friction and aggregate interlock
- \(V_{dw}\): dowel forces of the longitudinal reinforcement or stirrups

![Figure 2-1: Shear transfer actions in an RC member](image)

Direct strut action \(V_{cc}\) is usually considered in short beams or prestressed concrete beams. In slender beams, direct strut action is only possible if the bond stresses of the longitudinal reinforcement are negligible /Zin99/. In this context, it is actually contradictory to speak of a “shear” contribution from direct strut action in slender beams. Firstly, a member with unbonded reinforcement is not an Euler-Bernoulli beam as there is a significant slip of the reinforcement. Secondly, such a member behaves like a simple strut-and-tie in which no shear forces but only axial forces exist. A shear force only exists in the sense of a vertical component of the strut but not in the mechanical sense of beam theory.
2.2 Shear Transfer Actions

Nevertheless, direct strut action can significantly lower the shear force that needs to be carried by beam action. In case of diagonal cracks, stirrups enable the beam to transfer additional transverse forces, resulting in a stirrup contribution $V_{sy}$. While these first two shear transfer actions were known, others had to be investigated to understand the failure of beams without shear reinforcement. From early on it was thought that the compression zone of a beam would have to carry most of the shear forces. Due to the flexural deformations of the compression zone, the parts on either side of a diagonal crack would have to rotate relatively to one another and thus cause additional dowel action $V_{dw}$ in the longitudinal reinforcement /Tau60/. This basic mechanism was already illustrated by MÖRSCH /Mör08/ in the beginning of the 20th century (Figure 2-2).

Figure 2-2: Compression zone and dowel action resulting from diagonal cracking (reproduced from /Mör08/)

Later, it was considered that the concrete parts or concrete teeth would not be able to rotate freely, as proposed by KANI /Kan64/, but that there would be an additional resistance from interlocking of the aggregates on the interfaces of the shear cracks (Figure 2-3). A lot of research has been carried out to derive laws for the aggregate interlock component based on deformations (e.g. /Tay70/, /Fen68/), the most extensive being from WALRAVEN /Wal80/.

Figure 2-3: Crack patterns and shear stresses on the concrete tooth (reproduced from /Fen66/)

The introduction of a component $V_{CPZ}$ from residual tensile stresses at the crack tip was based on the introduction of fracture mechanics for the analysis of quasi-brittle materials /Hil76/, /Hil83/, /Gus88/ (Figure 2-4). By this, it was possible to determine the length over which the concrete would still be able to transfer tensile stresses once a crack al-
ready opened. In case of inclined cracks, these tensile forces result in an additional vertical action $V_{CPZ}$. While this action is relatively large for smaller members, it becomes less important for larger members due to the increased overall crack widths. This reduced contribution from the crack processing zone can also in part explain the phenomenon of the size effect /Zin99/.

While the general shear transfer actions themselves can be understood quite easily, the challenge lies within quantifying the distribution of internal forces on the different actions. This becomes increasingly difficult since there is also a redistribution of internal forces with respect to the stage of loading. In general, a mechanical model that attempts to account for the shear transfer actions explicitly will usually require an iterative process as there exists a circular dependency between external forces, internal forces and the deformation of the cross-section. The problem therefore lies within the task to provide a reasonable design model that accounts for the actual behavior of beams without or with shear reinforcement.
2.3 Failure Modes of Concrete Beams under Shear Loads

Giving exact definitions of the different shear failure modes that may occur in beams is not as trivial as it might seem. Within the literature, the same terms are sometimes used to describe different types of shear failure or different terms might be used to describe the same shear failure mode. Generally, shear failure has something to do with inclined cracks which distinguishes it from flexural failure. However, sometimes inclined cracks might just be inclined flexural cracks. A second characteristic of shear failure is therefore that it happens prior to flexural failure. The clearest type of shear failure occurs when diagonal cracks form without prior flexural cracking, which is usually the case in I-beams with thin webs or prestressed T-beams (Figure 2-5a). In web shear failure, a sudden rupture in the web occurs because principal tensile stresses exceed the concrete tensile strength. At the same time, the utilization of principal compressive stresses is relatively small in this failure mode.

Another common failure mode is flexural shear failure caused by shear cracks that originate from flexural cracks and form a distinct horizontal crack branch (Figure 2-5b). The rotation of the concrete tooth can also cause secondary cracks along the longitudinal reinforcement. This failure mode is typical for slender RC beams with distinct flexural cracking within their shear span.

![web shear crack](image1)
![flexural shear crack](image2)

**Figure 2-5:** a) web shear failure; b) flexural shear failure (adopted from /ACI73/)

In /ACI73/, a further distinction to “diagonal tension failure“ is made which is quite similar to flexural shear failure (Figure 2-6a). A possible distinction is that this failure mode occurs in less slender members so that the critical flexural crack leading to shear failure is the first major flexural crack next to the support. The shear failure is then induced by a diagonal splitting failure of the concrete compressive zone.

Sometimes the shear crack is also initiated by splitting cracks along to longitudinal reinforcement so that the secondary cracks of the flexural shear failure are now primary cracks (Figure 2-6b). This type of failure is also called shear-tension failure and can occur due to loss of bond or due to premature splitting of the concrete cover in beams with large amounts of flexural reinforcement in one layer /ACI73/.
2.3 Failure Modes of Concrete Beams under Shear Loads

Figure 2-6: a) diagonal tension failure; b) shear-tension failure

In shorter beams with large amounts of longitudinal reinforcement, shear stresses and longitudinal stresses are concentrated in the compression zone causing a brittle failure by crushing of the compression zone (Figure 2-7a). This type of failure is described as a shear-compression failure. Similarly, short beams can also exhibit significant compressive stresses perpendicular to the beam axis enabling direct strut action (Figure 2-7b). The high utilization of compressive strength reduces the residual tensile capacity of the concrete /Kup73/ and causes a brittle rupture of the concrete along the compressive force path, which can most accurately described as an arch-rib failure /ACI73/.

Figure 2-7: a) shear compression failure; b) arch rib failure

In beams with transverse reinforcement, shear failure is often induced by yielding or rupture of the stirrups (Figure 2-8a). For beams with with large amounts of stirrups, crushing of the concrete in the webs may occur before yielding of the transverse reinforcement (Figure 2-8b). This failure type is especially influenced by the transversal tensile strains that reduce the residual concrete compressive strength. For lower amounts of shear reinforcement, simultaneous failure of stirrups and crushing or just stirrup failure may occur. The failure of beams with shear reinforcement can also occur in combination with previously mentioned failure modes.

Figure 2-8: a) stirrup failure; b) crushing of the concrete strut (reproduced from /Zil10/)
2.4 Shear Strength of Beams without Shear Reinforcement

2.4.1 General

The different shear strength models will be uniformly presented in SI-Units. To determine the shear strength based on the concrete strength $f_c$, some proposals might refer to the characteristic concrete compressive strength $f_{ck}$ while others might refer to the mean concrete cylinder strength $f_{cm,cyl}$. For the sake of simplicity, no differentiation will therefore be made between the various types of concrete strengths $f_c$ in different approaches. The shear strength models presented in this section will mostly provide predictions about the ultimate shear strength at failure without safety coefficients, thus providing mean values of the shear strength of reinforced concrete members without shear reinforcement.

2.4.2 Empirical shear strength models

In the English language, the term “empirical” usually describes a theory that is based on direct observation or experimental research rather than on abstract theorizing or speculation. This definition could however be misleading in the context of shear strength models as virtually all so-called mechanical models are based on detailed observations of crack paths in tests or on general experimental research on shear failure mechanisms. In the context of strength models, an empirical formula is therefore generally seen as one being mainly derived from regressional analysis of test data. The word “mainly” shows that drawing a line between mechanical and empirical models is also subject to the opinion different researchers, as all of the existing shear strength models depend on fitting certain coefficients based on experimental results. Therefore, this chapter includes models which are recognized as empirical models by most researchers because their main parameters were fitted by regression analysis without putting much emphasis on a mechanical explanation. However, some models which could also be identified as empirical might be represented in other chapters.

While empirical formulas have some serious drawbacks, they have had an undeniably strong impact on the shear design models in the various codes. It was known from early stages of structural concrete research that various parameters besides the concrete strength affect the shear strength of concrete beams. In 1909, TALBOT stated, that the nominal shearing stress $v_c$ varies with respect to the amount of longitudinal reinforcement, the relative length of the beam and the quality and strength of the concrete /Tal09/. Although test results of 106 beams without shear reinforcement were evaluated by TALBOT with respect to different parameters (Figure 2-9), no specific formula was provided to describe the findings in mathematical terms /ACI62/. The nominal shear strength was thus determined only on the basis of the concrete strength $f_c$ for the decades to come.
2.4 Shear Strength of Beams without Shear Reinforcement

In 1945, MORETTO published test results of concrete beams (mainly with shear reinforcement) and derived an empirical formula that included the influence of the longitudinal reinforcement ratio $\rho$ on the shear capacity of a beam without shear reinforcement according to Eq. (2-1) /Mor45/. However, MORETTO stated that the experimental database was too small to proof the accuracy of his assumption.

$$V_c = (0.088 f_c + 30.4 \rho) b_n d$$  \hspace{1cm} (2-1)

Based on further shear tests on beams with and without shear reinforcement, CLARK introduced an empirical formula which included the influence of the longitudinal reinforcement ratio, the concrete strength $f_c$ and the shear span to depth ratio $a/d$ according to Eq. (2-2) /Cla51/. Though CLARK stated that the shear resistance was predicted rather well with his empirical formula, he emphasized that the formula was not for general application but rather valid for beams with diagonal tension failure within the tested range.

$$V_c = \left(48.3 \rho + 0.12 f_c \frac{d}{a}\right) b_n d$$  \hspace{1cm} (2-2)

One problem of utilizing the $a/d$-ratio for considering the shear slenderness was the restriction to point loaded single span beams. In the 1950s, MOODY ET AL. carried out test series on beams without stirrups that included tests on single span beams under point
loads /Moo54/ as well as single span beams with loaded cantilever arms /Moo55a/. The latter test series accounted for continuous support conditions, which led to the introduction of the $M/Vd$ ratio for the evaluation of the tests instead of the $a/d$-ratio, with $M$ being the maximum moment and $V$ being the maximum shear force in the investigated shear span. Some considered the $M/Vd$-ratio concept a breakthrough towards an empirical solution of shear and diagonal tension as a design problem /ACI62/. MOODY & VIEST seemed to be more critical, stating that within the studies performed “no single equation applicable to both simple and restrained beams could be established” /Moo55b/. Nevertheless, the ACI-ASCE Committee 326 recommended an update of the ACI 318-56 Building Code according to Eq. (2-3) /ACI62/. This formula is essentially still part of the ACI 318-08 Building Code /ACI08/, only slightly modified to account for lightweight concrete. The shear strength is not related to the concrete compressive strength $f_c$ but to an empirical approximation to the modulus of rupture by $f_c^{1/2}$, which is based on the works of MORROW /Mor57/ and KESLER /Kes54/. The $M/Vd$-ratio relates to the ratio of moment and the shear force in the respective control section.

$$V_c = \left(0.158\sqrt[3]{f_c} + 17.2\frac{Vd}{M}\right) b_v d \leq 0.29\sqrt[3]{f_c} b_v d$$  \hspace{1cm} (2-3)

The influence of the member size on the nominal shear strength, namely the size effect, was thought to be a minor influence on the shear capacity and was thus neglected. LEONHARDT & WALther /Leo62/ published their own findings on the shear behaviour of RC beams in 1962 based on the tests of MOODY and their own tests. Amongst others, they identified the concrete strength $f_c$, the reinforcement ratio $\rho$, the cross-section depth $d$, the cross-section geometry and the loading condition (point loads or uniformly distributed loads) as important parameters for shear strength. Based on the tests of MOODY, they concluded that the shear capacity was related to $f_c^{1/3}$. Pointing out that the relative shear strength decreased by increasing the depth of a cross-section, LEONHARDT & WALTHER concluded that the limit of the influence of the effective depth was at about 400 mm, based on tests results that included beams with a maximum height of 600 mm. This result seemed to be in accordance with investigations performed by FORSELL /For54/ and RÜSCH et al. /Rue62a/, who concluded that the limit of the size influence was at 300–400 mm or at 150–200 mm, respectively.

Apparently based on the aforementioned works, HEDMAN & LOSBERG /Hed78/ proposed Eq. (2-4) which was introduced into Model Code 1978 /MC78/ in similar form. It included a size effect factor that was constant for members with an effective cross-section depth above 600 mm.

$$V_c = 0.09(1.75 - 1.25d)(1 + 50\rho)\sqrt[3]{f_c} b_v d$$  \hspace{1cm} (2-4)

where
1.75 − 1.25d ≥ 1.0, d in m

Most of the previous equations are written in sum form. In 1968, a factorial approach was presented by ZSUTTY /Zsu68/. By a regressional evaluation of experimental results of 151 slender shear tests, he identified the concrete strength $f_c$, the longitudinal reinforcement ratio $\rho$ and the shear slenderness $a/d$ as the governing influences on the shear capacity according to Eq. (2-5). The main advantage of a factorial approach is that a dimensional space of a complex problem like shear can be reduced by forming dimensionless groups (cf. /Pha03/). As a result, ZSUTTY very much like LEONHARDT & WALTHER concluded that the shear strength of RC beams rather relates to $f_{c}^{1/3}$. He later added an equation to account for the arch action in short beams by Eq. (2-6) /Zsu71/.

$$V_c = 2.2 \left( f_c \frac{d}{a} \right)^{1/3} b_n d$$

(2-5)

$$V_c = 2.2 \left( f_c \frac{d}{a} \right)^{1/3} \left( \frac{2.5}{a/d} \right) b_n d \text{ for } a/d < 2.5$$

(2-6)

Obviously, these formulae also neglect the influence of the size effect on the shear capacity of members without transversal reinforcement, as the experimental database only included tests up to a cross-section height of 500 mm. An equation similar to ZSUTTY’s equation was later adopted in the Model Code 1990 /MC90/ by Eq. (2-7). As the size effect problem became more popular in the 1970s, this influence was added to ZSUTTY’s equation by a factor $\xi$.

$$V_c = 0.15 \cdot (3d/a_c)^{1/3} \cdot \xi (100f_c)^{1/3} b_n d$$

(2-7)

where

$$\xi = 1 + \sqrt{200/d}$$

Eq. (2-7) was introduced into Eurocode 2 and is still used to this day in form of Eq. (2-8) /EC211/, in which the influence of the shear slenderness is omitted.

$$V_c = \left( 0.18 \cdot k \cdot (100f_c)^{1/3} + 0.15\sigma_{y} \right) b_n d$$

(2-8)

where

$$k = 1 + \sqrt{200/d} \leq 2.0$$

In 1980, OKAMURA & HIGAI presented an equation based on an additive approach as they felt that a factorial approach would lead to extreme results if one of the factors had an unusual value /Oka80/. The approach was revised in 1986 by NIWA ET AL. since the

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1 Prior to ZSUTTY, KENNEDY published an empirical approach in 1967 including a size effect factor of $d^{-0.282}$, which was at the time the most progressive and most accurate size effect until the 1980s /Ken67/.
original equation was not applicable to depths over 1 m and small amounts of longitudinal reinforcement such as footings /Niw86/. The validation on 288 shear tests with Eq. (2-9) yielded a very good agreement and was consequently used within the Japanese Building Code JSCE, however, without the influence of \(a/d\). Here, the influence of the size effect law did not exhibit a lower limit as it was defined as \(d^{-1/4}\).

\[
V_c = 0.20 \left( \frac{f_c \rho}{d^{1/4}} \right)^{1/3} \left( 0.75 + \frac{1.40}{a/d} \right) b_n d
\]  

(2-9)

As the computational capabilities increased rapidly in the 1990s, researchers were no longer limited to simple regressive analysis to analyze data. Artificial Neural Networks (ANNs) were for example used by MARÍ & CLADERA /Cla04a/ or JUNG & KIM /Jun08/ to gain deeper insight into factors influencing shear behavior. ANNs are networks that are built to imitate the learning process of the human brain. The ANN is fed with input data from selected shear tests and learns to predict the shear capacity as accurately as possible. In /Cla04a/, this resulted in Eq. (2-10). A comparison of the equation with a much larger control database yielded lower COVs than other approaches available at the time.

\[
V_c = 0.13 \xi (100 \rho)^{1/2} f_c^{1/3} b_n d
\]  

(2-10)

where

\[
\xi = \frac{135000 \cdot f_c^{-1.1}}{s_x} \left( \frac{f_c}{75} \right)^{0.25} \leq 2.75
\]

\(25 \leq f_c \leq 100, f_c\) in MPa

Another computational method that was adopted for optimizing shear strength formulae is Genetic Programming, which is used for optimization of parametric problems /Ash03/. PÉREZ ET AL. used the Genetic Programming technique for optimizing the current EC2 formula for shear /Per12a/. Different optimized equations were computed with different scenarios, where Eq. (2-11) exhibited the best agreement with a shear database.

\[
V_c = 0.114 \left( 1 + \left( \frac{1600}{d} \right)^{0.42} \right) (100 \rho)^{0.37} f_c^{1/3} \left( \frac{Vd}{M} \right)^{0.21} b_n d
\]  

(2-11)

where

\[
\left( \frac{1600}{d} \right)^{0.42} \leq 5,0; \quad d \text{ in mm};
\]

\[
\frac{Vd}{M} \leq 1,0; \quad \rho \leq 0.08
\]
2.4 Shear Strength of Beams without Shear Reinforcement

There is certainly no deeper insight into the mechanical problem of shear failure by optimization of equations to fit an available dataset, which is also admitted by most authors. Computational methods might however not be as biased as the conceptions of individual researchers so that unexpected results from these investigations might actually lead to deeper insights if a mechanical explanation is sought. Based on the genesis of the presented empirical shear models, it can be concluded that no progress was made in the understanding of shear failure by an *a priori* theorization of shear failure. It was rather the unexpected behaviour of tests that *a posteriori* demanded researchers find comprehensive theoretical explanations.

2.4.3 Tooth or comb models

Although the main influence factors on the capacity of RC members without shear reinforcement were known in the 1950s and 1960s (longitudinal reinforcement, concrete strength, shear slenderness), a comprehensive mechanical model to explain the phenomenon of flexural shear failure was still lacking. At the time, providing a safe empirical formula was difficult due to the variety of parameters involved. Therefore, KANI attempted to investigate the internal mechanisms of shear failure in his 1964 paper titled “The Riddle of Shear failure and its solution” /Kan64/. According to KANI, the shape of a beam cracked in bending resembles a comb like structure (Figure 2-10a). Unlike a beam with unbonded flexural reinforcement, such a comb-like structure has to function as a beam to transfer shear forces as the inner lever arm $z$ remains constant. A beam with unbonded reinforcement can develop an arch-like mechanism that allows for far greater shear resistances. The concrete teeth can be interpreted as cantilever arms that are loaded by bond forces $\Delta T$ due to beam action. The member will thus only be able to function as a beam as long as the concrete tooth is able to carry the $\Delta T$ forces. The assumption was proven by measuring the horizontal out-of-plane displacement of a concrete tooth (Figure 2-10b) and thus confirming the bending effect of the bond force $\Delta T$.

![Figure 2-10: a) Comb-like structure for beams in bending; b) deformation of a concrete tooth acting as a cantilever (reproduced from /Kan64/)](image)

The model by KANI for assessing the shear capacity of beams is therefore based on calculating the flexural capacity of the concrete tooth, i.e. tensile failure at the edge of the tooth. The result of the theoretical investigation is the description of the well-known shear valley by KANI, that represents the ratio of the shear capacity and the flexural
2.4 Shear Strength of Beams without Shear Reinforcement

capacity in pure bending with respect to the shear slenderness. KANI concluded that for beams with small \(a/d\)-ratios, an arch-like failure will occur whereas beams with a larger \(a/d\)-ratio will fail in a sudden comb failure. However, KANI’s model assumes that the concrete teeth can rotate freely, which is not the case due to restraints resulting from dowel effects of the longitudinal reinforcement and friction between the surfaces of the flexural cracks. Also, if the concrete teeth were to rotate freely, the bond force moment would have to be entirely resisted by the concrete teeth /Tay74/. It was pointed out by MOE /Moe62/ and FENWICK & TAYLOR /Fen68/ that the flexural stresses on the crack tip of the concrete tooth were much too high to be resisted by concrete tensile stresses so that other mechanisms would have to be accounted for. It was estimated that the distribution of internal shear forces on the main shear transfer actions would amount to 20-40 % by the compression zone, 33-50 % by aggregate interlock and 15-25 % by dowel action /Tay74/.

A comprehensive mechanical model that considers the stress-states of the concrete tooth was developed by REINECK /Rei90/, /Rei91/. The model is based on a free-body cut of a concrete tooth in a slender beam with flexural cracks inclined at an angle of \(\beta_{cr} = 60^\circ\) (Figure 2-11a).

![Figure 2-11: Tooth model according to REINECK a) beam model; b) free cut of the compression zone; c) free cut of the concrete tooth (reproduced from /Rei91/)](image)

The concrete tooth is subjected to friction stresses in the flexural cracks and cantilever action from the anchorage in the compression zone (Figure 2-11c). The shear components are determined by various constitutive relationships with respect to the kinematic mechanism in the flexural cracks so that the sum of all internal components is equal to...
2.4 Shear Strength of Beams without Shear Reinforcement

The external shear force. A simplified design equation based on the mechanical model was given by Eq. (2-12).

\[
V_u = \frac{b_w d 0.4 f_{ct} - 0.16 \frac{f_{ct}}{f_c} \lambda \frac{z_c}{d} N + V_{dv}}{1 + 0.16 \frac{f_{ct}}{f_c} \lambda \left( \frac{x}{d} + 0.5 \right)}
\]

(2-12)

where \[ V_{dv} = 1.4 \frac{0.89 f_c^{\frac{1}{3}}}{d^{\frac{1}{3}}} b_w d \]

\[ \lambda = \frac{f_c}{E_s \rho} \cdot \frac{d}{\Delta n_u} \]

\[ \Delta n_u = 0.9 \text{ mm (critical crack width for friction capacity)} \]

Recently, a model by YANG /Yan14/ was proposed that allows for the calculation of the three main shear transfer actions from compression zone \( V_{cs} \), aggregate interlock \( V_{ai} \) and dowel action \( V_{dw} \) with respect to the vertical crack opening at the tensile reinforcement. The contribution by aggregate interlock \( V_{ai} \) is based on a simplified equation that was derived from WALRAVEN’s /Wal80/ model, which is subject to the horizontal and vertical crack opening and thus subject to the exterior forces so that the capacity of the member has to be determined in an iterative process. In this process, an initial shear force \( V_{Ed} \) and a corresponding initial concomitant moment are assumed from which a flexural crack width \( w \) can be calculated. It is then assumed that for shear failure to occur, a critical shear displacement \( \Delta_{cr} \) at the longitudinal reinforcement is required to induce a flexural shear failure. The contribution by dowel action \( V_{dw} \) is then equal to the maximum dowel capacity given by /Bau70/. From the critical shear displacement and the flexural crack width, the contribution from aggregate interlock \( V_{ai} \) is calculated from Eq. (2-13).

\[
V_{ai} = f_c^{0.56} (d-x_0) b_w \frac{0.03}{w-0.01} \left( -978 \Delta_{cr}^2 + 85 \Delta_{cr} - 0.27 \right)
\]

(2-13)

The contribution from the compression zone \( V_{cs} \) is equal to the sum of the average shear stresses \( V/bz \) over the compression zone depth. If all three components are added, the resulting capacity is usually different from the initial shear force \( V_{Ed} \). The calculation of the internal forces then has to be repeated until the two values converge.

As shown in this section, tooth models have been important for the understanding of basic shear mechanisms in flexural shear failure. However, the work performed in this area has not yet been adopted in a design code.
2.4 Shear Strength of Beams without Shear Reinforcement

2.4.4 Models based on the capacity of the compression zone

In contrast to comb-based models, other researchers see the explanation of shear failure in the resulting compressive force path of an RC member. At the point where the linear part of the compressive force is deflected towards the support, a tensile force near the crack tip should exist due to equilibrium (Figure 2-12). By definition, the corresponding critical shear crack is the first major crack next to the support. The effect of aggregate interlocking between crack surfaces is neglected or discarded as an infeasible mechanism /Kot14/.

![Figure 2-12: Compressive force path from load introduction to the support (reproduced from /Kot83/)](image)

Different models have been proposed to predict the ultimate capacity of members in such a failure mode. A model by ZARARIS & PAPADAKIS /Zar01/ was based on the splitting strength of the concrete compression zone. The geometry of the splitting surface is determined by connecting the resultant of the compressive force with the support (Figure 2-13).

![Figure 2-13: Splitting test analogy according to ZARARIS & PAPADAKIS (reproduced from /Zar01/)](image)

Based on the model, Eq. (2-14) was derived which included a combined slenderness and size effect factor.

\[
V_u = \left(1.2 - 0.2 \frac{a}{d} \right) \frac{x_0}{d} f_{ct} b_w d
\]  

(2-14)
2.4 Shear Strength of Beams without Shear Reinforcement

where

\[ 1,2 - 0,2 \frac{a}{d} \geq 0,65, \ a \text{ in m} \]

A similar but strain based approach was used by PARK ET AL. /Par06/ and CHOI ET AL. /Cho07a/, /Cho07b/. They investigated the shear failure of beams in the compression zone according to the free-body diagram in Figure 2-14a. A failure criterion according to Rankine was used to predict the shear capacity (Figure 2-14b).

![Figure 2-14](image)

**Figure 2-14:** a) free body of the considered failure region; b) failure criteria for compression-compression and compression tension failure (reproduced from /Cho07a/)

Based on the mechanical model, a simplified approach according to Eq. (2-15) was derived. As can be seen, the approach uses the size and slenderness factor by ZARARIS & PAPADAKIS so that the first part of Eq. (2-15) is quite similar to Eq. (2-14). However, as the approach accounts for the nonlinear stress-strain behavior of concrete, the equation also accounts for a region \( c_c \) in which crushing of the concrete takes place and where the shear stresses are concentrated.

\[
V_c = 0,52 \lambda_s \sqrt{f_c b_w (x_0 - c_c)} + 0,45 f_c b_w c_c
\]

(2-15)

where

\[
\lambda_s = 1,2 - 0,2 \frac{a}{d} \geq 0,65
\]

\[
c_c = (1 - 0,43 \frac{a}{d}) x_0
\]

Other well known models include /Kim99/, /Tur03/, /Khu01/, /Zin99/, /Kön99/. Recently, a refined approach was introduced by MARI ET AL. /Mar14/, /Mar15/ and CLADERA ET AL. /Cla15/, /Cla16/. The approach can be characterized as a compromise.
2.4 Shear Strength of Beams without Shear Reinforcement

between models with pure beam action (Figure 2-15b) and pure arch effects (Figure 2-15c). By considering additional effects from stresses which are transferred in the crack processing zone, a shear stress distribution according to Figure 2-15d is obtained. The failure in the compression zone is determined by the biaxial failure criterion of KUPFER /Kup73/. The model accounts for contributions from the compression zone \( v_c \), the crack tip of the inclined shear crack \( v_w \), dowel action \( v_l \) and stirrups \( v_s \) according to Eq. (2-16) /Cla15/. The model is thus applicable for beams without and with shear reinforcement. The beneficiary effect of a flange in compression is considered by an effective shear width \( b_{v,\text{eff}} \), whereas the influence of a flange in tension is considered by a coefficient \( K_T \). The size effect term \( \zeta \) is also identical to the one used by ZARARIS & PAPADAKIS.

\[
V_u = (v_c + v_w + v_l + v_s) f'_{ct} b_w d \leq \frac{\alpha_{eq} b_w d \cdot v_c f_c}{\cot \theta + \tan \theta}
\]

where

\[
v_c = \zeta \left( 0.70 + 0.18 K_T + \left( 0.2 + 0.5 \frac{b}{b_w} \right) v_s \right) \frac{x_0}{d} + 0.02 K_T \frac{b_{v,\text{eff}}}{b}
\]

\[
\zeta = 1.2 - 0.2 a \geq 0.65; \ a \text{ in m}
\]

\[
K_T = 0.1 + 0.9 \frac{b_w}{b} + 2.5 \frac{h_{cens}}{h} \left( \frac{b_{cens} - b_w}{b} \right)
\]

\[
v_w = 167 \frac{f'_{ct}}{E_c} \left( 1 + \frac{2E_c G_f}{f_{ct}^2 d} \right) b_w
\]

\[
v_l = 0.23 \frac{\alpha_{eq} b_w}{1 - x_0/d} \text{ if } v_s > 0; \ v_l = 0 \text{ if } v_s = 0
\]

\[
v_s = 0.85 K_o \rho w \frac{f_{yw}}{f_{ct}}
\]

Figure 2-15: Stress distribution according to MARI ET AL. /Mar15/ for b) beam action; c) arch effect; d) proposed stress distribution (reproduced from /Mar15/).
2.4 Shear Strength of Beams without Shear Reinforcement

\[
K_\theta = \frac{(d - h_f) + (h_f - x_0) b_f}{d - x_0}\]  

The model has shown a very good agreement in comparison with tests on beams without and with stirrups /Cla15/. In general, it seems that models based on the capacity of the compression zone show good results when compared to databanks. However, some of the initial assumptions of these models have to be assessed critically. One assumption is that the shear stresses between the crack interfaces are negligible so that the shear stresses are concentrated in the compression zone. Such an assumption would have to be confirmed experimentally, with recent research, however, indicating that aggregate interlock is crucial for beams in flexural shear /Cav15/. Another assumption is that a direct strut to the support is able to form behind the critical crack as shown in Figure 2-12. This is only possible if the critical crack is the first major crack next to the support, i.e. in diagonal tension failures but not in flexural shear failures. In favor of compression chord models, it can be said that the stress distributions in the crack as shown in Figure 2-15b and c will not be completely accurate and that a stress distribution according to Figure 2-15d and the resulting model, though contestable in some points, can be seen as one possible pragmatic solution to the problem of shear.

2.4.5 Models based on fracture mechanics

Since the phenomenon of flexural or diagonal shear cracking is generally considered a brittle mode of failure, principles from elastic fracture mechanics have been adopted to predict shear failure. Elastic fracture mechanics leads to the conclusion that a strong size effect is associated with the failure stress of a specimen. Generally, the scaling of shear failure in concrete beams was considered of minor importance until the 1970s, although the scaling of structural strength was already known in the Renaissance. A general overview on the history of scaling of structural failure is given in /Baz97/. In fracture mechanics, the energy demand of a crack and its potential energy release can be considered to determine whether a crack is stable. Since this is only true for materials that remain elastic until failure, the applicability of linear elastic fracture mechanics has been questioned /Hil79/. REINHARDT considered the approach of linear elastic fracture mechanics sound if the microcracking area was sufficiently small in comparison to the length of the crack /Rei81/. According to linear fracture mechanics, the nominal strength of a beam would then decrease proportional to \( \lambda^{1/2} \), in which \( \lambda \) represents the scale factor, contradicting the notion that the size effect phenomenon becomes irrelevant for larger beams. It then became clear, that concrete could not be treated as a brittle material as it possesses a large fracture process zone relative to its crack size /Baz84/. BAZANT & KIM /Baz84/ then proposed a structural size effect law according to Eq. (2-17) to describe the quasi-brittle nature of concrete according to nonlinear fracture mechanics.
2.4 Shear Strength of Beams without Shear Reinforcement

\[ \sigma_N = f' \phi(\lambda) \text{ and } \phi(\lambda) = \frac{1}{\sqrt{1 + \lambda/\lambda_0}} \]  

(2-17)

where

\[ \lambda = d / d_{ag} \]

\[ \lambda_0 = \text{constant} \]

In double-logarithmic scale, Eq. (2-17) provides a smooth transition between a plastic strength criterion and linear elastic fracture mechanics (Figure 2-16). BAZANT & KIM proposed Eq. (2-18) as a replacement for the code formulation of the ACI or the Model Code, which at the time did not or in an insufficient way consider size effects.

\[ V_c = \frac{83/\rho}{1 + \frac{25d_{ag}}{d}} \left[ 0.083 \sqrt{f_c} + 20.7 \sqrt{\rho \left( \frac{M}{V_d} \right)^5} \right] b_w d \]  

(2-18)

Figure 2-16: Size effect according to nonlinear fracture mechanics compared to a simple yield criterion (reproduced from /Baz84/)

Other contributions at the time were given by GUSTAFSSON & HILLERBORG /Gus88/, SO & KARIHALOO /So93/ or GASTEBLED & MAY /Gas01/. A comprehensive overview over the work on shear in fracture mechanics is given in XU & REINHARDT /Xu05/. The analytical model by GASTEBLAD & MAY used the basic idea, that the extra moment due to the rotation at the tip of the diagonal shear crack is equal to the fracture energy necessary to extend the unbonded length of longitudinal reinforcement (Figure 2-17) /Xu05/. The fracture mechanical approach is thus coupled to the strain effect in the longitudinal reinforcement.
Based on the model, Eq. (2-19) was derived to predict a diagonal shear type of failure (Mode II failure). The coefficient $K_{IIc}$ is a fracture toughness coefficient for mode II fracture failure that depends on the concrete strength and has to be determined experimentally.

$$V_c = \frac{0.446}{\sqrt{h}} \left( \frac{E_c}{E_s} \left( \frac{h}{a_s} \right) \right)^{1/3} \rho^{1/6} \left( 1 - \sqrt{\rho} \right)^{2/3} K_{IIc} bh$$  \hspace{1cm} (2-19)

Additional work was performed by BAŽANT ET AL. on the so-called Universal Size Effect Law (USEL) /Baz09/, /Hoo14/, that distinguishes the different types of statistical size effect (Type I) and energetic size effect (Type II). BAŽANT ET AL. also published work regarding the evaluation of test databanks with respect to validating size effect laws experimentally /Baz08/, /Yu15/. Although there are mechanical principles involved in models based on fracture mechanics, it seems that the emphasis of the models related to the explanation of size effects. Other influence from longitudinal reinforcement or shear slenderness are mostly considered on an empirical basis so that fracture mechanics based approaches should also be considered empirical.

### 2.4.6 Models based on plasticity theory

The lower bound theorem of the theory of plasticity uses strut-and-tie models to find an admissible state of equilibrium within a structure. Such strut-and-tie models can be applied to RC beams without shear reinforcement as illustrated in Figure 2-18.
A possible lower bound solution in case of yielding of the longitudinal reinforcement is given by Eq. (2-20) while the solution for strut failure is given by Eq. (2-21) /Nie78/. It is assumed that the stringer force is sufficiently anchored behind the support. The same solution can be obtained by an upper bound theorem with a straight yield line between the faces of the supports /Bra09/.

\[
V_c = \frac{1}{2} b_w h v f_c \left( \frac{a}{h} \right)^2 + \frac{4}{\sqrt{\nu}} \frac{f_y / f_c}{\sqrt{f_y / f_c} - \frac{a}{h}} \right) \quad \text{and} \quad \rho_i \frac{f_y}{f_c} \leq \frac{1}{2} v \]  

and

\[
V_c = \frac{1}{2} b_w h v f_c \left( \frac{a}{h} \right)^2 + 1 - \frac{a}{h} \right) \quad \text{and} \quad \rho_i \frac{f_y}{f_c} \geq \frac{1}{2} v \]  

(2-21)

Since the strut crosses flexural shear cracks, the compressive strength of the concrete cannot be fully activated. Therefore, an effectiveness factor \( v \) for the residual concrete compressive strength must be introduced. It was found that a constant factor of \( v \) was not applicable for all ratios of \( a/h \) but that it was rather dependent on the \( a/h \)-ratio /Nie11/. A possible solution was proposed with Eq. (2-22) by ROIKJÆR ET AL. /Roi79/.

\[
v = f_1 f_2 f_3 f_4 \]  

(2-22)

where

\[
f_1 = \frac{3.5}{f_c} \quad \text{and} \quad 5 < f_c < 60 MPa
\]

\[
f_2 = 0.27 \left( 1 + \frac{1}{\sqrt{h}} \right) \quad \text{and} \quad 0.08 < h < 0.7 m
\]

\[
f_3 = 15 \rho_i + 0.58 \quad \text{and} \quad \rho_i < 0.045
\]

\[
f_4 = 1.0 + 0.17 \left( \frac{a}{h} - 2.6 \right)^2 \quad \text{and} \quad a/h < 5.5
\]
2.4 Shear Strength of Beams without Shear Reinforcement

The equation account for influences from concrete strength, size effect, reinforcement ratio and shear slenderness on the effectiveness factor. The application of such direct strut approaches according to plasticity theory for slender members (i.e. B-regions) has been disputed and considered unsafe /ACI99/. REINECK /Rei91/ considers strut-and-tie approaches only applicable for members with \( a/d \leq 1.0 \) (i.e. the D-region of a beam). In fact, the approach loses its elegance by the introduction of a rather complicated \( \nu \)-factor that heavily relies on the calibration with test results.

MUTTONI /Mut90/ investigated the applicability of the plasticity theory on concrete design and came to the conclusion that for slender members in shear where flexural cracks disturb the direct strut, an alternative truss according to Figure 2-19 had to be used to account for the reduced ability of the critical crack to transmit shear stresses. The failure in the stress field occurs when the tensile strength is exceeded at point \( D \) in Figure 2-19, or if the strength of region \( E \) in shear compression is exceeded. The capacity according to the lower bound theorem of Figure 2-19 is smaller than according to a direct strut approach so that the latter should indeed be considered unsafe /Mut90/, /Rei91/, /ACI99/.

Recently, a new approach based on yield line theory was presented by FISKER & HAGSTEN /Fis16/. Here, instead of a single straight yield line, a yield line with three sections is assumed as shown in Figure 2-20a. The vertical deflection \( \delta_u \) activates shear components from sliding of plain concrete in the compression zone, sliding along the already existing shear crack and separation of the concrete layer along the longitudinal reinforcement. The rotation at the level of the longitudinal reinforcement causes additional dowel action and bending of the concrete layer /Fis16/. However, only the components from sliding of plain concrete and sliding along the crack are considered to occur simultaneously.
2.4 Shear Strength of Beams without Shear Reinforcement

Figure 2-20: a) yield line mechanism at failure, b) shear capacity $V_u$ with respect to the position of the critical crack (reproduced from /Fis16/)

The shear failure load is calculated according to Eq. (2-23). The crack friction coefficient $c_{cr}$ features a modified version of the aggregate interlock law by Vecchio & Collins /Vec86/. The value $w_0$ describes the critical crack width $w_0$, above which no interface shear stresses can be transmitted. The values $l_{cr}$ and $l_{pl}$ represent the lengths of the yield lines and are determined with respect to the failure location $x_{cr}$ and the crack geometry /Fis16/.

$$V_c = c_c \sqrt{k_{cr}} (1 - \sin \alpha_1) b_w l_{cr} + \frac{1}{2} f_c (1 - \sin \alpha_2) b_w l_{pl}$$

(2-23)

where

$$c_c = \frac{0.18 \sqrt{f_c}}{0.3 + \frac{24 w}{d_{ag} + 16}} \left(1 - \frac{w}{w_0}\right)$$

The failure location is unknown a priori and can be determined by finding the minimum capacity based on the position of the critical crack as shown in Figure 2-20b. In comparison with test results, the approach has shown convincing results. The phenomenon of the relatively decreased capacity for high strength concretes can also be understood in the light of the critical crack width value $w_0$. For high concrete strengths, the shear crack crosses the aggregates so that the $w_0$ value decreases significantly, thus reducing the shear capacity.

2.4.7 Models based on longitudinal strains

This section summarizes models which relate the shear capacity directly to the longitudinal reinforcement strain. The main problem of empirical shear methods is that influences from shear slenderness, size effect and reinforcement ratio are seen as isolated
parameters. Many shear test series were designed accordingly to investigate the influence of these parameters on the shear capacity. For instance, this led to the notion that increasing the reinforcement ratio \( \rho_l \) in a beam leads to a higher shear capacity. COLLINS ET AL. /Col08/ pointed out that this is in fact a fallacy. Rather, if the stress level in the longitudinal reinforcement is kept constant, the shear capacity of beams is independent from the longitudinal reinforcement ratio. According to /Col08/, the influences of longitudinal reinforcement ratio and shear slenderness on the shear strength can therefore only be understood in combination as a measure of the stress level in the reinforcement.

The shear model of Model Code 2010 /CEB10/ was developed based on the Modified Compression Field Theory (MCFT) /Vec86/. The MCFT accounts for strain compatibility and post-cracking behavior of concrete and can therefore also be applied on members without transverse reinforcement. Based on various simplifications made from the original model /Ben06/, Eq. (2-24) was derived as a code formulation.

\[
V_{Rd,c} = \frac{0.4}{1 + 1500 \varepsilon_x} \frac{1300}{1000 + k_{dg} z} \frac{f_{ck}}{\gamma_c} b_n z
\]  

(2-24)

where

\[
k_{dg} = \frac{32}{16 + d_g} \geq 0.75
\]

The strain \( \varepsilon_x \) is related to the longitudinal average strain between the top and bottom stringer of a beam according to Figure 2-21. An additional term is required to account for the size effect in shear failure by the inner lever arm \( z \).

A problem of the approach seems to be that judging from the second fraction in Eq. (2-24), the size effect is proportional to \( d^{-1} \), as pointed out by YU & BAŽANT /Yu15/. This would infer that very large beams cannot increase in shear strength if the member depth is increased which contradicts common sense on one side and is apparently “thermodynamically inadmissible” /Yu15/ on the other side. However, it should also be considered that the longitudinal strain \( \varepsilon_x \) also reduces with larger values of \( d \) so that within a practical range the actual size effect is less pronounced than assumed by YU & BAŽANT.
This is shown by the parametric study illustrated in Figure 2-22. The dimensionless shear capacity $v_{Rd}$ according to Eq. (2-24) is given for different longitudinal reinforcement ratios $\rho_l$ with respect to the effective depth $d$ in double-logarithmic scale. Here, the gradient of the size effect for large members ranges from 0.57 to 0.84 for a member depth of $d = 9000$ mm. However, if the diagram was extended further so that $d \to \infty$, all of the gradients tend towards 1.0 so that the basic assessment by $\text{YU} \& \text{BAŽANT} /\text{Yu}15/$ is correct.

Figure 2-22: Size effect according to the Level II approach of MC2010 /CEB10/

Another approach is based on the Critical Shear Crack Theory /Mut08/ and is implemented in the Swiss Code SIA 262 /SIA13/. The approach given by Eq. (2-25) has a quite simple appearance as it only depends on the concrete strength $f_{ck}$, the longitudinal strain in the reinforcement $\varepsilon_v$ and the maximum aggregate size $d_{ag}$.

\[
V_{Rd} = \frac{0.3 \sqrt{f_{ck}} / \gamma_c}{1 + \varepsilon_v d} \frac{b_w d}{48 \left(16 + d_{ag}\right)}
\]

where

\[
\varepsilon_v = \frac{f_{yd} M_{Ed}}{E_s M_{Rd}}
\]

Nevertheless, the approach has shown good agreement with test results /Mut03/ and is able to predict size effects in agreement with the predictions from nonlinear fracture mechanics /Rui15/. The main advantage of a strain based approach can therefore be seen in the fact that it is able to consider multiple influences from reinforcement ratio, slenderness and size effect within one strain variable.
2.5 Shear Strength of Beams with Shear Reinforcement

2.5.1 General

In this section, shear strength models of beams with shear reinforcement will be summarized briefly. Transverse reinforcement in beams crosses diagonal shear cracks, prevents a sudden shear failure and increases the shear capacity significantly. The truss analogy is generally accepted as a sound method to assess the shear capacity of beams with shear reinforcement. However, some progress has been made since the introduction of the 45°-truss analogy by RITTER and MÖRSCH. In chapter 2.5.2, models based on lower or upper bound solution of the theory of plasticity will be described. Truss models with an additional concrete contribution are presented in chapter 2.5.3. Finally, softened truss models are summarized in chapter 2.5.4.

2.5.2 Models based on plasticity theory

Lower Bound Solution

The capacity of a beam with transverse reinforcement can be determined by using a truss model according to Figure 2-23. The vertical force $V_{sy}$ that can be transferred by the stirrups depends on the angle of the strut $\theta$.

$$ V_{sy} = \frac{A_{sw}}{s} f_y z \cot \theta $$  \hspace{1cm} (2-26)

For flatter angles, more stirrups are activated so that $V_{sy}$ can be determined according to Eq. (2-26)

The capacity of the truss is limited by the compressive stresses in the inclined struts. For flat angles $\theta$ the compressive stresses increase. The maximum capacity of the strut $V_{\text{max}}$ is the result of Eq. (2-27).

$$ V_{\text{max}} = \frac{v \cdot f_c b_w z}{\tan \theta + \cot \theta} $$  \hspace{1cm} (2-27)
2.5 Shear Strength of Beams with Shear Reinforcement

The coefficient $\nu$ represents an effectiveness factor for the compressive strength of concrete. The full one-axial compressive strength of concrete cannot be activated due to transverse tensile strains from shear cracks. The size of $\nu$ is usually a value of $0.5 \sim 0.6$ (a detailed account on the development of the value is given in /Zil10/, /Nie11/ and chapter 5.6). The solution for the angle of the stress field $\theta$ is found if simultaneous yielding of strut and stirrups is assumed, i.e. $V_{sy} = V_{max}$. The value of cot $\theta$ is then given by Eq. (2-28).

$$\cot \theta = \frac{\nu f_{cy}}{\rho f_y} - 1$$  \hspace{1cm} (2-28)

Since a constant $\nu$-value is not valid for very small angles of $\theta$, a more or less arbitrary limitation of $\theta$ (e.g. cot $\theta \leq 2.5$) has to be introduced. In contrast, Model Code 2010 provides an effectiveness factor according to Eq. (2-29) that was derived from tests on reinforced concrete panels. Here, the effectiveness factor depends on the transverse strain $\varepsilon_1$ so that the model is also applicable for very low angles of $\theta$.

$$\nu = \eta_k k_\varepsilon$$  \hspace{1cm} (2-29)

where

$$\eta_k = \left( \frac{30}{f_{ck}} \right)^{1/3} \leq 1.0$$

$$k_\varepsilon = \frac{1}{1,2 + 55\varepsilon_1} = \frac{1}{1,2 + 55(\varepsilon_1 + (\varepsilon_x + 0.002)\cot^2 \theta)} \leq 0.65$$

The longitudinal strain $\varepsilon_x$ is defined according to Figure 2-21. Although Eq. (2-29) allows for a more general application of the plasticity truss model (especially for beams with low shear reinforcement ratios), a simple solution for the ideal strut inclination $\theta$ cannot be obtained anymore since the $\nu$-factor now depends on cot $\theta$ and $\varepsilon_x$, which itself depends on the shear force $V_{Ed}$ (which is equal to $V_{sy} = V_{max}$).

**Upper Bound Solution**

An upper bound solution for the shear strength of beams with shear reinforcement can be obtained by assuming a yield line in the shear span inclined at an angle $\beta_r$ (Figure 2-24). The external work from the load $V$ is equal to the internal work from stirrups and concrete according to Eq. (2-30). The concrete is defined as a modified Coulomb material with $f_{ct} = 0$ /Nie11/. 

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2.5 Shear Strength of Beams with Shear Reinforcement

\[ V \cdot \delta = W_y + W_c \]

\[ = \frac{A_{sw}}{s} f_y h \cot \beta_r \cdot \delta + \frac{1}{2} f_c b_w (1 - \cos \beta_r) \frac{h}{\sin \beta_r} \cdot \delta \]  

(2-30)

![Yield line mechanism for symmetrically loaded beams](image)

**Figure 2-24:**  Yield line mechanism for symmetrically loaded beams /Nie11/

The shear capacity according to the upper bound solution is then given by Eq. (2-31) /Nie11/. The upper bound theorem for beams with transverse reinforcement thus results in a stirrup and a concrete contribution.

\[ V = \frac{A_{sw}}{s} f_y h \cot \beta_r + \frac{1}{2} f_c b_w h (1 - \cos \beta_r) \frac{1}{\sin \beta_r} \]  

(2-31)

If Eq. (2-31) is solved for minimum values of \( V \) with respect to \( \beta_r \), the solution is identical to the upper bound solution. The angle \( \beta_r \) is however limited geometrically according to Eq. (2-32).

\[ \tan \beta_r \geq \frac{h}{a} \]  

(2-32)

This means that for low amounts of shear reinforcement, Eq. (2-33) is governing.

\[ V = \frac{1}{2} bh f_y \left[ \sqrt{1 + \left( \frac{a}{h} \right)^2} - \frac{a}{h} \right] + \frac{A_{sw}}{s} f_y \cdot a \]  

(2-33)

Figure 2-25 illustrates the different solutions of the upper bound theorem. The lower bound solution is illustrated by the plasticity circle and gives a shear capacity of zero for members without shear reinforcement. On the other hand, the upper bound solution according to Eq. (2-33) is a tangent to the plasticity circle and reaches a minimum value of \( v_c \) in case no shear reinforcement is provided. The upper bound solution will of course be different for other load types or other yield mechanisms (cf. /Nie11/). This example shows, however, that the plasticity theory can provide a consistent approach of assessing the shear strength of members without and with shear shear reinforcement.
2.5.3  Truss models with concrete contribution

Even before the upper bound theory of plasticity was developed it was clear that a simple 45°-truss model was insufficient to predict the stresses in transverse reinforcement and thus to predict the ultimate capacity of RC beams. According to /ACI62/, TALBOT stated that not the entire shear force is taken by the stirrups but that a portion of the shear should be attributed to the compression zone. RICHART /Ric27/ noted, that the stresses in the stirrups cannot account for the ultimate capacity of a beam with stirrups. Since the concrete in the diagonal struts is largely destroyed, this portion has to be carried by the uncracked compression zone.

The existence of the concrete contribution was also shown by test series of LEONHARDT & WALTHER who performed tests on beams with stirrups and different cross-section profiles (Figure 2-26a). It can be seen that a certain shear load is required to activate the stirrups after which the stirrups are stressed in a constant angle $\beta_r$. This truss angle $\beta_r$ is usually inclined at an angle of less than 45°. The concrete contribution is constantly available until the ultimate capacity of the beam is reached (Figure 2-26a). As shown in chapter 2.5.2, this is also in agreement with the upper bound solution of the plasticity theory in which a concrete contribution can be derived on the basis of a modified Mohr-Coulomb failure criterion which exists independently of the transverse reinforcement.
Different models have been proposed to account for the concrete contribution within a truss model. The ACI Code /ACI08/ uses a 45°-truss and a concrete contribution that is equal to the capacity of beams without shear reinforcement. For RC beams the capacity is given by Eq. (2-34).

\[
V_u = \frac{A_{sw}}{s} f_y d + \left( 0,158 f_c' + 17,2 \rho_t \frac{V_d}{M} \right) b_w d
\]  

(2-34)

In the Model Code 1978 /MC78/, a truss model with variable strut inclination and a concrete contribution is used. The concrete contribution \( V_c \) is implicitly reduced by a coefficient \( k_c \) (coefficient introduced by the author) with respect to the ratio of the acting shear force \( V_{Ed} \) and \( V_c \). That means that for very large shear forces the concrete contribution is neglected.

\[
V_u = k_c V_c + V_s = k_c \cdot \beta_1 \cdot 2,5 \cdot \tau_{Rd} b_w d + \frac{A_{sw}}{s} f_y 0,9 d \cot \theta \leq 0,3 f_c b_w d
\]  

(2-35)

where

\[
0 \leq k_c = 1,5 - 0,5 \frac{V_{Ed}}{V_c} \leq 1,5
\]

\[
\beta_1 = 1 + \frac{M_{dec}}{M_{Rd}} \leq 2
\]

\[
\frac{3}{5} \leq \cot \theta \leq \frac{5}{3}
\]
Another model by Hegger & Görtz /Heg06/, /Gör04/ was specifically developed for prestressed concrete beams. The model features a truss contribution $V_s$, a concrete contribution from the uncracked compression zone $V_{cs}$ and a direct strut component $V_{cp}$. The components can be added to determine the ultimate capacity $V_u$ of a beam. Similar to Model Code 1978, the concrete contributions are reduced by $\kappa$-coefficients which depend on the mechanical shear reinforcement ratio $\omega_{c,ct}$. This is supposed to account for the increased stiffness of the truss mechanism for beams with large amounts of shear reinforcement which prevents the internal redistribution of shear forces to softer shear transfer actions by concrete.

\[
V_u = V_s + \kappa_s V_{cs} + \kappa_p V_{cp}
\]  

(2-36)

where

\[
V_{cs} = \frac{2}{3} k_s f_{ct} \left( \frac{4d}{a} \right)^{0.25} \left( \frac{5l_{ch}}{d} \right)^{0.25} b_u d
\]

\[
V_{cp} = P \cdot z_{pl} / a
\]

\[
V_s = \frac{A_w}{s} f_y \cdot z \cot \beta_r
\]

\[
\kappa_s = 1 - \omega_{w,ct} / 3 \geq 0
\]

\[
\kappa_p = 1 - \omega_{w,ct} \geq 0
\]

\[
\omega_{w,ct} = \rho_w f_y / f_{ct}
\]

\[
\cot \beta_r = 1 + 0.15 / \omega_{w,ct} - 0.18 \sigma / f_{ct} \leq \frac{2.15}{a / d}
\]

Other models by CLADERA were derived empirically by genetic programming /Cla04b/, /Cla14/. The model given in the Level III approximation of Model Code 2010 /CEB10/ also consists of a truss and concrete contribution, where the concrete contribution is basically determined according to the Simplified MCFT /Ben06/. This model infers that the shear component of the concrete contribution can mainly attributed to aggregate interlock between the shear cracks. A very similar assumption was used in the truss model with crack friction by REINECK /Rei01/ that was used in the German concrete code DIN 1045-1 /DIN01/ and is still used in the German version of EC2 /DIN13/.

In his compression arch model, MAURER ET AL. /Mau13/ consider the concrete contribution as a result of the inclination of the longitudinal force from bending. The concrete contribution $V_c$ would then be equal to the component $V_{cc}$ which is usually considered in haunched beams. In their extended version of the compression arch model, GLEICH
ET AL. /Gle16/ provide a procedure for calculating the resulting inner lever arm \( z \) for RC and PC beams according to Figure 2-27.

![Figure 2-27: Truss model of a PC beam in bending and shear (reproduced from /Gle16/)](image)

As can be seen from Figure 2-27, the horizontal component of the shear force causes additional horizontal forces in the compression zone. To obtain equilibrium, a reduced inner lever arm \( z_{BV} \) can be calculated from Eq. (2-37). The inner lever arm \( z \) is determined from bending and can be taken as 0.9\( d \) for RC beams.

\[
z_{BV} = \frac{M_{Ed} + P \cdot \Delta e_p}{M_{Ed} + P \cdot \Delta e_p + \frac{1}{2} V \cdot z \cot \theta} \tag{2-37}
\]

The vertical component \( V_{cc} \) of the flexural compressive force is associated with the change of the inner lever arm \( z_{BV} \) along the beam axis. In a single span point loaded beam, the moment \( M_{Ed} \) can simply be calculated from the product of shear span \( a \) and shear force \( V \). The inclination of the flexural compressive force is then given by the first derivative of the inner lever \( z_{BV} \) arm according to Eq. (2-38).

\[
\frac{dz_{BV}}{da} = d \frac{V \cdot a}{V \cdot a + \frac{1}{2} V \cdot z \cot \theta} \cdot \frac{1}{2 \cot \theta} \cdot z = \frac{1}{2 \cot \theta} \cdot \left( \frac{a}{z} + \frac{1}{2} \cot \theta \right)^2 \tag{2-38}
\]

The ultimate capacity \( V_u \) of a beam is then given by the sum of the truss contribution \( V_{sy} \) with a variable strut inclination \( \theta \) and the contribution from the compression chord \( V_{cc} \) which itself is the product of the flexural compressive force and its inclination according to Eq. (2-39).

\[
V_{cc} = V_u \cdot \frac{a}{z} \cdot \frac{1}{2 \cot \theta} \cdot \left( \frac{a}{z} + \frac{1}{2} \cot \theta \right)^2 \tag{2-39}
\]
The ultimate capacity $V_u$ can then be determined in a closed form according to Eq. (2-40), resulting in an enhancement factor for the truss contribution $V_{sy}$.

$$V_u = \left(1 - \frac{1}{\frac{a}{z} \cot \theta} \right)^{-1} \cdot V_{sy}$$

(2-40)

The size of the shear enhancement value can be expressed by the ratio $V_u / V_{sy}$ and depends on the relative position of the control section $a/z$ (with $a/z = 0$ at the support) and the strut inclination $\cot \theta$ (Figure 2-28). As can be seen from Figure 2-28, the assumptions of the extended compressive arch model result in a shear enhancement for RC beams of $10 \sim 30 \%$ from the inclination of the flexural compressive alone. This enhancement is even larger for prestressed beams.

![Figure 2-28: Shear enhancement factor according to the extended compressive arch model](image)

As can be seen from this chapter, various models consider a concrete contribution that enhances the capacity of a truss. However, there are different opinions on whether the concrete contribution is a result of the contribution of the uncracked compressive zone, the interface and tensile stresses of concrete that are transmitted over shear cracks, arch action from an inclined flexural compressive force or combinations of all of these components.

### 2.5.4 Softened truss models

As KUPFER stated in 1964 /Kup64/, the robust shear design of the simple $45^\circ$ truss model by Mörsch allowed for a rapid success of reinforced concrete structures. However, increasing labor costs and the development of new corrugated steel gave room to advancements to new truss models. KUPFER gave three reasons to justify a truss analogy with angles below $45^\circ$/Kup64/:
• The actual shear crack angle in tests is often less than 45°
• The strut between to neighboring shear cracks is inclined at a smaller angle than the actual shear cracks.
• There exists a strong interlock between the crack interfaces so that stresses along the shear cracks can be transmitted.

KUPFER was reluctant to simply utilize the complete yield strength of the stirrups by lowering the strut angle further although from tests results values for \( \cot \theta \) of 4.0 would have been justified. He argued that the shear cracks would be too large to be verified in the SLS. From the principle of minimizing strain energy in the truss, Kupfer derived Eq. (2-41) to be able to determine a truss angle \( \theta \) with respect to the stresses in the longitudinal reinforcement \( \sigma_{sl} \), the flexural compression zone \( \sigma_c \) and the stirrups \( \sigma_{sy} \).

\[
\tan^3 \theta - \frac{\sigma_{sl}}{2\sigma_{sy}} \cdot \tan^2 \theta - \frac{\alpha_c \tau_c}{\sigma_{sy}} (1 - \tan^4 \theta) = 0
\]  

(2-41)

where

\[
\tau_c = \frac{V}{(b \cdot z)}
\]

The idea was that the stresses in the reinforcement could be freely chosen so that a corresponding strut angle \( \tan \theta \) could be read from a diagram to determine the amount of necessary reinforcement. By replacing stresses by average strains, Eq. (2-41) can also be expressed by Eq. (2-42) /Kup91/.

\[
\tan^2 \theta = \frac{\varepsilon_x - \varepsilon_c}{\varepsilon_y - \varepsilon_c}
\]  

(2-42)

where

\( \varepsilon_x \)  average axial strain of the cracked web
\( \varepsilon_y \)  average vertical strain of the beam
\( \varepsilon_c \)  principal compressive strain parallel to \( \sigma_2 \)

Eq. (2-42) can also be directly obtained from MOHR’s strain circle if it is assumed that the inclination of the shear cracks is equal to the inclination of the compression field and that there is no sliding in the crack plane /Kup91/. By introducing MOHR’s strain circle as a compatibility condition into shear design, the truss model is not treated as a rigid plastic truss anymore, hence the term “softened truss”. The softened truss approach ensures that the strain conditions in the truss are in agreement with the linear elastic stiffness of concrete and steel elements, similar to accounting for BERNOULLI’s hypothesis for flexural design.

Eq. (2-42) was also used by COLLINS in 1978 for his “diagonal compression field theory” /Col78/. The theory was developed to investigate symmetrical sections with a constant
web width, symmetrical reinforcement in longitudinal and transversal direction and without considerable moments or disturbances by point loads. While tensile stresses of the concrete were neglected, the theory considered elastic-plastic material laws for steel and nonlinear stress strain relationships for concrete. From evaluation of test results, COLLINS found that there exists a relationship between the limiting value $f_{du}$ of average principal compressive stresses and the strain conditions in the concrete according to Eq. (2-43).

\[
\frac{f_{du}}{f_c} = \frac{3.6}{1 + \frac{2\gamma_m}{\varepsilon_0}}
\]  

(2-43)

- $f_c$ concrete cylinder strength
- $\varepsilon_0$ peak cylinder strain at $f_c$
- $\gamma_m$ maximum value of average shear strains

A possible physical explanation was that with larger average shear strains and thus larger shear crack widths the concrete would lose its capability to transfer shear stresses over the cracks. In the following, Vecchio & Collins developed a modification of the original Diagonal Compression Field Theory which was simply called Modified Compression-Field Theory (MCFT) /Vec86/. In the modification of the theory, the tensile strength of concrete was accounted for as well as shear stresses transferred over cracks by aggregate interlock, while dowel action was neglected. The calculation of the stress-strain states was divided into the calculation of the average stresses in the concrete element and the local stresses at the crack. The constitutive relationships used for the model are summarized in Figure 2-29. The strain softening and tension stiffening response of concrete under transverse strains was studied on 30 reinforced concrete panel tests. Although the model is not suitable for calculations by hand, comprehensive solution techniques and tools are provided to solve the set of equations /Vec86/, /Ben00/. An application of the MCFT as a general shear design method for concrete members was presented in /Col96/.

Further developments or modifications of softened truss models, in some aspects quite similar to the MCFT, are the Rotating Angle – Softened Truss Model /Bel94/, /Bel95/, the Fixed Angle – Softened Truss Model /Pan96/, the Cracked Membrane Model /Kau98b/ or the Disturbed Stress Field Model /Vec00/, /Vec01/. An overview of different classes of design models with respect to their mechanical principles is given in Figure 2-30.
2.5 Shear Strength of Beams with Shear Reinforcement

Figure 2-29: Constitutive equations of the Modified Compression Field Theory (reproduced from /Ben06/)

\[
\begin{align*}
\text{Stress at Cracks:} \\
f_{cr} &= (f_x + v \cot \theta + v_y \cot \theta) / \rho_x \\
f_{cr} &= (f_x + v \tan \theta - v_y \tan \theta) / \rho_y
\end{align*}
\]

\[
\begin{align*}
\text{Crack Widths:} \\
w &= s_x \varepsilon_x \\
s &= 1 / \left( \frac{\sin \theta}{s_x} + \frac{\cos \theta}{s_y} \right)
\end{align*}
\]

Equilibrium:
Average Stresses:
\[
\begin{align*}
f_x &= \rho_x f_{nx} + f_x - v \cot \theta \\
f_x &= \rho_x f_{nx} + f_x - v \tan \theta \\
v &= (f_x + f_y) / (\tan \theta + \cot \theta)
\end{align*}
\]

Stress-Strain Relationships:
Reinforcement:
\[
\begin{align*}
f_x &= E_s \varepsilon_x \leq f_{x,s} \\
f_x &= E_s \varepsilon_x \leq f_{x,s}
\end{align*}
\]

Concrete:
\[
\begin{align*}
f_x &= \frac{f_{c}'}{0.8 + 170 \varepsilon_c} \left[ 2 \frac{\varepsilon_x}{\varepsilon_c} - \left( \frac{\varepsilon_x}{\varepsilon_c} \right)^2 \right] \\
f_x &= 0.33 \sqrt{f_{c}'} / (1 + \sqrt{500 \varepsilon_c}) \quad \text{MPa (13)}
\end{align*}
\]

Shear Stress on Crack:
\[
\nu_x \leq \frac{0.18 \sqrt{f_{c}'} wp}{0.31 + 24 w / d_y + 16} \quad \text{MPa, mm (15)}
\]

Figure 2-30: Overview of different design principles and application areas of design methods as proposed by Hsu/Hsu10/
3 Shear Strength of RC and PC Beams without Shear Reinforcement Based on a Mechanical Analysis

3.1 General

In this chapter, flexural shear failure and web shear failure of beams without shear reinforcement will be investigated based on a rational mechanical model. To achieve a basic understanding of the flexural shear failure of reinforced concrete beams without shear reinforcement, a simple mechanical procedure will be applied. The model is based on the assumption of pure beam action within the B-Region of a beam (Figure 3-1). The flexural cracks are assumed to be orthogonal to the beam axis so that the Bernoulli hypothesis is still valid. Since shear forces are defined as the first derivative of the bending moment, it follows from Eq. (3-1) that the shear forces are carried either by bond stresses along to longitudinal reinforcement or a varying inner lever arm $z$.

$$V = \frac{dM}{dx} = \frac{d (F_c \cdot z)}{dx} = \frac{d (F_s \cdot z)}{dx} = F_s \frac{dz}{dx} + F_s \frac{dz}{dx}$$

(3-1)

In case of unbonded reinforcement, the tensile force in the steel remains constant along the beam axis ($dF_s/dx = 0$) so that the entire shear results from a change of the inner lever arm $z$, i.e. a direct strut. In case of perfectly bonded reinforcement, shear is carried by bond stresses in the reinforcement, resulting in flexural cracks in the concrete. The flexural cracks prevent any arch action as the inner lever arm of RC beams with flexural cracks is independent of the concomitant moment.

Figure 3-1: Assumed crack pattern for pure beam action and free-cuts of a) concrete tooth, b) compression zone (partly adopted from /Col08/)
3.2 Model by Tue et al.

As a consequence, the entire shear is carried by beam action \((F_s dz/dx = 0)\). The force difference in the bottom stringer between two flexural cracks \(\Delta F_s\) can then be determined according to Eq. (3-2).

\[
V = \frac{dF_s}{dx} \Rightarrow \Delta F_s = V \frac{s_{set}}{z} \approx V \frac{(1-\xi)d}{z} \tag{3-2}
\]

From horizontal equilibrium it follows that the shear stresses \(\tau\) at the top of the concrete tooth are equal to \(V/bz\) if constant shear stresses are assumed (Figure 3-1a). Based on extensive experimental investigations it can further be assumed, that the surfaces of the cracks can transmit shear stresses by aggregate interlock and dowel action so that a constant average shear flow between top and bottom stringer results from equilibrium. The only principles needed to determine the stress distribution in case of a constant shear flow are therefore equilibrium of forces and the Bernoulli hypothesis. The used procedure is similar to a cross-sectional model for determining the shear capacity of RC beams by Tue et al. /Tue14/, /Tun16a/, in which shear failure is induced by a critical crack that forms in a critical shear band region. The model by Tue et al. is described in chapter 3.2. However, in this thesis the idea of the model will be used in a different manner to also account for the effects of limited capacity of aggregate interlock and dowel action. A modified procedure is therefore described in chapters 3.3 and 3.4. Based on the mechanical model, a simplified model for shear strength will also be derived in chapter 3.5 to allow for the prediction of shear strength within a calculation by hand. A mechanical model for the special case of web shear failure of prestressed I-beams is presented in chapter 0. The results of this chapter are summarized in chapter 0.

3.2 Model by Tue et al.

The model by Tue et al. /Tue14/, /Tun16a/ can be considered as a further development of Mörsch’s classical beam shear theory for beams cracked in bending.

![Figure 3-2: Flow of forces in a beam segment: a) shear transfer by various truss actions; b) average shear stresses according to the Euler-Bernoulli beam theory (reproduced from /Tun16a/)](image-url)
If the Bernoulli-hypothesis is assumed for a member cracked in bending, various actions like aggregate interlock and dowel action ensure a constant shear flow between compression and tensile chord as illustrated in Figure 3-2. In the model by TUE ET AL., a critical shear crack develops in a shear crack band with failure potential leading to flexural shear failure. The shear crack band is given by the concrete tension zone at the flexural crack tip. This shear crack can only develop if a mean value of principal tensile stresses within a critical width in the crack tip of the flexural crack reaches the concrete tensile strength (localization of damage).

![Critical crack formation in the model by TUE: a) formation of the critical shear crack in the failure band; b) distribution of normal and shear stresses in two different cross-sections; c) redistribution of internal forces after shear crack formation (reproduced from /Tun16a/)](image)
This damage localization has to occur not only in one section of a beam, but within a defined length along the beam axis as, according to Tue et al., flexural shear failure should be considered as a member failure and not merely as a cross-sectional failure (Figure 3-3). In Figure 3-3a, the formation of the critical shear crack in the failure band is shown. The width of the failure band depends on the length of the region of concrete in tension which results in different average tensile stresses at the flexural crack tip (Figure 3-3b). After the formation of a critical shear crack, a redistribution of internal forces into a strut and tie system according to Figure 3-3c might be possible. However, the distribution of internal forces can no longer be described by the Euler-Bernoulli beam theory.

The shear capacity in the model by Tue et al. is determined by integrating the shear stresses over the cross-sectional depth. For this, the depth of the compression zone \( x \) has to be determined as well as the length of the region of concrete in tension \( (x' \text{ and } x'') \) according to Figure 3-4. The average tensile stresses \( \sigma_{xm} \) are calculated as a mean value of the concrete tensile stresses within the empirically derived length of the critical shear band width \( d_{b,\text{crit}} \).

\[
\tau_{Rc} = \frac{2}{3} \tau_{\max} x + \frac{1}{2} (\tau_{\max} + \tau_u) x' + \tau_u (d - x - x') \left( \frac{1}{d} \right) \tag{3-3}
\]

As a result, the shear resistance stress \( \tau_{Rc} \) can be calculated according to Eq. (3-3).

The maximum value of shear stresses at the level of the neutral axis \( \tau_{\max} \) is calculated according to Eq. (3-4)

\[
\tau_{\max} = \frac{\tau_u}{1 - (x'/x)^2} \tag{3-4}
\]
The average shear stress in the cracked region of concrete $\tau_u$ is determined by Eq. (3-5).

$$\tau_u = \sqrt{f_{ct}} \left( f_{ct} - \sigma_{xm} \right)$$  \hspace{1cm} (3-5)

$\sigma_{xm}$ is calculated by Eq. (3-6).

$$\sigma_{xm} = f_{ct} \cdot \left( 1 - 0.5 \frac{d_{b,cr}}{x' + x''} \right) \text{ and } d_{b,cr} < x' + x''$$

$$= f_{ct} \cdot \left( 0.5 \frac{x' + x''}{d_{b,cr}} \right) \text{ and } d_{b,cr} \geq x' + x''$$ \hspace{1cm} (3-6)

The other values related to the previous equations are determined according to Eqs. (3-7) to (3-13).

$$x = \left[ \sqrt{(\alpha_r \rho)^2 + 2\alpha_r \rho - \alpha_r \rho} \right] d$$ \hspace{1cm} (3-7)

$$x' = \frac{f_{ct}}{E_x} (d - x)$$ \hspace{1cm} (3-8)

$$x'' = \frac{G_f}{w_k} (d - x - x')$$ \hspace{1cm} (3-9)

$$w_k = \frac{s_{rm}}{E_x} \left[ \sigma_s - 0.5 \frac{f_{ct}}{\rho_{p,eff}} (1 + \alpha_r \rho_{p,eff}) \right]$$ \hspace{1cm} (3-10)

$$\rho_{p,eff} \approx 4 \rho_s$$ \hspace{1cm} (3-11)

$$s_{rm} = 0.7d$$ \hspace{1cm} (3-12)

$$d_{b,cr} = 0.5 \frac{(100 \rho_s)^{0.9}}{f_c}$$ \hspace{1cm} (3-13)

The shear stresses due to external loading $\tau_{Ed}$ can be compared to the shear resistance $\tau_{Rc}$ as shown in Figure 3-5. Shear failure only occurs if the shear resistance is exceeded over a length $s_{rm}$, rather than local exceedance of the failure criterion. The model was compared to shear tests on beams under different loading conditions and yielded good results. For beams under point loading, 480 tests have been evaluated and compared to models of different codes. The model by TUE ET AL. resulted in a CoV of 17.7% whereas the model of the EC2 yielded a CoV of 19.5%.
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

3.3.1 General

In this section, the model by TUE ET AL. is modified and extended in some aspects. The distribution of shear stresses in the crack tip of a section is subject to the concomitant moment $M_{Ed}$ in the section which is associated with a shear force $V_{Ed}$. The calculated shear force resistance $V_{Rc}$ is not necessarily equal to the associated acting shear force $V_{Ed}$ so that equilibrium of internal and external forces would have to be determined by iteration. In this modification, the equilibrium of forces is considered for each level of loading. Also, in the original model the formation of the critical shear crack in the failure band by exceeding shear stresses in the failure band is considered the governing failure mechanism. Here, the possibility of dowel failure of the longitudinal reinforcement and failure of aggregate interlock is also considered. In total, the following five components of shear transfer are distinguished:

- $V_{cc}$ Contribution of the inclined concrete compressive force
- $V_{cs}$ Contribution of shear stresses in the compression zone
- $V_{CPZ}$ Contribution of the crack processing zone
- $V_{ai}$ Contribution by aggregate interlock
- $V_{dw}$ Contribution by dowel action

Figure 3-5: Analysis of shear stresses and calculated shear resistance (reproduced from /Tun16a/)
It is assumed that the acting shear force $V_{Ed}$ creates an evenly distributed average shear flow between the top and bottom chord of the cross-section as shown in Figure 3-1. In case the inner lever arm $z$ between two cracks differs (which can be the case if the section in the neighboring section remains uncracked), a direct strut component $V_{cc}$ results that is not considered within beam action. The contribution of the inclined compressive force $V_{cc}$ can therefore be subtracted from the shear force $V_{Ed}$ so that the average shear stress $\tau_{av}$ in the cross-section can be determined according to Eq. (3-14).

$$\tau_{av} = \frac{V_{Ed} - V_{cc}}{b_w z}$$  \hspace{1cm} (3-14)

Hence, the contributions of the remaining shear transfer actions can be determined for each load $V_{Ed}$ as shown in Figure 3-6. Aggregate interlock enables a constant shear flow over the crack surface. In case the resistance from aggregate interlock is insufficient, the stress deficit is compensated by dowel action to satisfy equilibrium (Figure 3-6b). The resistance of each component can be determined by defining appropriate failure criteria. If the shear force that is associated with one component, for instance the shear carried by aggregate interlock, exceeds its resistance, the mechanism fails and will consequently lead to overall shear failure. In the following, the failure criteria for each shear component will be defined.

**Figure 3-6:** Distribution of longitudinal and shear stresses in the cracked section

### 3.3.2 Contribution of the inclined compressive force $V_{cc}$

To determine the vertical component of an inclined compressive force $V_{cc}$, the change of the depth of the compression zone $x_0$ has to be determined. The depth of the compression zone of RC beams without axial forces can be computed quite easily if the influence of concrete tensile stresses is neglected. The equilibrium of forces and the Bernoulli hypothesis yield Eq. (3-15) for the depth of the compression zone.

$$F_{cc} = \frac{V_{cc}}{b_w z}$$  \hspace{1cm} (3-15)
As Eq. (3-15) is independent from the acting moment, the depth of the compression zone in areas cracked in bending remains constant which eliminates arch action. However, it can be observed, that the depth of flexural cracks varies depending on the acting moment, especially for beams with a small shear slenderness. To account for potential arch action in these beams, the influence of concrete tensile stresses on the depth of the compression zone is investigated. The first state of the member will be in the uncracked state (Figure 3-7). For simplicity, the concrete tensile stresses in the concrete cover are neglected. The depth of concrete tension zone $x_1$ is determined by Eq. (3-16).

$$x_1 = d - x_0$$  \hspace{1cm} (3-16)

As long as the concrete tensile stresses are smaller than $f_{ct}$, the resulting forces (cf. Figure 3-7) can be determined according to Eqs. (3-17) to (3-19).

$$F_{cc} = \frac{1}{2} \varepsilon_c E_{cm} b_w x_0$$  \hspace{1cm} (3-17)

$$F_{ct} = \frac{1}{2} \varepsilon_s E_{cm} b_w x_1 = \frac{1}{2} \varepsilon_c \left( \frac{1}{\xi} - 1 \right) E_{cm} b_w (d - x_0)$$  \hspace{1cm} (3-18)

$$F_s = \varepsilon_c E_s A_s = \varepsilon_c \left( \frac{1}{\xi} - 1 \right) E_s A_s$$  \hspace{1cm} (3-19)

Horizontal equilibrium then yields the depth of the compression zone according to Eq. (3-20) with respect to the reinforcement ratio $\alpha_e \rho_l$.

$$\xi = \frac{1/2 + \alpha_e \rho_l}{1 + \alpha_e \rho_l}$$  \hspace{1cm} (3-20)

It can be seen that Eq. (3-20) is independent from the concomitant moment. This is also the case for uncracked RC members, if the longitudinal reinforcement is not considered.
In that case, the relative depth of the compression zone would be 0,5. Instead, Eq. (3-20) yields values between 0,5 and 1,0. In the cracked state of the beam, the tensile stresses in the bottom fiber have already exceeded the concrete tensile strength \( f_{ct} \) (Figure 3-8).

Figure 3-8: Distribution of normal stresses in a cross-section after cracking

The depth of the tension zone \( x_1 \) is calculated according to Eq. (3-21).

\[
x_1 = \frac{\sigma_{ct}}{\sigma_{cc}} x_0 = \frac{f_{ct}}{E_{cm}} x_0
\]  

(3-21)

The horizontal forces in the cross-section are given by Eqs. (3-22) to (3-24).

\[
F_{cc} = \frac{1}{2} \sigma_c E_{cm} b_w x_0
\]  

(3-22)

\[
F_{ct} = \frac{1}{2} f_{ct} b_w x_1 = \frac{1}{2} f_{ct} b_w \frac{f_{ct}}{E_{cm}} x_0
\]  

(3-23)

\[
F_s = \varepsilon_c E_s A_s = \varepsilon_c \left( \frac{1}{\xi} - 1 \right) E_s A_s
\]  

(3-24)

Horizontal equilibrium of forces yields the concrete strain \( \varepsilon_c \). Since the squareroot in Eq. (3-25) is restricted to values > 0, the values for \( \xi \) have to be restricted accordingly. Interestingly, this restriction yields minimum values for \( \xi \) that are identical to the depth of the compression zone if concrete tensile stresses are not considered. The consideration of tensile stresses therefore yields larger values for \( \xi \).

\[
\varepsilon_c = \frac{f_{ct} / E_{cm} \xi^2}{\sqrt{\xi^2 - 2\alpha,\rho (1 - \xi)}} and \, \xi > \sqrt{(\alpha,\rho)^2 + 2\alpha,\rho - \alpha,\rho}
\]  

(3-25)

The equilibrium of moments in combination with Eq. (3-25) results in Eq. (3-26).

\[
\frac{M_{Eds}}{f_{ct} b_w d^2} = \mu_c = \frac{1}{f_{ct} b_w d^2} \left[ F_{cc} d \left(1 - \frac{\xi}{3}\right) - F_{ct} d \left(1 - \frac{2}{3} \frac{f_{ct}}{E_{cm} \xi}\right) \right]
\]  

(3-26)
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

The relative depth of the compression zone $\xi$ is then subject to the dimensionless cracking moment $\mu_{ct}$ and the reinforcement ratio $\alpha_e \rho$ according to Eq. (3-27).

$$\mu_{ct} = \frac{\xi^2 (1 - \xi/3) - (\xi^2 - 2\alpha_e \rho (1 - \xi)) \left[ 1 - \xi - \frac{2}{3} \sqrt{\xi^2 - 2\alpha_e \rho (1 - \xi)} \right]}{2\sqrt{\xi^2 - 2\alpha_e \rho (1 - \xi)}}$$

(3-27)

Eq. (3-27) can practically only be solved by iteration. A computational solution for a range of values of $\alpha_e \rho$ and $\mu_{ct}$ reveals that the concrete tensile stresses have only little influence on the depth of the compression zone. There is, however, no smooth transition of the compression zone depth from an uncracked to a cracked member but rather a sudden transition (Figure 3-9) which was already pointed out by HORDIJK/Hor91/. The point of transition does not depend only on the elastic cracking moment $M_{cr}$. If this was the case, there would be a sharp transition at $\mu_{ct} = 1/6$ which represents the dimensionless elastic cracking moment. This is not the case as can be seen in Figure 3-9, which illustrates that the point of transition also depends on the longitudinal reinforcement ratio.

![Figure 3-9: Relative depth of the compression zone with respect to the reinforcement ratio and the dimensionless moment](image)

This relation can also be derived mathematically. It can be assumed, that the unstable transition from a cracked to an uncracked state takes place when the strains in the tensile fibre $\varepsilon_s$ reach a value of $f_{cd}/E_{cm}$ which results in Eq. (3-28).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ \mu_{ct} = \frac{1}{2} \frac{\xi \left(1 - \frac{\xi}{3}\right)}{1/\xi - 1} - \frac{1}{2} \left(1 - \xi\right) \left(1 - \frac{\xi}{2} \left(1 - \frac{\xi}{3}\right)\right) \]

\[ \Leftrightarrow \frac{1}{3} \xi^3 + \frac{5}{6} \xi^2 - \frac{1}{6} \xi + \mu_{ct} (\xi - 1) = 0 \]

(3-28)

A very precise approximate solution (Coefficient of determination \( R^2 = 0.9998 \)) of Eq. (3-28) is given by Eq. (3-29).

\[ \xi \approx 0.82 \mu_{ct}^{0.275} \]

(3-29)

Eq. (3-29) in combination with Eq. (3-20), yields Eq. (3-30) which gives the limit value \( \mu_{ct,\text{lim}} \) of the dimensionless moment for which the member will remain uncracked.

\[ \mu_{ct,\text{lim}} = 2.06 \left(\frac{1/2 + \alpha_e \rho_l}{1 + \alpha_e \rho_l}\right)^{3.64} \]

(3-30)

If the moment in a cross-section is larger than this value, the depth of the compression zone for a cracked RC member can be determined according to Eq. (3-31). Otherwise, for an uncracked member \( \xi \) can be calculated according to Eq. (3-32).

\[ \xi = \sqrt{(\alpha_e \rho_l)^2 + 2 \alpha_e \rho_l - \alpha_e \rho} \quad \text{and} \quad \mu_{ct} > 2.06 \left(\frac{1/2 + \alpha_e \rho_l}{1 + \alpha_e \rho_l}\right)^{3.64} \]

\[ \xi = \frac{1/2 + \alpha_e \rho_l}{1 + \alpha_e \rho_l} \quad \text{and} \quad \mu_{ct} \leq 2.06 \left(\frac{1/2 + \alpha_e \rho_l}{1 + \alpha_e \rho_l}\right)^{3.64} \]

(3-31)

(3-32)

The contribution of the inclined concrete compressive force \( V_{cc} \) can then be determined according to Figure 3-10. For this, a linear path of the compressive force is assumed between two control sections “0” and “1”. If the dimensionless moment \( \mu_{ct} \) is smaller or larger than the critical value for both sections, the vertical component \( V_{cc} \) is zero.

\[ \frac{1}{3} \cdot x_{0,\gamma r} \]

\[ F_{cc} \]

\[ F_{ct} \]

\[ F_{s} \]

\[ F_{cc} + \Delta F_{cc} \]

\[ F_{ct} \cdot \Delta F_{ct} \approx 0 \]

\[ F_{s} + \Delta F_{s} \]

Figure 3-10: Vertical component \( V_{cc} \) between to neighboring flexural cracks

If the depth of the compression zone differs in the two sections, the vertical component can be determined according to Eq. (3-33).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ V_{ce,\gamma_1} = F_{ce} \frac{\Delta z}{s_{cr}} = \frac{M_{Eds,\gamma_1}}{z} \left( \frac{1}{3} \left( x_{0,\gamma_1} - x_{0,\gamma_2} \right) \right) = \frac{M_{Eds,\gamma_1}}{d - x_{0,\gamma_1}/3} \left( x_{0,\gamma_1} - x_{0,\gamma_2} \right) \]  \hspace{1cm} (3-33)

3.3.3 Shear in the compression zone \( V_{cs} \)

In the compression zone, a parabolic shear stress distribution is given that equals a constant shear stress from the point of the resultant compressive force as illustrated in Figure 3-11. Failure in the compression zone is induced when the principal tensile stresses exceed the residual tensile strength of concrete. The governing point of failure is in the neutral axis of the cross-section, because the longitudinal stresses are zero and the principal tensile stresses reach their peak value at this point. It then follows that the absolute value of the principal tensile and compressive stresses is equal to \( \tau_{av} \) according to Eq. (3-34).

\[ \sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\frac{\sigma_x^2}{4} + \tau_{av}^2} = \pm \tau_{av} \]  \hspace{1cm} (3-34)

According to the biaxial material laws for non-reinforced concrete by KUPFER /Kup73/, the principal compressive stresses in that point decrease the effective tensile strength according to Eq. (3-35).

\[ \sigma_{1,\text{max}} = \left( 1 + 0.8 \frac{\sigma_2}{f_c} \right) f_{ct} \]  \hspace{1cm} (3-35)

![Figure 3-11: Illustration of the shear flow according to the Euler-Bernoulli-beam theory](image)

Thus, the compression zone can be checked for shear failure according to Eq. (3-36).

\[ \tau_{av} \leq \left( 1 - 0.8 \frac{\tau_{av}}{f_{cm}} \right) f_{cm} \]  \hspace{1cm} (3-36)

The contribution of the compression zone \( V_{cs} \), and its respective failure criterion are then given by Eq. (3-37).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ V_{\text{cs}} = \frac{2}{3} \tau_{av} b_w x_0 \leq \frac{2}{3} \left( 1 - 0.8 \frac{\tau_{av}}{f_{cm}} \right) f_{cm} b_w x_0 \]  

(3-37)

3.3.4 Residual tensile stresses at the crack tip

In the region of the flexural crack tip, the shear stresses are transmitted by the residual tensile strength of concrete. Although the concrete tensile strength is exceeded at the tip of the crack, it can be assumed that a certain area of microcracking is necessary for damage localization /Hil83/. Therefore, a mean value of effective tensile stresses at the crack tip \( \sigma_{xm} \) can be determined as done by TUE ET AL. /Tue14/, /Tun16a/. In /Tun16a/, the length over which the horizontal crack branch is able to form was defined as \( d_{b,\text{crit}} \) as shown in Figure 3-12. However, the value of \( d_{b,\text{crit}} \) by TUE ET AL. was derived empirically through evaluation of selected tests on the basis of the respective model assumptions. Therefore, it seems out of place to simply adopt this value to a different model. From a mechanical point of view it is reasonable to assume that the length of the microcracking zone \( d_{b,\text{crit}} \) in which the crack localizes is proportional to the characteristic length \( l_{ch} \) of concrete according to Eq. (3-38) /Hil83/.

\[ d_{b,\text{crit}} \sim l_{ch} = \frac{E_c G_f}{f_{ct}^2} \]  

(3-38)

\[ a) \quad d_{b,\text{crit}} < x_1 + x_2 \]

\[ b) \quad d_{b,\text{crit}} \geq x_1 + x_2 \]

Figure 3-12: Calculation of mean stresses at the crack tip according to /Tun16a/

According to HILLERBORG /Hil83/, the length of the fracture zone is often in the order of 0.3 to 0.5 \( l_{ch} \) so that the principal tensile stresses can be averaged within this distance according to Figure 3-12. For this approach, it is therefore assumed that the critical crack length is approximately equal to 30 % of the characteristic length according to Eq. (3-39).

\[ d_{b,\text{crit}} \approx 0.3 \cdot \frac{E_c G_f}{f_{ct}^2} \]  

(3-39)

According to Eq. (3-40) it can actually be seen, that this value of \( d_{b,\text{crit}} \) is proportional to \( f_c^{-0.85} \) which is in part quite similar to the value given by TUE ET AL., which is \( f_c^{-1} \).
The mean residual tensile stress $\sigma_{xm}$ is then determined according to Eq. (3-41). In case the critical length $d_{b, crit}$ is smaller than the height of the crack processing zone $(x_1 + x_2)$, the mean value $\sigma_{xm}$ has to be calculated within the length $d_{b, crit}$, which makes it more likely that a critical horizontal crack forms. However, if the critical length exceeds the height of the crack processing zone, the formation of a critical shear crack becomes more unlikely as it indicates that there is no more crack growth at the tip of the crack.

\[
\sigma_{xm} = \begin{cases} 
 f_{ctm} \cdot \left(1 - 0.5 \frac{d_{b, crit}}{x_1 + x_2}\right) & \text{and } d_{b, crit} < x_1 + x_2 \\
 f_{ctm} \cdot \frac{x_1 + x_2}{d_{b, crit}} & \text{and } d_{b, crit} \geq x_1 + x_2 
\end{cases}
\]  

It has to be checked, if the average shear stress $\tau_{av}$ at the crack tip exceeds the maximum residual shear strength according to Eq. (3-42).

\[
\tau_{av} \leq \tau_{u, \text{max}} = \sqrt{f_{ct}(f_{ct} - \sigma_{xm})}
\]  

The distance $x_1$ from the neutral axis to the point of maximum tensile stresses can be determined from the Bernoulli hypothesis as given by Eq. (3-43).

\[
x_1 = \frac{e_{cr}}{e_s} (d - x_0) = \frac{f_{ct} / E_e}{e_s} (d - x_0) \leq d - x_0
\]  

The depth of the compression zone $x_0$ is given by Eq. (3-44).

\[
x_0 = \begin{cases} 
 \left(\sqrt{(\alpha e \rho)^2 + 2\alpha e \rho - \alpha e \rho}\right) d & \text{and } \mu_{\alpha} > 2.06 \left(\frac{1/2 + \alpha e \rho l}{1 + \alpha e \rho l}\right)^{3.64} \\
 \frac{1/2 + \alpha e \rho l}{1 + \alpha e \rho l} d & \text{and } \mu_{\alpha} \leq 2.06 \left(\frac{1/2 + \alpha e \rho l}{1 + \alpha e \rho l}\right)^{3.64} 
\end{cases}
\]

The tensile stresses reduce from the tensile strength $f_{ctm}$ to zero over a distance of $x_2$. The required crack width $w_1$ depends on the fracture energy $G_f$ of concrete. $G_f$ represents the area under the stress-displacement diagram in tension (Figure 3-13). Here, it is assumed that the tensile stresses in concrete decrease linearly with respect to the crack width $w$. 

\[
d_{b, crit} \approx \frac{E_G}{f_{ct}^2} \approx \frac{f_{ct}^{0.3} f_{ct}^{0.18}}{(f_{ct})^{2/3}} \approx f_{ct}^{0.85}
\]  

(3-40)
The value of $w_I$ can be determined according to Eq. (3-45).

$$G_f = \frac{1}{2} f_{cm} w_I \Rightarrow w_I = \frac{2 G_f}{f_{cm}}$$

(3-45)

With BERNOULLI’s hypothesis it can be assumed that the cracks open linearly so that the length $x_2$ can be calculated by Eq. (3-46). The crack opening $w_k$ at reinforcement level can be determined according to chapter 3.3.7.

$$x_2 = \frac{w_I}{w_k} (d - x_0 - x_i) = \frac{2G_f / f_{cm}}{w_k} (d - x_0 - x_i) \leq d - x_0 - x_i \text{ and } w_k > 0$$

$$x_2 = 0 \text{ and } w_k = 0$$

(3-46)

If the fracture energy $G_f$ of concrete has not been measured, its value can be estimated by Eq. (3-47) which is an empirical formula from /Mar15/.

$$G_f = 0.028 \cdot f_{cm}^{0.18} \cdot d_{ag}^{0.32}$$

(3-47)

The total shear contribution from the crack processing zone $V_{CPZ}$ and the respective failure criterion are given by Eq. (3-48).

$$V_{CPZ} = \tau_{cr} b_v (x_i + x_2) \leq \sqrt{f_{cr} (f_{cr} - \sigma_{cm})} \cdot b_v (x_i + x_2)$$

(3-48)

### 3.3.5 Aggregate interlock

Along the interfaces of flexural bending cracks, shear stresses can only be transferred by aggregate interlock. The average shear stresses in that region may therefore not exceed the shear stresses that can be resisted by aggregate interlock. Constitutive equations for determining aggregate interlock stresses based on crack widths $w$ and crack slips $\delta$ have for instance been proposed by WALRAVEN /Wal80/, GAMBAROVA /Gam83/, REINECK /Rei90/ or VECCHIO & LAI /Vec04/. Within this thesis, the crack slip $\delta$ will however not be considered since a corresponding kinematic mechanism has not been
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

established in the beginning. It is also a misconception that a considerable slip is necessary to “activate” aggregate interlock stresses. Rather, the gradual opening of the flexural cracks entails the simultaneous existence of interface shear stresses as the principal stresses of the concrete in tension are not normal to the crack surface even in the uncracked state (cf. /Baz79/). Moreover, it should be considered that a slip-free approach has some advantages for limit state design. BAŽANT & GAMBAROVA /Baz80/ concluded from experimental and theoretical evaluations that the ultimate loads determined by slip-free approaches corresponded with the beginning of a rapid increase of large crack widths and slips. BAŽANT & TSUBAKI /Baz79/ also stated that “By imposing a condition of no slip, one tries to avoid the consequences of the crack slip. These include crushing and breakage of asperities (surface humps, aggregate pieces) at their roots, spreading of secondary cracks from the original crack, and dowel splitting, which can lead to substantial damage to concrete”.

Several slip-free approaches can be considered to predict the capacity of this mechanism with respect to crack widths. Based on the work of WALRAVEN /Wal80/, VECCHIO & COLLINS /Vec86/ proposed Eq. (3-49) to account for the maximum aggregate interlock stresses of a crack surface if the compressive stresses on the surface are equal to zero. The value \( \tau_{ai,max} \) depends on the concrete compressive strength \( f_c \), the crack width \( w \) and the maximum aggregate size \( d_{ag} \).

\[
\tau_{ai,max} = \frac{0.18 \sqrt{f_c}}{0.31 + \frac{24w}{d_{ag} + 16}}
\]  
(3-49)

One problem of Eq. (3-49) seems to be that it will always provide values larger than zero for any given crack width \( w \). However, it would seem quite reasonable to assume that there is a finite critical crack width \( w_0 \), for which it is effectively not possible to transmit any further aggregate interlock stresses under the condition of no slip. Eq. (3-49) was therefore extended by FISKER & HAGSTEN /Fis16/ into Eq. (3-50) with a linearly decreasing term to impose an upper limit for \( w \) above which no aggregate interlock stresses exist.

\[
\tau_{ai}(w) = \frac{0.18 \sqrt{f_{ck}}}{0.31 + \frac{24w}{d_{ag} + 16}} \left(1 - \frac{w}{w_0}\right)
\]  
(3-50)

Such a linear decreasing term was also used by REINECK /Rei91/ within his mechanical tooth model. The value of \( w_0 \) should depend on the aggregate size but also on the concrete strength. For high strength concretes, crack surfaces tend to be smoother so that the value of \( w_0 \) decreases. In /Fis16/, a value of \( w_0 = 2 \text{ mm} \) was assumed for normal
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

strength concrete and a reduction of that value to 0.5 ~ 1.0 mm was proposed for concrete strength larger than 50 MPa. Within this thesis, Eq. (3-50) is adopted with a minor change. The coefficient of 0.18 is replaced by a variable \( k_{ai} \) according to Eq. (3-51), which will be subject to the evaluation of shear tests.

\[
\tau_{ai}(w) = \frac{k_{ai}\sqrt{f_{ck}}}{0.31 + \frac{24w}{d_{ag} + 16}} \left(1 - \frac{w}{w_0}\right) 
\]

The critical crack width \( w_0 \) will be decreased linearly from 2.0 to 1.0 mm for concrete strengths between 50 and 60 MPa according to Eq. (3-52).

\[
1.0 \leq w_0 = 2.0 - 0.1\left(f_{ck} - 50\right) \leq 2.0 \text{ (w}_0 \text{ in mm)}
\]

The crack width \( w \) varies along the flexural crack. The crack width \( w_k \) is defined as the crack width at reinforcement level. The crack width along the flexural crack can be determined by linear interpolation (Figure 3-14). Eq. (3-53) gives the maximum shear stress resisted by aggregate interlock at each point of the flexural crack subject to the vertical position \( z \) according to Figure 3-14.

\[
\tau_{ai}(z) = \frac{k_{ai}\sqrt{f_{ck}}}{0.31 + \frac{24w_k}{d_{ag} + 16(z - d_{x_0} - x_1)}} \left(1 - \frac{w_k}{w_0(d - x_0 - x_1)}\right) 
\]

As shown in Figure 3-14, the actual portion of the shear forces carried by aggregate interlock can be determined by calculating the blue area of \( V_{ai} \). If \( \tau_{ai}(z) \) is larger than the average stress on the cross-section \( \tau_{av} \), the capacity of aggregate interlock is not utilized to its full extent. From the point where \( \tau_{ai}(z) < \tau_{av} \), the excess shear stresses have to be compensated by dowel action which is indicated by the green area \( V_{dw} \) in Figure 3-14. The calculation of the the dowel contribution \( V_{dw} \) will be treated in chapter 3.3.6. The contribution of aggregate interlock \( V_{ai} \) is given by Eq. (3-54).

\[
V_{ai} = \tau_{av}b_w(d - x_0 - x_1 - x_2) - V_{dw} 
\]
If a transfer of aggregate interlock stresses to dowel stresses is assumed to be possible in an unlimited way, there would actually be no necessity to define a failure criterion for aggregate interlock failure. By definition, a failure in this region could only be induced by dowel failure. However, within the scope of this model it seems reasonable to limit the extent to which a redistribution of aggregate interlock stresses to dowel action is possible. If the crack surfaces are stress free, the bending moment of the concrete tooth has to be resisted by dowel action and the clamping support of the concrete compression zone which would certainly exceed the concrete tensile stresses. Also, the condition of no slip forbids an unlimited decrease of aggregate interlock stresses which is associated with large crack widths and a considerable crack slip. Such a state should therefore be avoided. A very convenient way to do so is to define a maximum allowable average shear stress for aggregate interlock $\tau_{ai,\text{max}}$ so that $\tau_{av} \leq \tau_{ai,\text{max}}$ is the corresponding failure criterion. The maximum shear force $V_{ai,\text{max}}$ that corresponds to $\tau_{ai,\text{max}}$ will be defined as the capacity of aggregate interlock over the whole flexural crack surface, which is the blue area of $V_{ai}$ combined with the red area in Figure 3-14. In other words, the total shear force over the crack surface may not exceed the capacity of aggregate interlock over the crack surface. Nevertheless, it is still possible that dowel failure occurs before aggregate interlock failure.

To determine the maximum shear force $V_{ai,\text{max}}$ that can be transferred by aggregate interlock by integration, a distinction of cases has to be made. In the first case, if the crack width $w_k$ is less than the critical crack width $w_0$ ($w_k \leq w_0$), the integral is given by Eq. (3-55). If $w_k > w_0$, then the aggregate interlock stress become zero along the crack and the upper limit of the integral has to be adjusted according to Eq. (3-56).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ V_{ai,\text{max}}(w) = k_{ai} \sqrt{f_{ck}} b_w \int_{x_2}^{d-x_0-x_1} \left( \frac{1}{w_k} \frac{z}{w_0 \left( d-x_0-x_1 \right)} \right) dz \quad \text{and} \quad w_k \leq w_0 \]  
(3-55)

\[ V_{ai,\text{max}}(w) = k_{ai} \sqrt{f_{ck}} b_w \int_{x_2}^{w_0/(d-x_0-x_1)} \left( \frac{1}{w_k} \frac{z}{w_0 \left( d-x_0-x_1 \right)} \right) dz \quad \text{and} \quad w_k > w_0 \]  
(3-56)

For \( w_k \leq w_0 \), the solved integral is given by Eq. (3-57).

\[ V_{ai,\text{max}} = k_{ai} \sqrt{f_{ck}} b_w \left( d-x_0-x_1 \right) \left( \frac{24w_k}{d_{ag}+16} \right) \left( \frac{1}{w_k} \frac{z}{w_0 \left( d-x_0-x_1 \right)} \right) \left( \frac{24w_k}{d_{ag}+16} + 0.31 \right) \ln \left( \frac{24w_k}{d_{ag}+16} + 0.31 \right) \]  
(3-57)

For \( w_k > w_0 \), the solved integral is given by Eq. (3-58).

\[ V_{ai,\text{max}} = k_{ai} \sqrt{f_{ck}} b_w \left( d-x_0-x_1 \right) \left( \frac{24w_k}{d_{ag}+16} \right) \left( \frac{1}{w_k} \frac{z}{w_0 \left( d-x_0-x_1 \right)} \right) \left( \frac{24w_k}{d_{ag}+16} + 0.31 \right) \ln \left( \frac{24w_k}{d_{ag}+16} + 0.31 \right) \]  
(3-58)

The maximum average shear stress \( \tau_{ai,\text{max}} \) that can be transferred by aggregate interlock along the flexural crack can be calculated by Eq. (3-59).

\[ \tau_{ai,\text{max}} = \frac{V_{ai,\text{max}}}{b_w \left( d-x_0-x_1-x_2 \right)} \]  
(3-59)

The failure criterion of aggregate interlock is defined as the point at which the average shear stresses on the cross-section \( \tau_{av} \) exceed the maximum average shear stresses by aggregate interlock \( \tau_{ai,\text{max}} \) (Eq. (3-60)).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ \tau_{av} = \frac{V_{Ed} - V_{cc}}{b_v z} \leq \tau_{av,\text{max}} = \frac{V_{av,\text{max}}}{b_v (d - x_0 - x_i - x_2)} \]  

(3-60)

### 3.3.6 Dowel Action

Since this model is equilibrium based, dowel action is a direct result of the loss of aggregate interlock due to large crack widths \( w \). From the point in the flexural crack where the average shear stresses \( \tau_{av} \) exceed the shear resistance by aggregate interlock \( \tau_{ai}(z) \), another shear transfer action must compensate for the loss of aggregate interlock to maintain internal equilibrium of forces. Since the loss of aggregate interlock is always initiated at the level of longitudinal reinforcement (where crack widths are the largest), it is reasonable to assume that the difference is primarily compensated by dowel action. The shear stresses that can be transferred by aggregate interlock decrease with larger flexural cracks \( w \). The vertical position \( x_{dw} \) for which the maximum aggregate interlock is smaller than the average shear stress (i.e. \( \tau_{ai}(z) < \tau_{av} \)) can be determined by using Eq. (3-53) which results in Eq. (3-61).

\[ x_{dw} = \frac{k_{ai} \sqrt{f_{ck}} - 0.31 \tau_{av}}{k_{ai} \sqrt{f_{ck}} w_k + \frac{24w_k}{d_{ag} + 16} \tau_{av}} (d - x_0 - x_i) \leq (d - x_0 - x_i) \]  

(3-61)

The shear force resisted by dowel action \( V_{dw} \) is the total shear force in the critical area minus the shear still resisted by aggregate interlock. For \( w_k \leq w_0 \) it can be determined by Eq. (3-62).

\[ V_{dw} = \tau_{av} (d - x_0 - x_i - x_{dw}) b_v \]

\[ -k_{ai} \sqrt{f_{ck}} b_v \left( \frac{24w_k}{d_{ag} + 16} \right)^2 \left( \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 w_k \right) \right) \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 \right) \]

\[ -k_{ai} \sqrt{f_{ck}} b_v \left( \frac{24w_k}{d_{ag} + 16} \right)^2 \left( \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 w_k \right) \right) \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 \right) \]

\[ + \frac{24w_k}{d_{ag} + 16} \left( \frac{24w_k}{d_{ag} + 16} + 0.31 \right) \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 \right) \]

\[ + \frac{24w_k}{d_{ag} + 16} \left( d - x_0 - x_i \right) \]

(3-62)

For \( w_k > w_0 \), the dowel force \( V_{dw} \) is calculated by Eq. (3-63).
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ V_{dw} = \tau_{av} \left( d - x_0 - x_1 - x_{dw} \right) b_w \]

\[ = -k_{ul} \sqrt{f_{ck}} b_w \left( \frac{24w_k}{d_{ag} + 16} \right)^2 \left[ \frac{24w_k}{d_{ag} + 16} + 0.31 \frac{w_k}{w_0} \right] \ln \left( \frac{24w_k}{d_{ag} + 16} + 0.31 \right) \]

\[ + \frac{24w_k}{d_{ag} + 16} \frac{w_k}{w_0} \frac{x_{dw}}{d_{ag} + 16 (d - x_0 - x_1)} \]

\[ \left( d - x_0 - x_1 \right) \]

(3-63)

It then has to be checked if the dowel force \( V_{dw} \) can be resisted by the dowel resistance \( V_{dw, max} \). The dowel forces cause bedding stresses in the surrounding concrete (Figure 3-15a). The dowel can then either fail due to tensile failure of the concrete in the reinforcement layer cover (Figure 3-15b) or by formation of a plastic hinge in the longitudinal reinforcement bar with simultaneous crushing of the concrete by bearing stresses (Figure 3-15c).

![Figure 3-15: a) bedding stresses in concrete due to dowel forces; b) tensile stresses in concrete due to dowel forces; c) plastic hinge and local crushing of concrete (reproduced from /Zil10/)](image)

The tensile failure of the concrete cover according to Figure 3-15b was investigated by BAUMANN & RÜSCH /Bau70/. The capacity of the concrete dowels for this mode of failure can be calculated by Eq. (3-64) (as given in /Yan14/).

\[ V_{dw,c} = 1,64b_w \phi \sqrt{f_c} \]

(3-64)

The dowel failure of the longitudinal reinforcement by forming a plastic hinge in combination with local crushing of the concrete which was investigated by RASMUSSEN /Ras62/ and VINTZELÉOU & TASSIOS /Vin86/. The coefficient of 1.18 in Eq. (3-65) for the ultimate dowel load for an individual bar was derived on the basis of test results /Ran05/. The stresses in the longitudinal reinforcement can be determined according to chapter 3.3.7.
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ V_{\text{dw},s,i} = 1,18 \cdot \phi_2^2 \cdot \sqrt{f_c f_{\text{y}}} \cdot \sqrt{1 - \left( \frac{\sigma_s}{f_{\text{y}}} \right)^2} \]  \hspace{1cm} (3-65)

The total dowel resistance \( V_{\text{dw,s}} \) is the sum of the dowel resistances of the individual bars according to Eq. (3-66).

\[ V_{\text{dw,s}} = \sum_i V_{\text{dw},s,i} \]  \hspace{1cm} (3-66)

The failure criterion of the dowel action can thus be defined according to Eq. (3-67).

\[ V_{\text{dw}} \leq V_{\text{dw,max}} = \min \left\{ V_{\text{dw,c}}; V_{\text{dw,s}} \right\} \]  \hspace{1cm} (3-67)

3.3.7 Crack widths and longitudinal reinforcement stresses

The crack width \( w_k \) on the reinforcement level can be calculated according to Eq. (3-68) /EC211/. The restriction of Eq. (3-68) to values larger than zero should be implemented for an automatic computation of the procedure.

\[ w_k = \frac{s_{\text{m}}}{E_s} \left[ \sigma_s - k_i \frac{f_{\text{ct}}}{\rho_{p,\text{eff}}} \left( 1 + \alpha_s \rho_{p,\text{eff}} \right) \right] \geq 0 \]  \hspace{1cm} (3-68)

The coefficient \( k_i \) is 0,4 for long term loading and 0,6 for short term loading /EC211/. For the evaluation of tests, it can be assumed that \( k_i \) is equal to 0,5. The effective reinforcement ratio \( \rho_{p,\text{eff}} \) is determined by Eq. (3-69).

\[ \rho_{p,\text{eff}} = \frac{A_p}{A_{c,\text{eff}}} \]  \hspace{1cm} (3-69)

The effective area of concrete is calculated according to Eq. (3-70).

\[ A_{c,\text{eff}} = b_w h_{c,\text{eff}} \text{ and } h_{c,\text{eff}} = \min \left\{ 2,5(h - d); \frac{(h - x)}{3}; \frac{h}{2} \right\} \]  \hspace{1cm} (3-70)

where

\( x \) depth of the compression zone in the uncracked state

The spacing between two fully developed flexural cracks \( s_{\text{rm}} \) is a critical factor for the shear capacity of members without shear reinforcement. It directly influences the crack width \( w_k \) at the reinforcement level and thereby the resistance by aggregate interlock. As a first approximation, it can be assumed that the crack spacing is approximately equal to the height of the fully developed flexural crack \((d - x_0)\) /Tue14/. In addition, the crack spacing can be calculated according to Eq. (3-71) of EC2 /EC211/.
3.3 Mechanical Model for Flexural Shear Strength of RC Beams

\[ s_{r,\text{max}} = k_3 \cdot c + k_1 k_2 k_4 \frac{\phi}{\rho_{\text{eff}}} \]  

(3-71)

The recommended values for \( k_3 \) and \( k_4 \) are 3.4 and 0.425, respectively. The coefficient \( k_1 \) describes the bond quality and is 0.8 for high bond and 1.6 for bars with plain surface. Here, \( k_1 \) is chosen as 1.0. The coefficient \( k_2 \) accounts for the distribution of strain and is 1.0 for pure tension and 0.5 for pure bending. A coefficient of 1.0 has to be chosen since the effective area \( A_{\text{c,eff}} \) surrounding the concrete is under pure tension. A coefficient of 0.5 for \( k_2 \) would be incoherent /Fin16/. The crack spacing can therefore be chosen as the maximum of the height of the flexural crack and the crack spacing according to Eq. (3-72). However, the crack spacing needs not be chosen larger than the length that is required to anchor the tensile stresses in the longitudinal reinforcement by bond stresses, which is expressed in Eq. (3-72).

\[ s_{r,m} = \max \left\{ d - x_0, 3.4 \cdot c + 0.425 \frac{\phi}{\rho_{\text{eff}}} \right\} \leq \frac{\sigma_s \cdot \phi}{3.6 \cdot f_{\text{cm}}} \]  

(3-72)

The mean tensile stresses in the longitudinal reinforcement can be calculated according to Eq. (3-73) if the section is cracked (\( \mu_{ct} > \mu_{ct,\text{lim}} \), see chapter 3.3.2).

\[ \sigma_s = \frac{M_{Ed}}{(d-x_0/3)A_i} \quad \text{and} \quad \mu_{ct} > \mu_{ct,\text{lim}} \]  

(3-73)

where

\[ x_0 = \xi \cdot d \quad \text{with} \quad \xi \quad \text{according to Eq. (3-31)} \]

\[ \mu_{ct} = \frac{M_{Ed}}{f_{ct} b_w d^2} \]

\[ \mu_{ct,\text{lim}} \quad \text{according to Eq. (3-30)} \]

If the cross-section is uncracked, the contribution of the concrete in tension has to be accounted for. From equilibrium of moments in the uncracked state (Figure 3-7), the concrete strains \( \varepsilon_c \) can be determined from Eqs. (3-17) and (3-18) according to Eq. (3-74).

\[ \varepsilon_c = \frac{\mu_{ct}}{1 - \frac{E_c}{6 E_{ct}} \left( \frac{1}{\xi} - 1 \right) \left( 1 - \frac{1}{\xi} \right)^2} \quad \text{and} \quad \mu_{ct} \leq \mu_{ct,\text{lim}} \]  

(3-74)

where

\[ \xi \quad \text{dimensionless compression zone depth according to Eq. (3-32)} \]

\[ \mu_{ct} = \frac{M_{Ed}}{f_{ct} b_w d^2} \]

\[ \mu_{ct,\text{lim}} \quad \text{according to Eq. (3-30)} \]
Using Eq. (3-19), the steel stress $\sigma_s$ can be calculated according to Eq. (3-75).

$$\sigma_s = \varepsilon_s E_s = \varepsilon_c \left( \frac{1}{\xi} - 1 \right) E_s \text{ and } \mu_{ct} \leq \mu_{ct,\text{lim}}$$

$$\Rightarrow \sigma_s = \frac{\mu_{ct} \left( \frac{1}{\xi} - 1 \right) E_s}{\frac{1}{2} E_c \xi \left( 1 - \frac{\xi}{3} \right) - \frac{1}{6} E_c \left( \frac{1}{\xi} - 1 \right) (1 - \xi)^2} \text{ and } \mu_{ct} \leq \mu_{ct,\text{lim}} \quad (3-75)$$

where

- $\xi$ dimensionless compression zone depth according to Eq. (3-32)
- $\mu_{ct} = M_{Ed} / (f_{ct} b_w d^2)$
- $\mu_{ct,\text{lim}}$ according to Eq. (3-30)

### 3.3.8 Control section

Often, the control section for members in shear is located at a distance of the effective depth $d$ from a point of discontinuity, which is either a support or a concentrated load. In general, a higher shear slenderness and thus higher bending moments are considered unfavorable for the shear capacity. This is supported by experimental results (e.g. /Kre66/, /Leo62/) and is also reflected in various available shear models for beams without shear reinforcement (e.g. /Mut08/, /Ben06/, /Zar01/, /Zin99/). For beams under distributed loads, a critical control section is therefore often not known in advance, as the highest shear forces are located at the support and the largest bending moments in the field. On the other side, for single span beams under point loads it seems clear that the critical control section should be located near the load introduction. The shear slenderness $\lambda$ in the control section would therefore amount to $a/d - 1$. However, according to EC2 Sec. 6.2.2(5) /EC211/, the $M_{Ed}$-line should be shifted over a distance $a_l = d$ to the unfavourable direction for the design of the longitudinal reinforcement, in the region cracked in flexure. This shift can be seen as an analogy to the additional tensile forces in the longitudinal reinforcement which must be considered in a truss analogy. This is also in agreement with the provisions of Model Code 2010 /CEB10/ for beams without shear reinforcement according to which an additional force of $V_{Ed}$ must be considered in the bottom stringer to determine the longitudinal strains. Alternatively, the shear slenderness in the control section can simply be increased by 1 which gives approximately equivalent results as the provisions of Model Code 2010 as shown in Eq. (3-76).

$$F_s = \frac{M_{Ed}}{z} + V_{Ed} = \frac{1}{z} (V_{Ed} \lambda d + V_{Ed} z) = \frac{V_{Ed} d}{z} \left( \frac{a}{d} - 1 + \frac{z}{d} \right) \approx \frac{V_{Ed} d}{z} a \quad (3-76)$$
3.4 Evaluation of the Mechanical Model

3.4.1 General

In this section, the developed mechanical model for reinforced concrete beams without shear reinforcement will be evaluated. First, a comparison with beam tests from shear databanks will be performed to assess the accuracy of the model. Then, the model will be evaluated concerning its predictions on the influence of different parameters on the shear capacity of RC beams. From this, conclusions on the requirements of a code model for RC beams can be drawn.

3.4.2 Evaluation of shear tests

To validate the mechanical model, tests from the ACI-DAfStb shear databanks for RC beams without shear reinforcement (/Rei12/, /Rei13/) are evaluated with the procedure presented in chapter 3.3. The original databank consists of 784 shear tests on RC beams under point and distributed loads. The tests were filtered further according to the following criteria:

- $a/d \geq 2.89$
- only rectangular cross-section ($b = b_w$)
- only point loaded tests
- maximum aggregate size $d_{ag}$ given
- number and diameter of reinforcement bars as flexural tensile reinforcement given

The shear slenderness of the evaluated tests is restricted to slender beams with $a/d \geq 2.89$. Although the databank also contains tests with $2.4 \leq a/d < 2.89$, these tests are sometimes biased by disproportional influence of direct strut action which can be more suitably described by compression chord models. Also, the evaluation is at this point limited to rectangular cross-sections and point loaded tests. To evaluate the shear strength of T-beams, additional assumptions or investigations are required for determining the effective shear width. As stated in chapter 3.3.8, the critical section of a beam under distributed loading might not be known in advance. So for the sake of simplicity as well as due to the fact, that the beams in the databank under distributed loads were “transformed“ into point loaded beams so that important information about the ultimate line load is missing, the evaluation is limited to point loaded beams. Since information on the maximum aggregate size $d_{ag}$ and the number and diameter of the reinforcement...
bars for the flexural reinforcement is crucial to determine the resistance of aggregate interlock and dowel action, only tests for which this information is provided are evaluated. With these filters, a total of 408 tests out of 784 remained for evaluation.

To evaluate a single test, the acting shear load $V_{\text{calc}}$ is increased until one of the failure criteria is reached. The four failure criteria are summarized in Eqs. (3-77) to (3-80).

$$V_{cs} = \frac{2}{3} \tau_{av} b_w x_0 \leq \frac{2}{3} \left(1 - 0.8 \frac{\tau_{av}}{f_{cm}} \right) f_{cm} b_w x_0$$  \hspace{1cm} (3-77)

$$V_{CPZ} = \tau_{av} b_w (x_1 + x_2) \leq \sqrt{f_{cs}(f_{cs} - \sigma_{\text{adm}})} \cdot b_w (x_1 + x_2)$$  \hspace{1cm} (3-78)

$$\tau_{av} \leq \tau_{ai,\text{max}} = \frac{V_{ai,\text{max}}}{b_w (d - x_0 - x_1 - x_2)}$$  \hspace{1cm} (3-79)

$$V_{dvw} \leq V_{dvw,\text{max}} = \min \left\{ V_{dvw,\text{c}} ; V_{dvw,\text{s}} \right\}$$  \hspace{1cm} (3-80)

In addition, the stresses in the longitudinal reinforcement are limited to the elastic range as yielding of the reinforcement and the consequent nonlinear determination of the strain distribution in the cross-section are not considered at this point. To account for excess strength in the reinforcement, the tensile stresses $\sigma_s$ are limited to 110% of the yield strength according to Eq. (3-81).

$$\sigma_s \leq 1.1 \cdot f_y$$  \hspace{1cm} (3-81)

One of the benefits of the model is that the contribution of each shear transfer action to the overall shear resistance for each load step can be determined. This is exemplarily illustrated for two individual tests in Figure 3-16. On the horizontal axis, the ratio of the applied theoretical load $V_{\text{Ew}}$ and the ultimate load $V_{\text{test}}$ in the test is displayed. On the vertical, axis the cumulated ratio of the different contributions $V_i$ and the applied load $V_{\text{Ew}}$ is shown. In this way, the distribution of the shear contributions is illustrated for each load stage.

![Figure 3-16: Evaluation of shear tests from the literature: a) Specimen 7-2 /Leo62/; b) Specimen 03 /Niw86/](image-url)
The first test in Figure 3-16a is a slender RC beam from LEONHARDT & WALThER /Leo62/ \((a/d = 4.9; h = 320 \text{ mm})\). In the uncracked state, the shear force is of course completely carried by concrete. Then, a first flexural crack forms at 10% of the ultimate load in the control section while the neighboring section remains uncracked. Therefore, a large part of the shear force is briefly carried by an inclined strut. This contribution disappears as flexural cracks form in the neighboring section as well. Note that aggregate interlock is not activated although the cross-section is cracked. This is due to the fact that the crack processing zone is as large as the crack height which is only possible for small cross-section depths. With increasing shear forces and therefore increasing crack widths, the length of the crack processing zone decreases and aggregate interlock is activated. Since the compression zone depth remains constant, the excess shear has to be carried by aggregate interlock. At failure, 30% of the shear force is carried in the compression zone while 25% is carried in the crack processing zone. Aggregate interlock contributes 43% while dowel action is limited to 2%.

The large size test by NIWA ET AL. /Niw86/ was evaluated in the same manner. Compared to LEONHARDT’s test, the test specimen is very large \((h = 1100 \text{ mm})\) while the shear slenderness is smaller \((a/d = 2.98)\). As before, the uncracked concrete carries all of the shear as the loading begins. When the crack in the control section opens, a direct strut and aggregate interlock are activated. When the flexural crack in the neighboring section opens as well, the excess shear force has to be carried by aggregate interlock as the contribution of the direct strut vanishes. As the capacity by aggregate interlock decreases with wider flexural cracks, the dowel forces in the reinforcement are activated. In the ultimate limit state, 38% of the shear force is carried by dowel action while 47% is carried by aggregate interlock. The remaining 9% and 6% are carried by the compression zone and the crack processing zone, respectively. As shown by these two examples, the distribution of the shear forces within a section strongly depends on the test parameters. It seems, however, that in the cracked state a larger part of the shear force must be carried by aggregate interlock.

In this way, all of the 408 tests in the selected databank were evaluated. The evaluation resulted in a mean ratio of test and theoretical loads of 0.99 and a coefficient of variation of 15.2% using a coefficient for aggregate interlock of \(k_{ai} = 0.14\). For a mechanical model this is a quite satisfying result, since trends regarding individual variables can not be as easily “corrected” as with a fully empirical model. The ratio of test and theoretical loads over the concrete strength \(f_{cm}\) and the longitudinal reinforcement ratio \(\rho_l\) is shown in Figure 3-17. There seem to be no particular trends concerning the concrete strength \(f_{cm}\). As can be seen, the model was applied to concrete strengths ranging from 13 to 111 Mpa so that normal strength as well as high strength concrete is included in the evaluation. Also, the influence of the longitudinal reinforcement ratio seems to be captured satisfactorily.
3.4 Evaluation of the Mechanical Model

The influence of the effective depth \(d\) and the shear slenderness \(a/d\) is shown in Figure 3-18a. The scatter for the smaller test specimens seems to be higher than for larger specimens which is due to the fact that small specimens may not exhibit a flexural-shear-like behaviour but can rather be described by plasticity analysis which was also pointed out in /Baz09/. The influence of the shear slenderness \(a/d\) in Figure 3-18b is captured very well. In fact, the scatter seems to be getting considerably smaller for beams with higher shear slenderness. This is most likely due to the fact that for a higher shear slenderness, the behaviour of the beam is mostly controlled by the crack opening at the level of the reinforcement. For smaller values of \(a/d\), influences from the crack processing zone, direct struts or other failure modes may occur. Also, the flexural cracks may be more inclined for small values of \(a/d\) which contradicts the assumption of the Bernoulli hypothesis for the model.

![Figure 3-17: Comparison of experimental and theoretical loads over a) concrete strength \(f_{cm}\); b) longitudinal reinforcement ratio \(\rho_l\)](image)

![Figure 3-18: Comparison of experimental and theoretical loads over a) effective depth \(d\); b) shear slenderness \(a/d\)](image)

In Figure 3-19, the percentages of the governing failure criterion of all 408 tests are illustrated. It can be seen that according to the model, a large majority of 79% of the
3.4 Evaluation of the Mechanical Model

tests exhibits aggregate interlock failure. The second most common failure type is dowel failure with 16 %. Failure of the crack processing zone and the compression zone are both below 3 %.

![Pie chart showing failure modes](image)

Figure 3-19: Failure modes in evaluated tests

3.4.3 Evaluation of influence factors

In this section, the influence of different parameter on the nominal shear capacity is investigated to draw some conclusions for deriving a code model. For this, a parameter of a representative beam is varied in isolation. The first investigated parameter is the longitudinal reinforcement ratio $\rho_l$. The evaluation was performed for a beam with an effective depth of $d = 800$ mm, a width of $b_w = 200$ mm, a concrete strength of $f_{cm} = 33.5$ MPa and three different values of the shear slenderness $a/d$. The results are shown in Figure 3-20.

![Graph showing correlation](image)

Figure 3-20: Correlation of the shear capacity with respect to the reinforcement ratio and the compression zone depth

On the vertical axis, the dimensionless shear capacity of the cross-section is shown. The shear capacity with respect to the longitudinal reinforcement ratio is illustrated for the
three different values of \( a/d \). As shown in the diagram, two modes of failure have to be distinguished. In case of yielding of the reinforcement, the strength of the beam increases nearly linearly with increasing longitudinal reinforcement. This is to be expected since the bending capacity also increases almost linearly if all other variables are kept constant as shown in Eq. (3-82).

\[
M_{rd} = A_p f_y z \Rightarrow \frac{V}{b_w d \sqrt{f_c}} = \frac{f_y}{\sqrt{f_c}} \frac{z/d}{a/d}
\]

When failure of aggregate interlock becomes governing, the curves in Figure 3-20 change into a parabolic shape. Here, the additional longitudinal reinforcement leads to a decreased crack width and thus to a higher contribution from aggregate interlock. This increase is however less pronounced than for the flexural capacity. Interestingly, the shear capacity seems to be proportional to the relative depth of the compression zone \( x_0/d \) (Figure 3-20). This is, however, comprehensible if one considers that the depth of the compression zone and the aggregate interlock law are both directly or indirectly coupled with the reinforcement ratio.

The influence of the shear slenderness \( a/d \) on the shear capacity is illustrated in Figure 3-21 for two different cross-section depths. In general, a higher shear slenderness leads to higher moments which cause larger crack widths. This should ultimately lead to a reduction of the dimensionless shear capacity, which is also the case within this model. In Figure 3-21, there is a distinction between a failure due to yielding of the longitudinal reinforcement and an “elastic” cross-section failure, for which flexural cracks are not able to localize \( (w_k < w_1) \). In between, failure of aggregate interlock becomes governing.

![Figure 3-21: Influence of the shear slenderness \( a/d \) on the shear capacity for an effective cross-section depth of a) \( d = 274 \text{ mm} \) and b) \( d = 1370 \text{ mm} \)](image)

For a small member \( (d = 274 \text{ mm}, \text{ Figure } 3-21a) \) there is a mild linear decrease of the shear capacity. For a larger member \( (d = 1370 \text{ mm}, \text{ Figure } 3-21b) \) the decrease of the shear capacity is much stronger which might be due to the fact that for smaller members
the influence of the depth of the crack processing zone is relatively more important than for larger members. This phenomenon can be regarded as one part of the size effect that weakens the impact of the larger moments on small members. It can therefore be concluded that the size effect and the influence of shear slenderness must be coupled in some way.

In Figure 3-22, the influence of the size effect is illustrated for a constant longitudinal reinforcement ratio of \( \rho_l = 2\% \), a constant number of reinforcing bars \( n = 2 \) and a shear slenderness of \( a/d = 6 \). For this study, also the ratio of cross-section width and height is kept constant with \( b_w/h = 0,6 \). Consequently, the ratio of the net width of the cross-section \( b_n \) to the total cross-section width \( b_w \) also remains approximately constant so that no dowel failure occurs. Another investigated parameter illustrated in Figure 3-22 is the concrete cover \( d_1 \) which is defined as a constant value or as 10 % of the effective depth \( d \). The concrete cover influences the tension stiffening effect of the longitudinal reinforcement. In the analysis, failure of aggregate interlock is governing for a large part of the investigated cross-sections depths (Figure 3-22a). To assess the exponent of the size effect, the graph is also illustrated in a double-logarithmic diagram (Figure 3-22b). The graph exhibits a more or less smooth transition from a horizontal line to a line with an inclination of 0,50 for a constant value of \( d_1 \). This value is in agreement with the theory of the energetic statistical size effect law by BAZANT /Baz84/. The specimen with a concrete cover of \( 0,1d \), however, exhibits a smaller slope of the size effect of 0,46 which shows that the tension stiffening effect influences the size effect.

![Figure 3-22: Influence of the size effect in a) regular scale; b) double logarithmic scale](image)

For the next study, the web width \( b_w \) is kept constant at 200 mm while only the cross-section height \( h \) is increased (Figure 3-23). To maintain the reinforcement ratio at a constant level, the diameter of the bars has to be increased with the cross-section height (i.e. the number of bars remains constant). For cross-section heights over 3800 mm, the longitudinal reinforcement does not fit within the width of the cross-section resulting in an infeasible geometry. This study is therefore limited to beams under 2500 mm. Due to
the decreasing net width of the cross-section, dowel failure of the concrete cover becomes governing for beams larger than 1400 mm (Figure 3-23a).

**Figure 3-23:** Size effect in case of transition from aggregate interlock failure to dowel failure in a) regular scale; b) double logarithmic scale

The mean gradient of the double-logarithmic curve is about 0.40 within the range where aggregate interlock is the governing failure mechanism (Figure 3-23b). In the range where dowel failure becomes governing, the gradient suddenly increases to about 0.70 and thus a much stronger size effect (Figure 3-23b). This increased size effect is more or less a “pseudo” size effect as the failure mode of the beams changes completely. This evaluation shows however, that size effect related test series where only the depth of the member is increased (which is the case for most size effect studies) might be biased due to a reduced dowel capacity. On the other hand, the area of the reinforcement bars in these test series is often not increased proportionally to the area of the cross-section which increases the nominal shear capacity. Another problem is, that the size of the aggregates $d_{ag}$ is often not increased proportionally to the beam size. In Figure 3-24, the size effects of beams with a constant aggregate size $d_{ag}$ and a constant critical crack width $w_{d0}$ are compared with scaled aggregate size and critical crack width.

**Figure 3-24:** Size effect for beams with a constant aggregate size $d_{ag}$ and a variable aggregate size in a) regular scale; b) double logarithmic scale
As can be seen, according to this mechanical model the size effect of beams with scaled aggregates and critical crack widths disappears entirely for very large beams resulting in a slope of zero in the double-logarithmic scale. This is not completely surprising as the size effect that is predicted based on the aggregate interlock law is not based on the same principles as the size effect due to energy release in the crack tip. If all parameters of the aggregate interlock failure criterion are scaled correctly with member size and a statistical size effect is ignored, no size effect should indeed be expected on a phenomenological level.

It can be concluded that apart from the concepts of the energetic-statistical size effect derived from fracture mechanics, there exist size effects on a phenomenological level (i.e. due to loss of aggregate interlock and dowel action) which can be determined from a mechanical representation of the actual components of shear transfer. Both approaches lead to very similar conclusions if realistic geometric boundaries are respected in the mechanical model, e.g. that aggregate size cannot be increased infinitely. According to the mechanical model, there exists a non-linear transition of the shear capacity from a horizontal to an inclined failure line in double-logarithmic scale for realistic beam structures. Also, there is no lower limit to the size effect as predicted by EC2 /EC211/ and the fractal statistical approach /Car95/.

As the flexural shear capacity of beams is governed by different parameters like shear slenderness, reinforcement ratio or cross-section depth, the question arises how these components can be included within a simple failure criterion. Since aggregate interlock failure is governing in most cases it is reasonable to assume that there exists a strong correlation between the parameters of the aggregate interlock law and the shear capacity. In Figure 3-25a, the dimensionless calculated shear resistances are illustrated with respect to the crack width coefficient $24w_k/(d_{ag}+16)$ according the aggregate interlock law by Vecchio & Collins /Vec86/.

![Figure 3-25: Comparison of calculated loads and failure criteria based on a) aggregate interlock law by /Vec86/; b) strain-based law similar to /Mut08/]
As can be seen, the correlation of calculated resistances and crack width coefficient can very well be described by the hyperbola given in Figure 3-25a. This correlation has basically already been described by MUTTONI /Mut03/, /Mut08/, the only difference being the representation of the crack width by a product of longitudinal strains and effective depth (Figure 3-25b). The failure criterion that can be obtained in this way is not a mechanical model as it is not a direct representation or a direct derivative of mechanical principles like equilibrium. It is, however, a failure criterion that is able to predict the shear strength of beams with a minimum number of parameters. As such, the equation possesses some appealing features. First, the method requires only one primary parameter to determine the shear strength which is the crack width (aggregate size can be regarded as a secondary parameter). Second, the method can illustrate clearly why there is an influence of the longitudinal reinforcement, the concomitant moment or shear slenderness and the size effect on the shear capacity as all of these parameters are directly coupled with the flexural crack width. However, this does not mean that the equations given in Figure 3-25 are necessarily suitable for the design of members, which will be discussed in chapter 4 of this work. They provide however a quite reasonable illustration of the correlation of parameters which most purely empirical equations are not able to give.

3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

3.5.1 General Critical Crack Width Model

The calculation procedure presented in chapter 3.3 is not suitable for calculations by hand due to the numbers of equations involved and the fact that a direct calculation of the ultimate shear strength is not possible. In this chapter, a simplified version of the model is derived that can be solved in a closed form. The main parameter of this solution will be the modified aggregate interlock law by FISKER & HAGSTEN /Fis16/ in which the critical crack width \( w_0 \), above which aggregate interlock stresses cannot be transferred, plays a key role. For this reason, the associated closed form solutions will be called Critical Crack Width Models (CCWM).

As a first simplification, only the contributions from compression zone \( V_{cs} \) and aggregate interlock \( V_{ai,max} \) are accounted for according to Eq. (3-83). The contribution of the crack processing zone is omitted and will be replaced by aggregate interlock (meaning that the tensile strength of concrete is not considered anymore). Dowel action is included implicitly as a redistribution of shear stresses from aggregate interlock to dowel action according to Figure 3-6 is still required to ensure a constant shear flow. However, the dowel failure criterion will be omitted assuming that the failure of aggregate interlock with \( V_{ai,max} \) is governing, which is the case for most test specimens (Figure 3-19). The
contribution from the compression zone can be determined directly from the compression zone depth and the average shear stresses between compression and tension chord according to Eq. (3-83).

\[ V_{\text{calc}} \approx V_{cs,x} + V_{ai,\text{max}} = \frac{2}{3} b_w \xi d V_{\text{calc}} + V_{ai,\text{max}} = \frac{2}{3} \frac{\xi}{(1-\xi/3)} V_{\text{calc}} + V_{ai,\text{max}} \]  

(3-83)

By that, Eq. (3-83) can be transformed into Eq. (3-84).

\[ V_{\text{calc}} = \frac{1}{1-\frac{2}{3} \frac{\xi}{(1-\xi/3)}} V_{ai,\text{max}} \]  

(3-84)

The modified aggregate interlock law from FISKER & HAGSTEN is used again to calculate the contribution from interface shear. However, the starting point of the integration is now the neutral axis so that the integrals are given by Eqs. (3-85) and (3-86). By this it is assumed that aggregate interlock is acting from the level of the neutral axis to the longitudinal reinforcement level. Despite these simplifications, equilibrium is still maintained within this approach.

\[ V_{ai,\text{max}}(w) = k_{ai} \sqrt{f_{ck} b_w} \int_{d-x_0}^{d-w_0} \left( 1 - \frac{w_k}{w_0} \frac{z}{(d-x_0)} \right) dz \quad \text{and} \quad w_k \leq w_0 \]  

(3-85)

\[ V_{ai,\text{max}}(w) = k_{ai} \sqrt{f_{ck} b_w} \int_{0}^{w_k/(d-x_0)} \left( 1 - \frac{w_k}{w_0} \frac{z}{d-x_0} \right) dz \quad \text{and} \quad w_k > w_0 \]  

(3-86)

For \( w_k \leq w_0 \), the solved integral is given by Eq. (3-87).

\[ V_{ai,\text{max}} = k_{ai} \sqrt{f_{ck} b_w} \left( \frac{d-x_0}{24 w_k} \right)^{1/2} \left[ \frac{24 w_k}{d_{ag} + 16} \ln \left( \frac{24 w_k}{d_{ag} + 16} + 0.31 \right) \right] \]  

For \( w_k > w_0 \), the solved integral is given by Eq. (3-88).
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

\[ V_{ai,\text{max}} = k_{ai} \sqrt{f_c b_w} \left[ \frac{(d-x_0)}{24} \right]^2 \left[ \frac{24w_k}{d_{ag} + 16} + 0.3 \frac{w_k}{w_0} \right] \ln \left( \frac{24w_k}{d_{ag} + 16} \frac{w_k}{w_0} + 0.31 \right) \]  

(3-88)

In their present form, Eqs. (3-87) and (3-88) are not suitable for practical use, especially since a distinction of cases with respect to \( w_0 \) has to be made. However, by plotting the analytical integral for different values of \( w_0 \) and \( d_{ag} \), a simplified approximation for the integral can be obtained by a regresional analysis which is given by Eq. (3-89). The right part of Eq. (3-89) resembles the aggregate interlock law by Vecchio & Collins /Vec86/, with the coefficient for the crack width being subject to the maximum crack opening \( w_0 \) and the aggregate size \( d_{ag} \).

\[ V_a\left(w_k, w_0, d_{ag}\right) \approx k_{ai} \sqrt{f_c b_w} d(1-\xi) \frac{3.23}{1+(10+1.9/w_0)} \left( \frac{24}{d_{ag} + 16} \right)^{-0.5} \frac{24w_k}{d_{ag} + 16} \]  

(3-89)

A comparison between the analytically solved integral and the approximation according to Eq. (3-89) is shown in Figure 3-26 for different values of \( w_0 \) and \( d_{ag} \). As can be seen, the approximation gives a reasonable agreement with the analytical solution. Figure 3-26 also illustrates the impact of the parameter \( w_0 \) on the shear capacity. For smaller values of \( w_0 \), the capacity from aggregate interlock reduces by a factor of 1~3, depending on the crack width.

![Figure 3-26](image)

*Figure 3-26: Comparison between the analytically integrated aggregate interlock function and its approximation for an aggregate size of a) 4 mm; b) 32 mm*

By inserting Eq. (3-89) into Eq. (3-84), the shear capacity \( V_{calc} \) can be determined according to Eq. (3-90). Interestingly, the contribution of the compression zone can be
cancelled out so that only the aggregate interlock equation remains. Judging from Eq. (3-90) it would seem that it is actually aggregate interlock that acts over the whole inner lever arm \( z = (1 - \xi/3) \cdot d \). This would however be a misinterpretation, as the contribution of the compression zone is subject to the definition of the failure stress level which in this case is the capacity of aggregate interlock.

\[
V_{\text{calc}} \approx k_{\text{ai}} \left[ \frac{1 - \xi}{1 - \frac{2}{3}(1 - \xi/3)} \right]^{3,23} \left[ 1 + \left( \frac{1,02 + \frac{1,88}{w_0}}{d_{\text{ag}} + 16} \right)^{0,5} \cdot \frac{24 w_k}{d_{\text{ag}} + 16} \right]
\]

(3-90)

For determining the flexural crack width \( w_k \), tension stiffening effects are neglected. The average crack spacing \( s_{\text{rm}} \) is assumed to be equal to \( 0,7d \) so that a simplified crack width \( w_k \) can be determined according to Eq. (3-91).

\[
w_k = \frac{s_{\text{rm}}}{E_s} \sigma_s \approx \frac{0,7d}{E_s A_s} F_s
\]

(3-91)

The tensile force in the longitudinal reinforcement \( F_s \) is equal to the moment divided by the inner lever arm \( z \). In case of an additional longitudinal force \( N \) (compression negative), the steel force \( F_s \) can be calculated according to Eq. (3-92).

\[
F_s = \frac{M_{\text{Ed}}}{z} + N = \frac{M_{\text{Ed}}}{z} + N \left( 1 - \frac{y_s - y_p}{z} \right)
\]

where

\[
y_s \quad \text{distance of mild reinforcement from center line of gravity}
\]

\[
y_p \quad \text{distance of tendons from center line of gravity}
\]

The crack width \( w_k \) is then given by Eq. (3-93).

\[
w_k = \frac{0,7d}{E_s A_s} \left[ \frac{M_{\text{Ed}}}{z} + N \left( 1 - \frac{y_s - y_p}{z} \right) \right]
\]

(3-93)

The statically indeterminate part of prestressing should be considered within the eccentricity of the tendons \( y_p \). A possible stress increase in the tendons should not be considered within the closed form model as the actual strains in the reinforcement are not explicitly determined. For beams in compression, the concomitant moment \( M_{\text{Ed}} \) is defined as the design moment in the ultimate limit state without the moments from prestresses. Only then, the moment \( M_{\text{Ed}} \) can be expressed as the product of the shear slenderness in
the control section and the shear resistance \((M_{Ed} = \lambda \cdot V_{calc} \cdot d)\). The reason for this is that to reach a value of the shear resistance \(V_{calc}\) in the ultimate limit state, the internal shear forces and moments from imposed loads have to be increased until the intersection point of resistance curve and load curve is reached. This is implicitly done by solving the recursive function for the main variable \(V_{calc}\). If only the internal forces from imposed loads are increased, then the moments from prestressing may not be considered in \(M_{Ed}\) as this would affect the shear slenderness \(\lambda\). The moments therefore have to be divided into moments from loading and prestressing as done by Eq. (3-93).

In case of axial tension, the normal forces usually result from imposed loads. In that case it is correct to consider effects from eccentricity in the moment \(M_{Ed}\). The crack width is then given by Eq. (3-94).

\[
W_k = \frac{0.7d}{E_A} \left[ \frac{M_{Ed}}{z} + N \right] 
\]  
(3-94)

For now, the formulae will be derived considering the case of compression. The case of axial tension will be treated in chapter 3.5.4. By inserting Eq. (3-93) into Eq. (3-90), the shear strength \(V_{calc}\) is given as a recursive function according to Eq. (3-95).

\[
V_{calc} = \frac{3.23 \cdot k_{ai} \sqrt{f_{ck} b_w z}}{1 + \left(1,0 + 1,9/w_0\right) \left(24 \left(\frac{d_{ag} + 16}{E_A} \right)^{0.3} \cdot 0.7d \left(\frac{V_{calc} \lambda d}{z} + P \left(1 - \frac{y_s - y_p}{z}\right)\right) \right)} 
\]  
(3-95)

\(V_{calc}\) can then be determined by solving the resulting quadratic equation given by Eq. (3-96). The parameters \(A\) and \(B\) are determined according to Eqs. (3-97) and (3-98). The product of \(3.23 \cdot k_{ai}\) is replaced by a coefficient \(k_c\).

\[
v_c^2 + (A + B) \cdot v_c - k_c \cdot A = 0 
\]  
(3-96)

\[
A = \frac{E_A \rho_l}{\lambda d} \sqrt{d_{ag} + 16} 
\]  
(3-97)

\[
B = \frac{P}{\lambda \sqrt{f_{ck} b_w d}} \left(1 - \frac{y_s - y_p}{z}\right) 
\]  
(3-98)

where

\[
v_c = \frac{V_{calc}}{\sqrt{f_{ck} b_w z}} 
\]

\(k_c = 0.42\) for mean value of test results

The closed form solution of Eq. (3-96) is given by Eq. (3-99).
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

\[
V_{\text{calc}} = \left[ \frac{1}{4} (A+B)^2 + A \cdot k_c - \frac{1}{2} (A+B) \right] \sqrt{f_{ck} b_w z} \tag{3-99}
\]

Eq. (3-99) can then be used to evaluate the test databank defined in chapter 3.4.2. The coefficient \( k_c \) was set to a value of 0.42 to fit the mean value of test results. Also note that the value of \( \lambda \) was taken equal to \( a/d \) to account for the effect of additional tensile forces in the reinforcement due to shear as discussed in section 3.3.8. The value \( w_0 \) was taken as 2.0 for \( f_{ck} \leq 50 \) MPa as proposed by FISKER & HAGSTEN /Fis16/. For concrete strengths between 50 and 60 MPa, \( w_0 \) is reduced linearly to 1.0 according to Eq. (3-100).

\[
1.0 \leq w_0 = 2.0 - 0.1 \cdot (f_{ck} - 50) \leq 2.0 \quad (w_0 \text{ in mm}) \tag{3-100}
\]

The databank evaluation is shown in Figure 3-27. The closed form approach shows a very good agreement with the tests which is demonstrated by the relatively low COV of 15.3%.

**Figure 3-27:** Comparison of experimental and theoretical loads using a closed form solution over a) concrete strength \( f_{cm} \); b) longitudinal reinforcement ratio \( \rho_1 \)

**Figure 3-28:** Comparison of experimental and theoretical loads using a closed form solution over a) effective depth \( d \); b) shear slenderness \( a/d \)
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

3.5.2 Simplified Critical Crack Width Model for RC beams without axial forces

Further simplifications of Eq. (3-99) seem to be reasonable for two reasons. First, the format of the equations might be counter-intuitive for designers. Second, the coefficient $k_c$ does not control the shear capacity in a way that it will be possible to determine a design level and to apply a partial safety factor on it. The design equation will therefore be further simplified for RC beams first. Since axial forces are represented by parameter $B$ it may be neglected so that the solution for $V_{calc}$ is given by Eq. (3-101).

$$V_{calc} = \left[ \frac{1}{4} A^2 + k_c A - \frac{1}{2} A \right] \sqrt{f_{ck} b_w z}$$  \hfill (3-101)

Eq. (3-101) depends on the parameter $A$ as illustrated in Figure 3-29.

![Figure 3-29: Illustration of an approximation to Eq. (3-101)](image)

A simplified function is given by Eq. (3-102), with which Eq. (3-101) can be approximated.

$$V_{calc} = 0.73k_c A^{1/3} \sqrt{f_{ck} b_w z} \leq k_c \sqrt{f_{ck} b_w z}$$  \hfill (3-102)

By inserting Eq. (3-97) into Eq. (3-102) and simplifying the equation as done in Eq. (3-103), a very brief design equation is obtained. The constant factors involving $k_c$ are replaced by a coefficient $C_{Rm,c}$ which is subject to evaluation of tests.
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

The simplified equation which is obtained in Eq. (3-103) has some similarities to existing empirical equations (e.g. ZSUTTY /Zsu68/). In fact, PLACAS & REGAN /Pla71/ derived Eq. (3-104) in 1971 based on the assumption that aggregate interlock would be dominant and that its stiffness would be inversely proportional to the widths of the flexural cracks. Eq. (3-104) is virtually identical to Eq. (3-103) with the exception that it does not account for the size effect.

An equation similar to Eq. (3-103) was recently derived by MUTTONI & RUIZ /Mut16/ on the basis of the Critical Shear Crack Theory. However, the model in this thesis incorporates a different law for aggregate interlock that account for a critical crack width parameter \( w_0 \) in the equation and a different power law for \( d_{ag} \). It should be noted that the original Eq. (3-101) automatically leads to an upper limit of the dimensionless shear capacity which has already been included in Eq. (3-103). The upper limit of the shear capacity can easily be found by the inequality used in Eq. (3-105).

The same result is obtained if a crack width of zero is inserted into Eq. (3-90). The upper limit of the shear capacity can thus be associated with the capacity of aggregate interlock of a cross-section with a flexural crack width of zero (which is not identical to the capacity of an uncracked cross-section). The simplified closed form solution of the original mechanical equations is then given by Eq. (3-106).
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

where

\[ k_c = 0.42 \]
\[ C_{Rm,c} = 0.22 \]

The coefficient \( C_{Rm,c} = 0.22 \) in Eq. (3-106) was determined to fit the mean value of test results. The shear slenderness \( \lambda \) was considered equal to \( a/d \) (cf. section 3.3.8). The evaluation of the tests from chapter 3.4.2 with the simplified closed form solution is shown in Figure 3-30 and Figure 3-31. The evaluation resulted in a mean value of 1.01 and a coefficient of variation of 14.8 % so that the agreement with tests is similar to the original mechanical calculation (COV = 15.3 %).

**Figure 3-30**: Comparison of experimental and theoretical loads using a simplified closed form solution over a) concrete strength \( f_{cm} \); b) longitudinal reinforcement ratio \( \rho_l \)

**Figure 3-31**: Comparison of experimental and theoretical loads using a simplified closed form solution over a) effective depth \( d \); b) shear slenderness \( a/d \)
3.5.3 Simplified Critical Crack Width Model for beams in compression

In the next step, the influence of prestressing is investigated to obtain a simplified closed form solution for this problem as well. For this, Eq. (3-107) is used with parameters $A$ and $B$ according to Eqs. (3-108) and (3-109). The prestressing force $N_{Pd}$ is considered negative for compression.

\[
V_{\text{calc}} = \left[ \frac{1}{4}(A+B)^2 + 0.42A - \frac{1}{2}(A+B) \right] \sqrt{f_{ck}b_nz} 
\]

(3-107)

\[
A = \frac{E_s \rho_l}{\lambda d} \frac{\sqrt{d_{\text{ag}} + 16}}{3.43 \sqrt{f_{ck} \left( 1.0 + 1.9/w_0 \right)}}
\]

(3-108)

\[
B = \frac{N_{Pd}}{\lambda \sqrt{f_{ck}b_n d}} \left( 1 - \frac{y_{s} - y_{p}}{z} \right)
\]

(3-109)

The solutions to Eq. (3-107) can be illustrated by a surface subject to the parameters $A$ and $B$ (Figure 3-32). To find a suitable way to approximate this surface, the values of the strength function for $B = 0$ ($V_{0,\text{calc}}$) are subtracted from Eq. (3-107) ($V_{\text{calc}}$), since the approximation for this part is already provided by Eq. (3-106). The additional capacity due to prestressing (i.e. $V_{P} = V_{\text{calc}} - V_{0,\text{calc}}$) with respect to the parameter $B$ is illustrated in Figure 3-33. The total capacity $V_{\text{calc}} + V_{P}$ is of course limited by the value $V_{c,\text{max}}$.

![Illustration of Eq. (3-107) for prestressed members](image)

Figure 3-32: Illustration of Eq. (3-107) for prestressed members
Figure 3-33: Additional shear capacity due to prestressing with respect to the parameter $B$

As can be seen, the additional capacity $V_p$ can quite precisely be approximated by a set of linear functions so that $V_p$ is the product of a gradient $\alpha_p$ and the parameter $B$ according to Eq. (3-110).

$$V_p = \alpha_p \cdot B \sqrt{f_{ck} b_w z}$$  \hspace{1cm} (3-110)

This gradient $\alpha_p$ is subject to parameter $A$. In Figure 3-34, the parameter $\alpha_p$ resulting from the original Eq. (3-107) is shown. A very reasonable approximation by Eq. (3-111) is also illustrated in Figure 3-34.

$$\alpha_p = -\frac{1}{\sqrt{1+20A}}$$  \hspace{1cm} (3-111)

By this, the total ultimate capacity $V_{calc}$ of a beam with or without prestressing can be determined according to Eq. (3-112).
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

\[
V_{\text{calc}} = C_{Rm,e} \left( \frac{E_s \rho_t}{\lambda \cdot d} \frac{\sqrt{d_{ag} + 16}}{(1,0 + 1,9/w_0)} f_{ck} \right)^{1/3} b_w \leq 0,42 \sqrt{f_{ck}} b_w
\]  

(3-112)

The contribution from prestressing \( V_P \) can be determined with Eqs. (3-113) and (3-114).

\[
V_P = \alpha_p \cdot N_{Pd} \frac{z}{\lambda d} \left( 1 - \frac{y_p}{y} \right) \tag{3-113}
\]

\[
\alpha_p = \frac{-1,0}{\sqrt{1 + 20 \frac{E_s \rho_t}{\lambda d} \frac{\sqrt{d_{ag} + 16}}{3,43 f_{ck} (1,0 + 1,9/w_0)}}} \tag{3-114}
\]

For design purposes, the coefficient \( \alpha_p \) can be simplified further by assuming conservative values for \( d_{ag} \) and \( w_0 \) according to Eq. (3-115).

\[
\alpha_p \approx \frac{-1,0}{\sqrt{1 + \frac{17 E_s \rho_t}{\lambda d} \sqrt{f_{ck}}}} \tag{3-115}
\]

To validate the simplified model for prestressed beams, tests on prestressed beams without shear reinforcement from the ACI-DAfStb shear databanks are evaluated /Rei16/. Here, only tests on rectangular and T-beam cross-sections are considered so that a total of 120 shear tests is evaluated.

![Figure 3-35: Evaluations of tests on PC beams without shear reinforcement](image)

- Rect.-Beams
- T-Beams

- \( n = 120 \)
- \( c_{cp} < 0 \)
- \( \rho_s = 0 \)
- MV = 1,575
- COV = 0,275

- MV = 1,029
- COV = 0,235

\( a/ \) without a correction factor \( \beta \); \( b/ \) with a correction factor \( \beta \)
The result of the evaluation is shown in Figure 3-35a with respect to the shear slenderness $a/d$. The shear strength of the members is generally underestimated with a mean value of test loads vs. point loads of 1,58. The coefficient of variation is fairly good with 27,5 %. However, it can be seen that there is a clear trend of strength predictions depending on the shear slenderness (Figure 3-35a). The reason for this is that an arch or a direct strut action is not considered as this model is supposed to predict the strength of slender beams. For RC beams, only beams with $a/d > 2,89$ were considered and no influence of direct strut action was observed (Figure 3-31b). However, in PC beams direct strut action is also possible for higher values of $a/d$ since the prestressing delays the formation of flexural cracks. For this reason, an empirical strength enhancement factor $\beta$ for loads close to supports can be introduced as shown in Eq. (3-116). This factor $\beta$ is similar to the factor $\beta$ in EC2 for loads close to the supports, the difference being that the $\beta$-factor is a reduction factor for point loads instead of an enhancement factor.

$$ V_{\text{calc}} = \beta \cdot (V_{c0} + V_p) = \left(1 + 3 \cdot e^{-2} \right) \cdot (V_{c0} + V_p) $$  \hspace{1cm} (3-116)

The evaluation of the simplified closed form solution including the strength enhancement factor is shown in Figure 3-35b. The mean value of test loads and theoretical loads is more accurate with a value of 1,03. The coefficient of variation is comparably low with 23,5 %, showing that the simplified model can also quite accurately predict the shear strength of prestressed concrete beams.

### 3.5.4 Simplified Critical Crack Width Model for beams in tension

The same evaluation performed in the previous chapter can be done to account for the influence of tensile forces on the shear capacity. For this, the diagram in Figure 3-32 has to be created for positive values of parameter $B$ which represents tensile forces. Figure 3-36 shows the dimensionless shear capacity with respect to parameters $A$ and $B$. 

![Illustration of solutions of Eq. (3-107) for members in tension](image)

Figure 3-36: Illustration of solutions of Eq. (3-107) for members in tension
It can be seen that the shear capacity decreases gradually with increasing tensile forces.

A very convenient way to describe this reduction of shear capacity is to define a reduction factor $\alpha_t$ which is applied to the shear capacity of beams without shear reinforcement $V_{0,\text{calc}}$ according to Eq. (3-117).

$$\alpha_t = \frac{V_{\text{calc}}}{V_{0,\text{calc}}}$$  \hspace{1cm} (3-117)

The reduction factor $\alpha_t$ is shown in Figure 3-37 with respect to parameter $B$.

Figure 3-37: Reduction factor $\alpha_t$ for the shear capacity with respect to the parameter $B$

The reduction factor $\alpha_t$ can be described by a set of hyperbolas that depend on an additional coefficient $\psi_t$ according to Eq. (3-118).

$$\alpha_t = \frac{1}{1 + \psi_t B}$$  \hspace{1cm} (3-118)

This coefficient $\psi_t$ depends on parameter $A$ as shown in Figure 3-38.

Figure 3-38: Coefficient $\psi_t$ with respect to the parameter $A$
An approximation of $\psi_t$ with a power function according to Eq. (3-119) is illustrated in Figure 3-38. As can be seen, the approximation with a power law is very precise.

$$\psi_t = \frac{0.65}{A^{2/3}} \quad (3-119)$$

The flexural shear capacity of a member in tension is then given by Eq. (3-120) with a reduction coefficient $\alpha_t$ according to Eq. (3-121).

$$V_{calc} = \alpha_t V_{0,calc} \quad (3-120)$$

$$\alpha_t = 1 + 0.65 \left[ 1 + \frac{N_T}{\lambda \sqrt{f_{ck}b_wd}} \left( \frac{E_c \rho_l}{\lambda d} \right)^{2/3} \left( \frac{d}{3,43 \sqrt[3]{f_{ck}} (1,0 + 1,9/w_0)} + 16 \right) \right]^{-1} \quad (3-121)$$

$$\alpha_t = 1 + 1.48 \frac{N_T}{b_wd (d_{ag} + 16)^{2/3} f_{ck}^{1/6}} \left( \frac{d}{E_c \rho_l \sqrt[3]{\lambda}} \right)^{2/3} \left( \frac{d}{3,43 \sqrt[3]{f_{ck}} (1,0 + 1,9/w_0)} + 16 \right) \quad (3-122)$$

Note that in Eq. (3-121), the effect from eccentricity is omitted. In case of tensile forces, the effects from eccentricity should be considered in the shear slenderness $\lambda$, the reason for this being that it should be assumed that for tension, normal forces are a direct result from imposed loading. If the loads are increased until the limit state of the structure, moments, shear forces and axial forces will increase equally so that $\lambda$ will be a constant. For design purposes, Eq. (3-121) should be simplified further by using common parameters for $d_{ag}$, $w_0$ and $f_{ck}$ so that the simplified Eq. (3-122) will be used.

$$\alpha_t = \frac{1}{1 + 0.6 \frac{N_T}{b_wd} \left( \frac{d}{E_c \rho_l \sqrt[3]{\lambda}} \right)^{2/3}} \quad (3-122)$$

The model can be verified by tests on rectangular RC beams under axial tensile forces from the literature (/Els57/, /Mat69/, /Reg71/, /Sør81/, /Jør13/; tests originally collected by FERNÁNDES-MONTES /Fer11/, /Fer15/). The ratio of ultimate test loads and theoretical loads is shown in Figure 3-39 for two different test groups with respect to the average cross-sectional tensile stress. The evaluation of the first test group with tests by /Els57/, /Mat69/, /Reg71/ and /Sør81/ yields a mean ratio of test and theoretical loads of 1,00 and a coefficient of variation of 16,6 % which can be seen as a very good result (Figure 3-39a). The second tests group with tests by /Jør13/ has an MV = 1,430 and COV = 24,0% so that in these tests the ultimate loads are underestimated for very high tensile forces as shown in Figure 3-39b. However, it should be pointed out that in these tests,
the tensile forces were applied by pulling of the symmetrically distributed reinforcement. Although high strength steel \((f_y = 1027 \text{ MPa})\) was used, tensile forces of up to 83 % (!) of the yield strength of the steel were applied. For this scenario, a transition to the activation of wire action to carry the transverse forces is very likely and thus not within the scope of the presented model. In general, however, it can be seen that the model is also applicable for beams in tension, even for very large tensile stresses of multiple times the concrete tensile strength.

![Figure 3-39: Evaluation of shear tests without shear reinforcement in tension with tests by a) /Els57/, /Mat69/, /Reg71/, /Sør81/; b) /Jør13/](image)

### 3.5.5 Minimum flexural shear capacity for design according to the Critical Crack Width Model

In design practice, it is often useful to have a simple equation for calculating a minimum shear capacity. This equation should not require information about the statical system (i.e. \(\lambda\)) or the reinforcement ratio \(\rho\). By this, designers can perform a shear check independent from the check for flexural bending which means they can first identify critical regions for the shear check before performing a more detailed check. A minimum shear capacity can be derived based on the assumption that the stresses in the longitudinal reinforcement have reached the yield strength which is equal to assuming that there is sufficient reinforcement to resist flexural bending. The reinforcement stresses \(\sigma\) in Eq. (3-91) can be replaced by the yield strength \(f_y\) which inserted into Eq. (3-90) yields Eq. (3-123).

\[
V_{c,\min} = k_c \sqrt{f_{ct}} b_w d \left(1 - \frac{\xi}{3}\right) \frac{1}{1 + \left(1 + 1.9/w_0\right) \left(\frac{24}{d_{ag} + 16}\right)^{0.5} \left(1 - \xi\right) d \frac{f_y}{E_s}}
\]  

(3-123)

Eq. (3-123) can then be further simplified with \((1 - \xi) d \approx 0.7 d\) so that the minimum shear capacity is given by Eqs. (3-124) and (3-125). In Eq. (3-125), the value of the coefficient in the denominator according to Eq. (3-123) was increased by a factor of 1.5
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

to 5.0. This was done after databank evaluations to ensure that Eq. (3-124) does not yield larger values than the actual shear strength equation.

\[ V_{c_{\text{min}}} = k_{c_{\text{min}}} \sqrt{f_{ck} b_c z} \]  
\[ k_{c_{\text{min}}} = \frac{0.42}{1 + 5 \frac{f_c d (1 + 1.9/w_0)}{E_s \sqrt{d_{ag} + 16}}} \]  

\[ (3-124) \]  
\[ (3-125) \]

In Figure 3-40, the proposed equation for \( V_{c_{\text{min}}} \) is compared to the provisions of EC2 /EC211/ and EC2 with German Annex /DIN13/ on a mean value level. Here, an aggregate size of \( d_{ag} = 16 \) mm, a maximum crack opening of \( w_0 = 2 \) mm and \( z/d = 0.9 \) are assumed. As can be seen, the proposed equation yields higher minimum capacities than the current EC2 for beams with \( d \leq 600 \) mm. This is particularly beneficiary for preliminary design of slabs in buildings or bridges. For higher effective depths, EC2 gives higher values which can be attributed to the size effect, which is underestimated in EC2.

![Figure 3-40: Comparison of proposed equation for minimum shear capacity with EC2 and EC2 NA(D)](image)

Note that this minimum shear capacity is only valid within the elastic range of the longitudinal reinforcement until it reaches its yield strain. In regions where maximum bending moments and shear forces coincide, yielding of the longitudinal reinforcement may occur as dimensionless diagrams for bending utilize yields strains of the reinforcement of up to 25 %. Significant yielding may also occur in the vicinity of plastic hinges over supports. In that case, the shear capacity \( V_{c_{\text{yield}}} \) should be calculated according to Eq. (3-126).
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

\[ V_{c,yield} = \frac{0.42 f'_{ck} b_w z}{1 + 5 \cdot \varepsilon_{sy, tot} d \left(1,0 + 1,9/w_0\right) \sqrt{d_{ag} +16}} \]  \hspace{1cm} (3-126)

where
\[ \varepsilon_{sy, tot} \] total reinforcement strain (elastic and inelastic strain)
\[ k_c = 0.42 \]

3.5.6 Design equations based on the Simplified Critical Crack Width Model

In this section, the Simplified Critical Crack Width Model for the flexural shear strength of beams will be rewritten and summarized in form of design equations. The design value for the flexural shear strength \( V_{Rd,c} \) of RC beams without axial forces is given by Eq. (3-127).

\[ V_{Rd,c} = \frac{C_{Rm,c} \left( E_{c} \rho_{c} \sqrt{d_{ag} +16} \lambda \cdot d \left(1,0 + 1,9/w_0\right) f'_{ck}\right)^{1/3}}{\gamma C} b_w z \leq \frac{0.42 f'_{ck} b_w z}{\gamma C} \]  \hspace{1cm} (3-127)

where
\[ C_{Rm,c} = 0.22 \]
\[ \gamma C = 1.65 \] (partial safety factor, cf. chapter 6.3.2)
\[ 1,0 \leq w_0 = 2,0 - 0,1 \cdot (f'_{ck} - 50) \leq 2,0 \]
\[ \lambda = \frac{M_{Ed}}{V_{Ed}} + 1, \] where \( M_{Ed} \) and \( V_{Ed} \) are the design moments and shear forces in the control section.

For a preliminary design of RC beams, PC beams or beams in tension with sufficient longitudinal reinforcement to resist flexural bending, a minimum value \( V_{Rd, c, min} \) may be calculated according to Eqs. (3-128) and (3-129).

\[ V_{Rd, c, min} = \frac{k_{c, min}}{\gamma C} \sqrt{f'_{ck} b_w z} \] \hspace{1cm} (3-128)

\[ k_{c, min} = \frac{0.42}{1 + 5 \cdot \varepsilon_{sy, d} d \left(1,0 + 1,9/w_0\right) \sqrt{d_{ag} +16}} \] \hspace{1cm} (3-129)

The design flexural shear strength of prestressed concrete beams \( V_{Rd, cp} \) is given by Eqs. (3-130) to (3-132). The moments or shear forces from prestressing \( N_{Pd} \) are not considered for the calculation of the shear slenderness \( \lambda \) (for determining \( V_{Pd} \) as well as
3.5 Derivation of Closed Form Solutions: Critical Crack Width Model

$V_{Rd,c}$). The reason is that it is assumed that the compressive forces results from prestressing and not from imposed loads and that the load vector to the limit state only implies moments and shear forces from imposed loading. In case the compressive forces directly result from imposed loading (e.g. compression in beams of concrete frames), the compressive forces may be considered for determining the shear slenderness.

$$V_{Rd,cr} = V_{Rd,c} + V_{pd} \leq \frac{0.42}{\gamma_C} \sqrt{f_{ck}} b_w z$$

$$V_{pd} = \frac{\alpha_p \cdot N_{pd}}{\gamma_C} \frac{z}{\lambda d} \left(1 - \frac{y_s - y_p}{z}\right)$$

$$\alpha_p = \frac{-1.0}{\sqrt{\frac{1+\frac{17}{E_s \rho_l}}{\sqrt{f_{ck}} \lambda d}}}$$

where

- $N_{pd}$ design compressive force from prestressing (compression negative)
- $y_s$ distance of the reinforcement layer to the center of gravity
- $y_p$ distance of the tendon layer to the center of gravity
- $\lambda = \frac{M_{Ed}}{V_{Ed} d} + 1$, where $M_{Ed}$ and $V_{Ed}$ are the design moments and shear forces in the control section without moments and shear forces from prestressing

The design flexural shear strength $V_{Rd,cr}$ of reinforced concrete beams in axial tension is given by Eqs. (3-133) and (3-134). Here, the moments or shear forces from $N_{Td}$ need to be considered for the calculation of the shear slenderness $\lambda$ (for determining $V_{Rd,c}$ as well as $\alpha_i$). The reason is that it is assumed that tensile forces result from imposed loading (e.g. silo walls) so that the load vector to the limit state implies axial forces, moments and shear forces equally.

$$V_{Rd,cr} = \alpha_i V_{Rd,c} \leq V_{Rd,c,\text{max}} = \frac{0.42}{\gamma_C} \sqrt{f_{ck}} b_w z$$

$$\alpha_i = \frac{1}{\sqrt{1+0.6 \frac{N_{td}}{b_w d} \left(\frac{d}{E_s \rho_l \sqrt{\lambda}}\right)^{2/3}}}$$

where

- $N_{Td}$ design tensile force from loading (considered positive)
\[ \lambda = \frac{M_{Ed}}{V_{Ed} \cdot d} + 1, \text{ where } M_{Ed} \text{ and } V_{Ed} \text{ are the design moments and forces in the control section including moments and shear forces from } N_{Td}. \]

In case yielding of the longitudinal reinforcement is expected, the design shear capacity \( V_{Rd,c,yield} \) can be calculated according to Eqs. (3-135) and (3-136).

\[ V_{Rd,c,yield} = \frac{k_{c,yield}}{\gamma_c} \sqrt{f_{ck} b_w z} \]  
\[ k_{c,yield} = \frac{0.42}{1 + 5 \cdot \varepsilon_{xy,tot} d \left(1,0 + 1.9/w_0\right)} \sqrt{d_{ag} + 16} \]

where

\( \varepsilon_{xy,tot} \) total strain in the longitudinal reinforcement without effects from tension stiffening. In case of prestressing steel, prestrains do not need to be considered. For combinations of different reinforcement materials and different layers, an average strain may be used.

### 3.6 Mechanical Model for Prestressed I-Beams without Flexural Cracking

The shear capacity of prestressed concrete beams which do not exhibit flexural cracks in the webs prior to failure can be more accurately predicted by a principal tensile stress criterion according to Eq. (3-137). Such an approach has in the past been used for the shear design of box girder bridges in Germany and still used for the assessment of existing bridges /Heg15/, /Heg14a/, /Heg14b/. Also, the principal tensile stress criterion is the applicable approach for the design of prestressed hollow core slabs according to EN 1168 /DIN11/, /Rog16a/, /Rog16b/.

\[ \sigma_1 = \frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + e_{xx}^2} \leq \sigma_{1,max} \]
\[ \sigma_1 = \frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \left(\frac{V_z S_y}{I_y b_w}\right)^2} \leq \sigma_{1,max} \]

Since rectangular and T-beam cross-sections usually exhibit flexural cracks prior to failure, the scope of application of a principal stress based formula is focused on I-beam sections. Due to their additional flanges in tension, the propagation of flexural cracks into the web is delayed. Shear failure is then induced by diagonal shear cracks in the webs that do not originate from flexural cracks (Figure 3-41a). For design purposes, a critical control section should be defined. In this control section, the critical shear force \( V_z \) for which the principal tensile stresses reach the concrete tensile strength is determined. To determine \( V_z \), the principal tensile stresses have to be determined over the
depth of the web since shear and longitudinal stresses are not constant and the location of the minimum \( V_z \) is unknown. The longitudinal stresses for different points on the web can be determined according to Eq. (3-138) with respect to the prestressing \( N_p \) and the moment \( M \). The moment \( M \) can be replaced by the product of the shear force \( V_z \) and the shear span \( a_x = \lambda \cdot d \). For single span beams, this shear span is equal to the distance of the control section from the support (Figure 3-41b).

If Eq. (3-138) is inserted into Eq. (3-137), Eq. (3-139) is obtained. The unknown shear force \( V_z \) from loading is replaced by the shear resistance \( V_{Rm,c} \). The maximum value of the principal tensile stress \( \sigma_{t,max} \) is replaced by the concrete tensile strength \( f_{ctm} \) according to the Rankine failure criterion.

\[
\frac{1}{2} \left( \frac{N_p}{A_c} \cdot \frac{V_{Rm,c} \cdot a_x + N_p \cdot \Delta \varepsilon_p}{I_y} \right) + \sqrt{\frac{1}{4} \left( \frac{N_p}{A_c} \cdot \frac{V_{Rm,c} \cdot a_x + N_p \cdot \Delta \varepsilon_p}{I_y} \right)^2 + \left( \frac{V_{Rm,c} S_{y,i}}{I_y b_{u,i}} \right)^2} = f_{ctm}
\]

In this context, the use of a biaxial failure criterion for unreinforced concrete according to KUPFER /Kup73/ would be more consistent as recently done by HUBER ET AL. /Hub16/. However, this would on one side make obtaining a closed form solution more difficult and is on the other side not necessary for a sufficient safety level based on the evaluation of test databanks. If Eq. (3-139) is solved for \( V_{Rm,c} \), Eq. (3-140) is obtained. An extended version of this formula was already used by the author for evaluating shear tests on prestressed hollow core slabs /Rog16a/. The shear capacity \( V_{Rm,c} \) in a particular point of the web can thus be calculated with this closed formulation without further iterations.

Figure 3-41: a) diagonal shear failure in the web prior to flexural crack propagation; b) definition of dimensions

\[
\sigma_{c,x} = \frac{N_p}{A_c} + \frac{M + N_p \cdot \Delta \varepsilon_p}{I_y} \cdot z
\]

\[
\sigma_{c,x} = \frac{N_p}{A_c} + \frac{V_z \cdot a_x + N_p \cdot \Delta \varepsilon_p}{I_y} \cdot z
\]

(3-138)
3.6 Mechanical Model for Prestressed I-Beams without Flexural Cracking

\[ V_{Rm,c}(z_i) = \frac{I_y \cdot b_{w,i}}{S_{y,i}} \left[ \sqrt{\frac{f_{ctm}^2 \left( \frac{a_x(z_{cog} - z_i)}{I_y} \cdot b_{w,i} \right)}{2I_y}} - N_p \left( \frac{1}{A_e} + \frac{(z_{cog} - z_i) \cdot \Delta e_p}{I_y} \right) \right] \]  \hspace{1cm} (3-140)

The capacity \( V_{Rm,c} \) is then given by the minimum shear capacity \( V_{Rm,c}(z_i) \) in the web according to Eq. (3-141).

\[ V_{Rm,c} = \min \left( h_{f,bot} \leq z_i \leq h - h_{f,bot}, V_{Rm,c}(z_i) \right) \] \hspace{1cm} (3-141)

\[ \sigma_{c,fl} \leq f_{ctm} \]  \hspace{1cm} (3-142)

An evaluation of 94 shear tests on prestressed I-beams from the ACI-DAfStb shear data-bank /Rei16/ is shown in Figure 3-43. The longitudinal stresses in the web \( \sigma_{c,fl} \) at failure load were smaller than \( f_{ctm} \) in all of the tests. The evaluation shows a good agreement of the proposed procedure with the test results. In general, the test loads tend to be underestimated with a mean ratio of test and calculated loads of 1,29 and a coefficient of variation of 21,4 %. Also, 72 tests on prestressed hollow core slabs have been evaluated with an extended version of Eq. (3-140) to account for shear stresses by prestress transfer /Rog16a/. The evaluation of the tests in Figure 3-44 showed a mean value of 1,06 and a coefficient of variation of 12,4 %.

The databank evaluations show that the COV of the principal tensile stress criterion is sufficiently small and that no further fitting of the function is needed as the mean value of test and theoretical loads was greater than 1,0 for both databanks.
The design value of the shear capacity in a control point \( i \) of a prestressed I-beam is therefore given by Eq. (3-143). The overall shear capacity then has to be determined according to Eq. (3-144).

\[
V_{\text{bd,}c}(z_i) = \frac{I_y \cdot b_{\text{wd},i}}{S_{y,i}} \cdot \left[ \frac{\lambda \cdot d \left( z_{\text{cog}} - z_i \right) I_y \cdot b_{\text{wd},i}}{2I_y} - 1 \right] - N_{pd} \left( 1 + \frac{z_{\text{cog}} - z_i}{I_y} \Delta e_p \right) f_{\text{cd}}
\]

\[V_{\text{bd,}c} = \min \left\{ h_{f,\text{bot}} \leq z_i \leq h - h_{f,\text{top}} \left| V_{\text{bd,}c}(z_i) \right| \right\} \text{ and } \sigma_c \left( h_{f,\text{bot}} \right) \leq f_{\text{cd}} \]

where

\[f_{\text{cd}} = f_{ck,0.05} / \gamma_c\]

\[\gamma_c = 1.8 \text{ (unreinforced concrete)}\]
3.7 Summary

In this chapter, the shear capacity of RC and PC members without reinforcement was investigated using mechanical procedures. In contrast to a similar model which was previously developed by TUE ET AL. /Tue14/, /Tun16a/, the influences of aggregate interlock and dowel action were included in the model. The following shear transfer actions were regarded as the main components of shear transfer in RC beams:

- \( V_{cc} \) Contribution of the inclined concrete compressive force
- \( V_{cs} \) Contribution of shear stresses in the compression zone
- \( V_{CPZ} \) Contribution of the crack processing zone
- \( V_{ai} \) Contribution by aggregate interlock
- \( V_{dw} \) Contribution by dowel action

For each of these components, a calculation procedure was provided in this chapter so that it is possible to determine the contribution of each component at every load stage by assuming a constant shear flow between the top and bottom force resultants of the cross-section. By this, the principle of equilibrium of external and internal forces is considered. The ultimate load of a beam was determined by definition of a failure criterion for each shear component. An evaluation of 408 selected shear tests from the ACI-DAf-Stb shear databanks /Rei13/ revealed a good agreement of tests and the mechanical model. With the validated theoretical model, additional evaluations were carried out which led to the following conclusions:

- The distribution of the shear transfer actions is strongly dependent on different parameters like reinforcement ratio, specimen depth or shear slenderness. In smaller members, the shear forces in the compression and crack processing zone can amount to over 50% of the overall capacity whereas in larger members aggregate interlock is dominant. In general, aggregate interlock is the largest single contributor to the overall shear capacity.
- The most common failure mode in the databank evaluation was aggregate interlock failure (79%) followed by dowel failure (16%).
- At ultimate limit state, there was no case where there still existed a contribution by an inclined compressive force or direct strut action in a slender RC beam. This action can be a large contributor within the loading history but is unlikely to remain until the ultimate limit state since flexural cracks are usually distributed over a large part of the member.
- The size effect in beams is the result of a degradation of aggregate interlock in large beams. The type of scaling used to investigate size effects also has a large influence. The predicted the size effect from the mechanical model is in complete
3.7 Summary

agreement with the predictions of the statistical-energetic size effect law by BAŽANT /Baz86/.

• In case the aggregate size \( d_{ag} \) and the critical crack width \( w_0 \) are also scaled with the beam size, the size effect disappears. In reality, the scaling of these parameters is however not possible for large structures.

• Relating the shear strength of slender beams to the flexural crack width leads to a failure criterion that provides a comprehensible explanation of the behavior of slender beams in flexural shear. Such a law has already been established by the Critical Shear Crack Theory /Mut03/, /Mut08/.

In the next step, the Critical Crack Width Model was derived as a simplified closed form solution of the mechanical model. The resulting equation for RC beams is very similar to an equation that was already presented by PLACAS & REGAN /Pla71/ and MUTTONI /Mut16/ and is able to predict the shear strength of RC beams without axial forces and RC beams in tension very precisely. Finally, effect of prestressing was included in the simplified closed form model. The databank evaluation revealed that the simplified model is also able to predict the shear strength of PC beams with rectangular and T-shaped cross-section very precisely, if an empirical shear enhancement factor for nonslender beams is introduced. For prestressed I-beams that do not exhibit flexural cracks in the web prior to failure, a closed form model based on the principal tensile stress criterion was presented which was already used by the author in /Rog16a/. This model has also shown a good agreement with test databanks with prestressed I-beams and prestressed hollow core slabs.
4 Semi-Empirical Shear Strength Model for Beams without Shear Reinforcement

4.1 General

In this chapter, a semi-empirical shear strength model for beams without shear reinforcement will be developed. In chapter 3, a mechanical model was derived based on the assumption of vertical flexural cracks and a constant shear flow between the resultants of the longitudinal forces. Contrary to this, the evaluation of shear tests on one side and mechanical considerations on the other side will be the basis of the semi-empirical model. The model is supposed to be practicable for design purposes while showing sufficiently precise results when compared to databanks at the same. Chapter 4.2 begins with the presentation of the theoretical background and the basic assumptions for the model. It is followed by the investigation of parameters known to affect the shear strength to conclusively incorporate them into the model. First, the influence of concrete properties on shear strength will be dealt with in chapter 4.3. The influence of the compression zone is regarded as an important part of this model. Determining the height of the compression zone for RC and PC members will be dealt with in chapter 4.4. The size effect for members in shear is treated in chapter 4.5. The influence of the shear slenderness or the moment to shear ratio is described in chapter 4.6. As most shear tests to this day have been performed on single span beams with point loads, this chapter directly affects the treatment of different statical systems, such as uniformly distributed loads or continuous beams. These will, however, be treated separately in chapter 4.7. Also, most models are derived assuming a rectangular cross-section. Additional flanges in the compression zone of previous tests have shown to be of importance to the shear strength of beams as they transfer additional shear stresses. The effect of the cross-section geometry will thus be treated in chapter 4.8. The final model for predicting the shear strength of beams without shear reinforcement will then be summarized in chapter 4.9.

4.2 Background

One possibility to determine the shear strength of a member is to attempt to determine the major components of the shear transfer directly by a mechanical analysis. However, as can be seen from chapter 3, such a mechanical model is usually not appropriate for design in its original form making simplifications necessary. A completely physical analysis of concrete structures, would require to account for equilibrium, crack formation, kinematic compatibility, interface shear transfer and (nonlinear) laws of material. None of these are accounted for by current design models for flexural shear failure. The main reason is that such a mechanical model would automatically require some sort of iterative process since there exists a circular dependency between member resistance,
4.2 Background

external forces, internal forces and the deformation of the beam. Therefore, simplifications have to be made. As a result, every mechanical model loses some of its precision and consistency or eventually ends up being an empirical model. The notion that every aspect of the physical reality of brittle shear failure can be represented by any sort of theory is very questionable to start with.

One way to provide a practicable solution for a shear strength predictor is to follow an empirical approach that provides reasonably precise results using sufficiently simple equations. A starting point for a reasonable empirical approach is given by the various models based on the capacity of the compression chord (see chapter 2.4.4). One advantage of such an approach is that the contribution of the compression zone is a component of shear transfer that can be described in a fairly simple manner. If the BERNULLI-hypothesis is assumed to be applicable for a cross-section in shear, a linear distribution of strains according to Figure 4-1a can be assumed.

This assumption will of course only be legitimate for sufficiently slender beams. In contrast to chapter 3, where a state of equilibrium prior to the formation of a horizontal branch of the flexural crack was investigated, it is now assumed that after formation of the critical horizontal shear crack, large flexural crack widths will significantly reduce possible aggregate interlock stresses across the crack surfaces. The shear stresses are then primarily transferred over the compression zone as shown in Figure 4-1c. Dowel action is required to prevent failure in this state of equilibrium but is implemented as an additional contributor to the shear capacity. The ultimate failure of the cross-section is induced when the residual tensile stresses in the compression zone exceed the concrete tensile strength $f_{ct}$. This will be the case on the level of the neutral axis where the shear stresses reach their maximum. PRUJSERS /Pru86/ introduced the term of the effective shear depth $h_t$ by considering an additional contribution of the crack processing zone. This is indicated by the decreasing shear stresses in the crack processing zone in Figure 4-1c.
A different free-body by ZINK/Zin99 is shown in Figure 4-2 in which the cut is made along the critical shear crack. According to ZINK, the length of the crack processing zone is approximately 0,3 to 0,5 times the characteristic length \( l_{ch} \) of the concrete. Over this length, the concrete is able to transfer additional tensile stresses in the ultimate limit state. ZINK introduced a separate contribution of the crack processing zone which was then summarized in a combined factor accounting for size effect and characteristic length.

![Figure 4-2: Free-body-cut at the ultimate limit state (reproduced from /Zin99/)](image)

For an empirical model, the effect of shear components apart from the contribution of the compression zone can be accounted for by considering such influence factors. Figure 4-3 summarizes the representation of the different components in the empirical approach. The arch or direct strut action is not accounted for in the basic model for slender beams. The shear capacity of the beam in the ultimate limit state is mainly governed by the failure of the compression zone which is subject to the concrete tensile strength. The contributions by the crack processing zone, aggregate interlock and dowel action are accounted for by size and slenderness factors.

![Figure 4-3: Representation of shear components in the empirical model](image)

The basic formulation of the shear strength is then given by Eq. (4-1). The equation is based on the shear capacity of the compression zone which is \( 2/3x_0b_wf_{ct} \) (Figure 4-3), with \( x_0 = \xi \cdot d \). The coefficient \( k_d \) accounts for the influence of the size effect and \( k_\lambda \) accounts for the influence of the shear slenderness. In the following chapters, each of the components of Eq. (4-1) will be determined.
4.3 Influence of Concrete Properties

4.3.1 Influence of the concrete strength

As stated before, it is assumed that the shear strength of beams is directly related to the concrete tensile strength $f_{ct}$. In many codes, the value of the concrete tensile strength $f_{ct}$ is given as a function of the compressive strength $f_c$, since the tensile strength itself is a value with a quite large variance. Since Eurocode 2 defines the concrete tensile strength as $0.3 \cdot f_{ck}^{2/3}$, some mechanical and empirical approaches relate the shear strength to $f_{ck}^{2/3}$. This, however, overestimates the tensile strength of high strength concrete with $f_{ck} > 50$ MPa for which EC2 defines the tensile strength as a logarithmic function. The relation of the shear strength to $f_{c}^{1/3}$ by ZSUTTY /Zsu68/ can be regarded as an empiric fit to the then available databank (which is identical to the simplified mechanical approach in chapter 3). However, the correlation to the actual tensile strength of concrete seems to be poor. As defining the basic strength value with two different functions for the model to be applicable to normal and high strength concrete seems to be unpractical, a simplification should be made. The classical estimate of $f_{ck}^{1/2}$ actually provides the best overall fit for all concrete strength classes up to $f_{ck} = 90$ MPa. If the coefficient is fitted to match the EC2 equation, Eq. (4-2) can be obtained. A comparison of the different approaches in shown in Figure 4-4.

\[
\begin{align*}
V_c = & \frac{2}{3} k_f k_d f_{ct} b_v d \\
\end{align*}
\]  \hspace{1cm} (4-1)

\[
\begin{align*}
\frac{f_{ctm}}{f_{ck}} & \approx 0.54 \sqrt{f_{ck}} \\
\end{align*}
\]  \hspace{1cm} (4-2)

Figure 4-4: Comparison of possible approximations for the mean concrete tensile strength $f_{ctm}$

However, an upper limit for the concrete strength $f_{ck}$ that is deployed for determining the shear strength should be considered. For high strength concrete it becomes more likely that shear cracks run through the aggregates, thus reducing the effect of aggregate
interlock. Investigations performed by CLADERA & MARI /Cla04a/, /Cla04b/ have confirmed this assumption. The authors propose a limitation of the adopted compressive strength \( f_{ck} \) to 60 MPa /Cla15/. Similar to the Model Code 2010 provisions, a limitation of \( f_{ck} \) to 64 MPa will be chosen so that \( f_{ck}^{1/2} \) should not be taken greater than 8.

### 4.3.2 Influence of the aggregate size

Determining the amount of shear stresses transferred over cracks and their influence on the shear carrying behaviour has been subject to many research efforts. Based on extensive experimental research and previous works (e.g. /Fen68/), a mechanical model for determining the amount of crack friction and aggregate interlock within a crack as a function of the flexural crack width \( w \) and the crack slip \( \Delta \) has been derived by WALRAVEN /Wal80/. The model is able to visualize the influence of various parameters on the shear stresses in a crack. In Figure 4-5, the influence of two different aggregate sizes (\( d_{max} = 32 \) mm and \( d_{max} = 16 \) mm) on the transfer of shear stresses in a crack is shown. As can be seen, the maximum aggregate size can have a significant impact on the shear stresses transferred in a crack, especially for larger crack widths. Some shear models therefore regard the influence of the aggregate size as the most important parameter after longitudinal strains, e.g. the Simplified Modified Compression Field Theory /Ben06/ or the Critical Shear Crack Theory /Mut08/. In these models, the effect of the aggregates is either directly coupled to the longitudinal reinforcement strains or the size effect. For the shear strength model proposed in this thesis based on a factorial approach, an additional factor \( k_{ag} \) would be needed to account for this effect.

![Figure 4-5: Shear stresses transferred by crack friction and aggregate interlock with respect to the vertical crack deflection \( \Delta \) and different aggregate sizes according to WALRAVEN’s model /Wal81/](image-url)
To include the influence of aggregates within an empirical approach, systematic test series on this influence have to be evaluated first. There exist, however, only few systematic shear test series on the influence of the maximum aggregate size. In the test series performed by SHERWOOD ET AL. /She07/ ten large-scale and ten small-scale shear tests were conducted on RC slab-strip specimens without shear reinforcement. The test results, shown in Figure 4-6, indeed indicate that there is an influence of the aggregate size on the shear capacity.

Figure 4-6: Test series by SHERWOOD ET AL. /She07/ on aggregate influence on the shear capacity for a) large scale specimen and b) small-scale specimen

For small-scale specimens in Figure 4-6b, the shear capacity seems to decrease for the largest aggregate size. The authors suggested that the “lower shear strength was most likely caused by the reduced aggregate interlock caused by aggregate fracturing“ /She07/. But only 1/5 of the aggregates were reportedly fractured. The simplest and most likely explanation of this effect is that the aggregates were too large for these small specimens, causing stress concentrations and bond problems for the longitudinal reinforcement with a concrete cover of only 30 mm for 51 mm aggregates. The influence of the aggregate size on the shear capacity can be represented by Eq. (4-3). The result is already shown in Figure 4-6. As indicated by the tests, a beneficial effect is not expected for aggregates larger than 32 mm. Theoretically, there should be no upper limit on the influence of the aggregate size if the overall member size is scaled accordingly. Due to certain design restraints like concrete cover and spacing of rebars the aggregate size cannot exceed certain practical limits so that an upper limit for $d_{ag}$ should be introduced.

$$k_{ag} = 1 + 0.45 \sqrt[1/2]{\frac{d_{ag}}{32}}$$

where

- $d_{ag}$ maximum aggregate size of the concrete. $d_{ag}$ should not be taken larger than 32 mm
4.3 Influence of Concrete Properties

ψ coefficient accounting for aggregate fracture

For higher concrete strengths, the beneficial effect of increased aggregate interlock decreases due to the fracture of aggregates. This is accounted for by the additional coefficient ψ that reduces the aggregate contribution for high strength concrete to a certain amount. Based on experimental investigations on the influence of concrete compressive strength and aggregate type on the concrete tensile strength, REMMEL /Rem94/ suggested that the mechanism fails for concrete strengths between 60 to 80 MPa. The coefficient ψ is therefore introduced as a reduction factor according to Eqs. (4-4) to (4-6).

\[ \psi = 1 \quad \text{and} \quad f_{\text{ck}} \leq 60 \]  
(4-4)

\[ \psi = 1 - \sqrt{\frac{f_{\text{ck}} - 60}{30}} \quad \text{and} \quad 60 < f_{\text{ck}} < 80 \]  
(4-5)

\[ \psi = 0,18 \quad \text{and} \quad f_{\text{ck}} \geq 80 \]  
(4-6)

It is clear from experimental investigations, that the beneficiary influence of the aggregate size factor should be limited for high strength concrete, due to splitting of the aggregates. The tests by ANGELAKOS ET AL. /Ang01/ are often referred to as proof. However, this particular test series seems to be unsuitable since an inadequate low strength grain (limestone) was used in the concrete, causing the aggregates to fracture prematurely. Tests performed by SAGASETA & VOLLM /Sag11/ also confirmed that limestone aggregates tend to fail at significantly lower loads in comparison to gravel aggregates. The authors, however, maintained that shear stresses can still be transferred over the crack surfaces due to the macro level roughness of the crack. Therefore, a lower limit for ψ of 0,18 was introduced. This fact is also reflected in the crack friction model by WALRAVEN in which the difference between combined aggregate interlock and crack friction and aggregate interlock without crack friction (μ = 0) can be visualized (Figure 4-7). The model shows, that a significant amount of shear stresses can still be transferred without aggregate interlock. In combination with the fact, that some consider aggregate interlock and crack friction minor contributors in the ultimate limit state of RC members /Baz05b/, a lower limit of ψ seems to be justified. However, the question remains if the influence of the aggregate size on the shear capacity should generally be included within a code proposal by introducing an aggregate factor k_{ag}. A practical concern might be, that engineers are used to design according to concrete strength and not concrete mixture. Accounting for aggregate sizes would for practical reasons be more valid for assessment of structures. The bigger concern should be that the value of the maximum aggregate size is simply not able to capture all important effects concerning crack friction and aggregate interlock. It should be pointed out, that the influence of the aggregates on the shear capacity also strongly depends on the aggregate type. REMMEL /Rem94/ reported that concrete with crushed basalt aggregates exhibited a 30 to 50 % higher fracture energy \( G_f \) compared to gravel aggregates of the same size. The friction curves shown in Figure 4-5 were for instance derived based on the so-called ideal Fuller
4.3 Influence of Concrete Properties

curves. WALRAVEN /Wal81/ also presented a comparison of shear stress diagrams with two allowable grading curves according to the Dutch code of Practice VB’74 /NEN84/ (Figure 4-8). The figure demonstrates very clearly that the predicted differences for the shear stresses between the two grading curves amounts up to 50%. In fact, it can be seen by comparison of Figure 4-8 to Figure 4-5 that the effect of the grading curves is much larger than the influence of the maximum aggregate size.

![Figure 4-7: Difference between shear stresses due crack friction and combined aggregate interlock and shear stresses without crack friction /Wal81/](image)

![Figure 4-8: Influence of the grading curve on the transfer of stresses in a crack for two different mixes /Wal81/](image)
It is therefore reasonable to conclude that, at this point, a factor $k_{ag}$ should rather not be included in an empirical design formula. More systematic experimental research is needed to account for the three important parameters aggregate size, aggregate type and grading curves in shear strength models. In contrast to chapter 3.3 in which the loss of shear strength for high strength concrete is directly covered by the parameter $w_0$, the decrease of crack friction for high strength concrete should rather be accounted for by limiting the nominal value of $f_{ck}$ in the design formula.

4.4 Influence of the Compression Zone

4.4.1 General approach

In a flexural-shear type of failure, the shear forces have to be transferred by aggregate interlock, by dowel action of the longitudinal reinforcement and as shear stresses over the uncracked compression zone of the cross section. Therefore, the height of the compression zone and the amount of longitudinal reinforcement are crucial parameters for the flexural-shear capacity. For RC beams, the height of the compression zone is mainly governed by the longitudinal reinforcement ratio while for PC beams, the degree of pre-stressing is an additional influence factor. The aim of this section is to derive an approach for determining the height of the compression zone by means of a mechanical approach.

The height of the compression zone is designated as $x_0$, whereas the ratio of the compression zone height $x_0$ and the effective depth $d$ is defined as the relative depth of the compression zone $\xi = x_0/d$. The simplest case for determining the depth of the compression zone of a non-prestressed member in bending is a rectangular cross-section with one layer of longitudinal reinforcement as shown in Figure 4-9. In case of a linear-elastic material behavior, the resulting concrete compressive force $F_c$ and the tensile force $F_s$ can be calculated according to Eqs. (4-7) and (4-8). Note that in this example, the concrete tensile strength is neglected as well as the nonlinear behavior of concrete. If additional concrete tensile stresses and the nonlinearity of the concrete stress-strain relationship are considered, the depth of the compression zone can only be determined by iteration. The simplifications made are not expected to be crucial to the precision of the formula for two reasons. Firstly, it is not expected that the concrete strains in compression prior to shear failure reach high values so that the nonlinearity is important, an assumption which is also justified by the fact that the longitudinal reinforcement ratio is usually limited accordingly. In fact, employing a parabola function for concrete with little strains leads to very unprecise results, as this function is designed for the ultimate limit state in flexural bending. Furthermore, the influence of the tensile stresses in the flexural crack tip become less important with increasing crack length and thus increasing moment. If a linear strain distribution is assumed and no concrete tensile stresses are
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considered, the depth of the compression zone of a rectangular cross-section in bending, as shown in Figure 4-9, can be determined by horizontal equilibrium. The horizontal forces in the compression zone $F_c$ and in the tensile reinforcement $F_s$ can be determined according to Eqs. (4-7) and (4-8), respectively.

\[ F_c = \frac{1}{2} E_c \varepsilon_c b_w \xi d \]  
\[ F_s = E_s \varepsilon_s A_s \]  

(4-7)  
(4-8)

Figure 4-9: Strain and stress distribution in an RC member

As the resulting axial force $N$ is zero in an RC section, $F_c$ and $F_s$ may be equalized according to Eq. (4-9). On the basis the Bernoulli-hypothesis, the reinforcement strain $\varepsilon_s$ can be replaced by a function of $\varepsilon_c$ and $\xi$ (i.e. $\varepsilon_s = \varepsilon_c (1 / \xi - 1)$).

\[ \frac{1}{2} E_c \varepsilon_c b_w \xi d = E_s \varepsilon_c \frac{1-\xi}{\xi} A_s \]  

(4-9)

If Eq. (4-9) is solved for $\xi$, Eq. (4-10) is obtained. In this equation, the depth of the compression zone depends on the strains $\varepsilon_s$ and $\varepsilon_c$ so that the acting moment has no influence of the depth of the compression zone. Eq. (4-10) is used within various different models that employ the depth of the compression zone for predicting the shear strength (e.g. /Tur03/, /Zin99/, /Mut08/, /Tue14/, /Rui15/, /Mar15/)

\[ \xi = \sqrt{\left(\alpha, \rho \right)^2 + 2 \alpha, \rho - \alpha, \rho} \]  
\[ = \frac{2}{1+\sqrt{1+2/\alpha, \rho}} \]  

(4-10)

where

\[ \alpha_c = E_s / E_c \]  
\[ \rho = A_s / (b_w \cdot d) \]

However, this equation is usually not found in practical design equations. The main reason is that Eq. (4-10) can be quite effectively approximated by Eq. (4-11) /Zin99/.
4.4 Influence of the Compression Zone

\( \xi = 0.783\sqrt{\alpha_c \rho} \)  \( (4-11) \)

A comparison of the analytical and the approximate solutions for \( \xi \) in Figure 4-10 reveals a very good agreement between the two functions. The comparison thus suggests that the height of the compression zone is affine to \( \rho^{1/3} \).

![Comparison of the analytical and approximate solution for \( \xi \)](image)

This would also explain why the flexural shear formula according to /Zsu68/ and /MC90/ is in quite good agreement with the test data for shear tests on RC beams. Eq. (4-11) is of course only valid for RC beams without prestressing. Axial forces are, however, a governing influence on the depth of the compression zone of PC beams. If an additional axial force is considered, the acting moment also should be accounted for (Figure 4-11).

![Stress and strain distribution on a PC member](image)

The horizontal equilibrium and the moment equilibrium lead to Eqs. (4-12) and (4-13), respectively.

\[
N_{Ed} = \frac{1}{2} E_c \varepsilon_c b_w \xi d - \varepsilon_s \left( \frac{1}{\xi} - 1 \right) E_s A_s
\]

(4-12)
4.4 Influence of the Compression Zone

\[ M_{Ed} = -\frac{1}{2} E_c \varepsilon_b h_c \xi^2 (1 - \xi / 3) \] (4-13)

If Eq. (4-12) is divided by Eq. (4-13), the concrete strain \( \varepsilon_c \) is cancelled so that the depth in the compression zone \( \xi \) can be calculated by solving Eq. (4-14).

\[ \frac{1}{6} \eta \xi^3 + \frac{1}{2} (1 - \eta) \xi^2 + \alpha_c \rho \xi - \alpha_c \rho = 0 \] (4-14)

where
\[ \eta = \frac{N_{Ed} \cdot d}{M_{Eds}} \]

This equation, however, cannot be solved in a closed form in a practicable manner. The easiest way to find solutions for each combination of \( \alpha_c \rho \) and \( \eta \) is to use computational methods. The resulting array of curves for members in compression is illustrated in Figure 4-12. It can be seen, that the function shown in Figure 4-10 is a special case for this array of curves for \( N_{Ed} = 0 \).

Figure 4-12: Array of curves for determining the height of the compression zone for members in compression

The graphs show that especially for beams with a high degree of prestressing and/or in regions with small flexural moments, the depth of the compression zone increases exponentially. Since no closed solution can be obtained for calculating the relative depth of the compression zone, an attempt should be made to find an approximate solution for this problem.

4.4.2 Approximation of \( \xi \) for members in compression (option 1)

The graphs shown in Figure 4-12a can be effectively approximated by an exponential function as given in Eq. (4-15). However, the parameters \( A \) and \( B \) vary for each value of the reinforcement ratio \( \alpha_c \rho \).

\[ \xi = A \left[ 1 + \exp (B \cdot \eta) \right] \] (4-15)
The functions illustrated in Figure 4-12a have consequently been approximated with Eq. (4-15). The original exact solutions and the approximated curved are compared in Figure 4-13a. The approximated functions seem to provide acceptable results. Only for very small amounts of longitudinal reinforcement ratios, the differences seem to be a little higher than for higher ratios. The parameters \( A \) and \( B \) were in consequence determined numerically for different values of \( \alpha_e \rho \) by minimizing the sum of error squares between exact and approximate solutions. The resulting parameters \( A \) and \( B \) are shown in Figure 4-13b over the reinforcement ratio \( \alpha_e \rho \).

![Figure 4-13: a) Illustration of the exact solution and approximation according to Eq. (4-15); b) corresponding values of coefficients \( A \) and \( B \) over the reinforcement ratio \( \alpha_e \rho \)](image)

The parameters \( A \) and \( B \) can then themselves be approximated as functions of \( \alpha_e \rho \) by regression analysis. However, the function for parameter \( A \) should be identical to the function chosen for RC beams without axial forces to obtain consistent equations. In that case, the basic value of the related compression zone depth \( \xi_{N=0} \) for RC beams can be multiplied with an exponential function as shown in Eq. (4-16).

\[
\xi = \xi_{N=0} \cdot 0,5 \left[ 1 + \exp \left( A \cdot \eta \right) \right] \quad (4-16)
\]

The approximation for the value \( A \) to reduce the error compared to the exact solution is shown in Figure 4-14a. By inserting the approximation for \( A \) into Eq. (4-16), Eq. (4-17) is obtained.

\[
\xi = \xi_{N=0} \cdot 0,5 \left[ 1 + \exp \left( \frac{0,5 \cdot \eta}{(\alpha_e \rho)^{0,25}} \right) \right] = 0,44 (\alpha_e \rho)^{0,4} \left[ 1 + \exp \left( \frac{0,5 \cdot \eta}{(\alpha_e \rho)^{0,25}} \right) \right] \quad (4-17)
\]

where

\[
\eta = \frac{N_e d}{M_{Eds}}
\]
4.4 Influence of the Compression Zone

As shown in Figure 4-14b, whilst catching the overall trend of the precise solution the agreement of the developed equation is less than perfect. The formula also seems to be less precise for very small amounts of longitudinal reinforcement, even leading to contradictory results due to overlapping of the functions in some cases. It is therefore considered to obtain an alternative solution for determining the related depth of the compression zone \( \xi \).

4.4.3 Approximation of \( \xi \) for members in compression (option 2)

One disadvantage of Eq. (4-17) is that it is quite sensitive with respect to the reinforcement ratio \( \alpha e \rho \). Furthermore, for the hypothetical case that there is no longitudinal reinforcement at all, the formula is not applicable as it will always yield a relative depth of the compression zone equal to zero. Therefore, a different method for approximating a formula for the related depth of the compression zone is investigated. A quite effective way to define the empirical function for \( \xi \) is to separate it into two parts. The general formula for \( \xi \) is thus represented by Eqs. (4-19) and (4-20), where the parameters \( A \) and \( B \) are subject to \( \eta \) (Figure 4-15a).

\[
\frac{1}{6} \eta \xi^3 + \frac{1}{2} (1-\eta) \xi^2 + A e \rho \xi - A e \rho = 0
\] (4-18)

If the degree of longitudinal reinforcement is assumed to be zero, Eq. (4-18) can be solved in closed form by Eq. (4-19).

\[
\xi = 3 \left( 1 - \frac{1}{\eta} \right) \text{ and } \rho = 0 \text{ and } 1,0 \leq \eta \leq 1,5
\] (4-19)

The solution according to Eq. (4-19) can be used as an additive term for values of \( \eta \) larger than 1.0. The general form of the equation for \( \xi \) is thus represented by Eqs. (4-19) and (4-20), where the parameters \( A \) and \( B \) are subject to \( \eta \) (Figure 4-15a).

\[
\xi = A \cdot (\alpha e \rho)^B \text{ and } 0 \leq \eta \leq 1,0
\] (4-20)
4.4 Influence of the Compression Zone

\[
\xi = A \cdot (\alpha, \rho)^\theta + 3 \left(1 - \frac{1}{\eta}\right) \text{ and } 1,0 \leq \eta \leq 1,5
\]

(4-21)

\[\eta = \frac{N_{Ed}d}{M_{Eds}}\]

Figure 4-15: a) numerical evaluation of the parameters \(A\) and \(B\) and their corresponding approximation; b) comparison of exact and approximated solution

In Figure 4-15a, the best approximates for parameters \(A\) and \(B\) are displayed. The comparison of the approximated solution and the exact solution in Figure 4-15b shows a very good agreement. The final form of the approximation for \(\xi\) is then given by Eqs. (4-22) and (4-23).

\[
\begin{align*}
\xi &= (0.86 + 0.14\eta) \cdot (\alpha, \rho)^{0.38 - 0.16\eta^7} \text{ and } 0 \leq \eta \leq 1,0 \quad (4-22) \\
\xi &= (3 - 2\eta) \cdot (\alpha, \rho)^{0.88\eta - 0.66} + 3(1 - 1/\eta) \text{ and } 1,0 \leq \eta \leq 1,5 \quad (4-23)
\end{align*}
\]

Although the fit of this approximation seems to be superior to the one of chapter 4.4.2 it should be noted that the application of this formula seems to be more complicated as a distinction of cases will be necessary. This should be considered when choosing an appropriate equation for design purposes.

4.4.4 Approximation of \(\xi\) for members in tension

In addition to defining a function for \(\xi\) for members in compression, the influence of tensile forces on the shear capacity has to be accounted for. Tensile forces might result from constraint action or due to tensile stresses in structures like containments or silos. The general equation for the related depth of the compression zone (Eq. (4-18)) has to be solved for negative values of \(\eta\). The array of curves for negative values of \(\eta\) is shown in Figure 4-16a and b.
4.4 Influence of the Compression Zone

Figure 4-16: Array of curves for determining the height of the compression zone for members in tension

The shape of the functions can be approximated by simply introducing an additional factor $A$ that is subject to the tensile force to moment ratio $\eta$ according to Eq. (4-24).

$$\xi = A \cdot 0.88 \cdot (\alpha_c, \rho)^{0.4}$$  \hspace{1cm} (4-24)

In Figure 4-17a, the exact solution of the parameter $A$ is shown along with two possible approximations. While the linear approximation seems to be conservative and easier to apply its serious drawback is, that the predicted shear capacity of members in tension and small bending moments will amount to zero. However, even beams that exhibit tensile crack over the whole depth of the cross-section will be able to transfer shear stresses via dowel action and some aggregate interlock.

A hyperbolic approach, as shown in Figure 4-17a, will on the other side give values above zero for all values of $\eta$. The function also provides a good fit for most of the regarded parameter field as shown in Figure 4-17b. The function for $\xi$ for members in tension is thus given by Eq. (4-25).
4.5 Influence of the Size Effect

Experimental investigations of shear tests on concrete beams have shown that concrete members exhibit a considerable size effect leading to a nominally decreased concrete strength for increasing cross-section depths. Early investigations assumed, that the size effect was only of concern for smaller members and that for instance only beams with a cross-section depth below 400 mm exhibited a size effect /Leo62/. This assumption was also reflected in the provisions of Model Code 1978 /MC78/, in which the size effect was limited to a cross-section depth of 600 mm. Since the 1980s, BAŽANT and various other researchers published work related to the size effect in concrete members that approached the problem from a theoretical perspective by regarding it as an energy related problem. From his perspective, BAŽANT lists a number of reasons for the occurrence of size effects /Baz94/. An extensive review on the history of the size effect is given in /Baz97/. As a result, BAŽANT advocated the so called energetic-statistical size effect law (SEL) according to Eq. (4-26) /Baz84/. This size effect law can be regarded as a special case of a combined size effect of the energy related fracture of notched specimens and the statistically (Weibull size effect) related failure of un-notched specimens. The combination of these two different types of size effect is reflected in the works on the so-called universal size effect law (USEL) by BAŽANT /Baz09/ and HOOVER /Hoo14/. For beams exhibiting flexural shear failure it is, however, sufficient to refer to the SEL as these beams can be regarded as notched specimens for which the statistical size effect is of minor importance.

\[
\frac{1}{\sqrt{1+d/d_0}} \tag{4-26}
\]

From a practical point of view, it is notable that this size effect law by BAŽANT according to Eq. (4-26) for notched specimens does not incorporate a lower limit for the size effect, opposite to the provisions of the Model Code 1978 /MC78/, Model Code 1990/MC90/ and the current EC2 /EC211/. Authors like CARPINTERI provided a theoretical basis for such an assumption based on the so called multi-fractal scaling law or fractal-statistical
In addition to a mechanical approach, the size effect phenomenon can be investigated with an empirical approach, as the conditions of an actual RC member might be somewhat different than in an idealized member anyway. From an empirical perspective, different exponents can be justified to fit the experimental data. It is quite difficult to verify a certain size effect law as the size range must be very large to distinguish between the exactness of different exponents. The reason for this is that the differences of the slopes of graphs with different exponents might seem small when compared in a linear scale (Figure 4-18a) and are only visible when visualized in a double logarithmic scale (Figure 4-18b). In such a scale, the exponents of the size effect law represent the slope of the respective asymptote (Figure 4-18b).

Validating these asymptotes would require to double the height of test specimens a considerable number of times to investigate the accuracy of different size effect laws. A statistical evaluation of tests to validate a size effect law might be biased anyway, as smaller members are usually overrepresented within databanks.

Therefore, numerical investigations with Abaqus on the size effect were carried out by HERBRAND ET AL. The aim was to validate the size effect prediction of the SEL for very large specimens by means of nonlinear Finite Element Analysis using a plastic damage material model for concrete. First, the FE model was verified on a test specimen of the literature (Figure 4-19). The plastic damage model for concrete was calibrated on the basis of investigations carried out by KUERES AT AL.
4.5 Influence of the Size Effect

The model was then scaled up to a cross-section height of over 10,000 mm. For each model, different configurations of the longitudinal reinforcement were used to minimize the scatter due to differences in the dowel effect. The results of the parametric study are shown in Figure 4-20. As shown in Figure 4-20b, this parametric study indicates that the size effect law by Bazant correctly represents the behavior of very large RC structures. It also shows that notable differences to EC2 are only to be expected for members with an effective depth of over 1000 mm.

As there are only few tests with such large specimens it is very hard to verify these size effect laws on the basis of experimental results, as stated before. This can also be seen when reviewing test series from the literature. In Figure 4-21, an evaluation of 69 tests from the literature is shown (/Ben05b/, /Bha68/, /Cha81/, /Kaw98/, /Kim94/, /Leo62/, /Niw86/, /Pod98/, /Shi90/, /Sne07/, /Tay72/, /Wal78/). All of the selected tests had a shear slenderness $a/d$ between 2.9 and 3.0 so that this influence was eliminated. The concrete strengths of these specimens ranges between 16 and 71 MPa.
4.5 Influence of the Size Effect

As can be seen in Figure 4-21, it is difficult to validate or invalidate a particular exponent of a size effect law on the basis of these tests. The size effect law of the current EC2 does, however, seem to be unsafe. A possibility to account for the uneven distribution of tests in databanks (“heteroscedasticity”) is to weigh the test data evenly, as done by YU ET AL. in /Yu15/. In case of investigating the size effect, the test data has divided into groups for different ranges of the effective depth \( d \). The spacing should be in logarithmic units, as shown in Figure 4-22b. Here, the 784 tests on RC beams without stirrups from the ACI-DAfStb databank were divided into seven subgroups. The number \( N_i \) of tests in each section is then counted. The reciprocal of that value \( N_i \) of a section is the weight for the associated tests. In Figure 4-22a, the weight factor \( 1/N_i \) of each test is represented by the size of the circles.

As shown in the diagram, the size effect law by BAZANT follows the path of the weighted tests very well. Some outliers can be seen above the value of 1.0. According to YU ET AL. /Yu15/, tests with a cross-section depth less than 250 mm should rather be analyzed
by plasticity calculation, as their results are often quite scattered. In summary, theoretical considerations as well as the evaluation of test data suggest that the size effect law by BA2ANT is the best choice to ensure that the shear strength model will be applicable for larger structures. In addition, it is also very practicable as the function itself does not require an arbitrary upper or a lower limit.

The factor \( d_0 \) in Eq. (4-26) is related to the characteristic length of concrete and can be adjusted empirically to the test data. A value of \( d_0 = 200 \text{ mm} \) is usually regarded as the best fit for RC members. However, it is clear that the size effect factor should also be influenced by the concrete strength or the aggregate size as the crack friction, dowel action and crack tip are key to the size effect. In /Yu15/, for example, the factor \( d_0 \) is determined according to Eq. (4-27) to account for the concrete strength \( f_c \).

\[
d_0 = \frac{c_a}{f_c^{0.7}} \tag{4-27}
\]

If it is assumed that the value of \( d_0 \) is proportional to the characteristic length of concrete \( l_{ch} \), then \( d_0 \) is subject to the Young’s modulus of concrete \( E_c \), the fracture energy \( G_f \) and the tensile strength \( f_{ct} \) according to Eq. (4-28).

\[
d_0 \sim l_{ch} = \frac{E_c G_f}{f_{ct}^2} \tag{4-28}
\]

According to Eurocode 2, the values for \( E_c \) and \( f_{ct} \) are subject to the concrete compressive strength, such that \( E_c \sim f_c^{0.3} \) and \( f_{ct} \sim f_c^{2/3} \). The value for the fracture energy is usually defined as a function of the concrete strength \( f_c \) (e.g. /CEB10/), the concrete tensile strength \( f_{ct} \) (e.g. /Rem94/) or the aggregate size \( d_{ag} \) /Wit02/. MARI ET AL. /Mar15/ proposed an empirical formula for the fracture energy that incorporates both the concrete strength \( f_c \) and the maximum aggregate size \( d_{max} \):

\[
G_f = 0.028 \cdot f_c^{0.18} \cdot d_{ag}^{0.32} \tag{4-29}
\]

The formula has shown a good agreement with available test data /Mar15/. If the Eurocode 2 formulae for \( E_c \) and \( f_{ct} \) and Eq. (4-29) are inserted into Eq. (4-28), Eq. (4-30) is obtained.

\[
d_0 \sim \frac{E_c G_f}{f_{ct}^2} = \frac{A \cdot f_c^{0.3} \cdot f_{ct}^{0.18} \cdot d_{ag}^{0.32}}{f_c^{2/3} \cdot \left( f_{ct}^{2/3} \right)^2} = \frac{A \cdot d_{ag}^{0.32}}{f_c^{0.85}} \tag{4-30}
\]

The parameter \( A \) can be adjusted according to test data. A comparison with the ACI-DAfStb databanks for RC beams without stirrups for the least error fit resulted in Eq. (4-31).
4.6 Influence of the Shear Slenderness

\[ d_0 = \frac{1600d_{ag}^{0.32}}{f_c^{0.85}} \]  \hspace{1cm} (4-31)

where

- \( d_{ag} \) maximum aggregate size in mm, with \( d_{ag} \leq 32 \) mm
- \( f_c \) concrete compressive strength in MPa, with \( f_c \leq 64 \) MPa

Eq. (4-31) provides a reasonable link of the characteristic length value to the mechanical properties of concrete. Regarding the accuracy when compared to large databanks, the formula does, however, not provide any advantage over a constant value for \( d_0 \). This is not surprising due to the large scatter of test results. Since the reproducibility of flexural shear tests is not very good altogether (COVs of repeated tests range from 6 to 12%) and the practically testable range of specimen is limited, it also seems not possible to verify such an approach for \( d_0 \) based on experimental results. For design practice, a constant value of \( d_0 = 200 \) mm is therefore preferable.

4.6 Influence of the Shear Slenderness

The shear slenderness of a beam is usually referred to as the shear-span-to-depth ratio \( a/d \) in conjunction with shear in point loaded single span beams. An increase of the shear span \( a \) leads to a larger \( a/d \) ratio, thus resulting in larger moments due to the constant shear forces. Larger moments cause larger crack widths due to an increased rotation, thus decreasing the stabilizing effect of aggregate interlock and the tensile stresses at the crack tip. Also, the flexural crack pattern that results from the \( a/d \)-ratio is crucial for the resulting load transfer actions. This can be seen from the well-known tests series by LEONHARDT & WALTHER /Leo62/ on the influence of the shear slenderness on the shear capacity (Figure 4-23).

![Test series by LEONHARDT & WALTHER on the influence of the shear slenderness for a/d values of a) 3.0 b) 4.0 c) 5.0 and d) 6.0](image)
The first two tests with \( a/d = 3 – 4 \) exhibit diagonal tension failure, as the critical crack results from a flexural crack next to the support. The shear capacity is significantly larger as there are no additional flexural cracks between the critical crack and the support disturbing the load transfer. For values of \( a/d = 5 – 6 \) there exist additional flexural cracks behind the critical shear crack so that the force in the compression zone due to bending is not inclined in any way at the critical shear failure plane. The shear has to be carried mostly by crack friction (Figure 4-24a), which is reduced due to larger crack widths, by tensile forces at the crack tip (Figure 4-24b) and by dowel action at the longitudinal reinforcement (Figure 4-24c). The loss of crack friction and the activation of dowel action leads to a rupture along the longitudinal reinforcement, as shown in Figure 4-23d. When the forces at the crack tip and cantilever bending (Figure 4-24a) are activated, a sudden shear failure due to horizontal cracking results.

Figure 4-24: Main shear transfer actions for slender beams a) cantilever action; b) aggregate interlock; c) dowel action; d) residual stresses at the crack tip (reproduced from /Rui15/)

Covering the complex shift of shear transfer actions in a direct way can be achieved by a mechanical model that explicitly accounts for all different transfer mechanisms. It has been pointed out that there is a strong interaction between the longitudinal reinforcement ratio \( \rho \) of a beam and the shear slenderness such that a constant ratio of the two will result in constant shear failure stresses /Col08/, /Kha13/. Another possibility to interpret this effect would be to regard it in terms of longitudinal strains in the reinforcement that are affected by both the reinforcement ratio as well as the shear slenderness of a beam. The main conclusion is that the shear slenderness is an influence factor that should be treated with the same attention as the reinforcement ratio, as neglecting the shear slenderness leads to an unsafe design for certain combinations of shear slenderness and reinforcement ratio, thus creating the need for limiting the nominal reinforcement ratio in the design formulae /Col08/.

The \( a/d \)-ratio has been used in different ways within an expressions of shear strength models as shown in section 2.4.2. For other static system like continuous beams under various loadings, the shear slenderness \( a/d \) can also be expressed by the \( M/Vd \) ratio in the control section. In code practice, considering the \( M/Vd \)-ratio instead of the shear span to depth ratio was considered an almost revolutionary step due to the universal applicability of such a criterion /ACI62/. If the shear slenderness is accounted for by a shear slenderness factor \( k_\lambda \), which incorporates the shear slenderness in terms of \( \lambda = \)
4.6 Influence of the Shear Slenderness

$M/Vd$, this coefficient will differ along the shear span even for a point loaded system with values ranging between zero at the support and $a/d$ beneath the point load. As stated before, a control section with a low $\lambda$-value will exhibit a higher shear capacity as it exhibits smaller flexural crack widths or no cracks at all. This influence can be estimated by evaluating tests series specifically performed to investigate the influence of the shear slenderness. Here, 112 tests from different authors ([/Kre66/, /Kan67/]) were used to derive an empirical approximation (Figure 4-25).

![Figure 4-25: Evaluation of tests by /Kre66/, /Kan67/ in on the influence of the shear slenderness on the nominal shear capacity](image)

For this evaluation, it was assumed that the shear check is performed at a distance of $d$ from the position of the point load. One empirical approximation for $k_\lambda$ is given by Eq. (4-32) (Figure 4-25a). The equation is of a linear rational style and has a very simple appearance while at the same time capturing the trend of the test results very good.

$$k_\lambda = \frac{1,8}{1+0,15\lambda} \geq 1,0 \quad (4-32)$$

One drawback of Eq. (4-32) is that a lower limit for the $k_\lambda$ value should be introduced for very slender beams. A distinction of cases has to be made during design and the equation itself can thus not be directly implemented into the design equation making the equation more unhandy. A different possibility to approximate the test results is by using an exponential type equation like Eq. (4-33). The major advantage is that the equation yields meaningful values of $k_\lambda$ for the full range of possible values for $\lambda$ in any statical system without limitations. A comparison with the test series in Figure 4-25b shows that Eq. (4-33) provides almost identical results as Eq. (4-32).

$$k_\lambda = 1 + 0,8 \cdot e^{-\lambda/2,5} \quad (4-33)$$
4.7 Influence of Loading Conditions and Statical Systems

4.7.1 General

Most shear tests have been conducted on single span beams under point loads. However, in design praxis, uniformly distributed loads (UDL) or a combination of point loads and UDLs seem to be more common. Also, in a typical high-rise building, continuous beams at interior columns are very common, raising the question of how to deal with a zero moment point in the vicinity of a column. In this chapter, tests on single span and continuous beams under uniformly distributed loads will therefore be evaluated.

4.7.2 Single span RC beams under UDL

In point loaded single span beams, shear failure occurs close to the point of load introduction where large shear forces coincide with large moments (Figure 4-23). This results in large values of $M/V_d$. In contrast, the failure of single span beams under UDL occurs somewhere in the vicinity of the support where large shear forces coincide with small flexural moments leading to smaller values of $M/V_d$ (e.g. Figure 4-26a). In single span beams under UDL, the critical shear crack usually results from a regular flexural crack and then changes its inclination to the direct line of compression to the support, as described by Zararis & Zararis /Zar08/ (Figure 4-26b).

![Figure 4-26: a) crack pattern of a test beam by Leonhardt & Walter /Leo62/ b) shear span of a member under UDL /Zar08/](image)

According to /Zar08/, it can be assumed that the splitting failure of the concrete compression zone occurs at a distance $a_i$ which represents the ideal shear span, similar to point loaded systems (Figure 4-26b). Within that idealized shear span, the uniform load can be represented by two statically equivalent point loads of $0.5 q a_i$ (Figure 4-27).
Since the shear force within the shear span has a constant value of $V = 0.5q(l-a_i)$, ZARARIS & ZARARIS conclude that their shear model for beams under point loads is also applicable for beams under distributed loads. If an ideal shear span of $a_i = 2.5d$ is assumed, Eq. (4-34) is yielded, which represents the shear failure load according to the splitting test analogy /Zar01/, /Zar08/.

$$V_{cr} = \left(1.2 - 0.2 \frac{a_i}{d}\right) \frac{x_0}{d} f_{\alpha, b_v, d} = (1,2 - 0.5d) \frac{x_0}{d} f_{\alpha, b_v, d} \quad (4-34)$$

If the shear force in the shear span $V = 0.5q(l-a_i)$ is introduced into Eq. (4-34), the ultimate distributed load $q_u$ can be obtained by Eq. (4-35) /Zar08/.

$$q_u = \frac{2h_w}{l/d - 2,5} \left(1,2 - 0.5d\right) \frac{x_0}{d} f_{\alpha} \quad (4-35)$$

Eq. (4-35) can of course also be rewritten to represent the shear force at the support as shown in Eq. (4-36), in which case the assumption of the critical shear span $a_i$ results in a shear enhancement factor $(1-2.5d/l)^1$. This factor shows, that the shear enhancement of beams under distributed loads also depends on the length to depth ratio $l/d$ of the members.

$$q_u l/2 = V_{ud} = V_{cr} = \frac{l/d}{l/d - 2,5} \left(1,2 - 0.5d\right) \frac{x_0}{d} f_{\alpha, b_v, d} \quad (4-36)$$

The comparison of the approach by ZARARIS & ZARARIS with 40 tests on slender RC beams from the literature has shown a very good agreement /Zar08/. Although the model of this thesis is not identical to the splitting test model, the basic idea of the approach for UDL beams can be adopted. First, the $k_\lambda$ factor, which was derived for pointed loaded beams should be considered. For a single span beam under UDL, the shear slenderness $\lambda$ differs along the span of the beams. Its value can be determined according to Eq. (4-37).
4.7 Influence of Loading Conditions and Statical Systems

\[ \lambda = \frac{M}{Vd} = \frac{q x/2(l-x)}{q(l/2-x)d} = \frac{x/2(l-x)}{(l/2-x)d} \]  

(4-37)

The length \( x \) is the distance of the control section from the support and should be equal to the critical shear span \( a_i \). In contrast to ZARARIS & ZARARIS /Zar08/, it is assumed that the critical shear span \( a_i \) is equal to the depth of the member \( d \). On the one hand it seems, that judging from crack patterns in tests, the section in which the rupture of the compression zone is initiated is closer to the support than \( 2.5d \) (Figure 4-28). On the other hand, it is very common in design praxis (e.g. in Germany) to perform a shear check at a distance of \( d \) from the support.

For the proposed shear model, it is assumed that the shear check is performed at a distance \( d \) from the support. Within that shear span, the beam is treated equally to a single span beam. The value of \( \lambda \) for \( x = d \) can then be determined according to Eq. (4-38). The shear slenderness in the control section is thus subject to the \( l/d \) ratio of the beam under distributed loads.

\[ \lambda = \frac{M}{Vd} = \frac{x/2(l-x)}{(l/2-x)d} = \frac{x/2(l-x)}{(l/2-x)d} = \frac{l/d - 1}{l/d - 2} \]  

(4-38)

As can be seen directly, the value of \( \lambda \) remains pretty much the same for common values of \( l/d \geq 10 \), so that the \( k_\lambda \) value barely changes for different values of \( l/d \). However, it should be considered, that by checking the shear force at a distance of \( d \) from the support, its value also has to be decreased accordingly. The design shear force at a distance \( d \) from the support is \( V_{ult} = q_{ult}(l/2-d) \). To compare this value to the maximum shear force at the support \( (q_{ult}l/2) \), the equation for checking shear can be rewritten into Eq. (4-39).
Here, it is shown that checking shear at a distance of $d$ from the support implicitly introduces a shear enhancement factor that is quite similar to the one of ZARARIS & ZARA-
RIS /Zar08/.

$$V_{Ed} = q_{ld}(l/2-d) = V_{ld} \Rightarrow q_{ld}l/2 = \frac{l/2}{l/2-d}V_{ld} = \frac{l/d}{l/d-2}V_{ld}$$ (4-39)

The proposal of this thesis was compared to the test data collected by ZARARIS & ZARA-
RIS /Zar08/ by comparing the shear failure load $V_{u,test}$ at a distance of $d$ from the support to the mean value of the shear strength $V_{Rm,c}$ at a distance of $d$. The proposal shows a very good agreement with the test data, with a mean value of 1,01 and a coefficient of variation of only 9.8 % (Figure 4-29).

This shows, that by the simple assumption that the critical control section is located at a distance of $d$ from the support, no additional coefficient is needed to account for the increased capacity of single span members under uniformly distributed loads.

### 4.7.3 Continuous RC beams under uniformly distributed loads

Only very few tests have been conducted so far to investigate the shear capacity of continuous RC beams under uniformly distributed loads (UDL). In theory, the behavior of continuous systems under UDL should be similar to the behavior of single span beams under point loads, as large bending moments over the column coincide with large shear forces at the face of the column. However, due to the usually high moment gradient in the shear span of the intermediate support, the actual failure section might be located near the point of counterflexure where there are only small bending moments. The behavior thus strongly depends on the actual crack formation in the tests as will be seen later. The two tests series known to the author have been conducted by TUNG & TUE /Tun16b/ and PÉREZ ET AL. /Per12b/. In /Tun16b/, a total of 15 shear tests were carried...
out on four different statical systems. The statical systems are illustrated in Figure 4-30 and included two tests on point loaded cantilevers (Figure 4-30a), two tests on single span beams under UDL (Figure 4-30b), six tests on cantilevers under UDL (Figure 4-30c) and five tests on continuous cantilever beams under UDL (Figure 4-30d).

The tests in /Per12b/ included four tests on point loaded cantilevers (similar to Figure 4-30a) and four tests on cantilevers under UDL (similar to Figure 4-30c), two of which were loaded with a triangular UDL. In addition, tests on tapered beams were carried out in /Per12b/ which are however not considered within this evaluation. In the first step, the tests were evaluated by placing the control section at a distance of \( d \) from the face of support as proposed in chapter 4.7.2. The results of the evaluation are shown in the diagram of Figure 4-31.

On the horizontal axis, the value of the shear slenderness \( \lambda \) in the control section is shown. On the vertical axis, the nominal shear capacity is shown in which all influences except for the factor \( k_\lambda \) have been eliminated. The axis thus represents experimentally measured value for \( k_\lambda \). A value of \( C_{Rm} = 0.79 \) based on the evaluation of point loaded single span beams was used to fit the mean level of tests. As can be seen, most of the tests results are located above the proposed function for \( k_\lambda \) which was derived based on
the evaluation in chapter 4.6 which means that it would be rather conservative for these tests. To get a better understanding about this discrepancy, one must account for the actual crack formation at failure. The crack patterns of the tests SV-6 and SV-7 by /Tun16b/ are shown in Figure 4-32. The critical shear cracks of specimens SV-7.1 and SV-7.2 are located in the span between the intermediate support and the point of inflection, which is where shear failure would be expected on the basis of the assumption of the control section being at a distance of \(d\) from the face of the support. However, in specimens SV-6.1 and SV-6.2 the critical shear crack is located in a region of positive moments on the other side of the point of inflection (Figure 4-32). In these tests, the point of inflection is located closer to the support so that the length of the shear span is insufficient to form a critical shear crack.

The evaluation of the tests was therefore also carried out considering the actual location of the critical shear crack based on the reported crack patterns which was assumed to be the location where the crack intersects with the center line of gravity. The results are shown in Figure 4-33. The alternative control section significantly changes the predicted shear strengths, especially in case of the test series SV-6. In general, the prediction of test results has become more accurate by considering the actual location of failure. Except for the tests with triangular UDL, the tests results lie within a reasonable range of the proposed function for \(k_\lambda\). To predict the failure location a priori, an analysis of the whole member would have to be performed. It is assumed that it is not sufficient to exceed the shear strength in one control section but that a certain critical length is required in which the flexural shear capacity is exceeded. In /Tun16b/ that length is assumed to be equal to the flexural crack spacing \(s_{cr}\).
4.7 Influence of Loading Conditions and Statical Systems

Figure 4-33: Evaluation of tests on continuous beams under UDL by /Tun16b/, /Per12b/ with a control section at an individual distance depending on the location of the critical shear crack

Figure 4-34 shows the evaluation of specimen SV-6.1 in the ultimate limit state. The first diagram illustrates the bending moment $M(x)$ along the beam axis as well the ratio of the dimensionless moments $\mu_{ct} = (M(x)/(f_{ct} b_w d^2))$ and cracking moments $\mu_{ct,lim}$ according to chapter 3.3.2. For values below 1,0, this ratio indicates areas which remain uncracked in bending. The second diagram shows the utilization of the shear capacity along the beam axis. A value above 1,0 indicates that the shear strength is exceeded. A length of $d$ on both sides of the column is considered a D-region in which no flexural shear failure occurs. Areas in red indicate a potential shear failure region. It can be seen that the area on the left side of the point of inflection (0,30 m) is much smaller than on the right side (0,91 m). Although the utilization is lower on the right side, the flexural shear failure is induced here since a necessary critical length is reached first here. For specimen SV-6.1, the theoretical crack spacing $s_{rm}$ amounts to $s_{rm} = d \cdot (1-\xi) = 0,30$ m. The assumption that a shear failure is required within a critical length can plausibly explain the behaviour of specimen SV-6.1.

The same evaluation for specimen SV-7.1 is shown in Figure 4-35. Due to the larger negative moment, the point of inflection moves further away from the column. In the positive moment region there is no critical section whereas in the shear span next to the column a large critical area of 0,76 m length is able to form. The crack pattern in Figure 4-35 confirms the prediction of the member analysis. In conclusion it can be noted that the empirical model is able to predict the shear failure of beams under uniformly distributed loads at interior columns well. The principles derived upon tests on single span point loaded beams can thus be applied on other statical systems. The procedure described in this section will therefore be useful for the necessary future investigations on these type of statical systems but might, however, be too elaborate for consideration in practical design. For the latter purpose, the current cross-sectional approach is sufficient as it is on the safe side.
4.7 Influence of Loading Conditions and Static Systems

Figure 4-34: Evaluation of the shear failure region for member SV-6.1 /Tun16b/

Figure 4-35: Evaluation of the shear failure region for member SV-7.1 /Tun16b/
4.8 Influence of the Cross-Section Geometry

The application of the equation for flexural shear failure in Eurocode 2 is limited to rectangular cross-sections. However, experimental and theoretical studies show that the flanges of a T-beams considerably increase the shear capacity of members. Figure 4-36 shows the results of an experimental study by PLACAS & REGAN /Pla71/ in which T-beams with a varying ratio of flange width and web width were tested. As can be seen, the shear capacity of the original rectangular beam increases by about 25% due to the different cross-section geometry. Also, the shear capacity does not increase significantly if the flange height is constant and only the flange width is increased (Figure 4-36). On the other side, the shear capacity increases if the height of the flange is increased. This leads to the conclusion that indeed there exists a considerable influence of the cross-section geometry on the shear strength, which is however limited to a certain effective shear width $b_v$ above which the beneficiary effect becomes negligible. Most researchers agree that this width is a function of the height of the flange $h_f$. GÖRTZ /Gör04/ carried out numerical investigations on the size of the effective width and came to the conclusion, that 30% of the flange height on each side could be added to the web width. The effective width $b_v$ is of course limited by the total width of the cross-section $b$. Additional work on this topic was recently published by CLADERA ET AL. /Cla15/. Based on nonlinear Finite Element analyses with a smeared-crack model and rotating cracks, a value for the effective shear width $b_v$ according to Eq. (4-40) was proposed.

$$b_v = b_w + 2h_f \leq b$$  \hspace{1cm} (4-40)

In this thesis, the approach of the effective shear width according to Eq. (4-40) will be applied on the basis of previous databank evaluations. If the neutral axis is located within the flange (i.e. $x_0 \leq h_f$), the web width $b_w$ of a rectangular member can be simply replaced by the effective shear width $b_v$ according to Eq. (4-40). The depth of the compression zone then has to be determined considering the effective flexural bending width of the
4.8 Influence of the Cross-Section Geometry

A T-beam can then be treated as a rectangular beam. If the depth of the compression zone exceeds the height of the flanges (i.e. \(x_0 > h_f\)), the depth of the compression zone has to be determined with respect to the varying web width. An analytical solution for RC T-beams was given in /Cla15/ according to Eq. (4-41).

\[
\xi = \frac{x_0}{d} = \left[ \delta (\beta - 1) + \beta \alpha_\rho \right] \sqrt{1 + \left( \frac{\delta^2 (\beta - 1) + 2 \beta \alpha_\rho}{\delta (\beta - 1) + \beta \alpha_\rho} \right)^{-2} - 1}
\]  

(4-41)

where
\[
\beta = b / b_w \\
\delta = h_f / d
\]

For prestressed concrete beams, the influence of axial forces should be accounted for to determine the depth of the compression zone of T-beams. The equilibrium of forces results in a cubic function according to Eq. (4-42). Unfortunately, it is not possible to derive a simplified closed form solution for PC beams as Eq. (4-42) consists of four independent variables. For design purposes either a solution to Eq. (4-42) must be found by computational methods or the compression zone depth of the cross-section has to be determined with a simplified average value for the width \(b\) neglecting the variable cross-section width.

\[
\xi^3 + 3 \left( \frac{1}{\eta} - 1 \right) \xi^2 - 6 \left[ \delta \left( \frac{1}{2} \delta - 1 + 2 \beta \left( 1 - \frac{1}{2} \delta \right) \right) - \frac{1}{\eta} (\beta \alpha_\rho + \delta - \beta \delta) \right] \xi \\
-6 \delta^2 \left( \frac{1}{2} - \frac{1}{3} \delta + \beta \left( \frac{2}{3} \delta - 1 \right) \right) - \frac{1}{\eta} (3 \beta \delta^2 + 3 \delta^2 - 6 \beta \alpha_\rho) = 0
\]

(4-42)

An equivalent effective shear width \(b_{v,\text{eff}}\) can then be determined by integrating the shear stresses in the compression zone over the different cross-section widths, which was done in /Cla15/. However, /Cla15/ uses a shear stress distribution in the compression zone that differs from the assumptions made in chapter 4.2. The value of \(b_{v,\text{eff}}\) should therefore be derived based on the assumptions made in this chapter. The shear stress distribution is defined according to Eq. (4-43). According to Figure 4-1, the shear stresses are zero in the top fiber of the cross-section \((y = 0)\) and reach their maximum value \(\tau_0\) in the neutral axis \((y = x_0)\).

\[
\tau(y) = \tau_0 \left( 2 \frac{y}{x_0} - \frac{y^2}{x_0^2} \right)
\]

(4-43)

The equivalent effective width \(b_{v,\text{eff}}\) is then defined according to Eq. (4-44) as the mean of the integral of shear stresses over the compression zone.
4.8 Influence of the Cross-Section Geometry

\[ \frac{2}{3} \tau_0 x_0 b_{v,\text{eff}} = b_f \int_0^{x_0} \tau(y) b_w dy + \int_{b_f}^{x_0} \tau(y) b_w dy \]  

(4-44)

Solving the integral in Eq. (4-44) for \( b_{v,\text{eff}} \) results in Eq. (4-45).

\[ b_{v,\text{eff}} = b_w \left[ 1 + \frac{3}{2} \left( \frac{h_f}{x_0} \right)^2 - \frac{1}{3} \left( \frac{h_f}{x_0} \right)^3 \right] \left( \frac{b_v}{b_w} - 1 \right) \]  

(4-45)

Eq. (4-45) can then be simplified by using the approximation according to Eq. (4-46).

\[ \frac{3}{2} \left( \frac{h_f}{x_0} \right)^2 - \frac{1}{3} \left( \frac{h_f}{x_0} \right)^3 \approx \left( \frac{h_f}{x_0} \right)^{1.7} \]  

(4-46)

As shown in Figure 4-37 this approximation is a very precise representation of the original equation. In /Cla16/, CLADERA ET. AL also carried out a very similar simplification of their approach for the equivalent effective width \( b_{v,\text{eff}} \) from /Cla15/.

![Figure 4-37: Approximation of parameters in Eq. (4-45)](image)

The final simplified equations for the equivalent effective width \( b_{v,\text{eff}} \) with respect to the position of the neutral axis are given by Eqs. (4-47) and (4-48).

\[ b_{v,\text{eff}} = b_v = b_w + 2h_f \leq b \quad \text{and} \quad x_0 \leq h_f \]  

(4-47)

\[ b_{v,\text{eff}} = b_v + (b_v - b_w) \left( \frac{h_f}{x_0} \right)^{1.7} \quad \text{and} \quad x_0 > h_f \]  

(4-48)
4.9 Compendium of Equations

The equations that were derived within this chapter are summarized briefly within this chapter. The flexural shear strength of RC and PC members without shear reinforcement can be determined by Eq. (4-49).

\[ V_{Rd,c} = C_{Rd,c} k_d k_\lambda \xi f_{ck} b_{v,eff} d \]  

(4-49)

where

\[ C_{Rd,c} = C_{Rm,c} / \gamma_C \]

\[ C_{Rm,c} = 0.766 \text{ (coefficient to fit the mean value of test loads and theoretical loads in a databank evaluation)} \]

\[ \gamma_C = 1.8 \text{ (cf. chapter 6.3.3), partial safety factor for uncertainties due to basic variables and model} \]

\[ k_d = \frac{1}{\sqrt{1+d/d_0}} \]

\[ d_0 = 200 \text{ mm} \]

\[ k_\lambda = 1 + 0.8 \exp(-\lambda/2.5) \]

\[ \lambda = M_{Ed} / (V_{Ed} d); M_{Ed} \text{ and } V_{Ed} \text{ in the considered control section} \]

\[ \xi = x_0 / d; \text{ relative depth of the compression zone} \]

\[ f_{ck} \text{ characteristic value of the concrete compressive strength, } \sqrt{f_{ck}} \leq 8 \text{ MPa} \]

\[ b_{v,eff} \text{ equivalent effective shear width} \]

\[ d \text{ effective depth of the cross-section} \]

For rectangular cross-sections in compression or tension, the relative depth of the compression zone \( \xi \) can approximatively be determined by Eqs. (4-50) to (4-52).

\[ \xi = (0.86 + 0.14\eta) \cdot (\alpha_e \rho)^{0.38-0.16\eta^2} \text{ and } 0 \leq \eta \leq 1.0 \]  

(4-50)

\[ \xi = (3-2\eta) \cdot (\alpha_e \rho)^{0.88\eta-0.66} + 3(1-1/\eta) \text{ and } 1.0 \leq \eta \leq 1.5 \]  

(4-51)

\[ \xi = 0.86 \cdot (\alpha_e \rho)^{0.38} \cdot \frac{1}{1-0.2\eta} \text{ and } \eta < 0 \]  

(4-52)

where

\[ \rho_l = A_{sl} / (b_w d) \]

\[ \alpha_e = E_s / E_{cm} \]

\[ \eta = \frac{N_{Ed} d}{M_{Eds}} \leq 1.5 \text{ (considered positive for compression)} \]

\[ N_{Ed} \text{ axial force in the center of gravity including effects from prestressing} \]
4.10 Summary

For rectangular beams, the equivalent effective shear width $b_{v,\text{eff}}$ is equal to the width of the web $b_w$. For I- and T-shaped beams, $b_{v,\text{eff}}$ can be determined according to Eqs. (4-53) and (4-54).

\[
b_{v,\text{eff}} = b_w = b_w + 2h_f \leq b \text{ and } x_0 \leq h_f \quad (4-53)
\]

\[
b_{v,\text{eff}} = b_w + (b_v - b_w) \left( \frac{h_f}{x_0} \right)^{1.7} \text{ and } x_0 > h_f \quad (4-54)
\]

4.10 Summary

In this chapter, a semi-empirical shear strength model for the prediction of the flexural shear strength of RC and PC beams was developed. The basic assumption for the model was that flexural cracks lose most of their capacity to transfer shear stresses across the crack surfaces in the ultimate limit state so shear stresses are mainly carried by the compression zone. Therefore, the first task was to derive equations for determining the depth of the compression zone with respect to axial forces and moments. The influence factor for the size effect was adopted from BAZANT and validated by shear tests from the ACI-DAFStb databank. The influence factor for the shear slenderness was derived empirically from the evaluation of point loaded single span beams from selected test series.

The model was then validated on test series performed on other statical systems like single span beams, cantilevers and continuous beams under UDL. It was shown that the semi-empirical model was also able to predict the behavior of these systems very precisely. The model can also be used to evaluate the behavior of the member as a whole to predict the location of failure if it is assumed that a critical length with exceeding shear forces along the beam axis is needed to induce flexural shear failure. By this, additional reserves in continuous members can be activated.

Finally, the influence of the cross-section geometry on the shear capacity was accounted for. An equivalent effective shear width based on the works by CLADERA ET AL. was derived that accounts for the beneficiary effect of flanges in T- or I-beams on the shear capacity.
5 Shear Strength of Beams with Shear Reinforcement

5.1 General

In this chapter, a model for predicting the shear strength of members with shear reinforcement is developed. The aim is to obtain a model with a comprehensive background, which is also able to provide a smooth transition between the behavior of members without transverse reinforcement to members with transverse reinforcement and that accounts for all relevant influence factors on the shear capacity. In chapter 5.2, the theoretical background of the proposed method will be explained first. The main idea is to have a flexural shear model with a stirrup contribution for members with a low amount of shear reinforcement and a truss model with a variable strut inclination for members with a larger amount of shear reinforcement. In chapter 5.3, the concrete contribution will be discussed. The contribution of the stirrups is dealt with in chapter 5.4 where the main task is to determine the shear crack angles for the truss model. In chapter 5.5, a criterion for a consistent transition between a flexural shear model with a stirrup contribution and a plasticity model will be derived. The capacity of members according to a variable strut inclination model which includes the maximum capacity of the struts is treated in chapter 5.6. Chapter 5.7 will deal with the question of the required minimum shear reinforcement ratio. Calculation procedures for the design of structures and the assessment of existing structures are given in chapter 5.8. The results of this chapter will be summarized in chapter 5.9.

5.2 Theoretical Background

The history and the development of truss models with a concrete contribution has already been discussed in chapter 2.5.3. Adding a concrete contribution which is related to the shear capacity of beams without shear reinforcement has been mostly seen as a pragmatic approach to be able to reproduce the test results of beams with a low amount of shear reinforcement. Truss models with a concrete contribution have shown to be more accurate for these type of members than models with a variable strut inclination angle (e.g. /Hed78/, /Gör04/, /Heg06/, /Cla14/, /Cla15/, /Her16a/). For this reason, a truss model with concrete contribution was implemented in the Model Code 1978 /MC78/ and Model Code 2010, Level III /CEB10/. For different reasons, lower bound models cannot work for members that do not possess the required minimum shear reinforcement ratio. Apart from theoretical problems like strain compatibilities there are also phenomenological issues of the shear failure mode (as discussed in /Her16c/, /Her16d/). If stress field models were also applicable to these type of members, the following statement would namely have to be correct:

- The compression zone depth along the beam axis remains constant so that the chords in the truss or stress field model remain parallel.
• The rupture of stirrups or the failure of the strut is preceded by an evenly distributed shear crack pattern with a constant crack spacing.

• The shear cracks run straight at a constant angle between compression and tension chord, there is no rotation of the shear crack tip.

Experimental investigations have shown that none of these assumptions are correct for members without shear reinforcement, as stated in /Her16c/, /Her16d/. Tests on prestressed continuous beams have shown, that the influence of the variable compression zone depth causes a significant contribution from inclined flexural compressive forces that should be accounted for /Her13/, /Her15/, /Heg06/, /Mau13/, /Mau14/. This is also the case for beams without prestressing /Gle16/. It can also be seen that beams with a low amount of shear reinforcement tend to fail due to single concentrated cracks instead of an even crack pattern. These cracks are also often indistinguishable from flexural shear cracks with rotating crack tips /Fro00/, /Her15/, /Res16/ so that the failure mechanism can be rather characterized as a flexural shear failure with a capacity of $V_c$ with an additional contribution by the stirrups $V_s$ as indicated by Eq. (5-1).

$$V_u = V_c + V_s$$ (5-1)

From common sense, it is clear that there must be a physical transition of the structural behavior of members without and with small amounts of shear reinforcement. This conclusion is also in agreement with the behavior predicted by the upper bound theory according to plasticity theory as shown in chapter 2.5.2. To avoid irritations concerning models with concrete contribution and plasticity models it might be helpful to just think of the truss models with concrete contribution from a different perspective. They can rather be thought of as flexural shear strength models for beams with an additional stirrup contribution, avoiding the term “truss” in this respect. For larger shear reinforcement ratios, there is a transition to actual truss models with a variable strut inclination angle.

If enough transversal reinforcement is provided, the member possesses a sufficient capability for internal redistribution of forces and actual rotation of shear crack angles since the shear crack pattern is evenly distributed with small crack widths. In the next chapter, an appropriate approach for the concrete contribution will be selected.

5.3 Concrete Contribution

As stated in chapter 5.2, the concrete contribution $V_c$ should reflect the capacity of members without shear reinforcement. Figure 5-1a shows the main shear transfer actions in members without shear reinforcement. For these members, an equation based on a mechanical cross-section analysis was developed in chapter 3.3. Once the beam reaches the flexural shear capacity $V_c$, a horizontal crack branch forms, leading to failure in beams without stirrups. The critical flexural shear crack can then be illustrated approximatively with two crack branches. If stirrups are added to the beam as shown in Figure 5-1b,
additional vertical tensile forces can be transferred in this state, resulting in a stirrup contribution $V_{sy}$. However, since the first crack branch of the flexural crack still has a very steep inclination, the effect of the stirrups on the flexural crack width is small. The concrete contribution $V_c$ can thus assumed to be very similar to a beam without stirrups. For larger shear reinforcement ratios, the two distinguishable crack branches disappear and the shear cracks are inclined at a constant angle. However, the basic shear transfer actions of the concrete contribution (compression zone, aggregate interlock, crack processing zone, dowel action) remain the same, although there now is some interaction between $V_c$ and the stirrup reinforcement (which was addressed in /Mar15/).

To determine $V_c$ by a flexural shear capacity model, any approach with sufficient agreement with shear tests on of beams without shear reinforcement is appropriate. In /Heg06/, /Her16a/ and /Her16d/, the flexural shear formula of EC2 (or Model Code 1990) was used as the concrete contribution leading to very convincing results. Within this thesis, the Simplified CCWM from chapter 3.5 will be used as a concrete contribution $V_c$. The main reason is that this model was derived based on the assumption that the shear transfer by aggregate interlock is crucial for the flexural shear capacity whereas the semi-empirical model from chapter 4 was based on a post-aggregate interlock state in which the compression zone is the main shear component. While both approaches might be admissible within their framework, it is clear that aggregate interlock becomes more relevant if transverse reinforcement reduces the crack widths. The SCCWM is therefore more adequate as a concrete contribution as it is based on the capacity of aggregate interlock. Another point is that in the SCCWM there is an additive contribution for the influence of prestressing $V_p$. For prestressed concrete members this results in a higher transparency as the different actions from aggregate interlock, prestressing and stirrups can easily be identified. The concrete contribution of a prestressed beam $V_{Rd,cp}$ can thus be calculated according to Eq. (5-2), with $V_{Rd,c}$ and $V_{pd}$ according to Eqs. (5-3) and (5-4), respectively.

![Comparison of shear transfer actions in ULS for a) flexural shear failure; b) flexural shear failure with stirrup contribution](image-url)
5.4 Stirrup Contribution $V_s$

5.4.1 General

The stirrup contribution $V_s$ results from inclined cracks that activate the vertical reinforcement. The main parameter for the stirrup contribution is the horizontal length over which the stirrups are activated in the ultimate limit state. This length can be calculated from the shear crack angle $\beta_r$. For a flexural shear crack with two branches, this constant crack angle $\beta_r$ is more or less a fictitious value that represents the mean of the two crack branches (Figure 5-2a). For straight cracks, the theoretical angle $\beta_r$ is equal to the actual shear crack angle (Figure 5-2b). This chapter presents two methods to estimate the value for $\beta_r$ for design or assessment purposes.

\[
V_{rd,\alpha p} = V_{rd,\alpha} + V_{rd} \leq \frac{0.42}{\gamma_C} \sqrt{f_{ck} b_w z}
\]

\[
V_{rd,c} = \frac{C_{fmc} E_s \rho_t}{\gamma_C} \left( \frac{1}{\lambda d} \left( \frac{d_w + 16}{w_c} \right) f_{ck} \right)^{1/3} \leq \frac{0.42}{\gamma_C} \sqrt{f_{ck} b_w z}
\]

\[
V_{rd} = \frac{\alpha_p}{\gamma_C} N_{vi} \frac{z}{\lambda d} \left( 1 - \frac{y_s - y_p}{z} \right)
\]

\[
\alpha_p = \frac{-1,0}{\sqrt{1 + \frac{17}{1 + \frac{E_s \rho_t}{f_{ck}}} \frac{1}{\lambda d}}}
\]

Figure 5-2: Stirrup contribution in concrete members governed by flexural shear failure

5.4.2 Definition of the shear crack angle $\beta_r$ for design

The main parameter for determining the contribution of the stirrups is the actual shear crack angle $\beta_r$ in a member. An approximate equation for $\beta_r$ can be derived on the basis of the principle tensile stress formula if the fictitious shear cracks are assumed to be approximately perpendicular to the principle tensile stresses /Kup89/, /Rei01/. The principle tensile stresses in an uncracked member can be calculated according to Eq. (5-6).
The angle of the principle tensile stresses in the center of gravity of a cross section can be calculated according to Eq. (5-7).

\[ \tan \phi_1 = \frac{\tau_{zx}}{\sigma_s - \sigma_2} = \frac{\tau_{zx}}{\sigma_1} \]  

(5-7)

Given that the shear crack angle \( \beta_r \) is perpendicular to the principle tensile stress angle \( \phi_1 \) (i.e. \( \phi_1 = \beta_r + \pi/2 \)), Eq. (5-7) can be transformed into Eq. (5-8).

\[ \tan^2 \phi_1 = (-\cot \beta_r)^2 = \left( \frac{\tau_{zx}}{\sigma_1} \right)^2 = \frac{\sigma_1^2 - \sigma_1 \cdot \sigma_x}{\sigma_1^2} \]  

(5-8)

where

\[ \tau_{zx}^2 = \sigma_1^2 - \sigma_1 \cdot \sigma_x \]

The shear crack angle \( \beta_r \) can thus be calculated according to Eq. (5-9).

\[ \cot \beta_r = \sqrt{1 - \sigma_c / f_{ctm}} \]  

(5-9)

For design purposes, Eq. (5-9) can also be approximated by linear equations, where Eq. (5-10) is a linearization for members in compression and Eq. (5-11) is a linearization for members in tension.

\begin{align*}
\cot \beta_r & = 1,2 - 0,2 \cdot \sigma_c / f_{ctm} \text{ and } \sigma_c \leq 0 \\
\cot \beta_r & = 1,2 - 1,2 \cdot \sigma_c / f_{ctm} \text{ and } \sigma_c > 0
\end{align*}  

(5-10)  

(5-11)

According to /Rei01/, the value of \( \cot \beta_r = 1,2 \) instead of 1,0 for \( \sigma_c = 0 \) can be explained due to microcracking in the uncracked state (cf. /Kup79/, /Har87/). The analytical solution for the shear crack angle \( \cot \beta_r \) and its linearization is illustrated in Figure 5-3.
To verify the theoretical approach for calculating the shear crack angle $\beta_r$, measured crack angles from shear tests can be compared to the theoretical approach. In /Gör04/, a databank including over 2000 shear tests on beams was collected. The shear crack angle was documented in a part of this databank so that a total of 114 tests with and without axial forces are shown in Figure 5-4 /Gör04/. Of these tests, 57 have an I-shaped cross-section and another 57 have either a Rectangular or a T-shaped cross-section.

Figure 5-4 shows that especially I-shaped beams often exhibit flatter crack angles than predicted. Additionally, it seems that the crack angles for members in tension do not behave according to Eq. (5-11). Instead, Eq. (5-10) seems to be applicable for members in compression as well as for member in tension, which is also reflected within the truss model with crack friction that was included in the German code DIN 1045-1. Here, the equation is based on a linearization according to Eq. (5-12). However, instead of the concrete tensile strength, the degree of prestressing was related to the design value of the concrete compressive strength, based on a concrete strength C30/37 (with $f_{ctm} = 2.9$ MPa, $f_{ck} = 30$ MPa, $f_{cd} = f_{ck} / 1.5 = 20$ MPa according to DIN 1045-1 (12/98), thus $f_{cd} / f_{ctm} = 6.9$). Therefore, Eq. (5-10) has been modified into Eq. (5-12) /Rei01/.

$$\cot \beta_r = 1.2 - 0.2 \sigma_{cp} / f_{ctm} = 1.2 - 0.2 \cdot 6.9 \sigma_{cp} / f_{cd} = 1.2 - 1.4 \sigma_{cp} / f_{cd}$$  \hspace{1cm} (5-12)

It should be noted, that in a later version of DIN 1045-1 /DIN01/, the design value for the concrete compressive strength was calculated with an additional coefficient $\alpha_c$ for long-term loading with $\alpha_c = 0.85$. This affects the ratio of $f_{cd} / f_{ctm}$ so that Eq. (5-12) should have been adjusted according to Eq. (5-13).

$$\cot \beta_r = 1.2 - 0.85 \cdot 1.4 \sigma_{cp} / f_{cd} = 1.2 - 1.2 \sigma_{cp} / f_{cd}$$  \hspace{1cm} (5-13)

The reason why this was not accounted for is unknown. But it shows, that the shear crack angle should not be related to design values of material properties as these values are subject to change for different values of $\alpha_c$ and $\gamma_c$. By defining the shear crack angle...
according to mean or characteristic values of material properties, irritations like these can be avoided. Eq. (5-12) only considers effects from axial forces, as this is the only parameter that would affect the shear crack angle according to the principle tensile stress criterion. GÖRTZ/Gör04/ additionally investigated the influence of the geometry and the shear reinforcement ratio on the shear crack angle $\beta$. Based on the evaluation of the experimentally measured crack angles, GÖRTZ proposed a formulation for the shear crack angles that includes the mechanical reinforcement ratio $\omega_{w,ct}$ of a member according to Eq. (5-14)

$$\cot \beta_r = 1,0 + \frac{0,15}{\omega_{w,ct}} - 0,18 \sigma_{cp} / f_{ctm} \leq 2,15$$ (5-14)

where

$$\omega_{w,ct} = \rho_{w} f_{yk} / f_{ctm} [-]$$

The formula by GÖRTZ according Eq. (5-14) was modified in the process of developing a refined shear model for recalculating existing bridge structures /Heg14a/, /Heg14b/. The modification was included in an update in the German Provision for the recalculation of existing bridges in 2015. In the process, Eq. (5-14) was adjusted to resemble Eq. (5-12) for reasons of continuity in codes. The modified formula is given in Eq. (5-15).

$$\cot \beta_r = 1,2 + \frac{f_{cd}}{70 \rho_{w} f_{yd}} - 1,4 \sigma_{cp} / f_{cd} \leq 2,25$$ (5-15)

For the uniform model that is developed within this thesis, a formulation based on characteristic values according to Eq. (5-16) will be employed to avoid future misunderstandings in the design process.
5.4 Stirrup Contribution Vs

\[ \cot \beta_r = 1,2 + \frac{f_{ck}}{150 \rho_w f_{sk}} - 2,4 \sigma_{cp} / f_{ck} \leq 2,25 \]  
(5-16)

For a researcher, the utilization of mean values like the uniaxial compressive strength of concrete \( f_{1c} \) would be more reasonable than using characteristic values, since the formula for \( \cot \beta_r \) is supposed to relate to the expected mean values of the shear crack angles. However, the designer usually operates with characteristic values of concrete strengths and would then be required to calculate mean values based on the concrete strength class that is assumed. For ease of use, the formulation might therefore as well be based on characteristic values.

For design purposes, the influence of the shear reinforcement ratio on the crack angle can be omitted, so that a simplified formula according to Eq. (5-17) is obtained that is similar to the existing equation for the crack angle in the German version of EC2.

\[ \cot \beta_r = 1,2 - 2,4 \sigma_{cp} / f_{ck} \leq 2,25 \]  
(5-17)

![Diagram](image)

**Figure 5-6:**  
(a) Relationship between concrete compressive and tensile strength according to EC2;  
(b) Comparison of measured and calculated shear crack angles according to Eq. (5-16)

### 5.4.3 Definition of the shear crack angle \( \beta_r \) for assessment

One question concerning the assessment of existing structures is how the shear crack angle \( \beta_r \) can be chosen in a more rational way that includes all relevant influence parameters. In case of the assessment of existing structures, information about the given shear reinforcement \( \rho_w \) and the longitudinal reinforcement \( \rho_l \) of the structure should be available. However, according to the direction of principal tensile stresses, the shear crack angle should be equal to 45° for RC beams regardless of the shear reinforcement ratio or the shear slenderness. However, GÖRTZ /Gör04/ had already shown by the evaluation of measures shear crack angles from beam tests that the angle also depends on the mechanical shear reinforcement ratio \( \omega_w \). These findings have also been validated.
by databank evaluations using Artificial Neural Networks, in which the resulting formula for the crack angle is subject to the shear reinforcement ratio and the ratio of shear slenderness and longitudinal reinforcement ratio /Cla14/. The reason for the varying shear crack angles in RC beams can be most likely attributed to the influence of the stiffness of diagonal struts and vertical ties. In the microcracking stage, the beams will tend to minimize their internal strain energy until yielding. The beams will thus prefer a state in which the strain energy reaches a minimum /Kup64/. The strain energy of a truss member can be calculated by Eq. (5-18).

\[ W_v = \frac{1}{2} \sum F_i \cdot e_i \cdot l_i = \frac{1}{2} \sum \frac{F_i^2}{E_i A_i} \cdot l_i \]  

(5-18)

Within a truss, the only way to minimize the strain energy is by adjusting the angle of the residual strut. If it is assumed that the actual or fictitious shear crack angle \( \beta_r \) is parallel to the residual strut, the minimum of the strain energy can be determined according to Eq. (5-19), where the first derivative of the strain energy with respect to \( \beta_r \) is equal to zero.

\[ \frac{dW_v}{d\beta_r} = 0 \quad \text{and} \quad \frac{d^2W_v}{d\beta_r^2} > 0 \]  

(5-19)

The free-body diagram of the internal forces of the hypothetic truss model is shown in Figure 5-7. The normal force \( N_{Ed} \) and the additional horizontal force from shear \( V_{Ed} \cdot \cot \beta_r \) are distributed equally to the top and bottom chord. For beams with a flexural shear type crack pattern with two branches, it is assumed that the fictitious strut is enabled by shear stresses in the crack processing zone and aggregate interlock.

![Free-body diagram of internal forces](image)

Figure 5-7: Free-body diagram of internal forces

The forces in the bottom chord and the diagonal and vertical struts are given in Eqs. (5-20) to (5-22). The strains in the compression chord are assumed to be negligible compared to the other components, which is also in line with the assumptions made in Model Code 2010 /CEB10/.

\[ F_{\text{bottom, chord}} = \frac{1}{2} N_{Ed} + \frac{M_{Ed}}{z} + \frac{1}{2} V_{Ed} \cot \beta_r \]  

(5-20)
5.4 Stirrup Contribution Vs

\[ F_{\text{diagonal}} = \frac{V}{\sin \beta_r} \]  
\[ F_{\text{vertical}} = V \]  
\( \text{(5-21)} \)

Using Eq. (5-18), the strain energy of the truss elements can be determined according to Eqs. (5-23) to (5-25).

\[ W_{\text{bottom, chord}} = \left( \frac{1}{2} N_{Ed} + \frac{M_{Ed}}{z} + \frac{1}{2} V \cot \beta_r \right)^2 \cdot \frac{2E_{s,l} \cdot A_{sl}}{z \cdot \cot \beta_r} \]  
\( \text{(5-23)} \)

\[ W_{\text{strut}} = \left( \frac{V}{\sin \beta_r} \right)^2 \cdot \frac{z}{2E_s b_w \sin \beta_r \cot \beta_r \sin \beta_r} \]  
\( \text{(5-24)} \)

\[ W_{\text{tie}} = \frac{V^2 z}{2E_{s,w} \rho_w b_w \cot \beta_r} \]  
\( \text{(5-25)} \)

For the moduli of elasticity of the vertical tie \( E_{s,w} \) and the bottom stringer \( E_{s,l} \), the influence of tension stiffening effects is neglected. The first derivative of the strain energy of the truss components is given by Eqs. (5-26) to (5-28).

\[ \frac{dW_{\text{bottom, chord}}}{d\theta_r} = \frac{\left( \frac{1}{2} N + \frac{M}{z} + \frac{1}{2} V \cot \beta_r \right) \left[ V \cot \beta_r + \left( \frac{1}{2} N + \frac{M}{z} + \frac{1}{2} V \cot \beta_r \right) \right] z}{2E_{s,l} \cdot A_{sl} \left( -\sin^2 \beta_r \right)} \]  
\( \text{(5-26)} \)

\[ \frac{dW_{\text{strut}}}{d\theta_r} = \frac{V^2}{2E_s b_w} \left( \frac{1}{\sin^2 \beta_r \cos^2 \beta_r} - \frac{3}{\sin^4 \beta_r} \right) \]  
\( \text{(5-27)} \)

\[ \frac{dW_{\text{tie}}}{d\theta_r} = \frac{V^2 z}{2E_{s,w} \rho_w b_w \cos^2 \beta_r} \]  
\( \text{(5-28)} \)

The minimum strain energy can be found if the sum of the first derivatives of the strain energy is equal to zero (Eq. (5-29)).

\[ \frac{dW_{\text{bottom, chord}}}{d\beta_r} + \frac{dW_{\text{strut}}}{d\beta_r} + \frac{dW_{\text{tie}}}{d\beta_r} = 0 \]  
\( \text{(5-29)} \)

After some transformations, Eq. (5-30) is obtained.

\[ \left( \frac{1}{2} N + \frac{M}{Vz} + \frac{1}{2} V \cot \beta_r \right) \left[ \cot \beta_r + \left( \frac{1}{2} N + \frac{M}{Vz} + \frac{1}{2} V \cot \beta_r \right) \right] \frac{z}{d} + \frac{2E_{s,l}}{E_s} \cdot \rho_l \left( -\sin^2 \beta_r \right) + \frac{1}{2} \left( \frac{1}{\sin^2 \beta_r \cos^2 \beta_r} - \frac{3}{\sin^4 \beta_r} \right) + \frac{1}{2} \frac{E_{s,w}}{E_s} \cdot \rho_w \cos^2 \beta_r = 0 \]  
\( \text{(5-30)} \)

For further simplification, the variables according to Eqs. (5-31) to (5-33) are introduced.
\[ \delta = \frac{1}{2} \frac{N}{V} + \frac{M}{V_2} + \frac{1}{2} \cot \beta_r \]  
\[ \alpha_{e,l} = E_{s,l} / E_c \]  
\[ \alpha_{e,w} = E_{s,w} / E_c \]  

Eq. (5-30) can then be written as Eq. (5-34).

\[ \delta \left[ \cot \beta_r + \delta \frac{z}{d} \right] + \frac{1}{2} \left( \frac{1}{\sin^2 \beta_r} - \frac{3}{\sin^2 \beta_r} \right) + \frac{1}{2} \frac{\alpha_{e,w} \rho_w \cos^2 \beta_r}{\alpha_{e,l} \rho_l} = 0 \]  

After further transformations, Eq. (5-35) is obtained. The equation can be solved for the shear crack angle \( \beta_r \) by a numerical solver.

\[ \delta \cot^3 \beta_r + \delta^2 \cot \beta_r + \frac{\alpha_{e,l} \rho_l}{z/d} \left( 3 \cot^2 \beta_r - 1 \right) + \frac{1}{z/d} \frac{\alpha_{e,w} \rho_w}{\alpha_{e,l} \rho_l} = 0 \]  

Since Eq. (5-35) consists of four independent variables \( (\delta, z/d, \alpha_{e,l} \rho_l, \alpha_{e,w} \rho_w) \), a simple closed form solution similar to the formula for the compression zone depth (chapter 4.4) is not easy to obtain. However, for the sake of practicability a simplified equation should be derived. For this, it is first assumed that \( z/d \) is equal to 0.85. Then, solutions for \( \cot \beta_r \) with respect to the remaining parameters can be obtained. This results in the nonlinear surfaces shown in Figure 5-8. These surfaces have to be created for different values of \( \alpha_{e,l} \rho_l \), resulting in an array of surfaces.

\[ \cot \beta_r = A \left( \frac{\alpha_{e,l} \rho_l}{\alpha_{e,w} \rho_w} \right)^{\theta} \]  

The parameters \( A \) and \( B \) then have to be defined as a function of the remaining parameters \( \delta \) and \( \alpha_{e,l} \rho_l \) to match the array of surfaces. A reasonable approximation is given by Eq. (5-37).
\[
\cot \beta_r = \frac{3.08 \cdot (\alpha_{c,i} \rho_i)^{0.195}}{\delta + 4.54 \cdot (\alpha_{c,i} \rho_i)^{0.448} \left( \frac{\alpha_{c,w} \rho_w}{\alpha_{c,i} \rho_i} \right)^{0.138 \sqrt{0.5 + 0.179}}} 
\]

Eq. (5-37) is of course not a closed form solution as the variable \( \delta \) also includes the shear crack angle \( \beta_r \). However, Eq. (5-37) can be utilized to determine a shear crack angle \( \beta_r \) by simple iteration as oppose to solving the nonlinear problem of Eq. (5-35). A comparison of the resulting shear crack angles according to the two equations is illustrated in Figure 5-9. For this, the theoretical shear crack angles of the 119 shear tests from the ACI-DAfStb databank on PC beams with shear reinforcement were compared. As can be seen, the approximate solution according to Eq. (5-37) yields quite precise results. Without including the six outliers, the mean value of exact vs. approximate solution is 0.95 with a coefficient of variation of 7.1 \%. The outliers are all originated from the same test series /Apa97/ and can be excluded since the test parameters are out of practical range (the shear reinforcement ratio amounts to 10 – 20 times the minimum shear reinforcement, unusually high values for the longitudinal reinforcement, unusually high prestressing).

![Figure 5-9: Comparison of theoretical shear crack angles of 119 tests determined by solving Eq. (5-35) (exact solution) and by using Eq. (5-37) (approximation)](image)

Within the databank of PC beams with shear reinforcement, a total of 25 tests out of 119 are provided with information about measured shear crack angles (not including the excluded tests from /Apa97/). A comparison of the experimental and theoretical shear crack angles according to Eq. (5-37) is shown in Figure 5-10. The correlation is rather good with a coefficient of variation of 14.5 \%. In comparison, the approach by GöRTZ according to Eq. (5-14) yields a slightly higher coefficient of variation of 19.4 \%.
The theoretical values are, however, on the safe side by a factor of 1.3 which might result from the various simplifications that were made in deriving the formula. Therefore, a preliminary empirical correction factor \( k_\beta \) can be introduced according to Eq. (5-38). In addition, an upper and a lower limit for the values of cot \( \beta_r \) should be introduced to avoid unrealistic values. As a lower limit, a value of 0.5 can be chosen, since only few combinations of parameters lead to values below 0.5 according to Figure 5-8. A value of 2.5 seems to be a reasonable choice as an upper limit for cot \( \beta_r \) for design purposes since there were no tests in the databank with angles lower than 21.8\(^\circ\) (although there are some tests in the literature which reportedly exhibit crack angles below 20\(^\circ\), e.g. /Rup13/).

\[
\cot \beta_r = k_\beta \frac{3.08 \cdot (\alpha_{e,\rho \beta})^{0.195}}{\delta + 4.54 \cdot (\alpha_{e,\rho \beta})^{0.448}} \left( \frac{\alpha_{e,\rho \beta}}{\alpha_{e,\rho \omega}} \right)^{0.138 \cdot 0.65 + 0.179} \begin{cases} \leq \cot \beta_{r,\text{max}} \\ \geq \cot \beta_{r,\text{min}} \end{cases}
\]

(5-38)

where

\( k_\beta = 1.3 \)

\( \cot \beta_{r,\text{max}} = 2.5 \)

\( \cot \beta_{r,\text{min}} = 0.5 \)

It should be noted, that the mechanical shear slenderness \( \delta \) is related to a particular shear force \( V \). That shear force \( V \) should be equal to the shear load at which the diagonal crack in the investigated section occurs. For this, two types of shear cracks can be distinguished. The first type of shear crack is a diagonal shear crack and forms independently of flexural cracks. This type of crack mostly occurs in I-shaped beams, box-sections or in regions adjacent the point of zero moment. In this case, the shear crack force \( V_{\text{crack}} \) can be determined by evaluating the the principal tensile stresses along the proposed crack line. The shear crack load \( V_{\text{crack}} \) then is the load at which the principal tensile
stresses in one point exceed the concrete tensile strength $f_{ct}$. As a simplification, the shear crack load can be evaluated in the center of gravity of the cross-section according to Eq. (5-40). The coefficient of 0.8 in Eq. (5-39) accounts for a reduced residual tensile strength of concrete due to longitudinal stresses from bending moments.

$$V_{crack} = 0.8 \frac{l \cdot b_w}{S} \sqrt{f_{ct}^2 - \sigma_{cp} f_{ct}}$$  \hspace{1cm} (5-39)

The second type of shear crack results from a flexural bending crack and can be described by a vertical and a horizontal crack branch. The formation of this shear crack can simply be related to the flexural shear capacity of the beam according to Eq. (5-40).

$$V_{ld,cp} = V_{ld,c} + V_{ld} \leq \frac{0.42}{\gamma_c} \sqrt{f_{ck} b_w z}$$  \hspace{1cm} (5-40)

### 5.5 Consistent Transition to a Truss Model with a Variable Strut Inclination

To acquire a consistent shear model for flexural shear failure with a stirrup contribution and a truss model with a variable strut inclination, a clear criterion for the scope of application for both models should be defined. Otherwise confusing or contradicting results might follow if the models are mixed up. The proposed method in this chapter was already presented in /Her16c/, /Her16d/. For slender members with low amounts of shear reinforcement, a failure of stirrups will occur so that yielding of the transversal reinforcement can be assumed in the ultimate limit state. Members with larger amounts of shear reinforcement will fail due to a combined failure of stirrups and the strut. For beams with very large amounts of shear reinforcement, a failure of the strut without yielding of the reinforcement is also possible. In this case, a larger amount of shear reinforcement would be uneconomic. As described in section 5.2, the shear failure of beams with small amounts of shear reinforcement is characterized as a flexural shear failure with a stirrup contribution according to Eq. (5-41).

$$V_{ld} = V_{ld,c} + \left( A_{sw} / s \right) \cdot f_{ywd} \cdot z \cdot \cot \beta_r$$  \hspace{1cm} (5-41)

To normalize Eq. (5-41) for an illustration within a dimensionless diagram, the dimensionless variables according to Eq. (5-42) are introduced.

$$v_r = v_c + \omega_v \cdot \cot \beta_r$$  \hspace{1cm} (5-42)

The dimensionless variables are related to the capacity of the strut according to Eqs. (5-43) to (5-45).

$$v_r = \frac{V_R}{b_w \cdot z \cdot v \cdot f_c}$$  \hspace{1cm} (5-43)
5.5 Consistent Transition to a Truss Model with a Variable Strut Inclination

\[
\omega_w = \frac{(A_w/s)f_{yw} \cdot z}{b_w \cdot z \cdot \nu \cdot f_c} = \rho_w f_{yw}/\nu \cdot f_c \tag{5-44}
\]

\[
v_c = \frac{V_{rd,c}}{b_w \cdot z \cdot \nu \cdot f_c} \tag{5-45}
\]

where

\[
\nu = \text{effectiveness factor for cracked concrete (cf. chapter 5.6)}
\]

For larger amounts of shear reinforcement, the shear capacity is determined with a truss model which is limited by the capacity of the strut according to Eq. (5-46). Here, \(\cot \theta\) describes the strut angle. The left side of Eq. (5-46) is illustrated as line (1) in Figure 5-11.

\[
v_R = \omega_w \cot \theta \leq \frac{1}{\tan \theta + \cot \theta} \tag{5-46}
\]

If it is assumed that failure of stirrups and struts occur simultaneous, Eq. (5-46) can be solved for the strut angle resulting in Eq. (5-47).

\[
\cot \theta = \sqrt{\frac{1}{\omega_w} - 1} \tag{5-47}
\]

Eq. (5-47) can then be inserted into Eq. (5-46) which yields the equation of the plasticity circle according to Eq. (5-48).

\[
v_R = \sqrt{\omega_w - \omega_w^2} \tag{5-48}
\]

The plasticity circle is illustrated in Figure 5-11. As can be seen, the equation yields very flat angles of the strut inclination for small amounts of shear reinforcement. These angles can usually not be observed within shear tests and are thus unrealistic. The problem is that the strains in the member are not compatible with a constant \(\nu\)-factor. Therefore, an arbitrary limitation of the strut angle is usually introduced (e.g. \(\cot \theta \leq 2.5\), shown as line (1) in Figure 5-11. As shown in Figure 5-11, the capacity of the plasticity circle as well as its limitation according to line (1) yield zero shear capacity for a member without shear reinforcement according to the lower bound solution. In contrast, the flexural shear model with an additional stirrup contribution according to Eq. (5-41) has a capacity of \(v_R = v_c\) for \(\omega_w = 0\), which is illustrated by line (2) in Figure 5-11. This diagram also illustrates that it is not necessary to perform a check of the strut for values of \(v_R\) that lie between point \(a\) and \(b\) in Figure 5-11 since the line is located inside the plasticity circle. The only necessary information is the position of the intersection point \(b\) for the transition from stirrup failure to combined failure of strut and stirrups.
5.5 Consistent Transition to a Truss Model with a Variable Strut Inclination

![Plasticity circle and concrete contribution](image)

**Figure 5-11:** Plasticity circle and concrete contribution

This intersection can be determined by Eq. (5-49).

\[ v_R = \sqrt{\omega_w - \omega_w^2} = \omega_w \cdot \cot \beta_r + v_c \quad (5-49) \]

Solving Eq. (5-49) for \( \omega_w \) results in Eq. (5-50) and reveals this two possible intersection points of line (2) with the plasticity circle.

\[
\omega_w = \frac{1 - 2v_c \cot \beta_r}{2(1 + \cot^2 \beta_r)} \pm \sqrt{\left(\frac{1 - 2v_c \cot \beta_r}{2(1 + \cot^2 \beta_r)}\right)^2 - \frac{v_c^2}{1 + \cot^2 \beta_r}} \\
= 0.5 - v_c \cot \beta_r \pm \sqrt{0.25 - v_c \cot \beta_r - v_c^2} \quad (5-50)
\]

Since only the second point of intersection is of interest, the limit value for the mechanical reinforcement ratio \( \omega_{w,\text{lim}} \) is defined according to Eq. (5-51).

\[
\omega_{w,\text{lim}} = \frac{0.5 - v_c \cot \beta_r + \sqrt{0.25 - v_c \cot \beta_r - v_c^2}}{1 + \cot^2 \beta_r} \quad (5-51)
\]

For values of \( \omega_w \leq \omega_{w,\text{lim}} \), the shear capacity can be calculated according to Eq. (5-41).

For values of \( \omega_w > \omega_{w,\text{lim}} \), the shear capacity can be calculated according to Eqs. (5-52) and (5-53), which represent the most economical solution according to plasticity theory.
5.6 Capacity of the Concrete Strut

$$v_R = \omega_w \cot \theta = \omega_w \sqrt{\frac{1}{\omega_w} - 1} \quad \text{and} \quad \omega_{w,lim} < \omega_w \leq 0.5 \quad (5-52)$$

$$v_R = \frac{1}{2} \quad \text{and} \quad \omega_w > 0.5 \quad (5-53)$$

This procedure presupposes that there is at least one point of intersection of line (2) with the plasticity circle in Figure 5-11. A parameter combination such that there is no intersection point according to line (3) might, however, also be feasible. In this case, the theoretical crack angle $\cot \beta_r$ must be limited so that at least one intersection with the plasticity circle exists. This is the case if line (2) becomes a tangent to the plasticity circle so that the discriminant of Eq. (5-50) becomes zero. The limitation of the shear crack angle $\cot \beta_r$ can thus be defined according to Eq. (5-54).

$$0.25 - v_c \cot \beta_r - v_c^2 = 0 \Rightarrow \cot \beta_r \leq \frac{0.25 - v_c^2}{v_c} \quad (5-54)$$

In summary, it can be noted that the procedure presented in this chapter allows for a clear distinction of the scope of application between the presented models for beams with shear reinforcement, which are a flexural shear model with stirrup contribution and a truss model with a variable strut inclination angle. How this procedure can be applied in practice for the design and the assessment of structures will be discussed in chapter 5.8.

5.6 Capacity of the Concrete Strut

In struts, the transversal tensile strains in the cracked concrete reduce its effective compressive strength. Based on tests by ROBINSON & DEMORIEUX /Rob69/, /Rob70/, a $\nu$-factor has been introduced to account for the effective concrete compressive strength. A large amount of experimental research has been performed on determining the size of the $\nu$-factor with respect to the crack formation /Zil10/. An overview is given in /Roo95/, /Har06/, /Nie11/. In general, a concrete membrane which is uncracked possesses the full compressive strength ($\nu = 1.0$; Figure 5-12a). If cracks form parallel to the direction of the compressive stresses, the effective strength is decreased by about 25% ($\nu \approx 0.75$; Figure 5-12b). If the struts rotate and thus cross the shear cracks, the effective strength is reduced further ($\nu \approx 0.5 – 0.6$; Figure 5-12c).

Based on experimental research, it was determined that the main influence factor on the $\nu$-factor is the concrete compressive strength. According to /Nie11/, Eq. (5-55) was proposed for the first edition of EC2 /EC291/, based on tests at the Technical University of Denmark.

$$\nu = 0.7 - \frac{f_c}{200} \geq 0.5 \quad (5-55)$$
Though the influence of prestressing was mostly not considered, CHEN came to the conclusion that prestressing had a favourable influence on the effective strength of concrete and proposed Eq. (5-56) /Che88/.

\[ v = \left( 0.8 - \frac{f_c}{200} \right) \left( 1 + 2.2 \frac{\sigma_{cp}}{f_c} \right) \leq 1.0 \]  
(5-56)

In the latest edition of EC2 /EC211/, a modified version of Eq. (5-55) was introduced accounting for the influence of prestresses by a combined effective strength factor \( \alpha_{cw} v_1 \) according to Eqs. (5-57) and (5-58).

\[
\begin{align*}
v_1 &= \begin{cases} 
0.6 & \text{for } f_{ck} \leq 60 \\
0.9 - f_{ck}/200 & \geq 0.5 & \text{for } f_{ck} \geq 60
\end{cases} \\
\alpha_{cw} &= \begin{cases} 
(1 + \sigma_{cp}/f_{cd}) & \text{for } 0 < \sigma_{cp} \leq 0.25 f_{cd} \\
1.25 & \text{for } 0.25 f_{cd} < \sigma_{cp} \leq 0.5 f_{cd} \\
2.5(1- \sigma_{cp}/f_{cd}) & \text{for } 0.5 f_{cd} < \sigma_{cp} \leq 1.0 f_{cd}
\end{cases}
\end{align*}
\]  
(5-57)  
(5-58)

In general, these rather simple formulas for the effective strength are only applicable for a certain range of strut angles and shear crack widths as they are mostly derived on the basis of beam tests. More general formulations have been derived on the basis of experimental and theoretical investigation on concrete membranes for example by VECCHIO & COLLINS /Vec82/, /Vec86/, KOLLEGGER & MEHLHORN /Kol90/, SCHÄFER ET AL. /Sch90/, KAUFMANN /Kau98a/ and SCHIESSL /Sch04/, /Sch05/. VECCHIO & COLLINS
5.7 Minimum Shear Reinforcement Ratio

To prevent sudden shear failure of beams that unlike slabs do not have the capability of transversal redistribution of loads, a minimum amount of shear reinforcement is required. The EC2 criterion for the minimum amount of reinforcement is most likely based on preventing a flexural shear type of failure in beams so that the shear reinforcement should be able to cover the flexural shear capacity of a beam. The minimum shear capacity according to EC2 is calculated according to Eq. (5-64).
5.7 Minimum Shear Reinforcement Ratio

\[ V_{fd,c,min} = 0.035 k^{3/2} f_{ck}^{1/2} b_w d \] (5-64)

The mean value of the minimum flexural shear capacity should be equal to the capacity of the stirrups of a truss with a minimum shear reinforcement according to Eq. (5-65). The assumed values for the size effect factor of EC2 are probably \( k = 2 \) and \( z = 0.9d \) for the inner lever arm.

\[ V_{fm,c,min} = 1.5 \cdot 0.035 \cdot 2^{3/2} \sqrt{f_{ck} b_w d} = \rho_{w,min} b_w f_{yk} z \cot \theta \] (5-65)

If \( \cot \theta \) is taken as 2.0, the minimum shear reinforcement required by EC2 is given by Eq. (5-66).

\[ \rho_{w,min} = 0.08 \frac{f_{ck}}{f_{yk}} \] (5-66)

In the German National Annex of EC2, a further distinction of failure modes is made for RC and PC beams based on the works of HEGGER ET AL. /Heg99/, /Heg03/. For RC beams, a value of \( \rho_{w,min} \) is derived according to Eq. (5-67) based on preventing a flexural shear type of failure, similar to the EC2 provisions. The concrete strength is accounted for in form of the mean value of the concrete tensile strength \( f_{ctm} \). The failure of PC beams with thin webs is assumed to be a shear tension type failure in the web. Therefore, the minimum shear capacity is determined in a simplified manner according to Eq. (5-68) /Gör04/.

\[ \rho_{w,min} = 0.16 \frac{f_{cm}}{f_{yk}} \] (5-67)

\[ V_{fm,c} \approx f_{ck,0.05} \cdot b_w \cdot z \Rightarrow \rho_{w,min} = 0.256 \frac{f_{cm}}{f_{yk}} \] (5-68)

For the proposal of this thesis, the required shear reinforcement ratio for beams shall be determined in a similar manner based on the minimum shear capacity of chapter 3.5.5. The minimum shear capacity can be determined by Eq. (5-69) with \( d_{ag} = 16 \text{ mm} \) and \( w_0 = 2 \text{ mm} \). The coefficient in the denominator was changed back to 3.43 according to the original model (cf. chapter 3.5.5).

\[ V_{c,min} = \frac{0.42 f_{ck} b_w z}{1 + 3.43 \left( 1.0 + 1.9/w_0 \right) E_s \sqrt{d_{ag} + 16}} \]

\[ = \frac{0.42 f_{ck} b_w z}{1 + 1.70 f_{yk} d} \] (5-69)
5.7 Minimum Shear Reinforcement Ratio

The stirrups of RC beams should then be able to resist the mean flexural shear capacity so that the minimum shear reinforcement ratio is defined according to Eq. (5-70) (with $\cot \theta = 2.5$).

$$V_{c,\text{min}} = \frac{0.42 \sqrt{f_{ck} b_w z}}{f_{yk} d} = \rho_{w,\text{min}} b_w f_{yk} z \cot \theta$$

$$\Rightarrow \rho_{w,\text{min}} = \frac{0.42 \sqrt{f_{ck}}}{1 + \frac{f_{yk} d}{170000}} \cdot \frac{1}{f_{yk} \cot \theta} = \frac{0.17 \sqrt{f_{ck}}}{f_{yk} + \frac{f_{yk}^2 d}{170000}}$$

(5-70)

For prestressed concrete beams, the beneficiary effect of the axial forces should be accounted for. To achieve simple rules, the minimum shear reinforcement is determined with respect to the maximum flexural shear capacity $V_{c,\text{max}}$. As this value is only reached for high degrees of prestressing, the corresponding strut angle is assumed to be $\cot \theta = 3.0$ so that the minimum shear reinforcement ratio can be determined according to Eq. (5-71).

$$V_{c,\text{max}} = 0.42 \sqrt{f_{ck} b_w z} = \rho_{w,\text{min}} b_w f_{yk} z \cot \theta$$

$$\Rightarrow \rho_{w,\text{min}} = \frac{0.42}{\cot \theta} \sqrt{f_{ck}} = 0.14 \frac{\sqrt{f_{ck}}}{f_{yk}}$$

(5-71)

In Figure 5-13a, the proposed definition of the minimum shear reinforcement ratio is compared to the provisions of EC2 /EC211/ according to Eq. (5-66) and EC2 NA(D) /DIN13/ according to Eqs. (5-67) and (5-68). As can be seen, the minimum shear reinforcement according to Eq. (5-70) is larger for small effective depths but decreases significantly for large structures. This seems quite comprehensive as the nominal shear capacity also decreases for larger structures due to the size effect.

Figure 5-13: Comparison of provisions for the minimum shear reinforcement of a) RC beams and b) PC beams
In Figure 5-13a, the comparison of the minimum shear reinforcement ratios for pre-stressed concrete beams is illustrated. EC2 /EC211/ provides the same definition for PC beams and RC beams while the German version EC2 NA(D) /DIN13/ provides significantly larger values due to the different failure scenario. The proposed Eq. (5-71) provides values that are larger than EC2 but similar to the German version. In conclusion, the derived criteria for the minimum shear reinforcement ratio have shown to provide results within a reasonable range. It should, however, be pointed out that the provisions in this chapter are intended to be simplified rules for determining a minimum shear reinforcement. For assessment, it can also be considered to calculate a precise value of the flexural shear capacity $V_c$ to determine the amount of necessary stirrups to prevent sudden failure.

5.8 Calculation Procedure for Design and Assessment of Structures

5.8.1 General

In this chapter, the derived model will be summarized and edited to be applicable for the design and assessment of structures. For the design of new structures, the design procedure is aimed at the calculation of the required shear reinforcement while for the assessment of an existing structure the procedure is used to determine the actual strength of the structure.

5.8.2 Procedure for the design of new structures

A transversal reinforcement is required in members for which the design shear forces $V_{Ed}$ exceed the shear capacity of a member without shear reinforcement $V_{Rd,c}$ which is given by Eq. (5-72). For the shear capacity of members in compression or tension, refer to chapter 3.5.6. In a simplified manner, the minimum shear capacity according to chapter 3.5.6 can also be taken as the concrete contribution.

$$V_{Rd,c} = C_{Rm,c} \left( \frac{E_s \rho_l}{\gamma_c} \left[ \frac{1}{\lambda} \cdot d \left( 1,0+1,9/W_0 \right) f_{ck} \right]^{1/3} \right)$$

The shear resistance of a member with shear reinforcement may be calculated with Eq. (5-73).

$$V_{Rd,y} = V_{Rd,c} + \frac{A_{sw}}{s} f_{yd} \cdot z \cdot \cot \beta_{f}$$

The required shear reinforcement $A_{sw,req}$ can be determined according to Eq. (5-74).

$$\frac{A_{sw,req}}{s} = \frac{V_{Ed} - V_{Rd,c}}{f_{yd} \cdot z \cdot \cot \beta_{f}}$$
where
\[ \cot \beta_r = 1.2 - 2.4 \sigma_y / f_{ck} \]

It then has to be checked according to Eq. (5-75) if the mechanical shear reinforcement ratio \( \omega_{w,\text{req}} \) exceeds the allowable limit for designing according to the flexural shear model with concrete contribution.

\[
\omega_{w,\text{req}} = \frac{(A_{w,\text{req}}/s) \cdot f_{yd}}{b_n \nu f_{cd}} \leq \omega_{w,\text{lim}} \tag{5-75}
\]

where
\[
\omega_{w,\lim} = \frac{0.5 - v_{Rd,e} \cot \beta_r + \sqrt{0.25 - v_{Rd,e} \cot \beta_r - v_{Rd,e}^2}}{1 + \cot^2 \beta_r}
\]

\[ v_{Rd,e} = \frac{V_{Rd,e}}{b_n z \cdot f_{cd}} \]

\[
\nu = \begin{cases} 
0.6 & \text{for } f_{ck} \leq 60 \\
0.9 - f_{ck} / 200 \geq 0.5 & \text{for } f_{ck} \geq 60 
\end{cases}
\]

If the limit for the mechanical shear reinforcement ratio is exceeded, the shear strength according Eq. (5-73) lies outside the plasticity circle so that the model is not applicable. To ensure a sufficient strut capacity, a check according to Eq. (5-76) should be performed. If the acting shear force \( V_{Ed} \) exceeds the capacity of the struts then the geometry or the concrete strength of the member have to be adjusted first.

\[
V_{Ed} \leq \frac{1}{2} b_n z \nu f_{cd} \tag{5-76}
\]

If the strength of the struts is sufficient, the shear strength of a member can be determined according to Eq. (5-77).

\[
V_{Rd,\text{sy}} = \frac{A_{w}}{s} f_{yd} \cdot z \cdot \cot \theta \tag{5-77}
\]

where
\[
\cot \theta = \sqrt{\frac{b_n \nu f_{cd}}{A_{w} / s \cdot f_{yd}}} - 1
\]

For design purposes, \( V_{Rd,\text{sy}} \) in Eq. (5-77) can be replaced by \( V_{Ed} \) and \( A_{w} \) by \( A_{w,\text{req}} \) so that the equation can be solved for \( A_{w,\text{req}} \) as shown in Eq. (5-78).

\[
\frac{A_{w,\text{req}}}{s} = b_n \nu f_{cd} \left[ 0.5 - \sqrt{0.25 - \left( \frac{V_{Ed}}{b_n \cdot z \cdot \nu \cdot f_{cd}} \right)^2} \right] \quad \text{and} \quad \omega_{w,\lim} < \omega_{w,\text{req}} \leq 0.5 \tag{5-78}
\]
5.8 Calculation Procedure for Design and Assessment of Structures

For structures like slabs and walls which have a sufficient ability for redistribution of internal forces, no minimum shear reinforcement needs to be applied. In beam-like structures, the required minimum amount of shear reinforcement can be determined according to Eq. (5-79) (RC beams) and Eq. (5-80) (PC beams).

\[ \rho_{w,\text{min}} = \frac{0.17 \sqrt{f_{ck}}}{f_{yk} + \frac{f_{ck}^2d}{170000}} \quad \text{and} \quad \sigma_{cp} \geq 0 \]  \hspace{1cm} (5-79)

\[ \rho_{w,\text{min}} = 0.14 \frac{\sqrt{f_{ck}}}{f_{yk}} \quad \text{and} \quad \sigma_{cp} < 0 \]  \hspace{1cm} (5-80)

As shown in this chapter, the developed procedure leads to a step-wise design process that provides a clear solution. Although the required effort to perform a shear check in this model is higher than in the current EC2 /EC211/, it should be considered that most of the times the shear check is performed using data sheets in computers. Here, the presented model provides a clear procedure to follow.

5.8.3 Procedure for the assessment of existing structures

For the assessment of existing structures, the design shear resistance can be determined from the given shear reinforcement in the section. In the first step, it should be checked on the basis of the given shear reinforcement which of the failure modes is governing. For this, the flexural shear capacity \( V_{Rd,c} \) of the member and the shear crack angle \( \cot \beta_r \) have to be determined. The flexural shear capacity can be determined according to Eq. (5-81). For the shear capacity of members in compression or tension, refer to chapter 3.5.6. In a simplified manner, the minimum shear capacity according to chapter 3.5.6 can also be taken as the concrete contribution.

\[ V_{Rd,c} = \frac{C_{hmc}}{\gamma_C} \left( \frac{E_s \rho_l \sqrt{d_{w}}}{(1.0+1.9/w_0)} f_{ck} \right)^{1/3} \leq \frac{0.42}{\gamma_C} \sqrt{f_{ck}} \]  \hspace{1cm} (5-81)

The shear crack angle \( \cot \beta_r \) can either be determined according to the simpler semi-empirical method given by Eq. (5-82) or the refined method according to Eq. (5-83), which requires additional iterations.

\[ \cot \beta_r = 1.2 + \frac{f_{ck}}{150 \rho_w f_{yk}} - 2.4 \sigma_{cp} / f_{ck} \leq 2.25 \]  \hspace{1cm} (5-82)

\[ \cot \beta_r = k_0 \cdot \frac{3.08 \cdot (\alpha_{e,l} \rho_l)^{0.195} \cdot (\alpha_{e,w} \rho_w)^{0.448} \cdot 0.138 \cdot 10^{0.5+0.179}}{\delta + 4.54 \cdot (\alpha_{e,l} \rho_l)^{0.195} \cdot (\alpha_{e,w} \rho_w)^{0.448}} \leq \cot \beta_{r,\text{max}} \geq \cot \beta_{r,\text{min}} \]  \hspace{1cm} (5-83)

where
5.8 Calculation Procedure for Design and Assessment of Structures

\[ k_\theta = 1,3 \]
\[ \cot \beta_{r,\text{max}} = 2,5 \]
\[ \cot \beta_{r,\text{min}} = 0,5 \]

\[ \delta = \frac{1}{2} N_{Ed} \frac{M_{cr}}{V_{cr}} + \frac{1}{2} \cot \beta_r \]

For determining the shear crack loads \( M_{cr} \) and \( V_{cr} \) cf. to chapter 5.4.3.

However, the shear crack angle must be limited according to Eq. (5-84) with respect to the flexural shear capacity \( V_{Rd,c} \).

\[ \cot \beta_r \leq \frac{0,25 - v_{Rd,c}^2}{v_{Rd,c}} \quad (5-84) \]

\[ v_{Rd,c} = \frac{V_{Rd,c}}{b_w z f_{cd}} \leq 0,5 \]

If the given mechanical shear reinforcement ratio \( \omega_w \) according to Eq. (5-85) is smaller than the limit value given by Eq. (5-86), a flexural shear failure with an additional stirrup contribution is governing. If the given shear reinforcement is larger than the limit value, a simultaneous failure of stirrups and strut is to be expected.

\[ \omega_w = \frac{A_{sw}}{s} f_{yd} \frac{f_{yd}}{b_w} \quad (5-85) \]

\[ \omega_{w,\text{lim}} = \frac{0,5 - v_c \cot \beta_r + \sqrt{0,25 - v_c \cot \beta_r - v_c^2}}{1 + \cot^2 \beta_r} \quad (5-86) \]

In the first case, the shear capacity can thus be determined according to Eq. (5-87).

\[ V_{Rd,xy} = V_{Rd,c} + \frac{A_{sw}}{s} f_{yd} \cdot z \cdot \cot \beta_r \quad \text{and} \quad \omega_w \leq \omega_{w,\text{lim}} \quad (5-87) \]

For larger reinforcement ratios, the shear capacity can be calculated according to Eq. (5-88).

\[ V_{Rd} = b_w z f_{cd} \sqrt{\omega_w - \omega_{w,\text{lim}}} \quad \text{and} \quad \omega_{w,\text{lim}} < \omega_w \leq 0,5 \quad (5-88) \]

If the mechanical shear reinforcement ratio \( \omega_w \) exceeds a value of 0,5 there is no further increase in shear capacity so that Eq. (5-89) applies.

\[ V_{Rd} = \frac{1}{2} b_w z f_{cd} \quad \text{and} \quad \omega_w > 0,5 \quad (5-89) \]

The presented procedure might be more elaborate than other existing shear models. However, it will provide a very good level of approximation and can quite easily be
implemented within spreadsheets since the procedure itself is very clear and comprehensive.

5.9 Summary and Conclusion

In this chapter, a model for determining the shear strength of beams with shear reinforcement was developed. The main idea was that the strength models should distinguish two cases. For low amounts of shear reinforcement, the shear capacity of beams is equal to the flexural shear capacity with an additional contribution from the stirrups. For larger amounts of shear reinforcement, a simultaneous failure of struts and stirrups is governing so that the shear capacity can be determined according to a truss model with a variable strut inclination. For this procedure, a clear criterion for the given reinforcement ratio was derived to define the applicable model.

In the following, shear verification procedures for design and assessment purposes were developed. The procedure for the assessment of existing structures features a refined approach for determining the shear crack angle based on the principle of minimum work. The presented model allows for a completely consistent design of members with no, little or large shear reinforcement so that it is suitable for the design as well as for the assessment of structures.
6 Probabilistic Safety Evaluation

6.1 General

In this chapter, the safety level of the developed shear strength models will be evaluated to determine partial safety factors for design. For this, shear tests on beams from shear databanks are evaluated in the first step to obtain information on the model uncertainty. Shear databanks for beams without shear reinforcement are evaluated in chapter 6.2.1 and databanks for beams with shear reinforcement in chapter 6.2.2. In addition, comparative evaluations with models from different code provision are presented. In chapter 6.3, partial safety factors $\gamma_C$ will be derived based on a probabilistic evaluation according to EN 1990, Annex D /DIN12/. The results of this chapter are summarized in chapter 6.4.

6.2 Evaluation of Shear Databanks

6.2.1 Evaluation of tests on beams without shear reinforcement

The evaluation of shear tests on beams without shear reinforcement is summarized in Table 6-1. The tests are divided into three different groups: tests on RC beams without axial forces, tests on prestressed concrete beams and tests on RC beams with axial tension. For each databank, the number of evaluated tests ($n_{\text{Test}}$), the mean ratio of test and theoretical shear capacity (MV) and the corresponding coefficient of variation (COV) are given for the different models. The considered shear models are the Simplified Critical Crack Width Model (SCCWM) according to chapter 3.5.6, the semi-empirical approach of chapter 4.9, the flexural shear formula of EC2 /EC211/, the Level II approximation of Model Code 2010 /CEB10/, which is based on the Simplified Modified Compression Field Theory /Ben06/ and the model from ACI 318-08 /ACI08/. The formulae for the last three approaches are summarized in Annex B. The SCCWM shows reasonable statistical results in all databanks. The semi-empirical method is equally precise for RC beams but shows higher coefficients of variation for prestressed beams and beams in tension. The flexural shear formula of EC2 shows comparable COVs for RC and PC beams but exhibits problems in predicting the shear capacity for beams in tension. The Level II approximation of Model Code 2010 has a rather mediocre performance in all of the databanks as does the model of ACI 318-08. The ACI 318-08 approach also has problems to predict the shear capacity of RC beams in tension so that it gave a negative shear capacity for 3 of 32 tests.
6.2 Evaluation of Shear Databanks

Table 6-1: Evaluation of shear databanks for beams without shear reinforcement

<table>
<thead>
<tr>
<th>Approach</th>
<th>Proposed mechanical model</th>
<th>Semi-Emp.</th>
<th>EC2, Eq. 6.4</th>
<th>MC2010 Level II</th>
<th>ACI 318-08</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>SCCWM Ch. 3.5.6</td>
<td>Ch. 0</td>
<td>/EC211/</td>
<td>/CEB10/</td>
<td>/ACI08/</td>
</tr>
<tr>
<td>RC with $N = 0$</td>
<td>$n_{\text{Test}}$</td>
<td>784</td>
<td>784</td>
<td>784</td>
<td>784</td>
</tr>
<tr>
<td>MV</td>
<td>1,059</td>
<td>1,001</td>
<td>1,074</td>
<td>1,618</td>
<td>1,342</td>
</tr>
<tr>
<td>COV</td>
<td>0.224</td>
<td>0.173</td>
<td>0.230</td>
<td>0.350</td>
<td>0.333</td>
</tr>
<tr>
<td>PC with $N &lt; 0$</td>
<td>$n_{\text{Test}}$</td>
<td>120</td>
<td>214</td>
<td>214</td>
<td>214</td>
</tr>
<tr>
<td>MV</td>
<td>1,029</td>
<td>1,476</td>
<td>1,721</td>
<td>3,617</td>
<td>1,468</td>
</tr>
<tr>
<td>COV</td>
<td>0.235</td>
<td>0.290</td>
<td>0.308</td>
<td>0.404</td>
<td>0.460</td>
</tr>
<tr>
<td>RC with $N \geq 0$</td>
<td>$n_{\text{Test}}$</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>29$^{1)}$</td>
</tr>
<tr>
<td>MV</td>
<td>1,000</td>
<td>0.89</td>
<td>1,366</td>
<td>1,487</td>
<td>2,056</td>
</tr>
<tr>
<td>COV</td>
<td>0.166</td>
<td>0.165</td>
<td>0.438</td>
<td>0.240</td>
<td>0.464</td>
</tr>
</tbody>
</table>

$^{1)}$ 3 tests could not be evaluated as the approach gave a negative shear capacity. These tests were omitted in the evaluation.

6.2.2 Evaluation of tests on beams with shear reinforcement

The evaluation of shear tests on beams with shear reinforcement is summarized in Table 6-2. The tests are divided into two different groups: tests on RC beams without axial forces and tests on prestressed concrete beams. For each databank, the number of evaluated tests ($n_{\text{Test}}$), the mean ratios of test and theoretical shear capacity (MV) and the corresponding coefficient of variation (COV) are given for the different models. The considered shear models are the proposed models for design and assessment according to chapter 5.8, the equilibrium based model with variable strut inclination of EC2 /EC211/, the Level II approximation of Model Code 2010 /CEB10/, which is based on a stress field approach /Kau98a/ and the model from ACI 318-08 /ACI08/. Again, the formulae for the last three approaches are summarized in Annex B. The proposed model from this thesis shows very good results for RC and PC beams with COVs of 17.7% and 16.4%, respectively. The plasticity model of EC2 has a rather mediocre agreement with a COV of 31.5% for RC beams and 56.2% for PC beams. This is mainly due to the fact, that the model is not applicable to beams with small amounts of shear reinforcement. The Level II approximation of Model Code 2010 is in good agreement with the tests on RC beams but exhibits a higher COV for PC beams. The approach by ACI 318-08 is less precise for RC beams with a COV of 32.2% but more precise for PC beams with a COV of 22.6%. The approach also has a relatively high safety margin with a MV of about 1.56 for both databanks.
Table 6-2: Evaluation of shear databanks for beams with shear reinforcement

<table>
<thead>
<tr>
<th>Approach</th>
<th>Proposed model</th>
<th>EC2</th>
<th>MC2010 Level II</th>
<th>ACI 318-08</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reference</td>
<td>Design</td>
<td>Assessment</td>
<td>/EC211/</td>
</tr>
<tr>
<td>RC with</td>
<td>nTest</td>
<td>170</td>
<td>170</td>
<td>170</td>
</tr>
<tr>
<td>N = 0</td>
<td>MV</td>
<td>1,173</td>
<td>1,046</td>
<td>1,555</td>
</tr>
<tr>
<td></td>
<td>COV</td>
<td>0,176</td>
<td>0,177</td>
<td>0,315</td>
</tr>
<tr>
<td>PC with</td>
<td>nTest</td>
<td>119</td>
<td>119</td>
<td>119</td>
</tr>
<tr>
<td>N &lt; 0</td>
<td>MV</td>
<td>1,167</td>
<td>1,154</td>
<td>1,901</td>
</tr>
<tr>
<td></td>
<td>COV</td>
<td>0,167</td>
<td>0,164</td>
<td>0,562</td>
</tr>
</tbody>
</table>

6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

6.3.1 Introduction

To determine partial safety factors for the derived shear strength models, the probabilistic evaluation methods according to EN 1990, Annex D, section 8 can be used. This section intends to provide the basis for calibrating resistance models and for deriving design equations. The procedure consists of different steps that are illustrated within a flowchart shown in Figure 6-1. First, a resistance function $r_t$ should be defined that includes all relevant basic variables $X$ that affect the resistance. Then, a mean value correction of the resistance function $r_t$ has to be performed to determine a coefficient of variation $V_δ$ of the resistance function in comparison with the test results. The check of the compatibility of the resistance function which is required by EN 1990 has already been done in section 6.2. Then, a coefficient $V_{rt}$ has to be determined that represents the influence of the variation of the basic variables on the results of the resistance function. Finally, these values can be used to calculate the characteristic level of the resistance function based on the 5%-fractile ($k_∞ = 1.64$). In this case, a commonly accepted partial safety factor for uncertainty of material properties $γ_m$ (e.g. $γ_c = 1.5$) is used to calculate design resistances. Alternatively, the design level of the resistance function can be calculated directly based on the 0,1%-fractile ($k_{d,∞} = 3.04$). Then a partial safety factor $γ_M$ (e.g. $γ_c = 1.8$) is determined, that includes uncertainties of model and material properties. In this thesis, the latter partial safety factors will be derived, as they are more common (except for Germany) and more transparent concerning the safety margin of an approach. Different fractile factors $k$ are applied depending on the number of available test results. As there are a lot of tests available in each of the shear databanks ($n > 100$), the fractile factors for an unlimited number of tests are applicable.
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

![Diagram of the statistical determination of a design model according to EN 1990 Annex D, methods a) and b)](image)

**Derivation of a model equation**

\[ V_c = r_i = g_{ri} \left( X_i \right) \]

**Mean value correction of the model equation**

\[ V_{u,\text{test}} = r_e = b \cdot r_t = b \cdot V_c \quad V_{Rm,c} = b \cdot r_i (X_{m}) = b \cdot g_{ri} (X_{m}) \delta \]

with \( b = \frac{\sum r_i r_t^2}{\sum r_i^2} \) and \( \delta_j = \frac{r_{ei}}{b r_{ti}} \)

**Estimated COV of the δ-variable**

\[ \Delta_j = \ln (\delta_i) \quad \bar{\Delta} = \frac{1}{n} \sum \Delta_i \quad s^2_{\delta} = \frac{1}{n-1} \sum (\Delta_i - \bar{\Delta})^2 \quad V_\delta = \sqrt{\exp(s^2_{\delta})} - 1 \]

**COV of the basic variables**

\[ \frac{V_r^2}{g_{ri}^2 (X_m)} = \frac{1}{g_{ri}^2 (X_m)} \times \sum \left( \frac{\partial g_{ri}}{\partial X_i} \times \sigma_i \right)^2 \quad V_r^2 = V_\delta^2 + V_{rt}^2 \]

\[ Q_n = \sqrt{\ln(V_r^2 + 1)} \quad Q_\delta = \sqrt{\ln(V_\delta^2 + 1)} \quad Q = \sqrt{\ln(V_r^2 + 1)} \quad \alpha_n = \frac{Q_n}{Q} \quad \alpha_\delta = \frac{Q_\delta}{Q} \]

**Calculation of the characteristic value**

\[ r_k = b \cdot g_n (X_n) \exp \left( -k_n Q - 0.5 Q^2 \right) \]

**Calculation of the design value**

\[ r_d = b \cdot g_n (X_n) \exp \left( -k_{d,a} \alpha_n Q_{n} - k_{d,a} \alpha_\delta Q_{\delta} - 0.5 Q^2 \right) \]

**Calculation of a partial safety factor**

\[ \gamma_m = r_k / r_d \quad \gamma_M = r_m / r_d \]

**Figure 6-1:** Flowchart of the statistical determination of a design model according to EN 1990 Annex D, methods a) and b)

The procedure of EN 1990, Annex D is based on the so-called Mean-Value First-Order Second-Moments (MVFOSM) method. That means that the basic variables are described by the first and second order static moment of their distribution (i.e. their mean value \( X_i \) and their variance \( \sigma_i^2 \)). The resistance function \( g_{ri} \) is linearized by a Taylor series and the variance of the linearized function is determined based on the variance
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

Another drawback of this simplified method is that the reliability index $\beta$ is not directly coupled to the failure probability of the structures. It is also not possible to include individual statistical distributions for the basic variables. In the procedure given by EN 1990, all distributions of the basic variables are assumed to be log-normal distributed. Nevertheless, the procedure provided by EN 1990, is used for the probabilistic evaluation of the strength model for several reasons. The first reason is the simple fact, that this method is provided by the code explicitly for assessing the safety level of strength model based on experimental results. The given value of $\beta = 3.04$ and the procedure itself are regarded as commonly accepted. Secondly, it can be seen that the terms in the strength models in this thesis that cannot be linearized are of minor importance to the end result. As stated before, the failure probability cannot not be calculated according to this method. However, it is possible to compare the derived partial safety factors to the currently applicable safety factors of EC2 as a relative comparison of the acceptable safety level.

6.3.2 Simplified Critical Crack Width Model for beams without shear reinforcement

In this section, a partial safety factor $\gamma_C$ for the Simplified Critical Crack Width Model (SCCWM) is derived according to the methods given by EN 1990, Annex D. For this, the influence of the variation of the basic variables is considered in addition to model uncertainties based on comparison with test databanks. The design value of the shear strength for beams without shear reinforcement is given by Eq. (6-1). The coefficient $C_{Rm,c}$ is adjusted to predict the mean value of test results and the partial safety factor $\gamma_C$ will be derived to meet the required safety level. Since the $C_{Rm,c}$ value has been derived using the selected databank in chapter 3.4.2, the corresponding (lower) COV will also be used for determining the partial safety factor.

$$V_{rd,c} = \frac{C_{Rm,c}}{\gamma_C} \left( \frac{E_f \rho_l}{\lambda \cdot d \left(1,0+1,9/w_0\right) f_{ck}} \right)^{\frac{1}{3}} b_0 z \leq \frac{0,42}{\gamma_C} \sqrt{f_{ck} b_w z}$$

(6-1)

In the first step, the formula can be further simplified. For practical reasons, the limitation to $V_{c,max}$ value is omitted. The reinforcement ratio $\rho_l$ consists of further basic variables and can be replaced by $A_s/(b_w d)$. The area of the longitudinal reinforcement $A_s$ usually comprises of multiple bars spaced at a distance of $s_{bar}$ so that $A_s/b_w = A_{s,bar} / s_{bar}$. While the area of a single reinforcement bar $A_{s,bar}$ usually does not vary, the spacing of reinforcement of for instance a slab might vary locally. The inner lever arm $z$ is estimated by $0,9d$. The mean value formula including all basic variables is then given by Eq. (6-2).
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

\[
V_{R_{m,c}} = C_{R_{m,c}} \left( \frac{E_s A_{s,\text{bar}}}{\lambda \cdot s_{\text{bar}}} \sqrt{\frac{d_{ag} + 16}{(1,0 + 1,9/w_0)}} f_{ck} \right)^{1/3} b_w 0,9d^{1/3}
\] (6-2)

In the next step, the variance of the model equation is determined by partially derivating the model equation for each basic variable. The basic variables themselves should be independent from one another which is the case in Eq. (6-2). The only exception is the value of the shear slenderness \( \lambda \) which consists of the moment to shear ratio \( M/V \) divided by the effective depth \( d \). However, the shear slenderness is not subject to change if a statical system is scaled or if the load level changes. One problem concerning the shear slenderness is that it couples the resistance side \( R_d \) with the actions on the structures \( E_d \), which according to the semi-probabilistic safety concept should be decoupled. For this evaluation, the shear slenderness will however simply be regarded as an independent variable as it is also only a minor influence on the final result. The critical crack width \( w_0 \) is not included in the probabilistic evaluation as it can be seen as a deterministic value which is also not continuously differentiable. The partial derivatives of the model equation for all basic variables are summarized in Table 6-3.

**Table 6-3:** Basic variables of the mechanical shear strength model and corresponding partial derivatives

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>( \frac{\partial V_{R_{m,c}}}{\partial X_i} )</th>
<th>Basic variable</th>
<th>( \frac{\partial V_{R_{m,c}}}{\partial X_i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{ck} )</td>
<td>( \frac{1}{3f_{ck}} \cdot V_{R_{m,c}} )</td>
<td>( A_{s,\text{bar}} )</td>
<td>( \frac{1}{3A_{s,\text{bar}}} \cdot V_{R_{m,c}} )</td>
</tr>
<tr>
<td>( d_{ag} )</td>
<td>( \frac{1}{6(d_{ag} + 16)} \cdot V_{R_{m,c}} )</td>
<td>( s_{\text{bar}} )</td>
<td>( -\frac{1}{3s_{\text{bar}}} \cdot V_{R_{m,c}} )</td>
</tr>
<tr>
<td>( E_s )</td>
<td>( \frac{1}{3E_s} \cdot V_{R_{m,c}} )</td>
<td>( b_w )</td>
<td>( \frac{1}{b_w} \cdot V_{R_{m,c}} )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>( -\frac{1}{3\lambda} \cdot V_{R_{m,c}} )</td>
<td>( d )</td>
<td>( \frac{1}{3d} \cdot V_{R_{m,c}} )</td>
</tr>
</tbody>
</table>

In the next step, information about the variation of the different basic variables is required. This is a crucial step in determining the safety level of a model as the results can be quite sensitive for some variables. Information about the standard deviation of different parameters can be acquired from codes or other literature. The first and most critical variable is the concrete compressive strength \( f_c \). EC2 implicitly assumes a COV for the concrete compressive strength that is subject to \( f_{cm} \) as shown in Eq. (6-3). This is generally comprehensible as investigations have shown that the coefficient of variation indeed decreases for higher concrete strengths /Rue62b/, /Shi00/.
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

\[
f_{ck} = f_{cm} - 8 = f_{cm} \left( 1 - 1.645 \cdot COV \right)
\]

\[
\Rightarrow COV = \frac{4.863}{f_{cm}}
\]

(6-3)

Eq. (6-3) gives a COV of 24.3 % for \( f_{cm} = 20 \) MPa which seems quite high. Other authors give values of the COV of concrete of 18 % (for \( f_{cm} = 27.5 \) MPa, /Mac76/) or 15 % (for \( f_{cm} = 22 \) MPa /Vro87/). Investigations by SHIMIZU ET AL. /Shi00/ suggest that the COV of concrete for newer buildings usually does not exceed 15–16 %. In agreement with /Vro87/, a COV of 15 % will be assumed for the concrete compressive strength.

The area of the reinforcement bar \( A_{s,\text{bar}} \) is assumed to deterministic. The dimensions \( s_{\text{bar}} \) and \( b_{w} \) will be associated with a constant standard deviation of 2 and 4 mm while for the effective depth \( d \) a standard deviation of 7 mm is assumed (values partly taken from /Vro87/). For \( E_{s} \), a constant COV of 4 % is assumed /Vro87/ which is also adopted for the aggregate size \( d_{ag} \) and the shear slenderness \( \lambda \). The results of the calculation are summarized in Table 6-4. In Table 6-4 it is indicated which of the variables have COVs which are independent from their mean value (i.e. constant COV) and which of the variables have constant standard deviations. For the latter, conservative mean values were chosen for the evaluation. The calculation yields a partial safety factor of \( \gamma_{C} = 1.65 \) in connection with a design fractile factor of \( k_{d,\infty} = 0.8 \cdot 3.8 = 3.04 \) (EN 1990, Annex C). For design purposes, \( \gamma_{C} = 1.65 \) would thus seem reasonable.

**Table 6-4:** Calculation of the partial safety factor \( \gamma_{C} \) for the simplified mechanical approach

<table>
<thead>
<tr>
<th>( X_{m} )</th>
<th>( V_{X} )</th>
<th>( \sigma_{i} )</th>
<th>( \partial V/\partial X_{i} )</th>
<th>( (\partial V/\partial X_{i} \cdot \sigma_{i})^{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{ck} )^{1}</td>
<td>30</td>
<td>0.150</td>
<td>4.500</td>
<td>0.0111</td>
</tr>
<tr>
<td>( A_{s,\text{bar}} )</td>
<td>12</td>
<td>0.000</td>
<td>0.000</td>
<td>0.0278</td>
</tr>
<tr>
<td>( s_{\text{bar}} )</td>
<td>50</td>
<td>0.040</td>
<td>2.000</td>
<td>-0.0067</td>
</tr>
<tr>
<td>( E_{s} )^{2}</td>
<td>200000</td>
<td>0.040</td>
<td>8.000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( d_{ag} )</td>
<td>32</td>
<td>0.040</td>
<td>1</td>
<td>0.0035</td>
</tr>
<tr>
<td>( \lambda )^{3}</td>
<td>2.5</td>
<td>0.040</td>
<td>0.100</td>
<td>-0.1333</td>
</tr>
<tr>
<td>( d )</td>
<td>200</td>
<td>0.035</td>
<td>7.000</td>
<td>0.0017</td>
</tr>
<tr>
<td>( b_{w} )</td>
<td>150</td>
<td>0.027</td>
<td>4.000</td>
<td>0.0067</td>
</tr>
</tbody>
</table>

\(^{1}\) COV is independent from \( X_{m} \)

\( \Sigma = 0.003900 \)

\( \sigma_{\lambda}^{2} = 0.02210 \)

\( Q_{rt} = 0.06239 \)

\( V_{rt} = 0.06245 \)

\( V_{s} = 0.14949 \)

\( Q = 0.16096 \)

\( V_{r}^{2} = 0.02625 \)

\( Q_{d} = 0.14866 \)

\( k_{d,\infty} = 3.04 \)

\( \gamma_{C} = 1.65246 \)

In comparison, the current international version of EC2 /EC211/ uses a partial safety factor of \( \gamma_{C} = C_{Rm,c} / C_{Rd,c} = 0.193 / 0.12 = 1.61 \) and the German version /DIN13/ uses
\( \gamma_C = 0,193 / 0,10 = 1,93 \) (The value of \( C_{Rm,c} = 0,193 \) for EC2 was determined for the databank evaluation in Table 6-1). The proposed value of \( \gamma_C = 1,65 \) can therefore be regarded as reasonably close to the currently applicable partial safety factor of EC2 /EC211/ (In fact the EC2 approach requires a partial safety factor of \( \gamma_C = 2,13 \) according to a probabilistic analysis analogous to Table 6-4 showing that this type of probabilistic analysis does not necessarily lead to progressive safety factors). The partial safety factor \( \gamma_C \) can however also be determined by using the COVs for concrete according to EC2 which is given by Eq. (6-3). \( \gamma_C \) is then subject to the concrete strength \( f_{cm} \). The value for \( \gamma_C \) with a COV of concrete according to EC2 in comparison with a constant COV of 15 % is illustrated in Figure 6-2. For high strength concrete, the partial safety factor decreases to 1,62, whereas for lower concrete strengths, \( \gamma_C \) is close to 1,72. Here, a partial safety factor of 1,65 can also be regarded as a reasonable mean between those values.

![Figure 6-2: Comparison of partial safety factors \( \gamma_C \) for different COVs of concrete](image)

**6.3.3 Semi empirical approach for beams without shear reinforcement**

In this chapter, the proposed semi-empirical formula will be evaluated according to EN 1990, Annex D. The mean value of the shear strength for beams without shear reinforcement is given by Eq. (6-4).

\[
V_{Rm,c} = C_{Rm,c} \frac{1+0,8e^{-3/2,5}}{\sqrt{1+d/200}} \sqrt{f_{ck} b_{veff} \xi d}
\]  

(6-4)

The formula can be simplified by replacing \( \rho_l \) by \( A_{s,bar} / (s_{bar} d) \). The design formula including all basic variables is given by Eq. (6-5).
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

\[
V_{Rm,c} = C_{Rm,c} \left( 1 + 0,8 e^{-2,5/2,5} \right) 0,86 \left( \frac{E_s A_s}{E_c b_d} \right)^{0,38} \sqrt{f_{ck} b_y d} \\
= C_{Rm,c} \left( 1 + 0,8 e^{-2,5/2,5} \right) 0,86 \left( \frac{E_s A_s}{E_c s_{bar}} \right)^{0,38} \sqrt{f_{ck} b_y d^{0,62}}
\]  

(6-5)

In the next step, the partial derivatives of all basic variables are determined. In general, the variables are identical to the SCCWM with the exception that the Young’s modulus of concrete \( E_{cm} \) is included in the equation and that the aggregate size \( d_{ag} \) is omitted. The partial derivatives of the model equation for all basic variables are summarized in Table 6-5.

Table 6-5: Basic variables of the shear strength model and corresponding partial derivatives

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>( \frac{\partial V_{Rm,c}}{\partial X_i} )</th>
<th>Basic variable</th>
<th>( \frac{\partial V_{Rm,c}}{\partial X_i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{ck} )</td>
<td>( \frac{1}{2f_{c}} V_{Rm,c} )</td>
<td>( A_{s,bar} )</td>
<td>( 0,38 \frac{V_{Rm,c}}{A_{s,bar}} )</td>
</tr>
<tr>
<td>( E_{cm} )</td>
<td>( \frac{0,38}{E_{cm}} V_{Rm,c} )</td>
<td>( s_{bar} )</td>
<td>( -0,38 \frac{V_{Rm,c}}{s_{bar}} )</td>
</tr>
<tr>
<td>( E_s )</td>
<td>( \frac{0,38}{E_s} V_{Rm,c} )</td>
<td>( b_w )</td>
<td>( \frac{1}{b_w} V_{Rm,c} )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>( \frac{-0,8 e^{-2,5/2,5}}{2,5 \left( 1 + 0,8 e^{-2,5/2,5} \right)} V_{Rm,c} )</td>
<td>( d )</td>
<td>( \frac{3d/5000 + 0,62}{d \left( 1 + d/200 \right)} V_{Rm,c} )</td>
</tr>
</tbody>
</table>

The information about the variation of the different basic variables is adopted from Ch. 6.3.2. For the COV of the Young’s modulus of concrete, a value of 10 % is assumed /Vro87/. The calculation of the partial safety factor \( \gamma_C \) is summarized in Table 6-4. The calculated partial safety factor of \( \gamma_C = 1,812 \) is a little higher than for the SCCWM but is also within the reasonable range of the partial safety factors of EC2 /EC211/ and the German version of EC2 /DIN13/. For design purposes, a partial safety factor equal to the one of Ch. 6.3.2 of \( \gamma_C = 1,8 \) should be sufficient.
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

### 6.3.4 Partial safety factors for beams with shear reinforcement

The shear capacity beams with shear reinforcement for which \( \omega_w \leq \omega_{w,\text{lim}} \) is defined as the sum of \( V_{Rd,c} \) and a stirrup contribution according to Eq. (6-6). Unfortunately, a sum approach cannot be investigated with the methods of EN 1990, Annex D. Since the stirrup contribution directly depends on the steel strength \( f_{yd} \), a partial safety factor for steel of \( \gamma_s = 1.15 \) is usually applied.

Using the partial safety factor \( \gamma_C \) for the concrete contribution and \( \gamma_s = 1.15 \) for the stirrup contribution can be considered safe. It might in fact be too conservative since a brittle flexural shear failure does not occur if a minimum shear reinforcement is available (which is defined in a way to prevent brittle failure). A reduced partial safety factor for flexural shear failure might therefore be adequate to ensure economic design (\( \gamma_C = 1.15 \sim 1.65 \)). However, additional probabilistic investigations have to be carried out to confirm an adequate \( \gamma_C \)-value.

The failure for \( \omega_w > \omega_{w,\text{lim}} \) according to plasticity theory is defined according to Eq. (6-7). Here, the failure directly depends on the design value of the concrete compressive strength \( f_{cd} \) and the design yield strength \( f_{yd} \) (by taking the mechanical reinforcement ratio \( \omega_w \) into account).

\[
V_{Rd} = b_w z V_{cd} \sqrt{\frac{\omega_w - \omega_{w,\text{lim}}^2}{\omega_w}} \quad \text{and} \quad \omega_{w,\text{lim}} < \omega_w \leq 0.5
\]  

### Table 6-6: Calculation of the partial safety factor \( \gamma_C \) for the simplified mechanical approach

<table>
<thead>
<tr>
<th>( X_m )</th>
<th>( V_x )</th>
<th>( \sigma_i )</th>
<th>( \partial V/\partial X_i )</th>
<th>( (\partial V/\partial X_i \cdot \sigma_i)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{ck} )</td>
<td>30</td>
<td>0,150</td>
<td>4,500</td>
<td>0,0167</td>
</tr>
<tr>
<td>( A_{s,\text{bar}} )</td>
<td>12</td>
<td>0,000</td>
<td>0,000</td>
<td>0,0317</td>
</tr>
<tr>
<td>( s_{\text{bar}} )</td>
<td>50</td>
<td>0,040</td>
<td>2,000</td>
<td>-0,0076</td>
</tr>
<tr>
<td>( E_{s} )</td>
<td>200000</td>
<td>0,040</td>
<td>8000</td>
<td>0,0000</td>
</tr>
<tr>
<td>( E_{c} )</td>
<td>30000</td>
<td>0,100</td>
<td>3000</td>
<td>0,0000</td>
</tr>
<tr>
<td>( \lambda_{s} )</td>
<td>2.5</td>
<td>0,040</td>
<td>0,100</td>
<td>-0,0910</td>
</tr>
<tr>
<td>( d )</td>
<td>200</td>
<td>0,035</td>
<td>7,000</td>
<td>0,0019</td>
</tr>
<tr>
<td>( b_w )</td>
<td>150</td>
<td>0,027</td>
<td>4,000</td>
<td>0,0067</td>
</tr>
</tbody>
</table>

\(^1\) COV is independent from \( X_m \) \quad \Sigma = \quad 0,008493

\[
\sigma_\delta^2 = 0,02773 \quad Q_{\delta} = 0,09196 \quad \nu_{\delta} = 0,09216
\]

\[
V_{\delta} = 0,16768 \quad Q = 0,18962 \quad \nu_{\delta}^2 = 0,03661
\]

\[
Q_{\delta} = 0,16652 \quad k_{d,\infty} = 3,04 \quad \gamma_C = 1,81197
\]
6.3 Probabilistic Safety Evaluation according to EN 1990 Annex D for RC Beams

The equation can be evaluated in a simplified manner by considering \( f_{cm} \), \( b_w \) and \( z \) as the main basic variables. The angle of the compression field and the \( \nu \)--value are considered to be deterministic. The partial derivatives are then given by Table 6-7.

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>( \frac{\partial V_{r_{m,c}}}{\partial X_i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{cm} )</td>
<td>( \frac{1}{f_{cm}} V_{Rm,c} )</td>
</tr>
<tr>
<td>( b_w )</td>
<td>( \frac{1}{b_w} V_{Rm,c} )</td>
</tr>
<tr>
<td>( z )</td>
<td>( \frac{1}{z} V_{Rm,c} )</td>
</tr>
</tbody>
</table>

The probabilistic evaluation can be performed assuming a constant COV for \( f_{cm} \) of 15 % as done for the previous models. The COV for the model uncertainty was obtained by evaluating tests on PC beams for which the plasticity model was applicable (i.e. \( \omega_w \geq \omega_{w,lim} \)). This was the case for 30 out of 119 tests for which the model yielded a COV of 13,5 %. The probabilistic evaluation of the model according to Annex D, EN 1990 yields a \( \gamma_C \) of 1,915 according to Table 6-8.

Table 6-8: Calculation of the partial safety factor \( \gamma_C \) for strut failure

<table>
<thead>
<tr>
<th>( f_{cm} )</th>
<th>( X_m )</th>
<th>( V_x )</th>
<th>( \sigma_i )</th>
<th>( \frac{\partial V}{\partial X_i} )</th>
<th>( (\frac{\partial V}{\partial X_i} \cdot \sigma_i)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{cm} )</td>
<td>30</td>
<td>0,150</td>
<td>4,500</td>
<td>0,0333</td>
<td>0,022500</td>
</tr>
<tr>
<td>( z )</td>
<td>150</td>
<td>0,047</td>
<td>7,000</td>
<td>0,0067</td>
<td>0,002178</td>
</tr>
<tr>
<td>( b_w )</td>
<td>150</td>
<td>0,027</td>
<td>4,000</td>
<td>0,0067</td>
<td>0,000711</td>
</tr>
</tbody>
</table>

\(^{*)}\) COV is independent from \( X_m \), \( \Sigma = 0,025389 \)

The same evaluation can be performed by calculating the COV of the concrete according to Eq. (6-3). The partial safety factor \( \gamma_C \) then depends on the concrete strength \( f_{cm} \) which is illustrated in Figure 6-3. The required partial safety factor decreases significantly for higher concrete strengths. The diagram also shows the partial safety which is implicitly given by EC2. It can simply be calculated by \( \gamma_{C,EC2} = \gamma_c \cdot f_{cm} / (f_{cm} - 8) \) with \( \gamma_c = 1,5 \).
6.4 Summary and Conclusions

In this chapter, a probabilistic safety evaluation of the derived models was performed. In the first step, shear databanks on beams without shear reinforcement were evaluated with the mechanical and the semi-empirical approach for flexural shear failure. In general, the derived models showed a better agreement with tests results when compared to the approaches of EC2, MC2010 or ACI 318-08. This was also the case for the databanks on beams with shear reinforcement. The proposed procedure yielded very good statistical results for RC as well as for PC beams.

On that basis, partial safety factors were derived for the flexural shear formulas. The required factors turned out to be slightly higher than according to the current EC2 /EC211/ but lower than its German version. For beams with shear reinforcement, a probabilistic evaluation for beams with large shear reinforcement was carried out. Here, the required safety level of the probabilistic evaluation was equal to the safety level provided by the current EC2 so that the strut capacity can be determined using the approach of EC2.

The partial safety factors according to EC2 and the probabilistic evaluation with a COV of concrete according to EC2 differ by less than 4% at the most so that it is sufficient to use the established partial safety factor $\gamma_m = \gamma_c = 1.5$ for determining the shear strength for beams with large shear reinforcement ratios.

### Figure 6-3: Partial safety factors $\gamma_c$ for strut failure according to EC2 and according to a probabilistic evaluation with a COV according to EC2 and a constant COV for concrete

![Graph showing partial safety factors for strut failure]
7 Summary and Conclusions

7.1 Summary

In this thesis, the shear strength of slender reinforced and prestressed members without and with shear reinforcement was investigated. The literature review illustrated the development of different approaches and different schools of thought attempting to solve the “riddle of shear failure”. Recently, a lot of work has been performed to develop models that rely stronger on mechanical principles than on pure empiricism. Therefore, a mechanical approach to determine the shear capacity of RC beams without shear reinforcement was developed in the first step to gain insights into their behavior. It was concluded that within the context of the model assumptions, aggregate interlock plays a key role for understanding flexural shear failure of RC beams. Phenomena like size effect, influence of shear slenderness and of longitudinal reinforcement can be directly linked to the flexural crack widths of beams and a critical crack width \( w_0 \) which limits aggregate interlock. Based on the mechanical model it was then possible to derive a closed form solution, the Critical Crack Width Model (CCWM). By linking the flexural shear capacity with the flexural crack widths, the influence of axial forces could also be accounted for consistently within the CCWM. The comparison of the model with test results showed a very good agreement and it can be concluded that all relevant influence parameters are considered correctly. A second semi-empirical model was then derived based on the assumption that the main shear contributor of beams without shear reinforcement is the uncracked compression zone. The main influence factors of size effect and shear slenderness were accounted for by adopting approaches from fracture mechanics and by regressional analysis of test data, respectively. It could be shown that the semi-empirical approach is also in agreement with test data and that it is especially capable of correctly predicting the behavior of beams with different load or support conditions, although it was calibrated using tests on point loaded single span beams. The comparison of the two approaches for flexural shear failure shows that there is most likely more than one admissible stress state or state of equilibrium in beams without shear reinforcement and that there exists thus more than one admissible solution to the problem of flexural shear failure.

In the following, the shear capacity of beams with shear reinforcement was investigated. The behavior of beams with very little shear reinforcement can be considered similar to the behavior of beams without shear reinforcement, but with an additional stirrup contribution. Here, the Simplified Critical Crack Width Model was chosen as a basis for the flexural shear capacity. To determine the number of activated stirrups, an equation for a fictitious shear crack angle was derived based on the principle of minimum work. For higher shear capacities, the behavior of the beams change so that it can be described by
7.2 Future Work

A plasticity truss model with simultaneous failure of stirrups and concrete struts. To distinguish these failure modes in a consistent manner, a criterion based on the mechanical shear reinforcement ratio of the beam was derived. Shear design procedures were then provided for the design of new structures for which the given shear reinforcement is not known a priori as well as for the assessment of existing structures for which the shear reinforcement is known in advance.

Ultimately, probabilistic evaluations of the derived models were carried out to derive partial safety factors for design. The safety factors for flexural shear design turned out to be comparable to the current EC2. For beams with small amounts of shear reinforcement it was proposed to adopt the partial safety factors for flexural shear failure for the concrete contribution until further probabilistic investigations are carried out. According to the probabilistic investigation, the partial safety factor of EC2 for strut failure is sufficient so that the approach of EC2 for the strut capacity was adopted. This thesis thus presents a comprehensive procedure for design and assessment of structures without and with shear reinforcement under shear loading. Judging from test evaluations it can be expected that the presented approaches will be especially beneficiary for the assessment of existing structures like bridges.

7.2 Future Work

After many decades of research, many questions regarding the shear behavior of beams still require further investigations. The main points in the view of the author are as follows:

- **Load and support conditions:** Although there are more and more tests carried out on continuous beams and under distributed loads, the number of tests is nowhere near the number of point loaded tests. Additional tests series under realistic loading conditions should therefore be performed.

- **Measurement techniques:** Measurement techniques in most test series have been pretty much the same in the last 60 years featuring load cells, extensometers, strain gauges and so on. Newer techniques based on Digital Image Correlation are supposed to measure crack kinematics with great precisions. At this time, it seems that there is no more decisive advancement in measuring techniques that will provide a whole new insight into what happens during shear failure. In the future, however, infrasound or MRT-based techniques might provide means to measure three dimensional stress/strain-states or crack patterns.

- **Tests on large beams:** Most of the tests so far have been conducted on small specimens. To validate the findings concerning the size effects of beams, additional tests on (correctly scaled) very large and slender beams are required.

- **Interaction with torsion or biaxial bending:** This thesis focused on the interaction of bending, shear and axial forces. In many structures like bridge girders, the
7.2 Future Work

simultaneous occurrence of additional torsion or bending and shear in other directions is quite common. Additional research is therefore required regarding tests on compact cross-sections like rectangular or T-beams and tests on thin-walled cross-sections like box girders.

- **Physical models:** For research it is still of interest to develop models that have a very high degree of physical consistency which means that such models would consider equilibrium of forces, nonlinear material laws for steel and concrete, concrete to concrete interaction (e.g. aggregate interlock), concrete to steel interaction (i.e. bond) and compatibility conditions (i.e. strains and cracks). For beams with shear reinforcement, very advanced models already exist. For beams without shear reinforcement, such methods would have to be developed.

- **Crack width based failure criterion for flexural shear failure:** In the Critical Crack Width Model, the flexural shear capacity is directly coupled to the mechanical properties of a flexural crack and its crack width. The crack widths were, however, determined in a simplified manner and are not necessarily equal to actual crack widths in structures. By a detailed analysis of shear tests regarding crack widths and aggregate interlock, a more refined failure criterion based on crack width measurements can be derived. The shear check of structures could then directly be coupled to a check of crack widths in SLS and ULS. For the assessment of structures, the shear check could also be performed based on in situ measurements of crack widths.

- **Other reinforcement materials:** In the future, other reinforcement materials like FRP bars or textile reinforcement grids will be used more frequently to build completely steel-free structures. Although many of the existing shear design methods have proven to be applicable to other reinforcement materials, investigations should be carried out continuously as the products using new reinforcement materials can significantly differ in quality and mechanical properties like bond behavior or stiffness.

- **Assessment of structures:** The motivation for this thesis mainly came from the problem of assessment of existing structures. Although many critical points have been addressed and solved for the assessment of structures, additional work is required, e.g. in the field of probabilistic analysis with respect to analytical models as well as Finite Element simulations. By more detailed probabilistic investigations, refined partial safety factors that for example account for the remaining service life of structures could be derived.

The aforementioned points on future work concerning the shear capacity of beams show that there still remains a lot of work to be done.
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<tr>
<th>Reference</th>
<th>Title</th>
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<tr>
<td>Baz08</td>
<td>BAZANT, Z.P.; YU, Q.: Minimizing Statistical Bias to Identify Size Effect from Beam Shear Database. ACI Structural Journal Vol. 105, Iss. 6, pp. 685–691, 2008</td>
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<td>Ben05b</td>
<td>BENTZ, E.C.; BUCKLEY, S.: Repeating a classic set of experiments on size effect in shear of members without stirrups. ACI Structural Journal Vol. 102, No. 6, pp. 832-838, 2005</td>
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Annex A – Shear tests on PC continuous Beams

**Test Series 1 (TB1 – TB3)**

**Concrete properties:**

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$f_{cm,cyl}$</th>
<th>$f_{cm,cube}$</th>
<th>$f_{cm,split}$</th>
<th>$f_{cm,cs}$</th>
<th>$E_{cm}$</th>
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<tbody>
<tr>
<td>TB1</td>
<td>36.9</td>
<td>42.9</td>
<td>2.94</td>
<td>2.71</td>
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</tr>
<tr>
<td>TB2</td>
<td>38.6</td>
<td>42.6</td>
<td>3.09</td>
<td>3.34</td>
<td>25100</td>
</tr>
<tr>
<td>TB3</td>
<td>39.6</td>
<td>42.0</td>
<td>2.92</td>
<td>3.14</td>
<td>24500</td>
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**Steel properties:**

<table>
<thead>
<tr>
<th>Diameter</th>
<th>$f_{p,0,1}$</th>
<th>$f_{p,0,2}$</th>
<th>$f_{y}$</th>
<th>$E_s$($E_p$)</th>
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<td>0.6&quot;</td>
<td>1729</td>
<td>(1764)</td>
<td>1950</td>
<td>(190000)</td>
</tr>
</tbody>
</table>

**Cross-section:**

**Reinforcement layout shear span:**

**Tendon profile:**
Annex A

Test setup:

1st test
external tendons
internal tendon

2nd test

Test setup and failure of TB1:

Crack patterns:

TB1 $\sigma_{cp,ext} = 0$ MPa

$\rho_v = 1,33 \%$

$\rho_w = 0,67 \%$

TB2 $\sigma_{cp,ext} = 1,5$ MPa

TB3 $\sigma_{cp,ext} = 2,5$ MPa
Annex A

**Prestressing:**

<table>
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<th>$P_{t,zt}$</th>
<th>$\Delta P_{CSR}$</th>
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</thead>
<tbody>
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<td>430</td>
<td>0</td>
<td>8,5</td>
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<td>0,195</td>
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<td>270</td>
<td>14,1</td>
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<tr>
<td>TB 3</td>
<td>---*</td>
<td>430</td>
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*not determined

**Shear crack loads and ultimate loads:**

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<td>$\rho_w = 0,67 %$</td>
<td></td>
<td></td>
<td></td>
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<td>208</td>
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<td>403</td>
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<tr>
<td>TB2</td>
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<td>366</td>
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<tr>
<td>TB3</td>
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<td>418</td>
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</table>

<table>
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<th>$\rho_w = 1,33 %$</th>
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</table>

**Load-deflection curves:**

[Graphs showing load-deflection curves for different specimens with annotations: $\rho_w = 0,67 \%$ and $\rho_w = 1,33 \%$]
Annex A

**Test Series 2 (DLT 1.1 and DLT 1.2)**

**Concrete properties:**

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<th>( f_{m,split} )</th>
<th>( f_{m,cs} )</th>
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<td>25 mm</td>
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<td>196840</td>
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<tr>
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<td>(1764)</td>
<td>1950</td>
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**Cross-section DLT1.2:**

- 6Ø25
- stirrups Ø10/25

**Tendon profile:**
Concrete properties:

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<th>$f_{c_{m,cube}}$</th>
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Steel properties:

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Tendon profile:
Annex A

Test setup:

Failure of DLT1.1 (left) and DLT1.2 (right):

Load-deflection curves:
Annex B – Shear Design Approaches

In this section, the shear design formulae used for databank evaluations in chapter 6.2 are summarized.

Shear Design according to ACI 318-08 /ACI08/

Shear capacity of beams without stirrups

\[
V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w\frac{V_u}{M_u}\right) b_u d \leq 3.5\lambda\sqrt{f'_c} b_u d
\]

(stresses in psi)

\[
V_c = \left(0.158\lambda\sqrt{f'_c} + 17.24\rho_w\frac{V_u}{M_u}\right) b_u d \leq 0.29\lambda\sqrt{f'_c} b_u d
\]

(stresses in MPa)

where

\[
\frac{V_u}{M_u} \leq 1,0
\]

\[
\lambda = 1,0 \text{ for normalweight concrete}
\]

\[
M_u \text{ and } V_u \text{ at section considered}
\]

Shear capacity of beams without stirrups subjected to axial compression

\[
V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w\frac{V_u}{M_m}\right) b_u d \leq 3.5\lambda\sqrt{f'_c} b_u d \sqrt{1 + \frac{N_u}{500A_g}}
\]

(stresses in psi)

\[
V_c = \left(0.158\lambda\sqrt{f'_c} + 17.24\rho_w\frac{V_u}{M_m}\right) b_u d \leq 0.29\lambda\sqrt{f'_c} b_u d \sqrt{1 + 0.29 \frac{N_u}{A_g}}
\]

(stresses in MPa)

where

\[
M_m = M_u - N_u \frac{4h-d}{8}
\]

\[
N_u \text{ is positive for compression and negative for tension}
\]

\[
A_g \text{ gross cross-section area}
\]
Shear capacity of beams without stirrups subjected to axial tension

\[ V_c = 2 \left( 1 + \frac{N_u}{500A_y} \right) \lambda \sqrt{f_c b_y d} \]  
(stresses in psi)

\[ V_c = 0.166 \left( 1 + 0.29 \frac{N_u}{A_y} \right) \lambda \sqrt{f_c b_y d} \]  
(stresses in MPa)

Nominal shear capacity of beams with stirrups

\[ V_n = V_c + V_s \]

where

\[ V_s = \frac{A_s f_y d_s}{s} \leq 8 \sqrt{f_c b_w d} \]  
(stresses in psi)

\[ V_s = \frac{A_s f_y d_s}{s} \leq 0.664 \sqrt{f_c b_w d} \]  
(stresses in MPa)

Shear Design according to Model Code 2010, Level II /CEB10/

Shear capacity of beams without stirrups

\[ V_{Ed,c} = k_v \sqrt{f_{ck}} b_w z \]

where

\[ \sqrt{f_{ck}} \leq 8 \text{ MPa} \]

\[ k_v = \frac{0.4}{1+1500 \varepsilon_x} \frac{1300}{1000+k_{dg} z} \]

\[ k_{dg} = \frac{32}{16+d_g} \geq 0.75 \]

\[ \varepsilon_x = \frac{M_{Ed} + V_{Ed} + N_{Ed} \frac{z_p - e_p}{z}}{2 \left( \frac{z}{z} E_s A_s + \frac{z}{z} E_p A_p \right)} \]
Shear capacity of beams with stirrups

\[ V_{rd,s} = \frac{A_{sw}}{s_w} z f_{ywd} \cot \theta \leq V_{rd,max} \]

where

\[ V_{rd,max} = k_e \frac{f_{ck}}{\gamma_c} b_w z \sin \theta \cos \theta \]

\[ k_e = k_c \eta_{fc} \]

\[ \eta_{fc} = \left( \frac{30}{f_{ck}} \right)^{\frac{2}{3}} \leq 1,0 \]

\[ k_e = \frac{1}{1.2 + 55 \left[ \varepsilon_x + \left( \varepsilon_x + 0,002 \cot \theta \right) \cot^2 \theta \right]} \leq 0,65 \]

\[ \theta \geq 20^\circ + 10000 \varepsilon_x \]

Shear Design according to Eurocode 2 /EC211/

Shear capacity of beams without stirrups

\[ V_{rd,c} = \left[ C_{rd,c} k \left( 100 \rho_i f_{ck} \right)^{\frac{1}{3}} + 0,15 \sigma_{cp} \right] b_w d \]

where

\[ C_{rd,c} = 0,18 / \gamma_C \]

\[ k = 1 + \sqrt{\frac{200}{d}} \leq 2,0 \]

\[ \rho_i = \frac{A_d}{b_i d} \leq 0,02 \]

\[ \sigma_{cp} = N_{Ed} / A_c < 0,2 f_{cd} \]

Shear capacity of beams with stirrups

\[ V_{rd,s} = \frac{A_{sw}}{s_w} z f_{ywd} \cot \theta \leq V_{rd,max} \]

where
Annex B

\[ V_{rd,\text{max}} = \frac{\alpha_{cw} b_v z v_i f_{cd}}{\cot \theta + \tan \theta} \]

\[ v_i = \begin{cases} 0.6 & \text{for } f_{ck} \leq 60 \\ 0.9 - f_{ck}/200 \geq 0.5 & \text{for } f_{ck} \geq 60 \end{cases} \]

\[ \alpha_{cw} = \begin{cases} \left(1 + \frac{\sigma_{cp}}{f_{cd}}\right) & \text{for } 0 < \sigma_{cp} \leq 0.25 f_{cd} \\ 1.25 & \text{for } 0.25 f_{cd} < \sigma_{cp} \leq 0.5 f_{cd} \\ 2.5\left(1 - \frac{\sigma_{cp}}{f_{cd}}\right) & \text{for } 0.5 f_{cd} < \sigma_{cp} \leq 1.0 f_{cd} \end{cases} \]

1.0 \leq \cot \theta \leq 2.5