Modular Portfolio Theory:
A General Framework with Risk and Utility Measures as well as Trading Strategies on Multi-period Markets

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Abstract

The following four modular building blocks are crucial in the context of portfolio theory: (a) the market model, (b) the "trading strategy" of the investor, (c) the risk and utility function and (d) the optimization problem. The setting of the so called "modern portfolio theory" by [Markowitz, John Wiley & Sons, Hoboken, NJ, 1959] and the capital asset pricing model (CAPM) by [Sharpe, The Journal of Finance, 19(3):425–442, 1964] consist of a one-period market model for (a) with the standard deviation as risk and the expected return as utility for (c). The trading strategies/portfolios are then just elements in the vector space $\mathbb{R}^n$ whose entries represent the corresponding asset in the portfolio, i.e. there is nothing to do in block (b). The optimization problem, in general, is parameter depend, because of a trade-off between risk and utility. It is known in the modern portfolio theory, that the risk-utility values of the solutions, i.e. of the so-called efficient portfolios, for different parameters are on the boundary of a convex set within the two-dimensional risk-utility space. In the CAPM they even lie on a straight line.

A generalization of such results is studied in [Rockafellar et al., Journal of Banking & Finance, 30(2):743–778, 2006] and [Maier-Paape and Zhu, Risks, 6(2):53, 2018]. In both literatures efficient portfolios are studied similar as in the modern portfolio theory and CAPM. In [Rockafellar et al., 2006] they use the same linear utility function but the risk is allowed to be a so-called generalized deviation measure, which e.g. requires convexity and also positive homogeneity. In [Maier-Paape and Zhu, 2018] a similar situation is discussed but the risk is defined with less assumptions, e.g. positive homogeneity is not required anymore. In addition, more general (concave) utility functions are allowed, which, however, still must be of a special form. Therein, block (a) again as one-period market and blocks (c) and (d) are studied. Again, the portfolios/trading strategies are just vectors in $\mathbb{R}^n$ because a one-period model allows no trading strategies.

This theory is now extended by the subdivision into the four modular building blocks (a) to (d) for general multi-period market models. The portfolios then are no longer represented by vectors in $\mathbb{R}^n$ but can be complex time dependent trading strategies for investing the wealth. Examples are the buy and hold strategy or a strategy which reallocates the invested money after each day to ensure constant weights. Reasonable properties of (a) and (b), which are required for (c) and (d), are studied as well. Furthermore, the (concave) utility now is of general form. With just a few reasonable assumptions on the market model, the trading strategy and the risk and utility functions, the existence and uniqueness of efficient portfolios and many similar results as in the references above are shown here for the modular setup as well.

The multi-period model is required e.g. to define risk functions based on the so-called drawdown. The drawdown of an equity curve is the (absolute or relative) difference between the maximum of this curve and its last value. Since this is an important value in praxis, the application of the optimization problem from above is studied also for a drawdown risk measure. However, the drawdown is difficult to evaluate. Therefore, some reasonable properties are derived in more detail for the absolute drawdown in case of an equity curve modeled by a Markov chain. For a general case of a Markov chain it is shown that the limit distribution for the absolute drawdown after $N \in \mathbb{N}$ time steps as N goes to infinity exists if and only if the expectation of a win or loss after each time step is positive. For some special cases, the corresponding distribution can even be expressed at least with an implicit formula. In the random walk case, an explicit expression can be given for the distribution after just $N \in \mathbb{N}$ time steps as well as in the limit case.

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Notation and Abbreviations

Notation

$\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$

$\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$

Signum function: $\operatorname{sgn}(\beta) := \begin{cases} -1, & \beta < 0, \\ 0, & \beta = 0, \\ 1, & \beta > 0. \end{cases}$

Kronecker delta: $\delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Abbreviations

a.s.: almost surely

gcd: greatest common devisor

iid: independent and identically distributed

TWR: terminal wealth relative

w.l.o.g.: without loss of generality

1 Introduction

The portfolio optimization involves more than the declaration and solution of some optimization problem. Assume an investor wants to build a portfolio based on $M \in \mathbb{N}$ different stocks. To decide how much to invest into each of these stocks depends on his preferences. In general one is interested in minimizing the risk and maximizing the (expected) return. The return is denoted by the more general expression utility in the remainder of this work because this function can be exchanged. In most cases, there is a trade-off between risk and utility. Hence, there is no solution for this problem. Instead it is possible to minimize risk such that the utility does not fall below a given but fixed threshold. Three natural questions and also one important practical question arise: (i) Which risk and utility measure should be used? (ii) How to evaluate them? (iii) Does there exist a (unique) solution? (iv) When a solution is found, how to realize the portfolio for the next period of time?

For this problem, Markowitz [36] published in the year 1952 a proposal known under the keyword "modern portfolio selection". Markowitz fixed the risk measure to the standard deviation of the portfolio and the utility is defined by the expected return, which is an arithmetic mean. Thus, questions (i) and also (ii) are answered. Under some assumptions, the answer to question (iii) is known to be positive in the Markowitz situation, i.e. in appropriate situations there exist unique solutions. One goal of the following is to introduce a large class of risk and utility measures and to answer question (iii) for new situations (see Chapter 2) in a modular way which includes the answer to question (iv). Another goal is to examine one special risk measure, the so-called drawdown, in more detail (see Chapter 3), which tackles questions (i) and (ii).

Content of Chapter 2

Let the $M \in \mathbb{N}$ assets from above supposed to be risky in some sense, which is characterized in detail in Chapter 2. Without loss of generality (w.l.o.g.) assume there is also a (risk-free) bond available which has index zero. In total this gives $M+1$ assets. The investor is interested in the "optimal" portfolio consisting of these financial products. Such a portfolio consists of the initial investment and a strategy to administrate the money over time, which can depend on the development of the equity curve. For instance, it is possible to reallocate the money after each day, e.g. to distribute winnings from the winner stocks to all others, or it is possible to just buy and hold the $M+1$ assets according to the initial investment.

Suppose the strategy is fixed, e.g. the investor wants to minimize reallocation costs and therefore follows the buy and hold approach. Then, only an "optimal" initial investment given by some vector $\mathbf{x} \in \mathbb{R}^{M+1}$ is required. The definition of "optimal" typically is based on a given risk measure $\mathfrak{r}\colon \mathbb{R}^{M+1} \to \mathbb{R}$ and a given utility measure $\mathfrak{u}\colon \mathbb{R}^{M+1} \to \mathbb{R}$, which both only depend on $\mathbf{x}$. The corresponding optimization problem can be formulated as

$$\min_{\mathbf{x} \in \mathbb{R}^{M+1}} \mathfrak{r}(\mathbf{x}) \quad \text{subject to } \mathfrak{u}(\mathbf{x}) \geq \mu,\ S^{(0)}\mathbf{x} = \beta, \tag{1.1}$$

where $\mu, \beta \in \mathbb{R}$ are given but fixed parameters, and $S^{(0)}\mathbf{x}$ denotes the total initial investment for the vector $\mathbf{x}$. The solution is the "optimal" initial investment.

Of course, the risk and utility in problem (1.1) should be based on the "trading strategy" used by the investor and on a reasonable model of the $M+1$ assets. Hence, an appropriate market model is required. The strategy of the investor should be known in advance, because this characterizes the behavior of the investor when having the initial distribution of money for the $M + 1$ assets. Using the market model and the strategy leads to the wealth of this portfolio. Note that this wealth is also a stochastic process because the market model consists of stochastic processes as well. Based on this, $\mathfrak{r}$ and $\mathfrak{u}$ and, consequently, (1.1) can be defined. Altogether, this gives the four modular building blocks, see also Figure 1.1:

(a) the multi-period market model,
(b) the trading strategy (called x-portfolio) of the investor (with the initial investment $\mathbf{x} \in \mathbb{R}^{M+1}$ as parameter),
(c) the definition of (convex) risk and (concave) utility functions based on (a) and (b),
(d) the optimization problem (see (1.1)): Find the initial investment such that it minimizes the risk under the constraint of a utility greater than or equal to a given threshold.

In the literature, authors often focus on just one or two of these four building blocks while fixing the others at the beginning, which are often restricted to special situations. Often a one-period model is used, i.e. only one future time step is modeled for each stock, see e.g. Markowitz [36, 37], the capital asset pricing model (CAPM) by Lintner, Mossin and Sharpe [31, 38, 52] (respectively), Rockafellar, Uryasev and Zabarankin [48], and Maier-Paape and Zhu [34]. In case of a one-period market model, the portfolio in (b) has only $M+1$ degrees of freedom and therefore is already fully represented by some vector $\mathbf{x} \in \mathbb{R}^{M+1}$. Hence, in all the above cited publications there is no further discussion for (b) because future time steps and, thus, the trading strategy of the investor cannot be modeled. In addition, the one-period market only allows to define risk measures independent of the “shape” of resulting equity curves and therefore limits the possibilities. Since the development of the wealth over time is an important factor for investors, a multi-period market model would be preferable, see e.g. Föllmer and Schied [18, Section 5.1] or Carr and Zhu [7, Chapter 3] for an introduction to such a model.

The superordinate goal of Chapter 2 is to fully characterize all four modular building blocks (a) to (d) in detail. This includes the connection of (a) and (b) to be able to define the risk and utility of block (c) in such a way that the existence and uniqueness for the solutions in (d) is guaranteed. To reach a high level of flexibility, it is tried to work with only few restrictions and assumptions for the market model, the trading strategy, the risk and utility measures and the optimization problem. To the best of the authors knowledge, the portfolio optimization has not been stated and discussed in such a detailed modular way and general form. Note that [48] and also [34] discuss (a) using a one-period market, use for (c) less general utility measures and prove similar results for (d). The discussion therein regarding building block (b), which is of high practical relevance, just restricts to the initial investment $\mathbf{x}$ because the strategy cannot be modeled appropriately (and is not important) for a one-period market model. However, the multi-period market model and in particular the more general and detailed look at the trading strategy allow a comparison of the functions used in the literature on a higher level which gives some new insights.

The general definition of a multi-period market model from (a) is introduced in Section 2.1 and the definition of a trading strategy (called x-portfolio) from (b) is given in Section 2.2. Both definitions are based on the work of Föllmer and Schied [18, Section 5.1]. In [18] there are given some reasonable properties for market models regarding what it means to have no arbitrage opportunities, i.e. there exist no trading strategies which produce winnings with positive probability and cannot lose for sure. This theory is extended (also in Section 2.2)

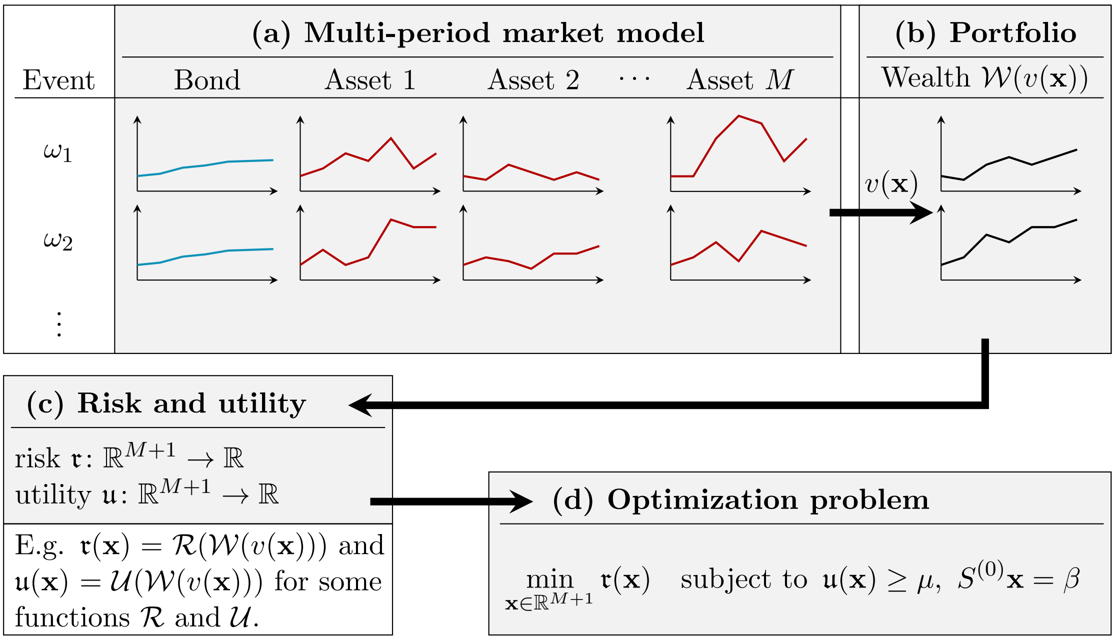


Figure 1.1.: Schematic diagram of the four modular building blocks (a) to (d), where $\mathbf{x} \in \mathbb{R}^{M+1}$ represents the initial investments in the $M+1$ assets (including the bond) and $v(\mathbf{x})$ represents the full trading strategy of the investor for given $\mathbf{x}$.

by the definition of non-redundancy (i.e. the bond cannot be replicated by a combination of risky assets) and by the notion of no nontrivial risk-free x-portfolio (the combination of no arbitrage and non-redundancy). These are properties for the market model and give some reasonable implications. For instance they ensure strict convexity/concavity of the risk/utility function, respectively, in some reasonable situations. This in turn ensures uniqueness of the solution of (1.1). For the remaining blocks (c) and (d), the trading strategy can be chosen without restrictions, except that it should only depend on the initial investment $\mathbf{x} \in \mathbb{R}^{M+1}$. This high degree of flexibility in general is not reached in the literature for such problems, see e.g. the recent work [34, 48]. Three important trading strategy examples, which are typically used in the literature, are discussed in this context as examples:

(i) *Buy and hold*, i.e. buy the assets according to the initial investment $\mathbf{x}$ and do not change the position sizes afterwards,
(ii) *Constant weight*, i.e. reallocate the money after each time step such that the fractions of wealth invested into each asset is constant in time,
(iii) *Constant invested money*, i.e. the amount of invested money is constant for each time step (do not reinvest gains and inject fresh capital after losses).

With this framework at hand, risk and utility measures for (c) can be defined. This is done in Section 2.3 in an axiomatic way, similar to the coherent risk measure from Artzner, Delbaen, Eber and Heath [2] or the generalized deviation measure from Rockafellar, Uryasev and Zabarankin [47]. In addition, some well-known examples are discussed in connection to (a) and especially to (b) in Section 2.3. Let $\mathbf{x} \in \mathbb{R}^{M+1}$ and $v(\mathbf{x})$ be the corresponding trading strategy (x-portfolio) for $\mathbf{x}$. Then, $v(\mathbf{x})$ has $\mathbf{x}$ as initial investment and contains also the strategy for further times steps. The wealth of the portfolio for $N \in \mathbb{N}$ future time steps is modeled by the stochastic process $\mathcal{W}(v(\mathbf{x})) := (\mathcal{W}^{(0)}(v(\mathbf{x})), \mathcal{W}^{(1)}(v(\mathbf{x})), \dots, \mathcal{W}^{(N)}(v(\mathbf{x})))$, where $\mathcal{W}^{(0)} = \mathcal{W}^{(0)}(v(\mathbf{x})) \in \mathbb{R}$ is the fixed initial wealth. A typical utility function has the form

$$\mathfrak{u}(\mathbf{x}) := \mathcal{U}(\mathcal{W}(v(\mathbf{x}))) := \mathrm{E}\left[\mathcal{W}^{(N)}(v(\mathbf{x})) - \mathcal{W}^{(0)}\right]. \tag{1.2}$$

Of course, the resulting utility function $\mathfrak{u}$ depends on the trading strategy given by the function v. For instance, if the trading strategy invests the same amount of money in each time step (see example (iii) above), function $\mathfrak{u}$ corresponds to a linear combination of the expected relative returns of the $M + 1$ assets, i.e. some arithmetic mean. Such a function is used e.g. by Rockafellar, Uryasev and Zabarankin [48]. If the weights for the assets are supposed to be constant in time (see example (ii) above) this gives the expected value of the so-called terminal wealth relative (TWR), which is introduced by Vince [53]. The TWR can be seen as a geometric mean. This shows that the utility functions typically used in the literature in essence are of the same nature, but differ only in the underlying trading strategy of the investor.

Section 2.4 finally discusses the above mentioned optimization problem in (d), see (1.1), including existence and uniqueness of a solution. For this, general utility and risk measures (involving almost arbitrary trading strategies), which can be defined on a general multi-period market model, with only a few assumptions, are allowed. Situations in a one-period market for special risk and utility measures are discussed e.g. by Markowitz [36, 37] with standard deviation and return for risk and utility, respectively. Also its extension falls into this category, namely the capital asset pricing model (CAPM) by Lintner, Mossin and Sharpe [31, 38, 52] (respectively), which allows to also invest into a bond (a risk-free asset with constant rate of returns). Generalizations are given by Rockafellar, Uryasev and Zabarankin [48] which allow more general risk measures, so-called deviation measures, but they still use the (arithmetic) return as utility, i.e. of a similar form as $\mathrm{E}\,[\mathcal{W}^{(1)}(v(\mathbf{x}))]$, cf. (1.2). Note that $\mathcal{W}^{(1)}(v(\mathbf{x}))$ in the corresponding situation is linear in $\mathbf{x}$. Recently, Maier-Paape and Zhu [34] extended the theory for more general utility functions, but still on a one-period market model, which in addition has a finite probability space. The utility therein is still defined as an expectation but in the more general form $\mathrm{E}\,[u(\mathcal{W}^{(1)}(v(\mathbf{x})))]$ (cf. (1.2), where $\mathcal{W}^{(1)}(v(\mathbf{x}))$ is linear as well and $u\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave), e.g. the mean of the logarithms of the returns can be used ($u := \ln$). Following [34], their so-called "general framework for portfolio theory" is here in Section 2.4 extended for the multi-period market model (with general and not necessarily finite or discrete distributions) with weaker assumptions as in [34] for the risk and utility. In addition, the utility is of general type and even does no longer need to be an expectation, e.g. as in (1.2) or in [34]. As in [34], the dependency of the solution (and its risk and utility values) to the parameter μ from (1.1) is discussed in detail in Section 2.4. It turns out, that most results from [34] also hold true in the more general setting and, consequently, the risk and utility values of the solutions for different μ lie on the boundary of a convex set in $\mathbb{R}^2$ and form an "efficient frontier". If the risk and utility functions have some additional properties like positive homogeneity, the efficient frontier is furthermore affine linear. This is a generalization of the one-fund theorem known from the CAPM. In [34], such a result has already been proven for the setting defined therein for positive homogeneous risk measures and in the special case that

u is the identity, i.e. when the utility is of the form $\mathrm{E}\,[\mathcal{W}^{(1)}(v(\mathbf{x}))]$. Such a situation is also discussed in [48].

Content of Chapter 3

A well-known quantity in finance is the drawdown, which gives the difference between the maximum of a given equity curve (where one should have stopped trading) to its current value. This difference is either being measured as the absolute or the relative loss since the maximum. It is easy to compute the drawdown for a given equity curve. For the portfolio theory discussed in Chapter 2, the equity curve for a given portfolio in general is not just one price series but a stochastic process depending on the market model and the trading strategy. Hence, a possible risk measure can be the expected value of the absolute or relative drawdown. Note, that the drawdown for one event of the portfolio's wealth needs a complete equity curve. Therefore, the market model must be a multi-period model. Since the absolute drawdown gives the absolute difference of the maximum of the equity to the current value, this quantity is independent of a vertical shift of the equity curve. For instance, if the maximum of the equity is \$ 1 000 and the current value is \$ 900, the absolute drawdown equals to \$ 100 while the relative drawdown is 10 %. If the equity has its maximum at \$ 10 000 and its current value is \$ 9 900, the absolute drawdown again equals to \$ 100 while in this case the relative drawdown is only 1 %. This already hints a possible problem of the absolute drawdown. The relative variant is a more significant quantity in this example. In a short term view, the equity curve should not vary much, in which case the absolute drawdown can be used without expecting such a big effect. But in a long term view, e.g. in a time horizon of 10 years, the problem illustrated with the example becomes much more relevant. Then the relative drawdown should be preferred over the absolute one.

In the literature, mostly the absolute drawdown is discussed, see e.g. Chekhlov, Uryasev and Zabarankin [8, 9], Goldberg and Mahmoud [20] and Zabarankin, Pavlikov and Uryasev [55], who introduce a measure which, roughly speaking, is the value at risk based on the absolute drawdown. The relative drawdown, however, is rarely studied in the literature. Only recently, Maier-Paape and Zhu [35] discussed properties of the expected value of the logarithm of the relative drawdown for a market model on a finite probability space with independent and identically distributed returns. They also examined some kind of linearization of the (logarithm of the) relative drawdown which defines another risk measure with more favorable properties. Section 3.1 generalizes some results from [35] for general multi-period market models from Chapter 2. In addition, the connection between the relative drawdown an its "linearization" is discussed from a different point of view which establishes a relationship to the absolute drawdown. Section 3.1 contains also the application of the results regarding the portfolio optimization from Chapter 2 to the expected drawdown as risk measure. The existence and uniqueness of efficient portfolios (for given parameters) for the logarithm of the relative drawdown as risk measure can be ensured. It turns out that the linearization of this risk measure, i.e. the expected absolute drawdown, can be used as an application of the generalized one-fund theorem of Section 2.4.

In Section 3.2, the market model and the trading strategy are assumed to be chosen such that the equity curve of the portfolio is characterized by a Markov chain on $\mathbb{Z}$. Then some more results can be shown for the absolute drawdown. For instance, under some mild and reasonable assumptions, the probability distribution of the absolute drawdown after N simulated days converges to a unique limit distribution as N goes to infinity. In the special case, where the equity can jump $\beta \in \mathbb{N}$ units up with probability p and one unit down with probability $q := 1 - p$, this limit distribution and also its expected value can easily be computed. To the best of the author's knowledge, this problem has not been discussed in the literature yet, except for the random walk case, i.e. the case $\beta = 1$, see e.g. Sericola [51, Section 3.1] and Graham [22, Section 3.2.4.3]. Note, that the absolute drawdown in this situation can be modeled by a Markov chain on $\mathbb{N}_0$ with a reflecting barrier at zero and no upper barrier. Markov chains (often just random walks) are discussed in the literature also in the situation with one absorbing lower barrier, i.e. the process sticks at such a barrier once reached, see e.g. Blasi [6], El-Shehawy [13], Feller [17], Gilliland, Levental and Xiao [19], Hardin and Sweet [24], Neuts [39] and Weesakul [54]. Note, that there is often also an absorbing or reflecting upper barrier in the literature, i.e. the state space of the Markov chain is finite. The situation with two reflecting barriers (one upper and one lower) has e.g. been discussed by El-Shehawy [14, 15] and Percus [41]. A random walk with only one reflecting barrier belongs to the so-called discrete-time birth death processes, see e.g. [51, Section 3.1]. In this situation, in most cases only the limit distribution is discussed, see e.g. [51, Section 3.1] and [22, Section 3.2.4.3].

In Section 3.3, for the random walk even the distribution after N steps is given by an explicit formula (which goes beyond the canonical computation when just applying the transition probability N times). Again, no such formula or discussion has been found in the literature so far.

Content of Chapter 4

A summary of the work and some conclusions are given in Chapter 4.

2 Portfolio Theory

Before applying the portfolio theory on a real application, there are three main steps required to encompass the problem fully in mathematical sense. A portfolio is a combination of finitely many assets. At first, a model for these markets is needed. In general, stochastic processes are taken to model the price values for the assets. Step two is to define a trading strategy on how to invest into these assets over time when knowing the initial allocation of wealth. An example could be the trading strategy of "buy and hold", i.e. without rebalancing investments over time. Note that the initial allocation is not part of the strategy itself but part of the asset allocation. Combining the market model and the trading strategy leads to the stochastic process for the equity curve of the complete portfolio, which of course is dependent on the parameters for the initial investment. The third and last step is the definition of an appropriate utility function and risk measure, which can be used for the portfolio optimization.

This section is organized as follows: Step one and two are discussed in Sections 2.1 and 2.2, respectively. Section 2.3 introduces the notion of utility functions and risk measures. Finally, Section 2.4 gives some results in the context of portfolio theory.

2.1 Market model

Possible future states for the market can be modeled by random variables on a reasonable probability space. For this purpose, let $(\Omega, \Sigma, \mathrm{P})$ be the probability space, where Ω represents the set of all possible elementary future events, Σ denotes the σ-algebra, e.g. the power set of Ω, and P is the corresponding probability measure. As set for the random variables in $\mathbb{R}$ the linear space $\mathcal{L}^2 := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}) := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R})$ is used, where the inner product in $\mathcal{L}^2$ is defined by $\langle X, Y\rangle_{\mathcal{L}^2} := \mathrm{E}[XY]$ for $X, Y \in \mathcal{L}^2$ and corresponding norm $\|X\|_{\mathcal{L}^2} := (\mathrm{E}[X^2])^{1/2}$. This space ensures a finite expectation and a finite variance for its elements. If not otherwise stated, equalities and inequalities between random variables are in the sense of holding true almost surely (a.s.). Furthermore, let $\mathcal{L}^2(\mathbb{R}_{>0}) := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R}_{>0}) \subset \mathcal{L}^2$ denote the set of all positive random variables, where $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$.

The same setting has been used e.g. by Rockafellar, Uryasev and Zabarankin [47], but of course there are other reasonable choices like $\mathcal{L}^\infty$ as in Goldberg and Mahmoud [20], see also the discussion in the introduction of [47]. The reason for the selection of $\mathcal{L}^2$ is to ensure the existence of expectation and variance, which is essential for some examples.

Since a family of assets is modeled in the following subsections, there is also need of random vectors. The set of random vectors with values in $\mathbb{R}^M$ for $M \in \mathbb{N}$ is denoted by $\mathcal{L}^2(\mathbb{R}^M) := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R}^M)$, where each component of this vector is an element of $\mathcal{L}^2$. Analogously the set of all random vectors with positive entries is denoted by $\mathcal{L}^2(\mathbb{R}^M_{>0}) := \mathcal{L}^2(\Omega, \Sigma, \mathrm{P}; \mathbb{R}^M_{>0})$.

However, in some cases it is sufficient to work on a bigger space. Therefore, the same notation is used for the space $\mathcal{L}^1$ which contains all random variables with finite expectation, i.e. $\|X\|_{\mathcal{L}^1} := \mathrm{E}[|X|] < \infty$, and also for the set of all random variables denoted by $\mathcal{L}^0$.

Using this structure, the market can be modeled either by a one-period model or by a multi-period model. In both cases, the market consists of $M + 1 \in \mathbb{N}$ financial instruments (or assets) numbered with $i = 0, 1, \ldots, M$. The instrument $i = 0$ is always assumed to be risk-free, which will be defined in more detail in this section. The remaining M assets for $i = 1, \dots, M$ are supposed to be risky and therefore their rates of returns are modeled by elements in $\mathcal{L}^2$ which in general are not constant.

The model for the one-period case is discussed in Subsection 2.1.1 while the multi-period case is introduced in Subsection 2.1.2. Some examples to obtain a market model from historical data are given in Subsection 2.1.3

2.1.1 One-period market model

Assume the current price, i.e. the price at time $n = 0$, of asset $i \in \{0, 1, \dots, M\}$ is known and equals to $S_i^{(0)} \in \mathbb{R}_{>0}$. The price at time step $n = 1$ is then given by the random variable $S_i^{(1)} = S_i^{(0)}(1 + T_i) \in \mathcal{L}^2(\mathbb{R}_{>0})$, where $T_i \in \mathcal{L}^2$, with $T_i \in [-1, \infty)$ almost surely, is the rate of return. Then, investing a total amount of $S_i^{(0)}$ into asset i, will lead to an absolute gain (or loss) of size $S_i^{(1)} - S_i^{(0)} = S_i^{(0)} T_i$ after one time step. The pair $(S_i^{(0)}, S_i^{(1)})$ is a model for the ith asset. Since the asset with index $i = 0$ is always assumed to be "risk-free", the following definition gives its characterization in the one-period setting.

Definition 2.1.1 (One-period risk-free asset) Let $(S_0^{(0)}, S_0^{(1)}) \in \mathbb{R}_{>0} \times \mathcal{L}^2(\mathbb{R}_{>0})$ be a model for an asset. $(S_0^{(0)}, S_0^{(1)})$ is called *risk-free* if $S_0^{(1)}$ is constant with $S_0^{(1)} \equiv c \in \mathbb{R}_{>0}$ a.s. and $c \geq S_0^{(0)}$.

The whole market is then defined as follows.

Definition 2.1.2 (One-period market model, cf. Maier-Paape and Zhu [34, Definition 1]) Let $M \in \mathbb{N}$ risky assets with indexes $1, \dots, M$ and one risk-free asset with index zero be given. The current prices for the $M + 1$ assets are given by $S^{(0)} := (S_0^{(0)}, S_1^{(0)}, \dots, S_M^{(0)}) \in \mathbb{R}_{>0}^{M+1}$. The future prices after one time step are modeled by

$$S^{(1)} := \left(S_0^{(1)}, S_1^{(1)}, \dots, S_M^{(1)}\right) \in \mathbb{R}_{>0} \times \mathcal{L}^2\left(\mathbb{R}_{>0}^M\right) \subset \mathcal{L}^2\left(\mathbb{R}_{>0}^{M+1}\right),$$

respectively, where in this case $\mathbb{R}_{>0}$ is identified with all constant random variables in $\mathcal{L}^2(\mathbb{R}_{>0})$. The stochastic process $S := (S^{(0)}, S^{(1)}) \in \mathbb{R}_{>0}^{M+1} \times \mathcal{L}^2(\mathbb{R}_{>0}^{M+1})$ with only one time step is called a *one-period market model* of size $M + 1$.

The rates of returns of market model S can directly be obtained from its prices by the random vector $T^{(1)} := (T_0^{(1)}, T_1^{(1)}, \dots, T_M^{(1)})$, where $T_i^{(1)} := S_i^{(1)}/S_i^{(0)} - 1$ for $i = 0, 1, \dots, M$. Hence, $T_i^{(1)} > -1$ almost surely and, furthermore, if $S^{(0)} = (1, \dots, 1)$, then $T^{(1)} = S^{(1)} - \mathbf{1}$, where $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^{M+1}$.

Of course, the one-period market model just models one future time step, which can be of different but fixed size, say a month. Thus, when a decision was made based on this kind of model and the investments were done in this way during the subsequent month, the procedure starts at the beginning, and a new decision is required. However, this market model is limited, because the upcoming future prices can be modeled by just one value per asset. There is no (natural) way to analyze the possible behavior of the portfolio's equity curve over time. As a consequence, important quantities, like drawdown, cannot be extracted from this kind of model. In order to apply more complex measures, a path dependent model might be required, which does not model only one future price, but a complete price development consisting of many time steps. Such a multi-period market model is introduced in the next subsection.

2.1.2 Multi-period market model

As in Section 2.1.1, there are $M+1$ financial instruments while the first one with index zero is supposed to be risk-free. As already defined in the one-period market model, the initial prices of the assets are denoted by $S^{(0)} := (S_0^{(0)}, S_1^{(0)}, \dots, S_M^{(0)}) \in \mathbb{R}_{>0}^{M+1}$. In this case, the stochastic process is supposed to model the future prices by $N \in \mathbb{N}$ time steps using the filtered probability space $(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\leq n\leq N}, \mathrm{P})$ for the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$ which satisfies

$$\{\emptyset, \Omega\} =: \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{N-1} \subset \mathcal{F}_N := \Sigma. \tag{2.1}$$

In addition, for $M \in \mathbb{N}$, let

$$\mathcal{L}^2\big(N;\mathbb{R}^M\big) := \mathcal{L}^2\big(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\leq n\leq N}, \mathrm{P};\mathbb{R}^M\big) := \bigtimes_{n=0}^{N} \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P};\mathbb{R}^M\big)$$

and $\mathcal{L}^2(N) := \mathcal{L}^2(N;\mathbb{R})$ with corresponding set of positive processes

$$\mathcal{L}^2\big(N;\mathbb{R}^M_{>0}\big) := \mathcal{L}^2\big(\Omega, \Sigma, \{\mathcal{F}_n\}_{0\leq n\leq N}, \mathrm{P};\mathbb{R}^M_{>0}\big) := \bigtimes_{n=0}^{N} \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P};\mathbb{R}^M_{>0}\big) \subset \mathcal{L}^2\big(N;\mathbb{R}^M\big).$$

Again, a risk-free asset is characterized to have no uncertainty and non-negative returns.

Definition 2.1.3 (Multi-period risk-free asset) For $N \in \mathbb{N}$, the model for an asset given by $(S_0^{(0)}, S_0^{(1)}, \dots, S_0^{(N)}) \in \mathcal{L}^2(N;\mathbb{R}_{>0})$ with $c^{(0)} := S_0^{(0)} > 0$ is called *risk-free* if $S_0^{(n)} \equiv c^{(n)} \in \mathbb{R}_{>0}$ is constant a.s. and $c^{(n)} \geq c^{(n-1)}$ for $n = 1, \dots, N$.

The definition of the multi-period market model then reads:

Definition 2.1.4 (Multi-period market model, cf. Föllmer and Schied [18, Section 5.1]) Let $M \in \mathbb{N}$ risky assets, indexed by $1, \dots, M$, and one risk-free asset, indexed with zero, be given. The current prices for the $M+1$ assets are given by $S^{(0)} := (S_0^{(0)}, S_1^{(0)}, \dots, S_M^{(0)}) \in \mathbb{R}_{>0}^{M+1}$. For $N \in \mathbb{N}$, the future prices after time step n, for $n = 1, \dots, N$, are modeled by

$$S^{(n)} := \Big(S_0^{(n)}, S_1^{(n)}, \dots, S_M^{(n)}\Big) \in \mathbb{R}_{>0} \times \mathcal{L}^2\big(\Omega, \mathcal{F}_n, \mathrm{P};\mathbb{R}^M_{>0}\big) \subset \mathcal{L}^2\Big(\Omega, \mathcal{F}_n, \mathrm{P};\mathbb{R}^{M+1}_{>0}\Big),$$

respectively, where $(S_0^{(0)}, S_0^{(1)}, \dots, S_0^{(N)})$ is risk-free.

The stochastic process $S := (S^{(n)})_{0\leq n\leq N} \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ is called a *multi-period market model* of size $M+1$ with N time steps.

Remark 2.1.5 A multi-period market model for $N = 1$ coincides with a one-period market model, i.e. $\mathcal{L}^2(1;\mathbb{R}^{M+1}_{>0}) = \mathcal{L}^2(\mathbb{R}^{M+1}_{>0})$, because of the following: For $N = 1$ it is $S = (S^{(0)}, S^{(1)}) \in \mathcal{L}^2(\Omega, \Sigma, \{\mathcal{F}_0, \mathcal{F}_1\}, \mathrm{P};\mathbb{R}^{M+1}_{>0})$, i.e. $S^{(0)} := (S_0^{(0)}, S_1^{(0)}, \dots, S_M^{(0)}) \in \mathbb{R}^{M+1}_{>0}$ is fixed (almost surely) and

$$S^{(1)} \in \mathbb{R}_{>0} \times \mathcal{L}^2\big(\Omega, \mathcal{F}_1, \mathrm{P};\mathbb{R}^M_{>0}\big) = \mathbb{R}_{>0} \times \mathcal{L}^2\big(\Omega, \Sigma, \mathrm{P};\mathbb{R}^M_{>0}\big) = \mathbb{R}_{>0} \times \mathcal{L}^2\big(\mathbb{R}^M_{>0}\big)$$

with $S_0^{(1)} \geq S_0^{(0)}$, i.e. $S := (S^{(0)}, S^{(1)})$ is also a one-period market according to Definition 2.1.2. Therefore, a multi-period market model is in the following just called a market model of given length $N \in \mathbb{N}$ and "multi-period" is omitted. ♦

The rates of returns for the multi-period market model S at time steps $n = 1, \dots, N$ are defined by

$$T^{(n)} := \left(T_0^{(n)}, T_1^{(n)}, \dots, T_M^{(n)}\right), \quad T_i^{(n)} := \frac{S_i^{(n)}}{S_i^{(n-1)}} - 1 > -1 \quad \text{a.s. for } i = 0, 1, \dots, M. \tag{2.2}$$

A more iterative representation of $S^{(n)}$ for $n = 1, \dots, N$ would be

$$S^{(n)} = S^{(0)} + \sum_{k=1}^{n} \left(S^{(k)} - S^{(k-1)}\right) \tag{2.3}$$

with absolute returns $S^{(k)} - S^{(k-1)}$. Alternatively, each of its items for $i = 0, \dots, M$ can be written as

$$S_i^{(n)} = S_i^{(n-1)} \left(1 + T_i^{(n)}\right) = S_i^{(0)} \prod_{k=1}^{n} \left(1 + T_i^{(k)}\right) \tag{2.4}$$

with relative returns $1+T_i^{(k)}$. The multi-period market model is much more flexible than the one-period model in Definition 2.1.2, because it allows to measure path dependent quantities like drawdown. Furthermore it allows to easily simulate investments with and also without reinvesting gains for the next time step, which is more challenging for the one-period market model.

2.1.3 Simple market models from historical data

From past price values of assets, at least a possible discrete distribution of the returns can be obtained. Assume there are $M \in \mathbb{N}$ risky assets and one risk-free asset having daily returns for $K \in \mathbb{N}$ days. Their absolute price values are given by the matrix

$$\widehat{S} := \left(\widehat{S}_i^{(n)}\right)_{\substack{0 \le n \le K \\ 0 \le i \le M}} \in \mathbb{R}^{(K+1)\times(M+1)}, \qquad \widehat{S}^{(n)} := \left(\widehat{S}_0^{(n)}, \widehat{S}_1^{(n)}, \dots, \widehat{S}_M^{(n)}\right) \in \mathbb{R}^{M+1}$$

for $n = 0, 1, \dots, K$, where $\widehat{S}_i^{(n)}$ is the absolute price value of the ith asset at time n in the historical data set. The rates of returns are then given by the matrix

$$\widehat{T} := \left(\widehat{T}_i^{(n)}\right)_{\substack{1 \le n \le K \\ 0 \le i \le M}} := \left(\frac{\widehat{S}_i^{(n)}}{\widehat{S}_i^{(n-1)}} - 1\right)_{\substack{1 \le n \le K \\ 0 \le i \le M}} \in \mathbb{R}^{K\times(M+1)},$$

$$\widehat{T}^{(n)} := \left(\widehat{T}_0^{(n)}, \widehat{T}_1^{(n)}, \dots, \widehat{T}_M^{(n)}\right) \in \mathbb{R}^{M+1} \qquad \text{for } n = 1, \dots, K.$$

There are several possibilities to build a market model from these data. Equations (2.3) and (2.4) already hint the distinction between an absolute and a relative version. Some simple models are given by the following examples.

Example 2.1.6 (One-period variant) Only one time step is modeled by the one-period version. Let $\Omega := \{1, \dots, K\}$ for market model $S = (S^{(0)}, S^{(1)})$, where $S^{(0)}$ is fixed, e.g. $S^{(0)} = \widehat{S}^{(0)}$ or $S^{(0)} = \widehat{S}^{(K)}$. There are two possible variants:

(a) *Absolute version:* Assume the absolute returns, seen in the historical data, are representative for the future returns independent on the current prices. Then define

$$S^{(1)}(\omega) := S^{(0)} + \left(\widehat{S}^{(\omega)} - \widehat{S}^{(\omega-1)}\right) \qquad \text{for } \omega \in \Omega, \tag{2.5}$$

according to (2.3) for $n = 1$.

(b) *Relative version:* Whenever the relative returns in $\widehat{T}$ are supposed to be significant, then define

$$S_i^{(1)}(\omega) := S_i^{(0)} \frac{\widehat{S}_i^{(\omega)}}{\widehat{S}_i^{(\omega-1)}} = S_i^{(0)}\left(1 + \widehat{T}_i^{(\omega)}\right) \qquad \text{for } \omega \in \Omega \text{ and } i = 0, 1, \ldots, M,$$

according to (2.4). Then, of course, it is $T_i^{(1)}(\omega) = \widehat{T}_i^{(\omega)}$. In this case it is reasonable to set $S_i^{(0)} = 1$ for $i = 0, 1, \ldots, M$, which leads to $S^{(1)}(\omega) = \mathbf{1} + \widehat{T}^{(\omega)}$.

In both cases it is $S \in \mathcal{L}^2(1; \mathbb{R}_{>0}^{M+1})$ because the historical data are positive and finite and consequently bounded from above and bounded away from zero. ♦

For the multi-period market model, the easiest possible method is to assume that there is only one path. In this case there would be no uncertainty.

Example 2.1.7 (Multi-period single path variant) Let $\Omega := \{1\}$, i.e. only one event can occur. Then $N := K$ and $S = (S^{(n)})_{0 \le n \le N} \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$, where $S^{(n)} := \widehat{S}^{(n)}$ for $n = 0, 1, \ldots, N$. Consequently, it is $T^{(n)} = \widehat{T}^{(n)}$ for $n = 1, \ldots, N$. ♦

A more complex multi-period model is the following.

Example 2.1.8 (Multi-period variant with different paths) Define the sample space by $\Omega := \{1, \ldots, K\}^N$ with elementary events $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$. The initial state for $S = (S^{(n)})_{0 \le n \le N}$ is assumed to be defined by $S^{(0)} := \widehat{S}^{(0)}$. Similar to Example 2.1.6, there are two possible variants:

(a) *Absolute version:* According to (2.3), define

$$S^{(n)}(\omega) := S^{(0)} + \sum_{k=1}^{n} \left(\widehat{S}^{(\omega_k)} - \widehat{S}^{(\omega_k - 1)}\right), \qquad \omega \in \Omega, \tag{2.6}$$

for $n = 1, \ldots, N$.

(b) *Relative version:* According to (2.4), define

$$S_i^{(n)}(\omega) := S_i^{(0)} \prod_{k=1}^{n} \left(1 + \widehat{T}_i^{(\omega_k)}\right), \qquad \omega \in \Omega,$$

for $n = 1, \ldots, N$ and $i = 0, 1, \ldots, M$. Then $T_i^{(n)}(\omega) = \widehat{T}_i^{(\omega_n)}$. Again, it is reasonable to set $S_i^{(0)} = 1$ for $i = 0, 1, \ldots, M$ in this case.

Since the historical data are positive and finite and therefore bounded from above and bounded away from zero, it is $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ in both cases. ♦

Of course, there are much more models which are also more complex and might be better representations for the market, see e.g. Gontis et al. [21] and the references therein. However, this would go far beyond the scope of this work. Modeling the market, e.g. from historical data, is a procedure behind the scene of portfolio theory. Once done, it is necessary to define a trading strategy for the portfolio to build its stochastic process. Then it is possible to measure chance and risk.

2.2 Portfolios

The notation for the (multi-period) market model for a set of assets was already introduced in Section 2.1. A portfolio consists of a combination of such assets. For this, the notion from Föllmer and Schied [18] is used. Note, that such a portfolio therein is called a trading strategy. This setting is often discussed in the context of dynamic arbitrage theory. Two central properties are firstly the so called self-financing portfolio, i.e. there is no cash flow in or out of the portfolio, and secondly the arbitrage opportunity. Both are introduced e.g. in [18, Section 5] and also in Carr and Zhu [7, Chapter 3]. Maier-Paape and Zhu [34] discussed a one-period market model with a strongly related notion of arbitrage opportunity. This definition is extended in [34] by the so called risk-free portfolio and bond replicating portfolio. In the subsequent subsection, this theory is discussed in detail and extended to the multi-period market model. It closes the bridge between the related results in [34] for the one-period market model and the basic results in [18], and gives some more insights regarding different characterizations of the portfolio in the multi-period case.

First, the general definition of a trading strategy, i.e. the combination of assets, is explained in Subsection 2.2.1. Of course, there are combinations that make no sense, such that Subsection 2.2.2 gives a minimum requirement on the strategy to be so-called admissible. Another property, a trading strategy often should meet, is to reinvest all gains to profit from compounded interest. The corresponding definition of self-financing portfolios is discussed in Subsection 2.2.3. For a market model, there should be no trading strategy, that replicates or even beats the risk-free asset almost surely without downside risk, i.e. there should be no arbitrage opportunity. The exact definition and results and some extensions are given in Subsection 2.2.4. Finally, Subsection 2.2.5 gives examples and also a definition for a function, which maps a vector to a complete trading strategy. Using such a definition is much more handy regarding portfolio optimization.

2.2.1 Definition of portfolio trading strategies

Next, the weights for the $M+1$ assets are defined. These definitions are closely related to [18, Section 5.1]. Note, that the vectors $S^{(n)}$ for $n = 0, \dots, N$ are row vectors, while in the following the vectors $\mathbf{x}^{(n)}, \boldsymbol{\varphi}^{(n)} \in \mathbb{R}^{M+1}$ are column vectors, i.e. for $\mathbf{x}^{(n)} := (x_0^{(n)}, x_1^{(n)}, \dots, x_M^{(n)})^\intercal$ it is $S^{(n)}\mathbf{x}^{(n)} = \sum_{i=0}^{M} S_i^{(n)} x_i^{(n)}$.

Definition 2.2.1 **(x-portfolio / trading strategy)** Let a market model $S := (S^{(n)})_{0 \le n \le N} \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ for $M, N \in \mathbb{N}$ according to Definition 2.1.4 be given. An *x-portfolio* (or trading strategy) for market model S is a time dependent vector

$$\mathbf{X} := \left(\mathbf{x}^{(n)}\right)_{1 \le n \le N} \in \mathcal{L}^0\big(N-1; \mathbb{R}^{M+1}\big) := \bigtimes_{n=0}^{N-1} \mathcal{L}^0\big(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1}\big),$$

for the same filtered probability space $(\Omega, \Sigma, \{\mathcal{F}_n\}_{0 \le n \le N}, \mathrm{P})$ from the market model, and

$$\mathbf{x}^{(n)} := \left(x_0^{(n)}, x_1^{(n)}, \dots, x_M^{(n)}\right)^\intercal \in \mathcal{L}^0\big(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}\big)$$

for $n = 1, \dots, N$.

The purpose and interpretation of $\mathbf{x}^{(n)}$ is the following: Each entry $x_i^{(n)}$ for $n = 1, \ldots, N$ and $i = 0, 1, \ldots, M$ of vector $\mathbf{x}^{(n)}$ corresponds to the number of shares invested into the ith asset from time step $n-1$ to n. The random variable $S_i^{(n-1)}\mathbf{x}_i^{(n)}$ then is the amount of money invested into the ith asset and $S_i^{(n)}\mathbf{x}_i^{(n)}$ is the absolute value of this investment after time step from $n-1$ to n. Hence, the combination of the assets using an x-portfolio gives the wealth of the corresponding portfolio.

Definition 2.2.2 (Wealth of x-portfolio) Let $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ be an arbitrary but fixed market model for $M, N \in \mathbb{N}$ and $\mathcal{W}^{(0)} \in \mathbb{R}_{>0}$ the fixed initial wealth from the investor. The corresponding *wealth* process for arbitrary x-portfolio $\mathbf{X} \in \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ is a stochastic process of the form $\mathcal{W}(\mathbf{X}) = (\mathcal{W}^{(n)}(\mathbf{X}))_{0 \le n \le N}$, where $\mathcal{W}^{(0)}(\mathbf{X}) := \mathcal{W}^{(0)}$ and

$$\mathcal{W}^{(n)}(\mathbf{X}) := \mathcal{W}^{(n-1)}(\mathbf{X}) + \left(S^{(n)} - S^{(n-1)}\right)\mathbf{x}^{(n)} = \mathcal{W}^{(0)} + \sum_{k=1}^{n} \left(S^{(k)} - S^{(k-1)}\right)\mathbf{x}^{(k)} \quad (2.7)$$

for $n = 1, \ldots, N$. Hence, $\mathcal{W}$ is an affine linear functional with domain $\mathcal{L}^0(N-1; \mathbb{R}^{M+1})$.

Note that $\mathbf{x}^{(n)}$ is $\mathcal{F}_{n-1}$ measurable in contrast to $S^{(n)}$, which is $\mathcal{F}_n$ measurable. This can be interpreted such that $\mathbf{x}^{(n)}$ is a "predictable" process, because its values from time step $n-1$ to n are already known after step $n-1$. This makes sense, because the number of shares invested from step $n-1$ to n must be known not later than time step $n-1$, but of course the price development from $n-1$ to n is not known before step n is reached.

The values of an x-portfolio can be seen as absolute values which are more related to the absolute representation (2.3). However, in some cases it is advantageous to work with fractions of wealth instead of numbers of shares, which leads to the next definition.

Definition 2.2.3 (φ-portfolio) Let market model $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ and an x-portfolio $\mathbf{X} \in \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ with wealth process $\mathcal{W}$ be given.

If $\mathcal{W}^{(n-1)}(\mathbf{X}) > 0$ a.s. for $n = 1, \ldots, N$, then the corresponding *φ-portfolio* of $\mathbf{X}$ is the time dependent relative weight vector $(\boldsymbol{\varphi}^{(n)})_{1 \le n \le N} \in \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ defined by

$$\boldsymbol{\varphi}^{(n)} := \left(\varphi_0^{(n)}, \varphi_1^{(n)}, \ldots, \varphi_M^{(n)}\right)^{\mathsf{T}}, \qquad \text{where } \varphi_i^{(n)} := \frac{S_i^{(n-1)} x_i^{(n)}}{\mathcal{W}^{(n-1)}(\mathbf{X})} \quad (2.8)$$

for $i = 0, 1, \ldots, M$ and $n = 1, \ldots, N$.

In contrast to $\mathbf{x}^{(n)}$, each entry $\varphi_i^{(n)}$ of vector $\boldsymbol{\varphi}^{(n)}$ for $n = 1, \ldots, N$ in (2.8) is the fraction of the invested money used for the ith asset from step $n-1$ to n. Note, that $\boldsymbol{\varphi}^{(n)}$ is $\mathcal{F}_{n-1}$ measurable for the same reason as $\mathbf{x}^{(n)}$.

The portfolio's wealth (2.7) can also be expressed in terms of $\boldsymbol{\varphi}^{(n)}$. From the definition in (2.8) and using $T^{(n)}$ from (2.2) one can conclude for $n = 1, \ldots, N$ that

$$\left(S^{(n)} - S^{(n-1)}\right)\mathbf{x}^{(n)} = \mathcal{W}^{(n-1)}(\mathbf{X}) \sum_{i=0}^{M} \left(S_i^{(n)} - S_i^{(n-1)}\right) \frac{\varphi_i^{(n)}}{S_i^{(n-1)}} = \mathcal{W}^{(n-1)}(\mathbf{X}) T^{(n)} \boldsymbol{\varphi}^{(n)}$$

and consequently

$$\mathcal{W}^{(n)}(\mathbf{X}) = \mathcal{W}^{(n-1)}(\mathbf{X})\left(1 + T^{(n)}\boldsymbol{\varphi}^{(n)}\right) = \mathcal{W}^{(0)} \prod_{k=1}^{n} \left(1 + T^{(k)}\boldsymbol{\varphi}^{(k)}\right). \quad (2.9)$$

Quantity	Interpretation
$x_i^{(n)}$	number of shares to buy.
$S_i^{(n-1)} x_i^{(n)}$	amount of money to invest.
$S_i^{(n)} x_i^{(n)}$	outcome of the trade.
$(S_i^{(n)} - S_i^{(n-1)}) x_i^{(n)}$	absolute win/loss of the trade.
$\varphi_i^{(n)}$	fraction of the current wealth to invest.
$T_i^{(n)} \varphi_i^{(n)}$	relative win/loss of the trade.

Table 2.1.: Interpretation for x-portfolio and φ-portfolio for a trade of ith asset from time step $n-1$ to n.

A summary of the interpretations for some quantities are listed in Table 2.1

Remark 2.2.4 From Remark 2.1.5 it is known, that a multi-period market model for $N = 1$ corresponds to a one-period market model. In this case, $\mathbf{x}^{(1)} \equiv \mathbf{x} \in \mathbb{R}^{M+1}$ and $\boldsymbol{\varphi}^{(1)} \equiv \boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ are fixed for some appropriate vectors $\mathbf{x}$ and $\boldsymbol{\varphi}$, and the φ-portfolio is always well-defined. ♦

Note, that by the Cauchy-Schwarz inequality $\mathrm{E}[XY]^2 \leq \mathrm{E}[X^2]\,\mathrm{E}[Y^2]$ holds. Hence, whenever $\mathbf{X} \in \mathcal{L}^2(N-1; \mathbb{R}^{M+1})$, then $\mathrm{E}[|\mathcal{W}^{(n)}(\mathbf{X})|] < \infty$ for $n = 1, \ldots, N$ follows, because the market model S is also defined in $\mathcal{L}^2$. This ensures at least a finite expectation of $\mathcal{W}(\mathbf{X})$, i.e. $\mathcal{W}(\mathbf{X}) \in \mathcal{L}^1(N)$, albeit it need not be in $\mathcal{L}^2(N)$.

2.2.2 Admissible portfolio

As the market model is defined on $\mathcal{L}^2$, it would be a natural requirement for the portfolios wealth to also be in $\mathcal{L}^2$. This property depends on $\mathbf{X}$ which leads to the following notion.

> **Definition 2.2.5 ($\mathcal{L}^2$-admissible x-portfolio)** Let market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ according to Definition 2.1.4 be given. An x-portfolio $\mathbf{X}$ from Definition 2.2.1 is called an *$\mathcal{L}^2$-admissible x-portfolio*, if $\mathcal{W}(\mathbf{X}) \in \mathcal{L}^2(N)$, where $\mathcal{W}$ is the wealth from Definition 2.2.2. The set of all $\mathcal{L}^2$-admissible x-portfolios is denoted by $\mathcal{A}^2 = \mathcal{A}^2(S) \subset \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$.

The overall goal is to compute "optimal" weights for a portfolio for the next period of time until another computation for new weights is required. For this it is essential to avoid ruin. At least for the market model, one can define admissible x-portfolios for which it is not possible of getting ruined.

> **Definition 2.2.6 (Admissible x-portfolio)** Let market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be given. An x-portfolio $\mathbf{X}$ is called an *admissible x-portfolio*, if $\mathcal{W}^{(n)}(\mathbf{X}) > 0$ a.s. for $n = 1, \ldots, N$. The set of all admissible x-portfolios is denoted by $\mathcal{A} = \mathcal{A}(S) \subset \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$.

Note that the definition of a φ-portfolio already requires that the corresponding x-portfolio is "almost" admissible; "almost" because the last value $\mathcal{W}^{(N)}$ is still allowed to be zero or even negative. Nevertheless, the following result gives sufficient and necessary conditions for the x-portfolio and φ-portfolio.

Proposition 2.2.7 Let market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ for $M, N \in \mathbb{N}$ be given. Then the following holds true:

(a) If an x-portfolio is admissible, then the corresponding φ-portfolio $(\boldsymbol{\varphi}^{(n)})_{1\leq n\leq N}$ exists and satisfies $1 + T^{(n)}\boldsymbol{\varphi}^{(n)} > 0$ a.s. for all $n = 1, \ldots, N$.

(b) If $(\boldsymbol{\varphi}^{(n)})_{1\leq n\leq N}$ satisfies $1+T^{(n)}\boldsymbol{\varphi}^{(n)} > 0$ a.s. for all $n = 1, \ldots, N$, then it corresponds to an x-portfolio according to relation (2.8), which is moreover admissible.

Proof. Proof of (a): From assumption, the x-portfolio, say $\mathbf{X}$, is admissible. This yields that $\mathcal{W}^{(n)}(\mathbf{X}) > 0$ a.s. for each $n = 0, 1, \ldots, N$ and that the φ-portfolio is well-defined. From (2.9) it is known that

$$\mathcal{W}^{(n)}(\mathbf{X}) = \mathcal{W}^{(n-1)}(\mathbf{X})\Big(1 + T^{(n)}\boldsymbol{\varphi}^{(n)}\Big).$$

Since $\mathcal{W}^{(n)}(\mathbf{X}) > 0$ for each $n = 0, 1, \ldots, N$ it must be $1 + T^{(n)}\boldsymbol{\varphi}^{(n)} > 0$ a.s. for all $n = 1, \ldots, N$, which proves (a).

Proof of (b): If $1 + T^{(n)}\boldsymbol{\varphi}^{(n)} > 0$ a.s. for all $n = 1, \ldots, N$, then $V^{(0)} := \mathcal{W}^{(0)} \in \mathbb{R}_{>0}$ and $V^{(n)} := V^{(n-1)}(1 + T^{(n)}\boldsymbol{\varphi}^{(n)}) > 0$ a.s. for $n = 1, \ldots, N$. For all $n = 1, \ldots, N$ and all $i = 0, 1, \ldots M$, by assumption $S_i^{(n-1)} > 0$ a.s. and

$$x_i^{(n)} := \frac{\varphi_i^{(n)}}{S_i^{(n-1)}} V^{(n-1)} \quad \text{is well-defined and equivalent to} \quad \varphi_i^{(n)} = \frac{S_i^{(n-1)} x_i^{(n)}}{V^{(n-1)}}. \tag{2.10}$$

This implies

$$T^{(n)}\boldsymbol{\varphi}^{(n)} = \sum_{i=0}^{M} T_i^{(n)} \frac{S_i^{(n-1)} x_i^{(n)}}{V^{(n-1)}} = \sum_{i=0}^{M} \frac{(S_i^{(n)} - S_i^{(n-1)}) x_i^{(n)}}{V^{(n-1)}} = \frac{(S^{(n)} - S^{(n-1)}) x^{(n)}}{V^{(n-1)}}. \tag{2.11}$$

Inserting (2.11) into the definition of $V^{(n)}$ leads to

$$V^{(n)} = V^{(n-1)} + \Big(S^{(n)} - S^{(n-1)}\Big) x^{(n)} = V^{(0)} + \sum_{k=1}^{n} \Big(S^{(k)} - S^{(k-1)}\Big) x^{(k)}.$$

Since $V^{(0)} = \mathcal{W}^{(0)}$, this is exactly the definition of $\mathcal{W}^{(n)}(\mathbf{X})$, see (2.7), i.e. $V^{(n)} = \mathcal{W}^{(n)}(\mathbf{X})$ for $n = 1, \ldots, N$. The relation (2.10) corresponds to (2.8) and therefore, $(\boldsymbol{\varphi}^{(n)})_{1\leq n\leq N}$ is the φ-portfolio of $\mathbf{X}$, which is admissible because $\mathcal{W}^{(n)}(\mathbf{X}) = V^{(n)} > 0$ almost surely for $n = 1, \ldots, N$. $\square$

Convexity and concavity of functions play an important role for optimization problems. For this property to make sense, the domain of a function needs to be a convex set.

Proposition 2.2.8 For a given market model, the set of all admissible x-portfolios $\mathcal{A}$ is convex. The set of all $\mathcal{L}^2$-admissible x-portfolios $\mathcal{A}^2$ is linear. In addition, $\mathcal{A}^2$ contains all constants (constant in $\omega \in \Omega$) and all $\mathbf{X} = (\mathbf{x}^{(n)})_{1\leq n\leq N} \in \mathcal{L}^2(N-1; \mathbb{R}^{M+1})$ with $x_i^{(n)} = 0$ a.s. for $n = 1, \ldots, N$ and $i = 1, \ldots, M$ (i.e. only $x_0^{(n)}$ may differ from zero).

Proof. Let $\lambda \in [0,1]$, $\mathbf{X} := (\mathbf{x}^{(n)})_{1\leq n\leq N} \in \mathcal{A}$ and $\mathbf{Y} := (\mathbf{y}^{(n)})_{1\leq n\leq N} \in \mathcal{A}$ be arbitrary. Then, for all $n = 1,\dots,N$ it is $\mathcal{W}^{(n)}(\mathbf{X}) > 0$ and $\mathcal{W}^{(n)}(\mathbf{Y}) > 0$, and consequently, because of (2.7),

$$\begin{aligned}\mathcal{W}^{(n)}(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) &= (\lambda + (1-\lambda))\mathcal{W}^{(0)} + \sum_{k=1}^{n}\left(S^{(k)} - S^{(k-1)}\right)\left(\lambda\mathbf{x}^{(k)} + (1-\lambda)\mathbf{y}^{(k)}\right)\\ &= \lambda\mathcal{W}^{(n)}(\mathbf{X}) + (1-\lambda)\mathcal{W}^{(n)}(\mathbf{Y}) > 0.\end{aligned}$$

Hence, $\lambda\mathbf{X} + (1-\lambda)\mathbf{Y} \in \mathcal{A}$ and thus $\mathcal{A}$ is convex. The upper equality also holds true for all $\lambda \in \mathbb{R}$ and all $\mathbf{X},\mathbf{Y} \in \mathcal{A}^2$ and therefore for all $n = 1,\dots,N$ one obtains

$$\mathcal{W}^{(n)}(\lambda\mathbf{X}) = \lambda\mathcal{W}^{(n)}(\mathbf{X}) + (1-\lambda)\mathcal{W}^{(n)}(0) = \lambda\mathcal{W}^{(n)}(\mathbf{X}) + (1-\lambda)\mathcal{W}^{(0)},$$

which implies that $\lambda\mathbf{X} \in \mathcal{A}^2$ for all $\lambda \in \mathbb{R}$ if $\mathbf{X} \in \mathcal{A}^2$. In addition, it follows that

$$\begin{aligned}4\,\mathrm{E}\left[|\mathcal{W}^{(n)}(\mathbf{X}+\mathbf{Y})|^2\right] &= \mathrm{E}\left[|\mathcal{W}^{(n)}(2\mathbf{X}) + \mathcal{W}^{(n)}(2\mathbf{Y})|^2\right]\\ &\leq \mathrm{E}\left[|\mathcal{W}^{(n)}(2\mathbf{X})|^2\right] + 2\,\mathrm{E}\left[|\mathcal{W}^{(n)}(2\mathbf{X})\mathcal{W}^{(n)}(2\mathbf{Y})|\right] + \mathrm{E}\left[|\mathcal{W}^{(n)}(2\mathbf{Y})|^2\right],\end{aligned}$$

which is finite because of Cauchy-Schwarz inequality for all $n = 1,\dots,N$ and all $\mathbf{X},\mathbf{Y} \in \mathcal{A}^2$. Hence, $\mathbf{X}+\mathbf{Y} \in \mathcal{A}^2$. If $\mathbf{X}$ is constant (in $\omega \in \Omega$), then $\mathbf{X} \in \mathcal{A}^2$ because $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$. If only the first component of $\mathbf{X}$ can differ from zero and is in $\mathcal{L}^2$, then $\mathbf{X} \in \mathcal{A}^2$ because $S_0^{(n)}$ is constant for $n = 0,1,\dots,N$ by assumption in Definition 2.1.4. □

2.2.3 Self-financing

Another question of a portfolio investment over different time periods is whether or not to reinvest gains. For reinvesting gains, the following definition is crucial.

Definition 2.2.9 (Self-financing, see Föllmer and Schied [18, Definition 5.4]) For $N \geq 2$ let market model $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be given. An x-portfolio $\mathbf{X} := (\mathbf{x}^{(n)})_{1\leq n\leq N}$ is called *self-financing*, if

$$S^{(n)}\mathbf{x}^{(n)} = S^{(n)}\mathbf{x}^{(n+1)} \tag{2.12}$$

for all $n = 1,\dots,N-1$.

The left hand side of (2.12) denotes the invested amount of money after the trade from step $n-1$ to n, i.e. the outcome of this trade, and the right hand side is the amount of money invested into the next trade from step n to $n+1$, which needs to be the same in each time step. In other words, self-financing means, that all the invested money remains invested without cash flows in either direction, i.e. without giving new money into or extracting money out of the portfolio.

Proposition 2.2.10 (Linear combination of self-financing x-portfolios) In the situation of Definition 2.2.9 let $\mathbf{X},\mathbf{Y}$ be two self-financing x-portfolios. Then $\alpha\mathbf{X} + \beta\mathbf{Y}$ is also self-financing for all $\alpha,\beta \in \mathbb{R}$.

Proof. This statement directly follows from definition (2.12) when inserting the linear combination. □

If $\mathbf{X}$ is self-financing, then the definition of wealth (2.7) of the portfolio at steps $n = 1, \ldots, N$ is a telescoping sum and thus simplifies to

$$\begin{aligned}\mathcal{W}^{(n)}(\mathbf{X}) &= \mathcal{W}^{(0)} + \sum_{k=1}^{n} S^{(k)}\mathbf{x}^{(k)} - \sum_{k=2}^{n} S^{(k-1)}\mathbf{x}^{(k-1)} - S^{(0)}\mathbf{x}^{(1)} \\ &= \mathcal{W}^{(0)} + \Big(S^{(n)}\mathbf{x}^{(n)} - S^{(0)}\mathbf{x}^{(1)}\Big). \end{aligned} \tag{2.13}$$

If in addition $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$, i.e. all wealth is invested, then $\mathcal{W}^{(n)}(\mathbf{X}) = S^{(n)}\mathbf{x}^{(n)}$. This leads to the following result.

Proposition 2.2.11 For $N \geq 2$ let $\mathbf{X}$ be an x-portfolio of market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$. Then $\mathbf{X}$ is self-financing, if and only if

$$\mathcal{W}^{(n)}(\mathbf{X}) = \mathcal{W}^{(0)} + \Big(S^{(n)}\mathbf{x}^{(n)} - S^{(0)}\mathbf{x}^{(1)}\Big) \tag{2.14}$$

for all $n = 1, \ldots, N$. Furthermore, if $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$, then (2.14) becomes $\mathcal{W}^{(n)}(\mathbf{X}) = S^{(n)}\mathbf{x}^{(n)}$.

Proof. The necessity has already shown with (2.13). It remains to show the opposite direction. For all $n = 1, \ldots, N-1$ it is from assumption (2.14)

$$\mathcal{W}^{(n+1)}(\mathbf{X}) = \mathcal{W}^{(0)} + \Big(S^{(n+1)}\mathbf{x}^{(n+1)} - S^{(0)}\mathbf{x}^{(1)}\Big),$$

and from definition of wealth together with assumption (2.14)

$$\begin{aligned}\mathcal{W}^{(n+1)}(\mathbf{X}) &= \mathcal{W}^{(n)}(\mathbf{X}) + \Big(S^{(n+1)} - S^{(n)}\Big)\mathbf{x}^{(n+1)} \\ &= \mathcal{W}^{(0)} + \Big(S^{(n)}\mathbf{x}^{(n)} - S^{(0)}\mathbf{x}^{(1)}\Big) + \Big(S^{(n+1)} - S^{(n)}\Big)\mathbf{x}^{(n+1)},\end{aligned}$$

respectively. Equating both yields $S^{(n)}\mathbf{x}^{(n)} = S^{(n)}\mathbf{x}^{(n+1)}$ for all $n = 1, \ldots, N-1$, i.e. the x-portfolio is self-financing. □

For a self-financing x-portfolio, there is a relation to its φ-portfolio as follows.

Proposition 2.2.12 For $N \geq 2$ and market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ let $\mathbf{X} := (\mathbf{x}^{(n)})_{1 \leq n \leq N}$ be an admissible x-portfolio with corresponding φ-portfolio $(\boldsymbol{\varphi}^{(n)})_{1 \leq n \leq N}$. Then $\mathbf{X}$ is self-financing, if and only if

$$\sum_{i=0}^{M} \varphi_i^{(n)} = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{S^{(n-1)}\mathbf{x}^{(n)} + \mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)}} \tag{2.15}$$

for all $n = 1, \ldots, N$. Furthermore, if $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$, then (2.15) becomes $\sum_{i=0}^{M} \varphi_i^{(n)} = 1$.

Proof. As already noted above, the φ-portfolio only requires the x-portfolio $\mathbf{X}$ to be admissible, which gives the existence.

If in addition $\mathbf{X}$ is self-financing, definition of the φ-portfolio (2.8) and equation (2.14) of Proposition 2.2.11 yields

$$\sum_{i=0}^{M} \varphi_i^{(n)} = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{\mathcal{W}^{(n-1)}(\mathbf{X})} = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{\mathcal{W}^{(0)} + S^{(n-1)}\mathbf{x}^{(n-1)} - S^{(0)}\mathbf{x}^{(1)}} = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{\mathcal{W}^{(0)} + S^{(n-1)}\mathbf{x}^{(n)} - S^{(0)}\mathbf{x}^{(1)}}$$

for all $n = 2, \ldots, N$. The case $n = 1$ is trivial, because the denominator of (2.15) then degenerates to $\mathcal{W}^{(0)}$. Hence, (2.15) holds true for all $n = 1, \ldots, N$.

The opposite direction is shown by mathematical induction over $N \in \mathbb{N}$ with the following induction hypothesis: If (2.15) holds true for $n = 1, \ldots, N$, then $\mathbf{X}$ is self-financing, i.e.

$$S^{(n)}\mathbf{x}^{(n)} = S^{(n)}\mathbf{x}^{(n+1)} \quad \text{for } n = 1, \ldots, N-1. \tag{2.16}$$

For the case $N = 1$, self-financing has no meaning. Therefore, the base case is $N = 2$: From definition in (2.7) it is $\mathcal{W}^{(1)}(\mathbf{X}) = \mathcal{W}^{(0)} + (S^{(1)}\mathbf{x}^{(1)} - S^{(0)}\mathbf{x}^{(1)})$ which, together with (2.8), gives

$$\sum_{i=0}^{M} \varphi_i^{(2)} = \frac{S^{(1)}\mathbf{x}^{(2)}}{\mathcal{W}^{(1)}(\mathbf{X})} = \frac{S^{(1)}\mathbf{x}^{(2)}}{\mathcal{W}^{(0)} + (S^{(1)}\mathbf{x}^{(1)} - S^{(0)}\mathbf{x}^{(1)})}.$$

On the other hand, from assumption (2.15) of the hypothesis it is

$$\sum_{i=0}^{M} \varphi_i^{(2)} = \frac{S^{(1)}\mathbf{x}^{(2)}}{S^{(1)}\mathbf{x}^{(2)} + \mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)}}.$$

Equating both expressions of the sum yields $S^{(1)}\mathbf{x}^{(1)} = S^{(1)}\mathbf{x}^{(2)}$, i.e. (2.16) for $n = 1$. Consequently the induction hypothesis holds true for $N = 2$.

The induction step from N to $N+1$ reads: From induction hypothesis it is already known that (2.16) holds true for $n = 1, \ldots, N-1$, i.e. $(\mathbf{x}^{(n)})_{0 \le n \le N}$ is self-financing when ignoring the last time step and $\mathbf{x}^{(N+1)}$. Therefore it remains to show $S^{(N)}\mathbf{x}^{(N)} = S^{(N)}\mathbf{x}^{(N+1)}$. Again from definition (2.8) and assumption (2.15) it is

$$\sum_{i=0}^{M} \varphi_i^{(N+1)} = \frac{S^{(N)}\mathbf{x}^{(N+1)}}{\mathcal{W}^{(N)}(\mathbf{X})} \qquad \text{and} \qquad \sum_{i=0}^{M} \varphi_i^{(N+1)} = \frac{S^{(N)}\mathbf{x}^{(N+1)}}{S^{(N)}\mathbf{x}^{(N+1)} + \mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)}},$$

respectively. Hence, the denominator of the right hand side of both formulas must be the same. Since Proposition 2.2.11 can be applied to $(\mathbf{x}^{(n)})_{0 \le n \le N}$ (on the shortened market model obtained by ignoring the time step from N to $N+1$), it is $\mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} + (S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)})$. Consequently, it must be $S^{(N)}\mathbf{x}^{(N)} = S^{(N)}\mathbf{x}^{(N+1)}$, which completes the proof. $\square$

In addition, each x-portfolio can be supplemented to a self-financing trading strategy in a natural way as follows.

Theorem 2.2.13 (Supplemented self-financing portfolio) For $N \geq 2$ let market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ as in Definition 2.1.4 and an arbitrary x-portfolio $\mathbf{X} := (\mathbf{x}^{(n)})_{1 \leq n \leq N}$ be given. The supplemented x-portfolio $\mathbf{Y} := (\mathbf{y}^{(n)})_{1 \leq n \leq N}$ defined by

$$\mathbf{y}^{(n)} := \left(x_0^{(n)} + \frac{\mathcal{W}^{(n-1)}(\mathbf{Y}) - S^{(n-1)}\mathbf{x}^{(n)}}{S_0^{(n-1)}}, x_1^{(n)}, \ldots, x_M^{(n)} \right)^{\mathsf{T}} \quad \text{for } n = 1, \ldots, N \qquad (2.17)$$

is self-financing with $S^{(0)}\mathbf{y}^{(1)} = \mathcal{W}^{(0)}$ and $\mathcal{W}^{(n)}(\mathbf{Y}) = S^{(n)}\mathbf{y}^{(n)}$ for $n = 1, \ldots, N$. In addition, the following two statements hold true:

(a) If $\mathbf{X} \in \mathcal{A}^2$, then $\mathbf{Y} \in \mathcal{A}^2$.

(b) If $S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X})$ a.s. for all $n = 1, \ldots, N$, then $\mathcal{W}^{(n)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X})$ a.s. and $\mathcal{W}^{(n)}(\mathbf{Y}) - \mathcal{W}^{(n-1)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X}) - \mathcal{W}^{(n-1)}(\mathbf{X})$ a.s. for all $n = 1, \ldots, N$.

Proof. For $n = 1, \ldots, N-1$ it is

$$S^{(n)}\mathbf{y}^{(n+1)} = S^{(n)}\mathbf{x}^{(n+1)} + S_0^{(n)} \frac{\mathcal{W}^{(n)}(\mathbf{Y}) - S^{(n)}\mathbf{x}^{(n+1)}}{S_0^{(n)}} = \mathcal{W}^{(n)}(\mathbf{Y})$$

and therefore, using (2.7), $\mathcal{W}^{(n+1)}(\mathbf{Y}) = \mathcal{W}^{(n)}(\mathbf{Y}) + (S^{(n+1)} - S^{(n)})\mathbf{y}^{(n+1)} = S^{(n+1)}\mathbf{y}^{(n+1)}$. Especially for $n = 0$ it is $S^{(0)}\mathbf{y}^{(1)} = \mathcal{W}^{(0)}$ and with (2.7) it is $\mathcal{W}^{(1)}(\mathbf{Y}) = S^{(1)}\mathbf{y}^{(1)}$. Hence, $\mathbf{Y}$ is self-financing and $\mathcal{W}^{(n)}(\mathbf{Y}) = S^{(n)}\mathbf{y}^{(n)}$ for $n = 1, \ldots, N$ because of Proposition 2.2.11.

Proof of (a): In case that $\mathbf{X} \in \mathcal{A}^2$, it must be $\mathbf{Y} \in \mathcal{A}^2$ because it is the sum of $\mathbf{X}$ and an x-portfolio which can only have values unequal to zero in the first component, see Proposition 2.2.8.

Proof of (b): Assume now, that $S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X})$ holds true for all $n = 1, \ldots, N$. The affine linearity of $\mathcal{W}$, see the second representation in (2.7), together with the definition of $\mathbf{Y}$ in (2.17), which is just a shifted variant of $\mathbf{X}$, yields

$$\mathcal{W}^{(n)}(\mathbf{Y}) = \mathcal{W}^{(n)}(\mathbf{X}) + \sum_{k=1}^{n} \left(S_0^{(k)} - S_0^{(k-1)} \right) \frac{\mathcal{W}^{(k-1)}(\mathbf{Y}) - S^{(k-1)}\mathbf{x}^{(k)}}{S_0^{(k-1)}}.$$

From this together with the assumption $S^{(k-1)}\mathbf{x}^{(k)} \leq \mathcal{W}^{(k-1)}(\mathbf{X})$ for $k = 1, \ldots, N$ and the fact that the asset with index zero is risk-free, i.e. $S_0^{(n)} \geq S_0^{(n-1)} > 0$ and consequently $S_0^{(k)} - S_0^{(k-1)} > 0$ for $k = 1, \ldots, N$, it follows

$$\mathcal{W}^{(n)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X}) + \sum_{k=1}^{n} \left(S_0^{(k)} - S_0^{(k-1)} \right) \frac{\mathcal{W}^{(k-1)}(\mathbf{Y}) - \mathcal{W}^{(k-1)}(\mathbf{X})}{S_0^{(k-1)}}.$$

For $n = 1$ it is $\mathcal{W}^{(1)}(\mathbf{Y}) \geq \mathcal{W}^{(1)}(\mathbf{X})$, because $\mathcal{W}^{(0)}(\mathbf{Y}) = \mathcal{W}^{(0)} = \mathcal{W}^{(0)}(\mathbf{X})$. Iteratively, it follows that $\mathcal{W}^{(n)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X})$ for $n = 1, \ldots, N$ because each summand is non-negative. The same way it can be shown that even $\mathcal{W}^{(n)}(\mathbf{Y}) - \mathcal{W}^{(n-1)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X}) - \mathcal{W}^{(n-1)}(\mathbf{X})$ for all $n = 1, \ldots, N$. $\square$

Because of Theorem 2.2.13, each x-portfolio can be improved by just using all unused cash for the risk-free asset. The required assumption $S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X})$ says, that no more money than available is invested, which is a natural assumption and should always be met, at least in a situation without leverage. Using the supplemented x-portfolio also does not increase the downside risk, but can improve the profit. Hence, it makes sense to focus on self-financing and fully investing x-portfolios.

2.2.4 Arbitrage opportunity and non-redundancy

Assume it is possible, to beat the risk-free market without downside risk. Then it is possible to use this inefficiency e.g. by "borrowing money from the risk-free market" (short-position) and spend all this money into the alternative portfolio which yields more money than it costs to borrow it. Such a strategy would generate money from close to scratch without risk and therefore is called an arbitrage opportunity. For comparison, the risk-free part is used, which in this case is the market with index zero. Hence, investing all available money $\mathcal{W}^{(0)} \in \mathbb{R}_{>0}$ into this part and hold the position until time step N is the benchmark. The corresponding x-portfolio is given by $\mathbf{Y} \in \mathcal{A}^2$ with

$$\mathbf{y}^{(n)} := \left(\frac{\mathcal{W}^{(0)}}{S_0^{(0)}}, 0, \ldots, 0 \right)^{\mathsf{T}} \qquad \text{for } n = 1, \ldots, N, \tag{2.18}$$

which is self-financing with $S^{(0)}\mathbf{y}^{(1)} = \mathcal{W}^{(0)}$. Proposition 2.2.11 implies

$$\mathcal{W}^{(n)}(\mathbf{Y}) = S^{(n)}\mathbf{y}^{(n)} = \mathcal{W}^{(0)} \frac{S_0^{(n)}}{S_0^{(0)}} \qquad \text{for } n = 1, \ldots, N, \tag{2.19}$$

which can be obtained by solely investing into the risk-free asset in the sense of Definition 2.1.3, i.e. $\mathcal{W}^{(n)}(\mathbf{Y}) \geq \mathcal{W}^{(n-1)}(\mathbf{Y})$ a.s. for $n = 1, \ldots, N$ and the wealth has no uncertainty. When comparing with $\mathbf{Y}$, it is not allowed to invest more money than available. Otherwise it would be trivial to construct a superior strategy, e.g. by just investing more than $\mathcal{W}^{(0)}$ into the risk-free market, which, of course, would lead to a higher value after time step N, if $\mathcal{W}^{(N)}(\mathbf{Y}) > \mathcal{W}^{(0)}$.

Definition 2.2.14 (Arbitrage opportunity and arbitrage-free market model) Let market model $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ be given for $M, N \in \mathbb{N}$. An x-portfolio $\mathbf{X}$ is called an *arbitrage opportunity* if

$$S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X}) \quad \text{a.s. for all } n = 1, \ldots, N, \tag{2.20a}$$

$$\mathcal{W}^{(N)}(\mathbf{X}) \geq \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}} \quad \text{a.s.,} \tag{2.20b}$$

$$\mathrm{P}\left(\mathcal{W}^{(N)}(\mathbf{X}) > \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}} \right) > 0. \tag{2.20c}$$

Market model S is called *arbitrage-free*, if there is no arbitrage opportunity.

Assume there is an arbitrage opportunity which is not self-financing. In other words, there is at least one point in time, where not all available money is invested. Then, this unused money can be invested into the risk-free asset, which at least will improve the portfolio without any risk, cf. Theorem 2.2.13. Hence, the resulting portfolio must also be an arbitrage opportunity, as is shown in the following result.

Lemma 2.2.15 Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a given market model for $M, N \in \mathbb{N}$ and $\mathbf{X}$ be an arbitrage opportunity. Then, the supplemented x-portfolio $\mathbf{Y}$ from Theorem 2.2.13 is also an arbitrage opportunity.

Proof. $\mathbf{Y}$ is self-financing because of Theorem 2.2.13 and thus $S^{(n)}\mathbf{y}^{(n+1)} = S^{(n)}\mathbf{y}^{(n)} = \mathcal{W}^{(n)}(\mathbf{Y})$ for $n = 1, \ldots, N-1$ and $S^{(0)}\mathbf{y}^{(1)} = \mathcal{W}^{(0)}$. Then, the first property (2.20a) of an arbitrage opportunity is fulfilled for $\mathbf{Y}$ by construction.

Since $\mathbf{X}$ is an arbitrage opportunity, (2.20a) holds true for $\mathbf{X}$ as well. Theorem 2.2.13 implies that $\mathcal{W}^{(n)}(\mathbf{Y}) \geq \mathcal{W}^{(n)}(\mathbf{X})$ for $n = 0, 1, \ldots, N$. From this, Theorem 2.2.13 (b) together with (2.20b) imply that

$$\mathcal{W}^{(N)}(\mathbf{Y}) \geq \mathcal{W}^{(N)}(\mathbf{X}) \geq \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}}$$

and (2.20c) gives

$$\mathrm{P}\left(\mathcal{W}^{(N)}(\mathbf{Y}) > \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}}\right) > 0,$$

i.e. $\mathbf{Y}$ is an arbitrage opportunity. □

The last result says, that if there is an arbitrage opportunity, there is always an arbitrage opportunity which is self-financing. Thus, for an arbitrage-free market model it suffices, that there is no self-financing arbitrage opportunity.

If the x-portfolio $\mathbf{X}$ is an arbitrage opportunity and in addition self-financing, then Proposition 2.2.11 is applicable and properties (2.20) in Definition 2.2.14 become

$$S^{(0)}\mathbf{x}^{(1)} \leq \mathcal{W}^{(0)}, \tag{2.21a}$$

$$S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)} \geq \mathcal{W}^{(0)}\left(\frac{S_0^{(N)}}{S_0^{(0)}} - 1\right) \quad \text{a.s.}, \tag{2.21b}$$

$$\mathrm{P}\left(S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)} > \mathcal{W}^{(0)}\left(\frac{S_0^{(N)}}{S_0^{(0)}} - 1\right)\right) > 0. \tag{2.21c}$$

In the book from Föllmer and Schied [18, Definition 5.10], an arbitrage opportunity is defined for self-financing x-portfolios, which is strongly related to Definition 2.2.14, but not exactly the same. To see the connection, the next result gives some equivalences.

Lemma 2.2.16 Let $S \in \mathcal{L}^2(N; \mathbb{R}_{>0}^{M+1})$ be a given market model for $M, N \in \mathbb{N}$ and $\mathbf{X}$ be a self-financing x-portfolio. Then the following three statements are equivalent:

(a) $\mathbf{X}$ is an arbitrage opportunity.

(b) $c\mathbf{X}$ is an arbitrage opportunity for all $1 \leq c \leq \mathcal{W}^{(0)}/(S^{(0)}\mathbf{x}^{(1)})$, if $S^{(0)}\mathbf{x}^{(1)} > 0$, and for all $c \geq 1$, if $S^{(0)}\mathbf{x}^{(1)} \leq 0$.

(c) $\mathbf{Z} := \mathbf{X} - \mathbf{Y}$, for $\mathbf{Y}$ from (2.18), defines a self-financing x-portfolio which satisfies

$$S^{(0)}\mathbf{z}^{(1)} \leq 0, \quad S^{(N)}\mathbf{z}^{(N)} \geq S^{(0)}\mathbf{z}^{(1)} \text{ a.s.}, \quad \text{and} \quad \mathrm{P}\left(S^{(N)}\mathbf{z}^{(N)} > S^{(0)}\mathbf{z}^{(1)}\right) > 0.$$

For the special case $\mathcal{W}^{(0)} = S^{(0)}\mathbf{x}^{(1)}$, the conditions in (c) become

$$S^{(0)}\mathbf{z}^{(1)} = 0, \qquad S^{(N)}\mathbf{z}^{(N)} \geq 0 \text{ a.s.}, \qquad \text{and} \qquad \mathrm{P}\left(S^{(N)}\mathbf{z}^{(N)} > 0\right) > 0. \tag{2.22}$$

Proof. The implication from (b) to (a) is trivial, since for $c := 1$ this corresponds to (a). The implication from (a) to (b) also holds true, because of the following: From (2.21b) it is known, that $S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)} \geq 0$ and consequently for all $c \geq 1$ it must be

$$S^{(N)}c\mathbf{x}^{(N)} - S^{(0)}c\mathbf{x}^{(1)} = c\left(S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)}\right) \geq S^{(N)}\mathbf{x}^{(N)} - S^{(0)}\mathbf{x}^{(1)},$$

i.e. (2.21b) and (2.21c) also hold true when replacing $\mathbf{X}$ by $c\mathbf{X}$. In addition, if $S^{(0)}\mathbf{x}^{(1)} > 0$, then for all $1 \leq c \leq \mathcal{W}^{(0)}/(S^{(0)}\mathbf{x}^{(1)})$, it is $0 < S^{(0)}c\mathbf{x}^{(1)} \leq \mathcal{W}^{(0)}$, i.e. (2.21a) for $c\mathbf{X}$. Analogously, if $S^{(0)}\mathbf{x}^{(1)} \leq 0$, then $S^{(0)}c\mathbf{x}^{(1)} \leq 0 < \mathcal{W}^{(0)}$ for all $c > 0$, especially for all $c \geq 1$.

It remains to show the equivalence of (a) and (c). First note, that $\mathbf{Y}$ is self-financing, because it is constant, and consequently $\mathbf{Z}$ is also self-financing because of Proposition 2.2.10. Now, it is

$$S^{(0)}\mathbf{z}^{(1)} = S^{(0)}\mathbf{x}^{(1)} - S_0^{(0)}\frac{\mathcal{W}^{(0)}}{S_0^{(0)}} \qquad \text{and} \qquad S^{(N)}\mathbf{z}^{(N)} = S^{(N)}\mathbf{x}^{(N)} - S_0^{(N)}\frac{\mathcal{W}^{(0)}}{S_0^{(0)}}. \tag{2.23}$$

Hence, (2.21a) for $\mathbf{X}$ is equivalent to

$$S^{(0)}\mathbf{z}^{(1)} \leq \mathcal{W}^{(0)} - S_0^{(0)}\frac{\mathcal{W}^{(0)}}{S_0^{(0)}} = 0$$

and, when adding $S^{(0)}\mathbf{x}^{(1)} - S_0^{(N)}\mathcal{W}^{(0)}/S_0^{(0)}$ on both sides of (2.21b) and using (2.23) in the last step yields

$$S^{(N)}\mathbf{z}^{(N)} \geq S^{(0)}\mathbf{x}^{(1)} + \mathcal{W}^{(0)}\left(\frac{S_0^{(N)}}{S_0^{(0)}} - 1\right) - S_0^{(N)}\frac{\mathcal{W}^{(0)}}{S_0^{(0)}} = S^{(0)}\mathbf{x}^{(1)} - \mathcal{W}^{(0)} = S^{(0)}\mathbf{z}^{(1)},$$

and for (2.21c) it is

$$\mathrm{P}\left(S^{(N)}\mathbf{z}^{(N)} > S^{(0)}\mathbf{z}^{(1)}\right) > 0,$$

respectively, which gives the equivalence of (a) and (c). The additional special case (2.22) for $\mathcal{W}^{(0)} = S^{(0)}\mathbf{x}^{(1)}$ is a direct consequence. $\square$

Remark 2.2.17 The definition of an arbitrage opportunity in [18, Definition 5.10] is only valid for self-financing x-portfolios, where such an x-portfolio $\mathbf{Z}$ is denoted an arbitrage opportunity if it satisfies

$$S^{(0)}\mathbf{z}^{(1)} \leq 0, \qquad S^{(N)}\mathbf{z}^{(N)} \geq 0 \text{ a.s.,} \qquad \text{and} \qquad \mathrm{P}\left(S^{(N)}\mathbf{z}^{(N)} > 0\right) > 0. \tag{2.24}$$

Note that in [18, Definition 5.10] they used scaled variants of $S^{(0)}\mathbf{z}^{(1)}$ and $S^{(N)}\mathbf{z}^{(N)}$. Since the factors are positive, this definition is equivalent to (2.24).

Both definitions are equivalent in the following sense: Assume there is an x-portfolio $\mathbf{Z}$ which is an arbitrage opportunity according to [18, Definition 5.10], i.e. which is self-financing and fulfills (2.24). Then it also meets the conditions in Lemma 2.2.16 (c) and consequently $\mathbf{X} := \mathbf{Z} + \mathbf{Y}$ with $\mathbf{Y}$ from (2.18) is an arbitrage opportunity according to Definition 2.2.14.

The other way around, assume there is an arbitrage opportunity according to Definition 2.2.14. Because of Lemma 2.2.15 there is also an arbitrage opportunity $\mathbf{X}$ which is self-financing and which meets the condition $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$. Because of the special case in Lemma 2.2.16, $\mathbf{Z} := \mathbf{X} - \mathbf{Y}$ satisfies (2.22) and therefore (2.24), i.e. $\mathbf{Z}$ it is an arbitrage opportunity according to the definition in [18, Definition 5.10]. ♦

As noted in the introduction of this subsection, there should exist no arbitrage opportunity. For this purpose, the following result gives necessary and sufficient conditions.

Proposition 2.2.18 (Arbitrage-free market model, cf. Föllmer and Schied [18, Proposition 5.11]) A market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ is arbitrage-free, if and only if for all $n = 1, \ldots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ it is

$$\mathrm{P}\left(\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} < 0\right) > 0 \quad \text{or} \quad \left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} = 0 \text{ a.s.}$$

Proof. Because of Remark 2.2.17 the existence of arbitrage opportunities is equivalent for both definitions. Therefore, the proof in [18, Proposition 5.11] holds true. The statement therein is, that there is an arbitrage opportunity, if and only if there exists $n \in \{1, \ldots, N\}$ and $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ such that

$$\left(\frac{S^{(n)}}{S_0^{(n)}} - \frac{S^{(n-1)}}{S_0^{(n-1)}}\right)\boldsymbol{\eta} \geq 0 \text{ a.s.} \qquad \text{and} \qquad \mathrm{P}\left(\left(\frac{S^{(n)}}{S_0^{(n)}} - \frac{S^{(n-1)}}{S_0^{(n-1)}}\right)\boldsymbol{\eta} > 0\right) > 0.$$

Multiplying the inequalities by $S_0^{(n)} > 0$ on both sides and negating the statement gives that for all $n = 1, \ldots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$

$$\mathrm{P}\left(\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} < 0\right) > 0 \quad \text{or} \quad \left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} \leq 0 \text{ a.s.}$$

holds. Whenever the expression in the second case is negative with positive probability it already fits into the first case. Hence, this situation does not need to be involved in the second case. It remains the situation when the expression equals to zero almost surely which completes the proof. □

This result directly gives necessary and sufficient conditions for a market model to be arbitrage-free. However, a market model that consists of M copies of the risk-free asset, i.e. $S_i^{(n)} = S_0^{(n)}$ for all $n = 0, \dots, N$ and all $i = 1, \dots, M$, is arbitrage-free, because it is not possible to construct a portfolio, which beats the risk-free market. Hence, this property does not prevent redundancies, which is important for uniqueness of x-portfolios and, consequently, for uniqueness of the solution for upcoming optimization problems. Therefore, a no redundancy property is needed. First, to be able to exclude copies of the risk-free part, the notion of bond replicating x-portfolio is introduced.

Definition 2.2.19 (Bond replicating and trivial x-portfolio) Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a market model. An x-portfolio $\mathbf{X} := (\mathbf{x}^{(n)})_{1 \le n \le N}$ is called *bond replicating* if

$$S^{(n-1)}\mathbf{x}^{(n)} \le \mathcal{W}^{(n-1)}(\mathbf{X}) \text{ a.s.} \quad \text{for all } n = 1, \dots, N \text{ and} \quad \mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}} \text{ a.s.}$$

A *trivial* x-portfolio $\mathbf{X}$ is an x-portfolio with $\widehat{\mathbf{X}} \equiv 0$ a.s., where $\widehat{\mathbf{X}} := (\widehat{\mathbf{x}}^{(n)})_{1 \le n \le N}$ with $\widehat{\mathbf{x}}^{(n)} := (0, x_1^{(n)}, \dots, x_M^{(n)})^\intercal$ denotes the risky part of $\mathbf{X}$.

If only trivial x-portfolios can be bond replicating, then market model S has *no nontrivial bond replicating x-portfolio* and is also called *non-redundant*.

Of course, the trivial x-portfolio $\mathbf{Y}$ from (2.18) is bond replicating, because it already represents the bond. In addition, each x-portfolio of the form $(c^{(n)}, 0, \dots, 0)^\intercal$ can be supplemented as in (2.17) such that it becomes bond replicating, or, to be more precise, it becomes $\mathbf{Y}$. Hence, portfolios of this form must be excluded, when requiring that there should be no bond replicating x-portfolio. Now, a similar result as Proposition 2.2.18 is the following for a market model with no nontrivial bond replicating x-portfolio, but only with necessary conditions.

Lemma 2.2.20 (Non-redundancy) Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a market model. Then the following statements hold true:

(a) If S is non-redundant, then, for the wealth in Definition 2.2.2,

$$\text{for all } \mathbf{X}, \mathbf{Y} \in \mathcal{L}^0\big(N-1; \mathbb{R}^{M+1}\big) \text{ with } \mathcal{W}(\mathbf{X}) = \mathcal{W}(\mathbf{Y}) \text{ a.s. it is } \widehat{\mathbf{X}} = \widehat{\mathbf{Y}} \text{ a.s.} \tag{2.25}$$

(b) If (2.25) holds true, then

$$\mathrm{P}\left(\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} \neq 0\right) > 0 \tag{2.26}$$

for all $n = 1, \dots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$, i.e. $\mathrm{P}(\widehat{\boldsymbol{\eta}} \neq 0) > 0$.

(c) If (2.25) holds true and $S_0^{(n)} > S_0^{(n-1)}$ for all $n = 1, \dots, N$, then $\mathcal{W}$ is injective.

Proof. Proof of (a): Suppose S is non-redundant. Assume there exist $\mathbf{X}, \mathbf{Y}$ with $\mathcal{W}(\mathbf{X}) = \mathcal{W}(\mathbf{Y})$ a.s. and $\mathrm{P}(\widehat{\mathbf{X}} \neq \widehat{\mathbf{Y}}) > 0$. Then, for all $n = 1, \dots, N$ it is $\mathcal{W}^{(n)}(\mathbf{X}) - \mathcal{W}^{(n-1)}(\mathbf{X}) = \mathcal{W}^{(n)}(\mathbf{Y}) - \mathcal{W}^{(n-1)}(\mathbf{Y})$ and therefore, by (2.7),

$$\left(S^{(n)} - S^{(n-1)}\right)\mathbf{x}^{(n)} = \mathcal{W}^{(n)}(\mathbf{X}) - \mathcal{W}^{(n-1)}(\mathbf{X}) = \left(S^{(n)} - S^{(n-1)}\right)\mathbf{y}^{(n)}. \tag{2.27}$$

Now, define $\mathbf{Z} \in \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$ by

$$\mathbf{z}^{(n)} := \begin{cases} \mathbf{x}^{(n)} - \mathbf{y}^{(n)}, & \text{if } S^{(n-1)}(\mathbf{x}^{(n)} - \mathbf{y}^{(n)}) < 0, \\ \mathbf{y}^{(n)} - \mathbf{x}^{(n)}, & \text{otherwise,} \end{cases}$$

for $n = 1, \ldots, N$. Then it is $\mathrm{P}\,(\widehat{\mathbf{Z}} \neq 0) > 0$ from construction and because of (2.7) and (2.27) it is

$$\mathcal{W}^{(n)}(\mathbf{Z}) = \mathcal{W}^{(0)} + \sum_{k=1}^{n} \left(S^{(k)} - S^{(k-1)}\right)\mathbf{z}^{(k)} = \mathcal{W}^{(0)} \qquad \text{and} \qquad S^{(n-1)}\mathbf{z}^{(n)} \leq 0$$

for all $n = 1, \ldots, N$. Next, define $\mathbf{Z}'$ by $\widehat{\mathbf{Z}}' := \widehat{\mathbf{Z}} \not\equiv 0$ and $z_0'^{(n)} := z_0^{(n)} + \mathcal{W}^{(0)}/S_0^{(0)}$ for $n = 1, \ldots, N$. This x-portfolio has the properties

$$\mathcal{W}^{(n)}(\mathbf{Z}') = \mathcal{W}^{(0)} + \sum_{k=1}^{n} \left(S_0^{(k)} - S_0^{(k-1)}\right)\frac{\mathcal{W}^{(0)}}{S_0^{(0)}} = \frac{S_0^{(n)}}{S_0^{(0)}}\mathcal{W}^{(0)},$$
$$S^{(n-1)}\mathbf{z}'^{(n)} = S^{(n-1)}\mathbf{z}^{(n)} + \frac{S_0^{(n-1)}}{S_0^{(0)}}\mathcal{W}^{(0)} \leq \frac{S_0^{(n-1)}}{S_0^{(0)}}\mathcal{W}^{(0)} = \mathcal{W}^{(n-1)}(\mathbf{Z}')$$

for all $n = 1, \ldots, N$. Hence, there exists a nontrivial bond replicating x-portfolio which contradicts the assumption. Therefore, (2.25) must be valid.

Proof of (b): Now (2.25) is supposed to hold true. Assume there is $n_0 \in \{1, \ldots, N\}$ and $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n_0-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$ not satisfying (2.26), i.e. it is

$$\left(S^{(n_0)} - \frac{S_0^{(n_0)}}{S_0^{(n_0-1)}}S^{(n_0-1)}\right)\boldsymbol{\eta} = 0 \text{ a.s.,} \quad \text{or equivalently} \quad S^{(n_0)}\boldsymbol{\eta} = \frac{S_0^{(n_0)}}{S_0^{(n_0-1)}}S^{(n_0-1)}\boldsymbol{\eta} \text{ a.s.}$$

Let $\mathbf{X}$ be the x-portfolio with $\mathbf{x}^{(n)} \equiv 0 \in \mathbb{R}^{M+1}$ for $n \neq n_0$ and $\mathbf{x}^{(n)} = \boldsymbol{\eta}$ for $n = n_0$. Furthermore, let $\mathbf{Y}$ be the corresponding supplemented x-portfolio according to Theorem 2.2.13 which is self-financing and for which $S^{(n-1)}\mathbf{y}^{(n)} = \mathcal{W}^{(n-1)}(\mathbf{Y})$ for $n = 1, \ldots, N$. In addition, for $n = 1, \ldots, n_0 - 1$ it is

$$\mathcal{W}^{(n)}(\mathbf{Y}) = S^{(n)}\mathbf{y}^{(n)} = S_0^{(n)}y_0^{(n)} = \frac{S_0^{(n)}}{S_0^{(n-1)}}\mathcal{W}^{(n-1)}(\mathbf{Y}) = \ldots = \frac{S_0^{(n)}}{S_0^{(0)}}\mathcal{W}^{(0)}(\mathbf{Y}) = \frac{S_0^{(n)}}{S_0^{(0)}}\mathcal{W}^{(0)}$$

and for $n = n_0$ because of (2.17) and the property of $\boldsymbol{\eta}$ it is

$$\begin{aligned} \mathcal{W}^{(n_0)}(\mathbf{Y}) &= S^{(n_0)}\mathbf{y}^{(n_0)} \\ &= S^{(n_0)}\boldsymbol{\eta} + \frac{S_0^{(n_0)}}{S_0^{(n_0-1)}}\left(\mathcal{W}^{(n_0-1)}(\mathbf{Y}) - S^{(n_0-1)}\boldsymbol{\eta}\right) \\ &= \frac{S_0^{(n_0)}}{S_0^{(n_0-1)}}S^{(n_0-1)}\boldsymbol{\eta} + \frac{S_0^{(n_0)}}{S_0^{(n_0-1)}}\left(\frac{S_0^{(n_0-1)}}{S_0^{(0)}}\mathcal{W}^{(0)} - S^{(n_0-1)}\boldsymbol{\eta}\right) = \frac{S_0^{(n_0)}}{S_0^{(0)}}\mathcal{W}^{(0)}. \end{aligned}$$

Hence, for $n = n_0 + 1, \ldots, N$ it is also $\mathcal{W}^{(n)}(\mathbf{Y}) = S_0^{(n)}/S_0^{(0)} \cdot \mathcal{W}^{(0)}$, i.e. $\mathbf{Y}$ is a nontrivial bond replicating x-portfolio.

The bond itself is represented by the trading strategy $\mathbf{Z} \in \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$ defined by $\mathbf{z}^{(n)} := (\mathcal{W}^{(0)}/S_0^{(0)}, 0, \dots, 0)^\intercal$ for all $n = 1, \dots, N$. For this, it is $\mathcal{W}(\mathbf{Z}) = \mathcal{W}(\mathbf{Y})$ a.s., see also (2.19). By assumption (2.25) it then must be $\widehat{\mathbf{Z}} = \widehat{\mathbf{Y}}$ a.s., which is a contradiction, because $\widehat{\mathbf{Z}} \equiv 0 \not\equiv \widehat{\mathbf{Y}}$. Hence, assertion (b) must hold true.

Proof of (c): Let $\mathbf{X}, \mathbf{Y} \in \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$ with $\mathcal{W}(\mathbf{X}) = \mathcal{W}(\mathbf{Y})$ a.s. be arbitrary. From (2.25) it is $\widehat{\mathbf{X}} = \widehat{\mathbf{Y}}$ a.s. and, together with (2.7), for all $n = 1, \dots, N$ it is

$$\begin{aligned} 0 = \mathcal{W}^{(n)}(\mathbf{X}) - \mathcal{W}^{(n)}(\mathbf{Y}) &= \Big(\mathcal{W}^{(n-1)}(\mathbf{X}) - \mathcal{W}^{(n-1)}(\mathbf{Y})\Big) + \Big(S^{(n)} - S^{(n-1)}\Big)\Big(\mathbf{x}^{(n)} - \mathbf{y}^{(n)}\Big) \\ &= \Big(S_0^{(n)} - S_0^{(n-1)}\Big)\Big(x_0^{(n)} - y_0^{(n)}\Big) + \Big(S^{(n)} - S^{(n-1)}\Big)\Big(\widehat{\mathbf{x}}^{(n)} - \widehat{\mathbf{y}}^{(n)}\Big) \\ &= \Big(S_0^{(n)} - S_0^{(n-1)}\Big)\Big(x_0^{(n)} - y_0^{(n)}\Big). \end{aligned}$$

Since $S_0^{(n)} > S_0^{(n-1)}$ it must be $x_0^{(n)} = y_0^{(n)}$ for all $n = 1, \dots, N$ and therefore $\mathbf{X} = \mathbf{Y}$. □

This result explains the name "non-redundant", because it does not allow nontrivial bond replicating x-portfolios and therefore, Lemma 2.2.20 implies, that $\mathcal{W}$ is "almost" injective, i.e. there is no redundancy up to the risk-free component.

Together with the notion of arbitrage-free, the following equivalence holds true. Note, that a similar result has been proven by Maier-Paape and Zhu [34, Theorem 2] for the one-period case with finite probability space.

Lemma 2.2.21 (Arbitrage-free and non-redundant) Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be the market model. Then S is arbitrage-free and non-redundant, if and only if

$$\mathrm{P}\left(\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} < 0\right) > 0 \tag{2.28}$$

for all $n = 1, \dots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P};\mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$.

Proof. Necessity of (2.28) directly follows from merging the properties in Proposition 2.2.18, restricted to all $\boldsymbol{\eta}$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$, and property (2.26) in Lemma 2.2.20, which gives (2.28).

It remains to show sufficiency of (2.28): First note, that for all $\boldsymbol{\eta}' \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P};\mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}}' = 0$ a.s., it is

$$\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta}' = S_0^{(n)}\eta_0' - S_0^{(n)}\eta_0' = 0 \quad \text{a.s.,}$$

i.e. all $\boldsymbol{\eta}'$ of this form satisfy the second property in Proposition 2.2.18. Since the remaining elements of $\mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P};\mathbb{R}^{M+1})$, i.e. those with $\widehat{\boldsymbol{\eta}}' \not\equiv 0$, satisfy (2.28) by assumption, the first property in Proposition 2.2.18 holds for them. Therefore, this proposition is applicable and implies, that S is arbitrage-free.

It remains to show that S is non-redundant. For this, assume there is a nontrivial bond replicating x-portfolio $\mathbf{X}$. Let $\mathbf{Y}$ be the trivial x-portfolio $\mathbf{Y}$ from (2.18), which satisfies (2.19). From $\mathbf{X}$ and $\mathbf{Y}$ it is possible to construct an arbitrage opportunity, see the following five steps, which contradicts the assumption of S being arbitrage-free.

(i) W.l.o.g. $\mathbf{X}$ is self-financing and $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$.
Proof: Assume $\mathbf{X}$ does not have these properties. Since $\mathbf{X}$ is bond replicating it is $S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X})$ a.s. for all $n = 1, \ldots, N$. Let $\mathbf{X}'$ be the corresponding supplemented x-portfolio according to Theorem 2.2.13 for which $S^{(0)}\mathbf{x}'^{(1)} = \mathcal{W}^{(0)}$. Because of Theorem 2.2.13 the x-portfolio $\mathbf{X}'$ is self-financing and $\mathcal{W}^{(n-1)}(\mathbf{X}') = S^{(n-1)}\mathbf{x}'^{(n-1)} = S^{(n-1)}\mathbf{x}'^{(n)}$ for $n = 2, \ldots, N$. Theorem 2.2.13 (b) implies that $\mathcal{W}^{(N)}(\mathbf{X}') \geq \mathcal{W}^{(N)}(\mathbf{X})$ a.s. holds true. Note, that $\mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} S_0^{(N)}/S_0^{(0)}$ a.s. because $\mathbf{X}$ is bond replicating. Then, either $\mathcal{W}^{(N)}(\mathbf{X}') = \mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} S_0^{(N)}/S_0^{(0)}$ a.s. or

$$\mathrm{P}\left(\mathcal{W}^{(N)}(\mathbf{X}') > \mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} S_0^{(N)}/S_0^{(0)}\right) > 0.$$

The second case implies that $\mathbf{X}'$ is an arbitrage opportunity, which contradicts the assumption that S is arbitrage-free. Hence the former case must hold true, which is $\mathbf{X}'$ is self-financing, $S^{(0)}\mathbf{x}'^{(1)} = \mathcal{W}^{(0)}$, nontrivial and bond replicating. In this case, set $\mathbf{X} := \mathbf{X}'$.

(ii) Because of (i), Proposition 2.2.11 implies $S^{(n)}\mathbf{x}^{(n)} = \mathcal{W}^{(n)}(\mathbf{X})$ and $S^{(n-1)}\mathbf{x}^{(n)} = \mathcal{W}^{(n-1)}(\mathbf{X})$ for $n = 1, \ldots, N$.

(iii) Because of (ii), the second property of Definition 2.2.19 implies

$$S^{(N)}\mathbf{x}^{(N)} = \mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}} = \mathcal{W}^{(N)}(\mathbf{Y}) \quad \text{a.s.}$$

(iv) Since $\widehat{\mathbf{X}} \not\equiv 0$, there exists $n_0 \in \{1, \ldots, N\}$ with $\widehat{\mathbf{x}}^{(n_0)} \not\equiv 0$. Let n_0 be minimal with that property. Then, for $\boldsymbol{\eta} := \widehat{\mathbf{x}}^{(n_0)}$ assumption (2.28) can be used and for $n < n_0$ the x-portfolio $\mathbf{X}$ equals the bond $\mathbf{Y}$, i.e.

$$\mathcal{W}^{(n)}(\mathbf{X}) = \mathcal{W}^{(n)}(\mathbf{Y}) \qquad \text{for } n = 1, \ldots, n_0 - 1.$$

Hence, it is possible with positive probability, that the equity of the bond $\mathbf{Y}$ is above the equity of $\mathbf{X}$, i.e. it must be

$$\mathrm{P}\left(\mathcal{W}^{(n_0)}(\mathbf{X}) < \mathcal{W}^{(n_0)}(\mathbf{Y})\right) > 0 \tag{2.29}$$

and together with (iii) it has to be

$$\mathrm{P}\left(\mathcal{W}^{(N)}(\mathbf{X}) > \frac{S_0^{(N)}}{S_0^{(n_0)}} \mathcal{W}^{(n_0)}(\mathbf{X})\right) > 0$$

to compensate a possible loss or too small gain.

(v) The x-portfolio $\mathbf{Z}$ defined by $\mathbf{z}^{(n)} := \mathbf{y}^{(n)}$ for $1 \leq n \leq n_0$, and

$$\mathbf{z}^{(n)} := \begin{cases} \mathbf{y}^{(n)}, & \text{if } \mathcal{W}^{(n_0)}(\mathbf{X}) \geq \mathcal{W}^{(n_0)}(\mathbf{Y}), \\ \mathbf{x}^{(n)}, & \text{if } \mathcal{W}^{(n_0)}(\mathbf{X}) < \mathcal{W}^{(n_0)}(\mathbf{Y}), \end{cases} \tag{2.30}$$

for $n_0 + 1 \leq n \leq N$, is an arbitrage opportunity.
Proof: For $n = 1, \ldots, n_0$ it is

$$S^{(n-1)}\mathbf{z}^{(n)} = S^{(n-1)}\mathbf{y}^{(n)} = \mathcal{W}^{(n-1)}(\mathbf{Y}) = \mathcal{W}^{(n-1)}(\mathbf{Z}),$$

which also holds true for the first case in (2.30) for $n = n_0 + 1, \dots, N$. Since $\mathbf{X}$ is bond replicating, for the second case

$$S^{(n-1)}\mathbf{z}^{(n)} = S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X}) \\ < \mathcal{W}^{(n_0)}(\mathbf{Y}) + \mathcal{W}^{(n-1)}(\mathbf{X}) - \mathcal{W}^{(n_0)}(\mathbf{X}) = \mathcal{W}^{(n-1)}(\mathbf{Z}),$$

holds for $n = n_0 + 1, \dots, N$. Thus, the first property (2.20a) is fulfilled for $\mathbf{Z}$. For the final wealth for $\mathbf{Z}$ in the first case of (2.30) it is

$$\mathcal{W}^{(N)}(\mathbf{Z}) = \mathcal{W}^{(N)}(\mathbf{Y}) = \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}}$$

and in the second case

$$\mathcal{W}^{(N)}(\mathbf{Z}) = \mathcal{W}^{(n_0)}(\mathbf{Y}) + \mathcal{W}^{(N)}(\mathbf{X}) - \mathcal{W}^{(n_0)}(\mathbf{X}) > \mathcal{W}^{(N)}(\mathbf{X}) = \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}},$$

where the second case occurs with positive probability due to (2.29). Consequently, (2.20b) as well as (2.20c) hold true and therefore $\mathbf{Z}$ is an arbitrage opportunity.

Since S is arbitrage-free, (v) is a contradiction. Hence, there is no nontrivial bond replicating x-portfolio. □

A combination of arbitrage opportunity and bond replicating x-portfolio gives the following.

Definition 2.2.22 (Risk-free x-portfolio and no nontrivial risk-free x-portfolio) Let a market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be given. An x-portfolio $\mathbf{X}$ is called *risk-free* if it is bond replicating or an arbitrage opportunity, i.e. if

$$S^{(n-1)}\mathbf{x}^{(n)} \leq \mathcal{W}^{(n-1)}(\mathbf{X}) \text{ a.s.} \quad \text{for all } n = 1, \dots, N \text{ and} \quad \mathcal{W}^{(N)}(\mathbf{X}) \geq \mathcal{W}^{(0)} \frac{S_0^{(N)}}{S_0^{(0)}} \text{ a.s.}$$

Market model S is said to have *no nontrivial risk-free x-portfolio* if only trivial x-portfolios, according to Definition 2.2.19, can be risk-free and all remaining x-portfolios are not risk-free.

From construction of the definitions, the next result is a consequence from the results shown before. Note, that similar results have been shown by Maier-Paape and Zhu [34, Proposition 4 and Theorem 2] for the one-period case with a discrete and finite probability space.

Theorem 2.2.23 (Multi-period market model with no nontrivial risk-free x-portfolio) Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be a given multi-period market model as in Definition 2.1.4 for $M, N \in \mathbb{N}$. Then the following assertions are equivalent:

(a) S has no nontrivial risk-free x-portfolio.

(b) S is arbitrage-free and non-redundant.

(c) For all $n = 1, \dots, N$ and all $\boldsymbol{\eta} \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\eta}} \not\equiv 0$ it is

$$\mathrm{P}\left(\left(S^{(n)} - \frac{S_0^{(n)}}{S_0^{(n-1)}} S^{(n-1)}\right)\boldsymbol{\eta} < 0\right) > 0.$$

(d) For all $n = 1, \dots, N$ and all $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_M)^\intercal \in \mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1})$ with $\widehat{\boldsymbol{\theta}} \not\equiv 0$ it is

$$\mathrm{P}\left(T^{(n)}\boldsymbol{\theta} < T_0^{(n)} \sum_{i=0}^{M} \theta_i\right) > 0.$$

(e) S is arbitrage-free and the wealth $\mathcal{W}$ from Definition 2.2.2 satisfies for all $\mathbf{X}, \mathbf{Y} \in \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$:

$$\mathcal{W}(\mathbf{X}) = \mathcal{W}(\mathbf{Y}) \text{ a.s. implies } \widehat{\mathbf{X}} = \widehat{\mathbf{Y}} \text{ a.s.}$$

Furthermore, if $S_0^{(n)} > S_0^{(n-1)}$ for all $n = 1, \dots, N$ and one of the assertions (a), (b), (c), (d) or (e) holds true, then $\mathcal{W}$ is injective.

Proof. The equivalence of (a) and (b) follows from Definitions 2.2.14, 2.2.19 and 2.2.22. The equivalence of (b) and (c) is exactly the statement of Lemma 2.2.21. The implication from (b) to (e) hold true because of Lemma 2.2.20 (a). The implication from (e) to (c) holds true, because in case (e) Proposition 2.2.18 and Lemma 2.2.20 (b) are applicable, i.e. the properties therein both must hold true, which leads to (c). The additional statement comes from Lemma 2.2.20 (c).

It remains to show the equivalence of (c) and (d): Let $n \in \{1, \dots, N\}$. For arbitrary $\boldsymbol{\eta}$ define $\boldsymbol{\theta}$ by $\theta_i := S_i^{(n-1)}\eta_i$ for $i = 0, \dots, M$ or, equivalently, for arbitrary $\boldsymbol{\theta}$ define $\boldsymbol{\eta}$ by $\eta_i := \theta_i / S_i^{(n-1)}$ for $i = 0, \dots, M$. Then, $\widehat{\boldsymbol{\eta}} \not\equiv 0$ if and only if $\widehat{\boldsymbol{\theta}} \not\equiv 0$. In addition, with $T^{(n)}$ from (2.2), it is

$$S^{(n)}\boldsymbol{\eta} = \sum_{i=0}^{M} S_i^{(n)}\eta_i = \sum_{i=0}^{M} \frac{S_i^{(n)}}{S_i^{(n-1)}}\theta_i = T^{(n)}\boldsymbol{\theta} + \sum_{i=0}^{M}\theta_i \qquad \text{and} \qquad S^{(n-1)}\boldsymbol{\eta} = \sum_{i=0}^{M}\theta_i.$$

Inserting this into (c) leads to (d), i.e. (d) is the same as (c) up to a substitution, which completes the proof. □

In the situation of Theorem 2.2.23 with $N = 1$ and Ω being finite, the following holds true.

Corollary 2.2.24 (One-period market model with no nontrivial risk-free x-portfolio) Let $\Omega := \{\omega_1, \ldots, \omega_K\}$ be finite with $K \in \mathbb{N}$ and $S \in \mathcal{L}^2(\mathbb{R}^{M+1}_{>0})$ with $N = 1$ and $M \in \mathbb{N}$ be a given one-period market model as in Definition 2.1.2. Define $R := S_0^{(1)}/S_0^{(0)} \geq 1$. Then the following assertions are equivalent:

(a) S has no nontrivial risk-free x-portfolio.

(b) S is arbitrage-free and non-redundant.

(c) For all $\mathbf{x} \in \mathbb{R}^{M+1}$ with $\widehat{\mathbf{x}} \neq 0$ there exists $\omega \in \Omega$ with $\mathrm{P}(\{\omega\}) > 0$ such that

$$\left(S^{(1)}(\omega) - RS^{(0)}\right)\mathbf{x} < 0.$$

(d) For all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ with $\widehat{\boldsymbol{\theta}} \neq 0$ there exists $\omega \in \Omega$ with $\mathrm{P}(\{\omega\}) > 0$ such that

$$\left(T^{(1)}(\omega) - (R-1) \cdot \mathbf{1}\right)\boldsymbol{\theta} < 0,$$

where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^{M+1}$.

(e) S is arbitrage-free and

$$G := \left(S_j^{(1)}(\omega_i) - S_j^{(0)}\right)_{\substack{1 \leq i \leq K \\ 0 \leq j \leq M}} \in \mathbb{R}^{K \times (M+1)},$$

has rank M if $R = 1$ and rank $M+1$ if $R > 1$.

(f) S is arbitrage-free and

$$\widehat{G}^* := \left(S_j^{(1)}(\omega_i) - RS_j^{(0)}\right)_{\substack{1 \leq i \leq K \\ 1 \leq j \leq M}} \in \mathbb{R}^{K \times M}.$$

has rank M, where $K \geq M$.

Proof. Theorem 2.2.23 can be applied to this special situation, i.e. a one-period market with finite probability space. Statements (a) and (b) from this corollary are exactly the same as in Theorem 2.2.23. Since $N = 1$, it is $\mathcal{L}^0(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}) = \mathbb{R}^{M+1}$. Using this in Theorem 2.2.23 gives (c) and (d). Hence, (a), (b), (c) and (d) are equivalent which in turn are equivalent to Theorem 2.2.23 (e).

For $\mathbf{x} \in \mathbb{R}^{M+1}$ it is $G\mathbf{x} = (\mathcal{W}^{(1)}(\mathbf{x})(\omega_i) - \mathcal{W}^{(0)})_{1 \leq i \leq K}$, i.e. $G\mathbf{x}$ fully characterizes the wealth of $\mathbf{x}$. Theorem 2.2.23 (e) here reads: S is arbitrage-free and $G\mathbf{x} = G\mathbf{y}$ implies $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. The latter is equivalent to the statement $G\mathbf{x} = 0$ implies $\widehat{\mathbf{x}} = 0$. If $R = 1$, then the first column of G equals to zero and therefore G must have rank M. If $R > 1$, then the special case at the end of Theorem 2.2.23 gives, that G defines an injective mapping, i.e. G must have rank $M + 1$. Consequently, Theorem 2.2.23 implies (e) of this corollary. The opposite direction is obvious and thus (a)–(e) are equivalent.

It remains to show the equivalence to (f). If $R = 1$, then $\widehat{G}^*$ coincides with G reduced by its first column, which is zero. Therefore, (e) and (f) state the same. In case $R > 1$, consider portfolios $\mathbf{x} \in \mathbb{R}^{M+1}$ with $S^{(0)}\mathbf{x} = 0$. Then, using again $\widehat{\mathbf{x}} := (0, x_1, \ldots, x_M)^\intercal$ for arbitrary $(x_1, \ldots, x_M)$, the bond part is fixed by $x_0 = -S^{(0)}\widehat{\mathbf{x}}/S_0^{(0)}$. Furthermore,

$$(G\mathbf{x})_i = \left(S_0^{(1)} - S_0^{(0)}\right)\left(-\frac{S^{(0)}\widehat{\mathbf{x}}}{S_0^{(0)}}\right) + \left(S^{(1)}(\omega_i) - S^{(0)}\right)\widehat{\mathbf{x}} = \left(S^{(1)}(\omega_i) - RS^{(0)}\right)\widehat{\mathbf{x}}.$$

Hence, this can also be written as a linear mapping with the risky part $(x_1, \dots, x_M)$ as input using the matrix $\widehat{G}^*$. Since this mapping is still injective when G is injective (as in (e) for $R > 1$), $\widehat{G}^*$ must have rank M, where now it suffices that $K \geq M$, because the dimension has been reduced by one. Hence, (e) implies (f). It remains to show the opposite direction for $R > 1$.

Let $R > 1$ and (f) hold true. Define $\widehat{G} := (0, \widehat{G}^*) \in \mathbb{R}^{K\times(M+1)}$ and let $\mathbf{x} \in \mathbb{R}^{M+1}$ be arbitrary with $\widehat{\mathbf{x}} \neq 0$. Since $\widehat{G}^*$ has full column rank, $\widehat{G}\mathbf{x} \neq 0$. Hence, there exists $\omega \in \Omega$ such that $(S^{(1)}(\omega) - RS^{(0)})\mathbf{x} \neq 0$. Next, show that (c) must hold true by indirect proof: Assume $(S^{(1)}(\omega) - RS^{(0)})\mathbf{x} \geq 0$ for all $\omega \in \Omega$. According to Proposition 2.2.18 for $N = n = 1$ and using that the market is arbitrage-free, this can only hold true, if $(S^{(1)}(\omega) - RS^{(0)})\mathbf{x} = 0$ for all $\omega \in \Omega$. This is a contradiction, because $(S^{(1)}(\omega) - RS^{(0)})\mathbf{x} \neq 0$ for at least one $\omega \in \Omega$. Hence, (c) holds true which closes the cycle. □

Remark 2.2.25 (Connection to Maier-Paape and Zhu [34, Section 3.1]) For $N = 1$ and $R := S_0^{(1)}/S_0^{(0)}$ it is $\mathbf{x}^{(1)} =: \mathbf{x}$ and for $S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}$ it follows from (2.7) that

$$\mathcal{W}^{(N)}(\mathbf{X}) - \mathcal{W}^{(0)}\frac{S_0^{(N)}}{S_0^{(0)}} = \mathcal{W}^{(0)} + \Big(S^{(1)} - S^{(0)}\Big)\mathbf{x} - R\mathcal{W}^{(0)} = \Big(S^{(1)} - RS^{(0)}\Big)\mathbf{x}.$$

Hence, for $S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}$ Definition 2.2.14 for arbitrage opportunity, Definition 2.2.19 for bond replicating x-portfolio and Definition 2.2.22 for risk-free x-portfolio are equivalent to the definitions in [34, Definition 4], respectively.

In case of a discrete and finite probability space with $\Omega := \{\omega_1, \dots, \omega_K\}$ for $K \in \mathbb{N}$, properties (c) and (f) in Corollary 2.2.24 are exactly the assertion in [34, Theorem 2 (ii) and (iii)]. Consequently, Theorem 2.2.23 can be seen as a generalization of [34, Theorem 2] for multi-period market models in case $R \geq 1$. Note, that (ii) in [34, Theorem 2] includes the assumption that S is arbitrage-free, which is not required in (c) of Corollary 2.2.23, because (c) implies that there is no arbitrage opportunity. ♦

2.2.5 Generating function and examples

Since the strategies behind a φ-portfolio or an x-portfolio have many degrees of freedom, it belongs to the user to define them. With regard to portfolio optimization, it is preferable to just compute one single vector $\mathbf{y} \in \mathbb{R}^{M+1}$ which then already fully describes the x-portfolio.

Definition 2.2.26 (x-portfolio generating function) Let $M, N \in \mathbb{N}$ and $A \subset \mathbb{R}^{M+1}$. A function $v\colon A \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ which maps a vector $\mathbf{y} \in A$ to an x-portfolio for a given market model $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ is called an *x-portfolio generating function*, where

$$v(\mathbf{y}) := \Big(v^{(1)}(\mathbf{y}), \dots, v^{(N)}(\mathbf{y})\Big), \quad v^{(n)}(\mathbf{y}) \in \mathcal{L}^0\big(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}\big) \text{ for } n = 1, \dots, N.$$

Furthermore, the set $A \subset \mathbb{R}^{M+1}$ is called *admissible*, if $v(\mathbf{y}) \in \mathcal{A}(S)$ is an admissible x-portfolio for all $\mathbf{y} \in A$, i.e. $v(A) \subset \mathcal{A}(S)$, and $A \subset \mathbb{R}^{M+1}$ is called *$\mathcal{L}^2$-admissible* if $v(A) \subset \mathcal{A}^2(S)$.

Regarding convexity and Proposition 2.2.8, the following holds true.

Corollary 2.2.27 Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be a market model with x-portfolio generating function $v\colon \mathbb{R}^{M+1} \to \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$ which is linear. Then $v^{-1}(\mathcal{A}(S)) \subset \mathbb{R}^{M+1}$ is admissible and convex and $v^{-1}(\mathcal{A}^2(S)) \subset \mathbb{R}^{M+1}$ is $\mathcal{L}^2$-admissible and linear.

Proof. If $A := v^{-1}(\mathcal{A}(S)) = \emptyset$ there is nothing to show. Let now $A \neq \emptyset$. By definition A is admissible. From Proposition 2.2.8 it is known, that $\mathcal{A} := \mathcal{A}(S)$ is convex. Let $\mathbf{x}, \widetilde{\mathbf{x}} \in A$ be arbitrary. Then for all $\lambda \in [0,1]$ it is $v(\lambda\mathbf{x} + (1-\lambda)\widetilde{\mathbf{x}}) = \lambda v(\mathbf{x}) + (1-\lambda)v(\widetilde{\mathbf{x}}) \in \mathcal{A}$, because $v(\mathbf{x}), v(\widetilde{\mathbf{x}}) \in \mathcal{A}$. Therefore, it must be $\lambda\mathbf{x} + (1-\lambda)\widetilde{\mathbf{x}} \in A$.

The same argument works for the set $v^{-1}(\mathcal{A}^2(S))$, even when replacing factor λ by $\alpha \in \mathbb{R}$ and factor $1-\lambda$ by $\beta \in \mathbb{R}$, because $\mathcal{A}^2(S)$ is linear, see Proposition 2.2.8. $\square$

Remark 2.2.28 Note that in the one-period case $N = 1$, function v can be set as identity, i.e. $v(\mathbf{x}) = \mathbf{x} =: \mathbf{x}^{(1)}$, which, of course, is linear. Hence, for $N = 1$ it is $A := v^{-1}(\mathcal{A}(S)) = \mathcal{A}(S)$, which is convex and admissible. Because of Proposition 2.2.8 the set $\mathcal{A}^2$ contains all constants which yields, that $\mathbb{R}^{M+1}$ is $\mathcal{L}^2$-admissible. ♦

Another useful result is the following.

Lemma 2.2.29 For $M, N \in \mathbb{N}$ let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be a market model (see Definition 2.1.4) with no nontrivial risk-free x-portfolio (see Definition 2.2.22) and $v\colon \mathbb{R}^{M+1} \to \mathcal{L}^0(N-1;\mathbb{R}^{M+1})$ be an x-portfolio generating function. For $\beta > 0$ fixed, let $A \subset A_\beta := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ be non-empty, admissible and convex, and assume $v^{(1)}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in A$. Then the set A is bounded.

Proof. The proof of this result is similar to the proof in Maier-Paape and Zhu [34, Lemma 2], which gives a related result for a specific example.

Indirect proof: Assume A is unbounded. Then, there exists a sequence $(\mathbf{x}^n)_{n\in\mathbb{N}} \subset A$ with $\|\mathbf{x}^n\| \to \infty$ as $n \to \infty$. Since all elements are in A_β it must be $\|\widehat{\mathbf{x}}^n\| \to \infty$ as $n \to \infty$ as well. Then there is a subsequence, w.l.o.g. the same sequence $(\mathbf{x}^n)_{n\in\mathbb{N}}$, such that $\widehat{\mathbf{x}}^n/\|\widehat{\mathbf{x}}^n\| \to \widehat{\mathbf{x}}^*$ with $\|\widehat{\mathbf{x}}^*\| = 1$ and

$$\lim_{n\to\infty} \frac{x_0^n}{\|\widehat{\mathbf{x}}^n\|} = \frac{1}{S_0^{(0)}} \lim_{n\to\infty} \left[\frac{\beta}{\|\widehat{\mathbf{x}}^n\|} - \frac{S^{(0)}\widehat{\mathbf{x}}^n}{\|\widehat{\mathbf{x}}^n\|}\right] = -\frac{S^{(0)}\widehat{\mathbf{x}}^*}{S_0^{(0)}} =: x_0^*.$$

Therefore, $\mathbf{x}^n/\|\widehat{\mathbf{x}}^n\| \to \mathbf{x}^*$ as $n \to \infty$ with $S^{(0)}\mathbf{x}^* = 0$. Since $v(\mathbf{x}^n)$ is admissible, see Definition 2.2.6, the definition of v and (2.7) give

$$\mathcal{W}^{(1)}(v(\mathbf{x}^n)) = \mathcal{W}^{(0)} + \left(S^{(1)} - S^{(0)}\right)v^{(1)}(\mathbf{x}^n) = \mathcal{W}^{(0)} + \left(S^{(1)} - S^{(0)}\right)\mathbf{x}^n > 0$$

a.s. for all $n \in \mathbb{N}$. Dividing this inequality by $\|\widehat{\mathbf{x}}^n\|$ and taking the limit as $n \to \infty$ yields $S^{(1)}\mathbf{x}^* \geq 0$ a.s. and consequently

$$\mathrm{P}\left(\left(S^{(1)} - \frac{S_0^{(1)}}{S_0^{(0)}}S^{(0)}\right)\mathbf{x}^* < 0\right) = \mathrm{P}\left(S^{(1)}\mathbf{x}^* < 0\right) = 0.$$

Since S has no nontrivial risk-free x-portfolio, this contradicts Theorem 2.2.23 (c). Consequently, A must be bounded. $\square$

Two very simple and also most common trading strategies in theory of portfolio optimization is to fix either $\boldsymbol{\varphi}^{(n)}$ or $\mathbf{x}^{(n)}$ over time.

Example 2.2.30 **($\mathbf{x}^{(n)} \equiv$ const)** Define the x-portfolio generating function v_{bnh}, where "bnh" stands for "buy and hold", by $v_{\mathrm{bnh}}(\mathbf{x}) := (\mathbf{x}, \ldots, \mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^{M+1}$, which obviously is linear. Then it is $\mathbf{x}^{(n)} := v_{\mathrm{bnh}}^{(n)}(\mathbf{x}) \equiv \mathbf{x} \in \mathbb{R}^{M+1}$ a.s. for all $n = 1, \ldots, N$, i.e. the number of shares for each asset does not change in time, which explains the name for this trading strategy. Furthermore, $v_{\mathrm{bnh}}(\mathbf{x})$ is self-financing for all $\mathbf{x} \in \mathbb{R}^{M+1}$, because

$$S^{(n)}\mathbf{x}^{(n)} = S^{(n)}\mathbf{x} = S^{(n)}\mathbf{x}^{(n+1)}$$

for $n = 1, \ldots, N-1$. Therefore from Definition 2.2.2 and also Proposition 2.2.11 it is

$$\mathcal{W}^{(n)}(v_{\mathrm{bnh}}(\mathbf{x})) = \mathcal{W}^{(0)} + \sum_{k=1}^{n} \left(S^{(k)} - S^{(k-1)}\right)\mathbf{x} = \mathcal{W}^{(0)} + \left(S^{(n)} - S^{(0)}\right)\mathbf{x} \tag{2.31}$$

for $n = 1, \ldots, N$. Because of Proposition 2.2.8, $\mathbb{R}^{M+1}$ is already $\mathcal{L}^2$-admissible. The admissible set $A := v_{\mathrm{bnh}}^{-1}(\mathcal{A}(S))$ is given by

$$A = \left\{\mathbf{x} \in \mathbb{R}^{M+1} \; : \; S^{(0)}\mathbf{x} - \mathcal{W}^{(0)} < S^{(n)}\mathbf{x} \quad \text{a.s.} \quad \text{for } n = 1, \ldots, N\right\},$$

i.e. $\mathbf{x} \in A$ if and only if $\mathcal{W}^{(n)}(v_{\mathrm{bnh}}(\mathbf{x})) > 0$ for $n = 1, \ldots, N$. ♦

Example 2.2.31 **($\boldsymbol{\varphi}^{(n)} \equiv$ const)** The goal is to have an x-portfolio such that for the corresponding φ-portfolio it is $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ a.s. for all $n = 1, \ldots, N$. In other words, the portfolio has constant weights. Note, that to guarantee fixed weights, it is required to reallocate the wealth after each time step. For instants $\varphi_i = 1/(M+1)$ for $i = 0, 1, \ldots, M$ corresponds to the equal weight portfolio. To ensure the existence of an admissible x-portfolio, it is sufficient to require that $1 + T^{(n)}\boldsymbol{\varphi} > 0$ a.s. for $n = 1, \ldots, N$, see Proposition 2.2.7 (b). Therefore, let

$$\Phi_{\mathrm{twr}} := \left\{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \; : \; 1 + T^{(n)}\boldsymbol{\varphi} > 0 \text{ a.s. for } n = 1, \ldots, N\right\}. \tag{2.32}$$

Now, let v_{twr} be the x-portfolio generating function which maps $\boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}$ to an x-portfolio whose φ-portfolio $(\boldsymbol{\varphi}^{(n)})_{1 \le n \le N}$ satisfies $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ a.s. for all $n = 1, \ldots, N$. The acronym "twr" means *terminal wealth relative* (TWR) and is used because the resulting wealth is a scaled variant of the TWR, see e.g. Vince [53]. For such v_{twr}, (2.9) gives

$$\mathcal{W}^{(n)}(v_{\mathrm{twr}}(\boldsymbol{\varphi})) = \mathcal{W}^{(0)} \prod_{k=1}^{n} \left(1 + T^{(k)}\boldsymbol{\varphi}\right), \qquad \boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}, \tag{2.33}$$

for $n = 1, \ldots, N$. Consequently, using the definition of φ_i in (2.8), it is

$$(v_{\mathrm{twr}})_i^{(1)}(\boldsymbol{\varphi}) := \frac{\varphi_i}{S_i^{(0)}} \mathcal{W}^{(0)}, \tag{2.34}$$

$$(v_{\mathrm{twr}})_i^{(n)}(\boldsymbol{\varphi}) := \frac{\varphi_i}{S_i^{(n-1)}} \mathcal{W}^{(0)} \prod_{k=1}^{n-1} \left(1 + T^{(k)}\boldsymbol{\varphi}\right) = \frac{\varphi_i}{S_i^{(n-1)}} \mathcal{W}^{(n-1)}(v_{\mathrm{twr}}(\boldsymbol{\varphi})) \tag{2.35}$$

for $n = 2, \dots, N$ and $i = 0, 1, \dots, M$. This function is well-defined and from construction satisfies (2.33).

In this case, function $(v_{\text{twr}})^{(1)}$ is linear, but in general, v_{twr} is nonlinear for $N \geq 2$. Note that $(v_{\text{twr}})_i^{(n)}(\boldsymbol{\varphi})$ is in general not constant over time, i.e. when the investment is really executed using this trading strategy, it might be necessary to reallocate the wealth after each time step. The property "self-financing" is discussed in the next proposition. ♦

Proposition 2.2.32 Let the x-portfolio generating function v_{twr} with domain Φ_{twr}, see (2.32), be defined as in Example 2.2.31 and $N \geq 2$. If for arbitrary but fixed $\boldsymbol{\varphi} \in \Phi_{\text{twr}}$ there exists $n \in \{1, \dots, N-1\}$ such that $\mathrm{P}\left(1 + T^{(n)}\boldsymbol{\varphi} \neq 1\right) > 0$, then the following three statements are equivalent:

(a) $v_{\text{twr}}(\boldsymbol{\varphi})$ is self-financing.
(b) $\sum_{i=0}^{M} \varphi_i = 1$.
(c) $\mathcal{W}^{(0)} = S^{(0)}(v_{\text{twr}})^{(1)}(\boldsymbol{\varphi})$.

Proof. For simplicity, denote $\mathbf{x}^{(n)} := (v_{\text{twr}})^{(n)}(\boldsymbol{\varphi})$ for all $n = 1, \dots, N$ and let $c := \sum_{i=0}^{M} \varphi_i$ for given $\boldsymbol{\varphi} \in \Phi_{\text{twr}}$.

The proof for the equivalence between (b) and (c) reads: From definition (2.8) of $\varphi_i^{(n)} := \varphi_i$ it follows that

$$\sum_{i=0}^{M} \varphi_i = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{\mathcal{W}^{(n-1)}(v_{\text{twr}}(\boldsymbol{\varphi}))} \qquad \text{for all } n = 1, \dots, N. \tag{2.36}$$

This expression equals to one if and only if $S^{(n-1)}\mathbf{x}^{(n)} = \mathcal{W}^{(n-1)}(v_{\text{twr}}(\boldsymbol{\varphi}))$. Since $\mathbf{x}^{(n)}$ is already defined, such that the corresponding φ-portfolio is independent on n, it suffices to require this for $n = 1$, i.e. $S^{(0)}\mathbf{x}^{(1)} = \mathcal{W}^{(0)}$.

It remains to show that (a) is equivalent to (b) and (c). From Proposition 2.2.12, statement (a) is equivalent to

$$c = \sum_{i=0}^{M} \varphi_i = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{S^{(n-1)}\mathbf{x}^{(n)} + \mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)}} \qquad \text{for all } n = 1, \dots, N. \tag{2.37}$$

Statements (b) and (c) directly imply (2.37). Hence, it must be shown, that (2.37) in return implies (b) and (c).

Note, that the denominator in (2.37) is always greater than zero, because the x-portfolio is self-financing from (a) and admissible by assumption and because of (2.13). Therefore, condition (2.37) implies that

$$c\Big(\mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)}\Big) = (1-c)S^{(n-1)}\mathbf{x}^{(n)} \qquad \text{for all } n = 1, \dots, N.$$

This in turn is equivalent to either one of the following two cases:

(i) $c = 1$ and $\mathcal{W}^{(0)} = S^{(0)}\mathbf{x}^{(1)}$, or
(ii) $c \neq 1$ and $S^{(n-1)}\mathbf{x}^{(n)} \equiv \gamma$ for all $n = 1, \dots, N$ with $\gamma := c(\mathcal{W}^{(0)} - S^{(0)}\mathbf{x}^{(1)})/(1-c)$.

Case (i) already states (b) and (c). Therefore, it remains to show, that case (ii) cannot occur.

For this assume, that the x-portfolio is self-financing. Then it is $S^{(n)}\mathbf{x}^{(n)} = S^{(n)}\mathbf{x}^{(n+1)} = S^{(0)}\mathbf{x}^{(1)} \equiv \gamma$ for all $n = 1, \ldots, N-1$. Consequently, for each $n = 1, \ldots, N-1$ it is $(S^{(n)} - S^{(n-1)})\mathbf{x}^{(n)} = 0$ and therefore, from definition of $\mathcal{W}$, it must be $\mathcal{W}^{(n)} = \mathcal{W}^{(n-1)} = \mathcal{W}^{(0)}$. From (2.33) it follows, that $1+T^{(n)}\boldsymbol{\varphi} = 1$ for all $n \in \{1, \ldots, N-1\}$, which contradicts the assumption on $\boldsymbol{\varphi}$. Hence, self-financing and (ii) cannot hold true at the same time, which completes the proof. □

Note, that the condition $\mathrm{P}\,(T^{(n)}\boldsymbol{\varphi}^{(n)} \neq 0) > 0$ for at least one $n \in \{1, \ldots, N-1\}$ means, that the wealth of this portfolio is a real multi-period model, because the first $N-1$ time periods cannot be omitted. Case (b) of Proposition 2.2.32 indeed is a condition typically used in applications, see e.g. Maier-Paape and Zhu [34, 35]. Nevertheless, the x-portfolio generating function for this example in general is nonlinear, which is also the case for $\mathcal{W}^{(n)}$ considered as a function in $\boldsymbol{\varphi}$.

Therefore, whenever the fractions $\boldsymbol{\varphi}$ are of interest, one often finds the following variant in the literature, see e.g. Chekhlov, Uryasev and Zabarankin [9] and Rockafellar, Uryasev and Zabarankin [48], which is linear, but lacks on the property of self-financing.

Example 2.2.33 **($S^{(n-1)}\mathbf{x}^{(n)} \equiv$ const)** Here, the amount of money invested for each time step is supposed to be constant over time, i.e. gains are not reinvested and losses must be adjusted with money from cash. For this, define the x-portfolio generating v_{cm}, with “cm” for “constant (invested) money”, for $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_M)^\intercal \in \mathbb{R}^{M+1}$ by

$$x_i^{(n)} := (v_{\mathrm{cm}})_i^{(n)}(\boldsymbol{\theta}) := \frac{\theta_i}{S_i^{(n-1)}}\mathcal{W}^{(0)}. \tag{2.38}$$

Then it is

$$\theta_i = \frac{S_i^{(n-1)}x_i^{(n)}}{\mathcal{W}^{(0)}}, \tag{2.39}$$

which differs from definition of a φ-portfolio, see (2.8) (cf. also (2.35) and (2.38)), only in the denominator, which is $\mathcal{W}^{(0)}$ instead of $\mathcal{W}^{(n-1)}(v_{\mathrm{cm}}(\boldsymbol{\theta}))$, because the same amount of money is invested or, equivalently, the same fraction of the initial wealth. Summing over θ_i yields

$$\sum_{i=0}^{M}\theta_i = \frac{S^{(n-1)}\mathbf{x}^{(n)}}{\mathcal{W}^{(0)}},$$

i.e. $S^{(n-1)}\mathbf{x}^{(n)}$ is constant in time. Inserting $x_i^{(n)}$ into (2.7) yields

$$\mathcal{W}^{(n)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) = \mathcal{W}^{(0)} + \sum_{k=1}^{n}\sum_{i=0}^{M}\left(S_i^{(k)} - S_i^{(k-1)}\right)\frac{\theta_i}{S_i^{(k-1)}}\mathcal{W}^{(0)} = \mathcal{W}^{(0)}\left(1 + \sum_{k=1}^{n}T^{(k)}\boldsymbol{\theta}\right) \tag{2.40}$$

with admissible set $v_{\mathrm{cm}}^{-1}(\mathcal{A}(S)) = \{\boldsymbol{\theta} \in \mathbb{R}^{M+1} : \sum_{k=1}^{n} T^{(k)}\boldsymbol{\theta} > -1 \text{ a.s. for } n = 1, \ldots, N\}$. In contrast to (2.33), the product is, more or less, replaced by a sum. Of course, the x-portfolio $v_{\mathrm{cm}}(\boldsymbol{\theta})$ is in general not self-financing, because $S^{(n-1)}\mathbf{x}^{(n)} \equiv$ const and therefore

$$\begin{aligned} S^{(n)}\mathbf{x}^{(n)} - S^{(n)}\mathbf{x}^{(n+1)} &= \mathcal{W}^{(n)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) - \mathcal{W}^{(n-1)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) + S^{(n-1)}\mathbf{x}^{(n)} - S^{(n)}\mathbf{x}^{(n+1)} \\ &= \mathcal{W}^{(n)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) - \mathcal{W}^{(n-1)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) \end{aligned}$$

in general is different from zero. ♦

A summary of these examples is given in Table 2.2.

Example	x-portfolio generating function	wealth
2.2.30	$v_{\mathrm{bnh}}^{(n)}(\mathbf{x}) \equiv \mathbf{x}$	$\mathcal{W}^{(n)}(v_{\mathrm{bnh}}(\mathbf{x})) = \mathcal{W}^{(0)} + (S^{(n)} - S^{(0)})\mathbf{x}$
	constant number of shares (buy and hold strategy)	
2.2.31	$(v_{\mathrm{twr}})_i^{(n)}(\boldsymbol{\varphi}) = \frac{\varphi_i}{S_i^{(n-1)}} \mathcal{W}^{(n-1)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))$	$\mathcal{W}^{(n)}(v_{\mathrm{twr}}(\boldsymbol{\varphi})) = \mathcal{W}^{(0)} \prod_{k=1}^{n} (1 + T^{(k)}\boldsymbol{\varphi})$
	constant weight (invest the same fraction of current wealth after each time step)	
2.2.33	$(v_{\mathrm{cm}})_i^{(n)}(\boldsymbol{\theta}) := \frac{\theta_i}{S_i^{(n-1)}} \mathcal{W}^{(0)}$	$\mathcal{W}^{(n)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) = \mathcal{W}^{(0)} \Big(1 + \sum_{k=1}^{n} T^{(k)}\boldsymbol{\theta}\Big).$
	constant total amount of money (invest the same amount for each time step)	

Table 2.2.: Summary of examples for some x-portfolio generating functions.

2.3 Utility and risk measures

The utility function is used to measure the possible outcome or the gain of an investment. Generally the convention is that a higher value of the utility function means higher expected profit, i.e. a higher value is preferable. Often the function $\mathbf{x} \mapsto \mathrm{E}\,[S^{(1)}\mathbf{x}]$ on top of a one-period market model is used, see e.g. Maier-Paape and Zhu [34], Markowitz [36] and Rockafellar, Uryasev and Zabarankin [48]. This function has some favorable properties like linearity. However, this choice might not cover all requirements. Another choice is the so-called Terminal Wealth Relative (TWR), which has been studied in detail e.g. by Hermes [25], Hermes and Maier-Paape [26] and Maier-Paape and Zhu [35].

Next to the utility function, the risk measure plays a crucial role. Risk has been studied since decades and has many different forms, like variance [36], (conditional) value at risk, see e.g. Rockafellar and Uryasev [45], and drawdown, see e.g. Chekhlov, Uryasev and Zabarankin [9], Goldberg and Mahmoud [20] and Maier-Paape and Zhu [35]. Often there is a trade-off between the values of utility and risk. Nevertheless, both quantities have the same input, which in general is the wealth of a parameter dependent portfolio. Consequently, both types of measures are of same form and have some basic properties in common. When defining properties for risk measures, it is conventional to do so in an axiomatic way, see e.g. the coherent risk measure from Artzner, Delbaen, Eber and Heath [2] or the generalized deviation measures from Rockafellar, Uryasev and Zabarankin [47]. The axioms given in these two works are defined for one-period market models. There are also extension to the multi-period case, see e.g. [20], and also in form of so-called dynamic risk measures, see e.g. Acciaio and Penner [1] and Riedel [42].

In the following, reasonable axioms are given in Subsection 2.3.1 for the multi-period market model. A comparison to the axioms in the literature is done in Subsection 2.3.2. Then, in Subsections 2.3.3 and 2.3.4, examples for utility and risk measures are given, respectively. The so-called drawdown is an important risk quantity, which is discussed in detail in Chapter 3.

2.3.1 Axioms

A risk or utility measure maps a given (multi-period) stochastic process in $\mathcal{L}^2(N;\mathbb{R}_{>0})$, which can be a model for the equity of a portfolio, like the wealth $\mathcal{W}$ in Definition 2.2.2, to $\mathbb{R}$. Note that for $N = 1$ this corresponds to a one-period model, see Remark 2.1.5.

Definition 2.3.1 (Axioms for risk and utility measures) Let $\mathcal{M} \subset \mathcal{L}^2(N)$ for $N \in \mathbb{N}$ be a convex set and $\mathcal{H}\colon \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ be a given functional. The axioms, which $\mathcal{H}$ can satisfy, are defined as follows:
Dependency on adding risk-free or constant assets:

($\mathcal{B}$1a) $\mathcal{H}$ is *invariant for risk-free assets*, i.e. $\mathcal{H}(W + cC) = \mathcal{H}(W) + c\mathcal{H}(C)$ for all $W, C \in \mathcal{M}$, where C is risk-free, and all $c \in \mathbb{R}$ such that $W + cC \in \mathcal{M}$,

($\mathcal{B}$1b) $\mathcal{H}$ is *translation invariant*, i.e. $\mathcal{H}(W + C) = \mathcal{H}(W) + \mathcal{H}(C)$ for all $W \in \mathcal{M}$ and all $C = (c, \dots, c) \in \mathbb{R}^{N+1}$ such that $C \in \mathcal{M}$ and $W + C \in \mathcal{M}$.

Dependency on positive factors:

($\mathcal{B}$2) $\mathcal{H}$ is *positive homogeneous*, i.e. $\mathcal{H}(\lambda W) = \lambda\mathcal{H}(W)$ for all $W \in \mathcal{M}$ and all $\lambda > 0$ with $\lambda W \in \mathcal{M}$.

Axioms related to convexity, in which case it is assumed that $\mathcal{H}\colon \mathcal{M} \to \mathbb{R} \cup \{\infty\}$:

($\mathcal{B}$3a) $\mathcal{H}$ is *subadditive*, i.e. $\mathcal{H}(W+V) \le \mathcal{H}(W)+\mathcal{H}(V)$ for all $W, V \in \mathcal{M}$ with $W+V \in \mathcal{M}$,

($\mathcal{B}$3b) $\mathcal{H}$ is *proper convex*, i.e. $\mathcal{H}(\lambda W+(1-\lambda)V) \le \lambda\mathcal{H}(W)+(1-\lambda)\mathcal{H}(V)$ for all $W, V \in \mathcal{M}$ and all $\lambda \in [0, 1]$ and there exists $W \in \mathcal{M}$ such that $\mathcal{H}(W) < \infty$,

($\mathcal{B}$3c) $\mathcal{H}$ is *strictly convex*, i.e. $\mathcal{H}(\lambda W + (1 - \lambda)V) < \lambda\mathcal{H}(W) + (1 - \lambda)\mathcal{H}(V)$ for all $W \neq V \in \mathcal{M}$ and all $\lambda \in (0, 1)$.

Axioms related to concavity, in which case it is assumed that $\mathcal{H}\colon \mathcal{M} \to \mathbb{R} \cup \{-\infty\}$:

($\mathcal{B}$4a) $\mathcal{H}$ is *superadditive*, i.e. $\mathcal{H}(W+V) \ge \mathcal{H}(W)+\mathcal{H}(V)$ for all $W, V \in \mathcal{M}$ with $W+V \in \mathcal{M}$,

($\mathcal{B}$4b) $\mathcal{H}$ is *proper concave*, i.e. $\mathcal{H}(\lambda W + (1 - \lambda)V) \ge \lambda\mathcal{H}(W) + (1 - \lambda)\mathcal{H}(V)$ for all $W, V \in \mathcal{M}$ and all $\lambda \in [0, 1]$ and there exists $W \in \mathcal{M}$ such that $\mathcal{H}(W) > -\infty$,

($\mathcal{B}$4c) $\mathcal{H}$ is *strictly concave*, i.e. $\mathcal{H}(\lambda W + (1 - \lambda)V) > \lambda\mathcal{H}(W) + (1 - \lambda)\mathcal{H}(V)$ for all $W \neq V \in \mathcal{M}$ and all $\lambda \in (0, 1)$.

Note that for instance $\mathcal{L}^2(N;\mathbb{R}_{>0})$ and $\mathcal{L}^2(N)$ are convex, such that axioms ($\mathcal{B}$3b), ($\mathcal{B}$3c), ($\mathcal{B}$4b) and ($\mathcal{B}$4c) are well-defined for theses domains.

Each axiom has its own meaning. ($\mathcal{B}$1a) and ($\mathcal{B}$1b) say the following: Adding (or subtracting) a fixed amount of money to the portfolio's value after each time step — not necessarily the same amount in ($\mathcal{B}$1a) for each step, but without uncertainty — is the same, in the sense of $\mathcal{H}$, as having two separated portfolios; one without the extra money and one consisting only on this extra money. Thus, certain cash flows have a linear effect on the function and can be separated from uncertainty.

The second axiom ($\mathcal{B}$2) is the positive homogeneity of $\mathcal{H}$, i.e. whenever only a fraction or even a multiple is invested into the portfolio, the same fraction or multiple is expected according to $\mathcal{H}$.

The next axioms can roughly be separated into the convex and concave case. A utility function is concave in many cases due to the St. Petersburg paradox, see e.g. [4], while a risk measure is often assumed to be convex. Axiom ($\mathcal{B}$2) together with ($\mathcal{B}$3a) imply ($\mathcal{B}$3b), i.e. convexity, if the function in addition is finite at some point, because for each $\lambda \in (0,1)$ and all $W, V \in \mathcal{L}^2(N; \mathbb{R}_{>0})$ it is

$$\mathcal{H}(\lambda W + (1-\lambda)V) \leq \mathcal{H}(\lambda W) + \mathcal{H}((1-\lambda)V) = \lambda\mathcal{H}(W) + (1-\lambda)\mathcal{H}(V). \tag{2.41}$$

Axiom ($\mathcal{B}$3a) and also ($\mathcal{B}$3b) mean that, combining two portfolios is not worse, i.e. has not higher risk, than two separated portfolios, because of diversification reasons.

Analogously, Axioms ($\mathcal{B}$2) and ($\mathcal{B}$4a) yield ($\mathcal{B}$4b), i.e. concavity, if the function is finite at some point. In contrast to convexity, this property says, that combining two portfolios is not worse, i.e. has not lower utility, than two separated portfolios.

The market model for the portfolio, see Definitions 2.1.2 and 2.1.4 is not directly involved in these axioms. Assume a multi-period market model S is given. As already hinted in Definition 2.2.26, an x-portfolio generating function v can be used, to reduce the choice of the x-portfolio to just one vector $\mathbf{x}$ of dimension $M+1$. Thus, given S and given v, the portfolio, and consequently its wealth $\mathcal{W}$ from Definition 2.2.2, only depends on $\mathbf{x}$. Now, a measure as a function depending on the x-portfolio may fulfill the following axioms. For this define for $\mathbf{x} = (x_0, x_1, \dots, x_M)^\intercal \in \mathbb{R}^{M+1}$ again the risky part by $\widehat{\mathbf{x}} := (0, x_1, \dots, x_M)^\intercal \in \mathbb{R}^{M+1}$ and sign function by $\operatorname{sgn}(\alpha)$ for $\alpha \in \mathbb{R}$, where $\operatorname{sgn}(\alpha) = 1$ if $\alpha > 0$ and $\operatorname{sgn}(\alpha) = -1$ if $\alpha < 0$ and $\operatorname{sgn}(\alpha) = 0$ if $\alpha = 0$.

Definition 2.3.2 (Axioms for x-portfolio measure) Let $A \subset \mathbb{R}^{M+1}$ be convex. The function $\mathfrak{h}\colon A \to \mathbb{R} \cup \{\pm\infty\}$ with domain $\operatorname{dom}(\mathfrak{h}) := \{\mathbf{x} : \mathfrak{h}(\mathbf{x}) \in \mathbb{R}\}$ may satisfy some of the following axioms:

- ($\mathfrak{b}$1) $\mathfrak{h}$ is *linear in the risk-free part*, i.e. $\mathfrak{h}(\mathbf{x}) = c_{\mathfrak{h}} x_0 + \mathfrak{h}(\widehat{\mathbf{x}})$ for all $\mathbf{x} \in A$ and a fixed constant $c_{\mathfrak{h}} \in \mathbb{R}$,
- ($\mathfrak{b}$2) $\mathfrak{h}$ is *positive homogeneous*, i.e. $\mathfrak{h}(\lambda\mathbf{x}) = \lambda\mathfrak{h}(\mathbf{x})$ for all $\mathbf{x} \in A$ and all $\lambda > 0$ with $\lambda\mathbf{x} \in A$,
- ($\mathfrak{b}$3a) $\mathfrak{h}$ is *subadditive*, i.e. $\mathfrak{h}(\mathbf{x}+\mathbf{x}') \leq \mathfrak{h}(\mathbf{x}) + \mathfrak{h}(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in A$ with $\mathbf{x}+\mathbf{x}' \in A$,
- ($\mathfrak{b}$3b) $\mathfrak{h}$ is *proper convex*, i.e. convex, $\operatorname{dom}(\mathfrak{h}) \neq \emptyset$ and $\mathfrak{h}(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in A$,
- ($\mathfrak{b}$3c) $\mathfrak{h}$ is *strictly convex*,
- ($\mathfrak{b}$3d) $\mathfrak{h}$ is *strictly convex in* $\widehat{\mathbf{x}}$, i.e. $\mathfrak{h}(\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') < \lambda\mathfrak{h}(\mathbf{x})+(1-\lambda)\mathfrak{h}(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in A$ with $\widehat{\mathbf{x}} \neq \widehat{\mathbf{x}}'$ and all $\lambda \in (0,1)$,
- ($\mathfrak{b}$4a) $\mathfrak{h}$ is *superadditive*, i.e. $\mathfrak{h}(\mathbf{x}+\mathbf{x}') \geq \mathfrak{h}(\mathbf{x}) + \mathfrak{h}(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in A$ with $\mathbf{x}+\mathbf{x}' \in A$,
- ($\mathfrak{b}$4b) $\mathfrak{h}$ is *proper concave*, i.e. concave, $\operatorname{dom}(\mathfrak{h}) \neq \emptyset$ and $\mathfrak{h}(\mathbf{x}) < \infty$ for all $\mathbf{x} \in A$,
- ($\mathfrak{b}$4c) $\mathfrak{h}$ is *strictly concave*.
- ($\mathfrak{b}$4d) $\mathfrak{h}$ is *strictly concave in* $\widehat{\mathbf{x}}$, i.e. $\mathfrak{h}(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}') > \lambda\mathfrak{h}(\mathbf{x}) + (1-\lambda)\mathfrak{h}(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in A$ with $\widehat{\mathbf{x}} \neq \widehat{\mathbf{x}}'$ and all $\lambda \in (0,1)$,

In addition, $\mathfrak{h}$ is *closed proper convex* if $\mathfrak{h}$ is proper convex and lower semi-continuous. If $\mathfrak{h}$ is proper concave and upper semi-continuous, then $\mathfrak{h}$ is *closed proper concave*.

As above, ($\mathfrak{b}$2) and ($\mathfrak{b}$3a) imply convexity. If the function never attains $-\infty$ and is finite at some point it even implies ($\mathfrak{b}$3b). Analogously, ($\mathfrak{b}$2) and ($\mathfrak{b}$4a) yield concavity and might give ($\mathfrak{b}$4b) depending on the function.

Remark 2.3.3 Note that, if ($\mathfrak{b}$3b) or ($\mathfrak{b}$4b) is satisfied, then the function is continuous in the relative interior of its domain, see Theorem A.1.1 in the appendix. ♦

Having a market model and a linear x-portfolio generating function, an x-portfolio measure can be obtained by a measure from Definition 2.3.1 as follows.

Lemma 2.3.4 Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ for $M, N \in \mathbb{N}$ be a market model according to Definition 2.1.4 and $v\colon A \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ for convex and $\mathcal{L}^2$-admissible $A \subset \mathbb{R}^{M+1}$ be an x-portfolio generating function, see Definition 2.2.26, with one or more of the following properties:

(i) v is linear, i.e. $v(\lambda\mathbf{x} + \mathbf{y}) = \lambda v(\mathbf{x}) + v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in A$ and all $\lambda \in \mathbb{R}$ such that $\lambda\mathbf{x} + \mathbf{y} \in A$,

(ii) for a fixed $\mathbf{c} := (c^{(1)}, \dots, c^{(N)}) \in \mathbb{R}^N_{>0}$ it is

$$v^{(n)}(\mathbf{x}) = \Big(x_0 c^{(n)}, v_1^{(n)}(\widehat{\mathbf{x}}), \dots, v_M^{(n)}(\widehat{\mathbf{x}})\Big) \in \mathcal{L}^0\big(\Omega, \mathcal{F}_{n-1}, \mathrm{P}; \mathbb{R}^{M+1}\big)$$

for all $\mathbf{x} \in A$ and all $n = 1, \dots, N$, and

(iii) matrix $B \in \mathbb{R}^{(M+1)\times(M+1)}$ with $v^{(1)}(\mathbf{x}) = B\mathbf{x} \in \mathbb{R}^{M+1}$ a.s. for all $\mathbf{x} \in A$ has full rank.

For a given functional $\mathcal{H}\colon \mathcal{L}^2(N) \to \mathbb{R}$ define

$$\mathfrak{h}\colon A \to \mathbb{R}, \quad \mathbf{x} \mapsto \mathcal{H}\Big(\mathcal{W}(v(\mathbf{x})) - \mathcal{W}^{(0)}\Big),$$

where $\mathcal{W}$ is the corresponding wealth from Definition 2.2.2. Then it is:

(a) If (i) holds and $\mathcal{H}$ fulfills one of the axioms ($\mathcal{B}$2), ($\mathcal{B}$3a), ($\mathcal{B}$3b), ($\mathcal{B}$4a), or ($\mathcal{B}$4b), then $\mathfrak{h}$ fulfills the corresponding axiom ($\mathfrak{b}$2), ($\mathfrak{b}$3a), ($\mathfrak{b}$3b), ($\mathfrak{b}$4a), or ($\mathfrak{b}$4b), respectively.

(b) If (ii) holds and $\mathcal{H}$ fulfills ($\mathcal{B}$1a), then $\mathfrak{h}$ fulfills ($\mathfrak{b}$1).

(c) If (i), (ii) and (iii) hold, S is non-redundant and $\mathcal{H}$ fulfills either ($\mathcal{B}$3c) or ($\mathcal{B}$4c), then $\mathfrak{h}$ satisfies ($\mathfrak{b}$3d) or ($\mathfrak{b}$4d), respectively.
If in addition $S_0^{(n)} > S_0^{(n-1)}$ for all $n = 1, \dots, N$, then $\mathfrak{h}$ fulfills ($\mathfrak{b}$3c) or ($\mathfrak{b}$4c), respectively.

Before proving this lemma note, that the existence of a matrix B in (iii) already follows from (i) and because $v^{(1)} \in \mathcal{L}^0(\Omega, \mathcal{F}_0, \mathrm{P}; \mathbb{R}^{M+1})$ is always fixed with no uncertainty, cf. (2.1). If furthermore (ii) holds, then the first row and the first column of B consist only of zeros except their first entry which must be equal to $c^{(1)} > 0$. In addition, $\mathfrak{h}$ is well-defined, because $\mathcal{W}(v(\mathbf{x})) \in \mathcal{L}^2(N)$ for all $\mathbf{x} \in A$ and $\mathcal{L}^2$-admissible $A \subset \mathbb{R}^{M+1}$, see Definition 2.2.26 and also Definition 2.2.5.

Proof of Lemma 2.3.4. Proof of (a): Since v is linear, the mapping $\mathbf{x} \mapsto \mathcal{W}(v(\mathbf{x})) - \mathcal{W}^{(0)} =: \mathcal{V}(\mathbf{x})$ is linear as well, cf. (2.7). Consequently, it is easy to show, that properties ($\mathcal{B}$2), ($\mathcal{B}$3a), ($\mathcal{B}$3b), ($\mathcal{B}$4a), ($\mathcal{B}$4b) imply ($\mathfrak{b}$2), ($\mathfrak{b}$3a), ($\mathfrak{b}$3b), ($\mathfrak{b}$4a), ($\mathfrak{b}$4b), respectively.

Proof of (b): Because of property (ii) of v it is

$$\Big(S^{(k)} - S^{(k-1)}\Big)v^{(k)}(\mathbf{x}) = x_0\Big(S_0^{(k)} - S_0^{(k-1)}\Big)c^{(k)} + \Big(S^{(k)} - S^{(k-1)}\Big)v^{(k)}(\widehat{\mathbf{x}}).$$

Using (2.7) then yields

$$\mathcal{V}^{(n)}(\mathbf{x}) = \sum_{k=1}^{n}\Big(S^{(k)} - S^{(k-1)}\Big)v^{(k)}(\mathbf{x}) = x_0\sum_{k=1}^{n}\Big(S_0^{(k)} - S_0^{(k-1)}\Big)c^{(k)} + \mathcal{V}^{(n)}(\widehat{\mathbf{x}}).$$

The value $\tilde{c}^{(n)} := \sum_{k=1}^{n}(S_0^{(k)} - S_0^{(k-1)})c^{(k)}$ is constant almost surely, because it consists of the risk-free part of the market model. Since $S_0^{(k)} \geq S_0^{(k-1)}$ and $c^{(k)} > 0$ for all $k = 1, \dots n$, it is $\tilde{c}^{(n)} \geq \tilde{c}^{(n-1)}$ for all $n = 1, \dots, N$, where $\tilde{c}^{(0)} := 0$. Consequently, for $\tilde{\mathbf{c}} := (\tilde{c}^{(0)}, \tilde{c}^{(1)}, \dots, \tilde{c}^{(N)})$ and $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^{N+1}$, the processes $\mathbf{1} + \tilde{\mathbf{c}}$ and also $\mathbf{1}$ are risk-free. Note, that $\tilde{c}$ is not risk-free according to Definition 2.1.3, because $\tilde{c}^{(0)}$ is not positive. Using

$$\mathcal{V}(\mathbf{x}) = x_0\tilde{\mathbf{c}} + \mathcal{V}(\widehat{\mathbf{x}}) = x_0(\mathbf{1} + \tilde{\mathbf{c}}) - x_0\mathbf{1} + \mathcal{V}(\widehat{\mathbf{x}}),$$

and ($\mathcal{B}$1a) several times leads to

$$\mathfrak{h}(\mathbf{x}) = \mathcal{H}(\mathcal{V}(\mathbf{x})) = x_0\mathcal{H}(\mathbf{1} + \tilde{\mathbf{c}}) - x_0\mathcal{H}(\mathbf{1}) + \mathcal{H}(\mathcal{V}(\widehat{\mathbf{x}})) = x_0\mathcal{H}(\tilde{\mathbf{c}}) + \mathcal{H}(\mathcal{V}(\widehat{\mathbf{x}})).$$

Consequently, ($\mathfrak{b}$1) holds true for $c_{\mathfrak{h}} := \mathcal{H}(\tilde{\mathbf{c}})$.

Proof of (c): Using that S is non-redundant, from Lemma 2.2.20 follows that $\mathcal{W}$ is injective in $\widehat{\mathbf{X}}$. Assumption (iii) implies that $v^{(1)}$ and therefore also v is injective. Then $\mathbf{x} \mapsto \mathcal{V}(\mathbf{x}) = \mathcal{W}(v(\mathbf{x})) - \mathcal{W}^{(0)}$ is injective in $\widehat{\mathbf{x}}$ and linear because of (i). Hence, for each $\mathbf{x}, \mathbf{y} \in A$ with $\widehat{\mathbf{x}} \neq \widehat{\mathbf{y}}$ it is $\widehat{v(\mathbf{x})} \neq \widehat{v(\mathbf{y})}$ because of (ii) and consequently $\mathcal{V}(\mathbf{x}) \not\equiv \mathcal{V}(\mathbf{y})$. Assumption ($\mathcal{B}$3c) then yields for all $\lambda \in (0, 1)$ that

$$\mathfrak{h}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \mathcal{H}(\lambda\mathcal{V}(\mathbf{x}) + (1-\lambda)\mathcal{V}(\mathbf{y})) < \lambda\mathcal{H}(\mathcal{V}(\mathbf{x})) + (1-\lambda)\mathcal{H}(\mathcal{V}(\mathbf{y})),$$

and hence $\mathfrak{h}$ is strictly convex in $\widehat{\mathbf{x}}$, i.e. ($\mathfrak{b}$3d). Analogously, if ($\mathcal{B}$4c) holds true, then $\mathfrak{h}$ is strictly concave in $\widehat{\mathbf{x}}$.

If in addition $S_0^{(n)} > S_0^{(n-1)}$, then, again from Lemma 2.2.20, $\mathcal{W}$ is injective in $\mathbf{X}$ and the same works for each $\mathbf{x}, \mathbf{y} \in A$ with $\mathbf{x} \neq \mathbf{y}$. Consequently, $\mathfrak{h}$ is strictly convex or concave, respectively, giving ($\mathfrak{b}$3c) or ($\mathfrak{b}$4c). □

2.3.2 Connection to literature

A survey of developments in modeling risk is given by Krokhmal, Zabarankin, and Uryasev [30] which was published in 2011. A small selection of the literature described in [30] and also some recent work which has been done since 2011 are discussed in the following.

In year 1999, Artzner, Delbaen, Eber and Heath [2] introduced reasonable axioms which define the so-called coherent risk measure on a one-period market. Adapted into the framework discussed in this section, the axioms read as follows.

Definition 2.3.5 (Coherent risk measure, see [2]) A functional of form $\mathcal{C}\colon \mathcal{L}^2 \to (-\infty, \infty]$ is called a *coherent risk measure* if it satisfies

($\mathcal{C}1$) $\mathcal{C}(X + C) = \mathcal{C}(X) - c$ for all $X, C \in \mathcal{L}^2$, where $C \equiv c \in \mathbb{R}$ is constant,
($\mathcal{C}2$) $\mathcal{C}(0) = 0$ and $\mathcal{C}(\lambda X) = \lambda\, \mathcal{C}(X)$ for all $X \in \mathcal{L}^2$ and all $\lambda > 0$,
($\mathcal{C}3$) $\mathcal{C}(X + X') \leq \mathcal{C}(X) + \mathcal{C}(X')$ for all $X, X' \in \mathcal{L}^2$,
($\mathcal{C}4$) $\mathcal{C}(X) \leq \mathcal{C}(X')$ for all $X, X' \in \mathcal{L}^2$ with $X \geq X'$ a.s.

It is obvious, that ($\mathcal{C}2$) and ($\mathcal{C}3$) are the same as ($\mathcal{B}2$) and ($\mathcal{B}3$a) for $N = 1$ from Definition 2.3.1, respectively. If in addition $\mathcal{H}(C) = -c$ for all $C \equiv c \in \mathbb{R}$, then for this $\mathcal{H}$ axioms ($\mathcal{C}1$) and ($\mathcal{B}1$a), and also ($\mathcal{B}1$b), are the same. Axiom ($\mathcal{C}4$) has no corresponding axiom in Definition 2.3.1. Such a monotonicity criterion can also be formulated for the multi-period case, but there are several different ways, because there is no unique order for such multi-period processes. For instance it can be meaningful to say that "$W \leq V$", if $W^{(n)} \leq V^{(n)}$ a.s. for $n = 0, 1, \dots, N$, i.e. the value of the equity curve is ordered. Another choice would be to order the absolute returns, i.e. to define "$W \leq V$" by $W^{(n)} - W^{(n-1)} \leq V^{(n)} - V^{(n-1)}$ for $n = 1, \dots, N$. Such a definition strongly depends on the application. For simplicity, this axiom is not translated to the multi-period case.

Rockafellar, Uryasev and Zabarankin introduced the so called general deviation measure in [47] and also in their working paper [46]. A general deviation measure is closely related to the coherent risk measure although they are not the same. In [47], they also give a dual characterization and some properties on the combination of finitely many deviation measures, like a convex combination. Explanation of the relation to coherent risk measures, which is one-to-one under few assumptions, can be found in [47, Theorem 2]. The axioms read as follows.

Definition 2.3.6 (Generalized deviation measure, see [47]) A functional $\mathcal{D}\colon \mathcal{L}^2 \to [0, \infty]$ is called a *deviation measure* if it satisfies

($\mathcal{D}1$) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all $X, C \in \mathcal{L}^2$, where $C \equiv c \in \mathbb{R}$ is constant,
($\mathcal{D}2$) $\mathcal{D}(0) = 0$ and $\mathcal{D}(\lambda X) = \lambda\, \mathcal{D}(X)$ for all $X \in \mathcal{L}^2$ and all $\lambda > 0$,
($\mathcal{D}3$) $\mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(X')$ for all $X, X' \in \mathcal{L}^2$,
($\mathcal{D}4$) $\mathcal{D}(X) \geq 0$ for all $X \in \mathcal{L}^2$ and $\mathcal{D}(X) > 0$ for all nonconstant $X \in \mathcal{L}^2$.

Again, axioms ($\mathcal{D}2$) and ($\mathcal{D}3$) correspond to ($\mathcal{B}2$) and ($\mathcal{B}3$a) for $N = 1$. Axiom ($\mathcal{D}1$) is fulfilled by $\mathcal{H}$, if $\mathcal{H}(C) = 0$ for all constants $C \in \mathcal{L}^2$ and ($\mathcal{B}1$a) or ($\mathcal{B}1$b) holds true. As in the previous case, axiom ($\mathcal{D}4$) has no counterpart in Definition 2.3.1.

Goldberg and Mahmoud [20, Definition 3.1] give a variant for a generalized path dependent deviation measure, i.e. an extension of Definition 2.3.6 to the multi-period case. Chekhlov, Uryasev and Zabarankin [9] used a very similar definition. The versions reads similar to Definition 2.3.6, when $\mathcal{L}^2$ is replaced by $\mathcal{L}^2(N)$, constants are defined by $C \equiv (c, \dots, c) \in \mathbb{R}^{N+1}$ and the condition, that $\mathcal{D}(X) > 0$ for all nonconstants, is omitted.

All upper definitions are based on the functional on top of the market model. The x-portfolio variant in Definition 2.3.2 can be compared to the risk measures defined by Maier-Paape and Zhu [34, 35].

Definition 2.3.7 (Admissible convex risk measure and additional properties, see [34, 35]) Let $\mathfrak{r}\colon A \to [0,\infty)$ be a function for a non-empty, convex and closed set $A \subset \mathbb{R}^{M+1}$. Define the following properties this function can fulfill:

(r0) $\mathfrak{r}$ is continuous and $\mathfrak{r}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in A$,

(r1) $\mathfrak{r}$ is independent on x_0, i.e. $\mathfrak{r}(\mathbf{x}) = \mathfrak{r}(\widehat{\mathbf{x}})$ for all $\mathbf{x} \in A$,

(r1n) there exists $\mathbf{x}' \in A$ with $\widehat{\mathbf{x}}' = 0$, furthermore, $\mathfrak{r}(\mathbf{x}) = 0$ if and only if $\widehat{\mathbf{x}} = 0$,

(r2) $\mathfrak{r}$ is convex,

(r2s) $\mathfrak{r}$ is strictly convex in $\widehat{\mathbf{x}}$,

(r3) $\mathfrak{r}$ satisfies (r1) and $\mathfrak{r}(\lambda\mathbf{x}) = \lambda\,\mathfrak{r}(\mathbf{x})$ for all $\mathbf{x} \in A$ and all $\lambda > 0$ with $\lambda\mathbf{x} \in A$,

(r3s) $\mathfrak{r}$ satisfies (r3) and for all $\mathbf{x}, \mathbf{x}' \in A$ with $\widehat{\mathbf{x}} \neq \widehat{\mathbf{x}}'$ as well as $\mathfrak{r}(\mathbf{x}) = \mathfrak{r}(\mathbf{x}') = 1$ and for all $\alpha \in (0,1)$ the function $\mathfrak{r}$ satisfies $\mathfrak{r}(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') < \alpha\,\mathfrak{r}(\mathbf{x}) + (1-\alpha)\,\mathfrak{r}(\mathbf{x}') = 1$.

(r4) for all $\mathbf{x} \in A$ with $\widehat{\mathbf{x}} \neq 0$ the mapping $\lambda \mapsto \mathfrak{r}(\lambda\mathbf{x})$, with $\lambda > 0$ such that $\lambda\mathbf{x} \in A$, is strictly increasing.

Function $\mathfrak{r}$ is called an *admissible convex risk measure*, if it satisfies (r0), (r1), (r2) and (r4).

Note, that in [34] axioms (r3) and (r3s) are originally stated for the function $\widehat{\mathfrak{r}}$ which, because of (r1), is defined by $\widehat{\mathfrak{r}}(x_1,\dots,x_M) := \mathfrak{r}(\widehat{\mathbf{x}})$ for all $\widehat{\mathbf{x}} = (0, x_1, \dots, x_M) \in A$. Since the first entry of the input variable $\mathbf{x}$ is irrelevant for $\mathfrak{r}$, the original axioms are equivalent to the variants in Definition 2.3.7.

Axiom (r1) corresponds to ($\mathfrak{b}$1) if $c_{\mathfrak{h}} = 0$. Of course, (r2) and (r3) are the same as ($\mathfrak{b}$3b) and ($\mathfrak{b}$2), respectively. If the function in Definition 2.3.2 is proper convex and finite, it is also continuous, which is the first part of (r0). If it is in addition non-negative, (r0) is fulfilled as well.

2.3.3 Utility measures

A utility measure can either be a (concave) function according to Definition 2.3.1 or a (concave) function according to Definition 2.3.2. However, in many applications, the utility function is of special form, which is often linear, see e.g. Maier-Paape and Zhu [34], Markowitz [36] and Rockafellar, Uryasev and Zabarankin [48]. In some cases, the utility function is also discussed in a more general form as by Maier-Paape and Zhu [34, 35]. Therein, the one-period market model is used, i.e. $N = 1$, and the utility function for an x-portfolio is defined by $\mathfrak{u}(\mathbf{x}) := \mathrm{E}[u(S^{(1)}\mathbf{x})]$ for some $u\colon \mathbb{R} \to \mathbb{R}$ with appropriate properties.

In the following, some examples are given, which all fit into the general form of Definition 2.3.2. Let $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ for $M \in \mathbb{N}$ with $N \in \mathbb{N}$ time steps be a market model according to Definition 2.1.4. Suppose the x-portfolio is generated by $v\colon A \to \mathcal{L}^0(N-1; \mathbb{R}^{M+1})$ for some convex and $\mathcal{L}^2$-admissible set $A \subset \mathbb{R}^{M+1}$, and the mapping $\mathcal{W}^{(N)}$ is the wealth at time step N according to Definition 2.2.2. One of the most reasonable and common utility functions in both variants is defined by

$$
\begin{aligned}
\mathcal{U}\colon \mathcal{L}^2(N) \to \mathbb{R}, &\qquad W \mapsto \mathcal{U}(\mathcal{W}) := \mathrm{E}\left[W^{(N)} - W^{(0)}\right],\\
\mathfrak{u}\colon A \to \mathbb{R}, &\qquad \mathbf{x} \mapsto \mathfrak{u}(\mathbf{x}) := \mathcal{U}(\mathcal{W}(v(\mathbf{x}))) = \mathrm{E}\left[\mathcal{W}^{(N)}(v(\mathbf{x})) - \mathcal{W}^{(0)}\right].
\end{aligned}
\tag{2.42}
$$

Function $\mathcal{U}$ is already well-defined and has no parameter. Since this function is linear, it is also invariant for risk-free assets ($\mathcal{B}$1a), translation invariant ($\mathcal{B}$1b), positive homogeneous ($\mathcal{B}2$), subadditive ($\mathcal{B}3$a), superadditive ($\mathcal{B}4$a), convex ($\mathcal{B}3$b) and concave ($\mathcal{B}4$b) at the same time.

On the other hand, function $\mathfrak{u}$ depends on the market model S and the x-portfolio generating function v, which, in this general form, are customizable. Therefore, different choices are discussed in the following.

Example 2.3.8 **($\mathbf{x}^{(n)} \equiv \mathbf{x}$)** Let the market model be based on historical data as described in Section 2.1.3, where again $\widehat{S} := (\widehat{S}_i^{(n)})_{0\leq n\leq K,\, 0\leq i\leq M} \in \mathbb{R}^{(K+1)\times(M+1)}$ is the matrix containing the historical data. In addition, let the occurrences of each elementary event in Ω have the same probability.

Assume that $\mathbf{x}^{(n)} := v_{\text{bnh}}^{(n)}(\mathbf{x}) \equiv \mathbf{x} \in A \subset \mathbb{R}^{M+1}$ a.s. for all $n = 1, \ldots, N$, see Example 2.2.30. This portfolio is always self-financing and equation (2.31) yields

$$\mathfrak{u}(\mathbf{x}) = \mathrm{E}\left[\mathcal{W}^{(N)}(v_{\text{bnh}}(\mathbf{x})) - \mathcal{W}^{(0)}\right] = \mathrm{E}\left[S^{(N)} - S^{(0)}\right]\mathbf{x}.$$

Now, for some different market models from historical data, one obtains the following:

(a) *One-period version:* If $N = 1$ it is in addition $\boldsymbol{\varphi}^{(1)} \equiv \boldsymbol{\varphi}$. When using the market model in Example 2.1.6 (a) with $\mathrm{P}(\{\omega\}) := 1/K$ for all $\omega \in \Omega = \{1, \ldots, K\}$, then (2.5) leads to

$$\mathfrak{u}(\mathbf{x}) = \left(\frac{1}{K}\sum_{n=1}^{K}\left(\widehat{S}^{(n)} - \widehat{S}^{(n-1)}\right)\right)\mathbf{x} = \frac{1}{K}\left(\widehat{S}^{(K)} - \widehat{S}^{(0)}\right)\mathbf{x}.$$

(b) *Multi-period single path version:* When using the market model from Example 2.1.7, where $\Omega = \{1\}$, with $N := K \in \mathbb{N}$, it is $S^{(0)}\mathbf{x} = \widehat{S}^{(0)}\mathbf{x}$ and

$$\mathfrak{u}(\mathbf{x}) = \left(\widehat{S}^{(N)} - \widehat{S}^{(0)}\right)\mathbf{x}.$$

(c) *Multi-period version:* In case of Example 2.1.8 (a) with $N \in \mathbb{N}$ simulated time step it is $\Omega := \{1, \ldots, K\}^N$. Assume the steps are independent and identically distributed (iid) with $\mathrm{P}(\{\omega\}) := 1/K^N$ for all $\omega \in \Omega$ and $\mathrm{P}(\{\omega = (\omega_1, \ldots, \omega_K) \in \Omega : \omega_n = k\}) = 1/K$ for $n = 1, \ldots, N$ and $k = 1, \ldots, K$. From (2.6) it follows that

$$\mathfrak{u}(\mathbf{x}) = \sum_{k=1}^{N}\left(\sum_{\omega_k=1}^{K}\frac{1}{K}\left(\widehat{S}^{(\omega_k)} - \widehat{S}^{(\omega_k-1)}\right)\right)\mathbf{x} = \frac{N}{K}\left(\widehat{S}^{(K)} - \widehat{S}^{(0)}\right)\mathbf{x}.$$

Since v_{bnh} is linear and obviously fulfills the assumptions in Lemma 2.3.4, it is known, that the function $\mathfrak{u}$ in all three cases inherits the properties from $\mathcal{U}$. This is obvious because $\mathfrak{u}$ itself is linear. Hence, $\mathfrak{u}$ is linear in the risk-free part ($\mathfrak{b}1$), positive homogeneous ($\mathfrak{b}2$), subadditive ($\mathfrak{b}3$a), superadditive ($\mathfrak{b}4$a), convex ($\mathfrak{b}3$b) and concave ($\mathfrak{b}4$b).

Note that $\mathfrak{u}$ is the same for all three variants up to a fixed scaling factor. This value is proportional to the absolute return of the historical data when holding a fixed number of shares of each asset given by $\mathbf{x}$. Thus, this would be the hypothetical value of the portfolio which would have been occurred, when investing in this portfolio in the past. ♦

Another well-known choice is to set the φ-portfolio constant over time, see Example 2.2.31. This leads to the so-called terminal wealth relative, see e.g. Vince [53].

Definition 2.3.9 (Terminal Wealth Relative (TWR)) Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be the market model and $\mathbf{X}$ be an admissible x-portfolio, with φ-portfolio defined by $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ a.s. for $n = 1, \ldots, N$. The *terminal wealth relative* (TWR) is defined by

$$\mathrm{TWR}(\boldsymbol{\varphi}) := \prod_{n=1}^{N} \left(1 + T^{(n)}\boldsymbol{\varphi}\right) \in \mathcal{L}^0(\mathbb{R}_{>0}).$$

For the x-portfolio generating function v_{twr} from Example 2.2.31, such that $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi}$ is constant in time, the observation in (2.33) for $\boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}$ with Φ_{twr} from (2.32) gives

$$\mathrm{TWR}(\boldsymbol{\varphi}) = \frac{\mathcal{W}^{(N)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))}{\mathcal{W}^{(0)}}, \tag{2.43}$$

i.e. the TWR is the final wealth divided by the initial wealth $\mathcal{W}^{(0)}$. Taking the expectation of TWR gives $\mathfrak{u}$ (writing now $\boldsymbol{\varphi}$ instead of $\mathbf{x}$) as defined in (2.42) up to an affine linear scaling (with factor $\mathcal{W}^{(0)}$) when restricted to an $\mathcal{L}^2$-admissible set A. Some special cases read as follows:

Example 2.3.10 ($\varphi^{(n)} \equiv \varphi$) As in Example 2.3.8 let the market model be based on historical data, see also Section 2.1.3. Assume $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi} \in A$ a.s. for all $n = 1, \ldots, N$, where it is important, that $A \subset \Phi_{\mathrm{twr}}$ is admissible and $\mathcal{L}^2$-admissible, see Example 2.2.31. Let $\mathfrak{u}$ be the corresponding utility function as defined in (2.42), i.e. $\mathfrak{u}(\boldsymbol{\varphi}) = \mathrm{E}\left[\mathcal{W}^{(N)}(v_{\mathrm{twr}}(\boldsymbol{\varphi})) - \mathcal{W}^{(0)}\right]$. Using this together with (2.43) it follows, that

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) := \mathrm{E}\left[\prod_{n=1}^{N} \left(1 + T^{(n)}\boldsymbol{\varphi}\right)\right] = \mathrm{E}\left[\mathrm{TWR}(\boldsymbol{\varphi})\right] = 1 + \frac{\mathfrak{u}(\boldsymbol{\varphi})}{\mathcal{W}^{(0)}} \tag{2.44}$$

for all $\boldsymbol{\varphi} \in A$. If in addition $T^{(1)}, \ldots, T^{(N)}$ are stochastically independent then

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = \prod_{n=1}^{N} \mathrm{E}\left[1 + T^{(n)}\boldsymbol{\varphi}\right] = \prod_{n=1}^{N} \left(1 + \mathrm{E}\left[T^{(n)}\right]\boldsymbol{\varphi}\right),$$

and if they are in addition identically distributed, it is

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = \left(1 + \mathrm{E}\left[T^{(1)}\right]\boldsymbol{\varphi}\right)^{N}. \tag{2.45}$$

Because of Proposition 2.2.32, this portfolio is self-financing, if and only if $\sum_{i=0}^{M} \varphi_i = 1$. For some different market models, this can also be written as follows:

(a) *One-period version:* In case $N = 1$ it is also $\mathbf{x}^{(1)} \equiv \mathbf{x}$ and from (2.44) it is

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = 1 + \mathrm{E}\left[T^{(1)}\right]\boldsymbol{\varphi}.$$

Using the market model in Example 2.1.6 (b) with $\mathrm{P}(\{\omega\}) = 1/K$ for all $\omega \in \Omega = \{1, \ldots, K\}$ leads to

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = 1 + \left(\frac{1}{K}\sum_{n=1}^{K} \widehat{T}^{(n)}\right)\boldsymbol{\varphi}. \tag{2.46}$$

(b) *Multi-period single path version:* Market model from Example 2.1.7 with $\Omega = \{1\}$ directly results in

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = \prod_{n=1}^{N} \left(1 + \widehat{T}^{(n)} \boldsymbol{\varphi}\right). \tag{2.47}$$

(c) *Multi-period version:* For Example 2.1.8 (b) with N simulated time steps, let the random variables $T^{(1)}, \ldots, T^{(N)}$ be iid and uniformly distributed. Equation (2.45) yields

$$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = \left[1 + \left(\frac{1}{K} \sum_{n=1}^{K} \widehat{T}^{(n)}\right) \boldsymbol{\varphi}\right]^{N}. \tag{2.48}$$

Version (a) defines an utility measure with properties ($\mathfrak{b}$1), ($\mathfrak{b}$3a), ($\mathfrak{b}$3b), ($\mathfrak{b}$4a) and ($\mathfrak{b}$4b) in Definition 2.3.2, because it is affine linear in $\boldsymbol{\varphi}$. For $N \geq 2$, the utility function $\mathfrak{u}$ in general is nonlinear, see (2.44), which is also the case with iid random variables (2.45). Of course, the same holds true for variant (b) and (c). Consequently, it lacks at least the property ($\mathfrak{b}$1). However, the Nth root of the function in variant (b) gives a concave function under some reasonable assumption on $\widehat{T}$, which can even be strictly concave, see e.g. Hermes [25, Lemma 4.2.5].

Note that (2.46) and (2.48) are almost the same up to the exponent N, which comes from the fact, that in (c) N steps of type (a) are simulated independently. ♦

Remark 2.3.11 (Connection to Vince [53]) The TWR introduced by Vince in [53] is of slightly different form. Therein, Ω is assumed to be finite in the one-period case $N = 1$. For this define

$$B := (1, B_1, \ldots, B_M) \qquad \text{with} \qquad B_i := \frac{S_i^{(1)} - S_i^{(0)}}{\max\limits_{\widetilde{\omega} \in \Omega} \left\{ S_i^{(0)} - S_i^{(1)}(\widetilde{\omega}) \right\}}, \; i = 1, \ldots, M,$$

where it is assumed that the biggest loss of the ith asset, namely $\max_{\widetilde{\omega} \in \Omega} \{S_i^{(0)} - S_i^{(1)}(\widetilde{\omega})\}$, is positive for $i = 1, \ldots, M$. For $\mathbf{f} := (f_0, f_1, \ldots, f_M)^{\intercal} \in \mathbb{R}^{M+1}$ it is

$$\mathrm{TWR}_{\mathrm{Vince}}(\mathbf{f}) := \left(\prod_{\omega \in \Omega} (1 + B(\omega)\mathbf{f})^{\mathrm{P}(\{\omega\})} \right)^{X}.$$

where X denotes the number of simulated time periods, see [53, (4.03)]. This variant is derived from the geometric mean of $1 + B\mathbf{f}$. Note, that B contains the absolute returns related to the biggest loss, which is always assumed to be positive. Since the denominator of the definition of B_i is independent on ω, it is

$$\mathrm{TWR}_{\mathrm{Vince}}(\mathbf{f}) = \left(\prod_{\omega \in \Omega} \left(1 + T^{(1)}(\omega) \boldsymbol{\varphi}\right)^{\mathrm{P}(\{\omega\})} \right)^{X} \quad \text{for} \quad \varphi_i := f_i \frac{S_i^{(0)}}{\max\limits_{\widetilde{\omega} \in \Omega} \left\{ S_i^{(0)} - S_i^{(1)}(\widetilde{\omega}) \right\}}.$$

Hence, **f** contains weights, which can be seen as fractions of wealth the investor is willing to risk for the corresponding asset. In addition, using the same market model and the TWR from Definition 2.3.9 for $N = 1$ it is

$$\mathrm{E}\left[\ln\left(\mathrm{TWR}(\boldsymbol{\varphi})\right)\right] = \sum_{\omega\in\Omega} \mathrm{P}\left(\{\omega\}\right) \ln\left(1 + T^{(1)}(\omega)\boldsymbol{\varphi}\right) = \ln\left(\prod_{\omega\in\Omega}\left(1 + T^{(1)}(\omega)\boldsymbol{\varphi}\right)^{\mathrm{P}(\{\omega\})}\right)$$

and therefore

$$\mathrm{E}\left[\ln\left(\mathrm{TWR}(\boldsymbol{\varphi})\right)\right] = \frac{1}{X}\ln\left(\mathrm{TWR}_{\mathrm{Vince}}(\mathbf{f})\right), \tag{2.49}$$

which shows the connection between both versions. ♦

As indicated with this remark, it is also common to take the expectation of $\ln(\mathcal{W}^{(N)})$, see (2.49), even though it is not always obvious. In TWR case $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi}$ the resulting utility function has some favorable properties like concavity as shown next, see also Maier-Paape and Zhu [34, Section 6] for some similar results for the one-period case with finite probability space.

Theorem 2.3.12 (Concavity of $\mathrm{E}[\ln(\mathrm{TWR})]$**)** Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be the market model such that $T^{(n)} \in \mathcal{L}^1(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$ for $n = 1,\dots,N$ and TWR from Definition 2.3.9. Assume the x-portfolio generating function v_{twr} is defined as in Example 2.2.31 with convex and admissible set Φ_{twr} from (2.32). Then, function

$$\mathfrak{u}_{\ln}\colon \mathbb{R}^{M+1} \to [-\infty,\infty), \qquad \boldsymbol{\varphi} \mapsto \begin{cases} \mathrm{E}\left[\ln\left(\mathrm{TWR}(\boldsymbol{\varphi})\right)\right], & \boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}, \\ -\infty, & \boldsymbol{\varphi} \notin \Phi_{\mathrm{twr}}, \end{cases}$$

is proper concave ($\mathfrak{b}$4b) with convex and non-empty effective domain

$$\mathrm{dom}(\mathfrak{u}_{\ln}) := \left\{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \,:\, \mathfrak{u}_{\ln}(\boldsymbol{\varphi}) > -\infty\right\} \subset \Phi_{\mathrm{twr}}.$$

The closure $\overline{\mathfrak{u}}_{\ln}$ of $\mathfrak{u}_{\ln}$, see (A.2) in the appendix, is closed proper concave.

If in addition S is non-redundant, then on $\Phi_\beta := \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \,:\, \sum_{i=0}^{M} \varphi_i = \beta\}$ for fixed $\beta > 0$ the following holds true:

(a) $\mathfrak{u}_{\ln}$ and $\overline{\mathfrak{u}}_{\ln}$ restricted to the convex set $\mathrm{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta$ are strictly concave ($\mathfrak{b}$4c).

(b) $\mathfrak{u}_{\ln}$ and $\overline{\mathfrak{u}}_{\ln}$ restricted to the convex set $\mathrm{dom}(\mathfrak{u}_{\ln}) \cap (\{0\} \times \mathbb{R}^M)$ are strictly concave ($\mathfrak{b}$4c).

(c) If $S_0^{(n)} > S_0^{(n-1)}$ for all $n = 1,\dots,N$, then $\mathfrak{u}_{\ln}$ and $\overline{\mathfrak{u}}_{\ln}$ are strictly concave ($\mathfrak{b}$4c) on $\mathrm{dom}(\mathfrak{u}_{\ln})$.

(d) If S has no nontrivial risk-free x-portfolio, then $\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu) := \{\mathbf{x} \in \Phi_\beta \,:\, \overline{\mathfrak{u}}_{\ln}(\mathbf{x}) \ge \mu\}$ is compact for all $\mu \in \mathbb{R}$.

Proof. Definition 2.3.9 of the TWR, properties of ln, and linearity of expectation yields

$$\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\ln\left(\prod_{n=1}^{N}\left(1 + T^{(n)}\boldsymbol{\varphi}\right)\right)\right] = \sum_{n=1}^{N}\mathrm{E}\left[\ln\left(1 + T^{(n)}\boldsymbol{\varphi}\right)\right] \tag{2.50}$$

for $\boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}$. Since $\ln(1+x) \le x$ for all $x > -1$, it is $\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) \le \sum_{n=1}^{N}\mathrm{E}\,[T^{(n)}]\boldsymbol{\varphi} < \infty$, because $T^{(n)}$ has finite expectation by assumption. For all $\boldsymbol{\varphi} \notin \Phi_{\mathrm{twr}}$ it is $\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) < \infty$ from definition.

Hence, $\mathfrak{u}_{\ln}$ is well-defined. In addition, $1 + T^{(n)}\boldsymbol{\varphi}$ is linear in $\boldsymbol{\varphi}$ and ln is strictly concave. Hence, $\boldsymbol{\varphi} \mapsto \ln(1 + T^{(n)}(\omega)\boldsymbol{\varphi})$ is concave for arbitrary but fixed $\omega \in \Omega$. The linearity and monotonicity of the expectation yields, that each summand in (2.50) must be concave, which then also must hold for $\mathfrak{u}_{\ln}$. From this, it directly follows, that its effective domain must be convex and $(0, \dots, 0)^\intercal \in \operatorname{dom}(\mathfrak{u}_{\ln}) \subset \Phi_{\text{twr}}$. Hence, $\mathfrak{u}_{\ln}$ is proper concave with non-empty domain. The closure of $\mathfrak{u}_{\ln}$ then must be closed proper concave, see Theorem A.1.1 and Remark A.1.5 in the appendix.

Proof of (a), (b) and (c): In situation (c), Lemma 2.2.20 yields, that $\mathcal{W}$ is injective. Since the same holds for v_{twr} by construction, see e.g. definition of $v_{\text{twr}}^{(1)}$ in (2.34), the mapping $\boldsymbol{\varphi} \mapsto \mathcal{W}(v_{\text{twr}}(\boldsymbol{\varphi}))$ must be injective, too. In situation (a) and (b), Lemma 2.2.20 yields that $\mathcal{W}$ is injective only in the risky part, i.e. for $\boldsymbol{\varphi}, \boldsymbol{\theta} \in \Phi_\beta$ in case (a) or $\boldsymbol{\varphi}, \boldsymbol{\theta} \in \operatorname{dom}(\mathfrak{u}_{\ln}) \cap (\{0\} \times \mathbb{R}^M)$ in case (b), $\mathbf{X} := v_{\text{twr}}(\boldsymbol{\varphi})$, $\mathbf{Y} := v_{\text{twr}}(\boldsymbol{\theta})$ with $\mathcal{W}(\mathbf{X}) = \mathcal{W}(\mathbf{Y})$ a.s. it is $\widehat{\mathbf{X}} = \widehat{\mathbf{Y}}$ a.s. From definition of $v_{\text{twr}}^{(1)}$ it then must be $\widehat{\boldsymbol{\varphi}} = \widehat{\boldsymbol{\theta}}$. In case (a), where $\boldsymbol{\varphi}, \boldsymbol{\theta} \in \Phi_\beta$, and also in case (b), it is has to be $\varphi_0 = \theta_0$ as well and therefore $\boldsymbol{\varphi} = \boldsymbol{\theta}$.

Hence, in all three cases (a), (b) and (c), for arbitrary $\boldsymbol{\varphi}, \boldsymbol{\theta}$ in the corresponding domain with $\boldsymbol{\varphi} \neq \boldsymbol{\theta}$, there exists $n \in \{1, \dots, N\}$, such that $T^{(n)}\boldsymbol{\varphi} \not\equiv T^{(n)}\boldsymbol{\theta}$. Therefore, for all $\lambda \in (0,1)$ from the strict concavity of ln follows

$$L := \ln\Big(1 + T^{(n)}(\lambda\boldsymbol{\varphi} + (1-\lambda)\boldsymbol{\theta})\Big) > \lambda \ln\Big(1 + T^{(n)}\boldsymbol{\varphi}\Big) + (1-\lambda)\ln\Big(1 + T^{(n)}\boldsymbol{\theta}\Big) =: R$$

with positive probability. In addition, there must exists $\varepsilon > 0$, such that $\mathrm{P}(L > R + \varepsilon) > 0$, because of the following: Assume $\mathrm{P}(L - R \le \varepsilon) = F_{L-R}(\varepsilon) = 1$ for all $\varepsilon > 0$, where F_{L-R} denotes the cumulative distribution function. Since F_{L-R} is upper semi-continuous, it is $\mathrm{P}(L \le R) = 1$ which is a contradiction to $\mathrm{P}(L > R) > 0$.

Let now $\varepsilon > 0$, such that $\mathrm{P}(L > R + \varepsilon) > 0$. In addition, it is already known that $L \ge R$ a.s., which gives

$$\mathrm{E}[L - R] \ge \varepsilon\, \mathrm{P}(L - R > \varepsilon) + 0 \cdot \mathrm{P}(L - R \le \varepsilon) > 0.$$

Consequently $\boldsymbol{\varphi} \mapsto 1 + T^{(n)}\boldsymbol{\varphi}$ and also $\mathfrak{u}_{\ln}$ are strictly concave in situations (a), (b) and (c).

Proof of (d): From Theorem A.1.2 and Remark A.1.5 it follows, that $\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)$ is closed and convex for all $\mu \in \mathbb{R}$. For boundedness it suffices to show that the relative interior $\operatorname{ri}(\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu))$ is bounded, because $\overline{\operatorname{ri}(\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu))} = \overline{\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)} = \mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)$, see e.g. [43, Theorem 6.3]. The functions $\overline{\mathfrak{u}}_{\ln}$ and $\mathfrak{u}_{\ln}$ differ at most at the relative boundary of $\operatorname{dom}(\mathfrak{u}_{\ln})$, see Theorem A.1.1, and $\operatorname{dom}(\mathfrak{u}_{\ln}) \subset \Phi_{\text{twr}}$. Consequently, $\operatorname{ri}(\operatorname{dom}(\overline{\mathfrak{u}}_{\ln})) \subset \operatorname{dom}(\mathfrak{u}_{\ln}) \subset \Phi_{\text{twr}}$ and therefore $\operatorname{ri}(\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)) \subset \Phi_{\text{twr}} \cap \Phi_\beta$. Hence, it suffices to show that $\Phi_{\text{twr}} \cap \Phi_\beta$ is bounded, which follows from Lemma 2.2.29 after a coordinate transformation as follows: Define the bijective transformation $\Psi\colon \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ by $\Psi(\boldsymbol{\varphi}) := v_{\text{twr}}^{(1)}(\boldsymbol{\varphi})$ where $(v_{\text{twr}})_i^{(1)}(\boldsymbol{\varphi}) = \varphi_i \mathcal{W}^{(0)}/S_i^{(0)}$ for $i = 0, 1, \dots, M$, see (2.34), and $v := v_{\text{twr}} \circ \Psi^{-1}$ with corresponding admissible set $A := \Psi(\Phi_{\text{twr}} \cap \Phi_\beta)$. Then, $v^{(1)}(\mathbf{x}) = \mathbf{x}$ and $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}\beta =: \widetilde{\beta}\} =: A_{\widetilde{\beta}}$ hold with A non-empty, admissible and convex. Lemma 2.2.29 applied to v and the set $A \subset A_{\widetilde{\beta}}$ gives boundedness of A. Hence, $\Phi_{\text{twr}} \cap \Phi_\beta = \Psi^{-1}(A)$ is bounded as well. $\square$

For a detailed discussion on TWR see also Hermes [25], Hermes and Maier-Paape [26] and Maier-Paape and Zhu [35]. Of course, in general TWR is nonlinear in $\boldsymbol{\varphi}$. Therefore, one often works with a linearization, for which (2.50) and $\ln(1+x) \le x$ for $x > -1$ implies that

$$\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) = \sum_{n=1}^{N} \mathrm{E}\left[\ln\left(1 + T^{(n)}\boldsymbol{\varphi}\right)\right] \le \sum_{n=1}^{N} \mathrm{E}\left[T^{(n)}\boldsymbol{\varphi}\right] = \left(\sum_{n=1}^{N} \mathrm{E}\left[T^{(n)}\right]\right)\boldsymbol{\varphi} =: \mathfrak{u}_{\mathrm{lin}}(\boldsymbol{\varphi}) \quad (2.51)$$

for all $\boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}$, where $\mathfrak{u}_{\mathrm{lin}}$ is this linearization for values $T^{(n)}\boldsymbol{\varphi}$ near zero. This already reminds on the expectation of the wealth (2.40) obtained in Example 2.2.33.

Example 2.3.13 **($S^{(n-1)}\mathbf{x}^{(n)} \equiv$ const and linearization of $\mathfrak{u}_{\ln}$)** Assume it is the situation in Example 2.2.33 with corresponding x-portfolio generation function v_{cm} and with the additional assumption that $T^{(n)} \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1}_{>0})$ for $n = 1, \dots, N$. Then, for all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ it follows from (2.40), (2.42) and (2.51) that

$$\mathfrak{u}(\boldsymbol{\theta}) = \mathrm{E}\left[\mathcal{W}^{(N)}(v_{\mathrm{cm}}(\boldsymbol{\theta})) - \mathcal{W}^{(0)}\right] = \mathcal{W}^{(0)} \sum_{n=1}^{N} \mathrm{E}\left[T^{(n)}\right]\boldsymbol{\theta} = \mathcal{W}^{(0)}\, \mathfrak{u}_{\mathrm{lin}}(\boldsymbol{\theta}) < \infty.$$

Hence, $\mathfrak{u}_{\mathrm{lin}}$ corresponds to $\mathfrak{u}$ in the case "$S^{(n-1)}\mathbf{x}^{(n)} \equiv$ const" up to the factor $\mathcal{W}^{(0)}$. Of course, in this case $\mathfrak{u}$ is linear and therefore linear in the risk-free part ($\mathfrak{b}1$), positive homogeneous ($\mathfrak{b}2$), superadditive ($\mathfrak{b}4$a) and proper concave ($\mathfrak{b}4$b). This function is even closed proper concave, because $\mathrm{dom}(\mathfrak{u}_{\mathrm{lin}}) = \mathbb{R}^{M+1}$ such that $\mathfrak{u}$ is continuous on $\mathbb{R}^{M+1}$, see Theorems A.1.3 and A.1.2 in the appendix. ♦

From this last example it can be seen, that the linearization $\mathfrak{u}_{\mathrm{lin}}$ is strongly related to the situation in Example 2.2.33. Hence, the input variable $\boldsymbol{\varphi}$ for $\mathfrak{u}_{\mathrm{lin}}$ can be interpreted by the weights $\boldsymbol{\theta}$ from Example 2.2.33, cf. (2.39), which realize a portfolio such that $S^{(n-1)}\mathbf{x}^{(n)} \equiv$ const instead of $\boldsymbol{\varphi}^{(n)} \equiv \boldsymbol{\varphi}$.

Remark 2.3.14 Interpreting the situations from above, $\mathfrak{u}_{\ln}$ discusses the case when reinvesting gains where $\boldsymbol{\varphi}$ corresponds to the φ-portfolio with $\varphi_i = (S_i^{(n-1)} x_i^{(n)})/\mathcal{W}^{(N-1)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))$, see Example 2.2.31, while its linearization $\mathfrak{u}_{\mathrm{lin}}$ discusses the case when always investing the same amount without reinvesting gains where $\boldsymbol{\theta}$ corresponds to $\theta_i = (S_i^{(n-1)} x_i^{(n)})/\mathcal{W}^{(0)}$, see Example 2.2.33. ♦

Table 2.3 summarizes the above examples, see also Table 2.2 for the corresponding x-portfolio generating functions and the wealth.

Examples	x-portfolio generating function	utility ($\mathfrak{u}(\mathbf{x}) = \mathrm{E}\,[\mathcal{W}^{(N)}(v(\mathbf{x})) - \mathcal{W}^{(0)}]$)
2.2.30 and 2.3.8	$v_{\mathrm{bnh}}^{(n)}(\mathbf{x}) \equiv \mathbf{x}$	$\mathfrak{u}(\mathbf{x}) = \mathrm{E}\,[S^{(N)} - S^{(0)}]\mathbf{x}$
	constant number of shares (buy and hold strategy)	
2.2.31 and 2.3.10	$(v_{\mathrm{twr}})_i^{(n)}(\boldsymbol{\varphi}) = \frac{\varphi_i}{S_i^{(n-1)}}\mathcal{W}^{(0)} \prod_{k=1}^{n-1}(1 + T^{(k)}\boldsymbol{\varphi})$	$\mathfrak{u}(\boldsymbol{\varphi}) = \mathcal{W}^{(0)}(\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) - 1)$
		$\mathfrak{u}_{\mathrm{TWR}}(\boldsymbol{\varphi}) = \mathrm{E}\left[\prod_{n=1}^{N}(1 + T^{(n)}\boldsymbol{\varphi})\right]$
	constant weight (invest the same fraction of current wealth after each time step)	
2.2.33 and 2.3.13	$(v_{\mathrm{cm}})_i^{(n)}(\boldsymbol{\theta}) := \frac{\theta_i}{S_i^{(n-1)}}\mathcal{W}^{(0)}$	$\mathfrak{u}(\boldsymbol{\theta}) = \mathcal{W}^{(0)} \sum_{n=1}^{N} \mathrm{E}\,[T^{(n)}]\boldsymbol{\theta}$
	constant total amount of money (invest the same amount for each time step)	

Table 2.3.: Summary of examples for some utility functions, see also Table 2.2.

2.3.4 Risk measures

A risk measure in general is a convex function. Some common examples are given in this section. Of course, the drawdown also falls into this category. However, this quantity is discussed in detail in Chapter 3.

Example 2.3.15 (Variance) Markowitz [36] defined risk in 1952 by the variance using a one-period market model. In this case, the risk-free asset in general is excluded, because it has zero variance. The covariance matrix is then defined by

$$\Sigma := \left(\mathrm{E}\left[\left(S_i^{(1)} - \mathrm{E}\left[S_i^{(1)}\right]\right)\left(S_j^{(1)} - \mathrm{E}\left[S_j^{(1)}\right]\right)\right]\right)_{1\leq i,j\leq M} \in \mathbb{R}^{M\times M}.$$

The risk-free asset would lead to a row and column of Σ which are zero and thus Σ would only be positive semi-definite. Without risk-free asset, Σ is positive definite under some mild assumptions, see Maier-Paape and Zhu [34, Corollary 1]. Then the standard deviation of the portfolio reads

$$\mathfrak{r}_\sigma(\mathbf{x}) := \sqrt{\overline{\mathbf{x}}^\intercal \Sigma \overline{\mathbf{x}}}$$

for $\mathbf{x} \in A$, where $\overline{\mathbf{x}} := (x_1, \ldots, x_M)^\intercal$.

Of course, ($\mathfrak{b}$1) is obviously fulfilled with $c_\mathfrak{r} = 0$. In addition, $\mathfrak{r}_\sigma$ is positive homogeneous ($\mathfrak{b}$2) and subadditive ($\mathfrak{b}$3a), because Σ is always positive semi-definite, and hence $\mathfrak{r}_\sigma$ is convex ($\mathfrak{b}$3b), see also [34, Hint after Corollary 1]. ♦

Since the variance does not only tell something about the downside risk but also about the chances, the value at risk (VaR) was defined to only consider downside risk and "high" losses.

Example 2.3.16 (Value at risk (VaR)) Let $W \in \mathcal{L}^2$ denote a one-period model for portfolios wealth with cumulative probability function F_W. The value at risk for one time step and $\alpha \in (0,1)$ is then defined by

$$\mathrm{VaR}_\alpha(W) := -\inf\{z \,:\, F_W(z) > \alpha\} = -\inf\{z \,:\, \mathrm{P}(W \leq z) > \alpha\}.$$

Let a random variable $C \equiv c \in \mathbb{R}$ be constant. Then it is $\mathrm{VaR}_\alpha(C) = -c$ and therefore

$$\mathrm{VaR}_\alpha(W + C) = -\inf\{z + c \,:\, \mathrm{P}(W \leq z) > \alpha\} = \mathrm{VaR}_\alpha(W) + \mathrm{VaR}_\alpha(C),$$

i.e. ($\mathcal{B}$1a) and ($\mathcal{B}$1b) for $N = 1$ hold true. Positive homogeneity ($\mathcal{B}$2) is obviously fulfilled. However, it is well-known, that VaR_α is not subadditive ($\mathcal{B}$3a), see e.g. Artzner, Delbaen, Eber and Heath [2, Section 3.3].

Let now $S := (S^{(n)})_{0\leq n\leq N} \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ be a multi-period market model for $M, N \in \mathbb{N}$ according to Definition 2.1.4. Furthermore, let v be an x-portfolio generating function with $\mathcal{L}^2$-admissible domain, see Definition 2.2.26. The value at risk of the x-portfolio characterized by $\mathbf{x}$ for N time steps and $\alpha \in (0,1)$ can then be defined by

$$\mathfrak{r}_{\mathrm{VaR},\alpha}(\mathbf{x}) := \mathrm{VaR}_\alpha\left(\mathcal{W}^{(N)}(v(\mathbf{x})) - \mathcal{W}^{(0)}\right) = -\inf\left\{z \,:\, \mathrm{P}\left(\mathcal{W}^{(N)}(v(\mathbf{x})) - \mathcal{W}^{(0)} \leq z\right) > \alpha\right\}.$$

In the situation of Lemma 2.3.4 it would follow, that $\mathfrak{r}_{\mathrm{VaR},\alpha}$ fulfills ($\mathfrak{b}$1) and ($\mathfrak{b}$2) because VaR_α fulfills ($\mathcal{B}$1a) and ($\mathcal{B}$2). ♦

Value at risk does not have the important property of convexity. A strongly related measure which is convex is the conditional value at risk (CVaR).

Example 2.3.17 (Conditional value at risk) In the situation of Example 2.3.16, the conditional value at risk for one time step and $\alpha \in (0,1)$ is defined by

$$\mathrm{CVaR}_\alpha(W) := -\mathrm{E}[W \mid W \leq -\mathrm{VaR}_\alpha(W)],$$

for $W \in \mathcal{L}^2(N)$. This definition might produce discontinuities. Therefore, there are approaches to obtain continuous variants, see e.g. the introduction in Rockafellar, Uryasev and Zabarankin [47]. There are variants which satisfy ($\mathcal{B}$1a), ($\mathcal{B}$1b), ($\mathcal{B}$2), ($\mathcal{B}$3a) and ($\mathcal{B}$3b), see e.g. [47] and also Rockafellar and Uryasev [45] for the details. From this it is again possible to construct a corresponding x-portfolio variant $\mathfrak{r}_{\mathrm{CVaR},\alpha}$ which, for instance in the situation of Lemma 2.3.4, is linear in the risk-free part ($\mathfrak{b}$1), positive homogeneous ($\mathfrak{b}$2) and convex ($\mathfrak{b}$3b). ♦

The evaluation of VaR and CVaR is not trivial. There are different widely common methods: One consists of inverting the cumulative distribution function, which often is assumed to be the normal distribution, see e.g. Deutsch [12]. Another one is to simulate the random variable via Monte Carlo simulation. The evaluation of VaR and CVaR can also be formulated as the solution of a minimization problem, which was first stated by Rockafellar and Uryasev [44]. Thereby the distribution of the returns can be unknown.

2.4 Optimization problem

Having a market model, an x-portfolio generating function, an utility measure and a risk measure, the goal is to compute an "optimal" x-portfolio. Often, optimality is interpreted in the sense of either minimizing risk such that the utility function stays above a given threshold, or maximizing the utility function subject to a maximum allowed risk level.

This task has been studied intensively in the past: Two well-known methods go back to portfolio selection from Markowitz [36, 37] and its extension, the capital asset pricing model (CAPM) by Lintner, Mossin and Sharpe [31, 38, 52] (respectively). In both cases, a linear utility measure as in Example 2.3.8 (a) or (2.46) and the standard deviation as risk are used. In the situation of Markowitz [36, 37] it is necessary, that all money is invested into the risky assets, i.e. there is no risk-free part. All efficient portfolios are uniquely determined by generalized convex combinations of two efficient portfolios, which is often called two-fund theorem. The extension in CAPM is the implementation of a risk-free asset, which leads to a linear efficiency curve which is tangential to the efficiency curve when excluding the risk-free part. The intersection point is called master fund or market portfolio, which already uniquely defines this linear efficiency curve. This is often called the one-fund theorem.

Similar results have been proved by Rockafellar, Uryasev, and Zabarankin [48] when replacing the standard deviation by a general deviation measure. The utility measure stays the same linear function. They prove existence of efficient portfolios and their homogeneity, similar to the case of Markowitz and CAPM, under reasonable assumptions, and give a generalization of the one-fund theorem. In [49], the same authors give optimality criteria for the solution of the portfolio optimization using general deviation measures.

Maier-Paape and Zhu [34] extend the theory of Markowitz and CAPM for even more general risk functions with the general deviation measure being a special case, and more general utility functions of the form $\mathbf{x} \mapsto \mathrm{E}[u(S^{(1)}\mathbf{x})]$ for a one-period market model and some concave function $u\colon \mathbb{R} \to \mathbb{R}$. They give an existence and uniqueness result for efficient portfolios. In case u is the identity, they prove a generalization of the one-fund theorem under some reasonable assumptions.

In the subsequent subsections an extension of the results from [34, Sections 3 and 5] to the multi-period case with risk and more general utility measure in x-portfolio form as in Definition 2.3.2 are provided under appropriate assumptions. The general setting and problem is stated in Subsection 2.4.1. The efficient frontier and its properties are discussed in Subsection 2.4.2. Using these results, the existence and uniqueness for efficient x-portfolios is proven in Subsection 2.4.3. Then a generalization of the one-fund theorem follows in Subsection 2.4.4.

2.4.1 Setting

The general settings of the three important aspects of Sections 2.1 (market model), Section 2.2 (x-portfolio / trading strategy), and Section 2.3 (utility and risk measure) are collected into the following.

Setting 2.4.1 Let the following be given for $M, N \in \mathbb{N}$:

(i) The *multi-period market model* is given by $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$, see Definition 2.1.4.

(ii) The *x-portfolio* is defined by an x-portfolio generating function v as in Definition 2.2.26, where the domain $A \subset \mathbb{R}^{M+1}$ is assumed to be non-empty and convex.

(iii) The *utility function* is represented by the proper concave function $\mathfrak{u}\colon A \to \mathbb{R}\cup\{-\infty\}$.

(iv) The *risk function* is represented by the proper convex function $\mathfrak{r}\colon A \to \mathbb{R}\cup\{\infty\}$.

Furthermore, assume that $\operatorname{dom}(\mathfrak{u}) \cap \operatorname{dom}(\mathfrak{r}) \neq \emptyset$.

From this, the optimization problem can be formulated.

Problem 2.4.2 Let Setting 2.4.1 be given, where it is assumed, that $v^{(1)}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in A$. Define two optimization problems:

(a) For $\beta > 0$ and $\mu \in \mathbb{R}$, the *minimum risk optimization problem* reads

$$\min_{\mathbf{x}\in A} \mathfrak{r}(\mathbf{x}) \quad \text{subject to } \mathfrak{u}(\mathbf{x}) \geq \mu,\ S^{(0)}\mathbf{x} = \beta. \tag{2.52}$$

(b) For $\beta > 0$ and $r \in \mathbb{R}$, the *maximum chance optimization problem* reads

$$\max_{\mathbf{x}\in A} \mathfrak{u}(\mathbf{x}) \quad \text{subject to } \mathfrak{r}(\mathbf{x}) \leq r,\ S^{(0)}\mathbf{x} = \beta. \tag{2.53}$$

Next to condition $S^{(0)}\mathbf{x} = \beta$, the market model and the x-portfolio generating function are not explicitly involved in these optimization problems. Nevertheless, they define the domain A, which may guarantee admissible portfolios independent of the risk and utility measure, and provide the framework for the realization of a real world portfolio for given $\mathbf{x} \in A$. In addition, it is also possible to either hide or eliminate the condition $S^{(0)}\mathbf{x} = \beta$. For example one could define $A_\beta := \{\mathbf{x} \in A : S^{(0)}\mathbf{x} = \beta\}$ and write

$$\min_{\mathbf{x}\in A_\beta} \mathfrak{r}(\mathbf{x}) \quad \text{subject to } \mathfrak{u}(\mathbf{x}) \geq \mu$$

instead of (2.52). Alternatively, one could substitute $x_0 = (\beta - S^{(0)}\widehat{\mathbf{x}})/S^{(0)}_0$.

Note, that the special form of the constraint $S^{(0)}\mathbf{x} = \beta$ already gives $\mathbf{x}$ the meaning of being the initial value of the x-portfolio, i.e. $v^{(1)}(\mathbf{x}) = \mathbf{x}$. If v depends on say $\boldsymbol{\varphi}$ as in Example 2.2.31, then $S^{(0)}\mathbf{x} = \beta$ can be replaced for instance by the equivalent linear equality constraint $\sum_{i=0}^{M} \varphi_i = \beta/\mathcal{W}^{(0)}$.

Lemma 2.4.3 (Problem 2.4.2 after coordinate transformation) Let Setting 2.4.1 and Problem 2.4.2 be given. Define the linear and bijective transformation $\Psi\colon \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ by $\Psi(\mathbf{y}) := D\mathbf{y}$, where $D \in \mathbb{R}^{(M+1)\times(M+1)}$ has full rank (e.g. a diagonal matrix with positive entries on the diagonal). Then, $v_\Psi := v \circ \Psi$ defines an x-portfolio generating function, with non-empty and convex domain $\Psi^{-1}(A) \subset \mathbb{R}^{M+1}$, and the following holds:

(a) $\mathbf{x}^*$ solves (2.52) if and only if $\mathbf{y}^* = \Psi^{-1}(\mathbf{x}^*)$ solves

$$\min_{\mathbf{y}\in\Psi^{-1}(A)} \mathfrak{r}(\Psi(\mathbf{y})) \quad \text{subject to } \mathfrak{u}(\Psi(\mathbf{y})) \geq \mu,\ S^{(0)}(D\mathbf{y}) = \beta. \tag{2.54}$$

(b) $\mathbf{x}^*$ solves (2.53) if and only if $\mathbf{y}^* = \Psi^{-1}(\mathbf{x}^*)$ solves

$$\max_{\mathbf{y}\in\Psi^{-1}(A)} \mathfrak{u}(\Psi(\mathbf{y})) \quad \text{subject to } \mathfrak{r}(\Psi(\mathbf{y})) \leq r,\ S^{(0)}(D\mathbf{y}) = \beta. \tag{2.55}$$

In both cases, the corresponding x-portfolios are the same for both solutions, i.e. $v(\mathbf{x}^*) = v_\Psi(\mathbf{y}^*)$.

Proof. The claim follows directly by inserting the bijective transformation Ψ in (2.52) and (2.53), see also the proof of Theorem 2.3.12 (d) which uses a specific transformation Ψ. □

Example 2.4.4 (Problem 2.4.2 in φ-portfolio form) In the sitation of Lemma 2.4.3 define for $\boldsymbol{\varphi} := \mathbf{y}$ the x-portfolio generation function $v_\Psi := v \circ \Psi$ with $(\Psi(\boldsymbol{\varphi}))_i := \varphi_i \mathcal{W}^{(0)}/S_i^{(0)}$ for $i = 0, 1, \ldots, M$. In this case, matrix D is a diagonal matrix with diagonal entries $D_{ii} = \mathcal{W}^{(0)}/S_i^{(0)}$. Then, the linear constraint in (2.54) and also in (2.55) becomes

$$\sum_{i=0}^{M} \varphi_i = \frac{\beta}{\mathcal{W}^{(0)}},$$

which directly follows when inserting this special diagonal matrix D. Note, that Ψ exactly corresponds to $\Psi(\boldsymbol{\varphi}) = v_{\text{twr}}^{(1)}(\boldsymbol{\varphi})$ from Example 2.2.31 for all $\boldsymbol{\varphi} \in \Phi_{\text{twr}}$ with Φ_{twr} from (2.32). ♦

However, albeit the optimization problems in Problem 2.4.2 and in Lemma 2.4.3 are equivalent (in the sense of this lemma), the remainder of this chapter focuses on the formulation in Setting 2.4.1 and Problem 2.4.2.

2.4.2 Efficient frontier

This subsection extends some results from Maier-Paape and Zhu [34] for the multi-period case, more general utility measures and risk measures, which are allowed to be negative as well. The efficiency curve in the one-fund and two-fund theorem typically denotes the curve obtained from efficient portfolios in the risk-expected utility space, in the following called risk-utility space. This efficient frontier contains all relevant x-portfolios and potential solutions of Problem 2.4.2. Therefore, some general properties and characterizations of this efficiency curve are given in this subsection. Nevertheless, at first it is necessary to define the risk-utility space itself, which is done similar to [34, Section 3.2].

Definition 2.4.5 (Risk-utility space) For Setting 2.4.1 define the sublevel and superlevel sets of $\mathfrak{r}$ and $\mathfrak{u}$ for thresholds $r, \mu \in \mathbb{R}$ by

$$\mathcal{B}_{\mathfrak{r},A}(r) := \{\mathbf{x} \in A \,:\, \mathfrak{r}(\mathbf{x}) \le r\} \subset \mathrm{dom}(\mathfrak{r}) \quad \text{and} \quad \mathcal{B}_{\mathfrak{u},A}(\mu) := \{\mathbf{x} \in A \,:\, \mathfrak{u}(\mathbf{x}) \ge \mu\} \subset \mathrm{dom}(\mathfrak{u}),$$

respectively, and its intersection

$$\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) := \{\mathbf{x} \in A \,:\, \mathfrak{r}(\mathbf{x}) \le r \text{ and } \mathfrak{u}(\mathbf{x}) \ge \mu\} = \mathcal{B}_{\mathfrak{r},A}(r) \cap \mathcal{B}_{\mathfrak{u},A}(\mu) \subset A.$$

Furthermore, define the set of valid risk and utility levels

$$\mathcal{G}(\mathfrak{r},\mathfrak{u};A) := \{(r,\mu) \,:\, \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) \ne \emptyset\} \subset \mathbb{R}^2 \tag{2.56}$$

in the risk-utility space.

According to the optimization problems in Problem 2.4.2, sets $\mathcal{B}_{\mathfrak{u},A}(\mu)$ and $\mathcal{B}_{\mathfrak{r},A}(r)$ contain all x-portfolios, which have utility of size μ or better (higher) and risk of size r or better (lower), respectively. Hence, its intersection $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ contains all x-portfolios with risk/utility value (r,μ) or more "efficient" x-portfolios. Essential properties for these sets are given in the next result, see also [34, Proposition 7].

Proposition 2.4.6 (Properties in risk-utility space) In the situation of Setting 2.4.1 the following holds true:

(a) $\mathfrak{u}$ is closed proper concave if and only if $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is closed for all $\mu \in \mathbb{R}$, and $\mathfrak{r}$ is closed proper convex if and only if $\mathcal{B}_{\mathfrak{r},A}(r)$ is closed for all $r \in \mathbb{R}$.

(b) If $\mathcal{B}_{\mathfrak{u},A}(\mu)$ and $\mathcal{B}_{\mathfrak{r},A}(r)$ are closed for all $\mu, r \in \mathbb{R}$ and either $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact for all $\mu \in \mathbb{R}$ or $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r \in \mathbb{R}$, then $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is convex and compact for all $r, \mu \in \mathbb{R}$.

(c) $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is convex and $(r,\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ implies, that $(r+k,\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ and $(r,\mu-k) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ for all $k > 0$.

(d) If $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$, then $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed.

Proof. Proof of (a): This is a direct consequence from Theorem A.1.2 in the appendix.

Proof of (b): Let $\mu, r \in \mathbb{R}$ be arbitrary. Since $\mathcal{B}_{\mathfrak{r},A}(r)$ and $\mathcal{B}_{\mathfrak{u},A}(\mu)$ are closed and as sublevel and superlevel sets of a proper convex and proper concave function, respectively, both sets are convex. Therefore, $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) = \mathcal{B}_{\mathfrak{r},A}(r) \cap \mathcal{B}_{\mathfrak{u},A}(\mu)$ is convex and closed as well. Since either $\mathcal{B}_{\mathfrak{u},A}(\mu)$ or $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact, $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is bounded and consequently compact.

Proof of (c): The proof of [34, Proposition 7] can easily be adapted for this case: The second property that $(r,\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ implies, that $(r+k,\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ and $(r,\mu-k) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ for all $k > 0$, follows from the definition of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. To show convexity, let $(r_1,\mu_1),(r_2,\mu_2) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ and $s \in [0,1]$. Then for $i = 1,2$, there exists $\mathbf{x}^1, \mathbf{x}^2 \in \mathrm{dom}(\mathfrak{r}) \cap \mathrm{dom}(\mathfrak{u})$ such that $\mathfrak{r}(\mathbf{x}^i) \le r_i$ and $\mathfrak{u}(\mathbf{x}^i) \ge \mu_i$. Convexity of $\mathfrak{r}$ and concavity of $\mathfrak{u}$ gives

$$\begin{aligned}
\mathfrak{r}(s\mathbf{x}^1 + (1-s)\mathbf{x}^2) &\le s\,\mathfrak{r}(\mathbf{x}^1) + (1-s)\,\mathfrak{r}(\mathbf{x}^2) \le sr_1 + (1-s)r_2, \\
\mathfrak{u}(s\mathbf{x}^1 + (1-s)\mathbf{x}^2) &\ge s\,\mathfrak{u}(\mathbf{x}^1) + (1-s)\,\mathfrak{u}(\mathbf{x}^2) \ge s\mu_1 + (1-s)\mu_2
\end{aligned}$$

respectively. Since $\mathrm{dom}(\mathfrak{r}) \cap \mathrm{dom}(\mathfrak{u})$ is convex, it then must be $s(r_1,\mu_1) + (1-s)(r_2,\mu_2) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$.

Proof of (d): Let $((r_n, \mu_n))_{n\in\mathbb{N}} \subset \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ be a sequence such that $(r_n, \mu_n) \to (r, \mu) \in \mathbb{R}^2$ as $n \to \infty$. Then, for all $n \in \mathbb{N}$ there exists $\mathbf{x}^n \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_n, \mu_n) \subset \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r})$ with $\mathfrak{r}(\mathbf{x}^n) \leq r_n$ and $\mathfrak{u}(\mathbf{x}^n) \geq \mu_n$. Furthermore, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $r_n < r + \varepsilon$ and $\mu_n > \mu - \varepsilon$ and consequently $\mathbf{x}^n \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + \varepsilon, \mu - \varepsilon)$ for all $n \geq n_0$. Since $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + \varepsilon, \mu - \varepsilon)$ is compact, there is a subsequence, w.l.o.g. the original sequence, with $\mathbf{x}^n \to \mathbf{x} \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r + \varepsilon, \mu - \varepsilon)$ as $n \to \infty$ for all $\varepsilon > 0$. Therefore, it must be $\mathbf{x} \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ which implies that $(r, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. □

As already mentioned, set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ contains all x-portfolios of given quality (r, μ) or better. In some cases it is possible to uniquely decide that one x-portfolio is better than another one in the sense of Problem 2.4.2, e.g. if it has lower risk and higher utility value. This leads to the notion of efficiency as in [34, Definition 5].

Definition 2.4.7 (Efficient portfolio and efficient frontier) In the situation of Setting 2.4.1, an x-portfolio $\mathbf{x} \in A$ is called *efficient*, if there is no $\mathbf{x}' \in A$ such that

$$\big[\mathfrak{r}(\mathbf{x}') \leq \mathfrak{r}(\mathbf{x}) \quad \text{and} \quad \mathfrak{u}(\mathbf{x}') > \mathfrak{u}(\mathbf{x})\big] \quad \text{or} \quad \big[\mathfrak{r}(\mathbf{x}') < \mathfrak{r}(\mathbf{x}) \quad \text{and} \quad \mathfrak{u}(\mathbf{x}') \geq \mathfrak{u}(\mathbf{x})\big].$$

The set

$$\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A) := \big\{(\mathfrak{r}(\mathbf{x}), \mathfrak{u}(\mathbf{x})) \in \mathbb{R}^2 \,:\, \mathbf{x} \in A \text{ is efficient}\big\} \subset \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$$

is called *efficient frontier*.

From definition of the sublevel and superlevel sets, an x-portfolio $\mathbf{x} \in A$ is efficient, if and only if

$$\big(\mathfrak{r}(\mathbf{x}'), \mathfrak{u}(\mathbf{x}')\big) = (\mathfrak{r}(\mathbf{x}), \mathfrak{u}(\mathbf{x})) \quad \text{for all } \mathbf{x}' \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}), \mathfrak{u}(\mathbf{x})). \tag{2.57}$$

Thus, if $\mathbf{x}$ is efficient, then all elements in $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}), \mathfrak{u}(\mathbf{x}))$ are efficient as well. Hence, roughly speaking, $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ contains the "best" possible risk and utility values which must lie in the "upper left boundary" of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ as is shown in the next result, see also Maier-Paape and Zhu [34, Theorem 3]. Note, that the set A is assumed to be closed in [34, Theorem 3], which is not necessary in the following result.

Theorem 2.4.8 (Properties of efficient frontier) Let Setting 2.4.1 be given.

(a) $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is located on the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ and has no vertical and no horizontal line segments.

(b) If $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$, then $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is non-empty and equals to the non-vertical and non-horizontal part of the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, i.e.

$$\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A) = \{(r, \mu) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \,:\, (r - k, \mu), (r, \mu + k) \notin \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \ \forall k > 0\}, \tag{2.58}$$

where $\partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ denotes the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ in $\mathbb{R}^2$.

(c) If $B \subset A$ is closed and convex, then $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A) \cap \mathcal{G}(\mathfrak{r}, \mathfrak{u}; B) \subset \mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; B)$.

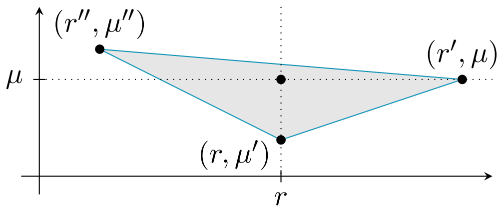
Proof. Proof of (a): The proof in [34, Theorems 3] also works in this case and reads: If $(r, \mu) := (\mathfrak{r}(\mathbf{x}), \mathfrak{u}(\mathbf{x})) \in \mathbb{R}^2$ for some $\mathbf{x} \in \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r}) \subset A$ is not on the boundary or is on the relative interior of the vertical or horizontal part of the boundary for $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, then for small $\varepsilon > 0$ either $(r-\varepsilon,\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ or $(r,\mu+\varepsilon) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. In this case $\mathbf{x}$ is not efficient.

Proof of (b): For the representation of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, the relation "$\subset$" holds true because of (a). The opposite direction "$\supset$" is satisfied because of the following: Because of Proposition 2.4.6 (d), $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed, i.e. $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \subset \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. Let $(r,\mu) \in \partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ be arbitrary but with $(r-k,\mu),(r,\mu+k) \notin \partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ for all $k > 0$. Because of Proposition 2.4.6 (c), $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is convex, unbounded to the right and not bounded from below. Hence, there is an element $(r',\mu) \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ with $r' > r$ and there is an element $(r,\mu') \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ with $\mu' < \mu$. Consequently, there cannot be an element $(r'',\mu'') \in \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ such that $r'' < r$ and $\mu'' > \mu$, because otherwise, (r,μ) would be in the interior of the convex hull of (r'',μ''), (r',μ) and (r,μ') and therefore would be in the interior of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ which is a contradiction, see Figure 2.1a for an illustration. Since (r,μ) is in the non-vertical and non-horizontal part of the boundary, it cannot be improved and therefore it must be $(r,\mu) \in \mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$.

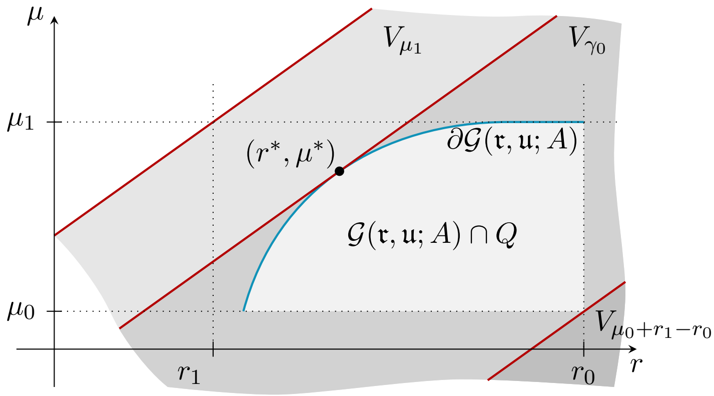
Furthermore, $\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A) \neq \emptyset$ because of the following: Since $\text{dom}(\mathfrak{u}) \cap \text{dom}(\mathfrak{r}) \subset A$ is non-empty by assumption in Setting 2.4.1, there exists $r_0,\mu_0 \in \mathbb{R}$ such that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_0,\mu_0) \neq \emptyset$. The closure $\overline{\mathfrak{r}}$ of $\mathfrak{r}$, see (A.2) in the appendix, is closed proper convex and the closure $\overline{\mathfrak{u}}$ of $\mathfrak{u}$ is closed proper concave. Then, $\overline{\mathfrak{r}}$ attains its global minimum r_1 and $\overline{\mathfrak{u}}$ its global maximum μ_1 in $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_0,\mu_0)$, see Theorem A.1.2 and Theorem A.1.6 in the appendix. Since $\overline{\mathfrak{r}} \le \mathfrak{r}$ and $\overline{\mathfrak{u}} \ge \mathfrak{u}$ (see the note after (A.2) in the appendix), functions $\mathfrak{r}$ and $\mathfrak{u}$ restricted to $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_0,\mu_0) \subset A$ then must be bounded and the image of $(\mathfrak{r},\mathfrak{u})$ restricted to $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_0,\mu_0)$ is in $[r_1,r_0] \times [\mu_0,\mu_1]$. Consequently, $\emptyset \neq \mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q \subset [r_1,r_0] \times [\mu_0,\mu_1]$ for $Q := \{(r',\mu') : r' \le r_0,\ \mu' \ge \mu_0\}$ must hold and therefore is bounded. Since $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is convex and closed, see Proposition 2.4.6 (c) and (d), so is $\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q$. It remains to show that there is at least one efficient x-portfolio, which, due to (2.58), has a risk-utility value located in the "upper left boundary" of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q$, which, of course, must be part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. For this, define the closed half space $V_\gamma := \{(r_1+a,\gamma+b) \in \mathbb{R}^2 : b \le a\}$. Then, $V_{\mu_1} \supset [r_1,r_0] \times [\mu_0,\mu_1] \supset \mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q \neq \emptyset$ holds true and there must exist a minimal $\gamma_0 \in [\mu_0+r_1-r_0,\mu_1]$ such that $\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q \subset V_{\gamma_0}$, cf. Figure 2.1b. Since $\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q$ and V_{γ_0} are closed, there exists at least one intersection point $(r^*,\mu^*) \in \partial(\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q) \cap \partial V_{\gamma_0}$. By construction $(r^*,\mu^*) \in \partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap Q \subset \partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ holds which cannot be in the relative interior of a vertical or horizontal part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ and therefore, again by (2.58), $(r^*,\mu^*) \in \mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)$.

Proof of (c): The proof in [34, Theorem 3] works in this case as well and reads: Since $\mathcal{G}(\mathfrak{r},\mathfrak{u};B) \subset \mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, each efficient x-portfolio in A must therefore also be efficient in B. □

Remark 2.4.9 (cf. Maier-Paape and Zhu [34, Remark 5]) Assume $\mathfrak{r}$ is decreasing in the first component (investing more into the risk-free asset does not worsen risk) and $\mathfrak{u}$ is strictly increasing in the first component (investing more into the risk-free asset increases utility). Let $\mathbf{x} \in A$ and assume $\mathbf{x}_\alpha := \mathbf{x} + (\alpha,0,\dots,0)^\intercal \in A$ for all $\alpha \ge 0$. Then, the initial costs of the portfolio $\mathbf{x}_\alpha$ given by $S^{(0)}\mathbf{x}_\alpha$ have no upper bound, because $S^{(0)}\mathbf{x}_\alpha$ is strictly increasing in α with no upper bound. On the other hand, it is $\mathfrak{r}(\mathbf{x}_\alpha) \le \mathfrak{r}(\mathbf{x})$ and $\mathfrak{u}(\mathbf{x}_\alpha) > \mathfrak{u}(\mathbf{x})$ for each $\alpha > 0$, i.e. the x-portfolio $\mathbf{x}$ can be improved by $\mathbf{x}_\alpha$. Hence, there exists no "optimal" value for α, because $\mathbf{x}_\alpha \in A$ for all $\alpha \in \mathbb{R}$. If this is possible for each $\mathbf{x} \in A$, then there is no efficient frontier, because there is no efficient x-portfolio. Note,



(a) Illustration to show $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is convex.



(b) Illustration to show $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \neq \emptyset$.

Figure 2.1.: Illustration for proof of Theorem 2.4.8.

that in this case the set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is not bounded for some $r, \mu \in \mathbb{R}$ and therefore does not contradict Theorem 2.4.8 (b).

However, it is not meaningful to allow arbitrary large initial costs. It is reasonable to reduce the size of A by bounding the initial costs. Then, there might be efficient x-portfolios, but each x-portfolio in the upper case would maximize α, which ends up in the situation of $S^{(0)}\mathbf{x}_\alpha$ being the maximum allowed value. Finally, one also could require $S^{(0)}\mathbf{x}$ to be a fixed value, e.g. $S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}$, which justifies the linear equality constraint in Problem 2.4.2. Since this constraint can also be forced by defining $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}\}$, the linear constraint will be omitted in the following. ⧫

The focus lies on efficient x-portfolios. Hence, the reasonable infimum and supremum of the utility and the risk function for these x-portfolios, i.e. the componentwise infimum and supremum of $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, are of interest. For this reason define

$$r_{\min} := \inf_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)} \{r\}, \qquad \mu_{\min} := \inf_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)} \{\mu\}, \tag{2.59a}$$

$$r_{\max} := \sup_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)} \{r\}, \qquad \mu_{\max} := \sup_{(r,\mu)\in\mathcal{G}_{\text{eff}}(\mathfrak{r},\mathfrak{u};A)} \{\mu\}. \tag{2.59b}$$

In terms of x-portfolios, these values can be written as follows.

Lemma 2.4.10 (Infima/Suprema of $\mathcal{G}_{\mathbf{eff}}(\mathfrak{r},\mathfrak{u};A)$) Let Setting 2.4.1 be given and assume $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu\in\mathbb{R}$. Then

$$r_{\min}=\inf_{\mathbf{x}\in\mathrm{dom}(\mathfrak{u})\cap\mathrm{dom}(\mathfrak{r})}\{\mathfrak{r}(\mathbf{x})\}<\infty,\qquad \mu_{\max}=\sup_{\mathbf{x}\in\mathrm{dom}(\mathfrak{u})\cap\mathrm{dom}(\mathfrak{r})}\{\mathfrak{u}(\mathbf{x})\}>-\infty,$$

and

$$\mu_{\min}=\begin{cases}\max\limits_{\mathbf{x}\in\mathcal{B}_{\mathfrak{r},A}(r_{\min})}\{\mathfrak{u}(\mathbf{x})\}, & \text{if } r_{\min}>-\infty \text{ and } \mathcal{B}_{\mathfrak{r},A}(r_{\min})\cap\mathrm{dom}(\mathfrak{u})\neq\emptyset,\\ -\infty, & \text{otherwise,}\end{cases}$$

$$r_{\max}=\begin{cases}\min\limits_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu_{\max})}\{\mathfrak{r}(\mathbf{x})\}, & \text{if } \mu_{\max}<\infty \text{ and } \mathcal{B}_{\mathfrak{u},A}(\mu_{\max})\cap\mathrm{dom}(\mathfrak{r})\neq\emptyset,\\ \infty, & \text{otherwise.}\end{cases}$$

If $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r\in\mathbb{R}$, then $r_{\min}>-\infty$ and $\mathcal{B}_{\mathfrak{r},A}(r_{\min})\neq\emptyset$. If $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact for all $\mu\in\mathbb{R}$, then $\mu_{\max}<\infty$ and $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max})\neq\emptyset$.

Proof. Let $B:=\mathrm{dom}(\mathfrak{u})\cap\mathrm{dom}(\mathfrak{r})$ which is non-empty due to Setting 2.4.1. Because of Theorem 2.4.8 (b), the set $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)$ is non-empty and equals the non-vertical and non-horizontal part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$.

Proof of $\mu_{\max}$: Note, that the vertical part of $\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$, if it exists, is located below $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)$, and the horizontal part does not change the supremum. In addition $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed and convex, see Proposition 2.4.6 (c) and (d). Therefore, it must be

$$\mu_{\max}=\sup_{(r,\mu)\in\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}=\sup_{(r,\mu)\in\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}=\sup_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{\mu\}=\sup_{\mathbf{x}\in B}\{\mathfrak{u}(\mathbf{x})\}>-\infty.$$

Proof of $r_{\min}$: The assertion for $r_{\min}$ can be shown using an analog argumentation such that

$$r_{\min}=\inf_{(r,\mu)\in\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\inf_{(r,\mu)\in\partial\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)}\{r\}=\inf_{\mathbf{x}\in B}\{\mathfrak{r}(\mathbf{x})\}<\infty.$$

Proof of $\mu_{\min}$: If $r_{\min}>-\infty$ and $\mathcal{B}_{\mathfrak{r},A}(r_{\min})\cap\mathrm{dom}(\mathfrak{u})\neq\emptyset$, there is an x-portfolio $\mathbf{x}\in\mathcal{B}_{\mathfrak{r},A}(r_{\min})$ with $\mathfrak{u}(\mathbf{x})\in\mathbb{R}$. Obviously, $(\mathfrak{r}(\mathbf{x}),\mathfrak{u}(\mathbf{x}))$ lies in $\{(r_{\min},\mu):\mu\leq\mathfrak{u}(\mathbf{x})\}$ which is contained in the vertical part of the boundary of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$. Since $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)\neq\emptyset$, the maximum of the μ component of this vertical part exists and must simultaneously be the minimum of the μ component of $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)$. Since neither the risk value nor the utility value of this maximum of the vertical part can be improved and $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed, it must belong to an efficient x-portfolio in A and it follows that

$$\mu_{\min}=\max_{\mathbf{x}\in\mathcal{B}_{\mathfrak{r},A}(r_{\min})}\{\mathfrak{u}(\mathbf{x})\}<\infty.$$

If $r_{\min}=-\infty$ or $\mathcal{B}_{\mathfrak{r},A}(r_{\min})\cap\mathrm{dom}(\mathfrak{u})=\emptyset$, then the infimum $r_{\min}$ of the risk $\mathfrak{r}$ is not attained within the risk-utility space and there exists a sequence $((r_n,\mu_n))_{n\in\mathbb{N}}\subset\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ with $r_n\to r_{\min}$ for $n\to\infty$. For each $n\in\mathbb{N}$, the set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_n,\mu_n)$ is non-empty, see (2.56), and compact. Then, there exists an efficient x-portfolio $\mathbf{x}^n$ in $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r_n,\mu_n)$, see the part of the proof of Theorem 2.4.8 (b) where $\mathcal{G}_{\mathrm{eff}}(\mathfrak{r},\mathfrak{u};A)\neq\emptyset$ was proved. Since $r_{\min}\leq\mathfrak{r}(\mathbf{x}^n)\leq r_n$ and $r_n\to r_{\min}$ as $n\to\infty$, it follows that $\mathfrak{r}(\mathbf{x}^n)\to r_{\min}$ as $n\to\infty$. Hence, w.l.o.g. it can

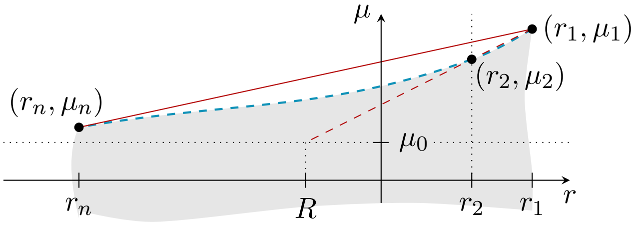


Figure 2.2.: Illustration for proof of Lemma 2.4.10.

be assumed that $(r_n, \mu_n) = (\mathfrak{r}(\mathbf{x}^n), \mathfrak{u}(\mathbf{x}^n))$ already is the corresponding value. Then it is $((r_n, \mu_n))_{n\in\mathbb{N}} \subset \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \subset \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ with $r_n \to r_{\min}$ for $n \to \infty$.

In addition, the sequence $(r_n)_{n\in\mathbb{N}}$ is, w.l.o.g., strictly decreasing, i.e. $r_n > r_{n+1}$ for all $n \in \mathbb{N}$. This implies, that $(\mu_n)_{n\in\mathbb{N}}$ is strictly decreasing as well, because otherwise there would be efficient portfolios $\mathbf{x}^{n_1}$ and $\mathbf{x}^{n_2}$ for $n_1, n_2 \in \mathbb{N}$, $n_1 < n_2$, such that $\mathfrak{r}(\mathbf{x}^{n_2}) < \mathfrak{r}(\mathbf{x}^{n_1})$ and $\mathfrak{u}(\mathbf{x}^{n_2}) \geq \mathfrak{u}(\mathbf{x}^{n_1})$ which is a contradiction, because then $\mathbf{x}^{n_1}$ would not be efficient. Hence, either $(\mu_n)_{n\in\mathbb{N}}$ converges from above to a finite value $\mu_0 \in \mathbb{R}$ with $\mu_0 \geq \mu_{\min}$ or this sequence diverges to $-\infty$. If the first case can be excluded then $\mu_{\min} = -\infty$ follows as claimed.

Thus, assume $(r_n, \mu_n) \to (r_{\min}, \mu_0)$ for $n \to \infty$ with $\mu_0 \in \mathbb{R}$ and $\mu_0 \geq \mu_{\min}$. In case $r_{\min} > -\infty$, since $r_{\min}$ is never attained, it is $(r_{\min}, \mu_0) \notin \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, which is a contradiction because $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed. Assume now $r_{\min} = -\infty$. Then for all $R \in \mathbb{R}$ there exists an $n_0 = n_0(R) \in \mathbb{N}$ such that $r_n < R$ for all $n \geq n_0$. Since $(r_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ both are strictly decreasing, it follows that $r_1 > r_2 > r_n$ and $\mu_1 > \mu_2 > \mu_n > \mu_0$ for all $n \geq 3$. Therefore, the line connecting (r_1, μ_1) and (r_2, μ_2) crosses the set $\{(r, \mu_0) : r \in \mathbb{R}\}$ at a finite point, say at $(R, \mu_0) \in \mathbb{R}^2$, see Figure 2.2. Then, for all $n > n_0(R)$ the inequality $r_n < R$ holds and consequently the line segment connecting $(r_n, \mu_n) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ and $(r_1, \mu_1) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ lies above (r_2, μ_2), see again Figure 2.2. This is a contradiction to $(r_2, \mu_2) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ because $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is convex and therefore all points lying below the line segment from (r_n, μ_n) to (r_1, μ_1) have to be in the interior of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, see Proposition 2.4.6 (c). Consequently it must be $\mu_{\min} = -\infty$.

The proof of $r_{\max}$ works analogously in a mirrored situation. If, in addition, $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r \in \mathbb{R}$, then $\mathfrak{r}$ is lower semi-continuous, see Theorem A.1.2 in the appendix. For arbitrary $\mathbf{x} \in \text{dom}(\mathfrak{r}) \neq \emptyset$ and $r := \mathfrak{r}(\mathbf{x})$ it is $\mathcal{B}_{\mathfrak{r},A}(r) \neq \emptyset$. Therefore, the minimum of $\mathfrak{r}$ is attained in $\mathcal{B}_{\mathfrak{r},A}(r)$, see Theorem A.1.6 in the appendix. Hence, it must be $r_{\min} > -\infty$ and $\mathcal{B}_{\mathfrak{r},A}(r_{\min}) \neq \emptyset$. If $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact for all $\mu \in \mathbb{R}$, one can argue analogously to obtain that $\mu_{\max} < \infty$ and $\mathcal{B}_{\mathfrak{u},A}(\mu_{\max}) \neq \emptyset$. $\square$

An application of Lemma 2.4.10 for a special case with non-negative risk function is the following, see also [34, Remark 7 (d)].

Corollary 2.4.11 Let Setting 2.4.1 be given with $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ for fixed $\beta > 0$ and $\mathbf{y} := (\beta/S_0^{(0)}, 0 \ldots, 0)^\intercal \in A$. Assume $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$. Furthermore, let $\mathfrak{r}(x) \geq 0$ for all $\mathbf{x} \in A$ and $\mathfrak{r}(x) = 0$ if and only if $\widehat{\mathbf{x}} = 0$. If $\mathfrak{u}(\mathbf{y}) \in \mathbb{R}$, then $(r_{\min}, \mu_{\min}) = (\mathfrak{r}(\mathbf{y}), \mathfrak{u}(\mathbf{y})) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ and $r_{\min} = 0$.

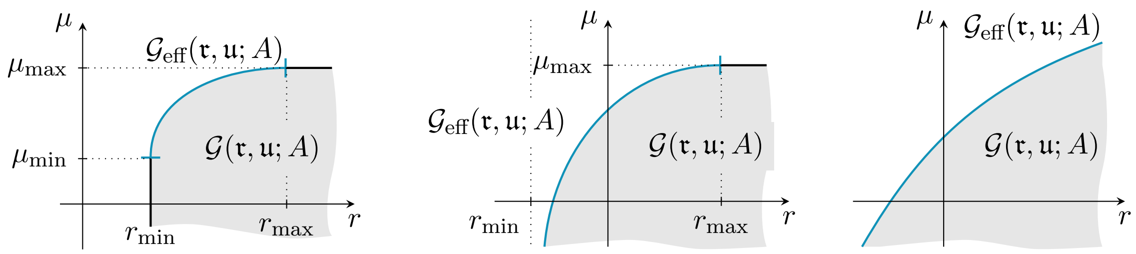
Proof. It is $\mathfrak{r}(\mathbf{y}) = 0 = r_{\min}$. In addition, only $\mathbf{y}$ has zero risky part in A and therefore $\mathfrak{r}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$. Hence, $\mathbf{y}$ is efficient and Lemma 2.4.10 yields $\mu_{\min} = \mathfrak{u}(\mathbf{y})$, which completes the proof. □

Definition 2.4.12 In the situation of Lemma 2.4.10, let J be one of the four intervals $[\mu_{\min}, \mu_{\max}]$, $[\mu_{\min}, \mu_{\max})$, $(-\infty, \mu_{\max}]$, or $(-\infty, \mu_{\max})$, depending on whether or not the corresponding boundary $\mu_{\min}$ or $\mu_{\max}$ is attained by $\mathfrak{u}$ for some $\mathbf{x} \in \operatorname{dom}(\mathfrak{u}) \cap \operatorname{dom}(\mathfrak{r})$, respectively. Analogously let I be one of the intervals $[r_{\min}, r_{\max}]$, $[r_{\min}, \infty)$, $(r_{\min}, r_{\max}]$, or $(r_{\min}, \infty)$ depending on the behavior of the function $\mathfrak{r}$ at the boundary.

Remark 2.4.13 Due to Lemma 2.4.10, exactly one of the following situation holds true depending on the situation:

- $I = [r_{\min}, r_{\max}]$ and $J = [\mu_{\min}, \mu_{\max}]$,
- $I = [r_{\min}, \infty)$ and $J = [\mu_{\min}, \mu_{\max})$, where $\mu_{\max} = \infty$ is possible,
- $I = (r_{\min}, r_{\max}]$ and $J = (-\infty, \mu_{\max}]$, where $r_{\min} = -\infty$ is possible,
- $I = (r_{\min}, \infty)$ and $J = (-\infty, \mu_{\max})$, where $\mu_{\max} = \infty$ and/or $r_{\min} = -\infty$ is possible.

Figure 2.3 shows some examples. ♦



(a) $[r_{\min}, r_{\max}]$, $[\mu_{\min}, \mu_{\max}]$ (b) $(r_{\min}, r_{\max}]$, $(-\infty, \mu_{\max}]$ (c) $(-\infty, \infty)$, $(-\infty, \infty)$

Figure 2.3.: Illustration of efficient frontiers for different cases of the intervals I and J, respectively.

From Theorem 2.4.8 it is known, that set $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ is located on the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ with bounds from Lemma 2.4.10. Between these values, the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ can be expressed by the two functions

$$\gamma\colon J \to \mathbb{R}, \qquad \mu \mapsto \inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)} \{r\} = \inf_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\}, \tag{2.60a}$$

$$\nu\colon I \to \mathbb{R}, \qquad r \mapsto \sup_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)} \{\mu\} = \sup_{\mathbf{x}\in\mathcal{B}_{\mathfrak{r},A}(r)} \{\mathfrak{u}(\mathbf{x})\}, \tag{2.60b}$$

where J and I are from Definition 2.4.12.

Remark 2.4.14 Note, that the first and second representation of γ in (2.60a) and also for ν in (2.60b), are the same because of the following (proof only for γ; proof for ν works similar): From definition of $\mathcal{B}_{\mathfrak{u},A}(\mu)$ it follows that

$$\inf_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\} = \inf_{(r,\mu')\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A),\ \mu'\geq\mu} \{r\}.$$

Proposition 2.4.6 (c) implies, that whenever $(r, \mu') \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ with $\mu' \geq \mu$, than $(r, \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Hence, one can ignore all elements $(r, \mu') \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ with $\mu' > \mu$, because these elements do not change the infimum, which gives the equality in (2.60a). ♦

In the situation of Proposition 2.4.6 (d), set $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed, and therefore, see the next proposition, the infimum and supremum in the definitions of γ and ν become minimum and maximum, respectively, and the two functions are well-defined as shown in the next result, see also Maier-Paape and Zhu [34, Proposition 8].

Proposition 2.4.15 (Functions related to efficient frontier) Let Setting 2.4.1 be given and assume $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu \in \mathbb{R}$. Then, functions γ and ν are well-defined and continuous with

$$\gamma(\mu) = \min_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)} \{r\} = \min_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\}, \qquad \text{for all } \mu \in J, \tag{2.61a}$$

$$\nu(r) = \max_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)} \{\mu\} = \max_{\mathbf{x}\in\mathcal{B}_{\mathfrak{r},A}(r)} \{\mathfrak{u}(\mathbf{x})\}, \qquad \text{for all } r \in I, \tag{2.61b}$$

where γ is increasing and convex and ν is increasing and concave.

Proof. Let $\mu \in J$ be arbitrary with J from Definition 2.4.12. From the characterization of $\mu_{\max}$ in Lemma 2.4.10, there is $\mathbf{x}' \in \operatorname{dom}(\mathfrak{u}) \cap \operatorname{dom}(\mathfrak{r})$ such that $\mathfrak{u}(\mathbf{x}') \geq \mu$, i.e. $\mathcal{B}_{\mathfrak{u},A}(\mu) \neq \emptyset$, and it must be $\mathfrak{r}(\mathbf{x}') \in \mathbb{R}$. Hence, $\mathbf{x}' \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu) \neq \emptyset$ and it is

$$\gamma(\mu) = \inf_{(r,\mu)\in\mathcal{G}(\mathfrak{r},\mathfrak{u};A)} \{r\} = \inf_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\} = \inf_{\mathbf{x}\in\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu)} \{\mathfrak{r}(\mathbf{x})\}. \tag{2.62}$$

When $\mathfrak{r}$ is restricted to the compact set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu) \subset \operatorname{dom}(\mathfrak{r})$, then for all $r \in \mathbb{R}$ the set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) \cap \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu) = \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\min\{r,\mathfrak{r}(\mathbf{x}')\},\mu)$ is a corresponding sublevel set which is closed. Hence, $\mathfrak{r}$ is lower semi-continuous on $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu)$ and therefore attains its global minimum in $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\mathfrak{r}(\mathbf{x}'),\mu)$, see Theorems A.1.2 and A.1.6 in the appendix. Therefore, the infimum in (2.62) becomes a minimum. Hence, this value is finite and γ is well-defined. In addition, for $\mu_1 < \mu_2$ it is $\mathcal{B}_{\mathfrak{u},A}(\mu_1) \supset \mathcal{B}_{\mathfrak{u},A}(\mu_2)$ and

$$\gamma(\mu_1) = \min_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu_1)} \{\mathfrak{r}(\mathbf{x})\} \leq \min_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu_2)} \{\mathfrak{r}(\mathbf{x})\} = \gamma(\mu_2).$$

Consequently, γ is increasing. The epigraph of γ is given by

$$\begin{aligned}\operatorname{epi}(\gamma) &= \left\{(\mu,r) \in J \times \mathbb{R} \,:\, r \geq \gamma(\mu) = \min_{\mathbf{x}\in\mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\}\right\} \\ &= \{(\mu,r) \in J \times \mathbb{R} \,:\, \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) \neq \emptyset\}\end{aligned}$$

which equals to $B := \mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap (\mathbb{R} \times J)$ up to interchanging coordinates. Since B is convex as intersection of convex sets, see Proposition 2.4.6 (c), function γ is a convex function. In addition, $B = \mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap (\mathbb{R} \times \mathbb{R})$ in case $\mu_{\min} = -\infty$ and $B = \mathcal{G}(\mathfrak{r},\mathfrak{u};A) \cap (\mathbb{R} \times [\mu_{\min},\infty))$ in case $\mu_{\min} > -\infty$, because $\mu_{\max}$ is already the natural supremum of $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ and therefore can be omitted. Since $\mathcal{G}(\mathfrak{r},\mathfrak{u};A)$ is closed from Proposition 2.4.6 (d), set B must be closed as well. Hence, γ is lower semi-continuous on J from Theorem A.1.2 in the appendix, while Theorem A.1.3 in the appendix yields continuity in the interior of J. Lower semi-continuity at the boundary of J (if it is contained in J) together with convexity of this one-dimensional function imply continuity on J.

Analogously, one can argue, that for each $r \in I$ there is $\mathbf{x}' \in \operatorname{dom}(\mathfrak{u}) \cap \operatorname{dom}(\mathfrak{r})$ such that $\mathfrak{r}(\mathbf{x}') \leq r$, i.e. $\mathcal{B}_{\mathfrak{r},A}(r) \neq \emptyset$. Then, $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mathfrak{u}(\mathbf{x}')) \neq \emptyset$ is compact and ν can be rewritten similar to γ in (2.62) and therefore is well-defined. The hypograph of ν equals to $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A) \cap (I \times \mathbb{R})$ and is convex. Hence, ν is an continuous, increasing and concave function. □

As in [34, Section 3.3], the functions γ and ν can be extended to $[\mu_{\max}, \infty) \cup J$ and $I \cup (-\infty, r_{\min}]$, respectively, by including the horizontal and vertical part of the boundary, respectively. However, for $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ this extension is not necessary.

From the last result, one can derive the representation of $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ by graphs of ν and γ, which gives a parametrization for the efficient frontier, see also [34, Theorem 4] for the setting therein.

Corollary 2.4.16 (Parametrization of efficient frontier) Assume Setting 2.4.1 is given and $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$. Then

$$\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) = \{(r, \nu(r)) \,:\, r \in I\} = \{(\gamma(\mu), \mu) \,:\, \mu \in J\}.$$

Furthermore ν and γ are strictly increasing and $\nu = \gamma^{-1}$, i.e. $\nu(\gamma(\mu)) = \mu$ for all $\mu \in J$ and $\gamma(\nu(r)) = r$ for all $r \in I$.

Proof. The domains of γ and ν are already the corresponding bounds of $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, cf. definition of theses bounds in (2.59), and these functions are well-defined by Proposition 2.4.15. At first, it has to be shown that for $\mu \in J$ always $(\gamma(\mu), \mu) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$. The first characterization of $\gamma(\mu)$ for $\mu \in J$ in (2.61a) from Proposition 2.4.15 implies, that $(\gamma(\mu), \mu) \in \mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$. Of course, it must be $(\gamma(\mu), \mu) \in \partial\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, because otherwise $\gamma(\mu)$ would not be the minimal risk value according to (2.61a) (note that $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$ is closed by Proposition 2.4.6 (d)). For the same reason, $(\gamma(\mu), \mu)$ cannot be on the relative interior of the horizontal part of the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, if it exists. Since $\mu \in J$, in particular $\mu \geq \mu_{\min}$, $(\gamma(\mu), \mu)$ cannot be in the relative interior of the vertical part of the boundary of $\mathcal{G}(\mathfrak{r}, \mathfrak{u}; A)$, if exists, because this part must lie below $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ and therefore must have utility values less than $\mu_{\min}$. Consequently, Theorem 2.4.8 (b) implies that $(\gamma(\mu), \mu) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ must hold. Since $\mu \in J$ was arbitrary, it follows that $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \supset \{(\gamma(\mu), \mu) \,:\, \mu \in J\}$. It remains to show the relation "$\subset$": $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \subset \mathbb{R} \times J$ by definition of J, see Definition 2.4.12 and (2.59). In addition, for fixed $\mu \in J$ there is at most one $r \in \mathbb{R}$ such that $(r, \mu) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$; otherwise it would contradict Theorem 2.4.8 (b), see (2.58). Hence, it also must be $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \subset \{(\gamma(\mu), \mu) \,:\, \mu \in J\}$ and, consequently, equality. The proof for ν works very similar.

Because the graph of ν is the same as the mirrored graph of γ, it has to be $\nu = \gamma^{-1}$. Hence, both functions are bijective and from Proposition 2.4.15 both are increasing. Consequently, they must be strictly increasing. □

Most requirements for the results in this and also in the two upcoming subsections are meaningful and mild. However, in most cases the set $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ has to be compact for all $r, \mu \in \mathbb{R}$, which is, at first glance, introduced for technical reason only, without a direct and reasonable relation to the market model or the utility and risk measure. Therefore, more fundamental assumptions are given in the following which guarantee the compactness of $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ in the situation of Problem 2.4.2 when A only contains admissible x-portfolios and, w.l.o.g., $A \subset A_\beta := \{\mathbf{x} \in \mathbb{R}^{M+1} \,:\, S^{(0)}\mathbf{x} = \beta\}$.

Lemma 2.4.17 Assume Setting 2.4.1 is given where S is assumed to have no nontrivial risk-free x-portfolio (cf. Definition 2.2.22) and $v^{(1)}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in A$. For $\beta > 0$ fixed, let $A \subset A_\beta := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ be admissible (see Definition 2.2.26) and convex. Then the sets A, $\mathcal{B}_{\mathfrak{u},A}(\mu)$, $\mathcal{B}_{\mathfrak{r},A}(r)$, and $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ are bounded. Furthermore, $\mathcal{B}_{\mathfrak{u},A}(\mu)$, $\mathcal{B}_{\mathfrak{r},A}(r)$ and $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ are compact for all $r,\mu \in \mathbb{R}$ if and only if $\mathfrak{u}$ is closed proper concave and $\mathfrak{r}$ is closed proper convex.

Proof. Lemma 2.2.29 implies that A is bounded. Since $\mathcal{B}_{\mathfrak{u},A}(\mu), \mathcal{B}_{\mathfrak{r},A}(r), \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) \subset A$, these sets are bounded as well. Proposition 2.4.6 (a) yields that $\mathfrak{u}$ and $\mathfrak{r}$ are closed if and only if $\mathcal{B}_{\mathfrak{u},A}(\mu)$ and $\mathcal{B}_{\mathfrak{r},A}(r)$, and consequently $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ are closed. □

Although there are much more assumptions in Lemma 2.4.17 to obtain the single property $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact, these assumptions are moderate and can easily be checked. Requiring that S has no nontrivial risk-free x-portfolio is a realistic assumption, because otherwise, it would be possible to invest into the risky part without any risk, which should not be assumed. The assumptions on v is for convenience and meets the requirements of Problem 2.4.2. In most cases, it should be possible to rewrite a given x-portfolio generating function depending on $M+1$ parameters, such that the new version needs the initial weights for the investment as input. Finally, the restriction on an admissible subset of A_β makes sense, because one should always be able to fill up the position by the bond without adding any risk such that $S^{(0)}\mathbf{x} = \beta$, and there is no demand for portfolios, where it is possible to become ruined or even end up with a debt. Hence, all these assumption are realistic and guarantee that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is bounded. If in addition both functions $\mathfrak{u}$ and $\mathfrak{r}$ are closed, then $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is even compact. This assumption can always be achieved when replacing the proper concave function $\mathfrak{u}$ and the proper convex function $\mathfrak{r}$ by their closures which always exist and only differ from their original function on the (relative) boundary of their domains, cf. Theorem A.1.1 in the appendix.

An alternative for Lemma 2.4.17 is the following result, which only has some requirements for the utility and risk function, which might implicitly depend on the market model S.

Lemma 2.4.18 Assume Setting 2.4.1 is given with $\mathfrak{u}$ being closed proper concave and $\mathfrak{r}$ being closed proper convex. Then $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r,\mu \in \mathbb{R}$, if either $\mathcal{B}_{\mathfrak{u},A}(\mu')$ is non-empty and bounded for at least one $\mu' \in \mathbb{R}$, or $\mathcal{B}_{\mathfrak{r},A}(r')$ is non-empty and bounded for at least one $r' \in \mathbb{R}$, e.g. if $A \subset A_\beta := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ for some fixed $\beta > 0$ and $\mathcal{B}_{\mathfrak{r},A}(0) = \{(\beta/S_0^{(0)}, 0, \ldots, 0)^\intercal\}$.

Proof. Theorem A.1.2 in the appendix gives, that $\mathcal{B}_{\mathfrak{u},A}(\mu)$ and $\mathcal{B}_{\mathfrak{r},A}(r)$ are closed for all $\mu, r \in \mathbb{R}$. Corollary A.1.4 in the appendix yields in the first case that $\mathcal{B}_{\mathfrak{u},A}(\mu)$ is compact for all $\mu \in \mathbb{R}$, and in the second case that $\mathcal{B}_{\mathfrak{r},A}(r)$ is compact for all $r \in \mathbb{R}$. Proposition 2.4.6 (b) then completes the proof. □

For this result to hold true, it suffices that functions $\mathfrak{u}$ and $\mathfrak{r}$ are closed, e.g. by just taking their closures, and that $\mathfrak{r}$ is zero for the bond portfolio and positive for all x-portfolios containing some risky part.

2.4.3 Existence and uniqueness

Corollary 2.4.16 already gives the existence but not necessarily uniqueness of efficient x-portfolios for a given risk level $r \in I$ or a given utility value $\mu \in J$. This already implies existence of solutions for Problem 2.4.2.

Theorem 2.4.19 (Existence for Problem 2.4.2) Assume Setting 2.4.1 is given with convex $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ for some fixed $\beta > 0$. Suppose $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$ and the intervals $I, J \subset \mathbb{R}$ are from Definition 2.4.12.

(a) For arbitrary, but fixed, $\mu \in J$ the minimum risk optimization problem (2.52) has a solution. Moreover, $\mathbf{x}_\mu$ solves (2.52) if and only if $\mathbf{x}_\mu$ is an efficient x-portfolio with $\mathfrak{u}(\mathbf{x}_\mu) = \mu$.

(b) For arbitrary, but fixed, $r \in I$ the maximum chance optimization problem (2.53) has a solution. Moreover, $\mathbf{y}_r$ solves (2.53) if and only if $\mathbf{y}_r$ is an efficient x-portfolio with $\mathfrak{r}(\mathbf{y}_r) = r$.

Proof. For arbitrary $\mu \in J$, Corollary 2.4.16 yields the existence of an efficient x-portfolio $\mathbf{x}_\mu$ with $\mathfrak{u}(\mathbf{x}_\mu) = \mu$ and $\mathfrak{r}(\mathbf{x}_\mu) = \gamma(\mu)$. The characterization of γ in (2.61a) exactly defines the optimization problem (2.52) for $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\} =: A_\beta$, i.e.

$$\mathfrak{r}(\mathbf{x}_\mu) = \gamma(\mu) = \min_{\mathbf{x} \in \mathcal{B}_{\mathfrak{u},A}(\mu)} \{\mathfrak{r}(\mathbf{x})\} = \min_{\mathbf{x} \in A \subset A_\beta} \{\mathfrak{r}(\mathbf{x}) : \mathfrak{u}(\mathbf{x}) \geq \mu\}.$$

This yields the existence of a solution of (2.52) and, furthermore, that each efficient x-portfolio with utility value μ solves (2.52). It remains to show that any solution of (2.52) is an efficient x-portfolio with utility value μ. For this, let $\mathbf{x}'_\mu$ be another solution of (2.52). Since $\mathbf{x}'_\mu$ is a solution of (2.52), $\mathfrak{r}(\mathbf{x}'_\mu) = \gamma(\mu) = \mathfrak{r}(\mathbf{x}_\mu)$ and $\mathfrak{u}(\mathbf{x}'_\mu) \geq \mu$ must hold. Furthermore, $\mathbf{x}_\mu$ is an efficient x-portfolio and therefore it must be $\mathfrak{u}(\mathbf{x}'_\mu) \leq \mathfrak{u}(\mathbf{x}_\mu) = \mu$. Consequently, $\mathfrak{u}(\mathbf{x}'_\mu) = \mu$ holds and, therefore, $\mathbf{x}'_\mu$ is efficient as well. Hence, each solution of (2.52) must be efficient with utility value μ and vice versa.

Assertion (b) follows analogously, when using $\operatorname{graph}(\nu) = \{(r, \nu(r)) : r \in I\}$ in Corollary 2.4.16 and its characterization in (2.61b), which corresponds to the optimization problem (2.53). □

Under some additional assumptions on risk and utility it can be shown, that for each point in $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ there exists exactly one corresponding efficient x-portfolio leading to this element in $\mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, which therefore gives uniqueness, see also Maier-Paape and Zhu [34, Theorem 5].

Theorem 2.4.20 (Uniqueness and efficient portfolio path) Assume Setting 2.4.1 is given with convex $A \subset \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$ for fixed $\beta > 0$ and let $I, J \subset \mathbb{R}$ be the intervals in Definition 2.4.12. Suppose $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$ and either $\mathfrak{u}$ is strictly concave in $\operatorname{dom}(\mathfrak{u})$ or $\mathfrak{r}$ is strictly convex in $\operatorname{dom}(\mathfrak{r})$. Then it is:

(a) For arbitrary, but fixed, $\mu \in J$ there is exactly one solution $\mathbf{x}_\mu \in A$ of (2.52), which is in addition the unique efficient x-portfolio in A such that $\mathfrak{u}(\mathbf{x}_\mu) = \mu$. Furthermore, the mapping $\widetilde{\boldsymbol{\gamma}} \colon J \to A,\ \mu \mapsto \mathbf{x}_\mu$ is continuous.

(b) For arbitrary, but fixed, $r \in I$ there is exactly one solution $\mathbf{y}_r \in A$ of (2.53), which is in addition the unique efficient x-portfolio in A such that $\mathfrak{r}(\mathbf{y}_r) = r$. Furthermore, the mapping $\widetilde{\boldsymbol{\nu}} \colon I \to A,\ r \mapsto \mathbf{y}_r$ is continuous.

Proof. Proof of (a): Let $\mu \in J$ be arbitrary but fixed. Theorem 2.4.19 (a) gives the existence of the solution of (2.52) where the utility value must equal to μ. Assume, there are two solutions $\mathbf{x}^0, \mathbf{x}^1 \in A$ of (2.52) with $\mathbf{x}^0 \neq \mathbf{x}^1$. Then $\mathfrak{u}(\mathbf{x}^0) = \mu = \mathfrak{u}(\mathbf{x}^1)$ and $\mathfrak{r}(\mathbf{x}^0) = \gamma(\mu) = \mathfrak{r}(\mathbf{x}^1)$, and therefore $\mathbf{x}^0, \mathbf{x}^1 \in \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r})$. Since $\mathrm{dom}(\mathfrak{u})$ and $\mathrm{dom}(\mathfrak{r})$ are convex, $\mathbf{x}^\lambda := \lambda \mathbf{x}^0 + (1-\lambda)\mathbf{x}^1 \in \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r})$ for all $\lambda \in (0,1)$ and, because $\mathfrak{u}$ is concave and $\mathfrak{r}$ is convex,

$$\mathfrak{u}(\mathbf{x}^\lambda) \geq \lambda\, \mathfrak{u}(\mathbf{x}^0) + (1-\lambda)\, \mathfrak{u}(\mathbf{x}^1) = \mu \quad \text{and} \quad \mathfrak{r}(\mathbf{x}^\lambda) \leq \lambda\, \mathfrak{r}(\mathbf{x}^0) + (1-\lambda)\, \mathfrak{r}(\mathbf{x}^1) = \gamma(\mu). \quad (2.63)$$

Therefore, $\mathbf{x}^\lambda$ also solves (2.52) and the inequalities in (2.63) must be equalities by Theorem 2.4.19 (a). This is a contradiction, because either $\mathfrak{u}$ is strictly concave or $\mathfrak{r}$ is strictly convex, which leads to at least one strict inequality in (2.63). Consequently, it must be $\mathbf{x}^0 = \mathbf{x}^1$.

From this, $\widetilde{\boldsymbol{\gamma}}(\mu)$ is well-defined for $\mu \in J$. It remains to show continuity. Suppose $\widetilde{\boldsymbol{\gamma}}$ is discontinuous at some point $\mu_0 \in J$. Then, for some $c > 0$, there exists a sequence $(\mu_n)_{n\in\mathbb{N}} \subset J$ with $\mu_n \to \mu_0$ for $n \to \infty$ such that for $(\mathbf{x}^n)_{n\in\mathbb{N}} := (\widetilde{\boldsymbol{\gamma}}(\mu_n))_{n\in\mathbb{N}}$ it is $\|\mathbf{x}^n - \widetilde{\boldsymbol{\gamma}}(\mu_0)\| \geq c$ for all $n \in \mathbb{N}$. In addition, γ is continuous, see Proposition 2.4.15, which yields $(\mathfrak{r}(\mathbf{x}^n), \mathfrak{u}(\mathbf{x}^n)) = (\gamma(\mu_n), \mu_n) \to (\gamma(\mu_0), \mu_0)$ as $n \to \infty$. Then, for all $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ (choose the smallest one) such that $\mathbf{x}^n \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+\varepsilon, \mu_0-\varepsilon)$ for all $n \geq n_0(\varepsilon)$. Choose the subsequence $(\mathbf{x}^{n_k})_{k\in\mathbb{N}} \subset \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$ with $n_k := n_0(1/k)+k$ for $k \in \mathbb{N}$ (the summand "$+k$" is added to ensure a valid subsequence). Since $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+1, \mu_0-1)$ is compact, there exists a convergent subsequence, w.l.o.g. the original subsequence $(\mathbf{x}^{n_k})_{k\in\mathbb{N}}$ itself, with limit $\mathbf{x}^*$, where $\mathbf{x}^* \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0)+\varepsilon, \mu_0-\varepsilon)$ for all $\varepsilon > 0$. Consequently, it must be $\mathbf{x}^* \in \mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(\gamma(\mu_0), \mu_0)$. Since $(\gamma(\mu_0), \mu_0) \in \mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A)$, the x-portfolio $\mathbf{x}^*$ is efficient with $(\mathfrak{r}(\mathbf{x}^*), \mathfrak{u}(\mathbf{x}^*)) = (\gamma(\mu_0), \mu_0)$, i.e. $\mathbf{x}^* = \widetilde{\boldsymbol{\gamma}}(\mu_0)$. This is a contradiction, because $\|\mathbf{x}^n - \widetilde{\boldsymbol{\gamma}}(\mu_0)\| \geq c$ is assumed for all $n \in \mathbb{N}$ and therefore $\|\mathbf{x}^* - \widetilde{\boldsymbol{\gamma}}(\mu_0)\| \geq c$. It follows, that $\widetilde{\boldsymbol{\gamma}}$ is continuous.

Again, the situation in (b) can be shown analogously: For $r \in I$, Theorem 2.4.19 (b) gives existence of the solution of problem (2.53), while strict convexity or concavity gives uniqueness. Continuity of $\widetilde{\boldsymbol{\nu}}$ can be shown as above, when using ν instead of γ and the fact, that ν must be continuous as well. □

Remark 2.4.21 (Connection to [34, Theorem 5]) The assumptions used in [34, Theorem 5] imply that the non-negative risk function $\mathfrak{r}$ is convex, the utility function $\mathbf{x} \mapsto \mathrm{E}[S^{(1)}\mathbf{x}] =: \mathfrak{u}(\mathbf{x})$ is concave and, when using Proposition 2.4.6 (b), that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$. Note, that convexity of $\mathfrak{r}$ also follows from assumption (c3) in [34, Theorem 5]. In this case it even follows, that $\mathfrak{r}^2$ is strictly convex. Since one also can use $\mathfrak{r}^2$ instead of $\mathfrak{r}$ in the optimization problems (with corresponding parameter r^2 instead of r), assumption (c3) implies (c1) after this transformation.

In addition, in [34, Theorem 5], $\mathfrak{r}$ is supposed to be independent on x_0. If $\mathfrak{r}$ is strictly convex in the risky part, then it is strictly convex in $A := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$, because for all $\mathbf{x}, \mathbf{x}' \in A$ with $\mathbf{x} \neq \mathbf{x}'$, it must be $\widehat{\mathbf{x}} \neq \widehat{\mathbf{x}}'$; otherwise it would be $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}'$ and

$$x_0 = \frac{\beta - S^{(0)}\widehat{\mathbf{x}}}{S_0^{(0)}} = \frac{\beta - S^{(0)}\widehat{\mathbf{x}}'}{S_0^{(0)}} = x_0'$$

and therefore $\mathbf{x} = \mathbf{x}'$ which is a contradiction. Function $\mathfrak{u}$ in [34, Theorem 5] is concave and must even be strictly concave, if $\mathfrak{r}$ (or $\mathfrak{r}^2$) is not strictly convex. Hence, the assumptions in [34, Theorem 5] imply the requirements of Theorem 2.4.20 and give similar assertions. Therefore, Theorem 2.4.20 is a generalization of [34, Theorem 5]. ♦

2.4.4 Generalized one-fund theorem

The main task of this section is to generalize the one-fund theorem similar as in Maier-Paape and Zhu [34, Theorem 10] and Rockafellar, Uryasev and Zabarankin [48]. In these two generalizations, the utility function is the expectation of the portfolio wealth in the one-period market model from Section 2.1.1, see e.g. Example 2.3.8 (a). In the following, the utility function is allowed to be of more general type. In essence, the problem for the one-fund theorem is of following form.

Problem 2.4.22 (cf. Problem 2.4.2) Let $f\colon \mathbb{R}^{M+1} \to \mathbb{R}$ and $g\colon \mathbb{R}^{M+1} \to \mathbb{R}$ be two positive homogeneous and convex functions. Furthermore, assume that both functions are linear in the first component, i.e. satisfy ($\mathfrak{b}1$) according to Definition 2.3.2, with the fixed constants $c_f, c_g \in \mathbb{R}$, respectively, i.e. $f(\mathbf{x}) = c_f x_0 + f(\widehat{\mathbf{x}})$ and $g(\mathbf{x}) = c_g x_0 + g(\widehat{\mathbf{x}})$.

Let $\mathbf{y} = (y_0, y_1, \ldots, y_M)^\intercal \in \mathbb{R}^{M+1}$ with $\mathbf{y} \neq 0$ be fixed. For given $(\alpha, \beta) \in \mathbb{R}^2$, find $\mathbf{x} \in \mathbb{R}^{M+1}$, which solves the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^{M+1}} f(\mathbf{x}) \quad \text{subject to } g(\mathbf{x}) \leq \alpha,\ \mathbf{y}^\intercal\mathbf{x} = \beta. \tag{2.64}$$

The corresponding set of solutions for given parameters (α, β) is denoted by $\mathbf{X}^*(\alpha, \beta) \subset \mathbb{R}^{M+1}$.

Remark 2.4.23 Note, that (2.52) for $A := \mathbb{R}^{M+1}$ is equivalent to

$$\min_{\mathbf{x}\in A} \mathfrak{r}(\mathbf{x}) \quad \text{subject to } [-\,\mathfrak{u}(\mathbf{x})] \leq -\mu,\ S^{(0)}\mathbf{x} = \beta,$$

and that (2.53) is equivalent to

$$\min_{\mathbf{x}\in A} [-\,\mathfrak{u}(\mathbf{x})] \quad \text{subject to } \mathfrak{r}(\mathbf{x}) \leq r,\ S^{(0)}\mathbf{x} = \beta,$$

where $-\,\mathfrak{u}$ is convex. Hence, with additional requirements on $\mathfrak{r}$ and $\mathfrak{u}$, both problems are of the type given in Problem 2.4.22 with $\mathbf{y}^\intercal := S^{(0)}$ and parameters $(\alpha, \beta) = (-\mu, \beta)$ or $(\alpha, \beta) = (r, \beta)$, respectively. ♦

The first observation is the dependency of the solutions on the parameter β.

Lemma 2.4.24 Assume Problem 2.4.22 is given. Then, for $\beta \neq 0$ it is

$$\mathbf{X}^*(\alpha, \beta) = |\beta|\mathbf{X}^*\left(\frac{\alpha}{|\beta|}, \operatorname{sgn}(\beta)\right),$$

where $\operatorname{sgn}(\beta)$ is the sign of β.

Proof. It is $\beta \neq 0$. For $\mathbf{x} \in \mathbb{R}^{M+1}$ let $\widetilde{\mathbf{x}} := \mathbf{x}/|\beta|$. Since f and g are positive homogeneous, it is

$$\begin{aligned}
&\mathbf{x}^* \text{ solves} && \min_{\mathbf{x}\in\mathbb{R}^{M+1}} f(\mathbf{x}) \quad \text{subject to } g(\mathbf{x}) \le \alpha,\ \mathbf{y}^\intercal\mathbf{x} = \beta \\
\Leftrightarrow\ & \widetilde{\mathbf{x}}^* := \frac{\mathbf{x}^*}{|\beta|} \text{ solves} && \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^{M+1}} f(|\beta|\widetilde{\mathbf{x}}) \quad \text{subject to } g(|\beta|\widetilde{\mathbf{x}}) \le \alpha,\ \mathbf{y}^\intercal\widetilde{\mathbf{x}} = \operatorname{sgn}(\beta) \\
\Leftrightarrow\ & \widetilde{\mathbf{x}}^* = \frac{\mathbf{x}^*}{|\beta|} \text{ solves} && \min_{\widetilde{\mathbf{x}}\in\mathbb{R}^{M+1}} f(\widetilde{\mathbf{x}}) \quad \text{subject to } g(\widetilde{\mathbf{x}}) \le \frac{\alpha}{|\beta|},\ \mathbf{y}^\intercal\widetilde{\mathbf{x}} = \operatorname{sgn}(\beta)
\end{aligned}$$

and consequently $[1/|\beta|]\mathbf{X}^*(\alpha,\beta) = \mathbf{X}^*(\alpha/|\beta|, \operatorname{sgn}(\beta))$. This includes the case when $\mathbf{X}^*(\alpha,\beta)$ is empty. □

Lemma 2.4.24 says, roughly speaking, that whenever Problem 2.4.22 is solvable for a fixed $\beta \neq 0$, then it is also solvable for all β' of same sign. The next result shows the dependency of the solutions on the parameter α, which turns out to be of affine linear type. Because of Lemma 2.4.24 this is focused on just one fixed value for β, namely $\beta := y_0$.

Theorem 2.4.25 Let Problem 2.4.22 be given with $y_0 \neq 0$. Then it is:

(a) Let $\mathbf{x}^1 \in \mathbf{X}^*(\alpha_1, y_0)$ with $g(\mathbf{x}^1) = \alpha_1 \neq c_g$ and $\mathbf{x}^0 := (1,0,\dots,0)^\intercal \in \mathbb{R}^{M+1}$. For $t \ge 0$ define $\mathbf{x}^t := t(\mathbf{x}^1 - \mathbf{x}^0) + \mathbf{x}^0$. Then for all $t \ge 0$ it is

$$f(\mathbf{x}^t) = t\big(f(\mathbf{x}^1) - c_f\big) + c_f, \qquad g(\mathbf{x}^t) = t(\alpha_1 - c_g) + c_g =: \alpha_t, \tag{2.65}$$

and $\mathbf{x}^t \in \mathbf{X}^*(\alpha_t, y_0)$, where $f(\mathbf{x}^t) \ge c_f$, if $\alpha_1 < c_g$, and $f(\mathbf{x}^t) \le c_f$, if $\alpha_1 > c_g$.

(b) Let $\widetilde{\mathbf{x}}^1 \in \mathbf{X}^*(\eta_1, -y_0)$ with $g(\widetilde{\mathbf{x}}^1) = \eta_1 \neq -c_g$ and $\widetilde{\mathbf{x}}^0 := (-1,0,\dots,0)^\intercal \in \mathbb{R}^{M+1}$. For $t \ge 0$ define $\widetilde{\mathbf{x}}^t := t(\widetilde{\mathbf{x}}^1 - \widetilde{\mathbf{x}}^0) + \widetilde{\mathbf{x}}^0$. Then for all $t \ge 0$ it is

$$f(\widetilde{\mathbf{x}}^t) = t\big(f(\widetilde{\mathbf{x}}^1) + c_f\big) - c_f, \qquad g(\widetilde{\mathbf{x}}^t) = t(\eta_1 + c_g) - c_g =: \eta_t,$$

and $\widetilde{\mathbf{x}}^t \in \mathbf{X}^*(\eta_t, -y_0)$, where $f(\widetilde{\mathbf{x}}^t) \ge -c_f$, if $\eta_1 < -c_g$, and $f(\widetilde{\mathbf{x}}^t) \le -c_f$, if $\eta_1 > -c_g$.

Proof. Note that (b) can be proven completely analogously to (a) with just using $\beta = -y_0$. Thus, only case (a) is shown, which is proven similar to the proof of Theorem 10 in Maier-Paape and Zhu [34]. For this, define $\beta := y_0$. For $\lambda := (\lambda_1, \lambda_2)$ the Lagrangian of (2.64) is defined by

$$L(\mathbf{x},\lambda) := f(\mathbf{x}) + \lambda_1(g(\mathbf{x}) - \alpha) + \lambda_2(\mathbf{y}^\intercal\mathbf{x} - \beta),$$

where $\lambda_1 \ge 0$ must hold true, see e.g. [34, Section 2.2]. Function $\mathbf{x} \mapsto \lambda_1 g(\mathbf{x})$ is convex, because g is convex and $\lambda_1 \ge 0$. Therefore, also $\mathbf{x} \mapsto L(\mathbf{x},\lambda)$ is convex in $\mathbf{x}$ for all $\lambda = (\lambda_1,\lambda_2)$ with $\lambda_1 \ge 0$.

For $\alpha \neq c_g$ let $\mathbf{x}^* := \mathbf{x}^*(\alpha,\beta) \in \mathbf{X}^*(\alpha,\beta)$ with $g(\mathbf{x}^*) = \alpha$ be a solution of (2.64) and let $\partial F(\mathbf{x}^*)$ denote the subdifferential of a function $F\colon \mathbb{R}^{M+1} \to \mathbb{R}$ with

$$\partial F(\mathbf{x}^*) := \big\{\mathbf{z} \in \mathbb{R}^{M+1} \ :\ F(\mathbf{x}) - F(\mathbf{x}^*) \ge \mathbf{z}^\intercal(\mathbf{x} - \mathbf{x}^*) \text{ for all } \mathbf{x} \in \mathbb{R}^{M+1}\big\}, \tag{2.66}$$

see e.g. [34, Definition 3]. The function g is positive homogeneous, linear in the first component and cannot be the trivial function because $g(\mathbf{x}^1) \neq c_g = g(\mathbf{x}^0)$ by assumption.

Let $s > 0$ be arbitrary and $\overline{\mathbf{x}}^s := s\mathbf{x}^* + (1-s)\mathbf{x}^0$. Then $g(\overline{\mathbf{x}}^s) = s\alpha + (1-s)c_g$ and $\mathbf{y}^\intercal(s\mathbf{x}^* + (1-s)\mathbf{x}^0) = y_0 = \beta$. If $\alpha < c_g$, then $g(\overline{\mathbf{x}}^2) < \alpha$, and if $\alpha > c_g$, then $g(\overline{\mathbf{x}}^{1/2}) < \alpha$. Hence, there is some $\overline{\mathbf{x}}$ such that $g(\overline{\mathbf{x}}) < \alpha$ and $\mathbf{y}^\intercal \overline{x} = \beta$. Then, there exist a Karush-Kuhn-Tucker vector $(\lambda_1^*, \lambda_2^*) := (\lambda_1^*(\alpha,\beta), \lambda_2^*(\alpha,\beta))$ with $\lambda_1^* \geq 0$, see Definition A.2.1 and Theorem A.2.2 in the appendix. Furthermore, $0 \in [\partial f(\mathbf{x}^*) + \lambda_1^* \partial g(\mathbf{x}^*) + \lambda_2^* \mathbf{y}]$ holds, see Theorem A.2.3 in the appendix, or, equivalently,

$$-\lambda_2^* \mathbf{y} \in \partial f(\mathbf{x}^*) + \lambda_1^* \partial g(\mathbf{x}^*).$$

Since f and g are linear in the first component x_0, it follows that $-\lambda_2^* y_0 = c_f + \lambda_1^* c_g$. Note, that $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbb{R}^{M+1}$ and therefore $\partial f(\mathbf{x}^*) + \lambda_1^* \partial g(\mathbf{x}^*) = \partial(f + \lambda_1^* g)(\mathbf{x}^*)$, cf. Rockafellar [43, Theorem 23.8]. It follows, that

$$\frac{1}{y_0}(c_f + \lambda_1^* c_g)\mathbf{y} \in \partial(f + \lambda_1^* g)(\mathbf{x}^*). \tag{2.67}$$

This yields for all $\mathbf{x} \in \mathbb{R}^{M+1}$ that

$$(f(\mathbf{x}) + \lambda_1^* g(\mathbf{x})) - (f(\mathbf{x}^*) + \lambda_1^* g(\mathbf{x}^*)) \geq \frac{1}{y_0}(c_f + \lambda_1^* c_g)\mathbf{y}^\intercal(\mathbf{x} - \mathbf{x}^*). \tag{2.68}$$

It is known, see e.g. [34, Equations (76) and (77)], that a positive homogeneous, convex function F has the property: If $\mathbf{z} \in \partial F(\mathbf{x})$, then $F(\mathbf{x}) = \mathbf{z}^\intercal \mathbf{x}$, because for all $t \in (-1, 1)$ it is

$$tF(\mathbf{x}) = F((1+t)\mathbf{x}) - F(\mathbf{x}) \geq \mathbf{z}^\intercal((1+t)\mathbf{x} - \mathbf{x}) = t\mathbf{z}^\intercal \mathbf{x}.$$

Since $\mathbf{x} \mapsto f(\mathbf{x}) + \lambda_1^* g(\mathbf{x})$ is positive homogeneous and convex, $\mathbf{y}^\intercal \mathbf{x}^* = \beta = y_0$, and $g(\mathbf{x}^*) = \alpha$ (by assumption), statement (2.67) gives

$$f(\mathbf{x}^*) + \lambda_1^* g(\mathbf{x}^*) = \frac{1}{y_0}(c_f + \lambda_1^* c_g)\mathbf{y}^\intercal \mathbf{x}^* \qquad \Leftrightarrow \qquad f(\mathbf{x}^*) = c_f + \lambda_1^*(c_g - \alpha). \tag{2.69}$$

Substituting the left hand side of (2.69) into (2.68) yields for all $\mathbf{x} \in \mathbb{R}^{M+1}$ that

$$f(\mathbf{x}) + \lambda_1^* g(\mathbf{x}) \geq \frac{1}{y_0}(c_f + \lambda_1^* c_g)\mathbf{y}^\intercal \mathbf{x}. \tag{2.70}$$

If in addition $\mathbf{x}$ satisfies $\mathbf{y}^\intercal \mathbf{x} = \beta = y_0$, then (2.70) becomes

$$f(\mathbf{x}) \geq c_f + \lambda_1^*(c_g - g(\mathbf{x})). \tag{2.71}$$

By assumption $\mathbf{x}^0 = (1, 0, \ldots, 0)^\intercal \in \mathbb{R}^{M+1}$ and $\mathbf{x}^1 \in \mathbf{X}^*(\alpha_1, y_0)$ exists with $g(\mathbf{x}^1) = \alpha_1$ for an arbitrary but fixed $\alpha_1 \neq c_g$. Denote the corresponding Karush-Kuhn-Tucker coefficient from above by $\lambda_1^1 := \lambda_1^*(\alpha_1, y_0) \geq 0$ and define $\mathbf{x}^t := t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \in \mathbb{R}^{M+1}$ for $t \geq 0$. Then

$$\mathbf{y}^\intercal \mathbf{x}^t = t\mathbf{y}^\intercal \mathbf{x}^1 + (1-t)y_0 = ty_0 + (1-t)y_0 = y_0 \tag{2.72}$$

and because g is positive homogeneous and linear in x_0 it is

$$g(\mathbf{x}^t) = tg(\mathbf{x}^1) + (1-t)c_g = t\alpha_1 + (1-t)c_g = c_g - t(c_g - \alpha_1) = \alpha_t. \tag{2.73}$$

For all $\mathbf{x} \in \mathbb{R}^{M+1}$ satisfying $\mathbf{y}^\mathsf{T}\mathbf{x} = \beta = y_0$ and $g(\mathbf{x}) \le c_g - t(c_g - \alpha_1)$, inequality (2.71) and the right hand side of (2.69) for $\mathbf{x}^* = \mathbf{x}^1$ gives

$$f(\mathbf{x}) \ge c_f + \lambda_1^1(c_g - g(\mathbf{x})) \ge c_f + t\lambda_1^1(c_g - \alpha_1) = c_f + t\big(f(\mathbf{x}^1) - c_f\big). \tag{2.74}$$

Let $\alpha < c_g$, if $\alpha_1 < c_g$, or $\alpha > c_g$, if $\alpha_1 > c_g$. Then for $t_\alpha := (c_g - \alpha)/(c_g - \alpha_1) > 0$ and for an arbitrary solution $\mathbf{x}^*(\alpha, y_0) \in \mathbf{X}^*(\alpha, y_0) \subset \mathbb{R}^{M+1}$, if it exists, holds

$$g(\mathbf{x}^*(\alpha, y_0)) \le \alpha = c_g - t_\alpha(c_g - \alpha_1)$$

and $\mathbf{y}^\mathsf{T}\mathbf{x}^*(\alpha, y_0) = y_0$, and (2.74) becomes

$$f(\mathbf{x}^*(\alpha, y_0)) \ge c_f + t_\alpha\big(f(\mathbf{x}^1) - c_f\big) = t_\alpha f(\mathbf{x}^1) + (1 - t_\alpha)c_f = f(\mathbf{x}^{t_\alpha}). \tag{2.75}$$

Since $\mathbf{x}^*(\alpha, y_0)$ is a solution and because $g(\mathbf{x}^{t_\alpha}) = \alpha$ (follows from (2.73) for $t = t_\alpha$) and $\mathbf{y}^\mathsf{T}\mathbf{x}^{t_\alpha} = y_0$ by (2.72), it must be $f(\mathbf{x}^*(\alpha, y_0)) \le f(\mathbf{x}^{t_\alpha})$. Consequently, $f(\mathbf{x}^*(\alpha, y_0)) = f(\mathbf{x}^{t_\alpha})$ and thus $\mathbf{x}^{t_\alpha} \in \mathbf{X}^*(\alpha, y_0) \ne \emptyset$ where due to (2.75) and (2.73)

$$f(\mathbf{x}^{t_\alpha}) = t_\alpha\big(f(\mathbf{x}^1) - c_f\big) + c_f \qquad \text{and} \qquad g(\mathbf{x}^{t_\alpha}) = t_\alpha(\alpha_1 - c_g) + c_g = \alpha.$$

Hence, $\mathbf{x}^t \in \mathbf{X}^*(\alpha_t, y_0)$ for $t > 0$ and (2.65) holds true. The limit case $t = 0$ gives the solution $\mathbf{x}^0 \in \mathbf{X}^*(c_g, y_0)$ with $f(\mathbf{x}^0) = c_f$ and $g(\mathbf{x}^0) = c_g$.

It remains to check, whether $f(\mathbf{x}^t) \le c_f$ or $f(\mathbf{x}^t) \ge c_f$: For $\alpha_t = t(\alpha_1 - c_g) + c_g$ assume $g(\mathbf{x}^t) = \alpha_t < c_g = g(\mathbf{x}^0)$, i.e. $\alpha_1 - c_g < 0$. Then it is

$$f(\mathbf{x}^t) = \min_{\substack{\mathbf{x} \in \mathbb{R}^{M+1} \\ g(\mathbf{x}) \le \alpha_t,\ \mathbf{y}^\mathsf{T}\mathbf{x} = y_0}} f(\mathbf{x}) \ge \min_{\substack{\mathbf{x} \in \mathbb{R}^{M+1} \\ g(\mathbf{x}) \le c_g = \alpha_0,\ \mathbf{y}^\mathsf{T}\mathbf{x} = y_0}} f(\mathbf{x}) = f(\mathbf{x}^0) = c_f.$$

If $g(\mathbf{x}^t) = \alpha_t > c_g = g(\mathbf{x}^0)$, then this holds true with "$\le$" instead of "$\ge$", which completes the proof. □

Remark 2.4.26 Theorem 2.4.25 (a) means, with the trivial solution $\mathbf{x}^0 = (1, 0, \dots, 0)^\mathsf{T} \in \mathbf{X}^*(\alpha_0 = c_g, y_0)$ and a nontrivial solution $\mathbf{x}^1 \in \mathbf{X}^*(\alpha_1, y_0)$ the whole ray $\{\mathbf{x}^t : t \ge 0\}$ starting at $\mathbf{x}^0$ and going through $\mathbf{x}^1$ contributes new solutions $\mathbf{x}^t \in \mathbf{X}^*(\alpha_t, y_0)$ for (2.64) with $\beta = y_0$. Theorem 2.4.25 (b) states a similar assertion for the parameter $\beta = -y_0$ and a ray starting in $\widetilde{\mathbf{x}}^0 := (-1, 0, \dots, 0)^\mathsf{T} \in \mathbb{R}^{M+1}$. ♦

A direct consequence from Theorem 2.4.25 together with Remark 2.4.23 is the following result.

Corollary 2.4.27 (Affine efficient frontier) Let Problem 2.4.2 with Setting 2.4.1 for fixed $\beta > 0$ and $A := \mathbb{R}^{M+1}$ be given and define $A_\beta := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \beta\}$. Assume:

(i) $\mathfrak{u}$ and $\mathfrak{r}$ are, in addition, positive homogeneous ($\mathfrak{b}2$) and linear in the first component with constants $c_\mathfrak{u}, c_\mathfrak{r} \in \mathbb{R}$ from ($\mathfrak{b}1$), respectively, ($\mathfrak{u}(\mathbf{x}) = c_\mathfrak{u} x_0 + \mathfrak{u}(\widehat{\mathbf{x}})$ and $\mathfrak{r}(\mathbf{x}) = c_\mathfrak{r} x_0 + \mathfrak{r}(\widehat{\mathbf{x}})$) and $\mathrm{dom}(\mathfrak{u}) = \mathrm{dom}(\mathfrak{r}) = \mathbb{R}^{M+1}$,

(ii) $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A_\beta}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$,

(iii) $\mathbf{x}^0 := (\beta/S_0^{(0)}, 0, \ldots, 0)^\intercal \in A_\beta$ is an efficient x-portfolio in A_β according to Definition 2.4.7, and

(iv) there exists $\mathbf{x}' \in A_\beta$ such that $\mathfrak{u}(\mathbf{x}') > \mathfrak{u}(\mathbf{x}^0) = c_\mathfrak{u}\beta/S_0^{(0)} =: \mu_0$.

Then, for all $\mu \geq \mu_0$ the following holds:

(a) There exists an efficient x-portfolio $\mathbf{x}_\mu \in A_\beta$ such that $\mathfrak{u}(\mathbf{x}_\mu) = \mu$.

(b) $\mathbf{x}^* \in A_\beta$ is an efficient x-portfolio with $\mathfrak{u}(\mathbf{x}^*) = \mu$ if and only if $\mathbf{x}^*$ solves (2.52).

(c) For $\mu_1 := \mu_0 + 1$ with solution $\mathbf{x}^1 := \mathbf{x}_{\mu_1}$ and $t \geq 0$, the vector $\mathbf{x}^t := \mathbf{x}^0 + t(\mathbf{x}^1 - \mathbf{x}^0)$ is efficient with $(\mathfrak{r}(\mathbf{x}^t), \mathfrak{u}(\mathbf{x}^t)) = (r_0 + t(\mathfrak{r}(\mathbf{x}^1) - r_0), \mu_0 + t) \in \mathcal{G}_{\mathrm{eff}}(\mathfrak{r}, \mathfrak{u}; A_\beta)$, where $r_0 := \mathfrak{r}(\mathbf{x}^0) = c_\mathfrak{r}\beta/S_0^{(0)}$. Thus, the efficient portfolios lie on rays and the related efficient frontier is also a ray in the risk-utility space.

Proof. Proof of (a) and (b): Let $\mathbf{x}' \in A_\beta$ such that $\mu' := \mathfrak{u}(\mathbf{x}') > \mu_0$, which exists by assumption (iv). Then, for all $t \geq 0$ it is $t\mathbf{x}' + (1-t)\mathbf{x}^0 \in A_\beta$ and, because of (i),

$$\mathfrak{u}(t\mathbf{x}' + (1-t)\mathbf{x}^0) = c_\mathfrak{u}\left(tx_0' + (1-t)\frac{\beta}{S_0^{(0)}}\right) + \mathfrak{u}(t\widehat{\mathbf{x}}') = \mathfrak{u}(t\mathbf{x}') + (1-t)\mu_0 = \mu_0 + t(\mu' - \mu_0)$$

which can be arbitrary large for t large enough. Hence, for all $\mu \geq \mu_0$ there exists an x-portfolio $\mathbf{x}$ with $\mathfrak{u}(\mathbf{x}) = \mu$ and it is $\mu_{\max} = \infty$. From assumption (iii) it must be $\mu_{\min} \leq \mu_0$. Theorem 2.4.19 then gives the existence of efficient x-portfolios for all $\mu \geq \mu_0$ and the equivalence in (b).

Proof of (c): Consequently, there exists a solution $\mathbf{x}^1 = \mathbf{x}_{\mu_1}$ with $\mathfrak{u}(\mathbf{x}^1) = \mu_1 > \mu_0$. Because of Remark 2.4.23, Lemma 2.4.24 and Theorem 2.4.25 can be applied. This yields, that the solutions, i.e. the efficient portfolios, are on a ray and also their risk/utility values are on a straight line connecting the corresponding values of $\mathbf{x}^0$ and $\mathbf{x}^1$, i.e. the line connecting $(\mathfrak{r}(\mathbf{x}^0), \mathfrak{u}(\mathbf{x}^0)) = (r_0, \mu_0)$, for $t = 0$, and $(\mathfrak{r}(\mathbf{x}^1), \mathfrak{u}(\mathbf{x}^1)) = (\mathfrak{r}(\mathbf{x}^1), \mu_0 + 1)$, for $t = 1$. □

Remark 2.4.28 ($\mathfrak{u}$ and $\mathfrak{r}$ are closed) Let the situation of Corollary 2.4.27 be given. Since $\mathfrak{u}$ and $\mathfrak{r}$ are positive homogeneous, it is $0 \in \mathrm{dom}(\mathfrak{u}) = \mathrm{dom}(\mathfrak{r}) = \mathbb{R}^{M+1}$. Then, for $\mathbf{x} := 0$ and for all $\mathbf{x}' \in \mathbb{R}^{M+1}$ it is

$$\lim_{\lambda \nearrow 1} \mathfrak{u}((1-\lambda)\mathbf{x} + \lambda\mathbf{x}') = \lim_{\lambda \nearrow 1} \lambda\,\mathfrak{u}(\mathbf{x}') = \mathfrak{u}(\mathbf{x}')$$

and

$$\lim_{\lambda \nearrow 1} \mathfrak{r}((1-\lambda)\mathbf{x} + \lambda\mathbf{x}') = \lim_{\lambda \nearrow 1} \lambda\,\mathfrak{r}(\mathbf{x}') = \mathfrak{r}(\mathbf{x}').$$

Consequently, $\mathfrak{u}$ and $\mathfrak{r}$ must be closed proper concave and closed proper convex, respectively, cf. equation (A.2) in the appendix. Furthermore, the level sets $\mathcal{B}_{\mathfrak{u},A}(\mu)$, $\mathcal{B}_{\mathfrak{r},A}(r)$ and $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ are at least closed for all $r, \mu \in \mathbb{R}$, see Theorem A.1.2 in the appendix. ♦

Remark 2.4.29 (Alternative assumptions in Corollary 2.4.27) Let the situation of Corollary 2.4.27 be given. For the four assumptions in this result, the following holds:

(a) Assumption (i) implies, that $\mathfrak{u}$ is closed proper concave and $\mathfrak{r}$ is closed proper convex, see Remark 2.4.28.

(b) Assume $\mathcal{B}_{\mathfrak{r},A_\beta}(0) = \{\mathbf{x}^0\}$. Lemma 2.4.18 then yields that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A_\beta}(r,\mu)$ is compact for all $r,\mu \in \mathbb{R}$. In addition, $\mathbf{x}^0 \in A_\beta$ must be efficient, because it has finite utility by assumption and defines the only vector in A_β with risk less than or equal to zero, i.e. $\mathbf{x}^0$ cannot be improved. Hence, $\mathcal{B}_{\mathfrak{r},A_\beta}(0) = \{\mathbf{x}^0\}$ implies assumptions (ii) and (iii) in Corollary 2.4.27.
Note, that $\mathcal{B}_{\mathfrak{r},A_\beta}(0) = \{\mathbf{x}^0\}$ means, that $\mathfrak{r}(\mathbf{x}^0) = 0$ and $\mathfrak{r}(\mathbf{x}) > 0$ for all $\mathbf{x} \in A_\beta$ with $\mathbf{x} \neq \mathbf{x}^0$. This is fulfilled for instance if $c_\mathfrak{r} = 0$ and $\mathfrak{r}(\widehat{\mathbf{x}}) > 0$ for all $\widehat{\mathbf{x}} \neq 0$.

(c) Assumption (iv) is equivalent to the existence of $\mathbf{x} \in A_{\beta'}$ for $\beta' := S_0^{(0)}$ such that $\mathfrak{u}(\mathbf{x}) > c_\mathfrak{u}$. In case that $\mathfrak{u}$ is a linear function, it suffices that there exists $\mathbf{x} \in A_{\beta'}$ such that $\mathfrak{u}(\mathbf{x}) \neq c_\mathfrak{u}$, because then either $\mathfrak{u}(\mathbf{x}) > c_\mathfrak{u}$ or $\mathfrak{u}(2\widetilde{\mathbf{x}}^0 - \mathbf{x}) > c_\mathfrak{u}$ for $\widetilde{\mathbf{x}}^0 := (1,0,\dots,0)^\intercal \in A_{\beta'}$.

Assumptions (i) to (iv) in Corollary 2.4.27 are stated to be able to state the complete result in a general form. As shown in this remark, some results can easily be obtained in special situations. ♦

Remark 2.4.30 (Connection to Maier-Paape and Zhu [34, Theorem 10]) In [34, Theorem 10], the authors discuss the case of discrete and finite probability space using a one-period model and a utility function defined by $\mathfrak{u}(\mathbf{x}) := \mathrm{E}[S^{(1)}\mathbf{x}]$, which is linear with $c_\mathfrak{u} = R$. The assumptions therein are, that $\mathfrak{r}$ is convex, positive homogeneous, independent on the risk-free part (i.e. $c_\mathfrak{r} = 0$), $\mathfrak{r}(\mathbf{x}) > 0$ if $\widehat{\mathbf{x}} \neq 0$ and $\mathfrak{r}(\mathbf{x}) = 0$ if and only if $\widehat{\mathbf{x}} = 0$. Hence assumptions (i), (ii) and (iii) directly hold true in Corollary 2.4.27, see Remark 2.4.29.

Assumption (iv) follows from [34, Equation (78)] because of the following: First note, that [34, Equation (78)] is equivalent to $\mathrm{E}\,[S^{(1)}] - RS^{(0)} \neq 0$, because the first component on the left hand side is always zero and is therefore omitted in [34]. Hence, there exists some $j \in \{1,\dots,M\}$ such that $\mathrm{E}\,[S_j^{(1)}] - RS_j^{(0)} \neq 0$. Furthermore, in [34] it is always $S_0^{(0)} = 1$ and $S_0^{(1)} = R$. Then, there exists some $\mathbf{x} \in A_{\beta'}$ with $\beta' := S_0^{(0)} = 1$ such that $0 \neq (\mathrm{E}\,[S^{(1)}] - RS^{(0)})\mathbf{x} = \mathfrak{u}(\mathbf{x}) - c_\mathfrak{u}$, e.g. $\mathbf{x} = (0,\dots,0,1/S_j^{(0)},0,\dots,0)^\intercal \in \mathbb{R}^{M+1}$ with only one non-zero entry at position j. Remark 2.4.29 (c) then implies assumption (iv).

Consequently, the assumptions in [34, Theorem 10] also fulfill the requirements of Corollary 2.4.27 and have the same assertions. Therefore, Corollary 2.4.27 is a generalization of [34, Theorem 10]. ♦

Remark 2.4.31 (Connection to Rockafellar, Uryasev and Zabarankin [48, Theorems 2 and 3]) The results in [48, Theorems 2 and 3] are also related to Corollary 2.4.27. Under the assumptions in [48], Property (i) is fulfilled from definition, because the utility function is a linear function as defined in Example 2.3.13, while the risk measure is a general deviation measure, see Definition 2.3.6, which is independent on the risk-free part. Property (ii) in Corollary 2.4.27 follows from [48, Proposition 4]. Because of ($\mathcal{D}4$) in Definition 2.3.6, the x-portfolio $\mathbf{x}^0$ obviously is efficient because it has zero risk. Another assumption in [48, Theorems 2 and 3] is the existence of a solution with higher utility than the bond, which therefore guarantees assumption (iv) in Corollary 2.4.27. Hence, the assumptions in [48, Theorems 2 and 3] meet the requirements of Corollary 2.4.27, where all these results state the same assertions. Consequently, Corollary 2.4.27 generalizes [48, Theorems 2 and 3] for more general risk and more general utility measures. ♦

Remark 2.4.32 (Master fund) A similar remark can be found in [34, Remark 9] for the one-period case and fixed linear utility function.

(a) Let the situation in Theorem 2.4.27 be given. As defined in [48], a master fund is a solution which has a zero first component, i.e. which fully invests into the risky assets. For this to exist for a fixed solution $\mathbf{x}^1$ it is necessary and sufficient, that there exists $t_M > 0$ such that $\mathbf{x}^{t_M} = \widehat{\mathbf{x}}^{t_M}$. This is the case if and only if $x_0^1 < \beta/S_0^{(0)}$ because of the following: Since $\beta, S_0^{(0)} > 0$, the inequality $x_0^1 < \beta/S_0^{(0)}$ is equivalent to $0 < \beta/(\beta - S_0^{(0)} x_0^1) =: t_M$. In addition, t_M is the unique positive parameter such that $\mathbf{x}^{t_M} = \mathbf{x}^0 + t_M(\mathbf{x}^1 - \mathbf{x}^0)$ with

$$x_0^{t_M} = \frac{\beta}{S_0^{(0)}} + t_M \left(x_0^1 - \frac{\beta}{S_0^{(0)}} \right) = 0.$$

and, therefore, $\mathbf{x}^{t_M} = t_M \widehat{\mathbf{x}}^1$.

(b) The following holds true: In the situation (a) with $x_0^1 < \beta/S_0^{(0)}$, the efficient frontier for A_β must be tangential to the efficient frontier when replacing A_β by $\widehat{A}_\beta := A_\beta \cap \{\mathbf{x} \in \mathbb{R}^{M+1} : \mathbf{x} = \widehat{\mathbf{x}}\}$, i.e. when only allowing the portfolios to fully invest in the risky assets.
Proof: Since $A_\beta \supset \widehat{A}_\beta$, the efficient frontier for A_β cannot lie below the efficient frontier for $\widehat{A}_\beta$. In addition, there is only one master fund. Its value in the risk-utility space is given by $(\mathfrak{r}(\mathbf{x}^{t_M}), \mathfrak{u}(\mathbf{x}^{t_M}))$ and must be the only intersection point of both efficient frontiers. Consequently, the efficient frontier for A_β must be tangential to the efficient frontier for $\widehat{A}_\beta$.

(c) In the one-period case with $\mathfrak{r} := \mathfrak{r}_\sigma$ from Example 2.3.15 and $\mathfrak{u}(\mathbf{x}) = \mathrm{E}[S^{(1)}\mathbf{x}]$, i.e. the situation in the CAPM, $\mathbf{x}^{t_M}$ corresponds to the market portfolio. Note, that in this case it is $\beta = 1$ and $S_0^{(0)} = 1$ and therefore $t_M = 1/(1 - x_0^1)$ and $\mathbf{x}^{t_M} = \widehat{\mathbf{x}}^1/(1 - x_0^1)$. This coincides with the observation in [34, Remark 9].
In the same situation but with more general risk, [34, Theorem 11] gives another sufficient and necessary condition for the existence of the master fund using a Fenchel conjugate and without involving the solution $\mathbf{x}^1$.

♦

3 Drawdown

The risk of investments is always an important factor. For financial investments a reasonable measure is the drawdown. It is a path dependent risk measure, where the order of the returns within the wealth process is crucial. Assume there are two paths of the wealth process given. One path contains a large (maximum) drawdown, i.e. there is a large movement from a peak to through within the equity curve. The other path is supposed to have the exact same returns but in different order, such there are only small drawdowns, cf. Figure 3.1. Although both paths have the same final outcome, the blue version is preferable, because of smaller drawdowns. The investor does not have to endure big retracements, but only small ones. The drawdown as risk measure takes this into account. Other risk measures, which are not path dependent, like the variance, would give both paths the same rating.

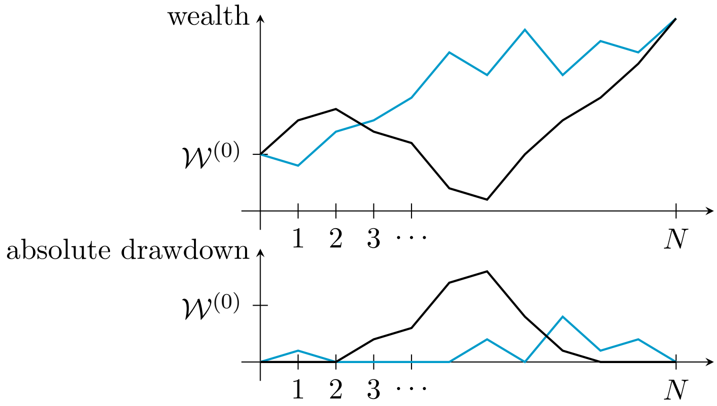


Figure 3.1.: Two possible equity curves (upper plot) with corresponding absolute drawdowns (lower plot).

However, drawdown measures are not completely understood yet and there are many open questions. For this reason, the drawdown is discussed in detail in this chapter. Section 3.1 gives its general definition in the context of Chapter 2, some basic properties, and a survey on related literature. In general it is hard to evaluate probabilities for the drawdown of a given stochastic process. In case of Markov chains, the drawdown and its behavior as the number of time steps goes to infinity is discussed in Section 3.2. For a special case of a Markov chain, the random walk, there exists an explicit expression for the probability mass function of the drawdown, which is shown in Section 3.3.

3.1 General case

In the recent work of Maier-Paape and Zhu [35], the so-called relative drawdown is discussed in detail in a finite probability space with iid drawings of a given random variable. They give reasonable properties required to ensure the existence and uniqueness of a solution e.g. for the minimum risk optimization (2.52) in Problem 2.4.2. Furthermore, the authors derive an approximation for this risk measure which has some additional preferable property, the positive homogeneity. The following subsections discuss the same risk measure but for general multi-period market models whose probability space does not need to be finite or discrete and whose steps do not need to be modeled by iid drawings of the same random variable. Furthermore, the relation between the risk measure (derived from the relative drawdown) and its approximation is studied in detail including their connection which involves the so-called absolute drawdown. In most of the literature, the absolute drawdown is discussed, see e.g. Chekhlov, Uryasev and Zabarankin [8, 9], Goldberg and Mahmoud [20] and Zabarankin, Pavlikov and Uryasev [55], because it has some favorable properties and is easier to handle. To the best of the author's knowledge, the properties of the relative drawdown have not been addressed in detail in the literature up to the work of Maier-Paape and Zhu [35].

A natural distinction for the definition of drawdown is, as already hinted above, an absolute and a relative variant, which is done in Subsection 3.1.1. Subsection 3.1.2 gives a more detailed overview on the literature within this context. However, both variants of the drawdown are related which is discussed in Subsection 3.1.3. Some important properties in the sense of the axioms in Subsection 2.3.1 follow in Subsection 3.1.4. Using the resulting risk measure in portfolio optimization with an appropriate utility function ensures the existence of a unique solution, which is shown in Subsection 3.1.5. Note, that the notation is used from Chapter 2, see especially the introduction of Section 2.1.

3.1.1 Definition

There are basically two different types of drawdown, the absolute and the relative drawdown. In both variants, the drawdown can be measured after each time step, i.e. the drawdown itself is a time dependent stochastic process defined as follows.

Definition 3.1.1 (Drawdown processes) For $N \in \mathbb{N}$ let $W := (W^{(n)})_{0\leq n\leq N} \in \mathcal{L}^2(N)$ model the wealth process.

(a) The *absolute drawdown process* is defined by

$$\mathcal{D}_{\text{abs}} \colon \mathcal{L}^2(N) \to \mathcal{L}^2(N), \qquad W \mapsto \left(\mathcal{D}_{\text{abs}}^{(0)}(W), \dots, \mathcal{D}_{\text{abs}}^{(N)}(W)\right), \tag{3.1a}$$

$$\mathcal{D}_{\text{abs}}^{(n)} \colon \mathcal{L}^2(N) \to \mathcal{L}^2(\Omega, \mathcal{F}_n, \mathrm{P}), \qquad W \mapsto \max_{0\leq \ell\leq n} \left\{W^{(\ell)}\right\} - W^{(n)}, \tag{3.1b}$$

for $n = 0, 1, \dots, N$, where $\mathcal{D}_{\text{abs}}^{(n)} \geq 0$ almost surely (a.s.).

(b) The *relative drawdown process* is defined for positive wealth processes by

$$\mathcal{D}_{\text{rel}} \colon \mathcal{L}^2(N; \mathbb{R}_{>0}) \to \mathcal{L}^\infty(N), \qquad W \mapsto \left(\mathcal{D}_{\text{rel}}^{(0)}(W), \dots, \mathcal{D}_{\text{rel}}^{(N)}(W)\right), \tag{3.2a}$$

$$\mathcal{D}_{\text{rel}}^{(n)} \colon \mathcal{L}^2(N; \mathbb{R}_{>0}) \to \mathcal{L}^\infty(\Omega, \mathcal{F}_n, \mathrm{P}), \quad W \mapsto \frac{\mathcal{D}_{\text{abs}}^{(n)}(W)}{\max\limits_{0\leq \ell\leq n} \left\{W^{(\ell)}\right\}} = 1 - \frac{W^{(n)}}{\max\limits_{0\leq \ell\leq n} \left\{W^{(\ell)}\right\}}, \tag{3.2b}$$

for $n = 0, 1, \dots, N$, which has values in $[0, 1)$ a.s.

3.1.2 State of the art

In 1993, Grossman and Zhou [23] studied an optimization problem under a drawdown constraint, where the maximal relative drawdown is used. The market model therein consists of one risk-free asset and one risky asset, which is modeled by a geometric Brownian motion with drift. Two years later, Cvitanić and Karatzas [11] extend the results for a more general setting which also allows several risky assets. In 2013, Cherny and Obłój [10] make further generalizations by using abstract semimartingale financial market and more general utility and risk functions.

Chekhlov, Uryasev, and Zabarankin [8] studied the absolute drawdown from Definition 3.1.1 in case of a single path as in Example 2.1.7, i.e. without uncertainty. They use the uncompounded variant for the x-portfolio as in Example 2.2.33, i.e. without reinvestments, and introduce the risk measure named conditional drawdown at risk (CDaR) from the drawdown curve. For this, they take a continuous version of the conditional value of risk for the values $\mathcal{D}^{(0)}_{\text{abs}}(\mathcal{W}), \dots, \mathcal{D}^{(N)}_{\text{abs}}(\mathcal{W})$, where $\mathcal{W} = \mathcal{W}(v_{\text{cm}}(\boldsymbol{\theta}))$ for $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ from Example 2.2.33. The average and the maximum drawdown are two special cases for this risk measure. For the portfolio optimization problem they use the value $\mathcal{W}^{(N)} = \mathcal{W}^{(N)}(v_{\text{cm}}(\boldsymbol{\theta}))$ as utility function, which is maximized such that the risk is below a given threshold. Since the conditional value at risk in this case can be written as the solution of a linear programming problem, the portfolio optimization also can be reformulated as a linear optimization problem, which is also stated in [8].

The same authors extend the CDaR in [9] for the case with multiple scenario as in Example 2.1.8 (b). This is done very similar, but now, the quantities $\mathcal{D}^{(n)}_{\text{abs}}(\mathcal{W})$ for $n = 0, 1, \dots, N$ are discrete random variables with uncertainty in a finite probability space. Again a variant of the conditional value at risk is used where the random variables $\mathcal{D}^{(0)}_{\text{abs}}(\mathcal{W}), \dots, \mathcal{D}^{(N)}_{\text{abs}}(\mathcal{W})$ are merged into just one random variable, which takes values of the N random variables with same probability. They call the resulting risk measure the conditional drawdown (CDD), and show some crucial properties like convexity and positive homogeneity closely related to the general deviation measure of Rockafellar, Uryasev and Zabarankin [47], see also Definition 2.3.6. However, it seems that the expression CDD only appears in the manuscript [9] and is called CDaR, even from the authors, when this manuscript is cited. Therefore, this risk measure is called CDaR in the following. Again the portfolio optimization problem is reduced as a linear programming problem, whereas now the utility function is the expectation of $\mathcal{W}^{(N)}$ from Example 2.2.33. In [9] they also propose a mixed conditional drawdown which is a convex combination of CDaR for different confidence level according to a risk profile, which can be obtained from the risk preferences of the investor. Krokhmal, Uryasev and Zrazhevsky [29] discussed an application of CDaR and CVaR for a real market.

About 9 years later in 2014, Zabarankin, Pavlikov, and Uryasev [55] give a slightly different definition of CDaR [8, 9] in the multiple scenario case. They propose to use the absolute drawdown on a rolling frame of size $\tau \in \mathbb{N}$, i.e. they use

$$\mathcal{D}^{(n)}_{\text{abs},\tau}(\mathcal{W}) := \max_{n_\tau \leq \ell \leq n} \left\{ \mathcal{W}^{(\ell)} \right\} - \mathcal{W}^{(n)}$$

for $n = 1, \dots, N$, where $n_\tau := \max\{1, n - \tau\}$. Again, the conditional value at risk is taken from these values as in [9], which defines the risk, and the utility function again is defined by the expectation of $\mathcal{W}^{(N)}$ from Example 2.2.33. For the corresponding portfolio optimization problem, they state a theorem for the beta from capital asset pricing model (CAPM) with CDaR.

In 2017, Goldberg and Mahmoud [20], introduce the so-called conditional expected Drawdown (CED), which is proven to be a multi-path variant of the general deviation measure. CED is similarly defined as CDaR. While CDaR is the conditional value at risk for the absolute drawdown at each point in time, the CED is the conditional value at risk of the path-wise maximum absolute drawdown of all scenarios. In addition, [20] concentrates on the continuous time case and more general stochastic processes, instead of a discrete and finite probability space as in [8, 9].

In the same year, Mahmoud [33] introduced a notation of path dependent risk measure which concentrates on the temporal component. Such risk measures do not quantify the risk in the sense of an absolute or relative amount of money, but on the risk in time, e.g. the time needed to get from a (local) maximum of the equity back to this level ("peak-to-peak") or how long the drawdown phases are ("peak-to-trough"). She gives axiomatic definitions, discusses their properties, and also gives some examples, mostly concerning drawdown and maximum drawdown.

Maier-Paape and Zhu [35] studied the expected value of the logarithm of the relative drawdown in a finite probability space and call the resulting risk measure the "current drawdown". They defined this risk measure based on a one-period market and simulate the multi-period market by $K \in \mathbb{N}$ iid drawings. In [35], the authors give reasonable properties according to the axioms in Definition 2.3.7 like convexity. Furthermore, an approximation – some kind of a linearization – is given which is even positive homogeneous. In the subsequent sections, similar results are proven for the more general multi-period market by using very different techniques as in [35] and also a comparison to this work is given. Moreover, it turns out that the approximation of the risk measure for the relative drawdown is strongly related to the absolute drawdown.

3.1.3 Relation between absolute and relative drawdown

The drawdown is often given as percentage relative to the maximum value of wealth, i.e. the value of the relative drawdown. One reason is that this value has no unit, like USD, and gives a percentaged loss. The absolute drawdown has a unit, namely the currency of wealth, where its size depends on the value of the wealth itself. A drawdown of size \$ 1000 is meaningless, if the wealth is not given, while a drawdown of size 10 % directly implies, that 10 % of the maximum wealth got lost. On the other side, the absolute variant has some preferable properties and can be seen as the maximum of linear functions. It is much easier to work with a linear function rather than a quotient as in the definition of the relative version. Therefore, a relation between both versions would be meaningful. The absolute and relative versions of the drawdown processes in Definition 3.1.1 are connected by

$$-\ln\left(1 - \mathcal{D}^{(n)}_{\text{rel}}(W)\right) = \max_{0 \le \ell \le n} \left\{\ln\left(W^{(\ell)}\right)\right\} - \ln\left(W^{(n)}\right) = \mathcal{D}^{(n)}_{\text{abs}}(\ln(W)), \tag{3.3}$$

for $n = 0, 1, \dots, N$ and $W \in \mathcal{L}^2(N; \mathbb{R}_{>0})$, where $\ln(W) := (\ln(W^{(n)}))_{0 \le n \le N}$. Note, that the logarithm of wealth has also been used in the context of TWR for the utility function, see Theorem 2.3.12, which is therefore related to this case. Using this transformation, it is possible to generate an absolute drawdown process out of a relative one. Nevertheless, the right hand side of (3.3) is still nonlinear in W because of the logarithm, which needs to be eliminated, cf. (2.51).

In most cases, the definition of the underlying wealth process, see (2.7) and (2.9), is related to the choice of the absolute or relative variant for the drawdown. Therefore, it is reasonable to first find a relation between the two main representations for the equity curve of the portfolio regarding (2.7) and (2.9), which involves the logarithm.

For this, let $N \in \mathbb{N}$ and $t^{(1)}, \dots, t^{(N)}$ be random variables with values in $(-1, \infty)$ to model rates of return, e.g. $t^{(n)} = T^{(n)}\boldsymbol{\varphi}^{(n)}$. The corresponding wealth process is then given by $W_{\text{rel}}(t) := (W_{\text{rel}}^{(n)}(t))_{0 \le n \le N}$, where $W_{\text{rel}}^{(0)} \in \mathbb{R}_{>0}$ is fixed and

$$W_{\text{rel}}^{(n)}(t) := W_{\text{rel}}^{(0)} \cdot \prod_{k=1}^{n} \left(1 + t^{(k)}\right) = W_{\text{rel}}^{(n-1)}(t) \cdot \left(1 + t^{(n)}\right) \tag{3.4}$$

for $n = 1, \dots, N$, cf. (2.9). Then, as in (2.50), it is

$$\ln\left(W_{\text{rel}}^{(n)}(t)\right) = \ln\left(W_{\text{rel}}^{(0)}\right) + \sum_{k=1}^{n} \ln\left(1 + t^{(k)}\right) \tag{3.5}$$

for $n = 1, \dots, N$, which corresponds to the wealth process with absolute returns of size $\ln(1 + t^{(n)})$ for $n = 1, \dots, N$ and initial wealth $\ln(W_{\text{rel}}^{(0)})$, cf. (2.7). Define the absolute wealth $W_{\text{abs}}(t) := (W_{\text{abs}}^{(n)}(t))_{0 \le n \le N}$ with $W_{\text{abs}}^{(0)} \in \mathbb{R}_{>0}$ fixed and

$$W_{\text{abs}}^{(n)}(t) := W_{\text{abs}}^{(0)} + \sum_{k=1}^{n} t^{(k)} = W_{\text{abs}}^{(n-1)}(t) + t^{(n)} \tag{3.6}$$

for $n = 1, \dots, N$. Assume $W_{\text{abs}}^{(0)} = \ln(W_{\text{rel}}^{(0)})$. Then, it is $\ln(1 + t^{(n)}) \le t^{(n)}$ a.s. and (3.5) gives

$$\ln\left(W_{\text{rel}}^{(n)}(t)\right) \le W_{\text{abs}}^{(n)}(t) \quad \text{a.s.} \tag{3.7}$$

for $n = 1, \dots, N$, where the upper bound is the linear approximation of $\ln(W_{\text{rel}}^{(n)}(t))$ for $t^{(1)}, \dots, t^{(n)}$ near zero. From this, it is possible to bound the relative wealth version by a linear version. To use this for estimating the relative and absolute drawdown, the following monotonicity criterion for the absolute drawdown is needed.

Proposition 3.1.2 (Monotonicity for absolute drawdown) Let $W, V \in \mathcal{L}^2(N)$ be arbitrary such that $W^{(n)} - W^{(n-1)} \ge V^{(n)} - V^{(n-1)}$ a.s. for all $n = 1, \dots, N$. Then $\mathcal{D}_{\text{abs}}^{(n)}(W) \le \mathcal{D}_{\text{abs}}^{(n)}(V)$ a.s. for $n = 0, 1, \dots, N$.

Proof. For all $0 \le \ell \le n \le N$ it is

$$W^{(n)} - W^{(\ell)} = \sum_{k=\ell+1}^{n} W^{(k)} - W^{(k-1)} \ge \sum_{k=\ell+1}^{n} V^{(k)} - V^{(k-1)} = V^{(n)} - V^{(\ell)}.$$

It follows that

$$\mathcal{D}_{\text{abs}}^{(n)}(W) = \max_{0 \le \ell \le n} \left\{W^{(\ell)} - W^{(n)}\right\} \le \max_{0 \le \ell \le n} \left\{V^{(\ell)} - V^{(n)}\right\} = \mathcal{D}_{\text{abs}}^{(n)}(V)$$

which completes the proof. □

From this, the following inequalities can be shown.

Proposition 3.1.3 For all $W \in \mathcal{L}^2(N; \mathbb{R}_{>0})$ it is

$$0 \le \mathcal{D}^{(n)}_{\mathrm{rel}}(W) \le -\ln\left(1 - \mathcal{D}^{(n)}_{\mathrm{rel}}(W)\right) \quad \text{a.s.}$$

for $n = 1, \dots, N$.

Let $t^{(n)} \in \mathcal{L}^0(\Omega, \mathcal{F}_n, \mathrm{P})$ with $t^{(n)} > -1$ a.s. for $n = 1, \dots, N$ such that $W_{\mathrm{abs}}(t) \in \mathcal{L}^2(N)$ from (3.6) with arbitrary $W^{(0)}_{\mathrm{abs}} \in \mathbb{R}$ and $W_{\mathrm{rel}}(t) \in \mathcal{L}^2(N; \mathbb{R}_{>0})$ from (3.4) with arbitrary $W^{(0)}_{\mathrm{rel}} \in \mathbb{R}_{>0}$. Then

$$0 \le \mathcal{D}^{(n)}_{\mathrm{abs}}(W_{\mathrm{abs}}(t)) \le -\ln\left(1 - \mathcal{D}^{(n)}_{\mathrm{rel}}(W_{\mathrm{rel}}(t))\right) \quad \text{a.s.} \tag{3.8}$$

Proof. From properties of the logarithm and because of $0 \le \mathcal{D}^{(n)}_{\mathrm{rel}}(W) < 1$ a.s., it follows that

$$-\ln\left(1 - \mathcal{D}^{(n)}_{\mathrm{rel}}(W)\right) = -\ln\left(1 + \left(-\mathcal{D}^{(n)}_{\mathrm{rel}}(W)\right)\right) \ge \mathcal{D}^{(n)}_{\mathrm{rel}}(W) \ge 0.$$

To show the second assertion, assume that $W^{(0)}_{\mathrm{abs}} = \ln(W^{(0)}_{\mathrm{rel}})$. For $r^{(n)} := \ln(1 + t^{(n)})$, $n = 1, \dots, N$, equation (3.5) yields $\ln(W_{\mathrm{rel}}(t)) = W_{\mathrm{abs}}(r)$ and because of (3.3) it follows

$$-\ln\left(1 - \mathcal{D}^{(n)}_{\mathrm{rel}}(W_{\mathrm{rel}}(t))\right) = \mathcal{D}^{(n)}_{\mathrm{abs}}(\ln(W_{\mathrm{rel}}(t))) = \mathcal{D}^{(n)}_{\mathrm{abs}}(W_{\mathrm{abs}}(r)).$$

In addition, for $n = 1, \dots, N$ it is

$$W^{(n)}_{\mathrm{abs}}(r) - W^{(n-1)}_{\mathrm{abs}}(r) = \ln\left(1 + t^{(n)}\right) \le t^{(n)} = W^{(n)}_{\mathrm{abs}}(t) - W^{(n-1)}_{\mathrm{abs}}(t).$$

Proposition 3.1.2 then implies that $\mathcal{D}^{(n)}_{\mathrm{abs}}(W_{\mathrm{abs}}(t)) \le \mathcal{D}^{(n)}_{\mathrm{abs}}(W_{\mathrm{abs}}(r))$. From Definition 3.1.1 of $\mathcal{D}^{(n)}_{\mathrm{abs}}$ and from W_{abs} in (3.6) it follows that $\mathcal{D}^{(n)}_{\mathrm{abs}}(W_{\mathrm{abs}})$ is independent on the initial wealth $W^{(0)}_{\mathrm{abs}}$. Hence, this value can be chosen arbitrary, which completes the proof. □

3.1.4 Properties

Because of (3.3), it is meaningful to first focus on the absolute drawdown. Basic properties, which are related to some axioms in Definition 2.3.1, for the absolute drawdown read as follows, see also Chekhlov, Uryasev and Zabarankin [9, Proposition 3.1] for the case of single-paths.

Proposition 3.1.4 For $n = 0, 1, \ldots, N$, the nth absolute drawdown $\mathcal{D}^{(n)}_{\text{abs}}$ of Definition 3.1.1 has the following properties:

(i) $\mathcal{D}^{(n)}_{\text{abs}}(0) = 0$ and $\mathcal{D}^{(n)}_{\text{abs}}(W) \geq 0$ for all $W \in \mathcal{L}^2(N)$,

(ii) $\mathcal{D}^{(n)}_{\text{abs}}(W + C) = \mathcal{D}^{(n)}_{\text{abs}}(W)$ for all $W, C \in \mathcal{L}^2(N)$, where $C \equiv (c, \ldots, c) \in \mathbb{R}^{N+1}$ is constant,

(iii) $\mathcal{D}^{(n)}_{\text{abs}}(\lambda W) = \lambda \mathcal{D}^{(n)}_{\text{abs}}(W)$ for all $W \in \mathcal{L}^2(N)$ and all $\lambda > 0$,

(iv) $\mathcal{D}^{(n)}_{\text{abs}}(W + V) \leq \mathcal{D}^{(n)}_{\text{abs}}(W) + \mathcal{D}^{(n)}_{\text{abs}}(V)$ for all $W, V \in \mathcal{L}^2(N)$,

(v) $\mathcal{D}^{(n)}_{\text{abs}}$ is convex.

Proof. The case $n = 0$ is trivial, because $\mathcal{D}^{(0)}_{\text{abs}} \equiv 0$ a.s. Let now $n \in \{1, \ldots, N\}$ be arbitrary. Properties (i), (ii) and (iii) directly follow from definition of $\mathcal{D}^{(n)}_{\text{abs}}$ and property (iv) holds true, because

$$\begin{aligned}
\mathcal{D}^{(n)}_{\text{abs}}(W + V) &= \max_{0 \leq \ell \leq n} \left\{ W^{(\ell)} + V^{(\ell)} \right\} - W^{(n)} - V^{(n)} \\
&\leq \max_{0 \leq \ell \leq n} \left\{ W^{(\ell)} \right\} + \max_{0 \leq \ell \leq n} \left\{ V^{(\ell)} \right\} - W^{(n)} - V^{(n)} = \mathcal{D}^{(n)}_{\text{abs}}(W) + \mathcal{D}^{(n)}_{\text{abs}}(V).
\end{aligned}$$

Convexity (v) is a direct consequence of (iii) and (iv), cf. (2.41). □

Note, that the absolute drawdown $\mathcal{D}_{\text{abs}}$ is a stochastic process and each time step is a random variable. Therefore, the reasonable properties in Proposition 3.1.4 are understood to hold true almost surely. There are different possibilities to build a risk measure from this drawdown process, which inherits the properties of $\mathcal{D}^{(n)}_{\text{abs}}$. Appropriate applications of the concept of CVaR, see Example 2.3.17, combined with the absolute drawdown can be found in [9] and Goldberg and Mahmoud [20].

The version in [9] can be interpreted as follows: Let $(\Omega', \Sigma', \mathrm{P}')$ with $\omega' = (n, \omega) \in \Omega' := \{1, \ldots, N\} \times \Omega$ and power set Σ' be an extended probability space. Define the random variable X by $X(\omega') = X(n, \omega) := \mathcal{D}^{(n)}_{\text{abs}}(\omega)$ with $\mathrm{P}'(\{(n, \omega)\}) = \mathrm{P}(\{\omega\})/N$ for $n = 1, \ldots, N$ and $\omega \in \Omega$. Then, X can be seen as the union of $\mathcal{D}^{(1)}_{\text{abs}}, \ldots, \mathcal{D}^{(N)}_{\text{abs}}$. In [9], they discuss the conditional value at risk applied to X, where the sample space Ω is discrete and finite. In [20], Goldberg and Mahmoud discuss the conditional value at risk applied to the maximum drawdown, which is defined by

$$\mathcal{D}_{\text{abs,max}} \colon \mathcal{L}^2(N) \to \mathcal{L}^2, \quad W \mapsto \max_{1 \leq n \leq N} \left\{ \mathcal{D}^{(n)}_{\text{abs}}(W) \right\}.$$

Both variants define non-negative risk measures, which are translation invariant ($\mathcal{B}$1b) with value zero for all constants, positive homogeneous ($\mathcal{B}$2) and proper convex ($\mathcal{B}$3b) (proper because e.g. the constant zero is in the domains), see Definition 2.3.1.

Another value of interest is the expected value of the absolute drawdown after the last time step. By Maier-Paape and Zhu [35, Theorem 9] for the case of a finite probability space a related risk measure is constructed as linearization of a relative drawdown.

Corollary 3.1.5 The functional

$$\overline{\mathcal{D}}^{(N)}_{\mathrm{abs}} \colon \mathcal{L}^2(N) \to [0,\infty), \quad W \mapsto \mathrm{E}\left[\mathcal{D}^{(N)}_{\mathrm{abs}}(W)\right]$$

is non-negative, translation invariant ($\mathcal{B}$1b) with $\overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}(C) = 0$, positive homogeneous ($\mathcal{B}$2) and proper convex ($\mathcal{B}$3b), see Definition 2.3.1.

Proof. Obviously, it is $\overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}(W) < \infty$ for all $W \in \mathcal{L}^2(N)$. The statement then is a direct consequence of Proposition 3.1.4 because of the linearity and monotonicity of the expectation. □

Another consequence is the following for a given x-portfolio generating function.

Corollary 3.1.6 For market model $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ such that $T^{(n)} \in \mathcal{L}^2(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$ for $n = 1,\dots,N$ let the x-portfolio generating function v_{cm} be defined as in Example 2.2.33 (invested amount of money constant over time). Then, function

$$\mathfrak{d}_{\mathrm{lin}} \colon \mathbb{R}^{M+1} \to [0,\infty),\ \boldsymbol{\theta} \mapsto \overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}(\mathcal{W}(v_{\mathrm{cm}}(\boldsymbol{\theta}))) = \mathcal{W}^{(0)}\,\mathrm{E}\left[\max_{0\le \ell\le N}\left\{\sum_{n=1}^{\ell} T^{(n)}\boldsymbol{\theta}\right\} - \sum_{n=1}^{N} T^{(n)}\boldsymbol{\theta}\right]$$

is non-negative, positive homogeneous ($\mathfrak{b}$2) and closed proper convex ($\mathfrak{b}$3b), see Definition 2.3.2.

If in addition S has no nontrivial risk-free x-portfolio and $S^{(n)}_0 = S^{(n-1)}_0$ for all $n = 1,\dots,N$, then for all $\boldsymbol{\theta} \in \Phi_\beta := \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} : \sum_{i=0}^{M} \varphi_i = \beta\}$ for fixed $\beta \in \mathbb{R}$ it is $\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) = 0$ if and only if $\boldsymbol{\theta} = (\beta,0,\dots,0)^\intercal$.

Proof. Function $\mathfrak{d}_{\mathrm{lin}}$ is finite on $\mathbb{R}^{M+1}$, because $T^{(n)} \in \mathcal{L}^2(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$, and consequently $\mathrm{dom}(\mathfrak{d}_{\mathrm{lin}}) = \{\boldsymbol{\theta} \in \mathbb{R}^{M+1} : \mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) < \infty\} = \mathbb{R}^{M+1}$. The x-portfolio generating function v_{cm} is linear and with (2.40)

$$\begin{aligned}\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) = \overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}(\mathcal{W}(v_{\mathrm{cm}}(\boldsymbol{\theta}))) &= \overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}\left(\mathcal{W}(v_{\mathrm{cm}}(\boldsymbol{\theta})) - \mathcal{W}^{(0)}\right)\\ &= \mathcal{W}^{(0)}\overline{\mathcal{D}}^{(N)}_{\mathrm{abs}}\left(\left[\sum_{k=1}^{n} T^{(k)}\boldsymbol{\theta}\right]_{0\le n\le N}\right).\end{aligned}$$

Hence, Lemma 2.3.4 (a) together with Corollary 3.1.5 imply that the function is positive homogeneous and proper convex. Since $\mathrm{dom}(\mathfrak{d}_{\mathrm{lin}}) = \mathbb{R}^{M+1}$, the function is continuous in $\mathbb{R}^{M+1}$ and therefore must be closed proper convex, see Theorems A.1.2 and A.1.3 in the appendix.

For the special case when S has no nontrivial risk-free x-portfolio and $S^{(n)}_0 = S^{(n-1)}_0$ for all $n = 1,\dots,N$, it also must be $T^{(n)}_0 = 0$ for all $n = 1,\dots,N$. Then, Theorem 2.2.23 (d) implies, that $T^{(N)}\boldsymbol{\theta} < 0$ with positive probability for all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ with $\widehat{\boldsymbol{\theta}} \neq 0$. Hence, the expression within the expectation in the definition of $\mathfrak{d}_{\mathrm{lin}}$ is positive with positive probability and therefore $\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) > 0$ if $\widehat{\boldsymbol{\theta}} \neq 0$. Of course, it is $\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) = 0$ if $\widehat{\boldsymbol{\theta}} = 0$. The fact, that $\widehat{\boldsymbol{\theta}} = 0$ for $\boldsymbol{\theta} \in \Phi_\beta$ if and only if $\boldsymbol{\theta} = (\beta,0,\dots,0)^\intercal$, completes the proof. □

A corresponding utility function for the same x-portfolio generating function has been defined in (2.51) denoted by $\mathfrak{u}_{\mathrm{lin}}$. Inserting this into the definition of $\mathfrak{d}_{\mathrm{lin}}$ yields

$$\mathfrak{u}_{\mathrm{lin}}(\boldsymbol{\theta}) + \frac{1}{\mathcal{W}^{(0)}}\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\theta}) = \mathrm{E}\left[\max_{0\le\ell\le N}\left\{\sum_{n=1}^{\ell} T^{(n)}\boldsymbol{\theta}\right\}\right].$$

Inequality (3.7) reminds on the inequality between ln(TWR) and its linearization, see (2.51). Hence, in the situation of Theorem 2.3.12 the following holds.

Theorem 3.1.7 (Properties of $\mathfrak{d}_{\mathrm{ln}}$) Let $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ be the market model. Assume the x-portfolio generating function v_{twr} is defined as in Example 2.2.31 (constant φ-portfolio, i.e. constant weights) with convex and admissible set

$$\Phi_{\mathrm{twr}} = \left\{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \,:\, 1 + T^{(n)}\boldsymbol{\varphi} > 0 \text{ a.s. for } n = 1,\dots,N\right\}$$

from (2.32). Then, the function

$$\mathfrak{d}_{\mathrm{ln}}\colon \mathbb{R}^{M+1} \to [0,\infty], \qquad \boldsymbol{\varphi} \mapsto \begin{cases} \mathrm{E}\left[-\ln\left(1 - \mathcal{D}^{(N)}_{\mathrm{rel}}(\mathcal{W}(v_{\mathrm{twr}}(\boldsymbol{\varphi})))\right)\right], & \boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}, \\ \infty, & \boldsymbol{\varphi} \notin \Phi_{\mathrm{twr}}, \end{cases}$$

is non-negative and proper convex ($\mathfrak{b}$3b) with convex and non-empty effective domain $\mathrm{dom}(\mathfrak{d}_{\mathrm{ln}}) = \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \,:\, \mathfrak{d}_{\mathrm{ln}}(\boldsymbol{\varphi}) < \infty\}$, and it is

$$0 \le \frac{1}{\mathcal{W}^{(0)}}\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\varphi}) \le \mathfrak{d}_{\mathrm{ln}}(\boldsymbol{\varphi})$$

for all $\boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ with $\mathfrak{d}_{\mathrm{lin}}$ from Corollary 3.1.6. The closure $\bar{\mathfrak{d}}_{\mathrm{ln}}$ of $\mathfrak{d}_{\mathrm{ln}}$, see (A.2) in the appendix, is closed proper convex.

If in addition S has no nontrivial risk-free x-portfolio, then the following holds true, where $\Phi_\beta := \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} \,:\, \sum_{i=0}^{M}\varphi_i = \beta\}$ for fixed $\beta > 0$:

(a) If $S_0^{(n)} = S_0^{(n-1)}$ for all $n = 1,\dots,N$, then $\mathfrak{d}_{\mathrm{ln}}$ and $\bar{\mathfrak{d}}_{\mathrm{ln}}$ restricted to the convex set $\mathrm{dom}(\mathfrak{d}_{\mathrm{ln}}) \cap \Phi_\beta$ are strictly convex ($\mathfrak{b}$3c). For all $\boldsymbol{\varphi} \in \Phi_\beta$ it is $\mathfrak{d}_{\mathrm{ln}}(\boldsymbol{\varphi}) = 0$ if and only if $\boldsymbol{\varphi} = (\beta, 0, \dots, 0)^{\intercal}$.

(b) If $S_0^{(n)} = S_0^{(n-1)}$ for all $n = 1,\dots,N$, then $\mathfrak{d}_{\mathrm{ln}}$ and $\bar{\mathfrak{d}}_{\mathrm{ln}}$ restricted to the convex set $\mathrm{dom}(\mathfrak{d}_{\mathrm{ln}}) \cap (\{0\}\times\mathbb{R}^M)$ are strictly convex ($\mathfrak{b}$3c). For all $\boldsymbol{\varphi} \in \{0\}\times\mathbb{R}^M$ it is $\mathfrak{d}_{\mathrm{ln}}(\boldsymbol{\varphi}) = 0$ if and only if $\boldsymbol{\varphi} = 0$.

(c) The sublevel set $\mathcal{B}_{\bar{\mathfrak{d}}_{\mathrm{ln}},\Phi_\beta}(r)$, see Definition 2.4.5, is compact and convex for all $r \ge 0$, and empty for $r < 0$ (and thus also compact).

Proof. Another representation of $\mathfrak{d}_{\mathrm{ln}}$ directly follows from (3.3), namely

$$\mathfrak{d}_{\mathrm{ln}}(\boldsymbol{\varphi}) = \mathrm{E}\left[-\ln\left(1 - \mathcal{D}^{(N)}_{\mathrm{rel}}(\mathcal{W}(v_{\mathrm{twr}}(\boldsymbol{\varphi})))\right)\right] = \mathrm{E}\left[\mathcal{D}^{(N)}_{\mathrm{abs}}(\ln(\mathcal{W}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))))\right] \tag{3.9}$$

for $\varphi \in \Phi_{\mathrm{twr}}$, which in addition implies that $\mathfrak{d}_{\mathrm{ln}} \ge 0$.

The inequality $\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\varphi})/\mathcal{W}^{(0)} \leq \mathfrak{d}_{\ln}(\boldsymbol{\varphi})$ follows from Proposition 3.1.3 because of the following: The wealth for $\mathfrak{d}_{\mathrm{lin}}$ is given by (2.40) in Example 2.2.33 (invested amount of money constant over time) and therefore

$$\frac{\mathcal{W}^{(n)}(v_{\mathrm{cm}}(\boldsymbol{\varphi}))}{\mathcal{W}^{(0)}} = 1 + \sum_{k=1}^{n} T^{(k)}\boldsymbol{\varphi} = W_{\mathrm{abs}}^{(n)}(T\boldsymbol{\varphi})$$

for W_{abs} from (3.6) with $W_{\mathrm{abs}}^{(0)} := 1$ and the wealth for $\mathfrak{d}_{\ln}$ is given by (2.33) in Example 2.2.31 (constant φ-portfolio, i.e. constant weights) with

$$\mathcal{W}^{(n)}(v_{\mathrm{twr}}(\boldsymbol{\varphi})) = \mathcal{W}^{(0)} \prod_{k=1}^{n} \left(1 + T^{(k)}\boldsymbol{\varphi}\right) = W_{\mathrm{rel}}^{(n)}(T\boldsymbol{\varphi})$$

for W_{rel} from (3.4) with $W_{\mathrm{rel}}^{(0)} := \mathcal{W}^{(0)}$. Since $\overline{\mathcal{D}}_{\mathrm{abs}}^{(N)}$ is positive homogeneous, see Corollary 3.1.5, it follows that $\mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\varphi})/\mathcal{W}^{(0)} = \overline{\mathcal{D}}_{\mathrm{abs}}^{(N)}(\mathcal{W}(v_{\mathrm{cm}}(\boldsymbol{\varphi}))/\mathcal{W}^{(0)})$. The inequality (3.8) in Proposition 3.1.3 together with linearity of the expectation gives the inequality $0 \leq \mathfrak{d}_{\mathrm{lin}}(\boldsymbol{\varphi})/\mathcal{W}^{(0)} \leq \mathfrak{d}_{\ln}(\boldsymbol{\varphi})$.

To show convexity ($\mathfrak{b}$3b), due to (3.9) it suffices to show that the function f defined by $f(\boldsymbol{\varphi}) := \mathcal{D}_{\mathrm{abs}}^{(N)}(\ln(\mathcal{W}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))))$ is convex, because the expectation is linear and monotone. From (3.1), (2.33) and the properties of the logarithm, it follows that

$$\begin{aligned} f(\boldsymbol{\varphi}) = \mathcal{D}_{\mathrm{abs}}^{(N)}(\ln(\mathcal{W}(v_{\mathrm{twr}}(\boldsymbol{\varphi})))) &= \max_{0\leq n\leq N} \left\{\ln\left(\mathcal{W}^{(n)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))\right) - \ln\left(\mathcal{W}^{(N)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))\right)\right\} \\ &= \max_{0\leq n\leq N-1} \max\left\{0, -\sum_{k=n+1}^{N} \ln\left(1 + T^{(k)}\boldsymbol{\varphi}\right)\right\} \geq 0. \end{aligned}$$

Since $1 + T^{(k)}\boldsymbol{\varphi}$ is linear in $\boldsymbol{\varphi}$ and ln is strictly concave, f must be convex as maximum and sum of convex functions. In addition, $\mathrm{dom}(\mathfrak{d}_{\ln})$ is convex and it is $0 \in \mathrm{dom}(\mathfrak{d}_{\ln})$ because $0 \in \Phi_{\mathrm{twr}}$ and $\mathfrak{d}_{\ln}(0) = 0$. Hence, $\mathfrak{d}_{\ln}$ is proper convex. Of course, its closure, which always exists, is closed proper convex.

Next the properties for the special case when S has no nontrivial risk-free x-portfolio and $S_0^{(n)} = S_0^{(n-1)}$ for all $n = 1, \dots, N$ are shown. Properties (a) and (b) are shown simultaneously. Because of $S_0^{(n)} = S_0^{(n-1)}$, it must be $T_0^{(n)} = 0$ for all $n = 1, \dots, N$. Then, Theorem 2.2.23 (d) implies, that $T^{(N)}\boldsymbol{\varphi} < 0$ with positive probability for all $\boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}$ with $\widehat{\boldsymbol{\varphi}} \neq 0$. Consequently, it must be $f(\boldsymbol{\varphi}) > 0$ with positive probability and therefore $\mathfrak{d}_{\ln}(\boldsymbol{\varphi}) > 0$ if $\widehat{\boldsymbol{\varphi}} \neq 0$. Of course, it is $f(\boldsymbol{\varphi}) = 0$ if $\widehat{\boldsymbol{\varphi}} = 0$. This gives the second assertion in (a) and also in (b).

The strict convexity is shown similar to the proof of strict concavity in Theorem 2.3.12. Let $\boldsymbol{\varphi}, \boldsymbol{\theta} \in \mathrm{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta$ or $\boldsymbol{\varphi}, \boldsymbol{\theta} \in \mathrm{dom}(\mathfrak{d}_{\ln}) \cap (\{0\} \times \mathbb{R}^M)$ be arbitrary with $\boldsymbol{\varphi} \neq \boldsymbol{\theta}$. In both cases, it must be $\widehat{\boldsymbol{\varphi}} \neq \widehat{\boldsymbol{\theta}}$. Theorem 2.2.23 (d) applied to $\boldsymbol{\varphi} - \boldsymbol{\theta}$, for which it is $\widehat{\boldsymbol{\varphi}} - \widehat{\boldsymbol{\theta}} \neq 0$, gives $\mathrm{P}(T^{(N)}\boldsymbol{\varphi} \neq T^{(N)}\boldsymbol{\theta}) \geq \mathrm{P}(T^{(N)}(\boldsymbol{\varphi} - \boldsymbol{\theta}) < 0) > 0$, using again that $T_0^{(n)} = 0$ for all $n = 1, \dots, N$. Then, for all $\lambda \in (0, 1)$ it is

$$\ln\left(1 + T^{(N)}(\lambda\boldsymbol{\varphi} + (1-\lambda)\boldsymbol{\theta})\right) > \lambda \ln\left(1 + T^{(N)}\boldsymbol{\varphi}\right) + (1-\lambda)\ln\left(1 + T^{(N)}\boldsymbol{\theta}\right)$$

with positive probability. Note that "$\geq$" instead of "$>$" always holds true because of concavity of ln. Hence, it is

$$R := f(\lambda\boldsymbol{\varphi} + (1-\lambda)\boldsymbol{\theta}) < \lambda f(\boldsymbol{\varphi}) + (1-\lambda) f(\boldsymbol{\theta}) =: L$$

with positive probability. As in the proof of Theorem 2.3.12, it follows that $\mathrm{E}[L-R] > 0$, i.e.

$$\mathfrak{d}_{\ln}(\lambda\boldsymbol{\varphi} + (1-\lambda)\boldsymbol{\theta}) = \mathrm{E}[R] < \mathrm{E}[L] = \lambda\mathfrak{d}_{\ln}(\boldsymbol{\varphi}) + (1-\lambda)\mathfrak{d}_{\ln}(\boldsymbol{\theta}).$$

Consequently, $\mathfrak{d}_{\ln}$ restricted to $\mathrm{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta$ and also restricted to $\mathrm{dom}(\mathfrak{d}_{\ln}) \cap (\{0\} \times \mathbb{R}^M)$ is strictly convex.

It remains to show (c): The compactness of the sublevel sets follows analog as in the proof of Theorem 2.3.12 (d). Of course, since $\mathfrak{d}_{\ln} \geq 0$ it follows $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\Phi_\beta}(r) = \emptyset$ for all $r < 0$. □

Remark 3.1.8 (Connection to Maier-Paape and Zhu [35]) The situation in [35] corresponds to the case in Example 2.3.10 (c), where the market model is discrete and finite and the rates of returns are iid. The function $\mathfrak{r}_{\mathrm{cur}}$ defined in [35, Theorem 8] equals to $\mathfrak{d}_{\ln}$ in Theorem 3.1.7 in this case. In addition, the approximation $\mathfrak{r}_{\mathrm{cur}X}$ in [35, Theorem 9] equals to $\mathfrak{d}_{\mathrm{lin}}$ in Corollary 3.1.6, which is the absolute drawdown of the linear equity in (2.40).

Hence, most results from Corollary 3.1.6 and Theorem 3.1.7 have been proven in [35] for this special case but using a completely different approach. However, the strict convexity at least on some subspaces has not been proven therein. Similarly, the remaining risk measure constructed from so-called down-trade series and the potential utility measures (see [35, Theorem 2 and Theorem 7]) defined in [35] have completely analog counterparts, and therefore their properties can be proven similarly as above. ♦

The definition of $\mathfrak{d}_{\ln}$ uses the x-portfolio generating function v_{twr} for the case $\boldsymbol{\varphi}^{(n)} \equiv \mathrm{const}$ from Example 2.2.31 (constant weights). The same function is used in the definition of TWR, see Definition 2.3.9, i.e. the same assumptions are used. In addition, there is the usage of logarithm for $\mathfrak{d}_{\ln}$ as well as for $\mathfrak{u}_{\ln}$ in Theorem 2.3.12. Hence, it seems likely that there is a connection between both measures.

Corollary 3.1.9 (Relation between $\mathfrak{d}_{\ln}$ and $\mathfrak{u}_{\ln}$) Assume that $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$ and $T^{(n)} \in \mathcal{L}^1(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$ for $n = 1,\dots,N$. Let $\mathfrak{d}_{\ln}$ be defined as in Theorem 3.1.7 and the expected logarithm of TWR given by $\mathfrak{u}_{\ln}$ from Theorem 2.3.12 with the x-portfolio generating function v_{twr} defined as in Example 2.2.31 (constant weights) and convex and admissible set $\Phi_{\mathrm{twr}} = \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} : 1 + T^{(n)}\boldsymbol{\varphi} > 0 \text{ a.s. for } n = 1,\dots,N\}$ from (2.32). Let

$$\mathrm{TWR}_1^n(\boldsymbol{\varphi}) := \prod_{k=1}^{n} \left(1 + T^{(k)}\boldsymbol{\varphi}\right) = \frac{\mathcal{W}^{(n)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))}{\mathcal{W}^{(0)}} \qquad \text{for } n = 1,\dots,N,$$

with $\mathrm{TWR}_1^0(\boldsymbol{\varphi}) := 1$. Then it is

$$\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) + \mathfrak{d}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\max_{0\leq n\leq N} \{\ln(\mathrm{TWR}_1^n(\boldsymbol{\varphi}))\}\right] \in [0,\infty) \qquad \text{for all } \boldsymbol{\varphi} \in \Phi_{\mathrm{twr}}. \tag{3.10}$$

Furthermore, it is $\mathrm{dom}(\mathfrak{u}_{\ln}) = \mathrm{dom}(\mathfrak{d}_{\ln})$.

Proof. Inserting the wealth (2.33) into the definition of $\mathfrak{d}_{\ln}$, see Theorem 3.1.7 (alternatively see (3.9) together with (3.1)) and canceling out the factor $\mathcal{W}^{(0)}$, which becomes a summand because of the logarithm, yields

$$\mathfrak{d}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\max_{0\le n\le N}\left\{\ln\left(\prod_{k=1}^{n}\left(1+T^{(k)}\boldsymbol{\varphi}\right)\right)\right\} - \ln\left(\prod_{k=1}^{N}\left(1+T^{(k)}\boldsymbol{\varphi}\right)\right)\right],$$

where $\mathcal{W}^{(0)}\prod_{k=1}^{0}(1+T^{(k)}\boldsymbol{\varphi}) = \mathcal{W}^{(0)}\,\mathrm{TWR}_1^0(\boldsymbol{\varphi}) = \mathcal{W}^{(0)}$ for convenience (giving the initial wealth when inserting $n=0$). Then, adding $\mathfrak{u}_{\ln}$ on both sides gives equality (3.10). The positiveness follows, because $\mathrm{TWR}_1^0(\boldsymbol{\varphi}) = 1$ and therefore the maximum must be non-negative. Since $\ln(1+x)\le x$ for all $x>-1$ and $T^{(n)}\in\mathcal{L}^1(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$, it is

$$\mathfrak{u}_{\ln}(\boldsymbol{\varphi}) + \mathfrak{d}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\max_{0\le n\le N}\left\{\sum_{k=1}^{n}\ln\left(1+T^{(k)}\boldsymbol{\varphi}\right)\right\}\right] \le \mathrm{E}\left[\sum_{k=1}^{N}\left|T^{(k)}\boldsymbol{\varphi}\right|\right] < \infty,$$

for all $\boldsymbol{\varphi}\in\Phi_{\mathrm{twr}}$. Hence, from (3.10), if $\mathfrak{d}_{\ln}(\boldsymbol{\varphi})=\infty$ it must be $\mathfrak{u}_{\ln}(\boldsymbol{\varphi})=-\infty$ and vice versa, and therefore $\mathrm{dom}(\mathfrak{u}_{\ln}) = \mathrm{dom}(\mathfrak{d}_{\ln})$. □

3.1.5 Application

After the definitions of the drawdown measures and discussion of their properties in the previous subsection, the goal is now to use theses functions for the optimization problems studied in Section 2.4. For this, first define the setting regarding the new measures.

Setting 3.1.10 Let the following be given for $M, N\in\mathbb{N}$:

(i) The *market model* $S\in\mathcal{L}^2(N;\mathbb{R}_{>0}^{M+1})$, see Definition 2.1.4, is assumed to have no nontrivial risk-free x-portfolio, see Definition 2.2.22. In addition, let $T^{(n)}\in\mathcal{L}^1(\Omega,\mathcal{F}_n,\mathrm{P};\mathbb{R}^{M+1})$, with $T^{(n)}$ from (2.2), for $n=1,\dots,N$.

(ii) The *x-portfolio* (or trading strategy) is defined by the x-portfolio generating function v_{twr} in Example 2.2.31 ($\boldsymbol{\varphi}^{(n)}\equiv\boldsymbol{\varphi}$ is constant in time; constant weights) with convex and admissible set $\Phi_{\mathrm{twr}} = \{\boldsymbol{\varphi}\in\mathbb{R}^{M+1} : 1+T^{(n)}\boldsymbol{\varphi}>0 \text{ a.s. for } n=1,\dots,N\}$ from (2.32).

(iii) The *utility function* is defined by the closed proper concave function $\overline{\mathfrak{u}}_{\ln}\colon\mathbb{R}^{M+1}\to[-\infty,\infty)$ from Theorem 2.3.12, where $\overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\ln\left(\mathrm{TWR}(\boldsymbol{\varphi})\right)\right]$, with $\mathrm{TWR}(\boldsymbol{\varphi}) = \prod_{n=1}^{N}(1+T^{(n)}\boldsymbol{\varphi})$ from Definition 2.3.9, for all $\boldsymbol{\varphi}$ in the relative interior $\mathrm{ri}(\mathrm{dom}(\overline{\mathfrak{u}}_{\ln}))$ and $-\infty$ outside the closure of $\mathrm{dom}(\overline{\mathfrak{u}}_{\ln})$. Note that the values on the relative boundary can be computed analog to (A.2) in the appendix for f replaced by $\mathfrak{u}_{\ln}$ from Theorem 2.3.12.

(iv) The *risk function* is represented by the closed proper convex function $\overline{\mathfrak{d}}_{\ln}\colon\mathbb{R}^{M+1}\to[0,\infty]$ from Theorem 3.1.7.

The corresponding problem states:

Problem 3.1.11 Let Setting 3.1.10 and fixed $\beta > 0$ be given. For $\Phi_\beta := \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} : \sum_{i=0}^{M} \varphi_i = \beta\}$ assume that $B \subset \operatorname{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta = \operatorname{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta \subset \mathbb{R}^{M+1}$ is non-empty, closed and convex. Define two optimization problems:

(a) For fixed $\mu \in \mathbb{R}$, the *minimum relative log drawdown optimization problem* reads

$$\min_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}) \quad \text{subject to } \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) \geq \mu,\ \sum_{i=0}^{M} \varphi_i = \beta. \tag{3.11}$$

(b) For fixed $r \in \mathbb{R}$, the *maximum log TWR optimization problem* reads

$$\max_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) \quad \text{subject to } \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}) \leq r,\ \sum_{i=0}^{M} \varphi_i = \beta. \tag{3.12}$$

One important property needed for most of the results in Section 2.4 is that $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu)$ from Definition 2.4.5 is compact.

Lemma 3.1.12 Let the situation in Problem 3.1.11 be given. Then $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\overline{\mathfrak{u}}_{\ln},B}(r,\mu)$ is compact for all $r, \mu \in \mathbb{R}$.

Proof. Theorem 2.3.12 (d) gives that $\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)$ is compact for all μ. Since $\overline{\mathfrak{d}}_{\ln}$ is closed proper convex, Theorem A.1.2 (a) in the appendix gives that $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\Phi_\beta}(r)$ is closed for all $r \in \mathbb{R}$. Altogether, $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\overline{\mathfrak{u}}_{\ln},B}(r,\mu) = \mathcal{B}_{\overline{\mathfrak{d}}_{\ln},B}(r) \cap \mathcal{B}_{\overline{\mathfrak{u}}_{\ln},B}(\mu) = B \cap \mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\Phi_\beta}(r) \cap \mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)$ is compact because $B \subset \Phi_\beta$ is closed. □

To be able to apply the theory in Section 2.4 for this setting, it is required that Setting 3.1.10 and Problem 3.1.11 are at least related to Setting 2.4.1 and Problem 2.4.2 (in the sense of Lemma 2.4.3).

Remark 3.1.13 (Connection to Setting 2.4.1 and Problem 2.4.2) Assume Setting 3.1.10 and Problem 3.1.11 are given. Let Ψ be the coordinate transformations from Example 2.4.4, i.e. $(\Psi(\boldsymbol{\varphi}))_i := \varphi_i \mathcal{W}^{(0)}/S_i^{(0)}$ for $i = 0, 1, \ldots, M$ and, therefore, $\Psi(\boldsymbol{\varphi}) = v_{\text{twr}}^{(1)}(\boldsymbol{\varphi})$, see Example 2.2.31. According to Theorem 2.4.20, the main theorem for the existence and uniqueness of a solution, define $A_{\widetilde{\beta}} := \{\mathbf{x} \in \mathbb{R}^{M+1} : S^{(0)}\mathbf{x} = \widetilde{\beta}\}$ for $\widetilde{\beta} > 0$. The connection between Settings 2.4.1 and 3.1.10 and between Problems 2.4.2 and 3.1.11 can then easily be established, see Tables 3.1 and 3.2.

Clearly, due to Lemma 3.1.12 $\mathcal{B}_{\mathfrak{r},\mathfrak{u},A}(r,\mu) = \Psi(\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\overline{\mathfrak{u}}_{\ln},B}(r,\mu))$ is compact for all $(r,\mu) \in \mathbb{R}^2$ since Ψ is just a coordinate transformation. Moreover, $\operatorname{dom}(\mathfrak{u}_{\ln}) = \operatorname{dom}(\mathfrak{d}_{\ln})$ because of Corollary 3.1.9.

Note that $\operatorname{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta \neq \emptyset$ (i.e. some non-empty $B \subset \operatorname{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta$ exists) because of the following: Each $\boldsymbol{\varphi} \in \mathbb{R}^{M+1}$ with $\widehat{\boldsymbol{\varphi}} = 0$ and $\varphi_0 \geq 0$ represents an x-portfolio which at most invests into the risk-free asset (which is constant with no uncertainty and non-negative and finite returns) and thus leads to a finite (and even non-negative) TWR and zero drawdown. Furthermore, $\overline{\mathfrak{u}}_{\ln} \geq \mathfrak{u}_{\ln}$ and $\overline{\mathfrak{d}}_{\ln} \leq \mathfrak{d}_{\ln}$, see the note after (A.2) and Remark A.1.5 in the appendix.

Setting 3.1.10	Setting 2.4.1
(i) $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$ has no nontrivial risk-free x-portfolio and $T^{(n)} \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ for $n = 1, \dots, N$.	(i) $S \in \mathcal{L}^2(N; \mathbb{R}^{M+1}_{>0})$; no further assumptions required.
(ii) v_{twr} with convex domain $\Phi_{\mathrm{twr}} \neq \emptyset$.	(ii) $v := v_{\mathrm{twr}} \circ \Psi^{-1}$ with convex domain $\Psi(\Phi_{\mathrm{twr}}) \neq \emptyset$.
(iii) $\overline{\mathfrak{u}}_{\ln}$ closed proper concave function.	(iii) $\mathfrak{u} := \overline{\mathfrak{u}}_{\ln} \circ \Psi^{-1}$ (closed) proper concave function.
(iv) $\overline{\mathfrak{d}}_{\ln}$ closed proper convex function.	(iv) $\mathfrak{r} := \overline{\mathfrak{d}}_{\ln} \circ \Psi^{-1}$ (closed) proper convex function.
Domain: $B \subset \mathrm{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta$ closed and convex.	$A := \Psi(B) \subset A_{\widetilde{\beta}} \subset \mathbb{R}^{M+1}$ with $\widetilde{\beta} = \mathcal{W}^{(0)}\beta$ non-empty and convex.
$B \cap \mathrm{dom}(\overline{\mathfrak{u}}_{\ln}) \cap \mathrm{dom}(\overline{\mathfrak{d}}_{\ln}) \neq \emptyset$.	$A \cap \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r}) \neq \emptyset$.

Table 3.1.: Setting 3.1.10 implies Setting 2.4.1 after coordinate transformation Ψ.

Problem 3.1.11	Problem 2.4.2
$v^{(1)}_{\mathrm{twr}}(\boldsymbol{\varphi}) = \Psi(\boldsymbol{\varphi})$.	$v^{(1)}(\mathbf{x}) = v^{(1)}_{\mathrm{twr}}(\Psi^{-1}(\mathbf{x})) = \mathbf{x}$, i.e. $\mathbf{x} = \Psi(\boldsymbol{\varphi})$.
$\mathcal{B}_{\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}, B}(r, \mu)$ is compact.	$\mathcal{B}_{\mathfrak{r}, \mathfrak{u}, A}(r, \mu) = \Psi(\mathcal{B}_{\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}, B}(r, \mu))$ is compact.
(a) $\min\limits_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi})$ s.t. $\overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) \geq \mu$, $\sum_{i=0}^{M} \varphi_i = \beta$.	(a) $\min\limits_{\mathbf{x} \in A} \mathfrak{r}(\mathbf{x})$ s.t. $\mathfrak{u}(\mathbf{x}) \geq \mu$, $S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}\beta = \widetilde{\beta}$.
(b) $\max\limits_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi})$ s.t. $\overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}) \leq r$, $\sum_{i=0}^{M} \varphi_i = \beta$.	(b) $\max\limits_{\mathbf{x} \in A} \mathfrak{u}(\mathbf{x})$ s.t. $\mathfrak{r}(\mathbf{x}) \leq r$, $S^{(0)}\mathbf{x} = \mathcal{W}^{(0)}\beta = \widetilde{\beta}$.

Table 3.2.: Problem 3.1.11 is embedded in Problem 2.4.2 according to transformation Ψ.

Therefore,

$$\begin{aligned} \infty > \mathfrak{u}(\Psi(\boldsymbol{\varphi})) = \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) \geq \mathfrak{u}_{\ln}(\boldsymbol{\varphi}) = \ln\left(\prod_{n=1}^{N}\left(1 + T_0^{(n)}\varphi_0\right)\right) \geq 0, \\ 0 \leq \mathfrak{r}(\Psi(\boldsymbol{\varphi})) = \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}) \leq \mathfrak{d}_{\ln}(\boldsymbol{\varphi}) = 0, \end{aligned}$$

and, consequently, $(\beta, 0, \dots, 0)^\intercal \in \mathrm{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta = \mathrm{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta$ for all $\beta \geq 0$. Note, that $\mathbf{x} := \Psi(\boldsymbol{\varphi})$ also satisfies $\widehat{\mathbf{x}} = 0$ and $x_0 \geq 0$, which implies that $A_{\widetilde{\beta}} \cap \mathrm{dom}(\mathfrak{u}) \cap \mathrm{dom}(\mathfrak{r}) \neq \emptyset$ for all $\widetilde{\beta} > 0$.

Altogether, Setting 3.1.10 and Problem 3.1.11 can be embedded in Setting 2.4.1 and Problem 2.4.2 in the sense of Lemma 2.4.3. ♦

For the risk measure in Setting 3.1.10, some properties follow from Theorem 3.1.7 and Corollary 3.1.9. Furthermore, the utility function in Setting 3.1.10 can be written as

$$\overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) = \mathrm{E}\left[\ln\left(\frac{\mathcal{W}^{(N)}(v_{\mathrm{twr}}(\boldsymbol{\varphi}))}{\mathcal{W}^{(0)}}\right)\right] = \sum_{n=1}^{N} \mathrm{E}\left[\ln\left(1 + T^{(n)}\boldsymbol{\varphi}\right)\right], \tag{3.13}$$

for $\boldsymbol{\varphi} \in \mathrm{ri}(\mathrm{dom}(\overline{\mathfrak{u}}_{\ln})) \subset \Phi_{\mathrm{twr}}$, cf. (2.33) and (2.50), and has the properties from Theorem 2.3.12. Using these results, it can be seen, that Setting 3.1.10 defines an application of Theorem 2.4.20. Consequently, for this setting the following result can be proven.

Corollary 3.1.14 (TWR and drawdown optimization, see also Maier-Paape and Zhu [34, Theorem 13]) Let Setting 3.1.10 and Problem 3.1.11 be given for fixed $\beta > 0$. Then, the following assertions hold true for I, J from Definition 2.4.12 (with $\mathfrak{r} := \overline{\mathfrak{d}}_{\ln} \circ \Psi^{-1}$ and $\mathfrak{u} := \overline{\mathfrak{u}}_{\ln} \circ \Psi^{-1}$, see Table 3.1):

(a) For arbitrary, but fixed, $\mu \in J$ there is exactly one solution $\boldsymbol{\varphi}_\mu \in B$ of (3.11), which is in addition the unique efficient x-portfolio in B such that $\overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}_\mu) = \mu$. Furthermore, the mapping $\widetilde{\boldsymbol{\gamma}}\colon J \to B,\ \mu \mapsto \boldsymbol{\varphi}_\mu$ is continuous.

(b) For arbitrary, but fixed, $r \in I$ there is exactly one solution $\widetilde{\boldsymbol{\varphi}}_r \in B$ of (3.12), which is in addition the unique efficient x-portfolio in B such that $\overline{\mathfrak{d}}_{\ln}(\widetilde{\boldsymbol{\varphi}}_r) = r$. Furthermore, the mapping $\widetilde{\boldsymbol{\nu}}\colon I \to B,\ r \mapsto \widetilde{\boldsymbol{\varphi}}_r$ is continuous.

Moreover, the corresponding efficient frontier $\mathcal{G}_{\text{eff}}(\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}; B) = \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A) \subset \mathbb{R}^2$ (with $\mathfrak{r}$, $\mathfrak{u}$ and A from Table 3.1) is the graph of a convex function with the corresponding representation as in Corollary 2.4.16.

Proof. First note, that $\overline{\mathfrak{u}}_{\ln}$ restricted to $\operatorname{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta \subset \Phi_{\text{twr}}$ is strictly concave because of Theorem 2.3.12 (a). Note that Theorem 2.3.12 (a) does not give strict concavity on $\operatorname{dom}(\overline{\mathfrak{u}}_{\ln}) \cap \Phi_\beta$. Moreover, $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}, B}(r, \mu)$ is compact for all $r, \mu \in \mathbb{R}$ because of Lemma 3.1.12. Because of Remark 3.1.13, Theorem 2.4.20 can be applied to the equivalent problem after transformation. Reversing the transformation afterwards completes the proof.

Corollary 2.4.16 and Proposition 2.4.15 yield that $\mathcal{G}_{\text{eff}}(\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}; B \cap \Phi_\beta)$ contains all values of efficient x-portfolios in the risk-utility space and is the graph of the concave and strictly increasing function ν. Note that $\mathcal{G}_{\text{eff}}(\overline{\mathfrak{d}}_{\ln}, \overline{\mathfrak{u}}_{\ln}; B) = \mathcal{G}_{\text{eff}}(\mathfrak{r}, \mathfrak{u}; A)$ holds true because the coordinate transformation Ψ does not affect the risk and utility values of the efficient x-portfolios. $\square$

Corollary 3.1.15 (Maximum log TWR and minimum log drawdown, see also Hermes and Maier-Paape [26] and Maier-Paape and Zhu [34, Theorem 12]) Let the situation of Corollary 3.1.14 be given, in particular with $\beta > 0$ and $\emptyset \neq B \subset \operatorname{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta = \operatorname{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta$ as in Problem 3.1.11. Define $r_{\min}, r_{\max}, \mu_{\min}, \mu_{\max} \in \mathbb{R} \cup \{-\infty, \infty\}$ from (2.59) (with $\mathfrak{r} := \overline{\mathfrak{d}}_{\ln} \circ \Psi^{-1}$ and $\mathfrak{u} := \overline{\mathfrak{u}}_{\ln} \circ \Psi^{-1}$ and A from Table 3.1). Then, the following optimization problems have a solution:

(a) $\max\limits_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}) \quad \text{subject to } \sum\limits_{i=0}^{M} \varphi_i = \beta, \quad \text{(solution is unique)},$

(b) $\min\limits_{\boldsymbol{\varphi} \in B} \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}) \quad \text{subject to } \sum\limits_{i=0}^{M} \varphi_i = \beta.$

Furthermore, for the unique solution $\boldsymbol{\varphi}_{\text{a}}^*$ from (a) it is

$$(\overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}_{\text{a}}^*), \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}_{\text{a}}^*)) = (r_{\max}, \mu_{\max}) \in [0, \infty) \times \mathbb{R}$$

and there exists exactly one solution $\boldsymbol{\varphi}_{\text{b}}^*$ of (b) such that

$$(\overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}_{\text{b}}^*), \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}_{\text{b}}^*)) = (r_{\min}, \mu_{\min}) \in [0, \infty) \times \mathbb{R}$$

holds. All remaining solutions of (b), if there are any, have utility values strictly less then $\mu_{\min}$.

Proof. Proof of (a): Since $\overline{\mathfrak{u}}_{\ln}$ is closed proper concave and $\mathcal{B}_{\overline{\mathfrak{u}}_{\ln},\Phi_\beta}(\mu)$ is compact for all $\mu \in \mathbb{R}$ (by Theorem 2.3.12 (d)), Theorem A.1.2 and Theorem A.1.6 in the appendix yields the existence of a maximizer $\boldsymbol{\varphi}_{\mathrm{a}}^* \in B \subset \Phi_\beta$ (note that $B \neq \emptyset$). Strict concavity of $\overline{\mathfrak{u}}_{\ln}$ restricted to $\mathrm{dom}(\mathfrak{u}_{\ln}) \cap \Phi_\beta$, see Theorem 2.3.12 (a), gives uniqueness. Furthermore, Lemma 2.4.10 yields that $\mu_{\max} = \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}_{\mathrm{a}}^*) \in \mathbb{R}$ and, since $B \subset \mathrm{dom}(\mathfrak{u}_{\ln}) = \mathrm{dom}(\mathfrak{d}_{\ln})$ (see Corollary 3.1.9), that $\mathfrak{r}_{\max} = \overline{\mathfrak{d}}_{\ln}(\boldsymbol{\varphi}_{\mathrm{a}}^*) \in [0,\infty)$.

Proof of (b): Since $\mathcal{B}_{\overline{\mathfrak{d}}_{\ln},\Phi_\beta}(r)$ is compact for all $r \in \mathbb{R}$ (by Theorem 3.1.7 (c)) and $B \neq \emptyset$, Theorem A.1.6 in the appendix gives the existence of a solution of problem (b). There might be more than one solution, because $\overline{\mathfrak{d}}_{\ln}$ does not necessarily need to be strictly convex on $\mathrm{dom}(\mathfrak{d}_{\ln}) \cap \Phi_\beta$.

Furthermore, this implies that $r_{\min} \in [0,\infty)$ and $\emptyset \neq \mathcal{B}_{\overline{\mathfrak{d}}_{\ln},B}(r_{\min}) \subset \mathrm{dom}(\mathfrak{d}_{\ln}) = \mathrm{dom}(\mathfrak{u}_{\ln})$. Lemma 2.4.10 then yields the existence of $\boldsymbol{\varphi}_{\mathrm{b}}^* \in \mathcal{B}_{\overline{\mathfrak{d}}_{\ln},B}(r_{\min})$ such that $\mu_{\min} = \overline{\mathfrak{u}}_{\ln}(\boldsymbol{\varphi}_{\mathrm{b}}^*) \in \mathbb{R}$ and therefore $\mu_{\min} \in J$. Thus, this solution is unique according to Corollary 3.1.14 (a) for $\mu = \mu_{\min}$, which completes the proof. $\square$

Corollary 3.1.14 gives the existence and uniqueness of the efficient x-portfolios for the optimization problem regarded the logarithm of TWR and the logarithm of the relative drawdown on a closed convex subset $B \subset \Phi_\beta$. This includes the case in which it is allowed to invest into the bond, and also the case $B \subset \{\mathbf{x} \in \Phi_\beta : x_0 = 0\}$, which excludes the bond, i.e. it has to be (fully) invested into the risky part only. Similar situations as in Corollary 3.1.15 (a) with discrete and finite probability space have also been discussed in Hermes and Maier-Paape [26] and Maier-Paape and Zhu [34, Section 6]. In the latter reference, the authors also discuss the optimization problems in Corollary 3.1.14 using a general risk measure.

The utility and risk measures in Setting 3.1.10 are nonlinear which makes it difficult to (numerically) solve the optimization problem. Hence, it is worthwhile to discuss Setting 3.1.10 when replacing the utility function $\mathfrak{u}_{\ln}$ by its linearization $\mathfrak{u}_{\mathrm{lin}}$, see (2.51) and Example 2.3.13, and the risk function $\mathfrak{d}_{\ln}$ by its linearization $\mathfrak{d}_{\mathrm{lin}}$, see Corollary 3.1.6. Note that in this case the x-portfolio generating function must be replaced by the one from Example 2.2.33 with corresponding admissible set.

Setting 3.1.16 Let the following be given for $M, N \in \mathbb{N}$:

(i) The *market model* $S \in \mathcal{L}^2(N;\mathbb{R}^{M+1}_{>0})$, see Definition 2.1.4, is assumed to have no nontrivial risk-free x-portfolio, see Definition 2.2.22. In addition, let $T^{(n)} \in \mathcal{L}^2(\Omega, \mathcal{F}_n, \mathrm{P}; \mathbb{R}^{M+1})$ for $n = 1, \ldots, N$ and assume that $\sum_{n=1}^N \mathrm{E}\,[T_j^{(n)}] \neq \sum_{n=1}^N \mathrm{E}\,[T_i^{(n)}]$ for some $i, j \in \{0, 1, \ldots, M\}$ and $\mathrm{P}\,(T^{(N)}\widehat{\boldsymbol{\theta}} < 0) > 0$ for all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ with $\widehat{\boldsymbol{\theta}} \neq 0$.

(ii) The *x-portfolio* is defined by the x-portfolio generating function v_{cm} such that $S^{(n-1)}\boldsymbol{\theta}^{(n)}$ is constant in time, see Example 2.2.33 (constant invested money).

(iii) The *utility function* is defined by the closed proper concave function $\mathfrak{u}_{\mathrm{lin}}$ from (2.51) on $\mathbb{R}^{M+1}$, cf. Example 2.3.13.

(iv) The *risk function* is given by the closed proper convex function $\mathfrak{r}_{\mathrm{lin}}$ defined by $\mathfrak{r}_{\mathrm{lin}}(\boldsymbol{\theta}) := \mathfrak{d}_{\mathrm{lin}}(\widehat{\boldsymbol{\theta}})$ for all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ with $\mathfrak{d}_{\mathrm{lin}}$ from Corollary 3.1.6, i.e. the risk-free asset is now ignored for the risk.

It is $\mathrm{dom}(\mathfrak{r}_{\mathrm{lin}}) = \mathrm{dom}(\mathfrak{u}_{\mathrm{lin}}) = \mathbb{R}^{M+1} \neq \emptyset$.

This gives an application for the one-fund theorem, see Corollary 2.4.27.

Corollary 3.1.17 (Optimization using linearized TWR and linearized drawdown) Let Setting 3.1.16 and fixed $\beta > 0$ be given. Then, for all $\mu \geq \mu_0 := \beta$ there exists an efficient x-portfolio $\boldsymbol{\theta}_\mu \in \Phi_\beta := \{\boldsymbol{\varphi} \in \mathbb{R}^{M+1} : \sum_{i=0}^{M} \varphi_i = \beta\}$ such that $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}_\mu) = \mu$ and the following holds true: $\boldsymbol{\theta}_\mu \in \Phi_\beta$ is efficient with $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}_\mu) = \mu$, if and only if $\boldsymbol{\theta}_\mu$ solves

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{M+1}} \mathfrak{r}_{\text{lin}}(\boldsymbol{\theta}) \quad \text{subject to } \mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}) \geq \mu,\ \sum_{i=0}^{M} \theta_i = \beta.$$

In addition, let $\boldsymbol{\theta}^0 := (\beta, 0, \ldots, 0)^\intercal \in \Phi_\beta$. Then for $\mu_1 := \mu_0 + 1$ with solution $\boldsymbol{\theta}^1 := \boldsymbol{\theta}_{\mu_1}$ and $t \geq 0$, the vector $\boldsymbol{\theta}^t := \boldsymbol{\theta}^0 + t(\boldsymbol{\theta}^1 - \boldsymbol{\theta}^0)$ is efficient with

$$\big(\mathfrak{r}_{\text{lin}}(\boldsymbol{\theta}^t), \mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^t)\big) = \big(t\, \mathfrak{r}_{\text{lin}}(\boldsymbol{\theta}^1), \mu_0 + t\big) \in \mathcal{G}_{\text{eff}}(\mathfrak{r}_{\text{lin}}, \mathfrak{u}_{\text{lin}}; \Phi_\beta).$$

Thus, the efficient portfolios lie on rays and the related efficient frontier is also a ray in the risk-utility space.

Proof. Using similar arguments and the same linear transformation Ψ as in Remark 3.1.13 gives the embedding in Setting 2.4.1 and Problem 2.4.2 in the sense of Lemma 2.4.3. Hence, the assertions directly follow from Corollary 2.4.27, if the requirements are fulfilled. Note, that it suffices that $\mathfrak{r}_{\text{lin}}$ and $\mathfrak{u}_{\text{lin}}$ satisfy analog assumptions because these imply (i) to (iv) of Corollary 2.4.27 after the linear coordinate transformation Ψ, respectively.

Assumption (i) of Corollary 2.4.27 (positive homogeneous and linear in the first component): The properties of $\mathfrak{u}_{\text{lin}}$ follow directly from Example 2.3.13 with constant $c_{\mathfrak{u}} = \sum_{n=1}^{N} T_0^{(n)}$ for linearity in the first component. Corollary 3.1.6 yields that $\mathfrak{r}_{\text{lin}}$ is positive homogeneous and closed proper convex. Since $\mathfrak{r}_{\text{lin}}$ is independent on the first component with index zero, it is linear in the risk-free part with $c_{\mathfrak{r}} = 0$. The domain of both functions $\mathfrak{u}_{\text{lin}}$ and $\mathfrak{r}_{\text{lin}}$ is $\mathbb{R}^{M+1}$, i.e. (i) is fulfilled even for the functions $\mathfrak{r} := \mathfrak{r}_{\text{lin}} \circ \Psi^{-1}$ and $\mathfrak{u} := \mathfrak{u}_{\text{lin}} \circ \Psi^{-1}$.

Assumptions (ii) of Corollary 2.4.27 (compactness of $\mathcal{B}_{\mathfrak{r}_{\text{lin}}, \mathfrak{u}_{\text{lin}}, \Phi_\beta}(r, \mu)$ or, equivalently, of $\mathcal{B}_{\mathfrak{r}, \mathfrak{u}, \Psi(\Phi_\beta)}(r, \mu)$): Because of Remark 2.4.29 (b) after using the linear coordinate transformation Ψ, it suffices to show that $\mathcal{B}_{\mathfrak{r}_{\text{lin}}, \Phi_\beta}(0) = \{\boldsymbol{\theta}^0\} \subset \mathbb{R}^{M+1}$ for $\boldsymbol{\theta}^0 := (\beta, 0, \ldots, 0)^\intercal \in \Phi_\beta$. For all $\boldsymbol{\theta} \in \mathbb{R}^{M+1}$ with $\widehat{\boldsymbol{\theta}} \neq 0$ it is $T^{(N)} \widehat{\boldsymbol{\theta}} < 0$ with positive probability from assumption (i) in Setting 3.1.16. As in the proof of Corollary 3.1.6 it then follows that $\mathfrak{r}_{\text{lin}}(\boldsymbol{\theta}) = \mathfrak{d}_{\text{lin}}(\widehat{\boldsymbol{\theta}}) > 0$. In addition it is $\mathfrak{r}_{\text{lin}}(\boldsymbol{\theta}^0) = 0$ for $\boldsymbol{\theta}^0 := (\beta, 0, \ldots, 0)^\intercal \in \Phi_\beta$ and for all $\boldsymbol{\theta} \in \Phi_\beta$ it is $\widehat{\boldsymbol{\theta}} = 0$ if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}^0$. Consequently, it must be $\mathcal{B}_{\mathfrak{r}_{\text{lin}}, \Phi_\beta}(0) = \{\boldsymbol{\theta}^0\}$.

Assumptions (iii) of Corollary 2.4.27 ($\boldsymbol{\theta}^0$ is efficient): Follows directly from $\mathcal{B}_{\mathfrak{r}_{\text{lin}}, \Phi_\beta}(0) = \{\boldsymbol{\theta}^0\}$.

Assumption (iv) of Corollary 2.4.27 (existence of portfolios with utility value greater than $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^0)$): The definition in (2.51) yields for $\mathbf{y} = (y_0, \ldots, y_M)^\intercal \in \mathbb{R}^{M+1}$ with $y_i := \sum_{n=1}^{N} \mathrm{E}\,[T_i^{(n)}]$ that $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}) = \mathbf{y}^\intercal \boldsymbol{\theta}$. Assumption (i) in Setting 3.1.16 gives the existence of some $i \in \{1, \ldots, M\}$ such that $y_0 \neq y_i$. Consequently, $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^0) = y_0 \beta \neq y_i \beta = \mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}')$ for $\boldsymbol{\theta}' := (0, \ldots, 0, \beta, 0, \ldots, 0)^\intercal$ with β at the ith position. In addition, either $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^0) < \mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}')$ or $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^0) < 2y_0\beta - y_i\beta = \mathfrak{u}_{\text{lin}}(2\boldsymbol{\theta}^0 - \boldsymbol{\theta}')$, i.e. there exists $\boldsymbol{\theta} \in \Phi_\beta$ such that $\mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}) > \mathfrak{u}_{\text{lin}}(\boldsymbol{\theta}^0)$.

Hence, all assumptions required for Corollary 2.4.27 are satisfied, which therefore completes the proof. $\square$

3.2 Markov chain on $\mathbb{Z}$

This subsection focuses on the absolute drawdown process in Definition 3.1.1 for discrete models of wealth processes. For a given time series, e.g. one path of the wealth process, it is easy to compute the absolute drawdown. If a stochastic model for the time series is known, there are more quantities of interest like the probability for an absolute drawdown of given size after some time step. For instance, in the game Roulette it is possible to bet on color, on single numbers, on a set of some numbers, a combination of all of them and so on. Each game ends up with a gain or loss of an amount of money depending on the strategy and the bet size. For fixed betting strategy, which is repeated for each game, the chance to win or loose a specific amount of money is always the same, because the games of Roulette can be assumed to be stochastically independent. Hence, such a game can be modeled by a discrete-time Markov chain with discrete state space.

Example 3.2.1 (French Roulette: Part 1/6) This example will be continued at the end of each subsequent subsection and is an example from French Roulette, i.e. there are 37 numbers with one green zero included.

Assume this game is played multiple times, where after each round (or time step), the same strategy is applied, which consists of a combination of the following two bets (for the same game):

(i) Six line bet on the numbers 1 to 6 (three numbers are red and three numbers are black); the payout is 5 to 1 and the winning probability is 6/37.

(ii) Bet red, i.e. bet on the 18 red numbers; the payout is 1 to 1 and probability to win is 18/37.

For $N \in \mathbb{N}$ games, the probability space $(\Omega, \Sigma, \mathrm{P})$ is given by $\Omega = \{0, 1, \ldots, 36\}^N$ with power set Σ and probability measure P. Let $X^{(n)}$, $n \in \mathbb{N}$, be the random variables giving the win/loss (in \$) for the nth game when betting \$ 2 (\$ 1 for (i) and \$ 1 for (ii); note that (i) and (ii) are, of course, stochastically dependent). The probabilities are then given by

$$\mathrm{P}\left(X^{(n)} = i\right) = \begin{cases} 3/37, & \text{if } i = 6, \\ 3/37, & \text{if } i = 4, \\ 15/37, & \text{if } i = 0, \\ 16/37, & \text{if } i = -2, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, the expectation for one round is negative and given by $\mathrm{E}\left[X^{(n)}\right] = -2/37$. Hence, this game is not fair. However, it can be assumed that the games are independent and identically distributed (iid). ♦

The goal is now to be able to compute the probability mass function of the absolute drawdown after given time step $N \in \mathbb{N}$. First the setting and stochastic model is defined in Subsection 3.2.1. Related topics in the literature are given afterwards in Subsection 3.2.2. Then, an explicit expression for the probability mass function is given in Subsection 3.2.3, which consists of matrix and vector multiplications. The discrete model allows the absolute drawdown to attain non-negative integers. However, there are cases, where not all values in $\mathbb{N}$ can be reached. Thus, a discussion of the state space follows in Subsection 3.2.4. When taking the limit $N \to \infty$, conditions can be found for existence and uniqueness of a corresponding limit distribution. This is done in Subsection 3.2.5 while an explicit expression for this distribution in some special cases is given in Subsection 3.2.6.

3.2.1 Definitions

Let K be a countable set where for each $i, j \in K$ there is a number $p_{i,j} \in [0,1]$ satisfying

$$\sum_{j \in K} p_{i,j} = 1, \quad \text{for all } i \in K.$$

A (discrete-time) Markov chain on a probability space $(\Omega, \Sigma, \mathrm{P})$ is a sequence of random variables $Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots$ having values in K and which fulfill the property

$$\mathrm{P}\left(Y^{(n+1)} = j \mid Y^{(0)} = i_0, \ldots, Y^{(n)} = i_n\right) = \mathrm{P}\left(Y^{(n+1)} = j \mid Y^{(n)} = i_n\right) := p_{i_n,j}$$

for all $n \in \mathbb{N}_0$ and all $i_0, \ldots, i_n \in K$ with $\mathrm{P}\left(Y^{(0)} = i_0, \ldots, Y^{(n)} = i_n\right) > 0$, see e.g. Billingsley [5, Section 8]. The numbers $p_{i,j}$ are called transition probabilities of the Markov chain. In the following, only the case of discrete time steps, i.e. Markov chains and no more general Markov processes are discussed. Therefore, the expression "discrete time" is omitted.

It can be concluded, that a Markov chain is a stochastic process on a countable set with "no memory", i.e. the next step only depends on the last position and not on the previous steps. For the game Roulette, this assumption makes sense because the outcome of the next game does not depend on past games. Next, the specific Markov chain for the relevant setting is defined.

Let $\alpha, \beta \in \mathbb{N}$ and $X^{(n)} \colon \Omega \to \{-\alpha, \ldots, -1, 0, 1, \ldots, \beta\} \subset \mathbb{Z}$, $n \in \mathbb{N}$, be iid random variables modeling the outcome of each game, i.e. for $n \in \mathbb{N}$ the random variable $X^{(n)}$ corresponds to the amount of money won or lost in the nth game. The probabilities are denoted by

$$\mathrm{P}\left(X^{(n)} = i\right) =: p_i \in [0,1] \qquad \text{for } i = 1, \ldots, \beta, \tag{3.14a}$$

$$\mathrm{P}\left(X^{(n)} = -i\right) =: q_i \in [0,1] \qquad \text{for } i = 1, \ldots, \alpha, \tag{3.14b}$$

$$\mathrm{P}\left(X^{(n)} = 0\right) = 1 - \sum_{i=1}^{\beta} p_i - \sum_{i=1}^{a} q_i =: r \in [0,1]. \tag{3.14c}$$

For instance, if $\alpha = \beta = 1$, the game could have three outcomes (win one, loose one, or tie).

Such a game can be interpreted by a particle on a line moving $i \in \{1, \ldots, \alpha\}$ positions to the left with probability q_i, moving $j \in \{1, \ldots, \beta\}$ positions to the right with probability p_j, or staying at the same position with probability r in the next step. The position of the particle after $N \in \mathbb{N}$ steps (or the sum of gains/losses after N games) is then given by the following Markov chain.

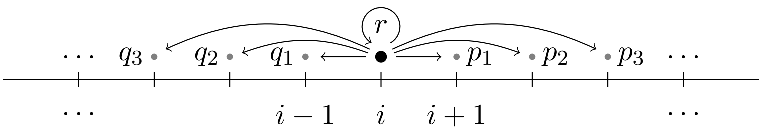


Figure 3.2.: Markov chain in Definition 3.2.2 for $\alpha = \beta = 3$, see also (3.16).

Definition 3.2.2 (Markov chain on $\mathbb{Z}$) Let $\alpha, \beta \in \mathbb{N}$ and $q_1, \ldots, q_\alpha, p_1, \ldots, p_\beta, r \in [0,1]$, where $r + \sum_{j=1}^{\alpha} q_j + \sum_{j=1}^{\beta} p_j = 1$. The *Markov chain* $(Z^{(n)})_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ (with the usual probability space) is defined by

$$Z^{(n)} := Z^{(n-1)} + X^{(n)} = Z^{(0)} + \sum_{k=1}^{n} X^{(n)} \in \mathbb{Z} \quad \text{for } n \in \mathbb{N} \tag{3.15}$$

with $X^{(n)}$ for $n \in \mathbb{N}$ from (3.14). The initial value $Z^{(0)} \in \mathbb{Z}$ can either be fixed, e.g. $Z^{(0)} \equiv 0$, or a non-constant random variable following a given distribution on $\mathbb{Z}$.

The probability of a particle starting at position $i \in \mathbb{Z}$ with $\mathrm{P}\left(Z^{(0)} = i\right) > 0$ and being at position $j \in \mathbb{Z}$ after step $n \in \mathbb{N}_0$ is denoted by

$$z_{i,j}^{(n)} := \mathrm{P}\left(Z^{(n)} = j \mid Z^{(0)} = i\right)$$

and for $i = 0$ by $z_j^{(n)} := z_{0,j}^{(n)}$.

This Markov chain can be seen as a model for the equity curve or wealth process of the games, cf. (2.7) in Definition 2.2.2. The conditional probabilities for the stochastic process $(Z^{(n)})_{n\in\mathbb{N}_0}$ are then given by

$$\mathrm{P}\left(Z^{(n+1)} = i + j \mid Z^{(n)} = i\right) = p_j, \qquad \text{for } i \in \mathbb{Z},\ j \in \{1, \ldots, \beta\}, \tag{3.16a}$$

$$\mathrm{P}\left(Z^{(n+1)} = i \mid Z^{(n)} = i\right) = r, \qquad \text{for } i \in \mathbb{Z}, \tag{3.16b}$$

$$\mathrm{P}\left(Z^{(n+1)} = i - j \mid Z^{(n)} = i\right) = q_j, \qquad \text{for } i \in \mathbb{Z},\ j \in \{1, \ldots, \alpha\}, \tag{3.16c}$$

$$\mathrm{P}\left(Z^{(0)} \in \mathbb{Z}\right) = 1 \tag{3.16d}$$

for all $n \in \mathbb{N}_0$, cf. Figure 3.2.

Remark 3.2.3 In the following, there are no further restrictions on the initial state $Z^{(0)}$. However, the study of state probabilities for a Markov chain starting at a given position, say at $i = 0$, can be realized by computing a conditional probability like $z_{0,j}^{(n)}$, having the initial position as condition. In this case it is assumed that $\mathrm{P}\left(Z^{(0)} = 0\right) > 0$, such that $z_{0,j}^{(n)}$ is well-defined.

In addition, a Markov chain has "no memory", i.e. the next position only depends on the current position. If the position is i at time step m, the probability of being at position j after n additional steps reads $\mathrm{P}\left(Z^{(m+n)} = j \mid Z^{(m)} = i\right)$. Since this does not depend on the

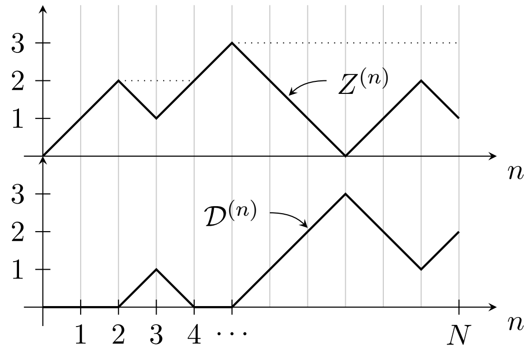


Figure 3.3.: Illustration of the absolute drawdown $\mathcal{D}^{(n)}$ obtained from a Markov chain $Z^{(n)}$, for $n = 1, \dots, N$.

position of the Markov chain at time steps $1, \dots, m-1$, such a Markov chain could also start directly from position i at time zero and walk n steps. Thus it can be written

$$\mathrm{P}\left(Z^{(m+n)} = j \mid Z^{(m)} = i\right) = \mathrm{P}\left(Z^{(n)} = j \mid Z^{(0)} = i\right) = z_{i,j}^{(n)}. \tag{3.17}$$

Note that (3.14) and (3.15) together with $\mathrm{P}\left(Z^{(0)} \in \mathbb{Z}\right) = 1$ is equivalent to (3.16), because of the following: To go from (3.15) to (3.16) is clear, because each step given by $X^{(n)}$ is independent on the current position i and the time index n. On the other hand, if (3.16) is known, one can find appropriate iid random variables $(X^{(n)})_{n\in\mathbb{N}_0}$ such that the Markov chain can be written as in (3.15), namely $X^{(n)} = Z^{(n)} - Z^{(n-1)}$ which gives (3.14). ♦

The absolute drawdown process from Definition 3.1.1 modeled by such a discrete-time Markov chain is then given as follows.

Definition 3.2.4 (Absolute drawdown; Markov chain with reflecting barrier at zero) Let $\alpha, \beta \in \mathbb{N}$ and $q_1, \dots, q_\alpha, p_1, \dots, p_\beta, r \in [0,1]$ where $r + \sum_{j=1}^{\alpha} q_j + \sum_{j=1}^{\beta} p_j = 1$ and $(Z^{(n)})_{n\in\mathbb{N}_0}$ from Definition 3.2.2 be given. The absolute drawdown represented as Markov chain $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ on $\mathbb{N}_0$ with reflecting barrier at zero is given by

$$\mathcal{D}^{(n)} := \max_{0\le k\le n}\left\{Z^{(k)}\right\} - Z^{(n)} = \max\left\{0, \mathcal{D}^{(n-1)} - X^{(n)}\right\} \in \mathbb{N}_0 \quad \text{for } n \in \mathbb{N}, \tag{3.18}$$

cf. Figure 3.3. $\mathcal{D}^{(0)} \in \mathbb{N}_0$ can either be fixed, e.g. $\mathcal{D}^{(0)} \equiv 0$, or a non-constant random variable following a distribution on $\mathbb{N}_0$.

The probability of a particle starting at position $i \in \mathbb{N}_0$ and being at position $j \in \mathbb{N}_0$ after $n \in \mathbb{N}_0$ steps is denoted by

$$d_{i,j}^{(n)} := \mathrm{P}\left(\mathcal{D}^{(n)} = j \mid \mathcal{D}^{(0)} = i\right) \qquad \text{whenever } \mathrm{P}\left(\mathcal{D}^{(0)} = i\right) > 0$$

and for $i = 0$ by $d_j^{(n)} := d_{0,j}^{(n)}$.

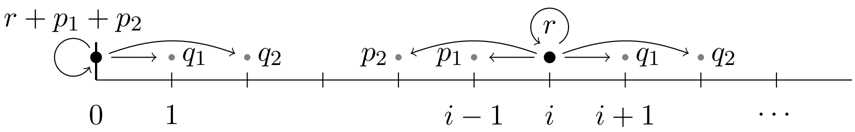


Figure 3.4.: Markov chain of Definition 3.2.4 with $\alpha = \beta = 2$ and reflecting barrier at 0.

The absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ walks in the opposite direction to $(Z^{(n)})_{n\in\mathbb{N}_0}$ but cannot become negative, cf. Figure 3.3. The transition probabilities are given by

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i+j \mid \mathcal{D}^{(n)} = i\right) = q_j, \qquad \text{for } i \in \mathbb{N}_0,\ j \in \{1,\dots,\alpha\}, \tag{3.19a}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i \mid \mathcal{D}^{(n)} = i\right) = r, \qquad \text{for } i \in \mathbb{N}, \tag{3.19b}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = 0 \mid \mathcal{D}^{(n)} = 0\right) = r + \sum_{j=1}^{\beta} p_j, \tag{3.19c}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = 0 \mid \mathcal{D}^{(n)} = i\right) = \sum_{j=i}^{\beta} p_j, \qquad \text{for } i \in \mathbb{N},\ 1 \le i \le \beta, \tag{3.19d}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i-j \mid \mathcal{D}^{(n)} = i\right) = p_j, \qquad \text{for } i \in \mathbb{N},\ i-j > 0,\ j \in \{1,\dots,\beta\}, \tag{3.19e}$$

$$\mathrm{P}\left(\mathcal{D}^{(0)} \in \mathbb{N}_0\right) = 1 \tag{3.19f}$$

and $\mathrm{P}\left(\mathcal{D}^{(n+1)} = j \mid \mathcal{D}^{(n)} = i\right) = 0$ in the remaining situations, cf. Figure 3.4.

Remark 3.2.5 Since the absolute drawdown in Definition 3.2.4 is a Markov chain, the same rules apply to $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ as described in Remark 3.2.3 for $(Z^{(n)})_{n\in\mathbb{N}_0}$, see (3.17). However, the reconstruction of $(X^{(n)})_{n\in\mathbb{N}_0}$ from (3.19) is more involved in this case because of the barrier at zero, but for $i > \beta$ it is $\mathrm{P}\left(\mathcal{D}^{(n+1)} = i-j \mid \mathcal{D}^{(n)} = i\right) = \mathrm{P}\left(X^{(n)} = j\right)$ for $j = -\alpha,\dots,0,\dots,\beta$, see (3.19) and (3.14). The remaining probabilities in (3.19) are related to (3.18) by construction. ♦

Remark 3.2.6 **($\alpha = \beta = 1$)** For $\alpha = \beta = 1$ the situation above corresponds to the random walk case and, in particular, it belongs to the class of discrete-time birth death processes (or birth-death chains), see e.g. Sericola [51, Section 3.1]. ♦

Example 3.2.7 **(French Roulette: Part 2/6; see Example 3.2.1)** The Markov chain of interest is the absolute drawdown in Definition 3.2.4. In the corresponding notation, for Example 3.2.1 it is $\alpha = 2$, $\beta = 6$ and

$$(q_2, q_1, r, p_1, p_2, p_3, p_4, p_5, p_6) = \left(\frac{16}{37}, 0, \frac{15}{37}, 0, 0, 0, \frac{3}{37}, 0, \frac{3}{37}\right)$$

which completely defines the absolute drawdown in Definition 3.2.4, see also (3.19). ♦

The (absolute) drawdown is an important quantity when analyzing risk of an investment or even of the game in Example 3.2.1. Therefore, the intention of the remainder of this section is to analyze the probability distributions of $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ for given time steps or rather its transition probabilities $d_{i,j}^{(n)}$.

3.2.2 State of the art

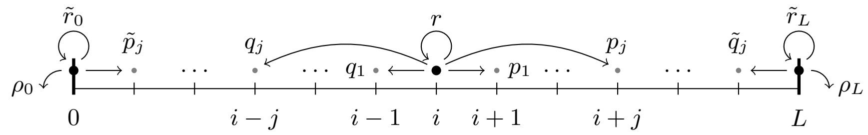
A lot of research has been done in the field of random walk and Markov chain with one or two barriers. While the drawdown can be modeled as a Markov chain with one reflecting barrier, there are also absorbing barriers used in the literature, where the particle stays at such a barrier with probability one once touched. In this context, one often meets the so-called (classical) gambler's ruin problem. Some results in this theory can help to prove properties for the (absolute) drawdown. Therefore, a literature review is given for this theory.

Assume the particle can move along $\mathbb{Z}$ with barriers on the left and on the right side. W.l.o.g. the particle moves in the bounded domain $\{0, 1, \ldots, L\} \subset \mathbb{N}_0$ for $L \in \mathbb{N}$. The figure in Table 3.3 shows the situation, where the probabilities are characterized as follows:

- q_j, p_j, r for $j \in \mathbb{N}$ denotes the probability of making one step of size j to the left, the right or stay at the same position, respectively, in case the particle is not at the boundary. This is separated into q_1, p_1 and q_j, p_j for $2 \leq j \leq M$, for fixed $2 \leq M \in \mathbb{N}$, to distinguish whether or not step sizes beyond one are possible with positive probability. In one case, $p_1 = p(i)$ and $q_1 = 1 - p(i)$, i.e. both probabilities depend on the current position $i \in \{1, \ldots, L-1\}$ and maximum step size equals to one.
- The particle cannot overshot the boundary, i.e. when the particle would overshot a barrier in one step, it moves to this barrier.
- ρ_0, ρ_L denotes the probability of absorption at the left or right barrier, respectively. Absorption means, the particle is removed from the system and the process stops. A value of one would therefore mean, that once this barrier is reached, the process stops immediately.
- $\tilde{r}_0, \tilde{r}_L$ denotes the probability of staying at the left or right barrier, respectively, with no absorption, i.e. the process does not stop and the particle might leave this position in some future time step.
- $\tilde{p}_j, \tilde{q}_j$ for $j \in \mathbb{N}$ denotes the probability of reflecting from left or right barrier, respectively, where j denotes the step size, which equals to one, if $q_j = p_j = 0$ for $j \geq 2$.
- N always denotes the number of time steps.

Of course, there are plenty of possible combinations for this general setup, like two fully absorbing barriers, the use of reflecting barriers, or just partially reflecting barriers where absorption can happen with positive probability less than one. Table 3.3 shows the situation discussed in the corresponding literature and the situation discussed for the rest of this section. Note, that in some cases, there is studied more than just one setting. As can be seen, even the random walk model with a step size of at most one, is still discussed even at the beginning of the 21st century. Most of the literature only discusses the case with two barriers, which makes the process finite. However, in the rest of Section 3.2 and Section 3.3, it is focused on the absolute drawdown, which is modeled with only one reflecting barrier on the left, and no barrier on the right. To the best of the author's knowledge, there is no other work for this specific setting in the literature.

In the following, let $Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots$ be the discrete-time Markov chain for the situation given in the corresponding literature from Table 3.3.



Reference	ρ_0	$\tilde r_0$	$\tilde p_j$	q_j	q_1	r	p_1	p_j	$\tilde q_j$	$\tilde r_L$	ρ_L
[6, 54]	1	0	0	0	$1-p$	0	p	0	$1-p$	p	0
[24]	1	0	0	0	p	$1-2p$	p	0	$1-p$	p	0
[13, 39]	1	0	0	0	q	$1-p-q$	p	0	$1-\omega$	ω	0
[6, 54]	1	0	0	0	$1-p$	0	p	0	–	–	–
[17]	1	0	0	0	$1-p$	0	p	0	0	0	1
[17]	1	0	0	0	$1-p$	0	p	0	–	–	–
[17, 19]	1	0	0	q_j	q_1	r	p_1	p_j	0	0	1
[41]	ρ	0	$1-\rho$	0	$1-p$	0	p	0	$1-\rho$	0	ρ
[14]	ρ	0	$1-\rho$	0	$1-p(i)$	0	$p(i)$	0	$1-\omega$	0	ω
[15]	ρ	0	$1-\rho$	0	q	$1-p-q$	p	0	$1-\omega$	0	ω
[22, 51]	0	$1-p$	p	0	q	$1-p-q$	p	0	–	–	–
Section 3.2	0	$\tilde r_0$	p_j	q_j	q_1	r	p_1	p_j	–	–	–
Section 3.3	0	$1-p$	p	0	q	$1-p-q$	p	0	–	–	–

Table 3.3.: Situation discussed in the corresponding reference (with slightly different notation in some cases), where the given probabilities are all values in $[0, 1]$.

As already noted in Remark 3.2.6, the $\alpha = \beta = 1$ case for the absolute drawdown in Definition 3.2.4, which is a random walk with one reflecting barrier at zero, corresponds to a discrete-time birth death process. This situation has been studied e.g. in Sericola [51, Section 3.1] and Graham [22, Section 3.2.4.3]. However, therein mainly the limit $N \to \infty$ is discussed and the corresponding limit distribution is given explicitly. The distribution for $N < \infty$ in general is just stated via the transition probability matrix. Furthermore, only the random walk case (and no step sizes beyond one) is discussed. To the best of the author's knowledge, the more general Markov chain has not been studied and no explicit formula (besides the one from the transition probability matrix) for the probability mass function after $N \in \mathbb{N}$ steps in the random walk case has been stated in the literature.

One of the first work for random walk with two absorbing barriers has been done by Weesakul [54], which has later been corrected by Blasi [6], see the first row in Table 3.3. For given initial position $0 < u \leq L$ they give an expression for

$$\mathrm{P}\left(Y^{(n)} = 0 \mid Y^{(0)} = u, Y^{(k)} > 0 \text{ for } k = 1, \ldots, n-1\right), \tag{3.20}$$

i.e. the probability of reaching 0 for the first time at time $n \in \mathbb{N}$ when starting at u. The resulting formula is implicit, because it needs the root of a polynomial, to obtain the probability generating function. The case for $L \to \infty$ is also discussed. In addition, an explicit expression for the expected duration until absorption is given.

In a similar situation as in [54], where the random walk is allowed to stay at the same position, Neuts [39] gives an implicit representation for probability (3.20), which also needs the root of a polynomial. However, Hadrin and Sweet [24] show, that in this situation for the special case $\omega \in \{p, 2p\}$, there is an explicit formula for probability (3.20). In addition, Neuts [39] discusses the case $N \to \infty$, where N denotes the number of time steps, and also gives an expression for expectation of duration until ruin.

In 1992, El-Shehawy [13] examines the situation in [54], for which the random walk is allowed to stay at the same position, as in Neuts [39]. He gives explicit expressions for the mean and variance of the time until absorption and an implicit expression for (3.20).

The situation with two partially reflecting barriers, i.e. absorption can happen with probability less than one, is discussed by Percus [41] and El-Shehawy [14, 15]. Note, that in [14] the probability of a move to the left or right depends on the current position. Percus [41] derives an explicit expression for the probability of a particle moving from position $u \in \{1, \ldots, L-1\}$ to position $v \in \{1, \ldots, L-1\}$ in $n \in \mathbb{N}$ steps, i.e.

$$\mathrm{P}\left(Y^{(n)} = v \mid Y^{(0)} = u\right). \tag{3.21}$$

El-Shehawy [15] also finds a closed form expression for (3.21), but therein the random walk is allowed to stay at the same position.

In [14], the same author computes mean for time of absorption. He also gives an expression for the probability of absorption at one of the two barriers for the first time at time n when starting at point u with $0 \leq u \leq L$.

In his book, Feller [17, Chapter XIV] discusses the classical gambler's ruin problem, which reads as follows:

Remark 3.2.8 (Gambler's ruin problem) When starting at position $0 < u < L$, what is the probability of being ruined (particle being absorbed at 0), before reaching the target (particle being absorbed at L)? There are two different main settings: The time horizon can either be finite and of given size or infinite time (ultimate ruin). ♦

In the finite time case, Feller [17, Chapter XIV] uses the probability generating function to obtain an explicit expression of this probability. He also gives an explicit expression for the case $L \to \infty$. In case of ultimate ruin, i.e. number of time steps are irrelevant, Feller in addition computes the probability distribution of the duration of the game. Note that the gambler's ruin problem is a classical problem in the theory of Markov chains and is also studied e.g. in [22, Section 2.3.1] and [51, Section 1.14.2].

Besides this random walk, Feller [17, Chapter XIV, Section 8] and Gilliland, Levental and Xiao [19] discuss the same situation but for the more general case of a Markov chain, which can have step size larger than one. Both expressions for probability of the gambler's ruin are again implicit. Upper and lower bounds for this probability, which are much easier to compute, are given by Feller [17, Chapter XIV, Section 8]. Ethier and Khoshnevisan [16] also gave bounds which are less accurate, but even more easier to compute and the Markov chain does not need to be inter-valued, i.e. more general processes walking on $\mathbb{R}$ instead of $\mathbb{Z}$ are allowed. Later, Hürlimann [27] improved these bounds which are more accurate than those in [16] but need a bit more computing time.

Another related but different setting is discussed by Zhang and Hadjiliadis [56]: The drawdown is the difference of the maximum of the equity to the current value. The authors define a rally by the opposite, i.e. by the difference of the minimum of the equity to the current value. They discuss the probability of that a rally of a given size occurs before a drawdown of the same size appears. They derive a formula when the stochastic process is modeled by a random walk and also the case of a Brownian motion.

Magdon-Ismail, Atiya, Pratap, and Abu-Mostafa [32] also studied the Brownian motion (with drift). They are interested in the maximum absolute drawdown and give a representation of its distribution.

3.2.3 Explicit matrix expression

For fixed number $N \in \mathbb{N}$ of time steps, one goal is an explicit expression for the discrete probability distribution of $\mathcal{D}^{(N)}$. The focus is on the absolute drawdown after N steps starting at zero, i.e. on $d_j^{(N)} = d_{0,j}^{(N)}$ of Definition 3.2.4 for $j \in \mathbb{N}_0$. Since the corresponding Markov chain does not have a right barrier, one needs to look at the full line of non-negative integers. However, only N trades with drawdown starting at position zero need to be considered and the largest loss is of size $\alpha \in \mathbb{N}$. Therefore, the overall possible maximum for the absolute drawdown equals to $A := N\alpha \in \mathbb{N}$. Hence, $d_j^{(N)} = 0$ for $j > N\alpha = A$. To ease the computation, it is reasonable to restrict the Markov chain from Figure 3.4 to the state space $\{0, 1, \ldots, A\} \subset \mathbb{N}_0$, because the Markov chain cannot exceed this state space within N steps.

Inserting the initial configuration for the absolute drawdown, e.g. $\mathcal{D}^{(0)} \equiv 0$, into the iterative scheme (3.19) for Definition 3.2.4 leads to the probability distribution after N steps. This can be expressed as matrix vector multiplications as follows: The matrix $\mathbf{Q}_m := (d_{i,j}^{(1)})_{0 \le i \le m,\, 0 \le j \le m+\alpha}$ for $m \in \mathbb{N}_0$ is called 1-step transition matrix and contains the probabilities $d_{i,j}^{(1)} = \mathrm{P}\left(\mathcal{D}^{(1)} = j \mid \mathcal{D}^{(0)} = i\right)$ of moving from any position between zero and m to all possible states on $\mathbb{N}_0$ in one time step. Of course, the absolute drawdown can only move to positions up to $m + \alpha$ because the largest loss equals to α. The matrix is of form

$$\mathbf{Q}_m = \begin{bmatrix} r + \sum\limits_{k=1}^{\beta} p_k & q_1 & \cdots & q_\alpha & & & & & & \\ \sum\limits_{k=1}^{\beta} p_k & r & q_1 & \cdots & q_\alpha & & & \mathbf{0} & & \\ \sum\limits_{k=2}^{\beta} p_k & p_1 & \ddots & \ddots & & \ddots & & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & & & \\ \sum\limits_{k=\beta}^{\beta} p_k & p_{\beta-1} & \cdots & p_1 & r & q_1 & \cdots & q_\alpha & & \\ & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\ \mathbf{0} & & p_\beta & p_{\beta-1} & \cdots & p_1 & r & q_1 & \cdots & q_\alpha \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+\alpha+1)}. \tag{3.22}$$

Assume the initial position is given by $\mathcal{D}^{(0)} \equiv 0$ with probability distribution $\mathbf{d}^{(0)} := [d_0^{(0)}] = [1] \in \mathbb{R}^{1\times 1}$, then the probability distribution $\mathbf{d}^{(1)} := [d_0^{(1)}, \ldots, d_\alpha^{(1)}] \in \mathbb{R}^{1\times(\alpha+1)}$ is obtained as column vector after one step by

$$\mathbf{d}^{(1)} = \mathbf{d}^{(0)}\mathbf{Q}_0 = \mathbf{Q}_0 = \left[r + \sum_{k=1}^{\beta} p_k, q_1, \ldots, q_\alpha\right] \in \mathbb{R}^{1\times(\alpha+1)}.$$

For $m = 1$ the matrix reads

$$\mathbf{Q}_1 = \begin{bmatrix} r + \sum\limits_{k=1}^{\beta} p_k & q_1 & q_2 & \dots & q_\alpha & 0 \\ \sum\limits_{k=1}^{\beta} p_k & r & q_1 & \dots & q_{\alpha-1} & q_\alpha \end{bmatrix} \in \mathbb{R}^{2\times(\alpha+2)}.$$

Assume now it is step $N \in \mathbb{N}$, again with initial state $\mathcal{D}^{(0)} \equiv 0$. Inductively one can compute $\mathbf{d}^{(N)} := [d_0^{(N)}, \dots, d_{N\alpha}^{(N)}] \in \mathbb{R}^{N\alpha+1}$ by

$$\mathbf{d}^{(N)} = \mathbf{d}^{(N-1)}\mathbf{Q}_{(N-1)\alpha} = \mathbf{d}^{(0)}\mathbf{Q}_0\mathbf{Q}_\alpha \cdot \dots \cdot \mathbf{Q}_{(N-1)\alpha} = \mathbf{d}^{(0)} \prod_{n=0}^{N-1} \mathbf{Q}_{n\alpha} = \prod_{n=0}^{N-1} \mathbf{Q}_{n\alpha},$$

which is well-defined by construction. Of course, this scheme is not restricted to the initial condition $\mathcal{D}^{(0)} \equiv 0$. One could also use a general initial distribution $\mathbf{d}^{(0)} = [d_0^{(0)}, \dots, d_m^{(0)}] \in \mathbb{R}^{m+1}$ for $m \in \mathbb{N}_0$ and end up with $\mathbf{d}^{(N)} = \mathbf{d}^{(0)} \prod_{n=0}^{N-1} \mathbf{Q}_{m+n\alpha} \in \mathbb{R}^{m+N\alpha+1}$.

Example 3.2.9 (French Roulette: Part 3/6; see Example 3.2.7) In this example, $\alpha = 2$ and $\beta = 6$ holds. The matrix (3.22) for $m \in \mathbb{N}$ reads

$$\mathbf{Q}_m = \frac{1}{37} \begin{bmatrix} 21 & 0 & 16 & & & & & & & & \\ 6 & 15 & 0 & 16 & & & & & & \mathbf{0} & \\ 6 & 0 & 15 & 0 & 16 & & & & & & \\ 6 & 0 & 0 & 15 & 0 & 16 & & & & & \\ 6 & 0 & 0 & 0 & 15 & 0 & 16 & & & & \\ 3 & 3 & 0 & 0 & 0 & 15 & 0 & 16 & & & \\ 3 & 0 & 3 & 0 & 0 & 0 & 15 & 0 & 16 & & \\ & \ddots & & & & & & & & \ddots & \\ \mathbf{0} & & 3 & 0 & 3 & 0 & 0 & 0 & 15 & 0 & 16 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+\alpha+1)}.$$

Note that each row sums up to one because $\mathbf{d}^{(n)}$ are still distributions for $n \in \mathbb{N}$ (if $\mathbf{d}^{(0)}$ is one). For $N = 2$ the distribution reads

$$\mathbf{d}^{(2)} = \mathbf{Q}_0\mathbf{Q}_2 = \frac{1}{37^2} \begin{bmatrix} 21 & 0 & 16 \end{bmatrix} \begin{bmatrix} 21 & 0 & 16 & 0 & 0 \\ 6 & 15 & 0 & 16 & 0 \\ 6 & 0 & 15 & 0 & 16 \end{bmatrix} = \frac{1}{37^2}[537, 0, 576, 0, 256].$$

Figure 3.5 shows the probability distribution $d_j^{(N)}$ for $N \in \{10, 10^2, 10^3, 10^4\}$ and for $0 \leq j \leq 100$. Since the expectation is negative for the game, it is no surprise that the higher the N the higher the absolute drawdown tends to be. The mass of the probability distribution seems to "deliquesce" in direction to infinity. This behavior is discussed later in Subsection 3.2.5. ♦

3.2.4 State space

The values of the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ in Definition 3.2.4 obviously are in $\mathbb{N}_0$. Nevertheless, it is not necessarily true, that each value in $\mathbb{N}_0$ can be reached. For instance, in case $\alpha = \beta = 3$, $p_3 = q_3 = 1/2$ and initial state $\mathcal{D}^{(0)} \equiv 0$, the Markov chain can only walk on the value $\{3n : n \in \mathbb{N}_0\} \subsetneq \mathbb{N}_0$. However, if $\mathrm{P}(X^{(n)} > 0) = \sum_{i=1}^{\beta} p_i > 0$, position zero can always be reached with positive probability independent on the initial position. Thus, define the state space when starting at the origin.

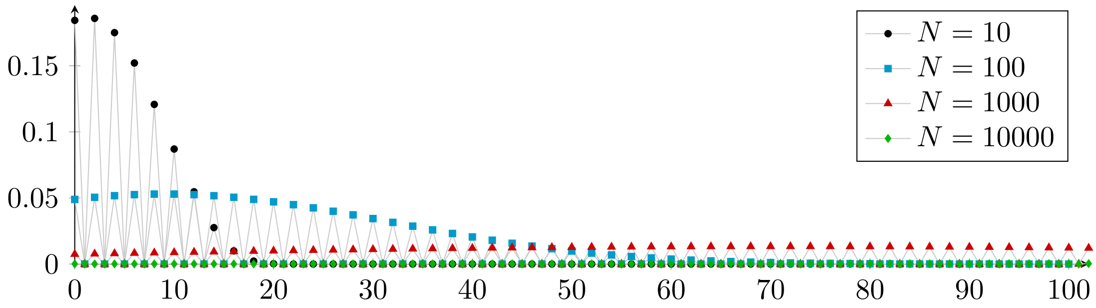


Figure 3.5.: Plot of $d_0^{(N)}, \dots, d_{100}^{(N)}$ for $N \in \{10, 10^2, 10^3, 10^4\}$ in the situation of Example 3.2.9.

Definition 3.2.10 (State space of absolute drawdown when starting at zero) Let the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4 with $\mathcal{D}^{(0)} \equiv 0$. We denote the set of all possible positions the absolute drawdown can reach in finite time with positive probability by the *state space* $K_0 \subset \mathbb{N}_0$.

Of course, for the absolute drawdown in Definition 3.2.4 it is $K_0 = \{0\}$ if and only if $\mathrm{P}\left(X^{(n)} < 0\right) = \sum_{i=1}^{\alpha} q_i = 0$ for $n \in \mathbb{N}$. Under reasonable assumptions it is $K_0 = \mathbb{N}_0$, as shown in the next result.

Theorem 3.2.11 ($K_0 = \mathbb{N}_0$) Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be the absolute drawdown as in Definition 3.2.4 with $\mathrm{P}\left(X^{(n)} > 0\right) = \sum_{i=1}^{\beta} p_i > 0$ and $\mathrm{P}\left(X^{(n)} < 0\right) = \sum_{i=1}^{\alpha} q_i > 0$. Then, for the state space K_0 from Definition 3.2.10 it is $K_0 = h\mathbb{N}_0 := \{hn \in \mathbb{N}_0 : n \in \mathbb{N}_0\}$ with smallest step size

$$h := \gcd\{k \in \mathbb{N} : p_k > 0 \text{ or } q_k > 0\} \in \mathbb{N}, \tag{3.23}$$

where, for a set $G \subset \mathbb{N}$, the expression $\gcd(G)$ denotes the greatest common divisor of all elements in G.

Proof. Let $M_{>0} := \{k \in \mathbb{N} : p_k > 0\}$ be the set of possible gains and $M_{<0} := \{k \in \mathbb{N} : q_k > 0\}$ the set of possible losses after one step. Both sets are non-empty because $\sum_{i=1}^{\beta} p_i > 0$ and $\sum_{i=1}^{\alpha} q_i > 0$, i.e. at least one p_i and one q_i is greater than zero. It is $h = \gcd\{k \in \mathbb{N} : p_k > 0 \text{ or } q_k > 0\} = \gcd(M_{>0} \cup M_{<0})$.

First assume that $h = \gcd(M_{>0} \cup M_{<0}) = 1$. It then has to be shown that $K_0 = \mathbb{N}_0$. A winning series is defined as a linear combination of elements from $M_{>0}$ using non-negative and integer-valued coefficients. The set of all winning series of arbitrary length is denoted by $\overline{M}_{>0} := \{\sum_{i=1}^{n} k_i \in \mathbb{N} : n \in \mathbb{N},\ k_1, \dots, k_n \in M_{>0}\}$, i.e. the closure under addition. Analogously, define all loss series, i.e. the closure under addition of $M_{<0}$, by $\overline{M}_{<0} := \{\sum_{i=1}^{n} k_i \in \mathbb{N} : n \in \mathbb{N},\ k_1, \dots, k_n \in M_{<0}\}$. Now, from Lemma A.5.3 in the appendix one can find winning and loss series $k_{>0} \in \overline{M}_{>0}$ and $k_{<0} \in \overline{M}_{<0}$, which are coprime, i.e. $\gcd\{k_{>0}, k_{<0}\} = \gcd(M_{>0} \cup M_{<0}) = 1$.

Bézout's identity, see e.g. Niven, Zuckerman and Montgomery [40, Theorem 1.3], gives us $r, s \in \mathbb{Z}$, such that $rk_{>0} + sk_{<0} = 1$. Since $k_{>0}, k_{<0} \geq 1$, it must be $r \cdot s \leq 0$, i.e. r and s

have different sign or one of them equals to zero. Thus, distinguish between the following four cases:

(a) If $r = 0$: Then it must be $s = k_{<0} = 1$ and consequently $k_{<0} \in M_{<0}$. In this case the absolute drawdown can make steps of size $+1$ with positive probability. Starting from zero one thus can reach each position in $\mathbb{N}$ stepwise. Hence, it is $K_0 = \mathbb{N}_0$.

(b) If $s = 0$: Now it is $r = k_{>0} = 1$, i.e. the absolute drawdown can make steps of size -1 with positive probability until it reaches position zero. Therefore, if the drawdown for instance reaches position i, it can walk to arbitrary position $j \in \mathbb{N}$ with $0 \le j \le i$. Since $\mathrm{P}\left(X^{(n)} < 0\right) > 0$, the Markov chain $\mathcal{D}$ can hit arbitrary large positions, i.e. for each $L \in \mathbb{N}$ there is a position $i \in \overline{M}_{<0}$ with $i \ge L$. Therefore it must be $K_0 = \mathbb{N}_0$.

(c) If $r < 0$ and $s > 0$: From construction of $k_{<0}$ and $k_{>0}$ it is $d^{(n)}_{i,i+k_{<0}} = d^{(n)}_{0,k_{<0}} > 0$ and $d^{(m)}_{L,L-k_{>0}} > 0$ for some $n, m \in \mathbb{N}$ and all $i, L \in \mathbb{N}_0$, $L \ge k_{>0}$. Now, one can concatenate $|s|$ times the loss series and $|r|$ times the winning series: For arbitrary $i \in \mathbb{N}_0$ the probability of moving from i to $i+1$ in $|s|n + |r|m$ steps is larger than or equal to the probability of first moving from i to $i + sk_{<0}$ in $|s|n$ steps and afterwards from $i + sk_{<0}$ to $i + sk_{<0} + rk_{>0} = i + 1$ in $|r|m$ steps. Hence,

$$d^{(|s|n+|r|m)}_{i,i+1} \ge d^{(|s|n)}_{i,i+sk_{<0}} \cdot d^{(|r|m)}_{i+sk_{<0},i+sk_{<0}+rk_{>0}} \ge \left(d^{(n)}_{0,k_{<0}}\right)^{|s|} \cdot \left(d^{(m)}_{L,L-k_{>0}}\right)^{|r|} > 0,$$

i.e. one can move one step to the right in finite time with positive probability. This can be repeated as long as needed. Starting from zero the absolute drawdown therefore can reach each positive integer, i.e. $K_0 = \mathbb{N}_0$.

(d) If $r > 0$ and $s < 0$: Use the same notation as in the case $r < 0$ and $s > 0$. Since it is $i - sk_{<0} - rk_{>0} = i - 1$, one obtains the same way as above (note that $-r < 0$ and $-s > 0$) that

$$d^{(|s|n+|r|m)}_{i,i-1} \ge d^{(|s|n)}_{i,i-sk_{<0}} \cdot d^{(|r|m)}_{i-sk_{<0},i-sk_{<0}-rk_{>0}} \ge \left(d^{(n)}_{0,k_{<0}}\right)^{|s|} \cdot \left(d^{(m)}_{L,L-k_{>0}}\right)^{|r|} > 0,$$

i.e. one can make one step to the left from each position. Analog to the case $s = 0$ one gets $K_0 = \mathbb{N}_0$, which completes the proof for $h = 1$.

Now assume that $1 < h = \gcd(M_{>0} \cup M_{<0}) \in \mathbb{N}$. Since h is a common divisor of $M_{>0}$ and also of $M_{<0}$ one can scale both sets by $\nu := 1/h$ and observe, that $\nu M_{>0} \subset \mathbb{N}$ and $\nu M_{<0} \subset \mathbb{N}$. Therefore, for all $i \in \mathbb{N}$ such that $i/h \notin \mathbb{N}$ it is known that $p_{i/h} = q_{i/h} = 0$. With other words, the scaled Markov chain can equivalently be examined when rescaling the results afterwards. For the scaled version it is $\tilde{p}_i := p_{hi}$, $1 \le i \le \beta/h$, and $\tilde{q}_i := q_{hi}$, $1 \le i \le \alpha/h$ with set of possible gains and losses given by $\nu M_{>0}$ and $\nu M_{<0}$, respectively. In this case it is $\gcd(\nu M_{>0} \cup \nu M_{<0}) = 1$. From the case $h = 1$ it is known, that the state space for this Markov chain equals to $\mathbb{N}_0$. Rescaling it back by multiplying with h yields $K_0 = h\mathbb{N}_0$. □

Example 3.2.12 (French Roulette: Part 4/6; see Example 3.2.9) For Theorem 3.2.11 applied to this example and using the values given in Example 3.2.7 it is

$$h = \gcd\{k \in \mathbb{N} : p_k > 0 \text{ or } q_k > 0\} = \gcd\{2, 4, 6\} = 2.$$

Hence, it is $K_0 = 2\mathbb{N}_0$, i.e. only even values can be attained by the absolute drawdown, which can be seen in Figure 3.5 where all probabilities at odd numbers are zero. ♦

3.2.5 Limit distribution: general case

For $N \in \mathbb{N}$ steps, the absolute drawdown in Definition 3.2.4 is finite. Hence, its probability distribution given by $d_{i,j}^{(N)}$ for all $j \in K_0$ and fixed initial position $i \in K_0$ can be computed, e.g. as described in Subsection 3.2.3. More questions could be: What happens, when then number of games N goes to infinity? Is there a limit distribution and is it dependent on the initial position i, if it exists?

Some well-known definitions and results on the limit distributions of Markov chains are introduced in the appendix in Section A.4. For Theorem A.4.3, which gives existence and uniqueness of the limit distribution under some assumptions, the Markov chain needs to be irreducible and aperiodic, see Definition A.4.2. Accordingly, the next result proves that the Markov chain build by the absolute drawdown is irreducible within its state space K_0.

Lemma 3.2.13 Let $(X^{(n)})_{n\in\mathbb{N}}$ be iid random variables and $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4. As in Definition 3.2.10 let K_0 be the state space when starting at position zero and assume $\mathrm{P}(\mathcal{D}^{(0)} \in K_0) = 1$.

Then the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is irreducible (see Definition A.4.2 (c)) if and only if $\mathrm{P}(X^{(n)} > 0) = \mathrm{P}(X^{(1)} > 0) = \sum_{i=1}^{\beta} p_i > 0$ or $K_0 = \{0\}$.

Proof. "$\Leftarrow$": Let $i, j \in K_0$ be arbitrary. It must be shown that $d_{i,j}^{(n)} > 0$ for some $n \in \mathbb{N}$. Case 1: If $K_0 = \{0\}$ the absolute drawdown can only attain one state (with probability one), which directly implies that $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is irreducible. Case 2: If $\mathrm{P}(X^{(n)} > 0) > 0$, the absolute drawdown always reaches position zero in finite time with positive probability $d_{i,0}^{(n)} > 0$ for some $n \in \mathbb{N}$. From zero the absolute drawdown reaches each position in K_0 because of its definition, i.e. $d_{0,j}^{(m)} > 0$ for some $m \in \mathbb{N}$. Thus it is $d_{i,j}^{(m+n)} \geq d_{i,0}^{(n)} d_{0,j}^{(m)} > 0$. Since $\mathrm{P}(\mathcal{D}^{(0)} \in K_0) = 1$, the absolute drawdown cannot be at a position which is not contained in K_0. Therefore it is irreducible.

"$\Rightarrow$": Assume $\mathrm{P}(X^{(n)} > 0) = 0$ and $K_0 \neq \{0\}$. Case 1: Assume $\mathrm{P}(X^{(n)} < 0) = 0$, and therefore $r = 1$. For this it is $K_0 = \{0\}$ because $\mathrm{P}(\mathcal{D}^{(0)} \in K_0) = 1$, which is a contradiction. Hence this case is already excluded. Case 2: If $\mathrm{P}(X^{(n)} < 0) > 0$, where w.l.o.g. $q_\alpha > 0$ for $\alpha \in \mathbb{N}$, it is $d_{0,\alpha}^{(1)} = q_\alpha > 0$ and, therefore, $0 < \alpha \in K_0$. Since $\mathrm{P}(X^{(n)} > 0) = 0$, the absolute drawdown cannot become smaller and therefore $d_{\alpha,0}^{(n)} = 0$ for all $n \in \mathbb{N}$. Thus it is not possible to move from $\alpha \in K_0$ to zero with positive probability. Therefore, the absolute drawdown cannot be irreducible. Hence, if $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is irreducible, then $\mathrm{P}(X^{(n)} > 0) > 0$ or $K_0 = \{0\}$ must hold. □

Note that the restriction onto the state space K_0 is important to be able to define a reasonable space for the initial position of the absolute drawdown, such that the Markov chain is irreducible, as shown by the following example.

Example 3.2.14 Assume $\mathrm{P}(X^{(n)} = 2) = p \in (0, 1)$ and $\mathrm{P}(X^{(n)} = -2) = 1 - p$. In this case it is $K_0 = \{2n : n \in \mathbb{N}_0\}$, i.e. the set of all even, non-negative integers. Restricted onto K_0 the absolute drawdown is irreducible because of Lemma 3.2.13.

If values outside of K_0 are allowed for the initial state of the absolute drawdown, e.g. $\mathcal{D}^{(0)} \equiv 5 \notin K_0$, the initial state can be an odd integer. Then the absolute drawdown can also reach each odd, non-negative integer, because it can move two positions to the right in one step with probability $1 - p > 0$ and of course also to the left before it reaches position zero. Once state zero is reached, the absolute drawdown can take only values in K_0 and it can never walk back to odd integers, i.e. it is trapped in K_0. In this special case, the state space is $\mathbb{N}$ but the Markov chain is not irreducible. ♦

As shown in this example, it is essential that the absolute drawdown is restricted onto K_0, for which it is necessary and sufficient to require for its initial state at time zero to be in K_0. Next, some conditions are given to be able to verify whether the absolute drawdown is transient or recurrent. A strongly related result for the case $\alpha = \beta = 1$ has been discussed in [17, Chapter XIV, End of Section 2].

Lemma 3.2.15 Let $(X^{(n)})_{n\in\mathbb{N}}$ be iid random variables and $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4. As in Lemma 3.2.13 assume $\mathrm{P}\left(\mathcal{D}^{(0)} \in K_0\right) = 1$ and in addition $\mathrm{P}\left(X^{(n)} > 0\right) > 0$, i.e. the absolute drawdown is irreducible. Then it is:

(a) If $\mathrm{E}\left[X^{(n)}\right] < 0$, then $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is transient (see Definition A.4.2 (b)).
(b) If $\mathrm{E}\left[X^{(n)}\right] \geq 0$, then $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is recurrent (see Definition A.4.2 (b)).

Proof. Let $f_{i,j}$ in Definition A.4.2 (a) now be the corresponding probability for $(Y^{(n)})_{n\in\mathbb{N}_0} := (\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$. Since the absolute drawdown is irreducible by Lemma 3.2.13, Theorem A.4.4 can be applied, and thus it suffices to show that state zero is transient (i.e. $f_{0,0} < 1$) if $\mathrm{E}\left[X^{(n)}\right] < 0$, or recurrent (i.e. $f_{0,0} = 1$) if $\mathrm{E}\left[X^{(n)}\right] \geq 0$, respectively. It is

$$f_{0,0} = r + \sum_{k=0}^{\beta} p_k + \sum_{k=1}^{\alpha} q_k f_{k,0} \overset{!}{=} 1 \quad \Leftrightarrow \quad f_{k,0} = 1 \text{ for all } k \in \mathbb{N} \text{ with } q_k > 0. \tag{3.24}$$

Assume that $f_{k,0} = 1$ for $k \in \mathbb{N}$ if and only if $\mathrm{E}\left[X^{(n)}\right] \geq 0$, where $(X^{(n)})_{n\in\mathbb{N}}$ are iid and $k \in \mathbb{N}$ is arbitrary and independent on q_k. This implies, that either $f_{k,0} = 1$ for all $k \in \mathbb{N}$ or $f_{k,0} < 1$ for all $k \in \mathbb{N}$. Consequently, (3.24) gives that $f_{0,0} = 1$ if and only if $\mathrm{E}\left[X^{(n)}\right] \geq 0$. For this, it remains to show that $f_{k,0} = 1$ if and only if $\mathrm{E}\left[X^{(n)}\right] \geq 0$.

Probability $f_{k,0}$ corresponds to the probability of reaching zero eventually when starting at k. This can be seen as the probability of getting absorbed at zero. When ignoring the actual purpose of absolute drawdown, $f_{k,0}$ can also be characterized by the probability of getting ruined when starting at k. Thus, this situation can be interpreted as the gambler's ruin problem, see Remark 3.2.8 in Section 3.2.2, where the "target" lies at infinity.

For this, assume $\mathcal{D}^{(0)} = k \in \mathbb{N}$ a.s. Then, the drawdown can be modeled similar to (3.18) by the corresponding Markov chain $(Z^{(n)})_{n\in\mathbb{N}_0}$ from Definition 3.2.2 with fixed initial value $Z^{(0)} \in \mathbb{Z}$ and having an imaginary maximum (before time 0) with value $Z^{(0)} + k$, i.e.

$$\mathcal{D}^{(N)} = \max\left\{Z^{(0)} + k, \max_{0\leq n\leq N}\left\{Z^{(n)}\right\}\right\} - Z^{(N)}, \tag{3.25}$$

for $N \geq 1$. Once the drawdown reaches the value zero for the first time it behaves like $\max_{0\leq n\leq N}\{Z^{(n)}\} - Z^{(N)}$, i.e. as the usual drawdown in (3.18). But before this, as long as the drawdown is above zero, $\mathcal{D}^{(N)}$ behaves like $Z^{(0)} + k - Z^{(N)}$ and it is

$$\begin{aligned}
&\mathrm{P}\left(\mathcal{D}^{(1)} > 0, \ldots, \mathcal{D}^{(N-1)} > 0, \mathcal{D}^{(N)} = 0 \mid \mathcal{D}^{(0)} = k\right) \\
&\qquad = \mathrm{P}\left(Z^{(1)} < Z^{(0)} + k, \ldots, Z^{(N-1)} < Z^{(0)} + k, Z^{(N)} \geq Z^{(0)} + k\right).
\end{aligned} \tag{3.26}$$

Consequently, for all $k \in \mathbb{N}$

$$\begin{aligned} f_{k,0} &= \mathrm{P}\left(\text{there is } N \in \mathbb{N} \text{ such that } \mathcal{D}^{(N)} = 0 \mid \mathcal{D}^{(0)} = k\right) \\ &= \mathrm{P}\left(\text{there is } N \in \mathbb{N} \text{ such that } Z^{(N)} \geq k + Z^{(0)}\right) \\ &= \mathrm{P}\left(\text{there is } N \in \mathbb{N} \text{ such that } -Z^{(N)} \leq 0 \mid -Z^{(0)} = k\right). \end{aligned} \tag{3.27}$$

Proposition A.3.1 estimates the probability $\mathrm{P}\left(Z^{(N_L)} \leq 0 \mid Z^{(0)} = k\right)$ from below and from above for $L \in \mathbb{N}$ with $L > k$ and $N_L = \min\{n \in \mathbb{N} : Z^{(n)} \leq 0 \text{ or } Z^{(n)} \geq L\}$, see (A.4). To show that $f_{k,0} = 1$, this result can be applied to $(-Z^{(n)})_{n\in\mathbb{N}_0}$.

In the following, replace $Z^{(n)}$ in Proposition A.3.1 by $-Z^{(n)}$ and consequently $X^{(n)}$ by $-X^{(n)}$. Then, for $N_L = \min\{n \in \mathbb{N} : -Z^{(n)} \leq 0 \text{ or } -Z^{(n)} \geq L\}$ define $A_L := \{-Z^{(N_L)} \leq 0\}$ where

$$A_L = \left\{\text{there is } N \in \mathbb{N} \text{ such that } -Z^{(0)} < L, \ldots, -Z^{(N-1)} < L, -Z^{(N)} \leq 0\right\},$$

and $A_L \subset A_{L+1}$ holds true for all $L > k$. Define $B_L := A_L \setminus A_{L-1}$ for $L > k+1$ and $B_{k+1} := A_{k+1}$, which gives a disjoint sequence $(B_L)_{L \geq k+1}$. It follows, that

$$\begin{aligned} f_{k,0} = \mathrm{P}\left(\bigcup_{L=k+1}^{\infty} A_L \mid -Z^{(0)} = k\right) &= \mathrm{P}\left(\bigcup_{L=k+1}^{\infty} B_L \mid -Z^{(0)} = k\right) \\ &= \lim_{L\to\infty} \sum_{\ell=k+1}^{L} \mathrm{P}\left(B_\ell \mid -Z^{(0)} = k\right) = \lim_{L\to\infty} \mathrm{P}\left(A_L \mid -Z^{(0)} = k\right). \end{aligned} \tag{3.28}$$

Hence, it suffices to discuss the limit of $\mathrm{P}\left(A_L \mid -Z^{(0)} = k\right) = \mathrm{P}\left(-Z^{(N_L)} \leq 0 \mid -Z^{(0)} = k\right)$ as L goes to infinity.

Applying now Proposition A.3.1 to $(-Z^{(n)})_{n\in\mathbb{N}_0}$ starting at $k \in \mathbb{N}$ and with target at $L \in \mathbb{N}$, such that $1 \leq k \leq L-1$, gives

$$\begin{aligned} \frac{L-k}{L+\beta-1} &\leq \mathrm{P}\left(-Z^{(N_L)} \leq 0 \mid -Z^{(0)} = k\right) \leq \frac{L+\alpha-k-1}{L+\alpha-1}, && \text{if } \mathrm{E}\left[-X^{(n)}\right] = 0, \\ \frac{\rho_1^L - \rho_1^k}{\rho_1^L - \rho_1^{-\beta+1}} &\leq \mathrm{P}\left(-Z^{(N_L)} \leq 0 \mid -Z^{(0)} = k\right) \leq \frac{\rho_1^{L+\alpha-1} - \rho_1^k}{\rho_1^{L+\alpha-1} - 1}, && \text{if } \mathrm{E}\left[-X^{(n)}\right] \neq 0, \end{aligned}$$

where in this case $\rho_1 > 1$, if $\mathrm{E}\left[-X^{(n)}\right] < 0$, and $\rho_1 \in (0,1)$, if $\mathrm{E}\left[-X^{(n)}\right] > 0$. This together with (3.28) yields that

$$f_{k,0} = \lim_{L\to\infty} \mathrm{P}\left(-Z^{(N_L)} \leq 0 \mid -Z^{(0)} = k\right) = 1$$

if and only if $\mathrm{E}\left[X^{(n)}\right] \geq 0$. □

Since Theorem A.4.3 also requires the Markov chain to be aperiodic (see Definition A.4.2 (d)), the next lemma gives appropriate conditions for the absolute drawdown.

Lemma 3.2.16 Let $(X^{(n)})_{n\in\mathbb{N}}$ be iid random variables and $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4. Assume $\mathrm{P}(\mathcal{D}^{(0)} \in K_0) = 1$ and $\mathrm{P}(X^{(n)} > 0) > 0$. Then the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is aperiodic, i.e. it has period one.

Proof. $\mathcal{D}^{(i)}$ is irreducible by Lemma 3.2.13. Let $d_{0,0}^{(n)} = \mathrm{P}(\mathcal{D}^{(n)} = 0 \mid \mathcal{D}^{(0)} = 0)$ as in Definition 3.2.4. Since $\mathrm{P}(X^{(n)} > 0) > 0$, it is known that $1 \in \{n \in \mathbb{N} : d_{0,0}^{(n)} > 0\}$, i.e. state zero has period one. Lemma A.4.5 completes the proof. □

Now, all requirements are given to apply Theorem A.4.3 to the absolute drawdown to obtain the existence of a limit distribution (see Definition A.4.1).

Theorem 3.2.17 Let $(X^{(n)})_{n\in\mathbb{N}}$ be iid random variables and $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4. As in Definition 3.2.10 let K_0 be the state space when starting at position zero and assume $\mathrm{P}(\mathcal{D}^{(0)} \in K_0) = 1$. In addition, let $\mathrm{P}(X^{(n)} > 0) > 0$ and $\mathrm{P}(X^{(n)} < 0) > 0$. Then, the absolute drawdown is irreducible, aperiodic, $K_0 \neq \{0\}$, and for $d_j := \lim_{N\to\infty} d_j^{(N)}$, where $d_j^{(N)} = d_{0,j}^{(N)}$, see Definition 3.2.4, the following holds:

(a) If $\mathrm{E}[X^{(i)}] \le 0$, then there is no limit distribution and it is $d_j = 0$ for all $j \in K_0$.
(b) If $\mathrm{E}[X^{(i)}] > 0$, then there exists a unique limit distribution and $d_j > 0$ for all $j \in K_0$ and $\sum_{k\in K_0} d_k = 1$. In particular, d_j does not depend on the initial state $\mathcal{D}^{(0)}$. In addition, $d_j = 1/\mu_j$ holds with the mean recurrence time μ_j.

Proof. Distinguish the following three cases.

(i) If $\mathrm{E}[X^{(i)}] < 0$, Lemma 3.2.15 yields that the Markov chain is transient. The transient case in Theorem A.4.3 implies that there is no limit distribution and that the limit equals to zero.
(ii) If $\mathrm{E}[X^{(i)}] > 0$, it is known from Lemma 3.2.15 that the absolute drawdown is recurrent. Now, use the notation from Definition A.4.2 for the case $(Y^{(n)})_{n\in\mathbb{N}_0} := (\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$. As in the proof of Lemma 3.2.15, let $(Z^{(n)})_{n\in\mathbb{N}_0}$ be the Markov chain from Definition 3.2.2 which belongs to $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$, cf. (3.25). As in (3.26) and (3.27), it can be shown that

$$f_{k,0}^{(n)} = \mathrm{P}\left(-Z^{(1)} > 0, \ldots, -Z^{(n-1)} > 0, -Z^{(n)} \le 0 \mid -Z^{(0)} = k\right)$$

for $k, n \in \mathbb{N}$, where $f_{k,0}^{(n)}$ is defined in Definition A.4.2 (a).
In the situation of Proposition A.3.2 applied to $(-Z^{(n)})_{n\in\mathbb{N}_0}$, the random variable N_L for $L \in \mathbb{N}$ is given by

$$N_L = \min\{n \in \mathbb{N} : -Z^{(n)} \le 0 \text{ or } -Z^{(n)} \ge L\}.$$

Define the (pointwise) limit of the monotone increasing sequence $(N_L)_{L\in\mathbb{N}}$ by

$$N_\infty := \lim_{L\to\infty} N_L = \min\{n \in \mathbb{N} : -Z^{(n)} \le 0\}, \tag{3.29}$$

which can be infinity in some cases. The monotone convergence theorem, see e.g. Billingsley [5, Theorem 16.2], then yields that

$$\lim_{L\to\infty} \mathrm{E}\left[N_L \mid -Z^{(0)} = k\right] = \mathrm{E}\left[\lim_{L\to\infty} N_L \mid -Z^{(0)} = k\right] = \mathrm{E}\left[N_\infty \mid -Z^{(0)} = k\right] \tag{3.30}$$

Next, define the mean time to walk from $k \in \mathbb{N}$ to 0 by

$$\mu_{k,0} := \sum_{n=1}^{\infty} n f_{k,0}^{(n)} = \mathrm{E}\left[N_\infty \mid -Z^{(0)} = k\right]. \tag{3.31}$$

Proposition A.3.2 in the limit case as $L \to \infty$ applied to $(-Z^{(n)})_{n\in\mathbb{N}_0}$ with $\mathrm{E}[-X^{(i)}] < 0$ and $-X^{(i)} \in \{-\alpha, \dots, \beta\}$, see also Ethier and Khoshnevisan [16, Corollary in Section 3], and (3.30) imply that

$$0 < -\frac{k}{\mathrm{E}[-X^{(i)}]} \le \lim_{L\to\infty} \mathrm{E}\left[N_L \mid -Z^{(0)} = k\right] = \mu_{k,0} \le -\frac{k+\beta-1}{\mathrm{E}[-X^{(i)}]} < \infty \tag{3.32}$$

for all $k \in \mathbb{N}$.

The mean recurrence time $\mu_0 = \sum_{n=1}^{\infty} n f_{0,0}^{(n)}$ in Definition A.4.2 (e) can be computed by splitting all possible situations into the paths reaching zero after just one step and the paths being at position $k \in \{1, \dots, \alpha\}$ after step one and then walking back to zero after at least one additional step. Linearity of the expectation then gives

$$\mu_0 = 1 \cdot \left(r + \sum_{k=1}^{\beta} p_k\right) + \sum_{k=1}^{\alpha} (1 + \mu_{k,0}) q_k \tag{3.33}$$

which is finite because of (3.32). Therefore, it must be the positive recurrent case of Theorem A.4.3, which proves the second part.

(iii) If $\mathrm{E}[X^{(i)}] = 0$, it is known again from Lemma 3.2.15 that the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ is recurrent. The drawdown constructed from the returns $\widetilde{X}^{(i)} := K \cdot X^{(i)} + 1 \in \{-K\beta + 1, \dots, K\alpha + 1\}$, $i \in \mathbb{N}$, for arbitrary but fixed $K \in \mathbb{N}$ (and corresponding Markov chain given by $\widetilde{Z}^{(n)} := K \cdot Z^{(n)} + n$ for $n \in \mathbb{N}_0$) fits into case (ii) (i.e. (b) holds for this drawdown). The left inequality (3.32) together with (3.30) (with $\ell \in \mathbb{N}$ instead of k) gives

$$-\frac{\ell}{\mathrm{E}[-\widetilde{X}^{(i)}]} \le \mathrm{E}\left[N_\infty^K \mid -\widetilde{Z}^{(0)} = \ell\right], \tag{3.34}$$

where

$$N_\infty^K := \min\left\{n \in \mathbb{N} : -\widetilde{Z}^{(n)} = -K \cdot Z^{(n)} - n \le 0\right\} = \min\left\{n \in \mathbb{N} : -Z^{(n)} \le \frac{n}{K}\right\}.$$

Note that $\mathrm{E}[\widetilde{X}^{(i)}] = K \cdot \mathrm{E}[X^{(i)}] + 1 = 1$, and $\widetilde{Z}^{(0)} = K \cdot Z^{(0)}$. For $\ell := K \cdot k$ the inequality (3.34) implies

$$K \cdot k \le \mathrm{E}\left[N_\infty^K \mid -Z^{(0)} = k\right].$$

Since $N_\infty^K \le \min\{n \in \mathbb{N} : -Z^{(n)} \le 0\} = N_\infty$ (see (3.29)) a.s. for all $k \in \mathbb{N}$ (under the condition that $-Z^{(0)} = k$) the same holds true for their conditional expected values, and therefore, together with (3.31),

$$K \cdot k \le \mathrm{E}\left[N_\infty^K \mid -Z^{(0)} = k\right] \le \mathrm{E}\left[N_\infty \mid -Z^{(0)} = k\right] = \mu_{k,0}$$

for all $K \in \mathbb{N}$. Taking the limit $K \to \infty$ implies, that $\mu_{k,0} = \infty$. Because of (3.33) it then must be $\mu_0 = \infty$. Therefore, it must be the null recurrent case (b) of Theorem A.4.3, which completes the proof.

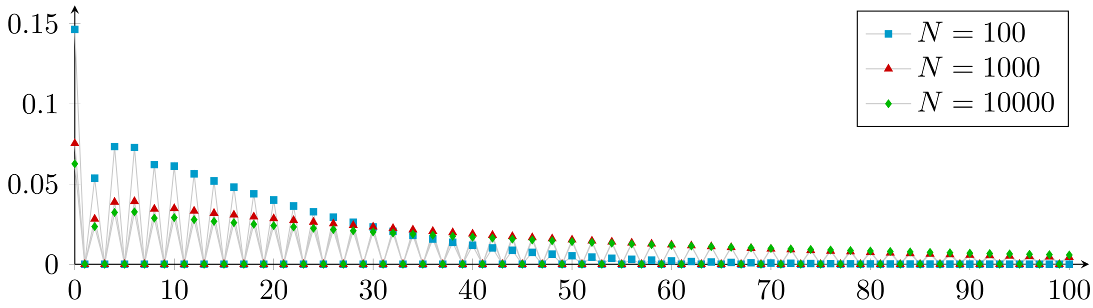


Figure 3.6.: Plot of $d_0^{(N)}, \dots, d_{100}^{(N)}$ for $N \in \{10^2, 10^3, 10^4\}$ in the situation of Example 3.2.19, cf. Example 3.2.9 and Figure 3.5.

□

Remark 3.2.18 ($\alpha = \beta = 1$) In case of a discrete-time birth death process, i.e. in case that $\alpha = \beta = 1$, see Remark 3.2.6, Theorem 3.2.17 already is proved e.g. by Sericola [51, Theorem 3.1 in Section 3.1] and Graham [22, Section 3.2.4.3] (note that the roles of p and q therein are interchanged). ♦

Example 3.2.19 (French Roulette: Part 5/6; see Example 3.2.12) In Example 3.2.1 it has already been noticed that $\mathrm{E}\left[X^{(n)}\right] = -2/37 < 0$. Theorem 3.2.17 states for this case that there is no limit distribution, which was already indicated by Figure 3.5. Therefore, arbitrary large absolute drawdowns can be expected if the game is played infinitely many often.

However, for the bank this game is much more lucrative, because the bank has a positive expectation and thus there exists a limit distribution for the absolute drawdown. The outcome of the nth game for the bank can be modeled by the random variable $\widetilde{X}^{(n)} := -X^{(n)}$ with $\mathrm{E}\left[\widetilde{X}^{(n)}\right] = 2/37 > 0$, $\tilde{\alpha} = 6$, $\tilde{\beta} = 2$, and

$$(q_6, q_5, q_4, q_3, q_2, q_1, r, p_1, p_2) = \left(\frac{3}{37}, 0, \frac{3}{37}, 0, 0, 0, \frac{15}{37}, 0, \frac{16}{37}\right)$$

The corresponding probability distribution $d_j^{(N)}$ for different values of j and N is shown in Figure 3.6, which indicates the limit distribution. ♦

3.2.6 Limit distribution: a special case

For the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ in Definition 3.2.4 consider the case $\beta \in \mathbb{N}$ and $\alpha = 1$, where $p := p_\beta$, $q := q_1$ and $p + q = 1$, see Figure 3.7. For this an implicit expression for the limit distribution $d_j := \lim_{N\to\infty} d_j^{(N)}$, $j \in K_0$ with $d_j^{(N)} = d_{0,j}^{(N)}$ from Definition 3.2.4, can be given, which involves to compute the root of a polynomial, as shown in the following.

In this case, there are iid random variables $X^{(n)} \colon \Omega \to \{-1, \beta\}$ for $n \in \mathbb{N}$, where $\mathrm{P}\left(X^{(n)} = \beta\right) = p$ is the probability of a winning and $P(X^{(n)} = -1) = q$ the probability of a loosing trade. If the expectation is positive, i.e. $\beta p - q > 0$, then Theorem 3.2.17 yields the existence and uniqueness of a limit distribution and $d_j > 0$ for all $j \in K_0$. Therefore the following result holds true.

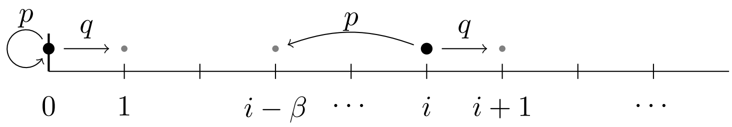


Figure 3.7.: Markov chain of Definition 3.2.4 with $\alpha = 1$, $\beta \in \mathbb{N}$, $p + q = 1$ (with $p := p_\beta$ and $q := q_1$) and reflecting barrier at 0.

Theorem 3.2.20 Let $(X^{(n)})_{n\in\mathbb{N}}$ be iid random variables and $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.2.4. Let $\beta \in \mathbb{N}$, $\alpha := 1$, $p := p_\beta > 0$, $q := q_1 > 0$ and $p + q = 1$, where $\mathrm{E}[X^{(n)}] = \beta p - q > 0$. Then the corresponding state space from Definition 3.2.10 is $K_0 = \mathbb{N}_0$ and for $d_j := \lim_{N\to\infty} d_j^{(N)}$ it is

$$d_0 = 1 - \lambda > 0, \tag{3.35a}$$

$$d_j = d_{j-1}\lambda = d_0\lambda^j = (1-\lambda)\lambda^j > 0, \qquad j \in \mathbb{N}, \tag{3.35b}$$

where λ is the unique real positive root unequal to one of $\rho = q + p\rho^{\beta+1}$ (in ρ), and it is $\lambda \in (0, 1)$.

Proof. Since $p, q > 0$, it is known from Theorem 3.2.11, that $K_0 = \mathbb{N}_0$, because $h = 1$ with h in (3.23). From Theorem 3.2.17 it is known that there exists a unique limit distribution, because $\mathrm{E}[X^{(n)}] = \beta p - q > 0$, which is equivalent to $q/p < \beta$.

Because of the definition of a stationary/limit distribution, see Definition A.4.1, the limit distribution $(d_j)_{j\in\mathbb{N}_0}$ must follow the recursion formula, which reads

$$d_0 = \sum_{\ell=0}^{\beta} p d_\ell, \tag{3.36a}$$

$$d_j = q d_{j-1} + p d_{j+\beta}, \qquad j \in \mathbb{N}, \tag{3.36b}$$

$$\sum_{j\in\mathbb{N}_0} d_j = 1. \tag{3.36c}$$

Inserting the ansatz $d_j = \rho^j$, where $\rho \neq 0$, directly into (3.36b) leads to

$$\rho^j = q\rho^{j-1} + p\rho^{j+\beta} \quad \Leftrightarrow \quad 1 = q\rho^{-1} + p\rho^\beta.$$

As in Feller [17, Chapter XIV, Section 8], see also Proposition A.3.1 in the appendix, the function ϕ, given by $\phi(\rho) := q\rho^{-1} + p\rho^\beta$, $\rho \in \mathbb{R}_{>0}$, has exactly two positive real values ρ_1 and ρ_2 such that $\phi(\rho_1) = \phi(\rho_2) = 1$, because of the following: It is $\phi(1) = p + q = 1$, $\phi(\rho) \to \infty$ as $\rho \to \infty$ and also as $\rho \searrow 0$, and ϕ is strictly convex and continuous as a linear combination of the strictly convex and continuous functions ρ^{-1} and ρ^β with positive coefficients q and p, respectively. Since $\phi'(1) = -q + \beta p = \mathrm{E}[X^{(n)}] > 0$, it is $\rho_1 = 1$ and $\lambda := \rho_2 \in (0, 1)$.

Using this, it is known that $A + B\lambda^j$ solves (3.36b) for all $A, B \in \mathbb{R}$. Now it is needed to choose A, B such that $A + B\lambda^j$, $j \in \mathbb{N}_0$, also fulfill (3.36a) and (3.36c). The next observation is that

$$\phi(\lambda) = 1 \quad \Leftrightarrow \quad 1 - p + p\lambda^{\beta+1} = \lambda \quad \Leftrightarrow \quad p \cdot \frac{\lambda^{\beta+1} - 1}{\lambda - 1} = 1 \quad \Leftrightarrow \quad \sum_{\ell=0}^{\beta} p\lambda^\ell = 1.$$

Inserting $d_j = A + B\lambda^j$ into (3.36a) yields

$$A + B = \sum_{\ell=0}^{\beta} p(A + B\lambda^\ell) = p(\beta+1)A + B\sum_{\ell=0}^{\beta} p\lambda^\ell = p(\beta+1)A + B,$$

which holds true if and only if $(p\beta - q)A = 0$. Since $p\beta - q = \mathrm{E}\,[X^{(n)}] > 0$ it must be $A = 0$. Hence, it is $d_j = B\lambda^j$. Inserting this into (3.36c) when choosing $B := d_0$ gives

$$1 = \sum_{k\in\mathbb{N}_0} d_k = d_0 \sum_{k\in\mathbb{N}_0} \lambda^k = \frac{d_0}{1-\lambda}$$

which holds true if and only if $d_0 = 1 - \lambda$. Hence, (3.35) defines a limit distribution, which in addition must be unique by Theorem 3.2.17 (b), see also Theorem A.4.3 (c) in the appendix. □

Remark 3.2.21 From (3.36a) and (3.36b) the pattern reads

$$\begin{aligned}
d_0 &= \sum_{\ell=0}^{\beta} pd_\ell && \Leftrightarrow && qd_0 = p\sum_{\ell=1}^{\beta} d_\ell,\\
d_1 &= qd_0 + pd_{1+\beta} = \sum_{\ell=1}^{\beta} pd_\ell + pd_{1+\beta} && \Leftrightarrow && qd_1 = p\sum_{\ell=1}^{\beta} d_{1+\ell},\\
&\vdots && && \vdots\\
d_j &= qd_{j-1} + pd_{j+\beta} = \sum_{\ell=1}^{\beta} pd_{j-1+\ell} + pd_{j+\beta} && \Leftrightarrow && qd_j = p\sum_{\ell=1}^{\beta} d_{j+\ell},
\end{aligned}$$

i.e. each value d_j equals to a scaled sum of its β neighbors to the right.

From inserting the ansatz $d_j = \rho^j$ into the right hand side one can conclude, that λ in Theorem 3.2.20 is also the unique real positive root (in ρ) of $q/p = \sum_{\ell=1}^{\beta} \rho^\ell$. One obtains the same result when $p\rho^{\beta+1} - \rho + q$ is divided by $(\rho - 1)$, i.e. by factoring out the root one. All together, the root λ only depends on β and the quotient of p and q. ♦

Remark 3.2.22 Note, that the limit distribution (3.35) in Theorem 3.2.20 is the well-known geometric distribution with values in $\mathbb{N}_0$, see e.g. Johnson, Kemp, and Kotz [28, Section 5.2] (with $k := 1$, $P := \lambda/(1-\lambda)$ and $Q := 1/(1-\lambda)$ in the literature). ♦

The result in Theorem 3.2.20 can be used also for the case that $P(X^{(n)} = 0) = r \neq 0$ as follows.

Corollary 3.2.23 Assume the situation in Theorem 3.2.20 is given but $p+q < 1$ (instead of $p+q = 1$) and $r := 1-p-q \in (0,1)$, i.e. it is also possible for the absolute drawdown to stay at the current position after the next step with probability r. Then the limit distribution $d_j := \lim_{N\to\infty} d_j^{(N)}$, $j \in \mathbb{N}_0 = K_0$, is defined by (3.35) where λ again is the unique real positive root of $q/p = \sum_{\ell=1}^{\beta} \rho^\ell$, see Remark 3.2.21, albeit the fact that $p + q < 1$.

Proof. The relation (3.36) reads in this more general setting

$$\begin{aligned} d_0 &= rd_0 + \sum_{\ell=0}^{\beta} pd_\ell, \\ d_j &= qd_{j-1} + rd_j + pd_{j+\beta}, \qquad j \in \mathbb{N}, \\ \sum_{j\in\mathbb{N}_0} d_j &= 1. \end{aligned}$$

which is equivalent to

$$\begin{aligned} d_0 &= \sum_{\ell=0}^{\beta} \frac{p}{1-r} d_\ell, \\ d_j &= \frac{q}{1-r} d_{j-1} + \frac{p}{1-r} d_{j+\beta}, \qquad j \in \mathbb{N}, \\ \sum_{j\in\mathbb{N}_0} d_j &= 1. \end{aligned}$$

where $\frac{q}{1-r} + \frac{p}{1-r} = 1$, and thus is equivalent to the case with $r = 0$ but with probabilities $\frac{p}{1-r}$ and $\frac{q}{1-r}$, see (3.36). Theorem 3.2.20 and Remark 3.2.21 gives the solution, which only depends on the quotient $(\frac{q}{1-r})/(\frac{p}{1-r}) = q/p$ and β. □

The expression in Theorem 3.2.20 and Corollary 3.2.23 for the limit distribution are of implicit type. However, in some special cases the value λ can be given in explicit form.

Corollary 3.2.24 In the situation of Theorem 3.2.20 and also of Corollary 3.2.23 one can give the value λ for the following special cases.

(a) If $\beta = 1$, it is $\lambda = q/p$.
(b) If $\beta = 2$, it is $\lambda = -1/2 + \sqrt{q/p + 1/4}$.

Proof. If $\beta = 1$, then λ is the positive root of $q/p = \rho$. If $\beta = 2$, then λ is the positive root of $q/p = \rho + \rho^2$ and therefore λ is the positive value of the two roots $\rho_{1,2} = -1/2 \pm \sqrt{q/p + 1/4}$. □

Remark 3.2.25 ($\alpha = \beta = 1$) If $\alpha = \beta = 1$ (see also Remark 3.2.18), Corollary 3.2.24 gives the limit distribution

$$d_j = \left(1 - \frac{q}{p}\right)\left(\frac{q}{p}\right)^j \qquad \text{for } j \in \mathbb{N}_0.$$

This is also proven e.g. by Sericola [51, end of Section 3.1] and Graham [22, Section 3.2.4.3] (note that the roles of p and q therein are interchanged). ♦

For the expectation of the limit distribution one obtains the following.

Corollary 3.2.26 In the situation of Theorem 3.2.20 and also of Corollary 3.2.23 it is

$$\mathrm{E}\left[\mathcal{D}^{(\infty)}\right] := \sum_{j\in\mathbb{N}_0} j d_j = \frac{\lambda}{1-\lambda}, \tag{3.37}$$

where λ is the unique real positive root of $q/p = \sum_{\ell=1}^{\beta} \rho^{\ell}$, see Remark 3.2.21. For $\beta \in \{1,2\}$ it is

$$\mathrm{E}\left[\mathcal{D}^{(\infty)}\right] = \frac{q}{p-q} \qquad \text{and} \qquad \mathrm{E}\left[\mathcal{D}^{(\infty)}\right] = \frac{\sqrt{4pq+p^2}+2q-p}{4p-2q},$$

respectively.

Proof. Since $(d_j)_{j\in\mathbb{N}_0}$ forms a geometric distribution, see Remark 3.2.22, the expected value in (3.37) is well-known, see e.g. Johnson, Kemp, and Kotz [28, Equation (5.5)] (for $k := 1$, $P := \lambda/(1-\lambda)$ and $Q := 1/(1-\lambda)$ in the literature). □

Example 3.2.27 (French Roulette: Part 6/6; see Example 3.2.19) Assume only bet (ii) in Example 3.2.1 is played with stake \$ 1. For the bank, the probability for a win of size 1 is $p = 19/37$ (with $\beta = 1$) and for a loss of size 1 is $q = 18/37$. Of course, the expected value is positive. Hence, Corollaries 3.2.24 and 3.2.26 can be applied and yield for infinite many games with $\lambda = q/p = 18/19$ the limit distribution

$$d_j = \frac{1}{19}\left(\frac{18}{19}\right)^j, \quad j \in \mathbb{N}_0, \qquad \text{and} \qquad \mathrm{E}\left[\mathcal{D}^{(\infty)}\right] = 18,$$

i.e. when playing infinitely many often, the expected value for the wealth of the bank goes to infinity while the expected value for the drawdown is just \$ 18. In this limit case the probability of a drawdown less than or equal to 18 is about 64 %.

Note, that bet (i) in Example 3.2.1 has negative expectation for the gambler and positive expectation for the bank. However, the bank can loose 5 or win 1 in one of such game, i.e. $\alpha = 5$ and $\beta = 1$. Hence, bet (i) does not fit into the special situation of this subsection. ♦

3.3 Random walk

Given a coin which does not necessarily need to be fair, the game "coin tossing" can be played. If the coin shows "heads" one wins \$ 1 (with probability $p \in [0,1]$), if it shows "tails" one looses \$ 1 (with probability $q = 1-p$). If this game is played $N \in \mathbb{N}$ times, what can be expected for the absolute drawdown after the Nth game, i.e. what is the absolute difference between the maximum of the wealth process/equity curve (where one should have stopped playing) and the wealth after the Nth game, cf. Figure 3.8?

This fits into the situation discussed in Section 3.2. In the following, a simpler stochastic process, or more precisely a special case, namely the random walk, is discussed. The random walk is a Markov chain on $\mathbb{Z}$, where the maximum allowed step size is one. As the probability distribution of the absolute drawdown for a Markov chain can explicitly be computed via matrix vector multiplications, see Section 3.2.3, the goal is now, to derive a simpler expression for the special case of a random walk.

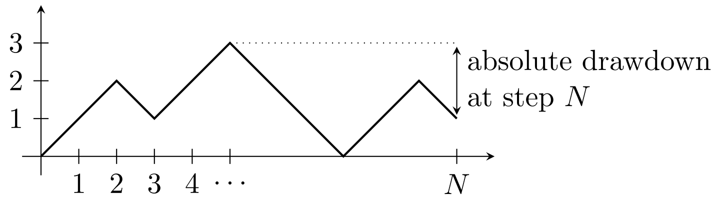


Figure 3.8.: Illustration of the absolute drawdown after N games for a given equity curve.

In Subsection 3.3.1 the random walk and its absolute drawdown is shortly introduced. For this process, different explicit expressions for the probability of the absolute drawdown are given: The first version derived in Subsection 3.3.2 is a more constructive variant which uses results from the literature. However, from some observations on the absolute drawdown for a given random walk, it is possible to derive a more compact expression for the probability mass function, which is done is Subsection 3.3.3. In the last section, see Subsection 3.2.5 and 3.2.6, the limit distribution has been discussed in detail in the more general Markov chain case, which can be applied also to the random walk case. To see the connection, the limit distribution is derived directly from the expression of the probability mass function in Subsection 3.3.4.

3.3.1 Definitions

Using the same notation as in Subsection 3.2.1, the random walk is just the special case $\alpha = \beta = 1$. For this, define $p := p_1$ and $q := q_1$. This leads to the following simplifications of Definitions 3.2.2 and 3.2.4.

Definition 3.3.1 (Random walk and absolute drawdown) Let $X^{(n)} \colon \Omega \to \{-1, 0, 1\}$ for $n \in \mathbb{N}$ be iid random variables, which have three outcomes (win/loose/tie) with probabilities

$$\mathrm{P}\left(X^{(n)} = 1\right) = p, \qquad \mathrm{P}\left(X^{(n)} = -1\right) = q, \qquad \mathrm{P}\left(X^{(n)} = 0\right) = 1 - p - q =: r,$$

where $p, q, r \in [0, 1]$. The position of the particle after $N \in \mathbb{N}$ steps, i.e. the wealth after N games, is given by the *random walk* $(Z^{(n)})_{n \in \mathbb{N}_0}$ as in Definition 3.2.2 with

$$Z^{(n)} := Z^{(n-1)} + X^{(n)} = Z^{(0)} + \sum_{k=1}^{n} X^{(n)} \in \mathbb{Z} \quad \text{for } n \in \mathbb{N}$$

and $\mathrm{P}\left(Z^{(0)} \in \mathbb{Z}\right) = 1$. The absolute drawdown $(\mathcal{D}^{(n)})_{n \in \mathbb{N}_0}$ is modeled as in Definition 3.2.4 with

$$\mathcal{D}^{(n)} := \max_{0 \le k \le n} \left\{ Z^{(k)} \right\} - Z^{(n)} = \max\left\{ 0, \mathcal{D}^{(n-1)} - X^{(n)} \right\} \in \mathbb{N}_0 \quad \text{for } n \in \mathbb{N}$$

and $\mathrm{P}\left(\mathcal{D}^{(0)} \in \mathbb{N}_0\right) = 1$.

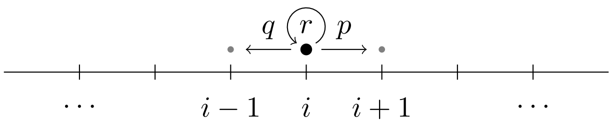


Figure 3.9.: Random walk in (3.38).

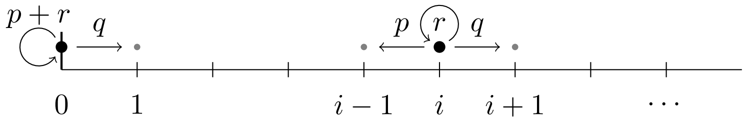


Figure 3.10.: Random walk in (3.39) with reflecting barrier at 0.

The transition probabilities of $(Z^{(n)})_{n\in\mathbb{N}_0}$ in (3.16) simplify to

$$\mathrm{P}\left(Z^{(n+1)} = i+1 \mid Z^{(n)} = i\right) = p, \qquad \mathrm{P}\left(Z^{(n+1)} = i \mid Z^{(n)} = i\right) = r, \tag{3.38a}$$

$$\mathrm{P}\left(Z^{(n+1)} = i-1 \mid Z^{(n)} = i\right) = q, \qquad \mathrm{P}\left(Z^{(0)} \in \mathbb{Z}\right) = 1 \tag{3.38b}$$

for all $n \in \mathbb{N}_0$ and all $i \in \mathbb{Z}$, cf. Figure 3.9. The probabilities of $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ in (3.19) become

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i+1 \mid \mathcal{D}^{(n)} = i\right) = q, \qquad \mathrm{P}\left(\mathcal{D}^{(n+1)} = 1 \mid \mathcal{D}^{(n)} = 0\right) = q, \tag{3.39a}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i \mid \mathcal{D}^{(n)} = i\right) = r, \qquad \mathrm{P}\left(\mathcal{D}^{(n+1)} = 0 \mid \mathcal{D}^{(n)} = 0\right) = p+r, \tag{3.39b}$$

$$\mathrm{P}\left(\mathcal{D}^{(n+1)} = i-1 \mid \mathcal{D}^{(n)} = i\right) = p, \qquad \mathrm{P}\left(\mathcal{D}^{(0)} \in \mathbb{N}_0\right) = 1 \tag{3.39c}$$

for all $n \in \mathbb{N}_0$ and all $i \in \mathbb{N}$, cf. Figure 3.10.

Now, explicit expressions for the discrete probability distribution of $\mathcal{D}^{(n)}$ for fixed $n \in \mathbb{N}$ are derived in the next subsections, while the limit case $n \to \infty$ is discussed afterwards in Subsection 3.3.4.

3.3.2 Probability mass function: Version 1

As in Subsection 3.2.3, the goal is to find an explicit expression to compute the discrete probability distribution of $\mathcal{D}^{(N)}$, where again the focus is on the case of the absolute drawdown starting at zero. However, the formulation of $d_j^{(N)} (= d_{0,j}^{(N)})$ for the random walk case should be simpler than the matrix variant in Subsection 3.2.3 for the more general case.

The idea of the following proof is to split each path of the finite wealth process $(Z^{(n)})_{0\le n\le N}$ leading to an absolute drawdown of given size, say of size j, into parts with some properties. If the equity ends at position m after N steps, it must have its maximum value at position $m + j$ to have an absolute drawdown of size j at the end. In this special situation, the upper bound of the path, which will be hit at some time step before N, the initial position, and the position at the end are known. Then the path can be split to a walk from the initial position to $j + m$ and a walk from $j + m$ to m. Afterwards one needs to sum over all possibilities.

To handle such bounded random walks, reasonable definitions and some consecutive results are given before obtaining the final theorem of this subsection.

Definition 3.3.2 (Probability of random walk being bounded) Let $(Z^{(n)})_{n\in\mathbb{N}_0}$ be a random walk as in Definition 3.3.1. For $N \in \mathbb{N}$ and $i, j, a, b \in \mathbb{Z}$ with $a \le i \le b$ and $a \le j \le b$ define

$$g_{i,j}^{(N)}(a,b) := \mathrm{P}\left(Z^{(N)} = j,\ a \le Z^{(n)} \le b \text{ for all } 0 \le n \le N \mid Z^{(0)} = i\right). \tag{3.40}$$

The following results are based on the work of Zhang and Hadjiliadis [56, mainly Lemma 1 and 2]. However, the results therein hold true for the case $r = 1 - p - q = 0$, but the case $r \neq 0$ is also considered in this subsection. First [56, Lemma 1] is extended. Please note, that the following lemma includes the case $r = 0$, which has already been proven in [56, Lemma 1].

Lemma 3.3.3 Let $(Z^{(n)})_{n\in\mathbb{N}_0}$ be a random walk as in Definition 3.3.1. Then:

(a) Case $r = 0$: Let $A \in \mathbb{N}$ and $k \in \mathbb{N}_0$. The number of possible paths of length k for the random walk that starts at $Z^{(0)} = i \in \{1, \dots, A\}$ and ends in $Z^{(k)} = j \in \{1, \dots, A\}$ while staying between 1 and A, i.e. $1 \le Z^{(n)} \le A$ for $0 \le n \le k$, is denoted by $c_{i,j}^{A,k}$. Then

$$c_{i,j}^{A,k} = \frac{2^{k+1}}{A+1}\sum_{\ell=1}^{A}\left[\cos\left(\frac{\pi\ell}{A+1}\right)\right]^k \sin\left(\frac{i\pi\ell}{A+1}\right)\sin\left(\frac{j\pi\ell}{A+1}\right) \in \mathbb{N}_0 \tag{3.41}$$

holds for all $A \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $i, j \in \{1, \dots, A\}$. Moreover, if $k \ge 1$, then

$$c_{i,j}^{A,k} = 0 \qquad \text{if } k + i + j \text{ is odd or } |j - i| > k \text{ or } A = 1, \tag{3.42}$$

and if $k = 0$, then

$$c_{i,j}^{A,0} = \delta_{i,j}, \quad \text{where } \delta_{i,j} \text{ is the Kronecker delta.} \tag{3.43}$$

(b) Case $r \in [0, 1)$: The probability $g_{i,j}^{(N)}(a,b)$ in (3.40) for $a := 1$ and $b := A$ for $A, N \in \mathbb{N}$ and $i, j \in \{1, \dots, A\}$ is determined by

$$g_{i,j}^{(N)}(1,A) = \sum_{k=|j-i|}^{N}\left[\binom{N}{k} c_{i,j}^{A,k} r^{N-k} p^{\frac{k-i+j}{2}} q^{\frac{k+i-j}{2}}\right]. \tag{3.44}$$

In the special case $A = 1$ it must be $i = j = 1$ and $g_{i,j}^{(N)}(1,A) = g_{1,1}^{(N)}(1,1) = r^N$.

Proof. The proof works similar to the proof in Zhang and Hadjiliadis [56, Lemma 1]. First, case (b) is shown assuming that (a) were already proved somewhere later in the proof.

Proof of (b): Let $r \in [0, 1)$ and $N \in \mathbb{N}$. If $A = 1$ it must be $i = j = 1$, which means that the random walk stays at position 1 after each step. Since $N \ge 1$ it is $g_{i,j}^{(N)}(1,A) = r^N$ which proves the special case after (3.44). It remains to show (3.44).

Next, let $A > 1$. The 1-step transition matrix for a random walk between 1 and A is given by the tridiagonal Toeplitz matrix

$$\mathbf{Q}_A := \begin{bmatrix} r & p & & 0 \\ q & \ddots & \ddots & \\ & \ddots & \ddots & p \\ 0 & & q & r \end{bmatrix} \in \mathbb{R}^{A\times A}, \tag{3.45}$$

where the (i,j)th entry, $1 \le i,j \le A$, of $\mathbf{Q}_A$ gives the probability of moving from position i to position j in one step. The N-step transition matrix is the Nth power of $\mathbf{Q}_A$ and the probability of (3.44) is the (i,j)th entry of this matrix. From Salkuyeh [50] it is known that the entries of $\mathbf{Z} := [\mathbf{Q}_A]^N = [g_{i,j}^{(N)}(1,A)]_{1\le i,j\le A}$ are given by

$$g_{i,j}^{(N)}(1,A) = \frac{2}{A+1}\left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sum_{\ell=1}^{A}\left[r + 2\sqrt{pq}\cos\left(\frac{\pi\ell}{A+1}\right)\right]^N \sin\left(\frac{i\pi\ell}{A+1}\right)\sin\left(\frac{j\pi\ell}{A+1}\right).$$

Using the binomial theorem gives

$$\left[r + 2\sqrt{pq}\cos\left(\frac{\pi\ell}{A+1}\right)\right]^N = \sum_{k=0}^{N}\binom{N}{k} r^{N-k} 2^k (pq)^{\frac{k}{2}}\left(\cos\left(\frac{\pi\ell}{A+1}\right)\right)^k$$

which yields that $g_{i,j}^{(N)}(1,A)$ equals to

$$\left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sum_{k=0}^{N}\left[\binom{N}{k} r^{N-k}(pq)^{\frac{k}{2}} \frac{2^{k+1}}{A+1}\sum_{\ell=1}^{A}\left(\cos\left(\frac{\pi\ell}{A+1}\right)\right)^k \sin\left(\frac{i\pi\ell}{A+1}\right)\sin\left(\frac{j\pi\ell}{A+1}\right)\right]. \tag{3.46}$$

Let the right hand side of (3.41) be defined by $\overline{c}_{i,j}^{A,k}$. This finally gives

$$g_{i,j}^{(N)}(1,A) = \left(\frac{q}{p}\right)^{\frac{i-j}{2}} \sum_{k=0}^{N}\left[\binom{N}{k} r^{N-k}(pq)^{\frac{k}{2}}\,\overline{c}_{i,j}^{A,k}\right] = \sum_{k=0}^{N}\left[\binom{N}{k}\overline{c}_{i,j}^{A,k} r^{N-k} p^{\frac{k-i+j}{2}} q^{\frac{k+i-j}{2}}\right]. \tag{3.47}$$

Note that the insertion of $A = 1$ and consequently $i = j = 1$ in (3.47) leads to

$$g_{1,1}^{(N)}(1,1) = \sum_{k=0}^{N}\left[\binom{N}{k}\overline{c}_{1,1}^{1,k} r^{N-k} p^{\frac{k}{2}} q^{\frac{k}{2}}\right],$$

where again by definition of $\overline{c}_{i,j}^{A,k}$ (defined by the right hand side of (3.41))

$$\overline{c}_{1,1}^{1,k} = \frac{2^{k+1}}{2}\left[\cos\left(\frac{\pi}{2}\right)\right]^k \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = \frac{2^{k+1}}{2}0^k = \begin{cases} 0, & \text{if } k \in \mathbb{N}, \\ 1, & \text{if } k = 0, \end{cases}$$

and, therefore, $g_{1,1}^{(N)}(1,1) = r^N$, which is known to be correct from above. Hence, (3.47) even holds true for $A = 1$ as well.

To complete the proof of (b) it remains to show, that the sum in (3.47) can also be started at $k = |j-i|$ which gives (3.44). Assume part (a), in particular (3.41) and (3.42), holds true (which still needs to be shown). Then it is known from (3.41) and (3.42) that $\overline{c}_{i,j}^{A,k} = c_{i,j}^{A,k} = 0$ if $|j-i| > k$. Hence, one can start the sum in (3.47) from $k = |j-i|$ and, consequently, after replacing $\overline{c}_{i,j}^{A,k}$ by $c_{i,j}^{A,k}$ in (3.47), the equality (3.44) holds, which completes the proof of (b). Thus, it remains to show (a) without using (3.44).

Proof of (a): Let $r = 0$. Since the proof of (b) needs part (a), (3.44) cannot be used. However, (3.47) already is proven for $A, N \in \mathbb{N}$ (the corresponding proof does not use (a)) and therefore can be used. Inserting $r = 0$ in (3.47) gives

$$g_{i,j}^{(N)}(1, A) = \overline{c}_{i,j}^{A,N} p^{\frac{N-i+j}{2}} q^{\frac{N+i-j}{2}} \tag{3.48}$$

for all $1 \leq i, j \leq A$. Note that this also is proven in [56, Lemma 1]. The occurrence of each relevant path for (3.40) has the same probability $p^{(N-i+j)/2} q^{(N+i-j)/2}$, because $(N-i+j)/2 + (N+i-j)/2 = N$ and $i + (N-i+j)/2 - (N+i-j)/2 = j$. Multiplying with the number of all allowed paths yields (3.48) (see (3.40)). Therefore for $N \geq 1$ the coefficient $\overline{c}_{i,j}^{A,N}$ corresponds exactly to the number of allowed paths and needs to be in $\mathbb{N}_0$. Hence $\overline{c}_{i,j}^{A,N} = c_{i,j}^{A,N}$ for $N \geq 1$. Replacing N by k, this proves (3.41) for $k \geq 1$.

Next show the special situation (3.42) when $k \geq 1$: If $|j-i| > k$, then the path can never reach j when starting at i in k steps. Therefore, $g_{i,j}^{(k)}(1, A) = 0$ must hold. Hence, in this case it must be $c_{i,j}^{A,k} = 0$. If $A = 1$, then $i = j = 1$ and the path must stay at position one for k steps, which is impossible because $r = 0$ and $k \geq 1$, i.e. $c_{i,j}^{1,k} = c_{1,1}^{1,k} = 0$.

To be able to walk from i to j in k steps, it must be $j = i + k_+ - k_-$ and $k_+ + k_- = k$, where k_+ denotes the number of steps to the right, and k_- the number of steps to the left. Hence it must be $k + i + j = 2k_+ + 2i$, i.e. $k + i + j$ must be even. Otherwise, there is no possible path. Therefore, $c_{i,j}^{A,k} = 0$ if $k + i + j$ is odd. Thus (3.42) holds.

It remains to show the special case $k = 0$ and $A \in \mathbb{N}$ in (3.41) together with (3.43): For this, let again $r \in [0, 1)$ be arbitrary. For $N = 1$ and arbitrary $A \in \mathbb{N}$ and $1 \leq i, j \leq A$, (3.47) gives

$$g_{i,j}^{(1)}(1, A) = \overline{c}_{i,j}^{A,0} r p^{\frac{-i+j}{2}} q^{\frac{+i-j}{2}} + \overline{c}_{i,j}^{A,1} p^{\frac{1-i+j}{2}} q^{\frac{1+i-j}{2}}. \tag{3.49}$$

By definition $g_{i,j}^{(1)}(1, A)$ gives the probability of the random walk moving from i to j in one time step (see (3.40)). Note that such paths automatically stay between 1 and A because $1 \leq i, j \leq A$. For such paths it must be

$$g_{i,j}^{(1)}(1, A) = \begin{cases} r, & j = i, \\ p, & j = i+1, \\ q, & j = i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.50}$$

Equating coefficients of (3.49) and (3.50) (as polynomials in r; note that $\overline{c}_{i,j}^{A,0}$ is independent of r) for the four cases yields that $\overline{c}_{i,j}^{A,0} = \delta_{i,j}$.

Since the coefficient $c_{i,j}^{A,0}$ by definition gives the number of paths which move from i to j in zero time steps, clearly $c_{i,j}^{A,0} = \delta_{i,j}$, because for $i = j$ there is only one such "path" and for $i \neq j$ there is none. Hence, $c_{i,j}^{A,0} = \delta_{i,j} = \overline{c}_{i,j}^{A,0}$ holds true which gives (3.41) and (3.43) for $k = 0$. □

Remark 3.3.4 Some notes for Lemma 3.3.3:

(a) The extension (3.44) to the case $r \neq 0$, is intuitive, because of the following: If $r \neq 0$, then it is possible that the random walk does not move for some steps, say for $N - k$ steps. Then it makes a total of k steps up and down. For $0 < k < N$ there are several possibilities when the k steps of the overall N steps are done. To be more precise there are $\binom{N}{k}$ possibilities. Multiplying the probabilities to the number of paths and summing over k gives (3.44).

(b) A mirrored version of the probability (3.44) can easily be obtained by looking at the negative of the random walk, i.e. replace $Z^{(n)}$ (still with probabilities from (3.38) in Definition 3.3.1) by $Y^{(n)} := -Z^{(n)}$, and thus by switching the roles of p and q. For $i, j, A, N \in \mathbb{N}$ with $1 \leq i, j \leq A$ as in Lemma 3.3.3, this leads to

$$\begin{aligned} g_{-i,-j}^{(N)}(-A,-1) &= \mathrm{P}\left(Z^{(N)} = -j,\ -A \leq Z^{(n)} \leq -1 \text{ for all } 0 \leq n \leq N \mid Z^{(0)} = -i\right) \\ &= \mathrm{P}\left(Y^{(N)} = j,\ 1 \leq Y^{(n)} \leq A \text{ for all } 0 \leq n \leq N \mid Y^{(0)} = i\right) \\ &= \sum_{k=|j-i|}^{N} \left[\binom{N}{k} c_{i,j}^{A,k} r^{N-k} q^{\frac{k-i+j}{2}} p^{\frac{k+i-j}{2}}\right]. \end{aligned} \tag{3.51}$$

♦

For the absolute drawdown, the upper bound A is not relevant. Hence, the coefficient $c_{i,j}^{A,k}$ in (3.41) is now discussed for large A. The next result follows from the reflection principle, see e.g. Feller [17, Chapter III, Section 1].

Lemma 3.3.5 Let $i, j, A, N \in \mathbb{N}$ such that $1 \leq i \leq A$ and $1 \leq j \leq A$. If $A \geq (N+i+j)/2$, coefficient $c_{i,j}^{A,N}$ in (3.41) is independent on A and it is $c_{i,j}^{A,N} = c_{i,j}^{N}$, where

$$c_{i,j}^{N} := \begin{cases} \binom{N}{\frac{N-i+j}{2}} - \binom{N}{\frac{N+i+j}{2}}, & \text{if } N+i+j \text{ is even, } |j-i| \leq N,\ |j+i| \leq N, \\ \binom{N}{\frac{N-i+j}{2}}, & \text{if } N+i+j \text{ is even, } |j-i| \leq N,\ |j+i| > N, \\ 0, & \text{otherwise, i.e. } N+i+j \text{ is odd or } |j-i| > N. \end{cases} \tag{3.52}$$

In particular it is $\lim_{A\to\infty} c_{i,j}^{A,N} = c_{i,j}^{N}$.

Proof. Let $\#\{\ldots\}$ denote the number of events within the set. For the case $r = 0$, when using the reflection principle, see e.g. Feller [17, Chapter III, Section 1], in the second step, cf. Figure 3.11, it is

$$\begin{aligned} c_{i,j}^{\infty,N} &:= \#\left\{Z^{(N)} = j,\ 1 \leq Z^{(n)} \text{ for all } 0 \leq n \leq N \mid Z^{(0)} = i\right\} \\ &= \#\left\{Z^{(N)} = j \mid Z^{(0)} = i\right\} \\ &\quad - \#\left\{Z^{(N)} = j,\ Z^{(n)} \leq 0 \text{ for at least one } n \in \{0,\ldots,N\} \mid Z^{(0)} = i\right\} \\ &= \#\left\{Z^{(N)} = j \mid Z^{(0)} = i\right\} - \#\left\{Z^{(N)} = j \mid Z^{(0)} = -i\right\}. \end{aligned} \tag{3.53}$$

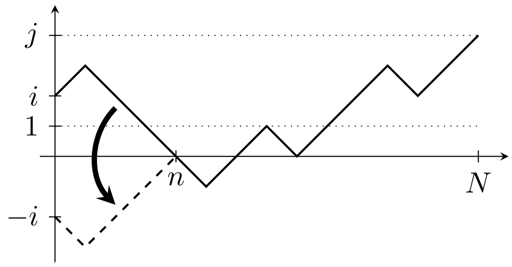


Figure 3.11.: Illustration of reflection principle: Each random walk from i to j crossing position 1 (here at step n) can be mirrored, so that it starts at position $-i$; the original and the mirrored random walk coincides from steps n to N.

The value $\#\{Z^{(N)} = j \mid Z^{(0)} = i\}$ is given by binomial coefficients. As shown in the proof of Lemma 3.3.3, the value $N + i + j$ must be even. Otherwise, there is no possible path. In addition, it must be $|j - i| \leq N$, because the step size equals to one. Hence, if $|j - i| \leq N$ and $N + i + j$ is even, then $\#\{Z^{(N)} = j \mid Z^{(0)} = i\} = \binom{N}{(N-i+j)/2} = \binom{N}{(N+i-j)/2}$. Inserting this into (3.53) gives $c_{i,j}^{\infty,N} = c_{i,j}^{N}$ with $c_{i,j}^{N}$ from (3.52).

When choosing $A \geq N/2 + (i + j)/2$ the random walk cannot overshoot the upper bound at position A and walks to position j within N steps. In this case both expression (3.41) and (3.53) must be the same, i.e. $c_{i,j}^{A,N} = c_{i,j}^{\infty,N}$, and thus the coefficients need to be equal, i.e. for A large enough it is $c_{i,j}^{A,N} = c_{i,j}^{N}$, which is independent of A. □

The next result gives the probability of a random walk having its maximum exactly at a given point. This result has been shown by Zhang and Hadjiliadis [56, Lemma 2] for the case $r = 0$, but here $r \in [0, 1]$ is arbitrary.

Lemma 3.3.6 Let $(Z^{(n)})_{n\in\mathbb{N}_0}$ be the random walk as in Definition 3.3.1, $j \in \mathbb{Z}$, $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$ with $m \leq N$. Define

$$h_j^{(N)}(m) := \mathrm{P}\left(Z^{(N)} = j,\ \max_{0\leq n\leq N} Z^{(n)} = j + m \mid Z^{(0)} = 0\right).$$

Then for $J_m := \{-m, -m + 1, \dots, N - 2m\}$ it is

$$h_j^{(N)}(m) = \begin{cases} \sum\limits_{k=j}^{N} \left[\binom{N}{k} c_{j+1,1}^{k} r^{N-k} q^{\frac{k-j}{2}} p^{\frac{k+j}{2}}\right], & \text{if } m = 0,\ j \in J_0, \\ \sum\limits_{k=j+2m}^{N} \binom{N}{k} r^{N-k} p^{\frac{k+j}{2}} q^{\frac{k-j}{2}} \sum\limits_{n=j+m}^{k-m} c_{j+m+1,1}^{n} c_{1,m}^{k-n-1}, & \text{if } m > 0,\ j \in J_m, \\ 0, & \text{otherwise,} \end{cases}$$

where the coefficients $c_{a,b}^{k}$ are from (3.52).

Proof. The proof can be done similar to the proof in [56]. The proof therein is more technical and lengthy which is not necessary in the case discussed here. A simpler proof is to proceed as described in Remark 3.3.4 (a). Following this description, first discuss the case $r = 0$ and $m > 0$:

Case $r = 0$ and $m \geq 1$: All paths considered in the probability $h_j^{(N)}(m)$ for $j \in \mathbb{Z}$ need to have its maximum at $j + m$ and reach position j afterwards. For this, the random walk needs at least $(j+m)+m = j+2m$ time steps to walk from zero to $j+m$ and from $j+m$ to j. Hence, $h_j^{(N)}(m) = 0$ if $N < j+2m$ (or equivalently if $j > N-2m$). Of course, if $j < -m$, then $j + m < 0 = Z^{(0)} = 0$ for the relevant paths, i.e. $j + m$ cannot be the maximum. Altogether, it follows that $h_j^{(N)}(m) = 0$ for $j \notin \{-m, -m+1, \ldots, N-2m\} = J_m$ in case $m \geq 1$ and $r = 0$.

Now, let $j \in J_m$ be arbitrary (and still $m \geq 1$ and $r = 0$). For each path considered in the probability $h_j^{(N)}(m)$, there is a point in time, say $n \in \{0, \ldots, N\}$, where this path touches the maximum $j + m \geq 0$ for the last time before walking to j. Using (3.51) for $r = 0$ together with Lemma 3.3.5 the corresponding probability of starting at 0 and ending at its maximum at $j + m$ after n steps is

$$\begin{aligned}
g_{0,j+m}^{(n)}(-\infty, j+m) &= \mathrm{P}\left(Z^{(n)} = j+m,\ -\infty \leq Z^{(k)} \leq j+m \text{ for all } 0 \leq k \leq n \mid Z^{(0)} = 0\right) \\
&= g_{-(j+m+1),-1}^{(n)}(-\infty, -1) \\
&= \lim_{A\to\infty} c_{j+m+1,1}^{A,n} q^{\frac{n-j-m}{2}} p^{\frac{n+j+m}{2}} \\
&= c_{j+m+1,1}^{n} q^{\frac{n-j-m}{2}} p^{\frac{n+j+m}{2}},
\end{aligned}$$

where it must be $n \geq |j+m| = j+m$.

From position $j+m$ at time n this path walks to position j in the remaining $N-n$ steps without touching $j+m$ again. For this, it must be $N-n \geq m$ or equivalently $n \leq N-m$ and the first step must go down. Again using (3.51) for $r = 0$ and Lemma 3.3.5 the probability for this last stage is

$$q \cdot g_{j+m-1,j}^{(N-n-1)}(-\infty, j+m-1) = q \cdot g_{-1,-m}^{(N-n-1)}(-\infty, -1) = c_{1,m}^{N-n-1} q^{\frac{N-n+m}{2}} p^{\frac{N-n-m}{2}}.$$

Note that this only works for $m \geq 1$ and $r = 0$. Both stages, i.e. the part from time 1 to n and from $n+1$ to N, are stochastically independent and the sets of all such paths for $n = j+m, \ldots, N-m$ are disjoint. Therefore, multiplying the probabilities for both stages and summing over n gives the desired probability by

$$h_j^{(N)}(m) = \sum_{n=j+m}^{N-m} c_{j+m+1,1}^{n} c_{1,m}^{N-n-1} p^{\frac{N+j}{2}} q^{\frac{N-j}{2}} = p^{\frac{N+j}{2}} q^{\frac{N-j}{2}} \sum_{n=j+m}^{N-m} c_{j+m+1,1}^{n} c_{1,m}^{N-n-1}$$

for $j \in J_m$, which completes the proof for $r = 0$, $m \geq 1$ and $j \in \mathbb{Z}$.

Case $r \neq 0$ and $m \geq 1$: Now, following Remark 3.3.4 (a), the probability for $r \neq 0$ is obtained by summing over k and multiplying the corresponding probabilities, where k is the number of steps the random walk goes up and down and $N-k$ the number of steps when the random walk stays at the same position. Note that it suffices to start this sum at $k = j+2m$ because the random walk needs to perform at least $j+m$ steps to reach the maximum and another m steps to reach the final position. For the two cases $j \in J_m$ and $j \notin J_m$, this leads to

$$h_j^{(N)}(m) = \sum_{k=j+2m}^{N} \binom{N}{k} r^{N-k} p^{\frac{k+j}{2}} q^{\frac{k-j}{2}} \sum_{n=j+m}^{k-m} c_{j+m+1,1}^{n} c_{1,m}^{k-n-1}$$

and $h_j^{(N)}(m) = 0$, respectively, which completes the proof for $r \neq 0$, $m \geq 1$ and $j \in \mathbb{Z}$.

Case $r \in [0,1]$ and $m = 0$: The proof from above for $m \geq 1$ does not work for $m = 0$. If $m = 0$ the random walk ends in its maximal value. Therefore, it is needed to explicitly demand that $\max_{0 \leq n \leq N} Z^{(n)} = j$. Using (3.51) and Lemma 3.3.5 it follows that

$$h_j^{(N)}(0) = g_{0,j}^{(N)}(-\infty, j) = g_{-(j+1),-1}^{(N)}(-\infty, -1) = \sum_{k=j}^{N} \left[\binom{N}{k} c_{j+1,1}^{k} r^{N-k} q^{\frac{k-j}{2}} p^{\frac{k+j}{2}} \right]$$

for all $j \geq 0$ (in particular for all $j \in J_0$). If $j > N$, then, obviously, $h_j^{(N)}(0) = 0$, and if $j < 0$, then j cannot be the maximum because the random walk starts at zero. Hence, $h_j^{(N)}(0) = 0$ for $j \notin J_0$, which completes the proof. □

Using Lemma 3.3.6 the final theorem can be concluded.

Theorem 3.3.7 Let $(\mathcal{D}^{(n)})_{n \in \mathbb{N}_0}$ be a random walk as in Definition 3.3.1 and $j, N \in \mathbb{N}$. The probability $d_j^{(N)} = d_{0,j}^{(N)} = \mathrm{P}\left(\mathcal{D}^{(N)} = j \mid \mathcal{D}^{(0)} = 0\right)$ that a drawdown of size j occurs exactly at step N when starting at zero is given by

$$d_j^{(N)} = \sum_{k=j}^{N} \binom{N}{k} r^{N-k} \sum_{s=\lceil \frac{k-j}{2} \rceil}^{k-j} p^s q^{k-s} \sum_{n=2s-k+j}^{k-j} c_{2s-k+j+1,1}^{n} c_{1,j}^{k-n-1},$$

where the coefficients $c_{a,b}^k$ are from (3.52). For the case $j = 0$ it is

$$d_0^{(N)} = \sum_{k=0}^{N} \binom{N}{k} r^{N-k} \sum_{s=\lceil \frac{k}{2} \rceil}^{k} p^s q^{k-s} \left[\binom{k}{s} - \binom{k}{s+1} \right].$$

Proof. The final position of the equity, such that the drawdown at time N equals j, must be greater than or equal to $-j$ (if its maximum is at zero) and cannot exceed $N - 2j$ (if its maximum is $N - j$). In each case, the maximum of the equity must be exactly j units above the final position. Therefore, Lemma 3.3.6 yields for $j \geq 1$ after rearranging the sum that

$$d_j^{(N)} = \sum_{s=-j}^{N-2j} h_s^{(N)}(j) = \sum_{k=j}^{N} \binom{N}{k} r^{N-k} \sum_{s=-j}^{k-2j} p^{\frac{k+s}{2}} q^{\frac{k-s}{2}} \sum_{n=s+j}^{k-j} c_{s+j+1,1}^{n} c_{1,j}^{k-n-1}$$

and for $j = 0$ that

$$d_0^{(N)} = \sum_{s=0}^{N} \sum_{k=s}^{N} \binom{N}{k} c_{s+1,1}^{k} r^{N-k} p^{\frac{k+s}{2}} q^{\frac{k-s}{2}} = \sum_{k=0}^{N} \binom{N}{k} r^{N-k} \sum_{s=0}^{k} c_{s+1,1}^{k} p^{\frac{k+s}{2}} q^{\frac{k-s}{2}}.$$

To obtain the representation of $d_j^{(N)}$ in the theorem, it is needed to use the fact that $c_{a,b}^t = 0$ whenever $t-a+b$ is odd, and $c_{a,b}^t$ can be nontrivial otherwise, see (3.52). Therefore $c_{s+j+1,1}^n c_{1,j}^{k-n-1} \neq 0$ is only possible if $n-s-j$ and $k-n+j$ are even. It follows that this product can only be nontrivial if s and k are either both even or both odd, but in any case it is $(k+s)/2 \in \mathbb{Z}$ and $(k-s)/2 \in \mathbb{Z}$. Now substituting $\tilde{s} = (k+s)/2 \in \{\lceil (k-j)/2 \rceil, \dots, k-j\}$ in the second sum and naming $\tilde{s}$ again s yields the second part of $d_j^{(N)}$. □

3.3.3 Probability mass function: Version 2

In Section 3.3.2, a formula is derived for the state probabilities $d_j^{(N)}$ for $j, N \in \mathbb{N}_0$ of the absolute drawdown $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ from Definition 3.3.1, see Theorem 3.3.7. Therein, the game has three outcomes (win/loose/tie), i.e. $p, q \in [0,1]$ and $r := 1-p-q \in [0,1]$. For convenience, let $d_j^{(n)} = d_{0,j}^{(n)} = \mathrm{P}\left(\mathcal{D}^{(n)} = j \mid \mathcal{D}^{(0)} = 0\right)$ be defined for $j \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. In the following, a similar result is derived but with a simpler formula and a different proof. For this, at first the transition probabilities of the absolute drawdown defined in Definition 3.3.1 for $r = 0$ are rewritten as the Chapman-Kolmogorov equation

$$d_j^{(n)} = p d_{j+1}^{(n-1)} + q d_{j-1}^{(n-1)} \tag{3.54a}$$

for $n, j \in \mathbb{N}$ together with the boundary condition

$$d_0^{(n)} = p d_1^{(n-1)} + p d_0^{(n-1)}, \tag{3.54b}$$

$$d_{-j}^{(n)} = 0 \tag{3.54c}$$

for $n, j \in \mathbb{N}$ and the stopping criterion

$$d_j^{(0)} = \delta_{0,j}, \tag{3.54d}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. Note that $\sum_{j\in\mathbb{N}_0} d_j^{(n)} = 1$ for $n \in \mathbb{N}_0$. The goal is again to find an explicit expression for $d_j^{(n)}$. The next step is to make an observation on the support of $d^{(N)} \colon \mathbb{N}_0 \to [0,1]$.

Lemma 3.3.8 Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be a random walk (the absolute drawdown) as in Definition 3.3.1 where $p \in [0,1]$ and $q = 1-p$, i.e. $r = 0$. If $\mathcal{D}^{(0)} \equiv 0$, then $\mathcal{D}^{(N)} \leq N$ for all $N \in \mathbb{N}_0$ and the following holds:

(a) For $p \in (0,1)$ it is $d_N^{(N)} = q^N$ for all $N \in \mathbb{N}_0$.
(b) If $p = 0$ then $q = 1$ and $d_j^{(N)} = \delta_{j,N}$, $j \in \mathbb{N}_0$, i.e. $\mathcal{D}^{(N)} \equiv N$ for all $N \in \mathbb{N}_0$.
(c) If $p = 1$ then $q = 0$ and $d_j^{(N)} = d_j^{(0)} = \delta_{j,0}$, $j \in \mathbb{N}_0$, i.e. $\mathcal{D}^{(N)} \equiv 0$ for all $N \in \mathbb{N}_0$.

Proof. Since the step size of the particle in each step is one and the particle starts at 0, it cannot reach positions beyond N and therefore $d_j^{(N)} = 0$ for $j > N$. To be at position N after step N the particle needs to move to the right in each step. The probability of moving N times to the right equals to q^N and therefore $d_N^{(N)} = q^N$. This holds true for $p \in [0,1]$, especially for $p \in (0,1)$.

N	0	1	2	3	4	5	6
0	1						
1	$1p$	$1q$					
2	$1pq+1p^2$	$1pq$	$1q^2$				
3	$2p^2q+1p^3$	$2pq^2+1p^2q$	$1pq^2$	$1q^3$			
4	$2p^2q^2+3p^3q+1p^4$	$3p^2q^2+1p^3q$	$3pq^3+1p^2q^2$	$1pq^3$	$1q^4$		
5	$5p^3q^2+4p^4q+1p^5$	$5p^2q^3+4p^3q^2+1p^4q$	$4p^2q^3+1p^3q^2$	$4pq^4+1p^2q^3$	$1pq^4$	$1q^5$	
6	$5p^3q^3+9p^4q^2+5p^5q$ $+1p^6$	$9p^3q^3+5p^4q^2+1p^5q$	$9p^2q^4+5p^3q^3$ $+1p^4q^2$	$5p^2q^4+1p^3q^3$	$5pq^5$ $+1p^2q^4$	$1pq^5$	$1q^6$

Table 3.4.: The probability $d_j^{(N)}$ for $N=0,1,\ldots,6$ and $j=0,1,\ldots,6$.

The two cases $p \in \{0,1\}$ are special cases. For $p=0$ the particle always moves to the right with probability $q=1$ and is always at position N after N steps. If $p=1$ the particle stays at position 0 in each step with probability $p=1$. □

Next, take a look at the formulas of the probabilities $d_j^{(N)}$ for small N and small j to be able to identify the scheme, see Table 3.4. From this, it is possible to observe some pattern that can be formulated as follows: The sum of the exponents of p and q always equals to N. The maximal exponent of p equals to $N-j$ and the lowest is found to be $\lceil (N-j)/2 \rceil$. The coefficients follow a recursion scheme similar to Pascal's triangle. Therefore, one can guess that

$$d_j^{(N)} = \sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} b_{N,s}(j), \tag{3.55}$$

where $j \in \{0,1,\ldots,N\}$, because in all other cases $d_j^{(N)} = 0$, and

$$b_{0,0}(0) = b_{1,1}(0) = 1, \tag{3.56a}$$

$$b_{n,s}(0) = b_{n-1,s}(0) + b_{n-1,s-1}(0), \qquad \text{for } n \geq 2 \text{ and } s \in \left\{ \left\lceil \frac{n}{2} \right\rceil, \ldots, n \right\}, \tag{3.56b}$$

$$b_{n,s}(0) = 0, \qquad \text{for } n \geq 0 \text{ and } s \notin \left\{ \left\lceil \frac{n}{2} \right\rceil, \ldots, n \right\}, \tag{3.56c}$$

$$b_{n,s}(j) = b_{n,s+1}(j-1) = \ldots = b_{n,s+j}(0), \qquad \text{for } n \geq 0,\ s \in \mathbb{Z} \text{ and } j \in \mathbb{N}. \tag{3.56d}$$

For simplicity, define $b_{n,s+j} := b_{n,s}(j) = b_{n,s+j}(0)$ and conclude the following result.

Lemma 3.3.9 For $p,q \in [0,1]$ with $p+q=1$, the expression

$$\tilde{d}_j^{(N)} := \begin{cases} \displaystyle\sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} b_{N,s+j}, & \text{if } j \in \{0,1,\ldots,N\}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.57}$$

for $j,N \in \mathbb{N}_0$, where the coefficients $b_{N,s+j} = b_{N,s}(j)$ fulfill (3.56), solve the recursion given by (3.54). Hence, $\tilde{d}_j^{(N)} = d_j^{(N)}$ for all $j,N \in \mathbb{N}_0$.

Proof. From Lemma 3.3.8 it is known that $d_j^{(N)} = 0$ for $j \notin \{0, 1, \dots, N\}$, which holds true for (3.57) as well. Therefore one only needs to consider values $j \in \{0, 1, \dots, N\}$ and it remains to check (3.54a), (3.54b) and (3.54d).

For $N = 0$ it is $\tilde{d}_0^{(0)} = b_{0,0} = 1$ such that the stopping criterion (3.54d) holds true. For $N \geq 1$ perform mathematical induction with respect to N to verify (3.54a) and (3.54b). The basis reads: For $N = 1$ it is $\tilde{d}_0^{(1)} = pb_{1,1} = p$ and $\tilde{d}_1^{(1)} = qb_{1,1} = q$. For $N = 2$ it is

$$\begin{aligned}
\tilde{d}_0^{(2)} &= pqb_{2,1} + p^2 b_{2,2} = pq + p^2, \\
\tilde{d}_1^{(2)} &= pqb_{2,2} + p^2 b_{2,3} = pq, \\
\tilde{d}_2^{(2)} &= q^2 b_{2,2} + pqb_{2,3} + p^2 b_{2,4} = q^2,
\end{aligned}$$

i.e. recursions (3.54a) and (3.54b) hold true for $N \in \{1, 2\}$, see also Table 3.4. Now perform the induction step from N to $N + 1$: Due to (3.56b) it is

$$\begin{aligned}
\tilde{d}_j^{(N+1)} &= \sum_{s=\lceil \frac{N+1-j}{2} \rceil}^{N+1-j} p^s q^{N+1-s} b_{N+1,s+j} = \sum_{s=\lceil \frac{N+1-j}{2} \rceil}^{N+1-j} p^s q^{N+1-s} (b_{N,s+j} + b_{N,s+j-1}) \\
&= \sum_{s=\lceil \frac{N+1-j}{2} \rceil}^{N+1-j} p^s q^{N+1-s} b_{N,s+j} + \sum_{s=\lceil \frac{N+1-j}{2} \rceil}^{N+1-j} p^s q^{N+1-s} b_{N,s+j-1} =: A + B.
\end{aligned}$$

Using the fact that $b_{N,N+1-j+j} = b_{N,N+1} = 0$, performing an index shift ($\tilde{s} := s - 1$) and using the induction hypothesis yields

$$A = p \sum_{s=\lceil \frac{N+1-j}{2} \rceil}^{N-j} p^{s-1} q^{N-(s-1)} b_{N,(s-1)+(j+1)} = p \sum_{s=\lceil \frac{N-(j+1)}{2} \rceil}^{N-(j+1)} p^s q^{N-s} b_{N,s+(j+1)} = p\tilde{d}_{j+1}^{(N)}.$$

For B, first take a look at the case $j > 0$. Using the induction hypothesis gives

$$B = q \sum_{s=\lceil \frac{N-(j-1)}{2} \rceil}^{N-(j-1)} p^s q^{N-s} b_{N,s+(j-1)} = q\tilde{d}_{j-1}^{(N)}$$

and therefore it is $\tilde{d}_j^{(N+1)} = A + B = p\tilde{d}_{j+1}^{(N)} + q\tilde{d}_{j-1}^{(N)}$, i.e. (3.54a) holds true for all $j > 0$.

Now, one only needs to verify the case $j = 0$. As above, using the fact that $\lceil (N-1)/2 \rceil \leq \lceil N/2 \rceil$ and $b_{N,s} = 0$ for $s < \lceil N/2 \rceil$, performing an index shift ($\tilde{s} := s - 1$) and using the induction hypothesis afterwards gives

$$B = p \sum_{s=\lceil \frac{N+1}{2} \rceil}^{N+1} p^{s-1} q^{N-(s-1)} b_{N,s-1} = p \sum_{s=\lceil \frac{N}{2} \rceil}^{N} p^s q^{N-s} b_{N,s} = p\tilde{d}_0^{(N)}.$$

Since $d_0^{(N+1)} = A + B = p\tilde{d}_1^{(N)} + p\tilde{d}_0^{(N)}$ this proves (3.54b). $\square$

$N=1$	1						$b_{1,1}$			
$N=2$	1	1					$b_{2,1}$	$b_{2,2}$		
$N=3$	2	1					$b_{3,2}$	$b_{3,3}$		
$N=4$	$2\vert 3$	3	1				$b_{4,2}\vert\binom{3}{1}$	$b_{4,3}$	$b_{4,4}$	
$N=5$	$5\vert 6$	4	1				$b_{5,3}\vert\binom{4}{2}$	$b_{5,4}$	$b_{5,5}$	
$N=6$	$5\vert 10$	$9\vert 10$	5	1			$b_{6,3}\vert\binom{5}{2}$	$b_{6,4}\vert\binom{5}{3}$	$b_{6,5}$	$\dots$
$N=7$	$14\vert 20$	$14\vert 15$	6	1			$b_{7,4}\vert\binom{6}{3}$	$b_{7,5}\vert\binom{6}{4}$	$b_{7,6}$	$\dots$
$N=8$	$14\vert 35$	$28\vert 35$	$20\vert 21$	7	1		$b_{8,4}\vert\binom{7}{3}$	$b_{8,5}\vert\binom{7}{4}$	$b_{8,6}$	$\dots$
$N=9$	$42\vert 70$	$48\vert 56$	$27\vert 28$	8	1		$b_{9,5}\vert\binom{8}{4}$	$b_{9,6}\vert\binom{8}{5}$	$b_{9,7}$	$\dots$
$N=10$	$42\vert 126$	$90\vert 126$	$75\vert 84$	$35\vert 36$	9	1	$b_{10,5}\vert\binom{9}{4}$	$b_{10,6}\vert\binom{9}{5}$	$b_{10,7}$	$\dots$

Table 3.5.: Each row shows the coefficients $b_{N,s}$ for $s = \lceil N/2 \rceil, \dots, N$; the second number for some entries are the corresponding binomial coefficient $\binom{N-1}{s-1}$ if it is different from $b_{N,s}$; see the right part of the table for explanation of the values.

Of course, with Lemma 3.3.9 there still is a recursion for the $b_{n,s}$ involved. However, the recursion does not depend on p and q anymore. Now, the goal is to find an explicit expression for $b_{N,s+j}$. Table 3.5 shows these coefficients for different time steps N. From this table one can see that the coefficients $b_{N,s}$ can be formulated as the difference of two binomial coefficients, i.e. by $b_{N,s} = \binom{N-1}{s-1} - \binom{N-1}{s+1}$. This leads to the following result.

Lemma 3.3.10 The coefficients $b_{N,s}$ in Lemma 3.3.9 can be written as

$$b_{N,s} = \begin{cases} \binom{N}{s} - \binom{N}{s+1}, & \text{if } s \in \{\lceil \frac{N}{2} \rceil, \dots, N\}, \\ 0, & \text{otherwise,} \end{cases} \tag{3.58}$$

where for convenience $\binom{N}{N+i} := 0$ for $i \geq 1$.

Proof. First note, that

$$\binom{N}{s} - \binom{N}{s+1} = \binom{N-1}{s-1} + \binom{N-1}{s} - \binom{N-1}{s} - \binom{N-1}{s+1} = \binom{N-1}{s-1} - \binom{N-1}{s+1}$$

to continue from the observation right before this lemma.

The case $s \notin \{\lceil N/2 \rceil, \dots, N\}$ clearly fulfills (3.56) for Lemma 3.3.9. Now, one only needs to check the case $s \in \{\lceil N/2 \rceil, \dots, N\}$.

Again, perform mathematical induction with respect to N: The basis reads: For $N = 0$ it is $b_{0,0} = \binom{0}{0} - \binom{0}{1} = 1 - 0 = 1$, for $N = 1$ it is $b_{1,1} = \binom{1}{1} - \binom{1}{2} = 1$ and for $N = 2$ it is $b_{2,1} = \binom{2}{1} - \binom{2}{2} = 1$ and $b_{2,2} = \binom{2}{2} - \binom{2}{3} = 1$. For the induction step from N to $N+1$ let $s \in \{\lceil (N+1)/2 \rceil, \dots, N+1\}$ be arbitrary. Then

$$\begin{aligned} b_{N+1,s} = \binom{N+1}{s} - \binom{N+1}{s+1} &= \binom{N}{s-1} + \binom{N}{s} - \binom{N}{s} - \binom{N}{s+1} \\ &= \underbrace{\binom{N}{s-1} - \binom{N}{s}}_{=:C} + \underbrace{\binom{N}{s} - \binom{N}{s+1}}_{=:D}. \end{aligned}$$

Depending on s, the following cases can occur:

(i) If $\lceil N/2 \rceil \le s-1 \le N$ the induction hypothesis yields $C = b_{N,s-1}$.
(ii) If $s = \lceil (N+1)/2 \rceil < \lceil N/2 \rceil + 1$, i.e. N is odd, $s = (N+1)/2$ and $s-1 < \lceil N/2 \rceil$, it is

$$C = \binom{N}{\frac{N-1}{2}} - \binom{N}{\frac{N+1}{2}} = 0 = b_{N,s-1},$$

because of the symmetry of the binomial coefficients.
(iii) If $\lceil N/2 \rceil \le s \le N$ then $D = b_{N,s}$ obtained from induction hypothesis.
(iv) If $s = N+1$, then

$$D = \binom{N}{N+1} - \binom{N}{N+2} = 0 = b_{N,s}.$$

It follows that for each $s \in \{\lceil (N+1)/2 \rceil, \dots, N+1\}$ it is $b_{N+1,s} = C + D = b_{N,s-1} + b_{N,s}$. □

Remark 3.3.11 Note, that $b_{N,s} = \binom{N}{s} - \binom{N}{s+1}$ also holds for all $s > N$ because of the convention used in Lemma 3.3.10. However, this is not true for $s < \lceil (N+1)/2 \rceil$. ♦

From Lemmas 3.3.9 and 3.3.10 one can conclude the following theorem.

Theorem 3.3.12 Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be a random walk (the absolute drawdown) as in Definition 3.3.1 where $p \in [0,1]$ and $q = 1-p$, i.e. $r = 0$. The probability $d_j^{(N)}$ for $j \in \mathbb{N}_0$ and $N \in \mathbb{N}$ can be written as

$$d_j^{(N)} = \begin{cases} \displaystyle\sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} \left[\binom{N}{s+j} - \binom{N}{s+j+1} \right] & \text{if } j \in \{0,1,\dots,N\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.59}$$

In case $r \neq 0$, it is for $N \in \mathbb{N}$ that

$$d_j^{(N)} = \sum_{k=j}^{N} \binom{N}{k} r^{N-k} \sum_{s=\lceil \frac{k-j}{2} \rceil}^{k-j} p^s q^{k-s} \left[\binom{k}{s+j} - \binom{k}{s+j+1} \right],$$

if $0 \le j \le N$, and $d_j^{(N)} = 0$ otherwise.

Proof. Equation (3.59) follows directly from Lemmas 3.3.9 and 3.3.10. The case $r \neq 0$ follows from the scheme in Remark 3.3.4 (a). □

Remark 3.3.13 (Connection to Theorem 3.3.7) For $p \in [0,1]$, $q = 1-p$ (i.e. $r = 0$), $N \in \mathbb{N}$ and $j \in \mathbb{N}_0$, Theorems 3.3.7 and 3.3.12 give

$$\begin{aligned} d_j^{(N)} &= \sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} \sum_{n=2s-N+j}^{N-j} c^{n}_{2s-N+j+1,1} c^{N-n-1}_{1,j} \\ &= \sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} \left[\binom{N}{s+j} - \binom{N}{s+j+1} \right]. \end{aligned}$$

Equating coefficients then yields that it must be

$$\sum_{n=2s-N+j}^{N-j} c^{n}_{2s-N+j+1,1} c^{N-n-1}_{1,j} = \binom{N}{s+j} - \binom{N}{s+j+1} \tag{3.60}$$

for all $N \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $s \in \{\lceil (N-j)/2 \rceil, \dots, N-j\}$. ♦

Remark 3.3.14 If $p = q = \frac{1}{2}$ in Theorem 3.3.12, the probabilities in (3.59) form a telescoping sum and therefore simplify to

$$d_j^{(N)} = \left(\frac{1}{2}\right)^N \sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} \left[\binom{N}{s+j} - \binom{N}{s+j+1}\right] = \left(\frac{1}{2}\right)^N \binom{N}{\lceil \frac{N+j}{2} \rceil}$$

for $j \in \{0, 1, \dots, N\}$. ♦

From Theorem 3.3.12, one also can formulate some implications.

Corollary 3.3.15 Assume the situation of Theorem 3.3.12 is given for some $N \in \mathbb{N}$, $p \in (0,1)$, $q = 1-p$ and $r = 0$. Let $Y^{(N)} \sim B(N,p)$ follow a binomial distribution with parameters N and p. It is well-known that

$$\begin{aligned} \mathrm{P}\left(Y^{(N)} = s\right) &= \binom{N}{s} p^s q^{N-s}, \qquad \text{for } s \in \mathbb{N}_0, \\ F_{Y^{(N)}}(y) := \mathrm{P}\left(Y^{(N)} \le y\right) &= \sum_{s=0}^{\lfloor y \rfloor} \binom{N}{s} p^s q^{N-s} \qquad \text{for } y \in \mathbb{R}, \end{aligned} \tag{3.61}$$

where for convenience $\sum_{s=0}^{K} [\dots] := 0$ if $K < 0$. Then, the equality

$$d_j^{(N)} = \left(\frac{q}{p}\right)^j \left[\frac{p-q}{p} + \frac{q}{p} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) - F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right)\right] \tag{3.62}$$

holds true for all $j \in \mathbb{N}_0$.

Proof. For the probabilities of the binomial distribution see e.g. Feller [17, Chapter VI, Section 2]. Setting $k := \lceil (N+j)/2 \rceil$ and again using the convention $\binom{N}{s} := 0$ for $s > N$ yields for $j = 0, 1, \dots, N$ (with Theorem 3.3.12)

$$\begin{aligned} d_j^{(N)} &= \sum_{s=\lceil \frac{N-j}{2} \rceil}^{N-j} p^s q^{N-s} \left[\binom{N}{s+j} - \binom{N}{s+j+1}\right] \\ &= \sum_{s=\lceil \frac{N-j}{2} \rceil + j}^{N} p^{s-j} q^{N-s+j} \binom{N}{s} - \sum_{s=\lceil \frac{N-j}{2} \rceil + j+1}^{N+1} p^{s-j-1} q^{N-s+j+1} \binom{N}{s} \\ &= \left(\frac{q}{p}\right)^j \sum_{s=k}^{N} p^s q^{N-s} \binom{N}{s} - \left(\frac{q}{p}\right)^{j+1} \sum_{s=k+1}^{N} p^s q^{N-s} \binom{N}{s} \\ &= \left(\frac{q}{p}\right)^j (1 - F_{Y^{(N)}}(k-1)) - \left(\frac{q}{p}\right)^{j+1} (1 - F_{Y^{(N)}}(k)), \end{aligned}$$

which proves (3.62) for $j \leq N$.

For all $y \geq N$ it must be $F_{Y^{(N)}}(y) = 1$, because of the binomial theorem (note that each of the summands with index $s > N$ vanishes because $\binom{N}{s} = 0$ for $s > N$). It follows, that for all $j > N$ it is $\left\lceil \frac{N+j}{2} \right\rceil > N$ and consequently

$$\left(\frac{q}{p}\right)^j \left[\frac{p-q}{p} + \frac{q}{p} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) - F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right)\right] = 0 = d_j^{(N)},$$

see Theorem 3.3.12. □

The next result gives a simple relation between $d_j^{(N)}$ and $d_{j+1}^{(N)}$.

Corollary 3.3.16 Assume the situation of Theorem 3.3.12 is given with $N \in \mathbb{N}$, $j \in \{0, 1, \ldots, N-1\}$, $p \in (0,1)$, $q = 1-p$ and $r = 0$. If $N - j$ is odd, then

$$d_{j+1}^{(N)} = \frac{q}{p} d_j^{(N)}. \tag{3.63}$$

If $N - j$ is even, then

$$d_{j+1}^{(N)} = \frac{q}{p} d_j^{(N)} + \left(\frac{q}{p}\right)^{j+1} \left[\frac{q}{p} \mathrm{P}\left(Y^{(N)} = \frac{N+j}{2} + 1\right) - \mathrm{P}\left(Y^{(N)} = \frac{N+j}{2}\right)\right], \tag{3.64}$$

where $Y^{(N)} \sim B(N, p)$ as in Corollary 3.3.15.

Proof. If $N - j$ is odd it is $\lceil (N-j+1)/2 \rceil = \lceil (N-j)/2 \rceil$. Using this fact right after an index shift ($\widetilde{s} := s+1$), Lemma 3.3.9 gives

$$d_{j+1}^{(N)} = \sum_{s=\lceil \frac{N-j-1}{2} \rceil}^{N-j-1} p^s q^{N-s} b_{N,s+j+1} = \sum_{s=\lceil \frac{N-j+1}{2} \rceil}^{N-j} p^{s-1} q^{N-s+1} b_{N,s+j} = \frac{q}{p} d_j^{(N)},$$

which proves (3.63) for all $j = 0, 1, \ldots, N-1$. If $N - j$ is even, then

$$\left\lceil \frac{N-j+1}{2} \right\rceil = \left\lceil \frac{N-j}{2} \right\rceil + 1 = \frac{N-j}{2} + 1 =: R + 1 \quad \text{and} \quad k := \left\lceil \frac{N+j}{2} \right\rceil = \frac{N+j}{2}.$$

Lemma 3.3.10 gives $b_{N,s+j} = \binom{N}{s+j} - \binom{N}{s+j+1}$ for $s + j \in \{\lceil \frac{N}{2} \rceil, \ldots, N\}$.
Again by Lemma 3.3.9 and because $k \geq \lceil \frac{N}{2} \rceil$ and $R + j = k$, it follows that

$$\begin{aligned}
d_{j+1}^{(N)} &= \sum_{s=\lceil \frac{N-j}{2} \rceil + 1}^{N-j} p^{s-1} q^{N-s+1} b_{N,s+j} \\
&= \frac{q}{p}\left(d_j^{(N)} - p^R q^{N-R} b_{N,R+j}\right) \\
&= \frac{q}{p}\left(d_j^{(N)} - p^{k-j} q^{N-k+j} \left[\binom{N}{k} - \binom{N}{k+1}\right]\right) \\
&= \frac{q}{p}\left(d_j^{(N)} - \left(\frac{q}{p}\right)^j \mathrm{P}\left(Y^{(N)} = k\right) + \left(\frac{q}{p}\right)^{j+1} \mathrm{P}\left(Y^{(N)} = k+1\right)\right)
\end{aligned}$$

holds for $j = 0, 1, \ldots, N-1$, which gives (3.64). □

3.3.4 Limit distribution

From Theorem 3.2.20 together with Corollary 3.2.24 (see also Remark 3.2.25) the limit distribution for the case of random walk with $r = 0$ is already known. However, to close the circle, the limit distribution is now derived directly from the probability mass function given from Theorem 3.3.12 or rather its consequence Corollary 3.3.15.

Theorem 3.3.17 Let the absolute drawdown as random walk $(\mathcal{D}^{(n)})_{n\in\mathbb{N}_0}$ be defined as in Definition 3.3.1 where $p \in [0,1]$, $q = 1-p$ and $r = 0$. If $p > \frac{1}{2}$ the limit distribution exists and is given by

$$d_0 = \frac{p-q}{p} > 0, \tag{3.65}$$

$$d_j = \frac{p-q}{p}\left(\frac{q}{p}\right)^j > 0 \qquad \text{for } j \in \mathbb{N}. \tag{3.66}$$

If $p \leq \frac{1}{2}$ then $d_j = 0$ for all $j \in \mathbb{N}_0$, i.e. there is no limit distribution.

Proof. Corollary 3.3.15 gives

$$d_j^{(N)} = \left(\frac{q}{p}\right)^j \left[\frac{p-q}{p} + \frac{q}{p} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) - F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right)\right]$$

for $j \in \mathbb{N}_0$ and $N \in \mathbb{N}$. Therefore, one can write

$$\begin{aligned}
d_j &= \lim_{N\to\infty} d_j^{(N)} \\
&= \lim_{N\to\infty} \left(\frac{q}{p}\right)^j \left[\frac{p-q}{p} + \frac{q}{p} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) - F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right)\right] \\
&= \left(\frac{q}{p}\right)^j \frac{p-q}{p} + \left(\frac{q}{p}\right)^j \lim_{N\to\infty} \left[\frac{q}{p} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) - F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right)\right].
\end{aligned} \tag{3.67}$$

Since $Y^{(N)} \in \mathbb{N}_0$, it follows that (see (3.61))

$$\begin{aligned}
F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) &= \mathrm{P}\left(Y^{(N)} \leq \left\lceil \frac{N+j}{2} \right\rceil\right) \\
&= \mathrm{P}\left(Y^{(N)} \leq \frac{N+j+1}{2}\right) \\
&= \mathrm{P}\left(\frac{1}{N} Y^{(N)} \leq \frac{1}{2} + \frac{j+1}{2N}\right).
\end{aligned} \tag{3.68}$$

The Markov chain $(Y^{(n)})_{n\in\mathbb{N}}$, where $Y^{(n)}$ gives the number of successes after n experiments, can be modeled by the sum of iid random variables $\tilde{X}^{(n)}$ with $\mathrm{P}(\tilde{X}^{(n)} = 1) = p$ and $\mathrm{P}(\tilde{X}^{(n)} = 0) = q$, i.e. $Y^{(N)} = \sum_{n=1}^N \tilde{X}^{(n)}$. It follows that $\mu := \mathrm{E}(\tilde{X}^{(n)}) = p$. The weak law of large number, see e.g. Feller [17, Chapter X, Section 1], yields for all $\varepsilon > 0$ that

$$\lim_{N\to\infty} \mathrm{P}\left(\left|\frac{1}{N} Y^{(N)} - \mu\right| > \varepsilon\right) = 0. \tag{3.69}$$

If $\mu > 1/2$ and $\varepsilon > 0$ small enough (and N large enough), such that $\mu - \varepsilon > 1/2 + (j+1)/(2N)$, then (3.68) and (3.69) yield

$$\begin{aligned} 0 = \lim_{N\to\infty} \mathrm{P}\left(\frac{1}{N} Y^{(N)} < \mu - \varepsilon\right) &\geq \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) \\ &\geq \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right) \geq 0. \end{aligned}$$

It follows that

$$\lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) = \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right) = 0$$

and (3.67) gives

$$d_j = \left(\frac{q}{p}\right)^j \frac{p-q}{p}.$$

Note, that in this case $q/p < 1$ and because of the geometric series it is

$$\sum_{j=0}^{\infty} d_j = \frac{p-q}{p} \cdot \frac{1}{1 - \frac{q}{p}} = 1,$$

i.e. the values $(d_j)_{j\in\mathbb{N}_0}$ form a distribution. If $\mu < 1/2$ and $\varepsilon > 0$ small enough, such that $\mu + \varepsilon < 1/2$, it is

$$\begin{aligned} 1 = \lim_{N\to\infty} \mathrm{P}\left(\frac{1}{N} Y^{(N)} < \mu + \varepsilon\right) &\leq \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right) \\ &\leq \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) \leq 1. \end{aligned}$$

Now, it is

$$\lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil\right) = \lim_{N\to\infty} F_{Y^{(N)}}\left(\left\lceil \frac{N+j}{2} \right\rceil - 1\right) = 1$$

and from (3.67)

$$d_j = \left(\frac{q}{p}\right)^j \frac{p-q}{p} + \left(\frac{q}{p}\right)^j \left(\frac{q}{p} - 1\right) = 0.$$

If $\mu = p = 1/2$, it is known from Remark 3.3.14 that

$$d_j^{(N)} = \left(\frac{1}{2}\right)^N \binom{N}{\lceil (N+j)/2 \rceil} \tag{3.70}$$

and Stirling's approximation, see e.g. [17, Chapter II, (9.15)], yields

$$\sqrt{2\pi} \cdot \frac{N^{N+\frac{1}{2}}}{e^N} \leq N! \leq e \cdot \frac{N^{N+\frac{1}{2}}}{e^N}.$$

Therefore, because of (3.70), it is

$$0 \leq d_j^{(N)} \leq d_0^{(N)} = \frac{N!}{\left(\left\lceil\frac{N}{2}\right\rceil!\right)^2 2^N} \leq \frac{e \cdot N^{N+\frac{1}{2}} \cdot e^{2\left\lceil\frac{N}{2}\right\rceil}}{2\pi \left\lceil\frac{N}{2}\right\rceil^{2\left\lceil\frac{N}{2}\right\rceil+1} \cdot e^N \cdot 2^N} \leq \frac{N^{N+\frac{1}{2}} \cdot e^{N+2}}{2\pi(\frac{N}{2})^{N+1} \cdot e^N \cdot 2^N} = \frac{e^2}{\pi\sqrt{N}},$$

which goes to zero as $N \to \infty$. Consequently it must be $d_j = 0$ for each $j \in \mathbb{N}_0$. □

Example 3.3.18 Let $p = 0.6$ and $q = 0.4$. Then $d_0 = 1 - \frac{2}{3} = \frac{1}{3}$. This means, that when playing this game very often, the probability of having an absolute drawdown of size zero tends to approximately 33 %. ♦

4 Conclusions

4.1 Summary

The main goal of this work is to divide a general framework for the portfolio theory into four modular building blocks, namely (a) the (multi-period) market model, (b) the trading strategy using x-portfolio generating functions, (c) the definition of risk and utility measures in an axiomatic way, and (d) the optimization problem.

Chapter 2 gives all necessary general definitions for all four points. Furthermore, reasonable properties for the market model and x-portfolios are discussed to ensure that there is no arbitrage opportunity and that the market model is non-redundant, i.e. one asset cannot be reproduced by a combination of the others. For instance the non-redundancy is an important property to be able to build utility and risk measures which are strictly concave and strictly convex, respectively, using the construction from Lemma 2.3.4. Examples are given for utility functions like the arithmetic return and the terminal wealth relative (TWR), which are both concave functions as required for the portfolio optimization. For the risk measure it is possible to use e.g. the standard deviation or conditional value at risk. Of course, the drawdown from Chapter 3 also fits into this setting.

If (a), (b), and (c) are given, then the risk-utility space can be characterized which, under few assumptions, is a closed convex set. The risk-utility values of efficient portfolios lie on the boundary of the risk-utility space. These efficient portfolios in addition are the solutions of the optimization problem: Minimize the risk such that the utility value is above or equal to a given threshold and such that the initial investment is fixed (e.g. it could be required to be fully invested). Existence and (with one additional assumption) also uniqueness of an efficient portfolio for a fixed threshold can be shown, see Theorems 2.4.19 and 2.4.20, respectively. If the risk and utility functions in addition have some more properties like positive homogeneity and linearity in the first component, a generalization of the one-fund theorem from the capital asset pricing model (CAPM) can be formulated which gives a similar result, see Corollary 2.4.27.

Chapter 3 discusses the drawdown in more detail. For a general multi-period market model, the expectation of the logarithm of the relative drawdown can be estimated by the expectation of the absolute drawdown. Note, that both variants correspond to different x-portfolio generating functions. The theory in Chapter 2 can be applied to both functions, see Corollaries 3.1.14 and 3.1.15 for the variant with the relative drawdown. The absolute drawdown gives an application of the generalized one-fund theorem, see Corollary 3.1.17.

These results only guarantee the existence and uniqueness of a solution. It remains unclear how to solve such optimization problems (e.g. numerically) and even how to evaluate the drawdown risk measure in general. Hence, the remainder of Chapter 3 focuses to a special situation for the absolute drawdown, in which the wealth can be modeled by a Markov chain. In this case, the distribution of the drawdown process after N time steps can be computed using the transition probability matrices, see Section 3.2.3. In the limit case as N goes to infinity, the existence and uniqueness of the limit distribution is guaranteed if the absolute returns of the wealth have a positive expectation, see Theorem 3.2.17. However, it is unclear how to evaluate the limit distribution and whether or not it has a finite expectation. For the special case, when it is only possible to loose one ($\alpha = 1$), to stay at the same position

($r \in [0, 1]$), or to win $\beta \in \mathbb{N}$ units, the limit distribution can be given implicitly and has finite expectation, see Section 3.2.6. For $\beta = 1$ and $\beta = 2$ an explicit expression is given as well, see Corollaries 3.2.24 and 3.2.26. The case $\alpha = \beta = 1$ corresponds to the random walk. For this special situation, furthermore, an explicit formula for the distribution of the drawdown after $N \in \mathbb{N}$ time steps is given, see Theorem 3.3.12.

4.2 Outlook

The main goal of this manuscript is to provide a modular framework for portfolio optimization and to fully incorporate the (absolute and relative) drawdown. However, there are still open questions and links worth to be discussed.

The examples for the multi-period market model in Chapter 2, see Section 2.1.3, are just defined on a discrete and finite probability space, although the here developed theory is not restricted to this case. A next step could be to examine more complex distributions in the same context. This could give better estimations for the "real" distributions of the returns and could allow to appropriately treat big losses.

For the x-portfolio generating functions, only the following three standard examples are given:

- v_{bnh} from Example 2.2.30 (x-portfolio constant over time; "buy and hold"),
- v_{twr} from Example 2.2.31 (φ-portfolio constant over time; constant weights),
- v_{cm} from Example 2.2.33 (invested amount of money constant over time).

This summarizes the typical settings used in the literature. However, an x-portfolio is a powerful building block which allows to fully characterize the way of trading the assets. It is also possible to define an x-portfolio generating function by arbitrary trading strategies, like simple strategies using moving averages of the price developments of the assets and so on. This would, of course, complicate the setting and make it harder to evaluate the functions and to solve the resulting optimization problems. Nevertheless, it opens the door to completely connect the ideas of traders directly to the risk management in the context of portfolio optimization.

Regarding the drawdown, more variants could be studied whether these can be used for the portfolio optimization from Chapter 2 or not. This includes e.g. the maximum drawdown and also the concepts of the conditional value at risk as in Chekhlov, Uryasev and Zabarankin [8, 9], Goldberg and Mahmoud [20] and Zabarankin, Pavlikov and Uryasev [55]. Chapter 3 mainly discusses the expectation of the absolute/relative drawdown. In case of the Markov chain, the existence and uniqueness of the limit distribution is shown. However, it remains unclear whether or not the expectation of this limit distribution exists, except for some special cases. If this could be shown, it might be possible to extend the results e.g. for more complex stochastic processes whose returns $X^{(n)}$ have a continuous distribution. This would allow to use the expectation of the limit distribution as a risk measure.

Last, but not least, real world applications to concrete financial markets should be studied to prove the benefits of these new concepts.

A Appendix

A.1 Convex analysis

This section summarizes some definitions and results from convex analysis, see Rockafellar [43], used throughout this manuscript. Let $A \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a convex set and $f\colon A \to \mathbb{R} \cup \{\infty\}$ be a given function. f is *convex*, if and only if its *epigraph*

$$\mathrm{epi}(f) = \{(\mathbf{x}, \mu) \in A \times \mathbb{R} \,:\, \mu \geq f(\mathbf{x})\},$$

cf. Figure A.1, is convex.

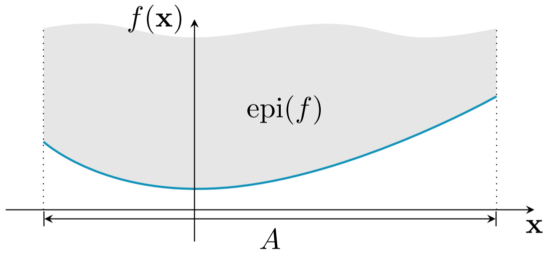


Figure A.1.: Example for a convex function f and its epigraph.

Since from definition it is $f(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in A$, the *(effective) domain* is given by

$$\mathrm{dom}(f) := \{\mathbf{x} \in A \,:\, f(\mathbf{x}) < \infty\} = \{\mathbf{x} \in A \,:\, f(\mathbf{x}) \in \mathbb{R}\}$$

which is a convex set. If $\mathrm{dom}(f) \neq \emptyset$ (and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in A$, which is already true by assumption) and f is convex, then f is called *proper convex*. If in addition the epigraph of f is closed, then function f is called a *closed proper convex* function. A function f can always be extended to $\mathbb{R}^n$ by $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined by

$$g(\mathbf{x}) := \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \in A, \\ \infty, & \text{if } \mathbf{x} \notin A. \end{cases} \tag{A.1}$$

If f is (closed) proper convex, then g is (closed) proper convex with $\mathrm{epi}(f) = \mathrm{epi}(g)$ and $\mathrm{dom}(f) = \mathrm{dom}(g)$. Hence, w.l.o.g. f is defined on $\mathbb{R}^n$.

In addition, each proper convex function f has a unique closure denoted by $\overline{f}$, which is the function whose epigraph is the closure of the epigraph of f, i.e. $\mathrm{epi}(\overline{f}) = \overline{\mathrm{epi}(f)}$. Both functions differ at most at the "boundary" of $\mathrm{dom}(f)$. To give an exact characterization, first define the *affine hull* of a set $M \subset \mathbb{R}^n$ by

$$\mathrm{aff}(M) := \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \,:\, k \in \mathbb{N},\ \mathbf{x}_1, \ldots, \mathbf{x}_k \in M,\ \lambda_1, \ldots, \lambda_k \in \mathbb{R},\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

and the ball with radius $\varepsilon > 0$ and center at $\mathbf{x}$ by $B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \,:\, \|\mathbf{y} - \mathbf{x}\| \leq \varepsilon\}$. This gives the definition of the *relative interior* of a set $M \subset \mathbb{R}^n$ by

$$\mathrm{ri}(M) := \{\mathbf{x} \in \mathrm{aff}(M) \,:\, B_\varepsilon(\mathbf{x}) \cap \mathrm{aff}(M) \subset M \text{ for at least one } \varepsilon > 0\}.$$

For example for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ fixed and $S := \{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \, : \, \alpha \in [0,1]\}$, i.e. the line segment from $\mathbf{x}$ to $\mathbf{y}$, the relative interior is $\mathrm{ri}(S) = \{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \, : \, \alpha \in (0,1)\}$, i.e. the same line segment but without the end points $\mathbf{x}$ and $\mathbf{y}$.

Then the closure of f differs from f at most at the *relative boundary* of $\mathrm{dom}(f)$ which is defined by $\overline{\mathrm{dom}(f)} \setminus \mathrm{ri}(\mathrm{dom}(f))$ as stated by the following result.

Theorem A.1.1 (cf. [43, Theorem 7.4 and Corollary 7.4.1]) If f is proper convex, then $\overline{f}$ is closed proper convex. In addition $\mathrm{dom}(\overline{f})$ differs from $\mathrm{dom}(f)$ at most at the relative boundary of $\mathrm{dom}(f)$ and $\overline{f}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ which are not on the relative boundary of $\mathrm{dom}(f)$.

Hence, it is possible to replace a proper convex function $f \colon A \to \mathbb{R} \cup \{\infty\}$ by its closure $\overline{f}$ which can be defined using arbitrary $\mathbf{x} \in \mathrm{ri}(\mathrm{dom}(f))$ by

$$\overline{f}(\mathbf{y}) = \lim_{\lambda \nearrow 1} f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \tag{A.2}$$

for all $\mathbf{y} \in A$, see [43, Theorems 7.5]. Then it is $\overline{f}(\mathbf{y}) \leq f(\mathbf{y})$ for all $\mathbf{y} \in A$. This closure has some preferable properties as given by the next result.

Theorem A.1.2 (cf. [43, Theorems 4.6 and 7.1]) Let $A \subset \mathbb{R}^n$ be non-empty and convex. Let $f \colon A \to \mathbb{R} \cup \{\infty\}$ be proper convex and extended to $\mathbb{R}^n$ as in (A.1), i.e. w.l.o.g. $A = \mathbb{R}^n$. Then the following statements are equivalent:

(a) the epigraph of f is closed, i.e. f is closed proper convex,

(b) the sublevel set $\{\mathbf{x} \in \mathbb{R}^n \, : \, f(\mathbf{x}) \leq \alpha\}$ is closed and convex for all $\alpha \in \mathbb{R}$,

(c) f is lower semi-continuous in $\mathbb{R}^n$.

Regarded continuity of f, the following result holds true.

Theorem A.1.3 (cf. [43, Theorem 10.1]) Let $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be proper convex. Then the restriction of f to $\mathrm{ri}(\mathrm{dom}(f))$ is continuous.

Another result is the following corollary, which gives conditions to obtain compact level sets.

Corollary A.1.4 (cf. [43, Corollary 8.7.1]) Let f be closed proper convex. If the level set $\{\mathbf{x} \, : \, f(\mathbf{x}) \leq \alpha\}$ is non-empty and bounded for one $\alpha \in \mathbb{R}$, then it is bounded for every $\alpha \in \mathbb{R}$. In this case the level sets are even compact.

Remark A.1.5 (Concave functions) For $A \subset \mathbb{R}^n$, $n \in \mathbb{N}$, a function $g \colon A \to \mathbb{R} \cup \{-\infty\}$ is called (proper) concave, if $-g$ is (proper) convex. Hence, the same results from above can be applied to $-g$ for concave function g, where e.g. Theorem A.1.2 applied to $-g$ states that the superlevel sets of g are closed and g is upper semi-continuous. ♦

In addition, the Weierstraß theorem for semi-continuous functions is also helpful and reads as follows.

Theorem A.1.6 (cf. Barbu and Precupanu [3, Theorem 2.8]) Let $M \subset \mathbb{R}^n$ be compact, $f \colon M \to \mathbb{R}$ be lower semi-continuous and $g \colon M \to \mathbb{R}$ be upper semi-continuous. Then f attains its global minimum on M and g attains its global maximum on M.

A.2 Convex programming

An ordinary convex program according to Rockafellar [43, Section 28] reads as follows: Let $C \subset \mathbb{R}^n$ be non-empty and convex with finite and convex functions $f, g_1, \dots, g_r$ on C and affine functions $h_1, \dots, h_m$ on C for $r, m \in \mathbb{N}$. Solve

$$\min_{\mathbf{x}\in C} f(\mathbf{x}) \quad \text{subject to} \quad g_1(\mathbf{x}) \le 0, \dots, g_r(\mathbf{x}) \le 0, \quad h_1(\mathbf{x}) = 0, \dots, h_m(\mathbf{x}) = 0. \tag{A.3}$$

Note, that $r = 0$ and also $m = 0$ is allowed which means there are no inequality or equality constraints, respectively.

The Lagrangian $L\colon \mathbb{R}^{r+m+n} \to \mathbb{R}$ of (A.3) is defined by

$$L(\lambda, \mathbf{x}) := \begin{cases} f(\mathbf{x}) + \sum_{i=1}^{r} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{m} \lambda_{r+j} h_j(\mathbf{x}), & \text{if } \mathbf{x} \in C \text{ and } \lambda_i \ge 0 \text{ for } i = 1, \dots, r, \\ -\infty, & \text{if } \mathbf{x} \in C \text{ and } \lambda_i < 0 \text{ for some } 1 \le i \le r, \\ +\infty, & \text{if } \mathbf{x} \notin C. \end{cases}$$

for $\mathbf{x} \in \mathbb{R}^n$ and Lagrange multipliers $\lambda := (\lambda_1, \dots, \lambda_{r+m}) \in \mathbb{R}^{r+m}$.

Definition A.2.1 (Karush-Kuhn-Tucker vector, cf. [43, Section 28]) A vector $\lambda := (\lambda_1, \dots, \lambda_{r+m}) \in \mathbb{R}^{r+m}$ is called a (Karush-)Kuhn-Tucker vector for (A.3) if $\lambda_i \ge 0$ for $i = 1, \dots, r$ and if the Lagrangian $L(\lambda, \cdot)$ has a finite infimum (in $\mathbf{x}$) which equals to the infimum of $\widetilde{f}$ which is given by

$$\widetilde{f}(\mathbf{x}) := \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \in C \text{ and } g_1(\mathbf{x}) \le 0, \dots, g_r(\mathbf{x}) \le 0, h_1(\mathbf{x}) = 0, \dots, h_m(\mathbf{x}) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

That means $\inf_{\mathbf{x}\in\mathbb{R}^n} L(\lambda, \mathbf{x}) = \inf_{\mathbf{x}\in\mathbb{R}^n} \widetilde{f}(\mathbf{x}) \in \mathbb{R}$.

The existence of such a Karush-Kuhn-Tucker vector is ensured by the following theorem.

Theorem A.2.2 (cf. [43, Theorem 28.2]) In the situation above let $I \subset \{1, \dots, r\}$ be the set of indexes such that g_i is not affine. Assume that the infimum of $\widetilde{f}$ is not $-\infty$ and that there exists an element in the relative interior $\mathrm{ri}(C)$ which satisfies all constraints in (A.3), with strict inequality for all the inequality constraints with index $i \in I$. Then a Karush-Kuhn-Tucker vector exists for (A.3).

The next result gives a connection between the ordinary convex program (A.3), the Karush-Kuhn-Tucker vector and the Lagrangian.

Theorem A.2.3 (cf. [43, Theorem 28.3]) Let the situation above be given and let $\lambda^* \in \mathbb{R}^{r+m}$ and $\mathbf{x}^* \in \mathbb{R}^n$. Then, λ^* is a Karush-Kuhn-Tucker vector for (A.3) and $\mathbf{x}^*$ is a solution of (A.3) if and only if $(\lambda^*, \mathbf{x}^*)$ is a saddle point of the Lagrangian of (A.3) (maximizing in λ and minimizing in $\mathbf{x}$, i.e. $L(\lambda, \mathbf{x}^*) \le L(\lambda^*, \mathbf{x}^*) \le L(\lambda^*, \mathbf{x})$ for all $\lambda \in \mathbb{R}^{r+m}$ and all $\mathbf{x} \in \mathbb{R}^n$). Moreover, this condition holds if and only if $\mathbf{x}^*$ and $\lambda^* = (\lambda_1^*, \dots, \lambda_{r+m}^*)$ satisfy

(a) $\lambda_i^* \ge 0$, $g_i(\mathbf{x}^*) \le 0$ and $\lambda_i^* g_i(\mathbf{x}^*) = 0$ for $i = 1, \dots, r$,
(b) $h_j(\mathbf{x}^*) = 0$ for $j = 1, \dots, m$,

(c) $0 \in [\partial f(\mathbf{x}^*) + \sum_{i=1}^{r} \lambda_i^* \partial g_i(\mathbf{x}^*) + \sum_{j=1}^{m} \lambda_{r+j}^* \partial h_j(\mathbf{x}^*)]$, where ∂F denotes the subdifferential of a function $F\colon \mathbb{R}^n \to \mathbb{R}$, i.e.

$$\partial F(\mathbf{x}^*) := \{\mathbf{z} \in \mathbb{R}^n \;:\; F(\mathbf{x}) - F(\mathbf{x}^*) \geq \mathbf{z}^\mathsf{T}(\mathbf{x} - \mathbf{x}^*) \text{ for all } \mathbf{x} \in \mathbb{R}^n\},$$

see also (2.66).

A.3 Gambler's ruin

The result in Feller [17] for bounds on the probability of the gambler's ruin problem, see Remark 3.2.8, is required for a proof regarding the drawdown (see Lemma 3.2.15) and reads as follows.

Proposition A.3.1 (Bounds for gambler's ruin problem, cf. [17, Chapter XIV, Section 8, Theorem]) Let $(Z^{(n)})_{n\in\mathbb{N}_0}$ be a Markov chain on $\mathbb{Z}$ as in Definition 3.2.2 defined by iid random variables $(X^{(n)})_{n\in\mathbb{N}}$ as in (3.14) and (3.15), where $X^{(n)} \sim X$. For fixed $1 < L \in \mathbb{N}$, the random variable which gives the number of steps, until the Markov chain either exceeds the upper barrier L or the lower barrier at zero, is denoted by

$$N_L := \min\{n \in \mathbb{N} \;:\; Z^{(n)} \leq 0 \text{ or } Z^{(n)} \geq L\}. \tag{A.4}$$

Assume $u \in \mathbb{N}$ with $1 \leq u \leq L-1$. If $\mathrm{E}[X] = 0$, then

$$\frac{L-u}{L+\alpha-1} \leq \mathrm{P}\left(Z^{(N_L)} \leq 0 \mid Z^{(0)} = u\right) \leq \frac{L+\beta-u-1}{L+\beta-1},$$

and if $\mathrm{E}[X] \neq 0$ then

$$\frac{\rho_1^L - \rho_1^u}{\rho_1^L - \rho_1^{-\alpha+1}} \leq \mathrm{P}\left(Z^{(N_L)} \leq 0 \mid Z^{(0)} = u\right) \leq \frac{\rho_1^{L+\beta-1} - \rho_1^u}{\rho_1^{L+\beta-1} - 1},$$

where $\rho_1 \in \mathbb{R}_{>0} \setminus \{1\}$ is the unique solution different from one of the equation $\phi(\rho) = 1$, where $\phi(\rho) := \mathrm{E}[\rho^X] = r + \sum_{k=1}^{\alpha} q_k \rho^{-k} + \sum_{k=1}^{\beta} p_k \rho^k$. It is $\rho_1 > 1$, if $\mathrm{E}[X] < 0$, and $\rho_1 \in (0,1)$, if $\mathrm{E}[X] > 0$.

In the case $\alpha = \beta = 1$ and $p := p_1 \neq q_1 =: q$, it is $\rho_1 = q/p$, independent on r, and, thus, Proposition A.3.1 gives that

$$\mathrm{P}\left(Z^{(N_L)} \leq 0 \mid Z^{(0)} = u\right) = \frac{(\frac{q}{p})^L - (\frac{q}{p})^u}{(\frac{q}{p})^L - 1},$$

which is a well-known result, see e.g. Feller [17, Chapter XIV, Section 2, (2.4)]. Another result, which is required in the sequel, can be found in Ethier and Khoshnevisan [16] (with slightly different assumptions and shifted random walk) and reads.

Proposition A.3.2 (Bounds for expected time until ruin, see [16, Corollary in Section 3]) In the situation of Proposition A.3.1 with same notation, let $1 \leq u \leq L-1$ be arbitrary. If $\mathrm{E}[X] < 0$, then $\rho_1 > 1$ and

$$\frac{L+\beta-1}{\mathrm{E}[X]} \frac{\rho_1^{u+\alpha-1}-1}{\rho_1^{L+\alpha-1}-1} - \frac{u}{\mathrm{E}[X]} \leq \mathrm{E}\left[N_L \mid Z^{(0)} = u\right] \leq \frac{L+\alpha-1}{\mathrm{E}[X]} \frac{\rho_1^{u}-1}{\rho_1^{L+\beta-1}-1} - \frac{u+\alpha-1}{\mathrm{E}[X]}$$

holds true.

Proof. The result follows exactly as in the proof in Ethier and Khoshnevisan [16, Corollary in Section 3] but with a shifted stochastic process. Therein, the Markov chain starts at zero, has lower barrier at $-L'$ (denoted by L in the reference) and upper barrier at W. Hence, the situation in Proposition A.3.1 shifted by $-u$ (i.e. the Markov chain just starts at 0 instead of u) gives $L' = u$ and $W = L - u$. Inserting this into the (inner bounds) of the result in [16, Corollary in Section 3] proves this proposition using Doob's optional sampling theorem, see e.g. Föllmer and Schied [18, Theoren 6.15]. To be more precise, for

$$\begin{aligned} g_1(\rho_1) &= \frac{1}{\mathrm{E}[X]}\left((L+\alpha-1)\frac{\rho_1^{u}-1}{\rho_1^{L+\beta-1}-1} - u - \alpha + 1\right), \\ g_2(\rho_1) &= \frac{1}{\mathrm{E}[X]}\left((L+\beta-1)\frac{\rho_1^{u+\alpha-1}-1}{\rho_1^{L+\alpha-1}-1} - u\right) \end{aligned}$$

the estimate

$$g_2(\rho_1) \leq \mathrm{E}\left[N_L \mid Z^{(0)} = u\right] \leq g_1(\rho_1)$$

holds. □

Also note, that the result in [16, Corollary in Section 3] has more assumptions on the Markov chain and in addition proves also less accurate bounds which are in explicit form and easier to compute. The additional assumptions are only necessary for these estimates, i.e. for some outer bounds around the bounds stated above.

A.4 Markov chain

Most of the required general theory on Markov chain can be found in the book of Billingsley [5, Section 8].

Definition A.4.1 (cf. [5, Section 8, Stationary/Limit Distributions]) Let a Markov chain $(Y^{(n)})_{n\in\mathbb{N}_0}$ with state space $K \subset \mathbb{Z}$ be given. For $i, j \in K$ and $n \in \mathbb{N}$ denote

$$p_{i,j}^{(n)} := \mathrm{P}\left(Y^{(m+n)} = j \mid Y^{(m)} = i\right), \qquad p_{i,j} := p_{i,j}^{(1)} \qquad \text{and} \qquad p_{i,j}^{(0)} := \delta_{i,j},$$

which is independent of $m \in \mathbb{N}_0$ and where $\delta_{i,j}$ is the Kronecker delta.
A distribution $(\pi_j)_{j\in K} \subset [0,1]$ is called *stationary*, if

$$\sum_{i\in K} \pi_i p_{i,j} = \pi_j \geq 0, \quad \text{for all } j \in K, \quad \text{and} \quad \sum_{i\in K} \pi_i = 1,$$

i.e. a stationary distribution (also called *limit distribution*) does not change in time under the Markov chain.

Using mathematical induction yields $\sum_{i\in K} \pi_i p_{i,j}^{(n)} = \pi_j$ for each $n \in \mathbb{N}$ and all $j \in K$, where the induction step reads

$$\sum_{i\in K} \pi_i p_{i,j}^{(n+1)} = \sum_{i\in K} \pi_i \sum_{k\in K} p_{i,k}^{(n)} p_{k,j} = \sum_{k\in K} p_{k,j} \sum_{i\in K} \pi_i p_{i,k}^{(n)} = \sum_{k\in K} p_{k,j} \pi_k = \pi_j.$$

The distribution $(\pi_j)_{j\in K}$ in Definition A.4.1 is also called the limit distribution, because it will be shown that, under some reasonable assumptions, this is the limit of the distributions after n steps as n goes to infinity. For this to show it is meaningful to define some additional properties of a Markov chain.

Definition A.4.2 (cf. [5, Section 8]) Let $(Y^{(n)})_{n\in\mathbb{N}_0}$ be a Markov chain with state space $K \subset \mathbb{Z}$. Make the following definitions:

(a) Define the probability of reaching $j \in K$ for the first time exactly at step n when starting at position $i \in K$ by

$$f_{i,j}^{(n)} := \mathrm{P}\left(Y^{(1)} \neq j, \ldots, Y^{(n-1)} \neq j, Y^{(n)} = j \mid Y^{(0)} = i\right)$$

and the probability of reaching $j \in K$ eventually when starting at position $i \in K$ by

$$f_{i,j} := \sum_{n=1}^{\infty} f_{i,j}^{(n)}.$$

(b) Position $i \in K$ is called *recurrent* (or *persistent*), if $f_{i,i} = 1$, and *transient*, if $f_{i,i} < 1$. If all states in K are recurrent then the Markov chain is called recurrent, and if all states are transient the Markov chain is called transient.

(c) A Markov chain is *irreducible*, if for each $i \in K$ and $j \in K$ there exists a number $n = n(i,j) \in \mathbb{N}$ such that $p_{i,j}^{(n)} > 0$, i.e. each position is reachable from any position in finite time with positive probability.

(d) State $j \in K$ is said to have period $t = t_j \in \mathbb{N}$, where t is defined by

$$t := \gcd\left\{n \in \mathbb{N} \,:\, p_{j,j}^{(n)} > 0\right\}.$$

If the period equals to one for all states ($t_j = 1$ for all $j \in K$), the Markov chain is called *aperiodic*.

(e) The mean recurrence time is defined by $\mu_j := \sum_{n=1}^{\infty} n f_{j,j}^{(n)}$ for $j \in K$.

Using these definitions, the following results for the existence and uniqueness of a stationary/limit distribution has been proved in Billingsley [5, Theorem 8.8].

Theorem A.4.3 (cf. [5, Theorem 8.8]) Let $(Y^{(n)})_{n\in\mathbb{N}_0}$ be an irreducible, aperiodic Markov chain with state space K. Then exactly one of the following three statements holds true:

(a) (Transient case) The Markov chain is transient. Then it is $\lim_{n\to\infty} p_{i,j}^{(n)} = 0$ for all $i,j \in K$, i.e. there is no limit distribution, and $\sum_{n=1}^{\infty} p_{i,j}^{(n)} < \infty$ for all $i,j \in K$.

(b) (Null recurrent case) The Markov chain is recurrent, but there is no limit distribution. For all $i,j \in K$ it is $\lim_{n\to\infty} p_{i,j}^{(n)} = 0$, $\sum_{n=1}^{\infty} p_{i,j}^{(n)} = \infty$ and $\mu_j = \infty$.

(c) (Positive recurrent case) The Markov chain is recurrent and there exists a unique limit distribution $(\pi_j)_{j\in K}$. Then for all $i,j \in K$ it is $\lim_{n\to\infty} p_{i,j}^{(n)} = \pi_j > 0$, independent of i, and the mean recurrence time satisfies $\mu_j = 1/\pi_j < \infty$.

This theorem guarantees the existence and also uniqueness of the limit distribution under some assumptions. Therein the Markov chain needs to be irreducible, aperiodic and recurrent. By Theorem A.4.3, if $\lim_{n\to\infty} p_{i,j}^{(n)} = \pi_j > 0$ for at least one pair i and j, or $\mu_j < \infty$ for at least one j, then there exists a unique limit distribution.

To verify, which case of Theorem A.4.3 holds true for a Markov chain, it is essential to be able to classify the Markov chain. Relevant for the case of the absolute drawdown in Section 3.2 are the following two results.

Theorem A.4.4 (cf. [5, Theorem 8.3]) Let $(Y^{(n)})_{n\in\mathbb{N}_0}$ be an irreducible Markov chain with state space K. Then exactly one of the following statements is true:

(a) All states are transient and $\sum_{n=1}^{\infty} p_{i,j}^{(n)} < \infty$ for all $i,j \in K$ (and thus the Markov chain is transient).

(b) All states are recurrent and $\sum_{n=1}^{\infty} p_{i,j}^{(n)} = \infty$ for all $i,j \in K$ (and thus the Markov chain is recurrent).

Lemma A.4.5 ([5, near equation (8.33)]) Let $(Y^{(n)})_{n\in\mathbb{N}_0}$ be an irreducible Markov chain with state space K. If state $j \in K$ has period $t \in \mathbb{N}$, then all states $i \in K$ have the same period t.

A.5 Number theory

The following three results from number theory are related to Bézout's identity, see e.g. Niven, Zuckerman and Montgomery [40, Theorem 1.3].

Lemma A.5.1 (see Billingsley [5, Proof of Appendix A21]) Let $M \subset \mathbb{N}$ be closed under addition. Then there exist $m, m' \in M$ such that $\gcd(M) = m - m'$.

Proof. Let $A := \{m, -m, m - m' \in \mathbb{Z} : m, m' \in M\}$, which is closed under addition and subtraction. Assume d is the smallest positive element of A. Each $n \in A$ can be written as $n = qd + r$, where $q \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $0 \le r < d$. Since $r = n - qd \in A$, it must be $r = 0$. Then A consists of multiples of d. Consequently d divides each number in A and thus also in $M \subset A$. Then it is $d = \gcd(M)$ and either $d \in M$ or there exist $m, m' \in M$ such that $d = m - m'$. But even in the first case d can be written as $d = 2d - d$, where $2d, d \in M$, because M is closed under addition. $\square$

Lemma A.5.2 (cf. [5, Appendix A21]) Let $M \subset \mathbb{N}$ be closed under addition and $\gcd(M) = 1$. Then there exists $n_0 \in \mathbb{N}$, such that $n \in M$ for all $n \geq n_0$.

Proof. Let $m, m' \in M$ such that $1 = m - m'$ (existence follows from Lemma A.5.1) and $n_0 := (m + m')^2$. Then each $n > n_0$ can be written as $n = q(m + m') + r$ where $q \in \mathbb{N}$ and $r \in \mathbb{N}$ with $0 \leq r < m + m'$. From $n > n_0 = (m + m')^2 \geq (r + 1)(m + m')$ it follows

$$q = \frac{n - r}{m + m'} > \frac{(r + 1)(m + m') - r}{m + m'} = r + 1 - \frac{r}{m + m'} > r.$$

In addition, it is $n = q(m + m') + r = q(m + m') + r(m - m') = (q + r)m + (q - r)m'$. Since M is closed under addition and $q + r \geq q - r > 0$, it must be $n \in M$. Note that $n_0 \in M$ as well because $n_0 = qm + qm'$ with $q = m + m' \in \mathbb{N}$. $\square$

Lemma A.5.3 Let $M_1, M_2 \subset \mathbb{N}$ be two finite sets. For $j \in \{1, 2\}$ the set

$$\overline{M}_j := \left\{ \sum_{i=1}^{n} k_i \in \mathbb{N} \,:\, n \in \mathbb{N},\ k_1, \ldots, k_n \in M_j \right\}$$

is the closure of M_j under addition. Then, there are elements $a_1 \in \overline{M}_1$ and $a_2 \in \overline{M}_2$ such that $\gcd\{a_1, a_2\} = \gcd(M_1 \cup M_2)$.

Proof. For $j \in \{1, 2\}$ let $d_j := \gcd(M_j)$. Define $A_j := \{m/d_j \,:\, m \in \overline{M}_j\} \subset \mathbb{N}$, which is closed under addition with $\gcd(A_j) = 1$. Lemma A.5.2 gives an integer $n_j \in \mathbb{N}$ such that $n \in A_j$ for all $n \geq n_j$. Consequently, it is $d_j n \in \overline{M}_j$ for all $n \geq n_j$. Choose two prime numbers $p_1, p_2 \in \mathbb{N}$, such that $p_1 \neq p_2$ and $p_1, p_2 > \max\{n_1, n_2, d_1, d_2\}$. Then it is $p_j \in A_j$ and $d_j p_j \in \overline{M}_j$ for $j \in \{1, 2\}$, and neither p_1 nor p_2 divide d_1 or d_2 (because $p_i > d_j$ for $i, j \in \{1, 2\}$). Hence, it is

$$\gcd(M_1 \cup M_2) = \gcd\{\gcd(M_1), \gcd(M_2)\} = \gcd\{d_1, d_2\} = \gcd\{d_1 p_1, d_2 p_2\},$$

which completes the proof. $\square$

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