

Left saturation closure

Theory and algorithms

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Abstract

Ore localization of rings and modules is a technique that is widely used throughout non-commutative algebra with applications in ring theory, dimension theory, non-commutative geometry, and many other areas.

In this thesis, we pursue several goals. First of all, we give a thorough introduction to Ore localization, starting from its axiomatic definition and working our way through the classical construction inspired by Ore's original work. While Ore localization is perfectly compatible with the addition operation of the ring, the presence of an additive structure is in no way required for the construction. With this in mind, we explore how the theory of Ore localization changes when we consider monoids instead of rings.

The main contribution of this thesis is the notion of *left saturation closure* or LSat and the applications thereof: for a left denominator set S in an arbitrary ring R , $\text{LSat}(S)$ represents a canonical form of S with respect to R -fixing isomorphisms of localizations of R . Furthermore, $\text{LSat}(S)$ is involved in many characterizations of left invertible elements, left ideals, and other structural properties of the localization $S^{-1}R$. We show that $\text{LSat}(S)$ is a left denominator set in R if and only if $S^{-1}R$ is Dedekind-finite, in which case $S^{-1}R$ is canonically isomorphic to $\text{LSat}(S)^{-1}R$ and $\text{LSat}(S)$ characterizes the unit group of $S^{-1}R$. We study classes of rings that satisfy this condition: for Ore domains we characterize maximal and pre-maximal left denominator set, and for factorization domains we prove that any saturated left denominator set is uniquely determined by the set of irreducible elements it contains. As an illustration we discuss several left denominator sets in Weyl algebras and their corresponding localizations.

Afterwards, we expand our structural investigations to Ore-localized modules, where we recognize the well-known concept of local closure of a submodule as another special case of left saturation closure. We identify local torsion, which is a generalization of the classical notion of torsion, as a special case of local closure and show further connections between the two concepts.

In the last part of this thesis, we give algorithmic solutions to several problems related to Ore localizations of G -algebras, or OLGAs for short. We describe our framework to perform basic arithmetic in OLGAs as well as our implementation in the computer algebra system SINGULAR:PLURAL. Lastly, we develop multiple algorithms to compute local closure in important special cases.

Kurzzusammenfassung

Die Theorie der Ore-Lokalisierung von Ringen und Moduln findet Verwendung in vielen Bereichen der nicht-kommutativen Algebra, zum Beispiel Ringtheorie, Dimensionstheorie, nicht-kommutative Geometrie und viele andere.

In dieser Dissertation verfolgen wir mehrere Ziele. Wir beginnen mit einer gründlichen Einführung zum Thema Ore-Lokalisierung, aufbauend auf einer axiomatischen Definition und der klassischen Konstruktion, die von Ores Arbeiten inspiriert ist. Ore-Lokalisierung ist problemlos mit der Ringaddition verträglich, allerdings ist deren Existenz für die Konstruktion überhaupt nicht notwendig. Aus diesem Grund untersuchen wir, wie sich die Theorie verändert, wenn wir statt Ringen Monoide betrachten.

Der Hauptbeitrag dieser Dissertation ist das Konzept des *Linkssaturationsabschlusses* oder kurz LSat und die sich daraus ergebenden Anwendungen: Für eine Linksnennermenge S in einem beliebigen Ring R stellt $\text{LSat}(S)$ eine kanonische Form von S bezüglich R -fixierender Homomorphismen von Lokalisierungen von R dar. Weiterhin spielt $\text{LSat}(S)$ eine bedeutende Rolle in Charakterisierungen von links-invertierbaren Elementen und Linksidealen sowie weiterer struktureller Eigenschaften der Lokalisierung $S^{-1}R$. Wir beweisen, dass $\text{LSat}(S)$ genau dann eine Linksnennermenge in R ist, wenn $S^{-1}R$ Dedekind-endlich ist. In diesem Fall ist $S^{-1}R$ auf kanonische Art und Weise isomorph zu $\text{LSat}(S)^{-1}R$ und zusätzlich liefert $\text{LSat}(S)$ eine vollständige Beschreibung der Einheitengruppe von $S^{-1}R$. Wir untersuchen Klassen von Ringen, die diese Eigenschaft erfüllen: In Ore-Bereichen charakterisieren wir maximale und prämaximale Linksnennermengen und in Faktorisierungsbereichen beweisen wir, dass jede saturierte Linksnennermenge eindeutig durch die darin enthaltenen irreduziblen Elemente bestimmt ist. Wir betrachten beispielhaft mehrere Linksnennermengen in Weyl-Algebren und die dazugehörigen Lokalisierungen.

Im Anschluss erweitern wir unsere strukturellen Untersuchungen auf Ore-lokalisierte Moduln. Insbesondere zeigen wir, dass das etablierte Konzept des lokalen Abschlusses von Untermoduln ein weiterer Spezialfall des Linkssaturationsabschlusses ist. Als Spezialfall des lokalen Abschlusses betrachten wir lokale Torsion, was wiederum eine Verallgemeinerung des klassischen Torsionsbegriffs ist, und zeigen weitere Verbindungen zwischen den zwei Konzepten auf.

Im letzten Teil dieser Dissertation geben wir algorithmische Lösungen für diverse Fragestellungen im Rahmen von Ore-Lokalisierungen von G -Algebren, kurz OLGAs. Wir beschreiben unser Konzept für grundlegende Arithmetik in OLGAs und deren Implementierung im Computeralgebra-System SINGULAR:PLURAL. Abschließend entwickeln wir mehrere Algorithmen, um den lokalen Abschluss in wichtigen Spezialfällen zu berechnen.

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Chapter 1.

Introduction

1.1. From commutative domains to arbitrary rings

Localizing commutative rings and especially domains is a fundamental tool throughout algebra which generalizes the construction of quotient fields. The name localization comes from algebraic geometry, where the concept is used to study algebraic varieties “locally”, that is, in neighborhoods of given points. In the realm of commutative algebra, localization is used to study rings, ideals, and modules by relating properties of such objects with properties of their localized counterparts.

The main idea behind localization is fairly simple: start with a domain R and a *multiplicative subset* S , which means that $1 \in S$ and the product of two elements in S is again contained in S . Then take the set $S^{-1}R$ of all objects of the form $\frac{r}{s}$ with numerator $r \in R$ and denominator $s \in S$ and call two such objects $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ equal if $s_2r_1 = s_1r_2$ as you would two rational numbers. Even more, if you define addition and multiplication for these objects exactly in the same way as for rational numbers, i.e.

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1 + s_1r_2}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2},$$

then $S^{-1}R$ is again a commutative domain in which R embeds naturally via the *localization map* $r \mapsto \frac{r}{1}$. The most common localizations of commutative domains R are the following:

- Let $S = \{f^k \mid k \in \mathbb{N}_0\}$ for some $f \in R \setminus \{0\}$, then $R_f := S^{-1}R = \left\{ \frac{r}{f^k} \mid r \in R, k \in \mathbb{N}_0 \right\}$.
- Let $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then $R_{\mathfrak{p}} := S^{-1}R = \left\{ \frac{r}{s} \mid r \in R, s \notin \mathfrak{p} \right\}$.
- Let $S = R \setminus \{0\}$, then $\text{Quot}(R) := S^{-1}R = \left\{ \frac{r}{s} \mid r, s \in R, s \neq 0 \right\}$.

The base ring R embeds into each of these localizations, and every localization can be embedded into the quotient field $\text{Quot}(R)$:

$$\begin{array}{ccccc} & & R_f & \hookrightarrow & \\ & \nearrow & & \searrow & \\ R & \hookrightarrow & & \hookrightarrow & \text{Quot}(R) \\ & \searrow & & \nearrow & \\ & & R_{\mathfrak{p}} & \hookrightarrow & \end{array}$$

In 1931, Øystein Ore gave a construction for (right) quotient fields of (not necessarily commutative) domains to study linear equations with coefficients in skew fields or division rings ([Ore31]). This initiated a search for even more general settings that still provide a theory of

localization in the spirit of Ore's work. After some intermediate results by Asano for sets S consisting only of regular elements ([Asa40; Asa49]), Gabriel gave a version of Ore localization for arbitrary rings, albeit with additional restrictions on S ([Gab62]).

Ore localization proved to be an important tool in the study of semi-prime rings ([Gol60]), Noetherian rings ([MR01; GW04]), dimension theory ([KL00]), non-commutative geometry ([Ško06]), differential equations ([Tsa00]), and many others.

In contrast to these works, where localization is a building block in another theory, the spotlight in this thesis belongs to the concept of localization itself.

1.2. From rings to monoids and back again

Axiomatically, a left Ore localization of a ring R at a multiplicative subset S is a pair (R_S, η) of a ring R_S and a homomorphism $\eta : R \rightarrow R_S$ satisfying three conditions:

- (i) For all $s \in S$, $\eta(s)$ is a unit in R_S .

This first condition is the entire reason behind localization: pass from R to another ring R_S in such a way that the image of the set S in R_S consists only of units. While we could stop here, the remaining two axioms are in place for a good reason: without them, the resulting localizations can be quite unreasonable ([Lam99; GW04]).

- (ii) For all $x \in R_S$, there exist $s \in S$ and $r \in R$ such that $x = \eta(s)^{-1}\eta(r)$.

The second condition requires the elements of the localization to at least appear like fractions in the sense that they have a denominator $\eta(s)$ and a numerator $\eta(r)$. Furthermore, we see here the reason why we have to make the distinction between left and right localizations, since we have to agree on which side we want to write the denominator: in general, the localization R_S can be non-commutative.

Lastly, we want to make sure that we do not lose too much by passing to the localization:

- (iii) $\ker(\eta) = \{r \in R \mid \exists s \in S : sr = 0\}$.

If there exists $s \in S$ such that $sr = 0$, then $\eta(r) = \eta(s)^{-1}\eta(s)\eta(r) = \eta(s)^{-1}\eta(sr) = 0$ since $\eta(s)$ is invertible by the first condition. Thus, this condition ensures that the kernel of η is as small as possible, since otherwise the zero ring would always be a perfectly valid (but still unhelpful) choice for R_S . Note that the third condition is equivalent to a kernel-free version:

- (iii') For all $r_1, r_2 \in R$, $\eta(r_1) = \eta(r_2)$ if and only if there exists $s \in S$ such that $sr_1 = sr_2$.

In this form we can see that the axioms (i), (ii), and (iii') represent requirements that are only concerned with the multiplicative structure of the ring. Thus it is a natural generalization to study (multiplicative) monoids and localizations thereof first; this will be the content of **Part I**. Then we will review the material again in **Part II**, but this time with the focus on rings, which are the main interest of this thesis.

While this certainly leads to many instances of results that are stated two times, where the only difference is that the word “monoid” has been replaced by the word “ring”, there are other advantages to this approach: firstly, by treating localization of monoids separately, we will gain

a deeper understanding of which results follow purely from the concept of localization and which require the presence of addition to make them work, similar to how we study non-commutative localization to gain insights into commutative localization. Furthermore, by proving the monoid-related versions of certain results first, we can reduce the amount and length of the proofs of the ring-related versions considerably, since we will only have to deal with properties related to addition.

For the convenience of the reader, we have included a multitude of cross-references between the monoid-theoretic and the ring-theoretic versions of the same result.

1.3. From definitions to results

In this section, R is always a monoid or a ring. Starting from the axiomatic definition discussed above, we give a thorough introduction to Ore localization in [Chapter 2](#) and [Chapter 5](#), respectively. This includes rigorous and complete elementary proofs, which are often omitted in the literature and left as an exercise for the reader. While certain results can be proved with faster methods ([\[MR01\]](#)), we firmly believe that the hands-on experience gained from working through these proofs the hard way is immensely helpful when dealing with concrete examples.

We use the universal property of localization to prove that the localization, if it exists, is always unique up to unique isomorphism. This leads into the construction of the localization inspired by Ore's work. In particular, we recover the fact that left Ore localization at S is possible if and only if S is a *left denominator set*: additionally to being a multiplicative set, S has to satisfy two additional properties, called the *left Ore condition* and *left reversibility*, respectively. Both of these requirements are always met for multiplicative sets consisting entirely of *central* elements, that is, elements commuting with every other element of the ring. At this point we also verify that the classical commutative localization is indeed a special case of Ore localization. Afterwards, we show that if both the left and the right Ore localization of R at a set S exist, then they must be isomorphic. We follow this up with a collection of helpful tools which aid in proving that a multiplicative set is a left denominator set. As a ring-exclusive bonus, we also look into how the localization of a graded ring can be graded itself.

In [Chapter 3](#) and [Chapter 6](#), respectively, we introduce the notion of *left saturation closure* in a general setting: for a subset P of a left R -module M , the left saturation closure at a subset T of R is defined as

$$\text{LSat}_T(P) := \{m \in M \mid \exists t \in T : tm \in P\}.$$

If T is a multiplicative set, then $P = \text{LSat}_T(P)$ if and only if P is *left T -saturated*: if $tm \in P$ for some $t \in T$ and $m \in M$, then $m \in P$. We give results to show that two sets share the same left saturation closure as well as general strategies to actually compute the closure for special cases of T .

In this thesis, we concentrate on two important special cases of left saturation closure. The first, which is discussed in [Chapter 4](#) and [Chapter 7](#), is concerned with closure of subsets of R at R itself, that is, we consider

$$\text{LSat}(P) = \{r \in R \mid \exists w \in R : wr \in P\}$$

for subsets P of R . It turns out that P satisfies the left Ore condition if and only if $\text{LSat}(P)$ does, so this naturally leads to the consideration of $\text{LSat}(S)$ for left denominator sets S . We prove that $\text{LSat}(S)$ is exactly the set of elements of R that become left invertible in the localization

$S^{-1}R$. Our investigations into the inner structure of Ore localizations show that for two left denominator sets S and T , $\text{LSat}(S) = \text{LSat}(T)$ is equivalent to the existence of a unique *R-fixing* isomorphism between $S^{-1}R$ and $T^{-1}R$, that is, the image of any $r \in R$ in $S^{-1}R$ is mapped to the image of r in $T^{-1}R$. We also look into a special case of iterated localizations: if $S \subseteq \text{LSat}(T)$, then $(S^{-1}T)^{-1}(S^{-1}R) \cong T^{-1}R$, where $S^{-1}T$ somewhat provocatively denotes the image of T in $S^{-1}R$ to draw a parallel to the classical rule for complex fractions: if a , b , and c are integers such that $b \neq 0 \neq c$, then $\frac{a}{b} / \frac{c}{b} = \frac{a}{c}$.

The theory of LSat works best when $\text{LSat}(S)$ happens to be a left denominator set, since then $S^{-1}R \cong \text{LSat}(S)^{-1}R$, which further justifies considering $\text{LSat}(S)$ as the *canonical form* of a left denominator set S . We give a characterization of this via *Dedekind-finite* monoids and rings: in these structures, $ab = 1$ implies $ba = 1$ for all elements a and b . Then $\text{LSat}(S)$ being a left denominator set in R is equivalent to $S^{-1}R$ being Dedekind-finite. There are two important special cases where $S^{-1}R$ is always Dedekind-finite: if R is commutative or a domain. For monoids, we introduce the non-standard notion of left (resp. right) *monoid domains* as a generalization of domains, which allows us to gain insight into what properties of being a domain yields which result. This also marks the end of [Chapter 4](#) and thus [Part I](#).

The remainder of [Chapter 7](#) deals with further types of rings. We consider *left Ore domains*, for which a (skew) field of left fractions exists. In *factorization domains*, where every non-zero non-unit has at least one decomposition into irreducible factors, we analyze the connection between left denominator sets and the irreducible elements they contain. Finally, we look at commutative rings, where we apply our theory to classify all localization of the integers at saturated multiplicative sets.

In [Chapter 8](#), we intensively study *Weyl algebras*, a family of non-commutative Noetherian domains used to study linear differential equations with polynomial coefficients from an algebraic viewpoint. We apply our results to prove the left Ore condition for a multitude of multiplicative subsets of the Weyl algebras and give exhaustive proofs for some of the examples scattered throughout the previous chapters.

1.4. From rings to modules

Given a left denominator set S in a ring R and a left R -module M , we define the left Ore localization $S^{-1}M$ of M at S axiomatically in [Chapter 9](#). The conditions are similar to the ones we discussed above for localization of a ring. We follow the same path via a universal property to prove uniqueness and give an analogous construction of $S^{-1}M$ to the one for $S^{-1}R$. As a further consequence of the structural results in [Chapter 7](#), we show that if $\text{LSat}(S) = \text{LSat}(T)$, then $S^{-1}M \cong T^{-1}M$ both as $S^{-1}R$ -modules as well as $T^{-1}R$ -modules. After verifying that localization is compatible with the standard module operations, we close this chapter with a functorial interpretation of localization. In particular, we see that $S^{-1}M \cong S^{-1}R \otimes_R M$, where the latter is the other established way of constructing the localization of M at S .

In [Chapter 10](#), we introduce the second important special case of left saturation closure, which is called *local closure*: for a submodule P of a left R -module M we consider

$$P^S := \text{LSat}_S(P) = \{m \in M \mid \exists s \in S : sm \in P\},$$

where S is a left denominator set in R . This is an already well-established object, especially in the study of functional equations. We show that P^S is the contraction of the submodule of

$S^{-1}M$ generated by the image of P and superficially discuss the submodule structure of $S^{-1}M$. As a special case, this includes the ideal structure of $S^{-1}R$.

Chapter 11 is concerned with *local torsion* as a generalization of classical torsion: instead of torsion with respect to regular elements, we allow arbitrary subsets Λ :

$$t_{\Lambda}(M) := \{m \in M \mid \exists \lambda \in \Lambda : \lambda m = 0\}.$$

Unsurprisingly, this works best if we consider torsion with respect to left denominator sets, in which case we recover most properties of the classical torsion module $t(M)$. Furthermore, we can identify local torsion as a special case of local closure via $t_S(M) = \{0\}^S$. By considering factor modules, we get $t_S(M/P) = P^S/P$, showing the intricate connection between the two notions. Finally, we discuss how annihilators interact with localization.

This marks the conclusion of **Part II**.

1.5. From theory to algorithms

Lastly, **Part III** explores how the theory presented up to this point can be turned into algorithms. In **Chapter 12**, we give an overview of the class of *G-algebras*, which will be the general setting for the remaining chapters. Apart from commutative polynomial rings, the most prominent family of *G-algebras* are the Weyl algebras, which are the focus of **Chapter 8** and a rich source of examples in general. While most *G-algebras* are non-commutative, they are still close enough to commutative polynomial rings to allow algorithmic treatment via a Gröbner basis theory. For the sake of completeness, we included the basic definitions and results of this theory.

In **Chapter 13**, we investigate how the basic arithmetic operations of a localization $S^{-1}R$ can be carried out algorithmically. The main obstacle that arises is the *intersection problem*: given a left denominator set S and a left ideal L , decide whether $L \cap S$ is empty, and if it is not empty, compute an element of the intersection. For the three most common types of localizations of *G-algebras* we give algorithmic solutions to this problem. We also discuss reducing of fractions, computations in right Ore localizations, and the computation of inverses. Finally, we document our implementation of these algorithms in the computer algebra system SINGULAR:PLURAL ([Gre+18]).

Chapter 14 is a collection of several algorithms to compute local closures, which all adhere to a common theme: they are in some sense “commutative enough”. Let A be a *G-algebra*, then for certain central multiplicative sets, we have *central saturation* of a submodule of A^k . For *G-algebras* with a commutative component, like $R := K[s_1, \dots, s_n] \otimes_K A$, we have the *central closure algorithm*, computing the closure of a submodule of R^k at $K[s_1, \dots, s_n] \setminus \{0\}$. If A is the tensor product of a Weyl algebra with a commutative polynomial ring, we can compute the *central Weyl closure* of a left ideal, a generalization of the famous *Weyl closure* ([Tsa00]). Lastly, we give an algorithm to compute the Θ -closure of ideals in the first complex Weyl algebra at a certain set Θ , which enjoys popularity in high energy physics.

1.6. From the pages to the reader

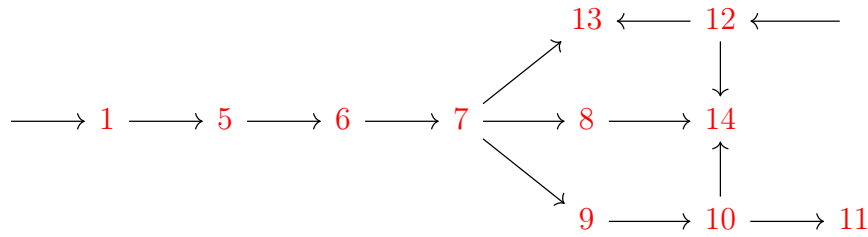
The somewhat modular format of this thesis allows for several approaches for the reader, apart from starting at the beginning and reading until the end of page 206.

The whole of **Part I** deals only with monoids and Ore localizations thereof. For the reader interested in rings who starts their reading with **Part II**, it still serves as a reference for the proofs omitted in some of the results for rings.

Chapter 5 can be seen as an introduction to left Ore localization and might be skipped by those familiar with the subject. **Chapter 6** provides the theory necessary for the development of the two applications of left saturation closure in **Chapter 7** and **Chapter 10**. **Chapter 7** explores the structure of Ore-localized rings. **Chapter 8** is an extensive treatment of the Weyl algebras with a focus on localization which showcases the theory developed up until this point. **Chapter 9** contains both an introduction to Ore-localized modules as well as a study into the structure of these objects, continuing the work of **Chapter 7**. **Chapter 10** interprets the well-known concept of local closure in the context of left saturation closure; the special case of local torsion is covered in **Chapter 11**.

Part III discusses several algorithmic problems related to Ore localization in the setting of G -algebras, which are overviewed in **Chapter 12**. **Chapter 13** discusses how basic arithmetic operations in Ore-localized rings can be made algorithmic. Lastly, **Chapter 14** contains several algorithms to compute local closures building on **Chapter 10**.

Thus, excluding **Part I**, the dependency graph of the chapters looks roughly like this, where the origin of an arrow is required for its target:



Part I.

Ore localization and monoids

Chapter 2.

Ore localization of monoids

To avoid any confusion concerning terminology, we recall the definition of a monoid:

Definition 2.1. Let Γ be a non-empty set with a binary operation

$$* : \Gamma \times \Gamma \rightarrow \Gamma, (\gamma_1, \gamma_2) \mapsto \gamma_1 * \gamma_2$$

and a distinguished element $e \in \Gamma$. The triple $(\Gamma, *, e)$ is called a *monoid*, if the following two requirements are met:

- The operation $*$ is associative: $\gamma_1 * (\gamma_2 * \gamma_3) = (\gamma_1 * \gamma_2) * \gamma_3$ holds for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$.
- The element e is a two-sided identity element: $\gamma * e = \gamma = e * \gamma$ holds for all $\gamma \in \Gamma$.

If $*$ and e are clear from the context, we identify $(\Gamma, *, e)$ with Γ . A monoid without identity is called a *semigroup*. $\diamond$

Definition 2.2. Let Γ be a monoid. An element $\gamma \in \Gamma$ is called

- *left invertible* in Γ if there exists $\alpha \in \Gamma$ such that $\alpha * \gamma = e$. The set of all left invertible elements in Γ is denoted $U_l(\Gamma)$.
- *right invertible* in Γ if there exists $\alpha \in \Gamma$ such that $\gamma * \alpha = e$. The set of all right invertible elements in Γ is denoted $U_r(\Gamma)$.
- *invertible* or a *unit* in Γ if there exists $\alpha \in \Gamma$ such that $\alpha * \gamma = e = \gamma * \alpha$. The set of all units in Γ is denoted $U(\Gamma)$.

A monoid in which every element is invertible is called a *group*. $\diamond$

Remark 2.3. Let Γ be a monoid.

- $U(\Gamma) = U_l(\Gamma) \cap U_r(\Gamma)$.
- $U_l(\Gamma)$ and $U_r(\Gamma)$ are submonoids of Γ .
- $U(\Gamma)$ is a subgroup of Γ . $\diamond$

2.1. Modules over monoids

To some extent, **Part I** serves as a large collection of preliminary results for **Part II**. Since we want to discuss localization of modules over rings later on, we also need to address modules over monoids now. Essentially, modules over monoids are what remains when we take the definition of a module over a ring and remove any requirement connected to addition:

Definition 2.4. Let $(\Gamma, *, e)$ be a monoid. A non-empty set M together with a map

$$\cdot : \Gamma \times M \rightarrow M, \quad (\gamma, m) \mapsto \gamma \cdot m$$

is called a *left Γ -module* if

- $e \cdot m = m$ for all $m \in M$ and
- $\gamma_1 \cdot (\gamma_2 \cdot m) = (\gamma_1 * \gamma_2) \cdot m$ for all $m \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$.

A non-empty subset N of a left Γ -module M is called *left Γ -submodule* of M if $\gamma \cdot n \in N$ for all $\gamma \in \Gamma$ and $n \in N$. $\diamond$

The basics of modules over monoids are the same as for modules over rings, apart from the fact that the intersection of submodules can be empty if there is no element common to all submodules like zero in the ring case:

Lemma 2.5. *Let Γ be a monoid and M a left Γ -module. Furthermore, let J be an arbitrary index set such that N_j is a left Γ -submodule of M for every $j \in J$. Then $N := \bigcap_{j \in J} N_j$ is either empty or a left Γ -submodule of M .*

Proof: Assume that $N \neq \emptyset$ and let $\gamma \in \Gamma$ as well as $n \in N$. Then $\gamma \cdot n \in N_j$ for all $j \in J$, thus $\gamma \cdot n \in N$, which shows that N is a left Γ -submodule of M . $\square$

Definition 2.6. Let Γ be a monoid and N a subset of a left Γ -module M . The left Γ -submodule of M generated by N , denoted ${}_{\Gamma}\langle N \rangle$, is the intersection of all left Γ -submodules of M containing N , or equivalently, ${}_{\Gamma}\langle N \rangle = \{\gamma \cdot n \mid \gamma \in \Gamma, n \in N\}$. $\diamond$

Definition 2.7. Let Γ be a monoid and M, N two left Γ -modules. A *homomorphism* of left Γ -modules is a map $\varphi : M \rightarrow N$ such that $\varphi(\gamma \cdot m) = \gamma \cdot \varphi(m)$ for all $\gamma \in \Gamma$ and $m \in M$. $\diamond$

Lemma 2.8. *Let Γ be a monoid and $\varphi : M \rightarrow N$ a homomorphism of left Γ -modules. If P is a left Γ -submodule of N , then $\varphi^{-1}(P)$ is a left Γ -submodule of M .*

Proof: Let $\gamma \in \Gamma$ and $x \in \varphi^{-1}(P)$, then $\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) \in P$, thus $\gamma \cdot x \in \varphi^{-1}(P)$. $\square$

Lemma 2.9. *Let $\varphi : \Gamma \rightarrow \Delta$ be a homomorphism of monoids and M a left Δ -module. Then M naturally becomes a left Γ -module via*

$$\bullet : \Gamma \times M \rightarrow M, \quad (\gamma, m) \mapsto \gamma \bullet m := \varphi(\gamma) \cdot m.$$

Proof: Let e_{Γ}, e_{Δ} be the neutral elements of Γ and Δ , respectively. For all $m \in M$ we have

$$e_{\Gamma} \bullet m = \varphi(e_{\Gamma}) \cdot m = e_{\Delta} \cdot m = m.$$

Now let $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$, then

$$\begin{aligned} \gamma_1 \bullet (\gamma_2 \bullet m) &= \gamma_1 \bullet (\varphi(\gamma_2) \cdot m) = \varphi(\gamma_1) \cdot (\varphi(\gamma_2) \cdot m) = (\varphi(\gamma_1) * \varphi(\gamma_2)) \cdot m \\ &= \varphi(\gamma_1 * \gamma_2) \cdot m = (\gamma_1 * \gamma_2) \bullet m. \end{aligned}$$

$\square$

2.2. Axiomatic definition, universal property and uniqueness

In our work, monoids mostly arise by taking a ring and temporarily forgetting about its addition. To ensure better compatibility with [Part II](#), we adjust the notation:

Convention 2.10. All further monoids will be written multiplicatively with identity 1. $\diamond$

As with commutative rings, any set of potential denominators should be a multiplicative set:

Definition 2.11. Let S be a subset of a monoid R .

- S is *multiplicatively closed* in R if for all $s_1, s_2 \in S$ we have $s_1 s_2 \in S$.
- S is a *multiplicative set* in R if S is multiplicatively closed and $1 \in S$. $\diamond$

Now we can define the localization of a monoid according to the considerations in [Chapter 1](#):

Definition 2.12. Let S be a multiplicative set in a monoid R . A pair (R_S, η) of a monoid R_S and a homomorphism $\eta : R \rightarrow R_S$ is called a *left Ore localization* of R at S if the following hold:

- (i) For all $s \in S$, $\eta(s) \in U(R_S)$.
- (ii) For all $x \in R_S$ there exist $s \in S$ and $r \in R$ such that $x = \eta(s)^{-1} \eta(r)$.
- (iii) For all $r_1, r_2 \in R$, $\eta(r_1) = \eta(r_2)$ if and only if there exists $s \in S$ such that $sr_1 = sr_2$. $\diamond$

If (R_S, η) is a localization of R at S , then for any other pair (T, φ) satisfying the first condition in [Definition 2.12](#), φ must factor uniquely through R_S :

Proposition 2.13 (Universal property, cf. [Proposition 5.9](#)). *Let S be a multiplicative set in a monoid R and (R_S, η) a left Ore localization of R at S . Further, let $\varphi : R \rightarrow T$ be a homomorphism of monoids such that $\varphi(S) \subseteq U(T)$.*

- (a) *If $r_1, r_2 \in R$ satisfy $\eta(r_1) = \eta(r_2)$, then we also have $\varphi(r_1) = \varphi(r_2)$.*
- (b) *There exists a unique monoid homomorphism $\psi : R_S \rightarrow T$ such that $\varphi = \psi \circ \eta$.*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & T \\ & \searrow \eta & \nearrow \psi \\ & R_S & \end{array}$$

Proof: (a) Let $r_1, r_2 \in R$ such that $\eta(r_1) = \eta(r_2)$, then there exists $s \in S$ such that $sr_1 = sr_2$. Applying φ yields $\varphi(s)\varphi(r_1) = \varphi(sr_1) = \varphi(sr_2) = \varphi(s)\varphi(r_2)$. Multiplication with $\varphi(s)^{-1}$ from the left implies $\varphi(r_1) = \varphi(r_2)$.

(b) Consider the map $\psi : R_S \rightarrow T$, $\eta(s)^{-1} \eta(r) \mapsto \varphi(s)^{-1} \varphi(r)$.

Well-definedness: Let $\eta(s_1)^{-1} \eta(r_1), \eta(s_2)^{-1} \eta(r_2) \in R_S$ such that

$$\eta(s_1)^{-1} \eta(r_1) = \eta(s_2)^{-1} \eta(r_2).$$

Since $\eta(s_1) \eta(s_2)^{-1} \in R_S$ there exists $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(\tilde{s})^{-1} \eta(\tilde{r}) = \eta(s_1) \eta(s_2)^{-1}$, which is equivalent to

$$\eta(\tilde{s} s_1) = \eta(\tilde{s}) \eta(s_1) = \eta(\tilde{r}) \eta(s_2) = \eta(\tilde{r} s_2).$$

By (a) we have $\varphi(\tilde{s})\varphi(s_1) = \varphi(\tilde{s}s_1) = \varphi(\tilde{r}s_2) = \varphi(\tilde{r})\varphi(s_2)$, which is equivalent to $\varphi(\tilde{s})^{-1}\varphi(\tilde{r}) = \varphi(s_1)\varphi(s_2)^{-1}$. On the other hand,

$$\eta(r_1) = \eta(s_1)\eta(s_2)^{-1}\eta(r_2) = \eta(\tilde{s})^{-1}\eta(\tilde{r})\eta(r_2)$$

implies $\eta(\tilde{s}r_1) = \eta(\tilde{s})\eta(r_1) = \eta(\tilde{r})\eta(r_2) = \eta(\tilde{r}s_2)$. By (a) we have

$$\varphi(\tilde{s})\varphi(r_1) = \varphi(\tilde{s}r_1) = \varphi(\tilde{r}s_2) = \varphi(\tilde{r})\varphi(s_2),$$

which is equivalent to $\varphi(r_1) = \varphi(\tilde{s})^{-1}\varphi(\tilde{r})\varphi(s_2)$. Now we have

$$\begin{aligned} \psi(\eta(s_1)^{-1}\eta(r_1)) &= \varphi(s_1)^{-1}\varphi(r_1) = \varphi(s_1)^{-1}\varphi(\tilde{s})^{-1}\varphi(\tilde{r})\varphi(s_2) \\ &= \varphi(s_1)^{-1}\varphi(s_1)\varphi(s_2)^{-1}\varphi(r_2) = \varphi(s_2)^{-1}\varphi(r_2) = \psi(\eta(s_2)^{-1}\eta(r_2)). \end{aligned}$$

Multiplicativity: We have

$$\psi(1_{R_S}) = \psi(\eta(1_R)^{-1}\eta(1_R)) = \varphi(1_R)^{-1}\varphi(1_R) = 1_T.$$

Let $x_1 := \eta(s_1)^{-1}\eta(r_1)$, $x_2 := \eta(s_2)^{-1}\eta(r_2) \in R_S$. Since $\eta(r_1)\eta(s_2)^{-1} \in R_S$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(r_1)\eta(s_2)^{-1} = \eta(\tilde{s})^{-1}\eta(\tilde{r})$, which is equivalent to

$$\eta(\tilde{s}r_1) = \eta(\tilde{s})\eta(r_1) = \eta(\tilde{r})\eta(s_2) = \eta(\tilde{r}s_2).$$

By (a) we have

$$\varphi(\tilde{s})\varphi(r_1) = \varphi(\tilde{s}r_1) = \varphi(\tilde{r}s_2) = \varphi(\tilde{r})\varphi(s_2),$$

which is equivalent to $\varphi(\tilde{s})^{-1}\varphi(\tilde{r}) = \varphi(r_1)\varphi(s_2)^{-1}$. Now

$$\begin{aligned} \psi(x_1 \cdot x_2) &= \psi(\eta(s_1)^{-1}\eta(r_1)\eta(s_2)^{-1}\eta(r_2)) \\ &= \psi(\eta(s_1)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\eta(r_2)) \\ &= \psi((\eta(\tilde{s}s_1))^{-1}\eta(\tilde{r}r_2)) \\ &= \varphi(\tilde{s}s_1)^{-1}\varphi(\tilde{r}r_2) \\ &= (\varphi(\tilde{s})\varphi(s_1))^{-1}\varphi(\tilde{r})\varphi(r_2) \\ &= \varphi(s_1)^{-1}\varphi(\tilde{s})^{-1}\varphi(\tilde{r})\varphi(r_2) \\ &= \varphi(s_1)^{-1}\varphi(r_1)\varphi(s_2)^{-1}\varphi(r_2) \\ &= \psi(\eta(s_1)^{-1}\eta(r_1)) \cdot \psi(\eta(s_2)^{-1}\eta(r_2)) \\ &= \psi(x_1) \cdot \psi(x_2). \end{aligned}$$

Composition: Let $r \in R$, then

$$(\psi \circ \eta)(r) = \psi(\eta(r)) = \psi(\eta(1)^{-1}\eta(r)) = \varphi(1)^{-1}\varphi(r) = \varphi(r),$$

thus $\psi \circ \eta = \varphi$.

Uniqueness: Let $\psi' : R_S \rightarrow T$ be a monoid homomorphism such that $\psi' \circ \eta = \varphi$. For all $s \in S$ we have

$$\psi'(\eta(s)^{-1}) \cdot \varphi(s) = \psi'(\eta(s)^{-1}) \cdot \psi'(\eta(s)) = \psi'(\eta(s)^{-1}\eta(s)) = \psi'(1_{R_S}) = 1_T$$

and analogously $\varphi(s) \cdot \psi'(\eta(s)^{-1}) = 1_T$, thus $\psi'(\eta(s)^{-1}) = \varphi(s)^{-1}$. For all $s \in S$ and $r \in R$ we thus have

$$\psi'(\eta(s)^{-1}\eta(r)) = \psi'(\eta(s)^{-1}) \cdot \psi'(\eta(r)) = \varphi(s)^{-1}\varphi(r) = \psi(\eta(s)^{-1}\eta(r)),$$

which implies $\psi' = \psi$. □

Applying the universal property twice to two localizations of R at S gives us the uniqueness:

Corollary 2.14 (Uniqueness, cf. [Corollary 5.10](#)). *Let S be a multiplicative set in a monoid R . If there exists a left Ore localization of R at S , then it is unique up to isomorphism.*

Proof: Let (R_S, η) and (R'_S, η') be two left Ore localizations of R at S . By [Proposition 2.13](#) there exist unique monoid homomorphisms $\psi : R_S \rightarrow R'_S$ (letting $T = R'_S$ and $\varphi = \eta'$ with respect to (R_S, η)) and $\psi' : R'_S \rightarrow R_S$ (letting $T = R_S$ and $\varphi = \eta$ with respect to (R'_S, η')) such that $\eta' = \psi \circ \eta$ and $\eta = \psi' \circ \eta'$.

Applying [Proposition 2.13](#) to $T = R_S$ and $\varphi = \eta$ with respect to (R_S, η) tells us that there exists a unique monoid homomorphism $\alpha : R_S \rightarrow R_S$ such that $\eta = \alpha \circ \eta$. On one hand we have $\alpha = \psi' \circ \psi$, since $\eta = \psi' \circ \eta' = \psi' \circ \psi \circ \eta$, on the other hand id_{R_S} clearly satisfies $\eta = \text{id}_{R_S} \circ \eta$, thus $\alpha = \text{id}_{R_S}$. This shows $\psi' \circ \psi = \alpha = \text{id}_{R_S}$ by uniqueness of α . By symmetry, we also have $\psi \circ \psi' = \text{id}_{R'_S}$, therefore $\psi : R_S \rightarrow R'_S$ is an isomorphism. $\square$

But when does the localization of R at S exist?

Definition 2.15. Let S be a subset of a monoid R .

- S satisfies the *left Ore condition* in R if for all $s \in S$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$. The tuple $(\tilde{s}, \tilde{r})$ is sometimes called *left Ore data* for s and r .
- S is a *left Ore set* in R if it is a multiplicative set satisfying the left Ore condition.
- S is *left reversible* in R if for all $r_1, r_2 \in R$ and $s \in S$ such that $r_1s = r_2s$ there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$.
- S is a *left denominator set* in R if S is a left reversible left Ore set in R . $\diamond$

Being a left denominator set is an immediate necessary condition for S :

Lemma 2.16 (cf. [Lemma 5.12](#)). *Let S be a multiplicative set in a monoid R and (R_S, η) a left Ore localization of R at S . Then S is a left denominator set in R .*

Proof: By assumption S is already a multiplicative set.

Left Ore condition: Let $s \in S$ and $r \in R$, then $\eta(r)\eta(s)^{-1} \in R_S$, thus there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(r)\eta(s)^{-1} = \eta(\tilde{s})^{-1}\eta(\tilde{r})$, which implies

$$\eta(\tilde{s}r) = \eta(\tilde{s})\eta(r) = \eta(\tilde{r})\eta(s) = \eta(\tilde{r}s).$$

Thus there exists $\hat{s} \in S$ such that $\hat{s}\tilde{s}r = \hat{s}\tilde{r}s$. Let $\mathring{s} := \hat{s}\tilde{s} \in S$ and $\mathring{r} := \hat{s}\tilde{r} \in R$, then $\mathring{s}r = \mathring{s}\tilde{s}r = \mathring{s}\tilde{r}s = \mathring{r}s$.

Left reversibility: Let $s \in S$ and $r_1, r_2 \in R$ such that $r_1s = r_2s$, then

$$\eta(r_1)\eta(s) = \eta(r_1s) = \eta(r_2s) = \eta(r_2)\eta(s)$$

implies $\eta(r_1) = \eta(r_2)$ by multiplication with $\eta(s)^{-1}$ from the right, thus there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$. $\square$

In the next section we will see that it is also sufficient.

2.3. Construction and existence

Remark 2.17. Note that some of the following constructions and results consider a left R -module M instead of the ring R itself. This saves us the trouble of proving essentially the same results again later when dealing with the localization of modules. However, for the remainder of this section it suffices to think of the special case $M = R$. $\diamond$

Starting from a left denominator set, the first step in constructing the localization is the following equivalence relation:

Proposition 2.18 (cf. [Proposition 5.15](#)). *Let S be a left denominator set in a monoid R and M a left R -module. The relation $\sim$ on $S \times M$, defined by*

$$(s_1, m_1) \sim (s_2, m_2) \quad :\Leftrightarrow \quad \exists \tilde{s} \in S, \tilde{r} \in R : \tilde{s}s_2 = \tilde{r}s_1 \text{ and } \tilde{s}m_2 = \tilde{r}m_1,$$

is an equivalence relation.

Proof: Let $(s, m), (s_1, m_1), (s_2, m_2), (s_3, m_3) \in S \times M$.

Reflexivity: Define $\tilde{s} := 1 \in S$ and $\tilde{r} := 1 \in R$, then $\tilde{s}s = 1 \cdot s = \tilde{r}s$ and $\tilde{s}m = 1 \cdot m = \tilde{r}m$, thus $(s, m) \sim (s, m)$.

Symmetry: Let $(s_1, m_1) \sim (s_2, m_2)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$\tilde{s}s_2 = \tilde{r}s_1, \tag{A1}$$

$$\tilde{s}m_2 = \tilde{r}m_1. \tag{A2}$$

Since $s_1 \in R$ and $s_2 \in S$ there exist $\hat{s} \in S$ and $\hat{r} \in R$ such that

$$\hat{s}s_1 = \hat{r}s_2. \tag{A3}$$

Furthermore, since $\hat{r} \in R$ and $\tilde{s} \in S$ there exist $\bar{s} \in S$ and $\bar{r} \in R$ such that

$$\bar{s}\hat{r} = \bar{r}\tilde{s}. \tag{A4}$$

By left reversibility,

$$\bar{r}\tilde{r}s_1 \stackrel{(A1)}{=} \bar{r}\tilde{s}s_2 \stackrel{(A4)}{=} \bar{s}\hat{r}s_2 \stackrel{(A3)}{=} \bar{s}\hat{s}s_1$$

implies that there exists $\check{s} \in S$ such that

$$\check{s}\bar{r}\tilde{r} = \check{s}\bar{s}\hat{s}. \tag{A5}$$

Now define $\mathring{s} := \check{s}\bar{s}\hat{s} \in S$ and $\mathring{r} = \check{s}\bar{r}\tilde{r} \in R$, then

$$\mathring{s}s_1 = \check{s}\bar{s}\hat{s}s_1 \stackrel{(A3)}{=} \check{s}\bar{s}\hat{r}s_2 \stackrel{(A4)}{=} \check{s}\bar{r}\tilde{s}s_2 = \mathring{r}s_2$$

and

$$\mathring{s}m_1 = \check{s}\bar{s}\hat{s}m_1 \stackrel{(A5)}{=} \check{s}\bar{r}\tilde{r}m_1 \stackrel{(A2)}{=} \check{s}\bar{r}\tilde{s}m_2 = \mathring{r}m_2$$

show that $(s_2, m_2) \sim (s_1, m_1)$.

Transitivity: Let $(s_1, m_1) \sim (s_2, m_2)$ and $(s_2, m_2) \sim (s_3, m_3)$, then there exist $\tilde{s}, \hat{s} \in S$ and $\tilde{r}, \hat{r} \in R$ such that

$$\tilde{s}s_2 = \tilde{r}s_1, \quad (\text{B1})$$

$$\tilde{s}m_2 = \tilde{r}m_1, \quad (\text{B2})$$

$$\hat{s}s_3 = \hat{r}s_2, \quad (\text{B3})$$

$$\hat{s}m_3 = \hat{r}m_2. \quad (\text{B4})$$

Furthermore, since $\tilde{s} \in S$ and $\hat{r} \in R$, there exist $\bar{s} \in S$ and $\bar{r} \in R$ such that

$$\bar{s}\hat{r} = \bar{r}\tilde{s}. \quad (\text{B5})$$

Now define $\circ s := \bar{s}\hat{s} \in S$ and $\circ r := \bar{r}\tilde{r} \in R$, then

$$\circ s s_3 = \bar{s}\hat{s}s_3 \stackrel{(\text{B3})}{=} \bar{s}\hat{r}s_2 \stackrel{(\text{B5})}{=} \bar{r}\tilde{s}s_2 \stackrel{(\text{B1})}{=} \bar{r}\tilde{r}s_1 = \circ r s_1$$

and

$$\circ s m_3 = \bar{s}\hat{s}m_3 \stackrel{(\text{B4})}{=} \bar{s}\hat{r}m_2 \stackrel{(\text{B5})}{=} \bar{r}\tilde{s}m_2 \stackrel{(\text{B2})}{=} \bar{r}\tilde{r}m_1 = \circ r m_1$$

show that $(s_1, m_1) \sim (s_3, m_3)$.

Therefore, $\sim$ is indeed an equivalence relation on $S \times M$. $\square$

In particular, we can expand and reduce on the left:

Lemma 2.19 (cf. [Lemma 5.16](#)). *Let S be a left denominator set in a monoid R , M a left R -module, $(s, m) \in S \times M$ and $t \in R$ such that $ts \in S$. Then $(s, m) \sim (ts, tm)$.*

Proof: Define $\tilde{s} := 1 \in S$ and $\tilde{r} := t \in R$, then

$$\tilde{s}ts = 1 \cdot ts = ts = \tilde{r}s \quad \text{and} \quad \tilde{s}tm = 1 \cdot tm = tm = \tilde{r}m$$

show that $(s, m) \sim (ts, tm)$. $\square$

The following characterization will be used implicitly in many of the following proofs:

Lemma 2.20 (cf. [Lemma 5.17](#)). *Let S be a left denominator set in a monoid R , M a left R -module and $(s_1, m_1), (s_2, m_2) \in S \times M$. The following are equivalent:*

- (1) $(s_1, m_1) \sim (s_2, m_2)$, that is, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}m_2 = \tilde{r}m_1$.
- (2) For all $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ there exists $\bar{s} \in S$ such that $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (3) There exist $\hat{s}, \bar{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ and $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (4) There exist $a, b \in R$ such that $as_2 = bs_1 \in S$ and $am_2 = bm_1$.

Proof: (1) $\Rightarrow$ (2): Let $\hat{s} \in S$ and $\hat{r} \in R$ such that

$$\hat{s}s_2 = \hat{r}s_1. \quad (\text{C1})$$

By assumption there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$\tilde{s}s_2 = \tilde{r}s_1, \quad (\text{C2})$$

$$\tilde{s}m_2 = \tilde{r}m_1. \quad (\text{C3})$$

By the left Ore condition on S there exist $s_a \in S$ and $r_a \in R$ such that

$$s_a\hat{s} = r_a\tilde{s}. \quad (\text{C4})$$

Now

$$s_a\hat{r}s_1 \stackrel{(\text{C1})}{=} s_a\hat{s}s_2 \stackrel{(\text{C4})}{=} r_a\tilde{s}s_2 \stackrel{(\text{C2})}{=} r_a\tilde{r}s_1$$

implies the existence of $s_b \in S$ such that

$$s_bs_a\hat{r} = s_br_a\tilde{r} \quad (\text{C5})$$

by left reversibility of S . Define $\bar{s} := s_bs_a \in S$, then

$$\bar{s}\hat{s}m_2 = s_bs_a\hat{s}m_2 \stackrel{(\text{C4})}{=} s_br_a\tilde{s}m_2 \stackrel{(\text{C3})}{=} s_br_a\tilde{r}m_1 \stackrel{(\text{C5})}{=} s_bs_a\hat{r}m_1 = \bar{s}\hat{r}m_1.$$

(2) $\Rightarrow$ (3): This is a special case since the left Ore condition on S guarantees the existence of at least one pair $\hat{s}, \hat{r}$ of left Ore data for s_1, s_2 .

(3) $\Rightarrow$ (4): Define $a := \bar{s}\hat{s} \in S$ and $b := \bar{s}\hat{r} \in R$, then $as_2 = \bar{s}\hat{s}s_2 \in S$ and

$$as_2 = \bar{s}\hat{s}s_2 = \bar{s}\hat{r}s_1 = bs_1 \quad \text{as well as} \quad am_2 = \bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1 = bm_1.$$

(4) $\Rightarrow$ (1): By [Lemma 2.19](#) we have $(s_1, m_1) \sim (as_1, am_1) = (bs_2, bm_2) \sim (s_2, m_2)$. $\square$

As a first application of this characterization, we show that all tuples of left Ore data are equivalent:

Lemma 2.21 (cf. [Lemma 5.18](#)). *Let S be a left denominator set in a monoid R and $(s, r) \in S \times R$. If there exist $(s_1, r_1), (s_2, r_2) \in S \times R$ such that $s_1r = r_1s$ and $s_2r = r_2s$, then $(s_1, r_1) \sim (s_2, r_2)$.*

Proof: Let $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$, then $\tilde{s}r_1s = \tilde{s}s_1r = \tilde{r}s_2r = \tilde{r}r_2s$ implies the existence of $\hat{s} \in S$ such that $\hat{s}\tilde{s}r_1 = \hat{s}\tilde{r}r_2$ by left reversibility. Now $(s_1, r_1) \sim (s_2, r_2)$ by [Lemma 2.20](#). $\square$

Definition 2.22. Let S be a left Ore set in a monoid R and M a left R -module. Then we define

- the set $S^{-1}M := (S \times M) / \sim$.
- the map

$$\mu_{S,R,M} : S^{-1}R \times S^{-1}M \rightarrow S^{-1}M, \quad ((s_1, r), (s_2, m)) \mapsto (\tilde{s}s_1, \tilde{r}m),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r = \tilde{r}s_2$.

By abuse of notation we denote the elements of $S^{-1}M$ again by (s, m) . If R and S are clear from the context, we also use $\cdot_M$ as an infix operator to have a shorter notation for the application of $\mu_{S,R,M}$. $\diamond$

Lemma 2.23. *Let S be a left denominator set in a monoid R and M a left R -module.*

- (a) *The map $\mu_{S,R,M}$ is well-defined.*
- (b) *For all $(s, m) \in S^{-1}M$ we have $(1, 1) \cdot_M (s, m) = (s, m)$.*
- (c) *For all $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ and $(s_3, m) \in S^{-1}M$ we have*

$$(s_1, r_1) \cdot_M ((s_2, r_2) \cdot_M (s_3, m)) = ((s_1, r_1) \cdot_M (s_2, r_2)) \cdot_M (s_3, m).$$

Proof: (a) Let $(s_1, r), (s'_1, r') \in S^{-1}R$ and $(s_2, m), (s'_2, m') \in S^{-1}M$ such that $(s_1, r) \sim (s'_1, r')$ and $(s_2, m) \sim (s'_2, m')$, then there exist $\hat{s}_1, \hat{s}_2 \in S$ and $\hat{r}_1, \hat{r}_2 \in R$ such that

$$\hat{s}_1 s_1 = \hat{r}_1 s'_1, \quad (\text{D1})$$

$$\hat{s}_1 r = \hat{r}_1 r', \quad (\text{D2})$$

$$\hat{s}_2 s_2 = \hat{r}_2 s'_2, \quad (\text{D3})$$

$$\hat{s}_2 m = \hat{r}_2 m'. \quad (\text{D4})$$

Then $(s_1, r) \cdot_M (s_2, m) = (\tilde{s}s_1, \tilde{r}m)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy

$$\tilde{s}r = \tilde{r}s_2, \quad (\text{D5})$$

and $(s'_1, r') \cdot_M (s'_2, m') = (\tilde{s}'s'_1, \tilde{r}'m')$, where $\tilde{s}' \in S$ and $\tilde{r}' \in R$ satisfy

$$\tilde{s}'r' = \tilde{r}'s'_2. \quad (\text{D6})$$

We need to show $(\tilde{s}s_1, \tilde{r}m) = (\tilde{s}'s'_1, \tilde{r}'m')$. By the left Ore condition there exist $s_a, s_b \in S$ and $r_a, r_b \in R$ such that

$$s_a \tilde{s}' s'_1 = r_a \tilde{s} s_1, \quad (\text{D7})$$

and

$$s_b r_a \tilde{s} = r_b \hat{s}_1. \quad (\text{D8})$$

By left reversibility,

$$s_b s_a \tilde{s}' s'_1 \stackrel{(\text{D7})}{=} s_b r_a \tilde{s} s_1 \stackrel{(\text{D8})}{=} r_b \hat{s}_1 s_1 \stackrel{(\text{D1})}{=} r_b \hat{r}_1 s'_1$$

implies that there exists $s_c \in S$ such that

$$s_c s_b s_a \tilde{s}' = s_c r_b \hat{r}_1. \quad (\text{D9})$$

Since $s_c s_b r_a \tilde{r} \in R$ and $\hat{s}_2 \in S$ there exist $s_d \in S$ and $r_d \in R$ such that

$$s_d s_c s_b r_a \tilde{r} = r_d \hat{s}_2. \quad (\text{D10})$$

Again by left reversibility,

$$\begin{aligned} s_d s_c s_b s_a \tilde{r}' s'_2 &\stackrel{(\text{D6})}{=} s_d s_c s_b s_a \tilde{s}' r'_1 \stackrel{(\text{D9})}{=} s_d s_c r_b \hat{r}_1 r'_1 \stackrel{(\text{D2})}{=} s_d s_c r_b \hat{s}_1 r_1 \stackrel{(\text{D8})}{=} s_d s_c s_b r_a \tilde{s} r_1 \\ &\stackrel{(\text{D5})}{=} s_d s_c s_b r_a \tilde{r} s_2 \stackrel{(\text{D10})}{=} r_d \hat{s}_2 s_2 \stackrel{(\text{D3})}{=} r_d \hat{r}_2 s'_2 \end{aligned}$$

implies the existence of $s_e \in S$ such that

$$s_e s_d s_c s_b s_a \tilde{r}' = s_e r_d \hat{r}_2. \quad (\text{D11})$$

Define $\mathring{s} := s_e s_d s_c s_b s_a \in S$ and $\mathring{r} := s_e s_d s_c s_b r_a \in R$, then

$$\mathring{s} \tilde{s}' s'_1 = s_e s_d s_c s_b s_a \tilde{s}' s'_1 \stackrel{(\text{D7})}{=} s_e s_d s_c s_b r_a \tilde{s} s_1 = \mathring{r} \tilde{s} s_1$$

and

$$\mathring{s} \tilde{r}' m' = s_e s_d s_c s_b s_a \tilde{r}' m' \stackrel{(\text{D11})}{=} s_e r_d \hat{r}_2 m' \stackrel{(\text{D4})}{=} s_e r_d \hat{s}_2 m \stackrel{(\text{D10})}{=} s_e s_d s_c s_b r_a \tilde{r} m = \mathring{r} \tilde{r} m$$

show that $(\tilde{s} s_1, \tilde{r} m) \sim (\tilde{s}' s'_1, \tilde{r}' m')$, thus $\cdot_M$ is well-defined.

(b) Let $(s, m) \in S^{-1}M$. Define $\tilde{s} := s \in S$ and $\tilde{r} := 1 \in R$, then $\tilde{s} \cdot 1 = s \cdot 1 = 1 \cdot s = \tilde{r} s$, thus

$$(1, 1) \cdot_M (s, m) = (\tilde{s} \cdot 1, \tilde{r} m) = (s \cdot 1, 1 \cdot m) = (s, m).$$

(c) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ and $(s_3, m) \in S^{-1}M$. We have

$$((s_1, r_1) \cdot_R (s_2, r_2)) \cdot_M (s_3, m) = (s_a s_1, r_a r_2) \cdot_M (s_3, m) = (s_b s_a s_1, r_b m)$$

and

$$(s_1, r_1) \cdot_M ((s_2, r_2) \cdot_M (s_3, m)) = (s_1, r_1) \cdot_M (s_c s_2, r_c m) = (s_d s_1, r_d r_c m),$$

where $s_a, s_b, s_c, s_d \in S$ and $r_a, r_b, r_c, r_d \in R$ satisfy

$$s_a r_1 = r_a s_2, \quad (\text{E1})$$

$$s_b r_a r_2 = r_b s_3, \quad (\text{E2})$$

$$s_c r_2 = r_c s_3, \quad (\text{E3})$$

$$s_d r_1 = r_d s_c s_2. \quad (\text{E4})$$

We have to show that $(s_b s_a s_1, r_b m) = (s_d s_1, r_d r_c m)$. By the left Ore condition there exist $s_e \in S$ and $r_e \in R$ such that

$$s_e s_d s_1 = r_e s_b s_a s_1. \quad (\text{E5})$$

By left reversibility there exists $s_f \in S$ such that

$$s_f s_e s_d = s_f r_e s_b s_a. \quad (\text{E6})$$

Again by left reversibility,

$$s_f s_e r_d s_c s_2 \stackrel{(\text{E4})}{=} s_f s_e s_d r_1 \stackrel{(\text{E6})}{=} s_f r_e s_b s_a r_1 \stackrel{(\text{E1})}{=} s_f r_e s_b r_a s_2$$

implies that there exists $s_g \in S$ such that

$$s_g s_f s_e r_d s_c = s_g s_f r_e s_b r_a, \quad (\text{E7})$$

and

$$s_g s_f s_e r_d r_c s_3 \stackrel{(\text{E3})}{=} s_g s_f s_e r_d s_c r_2 \stackrel{(\text{E7})}{=} s_g s_f r_e s_b r_a r_2 \stackrel{(\text{E2})}{=} s_g s_f r_e r_b s_3$$

implies the existence of $s_h \in S$ such that

$$s_h s_g s_f s_e r_d r_c = s_h s_g s_f r_e r_b. \quad (\text{E8})$$

Define $\mathring{s} := s_h s_g s_f s_e \in S$ and $\mathring{r} := s_h s_g s_f r_e \in R$, then

$$\mathring{s} s_d s_1 = s_h s_g s_f s_e s_d s_1 \stackrel{(\text{E5})}{=} s_h s_g s_f r_e s_b s_a s_1 = \mathring{r} s_b s_a s_1$$

and

$$\mathring{s} r_d r_c m = s_h s_g s_f s_e r_d r_c m \stackrel{(\text{E8})}{=} s_h s_g s_f r_e r_b m = \mathring{r} r_b m$$

show that $(s_b s_a s_1, r_b m) = (s_d s_1, r_d r_c m)$. $\square$

Putting the pieces together, we get the full construction:

Proposition 2.24 (Construction, cf. [Proposition 5.21](#)). *Let S be a left denominator set in a monoid R . Then $(S^{-1}R, \cdot, (1, 1))$ is a monoid with*

$$\cdot : S^{-1}R \times S^{-1}R \rightarrow S^{-1}R, \quad ((s_1, r_1), (s_2, r_2)) \mapsto (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$.

Proof: Since $\cdot$ is exactly $\mu_{S,R,R}$, [Lemma 2.23](#) tells us that $\cdot$ is well-defined and associative with left neutral element $(1, 1)$. It remains to show that $(1, 1)$ is right neutral: let $(s, r) \in S^{-1}R$. Define $\tilde{s} := 1 \in S$ and $\tilde{r} := r \in R$, then $\tilde{s}r = 1 \cdot r = r \cdot 1 = \tilde{r} \cdot 1$, thus

$$(s, r) \cdot (1, 1) = (\tilde{s}s, \tilde{r} \cdot 1) = (1 \cdot s, r \cdot 1) = (s, r). \quad \square$$

At this point, we could restate [Lemma 2.23](#) a lot shorter: $S^{-1}M$ is a left $S^{-1}R$ -module with scalar multiplication given by $\cdot_M$.

Example 2.25. Two of the defining properties of a *Garside monoid* are (left and right) cancellativity and the existence of (left and right) least common multiples. In particular, every Garside monoid M is a left denominator set in itself, and the resulting localization $M^{-1}M$ is called a *Garside group* ([[Deh15](#)]). $\diamond$

Lemma 2.26 (cf. [Lemma 5.23](#)). *Let S be a left denominator set in a monoid R and M a left R -module.*

(a) *Let $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) *We have $(s, r) = (1, 1)$.*
- (2) *There exists $\tilde{s} \in S$ such that $\tilde{s}r = \tilde{s}s$.*
- (3) *There exists $\hat{r} \in R$ such that $\hat{r}r = \hat{r}s$ and $\hat{r}s \in S$.*

(b) *Let $s_1, s_2 \in S$, $r \in R$ and $m \in M$ such that $s_2 r = r s_2$, then $(s_1, r) \cdot_M (s_2, m) = (s_2 s_1, r m)$.*

Proof: (a) Let $(s, r) \in S^{-1}R$.

(1) $\Rightarrow$ (2): Since $(s, r) = (1, 1)$ is equivalent to $(1, 1) = (s, r)$, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s = \tilde{r} \cdot 1 = \tilde{r}$ and $\tilde{s}r = \tilde{r} \cdot 1 = \tilde{r}$, thus $\tilde{s}s = \tilde{s}r$.

(2) $\Rightarrow$ (3): Obvious with $\hat{r} := \tilde{s}$.

(3) $\Rightarrow$ (1): Let $\hat{r} \in R$ such that $\hat{r}r = \hat{r}s$ and $\hat{r}s \in S$, then $(s, r) = (\hat{r}s, \hat{r}r) = (\hat{r}s, \hat{r}s) = (1, 1)$ by [Lemma 2.19](#).

(b) Let $\tilde{s} := s_2 \in S$ and $\tilde{r} := r \in R$, then $\tilde{s}r = s_2r = rs_2 = \tilde{r}s_2$ and thus

$$(s_1, r) \cdot (s_2, m) = (\tilde{s}s_1, \tilde{r}m) = (s_2s_1, rm). \quad \square$$

The last piece of the puzzle is the *localization map* or *structural homomorphism*:

Proposition 2.27 (cf. [Proposition 5.24](#)). *Let S be a left denominator set in a monoid R . The map*

$$\rho_{S,R} : R \rightarrow S^{-1}R, \quad r \mapsto (1, r)$$

is a homomorphism of monoids.

Proof: Let $\rho := \rho_{S,R}$, then $\rho(1_R) = (1_R, 1_R) = 1_{S^{-1}R}$ and for all $r_1, r_2 \in R$ we get

$$\rho(r_1 \cdot r_2) = (1, r_1 \cdot r_2) = (1, r_1) \cdot (1, r_2) = \rho(r_1) \cdot \rho(r_2). \quad \square$$

Lemma 2.28 (Existence, cf. [Lemma 5.25](#)). *Let S be a left denominator set in a monoid R . Then $(S^{-1}R, \rho_{S,R})$ is the left Ore localization of R at S .*

Proof: Let $\rho := \rho_{S,R}$. We verify the axioms of [Definition 2.12](#):

(i) Let $s \in S$. We have $(s, 1) \cdot \rho(s) = (s, 1) \cdot (1, s) = (s, s) = (1, 1)$. On the other hand, let $\tilde{s} := 1 \in S$ and $\tilde{r} := 1 \in R$, then $\tilde{s}s = 1 \cdot s = \tilde{r}s$, thus

$$\rho(s) \cdot (s, 1) = (1, s) \cdot (s, 1) = (\tilde{s} \cdot 1, \tilde{r} \cdot 1) = (\tilde{s}, \tilde{r}) = (1, 1).$$

This implies that $(s, 1)$ is the two-sided inverse of $\rho(s)$ in $S^{-1}R$.

(ii) Let $(s, r) \in S^{-1}R$, then $(s, r) = (s, 1) \cdot (1, r) = \rho(s)^{-1}\rho(r)$.

(iii) Let $r_1, r_2 \in R$. If $(1, r_1) = \rho(r_1) = \rho(r_2) = (1, r_2)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} = \tilde{s} \cdot 1 = \tilde{r} \cdot 1 = \tilde{r}$ and $\tilde{s}r_2 = \tilde{r}r_1$, thus $\tilde{s}r_1 = \tilde{r}r_1 = \tilde{s}r_2$. On the other hand, let $s \in S$ such that $sr_1 = sr_2$, then

$$\rho(r_1) = \rho(s)^{-1}\rho(s)\rho(r_1) = \rho(s)^{-1}\rho(sr_1) = \rho(s)^{-1}\rho(sr_2) = \rho(s)^{-1}\rho(s)\rho(r_2) = \rho(r_2). \quad \square$$

In particular, S being a left denominator set is also sufficient for the Ore localization of R at S to exist.

Proposition 2.29 (cf. [Theorem 5.26](#)). *Let S be a multiplicative set of a monoid R . The left Ore localization of R at S exists if and only if S is a left denominator set in R .*

Proof: Follows by combining [Lemma 2.16](#) and [Lemma 2.28](#). $\square$

In [Lemma 5.27](#) we will see that the Ore localization of a ring R at S is the zero ring if and only if $0 \in S$. For monoids, the situation is much more nuanced.

Definition 2.30. Let R be a monoid. An element $r \in R$ is called

- *left zero* in R if $r \cdot a = r$ for all $a \in R$. The set of all left zeros in R is denoted $N_l(R)$.

- *right zero* in R if $a \cdot r = r$ for all $a \in R$. The set of all right zeros in R is denoted $N_r(R)$.
- *zero* in R if r is both a left and a right zero. $\diamond$

Example 2.31. Take a non-empty set T and define $t_1 \cdot t_2 := t_1$ for all $t_1, t_2 \in T$. This operation is associative, since $(t_1 \cdot t_2) \cdot t_3 = t_1 \cdot t_2 = t_1 = t_1 \cdot (t_2 \cdot t_3)$, and thus makes T into semigroup, which is called the *left zero semigroup* over T . Take a symbol e not already contained in T and consider $T^1 := T \cup \{e\}$ with the extended operation

$$\cdot : T^1 \times T^1 \rightarrow T^1, \quad (t_1, t_2) \mapsto t_1 \cdot t_2 := \begin{cases} t_2, & \text{if } t_1 = e, \\ t_1, & \text{if } t_1 \neq e. \end{cases}$$

Then T^1 is a monoid in which any element except e is a left zero. $\diamond$

If R has both a left and a right zero, then they must be equal: let $a \cdot r = a$ and $r \cdot b = b$ for all $r \in R$, then $b = a \cdot b = a$. In particular, R has at most one zero. Thus, there are only four possible situations, which are mutually exclusive:

- R has neither a left nor a right zero (e. g. $\mathbb{N}$ with multiplication).
- R has at least one left zero, but no right zero ([Example 2.31](#)).
- R has at least one right zero, but no left zero (analogous to [Example 2.31](#)).
- R has exactly one zero z and $N_l(R) = \{z\} = N_r(R)$ (e. g. $\mathbb{N}_0$ with multiplication).

Only a monoid of the last type can be the multiplicative monoid of a ring.

Lemma 2.32. Let $\varphi : R \rightarrow T$ be a homomorphism of monoids.

- Let $l_0 \in R$. If $l_0 \in N_l(R)$, then $\varphi(l_0) \in N_l(\varphi(R))$.
- Let $r_0 \in R$. If $r_0 \in N_r(R)$, then $\varphi(r_0) \in N_r(\varphi(R))$.
- Let $z \in R$. If $z \in N(R)$, then $\varphi(z) \in N(\varphi(R))$.

If φ is injective, then the reverse implications hold as well.

Proof: Let $r \in R$. If $l_0 \in N_l(R)$, then $\varphi(l_0) \cdot \varphi(r) = \varphi(l_0 \cdot r) = \varphi(l_0)$, thus $\varphi(l_0) \in N_l(\varphi(R))$. If $\varphi(l_0) \in N_l(\varphi(R))$, then $\varphi(l_0 \cdot r) = \varphi(l_0) \cdot \varphi(r) = \varphi(l_0)$ implies $l_0 r = l_0$ if φ is injective, thus $l_0 \in N_l(R)$. The second part is analogous to the first part and the third part follows by combining the first two parts. $\square$

Lemma 2.33 (cf. [Lemma 5.27](#)). Let S be a left denominator set in a monoid R .

- If $S \cap N_l(R) \neq \emptyset$, then $S^{-1}R = \{1\}$.
- If $S^{-1}R = \{1\}$, then $N_r(R) \subseteq S$.

Proof: (a) Let $l_0 \in S \cap N_l(R)$ and $(s, r) \in S^{-1}R$, then $l_0 s = l_0 r$, thus $(s, r) = 1$ by [Lemma 2.26](#).

- Let $r_0 \in N_r(R)$, then $(1, r_0) = 1$ in $S^{-1}R$, thus by [Lemma 2.26](#) there exists $s \in S$ such that $r_0 = s r_0 = s \cdot 1 = s \in S$, thus $N_r(R) \subseteq S$. $\square$

If R is a ring and $0 \in S$, then $S^{-1}R$ collapses because 0 is a left zero. On the other hand, if the localization collapses, $0 \in S$ because 0 is a right zero.

2.4. Localization at central elements

Definition 2.34. Let R be a monoid. An element $r \in R$ is called *central* in R if $ra = ar$ for all $a \in R$. The set of all central elements in R is denoted $Z(R)$ and called the *center* of R . $\diamond$

If R is a monoid, then $Z(R)$ is a commutative submonoid of R . Localization at central multiplicative sets is always possible and particularly nice:

Proposition 2.35 (cf. [Proposition 5.31](#)). *Let R be a monoid and S a multiplicative subset of $Z(R)$.*

- (a) S is a left denominator set in R .
- (b) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) = (s_2, r_2)$ if and only if there exists $\tilde{s} \in S$ such that $\tilde{s}s_1r_2 = \tilde{s}s_2r_1$.
- (c) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) \cdot (s_2, r_2) = (s_1s_2, r_1r_2)$.
- (d) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ such that $r_1r_2 = r_2r_1$. Then $(s_1, r_1) \cdot (s_2, r_2) = (s_2, r_2) \cdot (s_1, r_1)$.

Proof: (a) Since $Z(R)$ is a submonoid of R we have $S \subseteq Z(R)$. Furthermore, S is a multiplicative set by construction. Let $s \in S$ and $r \in R$. Define $\tilde{s} := s \in S$ and $\tilde{r} := r \in R$, then $\tilde{s}r = sr = rs = \tilde{r}s$, thus S satisfies the left Ore condition in R . Now let $r_1, r_2 \in R$ such that $r_1s = r_2s$. Define $\tilde{s} := s \in S$, then $\tilde{s}r_1 = sr_1 = r_1s = r_2s = sr_2 = \tilde{s}r_2$ implies that S is left reversible in R .

- (b) Let $(s_1, r_1) = (s_2, r_2)$ in $S^{-1}R$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}r_2 = \tilde{r}r_1$, thus

$$\tilde{s}s_1r_2 = \tilde{s}r_2s_1 = \tilde{r}r_1s_1 = \tilde{r}s_1r_1 = \tilde{s}s_2r_1.$$

Now let $\tilde{s} \in S$ such that $\tilde{s}s_1r_2 = \tilde{s}s_2r_1$. Define $\hat{s} := \tilde{s}s_1 \in S$ and $\hat{r} := \tilde{s}s_2 \in S \subseteq R$, then

$$\hat{s}s_2 = \tilde{s}s_1s_2 = \tilde{s}s_2s_1 = \hat{r}s_1 \quad \text{as well as} \quad \hat{s}r_2 = \tilde{s}s_1r_2 = \tilde{s}s_2r_1 = \hat{r}r_1$$

show that $(s_1, r_1) = (s_2, r_2)$.

- (c) Define $\tilde{s} := s_2 \in S$ and $\tilde{r} := r_1 \in R$, then $\tilde{s}r_1 = s_2r_1 = r_1s_2 = \tilde{r}s_2$, thus

$$(s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2) = (s_2s_1, r_1r_2) = (s_1s_2, r_1r_2).$$

- (d) We have

$$(s_1, r_1) \cdot (s_2, r_2) = (s_1s_2, r_1r_2) = (s_2s_1, r_2r_1) = (s_2, r_2) \cdot (s_1, r_1). \quad \square$$

In particular, the localization of a commutative monoid is again commutative:

Corollary 2.36 (cf. [Corollary 5.33](#)). *Let S be a multiplicative set in a commutative monoid R .*

- (a) S is a left denominator set in R .
- (b) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) = (s_2, r_2)$ if and only if there exists $\tilde{s} \in S$ such that $\tilde{s}s_1r_2 = \tilde{s}s_2r_1$.
- (c) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) \cdot (s_2, r_2) = (s_1s_2, r_1r_2)$.
- (d) $S^{-1}R$ is a commutative monoid.

2.5. Left-right duality

If both the left and the right Ore localization of R at S exist, then they are isomorphic:

Proposition 2.37 (cf. [Proposition 5.34](#)). *Let S be a left and right denominator set in a monoid R . The map*

$$\varphi : RS^{-1} \rightarrow S^{-1}R, \quad (r, s) \mapsto (\tilde{s}, \tilde{r}),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r = \tilde{r}s$, is an isomorphism of monoids. In particular, the left Ore localization of R at S is isomorphic to the right Ore localization of R at S .

Proof: Since $1 \cdot 1 = 1 \cdot 1$ we clearly have $\varphi(1_{RS^{-1}}) = 1_{S^{-1}R}$. Let $(r_1, s_1), (r_2, s_2) \in RS^{-1}$.

Well-definedness: Let $(r_1, s_1) = (r_2, s_2)$ in RS^{-1} , then there exist $s_a \in S$ and $r_a \in R$ such that

$$s_1 s_a = s_2 r_a, \tag{F1}$$

$$r_1 s_a = r_2 r_a. \tag{F2}$$

By the left Ore condition there exist $\tilde{s}_1, \tilde{s}_2 \in S$ and $\tilde{r}_1, \tilde{r}_2 \in R$ such that

$$\tilde{s}_1 r_1 = \tilde{r}_1 s_1, \tag{F3}$$

$$\tilde{s}_2 r_2 = \tilde{r}_2 s_2. \tag{F4}$$

We have to show that $(\tilde{s}_1, \tilde{r}_1) = (\tilde{s}_2, \tilde{r}_2)$ in $S^{-1}R$. By the left Ore condition there exist $s_b \in S$ and $r_b \in R$ such that

$$s_b \tilde{s}_1 = r_b \tilde{s}_2. \tag{F5}$$

By left reversibility,

$$s_b \tilde{r}_1 s_1 s_a \stackrel{\text{(F3)}}{=} s_b \tilde{s}_1 r_1 s_a \stackrel{\text{(F5)}}{=} r_b \tilde{s}_2 r_1 s_a \stackrel{\text{(F2)}}{=} r_b \tilde{s}_2 r_2 r_a \stackrel{\text{(F4)}}{=} r_b \tilde{r}_2 s_2 r_a \stackrel{\text{(F1)}}{=} r_b \tilde{r}_2 s_1 s_a$$

implies the existence of $s_c \in S$ such that

$$s_c s_b \tilde{r}_1 = s_c r_b \tilde{r}_2. \tag{F6}$$

Define $\mathring{s} := s_c s_b \in S$ and $\mathring{r} := s_c r_b \in R$, then

$$\mathring{s} \tilde{s}_1 = s_c s_b \tilde{s}_1 \stackrel{\text{(F5)}}{=} s_c r_b \tilde{s}_2 = \mathring{r} \tilde{s}_2 \quad \text{and} \quad \mathring{s} \tilde{r}_1 = s_c s_b \tilde{r}_1 \stackrel{\text{(F6)}}{=} s_c r_b \tilde{r}_2 = \mathring{r} \tilde{r}_2$$

show that $(\tilde{s}_1, \tilde{r}_1) = (\tilde{s}_2, \tilde{r}_2)$ in $S^{-1}R$.

Multiplicativity: We have

$$\varphi((r_1, s_1)) \cdot \varphi((r_2, s_2)) = (\tilde{s}_1, \tilde{r}_1) \cdot (\tilde{s}_2, \tilde{r}_2) = (s_a \tilde{s}_1, r_a \tilde{r}_2),$$

and

$$\varphi((r_1, s_1) \cdot (r_2, s_2)) = \varphi((r_1 r_b, s_2 s_b)) = (s_c, r_c),$$

where $\tilde{s}_1, \tilde{s}_2, s_a, s_b, s_c \in S$ and $\tilde{r}_1, \tilde{r}_2, r_a, r_b, r_c \in R$ satisfy

$$\tilde{s}_1 r_1 = \tilde{r}_1 s_1, \tag{G1}$$

$$\tilde{s}_2 r_2 = \tilde{r}_2 s_2, \tag{G2}$$

$$s_a \tilde{r}_1 = r_a \tilde{s}_2, \tag{G3}$$

$$s_1 r_b = r_2 s_b, \tag{G4}$$

$$s_c r_1 r_b = r_c s_2 s_b. \tag{G5}$$

We have to show that $(s_a \tilde{s}_1, r_a \tilde{r}_2) = (s_c, r_c)$ in $S^{-1}R$. By the left Ore condition there exist $s_d \in S$ and $r_d \in R$ such that

$$s_d s_a \tilde{s}_1 = r_d s_c. \quad (\text{G6})$$

By left reversibility,

$$s_d r_a \tilde{r}_2 s_2 s_b \stackrel{(\text{G2})}{=} s_d r_a \tilde{s}_2 r_2 s_b \stackrel{(\text{G3})}{=} s_d s_a \tilde{r}_1 r_2 s_b \stackrel{(\text{G4})}{=} s_d s_a \tilde{r}_1 s_1 r_b \stackrel{(\text{G1})}{=} s_d s_a \tilde{s}_1 r_1 r_b \stackrel{(\text{G6})}{=} r_d s_c r_1 r_b \stackrel{(\text{G5})}{=} r_d r_c s_2 s_b$$

implies the existence of $s_e \in S$ such that

$$s_e s_d r_a \tilde{r}_2 = s_e r_d r_c. \quad (\text{G7})$$

Define $\mathring{s} := s_e s_d \in S$ and $\mathring{r} := s_e r_d \in R$, then

$$\mathring{s} s_a \tilde{s}_1 = s_e s_d s_a \tilde{s}_1 \stackrel{(\text{G6})}{=} s_e r_d s_c = \mathring{r} s_c \quad \text{and} \quad \mathring{s} r_a \tilde{r}_2 = s_e s_d r_a \tilde{r}_2 \stackrel{(\text{G7})}{=} s_e r_d r_c = \mathring{r} r_c$$

show that $(s_a \tilde{s}_1, r_a \tilde{r}_2) = (s_c, r_c)$ in $S^{-1}R$.

Injectivity: Let $(\tilde{s}_1, \tilde{r}_1) = \varphi((r_1, s_1)) = \varphi((r_2, s_2)) = (\tilde{s}_2, \tilde{r}_2)$ in $S^{-1}R$, where $\tilde{s}_1, \tilde{s}_2 \in S$ and $\tilde{r}_1, \tilde{r}_2 \in R$ satisfy

$$\tilde{s}_1 r_1 = \tilde{r}_1 s_1, \quad (\text{H1})$$

$$\tilde{s}_2 r_2 = \tilde{r}_2 s_2. \quad (\text{H2})$$

Then there exist $s_a \in S$ and $r_a \in R$ such that

$$s_a \tilde{s}_2 = r_a \tilde{s}_1, \quad (\text{H3})$$

$$s_a \tilde{r}_2 = r_a \tilde{r}_1. \quad (\text{H4})$$

We have to show $(r_1, s_1) = (r_2, s_2)$ in RS^{-1} . By the right Ore condition there exist $s_b \in S$ and $r_b \in R$ such that

$$s_1 s_b = s_2 r_b. \quad (\text{H5})$$

By right reversibility,

$$s_a \tilde{s}_2 r_1 s_b \stackrel{(\text{H3})}{=} r_a \tilde{s}_1 r_1 s_b \stackrel{(\text{H1})}{=} r_a \tilde{r}_1 s_1 s_b \stackrel{(\text{H5})}{=} r_a \tilde{r}_1 s_2 r_b \stackrel{(\text{H4})}{=} s_a \tilde{r}_2 s_2 r_b \stackrel{(\text{H2})}{=} s_a \tilde{s}_2 r_2 r_b$$

implies the existence of $s_c \in S$ such that

$$r_1 s_b s_c = r_2 r_b s_c. \quad (\text{H6})$$

Define $\mathring{s} := s_b s_c \in S$ and $\mathring{r} := r_b s_c \in R$, then

$$s_1 \mathring{s} = s_1 s_b s_c \stackrel{(\text{H5})}{=} s_2 r_b s_c = s_2 \mathring{r} \quad \text{and} \quad r_1 \mathring{s} = r_1 s_b s_c \stackrel{(\text{H6})}{=} r_2 r_b s_c = r_2 \mathring{r}$$

show that $(r_1, s_1) = (r_2, s_2)$ in RS^{-1} .

Surjectivity: Let $(\tilde{s}, \tilde{r}) \in S^{-1}R$. By the right Ore condition there exist $s \in S$ and $r \in R$ such that $\tilde{s}r = \tilde{r}s$, thus $(\tilde{s}, \tilde{r}) = \varphi((r, s)) \in \text{im}(\varphi)$. $\square$

2.6. Left denominator sets in monoids

The property of being a multiplicative set is preserved and reflected by homomorphisms, but the left Ore condition and especially left reversibility require additional assumptions:

Lemma 2.38 (cf. [Lemma 5.35](#)). *Let $\varphi : R \rightarrow R'$ be a homomorphism of monoids.*

- (a) *Let $S \subseteq R$.*
 - (i) *If S is multiplicatively closed in R , then $\varphi(S)$ is multiplicatively closed in R' .*
 - (ii) *If S is a multiplicative set in R , then $\varphi(S)$ is a multiplicative set in R' .*
 - (iii) *If φ is surjective and S satisfies the left Ore condition in R , then $\varphi(S)$ satisfies the left Ore condition in R' .*
 - (iv) *If φ is bijective and S is left reversible in R , then $\varphi(S)$ is left reversible in R' .*
 - (v) *If φ is bijective and S is a left denominator set in R , then $\varphi(S)$ is a left denominator set in R' .*
- (b) *Let $T \subseteq R'$.*
 - (i) *If T is multiplicatively closed in R' , then $\varphi^{-1}(T)$ is multiplicatively closed in R .*
 - (ii) *If T is a multiplicative set in R' , then $\varphi^{-1}(T)$ is a multiplicative set in R .*
 - (iii) *If φ is bijective and T satisfies the left Ore condition in R' , then $\varphi^{-1}(T)$ satisfies the left Ore condition in R .*
 - (iv) *If φ is bijective and T is left reversible in R' , then $\varphi^{-1}(T)$ is left reversible in R .*
 - (v) *If φ is bijective and T is a left denominator set in R' , then $\varphi^{-1}(T)$ is a left denominator set in R .*

Proof: (a) (i) Let $\varphi(s_1), \varphi(s_2) \in \varphi(S)$, then $\varphi(s_1) \cdot \varphi(s_2) = \varphi(s_1 s_2) \in \varphi(S)$ since $s_1 s_2 \in S$.
(ii) Since $1 \in S$, we have $1 = \varphi(1) \in \varphi(S)$, the rest follows from the previous statement.
(iii) Let $r' \in R'$ and $\varphi(s) \in \varphi(S)$. Since φ is surjective there exists $r \in R$ such that $\varphi(r) = r'$. By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$. Now

$$\varphi(\tilde{s})r' = \varphi(\tilde{s})\varphi(r) = \varphi(\tilde{s}r) = \varphi(\tilde{r}s) = \varphi(\tilde{r})\varphi(s).$$

- (iv) Let $r'_1, r'_2 \in R'$ and $\varphi(s) \in \varphi(S)$ such that $r'_1\varphi(s) = r'_2\varphi(s)$. Since φ is surjective there exist $r_1, r_2 \in R$ such that $\varphi(r_1) = r'_1$ and $\varphi(r_2) = r'_2$. Now

$$\varphi(r_1 s) = \varphi(r_1)\varphi(s) = r'_1\varphi(s) = r'_2\varphi(s) = \varphi(r_2)\varphi(s) = \varphi(r_2 s)$$

implies $r_1 s = r_2 s$ by injectivity of φ . Since S is left reversible there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$. Thus

$$\varphi(\tilde{s})r'_1 = \varphi(\tilde{s})\varphi(r_1) = \varphi(\tilde{s}r_1) = \varphi(\tilde{s}r_2) = \varphi(\tilde{s})\varphi(r_2) = \varphi(\tilde{s})r'_2.$$

- (v) Follows from the previous parts.

- (b) (i) Let $s_1, s_2 \in \varphi^{-1}(T)$, then $\varphi(s_1), \varphi(s_2) \in T$ and thus $\varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2) \in T$, which implies $s_1 s_2 \in \varphi^{-1}(T)$.
- (ii) Since $\varphi(1) = 1 \in T$, we have $1 \in \varphi^{-1}(T)$, the rest follows from the previous statement.
- (iii) Let $r \in R$ and $s \in \varphi^{-1}(T)$, then there exist $t \in T$ and $r' \in R'$ such that $t\varphi(r) = r'\varphi(s)$. Since φ is surjective there exist $\tilde{s} \in \varphi^{-1}(T)$ and $\tilde{r} \in R$ such that $\varphi(\tilde{s}) = t$ and $\varphi(\tilde{r}) = r'$, thus

$$\varphi(\tilde{s}r) = \varphi(\tilde{s})\varphi(r) = t\varphi(r) = r'\varphi(s) = \varphi(\tilde{r})\varphi(s) = \varphi(\tilde{r}s).$$

Since φ is injective, this implies $\tilde{s}r = \tilde{r}s$.

- (iv) Let $r_1, r_2 \in R$ and $s \in \varphi^{-1}(T)$ such that $r_1 s = r_2 s$, then

$$\varphi(r_1)\varphi(s) = \varphi(r_1 s) = \varphi(r_2 s) = \varphi(r_2)\varphi(s),$$

thus there exists $t \in T$ such that $t\varphi(r_1) = t\varphi(r_2)$. Since φ is surjective there exists $s \in \varphi^{-1}(T)$ such that $\varphi(s) = t$ and thus

$$\varphi(sr_1) = \varphi(s)\varphi(r_1) = t\varphi(r_1) = t\varphi(r_2) = \varphi(s)\varphi(r_2) = \varphi(sr_2).$$

Since φ is injective, this implies $sr_1 = sr_2$.

- (v) Follows from the previous parts. □

Definition 2.39. Let S be a subset of a monoid R . Then $[S]$ is the smallest submonoid of R with respect to inclusion that contains S , thus

$$[S] = \bigcup_{n \in \mathbb{N}_0} \{s_1 \cdot \dots \cdot s_n \mid s_i \in S \text{ for all } i \in \{1, \dots, n\}\}.$$

By the usual convention, the empty product in the case $n = 0$ corresponds to the element 1. We call S a *generating set* or *set of generators* for $[S]$ and $[S]$ the *multiplicative set generated* by S or the *multiplicative closure* of S . ◇

For a subset S of a monoid R we have $S = [S]$ if and only if S is a multiplicative set, which justifies the term “multiplicative closure”. In particular, every multiplicative set is a generating set of itself.

Example 2.40. Let S be a multiplicative set in a monoid R , then $S \cup \{0\}$ is a left denominator set in R . In particular, R is the largest left denominator set in R . In contrast, $[1] = \{1\}$ is the smallest left denominator set in R . ◇

Lemma 2.41 (cf. [Lemma 5.39](#)). *Let S and T be two multiplicatively closed subsets (resp. multiplicative subsets) of a monoid R , then $S \cap T$ is also multiplicatively closed (resp. a multiplicative set).*

Proof: Let $a, b \in S \cap T$, then $a, b \in S$ and thus $ab \in S$. Similarly, we get $ab \in T$ and therefore $ab \in S \cap T$. If additionally $1 \in S$ and $1 \in T$, then clearly $1 \in S \cap T$. □

To check whether a multiplicative set S is contained in another multiplicative set T , it suffices to check this on a generating set of S :

Lemma 2.42 (cf. Lemma 5.40). *Let T be a multiplicative set in a monoid R , I an arbitrary index set and $s_i \in R$ for all $i \in I$. Then $S := [\{s_i \mid i \in I\}] \subseteq T$ if and only if $s_i \in T$ for all $i \in I$.*

Proof: If $S \subseteq T$, then clearly $s_i \in S \subseteq T$ for all $i \in I$. On the other hand, let $s_i \in T$ for all $i \in I$. Since every element of S is a product of finitely many s_i and T is a multiplicative set we have $S \subseteq T$. $\square$

Similarly, it suffices to check left reversibility and the left Ore condition on generators:

Lemma 2.43 (cf. Lemma 5.41). *Let T be a subset of a monoid R and $S := [T]$ such that the following two conditions are satisfied:*

- (i) *For all $t \in T$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}t$.*
- (ii) *For all $t \in T$ and $r_1, r_2 \in R$ such that $r_1t = r_2t$ there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$.*

Then S is a left denominator set in R .

Proof: Without loss of generality assume that $1 \notin T$. The set S is a multiplicative set by construction, and any element in S can be written as a finite product of elements from T . Since the natural numbers are well-ordered, every element of $s \in S$ has a representation of this kind with a minimal number of factors. Consider the map $a : S \rightarrow \mathbb{N}_0$ that assigns this number to each element of S . This induces a partition $S = \bigsqcup_{n \in \mathbb{N}_0} S_n$, where $S_n := a^{-1}(\{n\})$. Let $S_{\leq n} := \bigsqcup_{i=0}^n S_i$, then $S_{\leq 1} = S_0 \sqcup S_1 = \{1\} \sqcup T$.

First we show the left Ore condition on S by induction on n :

Induction base: Let $r \in R$ and $s \in S_{\leq 1} = \{1\} \sqcup T$. Since the left Ore condition is trivially satisfied for $s = 1$, assume that $s \in T$. Then by condition (i) there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

Induction hypothesis: Assume that there is an $n \in \mathbb{N}$ with the following property: for all $s \in S_{\leq n}$ and all $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

Induction step: Let $r \in R$ and $s \in S_{n+1}$, then by construction there is a representation $s = \prod_{i=1}^{n+1} t_i$, where $t_i \in T$. Define $\tilde{s} := \prod_{i=2}^{n+1} t_i \in S_{\leq n}$, then $s = t_1\tilde{s}$. By the induction hypothesis there exist $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}r = \hat{r}\tilde{s}$. By the induction base there exist $\bar{s} \in S$ and $\bar{r} \in R$ such that $\bar{s}\tilde{s} = \bar{r}t_1$. Define $\mathring{s} := \bar{s}\hat{s} \in S$ and $\mathring{r} := \bar{r} \in R$, then

$$\mathring{s}r = \bar{s}\hat{s}r = \bar{s}\hat{r}\tilde{s} = \bar{r}t_1\tilde{s} = \mathring{r}s.$$

Lastly, we show left reversibility of S , again by an induction on $n \in \mathbb{N}$:

Induction base: Let $r_1, r_2 \in R$ and $s \in S_{\leq 1} = \{1\} \sqcup T$ such that $r_1s = r_2s$. If $s = 1$, then $r_1 = r_2$ and thus the claim holds, so again assume $s \in T$. Now by condition (ii) there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$.

Induction hypothesis: Assume that there is an $n \in \mathbb{N}$ with the following property: for all $s \in S_{\leq n}$ and all $r_1, r_2 \in R$ such that $r_1s = r_2s$ there exists $\tilde{s} \in S$ such that $\tilde{s}r_1 = \tilde{s}r_2$.

Induction step: Let $r_1, r_2 \in R$ and $s \in S_{n+1}$ such that $r_1 s = r_2 s$, then by construction there is a representation $s = \prod_{i=1}^{n+1} t_i$, where $t_i \in T$. Define $\tilde{s} := \prod_{i=2}^{n+1} t_i \in S_{\leq n}$, then $s = t_1 \tilde{s}$ and $r_1 t_1 \tilde{s} = r_2 t_2 \tilde{s}$. Since $r_1 t_1, r_2 t_2 \in R$ and $\tilde{s} \in S_{\leq n}$, by the induction hypothesis there exists $\hat{s} \in S$ such that $\hat{s} r_1 t_1 = \hat{s} r_2 t_2$. Since $\hat{s} r_1, \hat{s} r_2 \in R$ and $t_1 \in S_{\leq 1}$, by the induction base there exists $\bar{s} \in S$ such that $\bar{s} \hat{s} r_1 = \bar{s} \hat{s} r_2$. Define $\mathring{s} := \bar{s} \hat{s} \in S$, then $\mathring{s} r_1 = \bar{s} \hat{s} r_1 = \bar{s} \hat{s} r_2 = \mathring{s} r_2$. $\square$

Example 2.44 (cf. [Example 5.42](#)). Consider the *bicyclic monoid* $R := \langle p, q \mid pq = 1 \rangle$ and let $S := [p]$.

- Let $r = q^a p^b \in R$ and define $\tilde{s} := p^{a+1} \in S$ as well as $\tilde{r} := p^b \in R$, then

$$\tilde{s} r = p^{a+1} q^a p^b = p^{b+1} = p^b p = \tilde{r} p.$$

- Let $r \in R$ such that $rp = 0$, then $0 = rpq = r$, thus $pr = 0$.

By [Lemma 2.43](#), S is a left denominator set in R and we can consider the left Ore localization $S^{-1}R$, which will be done in [Example 4.19](#). $\diamond$

Corollary 2.45 (cf. [Corollary 5.43](#)). *Let S be a left reversible subset of a monoid R that satisfies the left Ore condition in R . Then $[S]$ is a left denominator set in R .*

Furthermore, a multiplicative set generated by left denominator sets is also a denominator set:

Corollary 2.46 (cf. [Corollary 5.44](#)). *Let R be a monoid and J a non-empty index set such that S_j is a left denominator set in R for every $j \in J$. Then $T := [\bigcup_{j \in J} S_j]$ is a left denominator set in R .*

Chapter 3.

Left saturation closure

Left saturation closure is a very abstract concept that generalizes many phenomena connected to left Ore localization. In this chapter, we discuss its basic properties before investigating an important special case in [Chapter 4](#).

3.1. Left saturation in general

Definition 3.1. Let T be a subset of a monoid R , M a left R -module and P, N two subsets of M . Then

$$\text{LSat}_T^N(P) := \{m \in N \mid \exists t \in T : tm \in P\} \subseteq N.$$

To keep the notation concise we use the shorthand $\text{LSat}_T(P) := \text{LSat}_T^M(P)$ if $M = N$. $\diamond$

Remark 3.2. If either T , N , or P is empty, then $\text{LSat}_T^N(P) = \emptyset$. Furthermore, LSat is inclusion-preserving in all three parameters by construction. $\diamond$

Lemma 3.3 (cf. [Lemma 6.3](#)). *Let T be a subset of a monoid R , M a left R -module and P, N two subsets of M .*

- (a) *We have $\text{LSat}_T^N(P) = \text{LSat}_T(P) \cap N$.*
- (b) *If $1 \in T$, then $P \subseteq \text{LSat}_T(P)$.*
- (c) *If $1 \in T$ and P is a left R -submodule of M , then $\text{LSat}_{\text{LSat}(T)}(P) = \text{LSat}_T(P)$.*
- (d) *If S is a subset of R , then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_{TS}^N(P).$$

- (e) *Let S be a subset of R such that $1 \in S$ and $S \subseteq T$. If T is multiplicatively closed, then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_T^N(P) = \text{LSat}_T^N(\text{LSat}_S^N(P)).$$

- (f) *Let S be a subset of R and Q a subset of M , then*

$$\text{LSat}_S^{\text{LSat}_T^N(Q)}(P) = \text{LSat}_T^N(Q) \cap \text{LSat}_S^N(P).$$

Proof: (a) We have

$$\begin{aligned} \text{LSat}_T^N(P) &= \{m \in N \mid \exists t \in T : tm \in P\} \\ &= \{m \in M \mid \exists t \in T : tm \in P\} \cap N = \text{LSat}_T(P) \cap N. \end{aligned}$$

- (b) Let $m \in P$. If $1 \in T$, then $1 \cdot m = m \in P$ shows $m \in \text{LSat}_T(P)$.
- (c) Since $1 \in T$ we have $T \subseteq \text{LSat}(T)$ and thus $\text{LSat}_T(P) \subseteq \text{LSat}_{\text{LSat}(T)}(P)$. Now let $m \in \text{LSat}_{\text{LSat}(T)}(P)$, then there exist $r \in \text{LSat}(T)$ such that $rm \in P$ and $w \in R$ such that $wr \in T$. Now $wrm \in P$ and thus $m \in \text{LSat}_T(P)$.
- (d) We have $\text{LSat}_S^N(\text{LSat}_T^N(P)) = \{m \in N \mid \exists s \in S \exists t \in T : tsm \in P\} = \text{LSat}_{TS}^N(P)$, since

$$m \in \text{LSat}_S^N(\text{LSat}_T^N(P)) \Leftrightarrow \exists s \in S \exists t \in T : tsm \in P \Leftrightarrow m \in \text{LSat}_{TS}^N(P).$$
- (e) Since $1 \in S$ we have $\text{LSat}_T^N(P) \subseteq \text{LSat}_S^N(\text{LSat}_T^N(P))$. The first equality now follows from

$$\begin{aligned} \text{LSat}_S^N(\text{LSat}_T^N(P)) &= \{m \in N \mid \exists s \in S \exists t \in T : tsm \in P\} \\ &\subseteq \{m \in N \mid \exists t_1, t_2 \in T : t_1 t_2 m \in P\} \\ &\subseteq \{m \in N \mid \exists t \in T : tm \in P\} \\ &= \text{LSat}_T^N(P), \end{aligned}$$

where the first inclusion is due to the fact that $S \subseteq T$ and the second inclusion relies on T being multiplicatively closed. Similarly, we get

$$\begin{aligned} \text{LSat}_T^N(\text{LSat}_S^N(P)) &= \{m \in N \mid \exists t \in T \exists s \in S : stm \in P\} \\ &\subseteq \{m \in N \mid \exists t_1, t_2 \in T : t_1 t_2 m \in P\} \\ &\subseteq \{m \in N \mid \exists t \in T : tm \in P\} \\ &= \text{LSat}_T^N(P). \end{aligned}$$

Since $1 \in S$ we have $P \subseteq \text{LSat}_S^N(P)$ and thus $\text{LSat}_T^N(P) \subseteq \text{LSat}_T^N(\text{LSat}_S^N(P))$ implies the second equality. $\square$

- (f) We have

$$\begin{aligned} \text{LSat}_S^{\text{LSat}_T^N(Q)}(P) &= \{m \in \text{LSat}_T^N(Q) \mid \exists s \in S : sm \in P\} \\ &= \{m \in N \mid \exists s \in S : sm \in P\} \cap \text{LSat}_T^N(Q) \\ &= \text{LSat}_S^N(P) \cap \text{LSat}_T^N(Q). \end{aligned}$$

Lemma 3.4 (cf. [Lemma 6.5](#)). *Let $\varphi : R \rightarrow T$ be a homomorphism of monoids and S, I two subsets of I . Then we have the following:*

- (a) $\text{LSat}_S(I) \subseteq \varphi^{-1}(\text{LSat}_{\varphi(S)}(\varphi(I)))$.
- (b) If φ is injective, then $\text{LSat}_S(I) = \varphi^{-1}(\text{LSat}_{\varphi(S)}(\varphi(I)))$.
- (c) If φ is bijective, then $\varphi(\text{LSat}_S(I)) = \text{LSat}_{\varphi(S)}(\varphi(I))$.

Proof: (a) Let $r \in \text{LSat}_S(I)$, then there exists $s \in S$ such that $sr \in I$. Applying φ yields $\varphi(s)\varphi(r) = \varphi(sr) \in \varphi(I)$, which implies $\varphi(r) \in \text{LSat}_{\varphi(S)}(\varphi(I))$ by definition and then finally $r \in \varphi^{-1}(\text{LSat}_{\varphi(S)}(\varphi(I)))$.

- (b) Let $r \in \varphi^{-1}(\text{LSat}_{\varphi(S)}(\varphi(I)))$, then $\varphi(r) \in \text{LSat}_{\varphi(S)}(\varphi(I))$, thus there exists $s \in S$ such that $\varphi(sr) = \varphi(s)\varphi(r) \in \varphi(I)$. Since φ is injective, we get $sr \in \varphi^{-1}(\varphi(I)) = I$ and therefore $r \in \text{LSat}_S(I)$.

- (c) Follows immediately from the previous part. $\square$

3.2. Left saturation at multiplicative sets

To explain the name of the concept, we have to look at the property of being *left saturated*:

Definition 3.5. Let T be a multiplicative set in a monoid R and M a left R -module. A subset P of M is called *left T -saturated* if for all $t \in T$ and $m \in M$, $tm \in P$ implies $m \in P$. $\diamond$

As one might expect, $\text{LSat}_T(P)$ is then the closure of P with respect to being left T -saturated:

Proposition 3.6 (cf. [Proposition 6.7](#)). *Let T be a multiplicative set in a monoid R , M a left R -module and P a subset of M .*

- (a) $\text{LSat}_T(P)$ is left T -saturated.
- (b) P is left T -saturated if and only if $P = \text{LSat}_T(P)$.
- (c) $\text{LSat}_T(P)$ is the smallest left T -saturated superset of P with respect to inclusion.

Proof: (a) Let $t \in T$ and $m \in M$ such that $tm \in \text{LSat}_T(P)$, then $\tilde{t}tm \in P$ for some $\tilde{t} \in T$. Since T is multiplicatively closed we have $\tilde{t}t \in T$, which implies $m \in \text{LSat}_T(P)$.

(b) Let $m \in \text{LSat}_T(P)$, then $tm \in P$ for some $t \in T$. If P is left T -saturated, we have $m \in P$, which implies $\text{LSat}_T(P) = P$. On the other hand, if $P = \text{LSat}_T(P)$, then P is left T -saturated by the result above.

(c) Let $Q \subseteq M$ be a left T -saturated set with $P \subseteq Q \subseteq \text{LSat}_T(P)$. Let $m \in \text{LSat}_T(P)$, then $tm \in P \subseteq Q$ for some $t \in T$. Since Q is left T -saturated, we have $m \in Q$ and therefore $\text{LSat}_T(P) = Q$. $\square$

Lemma 3.7 (cf. [Lemma 6.8](#)). *Let T be a multiplicative set in a monoid R and M a left R -module. If P_1 and P_2 are two left T -saturated subsets of M , then $P_1 \cap P_2$ is also left T -saturated.*

Proof: Let $t \in T$ and $m \in M$ such that $tm \in P_1 \cap P_2$, then in particular we have $tm \in P_1$. Since P_1 is left T -saturated this implies that $m \in P_1$. Analogously we get $m \in P_2$ and thus $m \in P_1 \cap P_2$, therefore $P_1 \cap P_2$ is left T -saturated. $\square$

Lemma 3.8 (cf. [Lemma 6.9](#)). *Let S, T be two multiplicative sets in a monoid R , M a left R -module and P, N two subsets of M . If $S \subseteq T$, then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_T^N(P) = \text{LSat}_T^N(\text{LSat}_S^N(P)).$$

Proof: Immediately from [Lemma 3.3](#). $\square$

In many cases it is sufficient to know if two sets share the same left saturation closure in contrast to actually knowing what their respective closures are. This can be aided by the following observations:

Lemma 3.9 (cf. [Lemma 6.10](#)). *Let T be a multiplicative set in a monoid R , M a left R -module and P_1, P_2 two subsets of M .*

- (a) We have $P_1 \subseteq \text{LSat}_T(P_2)$ if and only if $\text{LSat}_T(P_1) \subseteq \text{LSat}_T(P_2)$.
- (b) The following are equivalent:

- (1) $P_1 \subseteq \text{LSat}_T(P_2)$ and $P_2 \subseteq \text{LSat}_T(P_1)$.
- (2) $\text{LSat}_T(P_1) = \text{LSat}_T(P_2)$.

(c) If $P_1 \subseteq P_2 \subseteq \text{LSat}_T(P_1)$, then $\text{LSat}_T(P_1) = \text{LSat}_T(P_2)$.

Proof: (a) Let $P_1 \subseteq \text{LSat}_T(P_2)$ and $m \in \text{LSat}_T(P_1)$, then there exists $t \in T$ such that $tm \in P_1 \subseteq \text{LSat}_T(P_2)$. But then $t'tm \in P_2$ for some $t' \in T$, thus $m \in \text{LSat}_T(P_2)$ since $t't \in T$, which implies $\text{LSat}_T(P_1) \subseteq \text{LSat}_T(P_2)$. The other implication is obvious since $P_1 \subseteq \text{LSat}_T(P_1)$.

(b) Follows from applying the previous part twice.

(c) Follows from the previous part, since $P_1 \subseteq P_2$ implies $P_1 \subseteq \text{LSat}_T(P_2)$. $\square$

From [Corollary 2.46](#) we know that a multiplicative set S generated by a family of left denominator sets is again a left denominator set. In the same spirit, we can characterize the left saturation closure of P at S via the generating sets of S :

Lemma 3.10 (cf. [Lemma 6.12](#)). *Let R be a monoid, M a left R -module and P a left R -submodule of M . Further, let $\{S_i\}_{i \in I}$ be a family of multiplicative sets in R and $S := [\bigcup_{i \in I} S_i]$. Then $\text{LSat}_S(P) = P$ if and only if $\text{LSat}_{S_i}(P) = P$ all $i \in I$.*

Proof: If $\text{LSat}_S(P) = P$, then $P \subseteq \text{LSat}_{S_i}(P) \subseteq \text{LSat}_S(P) = P$, which implies $\text{LSat}_{S_i}(P) = P$ for all $i \in I$.

Now let $\text{LSat}_{S_i}(P) = P$ for all $i \in I$ and $m \in \text{LSat}_S(P)$, then there exists $s \in S$ such that $sm \in P$. By definition, $s = s_{i_1}s_{i_2} \cdots s_{i_k}$ for some $i_j \in I$ and $s_{i_j} \in S_{i_j}$. Since $\text{LSat}_{S_{i_j}}(P) = P$, from $s_{i_1}s_{i_2} \cdots s_{i_k}m \in P$ we get $s_{i_2} \cdots s_{i_k}m \in P$ by [Proposition 3.6](#). By induction we get $m \in P$ and thus $\text{LSat}_S(P) = P$. $\square$

Building on this result, we present the following recipe:

Proposition 3.11 (cf. [Proposition 6.13](#)). *Let R be a monoid, M a left R -module and P a left R -submodule of M . Further, let $\{S_i\}_{i \in I}$ be a family of multiplicative sets in R , $S := [\bigcup_{i \in I} S_i]$ and $f : \mathbb{N} \rightarrow I$ a map. Define $P_0 := P$ and $P_n := \text{LSat}_{S_{f(n)}}(P_{n-1})$ for all $n \in \mathbb{N}$.*

- (a) For all $n \in \mathbb{N}$, $P_n = \text{LSat}_{S_{f(1)}S_{f(2)} \cdots S_{f(n)}}(P)$.
- (b) For all $n \in \mathbb{N}$, $\text{LSat}_S(P_n) = \text{LSat}_S(P)$.
- (c) Let $l, m \in \mathbb{N}_0$ such that $\{f(l+j) \mid j \in \{1, \dots, m\}\} = I$ and $P_l = P_{l+m}$, then $P_l = \text{LSat}_S(P)$.
- (d) Let $I = \{1, \dots, k\}$, $S_iS_j = S_jS_i$ for all $i, j \in I$ and $f(I) = I$, then $P_k = \text{LSat}_S(P)$.

Proof: (a) Follows via induction, since by [Lemma 3.3](#) we have

$$P_2 = \text{LSat}_{S_{f(2)}}(P_1) = \text{LSat}_{S_{f(2)}}(\text{LSat}_{S_{f(1)}}(P)) = \text{LSat}_{S_{f(1)}S_{f(2)}}(P).$$

(b) Since $S_{f(1)}S_{f(2)} \cdots S_{f(n)} \subseteq S$ we have

$$P \subseteq P_n = \text{LSat}_{S_{f(1)}S_{f(2)} \cdots S_{f(n)}}(P) \subseteq \text{LSat}_S(P)$$

by [Lemma 3.3](#) and thus $\text{LSat}_S(P_n) = \text{LSat}_S(P)$ by [Lemma 3.9](#).

(c) By assumption, for any $i \in I$ there is a $j \in \{1, \dots, m\}$ such that $f(l+j) = i$. Then $P_l \subseteq P_{l+j} \subseteq P_{l+m} = P$, thus $P_l = P_{l+j} = \text{LSat}_{S_i}(P_{l+j-1})$ is left S_i -saturated. By [Lemma 3.10](#) we have $P_l = \text{LSat}_S(P_l) = \text{LSat}_S(P)$.

(d) If $S_i S_j = S_j S_i$, then

$$\text{LSat}_{S_i}(\text{LSat}_{S_j}(P)) = \text{LSat}_{S_j S_i}(P) = \text{LSat}_{S_i S_j}(P) = \text{LSat}_{S_j}(\text{LSat}_{S_i}(P))$$

by [Lemma 3.3](#), so by induction we have that P_k is left S_i -saturated for all $i \in I$. From [Lemma 3.10](#) we get $P_k = \text{LSat}_S(P_k) = \text{LSat}_S(P)$. □

Chapter 4.

A canonical form for left denominator sets

We now consider left saturation closure with respect to the monoid itself. Applying this to left denominator sets will give us a canonical form that will reveal the structure of the associated localization.

4.1. Left saturation at the monoid

Definition 4.1. Let P be a subset of a monoid R . The *left saturation closure* of P in R is

$$\text{LSat}(P) := \text{LSat}_R^R(P) = \{r \in R \mid \exists w \in R : wr \in P\}. \quad \diamond$$

For saturation with respect to the monoid itself we omit the prefix R :

Definition 4.2. Let S be a subset of a monoid R . Then S is called

- *left saturated* if for all $s, t \in R$, $st \in S$ implies $t \in S$.
- *right saturated* if for all $s, t \in R$, $st \in S$ implies $s \in S$.
- *saturated* if S is both left and right saturated. $\diamond$

Example 4.3. Consider again the bicyclic monoid $R := \langle p, q \mid pq = 1 \rangle$. From [Example 2.44](#) we know that $S := [p]$ is a left denominator set in R . Since $p^a \cdot q^a p^b = p^b \in [p]$ holds for all $a, b \in \mathbb{N}_0$, we find that $\text{LSat}(S) = R$. $\diamond$

Lemma 4.4 (cf. [Lemma 7.5](#)). *Let R be a monoid, then $\text{LSat}([1]) = U_l(R)$. In particular, we have $U_l(R) \subseteq \text{LSat}(P)$ for any non-empty subset P of R .*

Proof: By definition, $\text{LSat}([1]) = \{r \in R \mid \exists w \in R : wr = 1\} = U_l(R)$. Now let $m \in P$, then $m \cdot 1 = m \in P$, thus $[1] \subseteq \text{LSat}(P)$ and therefore $U_l(R) = \text{LSat}([1]) \subseteq \text{LSat}(P)$ by [Lemma 3.9](#). $\square$

For any subset of R , LSat preserves and reflects the left Ore condition:

Lemma 4.5 (cf. [Lemma 7.7](#)). *Let S be a subset of a monoid R . The following are equivalent:*

- (1) S satisfies the left Ore condition in R .
- (2) $\text{LSat}(S)$ satisfies the left Ore condition in R .

Proof: If $S = \emptyset$ then $\text{LSat}(S) = \emptyset$, thus both statements are trivially true, so assume $S \neq \emptyset$.

(1) $\Rightarrow$ (2): Let $r \in R$ and $x \in \text{LSat}(S)$, then $wx \in S$ for some $w \in R$. By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}wx$. Define $\hat{x} := \tilde{s} \in S \subseteq \text{LSat}(S)$ and $\hat{r} := \tilde{r}w \in R$, then $\hat{x}r = \tilde{s}r = \tilde{r}wx = \hat{r}\hat{x}$ shows that $\text{LSat}(S)$ satisfies the left Ore condition in R .

(2) $\Rightarrow$ (1): Let $r \in R$ and $s \in S \subseteq \text{LSat}(S)$. By the left Ore condition on $\text{LSat}(S)$ there exist $x \in \text{LSat}(S)$ and $\tilde{r} \in R$ such that $xr = \tilde{r}s$. Since $x \in \text{LSat}(S)$ we have $wx \in S$ for some $w \in R$. Define $\hat{s} := wx \in S$ and $\hat{r} := w\tilde{r} \in R$, then $\hat{s}r = wxr = w\tilde{r}s = \hat{r}\hat{s}$ shows that S satisfies the left Ore condition in R . $\square$

Corollary 4.6. *Let S and T be subsets of a monoid R such that $\text{LSat}(S) = \text{LSat}(T)$. Then S satisfies the left Ore condition in R if and only if T satisfies the left Ore condition in R .*

Furthermore, LSat preserves left Ore sets:

Lemma 4.7 (cf. [Lemma 7.9](#)). *Let S be a left Ore set in a monoid R . Then $\text{LSat}(S)$ is also a left Ore set in R .*

Proof: Since $S \subseteq \text{LSat}(S)$, $1 \in S$ implies $1 \in \text{LSat}(S)$. Now let $x_1, x_2 \in \text{LSat}(S)$, then $w_1x_1 \in S$ and $w_2x_2 \in S$ for some $w_1, w_2 \in R$. By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}w_2 = \tilde{r}w_1x_1$. Then $\tilde{r}w_1x_1x_2 = \tilde{s}w_2x_2 \in S$ since S is multiplicatively closed, thus $x_1x_2 \in \text{LSat}(S)$. The rest follows from [Lemma 4.5](#). $\square$

Lemma 4.8 (cf. [Lemma 7.10](#)). *Let $\varphi : R \rightarrow T$ be a homomorphism of monoids and $S \subseteq R$, then $\varphi(\text{LSat}(S)) \subseteq \text{LSat}(\varphi(S))$. If φ is an isomorphism, then $\varphi(\text{LSat}(S)) = \text{LSat}(\varphi(S))$.*

Proof: Let $t \in \varphi(\text{LSat}(S))$, then there exist $r \in \text{LSat}(S)$ and $w \in R$ such that $t = \varphi(r)$ and $wr \in S$, thus $\varphi(w)t = \varphi(w)\varphi(r) = \varphi(wr) \in \varphi(S)$ implies $t \in \text{LSat}(\varphi(S))$.

Now let φ additionally be bijective and take $t \in \text{LSat}(\varphi(S))$, then there exist $w \in T$ and $s \in S$ such that $wt = \varphi(s)$. Since φ is surjective, there exist $r, v \in R$ such that $\varphi(r) = t$ and $\varphi(v) = w$. Now $\varphi(vr) = \varphi(v)\varphi(r) = wt = \varphi(s)$ implies $vr = s$ by injectivity of φ , thus $r \in \text{LSat}(S)$ and $t = \varphi(r) \in \varphi(\text{LSat}(S))$. $\square$

As a first major result, we see that LSat gives us control over the left invertible elements in $S^{-1}R$:

Proposition 4.9 (cf. [Proposition 7.11](#)). *Let S be a left denominator set in a monoid R and $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) $(s, r) \in U_l(S^{-1}R)$.
- (2) $(1, r) \in U_l(S^{-1}R)$.
- (3) $r \in \text{LSat}(S)$.

Furthermore, if $(s, r) \in U_l(S^{-1}R)$, then a left inverse of (s, r) in $S^{-1}R$ is given by (wr, ws) , where $w \in R$ satisfies $wr \in S$.

Proof: We always have $(s, r) = (s, 1) \cdot (1, r)$, where $(s, 1) \in U_l(S^{-1}R)$ with inverse $(1, s)$.

(1) $\Rightarrow$ (2): Let $a \in S^{-1}R$ such that $a \cdot (s, r) = 1$. Then $a \cdot (s, 1) \in S^{-1}R$ is a left inverse of $(1, r)$, since $a \cdot (s, 1) \cdot (1, r) = a \cdot (s, r) = 1$, thus $(1, r) \in U_l(S^{-1}R)$.

(2) $\Rightarrow$ (3): Let $(1, r) \in U_l(S^{-1}R)$, then there exists $(s, w) \in S^{-1}R$ such that

$$(1, 1) = (s, w) \cdot (1, r) = (s, wr).$$

Now there exists $\tilde{s} \in S$ such that $\tilde{s}wr = \tilde{s}s \in S$, thus $r \in \text{LSat}(S)$.

(3) $\Rightarrow$ (1): Let $r \in \text{LSat}(S)$, then $wr \in S$ for some $w \in R$. Then $(wr, ws) \in S^{-1}R$ is a left inverse of (s, r) and thus $(s, r) \in U_l(S^{-1}R)$, since

$$(wr, ws) \cdot (s, r) = (wr, w) \cdot (1, s) \cdot (s, 1) \cdot (1, r) = (wr, w) \cdot (1, r) = (wr, wr) = (1, 1). \quad \square$$

4.2. Homomorphisms of localized monoids

Among all homomorphism between different localizations of the same monoid R , there are those who preserve the image of R :

Definition 4.10. Let S and T be left denominator sets in a monoid R . A homomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$ of monoids *fixes* R or is an *R -fixing homomorphism* if $\varphi \circ \rho_{S,R} = \rho_{T,R}$. The existence of an R -fixing isomorphism $S^{-1}R \rightarrow T^{-1}R$ is denoted by $S^{-1}R \cong_R T^{-1}R$. $\diamond$

Lemma 4.11 (cf. [Lemma 7.13](#)). *Let S and T be left denominator sets in a monoid R . If $\varphi : S^{-1}R \rightarrow T^{-1}R$ is an R -fixing isomorphism, then $\varphi^{-1} : T^{-1}R \rightarrow S^{-1}R$ is also an R -fixing isomorphism.*

Proof: We have $\varphi^{-1} \circ \rho_{T,R} = \varphi^{-1} \circ \varphi \circ \rho_{S,R} = \rho_{S,R}$. $\square$

We will now show that $\text{LSat}(S)$ is the best description of S with respect to R -fixing isomorphisms of the corresponding localizations. The key component is the following map:

Lemma 4.12 (cf. [Lemma 7.14](#)). *Let S and T be left denominator sets in a monoid R such that $S \subseteq \text{LSat}(T)$. For any $s \in S$ choose a $w_s \in R$ such that $w_s s \in T$. Then for any left R -module M the map*

$$\omega_{S,T,R,M} : S^{-1}M \rightarrow T^{-1}M, \quad (s, m) \mapsto (w_s s, w_s m),$$

is well-defined and independent of the choice of the w_s .

Proof: Let $(s_1, m_1), (s_2, m_2) \in S^{-1}M$ such that $(s_1, m_1) = (s_2, m_2)$, then there exist $s_a \in S$ and $r_a \in R$ such that

$$s_a s_2 = r_a s_1, \tag{I1}$$

$$s_a m_2 = r_a m_1. \tag{I2}$$

Furthermore, let $w_1, w_2 \in R$ such that $w_1 s_1, w_2 s_2 \in T$. We have to show that

$$(w_1 s_1, w_1 m_1) = (w_2 s_2, w_2 m_2)$$

in $T^{-1}M$. By the left Ore condition on T there exist $t_b \in T$ and $r_b \in R$ such that

$$t_b w_2 s_2 = r_b w_1 s_1. \tag{I3}$$

By the left Ore condition on S there exist $s_c \in S$ and $r_c \in R$ such that

$$s_c t_b w_2 = r_c s_a. \quad (\text{I4})$$

Since $s_c \in S \subseteq \text{LSat}(T)$ there exists $w_c \in R$ such that $w_c s_c \in T$. Now

$$w_c s_c r_b w_1 s_1 \stackrel{(\text{I3})}{=} w_c s_c t_b w_2 s_2 \stackrel{(\text{I4})}{=} w_c r_c s_a s_2 \stackrel{(\text{I1})}{=} w_c r_c r_a s_1$$

implies the existence of $s_d \in S$ such that

$$s_d w_c s_c r_b w_1 = s_d w_c r_c r_a \quad (\text{I5})$$

by left reversibility of S . Since $s_d \in S \subseteq \text{LSat}(T)$ there exists $w_d \in R$ such that $w_d s_d \in T$. Define $\mathring{t} := w_d s_d w_c s_c t_b \in T$ and $\mathring{r} := w_d s_d w_c s_c r_b \in R$, then

$$\mathring{t} w_2 s_2 = w_d s_d w_c s_c t_b w_2 s_2 \stackrel{(\text{I3})}{=} w_d s_d w_c s_c r_b w_1 s_1 = \mathring{r} w_1 s_1$$

and

$$\begin{aligned} \mathring{t} w_2 m_2 &= w_d s_d w_c s_c t_b w_2 m_2 \\ &\stackrel{(\text{I4})}{=} w_d s_d w_c r_c s_a m_2 \\ &\stackrel{(\text{I2})}{=} w_d s_d w_c r_c r_a m_1 \\ &\stackrel{(\text{I5})}{=} w_d s_d w_c s_c r_b w_1 m_1 \\ &= \mathring{r} w_1 m_1 \end{aligned}$$

show that $(w_1 s_1, w_1 m_1) = (w_2 s_2, w_2 m_2)$. $\square$

Remark 4.13. If $M = R$ in the situation of [Lemma 4.12](#) we write $\omega_{S,T,R}$ instead of $\omega_{S,T,R,M}$. $\diamond$

For two left denominator sets S and T , $S \subseteq \text{LSat}(T)$ is now equivalent to $\omega_{S,T,R}$ being a homomorphism from $S^{-1}R$ to $T^{-1}R$:

Proposition 4.14 (cf. [Proposition 7.16](#)). *Let S and T be left denominator sets in a monoid R . The following are equivalent:*

- (1) $S \subseteq \text{LSat}(T)$.
- (2) There exists some R -fixing monoid homomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$.
- (3) There exists a unique R -fixing monoid homomorphism from $S^{-1}R$ to $T^{-1}R$ which is given by

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ satisfies $ws \in T$.

$$\begin{array}{ccc} & R & \\ \rho_{S,R} \swarrow & & \searrow \rho_{T,R} \\ S^{-1}R & \xrightarrow{\omega_{S,T,R}} & T^{-1}R \end{array}$$

Proof: (3) $\Rightarrow$ (2): This is obvious.

(1) $\Rightarrow$ (3): The map $\omega := \omega_{S,T,R}$ is well-defined by [Lemma 4.12](#).

Neutral element/ R -fixing: Let $r \in R$. Since $1_R \in T$ we have

$$(\omega \circ \rho_{S,R})(r) = \omega(\rho_{S,R}(r)) = \omega((1_R, r)) = (1_R, r) = \rho_{T,R}(r).$$

In particular, we have $\omega(1_{S^{-1}R}) = \omega(\rho_{S,R}(1_R)) = \rho_{T,R}(1_R) = 1_{T^{-1}R}$.

Multiplicativity: Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$. We have

$$\omega((s_1, r_1) \cdot (s_2, r_2)) = \omega((s_a s_1, r_a r_2)) = (w_a s_a s_1, w_a r_a r_2)$$

and

$$\omega((s_1, r_1)) \cdot \omega((s_2, r_2)) = (w_1 s_1, w_1 r_1) \cdot (w_2 s_2, w_2 r_2) = (t_b w_1 s_1, r_b w_2 r_2)$$

where $s_a \in S$, $r_a, r_b, w_a, w_1, w_2 \in R$ and $t_b \in T$ satisfy

$$s_a r_1 = r_a s_2, \tag{J1}$$

$w_a s_a s_1 \in T$, $w_1 s_1 \in T$, $w_2 s_2 \in T$ and

$$t_b w_1 r_1 = r_b w_2 s_2. \tag{J2}$$

We have to show $(w_a s_a s_1, w_a r_a r_2) = (t_b w_1 s_1, r_b w_2 r_2)$ in $T^{-1}R$. By the left Ore condition on T there exist $t_c \in T$ and $r_c \in R$ such that

$$t_c t_b w_1 s_1 = r_c w_a s_a s_1, \tag{J3}$$

which implies the existence of $s_d \in S$ such that

$$s_d t_c t_b w_1 = s_d r_c w_a s_a \tag{J4}$$

by left reversibility of S . Since $s_d \in S \subseteq \text{LSat}(T)$ there exists $w_d \in R$ such that $w_d s_d \in T$. Now

$$w_d s_d t_c r_b w_2 s_2 \stackrel{\text{(J2)}}{=} w_d s_d t_c t_b w_1 r_1 \stackrel{\text{(J4)}}{=} w_d s_d r_c w_a s_a r_1 \stackrel{\text{(J1)}}{=} w_d s_d r_c w_a r_a s_2$$

implies the existence of $s_e \in S$ such that

$$s_e w_d s_d t_c r_b w_2 = s_e w_d s_d r_c w_a r_a \tag{J5}$$

by left reversibility of S . Since $s_e \in S \subseteq \text{LSat}(T)$ there exists $w_e \in R$ such that $w_e s_e \in T$. Define $\mathring{t} := w_e s_e w_d s_d t_c \in T$ and $\mathring{r} := w_e s_e w_d s_d r_c \in R$, then

$$\mathring{t} t_b w_1 s_1 = w_e s_e w_d s_d t_c t_b w_1 s_1 \stackrel{\text{(J3)}}{=} w_e s_e w_d s_d r_c w_a s_a s_1 = \mathring{r} w_a s_a s_1$$

and

$$\mathring{t} r_b w_2 r_2 = w_e s_e w_d s_d t_c r_b w_2 r_2 \stackrel{\text{(J5)}}{=} w_e s_e w_d s_d r_c w_a r_a r_2 = \mathring{r} w_a r_a r_2$$

show that $(w_a s_a s_1, w_a r_a r_2) = (t_b w_1 s_1, r_b w_2 r_2)$.

Uniqueness: Let $\varphi : S^{-1}R \rightarrow T^{-1}R$ be an R -fixing homomorphism of monoids and $(s, r) \in S^{-1}R$, then there exists $w \in R$ such that $ws \in T$. Now $(ws, w) \in T^{-1}R$ is the inverse of $(1, s) = \rho_{T,R}(s)$ in $T^{-1}R$, thus

$$\begin{aligned} \varphi((s, r)) &= \varphi((s, 1) \cdot (1, r)) = \varphi((s, 1)) \cdot \varphi((1, r)) = \varphi((s, 1)) \cdot (1, r) \\ &= \varphi((1, s)^{-1}) \cdot (1, r) = \varphi((1, s))^{-1} \cdot (1, r) = (1, s)^{-1} \cdot (1, r) \\ &= (ws, w) \cdot (1, r) = (ws, wr) = \omega((s, r)). \end{aligned}$$

(2) $\Rightarrow$ (1): Let $s \in S$, then there exist $t \in T$ and $r \in R$ such that $\varphi((s, 1)) = (t, r)$. Now

$$1_{T^{-1}R} = \varphi(1_{S^{-1}R}) = \varphi((s, s)) = \varphi((s, 1)) \cdot \varphi((1, s)) = (t, r) \cdot (1, s) = (t, rs),$$

since φ fixes R . By [Lemma 2.26](#) there exists $\tilde{t} \in T$ such that $\tilde{t}rs = \tilde{t}t \in T$, thus $s \in \text{LSat}(T)$, which implies $S \subseteq \text{LSat}(T)$. $\square$

Lemma 4.15 (cf. [Lemma 7.17](#)). *Let S and T be left denominator sets in a monoid R such that $\text{LSat}(S) = \text{LSat}(T)$. Then $\omega_{S,T,R,M}$ is a bijective map with inverse $\omega_{T,S,R,M}$.*

Proof: By applying [Lemma 4.12](#) twice we see that both $\omega_S := \omega_{S,T,R,M}$ and $\omega_T := \omega_{T,S,R,M}$ are well-defined. Let $(s, m) \in S^{-1}M$, then

$$\omega_T(\omega_S((s, m))) = \omega_T((ws, wm)) = (vws, vwm),$$

where $v, w \in R$ satisfy $ws \in T$ and $vws \in S$. Since $1 \cdot vws = vw \cdot s$ and $1 \cdot vwm = vw \cdot m$, we find that $(vws, vwm) = (s, m)$ in $S^{-1}M$ and thus $\omega_T \circ \omega_S = \text{id}_{S^{-1}M}$. By an analogous calculation, we have $\omega_S \circ \omega_T = \text{id}_{T^{-1}M}$. $\square$

As a direct consequence, we get the symmetric version of [Proposition 4.14](#):

Proposition 4.16 (cf. [Proposition 7.18](#)). *Let S and T be left denominator sets in a monoid R . The following are equivalent:*

- (1) $\text{LSat}(S) = \text{LSat}(T)$.
- (2) *There exists some R -fixing monoid isomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$.*
- (3) *There exists a unique R -fixing monoid isomorphism from $S^{-1}R$ to $T^{-1}R$ which is given by*

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ satisfies $ws \in T$, with inverse $\omega_{T,S,R}$.

$$\begin{array}{ccc} & R & \\ \rho_{S,R} \swarrow & & \searrow \rho_{T,R} \\ S^{-1}R & \xrightarrow[\omega_{T,S,R}]{\omega_{S,T,R}} & T^{-1}R \end{array}$$

Proof: (3) $\Rightarrow$ (2): This is obvious.

(1) $\Rightarrow$ (3): Since $S \subseteq \text{LSat}(S) = \text{LSat}(T)$ we have that $\omega := \omega_{S,T,R}$ is the unique R -fixing homomorphism from $S^{-1}R$ to $T^{-1}R$ by [Proposition 4.14](#), which is also bijective by [Lemma 4.15](#).

(2) $\Rightarrow$ (1): We have $S \subseteq \text{LSat}(T)$ by [Proposition 4.14](#). Since $\varphi^{-1} : T^{-1}R \rightarrow S^{-1}R$ is an R -fixing isomorphism by [Lemma 4.11](#) we can apply [Proposition 4.14](#) to φ^{-1} to get $T \subseteq \text{LSat}(S)$. Together, these two inclusions imply $\text{LSat}(S) = \text{LSat}(T)$ by [Lemma 3.9](#). $\square$

Remark 4.17. Let S and T be left denominator sets in a monoid R with $S \subseteq T$. Then we have $\omega_{S,T,R}((s, r)) = (s, r)$ for all $(s, r) \in S^{-1}R$. In particular, $\rho_{S,R} = \omega_{[1],S,R}$ if we identify R with $[1]^{-1}R$. $\diamond$

Finally, we show how a homomorphism between potentially different monoids can be lifted to a homomorphism of localizations:

Lemma 4.18 (Lifting, cf. [Lemma 7.22](#)). *Let $\phi : R \rightarrow T$ a homomorphism of monoids, S a left denominator set in R , and $\phi(S)$ a left denominator set in T . Consider*

$$\Phi : S^{-1}R \rightarrow \phi(S)^{-1}T, \quad (s, r) \mapsto (\phi(s), \phi(r)).$$

- (a) *The map Φ is a homomorphism of monoids.*
- (b) *We have $\Phi \circ \rho_{S,R} = \rho_{\phi(S),T} \circ \phi$.*
- (c) *If ϕ is surjective, then Φ is surjective.*
- (d) *If ϕ is bijective, then Φ is bijective.*

$$\begin{array}{ccc} R & \xrightarrow{\phi} & T \\ \rho_{S,R} \downarrow & & \downarrow \rho_{\phi(S),T} \\ S^{-1}R & \xrightarrow{\Phi} & \phi(S)^{-1}T \end{array}$$

Proof: (a) Let $a := (s_1, r_1), b := (s_2, r_2) \in S^{-1}R$.

Well-definedness: Let $a = b$ in $S^{-1}R$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}r_2 = \tilde{r}r_1$. But then $\phi(\tilde{s}) \in \phi(S)$ and $\phi(\tilde{r}) \in T$ satisfy

$$\phi(\tilde{s})\phi(s_2) = \phi(\tilde{s}s_2) = \phi(\tilde{r}s_1) = \phi(\tilde{r})\phi(s_1)$$

and

$$\phi(\tilde{s})\phi(r_2) = \phi(\tilde{s}r_2) = \phi(\tilde{r}r_1) = \phi(\tilde{r})\phi(r_1),$$

which shows that $\Phi((s_1, r_1)) = (\phi(s_1), \phi(r_1)) = (\phi(s_2), \phi(r_2)) = \Phi((s_2, r_2))$.

Neutral element: We have

$$\Phi(1_{S^{-1}R}) = \Phi((1_R, 1_R)) = (\phi(1_R), \phi(1_R)) = (1_T, 1_T) = 1_{\phi(S)^{-1}T}.$$

Multiplicativity: We have

$$\Phi(a \cdot b) = \Phi((s_1, r_1) \cdot (s_2, r_2)) = \Phi((s_a s_1, r_a r_2)) = (\phi(s_a s_1), \phi(r_a r_2))$$

and

$$\begin{aligned} \Phi(a) \cdot \Phi(b) &= \Phi((s_1, r_1)) \cdot \Phi((s_2, r_2)) \\ &= (\phi(s_1), \phi(r_1)) \cdot (\phi(s_2), \phi(r_2)) \\ &= (\phi(s_b)\phi(s_1), t_b\phi(r_2)), \end{aligned}$$

where $s_a, s_b \in S$, $r_a \in R$ and $t_b \in T$ satisfy

$$s_a r_1 = r_a s_2, \tag{K1}$$

$$\phi(s_b)\phi(r_1) = t_b\phi(s_2). \tag{K2}$$

We have to show $(\phi(s_a)\phi(s_1), \phi(r_a)\phi(r_2)) = (\phi(s_b)\phi(s_1), t_b\phi(r_2))$. By the left Ore condition on $\phi(S)$ there exist $s_c \in S$ and $t_c \in T$ such that

$$\phi(s_c)\phi(s_b)\phi(s_1) = t_c\phi(s_a)\phi(s_1). \quad (\text{K3})$$

By left reversibility of $\phi(S)$ there exists $s_d \in S$ such that

$$\phi(s_d)\phi(s_c)\phi(s_b) = \phi(s_d)t_c\phi(s_a). \quad (\text{K4})$$

Again by left reversibility of $\phi(S)$,

$$\begin{aligned} \phi(s_d)\phi(s_c)t_b\phi(s_2) &\stackrel{(\text{K2})}{=} \phi(s_d)\phi(s_c)\phi(s_b)\phi(r_1) \stackrel{(\text{K4})}{=} \phi(s_d)t_c\phi(s_a)\phi(r_1) = \phi(s_d)t_c\phi(s_ar_1) \\ &\stackrel{(\text{K1})}{=} \phi(s_d)t_c\phi(r_as_2) = \phi(s_d)t_c\phi(r_a)\phi(s_2) \end{aligned}$$

implies the existence of $s_e \in S$ such that

$$\phi(s_e)\phi(s_d)\phi(s_c)t_b = \phi(s_e)\phi(s_d)t_c\phi(r_a). \quad (\text{K5})$$

Define $\mathring{s} := s_es_ds_c \in S$ and $\mathring{t} := \phi(s_e)\phi(s_d)t_c \in T$, then

$$\phi(\mathring{s})\phi(s_b)\phi(s_1) = \phi(s_e)\phi(s_d)\phi(s_c)\phi(s_b)\phi(s_1) \stackrel{(\text{K3})}{=} \phi(s_e)\phi(s_d)t_c\phi(s_a)\phi(s_1) = \mathring{t}\phi(s_a)\phi(s_1)$$

and

$$\phi(\mathring{s})t_b\phi(r_2) = \phi(s_e)\phi(s_d)\phi(s_c)t_b\phi(r_2) \stackrel{(\text{K5})}{=} \phi(s_e)\phi(s_d)t_c\phi(r_a)\phi(r_2) = \mathring{t}\phi(r_a)\phi(r_2)$$

show that $(\phi(s_a)\phi(s_1), \phi(r_a)\phi(r_2)) = (\phi(s_b)\phi(s_1), t_b\phi(r_2))$.

(b) Let $r \in R$, then

$$\Phi(\rho_{S,R}(r)) = \Phi((1, r)) = (\phi(1), \phi(r)) = (1, \phi(r)) = \rho_{\phi(S), T}(\phi(r)).$$

(c) Let $(\phi(s), t) \in \phi(S)^{-1}T$. If ϕ is surjective, then there exists $r \in R$ such that $\phi(r) = t$. Thus $(\phi(s), t) = (\phi(s), \phi(r)) = \Phi((s, r)) \in \text{im}(\Phi)$, hence Φ is surjective.

(d) It remains to show injectivity of Φ : let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ such that

$$(\phi(s_1), \phi(r_1)) = \Phi((s_1, r_1)) = \Phi((s_2, r_2)) = (\phi(s_2), \phi(r_2)),$$

then there exist $\tilde{s} \in S$ and $\tilde{t} \in T$ such that $\phi(\tilde{s})\phi(s_2) = \tilde{t}\phi(s_1)$ and $\phi(\tilde{s})\phi(r_2) = \tilde{t}\phi(r_1)$. By surjectivity of ϕ there exists $\hat{r} \in R$ such that $\phi(\hat{r}) = \tilde{t}$. Then

$$\phi(\tilde{s}s_2) = \phi(\tilde{s})\phi(s_2) = \tilde{t}\phi(s_1) = \phi(\hat{r})\phi(s_1) = \phi(\hat{r}s_1)$$

implies $\tilde{s}s_2 = \hat{r}s_1$ by injectivity of ϕ . Similarly, we have $\phi(\tilde{s}r_2) = \phi(\hat{r}r_1)$, which implies $\tilde{s}r_2 = \hat{r}r_1$. Together, this shows $(s_1, r_1) = (s_2, r_2)$, thus Φ is injective. $\square$

Example 4.19 (cf. [Example 7.23](#)). Consider again the bicyclic monoid $R := \langle p, q \mid pq = 1 \rangle$, which contains the free monoid $\langle p \rangle := \{p^k \mid k \in \mathbb{N}_0\}$ as a submonoid, thus we can consider the canonical embedding $\phi : \langle p \rangle \rightarrow R$, $r \mapsto r$. Since $\langle p \rangle$ is Abelian, $S := [p]$ is a left denominator set in $\langle p \rangle$. From [Example 2.44](#) we know that $S = \phi(S)$ is also a left denominator set in R . By [Lemma 4.18](#), the map

$$\Phi : S^{-1}\langle p \rangle \rightarrow S^{-1}R, \quad (s, r) \mapsto (\phi(s), \phi(r)) = (s, r)$$

is a homomorphism of monoids.

Surjectivity: Let $(p^l, q^a p^b) \in S^{-1}R$, then $(p^l, q^a p^b) = (p^a p^l, p^a q^a p^b) = (p^{a+l}, p^b) = \Phi((p^{a+l}, p^b))$.

Injectivity: Let $(p^{l_1}, p^{b_1}), (p^{l_2}, p^{b_2}) \in S^{-1}\langle p \rangle$ such that

$$(p^{l_1}, p^{b_1}) = \Phi((p^{l_1}, p^{b_1})) = \Phi((p^{l_2}, p^{b_2})) = (p^{l_2}, p^{b_2}),$$

then there exist $l, a, b \in \mathbb{N}_0$ such that

$$p^l p^{l_1} = q^a p^b p^{l_2} \quad \text{and} \quad p^l p^{b_1} = q^a p^b p^{b_2}.$$

Multiplying both equations on the left with p^a yields

$$p^{a+l} p^{l_1} = p^b p^{l_2} \quad \text{and} \quad p^{a+l} p^{b_1} = p^b p^{b_2},$$

thus $(p^{l_1}, p^{b_1}) = (p^{l_2}, p^{b_2})$ in $S^{-1}\langle p \rangle$.

Thus, $S^{-1}R$ is isomorphic to $S^{-1}\langle p \rangle$, which is the free group $\langle p, p^{-1} \rangle = \{p^k \mid k \in \mathbb{Z}\}$ and thus isomorphic to the additive group of the integers. $\diamond$

In particular, the localization of a non-commutative monoid can be a non-trivial Abelian group.

4.3. Iterated localizations

If $S \subseteq T$ is an inclusion of left denominator sets, then localizing $S^{-1}R$ at the image of T yields $T^{-1}R$. This also holds under the weaker requirement $S \subseteq \text{LSat}(T)$:

Lemma 4.20 (cf. [Lemma 7.24](#)). *Let S, T be two left denominator sets in a monoid R with $S \subseteq \text{LSat}(T)$. Then $\rho_{S,R}(T)$ is a left Ore set in $S^{-1}R$.*

Proof: Let $\rho := \rho_{S,R}$. Then $\rho(T)$ is a multiplicative set in $S^{-1}R$ by [Lemma 2.38](#).

Left Ore condition: Let $t \in T$, $s \in S$ and $r \in R$, we have to show that there exist $\mathring{t} \in T$, $\mathring{s} \in S$ and $\mathring{r} \in R$ such that $(1, \mathring{t}) \cdot (s, r) = (\mathring{s}, \mathring{r}) \cdot (1, t)$. Since $s \in S \subseteq \text{LSat}(T)$ there exists $w \in R$ such that $ws \in T$. By the left Ore condition on T there exist $\tilde{t} \in T$ and $\tilde{r} \in R$ such that $\tilde{t}wr = \tilde{r}t$. Now let $\mathring{t} := \tilde{t}ws \in T$, $\mathring{s} := 1 \in S$ and $\mathring{r} := \tilde{r} \in R$, then

$$(1, \mathring{t}) \cdot (s, r) = (1, \tilde{t}ws) \cdot (s, r) = (1, \tilde{t}wr) = (1, \tilde{r}t) = (1, \tilde{r}) \cdot (1, t) = (\mathring{s}, \mathring{r}) \cdot (1, t)$$

Left reversibility: Let $t \in T$, $s_1, s_2 \in S$ and $r_1, r_2 \in R$ such that $(s_1, r_1) \cdot (1, t) = (s_2, r_2) \cdot (1, t)$, we have to show that there exists $\tilde{t} \in T$ such that $(1, \tilde{t}) \cdot (s_1, r_1) = (1, \tilde{t}) \cdot (s_2, r_2)$. Since $(s_1, r_1 t) = (s_2, r_2 t)$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} s_1 = \tilde{r} s_2$ and $\tilde{s} r_1 t = \tilde{r} r_2 t$. Since $\tilde{s} s_1 \in S \subseteq \text{LSat}(T)$ there exists $w \in R$ such that $w \tilde{s} s_1 \in T$. Now $w \tilde{s} r_1 t = w \tilde{r} r_2 t$ and by left reversibility of T there exists $\tilde{t} \in T$ such that $\tilde{t} w \tilde{s} r_1 = \tilde{t} w \tilde{r} r_2$. Now let $\tilde{t} := \tilde{t} w \tilde{s} s_1 \in T$, then

$$\begin{aligned} (1, \tilde{t}) \cdot (s_1, r_1) &= (1, \tilde{t} w \tilde{s} s_1) \cdot (s_1, r_1) = (1, \tilde{t} w \tilde{s} r_1) = (1, \tilde{t} w \tilde{r} r_2) \\ &= (1, \tilde{t} w \tilde{r} s_2) \cdot (s_2, r_2) = (1, \tilde{t} w \tilde{s} s_1) \cdot (s_2, r_2) = (1, \tilde{t}) \cdot (s_2, r_2). \end{aligned} \quad \square$$

Proposition 4.21 (cf. **Proposition 7.25**). *Let S and T be two left denominator sets in a monoid R . The following are equivalent:*

- (1) $S \subseteq \text{LSat}(T)$.
- (2) *There exists some monoid homomorphism $\psi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R$ satisfying $\psi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{T,R}$.*
- (3) *There exists a unique monoid isomorphism $\varphi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R$ satisfying $\varphi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{T,R}$ which is given by*

$$\varphi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R, \quad ((1, t), (s, r)) \mapsto (wst, wr).$$

$$\begin{array}{ccccc} & & R & & \\ & \swarrow \rho_{S,R} & & \searrow \rho_{T,R} & \\ S^{-1}R & & & & T^{-1}R \\ & \xrightarrow{\omega_{S,T,R}} & & & \\ & \searrow \rho_{\rho_{S,R}(T), S^{-1}R} & & \nearrow \varphi \text{ resp. } \psi & \\ & & \rho_{S,R}(T)^{-1}(S^{-1}R) & & \end{array}$$

Proof: (3) $\Rightarrow$ (2): This is obvious.

(1) $\Rightarrow$ (3): Let $L := (\rho_{S,R}(T))^{-1}(S^{-1}R)$, $a_1 := ((1, t_1), (s_1, r_1))$, $a_2 := ((1, t_2), (s_2, r_2)) \in L$ and $\omega := \omega_{S,T,R}$.

Well-definedness: Let $a_1 = a_2$, then there exist $\tilde{t} \in T$, $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$(1, \tilde{t} t_1) = (1, \tilde{t}) \cdot (1, t_1) = (\tilde{s}, \tilde{r}) \cdot (1, t_2) = (\tilde{s}, \tilde{r} t_2),$$

and

$$(1, \tilde{t}) \cdot (s_1, r_1) = (\tilde{s}, \tilde{r}) \cdot (s_2, r_2).$$

Now

$$\begin{aligned}
 (w_1 s_1 t_1, w_1 r_1) &= (t_1, 1) \cdot (w_1 s_1, w_1 r_1) = (t_1, 1) \cdot \omega(s_1, r_1) = (t_1, 1) \cdot (\tilde{t}, 1) \cdot (1, \tilde{t}) \cdot \omega((s_1, r_1)) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((1, \tilde{t})) \cdot \omega((s_1, r_1)) = (\tilde{t} t_1, 1) \cdot \omega((1, \tilde{t}) \cdot (s_1, r_1)) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r}) \cdot (s_2, r_2)) = (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot \omega((s_2, r_2)) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot (w_2 s_2, w_2 r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot (1, t_2) \cdot (t_2, 1) \cdot (w_2 s_2, w_2 r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot \omega((1, t_2)) \cdot (w_2 s_2 t_2, w_2 r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r}) \cdot (1, t_2)) \cdot (w_2 s_2 t_2, w_2 r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((1, \tilde{t} t_1)) \cdot (w_2 s_2 t_2, w_2 r_2) = (\tilde{t} t_1, 1) \cdot (1, \tilde{t} t_1) \cdot (w_2 s_2 t_2, w_2 r_2) \\
 &= (w_2 s_2 t_2, w_2 r_2).
 \end{aligned}$$

Neutral element/commuting diagram: Let $r \in R$. Since $1_R \in T$ we have

$$\begin{aligned}
 (\varphi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R})(r) &= \varphi(\rho_{\rho_{S,R}(T), S^{-1}R}(\rho_{S,R}(r))) = \varphi(\rho_{\rho_{S,R}(T), S^{-1}R}(1_R, r)) \\
 &= \varphi(((1_R, 1_R), (1_R, r))) = (1_R, r) = \rho_{T,R}(r).
 \end{aligned}$$

In particular, we have

$$\begin{aligned}
 \varphi(1_L) &= \varphi(1_{S^{-1}R}, 1_{S^{-1}R}) = \varphi(\rho_{\rho_{S,R}(T), S^{-1}R}(1_{S^{-1}R})) = \varphi(\rho_{\rho_{S,R}(T), S^{-1}R}(\rho_{S,R}(1_R))) \\
 &= (\varphi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R})(1_R) = \rho_{T,R}(1_R) = 1_{T^{-1}R}.
 \end{aligned}$$

Multiplicativity: We have

$$\begin{aligned}
 a_1 \cdot a_2 &= ((1, t_1), (s_1, r_1)) \cdot ((1, t_2), (s_2, r_2)) \\
 &= ((1, \tilde{t}) \cdot (1, t_1), (\tilde{s}, \tilde{r}) \cdot (s_2, r_2)) = ((1, \tilde{t} t_1), (\hat{s} \tilde{s}, \hat{r} r_2)),
 \end{aligned}$$

where $\tilde{t} \in T$, $\tilde{s}, \hat{s} \in S$ and $\tilde{r}, \hat{r} \in R$ satisfy

$$(1, \tilde{t}) \cdot (s_1, r_1) = (\tilde{s}, \tilde{r}) \cdot (1, t_2) \quad \text{and} \quad \hat{s} \tilde{r} = \hat{r} s_2.$$

Let $w, w_1, w_2 \in R$ such that $w \hat{s} \tilde{s}, w_1 s_1, w_2 s_2 \in T$, then

$$\begin{aligned}
 \varphi(a_1 \cdot a_2) &= \varphi(((1, \tilde{t} t_1), (\hat{s} \tilde{s}, \hat{r} r_2))) = (w \hat{s} \tilde{s} \tilde{t} t_1, w \hat{r} r_2) = (\tilde{t} t_1, 1) \cdot (w \hat{s} \tilde{s}, w \hat{r} r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\hat{s} \tilde{s}, \hat{r} r_2)) = (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r}) \cdot (s_2, r_2)) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot \omega((s_2, r_2)) = (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot (1, t_2) \cdot (t_2, 1) \cdot \omega((s_2, r_2)) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r})) \cdot \omega((1, t_2)) \cdot (t_2, 1) \cdot (w_2 s_2, w_2 r_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((\tilde{s}, \tilde{r}) \cdot (1, t_2)) \cdot (w_2 s_2 t_2, w_2 r_2) = (\tilde{t} t_1, 1) \cdot \omega((1, \tilde{t}) \cdot (s_1, r_1)) \cdot \varphi(a_2) \\
 &= (\tilde{t} t_1, 1) \cdot \omega((1, \tilde{t})) \cdot \omega((s_1, r_1)) \cdot \varphi(a_2) = (\tilde{t} t_1, 1) \cdot (1, \tilde{t}) \cdot (w_1 s_1, w_1 r_1) \cdot \varphi(a_2) \\
 &= (w_1 s_1 t_1, w_1 r_1) \cdot \varphi(a_2) = \varphi(a_1) \cdot \varphi(a_2).
 \end{aligned}$$

Injectivity: Let $\varphi(a_1) = \varphi(a_2)$ and $w_1, w_2 \in R$ such that $w_1 s_1, w_2 s_2 \in T$, then

$$(w_1 s_1 t_1, w_1 r_1) = \varphi(a_1) = \varphi(a_2) = (w_2 s_2 t_2, w_2 r_2)$$

implies the existence of $\tilde{t} \in T$ and $\tilde{r} \in R$ such that

$$\tilde{t}w_1s_1t_1 = \tilde{r}w_2s_2t_2 \quad \text{and} \quad \tilde{t}w_1r_1 = \tilde{r}w_2r_2.$$

Now let $\mathring{t} := \tilde{t}w_1s_1 \in T$, $\mathring{s} := 1 \in S$ and $\mathring{r} := \tilde{r}w_2s_2 \in R$, then $a_1 = a_2$ follows from

$$(1, \mathring{t}) \cdot (1, t_1) = (1, \mathring{t}t_1) = (1, \tilde{t}w_1s_1t_1) = (1, \tilde{r}w_2s_2t_2) = (1, \mathring{r}t_2) = (\mathring{s}, \mathring{r}) \cdot (1, t_2)$$

and

$$\begin{aligned} (1, \mathring{t}) \cdot (s_1, r_1) &= (1, \tilde{t}w_1s_1) \cdot (s_1, r_1) = (1, \tilde{t}w_1) \cdot (1, s_1) \cdot (s_1, 1) \cdot (1, r_1) \\ &= (1, \tilde{t}w_1) \cdot (1, r_1) = (1, \tilde{t}w_1r_1) = (1, \tilde{r}w_2r_2) = (1, \tilde{r}w_2) \cdot (1, r_2) \\ &= (1, \tilde{r}w_2) \cdot (1, s_2) \cdot (s_2, 1) \cdot (1, r_2) = (1, \tilde{r}w_2s_2) \cdot (s_2, r_2) = (\mathring{s}, \mathring{r}) \cdot (s_2, r_2). \end{aligned}$$

Surjectivity: Let $(t, r) \in T^{-1}R$, then $(t, r) = \varphi((1, t), (1, r)) \in \varphi(L)$, therefore φ is surjective.

Uniqueness: Let $\psi : L \rightarrow T^{-1}R$ be a monoid isomorphism satisfying $\psi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{T,R}$ and $((1, t), (s, r)) \in L$, then there exists $w \in R$ such that $ws \in T$. Now $(ws, w) \in T^{-1}R$ is the inverse of $(1, s) = \rho_{T,R}(s)$ in $T^{-1}R$. Furthermore, we have

$$(1_{S^{-1}R}, (s, 1_R)) = (1_{S^{-1}R}, (1_R, s))^{-1} \quad \text{and} \quad ((1_R, t), 1_{S^{-1}R}) = (1_{S^{-1}R}, (1_R, t))^{-1},$$

thus

$$\begin{aligned} \psi(((1_R, t), (s, r)))) &= \psi(((1_R, t), 1_{S^{-1}R})) \cdot \psi((1_{S^{-1}R}, (s, 1_R))) \cdot \psi((1_{S^{-1}R}, (1_R, r))) \\ &= \psi((1_{S^{-1}R}, (1_R, t))^{-1}) \cdot \psi((1_{S^{-1}R}, (1_R, s))^{-1}) \cdot (1_R, r) \\ &= \psi((1_{S^{-1}R}, (1_R, t)))^{-1} \cdot \psi((1_{S^{-1}R}, (1_R, s)))^{-1} \cdot (1_R, r) \\ &= (1_R, t)^{-1} \cdot (1_R, s)^{-1} \cdot (1_R, r) \\ &= (t, 1_R) \cdot (ws, w) \cdot (1_R, r) \\ &= (wst, wr) = \varphi(((1_R, t), (s, r))). \end{aligned} \quad \square$$

(2) $\Rightarrow$ (1): Let $\rho_\rho := \rho_{\rho_{S,R}(T), S^{-1}R}$, $\rho_S := \rho_{S,R}$, and $\rho_T := \rho_{T,R}$. Let $s \in S$, then there exist $t \in T$ and $r \in R$ such that $\psi(\rho_\rho((s, 1))) = (t, r)$. Now in $T^{-1}R$ we have

$$\begin{aligned} (t, r) \cdot (1, s) &= \psi(\rho_\rho((s, 1))) \cdot \rho_T(s) = \psi(\rho_\rho((s, 1))) \cdot \psi(\rho_\rho(\rho_S(s))) \\ &= \psi(\rho_\rho((s, 1) \cdot \rho_S(s))) = \psi(\rho_\rho(1)) = 1, \end{aligned}$$

thus $(1, s) \in U_l(T^{-1}R)$ and therefore $s \in \text{LSat}(T)$ by [Proposition 4.9](#).

4.4. The canonical form

While $\text{LSat}(S)$ determines the isomorphism class of $S^{-1}R$ under R -fixing homomorphisms, we would like $\text{LSat}(S)$ to be a left denominator set as well:

Proposition 4.22 (cf. [Proposition 7.27](#)). *Let S be a left denominator set in a monoid R . If $\text{LSat}(S)$ is a left denominator set in R , then $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.*

Proof: Follows from [Proposition 4.16](#), since $\text{LSat}(S) = \text{LSat}(\text{LSat}(S))$. $\square$

Important in this context is the following notion:

Definition 4.23. A monoid R is called *Dedekind-finite* if for all $a, b \in R$, $a \cdot b = 1$ implies $b \cdot a = 1$. The monoids that are not Dedekind-finite are sometimes called *Dedekind-infinite*. $\diamond$

Lemma 4.24 (cf. [Lemma 7.29](#)). *Let $\varphi : R \rightarrow T$ be an injective homomorphism of monoids. If T is Dedekind-finite, then so is R .*

Proof: Let $a, b \in R$ such that $a \cdot b = 1$, then $\varphi(a) \cdot \varphi(b) = \varphi(a \cdot b) = \varphi(1) = 1$. Since T is Dedekind-finite, we have $\varphi(b \cdot a) = \varphi(b) \cdot \varphi(a) = 1 = \varphi(1)$. Since φ is injective, we have $b \cdot a = 1$, thus R is Dedekind-finite. $\square$

By construction, Dedekind-finite monoids have saturated unit groups:

Lemma 4.25 (cf. [Lemma 7.30](#)). *Let R be a monoid. Then the following are equivalent:*

- (1) $U(R)$ is saturated.
- (2) R is Dedekind-finite.
- (3) $U_l(R) = U(R) = U_r(R)$.

Proof: (1) $\Rightarrow$ (2): Let $a, b \in R$ such that $a \cdot b = 1 \in U(R)$. Since $U(R)$ is saturated, we have $a, b \in U(R)$, thus $b \cdot a = 1$ by uniqueness of inverses.

(2) $\Rightarrow$ (3): Let $a \in U_l(R)$, then there exists $b \in R$ such that $b \cdot a = 1$. Since R is Dedekind-finite we have $a \cdot b = 1$, thus $a \in U(R)$, which implies $U_l(R) = U(R)$. The other equality follows analogously.

(3) $\Rightarrow$ (1): Let $a, b \in R$ such that $a \cdot b \in U(R)$, then $u \cdot a \cdot b = 1 = a \cdot b \cdot u$ for some $u \in U(R)$, thus $b \in U_l(R) = U(R)$ and $a \in U_r(R) = U(R)$. $\square$

This allows the following improvements of previous results:

Lemma 4.26 (cf. [Lemma 7.31](#)). *Let R be a Dedekind-finite monoid, then $\text{LSat}([1]) = U(R)$.*

Proof: Follows from [Lemma 4.4](#) and [Lemma 4.25](#). $\square$

Proposition 4.27 (cf. [Proposition 7.32](#)). *Let S be a left denominator set in a monoid R such that $S^{-1}R$ is Dedekind-finite and $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) $(s, r) \in U(S^{-1}R)$.
- (2) $(1, r) \in U(S^{-1}R)$.
- (3) $r \in \text{LSat}(S)$.

Furthermore, if $(s, r) \in U(S^{-1}R)$, then the inverse of (s, r) in $S^{-1}R$ is given by (wr, ws) , where $w \in R$ satisfies $wr \in S$.

Proof: Since $S^{-1}R$ is Dedekind-finite, we have $U_l(S^{-1}R) = U(S^{-1}R)$, thus the statement follows from [Proposition 4.9](#). $\square$

This leads into our main result, which characterizes left denominator sets with an agreeable left saturation closure:

Proposition 4.28 (cf. [Proposition 7.33](#)). *Let S be a left denominator set in a monoid R . Then the following are equivalent:*

- (1) $S^{-1}R$ is Dedekind-finite.
- (2) $\text{LSat}(S)$ is left reversible in R .
- (3) $\text{LSat}(S)$ is a left denominator set in R .
- (4) $\text{LSat}(S)$ is saturated in R .

If the equivalent conditions are satisfied, then we also have $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.

Proof: We prove the equivalence first:

- (1) $\Rightarrow$ (2): Let $\rho := \rho_{S,R}$ and consider $r_1, r_2 \in R$ as well as $x \in \text{LSat}(S)$ such that $r_1x = r_2x$. Since $S^{-1}R$ is Dedekind-finite, we have $\rho(x) \in U_l(S^{-1}R) = U(S^{-1}R)$, thus

$$\rho(r_1)\rho(x) = \rho(r_1x) = \rho(r_2x) = \rho(r_2)\rho(x)$$

implies $\rho(r_1) = \rho(r_2)$. But then there exists $s \in S \subseteq \text{LSat}(S)$ such that $sr_1 = sr_2$ by [Lemma 2.28](#).

- (2) $\Rightarrow$ (3): By [Lemma 4.7](#), $\text{LSat}(S)$ is a left Ore set in R , thus $\text{LSat}(S)$ is a left denominator set in R .

- (3) $\Rightarrow$ (4): Since $\text{LSat}(S)$ is a left denominator set, the localization $\text{LSat}(S)^{-1}R$ and its localization map $\rho_L := \rho_{\text{LSat}(S),R}$ both exist. Let $a, b \in R$ such that $ab \in \text{LSat}(S)$. Since $\text{LSat}(S)$ is left saturated by construction, we have $b \in \text{LSat}(S)$. Now $\rho_L(ab), \rho_L(b) \in U(\text{LSat}(S)^{-1}R)$ by construction of $\text{LSat}(S)^{-1}R$, thus

$$\rho_L(a) = \rho_L(ab)\rho_L(b)^{-1} \in U(\text{LSat}(S)^{-1}R) \subseteq U_l(\text{LSat}(S)^{-1}R),$$

which implies $a \in \text{LSat}(\text{LSat}(S)) = \text{LSat}(S)$ by [Proposition 4.9](#) and [Lemma 3.8](#).

- (4) $\Rightarrow$ (1): Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ such that $1 = (s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$. Since $(\tilde{s}s_1, \tilde{r}r_2) = 1 \in U_l(S^{-1}R)$, we have $\tilde{r}r_2 \in \text{LSat}(S)$ by [Proposition 4.9](#). Now $\text{LSat}(S)$ is saturated by assumption, thus $\tilde{r} \in \text{LSat}(S)$, which implies that there exists $w \in R$ such that $w\tilde{r} \in S$. From $w\tilde{s}r_1 = w\tilde{r}s_2 \in S$ we get that $r_1 \in \text{LSat}(S)$, therefore again by [Proposition 4.9](#) we have $(s_1, r_1) \in U_l(S^{-1}R)$. Since $(s_1, r_1) \in U_r(S^{-1}R)$ holds by assumption, (s_1, r_1) is a unit in $S^{-1}R$, but then $(s_2, r_2) \cdot (s_1, r_1) = 1$ by uniqueness of inverses.

Lastly, if $\text{LSat}(S)$ is a left denominator set in R , then $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$ by [Proposition 4.22](#). $\square$

Note that if the localization is Dedekind-finite, this alone has no consequence for the base ring, which can be Dedekind-infinite:

Example 4.29 (cf. [Example 7.34](#)). Consider again the bicyclic monoid $R := \langle p, q \mid pq = 1 \rangle$. By construction, R is not Dedekind-finite, since $qp \neq 1 = pq$. From [Example 4.19](#) we know that the localization of R at $[p]$ is isomorphic to the free group $\langle p, p^{-1} \rangle = \{p^k \mid k \in \mathbb{Z}\}$, which is Abelian and thus Dedekind-finite. According to [Lemma 4.24](#), this implies that the localization map $\rho := \rho_{[p], R}$ is not injective, which can also be seen from

$$\rho(qp) = (1, qp) = (p, pqp) = (p, p) = 1 = \rho(1). \quad \diamond$$

For us it is an open question whether all localizations of a Dedekind-finite monoid are again Dedekind-finite.

4.5. Localization of monoid domains

To prepare for the discussion on localization of rings that are also domains, we consider a corresponding non-standard concept for monoids: left and right monoid domains. Similar to the discussion on left and right zeros leading up to [Lemma 2.33](#), we will gain an insight into which properties of a ring R being a domain are responsible for which results.

Since we do not have a zero in an arbitrary monoid, we have to find a different defining property for a monoid domain:

Definition 4.30. Let R be a monoid. An element $r \in R$ is called

- *left cancellative* in R if for all $a, b \in R$, $r \cdot a = r \cdot b$ implies $a = b$. A subset of R is called *left cancellative* if all of its elements are left cancellative in R .
- *right cancellative* in R if for all $a, b \in R$, $a \cdot r = b \cdot r$ implies $a = b$. A subset of R is called *right cancellative* if all of its elements are right cancellative in R . $\diamond$

Note that a left/right invertible element is also left/right cancellative. Furthermore, cancellativity can be related to reversibility:

Lemma 4.31. Let S be a subset of a monoid R .

- (a) If S is right cancellative in R , then S is left reversible in R . In particular, a right cancellative left Ore set is a left denominator set.
- (b) If S is left cancellative and left reversible, then S is right cancellative. In particular, a left cancellative left denominator set is right cancellative.

Proof: (a) Let $s \in S$ and $r_1, r_2 \in R$ such that $r_1 s = r_2 s$, then $r_1 = r_2$ by right cancellativity of S , thus $sr_1 = sr_2$ shows that S is left reversible.

- (b) Let $s \in S$ and $r_1, r_2 \in R$ such that $r_1 s = r_2 s$. Since S is left reversible there exists $\tilde{s} \in S$ such that $\tilde{s} r_1 = \tilde{s} r_2$. Since S is left cancellative, this implies $r_1 = r_2$, thus S is right cancellative in R . $\square$

Definition 4.32. A monoid R is called

- a *left monoid domain* if $R \neq \{1\}$ and $R \setminus N_l(R)$ is left cancellative in R .
- a *right monoid domain* if $R \neq \{1\}$ and $R \setminus N_r(R)$ is right cancellative in R . $\diamond$

This definition has the advantage that it applies to all monoids. In [Lemma 7.38](#) we will see that for a ring R , either one of these properties is equivalent to R being a domain. Note that for a left or right monoid domain with zero 0 , we have the classical domain property: $ab = 0$ if and only if $a = 0$ or $b = 0$. But the reverse is not true:

Example 4.33. Consider the *maximum monoid* $M_n := \{1, \dots, n\}$ with identity 1 , where the operation is given by $(a, b) \mapsto \max\{a, b\}$. Since M_n is commutative, we have that n is the unique zero in M_n . Furthermore, $\max\{a, b\} = n$ implies $a = n$ or $b = n$. But for $n \geq 3$, 2 is not left cancellative, since $\max\{2, 1\} = 2 = \max\{2, 2\}$, but $1 \neq 2$. In particular, M_3 is not a left monoid domain. $\diamond$

Lemma 4.34 (cf. [Lemma 7.39](#)). *Let $\varphi : R \rightarrow T$ be an injective homomorphism of monoids and $R \neq \{1\}$.*

- (a) *If T is a left monoid domain, then R is also a left monoid domain.*
- (b) *If T is a right monoid domain, then R is also a right monoid domain.*

Proof: Let $r \in R \setminus N_l(R)$ and $a, b \in R$ such that $r \cdot a = r \cdot b$. By [Lemma 2.32](#) we have that $\varphi(r) \in \varphi(R \setminus N_l(R)) \subseteq T \setminus N_l(T)$, thus $\varphi(r)$ is left cancellative in T . From the equation $\varphi(r) \cdot \varphi(a) = \varphi(r) \cdot \varphi(b)$ we get $\varphi(a) = \varphi(b)$, which implies $a = b$ by injectivity of φ . The other statement follows analogously. $\square$

Lemma 4.35 (cf. [Lemma 7.40](#)). *Any left or right monoid domain is Dedekind-finite.*

Proof: Let R be a left monoid domain and $a, b \in R$ such that $a \cdot b = 1$, then $a \cdot b \cdot a = a$. If $a \in N_l(R)$, then $1 = a \cdot b = a \in N_l(R)$ implies the contradiction $R = \{1\}$. Thus $a \notin N_l(R)$ is left cancellative and therefore $b \cdot a = 1$. The other statement follows analogously. $\square$

Lemma 4.36. *Let S be a left denominator set in a right monoid domain R . If $r \in N_r(R)$, then $(s, r) \in N_r(S^{-1}R)$ for all $s \in S$.*

Proof: Let $s, t \in S$ and $w \in R$, then $(t, w)(s, r) = (\tilde{s}t, \tilde{r}r) = (\tilde{s}t, r)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}w = \tilde{r}s$. Let $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s = \hat{r}\tilde{s}t$, then $\hat{s}r = r = \hat{r}r$ and therefore we have $(t, w)(s, r) = (\tilde{s}t, r) = (s, r)$. $\square$

Now we see the influence of R being a left or right monoid domain on its localizations:

Proposition 4.37. *Let S be a left denominator set in a monoid R .*

- (a) *If R is a right monoid domain, then $S^{-1}R$ is also a right monoid domain.*
- (b) *If S is left cancellative, then $\rho_{S,R}$ is injective. In particular, if R is a left monoid domain and $S \cap N_l(R) = \emptyset$, then $\rho_{S,R}$ is injective.*

Proof: (a) Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ and $(s_3, r_3) \in S^{-1}R \setminus N_r(S^{-1}R)$ such that

$$(s_a s_1, r_a r_3) = (s_1, r_1) \cdot (s_3, r_3) = (s_2, r_2) \cdot (s_3, r_3) = (s_b s_2, r_b r_3),$$

where $s_a, s_b \in S$ and $r_a, r_b \in R$ satisfy

$$s_a r_1 = r_a s_3, \tag{L1}$$

$$s_b r_2 = r_b s_3. \tag{L2}$$

Furthermore, there exist $s_c \in S$ and $r_c \in R$ such that

$$s_c s_b s_2 = r_c s_a s_1, \quad (\text{L3})$$

$$s_c r_b r_3 = r_c r_a r_3. \quad (\text{L4})$$

By [Lemma 4.36](#) we have $r_3 \notin N_{\tau}(R)$, thus the last equation (L4) implies

$$s_c r_b = r_c r_a \quad (\text{L5})$$

since R is a right monoid domain. Define $\mathring{s} := s_c s_b \in S$ and $\mathring{r} := r_c s_a$, then

$$\mathring{s} s_2 = s_c s_b s_2 \stackrel{(\text{L3})}{=} r_c s_a s_1 = \mathring{r} s_1$$

and

$$\mathring{s} r_2 = s_c s_b r_2 \stackrel{(\text{L2})}{=} s_c r_b s_3 \stackrel{(\text{L5})}{=} r_c r_a s_3 \stackrel{(\text{L1})}{=} r_c s_a r_1 = \mathring{r} r_1$$

show that $(s_1, r_1) = (s_2, r_2)$, thus $S^{-1}R$ is a right monoid domain.

- (b) Let $\rho := \rho_{S,R}$ and $r_1, r_2 \in R$ such that $\rho(r_1) = \rho(r_2)$. By [Definition 2.12](#) there exists $s \in S$ such that $sr_1 = sr_2$, which implies $r_1 = r_2$ by the left cancellation property of S , thus ρ is injective. $\square$

Proposition 4.38. *Let S be a left Ore set in a right monoid domain R such that $S \cap N_{\tau}(R) = \emptyset$.*

- (a) S is a left denominator set in R .
- (b) $\text{LSat}(S)$ is a saturated left denominator set in R .
- (c) $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.

Proof: (a) By assumption, S is right cancellative in R , thus S is left reversible in R by [Lemma 4.31](#).

- (b) Since R is a right monoid domain, $S^{-1}R$ is a right monoid domain by [Proposition 4.37](#) and thus Dedekind-finite by [Lemma 4.35](#). Then the claim follows from [Proposition 4.28](#).
- (c) Since both S and $\text{LSat}(S)$ are left denominator sets in R , this follows from [Proposition 4.28](#). $\square$

Only if R is both a left and a right monoid domain, we get the full picture:

Corollary 4.39 (cf. [Proposition 7.44](#)). *Let S be a left Ore set in a left and right monoid domain such that S contains neither left zeros nor right zeros.*

- (a) S is a left denominator set in R .
- (b) $S^{-1}R$ is a right monoid domain.
- (c) $\rho_{S,R}$ is injective.
- (d) $\text{LSat}(S)$ is a saturated left denominator set in R .
- (e) $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.

Part II.

Ore localization and rings

Chapter 5.

Ore localization of rings

Convention 5.1. All rings are associative and unital. $\diamond$

We will make use of the following notation:

Definition 5.2. Let R be a ring. An element $r \in R$ is called

- *left invertible* in R if there exists $a \in R$ such that $a \cdot r = 1$. The set of all left invertible elements of R is denoted $U_l(R)$.
- *right invertible* in R if there exists $a \in R$ such that $r \cdot a = 1$. The set of all right invertible elements of R is denoted $U_r(R)$.
- *invertible* or a *unit* in R if there exists $a \in R$ such that $r \cdot a = 1 = a \cdot r$. The set of all units of R is denoted $U(R)$.
- a *left zero-divisor* in R if there exists $a \in R \setminus \{0\}$ such that $r \cdot a = 0$.
- a *right zero-divisor* in R if there exists $a \in R \setminus \{0\}$ such that $a \cdot r = 0$.
- a *zero-divisor* in R if r is a left or a right zero-divisor in R .
- *regular* in R if r is not a zero divisor in R . $\diamond$

Remark 5.3. Let R be a ring.

- $U(R) = U_l(R) \cap U_r(R)$.
- $U_l(R)$ and $U_r(R)$ are submonoids of $(R, \cdot, 1)$.
- $U(R)$ is a subgroup of $(R, \cdot, 1)$. $\diamond$

Definition 5.4. Let R be a non-zero ring. Then R is called a *domain* if 0 is the only zero divisor in R , or equivalently, if whenever $a \cdot b = 0$ for $a, b \in R$, we have $a = 0$ or $b = 0$. $\diamond$

Throughout the following chapters, we will give examples to demonstrate our results, which often include the Weyl algebra:

Definition 5.5. Let K be a field. The first (*polynomial*) *Weyl algebra* over K is

$$A_1 := A_1(K) := K\langle x, \partial \mid \partial x = x\partial + 1 \rangle.$$

It can also be seen as the quotient of the free algebra $K\langle x, \partial \rangle$ and the two-sided ideal generated by $\partial x - x\partial - 1$. $\diamond$

5.1. Axiomatic definition, universal property and uniqueness

As in the commutative case, we start with the minimal requirements for the set S of our denominators-to-be: 1 should always be a denominator, and for every pair s, t of denominators, their product st should also be a denominator. In the literature, there are two predominant names for such sets: some authors call them “multiplicatively closed sets” ([KL00; MR01]), while others use the term “multiplicative set” ([Lam99; BGV03; GW04; Ško06]). We propose the following compromise:

Definition 5.6. Let S be a subset of a ring R .

- S is *multiplicatively closed* in R if for all $s_1, s_2 \in S$ we have $s_1 s_2 \in S$.
- S is a *multiplicative set* in R if S is multiplicatively closed and $1 \in S$. ◇

Now we can formalize what a localization should be:

Definition 5.7. Let S be a multiplicative set in a ring R . A pair (R_S, η) of a ring R_S and a homomorphism $\eta : R \rightarrow R_S$ is called a *left Ore localization* of R at S if the following hold:

- (i) $\eta(S) \subseteq U(R_S)$.
- (ii) For all $x \in R_S$ there exist $s \in S$ and $r \in R$ such that $x = \eta(s)^{-1} \eta(r)$.
- (iii) $\ker(\eta) = \{r \in R \mid \exists s \in S : sr = 0\}$. ◇

Remark 5.8. If $0 \in S$, then $0 = \eta(0)$ becomes a unit in R_S , thus the localization R_S implodes and becomes the zero ring. For this reason, many authors additionally require multiplicative sets to exclude 0. ◇

Purely from the axiomatic definition of localization we get the following universal property, which relates a left Ore localization with other pairs of rings and homomorphisms satisfying the first condition in [Definition 5.7](#):

Proposition 5.9 (Universal property, cf. [Proposition 2.13](#)). *Let S be a multiplicative set in a ring R and (R_S, η) a left Ore localization of R at S . Further, let $\varphi : R \rightarrow T$ be a homomorphism of rings such that $\varphi(S) \subseteq U(T)$.*

- (a) *If $r_1, r_2 \in R$ satisfy $\eta(r_1) = \eta(r_2)$, then we also have $\varphi(r_1) = \varphi(r_2)$.*
- (b) *There exists a unique ring homomorphism $\psi : R_S \rightarrow T$ such that $\varphi = \psi \circ \eta$.*

$$\begin{array}{ccc}
 R & \xrightarrow{\varphi} & T \\
 \eta \searrow & & \nearrow \psi \\
 & R_S &
 \end{array}$$

Proof: The only thing left to show is additivity of

$$\psi : R_S \rightarrow T, \quad \eta(s)^{-1} \eta(r) \mapsto \varphi(s)^{-1} \varphi(r),$$

the rest follows from **Proposition 2.13**: let $x_1 := \eta(s_1)^{-1}\eta(r_1)$, $x_2 := \eta(s_2)^{-1}\eta(r_2) \in S^{-1}R$. Since $\eta(s_2)\eta(s_1)^{-1} \in R_S$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(\tilde{s})^{-1}\eta(\tilde{r}) = \eta(s_2)\eta(s_1)^{-1}$, which is equivalent to

$$\eta(\tilde{r}s_1) = \eta(\tilde{r})\eta(s_1) = \eta(\tilde{s})\eta(s_2) = \eta(\tilde{s}s_2).$$

By (a) we have

$$\varphi(\tilde{r})\varphi(s_1) = \varphi(\tilde{r}s_1) = \varphi(\tilde{s}s_2) = \varphi(\tilde{s})\varphi(s_2),$$

which is equivalent to $\varphi(\tilde{s})^{-1}\varphi(\tilde{r}) = \varphi(s_2)\varphi(s_1)^{-1}$. Now

$$\begin{aligned} \psi(x_1 + x_2) &= \psi(\eta(s_1)^{-1}\eta(r_1) + \eta(s_2)^{-1}\eta(r_2)) \\ &= \psi(\eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\eta(r_1) + \eta(s_2)^{-1}\eta(r_2)) \\ &= \psi(\eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\eta(r_1) + \eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{s})\eta(r_2)) \\ &= \psi(\eta(s_2)^{-1}\eta(\tilde{s})^{-1}(\eta(\tilde{r})\eta(r_1) + \eta(\tilde{s})\eta(r_2))) \\ &= \psi(\eta(\tilde{s}s_2)^{-1}\eta(\tilde{r}r_1 + \tilde{s}r_2)) \\ &= \varphi(\tilde{s}s_2)^{-1}\varphi(\tilde{r}r_1 + \tilde{s}r_2) \\ &= \varphi(s_2)^{-1}\varphi(\tilde{s})^{-1}\varphi(\tilde{r})\varphi(r_1) + \varphi(s_2)^{-1}\varphi(\tilde{s})^{-1}\varphi(\tilde{s})\varphi(r_2) \\ &= \varphi(s_2)^{-1}\varphi(s_2)\varphi(s_1)^{-1}\varphi(r_1) + \varphi(s_2)^{-1}\varphi(r_2) \\ &= \varphi(s_1)^{-1}\varphi(r_1) + \varphi(s_2)^{-1}\varphi(r_2) \\ &= \psi(\eta(s_1)^{-1}\eta(r_1)) + \psi(\eta(s_2)^{-1}\eta(r_2)) \\ &= \psi(x_1) + \psi(x_2). \end{aligned}$$

□

As an immediate application of the universal property we get the uniqueness of the Ore localization:

Corollary 5.10 (Uniqueness, cf. **Corollary 2.14**). *Let S be a multiplicative set in a ring R . If there exists a left Ore localization of R at S , then it is unique up to isomorphism.*

The next task is to characterize for which sets S the localization R_S exists.

Definition 5.11. Let S be a subset of a ring R .

- S satisfies the *left Ore condition* in R if for all $s \in S$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$. The tuple $(\tilde{s}, \tilde{r})$ is sometimes called *left Ore data* for s and r .
- S is a *left Ore set* in R if it is a multiplicative set satisfying the left Ore condition.
- S is *left reversible* in R if for all $r \in R$ and $s \in S$ such that $rs = 0$ there exists $\tilde{s} \in S$ such that $\tilde{s}r = 0$.
- S is a *left denominator set* in R if S is a left reversible left Ore set in R . ◇

As the name strongly implies, we will see in **Theorem 5.26** that the localization at S exists if and only if S is a left denominator set. While the “if”-part will take a few more intermediate steps to prove, we at least get the necessary condition of S being a left denominator set immediately from **Definition 5.7**:

Lemma 5.12 (cf. **Lemma 2.16**). *Let S be a multiplicative set in a ring R and (R_S, η) a left Ore localization of R at S . Then S is a left denominator set in R .*

Note that left reversibility is a trivial condition if there are no zero-divisors present:

Lemma 5.13. *Let S be a subset of a ring R . If S contains no right zero-divisor of R , then S is left reversible in R . In particular, any left Ore set in a domain is a left denominator set.*

Proof: Let $s \in S$ and $r \in R$ such that $rs = 0$, then $r = 0$ since s is not a right zero-divisor, thus $sr = 0$ or even $1 \cdot r = 0$ shows that S is left reversible. $\square$

In commutative rings, both the left Ore condition and left reversibility are satisfied by every subset, which can be seen by setting $\tilde{s} := s$ and $\tilde{r} := r$.

5.2. Construction and existence

Remark 5.14. Note that some of the following constructions and results consider a left R -module M instead of the ring R itself. This saves us the trouble of proving essentially the same results again later when dealing with the localization of modules. However, for the remainder of this section it suffices to think of the special case $M = R$. $\diamond$

To actually construct the Ore localization at a denominator set S , we start with an equivalence relation $\sim$ on $S \times R$ which will later on determine when two tuples should represent the same “left fraction”.

Proposition 5.15 (cf. [Proposition 2.18](#)). *Let S be a left denominator set in a ring R and M a left R -module. The relation $\sim$ on $S \times M$, defined by*

$$(s_1, m_1) \sim (s_2, m_2) \quad :\Leftrightarrow \quad \exists \tilde{s} \in S, \tilde{r} \in R : \tilde{s}s_2 = \tilde{r}s_1 \text{ and } \tilde{s}m_2 = \tilde{r}m_1,$$

is an equivalence relation.

Intuitively, this means that (s_1, m_1) and (s_2, m_2) represent the same “left fraction” if and only if we can “expand” (s_1, m_1) on the left by multiplication with $\tilde{r}$ and analogously (s_2, m_2) with $\tilde{s}$, such that the two resulting representations $(\tilde{r}s_1, \tilde{r}m_1)$ and $(\tilde{s}s_2, \tilde{s}m_2)$ have the same “numerator” and “denominator”.

In support of this intuition we can see that the equivalence relation $\sim$ indeed allows expanding and reducing on the left:

Lemma 5.16 (cf. [Lemma 2.19](#)). *Let S be a left denominator set in a ring R , M a left R -module, $(s, m) \in S \times M$ and $t \in R$ such that $ts \in S$. Then $(s, m) \sim (ts, tm)$.*

The equivalence relation $\sim$ defined in [Proposition 5.15](#) is inspired by [\[Ško06\]](#). Even in the special case $M = R$, it seems to be different from

$$(s_1, r_1) \approx (s_2, r_2) \quad :\Leftrightarrow \quad \exists a, b \in R : as_2 = bs_1 \in S \text{ and } ar_2 = br_1,$$

which is the one given in many textbooks (e.g., [\[BGV03; GW04; KL00\]](#)). Note that instead of requiring $\tilde{s} \in S$, we need to include the condition that $as_2 = bs_1$ is contained in S . Nevertheless, the two relations share the same equivalence classes. Indeed, there exist several characterizations of the equivalence relation behind Ore localization:

Lemma 5.17 (cf. [Lemma 2.20](#)). *Let S be a left denominator set in a ring R , M a left R -module and $(s_1, m_1), (s_2, m_2) \in S \times M$. The following are equivalent:*

- (1) $(s_1, m_1) \sim (s_2, m_2)$, that is, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}m_2 = \tilde{r}m_1$.
- (2) For all $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ there exists $\bar{s} \in S$ such that $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (3) There exist $\hat{s}, \bar{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ and $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (4) $(s_1, m_1) \approx (s_2, m_2)$, that is, there exist $a, b \in R$ such that $as_2 = bs_1 \in S$ and $am_2 = bm_1$.

All tuples of left Ore data are equivalent in the sense of [Proposition 5.15](#):

Lemma 5.18 (cf. [Lemma 2.21](#)). *Let S be a left denominator set in a ring R and $(s, r) \in S \times R$. If there exist $(s_1, r_1), (s_2, r_2) \in S \times R$ such that $s_1r = r_1s$ and $s_2r = r_2s$, then $(s_1, r_1) \sim (s_2, r_2)$.*

As the next step towards the construction we define the underlying set of the localization as the equivalence classes of $S \times R$ with respect to the equivalence relation $\sim$, as well as the maps that will become addition and multiplication. Slightly generalized, this reads as follows:

Definition 5.19. Let S be a left denominator set in a ring R and M a left R -module. Then we define

- the set $S^{-1}M := (S \times M) / \sim$.
- the map

$$\mu_{S,R,M} : S^{-1}R \times S^{-1}M \rightarrow S^{-1}M, \quad ((s_1, r), (s_2, m)) \mapsto (\tilde{s}s_1, \tilde{r}m),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r = \tilde{r}s_2$.

- the map

$$\alpha_{S,R,M} : S^{-1}M \times S^{-1}M \rightarrow S^{-1}M, \quad ((s_1, m_1), (s_2, m_2)) \mapsto (\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$.

By abuse of notation, we denote the elements of $S^{-1}M$ again in the form (s, m) . If R and S are clear from the context, we also use $+_M$ (resp. $\cdot_M$) as an infix operator to have a shorter notation for the application of $\alpha_{S,R,M}$ (resp. $\mu_{S,R,M}$).

◇

From [Lemma 2.23](#) we know that $\mu_{S,R,M}$ is well-defined and induces the monoid operation of $(S^{-1}R, \cdot, (1, 1))$ as proven in [Proposition 2.24](#). Thus, we only have to deal with addition and the interplay between addition and multiplication:

Proposition 5.20. *Let S be a left denominator set in a ring R and M a left R -module.*

- (a) $(S^{-1}M, +_M, (1, 0))$ is an Abelian group.
- (b) For all $(s_1, r_1) \in S^{-1}R$ and all $(s_2, m_2), (s_3, m_3) \in S^{-1}M$ we have

$$(s_1, r_1) \cdot_M ((s_2, m_2) +_M (s_3, m_3)) = ((s_1, r_1) \cdot_M (s_2, m_2)) +_M ((s_1, r_1) \cdot_M (s_3, m_3)).$$

- (c) For all $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ and all $(s_3, m_3) \in S^{-1}M$ we have

$$((s_1, r_1) +_R (s_2, r_2)) \cdot_M (s_3, m_3) = ((s_1, r_1) \cdot_M (s_3, m_3)) +_M ((s_2, r_2) \cdot_M (s_3, m_3)).$$

Proof: Since the verification of these statements takes up multiple pages, the proof can be found in [Appendix A](#). $\square$

The only thing remaining is to put the all the pieces together:

Proposition 5.21 (Construction, cf. [Proposition 2.24](#)). *Let S be a left denominator set in a ring R . Then*

$$(S^{-1}R, +, (1, 0), \cdot, (1, 1))$$

is a ring with

$$+ : S^{-1}R \times S^{-1}R \rightarrow S^{-1}R, \quad ((s_1, r_1), (s_2, r_2)) \mapsto (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$, and

$$\cdot : S^{-1}R \times S^{-1}R \rightarrow S^{-1}R, \quad ((s_1, r_1), (s_2, r_2)) \mapsto (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$.

Proof: From [Proposition 2.24](#) we already know that $(S^{-1}R, \cdot, (1, 1))$ is a monoid. Furthermore, by [Proposition 5.20](#) we know that $(S^{-1}R, +, (1, 0))$ is an Abelian group. Lastly, the distributive laws follow from [Proposition 5.20](#) as the special case $M = R$. $\square$

From the rational numbers we are used to the fact that a fraction is zero if and only if its numerator is zero, but this does not hold in general in the presence of zero-divisors:

Example 5.22. In the localization of $\mathbb{Z}/6\mathbb{Z}$ at the multiplicative set $\{1, 2, 4\}$, $\frac{3}{1} = \frac{2 \cdot 3}{2 \cdot 1} = 0$. $\diamond$

The universal characterization for a fraction to be zero and a few selected other tools to simplify working with non-commutative fractions are given below:

Lemma 5.23 (cf. [Lemma 2.26](#)). *Let S be a left denominator set in a ring R and M a left R -module.*

(a) *Let $(s, m) \in S^{-1}M$. The following are equivalent:*

- (1) *We have $(s, m) = (1, 0)$.*
- (2) *There exists $\tilde{s} \in S$ such that $\tilde{s}m = 0$.*
- (3) *There exists $\hat{r} \in R$ such that $\hat{r}m = 0$ and $\hat{r}s \in S$.*

(b) *Let $m_1, m_2 \in M$ and $s \in S$, then $(s, m_1) + (s, m_2) = (s, m_1 + m_2)$.*

(c) *Let $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) *We have $(s, r) = (1, 1)$.*
- (2) *There exists $\tilde{s} \in S$ such that $\tilde{s}r = \tilde{s}s$.*
- (3) *There exists $\hat{r} \in R$ such that $\hat{r}r = \hat{r}s$ and $\hat{r}s \in S$.*

(d) *Let $s_1, s_2 \in S$, $r \in R$ and $m \in M$ such that $s_2r = rs_2$, then $(s_1, r) \cdot_M (s_2, m) = (s_2s_1, rm)$.*

Proof: The last two statements follow from [Lemma 2.26](#).

(a) Let $(s, m) \in S^{-1}M$.

(1) $\Rightarrow$ (2): Since $(s, m) = (1, 0)$ is equivalent to $(1, 0) = (s, m)$, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s = \tilde{r} \cdot 1 = \tilde{r}$ and $\tilde{s}m = \tilde{r} \cdot 0 = 0$.

(2) $\Rightarrow$ (3): Obvious with $\hat{r} := \tilde{s}$.

(3) $\Rightarrow$ (1): Let $\hat{r} \in R$ such that $\hat{r}m = 0$ and $\hat{r}s \in S$, then

$$(s, m) = (\hat{r}s, \hat{r}m) = (\hat{r}s, 0) = (1, 0).$$

(b) Let $\tilde{s} := 1 \in S$ and $\tilde{r} := 1 \in R$, then $\tilde{s}s = 1 \cdot s = \tilde{r}s$, thus

$$(s, m_1) + (s, m_2) = (\tilde{s}s, \tilde{s}m_1 + \tilde{r}m_2) = (1 \cdot s, 1 \cdot m_1 + 1 \cdot m_2) = (s, m_1 + m_2). \quad \square$$

The last ingredient is the *localization map* or *structural homomorphism*, which provides a way to get from R to $S^{-1}R$:

Proposition 5.24 (cf. [Proposition 2.27](#)). *Let S be a left denominator set in a ring R . The map*

$$\rho_{S,R} : R \rightarrow S^{-1}R, \quad r \mapsto (1, r)$$

is a homomorphism of rings satisfying $\ker(\rho) = \{r \in R \mid \exists s \in S : sr = 0\}$.

Proof: Since $\rho := \rho_{S,R}$ is already a homomorphism of (multiplicative) monoids by [Proposition 2.27](#), we only have to show additivity: let $r_1, r_2 \in R$, then

$$\rho(r_1 + r_2) = (1, r_1 + r_2) = (1, r_1) + (1, r_2) = \rho(r_1) + \rho(r_2).$$

The description of the kernel is immediate from [Lemma 5.23](#). $\square$

Now the pair $(S^{-1}R, \rho_{S,R})$ is a realization of the left Ore localization (R_S, η) :

Lemma 5.25 (Existence, cf. [Lemma 2.28](#)). *Let S be a left denominator set in a ring R . Then $(S^{-1}R, \rho_{S,R})$ is the left Ore localization of R at S .*

Since all localizations of R at S are isomorphic, we will call $(S^{-1}R, \rho_{S,R})$ the localization of R at S . The localization map is usually suppressed in the notation and $S^{-1}R$ alone is called the localization of R at S .

Combining the results so far we arrive at the foreshadowed characterization of all sets S permitting Ore localization:

Theorem 5.26 ([[Gab62](#), Prop. 5, p. 415], cf. [Proposition 2.29](#)). *Let S be a multiplicative set of a ring R . The left Ore localization of R at S exists if and only if S is a left denominator set in R .*

In contrast to the localization of monoids we have a clear-cut characterization of trivial localizations:

Lemma 5.27 (cf. [Lemma 2.33](#)). *Let S be a left denominator set in a ring R . Then $S^{-1}R = \{0\}$ if and only if $0 \in S$.*

Proof: Let $0 \in S$ and $(s, r) \in S^{-1}R$. Define $\tilde{s} := 0 \in S$ and $\tilde{r} := 0 \in R$, then

$$\tilde{s} \cdot 1 = 0 \cdot 1 = 0 \cdot s = \tilde{r}s \quad \text{and} \quad \tilde{s} \cdot 0 = 0 \cdot 0 = 0 \cdot r = \tilde{r}r$$

show that $(s, r) = (1, 0)$, thus $S^{-1}R = \{0\}$.

On the other hand, let $S^{-1}R = \{0\}$, then $(1, 0) = (1, 1)$, thus there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} = \tilde{s} \cdot 1 = \tilde{r} \cdot 1 = \tilde{r}$ and $\tilde{s} = \tilde{s} \cdot 1 = \tilde{r} \cdot 0 = 0$, thus $0 = \tilde{s} \in S$. $\square$

Generalizing the examples of commutative localizations presented in [Chapter 1](#), we arrive at the following list of special types of Ore localizations:

Definition 5.28. Let S be a left denominator set in a ring R . Then S (and by extension, the localization $S^{-1}R$) might belong to one (or multiple) of the following types:

Monoidal: S is generated as a multiplicative monoid by at most countably many elements.

Geometric: $S = T \setminus \mathfrak{p}$ for a commutative subring T of R with $\mathfrak{p}$ being a prime ideal in T .

Rational: $\hat{S} := S \cup \{0\}$ is a subring of R . Note that since S is multiplicatively closed, $\hat{S}$ must be a domain. $\diamond$

Consider the following examples of left Ore localizations. Note that it is not obvious at all that the multiplicative sets in question are indeed left Ore sets, this will be shown later in a more general version in [Corollary 8.11](#).

Example 5.29. Let K be a field, $A := A_1(K)$, and consider the following localizations:

- Taking $S = \{x^k \mid k \in \mathbb{N}_0\}$ leads to the monoidal localization

$$S^{-1}A \cong K\langle x, x^{-1}, \partial \mid \partial x = x\partial + 1, \partial x^{-1} = x^{-1}\partial + x^{-2}, xx^{-1} = x^{-1}x = 1 \rangle,$$

which can be seen as a Weyl algebra with Laurent polynomial coefficients.

- Let $p \in K$ and consider the maximal ideal $\mathfrak{m}_p := \langle x - p \rangle$ in the commutative subring $K[x]$ of A . The geometric localization

$$(K[x] \setminus \mathfrak{m}_p)^{-1}A \cong K[x]_{\mathfrak{m}_p} \langle \partial \mid \partial f = f\partial + \frac{df}{dx} \text{ for all } f \in K[x]_{\mathfrak{m}_p} \subseteq K(x) \rangle$$

is a so-called *local (algebraic) Weyl algebra*, which is of importance in D -module theory.

- Rational localization at $K[x] \setminus \{0\}$ allows us to pass from the polynomial Weyl algebra A to the rational Weyl algebra

$$(K[x] \setminus \{0\})^{-1}A \cong K(x) \left\langle \partial \mid \partial f = f\partial + \frac{df}{dx} \text{ for all } f \in K(x) \right\rangle.$$

$\diamond$

5.3. Localization at central elements

Definition 5.30. , Let R be a ring. An element $r \in R$ is called *central* in R if $r \cdot \tilde{r} = \tilde{r} \cdot r$ for all $\tilde{r} \in R$. The set of all central elements in R is denoted $Z(R)$ and called the *center* of R . $\diamond$

If R is a ring, then $Z(R)$ is a commutative subring of R . Localizing R at a set of central elements greatly simplifies both the equivalence relation and the operations themselves to the point where they closely resemble the commutative situation:

Proposition 5.31 (cf. [Proposition 2.35](#)). *Let R be a ring and S a multiplicative subset of $Z(R)$.*

- (a) *S is a left (and right) denominator set in R .*
- (b) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) = (s_2, r_2)$ if and only if there exists $\tilde{s} \in S$ such that $\tilde{s}(s_1 r_2 - s_2 r_1) = 0$.*
- (c) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) \cdot (s_2, r_2) = (s_1 s_2, r_1 r_2)$.*
- (d) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ such that $r_1 r_2 = r_2 r_1$. Then $(s_1, r_1) \cdot (s_2, r_2) = (s_2, r_2) \cdot (s_1, r_1)$.*
- (e) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) + (s_2, r_2) = (s_1 s_2, s_2 r_1 + s_1 r_2)$.*

Proof: The only thing left to show is the last part, the rest follows from [Proposition 2.35](#): define $\tilde{s} := s_2 \in S$ and $\tilde{r} := s_1 \in S$, then $\tilde{s} s_1 = s_2 s_1 = s_1 s_2 = \tilde{r} s_2$, thus

$$(s_1, r_1) + (s_2, r_2) = (\tilde{s} s_1, \tilde{s} r_1 + \tilde{r} r_2) = (s_2 s_1, s_2 r_1 + s_1 r_2) = (s_1 s_2, s_2 r_1 + s_1 r_2). \quad \square$$

Example 5.32. Consider the non-commutative ring $R := \mathbb{Z}^{2 \times 2}$ of 2-by-2 integer matrices. Then the set $S := \{\text{diag}(z, z) \mid z \in \mathbb{Z} \setminus \{0\}\} \subseteq Z(R)$ of diagonal matrices with the same non-zero diagonal entries is a central multiplicative set in R and the localization $S^{-1}R$ is canonically isomorphic to $\mathbb{Q}^{2 \times 2}$. $\diamond$

Consequently, for a commutative ring we can see that the classical localization is indeed a special case of Ore localization:

Corollary 5.33 (cf. [Corollary 2.36](#)). *Let S be a multiplicative set in a commutative ring R .*

- (a) *S is a left (and right) denominator set in R .*
- (b) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) = (s_2, r_2)$ if and only if there exists $\tilde{s} \in S$ such that $\tilde{s}(s_1 r_2 - s_2 r_1) = 0$.*
- (c) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) \cdot (s_2, r_2) = (s_1 s_2, r_1 r_2)$.*
- (d) *$S^{-1}R$ is a commutative ring.*
- (e) *Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$, then $(s_1, r_1) + (s_2, r_2) = (s_1 s_2, s_2 r_1 + s_1 r_2)$.*

If we additionally require R to be a domain, then we arrive at the classical construction presented in [Chapter 1](#).

5.4. Left-right duality

In the commutative case we do not have to make a distinction between left and right localizations, since they are obviously isomorphic via $S^{-1}R \rightarrow RS^{-1}$, $(s, r) \mapsto (r, s)$. A similar result holds in arbitrary rings, if both localizations at S exist:

Proposition 5.34 (cf. [Proposition 2.37](#)). *Let S be a left and right denominator set in a ring R . The map*

$$\varphi : RS^{-1} \rightarrow S^{-1}R, \quad (r, s) \mapsto (\tilde{s}, \tilde{r}),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r = \tilde{r}s$, is an isomorphism of rings. In particular, the left Ore localization of R at S is isomorphic to the right Ore localization of R at S .

Proof: The only thing left to show is additivity of φ , the rest follows from [Proposition 2.37](#): let $(r_1, s_1), (r_2, s_2) \in RS^{-1}$, then

$$\varphi((r_1, s_1)) + \varphi((r_2, s_2)) = (\tilde{s}_1, \tilde{r}_1) + (\tilde{s}_2, \tilde{r}_2) = (s_a \tilde{s}_1, s_a \tilde{r}_1 + r_a \tilde{r}_2)$$

and

$$\varphi((r_1, s_1) + (r_2, s_2)) = \varphi((r_1 s_b + r_2 r_b, s_1 s_b)) = (s_c, r_c),$$

where $\tilde{s}_1, \tilde{s}_2, s_a, s_b, s_c \in S$ and $\tilde{r}_1, \tilde{r}_2, r_a, r_b, r_c \in R$ satisfy

$$\tilde{s}_1 r_1 = \tilde{r}_1 s_1, \tag{M1}$$

$$\tilde{s}_2 r_2 = \tilde{r}_2 s_2, \tag{M2}$$

$$s_a \tilde{s}_1 = r_a \tilde{s}_2, \tag{M3}$$

$$s_1 s_b = s_2 r_b, \tag{M4}$$

$$s_c(r_1 s_b + r_2 r_b) = r_c s_1 s_b. \tag{M5}$$

We have to show that $(s_a \tilde{s}_1, s_a \tilde{r}_1 + r_a \tilde{r}_2) = (s_c, r_c)$ in $S^{-1}R$. By the left Ore condition there exist $s_d \in S$ and $r_d \in R$ such that

$$s_d s_a \tilde{s}_1 = r_d s_c. \tag{M6}$$

By left reversibility,

$$\begin{aligned} s_d(s_a \tilde{r}_1 + r_a \tilde{r}_2) s_1 s_b &= s_d s_a \tilde{r}_1 s_1 s_b + s_d r_a \tilde{r}_2 s_1 s_b \\ &\stackrel{(M1)}{=} s_d s_a \tilde{s}_1 r_1 s_b + s_d r_a \tilde{r}_2 s_1 s_b \\ &\stackrel{(M4)}{=} s_d s_a \tilde{s}_1 r_1 s_b + s_d r_a \tilde{r}_2 s_2 r_b \\ &\stackrel{(M2)}{=} s_d s_a \tilde{s}_1 r_1 s_b + s_d r_a \tilde{s}_2 r_2 r_b \\ &\stackrel{(M3)}{=} s_d s_a \tilde{s}_1 r_1 s_b + s_d s_a \tilde{s}_1 r_2 r_b \\ &= s_d s_a \tilde{s}_1 (r_1 s_b + r_2 r_b) \\ &\stackrel{(M6)}{=} r_d s_c (r_1 s_b + r_2 r_b) \\ &\stackrel{(M5)}{=} r_d r_c s_1 s_b \end{aligned}$$

implies the existence of $s_e \in S$ and $r_e \in R$ such that

$$s_e s_d (s_a \tilde{r}_1 + r_a \tilde{r}_2) = s_e r_d r_c s_1 s_b. \tag{M7}$$

Define $\mathring{s} := s_e s_d \in S$ and $\mathring{r} := s_e r_d \in R$, then

$$\mathring{s} s_a \tilde{s}_1 = s_e s_d s_a \tilde{s}_1 \stackrel{(\text{M6})}{=} s_e r_d s_c = \mathring{r} s_c$$

and

$$\mathring{s}(s_a \tilde{r}_1 + r_a \tilde{r}_2) = s_e s_d (s_a \tilde{r}_1 + r_a \tilde{r}_2) \stackrel{(\text{M7})}{=} s_e r_d r_c s_1 s_b = \mathring{r} r_c s_1 s_b$$

show that $(s_a \tilde{s}_1, s_a \tilde{r}_1 + r_a \tilde{r}_2) = (s_c, r_c)$ in $S^{-1}R$. $\square$

5.5. Left denominator sets in rings

In this section, we have a closer look at the defining properties of left denominator sets. The property of being a multiplicative set is preserved and reflected by homomorphisms, but the left Ore condition and especially left reversibility require additional assumptions:

Lemma 5.35 (cf. Lemma 2.38). *Let $\varphi : R \rightarrow R'$ be a homomorphism of rings.*

- (a) *Let $S \subseteq R$.*
 - (i) *If S is multiplicatively closed in R , then $\varphi(S)$ is multiplicatively closed in R' .*
 - (ii) *If S is a multiplicative set in R , then $\varphi(S)$ is a multiplicative set in R' .*
 - (iii) *If φ is surjective and S satisfies the left Ore condition in R , then $\varphi(S)$ satisfies the left Ore condition in R' .*
 - (iv) *If φ is bijective and S is left reversible in R , then $\varphi(S)$ is left reversible in R' .*
 - (v) *If φ is bijective and S is a left denominator set in R , then $\varphi(S)$ is a left denominator set in R' .*
- (b) *Let $T \subseteq R'$.*
 - (i) *If T is multiplicatively closed in R' , then $\varphi^{-1}(T)$ is multiplicatively closed in R .*
 - (ii) *If T is a multiplicative set in R' , then $\varphi^{-1}(T)$ is a multiplicative set in R .*
 - (iii) *If φ is bijective and T satisfies the left Ore condition in R' , then $\varphi^{-1}(T)$ satisfies the left Ore condition in R .*
 - (iv) *If φ is bijective and T is left reversible in R' , then $\varphi^{-1}(T)$ is left reversible in R .*
 - (v) *If φ is bijective and T is a left denominator set in R' , then $\varphi^{-1}(T)$ is a left denominator set in R .*

Example 5.36. The first Weyl algebra A_1 contains the commutative polynomial subring $K[x\partial]$. Now $[x\partial]$ is a left denominator set in $K[x\partial]$, but not in A_1 : assume otherwise, then there exist $r \in A_1$ and $k \in \mathbb{N}_0$ such that

$$r \cdot x\partial = (x\partial)^k x = x(\partial x)^k = x(x\partial + 1)^k = x \left(\sum_{j=1}^k \binom{k}{j} (x\partial)^{j-1} \right) x\partial + x,$$

which implies the contradiction $x \in_{A_1} \langle \partial \rangle$. $\diamond$

Definition 5.37. Let S be a subset of a ring $(R, +, 0, \cdot, 1)$. Then $[S]$ is the smallest submonoid of $(R, \cdot, 1)$ with respect to inclusion that contains S , thus

$$[S] = \bigcup_{n \in \mathbb{N}_0} \{s_1 \cdot \dots \cdot s_n \mid s_i \in S \text{ for all } i \in \{1, \dots, n\}\}.$$

By the usual convention, the empty product in the case $n = 0$ corresponds to the element 1. We call S a *generating set* or *set of generators* for $[S]$ and $[S]$ the multiplicative set *generated* by S or the *multiplicative closure* of S . $\diamond$

For a subset S of a ring R we have $S = [S]$ if and only if S is a multiplicative set, which justifies the term “multiplicative closure”. In particular, every multiplicative set is a generating set of itself.

Example 5.38. Let S be a multiplicative set in a ring R , then $S \cup \{0\}$ is a left denominator set in R . In particular, R is the largest left denominator set in R . In contrast, $[1] = \{1\}$ is the smallest left denominator set in R . $\diamond$

Note that in the presence of zero-divisors, $[S]$ might contain 0 even if $0 \notin S$.

Lemma 5.39 (cf. [Lemma 2.41](#)). *Let S and T be two multiplicatively closed subsets (resp. multiplicative subsets) of a ring R , then $S \cap T$ is also multiplicatively closed (resp. a multiplicative set).*

Just as with ideals, it suffices to check inclusion of multiplicative sets on generators:

Lemma 5.40 (cf. [Lemma 2.42](#)). *Let T be a multiplicative set in a ring R , I an arbitrary index set and $s_i \in R$ for all $i \in I$. Then $S := [\{s_i \mid i \in I\}] \subseteq T$ if and only if $s_i \in T$ for all $i \in I$.*

Even better, this also holds for the left Ore condition and left reversibility:

Lemma 5.41 (cf. [Lemma 2.43](#)). *Let T be a subset of a ring R and $S := [T]$ such that the following two conditions are satisfied:*

- (i) *For all $t \in T$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}t$.*
- (ii) *For all $t \in T$ and $r \in R$ such that $rt = 0$ there exists $\tilde{s} \in S$ such that $\tilde{s}r = 0$.*

Then S is a left denominator set in R .

Example 5.42 (cf. [Example 2.44](#)). Let K be a field and consider the algebra of one-sided inverses, which is defined by $R := K\langle x, y \mid xy = 1 \rangle$. Note that R is not a domain, since $x(yx - 1) = 0$, but $x \neq 0$ and $yx \neq 1$. In fact, as a K -vector space we have $R = \bigoplus_{(a,b) \in \mathbb{N}_0^2} Ky^a x^b$. Now let $S := [x]$.

- Let $r = \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} y^a x^b \in R$ for some $c_{a,b} \in K$. Define the maximal a -value

$$\bar{a} := \max \{a \in \mathbb{N}_0 \mid \exists b \in \mathbb{N}_0 : c_{a,b} \neq 0\}.$$

Now let $\tilde{s} := x^{\bar{a}+1} \in S$ and $\tilde{r} := \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} x^{b+\bar{a}-a} \in R$, then

$$\tilde{s}r = x^{\bar{a}+1} \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} y^a x^b = \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} x^{\bar{a}+1} y^a x^b = \left(\sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} x^{b+\bar{a}-a} \right) x = \tilde{r}x.$$

- Let $r \in R$ such that $rx = 0$, then $0 = rxy = r$, thus $xr = 0$.

By [Lemma 5.41](#), S is a left denominator set in R and we can consider the left Ore localization $S^{-1}R$, which will be done in [Example 7.23](#). $\diamond$

Corollary 5.43 (cf. [Corollary 2.45](#)). *Let S be a left reversible subset of a monoid R that satisfies the left Ore condition in R . Then $[S]$ is a left denominator set in R .*

As a second consequence, this gives us a method to build larger left denominator sets from smaller ones:

Corollary 5.44 (cf. [Corollary 2.46](#)). *Let R be a ring and J a non-empty index set such that S_j is a left denominator set in R for every $j \in J$. Then $T := [\bigcup_{j \in J} S_j]$ is a left denominator set in R .*

As we have seen in [Lemma 5.13](#), all left Ore sets in R which do not contain right zero-divisors are left denominator sets. This also holds for other important classes of rings:

Lemma 5.45. *Let S be a left Ore set in a ring R . If either of the following conditions hold, then S is also left reversible and thus a left denominator set in R :*

- (a) R is left Noetherian.
- (b) R does not contain any non-zero nilpotent elements.

Proof: Let $s \in S$ and $r \in R$ such that $rs = 0$.

- (a) Consider the ascending chain

$$\text{LAnn}_R(s) \subseteq \text{LAnn}_R(s^2) \subseteq \dots$$

of left annihilators $\text{LAnn}_R(s) := \{r \in R \mid rs = 0\}$, which are left ideals in R . By assumption there exists $n \in \mathbb{N}$ such that $\text{LAnn}_R(s^n) = \text{LAnn}_R(s^{n+1})$. By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s^n$. Now $\tilde{r}s^{n+1} = \tilde{s}rs = 0$ implies $\tilde{r} \in \text{LAnn}_R(s^{n+1}) = \text{LAnn}_R(s^n)$ and thus $\tilde{s}r = \tilde{r}s^n = 0$, which shows that S is left reversible.

- (b) Since $(sr)^2 = sr sr = s \cdot 0 \cdot r = 0$ we have that sr is nilpotent and thus $sr = 0$. $\square$

5.6. Localization of graded rings

Graded rings have an additional structure that can be exploited to a large degree in proofs and algorithms. In this section we will explore the interplay between gradings and localizations.

Definition 5.46. Let $(\Gamma, *, e)$ be a monoid.

- A ring R is called Γ -graded if there exists a family $\{R_\gamma\}_{\gamma \in \Gamma}$ of Abelian subgroups of R with respect to addition such that $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$, $R_\gamma R_\delta \subseteq R_{\gamma*\delta}$ for all $\gamma, \delta \in \Gamma$ and $1 \in R_e$.
- An element $r \in R_\gamma$ is called *homogeneous* of *degree* γ and we write $\deg(r) = \gamma$.
- The set of all homogeneous elements of R is $h(R) := \bigcup_{\gamma \in \Gamma} R_\gamma \subseteq R$.

- A subset M of R is called *homogeneous* if $M \subseteq h(R)$.
- Any $r \in R$ has a unique decomposition $r = \sum_{\gamma \in \Gamma} a_\gamma$ into its homogeneous parts, where $a_\gamma \in R_\gamma$. The unique homogeneous part of r corresponding to $\gamma \in \Gamma$ is denoted as $r_\gamma := a_\gamma$.
- The *graded support* of r is the finite set $\text{gsupp}(r) := \{\gamma \in \Gamma \mid r_\gamma \neq 0\} \subseteq \Gamma$.
- The *graded length* of r is $\text{gl}(r) := |\text{gsupp}(r)| \in \mathbb{N}_0$. $\diamond$

The group R_e is a subring of R . Furthermore, an element $r \in R \setminus \{0\}$ is homogeneous if and only if its graded support is a singleton, which is equivalent to $\text{gl}(r) = 1$.

Remark 5.47. Similarly, one can define a Γ -graded *left module* over a Γ -graded ring R as a left R -module M with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ into Abelian subgroups M_γ of M satisfying $R_\gamma M_\delta \subseteq M_{\gamma*\delta}$ for all $\gamma, \delta \in \Gamma$. $\diamond$

Lemma 5.48. Let $(\Gamma, *, e)$ be a monoid, R a Γ -graded domain, $\gamma \in \Gamma$, $a \in R_\gamma \setminus \{0\}$ and $r \in R$.

- We have $\text{gsupp}(ra) = \text{gsupp}(r) * \gamma$.
- If Γ is right-cancellative, then $\text{gl}(ra) = \text{gl}(r)$.
- If Γ is right-cancellative and $ra \in h(R)$, then $r \in h(R)$.
- We have $\text{gsupp}(ar) = \gamma * \text{gsupp}(r)$.
- If Γ is left-cancellative, then $\text{gl}(ar) = \text{gl}(r)$.
- If Γ is left-cancellative and $ar \in h(R)$, then $r \in h(R)$.

Proof: Let $G := \text{gsupp}(r)$ and $r = \sum_{\tau \in G} r_\tau$. Assume that $r \neq 0$, since the statements above are trivially true for $r = 0$. Then $G \neq \emptyset$ and $\text{gl}(r) \geq 1$.

- We have $ra = \sum_{\tau \in G} r_\tau a$ with $\deg(r_\tau a) = \tau * \gamma$ and $r_\tau a \neq 0$, since R is a domain. Now

$$\text{gsupp}(ra) = \{\tau * \gamma \mid \tau \in G \text{ and } r_\tau a \neq 0\} = \{\tau * \gamma \mid \tau \in G\} = G * \gamma = \text{gsupp}(r) * \gamma.$$

- We have

$$\text{gl}(ra) = |\text{gsupp}(ra)| = |\text{gsupp}(r) * \gamma| \leq |\text{gsupp}(r)| = \text{gl}(r).$$

Now let $\tau, \sigma \in G$ such that $\tau * \gamma = \sigma * \gamma$. Since Γ is right-cancellative this implies $\tau = \sigma$ and thus $\text{gl}(ra) = \text{gl}(r)$.

- Since Γ is right-cancellative, we have $\text{gl}(r) = \text{gl}(ra) = 1$ by the result above, thus $r \in h(R)$.

The remaining results follow analogously. $\square$

The most commonly used monoids for grading are $\mathbb{N}_0^n$ and $\mathbb{Z}^n$, partly due to the fact that they are *ordered*:

Definition 5.49. An *ordered semigroup* is a semigroup $(\Gamma, *)$ together with a total ordering $\leq$ on Γ that is compatible with the semigroup operation $*$: for all $x, y, z \in \Gamma$, $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$. An *ordered monoid* is a monoid that is ordered as a semigroup. $\diamond$

Lemma 5.50. *Let $(\Gamma, *, e)$ be an ordered monoid with respect to $\leq$ and R a Γ -graded domain. Then $h(R) \setminus \{0\}$ is saturated.*

Proof: Let $a, b \in R$ with $ab \in R_\gamma \setminus \{0\}$ for some $\gamma \in \Gamma$, then $a \neq 0 \neq b$. Let $n := \text{gl}(a)$, $m := \text{gl}(b)$ and write $a = \sum_{i=1}^n a_{\tau_i}$ and $b = \sum_{j=1}^m b_{\sigma_j}$, where $\tau_i, \sigma_j \in \Gamma$ as well as $\tau_i < \tau_{i+1}$ and $\sigma_j < \sigma_{j+1}$ for all $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, m-1\}$. Now

$$(ab)_{\tau_1 * \sigma_1} = a_{\tau_1} b_{\sigma_1} \neq 0 \quad \text{and} \quad (ab)_{\tau_n * \sigma_m} = a_{\tau_n} b_{\sigma_m} \neq 0,$$

thus $\tau_1 * \sigma_1 = \gamma = \tau_n * \sigma_m$. Assume $n > 1$, then $\tau_1 < \tau_n$ implies $\tau_1 * \sigma_1 < \tau_n * \sigma_1 \leq \tau_n * \sigma_m = \tau_1 * \sigma_1$, a contradiction. Thus $n = 1$ and similarly we get $m = 1$. Now $\text{gl}(a) = n = 1 = m = \text{gl}(b)$ implies $a, b \in h(R) \setminus \{0\}$. $\square$

Note that **Lemma 5.50** does not hold for arbitrary Γ -gradings:

Remark 5.51. Consider $R = \mathbb{Z}[x]$ and $\Gamma = \mathbb{Z}/2\mathbb{Z}$, where $R_0 = \bigoplus_{i \in \mathbb{N}_0} \mathbb{Z}x^{2i}$ and $R_1 = \bigoplus_{i \in \mathbb{N}_0} \mathbb{Z}x^{2i+1}$. Then $(x+1)(x-1) = x^2 - 1 \in R_0$, but neither $x+1$ nor $x-1$ is homogeneous. $\diamond$

Nevertheless, $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces, so-called *super vector spaces*, appear in theoretical physics in the study of supersymmetry.

For homogeneous sets S it suffices to check the left Ore condition inside $h(R)$:

Lemma 5.52. *Let Γ be a monoid, R a Γ -graded ring and S a multiplicative set in R with $S \subseteq h(R)$.*

- (a) *If S satisfies the left Ore condition in $h(R)$, that is, for all $s \in S$ and $r \in h(R)$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$, then S satisfies the left Ore condition in R .*
- (b) *Let Γ be right-cancellative, $s \in S$ and $r \in h(R)$. If there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$, then $\tilde{r} \in h(R)$.*

Proof: (a) Let $s \in S$ and $r \in R \setminus \{0\}$. Induction on $n := \text{gl}(r) \in \mathbb{N}$:

Induction base: If $n = 1$, then $r \in h(R)$ and the statement follows from the assumption.

Induction hypothesis: Assume that there is an $n \in \mathbb{N}$ with the following property: for all $s \in S$ and all $r \in R$ such that $\text{gl}(r) \leq n$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

Induction step: Let $\text{gl}(r) = n + 1$ and $\gamma \in G := \text{gsupp}(r)$.

- By the induction hypothesis there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$\tilde{s} \sum_{\tau \in G \setminus \{\gamma\}} r_\tau = \tilde{r}s.$$

- By the induction base there exist $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}r_\gamma = \hat{r}s$.
- Again by the induction base there exist $\bar{s} \in S$ and $\bar{r} \in R$ such that $\bar{s}\tilde{s} = \bar{r}\tilde{r}$.

Now define $\mathring{s} := \bar{s}\tilde{s} \in S$ and $\mathring{r} := \bar{s}\tilde{r} + \bar{r}\hat{r} \in R$, then

$$\mathring{s}r = \mathring{s} \sum_{\tau \in G} r_\tau = \bar{s}\tilde{s} \sum_{\tau \in G \setminus \{\gamma\}} r_\tau + \bar{s}\tilde{s}r_\gamma = \bar{s}\tilde{r}s + \bar{r}\hat{r}s = (\bar{s}\tilde{r} + \bar{r}\hat{r})s = \mathring{r}s.$$

(b) We have $\tilde{r}s = \tilde{s}r \in h(R)$ and $s \in h(R)$. Now [Lemma 5.48](#) implies $\tilde{r} \in h(R)$. $\square$

If R is graded by a group, like $\mathbb{Z}^n$ for example, then any localization at a homogeneous left denominator set inherits the grading of the base ring R :

Proposition 5.53. *Let $(G, \cdot, e)$ be a group, R a G -graded ring and $S \subseteq h(R)$ a left denominator set in R . Then $S^{-1}R$ is a G -graded ring with*

$$(S^{-1}R)_g := \{(s, r) \in S^{-1}R \mid r \in h(R) \text{ and } \deg((s, r)) := \deg(s)^{-1} \deg(r) = g\} \cup \{0\}.$$

Proof: Let $g, h \in G$.

- Since $0 \in (S^{-1}R)_g$, $(S^{-1}R)_g$ is non-empty.
- Let $(s, r) \in (S^{-1}R)_g$, then $r \in h(R)$. Let $h \in G$ such that $r \in R_h$, then $-r \in R_h$ since R_h is closed under taking additive inverses. Now

$$\deg(-(s, r)) = \deg((s, -r)) = \deg(s)^{-1} \deg(-r) = \deg(s)^{-1} \deg(r) = g,$$

thus $-(s, r) \in (S^{-1}R)_g$, which shows that $(S^{-1}R)_g$ is closed under taking additive inverses.

- Let $(s_1, r_1), (s_2, r_2) \in (S^{-1}R)_g$, then $s_1, s_2, r_1, r_2 \in h(R)$ and

$$\deg(s_1)^{-1} \deg(r_1) = g = \deg(s_2)^{-1} \deg(r_2).$$

We have $(s_1, r_1) + (s_2, r_2) = (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$. Now G is right-cancellative since it is a group, thus [Lemma 5.52](#) implies $\tilde{r} \in h(R)$ and we get

$$\deg(\tilde{s}) \deg(s_1) = \deg(\tilde{s}s_1) = \deg(\tilde{r}s_2) = \deg(\tilde{r}) \deg(s_2).$$

From

$$\begin{aligned} \deg(\tilde{s}r_1) &= \deg(\tilde{s}) \deg(r_1) \\ &= \deg(\tilde{s}) \deg(s_1) \deg(s_1)^{-1} \deg(r_1) \\ &= \deg(\tilde{r}) \deg(s_2) \deg(s_2)^{-1} \deg(r_2) \\ &= \deg(\tilde{r}) \deg(r_2) \\ &= \deg(\tilde{r}r_2) \end{aligned}$$

we can see that $\tilde{s}r_1 + \tilde{r}r_2$ is homogeneous of degree $\deg(\tilde{s}r_1)$. Finally,

$$\begin{aligned} \deg(\tilde{s}s_1)^{-1} \deg(\tilde{s}r_1 + \tilde{r}r_2) &= \deg(\tilde{s}s_1)^{-1} \deg(\tilde{s}r_1) \\ &= \deg(s_1)^{-1} \deg(\tilde{s})^{-1} \deg(\tilde{s}) \deg(r_1) \\ &= \deg(s_1)^{-1} \deg(r_1) \\ &= g \end{aligned}$$

shows that $(S^{-1}R)_g$ is closed under addition.

- Let $(s_1, r_1) \in (S^{-1}R)_g$ and $(s_2, r_2) \in (S^{-1}R)_h$, then $(s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$. Again by [Lemma 5.52](#) we have $\tilde{r} \in h(R)$. Now

$$\deg(\tilde{s}) \deg(r_1) = \deg(\tilde{s}r_1) = \deg(\tilde{r}s_2) = \deg(\tilde{r}) \deg(s_2),$$

thus

$$\begin{aligned} \deg(\tilde{s}s_1)^{-1} \deg(\tilde{r}r_2) &= \deg(s_1)^{-1} \deg(\tilde{s})^{-1} \deg(\tilde{r}) \deg(r_2) \\ &= \deg(s_1)^{-1} \deg(\tilde{s})^{-1} \deg(\tilde{r}) \deg(s_2) \deg(s_2)^{-1} \deg(r_2) \\ &= \deg(s_1)^{-1} \deg(\tilde{s})^{-1} \deg(\tilde{s}) \deg(r_1) \deg(s_2)^{-1} \deg(r_2) \\ &= \deg(s_1)^{-1} \deg(r_1) \deg(s_2)^{-1} \deg(r_2) \\ &= gh \end{aligned}$$

shows that $(S^{-1}R)_g(S^{-1}R)_h \subseteq (S^{-1}R)_{gh}$.

- Let $(s, r) \in S^{-1}R$ and $r = \sum_{g \in G} r_g$, then

$$(s, r) = \left(s, \sum_{g \in G} r_g \right) = \sum_{g \in G} (s, r_g) \in \sum_{g \in G} (S^{-1}R)_{\deg(s)^{-1}g} \subseteq \sum_{g \in G} (S^{-1}R)_g,$$

thus $S^{-1}R = \sum_{g \in G} (S^{-1}R)_g$. Now let $0 \neq (s, r) \in (S^{-1}R)_g \cap (S^{-1}R)_h$, then we have $g = \deg(s)^{-1} \deg(r) = h$, which shows that $(S^{-1}R)_g \cap (S^{-1}R)_h = \{0\}$ and therefore $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$. $\square$

Chapter 6.

Left saturation closure

Now it is time to introduce the central object of this thesis: the left saturation closure. The idea behind it is to define a universal concept describing all elements of a module such that certain left multiples of them are contained in a given subset of the module. A well-known example is the torsion submodule of a left module M over a ring R , consisting of all elements of M that become zero when multiplied on the left with some regular element of R .

In this chapter, we will discuss the basic algebraic properties.

Legal note. The definition of left saturation closure and some of the basic properties up to and including [Proposition 6.7](#) were already part of the author's master thesis ([\[Hof16\]](#)).

6.1. General construction

In its most general version, left saturation closure is defined as follows:

Definition 6.1. Let T be a subset of a ring R , M a left R -module and P, N two subsets of M . The *left saturation closure* of P at T in N is

$$\text{LSat}_T^N(P) := \{m \in N \mid \exists t \in T : tm \in P\} \subseteq N.$$

To keep the notation concise we use the shorthand $\text{LSat}_T(P) := \text{LSat}_T^M(P)$ if $M = N$. $\diamond$

Remark 6.2. If either T , N , or P is empty, then $\text{LSat}_T^N(P) = \emptyset$. Furthermore, LSat is inclusion-preserving in all three parameters by construction. $\diamond$

Lemma 6.3 (cf. [Lemma 3.3](#)). *Let T be a subset of a ring R , M a left R -module and P, N two subsets of M .*

- (a) *We have $\text{LSat}_T^N(P) = \text{LSat}_T(P) \cap N$.*
- (b) *If $1 \in T$, then $P \subseteq \text{LSat}_T(P)$.*
- (c) *If $1 \in T$ and P is a left R -submodule of M , then $\text{LSat}_{\text{LSat}(T)}(P) = \text{LSat}_T(P)$.*
- (d) *If S is a subset of R , then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_{TS}^N(P).$$

- (e) *Let S be a subset of R such that $1 \in S$ and $S \subseteq T$. If T is multiplicatively closed, then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_T^N(P) = \text{LSat}_T^N(\text{LSat}_S^N(P)).$$

(f) Let S be a subset of R and Q a subset of M , then

$$\text{LSat}_S^{\text{LSat}_T^N(Q)}(P) = \text{LSat}_T^N(Q) \cap \text{LSat}_S^N(P).$$

Lemma 6.4. Let T be a subset of a ring R , M a left R -module and P a subset of M .

(a) If $T \neq \emptyset$, then $0 \in P$ if and only if $0 \in \text{LSat}_T(P)$.

(b) If $0 \in T$, then the following are equivalent:

- (1) $\text{LSat}_T(P) = M$.
- (2) $0 \in \text{LSat}_T(P)$.
- (3) $0 \in P$.

(c) If $0 \notin P$, then $\text{LSat}_T(P) = \text{LSat}_{T \setminus \{0\}}(P)$.

Proof: (a) If $0 \in P$, then $t \cdot 0 = 0 \in P$ for any $t \in T$, thus $0 \in \text{LSat}_T(P)$.

If $0 \in \text{LSat}_T(P)$, then there is some $t \in T$ such that $0 = t \cdot 0 \in P$, thus $0 \in P$.

(b) Since $0 \in T$ implies $T \neq \emptyset$, the equivalence of (2) and (3) holds by the result above. The implication from (1) to (2) is obvious. To complete the proof, we show that (3) implies (1): let $0 \in P$ and $m \in M$, then $0 \cdot m = 0 \in P$ shows that $m \in \text{LSat}_T(P)$ since $0 \in T$.

(c) Let $m \in \text{LSat}_T^M(P)$, then $tm \in P$ for some $t \in T$. Since $0 \notin P$ we have $tm \neq 0$, which implies $t \neq 0$, thus $m \in \text{LSat}_{T \setminus \{0\}}^M(P)$. $\square$

Lemma 6.5 (cf. [Lemma 3.4](#)). Let $\varphi : R \rightarrow R'$ be a homomorphism of rings and T, P two subsets of R . Then we have the following:

(a) $\text{LSat}_T(P) \subseteq \varphi^{-1}(\text{LSat}_{\varphi(T)}(\varphi(P)))$.

(b) If φ is injective, then $\text{LSat}_T(P) = \varphi^{-1}(\text{LSat}_{\varphi(T)}(\varphi(P)))$.

(c) If φ is bijective, then $\varphi(\text{LSat}_T(P)) = \text{LSat}_{\varphi(T)}(\varphi(P))$.

6.2. Left saturation at multiplicative sets

A subset S of a commutative ring R is called *saturated* if $ab \in S$ implies $a \in S$ and $b \in S$ for all $a, b \in R$. At first glance, this seems to be a dual concept to multiplicatively closed sets, but they behave differently under ring extensions: let $R \subseteq R'$ be an inclusion of commutative rings, then every multiplicatively closed subset of R is also multiplicatively closed as a subset of R' . In contrast, the set $\{1, -1\}$ is saturated in $\mathbb{Z}$, but not in $\mathbb{Q}$, since $1 = 2 \cdot 2^{-1}$.

As usual, for non-commutative rings we have to make a distinction between the two sides: S is called *left saturated* if whenever $ab \in S$, then $b \in S$. To see the origin of the name “left saturation closure” we have to examine the following generalization of left saturation:

Definition 6.6. Let T be a multiplicative set in a ring R and M a left R -module. A subset P of M is called *left T -saturated* if for all $t \in T$ and $m \in M$, $tm \in P$ implies $m \in P$. $\diamond$

Restricting ourselves to multiplicative sets T guarantees that $P \subseteq \text{LSat}_T(P)$. With this, we can prove what has been foreshadowed by the notation already:

Proposition 6.7 (cf. [Proposition 3.6](#)). *Let T be a multiplicative set in a ring R , M a left R -module and P a subset of M .*

- (a) $\text{LSat}_T(P)$ is left T -saturated.
- (b) P is left T -saturated if and only if $P = \text{LSat}_T(P)$.
- (c) $\text{LSat}_T(P)$ is the smallest left T -saturated superset of P with respect to inclusion.

If two sets P_1 and P_2 are left T -saturated, then so is their intersection:

Lemma 6.8 (cf. [Lemma 3.7](#)). *Let T be a multiplicative set in a monoid R and M a left R -module. If P_1 and P_2 are two left T -saturated subsets of M , then $P_1 \cap P_2$ is also left T -saturated.*

Applying left saturation at T absorbs the effect of left saturation at any subset of T :

Lemma 6.9 (cf. [Lemma 3.8](#)). *Let S, T be two multiplicative sets in a ring R , M a left R -module and P, N two subsets of M . If $S \subseteq T$, then*

$$\text{LSat}_S^N(\text{LSat}_T^N(P)) = \text{LSat}_T^N(P) = \text{LSat}_T^N(\text{LSat}_S^N(P)).$$

Actually computing the left saturation closure, either by hand or with a computer algebra system, can be very hard, as we will see in [Proposition 8.27](#). In contrast, checking whether two sets P_1 and P_2 share the same left saturation closure at T is typically easier, especially if we combine [Lemma 5.40](#) with the following characterization:

Lemma 6.10 (cf. [Lemma 3.9](#)). *Let T be a multiplicative set in a ring R , M a left R -module and P_1, P_2 two subsets of M .*

- (a) *The following are equivalent:*
 - (1) $P_1 \subseteq \text{LSat}_T(P_2)$.
 - (2) $\text{LSat}_T(P_1) \subseteq \text{LSat}_T(P_2)$.
- (b) *The following are equivalent:*
 - (1) $P_1 \subseteq \text{LSat}_T(P_2)$ and $P_2 \subseteq \text{LSat}_T(P_1)$.
 - (2) $\text{LSat}_T(P_1) = \text{LSat}_T(P_2)$.
- (c) *If $P_1 \subseteq P_2 \subseteq \text{LSat}_T(P_1)$, then $\text{LSat}_T(P_1) = \text{LSat}_T(P_2)$.*

Example 6.11. Consider the multiplicative sets $V := [x, \partial]$ and $\Theta := [\{x\partial + z \mid z \in \mathbb{Z}\}]$ in the first Weyl algebra A_1 over a field of characteristic 0, then by definition $\partial \in \text{LSat}(\Theta)$ since $x \cdot \partial = x\partial + 0 \in \Theta$. But we also have $\partial \cdot x = x\partial + 1 \in \Theta$ and thus $x \in \text{LSat}(\Theta)$. Together, this implies $\text{LSat}(V) \subseteq \text{LSat}(\Theta)$ by [Lemma 5.40](#) and [Lemma 6.10](#). In fact, we even have $\text{LSat}(V) = \text{LSat}(\Theta)$, which will be proven in [Lemma 8.25](#) in a more general setting. $\diamond$

A multiplicative set generated by smaller multiplicative sets is left saturated if and only if its building blocks are:

Lemma 6.12 (cf. [Lemma 3.10](#)). *Let R be a ring, M a left R -module and P a left R -submodule of M . Further, let $\{S_i\}_{i \in I}$ be a family of multiplicative sets in R and $S := [\bigcup_{i \in I} S_i]$. Then $\text{LSat}_S(P) = P$ if and only if $\text{LSat}_{S_i}(P) = P$ all $i \in I$.*

This observation immediately suggests multiple strategies for computing the left saturation closure at a multiplicative set built in this way if we are able to compute the closure at its generating sets:

Proposition 6.13 (cf. [Proposition 3.11](#)). *Let R be a ring, M a left R -module and P a left R -submodule of M . Further, let $\{S_i\}_{i \in I}$ be a family of multiplicative sets in R , $S := [\bigcup_{i \in I} S_i]$ and $f : \mathbb{N} \rightarrow I$ a map. Define $P_0 := P$ and $P_n := \text{LSat}_{S_{f(n)}}(P_{n-1})$ for all $n \in \mathbb{N}$.*

- (a) *For all $n \in \mathbb{N}$, $P_n = \text{LSat}_{S_{f(1)}S_{f(2)} \cdots S_{f(n)}}(P)$.*
- (b) *For all $n \in \mathbb{N}$, $\text{LSat}_S(P_n) = \text{LSat}_S(P)$.*
- (c) *Let $l, m \in \mathbb{N}_0$ such that $\{f(l+j) \mid j \in \{1, \dots, m\}\} = I$ and $P_l = P_{l+m}$, then $P_l = \text{LSat}_S(P)$.*
- (d) *Let $I = \{1, \dots, k\}$, $S_i S_j = S_j S_i$ for all $i, j \in I$ and $f(I) = I$, then $P_k = \text{LSat}_S(P)$.*

The Θ -closure algorithm presented in [Algorithm 14.19](#) is an application of this recipe.

Chapter 7.

A canonical form for left denominator sets

The theory of left saturation closure encompasses at least two important special cases. The first one arises when we set $T = R$ and thus consider left saturation closure at the ring itself, which will lead us to a canonical form of left denominator sets and further structural insights into the localization itself.

7.1. Left saturation at the ring

Definition 7.1. Let P be a subset of a ring R . The *left saturation closure* of P in R is

$$\text{LSat}(P) := \text{LSat}_R^R(P) = \{r \in R \mid \exists w \in R : wr \in P\}.$$

◇

For left saturation with respect to the ring itself we omit the prefix R :

Definition 7.2. Let S be a subset of a ring R . Then S is called

- *left saturated* if for all $s, t \in R$, $st \in S$ implies $t \in S$.
- *right saturated* if for all $s, t \in R$, $st \in S$ implies $s \in S$.
- *saturated* if S is both left and right saturated.

◇

Example 7.3. Let L be a left ideal in a ring R , then $R \setminus L$ is left saturated: let $a, b \in R$ such that $ab \in R \setminus L$. Assume $b \in L$, then $ab \in L$, which is a contradiction. Thus $b \notin L$, which implies $b \in R \setminus L$.

◇

By [Lemma 6.4](#) we have $\text{LSat}_R(L) = R$ for any left ideal L in R , thus we will turn to multiplicative sets and later on left denominator sets to have something interesting to saturate.

Remark 7.4. Let S and T be two multiplicative sets in a monoid R . Sometimes S is called *left dense* in T if $S \subseteq T \subseteq \text{LSat}(S)$ ([Bav17](#)).

◇

As a first observation we see that any left saturation of a non-empty subset includes all left invertible elements:

Lemma 7.5 (cf. [Lemma 4.4](#)). *Let R be a ring, then $\text{LSat}([1]) = U_l(R)$. In particular, we have $U_l(R) \subseteq \text{LSat}(P)$ for any non-empty subset P of R .*

Example 7.6 (cf. [Example 5.32](#)). Consider again the multiplicative subset S of the matrix ring $R = \mathbb{Z}^{2 \times 2}$ given by $S = \{\text{diag}(z, z) \mid z \in \mathbb{Z} \setminus \{0\}\}$. Since $\text{diag}(z, z) = \text{diag}(z, -z) \cdot \text{diag}(-z, z)$ holds for every integer z , we have $\text{LSat}(S) = \{\text{diag}(a, b) \mid a, b \in \mathbb{Z} \setminus \{0\}, |a| = |b|\}$.

◇

For any subset of R , LSat preserves and reflects the left Ore condition:

Lemma 7.7 (cf. [Lemma 4.5](#)). *Let S be a subset of a ring R . The following are equivalent:*

- (1) S satisfies the left Ore condition in R .
- (2) $\text{LSat}(S)$ satisfies the left Ore condition in R .

Corollary 7.8. *Let S and T be subsets of a ring R such that $\text{LSat}(S) = \text{LSat}(T)$. Then S satisfies the left Ore condition in R if and only if T satisfies the left Ore condition in R .*

Even more, LSat preserves left Ore sets:

Lemma 7.9 (cf. [Lemma 4.7](#)). *Let S be a left Ore set in R . Then $\text{LSat}(S)$ is also a left Ore set in R .*

Lemma 7.10 (cf. [Lemma 4.8](#)). *Let $\varphi : R \rightarrow T$ be a homomorphism of rings and $S \subseteq R$, then $\varphi(\text{LSat}(S)) \subseteq \text{LSat}(\varphi(S))$. If φ is an isomorphism, then $\varphi(\text{LSat}(S)) = \text{LSat}(\varphi(S))$.*

The first main result of this theory concerns the left invertible elements in the localization $S^{-1}R$, which can be fully characterized by LSat :

Proposition 7.11 (cf. [Proposition 4.9](#)). *Let S be a left denominator set in a ring R and $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) $(s, r) \in U_l(S^{-1}R)$.
- (2) $(1, r) \in U_l(S^{-1}R)$.
- (3) $r \in \text{LSat}(S)$.
- (4) ${}_R\langle r \rangle \cap S \neq \emptyset$.

Furthermore, if $(s, r) \in U_l(S^{-1}R)$, then a left inverse of (s, r) in $S^{-1}R$ is given by (wr, ws) , where $w \in R$ satisfies $wr \in S$.

The question whether ${}_R\langle r \rangle \cap S$ is empty or not is a special case of the *intersection problem* discussed in [Section 13.2](#).

7.2. Homomorphisms of localized rings

Next, we investigate homomorphisms between different localizations of the same ring. In particular, we are interested in homomorphisms that preserve the image of R in the respective localizations:

Definition 7.12. Let S and T be left denominator sets in a ring R . A homomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$ of rings *fixes R* or is an *R -fixing homomorphism* if $\varphi \circ \rho_{S,R} = \rho_{T,R}$. The existence of an R -fixing isomorphism $S^{-1}R \rightarrow T^{-1}R$ is denoted by $S^{-1}R \cong_R T^{-1}R$. $\diamond$

Lemma 7.13 (cf. [Lemma 4.11](#)). *Let S and T be left denominator sets in a ring R . If $\varphi : S^{-1}R \rightarrow T^{-1}R$ is an R -fixing isomorphism, then $\varphi^{-1} : T^{-1}R \rightarrow S^{-1}R$ is also an R -fixing isomorphism.*

The centerpiece for this theory is the following map:

Lemma 7.14 (cf. [Lemma 4.12](#)). *Let S and T be left denominator sets in a ring R such that $S \subseteq \text{LSat}(T)$. For any $s \in S$ choose a $w_s \in R$ such that $w_s s \in T$. Then for any left R -module M the map*

$$\omega_{S,T,R,M} : S^{-1}M \rightarrow T^{-1}M, \quad (s, m) \mapsto (w_s s, w_s m),$$

is well-defined, independent of the choice of the w_s , and additive.

Proof: The only thing left to show is additivity of $\omega := \omega_{S,T,R,M}$, the rest follows from [Lemma 4.12](#): let $(s_1, m_1), (s_2, m_2) \in S^{-1}M$, then

$$\omega((s_1, m_1) + (s_2, m_2)) = \omega((s_a s_1, s_a m_1 + r_a m_2)) = (w_a s_a s_1, w_a s_a m_1 + w_a r_a m_2)$$

and

$$\omega((s_1, m_1)) + \omega((s_2, m_2)) = (w_1 s_1, w_1 m_1) + (w_2 s_2, w_2 m_2) = (t_b w_1 s_1, t_b w_1 m_1 + r_b w_2 m_2),$$

where $s_a \in S$, $r_a, r_b, w_a, w_1, w_2 \in R$ and $t_b \in T$ satisfy

$$s_a s_1 = r_a s_2, \tag{N1}$$

$w_a s_a s_1 \in T$, $w_1 s_1 \in T$, $w_2 s_2 \in T$ and

$$t_b w_1 s_1 = r_b w_2 s_2. \tag{N2}$$

We have to show $(w_a s_a s_1, w_a s_a m_1 + w_a r_a m_2) = (t_b w_1 s_1, t_b w_1 m_1 + r_b w_2 m_2)$ in $T^{-1}M$. By the left Ore condition on T there exist $t_c \in T$ and $r_c \in R$ such that

$$t_c t_b w_1 s_1 = r_c w_a s_a s_1, \tag{N3}$$

which implies the existence of $s_d \in S$ such that

$$s_d t_c t_b w_1 = s_d r_c w_a s_a \tag{N4}$$

by left reversibility of S . Since $s_d \in S \subseteq \text{LSat}(T)$ there exists $w_d \in R$ such that $w_d s_d \in T$. Now

$$w_d s_d t_c r_b w_2 s_2 \stackrel{(N2)}{=} w_d s_d t_c t_b w_1 s_1 \stackrel{(N3)}{=} w_d s_d r_c w_a s_a s_1 \stackrel{(N1)}{=} w_d s_d r_c w_a r_a s_2$$

implies the existence of $s_e \in S$ such that

$$s_e w_d s_d t_c r_b w_2 = s_e w_d s_d r_c w_a r_a \tag{N5}$$

by left reversibility of S . Since $s_e \in S \subseteq \text{LSat}(T)$ there exists $w_e \in R$ such that $w_e s_e \in T$. Define $\mathring{t} := w_e s_e w_d s_d t_c \in T$ and $\mathring{r} := w_e s_e w_d s_d r_c \in R$, then

$$\mathring{t} t_b w_1 s_1 = w_e s_e w_d s_d t_c t_b w_1 s_1 \stackrel{(N3)}{=} w_e s_e w_d s_d r_c w_a s_a s_1 = \mathring{r} w_a s_a s_1$$

and

$$\begin{aligned} \mathring{t}(t_b w_1 m_1 + r_b w_2 m_2) &= w_e s_e w_d s_d t_c t_b w_1 m_1 + w_e s_e w_d s_d t_c r_b w_2 m_2 \\ &\stackrel{(N4)}{=} w_e s_e w_d s_d r_c w_a s_a m_1 + w_e s_e w_d s_d t_c r_b w_2 m_2 \\ &\stackrel{(N5)}{=} w_e s_e w_d s_d r_c w_a s_a m_1 + w_e s_e w_d s_d r_c w_a r_a m_2 \\ &= \mathring{r}(w_a s_a m_1 + w_a r_a m_2) \end{aligned}$$

show that $(w_a s_a s_1, w_a s_a m_1 + w_a r_a m_2) = (t_b w_1 s_1, t_b w_1 m_1 + r_b w_2 m_2)$, thus ω is additive. $\square$

Remark 7.15. If $M = R$ in the situation of [Lemma 7.14](#) we write $\omega_{S,T,R}$ instead of $\omega_{S,T,R,M}$. $\diamond$

Now we can see that LSat also characterizes whether an R -fixing homomorphism from $S^{-1}R$ to $T^{-1}R$ can exist:

Proposition 7.16 (cf. [Proposition 4.14](#)). *Let S and T be left denominator sets in a ring R . The following are equivalent:*

- (1) $S \subseteq \text{LSat}(T)$.
- (2) *There exists some R -fixing ring homomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$.*
- (3) *There exists a unique R -fixing ring homomorphism from $S^{-1}R$ to $T^{-1}R$ which is given by*

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ satisfies $ws \in T$.

$$\begin{array}{ccc} & R & \\ \rho_{S,R} \swarrow & & \searrow \rho_{T,R} \\ S^{-1}R & \xrightarrow{\omega_{S,T,R}} & T^{-1}R \end{array}$$

Proof: Follows from [Proposition 4.14](#) in combination with [Lemma 7.14](#). $\square$

Lemma 7.17 (cf. [Lemma 4.15](#)). *Let S and T be left denominator sets in a ring R such that $\text{LSat}(S) = \text{LSat}(T)$. Then $\omega_{S,T,R,M}$ is a bijective map with inverse $\omega_{T,S,R,M}$.*

The symmetric version of [Proposition 7.16](#) yields the corresponding result for isomorphisms:

Proposition 7.18 (cf. [Proposition 4.16](#)). *Let S and T be left denominator sets in a ring R . The following are equivalent:*

- (1) $\text{LSat}(S) = \text{LSat}(T)$.
- (2) *There exists some R -fixing ring isomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$.*
- (3) *There exists a unique R -fixing ring isomorphism from $S^{-1}R$ to $T^{-1}R$ which is given by*

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ satisfies $ws \in T$, with inverse $\omega_{T,S,R}$.

$$\begin{array}{ccc} & R & \\ \rho_{S,R} \swarrow & & \searrow \rho_{T,R} \\ S^{-1}R & \xrightarrow{\omega_{S,T,R}} & T^{-1}R \\ & \xleftarrow{\omega_{T,S,R}} & \end{array}$$

Proof: Follows from [Proposition 4.16](#) in combination with [Proposition 7.16](#). $\square$

Example 7.19. Let K be a field of characteristic zero, $A := A_1(K)$ and consider the sets $V := [x, \partial]$ and $\Theta := [\{x\partial + z \mid z \in \mathbb{Z}\}]$. Again, it is not obvious that V and Θ are left Ore sets, which will follow from [Corollary 8.11](#) and [Proposition 8.20](#), respectively. Neither set is a subset of the other, but it is relatively easy to show that $\text{LSat}(V) = \text{LSat}(\Theta)$. Now [Proposition 7.18](#) directly implies $V^{-1}A \cong_A \Theta^{-1}A$, which is not so easy to see up front. $\diamond$

Remark 7.20. Let S and T be left denominator sets in a ring R with $S \subseteq T$. Then we have $\omega_{S,T,R}((s, r)) = (s, r)$ for all $(s, r) \in S^{-1}R$. In particular, $\rho_{S,R} = \omega_{[1],S,R}$ if we identify R with $[1]^{-1}R$. $\diamond$

Note that with LSat we can say nothing about other homomorphisms:

Example 7.21. Let K be a field. There are two obvious localizations of $K[x, y]$ resulting in a polynomial ring in one variable over a larger field, namely $S_x^{-1}K[x, y] \cong K(x)[y]$ and $S_y^{-1}K[x, y] \cong K(y)[x]$ for $S_x := K[x] \setminus \{0\}$ and $S_y := K[y] \setminus \{0\}$. By a simple degree-argument, both S_x and S_y are saturated, thus $\text{LSat}(S_x) \cap \text{LSat}(S_y) = K \setminus \{0\}$, which means that S_x and S_y are as different as they could possibly be. In particular, there is no $K[x, y]$ -fixing isomorphism between the two localizations. Nevertheless, $K(x)[y] \cong K(y)[x]$ by exchange of variables. $\diamond$

Finally, we show how a homomorphism between potentially different rings can be lifted to a homomorphism of localizations:

Lemma 7.22 (Lifting, cf. [Lemma 4.18](#)). *Let $\phi : R \rightarrow T$ be homomorphism of rings, S a left denominator set in R , and $\phi(S)$ a left denominator set in T . Consider*

$$\Phi : S^{-1}R \rightarrow \phi(S)^{-1}T, \quad (s, r) \mapsto (\phi(s), \phi(r)).$$

- (a) *The map Φ is a homomorphism of rings.*
- (b) *We have $\Phi \circ \rho_{S,R} = \rho_{\phi(S),T} \circ \phi$.*
- (c) *If ϕ is surjective, then Φ is surjective.*
- (d) *If ϕ is injective, then Φ is injective.*
- (e) *If Φ and $\rho_{S,R}$ are injective, then ϕ is injective.*

$$\begin{array}{ccc} R & \xrightarrow{\phi} & T \\ \rho_{S,R} \downarrow & & \downarrow \rho_{\phi(S),T} \\ S^{-1}R & \xrightarrow{\Phi} & \phi(S)^{-1}T \end{array}$$

Proof: (a) The only thing left to show is additivity of Φ , the rest follows from [Lemma 4.18](#): let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$. We have

$$\Phi((s_1, r_1) + (s_2, r_2)) = \Phi((s_a s_1, s_a r_1 + r_a r_2)) = (\phi(s_a s_1), \phi(s_a r_1 + r_a r_2))$$

and

$$\begin{aligned} \Phi((s_1, r_1)) + \Phi((s_2, r_2)) &= (\phi(s_1), \phi(r_1)) + (\phi(s_2), \phi(r_2)) \\ &= (\phi(s_b)\phi(s_1), \phi(s_b)\phi(r_1) + t_b\phi(r_2)), \end{aligned}$$

where $s_a, s_b \in S$, $r_a \in R$ and $t_b \in T$ satisfy

$$s_a s_1 = r_a s_2, \tag{O1}$$

$$\phi(s_b)\phi(s_1) = t_b\phi(s_2). \tag{O2}$$

We have to show $(\phi(s_a s_1), \phi(s_a r_1 + r_a r_2)) = (\phi(s_b) \phi(s_1), \phi(s_b) \phi(r_1) + t_b \phi(r_2))$. By the left Ore condition on $\phi(S)$ there exist $s_c \in S$ and $t_c \in T$ such that

$$\phi(s_c) \phi(s_b) \phi(s_1) = t_c \phi(s_a) \phi(s_1). \quad (\text{O3})$$

By left reversibility of $\phi(S)$ there exists $s_d \in S$ such that

$$\phi(s_d) \phi(s_c) \phi(s_b) = \phi(s_d) t_c \phi(s_a). \quad (\text{O4})$$

Again by left reversibility of $\phi(S)$,

$$\phi(s_c) t_b \phi(s_2) \stackrel{(\text{O2})}{=} \phi(s_c) \phi(s_b) \phi(s_1) \stackrel{(\text{O3})}{=} t_c \phi(s_a) \phi(s_1) \stackrel{(\text{O1})}{=} t_c \phi(r_a) \phi(s_2)$$

implies the existence of $s_e \in S$ such that

$$\phi(s_e) \phi(s_c) t_b = \phi(s_e) t_c \phi(r_a). \quad (\text{O5})$$

By the left Ore condition on S there exist $s_f \in S$ and $r_f \in R$ such that

$$s_g := s_f s_d = r_f s_e \in S. \quad (\text{O6})$$

Define $\mathring{s} := s_g s_c \in S$ and $\mathring{t} := \phi(s_g) t_c \in T$, then

$$\phi(\mathring{s}) \phi(s_b) \phi(s_1) = \phi(s_g) \phi(s_c) \phi(s_b) \phi(s_1) \stackrel{(\text{O3})}{=} \phi(s_g) t_c \phi(s_a) \phi(s_1) = \mathring{t} \phi(s_a) \phi(s_1)$$

and

$$\begin{aligned} \phi(\mathring{s})(\phi(s_b) \phi(r_1) + t_b \phi(r_2)) &= \phi(s_g) \phi(s_c) \phi(s_b) \phi(r_1) + \phi(s_g) \phi(s_c) t_b \phi(r_2) \\ &\stackrel{(\text{O6})}{=} \phi(s_f) \phi(s_d) \phi(s_c) \phi(s_b) \phi(r_1) + \phi(r_f) \phi(s_e) \phi(s_c) t_b \phi(r_2) \\ &\stackrel{(\text{O4})}{=} \phi(s_f) \phi(s_d) t_c \phi(s_a) \phi(r_1) + \phi(r_f) \phi(s_e) \phi(s_c) t_b \phi(r_2) \\ &\stackrel{(\text{O5})}{=} \phi(s_f) \phi(s_d) t_c \phi(s_a) \phi(r_1) + \phi(r_f) \phi(s_e) t_c \phi(r_a) \phi(r_2) \\ &\stackrel{(\text{O6})}{=} \phi(s_g) t_c \phi(s_a) \phi(r_1) + \phi(s_g) t_c \phi(r_a) \phi(r_2) \\ &= \mathring{t} (\phi(s_a) \phi(r_1) + \phi(r_a) \phi(r_2)) \\ &= \mathring{t} (\phi(s_a r_1 + r_a r_2)) \end{aligned}$$

show that $(\phi(s_a s_1), \phi(s_a r_1 + r_a r_2)) = (\phi(s_b) \phi(s_1), \phi(s_b) \phi(r_1) + t_b \phi(r_2))$.

(b) By [Lemma 4.18](#).

(c) Let $(s, r) \in S^{-1}R$ such that $0 = \Phi((s, r)) = (\phi(s), \phi(r))$, then by [Lemma 5.23](#) there exists $\tilde{s} \in S$ such that $\phi(\tilde{s}r) = \phi(\tilde{s})\phi(r) = 0$. If ϕ is injective this implies $\tilde{s}r = 0$. Again by [Lemma 5.23](#) we have $(s, r) = 0$, thus Φ is injective.

(d) Let $r \in R$ such that $\phi(r) = 0$, then

$$\Phi(\rho_{S,R}(r)) = \rho_{\phi(S),T}(\phi(r)) = \rho_{\phi(S),T}(0) = (1, 0)$$

implies $\rho_{S,R}(r) = 0$ and thus $r = 0$ by injectivity of Φ and $\rho_{S,R}$. □

Example 7.23 (cf. [Example 4.19](#)). Let K be a field and $R := K\langle x, y \mid xy = 1 \rangle$ the algebra of one-sided inverses, which contains the polynomial ring $K[x]$ as a subring, thus we can consider the canonical embedding $\phi : K[x] \rightarrow R$, $r \mapsto r$. Since $K[x]$ is a commutative ring, $S := [x]$ is a left denominator set in $K[x]$. From [Example 5.42](#) we know that $S = \phi(S)$ is also a left denominator set in R . By [Lemma 7.22](#), the map

$$\Phi : S^{-1}K[x] \rightarrow S^{-1}R, \quad (s, r) \mapsto (\phi(s), \phi(r)) = (s, r)$$

is an injective homomorphism of rings. In this case, Φ is surjective even if ϕ is not: let $(x^l, \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} y^a x^b) \in S^{-1}R$ and consider again $\bar{a} := \max \{a \in \mathbb{N}_0 \mid \exists b \in \mathbb{N}_0 : c_{a,b} \neq 0\}$, then

$$(x^l, \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} y^a x^b) = (x^{\bar{a}+l}, \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} x^{b+\bar{a}-a}) = \Phi((x^{\bar{a}+l}, \sum_{(a,b) \in \mathbb{N}_0^2} c_{a,b} x^{b+\bar{a}-a})).$$

Thus, $[x]^{-1}K\langle x, y \mid xy = 1 \rangle \cong [x]^{-1}K[x] \cong K[x, x^{-1}]$. $\diamond$

In particular, the localization of a non-commutative ring with zero-divisors can be a non-zero commutative domain.

7.3. Iterated localizations

A difficult and thus interesting question is whether a given localization can be interpreted as a sequence or an iteration of localizations. As one would expect, if $S \subseteq T$, the localization of $S^{-1}R$ at the image of T is isomorphic to $T^{-1}R$. With LSat, we can give a slightly more general result.

Lemma 7.24 (cf. [Lemma 4.20](#)). *Let S and T be two left denominator sets in a ring R with $S \subseteq \text{LSat}(T)$. Then $\rho_{S,R}(T)$ is a left denominator set in $S^{-1}R$.*

A short way to think of the next proposition is the following: if $S \subseteq \text{LSat}(T)$, then $(S^{-1}T)^{-1}(S^{-1}R) \cong T^{-1}R$, where we set $S^{-1}T := \rho_{S,R}(T)$.

Proposition 7.25 (cf. [Proposition 4.21](#)). *Let S and T be two left denominator sets in a ring R . The following are equivalent:*

- (1) $S \subseteq \text{LSat}(T)$.
- (2) *There exists some ring homomorphism $\psi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R$ satisfying $\psi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{T,R}$.*
- (3) *There exists a unique ring isomorphism $\varphi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R$ satisfying $\varphi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{T,R}$ which is given by*

$$\varphi : (\rho_{S,R}(T))^{-1}(S^{-1}R) \rightarrow T^{-1}R, \quad ((1, t), (s, r)) \mapsto (wst, wr).$$

$$\begin{array}{ccccc}
 & & R & & \\
 & \swarrow \rho_{S,R} & & \searrow \rho_{T,R} & \\
 S^{-1}R & & & & T^{-1}R \\
 & \xleftarrow{\omega_{S,T,R}} & & \xrightarrow{\omega_{S,T,R}} & \\
 & \searrow \rho_{\rho_{S,R}(T), S^{-1}R} & & \nearrow \varphi \text{ resp. } \psi & \\
 & & \rho_{S,R}(T)^{-1}(S^{-1}R) & &
 \end{array}$$

Proof: The only thing left to show is that $S \subseteq \text{LSat}(T)$ implies the additivity of φ , the rest follows from [Proposition 4.21](#): let $\omega := \omega_{S,T,R}$ and

$$a_1 := ((1, t_1), (s_1, r_1)), a_2 := ((1, t_2), (s_2, r_2)) \in (\rho_{S,R}(T))^{-1}(S^{-1}R).$$

Consider $t_a \in T$, $s_a, s_b, s_c, s_d \in S$ and $r_a, r_b, r_c, r_d, w, w_1, w_2 \in R$ such that

$$(1, t_a t_1) = (1, t_a) \cdot (1, t_1) = (s_a, r_a) \cdot (1, t_2) = (s_a, r_a t_2)$$

as well as

$$w_1 s_1 \in T, \quad w_2 s_2 \in T, \quad s_b t_a = r_b s_1, \quad s_c r_a = r_c s_2, \quad s_d s_b = r_d s_c s_a, \quad \text{and} \quad w s_d s_b \in T,$$

then

$$\begin{aligned} \varphi(a_1 + a_2) &= \varphi(((1, t_1), (s_1, r_1)) + ((1, t_2), (s_2, r_2))) \\ &= \varphi(((1, t_a) \cdot (1, t_1), (1, t_a) \cdot (s_1, r_1) + (s_a, r_a) \cdot (s_2, r_2))) \\ &= \varphi(((1, t_a t_1), (s_b, r_b r_1) + (s_c s_a, r_c r_2))) \\ &= \varphi(((1, t_a t_1), (s_d s_b, s_d r_b r_1 + r_d r_c r_2))) \\ &= (w s_d s_b t_a t_1, w(s_d r_b r_1 + r_d r_c r_2)) \\ &= (t_a t_1, 1) \cdot (w s_d s_b, w(s_d r_b r_1 + r_d r_c r_2)) \\ &= (t_a t_1, 1) \cdot \omega((s_d s_b, s_d r_b r_1 + r_d r_c r_2)) \\ &= (t_a t_1, 1) \cdot \omega((s_b, r_b r_1) + (s_c s_a, r_c r_2)) \\ &= (t_a t_1, 1) \cdot \omega((s_b, r_b r_1)) + (t_a t_1, 1) \cdot \omega((s_c s_a, r_c r_2)) \\ &= (t_a t_1, 1) \cdot \omega((1, t_a) \cdot (s_1, r_1)) + (t_a t_1, 1) \cdot \omega((s_a, r_a) \cdot (s_2, r_2)) \\ &= (t_a t_1, 1) \cdot \omega((1, t_a)) \cdot \omega((s_1, r_1)) + (t_a t_1, 1) \cdot \omega((s_a, r_a)) \cdot \omega((s_2, r_2)) \\ &= (t_a t_1, 1) \cdot (1, t_a) \cdot (w_1 s_1, w_1 r_1) + (t_a t_1, 1) \cdot \omega((s_a, r_a)) \cdot (w_2 s_2, w_2 r_2) \\ &= (t_1, 1) \cdot (w_1 s_1, w_1 r_1) + (t_a t_1, 1) \cdot \omega((s_a, r_a)) \cdot (1, t_2) \cdot (t_2, 1) \cdot (w_2 s_2, w_2 r_2) \\ &= (w_1 s_1 t_1, w_1 r_1) + (t_a t_1, 1) \cdot \omega((s_a, r_a)) \cdot \omega((1, t_2)) \cdot (w_2 s_2 t_2, w_2 r_2) \\ &= \varphi(a_1) + (t_a t_1, 1) \cdot \omega((s_a, r_a) \cdot (1, t_2)) \cdot \varphi(a_2) \\ &= \varphi(a_1) + (t_a t_1, 1) \cdot \omega((1, t_a t_1)) \cdot \varphi(a_2) \\ &= \varphi(a_1) + (t_a t_1, 1) \cdot (1, t_a t_1) \cdot \varphi(a_2) \\ &= \varphi(a_1) + \varphi(a_2). \end{aligned}$$

□

As an application, we see that it does not matter whether we localize $K[x, y]$ or $K(y)[x]$ at the (complement of the) prime ideal generated by x :

Corollary 7.26. *Let K be a field and consider $K[X, Y]$ with two finite disjoint sets of variables $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. Then $K[X, Y]_{\langle X \rangle} \cong K(Y)[X]_{\langle X \rangle}$.*

Proof: Let $R := K[X, Y]$, $S := K[Y] \setminus \{0\}$, $T := K[X, Y] \setminus \langle X \rangle$ and $\rho := \rho_{S,R}$, then $S \subseteq T$ and

$$\rho(T)^{-1}(K(Y)[X]) = \rho(T)^{-1}(S^{-1}R) \cong T^{-1}R = K[X, Y]_{\langle X \rangle}$$

by [Proposition 7.25](#).

Now let $A := \rho(T) \subseteq K(Y)[X]$ and $B := K(Y)[X] \setminus \langle X \rangle \subseteq K(Y)[X]$. For any $a \in A$ we have $a = \rho\left(\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha X^\alpha\right)$ for some $c_\alpha \in K[Y]$ which are zero for almost all α , but $c_0 \neq 0$. Thus $a \in B$ and $A \subseteq B$.

Let $b \in B$, then $b = \sum_{\alpha \in \mathbb{N}_0^n} \frac{f_\alpha}{g_\alpha} X^\alpha$ for some $f_\alpha, g_\alpha \in K[Y]$, such that $g_\alpha \neq 0$ for all α and $f_\alpha = 0$ for almost all α , but $f_0 \neq 0$. Let $g := \prod_{\substack{\alpha \in \mathbb{N}_0^n \\ f_\alpha \neq 0}} g_\alpha$, then $f_0 \frac{g}{g_0} \in K[Y] \setminus \{0\}$ and thus

$$gb = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha \frac{g}{g_\alpha} X^\alpha = \rho\left(\sum_{\alpha \in \mathbb{N}_0^n} f_\alpha \frac{g}{g_\alpha} X^\alpha\right) \in \rho(K[X, Y] \setminus \langle X \rangle) = \rho(T) = A,$$

which implies $B \subseteq \text{LSat}(A)$ and thus $\text{LSat}(A) = \text{LSat}(B)$ by [Lemma 6.10](#). From [Proposition 7.18](#) we finally get

$$K[X, Y]_{\langle X \rangle} = T^{-1}R \cong A^{-1}(S^{-1}R) \cong B^{-1}(S^{-1}R) = K(Y)[X]_{\langle X \rangle}. \quad \square$$

7.4. The canonical form

From [Proposition 7.18](#) we get that $\text{LSat}(S)$ determines the isomorphism class of $S^{-1}R$ under R -fixing homomorphisms, which already demonstrates the importance of this construction. But ideally, we would like $\text{LSat}(S)$ to be a left denominator set as well:

Proposition 7.27 (cf. [Proposition 4.22](#)). *Let S be a left denominator set in a ring R . If $\text{LSat}(S)$ is a left denominator set in R , then $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.*

Proof: Follows from [Proposition 7.18](#), since $\text{LSat}(S) = \text{LSat}(\text{LSat}(S))$. $\square$

Towards a characterization of the left denominator sets S satisfying this property, we look towards Dedekind-finite rings, which are also called *von Neumann finite* and *directly finite* in the literature ([\[Coh06\]](#)):

Definition 7.28. A ring R is called *Dedekind-finite* if for all elements $a, b \in R$, $a \cdot b = 1$ implies $b \cdot a = 1$. The rings that are not Dedekind-finite are sometimes called *Dedekind-infinite*. $\diamond$

Dedekind-finite rings are not to be confused with Dedekind domains or Dedekind rings: those are commutative domains in which every non-zero proper ideal factors into prime ideals.

A ring embedding into a Dedekind-finite ring has to be Dedekind-finite itself:

Lemma 7.29 (cf. [Lemma 4.24](#)). *Let $\varphi : R \rightarrow T$ be an injective homomorphism of rings. If T is Dedekind-finite, then so is R .*

In Dedekind-finite rings we do not have to distinguish between left units and right units:

Lemma 7.30 (cf. [Lemma 4.25](#)). *Let R be a ring. Then the following are equivalent:*

- (1) $U(R)$ is saturated.
- (2) R is Dedekind-finite.
- (3) $U_l(R) = U(R) = U_r(R)$.

As a consequence, we get the following improved results:

Lemma 7.31 (cf. [Lemma 4.26](#)). *Let R be a Dedekind-finite ring, then $\text{LSat}([1]) = U(R)$.*

Proposition 7.32 (cf. [Proposition 4.27](#)). *Let S be a left denominator set in a ring R such that $S^{-1}R$ is Dedekind-finite and $(s, r) \in S^{-1}R$. The following are equivalent:*

- (1) $(s, r) \in U(S^{-1}R)$.
- (2) $(1, r) \in U(S^{-1}R)$.
- (3) $r \in \text{LSat}(S)$.

Furthermore, if $(s, r) \in U(S^{-1}R)$, then the inverse of (s, r) in $S^{-1}R$ is given by (wr, ws) , where $w \in R$ satisfies $wr \in S$.

Now we can characterize when $\text{LSat}(S)$ is again a left denominator set:

Proposition 7.33 (cf. [Proposition 4.28](#)). *Let S be a left denominator set in a ring R . Then the following are equivalent:*

- (1) $S^{-1}R$ is Dedekind-finite.
- (2) $\text{LSat}(S)$ is left reversible in R .
- (3) $\text{LSat}(S)$ is a left denominator set in R .
- (4) $\text{LSat}(S)$ is saturated in R .

If the equivalent conditions are satisfied, then we also have $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.

Note that the localization of a Dedekind-infinite ring might be Dedekind-finite:

Example 7.34 (cf. [Example 4.29](#)). Let K be a field and consider again $R := K\langle x, y \mid xy = 1 \rangle$, the algebra of one-sided inverses. As the name suggests, R is not Dedekind-finite, since $yx \neq 1 = xy$. From [Example 7.23](#) we know that the localization of R at $[x]$ is isomorphic to the Laurent polynomial ring $K[x, x^{-1}]$, which is commutative and thus Dedekind-finite. According to [Lemma 7.29](#), this implies that the localization map $\rho := \rho_{[x], R}$ is not injective, which can also be seen from $\rho(yx) = (1, yx) = (x, xyx) = (x, x) = 1 = \rho(1)$. $\diamond$

To the best of our knowledge, the following is an open problem:

Question 7.35. Is every left Ore localization of a Dedekind-finite ring Dedekind-finite? $\diamond$

We know that a ring R in which $\text{LSat}(S)$ is a left denominator set for all left denominator sets S must be Dedekind-finite by identifying R with its localization $[1]^{-1}R$, and a sufficient condition is given by the following:

Definition 7.36. A ring R is called *zero-insertive* or *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $arb = 0$ for all $r \in R$. $\diamond$

Lemma 7.37. *Let S be a left denominator set in a zero-insertive ring R , then $\text{LSat}(S)$ is also a left denominator set.*

Proof: By [Proposition 7.33](#) it suffices to show left reversibility of $\text{LSat}(S)$: let $r \in R$ and $x \in \text{LSat}(S)$ such that $rx = 0$. By definition of $\text{LSat}(S)$, there exists $w \in R$ such that $wx \in S$. Since R is zero-insertive, we know that $rw x = 0$ must hold as well. But then there exists $s \in S \subseteq \text{LSat}(S)$ satisfying $sr = 0$ by left reversibility of S . $\square$

Many rings of (especially practical) interest are zero-insertive, including domains and commutative rings. We will look at both of these special instances in the following sections.

7.5. Localization of domains

Recall that a domain is a ring with no non-zero zero-divisors. In contrast to the case for monoids, the property of being a domain is two-sided:

Lemma 7.38. *Let R be a ring. The following are equivalent:*

- (1) R is a domain.
- (2) $(R, \cdot, 1)$ is a left monoid domain.
- (3) $(R, \cdot, 1)$ is a right monoid domain.

Proof: We only show the equivalence to the left-sided version, the rest follows analogously.

(1) $\Rightarrow$ (2): Let $a, b, c \in R$ such that $ab = ac$, which is equivalent to $a(b - c) = 0$. If $a \neq 0$ we have $b - c = 0$ and thus $b = c$, since R is a domain.

(2) $\Rightarrow$ (1): Let $a, b \in R$ such that $ab = 0$. If $a \neq 0$, then $a \cdot b = 0 = a \cdot 0$ implies $b = 0$ by the left cancellation property on $R \setminus \{0\}$. $\square$

Just like with Dedekind-finite, a ring embedding into a domain is also a domain:

Lemma 7.39 (cf. [Lemma 4.34](#)). *Let $\varphi : R \rightarrow T$ be an injective homomorphism of rings. If T is a domain, then so is R .*

Proof: Let $r_1, r_2 \in R$ such that $r_1 r_2 = 0$, then $0_T = \varphi(0_R) = \varphi(r_1 r_2) = \varphi(r_1) \varphi(r_2)$ implies $\varphi(r_1) = 0$ or $\varphi(r_2) = 0$ since T is a domain. Injectivity of φ gives us $r_1 = 0$ or $r_2 = 0$, thus R is a domain. Alternatively, this is a consequence of [Lemma 4.34](#) by applying [Lemma 7.38](#) twice. $\square$

In fact, being a domain is the stronger property:

Lemma 7.40 (cf. [Lemma 4.35](#)). *Any domain is zero-insertive and thus Dedekind-finite.*

Proof: Let R be a domain and $a, b \in R$ such that $ab = 0$, then $a = 0$ or $b = 0$. Both cases imply $arb = 0$ for all $r \in R$, thus R is zero-insertive. $\square$

Now we turn back towards localization:

Lemma 7.41. *Let R be a ring. The following are equivalent:*

- (1) R is a domain.
- (2) $R \setminus \{0\}$ is a multiplicative set.
- (3) For any multiplicative subset S of R , $S \setminus \{0\}$ is a multiplicative set.
- (4) For any subset S of R satisfying $0 \notin S$ we have $0 \notin [S]$.

Proof: (1) $\Leftrightarrow$ (2): Obvious, since R has no non-zero zero-divisors if and only if the product of two non-zero elements is again non-zero.

(2) $\Rightarrow$ (4): If $0 \notin S$, then $S \subseteq R \setminus \{0\}$, thus $[S] \subseteq R \setminus \{0\}$ since $R \setminus \{0\}$ is a multiplicative set.

(4) $\Rightarrow$ (2): Let $S = R \setminus \{0\}$, then $R \setminus \{0\} \subseteq [S] \subseteq R \setminus \{0\}$ since $0 \notin [S]$, thus $[S] = R \setminus \{0\}$.

(2) $\Rightarrow$ (3): Since $1 \in S$ and $1 \in R \setminus \{0\}$ we also have $1 \in S \setminus \{0\}$. Let S be a multiplicative subset of R and $a, b \in S \setminus \{0\} \subseteq R \setminus \{0\}$, then $ab \in S \cap (R \setminus \{0\}) = S \setminus \{0\}$. Furthermore, $ab \in R \setminus \{0\}$, since R is a domain.

(3) $\Rightarrow$ (2): Special case, since R is a ring and thus a multiplicative subset of itself. $\square$

In light of this lemma, the following is not an enormous restriction:

Convention 7.42. In domains, we only consider multiplicative sets that do not contain 0. $\diamond$

Recall that by [Lemma 5.13](#), any left Ore set in a domain is also a left denominator set. If R is a domain, then its localizations behave even more like the rational numbers:

Lemma 7.43. Let S be a left Ore set in a domain R and $(s, r) \in S^{-1}R$.

- (a) We have $(s, r) = 1$ if and only if $s = r$.
- (b) We have $(s, r) = 0$ if and only if $r = 0$.
- (c) Let $p \in R$, then $(1, p) = (s, r)$ if and only if $sp = r$.

Proof: (a) Let $(s, r) = 1$, then by [Lemma 5.23](#) there exists $\tilde{s} \in S$ such that $\tilde{s}s = \tilde{s}r$, which is equivalent to $\tilde{s}(s - r) = 0$. Then $s = r$ since $\tilde{s} \neq 0$ and R is a domain. On the other hand, if $s = r$, then $1 \cdot s = 1 \cdot r$ with $1 \in S$, thus $(s, r) = 1$ by [Lemma 5.23](#).

(b) Let $(s, r) = 0$, then by [Lemma 5.23](#) there exists $\tilde{s} \in S$ such that $\tilde{s}r = 0$. Then $r = 0$ since $\tilde{s} \neq 0$ and R is a domain. On the other hand, if $r = 0$, then $1 \cdot r = r = 0$ with $1 \in S$, thus $(s, r) = 0$ by [Lemma 5.23](#).

(c) If $sp = r$, then $(1, p) = (s, sp) = (s, r)$. On the other hand, let $(1, p) = (s, r)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s = \tilde{r} \cdot 1 = \tilde{r}$ and $\tilde{s}r = \tilde{r}p = \tilde{s}sp$, which implies $sp = r$ since R is a domain and $\tilde{s} \neq 0$. $\square$

Since any domain R is zero-insertive, we know that all left Ore localizations of R are Dedekind-finite. In fact, they are even domains themselves:

Proposition 7.44 (cf. [Corollary 4.39](#)). Let S be a left denominator set in a ring R . Then the following are equivalent:

- (1) R is a domain.
- (2) $S^{-1}R$ is a domain and $\rho_{S,R}$ is injective.

Proof: Let $\rho := \rho_{S,R}$.

(1) $\Rightarrow$ (2): Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ such that $(1, 0) = (s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$. By [Lemma 7.43](#) we have $\tilde{r}r_2 = 0$, thus $\tilde{r} = 0$ or $r_2 = 0$ since R is a domain. If $r_2 = 0$ then $(s_2, r_2) = 0$, so assume $\tilde{r} = 0$. But then $\tilde{s}r_1 = \tilde{r}s_2 = 0$ implies $r_1 = 0$ and $(s_1, r_1) = 0$ since $\tilde{s} \neq 0$ and R is a domain, therefore $S^{-1}R$ is a domain. By [Proposition 5.24](#) we have $\ker(\rho) = \{r \in R \mid \exists s \in S : sr = 0\}$. Now $sr = 0$ and $s \neq 0$ imply $r = 0$, thus $\ker(\rho) = \{0\}$, hence ρ is injective.

(2) $\Rightarrow$ (1): This is a special case of [Lemma 7.39](#). $\square$

Based on [Proposition 7.44](#) we can state the following adapted version of [Proposition 7.33](#):

Proposition 7.45. *Let S be a left Ore set in a domain R . Then $\text{LSat}(S)$ is a saturated left Ore set in R and $S^{-1}R \cong_R \text{LSat}(S)^{-1}R$.*

The fact that $\text{LSat}(S)$ is saturated on both sides gives us further insight into the defining equivalence relation of Ore localization and its variants described in [Lemma 5.17](#):

Lemma 7.46. *Let S be a left Ore set in a domain R , M a left R -module and consider $(s_1, m_1), (s_2, m_2) \in S \times M$. The following are equivalent:*

- (1) $(s_1, m_1) \sim (s_2, m_2)$, that is, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}m_2 = \tilde{r}m_1$.
- (2) For all $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ there exists $\bar{s} \in S$ such that $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (3) There exist $\hat{s}, \bar{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}s_2 = \hat{r}s_1$ and $\bar{s}\hat{s}m_2 = \bar{s}\hat{r}m_1$.
- (4) $(s_1, m_1) \approx (s_2, m_2)$, that is, there exist $a, b \in R$ such that $as_2 = bs_1 \in S$ and $am_2 = bm_1$.
- (5) There exist $\dot{s} \in S$ and $\dot{x} \in \text{LSat}(S)$ such that $\dot{s}s_2 = \dot{x}s_1$ and $\dot{s}m_2 = \dot{x}m_1$.

Proof: Since (5) is a special case of (1), the only thing to show is that (1) implies (5), the rest follows from [Lemma 5.17](#): from $\dot{s} := \tilde{s} \in S$ we know that $\tilde{r}s_1 = \tilde{s}s_2 \in S$ and therefore $\dot{x} := \tilde{r} \in \text{LSat}(S)$. $\square$

In particular, if S is already saturated, then for any pair $s_1, s_2 \in S$ we can find Ore data $(\tilde{s}_1, \tilde{s}_2)$ contained in $S \times S$.

Recall that the existence of an R -fixing homomorphism between $S^{-1}R$ and $T^{-1}R$ is equivalent to $S \subseteq \text{LSat}(T)$. In a domain we additionally have that any such homomorphism must be injective:

Proposition 7.47 (cf. [Proposition 7.16](#)). *Let S and T be left Ore sets in a domain R . The following are equivalent:*

- (1) $S \subseteq \text{LSat}(T)$.
- (2) There exists some R -fixing ring homomorphism $\varphi : S^{-1}R \rightarrow T^{-1}R$.
- (3) There exists a unique injective R -fixing ring homomorphism from $S^{-1}R$ to $T^{-1}R$ which is given by

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ satisfies $ws \in T$.

Proof: The only thing left to show is injectivity of $\omega := \omega_{S,T,R}$, the rest follows from [Proposition 7.16](#): let $(s, r) \in \ker(\omega)$. Since $(s, 1) \in U(S^{-1}R)$ we have $\omega((s, 1)) \in U(T^{-1}R)$, in particular we have $\omega((s, 1)) \neq 0$. Now $0 = \omega((s, r)) = \omega((s, 1)) \cdot \omega((1, r)) = \omega((s, 1)) \cdot (1, r)$, because ω is R -fixing. Since $T^{-1}R$ is a domain and $\omega((s, 1)) \neq 0$, we have $(1, r) = 0$, which implies $r = 0$ and thus $(s, r) = 0$. Therefore, ω is injective. $\square$

7.5.1. Left Ore domains

The following is a generalization of the classical quotient field construction:

Definition 7.48. A domain R is called a *left Ore domain* if $R \setminus \{0\}$ is a left Ore set in R . The associated localization $\text{Quot}(R) := (R \setminus \{0\})^{-1}R$ is called the *left quotient (skew) field*, *left (skew) field of fractions* or *left division ring* of R . $\diamond$

Therefore, any left Ore domain can be embedded into a division ring. In particular, this holds for any left Noetherian domain:

Lemma 7.49. *Let R be a left Noetherian domain. Then R is a left Ore domain.*

Proof: Let $r \in R$ and $s \in S$. If $r = 0$, then $\tilde{r} := 0 \in R$ and $\tilde{s} := 1 \in S$ satisfy $\tilde{s}r = 0 = \tilde{r}s$, so let $r \neq 0$. For $n \in \mathbb{N}_0$ define the left ideal $I_n := \sum_{i=0}^n {}_R \langle rs^i \rangle$. Since R is left Noetherian, the ascending chain $I_0 \subseteq I_1 \subseteq \dots$ has to become stationary. Let $n \in \mathbb{N}_0$ be minimal with the property $I_n = I_{n+1}$. Now $rs^{n+1} \in I_n$, thus there exist $a_i \in R$ such that $rs^{n+1} = \sum_{i=0}^n a_i rs^i$. This gives us

$$a_0 r = rs^{n+1} - \sum_{i=1}^n a_i rs^i = \underbrace{\left(rs^n - \sum_{i=1}^n a_i rs^{i-1} \right)}_{=: \tilde{r}} s.$$

If $a_0 = 0$ then also $\tilde{r} = 0$ since R is a domain, which implies

$$rs^n = \sum_{i=1}^n a_i rs^{i-1} = \sum_{i=0}^{n-1} a_{i+1} rs^i \in I_{n-1}.$$

Now $I_n = I_{n-1}$, which contradicts the minimality of n , thus $a_0 \in S$. $\square$

If a domain R has a left quotient field, then this also holds for any localization of R :

Lemma 7.50. *Let S be a left Ore set in a left Ore domain R . Then $S^{-1}R$ is a left Ore domain.*

Proof: Let $(s_1, r_1) \in S^{-1}R$ and $(s_2, r_2) \in S^{-1}R \setminus \{0\}$, then $r_2 \in R \setminus \{0\}$. Since R is a left Ore domain there exist $x' \in R \setminus \{0\}$ and $y' \in R$ such that $x'r_1 = y'r_2$. Define $x := (1, x') \cdot (1, s_1) \in S^{-1}R \setminus \{0\}$ and $y := (1, y') \cdot (1, s_2) \in S^{-1}R$, then

$$x(s_1, r_1) = (1, x') \cdot (1, s_1) \cdot (s_1, r_1) = (1, x'r_1) = (1, y'r_2) = (1, y') \cdot (1, s_2) \cdot (s_2, r_2) = y(s_2, r_2). \quad \square$$

A nice consequence of the left Ore condition on $R \setminus \{0\}$ is that two non-zero left ideals always have a non-zero intersection:

Lemma 7.51. *Let R be a left Ore domain, $S := R \setminus \{0\}$ and I_1, I_2 two left ideals in R with $I_1 \neq \{0\} \neq I_2$. Then $I_1 \cap I_2 \neq \{0\}$.*

Proof: Let $f_1 \in I_1 \setminus \{0\}$ and $f_2 \in I_2 \setminus \{0\}$. Since $f_1, f_2 \in S$ we have $Rf_1 \cap Sf_2 \neq \emptyset$, which implies $\{0\} \neq Rf_1 \cap Rf_2 \subseteq I_1 \cap I_2$. $\square$

Since in a left Ore domain the quotient field $\text{Quot}(R)$ always exists, we can pose the question which left Ore sets also lead to $\text{Quot}(R)$ and which are just one step shy:

Definition 7.52. Let S be a left Ore set in a left Ore domain R . We call S

- *maximal*, if $S^{-1}R \cong_R \text{Quot}(R)$.
- *pre-maximal*, if S is not maximal, but any left Ore set T with $\text{LSat}(S) \subsetneq \text{LSat}(T)$ is maximal. $\diamond$

From **Proposition 7.18** we get an immediate characterization of maximality:

Proposition 7.53. *Let S be a left Ore set in a left Ore domain R . Then the following are equivalent:*

- (1) S is a maximal left Ore set.
- (2) $S^{-1}R \cong_R \text{Quot}(R)$.
- (3) $\text{LSat}(S)$ is a maximal left Ore set.
- (4) $\text{LSat}(S) = R \setminus \{0\}$.

Example 7.54. Let K be a field and consider the Laurent polynomial ring $R = K[x, x^{-1}]$. The subset $S = K[x] \setminus \{0\}$ is a multiplicative set, but not saturated, since $x \cdot x^{-1} = 1 \in S$, but $x^{-1} \notin S$. Now let $r \in R \setminus \{0\}$, then there is a representation $r = x^{-k}f$, where $f \in S$ and $k \in \mathbb{N}_0$. Then $x^k r = x^k x^{-k} f = f \in S$ implies $r \in \text{LSat}(S)$, thus $\text{LSat}(S) = R \setminus \{0\}$ and so S is maximal. Since $x^k \in U(R)$, we also have the stronger property $\text{LSat}_{U(R)}(S) = R \setminus \{0\}$. $\diamond$

Corollary 7.55. *Let S be a left Ore set in a left Ore domain R . Then the following are equivalent:*

- (1) S is pre-maximal.
- (2) S is not maximal and for all non-zero $r \in R \setminus \text{LSat}(S)$, any left Ore set containing both S and r is maximal.

Proof: (1) $\Rightarrow$ (2): Let $r \in R \setminus \text{LSat}(S)$ be non-zero and T a left Ore set containing both S and r . Then $\text{LSat}(S) \subsetneq \text{LSat}(T)$, thus $\text{LSat}(T)$ and therefore T is maximal.

(2) $\Rightarrow$ (1): Let T be a left Ore set with $\text{LSat}(S) \subsetneq T$, then T contains a non-zero element $r \in R \setminus \text{LSat}(S)$. Since T contains S and r , T is maximal. $\square$

7.5.2. Factorization domains

For domains, we can generalize the classical notion of irreducibility:

Definition 7.56. Let R be a domain and $r \in R \setminus (\{0\} \cup U(R))$. Then r is called

- *reducible* if r can be written as a product of two non-units, that is, $r = pq$ for some $p, q \in R \setminus U(R)$.
- *irreducible* if r is not reducible. $\diamond$

Proposition 7.57. *Let $S \subseteq R$ be a left Ore set in a domain R , $p, q \in R \setminus \{0\}$ and $r = pq$.*

- (a) *We have $|\{p, q\} \cap \text{LSat}(S)| = 2$ if and only if $(1, r) \in U(S^{-1}R)$.*
- (b) *If $|\{p, q\} \cap \text{LSat}(S)| = 0$, then $(1, r)$ is reducible in $S^{-1}R$ and r is reducible in R .*

(c) If $(1, r)$ is irreducible in $S^{-1}R$, then $|\{p, q\} \cap \text{LSat}(S)| = 1$.

Proof: We always have the induced factorization $(1, r) = (1, p) \cdot (1, q)$ in $S^{-1}R$.

(a) By [Proposition 7.32](#), $|\{p, q\} \cap \text{LSat}(S)| = 2$ is equivalent to $(1, p), (1, q) \in U(S^{-1}R)$. Since $S^{-1}R$ is a domain this is equivalent to $(1, r) \in U(S^{-1}R)$ by [Lemma 7.30](#).

(b) Since $U(R) \subseteq \text{LSat}(S)$ by [Lemma 7.5](#), we have that $r = pq$ is the product of two non-zero non-units of R , thus r is reducible in R . Furthermore we get that both $(1, p)$ and $(1, q)$ are non-zero non-units in $S^{-1}R$ by [Proposition 7.32](#), thus their product $(1, r)$ is reducible in $S^{-1}R$.

(c) Follows from combining the previous parts. □

Definition 7.58. If every non-zero non-unit of a domain R can be written as a product of finitely many irreducible elements we call R a *factorization domain*. Two elements a and b in a factorization domain R are *left associated* if there exists $u \in U(R)$ such that $a = ub$. ◇

This is of course a generalization of the classical notion of a *unique factorization domain* (UFD), where any non-zero non-unit has exactly one factorization into irreducible elements.

Example 7.59. The first Weyl algebra A_1 is a factorization domain. In fact, together with the whole class of G -algebras, which provide the setting of [Part III](#), A_1 is even a *finite factorization domain* (FFD): any non-zero non-unit of a G -algebra has at least one but at most finitely many distinct factorizations into irreducible elements up to multiplication of the irreducible factors by central units in the algebra ([\[BHL17\]](#)). ◇

Lemma 7.60. Let S be a left Ore set in a factorization domain R . Up to units, $\text{LSat}(S)$ is generated by a set of irreducible elements of R that are pairwise not left associated.

Proof: Start with $T := \text{LSat}(S)$ as a trivial generating set of itself. Let $s \in T$ be reducible. Since R is a factorization domain, s can be factorized into finitely many irreducible elements $r_1, \dots, r_k$. But $\text{LSat}(S)$ is saturated, thus all r_i already belong to $\text{LSat}(S)$ and we can omit s from T . Iteratively we can remove all reducible elements from T and are left with only irreducible elements and units. Similarly, given two left associated irreducible elements in $\text{LSat}(S)$ one is a unit left multiple of the other. Since all units are contained in T we can remove one of the two irreducible elements from T and still retain a generating set. □

Thus any saturated left Ore set is uniquely determined by the set of irreducible elements it contains:

Proposition 7.61. Let R be a factorization domain. Let $\mathcal{M}$ be a set of representatives of the equivalence classes of irreducible elements in R with respect to left associativity. Let $\mathcal{S}$ be a set of representatives of the equivalence classes of saturated left Ore sets in R with respect to R -fixing isomorphisms of the induced localizations. Then $\varphi : \mathcal{S} \rightarrow \text{Pot}(\mathcal{M})$, $S \mapsto S \cap \mathcal{M}$ is an injective map.

Proof: Any saturated left Ore set in R is generated by $(S \cap \mathcal{M}) \cup U(R)$, thus S can be uniquely reconstructed from $S \cap \mathcal{M}$. □

This induces a bijection between $\mathcal{S}$ and the subsets of $\mathcal{M}$ that generate a saturated left Ore set.

Corollary 7.62. *Let R be a commutative UFD. Let $\mathcal{M}$ be a set of representatives of the equivalence classes of irreducible elements in R with respect to associativity. Let $\mathcal{S}$ be a set of representatives of the equivalence classes of saturated multiplicatively closed sets in R with respect to natural isomorphisms of the induced localizations. Then there is a bijection $\varphi : \mathcal{S} \rightarrow \text{Pot}(\mathcal{M})$, $S \mapsto S \cap \mathcal{M}$.*

Proof: Let M be a subset of $\mathcal{M}$, then $S := [M \cup U(R)]$ is a multiplicatively closed set in R that satisfies $M = S \cap \mathcal{M}$. Since any element of S up to associativity factors uniquely only into units and irreducible elements contained in M we have that S is saturated and thus $S \in \mathcal{S}$. $\square$

7.6. Localization of commutative rings

The following characterization of saturated multiplicatively closed sets in commutative rings is a classical result that has been part of many exercise sheets, e.g. taken from Exercise 7 in Chapter 3 in [AM69]:

Lemma 7.63. *Let S be a multiplicative set in a commutative ring R . Then the following are equivalent:*

- (1) $S = \text{LSat}(S)$.
- (2) $S = R \setminus \bigcup_{\mathfrak{p} \in P} \mathfrak{p}$, where P is a set of prime ideals in R .

Proof: (1) $\Rightarrow$ (2): Let $r \in R \setminus S$, then it suffices to show that r is contained in some prime ideal $\mathfrak{p}$ in R with $\mathfrak{p} \cap S = \emptyset$. Since S is saturated, no multiple of r can be contained in S and therefore $\langle r \rangle \cap S = \emptyset$. Consider the set

$$M := \{I \mid I \text{ ideal in } R, r \in I, I \cap S = \emptyset\},$$

which is partially ordered by inclusion and non-empty since $\langle r \rangle \in M$. Let Q be a totally ordered subset of M , then $I_Q := \bigcup_{I \in Q} I$ is an ideal in R which contains r . Since $I \cap S = \emptyset$ for all $I \in Q$ we have $I_Q \cap S = \emptyset$, thus $I_Q \in M$ is an upper bound for Q . By Zorn's lemma there exists a maximal element $\mathfrak{p} \in M$ with respect to inclusion. To show that $\mathfrak{p}$ is a prime ideal, let $x, y \in R \setminus \mathfrak{p}$, then $\mathfrak{p} \subsetneq \mathfrak{p} + \langle x \rangle$, thus $\mathfrak{p} + \langle x \rangle$ does not belong to M due to the maximality of $\mathfrak{p}$. But then $(\mathfrak{p} + \langle x \rangle) \cap S \neq \emptyset$ and analogously $(\mathfrak{p} + \langle y \rangle) \cap S \neq \emptyset$. Let $s \in (\mathfrak{p} + \langle x \rangle) \cap S$ and $t \in (\mathfrak{p} + \langle y \rangle) \cap S$, then $st \in (\mathfrak{p} + \langle xy \rangle) \cap S$, which implies $(\mathfrak{p} + \langle xy \rangle) \cap S \neq \emptyset$ and therefore $xy \notin \mathfrak{p}$.

(2) $\Rightarrow$ (1): Let $s, t \in R$ such that $st \in S$. Assume that $s \notin S$, then $s \in R \setminus S$ is contained in a prime ideal $\mathfrak{p}$ in R . This implies the contradiction $st \in \mathfrak{p} \subseteq R \setminus S$, therefore $s \in S$. By commutativity, the same reasoning applies to t . $\square$

Thus we can also characterize the pre-maximal sets with prime ideals:

Proposition 7.64. *Let S be a multiplicatively closed subset of a commutative domain R . Then the following are equivalent:*

- (1) S is pre-maximal.

- (2) $\text{LSat}(S) = R \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ in R of height 1 (in other words, a prime ideal that is minimal among all non-zero prime ideals with respect to inclusion).

Proof: Note that by [Lemma 7.63](#), the saturated multiplicative sets in R are exactly the complements of unions of prime ideals.

(1) $\Rightarrow$ (2): Let S be pre-maximal, then $\text{LSat}(S) \subsetneq R \setminus \{0\}$, thus $R \setminus \text{LSat}(S) = \bigcup_{\mathfrak{q} \in P} \mathfrak{q}$, where P is a non-empty set of non-zero prime ideals. Let $\mathfrak{p} \in P$, then $T := R \setminus \bigcup_{\mathfrak{q} \in P \setminus \{\mathfrak{p}\}} \mathfrak{q}$ is a saturated multiplicatively closed set with $\text{LSat}(S) \subsetneq T$, thus $T = R \setminus \{0\}$ which implies $P = \{\mathfrak{p}\}$ and $R \setminus \text{LSat}(S) = \mathfrak{p}$. Since for any prime ideal $\mathfrak{q}$ satisfying $\{0\} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$ we would have a chain $\text{LSat}(S) = R \setminus \mathfrak{p} \subsetneq R \setminus \mathfrak{q} \subsetneq R \setminus \{0\}$ of saturated multiplicatively closed set in contradiction to S being pre-maximal we have that $\mathfrak{p}$ must be a prime ideal of height 1.

(2) $\Rightarrow$ (1): Now let $\mathfrak{p} := R \setminus \text{LSat}(S)$ be a prime ideal of height 1. Then $\{0\} \subsetneq \mathfrak{p}$ and therefore $\text{LSat}(S) = R \setminus \mathfrak{p} \subsetneq R \setminus \{0\}$, thus S is not maximal. Let T be a multiplicatively closed subset of R such that $R \setminus \mathfrak{p} = \text{LSat}(S) \subsetneq T \subseteq \text{LSat}(T)$. In particular, $R \setminus \text{LSat}(T)$ is a union of prime ideals strictly contained in a prime ideal of height one, which implies $R \setminus \text{LSat}(T) = \{0\}$. Therefore we have $\text{LSat}(T) = R \setminus \{0\}$ or equivalently that T is maximal. Thus, S is pre-maximal. $\square$

Lemma 7.65. *Let I be a non-zero ideal in a commutative domain R . Then $\hat{I} := (I \setminus \{0\}) \cup \{1\}$ is a maximal multiplicative set.*

Proof: Let $a, b \in \hat{I}$. If $a = 1$, then $ab = b \in \hat{I}$. If $a \neq 1$, then $a \in I$ and thus $ab \in I$. Since R is a domain we have $ab \neq 0$ and thus $ab \in \hat{I}$. Let $f \in I \setminus \{0\}$. For all $r \in R \setminus \{0\}$ we have $fr \in I \setminus \{0\}$ and thus $r \in \text{LSat}(\hat{I})$, which implies the maximality of $\hat{I}$ by [Proposition 7.53](#). $\square$

For commutative domains, we have the following variant of [Proposition 7.25](#) concerning iterated localizations:

Lemma 7.66. *Let A and B subsets of a commutative domain such that $W := [A \cup B]$ is a multiplicative set. Then $S := [A]$ and $T := [B]$ are multiplicative sets in R , while $\rho_{S,R}(T)$ is a multiplicative set in $S^{-1}R$. Furthermore, the map*

$$\varphi : \rho_{S,R}(T)^{-1}(S^{-1}R) \rightarrow W^{-1}R, \quad ((1, t), (s, r)) \mapsto (st, r)$$

is an isomorphism of rings such that $\varphi \circ \rho_{\rho_{S,R}(T), S^{-1}R} \circ \rho_{S,R} = \rho_{W,R}$.

Proof: Elementary calculations in commutative localizations. $\square$

Corollary 7.67. *Let S be a multiplicative subset of a commutative domain R and $p \in R \setminus \{0\}$. Then*

$$\varphi : \left[\frac{p}{1} \right]^{-1}(S^{-1}R) = [\rho_{S,R}(p)]^{-1}(S^{-1}R) \rightarrow [S \cup \{p\}]^{-1}R, \quad ((1, p^k), (s, r)) \mapsto (sp^k, r)$$

is an isomorphism of rings such that $\varphi \circ \rho_{[p], S^{-1}R} \circ \rho_{S,R} = \rho_{[S \cup \{p\}], R}$.

This means there are two equivalent ways of enlarging a localization by a single element: assume we are given a commutative domain R and a multiplicatively closed subset S . Furthermore, there is an element $p \in R$ that should additionally become invertible. Then it does not matter if we localize R at $[S \cup \{p\}]$ or if we localize R at S and then localize the result again at $[\rho_{S,R}(p)]$.

As an example, we look at the the integers: the prime numbers $\mathbb{P}$ are a set of representatives of irreducible elements in the UFD $\mathbb{Z}$. Thus any saturated multiplicative set in $\mathbb{Z}$ can be identified with a subset P of $\mathbb{P}$ by determining which primes become units in the localization. Note that any such set belongs to exactly one of the following three types:

- (i) P is finite and thus $\mathbb{P} \setminus P$ is infinite, meaning that only finitely many primes are invertible in the localization.

Special case $P = \emptyset$: This corresponds to the multiplicative set $U(\mathbb{Z}) = \{1, -1\}$, which induces the largest localization of $\mathbb{Z}$ which is still isomorphic to $\mathbb{Z}$.

Special case $|P| = 1$: These are exactly the smallest non-trivial saturated localizations of $\mathbb{Z}$, which are given by $\mathbb{Z}[\frac{1}{p}] := [p, -1]^{-1}\mathbb{Z} \cong_{\mathbb{Z}} [p]^{-1}\mathbb{Z}$, where $p \in \mathbb{P}$.

- (ii) P and $\mathbb{P} \setminus P$ are infinite.

- (iii) P is infinite and $\mathbb{P} \setminus P$ is finite, meaning that only finitely many primes are not invertible in the localization.

Special case $|\mathbb{P} \setminus P| = 1$: These are exactly the localizations at pre-maximal multiplicative sets, which by [Proposition 7.64](#) are given by $\mathbb{Z} \setminus p\mathbb{Z}$ for $p \in \mathbb{P}$. The corresponding localizations are thus of the form $\mathbb{Z}_{(p)} := (\mathbb{Z} \setminus p\mathbb{Z})^{-1}\mathbb{Z}$.

Special case $P = \mathbb{P}$: Here we find the localizations at the maximal multiplicatively closed subsets of $\mathbb{Z}$, which include the sets $(n\mathbb{Z} \setminus \{0\}) \cup \{1\}$ where $n \in \mathbb{N}$.

A representation of all saturated localizations of $\mathbb{Z}$ as a binary tree is sketched in [Figure 7.1](#).

Furthermore, from [Proposition 7.16](#) we can see that the saturated localizations of $\mathbb{Z}$ form a bounded lattice with respect to inclusion, where $\mathbb{Z}$ is the minimal element and $\mathbb{Q}$ is the maximal element. A visualization of a small part of this lattice is given in [Figure 7.2](#).

Summing up, our machinery shows that the behavior of any localization of the integers depends (up to $\mathbb{Z}$ -fixing isomorphisms) only on which prime number becomes invertible.

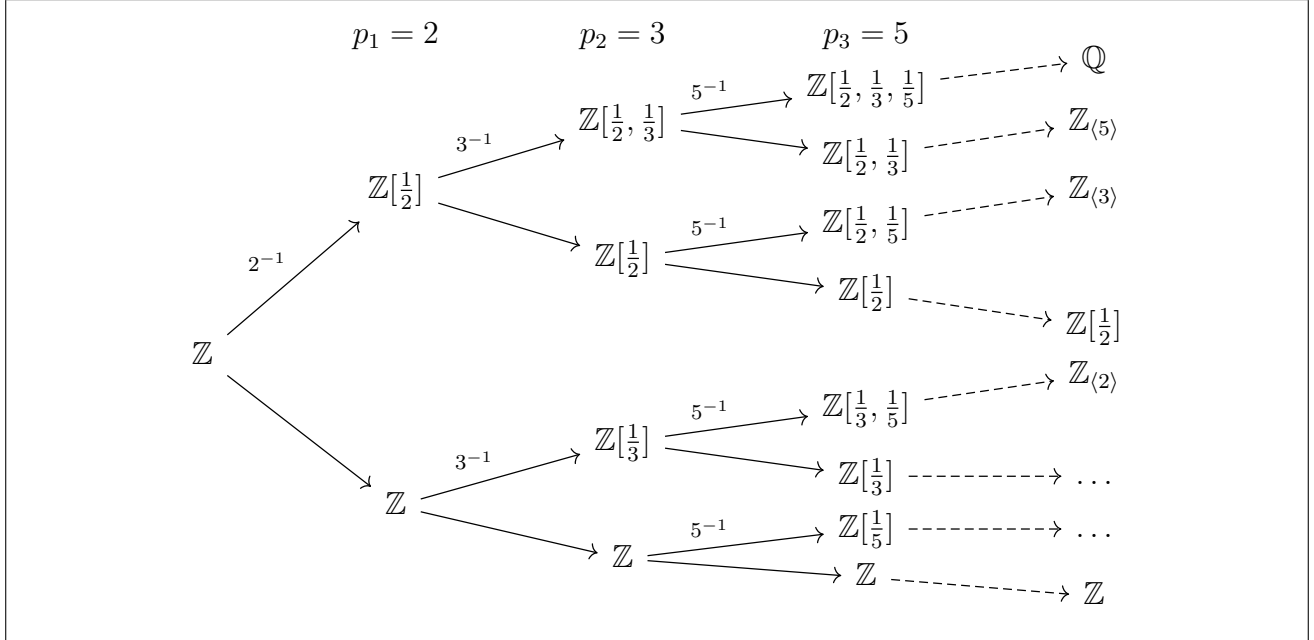


Figure 7.1.: Localizations of $\mathbb{Z}$ at saturated left Ore sets as a binary tree, where each arrow represents an embedding. Each level i corresponds to either making the i -th prime p_i invertible or not.

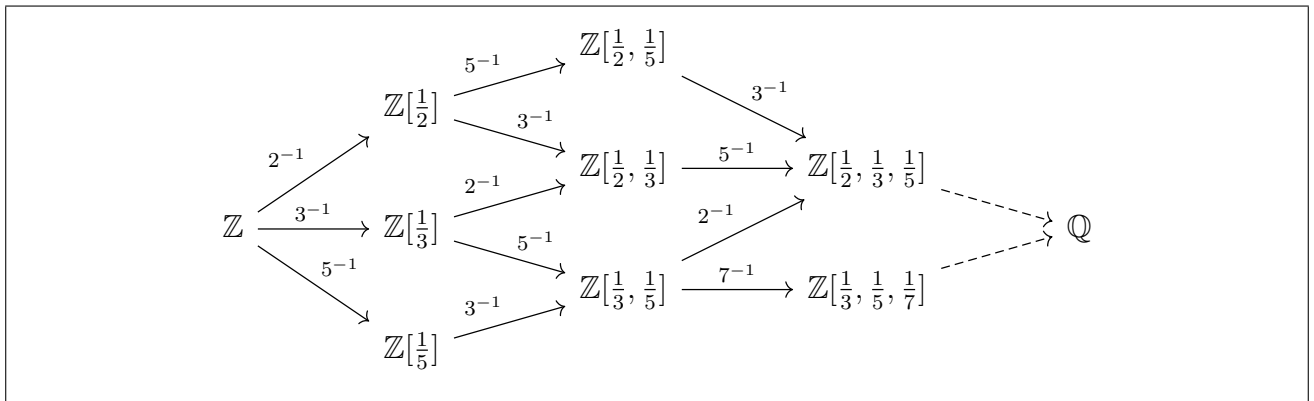


Figure 7.2.: Localizations of $\mathbb{Z}$ at saturated left Ore sets as a bounded lattice with minimum $\mathbb{Z}$ and maximum $\mathbb{Q}$, where the partial order is given by embeddings.

Chapter 8.

Weyl algebras

In the previous chapters, we already utilized the first Weyl algebra $K\langle x, \partial \mid \partial x = x\partial + 1 \rangle$ as a source for many non-trivial, but still easily understandable examples. The construction can be generalized in a natural way:

Definition 8.1. Let K be a field. The n -th (*polynomial*) *Weyl algebra* over K is

$$A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid Q \rangle$$

with the set of relations

$$Q := \{x_j x_i = x_i x_j, \partial_j \partial_i = \partial_i \partial_j, \partial_j x_i = x_i \partial_j + \delta_{i,j} \mid i, j \in \{1, \dots, n\}\}.$$

The n -th *rational Weyl algebra* over K is

$$B_n := K(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \mid Q \cup \tilde{Q} \rangle$$

with the additional relations

$$\tilde{Q} := \left\{ \partial_i f = f \partial_i + \frac{\partial f}{\partial x_i} \mid f \in K(x_1, \dots, x_n), i \in \{1, \dots, n\} \right\}.$$

Another way to view A_n is as an n -fold tensor product of A_1 over K . ◇

Remark 8.2. The polynomial rings $K[x] := K[x_1, \dots, x_n]$ and $K[\partial] := K[\partial_1, \dots, \partial_n]$ are commutative subrings of A_n . In [Lemma 13.24](#) we will see that they are saturated in A_n . ◇

The polynomial Weyl algebras provide an algebraic interpretation of linear partial differential equations with polynomial coefficients. Their action $\bullet$ on an element f in an appropriate left module of functions is induced by $x_i \bullet f := x_i \cdot f$ and $\partial_i \bullet f := \frac{\partial f}{\partial x_i}$.

The polynomial Weyl algebras belong to the class of G -algebras, which provide the setting of [Part III](#). In particular, they are Noetherian domains, thus any left Ore set in a Weyl algebra is also a left denominator set.

In the following, we deal with the cases of characteristic zero and positive characteristic simultaneously. To ease notation, we identify elements of $\mathbb{Z}$ with their image in the ground field K . Without going into further detail, we want to mention that the Weyl algebras behave quite differently depending on the characteristic. As an example, note that

$$Z(A_1) = \begin{cases} K, & \text{if } \text{char}(K) = 0, \\ K[x^p, \partial^p], & \text{if } \text{char}(K) = p \in \mathbb{P}. \end{cases}$$

8.1. Classical left Ore sets

In this section we prove that many sets we would consider localizing $K[x, \partial]$ at are also left Ore sets in A_n , for example $[x_i]$ and $K[x_i] \setminus \{0\}$. Towards this goal, we investigate several “commutation” or rewriting rules which will help us in pursuit of the left Ore condition:

Lemma 8.3. *Let $i \in \{1, \dots, n\}$.*

- (a) *For all $k \in \mathbb{N}$ we have $x_i^k \partial_i = \partial_i x_i^k - k x_i^{k-1}$.*
- (b) *For all $\alpha \in \mathbb{N}_0^n$ we have $x^{\alpha+e_i} \partial_i = \partial_i x^{\alpha+e_i} - (\alpha_i + 1) x^\alpha$.*
- (c) *Let $f \in K[x]$, then $f \partial_i = \partial_i f - \frac{\partial f}{\partial x_i}$.*

Proof: (a) Induction on $k \in \mathbb{N}$:

Induction base: Let $k = 1$, then $x_i^1 \partial_i = x_i \partial_i = \partial_i x_i - 1 = \partial_i x_i^1 - 1 \cdot x_i^{1-1}$.

Induction step: $k \rightsquigarrow k + 1$: We have

$$\begin{aligned} x_i^{k+1} \partial_i &= x_i x_i^k \partial_i \stackrel{(\text{IH})}{=} x_i (\partial_i x_i^k - k x_i^{k-1}) = x_i \partial_i x_i^k - k x_i^k \\ &\stackrel{(\text{IB})}{=} (\partial_i x_i - 1) x_i^k - k x_i^k = \partial_i x_i^{k+1} - (k + 1) x_i^k. \end{aligned}$$

(b) We have

$$x^{\alpha+e_i} \partial_i = \left(\prod_{\substack{j=1 \\ j \neq i}}^n x^{\alpha_j} \right) x_i^{\alpha_i+1} \partial_i = \left(\prod_{\substack{j=1 \\ j \neq i}}^n x^{\alpha_j} \right) (\partial_i x_i^{\alpha_i+1} - (\alpha_i + 1) x_i^{\alpha_i}) = \partial_i x^{\alpha+e_i} - (\alpha_i + 1) x^\alpha.$$

(c) Let $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha$ with $c_\alpha \in K$, then $\frac{\partial f}{\partial x_i} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i > 0}} c_\alpha \alpha_i x^{\alpha-e_i}$, thus

$$\begin{aligned} f \partial_i &= \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha \partial_i = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i > 0}} c_\alpha x^\alpha \partial_i + \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i = 0}} c_\alpha x^\alpha \partial_i = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i > 0}} c_\alpha (\partial_i x^\alpha - \alpha_i x^{\alpha-e_i}) + \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i = 0}} c_\alpha \partial_i x^\alpha \\ &= \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i > 0}} c_\alpha \partial_i x^\alpha + \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i = 0}} c_\alpha \partial_i x^\alpha - \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_i > 0}} c_\alpha \alpha_i x^{\alpha-e_i} = \partial_i \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha - \frac{\partial f}{\partial x_i} = \partial_i f - \frac{\partial f}{\partial x_i}. \end{aligned}$$

□

Definition 8.4. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, let $|\alpha| := \sum_{i=1}^n \alpha_i$. ◇

Lemma 8.5. *Let $f \in K[x]$. For all $\beta \in \mathbb{N}_0^n$ we have*

$$\partial^\beta f = \sum_{\alpha \leq_c \beta} \binom{\beta}{\alpha} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\beta-\alpha}, \quad \text{where} \quad \binom{\beta}{\alpha} := \prod_{i=1}^n \binom{\beta_i}{\alpha_i}.$$

Proof: If $|\beta| = 0$, then $\beta = 0$, thus $\partial^\beta f = f = \binom{0}{0} \frac{\partial^{0|} f}{\partial x^0} \partial^{0-0}$. Induction on $|\beta| \in \mathbb{N}$:

Induction base: $|\beta| = 1$, then $\beta = e_j$ for some $j \in \{1, \dots, n\}$, thus $\partial^\beta f = \partial_j f = f \partial_j + \frac{\partial f}{\partial x_j}$.

Induction step: Let $\beta \in \mathbb{N}_0^n \setminus \{0\}$, then $\beta = \gamma + e_j$ for some $j \in \{1, \dots, n\}$ and $\gamma \in \mathbb{N}_0^n$. For $\alpha \in \mathbb{N}_0^n$ with $\alpha \leq_c \beta$ let $k_\alpha^\beta(f) := \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\beta-\alpha}$. Now

$$\begin{aligned}
 \partial^\beta f &= \partial_j \partial^\gamma f \stackrel{(\text{IH})}{=} \partial_j \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\gamma-\alpha} = \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \partial_j \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\gamma-\alpha} \\
 &= \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial_j + \frac{\partial}{\partial x_j} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right) \partial^{\gamma-\alpha} = \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial_j + \frac{\partial^{|\alpha+e_j|} f}{\partial x^{\alpha+e_j}} \right) \partial^{\gamma-\alpha} \\
 &= \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial_j \partial^{\gamma-\alpha} + \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} \frac{\partial^{|\alpha+e_j|} f}{\partial x^{\alpha+e_j}} \partial^{\gamma-\alpha} \\
 &= \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} k_\alpha^\beta(f) + \sum_{\alpha+e_j \leq_c \gamma+e_j} \binom{\gamma}{\alpha} \frac{\partial^{|\alpha+e_j|} f}{\partial x^{\alpha+e_j}} \partial^{\beta-(\alpha+e_j)} \\
 &= \sum_{\alpha \leq_c \gamma} \binom{\gamma}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \gamma+e_j \\ \alpha_j > 0}} \binom{\gamma}{\alpha - e_j} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\beta-\alpha} \\
 &= \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j = 0}} \binom{\gamma}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j > 0}} \binom{\gamma}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j > 0}} \binom{\gamma}{\alpha - e_j} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \beta \\ \alpha_j = \gamma_j + 1}} \binom{\gamma}{\alpha - e_j} k_\alpha^\beta(f) \\
 &= \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j = 0}} \binom{\gamma}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j > 0}} \underbrace{\left(\binom{\gamma}{\alpha} + \binom{\gamma}{\alpha - e_j} \right)}_{=\binom{\beta}{\alpha}} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \beta \\ \alpha_j = \gamma_j + 1}} \underbrace{\binom{\gamma}{\alpha - e_j}}_{=\binom{\beta}{\alpha}} k_\alpha^\beta(f) \\
 &= \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j = 0}} \binom{\beta}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \gamma \\ \alpha_j > 0}} \binom{\beta}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \beta \\ \alpha_j = \gamma_j + 1}} \binom{\beta}{\alpha} k_\alpha^\beta(f) \\
 &= \sum_{\alpha \leq_c \gamma} \binom{\beta}{\alpha} k_\alpha^\beta(f) + \sum_{\substack{\alpha \leq_c \beta \\ \alpha_j = \gamma_j + 1}} \binom{\beta}{\alpha} k_\alpha^\beta(f) \\
 &= \sum_{\alpha \leq_c \beta} \binom{\beta}{\alpha} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \partial^{\beta-\alpha}.
 \end{aligned}$$

□

Lemma 8.6. Let $f \in K[x]$ and $j \in \{1, \dots, n\}$. For all $i \in \mathbb{N}_0$ we have

$$f^{i+1} \partial_j = \left(\partial_j f - (i+1) \frac{\partial f}{\partial x_j} \right) f^i.$$

Proof: Induction on $i \in \mathbb{N}_0$:

Induction base: Let $i = 0$, then $f^1 \partial_j = f \partial_j = \partial_j f - 1 \cdot \frac{\partial f}{\partial x_j} f^0$.

Induction step: $i \rightsquigarrow i + 1$: We have

$$\begin{aligned} f^{i+2}\partial_j &= f f^{i+1}\partial_j \stackrel{(\text{IH})}{=} f \left(\partial_j f - (i+1) \frac{\partial f}{\partial x_j} \right) f^i = \left(f \partial_j f - (i+1) f \frac{\partial f}{\partial x_j} \right) f^i \\ &= \left(f \partial_j - (i+1) \frac{\partial f}{\partial x_j} \right) f^{i+1} \stackrel{(\text{IB})}{=} \left(\partial_j f - \frac{\partial f}{\partial x_j} - (i+1) \frac{\partial f}{\partial x_j} \right) f^{i+1} \\ &= \left(\partial_j f - (i+2) \frac{\partial f}{\partial x_j} \right) f^{i+1}. \end{aligned} \quad \square$$

Definition 8.7. Let $r = \sum_{\alpha, \beta \in \mathbb{N}_0^n} c_{\alpha, \beta} x^\alpha \partial^\beta \in A_n$, then

$$\text{tdeg}_\partial(r) := \max \{ |\beta| \in \mathbb{N}_0 \mid \beta \in \mathbb{N}_0^n \text{ such that } c_{\alpha, \beta} \neq 0 \}. \quad \diamond$$

Lemma 8.8. Let $f \in K[x]$, $i \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$ such that $i+1 \geq |\beta|$. Then there exists $v_{i+1, \beta} \in A_n$ such that

- (i) $\text{tdeg}_\partial(v_{i+1, \beta}) < |\beta|$,
- (ii) $v_{i+1, \beta}$ only contains partial derivatives of f of the form $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}$, where $\beta - \alpha \in \mathbb{N}_0^n$ and
- (iii) $f^{i+1} \partial^\beta = (\partial^\beta f^{|\beta|} + v_{i+1, \beta}) f^{i+1-|\beta|}$.

Proof: Induction on $|\beta| \in \mathbb{N}_0$:

Induction base: $|\beta| = 0$, then $\beta = 0$. Set $v_{i+1, 0} := 0$, then

$$f^{i+1} \partial^\beta = f^{i+1} = (\partial^\beta f^{|\beta|} + v_{i+1, 0}) f^{i+1-|\beta|}.$$

Induction step: Let $\beta \in \mathbb{N}_0^n \setminus \{0\}$, then $\beta = \alpha + e_j$ for some $j \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < |\beta|$. Now

$$\begin{aligned} f^{i+1} \partial^\beta &= f^{i+1} \partial^\alpha \partial_j \stackrel{(\text{IH})}{=} (\partial^\alpha f^{|\alpha|} + v_{i+1, \alpha}) f^{i+1-|\alpha|} \partial_j = \partial^\alpha f^{i+1} \partial_j + v_{i+1, \alpha} f^{i+1-|\alpha|} \partial_j \\ &= \partial^\alpha \left(\partial_j f - (i+1) \frac{\partial f}{\partial x_j} \right) f^i + v_{i+1, \alpha} \left(\partial_j f - (i+1-|\alpha|) \frac{\partial f}{\partial x_j} \right) f^{i-|\alpha|} \\ &= \left(\partial^\alpha \partial_j f^{1+|\alpha|} - (i+1) \partial^\alpha \frac{\partial f}{\partial x_j} f^{|\alpha|} + v_{i+1, \alpha} \partial_j f - (i+1-|\alpha|) v_{i+1, \alpha} \frac{\partial f}{\partial x_j} \right) f^{i-|\alpha|} \\ &= (\partial^\beta f^{|\beta|} + v_{i+1, \beta}) f^{i+1-|\beta|}, \end{aligned}$$

where $v_{i+1, \beta} := -(i+1) \partial^\alpha \frac{\partial f}{\partial x_j} f^{|\alpha|} + v_{i+1, \alpha} \partial_j f - (i+1-|\alpha|) v_{i+1, \alpha} \frac{\partial f}{\partial x_j}$ satisfies the conditions above. $\square$

Lemma 8.9. Let $f \in K[x]$, $r \in A_n$, $d := \text{tdeg}_\partial(r)$ and $k \in \mathbb{N}_0$. Then there exist $\tilde{r}, \hat{r} \in A_n$ such that

$$f^{d+k} \cdot r = \tilde{r} \cdot f^k \quad \text{and} \quad r \cdot f^{d+k} = f^k \cdot \hat{r}.$$

Proof: Let $r = \sum_{\beta \in \mathbb{N}_0^n} b_\beta \partial^\beta$, where $b_\beta \in K[x]$. Since $d + k \geq |\beta|$ for all $\beta \in \text{supp}_\partial(r)$, by [Lemma 8.8](#) there exist $v_{d+k,\beta} \in A_n$ such that

$$f^{d+k} \partial^\beta = (\partial^\beta f^{|\beta|} + v_{d+k,\beta}) f^{d+k-|\beta|} = (\partial^\beta f^{|\beta|} + v_{d+k,\beta}) f^{d-|\beta|} f^k.$$

Define

$$\tilde{r} := \sum_{\beta \in \mathbb{N}_0^n} b_\beta (\partial^\beta f^{|\beta|} + v_{d+k,\beta}) f^{d-|\beta|},$$

then

$$f^{d+k} \cdot r = f^{d+k} \cdot \sum_{\beta \in \mathbb{N}_0^n} b_\beta \partial^\beta = \sum_{\beta \in \mathbb{N}_0^n} b_\beta f^{d+k} \partial^\beta = \sum_{\beta \in \mathbb{N}_0^n} b_\beta (\partial^\beta f^{|\beta|} + v_{d+k,\beta}) f^{d-|\beta|} f^k = \tilde{r} \cdot f^k.$$

The other statement can be shown analogously using a right-sided version of [Lemma 8.8](#). $\square$

Proposition 8.10. *Let S be a multiplicative set in $K[x]$ and T a multiplicative set in $K[\partial]$. Then S and T are left and right Ore sets in A_n .*

Proof: Since S is a multiplicative set in $K[x]$ it is also a multiplicative set in A_n . Let $r \in A_n$ and $s \in S$. By [Lemma 8.9](#) there exist $\tilde{r}, \hat{r} \in A_n$ such that $s^{d+1} \cdot r = \tilde{r} \cdot s$ and $r \cdot s^{d+1} = s \cdot \hat{r}$, where $d := \text{tdeg}_\partial(r)$. Since S is multiplicatively closed we have $s^{d+1} \in S$, therefore S satisfies the left and the right Ore condition in A_n , thus S is a left and right Ore set in A_n . The statement for T follows from analogous calculations. $\square$

Corollary 8.11. *The following are left and right Ore sets in A_n :*

- $[f] = \{f^k \mid k \in \mathbb{N}_0\}$ for any $f \in (K[x] \cup K[\partial]) \setminus \{0\}$ (monoidal).
- $[M]$ for any $M \subseteq (K[x] \cup K[\partial]) \setminus \{0\}$ (type depends on M).
- $K[x] \setminus \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subseteq K[x]$ (geometric).
- $K[x_{i_1}, \dots, x_{i_k}] \setminus \{0\}$ for any $1 \leq i_1 < \dots < i_k \leq n$ (rational).
- $K[\partial_{i_1}, \dots, \partial_{i_k}] \setminus \{0\}$ for any $1 \leq i_1 < \dots < i_k \leq n$ (rational).
- $A_n \setminus \{0\}$ (rational).

Proof: Follows from [Proposition 8.10](#) in combination with [Corollary 5.44](#) and [Lemma 7.49](#). $\square$

Remark 8.12. Prior knowledge of the localized ring can make it easier to prove that a given set is indeed a left Ore set. For example, the polynomial Weyl algebra A_n embeds into B_n in a way that every element of $K[x] \setminus \{0\}$ becomes invertible. Furthermore, by raising every term of an element in B_n to a common denominator in $K[x] \setminus \{0\}$, one can see that B_n is a left Ore localization of A_n at $K[x] \setminus \{0\}$. Thus by [Lemma 5.12](#), $K[x] \setminus \{0\}$ is a left Ore set in A_n . $\diamond$

8.2. Homogeneous left Ore sets and the Euler operators

Given $\omega \in \mathbb{Z}^n \setminus \{0\}$, there exists a non-trivial $\mathbb{Z}^n$ -grading on the n -th Weyl algebra utilizing the weight vector $[-\omega \ \omega]$ on the variables $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, assigning to the monomial $x^\alpha \partial^\beta$ for $\alpha, \beta \in \mathbb{N}_0^n$ the degree $\gamma \in \mathbb{Z}^n$ with $\gamma_i := \omega_i(\beta_i - \alpha_i)$. In the following we consider the grading induced by $\omega = [1 \ \dots \ 1]^T$. Formally, for $\gamma \in \mathbb{Z}^n$ we have

$$(A_n)_\gamma = \left\{ \sum_{\alpha, \beta \in \mathbb{N}_0^n} c_{\alpha, \beta} x^\alpha \partial^\beta \in A_n \mid c_{\alpha, \beta} \in K \text{ and } (\beta - \alpha = \gamma \text{ or } c_{\alpha, \beta} = 0) \text{ for all } \alpha, \beta \in \mathbb{N}_0^n \right\}.$$

Definition 8.13. Let $i \in \{1, \dots, n\}$. The element $\theta_i := x_i \partial_i \in A_n$ is the i -th *Euler operator*. An *integer shift* of the i -th Euler operator is an element of $\theta_i + \mathbb{Z} \subseteq A_n$. The commutative polynomial subring of A_n generated by the Euler operators is $K[\theta] := K[\theta_1, \dots, \theta_n]$. $\diamond$

The Euler operators and all their integer shifts are examples of homogeneous elements of degree $0 \in \mathbb{Z}^n$ in A_n . They play an important role in high energy physics in the study of Feynman integrals ([Bit+18]). In fact, the set of all non-zero homogeneous elements is given by $h(A_n) \setminus \{0\} = [K[\theta] \setminus \{0\} \cup \{x, \partial\}]$, which is a saturated set by [Lemma 5.50](#).

Definition 8.14. For $f \in K[\theta]$ and $\alpha \in \mathbb{Z}^n$ let $f(\theta + \alpha) := f(\theta_1 + \alpha_1, \dots, \theta_n + \alpha_n)$. Similarly, for any $A \subseteq \mathbb{Z}^n$ define $f(\theta + A) := \{f(\theta + \alpha) \mid \alpha \in A\}$. $\diamond$

With this compact notation, the following proofs are still tedious, but shorter:

Lemma 8.15. Let $z \in \mathbb{Z}$ and $i \in \{1, \dots, n\}$.

(a) For all $k, m \in \mathbb{N}$ we have

$$(\theta_i + z)^m x_i^k = x_i^k (\theta_i + z + k)^m \quad \text{and} \quad \partial_i^k (\theta_i + z)^m = (\theta_i + z + k)^m \partial_i^k.$$

(b) For all $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ we have

$$(\theta_i + z)^m x^\alpha = x^\alpha (\theta_i + z + \alpha_i)^m \quad \text{and} \quad \partial^\alpha (\theta_i + z)^m = (\theta_i + z + \alpha_i)^m \partial^\alpha.$$

(c) Let $\gamma \in \mathbb{Z}^n$ and $r \in (A_n)_\gamma$. Then $(\theta_i + z + \gamma_i)r = r(\theta_i + z)$. In particular, we have $(\theta_i + z)(\theta_j + w) = (\theta_j + w)(\theta_i + z)$ for all $w \in \mathbb{Z}$ and $j \in \{1, \dots, n\}$.

(d) For all $f \in K[\theta]$ and $\alpha \in \mathbb{N}_0^n$ we have

$$f(\theta)x^\alpha = x^\alpha f(\theta + \alpha) \quad \text{and} \quad \partial^\alpha f(\theta) = f(\theta + \alpha)\partial^\alpha.$$

(e) Let $\gamma \in \mathbb{Z}^n$, $r \in (A_n)_\gamma$ and $f \in K[\theta]$. Then

$$r \cdot f(\theta) = f(\theta + \gamma) \cdot r.$$

Proof: (a) We have

$$(\theta_i + z)x_i = x_i \partial_i x_i + zx_i = x_i(x_i \partial_i + 1) + x_i z = x_i(\theta_i + z + 1)$$

and

$$\partial_i(\theta_i + z) = \partial_i x_i \partial_i + \partial_i z = (x_i \partial_i + 1)\partial_i + z\partial_i = (\theta_i + z + 1)\partial_i.$$

The full statement follows by induction on m and k .

(b) We have

$$(\theta_i + z)x^\alpha = x^{\alpha - \alpha_i e_i}(\theta_i + z)x_i^{\alpha_i} = x^{\alpha - \alpha_i e_i}x_i^{\alpha_i}(\theta_i + z + \alpha_i) = x^\alpha(\theta_i + z + \alpha_i)$$

and

$$\partial^\alpha(\theta_i + z) = \partial_i^{\alpha_i}(\theta_i + z)\partial^{\alpha - \alpha_i e_i} = (\theta_i + z + \alpha_i)\partial_i^{\alpha_i}\partial^{\alpha - \alpha_i e_i} = (\theta_i + z + \alpha_i)\partial^\alpha.$$

The full statement follows by induction on m .

(c) Let $r = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta$, then

$$\begin{aligned} (\theta_i + z + \gamma_i)r &= (\theta_i + z + \gamma_i) \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} (\theta_i + z + \gamma_i) x^\alpha \partial^\beta \\ &= \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha (\theta_i + z + \gamma_i + \alpha_i) \partial^\beta \\ &= \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta (\theta_i + z + \gamma_i + \alpha_i - \beta_i) \\ &= \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta (\theta_i + z) \\ &= r(\theta_i + z). \end{aligned}$$

Now let $w \in \mathbb{Z}$ and $j \in \{1, \dots, n\}$. Then $(\theta_i + z)(\theta_j + w) = (\theta_j + w)(\theta_i + z)$, since $\theta_j + w \in (A_n)_0$.

(d) Let $f = \sum_{\beta \in \mathbb{N}_0^n} c_\beta \theta^\beta$, then

$$\begin{aligned} f(\theta)x^\alpha &= \sum_{\beta \in \mathbb{N}_0^n} c_\beta \theta^\beta x^\alpha = \sum_{\beta \in \mathbb{N}_0^n} c_\beta \prod_{i=1}^n \theta_i^{\beta_i} x_i^{\alpha_i} = \sum_{\beta \in \mathbb{N}_0^n} c_\beta \prod_{i=1}^n x_i^{\alpha_i} (\theta_i + \alpha_i)^{\beta_i} \\ &= \sum_{\beta \in \mathbb{N}_0^n} c_\beta x^\alpha \prod_{i=1}^n (\theta_i + \alpha_i)^{\beta_i} = x^\alpha f(\theta + \alpha) \end{aligned}$$

and analogously

$$\begin{aligned} \partial^\alpha f(\theta) &= \partial^\alpha \sum_{\beta \in \mathbb{N}_0^n} c_\beta \theta^\beta = \sum_{\beta \in \mathbb{N}_0^n} c_\beta \partial^\alpha \theta^\beta = \sum_{\beta \in \mathbb{N}_0^n} c_\beta \prod_{i=1}^n \partial_i^{\alpha_i} \theta_i^{\beta_i} \\ &= \sum_{\beta \in \mathbb{N}_0^n} c_\beta \prod_{i=1}^n (\theta_i + \alpha_i)^{\beta_i} \partial_i^{\alpha_i} = \left(\sum_{\beta \in \mathbb{N}_0^n} c_\beta \prod_{i=1}^n (\theta_i + \alpha_i)^{\beta_i} \right) \partial^\alpha \\ &= f(\theta + \alpha) \partial^\alpha. \end{aligned}$$

(e) Let $r = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta$, then by the previous parts we have

$$\begin{aligned}
 r \cdot f(\theta) &= \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta f(\theta) = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha f(\theta + \beta) \partial^\beta \\
 &= \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} f(\theta + \beta - \alpha) x^\alpha \partial^\beta = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} f(\theta + \gamma) x^\alpha \partial^\beta \\
 &= f(\theta + \gamma) \cdot \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ \beta - \alpha = \gamma}} c_{\alpha, \beta} x^\alpha \partial^\beta = f(\theta + \gamma) \cdot r.
 \end{aligned}$$

□

Lemma 8.16. *Let $f \in K[\theta]$ and $r \in A_n$. Then there exist $\tilde{r} \in A_n$ and $s \in [f(\theta + \mathbb{Z}^n)] \subseteq K[\theta]$ such that $\tilde{s}r = \tilde{r}f$.*

Proof: Let $G := \text{gsupp}(r) \subseteq \mathbb{Z}^n$ and $r = \sum_{\gamma \in G} r_\gamma$, where $r_\gamma \in (A_n)_\gamma$. Define

$$\tilde{s} := \prod_{\delta \in G} f(\theta + \delta) \in [f(\theta + \mathbb{Z}^n)] \quad \text{and} \quad \tilde{r} := \sum_{\gamma \in G} \left(\prod_{\delta \in G \setminus \{\gamma\}} f(\theta + \delta) \right) r_\gamma \in A_n,$$

then by [Lemma 8.15](#) we have

$$\begin{aligned}
 \tilde{s}r &= \prod_{\delta \in G} f(\theta + \delta) \sum_{\gamma \in G} r_\gamma = \sum_{\gamma \in G} \left(\prod_{\delta \in G} f(\theta + \delta) \right) r_\gamma \\
 &= \sum_{\gamma \in G} \left(\prod_{\delta \in G \setminus \{\gamma\}} f(\theta + \delta) \right) f(\theta + \gamma) r_\gamma \\
 &= \sum_{\gamma \in G} \left(\prod_{\delta \in G \setminus \{\gamma\}} f(\theta + \delta) \right) r_\gamma f(\theta) = \tilde{r}f.
 \end{aligned}$$

□

Proposition 8.17. *Let $f \in K[\theta] \setminus \{0\}$. Then $S := [f(\theta + \mathbb{Z}^n)]$ is a left Ore set in A_n .*

Proof: Let $\alpha, \beta \in \mathbb{Z}^n$, $r \in A_n$ and consider $f(\theta + \alpha)f(\theta + \beta) \in S$. By [Lemma 8.16](#) there exist $r_\beta \in A_n$ and $s_\beta \in [f(\theta + \beta + \mathbb{Z}^n)] = [f(\theta + \mathbb{Z}^n)] = S$ such that $s_\beta r = r_\beta f(\theta + \beta)$. Again by [Lemma 8.16](#) there exist $r_\alpha \in A_n$ and $s_\alpha \in [f(\theta + \alpha + \mathbb{Z}^n)] = S$ such that $s_\alpha r_\beta = r_\alpha f(\theta + \alpha)$. Then $s_\alpha s_\beta \in S$ and $r_\alpha \in A_n$ satisfy

$$s_\alpha s_\beta r = s_\alpha r_\beta f(\theta + \beta) = r_\alpha f(\theta + \alpha) f(\theta + \beta).$$

Since any element of S has the form $\prod_{\gamma \in G} f(\theta + \gamma)$ for some finite subset G of $\mathbb{Z}^n$, the claim follows by induction. □

Corollary 8.18. *Let $F \subseteq K[\theta] \setminus \{0\}$. Then $\left[\bigcup_{f \in F} [f(\theta + \mathbb{Z}^n)] \right]$ is a left Ore set in A_n .*

Proof: Follows from [Proposition 8.17](#) and [Corollary 5.44](#). □

Definition 8.19. For $i \in \{1, \dots, n\}$ define $\Theta_i := [\theta_i + \mathbb{Z}]$, the multiplicative closure of all integer shifts of θ_i . Furthermore, define $\Theta := [\bigcup_{i=1}^n \Theta_i] = [\bigcup_{i=1}^n (\theta_i + \mathbb{Z})]$. $\diamond$

Proposition 8.20. *The following are homogeneous left Ore sets in A_n , but not saturated:*

- $\Theta_i = [\theta_i + \mathbb{Z}] = [f(\theta + \mathbb{Z}^n)]$ for $f(\theta) = \theta_i$ (monoidal).
- $\Theta = [\bigcup_{i=1}^n \Theta_i] = [\bigcup_{i=1}^n (\theta_i + \mathbb{Z})]$ (monoidal).
- $K[\theta] \setminus \{0\} = \left[\bigcup_{f \in K[\theta] \setminus \{0\}} [f(\theta + \mathbb{Z}^n)] \right]$ (rational).

Proof: All three sets are contained in $K[\theta] \setminus \{0\}$ and are thus homogeneous. They all contain $\theta_i = x_i \partial_i$ for some $i \in \{1, \dots, n\}$, but neither of them contains x_i or ∂_i , thus they are not saturated. Finally, they are left Ore sets by [Proposition 8.17](#) and [Corollary 8.18](#). $\square$

Note that $\theta_i + \mathbb{Z}$ is a finite set if the characteristic of K is positive, hence Θ_i is finitely generated in this case.

Corollary 8.21. *The set $h(A_n) \setminus \{0\} = [K[\theta] \setminus \{0\} \cup \{x, \partial\}]$ is the largest homogeneous left Ore set in A_n .*

Now we will generalize [Example 7.19](#), where we have claimed that $\text{LSat}([x, \partial]) = \text{LSat}(\Theta)$ in A_1 and thus $[x, \partial]^{-1} A_1 \cong_{A_1} \Theta^{-1} A_1$:

Definition 8.22. For $i \in \{1, \dots, n\}$ define $V_i := [x_i, \partial_i]$, the multiplicative closure of the i -th variables. Furthermore, define $V := [\bigcup_{i=1}^n V_i] = [\bigcup_{i=1}^n \{x_i, \partial_i\}]$. $\diamond$

Remark 8.23. The sets V_i and V are monoidal left Ore sets in A_n by [Corollary 8.11](#). Furthermore, they are homogeneous and not saturated: we have $(x_i \partial_i + 2) \partial_i = \partial_i^2 x_i \in V_i$, but $x_i \partial_i + 2 \notin V$. $\diamond$

Lemma 8.24. *We have $x_i \notin \text{LSat}([\theta_i])$ for all $i \in \{1, \dots, n\}$.*

Proof: By induction we get $\text{LeftNF}(\theta_i^k | \langle x_i \rangle) = (-1)^k$, which shows that for any $w \in A_n$ and $k \in \mathbb{N}_0$, $w x_i \neq \theta_i^k$ and thus $x_i \notin \text{LSat}([\theta_i])$. $\square$

In particular, $\text{LSat}([\theta_i])$ is not saturated since $x_i \partial_i = \theta_i \in [\theta_i]$, but it is left reversible since A_n is a domain. By [Proposition 7.33](#), this implies that $[\theta_i]$ is not a left Ore set in A_n . In fact, the smallest left Ore set containing θ_i is given by Θ_i .

Lemma 8.25. *For all $i \in \{1, \dots, n\}$ we have $\text{LSat}(\Theta_i) = \text{LSat}(V_i)$.*

Proof: By [Lemma 6.10](#) and [Lemma 5.40](#) it suffices to show the inclusions $\{x_i, \partial_i\} \subseteq \text{LSat}(\Theta_i)$ and $\theta_i + \mathbb{Z} \subseteq \text{LSat}(V_i)$. The first is obvious from $x_i \partial_i = \theta_i \in \Theta_i$ and $\partial_i x_i = \theta_i + 1 \in \Theta_i$, to see the second let $z \in \mathbb{Z}$:

Case 1: $z \in \mathbb{N}_0$, then $x_i^z (\theta_i + z) = \theta_i x_i^z = x_i \partial_i x_i^z \in V_i$.

Case 2: $-z \in \mathbb{N}$, then $\partial_i^{-z} (\theta_i + z) = (\theta_i + z - z) \partial_i^{-z} = \theta_i \partial_i^{-z} = x_i \partial_i^{1-z} \in V_i$.

In both cases, there exists $r \in V_i \subseteq A_n$ such that $r(\theta_i + z) \in V_i$, thus $\theta_i + z \in \text{LSat}(V_i)$, which shows $\theta_i + \mathbb{Z} \subseteq \text{LSat}(V_i)$. $\square$

Corollary 8.26. *We have $\text{LSat}(\Theta) = \text{LSat}(V)$.*

Proposition 8.27. *We have*

$$\text{LSat}(\Theta_i) = [\Theta_i \cup \{x, \partial\} \cup U(A_n)] = [(\theta_i + \mathbb{Z}) \cup \{x_i, \partial_i\} \cup (K \setminus \{0\})].$$

Proof: Let $S := [\Theta_i \cup \{x, \partial\} \cup U(A_n)]$. By [Lemma 7.5](#) all units of A_n are contained in $\text{LSat}(\Theta_i)$, by [Lemma 8.25](#) we also have $\{x_i, \partial_i\} \subseteq \text{LSat}(\Theta_i)$. Now $\Theta_i \cup \{x_i, \partial_i\} \cup U(A_n) \subseteq \text{LSat}(\Theta_i)$, which implies $S \subseteq \text{LSat}(\Theta_i)$ by [Lemma 5.40](#).

To see the other inclusion consider that by [Lemma 8.15](#) every element $s \in S$ can be written in the form $s = uty^n$, where $u \in U(A_n) = K \setminus \{0\}$, $t \in \Theta_i$, $y \in \{x, \partial\}$ and $n \in \mathbb{N}_0$. Now [Lemma 2.7](#) in [\[HL18a\]](#) implies that every other non-trivial factorization of s can be derived by using the commutation rules given in [Lemma 8.15](#) and rewriting θ_i respectively $\theta_i + 1$ as $x_i \partial_i$ respectively $\partial_i x_i$. But the only factors that can occur in these other factorizations are x_i , ∂_i , integer shifts of θ_i or units, which are already contained in S , thus $\text{LSat}(\Theta_i) \subseteq S$. $\square$

Corollary 8.28. *We have*

$$\text{LSat}(\Theta) = \left[\bigcup_{i=1}^n \text{LSat}(\Theta_i) \right] = \left[U(A_n) \cup \bigcup_{i=1}^n ((\theta_i + \mathbb{Z}) \cup \{x_i, \partial_i\}) \right].$$

Proof: Let $S := [\bigcup_{i=1}^n \text{LSat}(\Theta_i)]$, then clearly $S \subseteq \text{LSat}(\Theta)$, since $\text{LSat}(\Theta_i) \subseteq \text{LSat}(\Theta)$ for all i . The other inclusion is an application of [Lemma 4](#) in [\[GHL16\]](#) similar to the one in the proof of [Proposition 8.27](#). $\square$

Being homogeneous is such a nice property that everything immediately falls apart if we add a non-homogeneous element:

Example 8.29. Consider the inhomogeneous set $S = [\Theta_1 \cup \{\partial_1 - 1\}]$, which is a left Ore set in A_1 by [Corollary 8.11](#). Makar-Limanov showed in [\[Mak83\]](#) that the skew field of fractions of the first Weyl algebra, which is the localization $(A_1 \setminus \{0\})^{-1} A_1$, contains a free subalgebra generated by the elements $(\partial_1 x_1, 1)$ and $(\partial_1 x_1, 1) \cdot (1 - \partial_1, 1)$. These two elements can also be found in the (smaller) localization $S^{-1} A_1$. In contrast to $\text{LSat}(\Theta)$, $\text{LSat}(S)$ is inhomogeneous and thus much harder to describe, for example, for all $n \in \mathbb{N}$ and all $z \in \mathbb{Z}$ we have

$$\left(\prod_{i=1}^n (\theta_1 + z + i) \right) (\partial_1 - 1) = (x_1 \partial_1^2 - x_1 \partial_1 + (z + 1) \partial_1 - (z + n)) \prod_{i=1}^{n-1} (\theta_1 + z + i)$$

as well as

$$(\partial_1 - 1) \prod_{i=1}^n (\theta_1 + z + i) = \left(\prod_{i=2}^n (\theta_1 + z + i) \right) (x_1 \partial_1^2 - x_1 \partial_1 + (z + n + 1) \partial_1 - (z + 1)),$$

thus $x_1 \partial_1^2 - x_1 \partial_1 + (z + 1) \partial_1 - (z + n)$ and $x_1 \partial_1^2 - x_1 \partial_1 + (z + n + 1) \partial_1 - (z + 1)$ are contained in $\text{LSat}(S)$. $\diamond$

Chapter 9.

Ore localization of modules over rings

Just like in the commutative case it is only a small step from localizing rings to localizing modules, or more precisely, from the left Ore localization of rings to the left Ore localization of left modules.

9.1. Axiomatic definition, universal property and uniqueness

Just as with rings, we start with an axiomatic definition of what the localization of a module should be:

Definition 9.1. Let S be a multiplicative set in a ring R , M a left R -module and (R_S, η) a left Ore localization of R at S . A left R_S -module M_S together with a homomorphism $\tau : M \rightarrow M_S$ of left R -modules is called a *left Ore localization* of M at S with respect to (R_S, η) if the following hold:

- (i) For all $r \in R$ and $m \in M$, $\eta(r)\tau(m) = \tau(rm)$.
- (ii) For all $x \in M_S$ there exist $s \in S$ and $m \in M$ such that $x = \eta(s)^{-1}\tau(m)$.
- (iii) $\ker(\tau) = \{m \in M \mid \exists s \in S : sm = 0\}$. ◇

Here, the first condition provides compatibility of the localization of the module with the localization of the underlying ring. The second condition imposes a fraction-like structure on the elements of M_S with denominators on the left, while the third condition again assures that the kernel of τ is as small as possible: if there exists $s \in S$ such that $sm = 0$, then

$$\tau(m) = \eta(s)^{-1}\eta(s)\tau(m) = \tau(s)^{-1}\tau(sm) = 0$$

by the first condition, thus $m \in \ker(\tau)$.

Remark 9.2. We can also study Ore localizations of modules over monoids by replacing the third condition in [Definition 9.1](#) with

- (iii') For all $r_1, r_2 \in R$, $\tau(r_1) = \tau(r_2)$ if and only if there exists $s \in S$ such that $sr_1 = sr_2$.

With similar modifications, most of the theory presented here also applies to modules over monoids, and the differences are not big enough to warrant a separate treatment in this work. ◇

The corresponding universal property reads as follows:

Proposition 9.3 (Universal property). *Let S be a multiplicative set in a ring R , M a left R -module, (R_S, η) a left Ore localization of R at S and (M_S, τ) a left Ore localization of M at S with respect to (R_S, η) . Further, let $\mathcal{N}$ be a left R_S -module and $\varphi : M \rightarrow \mathcal{N}$ a homomorphism of left R -modules such that $\eta(r)\varphi(m) = \varphi(rm)$ for all $r \in R$ and $m \in M$. Then there exists a unique homomorphism $\psi : M_S \rightarrow \mathcal{N}$ of left R_S -modules such that $\varphi = \psi \circ \tau$.*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & \mathcal{N} \\ & \searrow \tau & \nearrow \psi \\ & M_S & \end{array}$$

Proof: Define $\psi : M_S \rightarrow \mathcal{N}$, $\eta(s)^{-1}\tau(m) \mapsto \eta(s)^{-1}\varphi(m)$.

Well-definedness: Let $\eta(s_1)^{-1}\tau(m_1), \eta(s_2)^{-1}\tau(m_2) \in M_S$ such that

$$\eta(s_1)^{-1}\tau(m_1) = \eta(s_2)^{-1}\tau(m_2).$$

Since $\eta(s_1)\eta(s_2)^{-1} \in R_S$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(s_1)\eta(s_2)^{-1} = \eta(\tilde{s})^{-1}\eta(\tilde{r})$. We have

$$\tau(m_1) = \eta(s_1)\eta(s_2)^{-1}\tau(m_2) = \eta(\tilde{s})^{-1}\eta(\tilde{r})\tau(m_2),$$

thus

$$\tau(\tilde{s}m_1) = \eta(\tilde{s})\tau(m_1) = \eta(\tilde{r})\tau(m_2) = \tau(\tilde{r}m_2),$$

implies the existence of $\hat{s} \in S$ such that $\hat{s}\tilde{s}m_1 = \hat{s}\tilde{r}m_2$. From

$$\eta(\hat{s})\varphi(\tilde{s}m_1) = \varphi(\hat{s}\tilde{s}m_1) = \varphi(\hat{s}\tilde{r}m_2) = \eta(\hat{s})\varphi(\tilde{r}m_2).$$

we get $\varphi(\tilde{s}m_1) = \varphi(\tilde{r}m_2)$. Now

$$\begin{aligned} \psi(\eta(s_1)^{-1}\tau(m_1)) &= \eta(s_1)^{-1}\varphi(m_1) \\ &= \eta(s_1)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{s})\varphi(m_1) \\ &= \eta(s_1)^{-1}\eta(\tilde{s})^{-1}\varphi(\tilde{s}m_1) \\ &= \eta(s_1)^{-1}\eta(\tilde{s})^{-1}\varphi(\tilde{r}m_2) \\ &= \eta(s_1)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\varphi(m_2) \\ &= \eta(s_1)^{-1}\eta(s_1)\eta(s_2)^{-1}\varphi(m_2) \\ &= \eta(s_2)^{-1}\varphi(m_2) \\ &= \psi(\eta(s_2)^{-1}\tau(m_2)). \end{aligned}$$

Additivity: Let $x_1 := \eta(s_1)^{-1}\tau(m_1), x_2 := \eta(s_2)^{-1}\tau(m_2) \in M_S$. Since $\eta(s_2)\eta(s_1)^{-1} \in R_S$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(s_2)\eta(s_1)^{-1} = \eta(\tilde{s})^{-1}\eta(\tilde{r})$, which is equivalent to

$\eta(s_1)^{-1} = \eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})$. Now

$$\begin{aligned}
 \psi(x_1 + x_2) &= \psi(\eta(s_1)^{-1}\tau(m_1) + \eta(s_2)^{-1}\tau(m_2)) \\
 &= \psi(\eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\tau(m_1) + \eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{s})\tau(m_2)) \\
 &= \psi(\eta(s_2)^{-1}\eta(\tilde{s})^{-1}(\eta(\tilde{r})\tau(m_1) + \eta(\tilde{s})\tau(m_2))) \\
 &= \psi(\eta(\tilde{s}s_2)^{-1}(\tau(\tilde{r}m_1) + \tau(\tilde{s}m_2))) \\
 &= \psi(\eta(\tilde{s}s_2)^{-1}\tau(\tilde{r}m_1 + \tilde{s}m_2)) \\
 &= \eta(\tilde{s}s_2)^{-1}\varphi(\tilde{r}m_1 + \tilde{s}m_2) \\
 &= \eta(s_2)^{-1}\eta(\tilde{s})^{-1}(\varphi(\tilde{r}m_1) + \varphi(\tilde{s}m_2)) \\
 &= \eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\varphi(m_1) + \eta(s_2)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{s})\varphi(m_2) \\
 &= \eta(s_2)^{-1}\eta(s_2)\eta(s_1)^{-1}\varphi(m_1) + \eta(s_2)^{-1}\varphi(m_2) \\
 &= \eta(s_1)^{-1}\varphi(m_1) + \eta(s_2)^{-1}\varphi(m_2) \\
 &= \psi(\eta(s_1)^{-1}\tau(m_1)) + \psi(\eta(s_2)^{-1}\tau(m_2)) \\
 &= \psi(x_1) + \psi(x_2).
 \end{aligned}$$

Scalar multiplicativity: Let $\eta(s_1)^{-1}\eta(r_1) \in R_S$ and $\eta(s_2)^{-1}\tau(m_2) \in M_S$. Since $\eta(r_1)\eta(s_2)^{-1} \in R_S$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\eta(r_1)\eta(s_2)^{-1} = \eta(\tilde{s})^{-1}\eta(\tilde{r})$. Now

$$\begin{aligned}
 \psi(\eta(s_1)^{-1}\eta(r_1) \cdot \eta(s_2)^{-1}\tau(m_2)) &= \psi(\eta(s_1)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\tau(m_2)) \\
 &= \psi(\eta(\tilde{s}s_1)^{-1}\tau(\tilde{r}m_2)) \\
 &= \eta(\tilde{s}s_1)^{-1}\varphi(\tilde{r}m_2) \\
 &= \eta(s_1)^{-1}\eta(\tilde{s})^{-1}\eta(\tilde{r})\varphi(m_2) \\
 &= \eta(s_1)^{-1}\eta(r_1)\eta(s_2)^{-1}\varphi(m_2) \\
 &= \eta(s_1)^{-1}\eta(r_1) \cdot \psi(\eta(s_2)^{-1}\tau(m_2)).
 \end{aligned}$$

Composition: Let $m \in M$, then

$$(\psi \circ \tau)(m) = \psi(\tau(m)) = \psi(\eta(1)^{-1}\tau(m)) = \eta(1)^{-1}\varphi(m) = \varphi(m),$$

thus $\psi \circ \tau = \varphi$.

Uniqueness: Let $\psi' : M_S \rightarrow \mathcal{N}$ be a homomorphism of left R_S -modules such that $\varphi = \psi' \circ \tau$. Let $\eta(s)^{-1}\tau(m) \in M_S$, then

$$\psi'(\eta(s)^{-1}\tau(m)) = \eta(s)^{-1}\psi'(\tau(m)) = \eta(s)^{-1}\varphi(m) = \psi(\eta(s)^{-1}\tau(m)),$$

which implies $\psi' = \psi$. □

Following the usual course of action, we now get the uniqueness of the localization of a module if it exists:

Corollary 9.4 (Uniqueness). *Let S be a multiplicative set in a ring R , M a left R -module and (R_S, η) a left Ore localization of R at S . If there exists a left Ore localization of M at S with respect to (R_S, η) , then it is unique up to isomorphism.*

Proof: Let (M_S, τ) and (M'_S, τ') be two left Ore localizations of M at S with respect to (R_S, η) . By [Proposition 9.3](#) there exist unique homomorphisms $\psi : M_S \rightarrow M'_S$ (letting $\mathcal{N} = M'_S$ and $\varphi = \tau'$ with respect to (M_S, τ)) and $\psi' : M'_S \rightarrow M_S$ (letting $\mathcal{N} = M_S$ and $\varphi = \tau$ with respect to (M'_S, τ')) of left R_S -modules such that $\tau' = \psi \circ \tau$ and $\tau = \psi' \circ \tau'$.

Applying [Proposition 9.3](#) to $\mathcal{N} = M_S$ and $\varphi = \tau$ with respect to (M_S, τ) tells us that there exists a unique homomorphism $\alpha : M_S \rightarrow M_S$ of left R_S -modules such that $\tau = \alpha \circ \tau$.

$$\begin{array}{ccc}
 & M & \\
 \tau \swarrow & & \searrow \tau' \\
 M_S & \xrightarrow{\phi} & M'_S \\
 \alpha \circ \tau \swarrow & & \searrow \tau' \\
 M_S & \xleftarrow{\phi'} & M'_S
 \end{array}$$

On one hand we have $\alpha = \psi' \circ \psi$, since $\tau = \psi' \circ \tau' = \psi' \circ \psi \circ \tau$, on the other hand id_{M_S} clearly satisfies $\tau = \text{id}_{M_S} \circ \tau$, thus $\alpha = \text{id}_{M_S}$. This shows $\psi' \circ \psi = \alpha = \text{id}_{M_S}$ by uniqueness of α . By symmetry, we also have $\psi \circ \psi' = \text{id}_{M'_S}$, therefore $\psi : M_S \rightarrow M'_S$ is an isomorphism of left R_S -modules. $\square$

9.2. Construction and existence

Since we did most of the work already when constructing $S^{-1}R$, we can immediately conclude that $S^{-1}M$ is indeed a left $S^{-1}R$ -module with addition $\alpha_{S,R,M}$ and scalar multiplication $\mu_{S,R,M}$:

Proposition 9.5 (Construction). *Let S be a left denominator set in a ring R and M a left R -module. Then $S^{-1}M$ is a left $S^{-1}R$ -module with the operations*

$$+ : S^{-1}M \times S^{-1}M \rightarrow S^{-1}M, \quad ((s_1, m_1), (s_2, m_2)) \mapsto (\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$, and

$$\cdot : S^{-1}R \times S^{-1}M \rightarrow S^{-1}M, \quad ((s_1, r), (s_2, m)) \mapsto (\tilde{s}s_1, \tilde{r}m),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$.

Proof: Follows from [Lemma 2.23](#) and [Proposition 5.20](#). $\square$

Similar to the ring case, there is a *localization map* or *structural homomorphism* of $S^{-1}M$:

Proposition 9.6. *Let S be a left denominator set in a ring R and M a left R -module. The map*

$$\varepsilon_{S,R,M} : M \rightarrow S^{-1}M, \quad m \mapsto (1, m)$$

is a homomorphism of left R -modules.

Proof: Let $r \in R$ and $m, m_1, m_2 \in M$, then $\varepsilon(rm) = (1, rm) = (1, r) \cdot (1, m) = r\varepsilon(m)$ and $\varepsilon(m_1 + m_2) = (1, m_1 + m_2) = (1, m_1) + (1, m_2) = \varepsilon(m_1) + \varepsilon(m_2)$. $\square$

In the special case $M = R$ we recover the localization map of $S^{-1}R$: $\varepsilon_{S,R,R} = \rho_{S,R}$.

Lemma 9.7 (Existence). *Let S be a left denominator set in a ring R and M a left R -module. Then $(S^{-1}M, \varepsilon_{S,R,M})$ is the left Ore localization of M at S with respect to $(S^{-1}R, \rho_{S,R})$.*

Proof: Let $\varepsilon := \varepsilon_{S,R,M}$ and $\rho := \rho_{S,R}$. We verify the axioms of [Definition 9.1](#):

- (i) Let $r \in R$ and $m \in M$, then $\rho(r)\varepsilon(m) = (1, r) \cdot (1, m) = (1, rm) = \varepsilon(rm)$.
- (ii) Let $(s, m) \in S^{-1}M$, then $(s, m) = (s, 1) \cdot (1, m) = \rho(s)^{-1}\varepsilon(m)$.
- (iii) Let $m \in M$ such that $(1, m) = \varepsilon(m) = (1, 0)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} \cdot 1 = \tilde{r} \cdot 1$ and $\tilde{s}m = \tilde{r} \cdot 0 = 0$.

On the other hand, let $s \in S$ such that $sm = 0$, then

$$\varepsilon(m) = \rho(s)^{-1}\rho(s)\varepsilon(m) = \rho(s)^{-1}\varepsilon(sm) = 0. \quad \square$$

In particular, if the left Ore localization of R at S exists, then so does the left Ore localization of M at S for any left R -module M .

Proposition 9.8. *Let S, T be two left denominator sets in a ring R such that $S \subseteq \text{LSat}(T)$ and M a left R -module.*

- (a) $T^{-1}M$ becomes a left $S^{-1}R$ -module via

$$\bullet : S^{-1}R \times T^{-1}M \rightarrow T^{-1}M, \quad ((s, r), (t, m)) \mapsto (s, r) \bullet (t, m) := \omega_{S,T,R}((s, r)) \cdot (t, m).$$

- (b) The map

$$\omega_{S,T,R,M} : S^{-1}M \rightarrow T^{-1}M, \quad (s, m) \mapsto (ws, wm),$$

where $w \in R$ satisfies $ws \in T$, is the unique homomorphism of left $S^{-1}R$ -modules satisfying $\omega_{S,T,R,M} \circ \varepsilon_{S,R,M} = \varepsilon_{T,R,M}$.

$$\begin{array}{ccc} & M & \\ \varepsilon_{S,R,M} \swarrow & & \searrow \varepsilon_{T,R,M} \\ S^{-1}M & \xrightarrow{\omega_{S,T,R,M}} & T^{-1}M \end{array}$$

Proof: Let $\varepsilon_S := \varepsilon_{S,R,M}$, $\varepsilon_T := \varepsilon_{T,R,M}$, $\rho_S := \rho_{S,R}$, $\rho_T := \rho_{T,R}$ and $\omega_M := \omega_{S,T,R,M}$.

- (a) Since $S \subseteq \text{LSat}(T)$ the map

$$\omega_{S,T,R} : S^{-1}R \rightarrow T^{-1}R, \quad (s, r) \mapsto (ws, wr),$$

where $w \in R$ such that $ws \in T$, is a homomorphism of rings by [Proposition 7.16](#), thus any left $T^{-1}R$ -module can be turned into a left $S^{-1}R$ -module in this fashion.

- (b) By [Lemma 7.14](#), ω_M is well-defined and additive.

Composition: Let $m \in M$, then

$$(\omega \circ \varepsilon_{S,R,M})(m) = \omega(\varepsilon_{S,R,M}(m)) = \omega((1, m)) = (1, m) = \varepsilon_{T,R,M}(m).$$

Scalar multiplicativity: Let $(s_r, r) \in S^{-1}R$ and $(s_m, m) \in S^{-1}M$, then

$$\omega((s_r, r) \cdot (s_m, m)) = \omega((s_a s_r, r_a m)) = (w_a s_a s_r, w_a r_a m)$$

and

$$(s_r, r) \cdot \omega((s_m, m)) = (w_r s_r, w_r r) \cdot (w_m s_m, w_m m) = (t_b w_r s_r, r_b w_m m),$$

where $s_a \in S$, $t_b \in T$ and $r_a, r_b, w_a, w_r, w_m \in R$ satisfy

$$s_a r = r_a s_m, \tag{P1}$$

$w_a s_a s_r \in T$, $w_r s_r \in T$, $w_m s_m \in T$ and

$$t_b w_r r = r_b w_m s_m. \tag{P2}$$

We have to show $(w_a s_a s_r, w_a r_a m) = (t_b w_r s_r, r_b w_m m)$. By the left Ore condition on T there exist $t_c \in T$ and $r_c \in R$ such that

$$t_c t_b w_r s_r = r_c w_a s_a s_r, \tag{P3}$$

which implies the existence of $s_d \in S$ such that

$$s_d t_c t_b w_r = s_d r_c w_a s_a \tag{P4}$$

by left reversibility of S . Since

$$s_d t_c r_b w_m s_m \stackrel{(P2)}{=} s_d t_c t_b w_r r \stackrel{(P4)}{=} s_d r_c w_a s_a r \stackrel{(P1)}{=} s_d r_c w_a r_a s_m$$

there exists $s_e \in S$ such that

$$s_e s_d t_c r_b w_m = s_e s_d r_c w_a r_a \tag{P5}$$

again by left reversibility of S . Since $s_e s_d \in S \subseteq \text{LSat}(T)$ there exists $w_e \in R$ such that $w_e s_e s_d \in T$. Define $\dot{t} := w_e s_e s_d t_c \in T$ and $\dot{r} := w_e s_e s_d r_c \in R$, then

$$\dot{t} t_b w_r s_r = w_e s_e s_d t_c t_b w_r s_r \stackrel{(P3)}{=} w_e s_e s_d r_c w_a s_a s_r = \dot{r} w_a s_a s_r$$

and

$$\dot{t} r_b w_m m = w_e s_e s_d t_c r_b w_m m \stackrel{(P5)}{=} w_e s_e s_d r_c w_a r_a m = \dot{r} w_a r_a m$$

show that $(w_a s_a s_r, w_a r_a m) = (t_b w_r s_r, r_b w_m m)$.

Uniqueness: Let $\varphi : S^{-1}R \rightarrow T^{-1}R$ be a homomorphism of left $S^{-1}R$ -modules satisfying $\varphi \circ \varepsilon_S = \varepsilon_T$ and consider $(s, m) \in S^{-1}M$. Since $S \subseteq \text{LSat}(T)$ there exists $w \in R$ such that $ws \in T$. In particular, $(1, s)$ is left invertible in $T \in R$ with left inverse (ws, w) by [Proposition 4.9](#). Thus,

$$\begin{aligned} \varphi((s, m)) &= \varphi((s, 1) \cdot (1, m)) = \omega_R((s, 1)) \cdot \varphi((1, m)) = (ws, s) \cdot \varphi(\varepsilon_S(m)) \\ &= (ws, s) \cdot \varepsilon_T(m) = (ws, s) \cdot (1, m) = (ws, sm) = \omega_M((s, m)). \end{aligned} \quad \square$$

Proposition 9.9. *Let S, T be two left denominator sets in a ring R such that $\text{LSat}(S) = \text{LSat}(T)$ and M a left R -module.*

(a) $T^{-1}M$ becomes a left $S^{-1}R$ -module via

$$\bullet : S^{-1}R \times T^{-1}M \rightarrow T^{-1}M, \quad ((s, r), (t, m)) \mapsto (s, r) \bullet (t, m) := \omega_{S,T,R}((s, r)) \cdot (t, m).$$

Similarly, $S^{-1}M$ becomes a left $T^{-1}R$ -module with scalar multiplication induced by $\omega_{T,S,R}$.

(b) The map

$$\omega_{S,T,R,M} : S^{-1}M \rightarrow T^{-1}M, \quad (s, m) \mapsto (ws, wm),$$

where $w \in R$ satisfies $ws \in T$, is the unique isomorphism of left $S^{-1}R$ -modules and left $T^{-1}R$ -modules satisfying $\omega_{S,T,R,M} \circ \varepsilon_{S,R,M} = \varepsilon_{T,R,M}$, with inverse given by $\omega_{T,S,R,M}$. In particular, $S^{-1}M \cong T^{-1}M$ both as left $S^{-1}R$ -modules and as left $T^{-1}R$ -modules.

$$\begin{array}{ccc} & M & \\ \varepsilon_{S,R,M} \swarrow & & \searrow \varepsilon_{T,R,M} \\ S^{-1}M & \xrightarrow[\omega_{T,S,R,M}]{\omega_{S,T,R,M}} & T^{-1}M \end{array}$$

Proof: (a) Immediate by applying [Proposition 9.8](#) twice.

(b) Since by [Lemma 7.17](#), $\omega_{T,S,R,M}$ is the inverse of $\omega := \omega_{S,T,R,M}$, it suffices to show that ω is a homomorphism of $T^{-1}R$ -modules, the rest follows by symmetry: let $(t, r) \in T^{-1}R$ and $(s, m) \in S^{-1}M$. Since the $T^{-1}R$ -module structure of $S^{-1}M$ is induced by $\omega_{T,S,R}$, we have

$$\begin{aligned} \omega((t, r) \cdot (s, m)) &= \omega(\omega_{T,S,R}((t, r)) \cdot (s, m)) = \omega((w_t t, w_t r) \cdot (s, m)) \\ &= \omega((s_a w_t t, r_a m)) = (w_a s_a w_t t, w_a r_a m), \end{aligned}$$

where $s_a \in S$ and $r_a, w_a, w_t \in R$ satisfy $w_t t \in S$,

$$s_a w_t r = r_a s, \tag{Q1}$$

and $w_a s_a w_t t \in T$. On the other hand,

$$(t, r) \cdot \omega((s, m)) = (t, r) \cdot (w_s s, w_s m) = (t_b t, r_b w_s m),$$

where $t_b \in T$ and $r_b, w_s \in R$ satisfy $w_s s \in T$ and

$$t_b r = r_b w_s s. \tag{Q2}$$

We have to show $(w_a s_a w_t t, w_a r_a m) = (t_b t, r_b w_s m)$. By the left Ore condition on T there exist $t_c \in T$ and $r_c \in R$ such that

$$t_c t_b t = r_c w_a s_a w_t t, \tag{Q3}$$

which implies the existence of $t_d \in T$ such that

$$t_d t_c t_b = t_d r_c w_a s_a w_t \tag{Q4}$$

by left reversibility of T . Since

$$t_d t_c r_b w_s s \stackrel{(P2)}{=} t_d t_c t_b r \stackrel{(P4)}{=} t_d r_c w_a s_a w_t r \stackrel{(P1)}{=} t_d r_c w_a r_a s$$

there exists $s_e \in S$ such that

$$s_e t_d t_c r_b w_s = s_e t_d r_c w_a r_a \quad (\text{Q5})$$

by left reversibility of S . Since $s_e \in S \subseteq \text{LSat}(T)$ there exists $w_e \in R$ such that $w_e s_e \in T$. Define $\dot{t} := w_e s_e t_d t_c \in T$ and $\dot{r} := w_e s_e t_d r_c \in R$, then

$$\dot{t} t_b t = w_e s_e t_d t_c t_b t \stackrel{(\text{P3})}{=} w_e s_e t_d r_c w_a s_a w_t t = \dot{r} w_a s_a w_t t$$

and

$$\dot{t} r_b w_s m = w_e s_e t_d t_c r_b w_s m \stackrel{(\text{P5})}{=} w_e s_e t_d r_c w_a r_a m = \dot{r} w_a r_a m$$

show that $(w_a s_a w_t t, w_a r_a m) = (t_b t, r_b w_s m)$. $\square$

In analogy to [Definition 7.12](#), we can call $\omega_{S,T,R,M}$ an M -fixing homomorphism if $S \subseteq \text{LSat}(T)$.

If P is a submodule of M , then the question whether an element (s, m) of $S^{-1}M$ is contained in the submodule $S^{-1}P$ can be framed via left saturation:

Lemma 9.10. *Let S be a left denominator set in a ring R , M a left R -module, P a left R -submodule of M and $(s, m) \in S^{-1}M$. Then the following are equivalent:*

$$(1) \ (s, m) \in S^{-1}P.$$

$$(2) \ m \in \text{LSat}_S(P).$$

Proof: (1) $\Rightarrow$ (2): Let $(s, m) \in S^{-1}P$, then there exist $t \in S$ and $p \in P$ such that $(s, m) = (t, p)$ in $S^{-1}M$, thus there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s = \tilde{r}t$ and $\tilde{s}m = \tilde{r}p \in P$, thus $m \in \text{LSat}_S(P)$.

(2) $\Rightarrow$ (1): Let $m \in \text{LSat}_S(P)$, then there exists $\tilde{s} \in S$ such that $\tilde{s}m \in P$, thus we have $(s, m) = (\tilde{s}s, \tilde{s}m) \in S^{-1}P$. $\square$

9.3. Compatibility of localization with module constructions

Localization of modules is compatible with all popular module operations:

Proposition 9.11. *Let S be a left denominator set in a ring R , M and N two left R -modules, and P, P_1, P_2 three left R -submodules of M . Then we have the following:*

$$(a) \ S^{-1}(M \times N) \cong S^{-1}M \times S^{-1}N.$$

$$(b) \ S^{-1}(P_1 \cap P_2) = S^{-1}P_1 \cap S^{-1}P_2.$$

$$(c) \ S^{-1}(P_1 + P_2) = S^{-1}P_1 + S^{-1}P_2.$$

$$(d) \ S^{-1}(P_1 \oplus P_2) = S^{-1}P_1 \oplus S^{-1}P_2.$$

$$(e) \ S^{-1}(M/P) \cong S^{-1}M/S^{-1}P.$$

Proof: (a) Consider the map

$$\varphi : S^{-1}(M \times N) \rightarrow S^{-1}M \times S^{-1}N, \quad (s, (m, n)) \mapsto ((s, m), (s, n)).$$

Well-definedness: Let $(s_1, (m_1, n_1)), (s_2, (m_2, n_2)) \in S^{-1}(M \times N)$ such that we have $(s_1, (m_1, n_1)) = (s_2, (m_2, n_2))$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and

$$(\tilde{s}m_2, \tilde{s}n_2) = \tilde{s}(m_2, n_2) = \tilde{r}(m_1, n_1) = (\tilde{r}m_1, \tilde{r}n_1).$$

The latter is equivalent to $\tilde{s}m_2 = \tilde{r}m_1$ and $\tilde{s}n_2 = \tilde{r}n_1$, which together with the former implies $(s_1, m_1) = (s_2, m_2)$ and $(s_1, n_1) = (s_2, n_2)$ and thus

$$\varphi((s_1, (m_1, n_1))) = ((s_1, m_1), (s_1, n_1)) = ((s_2, m_2), (s_2, n_2)) = \varphi((s_2, (m_2, n_2))).$$

Additivity: Let $(s_1, (m_1, n_1)), (s_2, (m_2, n_2)) \in S^{-1}(M \times N)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and thus

$$\begin{aligned} \varphi((s_1, (m_1, n_1)) + (s_2, (m_2, n_2))) &= \varphi((\tilde{s}s_1, \tilde{s}(m_1, n_1) + \tilde{r}(m_2, n_2))) \\ &= \varphi((\tilde{s}s_1, (\tilde{s}m_1, \tilde{s}n_1) + (\tilde{r}m_2, \tilde{r}n_2))) \\ &= \varphi((\tilde{s}s_1, (\tilde{s}m_1 + \tilde{r}m_2, \tilde{s}n_1 + \tilde{r}n_2))) \\ &= ((\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2), (\tilde{s}s_1, \tilde{s}n_1 + \tilde{r}n_2)) \\ &= (((s_1, m_1) + (s_2, m_2)), ((s_1, n_1) + (s_2, n_2))) \\ &= ((s_1, m_1), (s_1, n_1)) + ((s_2, m_2), (s_2, n_2)) \\ &= \varphi((s_1, (m_1, n_1))) + \varphi((s_2, (m_2, n_2))). \end{aligned}$$

Scalar multiplicativity: Let $(t, r) \in S^{-1}R$ and $(s, (m, n)) \in S^{-1}(M \times N)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$ and thus

$$\begin{aligned} \varphi((t, r) \cdot (s, (m, n))) &= \varphi((\tilde{s}t, \tilde{r}(m, n))) \\ &= \varphi((\tilde{s}t, (\tilde{r}m, \tilde{r}n))) \\ &= ((\tilde{s}t, \tilde{r}m), (\tilde{s}t, \tilde{r}n)) \\ &= ((t, r) \cdot (s, m), (t, r) \cdot (s, n)) \\ &= (t, r) \cdot ((s, m), (s, n)) \\ &= (t, r) \cdot \varphi((s, (m, n))). \end{aligned}$$

Surjectivity: Let $((s_m, m), (s_n, n)) \in S^{-1}M \times S^{-1}N$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_m = \tilde{r}s_n$. Defining $s := \tilde{s}s_m \in S$ we get

$$((s_m, m), (s_n, n)) = ((\tilde{s}s_m, \tilde{s}m), (\tilde{r}s_n, \tilde{r}n)) = ((s, \tilde{s}m), (s, \tilde{r}n)) = \varphi((s, (\tilde{s}m, \tilde{r}n))) \in \text{im}(\varphi).$$

Injectivity: Let $(s_1, (m_1, n_1)), (s_2, (m_2, n_2)) \in S^{-1}(M \times N)$ such that

$$((s_1, m_1), (s_1, n_1)) = \varphi((s_1, (m_1, n_1))) = \varphi((s_2, (m_2, n_2))) = ((s_2, m_2), (s_2, n_2)).$$

This is equivalent to $(s_1, m_1) = (s_2, m_2)$ and $(s_1, n_1) = (s_2, n_2)$, thus there exist $s_m, s_n \in S$ and $r_m, r_n \in R$ such that

$$s_m s_2 = r_m s_1, \quad (\text{R1})$$

$$s_m m_2 = r_m m_1, \quad (\text{R2})$$

$$s_n s_2 = r_n s_1, \quad (\text{R3})$$

$$s_n n_2 = r_n n_1. \quad (\text{R4})$$

By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$\tilde{s} s_m = \tilde{r} s_n. \quad (\text{R5})$$

Now

$$\tilde{s} r_m s_1 \stackrel{(\text{R1})}{=} \tilde{s} s_m s_2 \stackrel{(\text{R5})}{=} \tilde{r} s_n s_2 \stackrel{(\text{R3})}{=} \tilde{r} r_n s_1$$

implies the existence of $\hat{s} \in S$ such that

$$\hat{s} \tilde{s} r_m = \hat{s} \tilde{r} r_n \quad (\text{R6})$$

by left reversibility of S . Define $\mathring{s} := \hat{s} \tilde{s} s_m \in S$ and $\mathring{r} := \hat{s} \tilde{r} r_n \in R$, then

$$\mathring{s} s_2 = \hat{s} \tilde{s} s_m s_2 \stackrel{(\text{R1})}{=} \hat{s} \tilde{s} r_m s_1 = \mathring{r} s_1$$

and

$$\begin{aligned} \mathring{s}(m_2, n_2) &= (\mathring{s} m_2, \mathring{s} n_2) = (\hat{s} \tilde{s} s_m m_2, \hat{s} \tilde{s} s_m n_2) \\ &\stackrel{(\text{R2})}{=} (\hat{s} \tilde{s} r_m m_1, \hat{s} \tilde{s} s_m n_2) \\ &\stackrel{(\text{R5})}{=} (\hat{s} \tilde{s} r_m m_1, \hat{s} \tilde{r} s_n n_2) \\ &\stackrel{(\text{R4})}{=} (\hat{s} \tilde{s} r_m m_1, \hat{s} \tilde{r} r_n n_1) \\ &\stackrel{(\text{R6})}{=} (\hat{s} \tilde{s} r_m m_1, \hat{s} \tilde{r} r_n n_1) \\ &= (\mathring{r} m_1, \mathring{r} n_1) = \mathring{r}(m_1, n_1) \end{aligned}$$

show that $(s_1, (m_1, n_1)) = (s_2, (m_2, n_2))$.

- (b) Let $x \in S^{-1}(P_1 \cap P_2)$, then $x = (s, m)$ for some $s \in S$ and $m \in P_1 \cap P_2$, thus $x \in S^{-1}P_1 \cap S^{-1}P_2$. On the other hand, let $x \in S^{-1}P_1 \cap S^{-1}P_2$, then x has two representations $x = (s_1, p_1) \in S^{-1}P_1$ and $x = (s_2, p_2) \in S^{-1}P_2$. Now $(s_1, p_1) = x = (s_2, p_2)$ in $S^{-1}M$, thus there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} s_2 = \tilde{r} s_1$ and $\tilde{s} p_2 = \tilde{r} p_1$. Define $s := \tilde{s} s_2 \in S$ and $m := \tilde{s} p_2 = \tilde{r} p_1 \in P_1 \cap P_2$, then $x = (s_2, p_2) = (\tilde{s} s_2, \tilde{s} p_2) = (s, m) \in S^{-1}(P_1 \cap P_2)$.
- (c) Let $x \in S^{-1}(P_1 + P_2)$, then $x = (s, p_1 + p_2)$ for some $s \in S$, $p_1 \in P_1$ and $p_2 \in P_2$, thus $x = (s, p_1 + p_2) = (s, p_1) + (s, p_2) \in S^{-1}P_1 + S^{-1}P_2$.

On the other hand, let $x \in S^{-1}P_1 + S^{-1}P_2$, then $x = (s_1, p_1) + (s_2, p_2)$ for some $(s_1, p_1) \in S^{-1}P_1$ and $(s_2, p_2) \in S^{-1}P_2$. By the left Ore condition on S there exist $s \in S$ and $r \in R$ such that $ss_1 = rs_2$, thus

$$x = (s_1, p_1) + (s_2, p_2) = (ss_1, sp_1) + (rs_2, rp_2) = (ss_1, sp_1 + rp_2) \in S^{-1}(P_1 + P_2).$$

(d) Follows from the two previous statements.

(e) Consider the map

$$\varphi : S^{-1}(M/P) \rightarrow S^{-1}M/S^{-1}P, \quad (s, [m]) \mapsto [(s, m)],$$

where the square brackets denote the respective equivalence classes.

Well-definedness: Let $(s_1, [m_1]), (s_2, [m_2]) \in S^{-1}(M/P)$ such that $(s_1, [m_1]) = (s_2, [m_2])$, then there exist $s \in S$ and $r \in R$ such that $ss_2 = rs_1$ and $[sm_2] = s[m_2] = r[m_1] = [rm_1]$. The latter equation implies $sm_2 = rm_1 + p$ for some $p \in P$. Now

$$\begin{aligned} (s_2, m_2) &= (ss_2, sm_2) = (ss_2, rm_1 + p) = (ss_2, rm_1) + (ss_2, p) \\ &= (rs_1, rm_1) + (ss_2, p) = (s_1, m_1) + (ss_2, p) \end{aligned}$$

implies $\varphi((s_1, [m_1])) = [(s_1, m_1)] = [(s_2, m_2)] = \varphi((s_2, [m_2]))$, since $(ss_2, p) \in S^{-1}P$.

Additivity: Let $(s_1, [m_1]), (s_2, [m_2]) \in S^{-1}(M/P)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$, thus

$$\begin{aligned} \varphi((s_1, [m_1]) + (s_2, [m_2])) &= \varphi((\tilde{s}s_1, \tilde{s}[m_1] + \tilde{r}[m_2])) = \varphi((\tilde{s}s_1, [\tilde{s}m_1 + \tilde{r}m_2])) \\ &= [(\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2)] = [(s_1, m_1) + (s_2, m_2)] \\ &= [(s_1, m_1)] + [(s_2, m_2)] \\ &= \varphi((s_1, [m_1])) + \varphi((s_2, [m_2])). \end{aligned}$$

Scalar multiplicativity: Let $(t, r) \in S^{-1}R$ and $(s, [m]) \in S^{-1}(M/P)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$, thus

$$\begin{aligned} \varphi((t, r) \cdot (s, [m])) &= \varphi((\tilde{s}t, \tilde{r}[m])) = \varphi((\tilde{s}t, [\tilde{r}m])) = [(\tilde{s}t, \tilde{r}m)] \\ &= [(t, r) \cdot (s, m)] = (t, r) \cdot [(s, m)] \\ &= (t, r) \cdot \varphi((s, [m])). \end{aligned}$$

Surjectivity: Let $[(s, m)] \in S^{-1}M/S^{-1}P$, then $[(s, m)] = \varphi((s, [m])) \in \text{im}(\varphi)$.

Injectivity: Let $(s_1, [m_1]), (s_2, [m_2]) \in S^{-1}(M/P)$ such that

$$[(s_1, m_1)] = \varphi((s_1, [m_1])) = \varphi((s_2, [m_2])) = [(s_2, m_2)],$$

then there exists $(s, p) \in S^{-1}P$ such that $(s_1, m_1) = (s_2, m_2) + (s, p)$. By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that

$$(s_1, m_1) = (s_2, m_2) + (s, p) = (\tilde{s}s_2, \tilde{s}m_2 + \tilde{r}p),$$

which implies the existence of $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}\tilde{s}s_2 = \hat{r}s_1$ and $\hat{s}(\tilde{s}m_2 + \tilde{r}p) = \hat{r}m_1$, thus

$$\begin{aligned} (s_1, [m_1]) &= (\hat{r}s_1, \hat{r}[m_1]) = (\hat{r}s_1, [\hat{r}m_1]) = (\hat{s}\tilde{s}s_2, [\hat{s}(\tilde{s}m_2 + \tilde{r}p)]) = (\hat{s}\tilde{s}s_2, \hat{s}[\tilde{s}m_2 + \tilde{r}p]) \\ &= (\tilde{s}s_2, [\tilde{s}m_2 + \tilde{r}p]) = (\tilde{s}s_2, [\tilde{s}m_2]) = (\tilde{s}s_2, \tilde{s}[m_2]) = (s_2, [m_2]). \end{aligned} \quad \square$$

9.4. Functorial interpretation

Apart from the direct approach of defining the localization of a module via equivalence relations presented here, there is another way utilizing the tensor product $S^{-1}R \otimes_R M$. In contrast to general tensor products where we always have to consider finite sums of elementary tensors, this is not necessary here thanks to the left Ore condition:

Lemma 9.12. *Let S be a left denominator set in a ring R and M a left R -module. Every element of the left $S^{-1}R$ -module $S^{-1}R \otimes_R M$ has a representation $(s, 1) \otimes m$ for some $s \in S$ and $m \in M$.*

Proof: Let $\sum_{i=1}^n (s_i, r_i) \otimes m_i \in S^{-1}R \otimes_R M$. We have $(s_i, r_i) \otimes m_i = (s_i, 1) \otimes r_i m_i$ for all i , so we can assume $r_i = 1$ for all i without loss of generality. Now let $n \geq 2$. By the left Ore condition on S there exist $s \in S$ and $r \in R$ such that $ss_1 = rs_2$. Then

$$\begin{aligned} (s_1, 1) \otimes m_1 + (s_2, 1) \otimes m_2 &= (ss_1, s) \otimes m_1 + (rs_2, r) \otimes m_2 \\ &= (ss_1, 1) \otimes sm_1 + (rs_2, 1) \otimes rm_2 \\ &= (ss_1, 1) \otimes (sm_1 + rm_2). \end{aligned}$$

The rest follows by induction on n . □

It should not be surprising that this results in another way to construct the left Ore localization of M at S :

Proposition 9.13. *Let S be a left denominator set in a ring R and M a left R -module. Then $S^{-1}M$ and $S^{-1}R \otimes_R M$ are isomorphic as left $S^{-1}R$ -modules.*

Proof: Consider the map $\varphi : S^{-1}M \rightarrow S^{-1}R \otimes_R M$, $(s, m) \mapsto (s, 1) \otimes m$.

Well-definedness: Let $(s_1, m_1), (s_2, m_2) \in S^{-1}M$ such that $(s_1, m_1) = (s_2, m_2)$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}s_1$ and $\tilde{s}m_2 = \tilde{r}m_1$. Then

$$\begin{aligned} \varphi((s_2, m_2)) &= (s_2, 1) \otimes m_2 = (\tilde{s}s_2, \tilde{s}) \otimes m_2 = (\tilde{s}s_2, 1) \otimes \tilde{s}m_2 \\ &= (\tilde{r}s_1, 1) \otimes \tilde{r}m_1 = (\tilde{r}s_1, \tilde{r}) \otimes m_1 = (s_1, 1) \otimes m_1 = \varphi((s_1, m_1)). \end{aligned}$$

Additivity: Let $(s_1, m_1), (s_2, m_2) \in S^{-1}M$. Let $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$, then

$$\begin{aligned} \varphi((s_1, m_1) + (s_2, m_2)) &= \varphi((\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2)) \\ &= (\tilde{s}s_1, 1) \otimes (\tilde{s}m_1 + \tilde{r}m_2) \\ &= (\tilde{s}s_1, 1) \otimes \tilde{s}m_1 + (\tilde{s}s_1, 1) \otimes \tilde{r}m_2 \\ &= (\tilde{s}s_1, \tilde{s}) \otimes m_1 + (\tilde{s}s_1, \tilde{r}) \otimes m_2 \\ &= (s_1, 1) \otimes m_1 + (\tilde{s}s_1, \tilde{r}) \cdot (1, s_2) \cdot (s_2, 1) \otimes m_2 \\ &= \varphi((s_1, m_1)) + (\tilde{s}s_1, \tilde{r}s_2) \cdot (s_2, 1) \otimes m_2 \\ &= \varphi((s_1, m_1)) + (\tilde{s}s_1, \tilde{s}s_1) \cdot \varphi((s_2, m_2)) \\ &= \varphi((s_1, m_1)) + \varphi((s_2, m_2)). \end{aligned}$$

Scalar multiplicativity: Now let $(s_1, r_1) \in S^{-1}R$ and $(s_2, m_2) \in S^{-1}M$. Let $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r_1 = \tilde{r}s_2$, then

$$\begin{aligned} \varphi((s_1, r_1) \cdot (s_2, m_2)) &= \varphi((\tilde{s}s_1, \tilde{r}m_2)) \\ &= (\tilde{s}s_1, 1) \otimes \tilde{r}m_2 \\ &= (\tilde{s}s_1, \tilde{r}) \otimes m_2 \\ &= (\tilde{s}s_1, \tilde{r}) \cdot (1, s_2) \cdot (s_2, 1) \otimes m_2 \\ &= (\tilde{s}s_1, \tilde{r}s_2) \cdot (s_2, 1) \otimes m_2 \\ &= (\tilde{s}s_1, \tilde{s}r_1) \cdot \varphi((s_2, m_2)) \\ &= (s_1, r_1) \cdot \varphi((s_2, m_2)). \end{aligned}$$

Injectivity: Let $(s, m) \in S^{-1}M$ such that $0 = \varphi((s, m)) = (s, 1) \otimes m$, then there exists $r \in R$ satisfying $rs \in S$ and $rm = 0$. Now $(s, m) = (rs, rm) = (rs, 0) = (1, 0) = 0_{S^{-1}M}$.

Surjectivity: Let $x \in S^{-1}R \otimes_R M$, then by [Lemma 9.12](#) there exist $s \in S$ and $m \in M$ such that $x = (s, 1) \otimes m$, then $x = (s, 1) \otimes m = \varphi((s, m)) \in \text{im}(\varphi)$. $\square$

Alternatively, one could easily show that $S^{-1}R \otimes_R M$ together with the homomorphism

$$M \rightarrow S^{-1}R \otimes_R M, \quad m \mapsto (1, 1) \otimes m,$$

satisfies the axioms for a localization of M at S given in [Definition 9.1](#), then from the uniqueness property in [Corollary 9.4](#) we immediately get $S^{-1}M \cong S^{-1}R \otimes_R M$.

Remark 9.14. Let S be a left denominator set in a ring R and $\varphi : M \rightarrow N$ a homomorphism of left R -modules. Then $S^{-1}\cdot$ becomes a covariant functor from the category of left R -modules to the category of left $S^{-1}R$ -modules via $S^{-1}\varphi : S^{-1}M \rightarrow S^{-1}N$, $(s, m) \mapsto (s, \varphi(m))$, which is called *localization functor*. Furthermore, consider the covariant functor $S^{-1}R \otimes_R \cdot$ also from the category of left R -modules to the category of left $S^{-1}R$ -modules via

$$S^{-1}R \otimes_R \varphi := \text{id} \otimes \varphi : S^{-1}R \otimes_R M \rightarrow S^{-1}R \otimes_R N, \quad (s, 1) \otimes m \mapsto (s, 1) \otimes \varphi(m).$$

The functors $S^{-1}\cdot$ and $S^{-1}R \otimes_R \cdot$ are naturally isomorphic with the functorial isomorphism induced by the maps from [Proposition 9.13](#). $\diamond$

Proposition 9.15. *Let S be a left denominator set in a ring R , then the functor $S^{-1}\cdot$ is exact. Furthermore, $S^{-1}R$ is flat as a right R -module.*

Proof: Let $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3$ be an exact sequence of left R -modules. We have to show the exactness of the sequence $S^{-1}M_1 \xrightarrow{S^{-1}\varphi} S^{-1}M_2 \xrightarrow{S^{-1}\psi} S^{-1}M_3$, which is equivalent to $\text{im}(S^{-1}\varphi) = \ker(S^{-1}\psi)$.

- By assumption we have $\psi \circ \varphi = 0$, which implies $(S^{-1}\psi) \circ (S^{-1}\varphi) = S^{-1}(\psi \circ \varphi) = S^{-1}0 = 0$ and thus $\text{im}(S^{-1}\varphi) \subseteq \ker(S^{-1}\psi)$.
- Let $(s, m_2) \in \ker(S^{-1}\psi) \subseteq S^{-1}M_2$. Now $0 = (S^{-1}\psi)((s, m_2)) = (s, \psi(m_2))$, thus there exists $\tilde{s} \in S$ such that $0 = \tilde{s}\psi(m_2) = \psi(\tilde{s}m_2)$. Since $\tilde{s}m_2 \in \ker(\psi) = \text{im}(\varphi)$ there exists $m_1 \in M_1$ such that $\varphi(m_1) = \tilde{s}m_2$. Now we get $\ker(S^{-1}\psi) \subseteq \text{im}(S^{-1}\varphi)$ from

$$(s, m_2) = (\tilde{s}s, \tilde{s}m_2) = (\tilde{s}s, \varphi(m_1)) = (S^{-1}\varphi)(\tilde{s}s, m_1) \in \text{im}(S^{-1}\varphi).$$

Since $S^{-1}\cdot$ is exact, so is $S^{-1}R \otimes_R \cdot$, which is equivalent to flatness of $S^{-1}R$ as a right R -module. $\square$

This shows that the localization of a finitely presented module is induced by the same matrix, albeit considered as a matrix over the localized ring:

Proposition 9.16. *Let S be a left denominator set in a ring R and $P \in R^{m \times n}$ a presentation matrix of the finitely presented R -module $M \cong R^{1 \times n} / R^{1 \times m} P$. Then*

$$S^{-1}M \cong (S^{-1}R)^{1 \times n} / (S^{-1}R)^{1 \times m} P.$$

Proof: The sequence $R^{1 \times m} \xrightarrow{\cdot P} R^{1 \times n} \longrightarrow M \rightarrow 0$ is exact. By exactness of $S^{-1}\cdot$,

$$S^{-1}(R^{1 \times m}) \xrightarrow{S^{-1}(\cdot P)} S^{-1}(R^{1 \times n}) \longrightarrow S^{-1}M \rightarrow 0$$

is also exact. We have $S^{-1}(R^{1 \times k}) \cong (S^{-1}R)^{1 \times k}$ for all $k \in \mathbb{N}$ by the left Ore condition on S . Now the homomorphism theorem gives us

$$S^{-1}M \cong S^{-1}(R^{1 \times n}) / \text{im}(S^{-1}(\cdot P)) \cong (S^{-1}R)^{1 \times n} / (S^{-1}R)^{1 \times m} P. \quad \square$$

Chapter 10.

Local closure

In this chapter we see the second important special case of left saturation closure. In contrast to the first application, where we considered saturation at the ring itself, here we will discuss saturation of submodules.

10.1. Left saturation of submodules at left denominator sets

For arbitrary sets T , LSat_T is hardly compatible with even the most basic module operations:

Lemma 10.1. *Let T be a subset of a ring R and M_1, M_2 left R -modules with left R -submodules $P_1 \subseteq M_1$ and $P_2 \subseteq M_2$.*

(a) *We have $\text{LSat}_T^{M_1 \times M_2}(P_1 \times P_2) \subseteq \text{LSat}_T^{M_1}(P_1) \times \text{LSat}_T^{M_2}(P_2)$.*

(b) *If $M_1 = M_2$, then $\text{LSat}_T(P_1 \cap P_2) \subseteq \text{LSat}_T(P_1) \cap \text{LSat}_T(P_2)$.*

Proof: (a) Let $(m_1, m_2) \in \text{LSat}_T^{M_1 \times M_2}(P_1 \times P_2) \subseteq M_1 \times M_2$, then there exists $t \in T$ such that $t(m_1, m_2) = (tm_1, tm_2) \in P_1 \times P_2$. For $i \in \{1, 2\}$ we have $tm_i \in P_i$, thus $m_i \in \text{LSat}_T^{M_i}(P_i)$ and $(m_1, m_2) \in \text{LSat}_T^{M_1}(P_1) \times \text{LSat}_T^{M_2}(P_2)$.

(b) Let $m \in \text{LSat}_T(P_1 \cap P_2) \subseteq M$, then there exists $t \in T$ such that $tm \in P_1 \cap P_2$. For $i \in \{1, 2\}$ we have $tm \in P_i$, thus $m \in \text{LSat}_T(P_i)$ and $m \in \text{LSat}_T^{M_1}(P_1) \cap \text{LSat}_T^{M_2}(P_2)$. $\square$

The compatibility with intersection does not even hold over commutative rings:

Example 10.2. Let $R = \mathbb{Z}$, $M = (\mathbb{Z}/6\mathbb{Z})^2$, $P_1 = \mathbb{Z}/6\mathbb{Z} \times \{0\}$, $P_2 = \{0\} \times \mathbb{Z}/6\mathbb{Z}$ and $T = \{2, 3\}$. Then

$$(2, 3) \notin \text{LSat}_T(\{(0, 0)\}) = (2\mathbb{Z}/6\mathbb{Z})^2 \cup (3\mathbb{Z}/6\mathbb{Z})^2,$$

but $2 \cdot (2, 3) = (4, 0) \in P_1$ and $3 \cdot (2, 3) = (0, 3) \in P_2$, thus $(2, 3) \in \text{LSat}_T(P_1) \cap \text{LSat}_T(P_2)$. $\diamond$

While this can be fixed if we require T to be a multiplicative set, this still is not enough for non-commutative rings:

Example 10.3. Let K be a field, $R = K\langle x, y \rangle$ the free algebra over K in the two non-commuting variables x and y , and $T = [x, y]$. Then the two left ideals $L_1 := {}_R\langle y^2 \rangle$ and $L_2 := {}_R\langle xy \rangle$ intersect only in 0, thus $\text{LSat}_T(L_1 \cap L_2) = \{0\}$ since R is a domain. But $y \cdot y \in L_1$ and $x \cdot y \in L_2$ show that $y \in \text{LSat}_T(L_1) \cap \text{LSat}_T(L_2)$. $\diamond$

For this reason, we will consider the left saturation closure of submodules at left denominator sets, which is compatible with the operations we have seen above. Note that the following result also holds for left Ore sets.

Lemma 10.4. *Let S be a left denominator set in a ring R and M_1, M_2 left R -modules with left R -submodules $P_1 \subseteq M_1$ and $P_2 \subseteq M_2$.*

- (a) *We have $\text{LSat}_S^{M_1 \times M_2}(P_1 \times P_2) = \text{LSat}_S^{M_1}(P_1) \times \text{LSat}_S^{M_2}(P_2)$.*
- (b) *If $M_1 = M_2$, then $\text{LSat}_S(P_1 \cap P_2) = \text{LSat}_S(P_1) \cap \text{LSat}_S(P_2)$.*
- (c) *If $M_1 = M_2$, then $\text{LSat}_S(P_1 + P_2) \supseteq \text{LSat}_S(P_1) + \text{LSat}_S(P_2)$.*

Proof: (a) Let $(m_1, m_2) \in \text{LSat}_S^{M_1 \times M_2}(P_1 \times P_2)$, thus for all $i \in \{1, 2\}$ we have that $m_i \in \text{LSat}_S^{M_i}(P_i)$, which implies that there exists $s_i \in S$ such that $s_i m_i \in P_i$. By the left Ore condition there exists a common left multiple $s \in S$ of s_1 and s_2 , thus $sm_i \in P_i$. Now $s(m_1, m_2) = (sm_1, sm_2) \in P_1 \times P_2$ implies $(m_1, m_2) \in \text{LSat}_S^{M_1 \times M_2}(P_1 \times P_2)$. The other inclusion is a special case of [Lemma 10.1](#).

(b) Let $m \in \text{LSat}_S(P_1) \cap \text{LSat}_S(P_2)$. For $i \in \{1, 2\}$ we have $m \in \text{LSat}_S(P_i)$, thus there exists $s_i \in S$ such that $s_i m \in P_i$. By the left Ore condition there exists a common left multiple $s \in S$ of s_1 and s_2 , thus $sm \in P_i$. Now $sm \in P_1 \cap P_2$ implies $m \in \text{LSat}_S(P_1 \cap P_2)$. The other inclusion is a special case of [Lemma 10.1](#).

(c) Let $m \in \text{LSat}_S(P_1) + \text{LSat}_S(P_2)$, then there exist $m_1 \in \text{LSat}_S(P_1)$ and $m_2 \in \text{LSat}_S(P_2)$ such that $m = m_1 + m_2$. For all $i \in \{1, 2\}$ there exists $s_i \in S$ such that $s_i m_i \in P_i$. By the left Ore condition there exists a common left multiple $s \in S$ of s_1 and s_2 , thus $sm_i \in P_i$. Now $sm = sm_1 + sm_2 \in P_1 + P_2$ implies $m \in \text{LSat}_S(P_1 + P_2)$. $\square$

The last inclusion can be strict even in the nicest of settings:

Example 10.5. Let K be a field and consider $R = M = K[x, y]$, $P_1 = \langle x \rangle$, $P_2 = \langle y \rangle$ and $S = [x + y]$. Since x is prime in R and $x \nmid x + y$ we have

$$\text{LSat}(P_1) = \text{LSat}_{[x+y]}(\langle x \rangle) = \{r \in R : x \mid (x + y)^k r \text{ for some } k \in \mathbb{N}_0\} = \langle x \rangle = P_1$$

and analogously $\text{LSat}(P_2) = P_2$, thus

$$\text{LSat}(P_1) + \text{LSat}(P_2) = P_1 + P_2 = \langle x, y \rangle \neq K[x, y] = R.$$

On the other hand,

$$x + y \in (P_1 + P_2) \cap S = (\langle x \rangle + \langle y \rangle) \cap [x + y],$$

therefore $\text{LSat}(P_1 + P_2) = R \supsetneq \text{LSat}(P_1) + \text{LSat}(P_2)$. $\diamond$

Another application of LSat turns this into an equality:

Corollary 10.6. *Let S be a left denominator set in a ring R and P_1, P_2 two left R -submodules of a left R -module M . Then $\text{LSat}_S(P_1 + P_2) = \text{LSat}(\text{LSat}_S(P_1) + \text{LSat}_S(P_2))$.*

Proof: Since $P_1 + P_2 \subseteq \text{LSat}_S(P_1) + \text{LSat}_S(P_2) \subseteq \text{LSat}_S(P_1 + P_2)$ by [Lemma 10.4](#), this follows from [Lemma 6.10](#). $\square$

Note that the left Ore condition of a set S can be characterized in terms of LSat_S :

Lemma 10.7. *Let S be a subset of a ring R .*

- (a) *Let $s \in S$ and $r \in R$. The following are equivalent:*

- (1) There exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.
- (2) $r \in \text{LSat}_S({}_R\langle s \rangle)$.
- (b) Let $s \in S$. The following are equivalent:
 - (1) For all $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.
 - (2) $\text{LSat}_S({}_R\langle s \rangle) = R$.
- (c) The following are equivalent:
 - (1) S satisfies the left Ore condition in R .
 - (2) For all left ideals I in R , $I \cap S \neq \emptyset$ implies $\text{LSat}_S(I) = R$.
 - (3) For all $s \in S$ we have $\text{LSat}_S({}_R\langle s \rangle) = R$.

Proof: (a) By definition.

(b) Immediately from the previous statement.

(c) (1) $\Rightarrow$ (2): By assumption there exists $s \in I \cap S$. Let $r \in R$, then by the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s \in I$, thus $r \in \text{LSat}_S(I)$.

(2) $\Rightarrow$ (3): Special case.

(3) $\Rightarrow$ (1): Follows from the previous statement. $\square$

10.2. Local closure: contraction of the extension

Definition 10.8. Let S be a left denominator set in a ring R , M a left R -module and P a left R -submodule of M . The S -closure or *local closure* of P in M is defined as

$$P^S := \text{LSat}_S^M(P) = \{m \in M \mid \exists s \in S : sm \in P\}.$$

Furthermore, P is called *left S -closed* if $P = P^S$. $\diamond$

The local closure of a submodule is a well-known concept in the area of algebraic analysis of functional equations:

Remark 10.9. An element $r = \sum_{i=1}^n p_i \partial^i$ with $p_i \in K[x]$ in the first polynomial Weyl algebra A_1 corresponds to an ordinary differential equation. A solution of this equation can only have singularities at roots of the polynomial p_n , but there might be roots of p_n where no solution is singular. The latter roots are called *apparent singularities* and since they are somewhat misleading, we may want to get rid of them. Algebraically, we are interested in an element $t \in \langle r \rangle^{K[x] \setminus \{0\}}$ such that as many apparent singularities of r as possible are not present in t . This process is called *desingularization*. A typical algorithmic approach is to first compute the local closure of the left ideal generated by the operator r and then find a desingularized operator contained in the closure. $\diamond$

Example 10.10 ([BM15]). Let $r = x\partial^2 - (x+2)\partial + 2 \in A_1(\mathbb{Q})$, then $\langle r \rangle^{\mathbb{Q}[x] \setminus \{0\}} = \langle r, \partial^4 - \partial^3 \rangle$, thus $\partial^4 - \partial^3$ is a desingularized operator for r . $\diamond$

The local closure absorbs the local closure with respect to any subset:

Lemma 10.11. *Let S, T be two left denominator sets in a ring R , M a left R -module and P a left R -submodule of M . If $S \subseteq T$, then $(P^S)^T = P^T = (P^T)^S$.*

Proof: By Lemma 6.3 we have

$$(P^S)^T = \text{LSat}_T(\text{LSat}_S(P)) = \text{LSat}_T(P) = P^T = \text{LSat}_T(P) = \text{LSat}_S(\text{LSat}_T(P)) = (P^T)^S. \quad \square$$

Another way to arrive at local closure is via extension and contraction:

Definition 10.12. Let R, T be rings, M a left R -module and N a left T -module. Additionally, let N be a left R -module and $\varphi : M \rightarrow N$ a homomorphism of left R -modules.

- The *extension* of a left R -submodule P of M to T with respect to φ is $P^e := {}_T\langle\varphi(P)\rangle$.
- The *contraction* of a left T -submodule Q of N with respect to φ is $Q^c := \varphi^{-1}(Q)$. $\diamond$

This naturally leads to the question under which circumstances the extension of the contraction and the contraction of the extension result in same module we started with. Generically, we only have two inclusions:

Lemma 10.13. *Let R, T be rings, M a left R -module and N a left T -module. Additionally, let N be a left R -module and $\varphi : M \rightarrow N$ a homomorphism of left R -modules.*

- (a) *Let P be a left R -submodule of M , then $P \subseteq (P^e)^c$ with respect to φ .*
- (b) *Let Q be a left T -submodule of N , then $(Q^c)^e \subseteq Q$ with respect to φ .*

Proof: (a) We have $P \subseteq \varphi^{-1}(\varphi(P)) \subseteq \varphi^{-1}({}_T\langle\varphi(P)\rangle) = (P^e)^c$.

(b) We have $(Q^c)^e = {}_T\langle\varphi(\varphi^{-1}(Q))\rangle \subseteq {}_T\langle Q \rangle = Q$. $\square$

But if we consider these operations with respect to the localization map, then the extension of a module is just its localization. Furthermore, extending after contracting becomes the identity:

Lemma 10.14. *Let S be a left denominator set in a ring R , M a left R -module and $\varepsilon := \varepsilon_{S,R,M}$.*

- (a) *Let P be a left R -submodule of M . Then $P^e = S^{-1}P$ with respect to ε .*
- (b) *Let Q be a left $S^{-1}R$ -submodule of $S^{-1}M$. Then $Q = (Q^c)^e$ with respect to ε .*
- (c) *Any left $S^{-1}R$ -submodule of $S^{-1}M$ is the extension of a left R -submodule of M with respect to ε .*

Proof: (a) Let $(s, p) \in S^{-1}P$, then

$$(s, p) = (s, 1) \cdot (1, p) = (s, 1) \cdot \varepsilon(p) \in {}_{S^{-1}R}\langle\varepsilon(P)\rangle = P^e.$$

On the other hand, let $x \in P^e$, then $x = \sum_{i=1}^n (s_i, r_i) \cdot \varepsilon(p_i)$ for some $s_i \in S$, $r_i \in R$ and $p_i \in P$. By the left Ore condition on S there exist $a_i \in R$ and $s \in S$ such that $a_i s_i = s$ for all i . Now

$$x = \sum_{i=1}^n (s_i, r_i) \cdot \varepsilon(p_i) = \sum_{i=1}^n (a_i s_i, a_i r_i) \cdot (1, p_i) = \sum_{i=1}^n (s, a_i r_i p_i) = \left(s, \sum_{i=1}^n a_i r_i p_i \right) \in S^{-1}P. \quad \square$$

- (b) Let $(s, m) \in Q$, then $\varepsilon(m) = (1, m) = (1, s) \cdot (s, m) \in Q$, thus $m \in \varepsilon^{-1}(Q) = Q^c$, which implies $\varepsilon(m) \in \varepsilon(Q^c)$. Now $(s, m) = (s, 1) \cdot (1, m) = (s, 1) \cdot \varepsilon(m) \in {}_{S^{-1}R}\langle \varepsilon(Q^c) \rangle = (Q^c)^e$.
- (c) Let Q be a left $S^{-1}R$ -submodule of $S^{-1}M$, then Q^c is a left R -submodule of M and $Q = (Q^c)^e$ by the previous part.

In particular, properties concerning chains of submodules are preserved under localization:

Lemma 10.15. *Let S be a left denominator set in a ring R and M a left R -module.*

- (a) *If M is Noetherian, then $S^{-1}M$ is also Noetherian.*
- (b) *If M is Artinian, then $S^{-1}M$ is also Artinian.*

Proof: Let $N_1 \subseteq N_2 \subseteq \dots$ be an ascending chain of left $S^{-1}R$ -submodules of $S^{-1}M$, then $N_1^c \subseteq N_2^c \subseteq \dots$ is an ascending chain of left R -submodules of M with respect to $\varepsilon := \varepsilon_{S,R,M}$. Since M is Noetherian this chain becomes stationary: there exists $k \in \mathbb{N}$ such that $(N_k)^c = (N_{k+m})^c$ for all $m \in \mathbb{N}$. But then $N_k = ((N_k)^c)^e = ((N_{k+m})^c)^e = N_{k+m}$ for all $m \in \mathbb{N}$ by [Lemma 10.14](#), which shows that the original chain $N_1 \subseteq N_2 \subseteq \dots$ becomes stationary, thus $S^{-1}M$ is Noetherian. The second statement follows analogously. $\square$

The last piece of the puzzle is the connection between the local closure and the contraction of the extension:

Proposition 10.16. *Let S be a left denominator set in a ring R , M a left R -module and P a left R -submodule of M . Then $P^S = (P^e)^c$ with respect to $\varepsilon_{S,R,M}$. In particular, P^S is a left R -submodule of M .*

Proof: Let $\varepsilon := \varepsilon_{S,R,M}$.

- Let $m \in (P^e)^c$, then $\varepsilon(m) \in P^e = S^{-1}P$, thus there exist $s \in S$ and $p \in P$ such that $(1, m) = \varepsilon(m) = (s, p)$. This implies the existence of $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} \cdot 1 = \tilde{r}s$ and $\tilde{s}m = \tilde{r}p \in P$, which implies $m \in \text{LSat}_S(P) = P^S$.
- Let $m \in P^S = \text{LSat}_S(P)$, then there exists $s \in S$ such that $sm \in P$. Now

$$\varepsilon(m) = (1, m) = (s, sm) \in S^{-1}P = P^e,$$

thus $m \in \varepsilon^{-1}(P^e) = (P^e)^c$. $\square$

This additional structural knowledge can be exploited for new algorithms to compute local closures. Some applications can be found in [Chapter 14](#).

Lastly, we give a result concerning a situation in which the preimage of a left saturated submodule will also be left saturated:

Lemma 10.17. *For $i \in \{1, 2\}$, let S_i be a multiplicative set in a ring R_i and M_i a left R_i -module. Let $\varphi : R_1 \rightarrow R_2$ be a homomorphism of rings such that $\varphi(S_1) \subseteq S_2$. Furthermore, let $\psi : M_1 \rightarrow M_2$ be a homomorphism of left R_1 -modules, where the left R_1 -module structure of M_2 is given by $\bullet : R_1 \times M_2 \rightarrow M_2, (r_1, m_2) \mapsto r_1 \bullet m_2 := \varphi(r_1)m_2$. Let P_2 be a left R_2 -submodule of M_2 such that P_2 is left S_2 -closed. Then $P_1 := \psi^{-1}(P_2)$ is left S_1 -closed.*

Proof: Let $x_1 \in \text{LSat}_{S_1}^{M_1}(P_1)$, then there exists $s_1 \in S_1$ such that $s_1 x_1 \in P_1$. Now

$$\varphi(s_1)\psi(x_1) = s_1 \bullet \psi(x_1) = \psi(s_1 x_1) \in P_2 = P_2^{S_2},$$

thus $\psi(x_1) \in P_2$, since $\varphi(s_1) \in S_2$. But then $x_1 \in \psi^{-1}(P_2) = P_1$, thus $P_1 = \text{LSat}_{S_1}^{M_1}(P_1)$. $\square$

10.3. Local closure of ideals

In the special case where $M = R$, we get the corresponding results for the local closure of left ideals:

Corollary 10.18. *Let S be a left denominator set in a ring R and $\rho := \rho_{S,R}$.*

- (a) *Let L be a left ideal in R . Then $S^{-1}L = L^e$ with respect to ρ .*
- (b) *Let J be a left ideal in $S^{-1}R$. Then $(J^e)^e = J$ with respect to ρ .*
- (c) *Any left ideal in $S^{-1}R$ is the extension of a left ideal in R with respect to ρ .*

Corollary 10.19. *Let S be a left denominator set in a ring R and L a left ideal in R . Then $L^S = (L^e)^c$ with respect to $\rho_{S,R}$. In particular, L^S is a left ideal in R .*

Corollary 10.20. *Let S be a left Ore set in a ring R .*

- (a) *If R is left Noetherian, then $S^{-1}R$ is also left Noetherian.*
- (b) *If R is left Artinian, then $S^{-1}R$ is also left Artinian.*

As a bonus, we get a characterization of all non-proper left ideals in the localization $S^{-1}R$: they are exactly the extensions of left ideals in R that have a non-trivial intersection with S .

Lemma 10.21. *Let S be a left Ore set in a ring R and L a left ideal in R . The following are equivalent:*

- (1) $S^{-1}L = S^{-1}R$.
- (2) $L \cap \text{LSat}(S) \neq \emptyset$.
- (3) $L \cap S \neq \emptyset$.

Proof: (1) $\Rightarrow$ (2): Since $1_{S^{-1}R} \in S^{-1}L$ there exist $s \in S$ and $l \in L$ such that $1_{S^{-1}R} = (s, l)$, thus $\tilde{s}l = \tilde{s}s \in S$ for some $\tilde{s} \in S$, which implies $l \in \text{LSat}(S)$.

(2) $\Rightarrow$ (3): Let $x \in L \cap \text{LSat}(S)$, then there exists $w \in R$ such that $wx \in S$, which implies $wx \in L \cap S$, since L is a left ideal.

(3) $\Rightarrow$ (1): Let $x \in L \cap S$, then $1_{S^{-1}R} = (x, x) \in S^{-1}L$. □

Lemma 10.22. *For $i \in \{1, 2\}$, let S_i be a left denominator set in a ring R_i such that $R_1 \subseteq R_2$ and $S_1 \subseteq S_2$. Furthermore, let L_2 be a left ideal of R_2 and $L_1 := L_2 \cap R_1$. If $L_2 = L_2^{S_2}$, then $L_1 = L_1^{S_1}$.*

Proof: Immediately from [Lemma 10.17](#). □

In a commutative ring R , the *radical* of an ideal I is $\sqrt{I} := \{r \in R \mid \exists n \in \mathbb{N}_0 : r^n \in I\}$. In this setting, both localization and saturation are compatible with taking radicals:

Lemma 10.23. *Let S be a multiplicative set in a commutative ring R and I an ideal in R .*

- (a) $\text{LSat}_S(\sqrt{I}) = \sqrt{\text{LSat}_S(I)}$.

$$(b) \ S^{-1}\sqrt{I} = \sqrt{S^{-1}I}.$$

Proof: (a) Let $r \in \text{LSat}_S(\sqrt{I})$, then there exists $s \in S$ such that $sr \in \sqrt{I}$. This implies the existence of $n \in \mathbb{N}$ such that $s^n r^n = (sr)^n \in I$ and thus $r^n \in \text{LSat}_S(I)$ since $s^n \in S$. Therefore we have $r \in \sqrt{\text{LSat}_S(I)}$.

Now let $r \in \sqrt{\text{LSat}_S(I)}$, then there exists $n \in \mathbb{N}$ such that $r^n \in \text{LSat}_S(I)$. This implies the existence of $s \in S$ such that $sr^n \in I$ and thus $(sr)^n = s^n r^n \in I$ since I is an ideal. Therefore we have $sr \in \sqrt{I}$ and thus $r \in \text{LSat}_S(\sqrt{I})$.

(b) Let $(s, r) \in S^{-1}\sqrt{I}$ with $r \in \sqrt{I}$, then there exists $n \in \mathbb{N}$ such that $r^n \in I$, thus $(s, r)^n = (s^n, r^n) \in S^{-1}I$, which implies $(s, r) \in \sqrt{S^{-1}I}$.

Now let $(s, r) \in \sqrt{S^{-1}I}$, then there exists $n \in \mathbb{N}$ such that $(s^n, r^n) = (s, r)^n \in S^{-1}I$. By [Lemma 9.10](#), $r^n \in \text{LSat}_S(I)$, which implies $r \in \sqrt{\text{LSat}_S(I)} = \text{LSat}_S(\sqrt{I})$ by the previous part and finally $(s, r) \in S^{-1}\sqrt{I}$ again by [Lemma 9.10](#). $\square$

Chapter 11.

Local torsion

From the axiomatic definition of localization of modules in [Definition 9.1](#) we know that

$$\ker(\varepsilon_{S,R,M}) = \{m \in M \mid \exists s \in S : sm = 0\}.$$

If R is a commutative domain and $S = R \setminus \{0\}$, this is exactly the definition of the torsion submodule of M . Inspired by this, $\ker(\varepsilon_{S,R,M})$ is also called the S -torsion submodule of M and denoted $t_S(M)$. Similarly to local closure, this torsion is also called *local torsion*. In fact, we will see that for finitely presented modules, local closure and local torsion are intrinsically tied to each other. Before we go there, however, we will make a detour and investigate the properties of torsion with respect to arbitrary subsets Λ of the ring R , denoted $t_\Lambda(M)$. By exploring $t_\Lambda(M)$ first, we will attain a deeper understanding of which properties of $t_S(M)$ hold by construction and which are due to S additionally being a left denominator set.

11.1. General torsion

The classical torsion is concerned with torsion with respect to regular elements:

Definition 11.1 (Classical torsion). Let R be a ring and M a left R -module.

- An element $m \in M$ is called *torsion element* if there exists a regular $r \in R$ such that $rm = 0$.
- The set of all torsion elements is $t(M) := \{m \in M \mid m \text{ is a torsion element}\}$.
- The elements of $M \setminus t(M)$ are called *regular*.
- M is called *torsion* or a *torsion module* if $t(M) = M$.
- M is called *torsion-free* if $t(M) = \{0\}$. ◇

Remark 11.2. Suppose that R is an algebra of linear functional operators like the Weyl algebras, and $\mathcal{F}$ a left R -module. In systems and control theory, to any matrix $A \in R^{g \times q}$ we can associate the *abstract linear system* $\mathcal{B} := \{w \in \mathcal{F}^q \mid Aw = 0\}$ and the corresponding *system module* $M := R^{1 \times q} / R^{1 \times q} A$. In this situation, $t(M)$ is a submodule of M , and the quotient module $M/t(M)$ corresponds to the *controllable part* of the system $\mathcal{B}$. In particular, $\mathcal{B}$ is controllable if and only if M is torsion-free, and *autonomous* if and only if M is torsion. ◇

But where there is a definition, there is also a generalization, in this case we replace the set of regular elements of R with an arbitrary set Λ :

Definition 11.3 (General torsion). Let Λ be a non-empty subset of a ring R and M a left R -module.

- An element $m \in M$ is called Λ -torsion element if there exists $\lambda \in \Lambda$ such that $\lambda m = 0$.
- The set of all Λ -torsion elements is

$$t_\Lambda(M) := \{m \in M \mid \exists \lambda \in \Lambda : \lambda m = 0\} = \text{LSat}_\Lambda^M(\{0\}).$$

- The elements of $M \setminus t_\Lambda(M)$ are called Λ -regular.
- M is called Λ -torsion or a Λ -torsion module if $t_\Lambda(M) = M$.
- M is called Λ -torsion-free if $t_\Lambda(M) = \{0\}$. $\diamond$

In this generality, $t_\Lambda(M)$ is not necessarily a submodule of M :

Example 11.4. Consider $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$ -module. For $\Lambda = \{2, 3\}$ we have $2 \cdot 3 = 0$ and $3 \cdot 2 = 0$, but $2 \cdot (2 + 3) = 2 \cdot 5 = 4 \neq 0 \neq 3 = 3 \cdot 5 = 3 \cdot (2 + 3)$. $\diamond$

Nevertheless, t_Λ enjoys some nice properties:

Lemma 11.5. Let Λ be a non-empty subset of a ring R and M a left R -module.

- $t_\Lambda(M) \subseteq M$.
- $t_\Lambda(t_\Lambda(M)) = t_\Lambda(M)$.
- Let Σ be non-empty subset of R such that $\Lambda \subseteq \Sigma$, then $t_\Lambda(M) \subseteq t_\Sigma(M)$.
- Let N be a left R -module such that $M \subseteq N$, then $t_\Lambda(M) \subseteq t_\Lambda(N)$.

Proof: Obvious from the definition. $\square$

Lemma 11.6. Let Λ, Σ be non-empty subsets of a ring R and M a left R -module.

- We have

$$t_\Lambda(t_\Sigma(M)) = t_\Lambda(M) \cap t_\Sigma(M) = t_\Sigma(t_\Lambda(M)).$$

In particular, $\Lambda \subseteq \Sigma$ implies $t_\Lambda(t_\Sigma(M)) = t_\Lambda(M) = t_\Sigma(t_\Lambda(M))$.

- We have

$$t_{\Lambda \cap \Sigma}(M) \subseteq t_\Lambda(M) \cap t_\Sigma(M) \subseteq t_{\Lambda \cup \Sigma}(M).$$

Proof: (a) By [Lemma 6.3](#) we have

$$t_\Lambda(t_\Sigma(M)) = \text{LSat}_\Lambda^{\text{LSat}_\Sigma^M(\{0\})}(\{0\}) = \text{LSat}_\Lambda^M(\{0\}) \cap \text{LSat}_\Sigma^M(\{0\}) = t_\Lambda(M) \cap t_\Sigma(M),$$

the rest follows by symmetry and [Lemma 11.5](#).

- Let $m \in t_{\Lambda \cap \Sigma}(M)$, then there exists $\lambda \in \Lambda \cap \Sigma$ such that $\lambda m = 0$, thus $m \in t_\Lambda(M) \cap t_\Sigma(M)$. Now let $m \in t_\Lambda(M) \cap t_\Sigma(M)$, then there exists $\lambda \in \Lambda \subseteq \Lambda \cup \Sigma$ such that $\lambda m = 0$, thus $m \in t_{\Lambda \cup \Sigma}(M)$. $\square$

Lemma 11.7. *Let Λ be a non-empty subset of a ring R and $\varphi : M \rightarrow N$ a homomorphism of left R -modules.*

- (a) *We have $\varphi(t_\Lambda(M)) \subseteq t_\Lambda(N)$.*
- (b) *We have $t_\Lambda(\ker(\varphi)) = t_\Lambda(M) \cap \ker(\varphi)$.*
- (c) *If N is Λ -torsion-free, then $t_\Lambda(M) \subseteq \ker(\varphi)$ and $t_\Lambda(M) = t_\Lambda(\ker(\varphi))$.*
- (d) *If φ is injective, then $\varphi^{-1}(t_\Lambda(N)) \subseteq t_\Lambda(M)$.*

Proof: (a) Let $m \in t_\Lambda(M)$, then there exists $\lambda \in \Lambda$ such that $\lambda m = 0$. Then $\lambda \cdot \varphi(m) = \varphi(\lambda m) = \varphi(0) = 0$, thus $\varphi(m) \in t_\Lambda(N)$.

(b) We have $t_\Lambda(\ker(\varphi)) = \{m \in M \mid \exists \lambda \in \Lambda : \lambda m = 0 \text{ and } m \in \ker(\varphi)\} = t_\Lambda(M) \cap \ker(\varphi)$.

(c) Let N be Λ -torsion-free, then $\varphi(t_\Lambda(M)) \subseteq t_\Lambda(N) = \{0\}$, thus $t_\Lambda(M) \subseteq \ker(\varphi)$. Furthermore, we have $t_\Lambda(\ker(\varphi)) = t_\Lambda(M) \cap \ker(\varphi) = t_\Lambda(M)$.

(d) Let $m \in M$ such that $n := \varphi(m) \in t_\Lambda(N)$, then there exists $\lambda \in \Lambda$ such that $\lambda n = 0$. Now $0 = \lambda n = \lambda \varphi(m) = \varphi(\lambda m)$ implies $\lambda m = 0$ by injectivity of φ , thus $m \in t_\Lambda(M)$. $\square$

As we have seen in the run-up to [Lemma 10.4](#), many module constructions are more compatible with local closure at left denominator sets than at arbitrary subsets of R . The same phenomenon can be observed in the context of local torsion:

Lemma 11.8. *Let Λ be a non-empty subset of a ring R , S a left denominator set in R and M a left R -module. Further, let P_1, P_2 be left R -submodules of M .*

- (a) $t_\Lambda(P_1 \cap P_2) = t_\Lambda(P_1) \cap t_\Lambda(P_2)$.
- (b) $t_\Lambda(P_1 \oplus P_2) \subseteq t_\Lambda(P_1) \oplus t_\Lambda(P_2)$.
- (c) $t_S(P_1 \oplus P_2) = t_S(P_1) \oplus t_S(P_2)$.
- (d) $t_S(P_1 + P_2) \supseteq t_S(P_1) + t_S(P_2)$.

Proof: (a) By [Lemma 6.3](#) we have

$$\begin{aligned} t_\Lambda(P_1 \cap P_2) &= \text{LSat}_\Lambda^{P_1 \cap P_2}(\{0\}) = \text{LSat}_\Lambda^M(\{0\}) \cap P_1 \cap P_2 \\ &= \text{LSat}_\Lambda^{P_1}(\{0\}) \cap \text{LSat}_\Lambda^{P_2}(\{0\}) = t_\Lambda(P_1) \cap t_\Lambda(P_2). \end{aligned}$$

(b) Let $p_1 \oplus p_2 \in t_\Lambda(P_1 \oplus P_2)$, then there exists $\lambda \in \Lambda$ such that $\lambda p_1 \oplus \lambda p_2 = \lambda(p_1 \oplus p_2) = 0$, which implies $\lambda p_1 = 0 = \lambda p_2$, thus $p_1 \oplus p_2 \in t_\Lambda(P_1) \oplus t_\Lambda(P_2)$.

(c) Let $p_1 \oplus p_2 \in t_S(P_1) \oplus t_S(P_2)$, then there exist $s_1, s_2 \in S$ such that $s_1 p_1 = 0$ in P_1 and $s_2 p_2 = 0$ in P_2 . By the left Ore condition on S there exists a common left multiple $s \in S$ of s_1 and s_2 . Now $s(p_1 \oplus p_2) = s p_1 \oplus s p_2 = 0$ in $P_1 \oplus P_2$, thus $p_1 \oplus p_2 \in t_S(P_1 \oplus P_2)$.

(d) Let $p_1 + p_2 \in t_S(P_1) + t_S(P_2)$, then there exist $s_1, s_2 \in S$ such that $s_1 p_1 = 0 = s_2 p_2$. By the left Ore condition on S there exists a common left multiple $s \in S$ of s_1 and s_2 . Now $s(p_1 + p_2) = s p_1 + s p_2 = 0$ implies $p_1 + p_2 \in t_S(P_1 + P_2)$. $\square$

11.2. Local torsion with respect to left denominator sets

Now we look at local torsion with respect to left denominator sets. The basic structural properties are the same as in the classical case of torsion with respect to regular elements:

Proposition 11.9 (Structural theorem of local torsion). *Let S be a left denominator set in a ring R and M a left R -module.*

- (a) *We have $t_S(M) = \ker(\varepsilon_{S,R,M})$. In particular, $t_S(M)$ is a left R -submodule of M .*
- (b) *The module $M/t_S(M)$ is S -torsion-free.*
- (c) *The map $\varepsilon_{S,R,M/t_S(M)}$ is injective.*
- (d) *We have $S^{-1}t_S(M) = \{0\}$.*
- (e) *We have $S^{-1}M \cong S^{-1}(M/t_S(M))$.*
- (f) *The module M is S -torsion if and only if $S^{-1}M = \{0\}$.*

Proof: (a) We have $t_S(M) = \{m \in M \mid \exists s \in S : sm = 0\} = \ker(\varepsilon_{S,R,M})$.

- (b) Let $m + t_S(M) \in t_S(M/t_S(M))$ where $m \in M$. By definition there exists $s \in S$ such that $sm + st_S(M) = s(m + t_S(M)) \in t_S(M)$. Since $st_S(M) \subseteq t_S(M)$ we have $sm \in t_S(M)$, which in turn implies the existence of $\tilde{s} \in S$ such that $\tilde{s}sm = 0$. Now $m \in t_S(M)$, thus $t_S(M/t_S(M)) = \{0\}$ and $M/t_S(M)$ is S -torsion-free.

- (c) By the previous parts we have $\ker(\varepsilon_{S,R,M/t_S(M)}) = t_S(M/t_S(M)) = \{0\}$.

- (d) Let $(s, m) \in S^{-1}t_S(M)$, then there exists $\tilde{s} \in S$ such that $\tilde{s}m = 0$. Now

$$(s, m) = (\tilde{s}s, \tilde{s}m) = (\tilde{s}s, 0) = (1, 0) = 0_{S^{-1}M}.$$

- (e) The canonical sequence $0 \rightarrow t_S(M) \rightarrow M \rightarrow M/t_S(M) \rightarrow 0$ is exact. Since $S^{-1}\cdot$ is exact, so is $0 \rightarrow S^{-1}t_S(M) \rightarrow S^{-1}M \rightarrow S^{-1}(M/t_S(M)) \rightarrow 0$. By the previous part we have $S^{-1}t_S(M) = \{0\}$, thus $S^{-1}M \cong S^{-1}(M/t_S(M))$.

- (f) If M is S -torsion, then $t_S(M) = M$ and $S^{-1}M = S^{-1}t_S(M) = \{0\}$. On the other hand, let $S^{-1}M = \{0\}$ and $m \in M$. Then $\varepsilon_{S,R,M}(m) = (1, m) = 0$, thus there exists $s \in S$ such that $sm = 0$, which implies $m \in t_S(M)$ and therefore $t_S(M) = M$. $\square$

In fact, we also have a functorial interpretation of local torsion:

Proposition 11.10. *Let S be a left denominator set in a ring R . Then $t_S(\cdot)$ becomes a covariant left-exact functor from the category of left R -modules to the category of left S -torsion R -modules via $t_S(\varphi) := \varphi|_{t_S(M)}$ for any homomorphism $\varphi : M \rightarrow N$ of left R -modules.*

Proof: The action of $t_S(\cdot)$ on homomorphism of left R -modules is well-defined by Lemma 11.7 and thus $t_S(\cdot)$ becomes a covariant functor. Now let $0 \rightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3$ be an exact sequence of left R -modules. We have to show the exactness of the induced sequence

$$0 \rightarrow t_S(M_1) \xrightarrow{t_S(\varphi)} t_S(M_2) \xrightarrow{t_S(\psi)} t_S(M_3),$$

which is equivalent to $\ker(t_S(\varphi)) = \{0\}$ and $\text{im}(t_S(\varphi)) = \ker(t_S(\psi))$.

- By assumption we have $\ker(\varphi) = \{0\}$, thus $\ker(t_S(\varphi)) = \ker(\varphi|_{t_S(M)}) \subseteq \ker(\varphi) = \{0\}$.
- By assumption we have $\psi \circ \varphi = 0$, which implies $t_S(\psi) \circ t_S(\varphi) = t_S(\psi \circ \varphi) = t_S(0) = 0$, thus $\operatorname{im}(t_S(\varphi)) \subseteq \ker(t_S(\psi))$.
- Let $m_2 \in \ker(t_S(\psi)) \subseteq t_S(M_2)$, then $0 = t_S(\psi)(m_2) = \psi(m_2)$. Now $m_2 \in \ker(\psi) = \operatorname{im}(\varphi)$, thus there exists $m_1 \in M_1$ such that $\varphi(m_1) = m_2$. By [Lemma 11.7](#) we have $m_1 \in t_S(M_1)$, since φ is injective by assumption. This implies $m_2 = \varphi(m_1) = t_S(\varphi)(m_1) \in \operatorname{im}(t_S(\varphi))$, thus $\ker(t_S(\psi)) \subseteq \operatorname{im}(t_S(\varphi))$. $\square$

Example 11.11. The functor $t_S(\cdot)$ is not right-exact in general: consider $M = R = \mathbb{Z}$, $n \in \mathbb{N} \setminus \{1\}$, $N = \mathbb{Z}/n\mathbb{Z}$ and the canonical surjection $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. With $S = [n]$ we get $n \in S \cap n\mathbb{Z}$. But then $t_S(M) = t_S(\mathbb{Z}) = \{0\}$ and $t_S(N) = t_S(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \neq \{0\}$, thus $t_S(M) \rightarrow t_S(N)$ is not surjective. $\diamond$

Corollary 11.12. Let S be a left denominator set in a ring R and $0 \rightarrow L \rightarrow M \rightarrow N$ an exact sequence of left R -modules. If M is S -torsion-free, so is L .

Proof: By left-exactness of $t_S(\cdot)$, the sequence $0 \rightarrow t_S(L) \rightarrow t_S(M) \rightarrow t_S(N)$ is also exact. Now $t_S(M) = \{0\}$ implies $t_S(L) = \{0\}$. $\square$

Now we can give another connection between local closure and local torsion:

Proposition 11.13. Let S be a left denominator set in a ring R and P a left R -submodule of a left R -module M .

- (a) $t_S(M/P) = P^S/P$.
- (b) P is left S -saturated if and only if M/P is S -torsion-free.

Proof: (a) Let $m + P \in t_S(M/P)$, then there exists $s \in S$ such that $sm + sP \subseteq P$. Then $sm \in P$, thus $m \in P^S$, which implies $m + P \in P^S/P$.

On the other hand, let $m + P \in P^S/P$ with $m \in P^S$, then there exists $s \in S$ such that $sm \in P$. Now $s(m + P) = sm + sP \subseteq P$ and thus $m + P \in t_S(M/P)$.

- (b) By definition, P is left S -saturated if and only if $P = P^S$, or, equivalently, $P^S/P = \{0\}$. Since $t_S(M/P) = P^S/P$, this is equivalent to $t_S(M/P) = \{0\}$, which in turn holds by definition if and only if M/P is S -torsion-free. $\square$

11.3. Annihilators

No discussion of torsion is complete without at least mentioning annihilators:

Definition 11.14. Let R be a ring and M a left R -module.

- The (left) annihilator of $m \in M$ is $\operatorname{Ann}_R^M(m) := \{r \in R \mid rm = 0\}$.
- The annihilator of M is $\operatorname{Ann}_R(M) := \{r \in R \mid \forall m \in M : rm = 0\}$.

- Let M be finitely generated via $M \cong R^n/P$ for some left R -submodule P of R^n , then the (left) pre-annihilator of M with respect to the given presentation is

$$\text{preAnn}_R(M) := \bigcap_{j=1}^n \text{Ann}_R^M(e_j),$$

where $e_1, \dots, e_n$ are the images of the canonical standard basis of R^n in M . $\diamond$

For every $m \in M$, $\text{Ann}_R^M(m)$ is a left ideal of R . Furthermore, $\text{Ann}_R(M)$ and $\text{preAnn}_R(M)$ are left ideals as intersections of left ideals, since $\text{Ann}_R(M) = \bigcap_{m \in M} \text{Ann}_R^M(m)$. But $\text{Ann}_R(M)$ is even a two-sided ideal of R : let $r \in R$, $a \in \text{Ann}_R(M)$ and $m \in M$, then $(ar)m = a(rm) = 0$, since $rm \in M$.

Remark 11.15. While $\text{Ann}_R(M)$ is an invariant of M , $\text{preAnn}_R(M)$ does depend on the presentation of M . In the following, when talking about pre-annihilators we implicitly assume that we have fixed a representation of M and consider $\text{preAnn}_R(M)$ with respect to this fixed representation. $\diamond$

Remark 11.16. If M is finitely generated, we always have

$$\text{Ann}_R(M) = \bigcap_{m \in M} \text{Ann}_R^M(m) \subseteq \bigcap_{j=1}^n \text{Ann}_R^M(e_j) = \text{preAnn}_R(M).$$

If R is commutative, then $\text{Ann}_R(M) = \text{preAnn}_R(M)$: let $a \in \text{preAnn}_R(M)$ and $m = \sum_{j=1}^n c_j e_j$, then

$$am = a \sum_{j=1}^n c_j e_j = \sum_{j=1}^n ac_j e_j = \sum_{j=1}^n c_j a e_j = 0,$$

thus $a \in \text{Ann}_R(M)$. $\diamond$

Localizing the annihilator of a single module element is the same as taking the annihilator of the image of the element in the localized module with respect to the localized ring:

Proposition 11.17. Let S be a left denominator set in a ring R , M a left R -module, $m \in M$, $\rho := \rho_{S,R}$ and $\varepsilon := \varepsilon_{S,R,M}$. Then

$$S^{-1} \text{Ann}_R^M(m) = \text{Ann}_{S^{-1}R}^{S^{-1}M}(\varepsilon(m))$$

and thus

$$\text{Ann}_R^M(m) \subseteq (\text{Ann}_R^M(m))^S = \rho^{-1}(S^{-1} \text{Ann}_R^M(m)) = \rho^{-1}(\text{Ann}_{S^{-1}R}^{S^{-1}M}(\varepsilon(m))).$$

Proof: First, let $x \in S^{-1} \text{Ann}_R^M(m)$, then $x = (s, r)$ for some $s \in S$ and $r \in \text{Ann}_R^M(m)$. Now

$$x \cdot \varepsilon(m) = (s, r) \cdot (1, m) = (s, rm) = (s, 0) = 0,$$

thus $x \in \text{Ann}_{S^{-1}R}^{S^{-1}M}(\varepsilon(m))$. On the other hand, let $y = (s, r) \in \text{Ann}_{S^{-1}R}^{S^{-1}M}(\varepsilon(m))$, then

$$0 = y \cdot \varepsilon(m) = (s, r) \cdot (1, m) = (s, rm).$$

By [Lemma 5.23](#) there exists $\tilde{s} \in S$ such that $\tilde{s}rm = 0$. Now $\tilde{s}r \in \text{Ann}_R^M(m)$ and

$$y = (s, r) = (\tilde{s}s, \tilde{s}r) \in S^{-1} \text{Ann}_R^M(m). \quad \square$$

Corollary 11.18. *Let S be a left Ore set in a domain R , M a left R -module, $m \in M$ and $\varepsilon := \varepsilon_{S,R,M}$. If $\varepsilon(m)$ is a torsion element in $S^{-1}M$, then m is a torsion element of M .*

The same kind of compatibility with localization holds for pre-annihilators:

Proposition 11.19. *Let S be a left denominator set in a ring R and $M \cong R^n/P$ a finitely generated module for some left R -submodule P of R^n . Then*

$$S^{-1} \text{preAnn}_R(M) = \text{preAnn}_{S^{-1}R}(S^{-1}M).$$

Proof: We have

$$\begin{aligned} S^{-1} \text{preAnn}_R(M) &= S^{-1} \bigcap_{j=1}^n \text{Ann}_R^M(e_j) \stackrel{9.11}{=} \bigcap_{j=1}^n S^{-1} \text{Ann}_R^M(e_j) \\ &\stackrel{11.17}{=} \bigcap_{j=1}^n \text{Ann}_{S^{-1}R}^{S^{-1}M}(\varepsilon(e_j)) \stackrel{9.16}{=} \text{preAnn}_{S^{-1}R}(S^{-1}M). \quad \square \end{aligned}$$

This results in the following characterization of S -torsion modules via pre-annihilators:

Proposition 11.20. *Let S be a left denominator set in a ring R , P a left R -submodule of R^n and $M = R^n/P$. Then the following are equivalent:*

- (1) M is an S -torsion module.
- (2) $S^{-1}M = \{0\}$.
- (3) $S^{-1} \text{preAnn}_R(M) = S^{-1}R$.
- (4) $\text{preAnn}_R(M) \cap \text{LSat}(S) \neq \emptyset$.
- (5) $\text{preAnn}_R(M) \cap S \neq \emptyset$.

Proof: The equivalence of the last three statements is a special case of [Lemma 10.21](#).

(1) $\Rightarrow$ (2): Since M is S -torsion, we have $M = t_S(M)$. Now $S^{-1}M = S^{-1}t_S(M) = \{0\}$ by [Proposition 11.9](#).

(2) $\Rightarrow$ (1): Let $m \in M$, then $(1, m) \in S^{-1}M = \{0\}$ implies $(1, m) = 0$. Thus there exists $s \in S$ such that $sm = 0$, therefore $m \in t_S(M)$.

(2) $\Rightarrow$ (3): By [Proposition 11.19](#) we have

$$S^{-1} \text{preAnn}_R(M) = \text{preAnn}_{S^{-1}R}(S^{-1}M) = \text{preAnn}_{S^{-1}R}(\{0\}) = S^{-1}R.$$

(3) $\Rightarrow$ (2): Again by [Proposition 11.19](#) we have

$$S^{-1}R = S^{-1} \text{preAnn}_R(M) = \text{preAnn}_{S^{-1}R}(S^{-1}M).$$

But then $S^{-1}M = \{0\}$, since $1_{S^{-1}R}$ is central and annihilates the basis of $S^{-1}M$. $\square$

Corollary 11.21. *Let R be a left Ore domain, P a left R -submodule of R^n and $M = R^n/P$. Then the following are equivalent:*

- (1) M is a torsion module.
- (2) $(R \setminus \{0\})^{-1}M = \{0\}$.
- (3) $(R \setminus \{0\})^{-1}\text{preAnn}_R(M) = \text{Quot}(R)$.
- (4) $\text{preAnn}_R(M) \neq \{0\}$.

Part III.

Algorithms

Chapter 12.

G -algebras

G -algebras are a generalization of classical commutative polynomial rings. While (mostly) non-commutative, they are close enough to the commutative case that many properties, concepts, and algorithms can be carried over. In this chapter we give a brief overview over their basic properties and just enough information on their Gröbner basis theory to make sense of the algorithms presented in the later chapters. More details, including the proofs omitted here, can be found in [Lev05].

12.1. Orderings on exponents

One of the main ingredients in the theory of Gröbner bases is the concept of orderings, which allows us to talk about notions like degree and leading monomial. Note that the results in this section will also be used to prove correctness of [Algorithm 14.11](#).

Convention 12.1. In this section, let $n, m, r \in \mathbb{N}$. ◇

Definition 12.2. The (partial) ordering $\leq_{cw}$ on $\mathbb{N}_0^n$, defined via

$$\alpha \leq_{cw} \beta \quad :\Leftrightarrow \quad \alpha_i \leq \beta_i \text{ for all } i \in \{1, \dots, n\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, is called the *component-wise ordering*. Equivalently, we have $\alpha \leq_{cw} \beta$ if and only if $\beta \in \alpha + \mathbb{N}_0^n$. ◇

Definition 12.3. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. The *support* of α is

$$\text{supp}(\alpha) := \{i \in \{1, \dots, n\} \mid \alpha_i \neq 0\} \subseteq \{1, \dots, n\}. \quad \diamond$$

Lemma 12.4. Let $\alpha, \beta \in \mathbb{N}_0^n$. The following are equivalent:

- (1) $\text{supp}(\alpha) \subseteq \text{supp}(\beta)$.
- (2) There exists $k \in \mathbb{N}$ such that $\alpha \leq_{cw} k\beta$.

Proof: (2) $\Rightarrow$ (1): Let $i \in \text{supp}(\alpha)$, then $0 < \alpha_i \leq_{cw} k\beta_i$ implies $\beta_i > 0$ and thus $i \in \text{supp}(\beta)$.

(1) $\Rightarrow$ (2): For $i \in \{1, \dots, n\}$ define

$$k_i := \begin{cases} \lceil \frac{\alpha_i}{\beta_i} \rceil, & \text{if } i \in \text{supp}(\alpha), \\ 1, & \text{else} \end{cases}$$

and $k := \max\{k_i \mid i \in \{1, \dots, n\}\} \in \mathbb{N}$. Now let $i \in \{1, \dots, n\}$: if $i \in \text{supp}(\alpha)$, then $\alpha_i = \frac{\alpha_i}{\beta_i} \beta_i \leq k_i \beta_i \leq k \beta_i$, if $i \notin \text{supp}(\alpha)$, then $\alpha_i = 0 \leq k \beta_i$, since $k \in \mathbb{N}$ and $\beta_i \in \mathbb{N}_0$. Now $\alpha_i \leq k \beta_i$ for all i , thus $\alpha \leq_{cw} k\beta$. □

Definition 12.5. A total ordering $\leq$ on $\mathbb{N}_0^n$ with least element 0 is called *admissible* if $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$. $\diamond$

Lemma 12.6. Any admissible ordering $\leq$ on $\mathbb{N}_0^n$ is a refinement of the component-wise ordering, that is, $\alpha \leq_{cw} \beta$ implies $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}_0^n$.

Proof: Let $\alpha, \beta \in \mathbb{N}_0^n$ such that $\alpha \leq_{cw} \beta$, then $\beta - \alpha \in \mathbb{N}_0^n$. Since $\leq$ is admissible, we have $0 \leq \beta - \alpha$ and thus $\alpha = \alpha + 0 \leq \alpha + \beta - \alpha = \beta$. $\square$

Definition 12.7. Let $\leq_1$ resp. $\leq_2$ be an admissible ordering on $\mathbb{N}_0^n$ resp. $\mathbb{N}_0^m$. The ordering $\leq := (\leq_1, \leq_2)$ on $\mathbb{N}_0^{n+m}$, defined via

$$\alpha \leq \beta \quad :\Leftrightarrow \quad \begin{cases} \alpha_2 <_2 \beta_2 \text{ or} \\ \alpha_2 = \beta_2 \text{ and } \alpha_1 \leq_1 \beta_1, \end{cases}$$

for $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^{n+m}$, where $\alpha_1, \beta_1 \in \mathbb{N}_0^n$ and $\alpha_2, \beta_2 \in \mathbb{N}_0^m$, is called an (n, m) -antiblock ordering. $\diamond$

Remark 12.8. Let $\leq = (\leq_1, \leq_2)$ be an (n, m) -antiblock ordering on $\mathbb{N}_0^{n+m}$. Then $\leq$ is an admissible ordering on $\mathbb{N}_0^{n+m}$. $\diamond$

Definition 12.9. Let $\leq$ be an admissible ordering on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$. We call $\leq$ an *elimination ordering* for the last m components if for all $\alpha, \beta \in \mathbb{N}_0^{n+m}$, $\beta \in \mathbb{N}_0^n \times \{0\}$ and $\alpha \leq \beta$ imply $\alpha \in \mathbb{N}_0^n \times \{0\}$. $\diamond$

Lemma 12.10. Let $\leq = (\leq_1, \leq_2)$ be an (n, m) -antiblock ordering on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$. Then $\leq$ is an elimination ordering for the last m components.

Proof: Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^{n+m}$ and $\beta = (\beta_1, 0) \in \mathbb{N}_0^n \times \{0\}$ such that $\alpha \leq \beta$. If $\alpha = \beta$ then there is nothing to show, so assume $\alpha \neq \beta$. Now either $\alpha_2 <_2 0$, which is a contradiction to $\leq_2$ being admissible, or $\alpha_2 = 0$ and $\alpha_1 <_1 \beta_1$, which implies $\alpha \in \mathbb{N}_0^n \times \{0\}$. $\square$

Remark 12.11. The set $\underline{r} \times \mathbb{N}_0^n$ is a (left) $\mathbb{N}_0^n$ -monoideal via

$$+ : \mathbb{N}_0^n \times (\underline{r} \times \mathbb{N}_0^n), \quad (\beta, (i, \alpha)) \mapsto \beta + (i, \alpha) := (i, \beta + \alpha). \quad \diamond$$

Definition 12.12. Let $\leq$ be an admissible ordering on $\mathbb{N}_0^n$. A total ordering $\preceq$ on $\underline{r} \times \mathbb{N}_0^n$ is an *admissible ordering extending $\leq$* if the following conditions hold:

- Extending: for all $\alpha, \beta \in \mathbb{N}_0^n$ and $i \in \underline{r}$, $\alpha \leq \beta$ implies $(i, \alpha) \preceq (i, \beta)$.
- Admissible: for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ and $i, j \in \underline{r}$, $(i, \alpha) \preceq (j, \beta)$ implies $\gamma + (i, \alpha) \preceq \gamma + (j, \beta)$. $\diamond$

Definition 12.13. Let $\leq$ be an admissible ordering on $\mathbb{N}_0^n$. The *(ascending) position-over-term ordering* extending $\leq$ is the ordering $\leq^{\text{POT}}$ on $\underline{r} \times \mathbb{N}_0^n$, defined via

$$(i, \alpha) \leq^{\text{POT}} (j, \beta) \quad :\Leftrightarrow \quad \begin{cases} i < j \text{ or} \\ i = j \text{ and } \alpha \leq \beta \end{cases}$$

for $\alpha, \beta \in \mathbb{N}_0^n$ and $i, j \in \underline{r}$. $\diamond$

Lemma 12.14. *Let $\leq = (\leq_1, \leq_2)$ be an (n, m) -antiblock ordering on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$ and $\preceq$ an admissible ordering on $\underline{r} \times \mathbb{N}_0^{n+m}$ extending $\leq$. Then the ordering $\preceq_2$ on $\underline{r} \times \mathbb{N}_0^m$, defined via*

$$(i, \alpha) \preceq_2 (j, \beta) \quad :\Leftrightarrow \quad (i, (0, \alpha)) \preceq (j, (0, \beta))$$

for $\alpha, \beta \in \mathbb{N}_0^m$ and $i, j \in \underline{r}$, is an admissible ordering on $\underline{r} \times \mathbb{N}_0^m$ extending $\leq_2$. In particular, if $\preceq = \leq^{\text{POT}}$, then $\preceq_2 = \leq_2^{\text{POT}}$.

Proof: By construction, $\preceq_2$ is a total ordering on $\underline{r} \times \mathbb{N}_0^m$. Let $\alpha, \beta, \gamma \in \mathbb{N}_0^m$ and $i, j \in \underline{r}$.

- Let $\alpha \leq_2 \beta$, then $(0, \alpha) \leq (0, \beta)$, since $\leq$ is an (n, m) -antiblock ordering. Now $(i, (0, \alpha)) \preceq (i, (0, \beta))$, since $\preceq$ extends $\leq$, which implies $(i, \alpha) \preceq_2 (i, \beta)$ by definition of $\preceq_2$.
- Let $(i, \alpha) \preceq_2 (j, \beta)$, then $(i, (0, \alpha)) \preceq (j, (0, \beta))$ by definition of $\preceq_2$. Now

$$(i, (0, \alpha + \gamma)) = (0, \gamma) + (i, (0, \alpha)) \preceq (0, \gamma) + (j, (0, \beta)) = (j, (0, \beta + \gamma)),$$

since $\preceq$ is admissible, which implies

$$\gamma + (i, \alpha) = (i, \alpha + \gamma) \preceq_2 (j, \beta + \gamma) = \gamma + (j, \beta)$$

by definition of $\preceq_2$.

Thus, $\preceq_2$ is an admissible ordering extending $\leq_2$. Finally, let $\preceq = \leq^{\text{POT}}$, then

$$\begin{aligned} (i, \alpha) \preceq_2 (j, \beta) &\Leftrightarrow (i, (0, \alpha)) \preceq (j, (0, \beta)) \\ &\Leftrightarrow (i, (0, \alpha)) \leq^{\text{POT}} (j, (0, \beta)) \\ &\Leftrightarrow \begin{cases} i < j \text{ or} \\ i = j \text{ and } (0, \alpha) \leq (0, \beta) \end{cases} \\ &\Leftrightarrow \begin{cases} i < j \text{ or} \\ i = j \text{ and } \alpha \leq_2 \beta \end{cases} \\ &\Leftrightarrow (i, \alpha) \leq_2^{\text{POT}} (j, \beta). \end{aligned}$$

□

Lemma 12.15. *Let $\leq = (\leq_1, \leq_2)$ be an (n, m) -antiblock ordering on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$ and $\preceq := \leq^{\text{POT}}$. Then for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{N}_0^{n+m}$ and $i, j \in \underline{r}$, $(i, (\alpha_1, \alpha_2)) \preceq (j, (\beta_1, \beta_2))$ implies $(i, \alpha_2) \leq_2^{\text{POT}} (j, \beta_2)$.*

Proof: We have

$$\begin{aligned} (i, (\alpha_1, \alpha_2)) \preceq (j, (\beta_1, \beta_2)) &\Leftrightarrow (i, (\alpha_1, \alpha_2)) \leq^{\text{POT}} (j, (\beta_1, \beta_2)) \\ &\Leftrightarrow \begin{cases} i < j \text{ or} \\ i = j \text{ and } (\alpha_1, \alpha_2) \leq (\beta_1, \beta_2) \end{cases} \\ &\Rightarrow \begin{cases} i < j \text{ or} \\ i = j \text{ and } \alpha_2 \leq_2 \beta_2 \end{cases} \\ &\Leftrightarrow (i, \alpha_2) \leq_2^{\text{POT}} (j, \beta_2). \end{aligned}$$

□

12.2. Definition and properties

Definition 12.16. Let K be a field and A a K -algebra generated by $x_1, \dots, x_n$. The set of *standard monomials* of A is

$$\text{Mon}(A) := \{x^\alpha \mid \alpha \in \mathbb{N}_0^n\} := \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}_0\}. \quad \diamond$$

Definition 12.17. Let K be a field and A a K -algebra generated by $x_1, \dots, x_n$. Further, let $\leq$ be an admissible ordering on $\mathbb{N}_0^n$. Any $f \in {}_K\langle \text{Mon}(A) \rangle \setminus \{0\}$ has a unique representation $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha$ for some $c_\alpha \in K$, where $c_\alpha = 0$ for almost all α , but $c_\alpha \neq 0$ for at least one α . Now we define

- $\mathcal{N}(f) := \{\alpha \in \mathbb{N}_0^n \mid c_\alpha \neq 0\} \subseteq \mathbb{N}_0^n$, the *Newton diagram* of f ,
- $\text{le}_\leq(f) := \max_\leq(\mathcal{N}(f)) \in \mathbb{N}_0^n$, the *leading exponent* of f with respect to $\leq$,
- $\text{lc}_\leq(f) := c_{\text{le}_\leq(f)} \in K \setminus \{0\}$, the *leading coefficient* of f with respect to $\leq$,
- $\text{lm}_\leq(f) := x^{\text{le}_\leq(f)} \in \text{Mon}(A)$, the *leading monomial* of f with respect to $\leq$. $\diamond$

Definition 12.18. Let K be a field and consider the K -algebra

$$A := K\langle x_1, \dots, x_n \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\}_{1 \leq i < j \leq n} \rangle,$$

where $c_{i,j} \in K \setminus \{0\}$ and $d_{i,j} \in {}_K\langle \text{Mon}(A) \rangle$. We call A a G -algebra if the following two conditions are met:

- Ordering condition: there exists an admissible ordering on $\mathbb{N}_0^n$ such that

$$\text{le}_\leq(d_{i,j}) < e_i + e_j = \text{le}_\leq(x_i x_j) \quad \text{for all } 1 \leq i < j \leq n \quad \text{where } d_{i,j} \neq 0.$$

- Non-degeneracy condition: for all $1 \leq i < j < k \leq n$ we have

$$\mathcal{NDC}_{i,j,k} := c_{i,k} c_{j,k} d_{i,j} x_k - x_k d_{i,j} + c_{j,k} x_j d_{i,k} - c_{i,j} d_{i,k} x_j + d_{j,k} x_i - c_{i,j} c_{i,k} x_i d_{j,k} = 0. \quad \diamond$$

Definition 12.19. Let K be a field and A a K -algebra generated by $x_1, \dots, x_n$. Then A has a *Poincaré-Birkhoff-Witt basis* (or *PBW basis* for short) if $\text{Mon}(A)$ is a K -basis of A . $\diamond$

Theorem 12.20. Let A be a G -algebra.

- (a) A has a PBW basis.
- (b) A is (left and right) Noetherian.
- (c) A is a domain.

G -algebras [LS03; Lev05] are also known as *algebras of solvable type* [KW90; Kre93; Kre15] and as *PBW algebras* [BGV03]. They encompass algebras of linear functional operators modeling difference and differential equations.

Example 12.21. Let K be a field, $n \in \mathbb{N}$, $q_i \in K \setminus \{0\}$ and $q := [q_1 \ \dots \ q_n] \in K^n$. Common G -algebras include the following examples, where only the relations between non-commutating variables are listed:

- The commutative polynomial ring $K[x_1, \dots, x_n]$.
- The n -th Weyl algebra $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ with $\partial_i x_i = x_i \partial_i + 1$.
- The n -th shift algebra $S_n := K\langle x_1, \dots, x_n, s_1, \dots, s_n \rangle$ with $s_i x_i = x_i s_i + s_i = (x_i + 1)s_i$.
- The n -th q -shift algebra $S_n^{(q)} := K\langle x_1, \dots, x_n, s_1, \dots, s_n \rangle$ with $s_i x_i = q_i x_i s_i$.
- The n -th q -Weyl algebra $A_n^{(q)} := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ with $\partial_i x_i = q_i x_i \partial_i + 1$.
- The n -th integration algebra $K\langle x_1, \dots, x_n, I_1, \dots, I_n \rangle$ with $I_i x_i = x_i I_i + I_i^2$.

The examples above are all algebras of linear operators that are used to study functional equations from an algebraic viewpoint. For example, the Weyl algebras model linear differential equations with polynomial coefficients, and the shift algebras are their difference analogon.

Tensor products of G -algebras over a common ground field are again G -algebras, like the algebra $A_1 \otimes_K S_1$ of mixed difference and differential operators.

Furthermore, universal enveloping algebras of finite-dimensional Lie algebras and many quantum groups are also G -algebras. $\diamond$

12.3. Gröbner bases in G -algebras

An unnecessarily short summary of this section would be the following: apart from some shortcomings mostly concerning incompatible orderings, the Gröbner basis theory for G -algebras provides the same tools as the classical theory for commutative polynomial rings.

Convention 12.22. In this section, let K be a field, $n, r \in \mathbb{N}$ and

$$A := K\langle x_1, \dots, x_n \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\}_{1 \leq i < j \leq n} \rangle$$

a G -algebra over K . We will consider the free left A -module A^r and $e_i \in A^r$, where $(e_i)_j := \delta_{i,j}$ for all i and j . Furthermore, let $\leq$ be an admissible ordering on $\mathbb{N}_0^n$ and $\preceq$ an admissible ordering on $\underline{r} \times \mathbb{N}_0^n$ extending $\leq$. $\diamond$

Definition 12.23. Let $\alpha \in \mathbb{N}_0^n$ and $i \in \underline{r}$, then $x^\alpha e_i \in A^r$ is a *monomial* in *component* i and

$$\text{Mon}(A^r) := \{x^\alpha e_i \mid \alpha \in \mathbb{N}_0^n, i \in \underline{r}\} \cong \underline{r} \times \mathbb{N}_0^n$$

is the set of (*standard*) *monomials* of A^r . $\diamond$

Remark 12.24. $\text{Mon}(A^r)$ is a K -basis of A^r . $\diamond$

Definition 12.25. Let $m_1 := x^\alpha e_i, m_2 := x^\beta e_j \in \text{Mon}(A^r)$. Then m_1 *divides* m_2 , denoted $m_1 \mid m_2$, if $i = j$ and $\alpha \leq_{cw} \beta$. $\diamond$

Definition 12.26. Any $f \in A^r \setminus \{0\}$ has a unique representation $f = \sum_{(i,\alpha) \in \underline{r} \times \mathbb{N}_0^n} c_{(i,\alpha)} x^\alpha e_i$ for some $c_{(i,\alpha)} \in K$, where $c_{(i,\alpha)} = 0$ for almost all (i, α) , but $c_{(i,\alpha)} \neq 0$ for at least one (i, α) . Define

- $\text{le}_\preceq(f) := \max_\preceq(\{(i, \alpha) \in \underline{r} \times \mathbb{N}_0^n \mid c_{(i,\alpha)} \neq 0\}) \in \underline{r} \times \mathbb{N}_0^n$, the *leading exponent* of f with respect to $\preceq$,
- $\text{lc}_\preceq(f) := c_{\text{le}_\preceq(f)} \in K \setminus \{0\}$, the *leading coefficient* of f with respect to $\preceq$,

- $\text{lm}_{\preceq}(f) := x^\alpha e_i \in \text{Mon}(A^r)$, where $(i, \alpha) = \text{le}_{\preceq}(f)$, the *leading monomial* of f with respect to $\preceq$.

Furthermore, let $B \subseteq A^r$ and define

- $\mathcal{L}_{\preceq}(B) := \text{Exp}_{\preceq}(B) := \mathbb{N}_0^n + \{(i, \alpha) \in \underline{r} \times \mathbb{N}_0^n \mid \exists f \in B : \text{le}_{\preceq}(f) = (i, \alpha)\} \subseteq \underline{r} \times \mathbb{N}_0^n$, the *monoidal of leading exponents* of B with respect to $\preceq$,
- $L_{\preceq}(B) := {}_K\langle \{x^\alpha e_i \in \text{Mon}(A^r) \mid (i, \alpha) \in \text{Exp}_{\preceq}(B)\} \rangle \subseteq A^r$, the *span of leading monomials* of B with respect to $\preceq$. $\diamond$

Definition 12.27. Let I be a left A -submodule of A^r and G a finite subset of I . Then G is a *left Gröbner basis* of I with respect to $\preceq$ if for all $f \in I \setminus \{0\}$ there exists $g \in G$ such that $\text{lm}_{\preceq}(g) \mid \text{lm}_{\preceq}(f)$. $\diamond$

Theorem 12.28. Let I be a left A -submodule of A^r and G a finite subset of I . Then the following are equivalent:

- (1) G is a left Gröbner basis of I with respect to $\preceq$.
- (2) $L_{\preceq}(G) = L_{\preceq}(I)$ as K -vector spaces.
- (3) $\text{Exp}_{\preceq}(G) = \text{Exp}_{\preceq}(I)$ as $\mathbb{N}_0^n$ -monoids.

Definition 12.29. Let $\mathcal{G}$ be the set of all finite subsets of A^r .

- A map $\text{NF} := \text{NF}_{\preceq} : A^r \times \mathcal{G} \rightarrow A^r$, $(f, G) \mapsto \text{NF}(f|G)$ is a *left normal form* on A^r with respect to $\preceq$ if for all $f \in A^r$ and $G \in \mathcal{G}$
 - (i) $\text{NF}(0|G) = 0$,
 - (ii) $\text{NF}(f|G) \neq 0$ implies $\text{lm}_{\preceq}(\text{NF}(f|G)) \notin L(G)$,
 - (iii) $f - \text{NF}(f|G) \in {}_A\langle G \rangle$.
- Let $G = \{g_1, \dots, g_s\} \in \mathcal{G}$. A *left Gröbner representation* of $f \in {}_A\langle G \rangle$ with respect to G and $\preceq$ is a representation $f = \sum_{i=1}^s a_i g_i$ with $a_i \in A$ satisfying $\text{le}_{\preceq}(a_i g_i) \preceq \text{le}_{\preceq}(f)$ for all i such that $a_i g_i \neq 0$. $\diamond$

Lemma 12.30. Let I be a left A -submodule of A^r , G a left Gröbner basis of I with respect to $\preceq$ and $\text{NF}(\cdot|G)$ a left normal form on A^r with respect to G and $\preceq$.

- (a) For any $f \in A^r$ we have $f \in I$ if and only if $\text{NF}(f|G) = 0$.
- (b) If J is a left A -submodule of A^r with $I \subseteq J$, then $L(I) = L(J)$ implies $I = J$. In particular, $I = {}_A\langle G \rangle$.

Algorithm 12.31 (LEFTNF).

Input: $f \in A^r$, a finite subset G of A^r

Output: $h \in A^r$, a left normal form of f with respect to G and $\preceq$.

```

1 begin
2    $h := f$ ;
3   while  $h \neq 0$  and  $G_h := \{g \in G : \text{lm}_{\preceq}(g) \mid \text{lm}_{\preceq}(h)\} \neq \emptyset$  do
4     choose  $g \in G_h$ ;
5      $(i, \alpha) := \text{le}_{\preceq}(h)$ ;
6      $(i, \beta) := \text{le}_{\preceq}(g)$ ;
7      $h := \text{LeftSpoly}(h, g) := h - \frac{\text{lc}_{\preceq}(h)}{\text{lc}_{\preceq}(x^{\alpha-\beta}g)} x^{\alpha-\beta} g$ ;
8   end
9   return  $h$ ;
10 end
```

Theorem 12.32. *Let I be a left A -submodule of A^r , G a finite subset of I and $\text{LeftNF}(\cdot|G)$ a left normal form on A^r with respect to G and $\preceq$. Then the following are equivalent:*

- (1) G is a left Gröbner basis of I with respect to $\preceq$.
- (2) $\text{LeftNF}(f|G) = 0$ for all $f \in I$.
- (3) Each $f \in I$ has a standard left representation with respect to G and $\preceq$.

Definition 12.33. Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$. Define $\mu(\alpha, \beta) \in \mathbb{N}_0^n$ via $\mu(\alpha, \beta)_i := \max(\{\alpha_i, \beta_i\})$. $\diamond$

Algorithm 12.34 (LEFTGRÖBNERBASIS).**Input:** a finite subset F of A^r **Output:** a left Gröbner basis G of ${}_A\langle F \rangle$ with respect to $\preceq$

```
1 begin
2    $G := F$ ;
3    $P := \{(f, g) \in G \times G \mid f \neq g\}$ ;
4   while  $P \neq \emptyset$  do
5     choose  $(f, g) \in P$ ;
6      $P := P \setminus \{(f, g)\}$ ;
7      $(i, \alpha) := \text{le}_{\preceq}(f)$ ;
8      $(j, \beta) := \text{le}_{\preceq}(g)$ ;
9      $\gamma := \mu(\alpha, \beta)$ ;
10     $h := 0$ ;
11    if  $i = j$  then
12       $t := x^{\gamma-\alpha}f - \frac{\text{lc}_{\preceq}(x^{\gamma-\alpha}f)}{\text{lc}_{\preceq}(x^{\gamma-\beta}g)}x^{\gamma-\beta}g$ ;
13       $h := \text{LeftNF}(t|G)$ ;
14    end
15    if  $h \neq 0$  then
16       $P := P \cup \{(h, f) \mid f \in G\}$ ;
17       $G := G \cup \{h\}$ ;
18    end
19  end
20  return  $G$ ;
21 end
```

Chapter 13.

Constructive computations in Ore-localized G -algebras

This section is concerned with the following problem: given a left Ore set in a domain, how can we treat the basic arithmetic operations addition and multiplication constructively?

Legal note. The material presented in this chapter is an extended version of our 2017 ISSAC paper ([HL17]), which in turn is a highly improved version of the author's bachelor thesis ([Hof14]). The theory, algorithms, and implementation are the author's original work, with some remarks and suggestions contributed by his co-author.

Recall the construction of the left Ore localization given in [Proposition 5.21](#):

Proposition 5.21 (Construction, cf. [Proposition 2.24](#)). *Let S be a left denominator set in a ring R . Then*

$$(S^{-1}R, +, (1, 0), \cdot, (1, 1))$$

is a ring with

$$+ : S^{-1}R \times S^{-1}R \rightarrow S^{-1}R, \quad ((s_1, r_1), (s_2, r_2)) \mapsto (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$, and

$$\cdot : S^{-1}R \times S^{-1}R \rightarrow S^{-1}R, \quad ((s_1, r_1), (s_2, r_2)) \mapsto (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$.

A close examination shows us that we require two things for a constructive treatment of these operations: firstly, we have to be able to perform addition and multiplication in the base ring R , which should not be a restriction for practical purposes. But more importantly, we have to be able to compute left Ore data for any $s \in S$ and any $r \in R$. Note that the left Ore condition itself is inherently unconstructive, since it only guarantees the existence of left Ore data. Thus we have to find an algorithmic way by other means.

13.1. A constructive approach to the left Ore condition

Given a left Ore set S in a domain R and $(s, r) \in S \times R$, we will consider the following sub-problems:

- (1) Find $\tilde{s} \in S$ such that there exists $\tilde{r} \in R$ satisfying $\tilde{s}r = \tilde{r}s$.

- (2) Find the set of all $\tilde{s} \in S$ such that there exists $\tilde{r} \in R$ satisfying $\tilde{s}r = \tilde{r}s$.
- (3) Given $\tilde{s} \in S$ such that there exists $\tilde{r} \in R$ satisfying $\tilde{s}r = \tilde{r}s$, find such $\tilde{r}$.

In [Lemma 5.18](#) we have seen that all pairs of left Ore data represent the same left fraction in $S^{-1}R$. Nevertheless, the algorithms presented later strive to compute a solution to problem (2) above, since even checking the equivalence of two left fractions is a non-trivial task.

Furthermore, the solution to problem (3) is unique:

Lemma 13.1. *Let S be a left Ore set in a domain R , $s, \tilde{s} \in S$ and $r \in R$. Then there is at most one $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.*

Proof: Let $r_1, r_2 \in R$ such that $r_1s = \tilde{s}r = r_2s$, then $(r_1 - r_2)s = 0$, which implies $r_1 = r_2$ since R is a domain and $s \neq 0$. $\square$

The kernel technique

Let S be a multiplicatively closed subset of a domain R . To the best of our knowledge there is no algorithm to decide whether S is a left Ore set in R . Usually such facts are established by the means of theoretical proofs like the ones presented in [Chapter 8](#).

To deal with this issue, consider the map

$$\varphi_{s,r} : R \xrightarrow{r} R/Rs, \quad x \mapsto xr + Rs.$$

It is a homomorphism of left R -modules and the intersection of its kernel with S is exactly the solution of (2) from above. This immediately gives us the following characterization of the left Ore property:

Proposition 13.2. *Let S be a multiplicative set in a domain R . The following are equivalent:*

- (1) S satisfies the left Ore condition in R .
- (2) For all $(s, r) \in S \times R$, $\ker(\varphi_{s,r}) \cap S \neq \emptyset$.

Remark 13.3. Provided we can algorithmically check whether $\ker(\varphi_{s,r}) \cap S$ is empty, [Proposition 13.2](#) also allows us to prove that a given set S is *not* a left Ore set in R , if we (correctly) suspect that no left Ore data exists for s and r (an example of this can be found in [Example 13.16](#)). $\diamond$

After choosing a $\tilde{s} \in \ker(\varphi_{s,r}) \cap S$, $\tilde{r}$ is the unique solution of the equation $\tilde{s}r = \tilde{r}s$. These considerations are combined in the procedure LEFTORE.

Algorithm 13.4 (LEFTORE).

Input: $(s, r) \in S \times R$.

Output: $(\tilde{s}, \tilde{r}) \in S \times R$ such that $\tilde{s}r = \tilde{r}s$ and J , the set of all $\hat{s} \in S$ such that there exists $\hat{r} \in R$ satisfying $\hat{s}r = \hat{r}s$.

```

1 begin
2   let  $\varphi_{s,r} : R \rightarrow R/Rs, x \mapsto xr + Rs$ ;
3   compute  $\ker(\varphi_{s,r})$ ;
4   compute  $J := \ker(\varphi_{s,r}) \cap S$ ;
5   choose  $\tilde{s} \in J$ ;
6   compute the unique solution  $\tilde{r} \in R$  of the equation  $\tilde{s}r = \tilde{r}s$ ;
7   return  $(\tilde{s}, \tilde{r}), J$ 
8 end

```

At this point this is nothing but a recipe. To apply it, we need to choose a setting where all the intermediate steps can actually be carried out. Before we deal with this we will discuss alternative approaches to computing left Ore data algorithmically:

Remark 13.5. Instead of $\varphi_{s,r}$ one could also consider the seemingly symmetric map

$$\psi_{s,r} : R \xrightarrow{\cdot s} R/Rr, \quad x \mapsto xs + Rr.$$

Here we avoid intersecting $\ker(\psi_{s,r})$ with S at first, but this kernel may contain numerous false candidates for $\tilde{r}$: although there exists $y \in R$ such that $yr = \tilde{r}s$, there is no guarantee that we can find such y in S . Therefore we need to intersect $M := \{(\tilde{r}, \tilde{s}) \in \ker(\psi_{s,r}) \times R \mid \tilde{r}s = \tilde{s}r\}$ with $R \times S$, which poses the same problems as intersecting $\ker(\varphi_{s,r})$ with S directly. $\diamond$

Remark 13.6. Another approach can be based on *syzygies*: consider the left syzygy module of $\begin{bmatrix} r & -s \end{bmatrix}^T \in R^{2 \times 1}$, given by

$$\text{LSyz}\left(\begin{bmatrix} r & -s \end{bmatrix}^T\right) := \left\{ \begin{bmatrix} r_1 & r_2 \end{bmatrix} \in R^{1 \times 2} : r_1r - r_2s = 0 \right\}.$$

While this gives us candidates for $\tilde{s}$ and $\tilde{r}$ in one computation, we still have to intersect the syzygy module with $S \times R$. $\diamond$

In the following we restrict ourselves to localizing G -algebras, since they provide a computation-friendly setting in which we can actually carry out the computations from the procedure `LEFTORE`. Nevertheless, the basic concept of `LEFTORE` can be adapted to other settings, provided the individual steps can be carried out effectively.

Convention 13.7. For the remainder of this chapter, let A be a G -algebra over a field K . $\diamond$

Remark 13.8. Except for the computation of $\ker(\varphi_{s,r}) \cap S$, all steps of [Algorithm 13.4](#) are algorithmic in G -algebras with standard Gröbner techniques, since they boil down to computing kernel of module homomorphisms and performing division with remainder, see [\[Lev05\]](#) for details. $\diamond$

Computable localization types

To get more exploitable structure we will only consider the following types of left Ore sets, which represent a restriction of the more general setting in [Definition 5.28](#):

Definition 13.9. A left Ore set S in A is called a *computable* Ore set if it belongs to one (or multiple) of the following types:

Monoidal: S is generated as a monoid by finitely many elements contained in a commutative polynomial subring of A generated by a subset of the variables.

More precisely, let A be generated by variables $x_1, \dots, x_n$ and $1 \leq i_1 < \dots < i_k \leq n$ such that $P = K[x_{i_1}, \dots, x_{i_k}]$ is a commutative polynomial subring of A . Then $S = [f_1, \dots, f_m]$ for some $f_j \in P$.

Geometric: S is the complement of a prime ideal in a polynomial subring of a Weyl algebra.

More precisely, let $A = A_n$ and $\mathfrak{p}$ a prime ideal in $P = K[x_1, \dots, x_n] \subsetneq A_n$. Then $S = P \setminus \mathfrak{p}$.

Rational: $S \cup \{0\}$ is a sub- G -algebra of A generated by a subset of the variables.

More precisely, let A be generated by variables $x_1, \dots, x_n$ and $1 \leq i_1 < \dots < i_k \leq n$ such that the variables $x_{i_1}, \dots, x_{i_k}$ generate a sub- G -algebra T of A . Then $S = T \setminus \{0\}$. For algorithmic purposes, we require the existence of an elimination ordering for $x_{i_1}, \dots, x_{i_k}$ which is compatible with the ordering conditions on T as a G -algebra. $\diamond$

The left Ore sets given in [Example 5.29](#) are all computable, while the sets in [Proposition 8.20](#) and [Definition 8.22](#) are not computable by the definition above.

13.2. The intersection problem

This section provides the theory and algorithms to compute a representation of the intersection of a left ideal I with a left Ore set S , where S belongs to one of the three types stated above. To avoid rather trivial cases we will assume that I is neither the zero ideal nor the whole algebra, i.e. $\{0\} \subsetneq I \subsetneq A$.

Definition 13.10. Let S be a left denominator set in a ring R and I a left ideal in R . The *intersection problem* is to decide whether $I \cap S = \emptyset$ and to compute an element contained in this intersection when the answer is negative. $\diamond$

Remark 13.11. Another motivation for studying the intersection problem is given in [\[Pos18\]](#), where it is revealed to be an important ingredient for solving linear systems over localizations of commutative rings. In our recent paper ([\[HL18b\]](#)) we gave algorithmic solutions in the case where the ring R is a polynomial algebra, i.e. $R = K[x_1, \dots, x_n]/J$ for a field K and an ideal J in $K[x_1, \dots, x_n]$. $\diamond$

Monoidal localizations

The first step is to reduce the problem to the case where S is generated by one element only. To this end, we generalize the following classical result: let $f_1, \dots, f_k \in R$, where R is a commutative ring, $S := [f_1, \dots, f_k]$ and $T = [f_1 \cdot \dots \cdot f_k]$, then $S^{-1}R \cong T^{-1}R$.

Lemma 13.12. Let R be a domain and $g_1, \dots, g_k \in R \setminus \{0\}$ such that $g_i g_j = g_j g_i$ for all i and j . Consider $S = [g_1, \dots, g_k]$ and $T = [g]$ for $g := g_1 \cdot \dots \cdot g_k$.

- (a) S is a left Ore set in R if and only if T is a left Ore set in R .
- (b) If S and T are left Ore sets, then $S^{-1}R \cong_R T^{-1}R$.

Proof: By construction, S and T are multiplicative sets such that $T \subseteq S$. Since the g_i commute we have

$$(g_1 \cdot \dots \cdot g_{j-1} \cdot g_{j+1} \cdot \dots \cdot g_k) g_j = g \in T$$

which implies $g_j \in \text{LSat}(T)$ and thus $S \subseteq \text{LSat}(T)$, since the g_j generate S as a monoid. Together with $T \subseteq S \subseteq \text{LSat}(S)$ we get $\text{LSat}(S) = \text{LSat}(T)$ by applying [Lemma 6.10](#). Now the first part follows from [Corollary 7.8](#), the second from [Proposition 7.18](#). $\square$

On the one hand, this allows us to assume without loss of generality that the generators of a given computable monoidal left Ore set are irreducible in $K[x_{i_1}, \dots, x_{i_k}]$, which can be helpful for theoretical proofs.

On the other (and for our purposes, more important) hand, we may assume that a given computable monoidal left Ore set is generated by only one (possibly reducible) element: let $S = [g]$ for some $g \in A \setminus K$. To the best of our knowledge it is not possible in general to decide whether $I \cap S$ is empty, but in some cases we can give a positive answer:

Remark 13.13. Consider $K[S] = K[g] \subseteq A$, the K -monoid algebra of S , which is a domain as a subring of the domain A . Assume that we are able to compute $L := I \cap K[S]$. If $L = \{0\}$, then in particular $I \cap S = \emptyset$ since $I \cap S = L \cap S \subsetneq L = I \cap K[S] = \{0\}$. Details on the computation of $I \cap K[S]$ can be found in [And+10]. $\diamond$

In the following we will assume $I \cap S \neq \emptyset$. For our purposes this is not a restriction since we are mostly interested in the case where $I = \ker(\varphi_{s,r})$ for some $s \in S$ and $r \in R$. Then $I \cap S \neq \emptyset$ follows from Proposition 13.2.

If $I \cap S$ is non-empty, then it has the structure of a principal monoid ideal:

Lemma 13.14. *Let $g \in A \setminus K$. If $I \cap [g] \neq \emptyset$, then there exists $m \in \mathbb{N}$ such that $I \cap [g] = g^m \cdot [g]$.*

Proof: If $I \cap [g] \neq \emptyset$, then there exists a minimal $m \in \mathbb{N}$ such that $g^m \in I$ and $g^k \notin I$ for all $k < m$. Now $g^j = g^{j-m} \cdot g^m \in I$ for any $j \geq m$, thus $I \cap [g] = g^m \cdot [g]$. $\square$

The natural thing to do is to iterate over the natural numbers to find the smallest m among them such that $g^m \in I$. This membership test can be done by computing normal forms in the Gröbner sense: in Algorithm 13.15, $\text{NF}(g^m|F)$ denotes the normal form of g^m with respect to F .

To avoid unnecessary expensive normal form computations we use another fact from Gröbner basis theory: the leading monomial of any element of I must be divisible by the leading monomial of an element of a Gröbner basis of I . Since we know that at least one power of g is contained in I , we find the minimal m such that $\text{lm}(g^m)$ is divisible by the leading monomial of any basis element, which can be done by comparing the leading exponents. Note that further enhancements of such computations are available ([And+10]).

Algorithm 13.15 (MONOIDALINTERSECTION).

Input: Gröbner basis $F = \{f_1, \dots, f_k\}$ of I and $g \in A \setminus K$ with $I \cap [g] \neq \emptyset$.

Output: $m \in \mathbb{N}$ such that $I \cap [g] = g^m \cdot [g]$.

```

1 begin
2   foreach  $1 \leq i \leq k$  do
3      $m_i := \min \{k \in (\mathbb{N} \cup \{\infty\}) : \text{lm}(f_i) \mid \text{lm}(g)^k\};$ 
4   end
5    $m := \min \{m_i \mid 1 \leq i \leq k\};$ 
6   while  $\text{NF}(g^m|F) \neq 0$  do
7      $m := m + 1;$ 
8   end
9   return  $m;$ 
10 end
```

Geometric localization

As we have seen in Corollary 8.11, the n -th Weyl algebra A_n contains a multitude of left Ore sets. In particular, we have that geometric localization of A_n is possible at the complement of any prime ideal $\mathfrak{p}$ in $R := K[x_1, \dots, x_n]$. But even in closely related G -algebras like the shift algebra this does not need to hold, as the following example demonstrates:

Example 13.16. Consider the prime ideal $\mathfrak{p} = \langle x + 1 \rangle$ in $K[x]$ as a subalgebra of the first shift algebra S_1 , then for the pair $(x, s) \in (K[x] \setminus \mathfrak{p}) \times S_1$ a simple computation delivers $\ker(\varphi_{x,s}) = \langle x + 1 \rangle = \mathfrak{p}$. Therefore $\ker(\varphi_{x,s}) \cap S = \emptyset$, so $K[x] \setminus \mathfrak{p}$ is not a left Ore set in S_1 by [Proposition 13.2](#). $\diamond$

Thus the main application of the geometric type is localizing A_n at the left Ore set $S := R \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in R . The most common occurrence of this localization is the special case where we replace $\mathfrak{p}$ by the maximal ideal $\mathfrak{m}_p$ in R corresponding to a point $p \in K^n$. The result is the so-called *local (algebraic) Weyl algebra* $A_{n,p} := (R \setminus \mathfrak{m}_p)^{-1} A_n$, which is important in D -module theory.

In contrast to the two other types of localizations, the intersection of I and S has no exploitable additional structure: while it is a multiplicatively closed set without 1, it need not be finitely generated.

Therefore, in [Algorithm 13.17](#), we return essentially the intersection of I and R , which can be computed via Gröbner-driven elimination of variables.

Algorithm 13.17 (GEOMETRICINTERSECTION).

Input: A left ideal I of A_n , S , R and $\mathfrak{p}$ as above.

Output: A left ideal J in R such that $I \cap S = J \setminus \mathfrak{p}$.

```

1 begin
2   compute  $\tilde{I} := I \cap R = \langle m_1, \dots, m_k \rangle$ ;
3   foreach  $1 \leq i \leq k$  do
4     | let  $\tilde{m}_i$  be the normal form of  $m_i$  with respect to  $\mathfrak{p}$ ;
5   end
6   return  $J := \langle \tilde{m}_1, \dots, \tilde{m}_k \rangle$ ;
7 end
```

An element $f \in I \cap R$ is an element of $I \cap S$ if and only if $f \notin \mathfrak{p}$, which can be checked by computing the normal form of f with respect to $\mathfrak{p}$.

Proposition 13.18. *In the situation of [Algorithm 13.17](#), $I \cap S = \emptyset$ if and only if $\tilde{m}_i = 0$ for all i .*

Proof: By construction, $\tilde{m}_i = 0$ for all i if and only if $m_i \in \mathfrak{p}$ for all i , which is equivalent to $I \cap R \subseteq \mathfrak{p}$. From

$$I \cap R = I \cap ((R \setminus \mathfrak{p}) \cup \mathfrak{p}) = I \cap (S \cup \mathfrak{p}) = (I \cap S) \cup (I \cap \mathfrak{p})$$

we can see that this is equivalent to $I \cap S = \emptyset$, since $I \cap S \subseteq R \setminus \mathfrak{p}$. $\square$

Thus, if $I \cap S \neq \emptyset$, a member of this intersection can be found among the non-zero generators of J .

Rational localization

In algebras of linear operators, rational localization provides the formal mechanism of passing from polynomial to rational coefficients, for example from the polynomial Weyl algebra A_n to the rational Weyl algebra $B_n = (K[x] \setminus \{0\})^{-1} A_n$.

To set the scene, let A be generated as a G -algebra by the variables $x_1, \dots, x_n$ and let $V \subseteq \{1, \dots, n\}$ such that $\{x_i \mid i \in V\}$ generate a subalgebra B of A and $S := B \setminus \{0\}$ is a left

Ore set in A . If we can eliminate the variables $\{x_i \mid i \in \{1, \dots, n\} \setminus V\}$ with Gröbner-driven elimination, then [Algorithm 13.19](#) computes the intersection of S and I . In contrast to the commutative case, this elimination is not always possible, see [\[Lev05; Lev06\]](#).

Algorithm 13.19 (RATIONALINTERSECTION).

Input: A left ideal I of A , B as above.

Output: The intersection $I \cap S$.

```

1 begin
2   | compute  $J := I \cap B$  via elimination;
3   | return  $J \setminus \{0\}$ ;
4 end
```

13.3. Canceling a fraction

Given a representation (s, r) of a fraction it is a natural question to ask whether there is a simpler representation (s', r') of the same fraction, for example a representation where the total degree of the denominator s' is smaller than the one of s . Canceling a fraction between other computation steps can have a significant impact on the total computation time. Given that we are not in a unique factorization domain there may be many different representations that we could call simpler than the initial one, thus it is also of interest to find all simpler representations.

We present two approaches to this problem. To this end, let $(s, r) \in S \times R$ be a representation of a fraction in a left Ore localization $S^{-1}R$ of a G -algebra R . We want to compute (at least)

$$C_{s,r} := \{(\hat{s}, \hat{r}) \in S \times R \mid \exists f \in R : f\hat{s} = s \text{ and } f\hat{r} = r\},$$

the set of all representations of (s, r) that can be constructed from (s, r) by left canceling.

Syzygy-based canceling

The first approach is based on computing right and left syzygies (denoted RSyz resp. LSyz below), which can be done with Gröbner-driven algorithms. Let

$$M := \text{RSyz}([s \ r]) := \left\{ [a \ b]^T \in R^{2 \times 1} \mid sa + rb = 0 \right\}.$$

Note that $M \neq \{0\}$ since any G -algebra is right Noetherian and thus a right Ore domain. For any $[a \ b]^T \in M$ let $N_{a,b} := \text{LSyz}([a \ b]^T) = \{[q \ p] \in R^{1 \times 2} \mid qa + pb = 0\}$. We also have $N_{a,b} \neq \{0\}$ since $[s \ r] \in N_{a,b}$. Furthermore, M and $N_{a,b}$ are finitely generated since R is Noetherian.

Lemma 13.20. *Let $[a \ b]^T \in M \setminus \{0\}$.*

- (a) *We have $C_{s,r} \subseteq N_{a,b}$.*
- (b) *Let $(q, p) \in N_{a,b}$ with $q \in S$. Then $(q, p) = (s, r)$ in $S^{-1}R$.*

Proof: (a) Let $[\hat{s} \ \hat{r}] \in C_{s,r}$, then there exists $f \in R \setminus \{0\}$ such that $s = f\hat{s}$ and $r = f\hat{r}$. Since $[a \ b] \in M$ we have $0 = sa + rb = f\hat{s}a + f\hat{r}b = f(\hat{s}a + \hat{r}b)$, which implies $\hat{s}a + \hat{r}b = 0$ and thus $[\hat{s} \ \hat{r}] \in N_{a,b}$.

- (b) By the left Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}q = \tilde{r}s$. Now we have $\tilde{s}pb = -\tilde{s}qa = -\tilde{r}sa = \tilde{r}rb$. Since $b = 0$ would imply the contradiction $a = 0$ we can infer that $\tilde{s}p = \tilde{r}r$, which implies $(q, p) = (s, r)$ in $S^{-1}R$. $\square$

Thus $\tilde{N}_{a,b} := N_{a,b} \cap (S \times R)$ is a superset of $C_{s,r}$ consisting of representations of (s, r) , which immediately leads to the algorithm SYZCANCEL.

Algorithm 13.21 (SYZCANCEL).

Input: A left fraction $(s, r) \in S^{-1}R$.

Output: A set of representations of (s, r) containing $C_{s,r}$.

```

1 begin
2   compute  $M := \text{RSyz}([s \ r]) = \left\{ \begin{bmatrix} a & b \end{bmatrix}^T \in R^{2 \times 1} \mid sa + rb = 0 \right\}$ ;
3   choose any non-zero  $\begin{bmatrix} a & b \end{bmatrix}^T \in M$ ;
4   compute  $N := \text{LSyz}([a \ b]^T) = \left\{ \begin{bmatrix} q & p \end{bmatrix} \in R^{1 \times 2} \mid qa + pb = 0 \right\}$ ;
5   compute  $\tilde{N} := N \cap (S \times R)$ ;
6   return  $\tilde{N}$ ;
7 end
```

Factorization-based canceling

Since G -algebras are finite factorization domains ([BHL17]) there are only finitely many factorizations of the denominator s . Thus we can compute $C_{s,r}$ as follows:

1. Set $M := \emptyset$.
2. Compute all factorizations of $s \in S \subsetneq R$ of the form $s = f_i s_i$, where f_i is irreducible, s_i a non-unit and $i \in I$, where I is a suitable index set for keeping track of these different factorizations. If $I = \emptyset$ return $\{(s, r)\}$.
3. Compute the index set $J := \{i \in I \mid \exists r_i \in R : r = f_i r_i\}$, where r_i can be obtained by right division: $j \in J$ if and only if the right normal form $\text{RightNF}(r, f_j) = 0$. This can only be the case if $\text{tdeg}(r) \geq \text{tdeg}(f_j)$. If $J = \emptyset$ return $\{(s, r)\}$.
4. For every $j \in J$ apply the procedure recursively to (s_j, r_j) and add the results to M .

After finitely many steps we obtain a list of all fully canceled representations of (s, r) . Apart from (s, r) itself they all have denominators with total degree strictly smaller than $\text{tdeg}(s)$, since $\text{tdeg}(s_i) = \text{tdeg}(s) - \text{tdeg}(f_i) \leq \text{tdeg}(s) - 1$. Still there can be several representatives with minimal total degree of the denominator:

Example 13.22. Consider again the localization $\text{LSat}(V)^{-1}A_1$ from Definition 8.22. Then the elements $(x^2, x\partial - 1)$ and $(x\partial + 2, \partial^2)$ represent the same fraction in $\text{LSat}(V)^{-1}A_1$ since

$$(x^2, x\partial - 1) = (\partial x^2, \partial(x\partial - 1)) = (x(x\partial + 2), x\partial^2) = (x\partial + 2, \partial^2).$$

Both denominators have total degree 2 and cannot be canceled further. $\diamond$

13.4. Further algorithmic aspects

Computing in right OLGAs

While we concentrate mostly on the left-sided version of non-commutative structures, the right-sided analogues of the given definitions and results hold as well, which can also be seen by considering *opposite structures*:

Definition 13.23. Let $(R, +, \cdot)$ be a ring, then the *opposite ring* of R is $R^{\text{op}} := (R, +, *)$, where $a * b := b \cdot a$ for all $a, b \in R$. $\diamond$

If K is a field and A is a K -algebra, then A^{op} is also a K -algebra with the same vector space structure as A and is consequently called *opposite algebra* of A . In particular, if A is a G -algebra, we can construct an involution $(\cdot)^{\text{op}} : A \rightarrow A$ by inverting the order of the variables, which realizes A^{op} on the level of individual elements (details for this construction can be found in [Lev05]). Most algorithms for non-commutative structures in SINGULAR:PLURAL are only implemented for the left-sided versions, while right-sided computations are carried out in a left-sided setting in the opposite ring.

For our purposes, the machinery of opposite structures allows us to compute right Ore data by opposing the setting, computing left Ore data, and then opposing again.

The left-right conundrum

Assume that S is a left and right Ore set in a domain R . According to Proposition 5.34, the left Ore localization $S^{-1}R$ is isomorphic to the right Ore localization RS^{-1} via

$$RS^{-1} \rightarrow S^{-1}R, \quad rs^{-1} \mapsto \tilde{s}^{-1}\tilde{r},$$

where $\tilde{s}r = \tilde{r}s$. Given a right fraction $rs^{-1} \in RS^{-1}$, finding a corresponding left fraction $\tilde{s}^{-1}\tilde{r} \in S^{-1}R$ is therefore just another application of the left Ore condition, while computing the inverse image of a left fraction requires the right Ore condition.

Computing inverses

Additive inverses are given by $-(s, r) = (s, -r)$, but, as we have seen earlier, multiplicative inverses are immensely more complicated. Proposition 7.32 tells us that a fraction (s, r) is invertible if and only if $r \in \text{LSat}(S)$, thus deciding invertibility of a fraction is not harder than computing $\text{LSat}(S)$.

Fortunately, in the case of geometric localizations at a prime ideal $\mathfrak{p}$ we are in the fortunate situation that S is already saturated:

Lemma 13.24. Let $\mathfrak{p}$ be a prime ideal in $R := K[x_1, \dots, x_n]$. Then both $R \setminus \{0\}$ and $S := R \setminus \mathfrak{p}$ are saturated in A_n .

Proof: For the first part, consider a global monomial ordering where $\partial_i > x_j$ for all i, j . Let $a, b \in A_n$ such that $a \cdot b \in R \setminus \{0\}$, then a and b are non-zero, thus $\text{tdeg}_{\partial}(a) + \text{tdeg}_{\partial}(b) = \text{tdeg}_{\partial}(a \cdot b) = 0$. Therefore both a and b are contained in $R \setminus \{0\}$. The last part is a consequence of the following result. $\square$

Lemma 13.25. Let K be a field and R a K -algebra and a domain. Let further $K[x] := K[x_1, \dots, x_n] \subseteq R$ and $\mathfrak{p}$ a prime ideal in $K[x]$. If $K[x]$ is saturated in R , then $S := K[x] \setminus \mathfrak{p}$ is saturated in R as well.

Proof: Let $a, b \in R$. Then $ab \in K[x] \setminus \mathfrak{p}$ if and only if $ab \in K[x]$ and $ab \notin \mathfrak{p}$. Since $K[x]$ is saturated in R and $\mathfrak{p}$ is a prime ideal, this is equivalent to $a, b \in K[x]$ and $a, b \notin \mathfrak{p}$ and thus $a, b \in K[x] \setminus \mathfrak{p}$. $\square$

Thus we have that a fraction (s, r) in a geometric localization of A_n at $\mathfrak{p}$ is invertible if and only if $r \in K[x_1, \dots, x_n] \setminus \mathfrak{p}$; the latter condition can be checked algorithmically with commutative Gröbner methods.

Unfortunately, S will not be saturated in general when we consider the other localization types.

In [Lemma 6.5](#) we have seen that $r \in \text{LSat}(S)$ if and only if ${}_R\langle r \rangle \cap S \neq \emptyset$. Therefore we can decide invertibility of a given fraction if we can decide non-emptiness of the intersection of S with a principal left ideal. For rational localizations this can be checked with the usual Gröbner tools, but for monoidal localizations this is still an open problem as stated before. Some non-units might be identified with the technique described in [Remark 13.13](#).

13.5. Implementation

In this section we outline the structure of our implementation of the algorithms developed above in the computer algebra system SINGULAR:PLURAL ([\[Gre+18\]](#)). Our algorithmic framework strives for broad generality over speed: to the best of our knowledge, apart from JAS ([\[Kre00\]](#)), which performs similar computations even over *parametric solvable polynomial rings* [\[Kre15\]](#), no other package can offer constructive computations on a similar level of generality. However, the price we pay for this is high: generally, Gröbner bases over the base ring are invoked for manipulations with fractions both in PLURAL and in JAS.

The advantage of the general approach is that further types of localizations at a left Ore set S in a G -algebra R can easily be added, provided the following two problems can be solved algorithmically:

1. the submonoid membership problem (i. e. whether $r \in S$ for a given $r \in R$),
2. the intersection of a left ideal in R with a submonoid S .

There are several packages for computer algebra systems dealing with similar situations, most notably ORETOOLS [\[ALL03\]](#) and OREALGEBRA [\[CS98\]](#) in MAPLE, ORE_ALGEBRA [\[KJJ13\]](#) in SAGE, and HOLONOMICFUNCTIONS [\[Kou10\]](#) in MATHEMATICA.

The main part of our framework is the library `olga.lib`, which contains the algorithms themselves and allows the user to perform computations in Ore localizations of G -algebras (in short, OLGAs) by working explicitly with standard data types.

The libraries `ncloc.lib` and `ncfrac.lib` utilize the ability to create user-defined data types introduced in SINGULAR from version 4 on to provide an object-oriented interface to `olga.lib`.

All three libraries are included in the distribution of SINGULAR.

Setup, conventions and restrictions

For now `olga.lib` (and therefore also `ncloc.lib` and `ncfrac.lib`) can perform computations in the following situations:

- For monoidal localizations, consider a G -algebra A generated by the variables $x_1, \dots, x_n$ and let $1 \leq k \leq n$ such that $R := K[x_1, \dots, x_k]$ is a commutative polynomial subring of A . Further, let $g_1, \dots, g_t \in R \setminus \{0\}$ such that $S := [g_1, \dots, g_t]$ is a left Ore set in A .
- Geometric localizations are only implemented for Weyl-like algebras A , consisting of $2n$ variables, where the first n variables $x_1, \dots, x_n$ generate a commutative polynomial subring $R := K[x_1, \dots, x_n]$ of A : let $\mathfrak{p}$ be a prime ideal in $R \subsetneq A$ and set $S := R \setminus \mathfrak{p}$.
- For rational localizations, consider a G -algebra A generated by the variables $x_1, \dots, x_n$ and let $1 \leq i_1 < \dots < i_k \leq n$ such that $x_{i_1}, \dots, x_{i_k}$ generate a sub- G -algebra B of A and $S := B \setminus \{0\}$ is a left Ore set in A .

In any of these cases we can perform basic arithmetic in the localization at S constructively.

Remark 13.26. In the monoidal case, the restriction for $g_1, \dots, g_t$ to be contained in a commutative polynomial ring is due to the existence of a unique factorization into irreducible elements there, which easily allows to check whether a given element s is contained in S or not. For the computation of the left Ore condition it suffices if the g_i commute pairwise, see [Lemma 13.12](#).

All computations will actually be carried out in the localization at $[g]$, where g is the square-free part of $g_1 \cdot \dots \cdot g_t$, which is isomorphic to the localization at $S = [g_1, \dots, g_t]$ again by [Lemma 13.12](#). $\diamond$

Remark 13.27. In the rational case we also need the existence of an elimination ordering for the variables not indexed by $i_1, \dots, i_k$ to compute the intersection of $\ker(\varphi_{s,r})$ with the subalgebra B . This technical condition is satisfied in many applications, especially in the transition from polynomial to rational coefficients in the setting of linear functional operators. Total rational localizations, that is, computations in the quotient field of a G -algebra, also satisfy this condition since no elimination is required. $\diamond$

Remark 13.28. While we know that geometric localization is well-defined at any prime ideal $\mathfrak{p}$ in the Weyl setting, in the other situations we have no automatic way to check if the given input indeed represents a left Ore set in A . If it is not left Ore, then the behavior of the algorithms is unspecified: computations may yield a plausible result or fail with an error. $\diamond$

Remark 13.29. Computing a left representation from a right representation is based on the computation of a left Ore condition. Analogously, computing a right representation from a left representation requires the right Ore property. If the respective Ore property does not hold, the corresponding algorithms might fail. $\diamond$

The library `olga.lib`

Data structure

A non-commutative fraction x is represented by a **vector** $[s, r, p, t]$ with entries of type `poly`, where $(s, r) = s^{-1}r$ is a representation of x as a left fraction, while $(p, t) = pt^{-1}$ is a representation of x as a right fraction. If $s = 0$ or $t = 0$ then the corresponding representation is considered as not yet known. If both are zero, the fraction is not valid. If both s and t are non-zero, the two representations have to agree, that is, $rt = sp$. A **vector** adhering to this specifications will be called a *fraction vector* below.

To interpret x in the context of a localization we have to specify a localization type, which is an `int` with value 0 for monoidal, 1 for geometric and 2 for rational localization, as well as

some localization data which is stored in an object of the universal type `def` to accommodate the different settings:

- Monoidal: list $g_1, \dots, g_t$ with entries of type `poly`.
- Geometric: `ideal p` in $K[x_1, \dots, x_n]$.
- Rational: `intvec` containing $i_1, \dots, i_k$.

Procedures

Apart from some auxiliary functions, `olga.lib` contains the following procedures, which – additionally to the parameters listed specifically below – require two parameters specifying a left Ore S set via an `int locType` and a `def locData` as described in the section above, which have to be appended at the end of the parameter lists. For example, a valid call of the function `leftOre` would require the parameter list consisting of `s`, `r`, `locType`, and `locData`.

`locStatus()`: reports on the status resp. validity of the given localization data via a list (i, w) , where i is an `int` taking values 1 or 0 depending on whether the data is valid or not. The `string w` contains information on the nature of the first problem that occurred while checking the data.

`testLocData()`: tests if the given data specifies valid localization data via `locStatus`, throwing an error if this is not the case.

`isInS(poly p)`: determines whether the polynomial `p` is contained in the specified left Ore set or not.

`fracStatus(vector frac)`: reports on whether `frac` is a valid representation of a fraction in the given localization via a list (i, w) , where i is an `int` taking values 1 or 0 depending on whether the data is valid or not. The `string w` contains information on the nature of the first problem that occurred while checking the data.

`testFraction(vector frac)`: tests if `frac` specifies a valid representation of a fraction in the given localization via `fracStatus`, throwing an error if this is not the case.

`leftOre(poly s, poly r)`: computes a list (v, J) , where v is a `vector` with entries $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$ and J is an `ideal` describing all possible choices for $\tilde{s}$ as specified in [Section 13.2](#).

`rightOre(poly s, poly r)`: computes a list (v, J) , where v is a `vector` with entries $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $r\tilde{s} = s\tilde{r}$ and J is an `ideal` describing all possible choices for $\tilde{s}$ as specified in [Section 13.2](#).

`convertRightToLeftFraction(vector frac)`: computes a representation of `frac` as a left fraction.

`convertLeftToRightFraction(vector frac)`: computes a representation of `frac` as a right fraction.

`isZeroFraction(vector frac)`: determines whether `frac` represents the left fraction $(1, 0)$.

`isOneFraction(vector frac)`: determines whether `frac` represents the left fraction $(1, 1)$.

`addLeftFractions(vector a, vector b)`: computes the sum of the fractions `a` and `b` in the specified localization.

`multiplyLeftFractions(vector a, vector b)`: computes the product of the fractions `a` and `b` in the specified localization.

`areEqualLeftFractions(vector a, vector b)`: determines whether `a` and `b` represent the same fractions in the specified localization.

`isInvertibleLeftFraction(vector frac)`: determine whether `frac` represents an invertible fraction in the specified localization.

`invertLeftFraction(vector frac)`: computes the multiplicative inverse of the fraction represented by `frac` in the specified localization. Returns the fraction representing zero if `isInvertibleLeftFraction(frac)` returns 0.

The last two procedures are subject to the problems mentioned in [Section 13.4](#): a positive result is always correct, but a negative result might not be.

Note that all arithmetic operations are only implemented explicitly for *left* fractions. At this point it is only possible to compute with *right* fractions if the right Ore set in question is also a left Ore set. For example, adding two right fractions can then be done by converting them into left fractions with `convertRightToLeftFraction`, adding the left fractions with `addLeftFractions`, and then converting the result back into a right fraction via `convertLeftToRightFraction`.

Examples

The first example demonstrates a left-to-right conversion in the second rational q -shift algebra A , which is $\mathbb{Q}(q)(x, y)\langle Q_x, Q_y \mid F \rangle$, where the set F consists of the relations (cf. also [Example 12.21](#))

$$Q_x g(x, y) = g(qx, y) Q_x, \quad Q_y g(x, y) = g(x, qy) Q_y, \quad \text{and} \quad Q_y Q_x = Q_x Q_y$$

for all $g(x, y) \in \mathbb{Q}(q)(x, y) \setminus \mathbb{Q}(q)$.

```
LIB "olga.lib";
ring Q = (0,q),(x,y,Qx,Qy),dp; // comm. polynomial ring
matrix C[4][4] = UpOneMatrix(4); // sets non-comm.
C[1,3] = q; C[2,4] = q; // relations
def ncQ = nc_algebra(C,0); // creates A from Q
setring ncQ;
intvec v = 1,2;
poly f = Qx+Qy; poly g = x^2+1;
vector frac = [g,f,0,0];
vector result = convertLeftToRightFraction(frac,2,v);
```

Now `result` contains the left representation $(x^2 + 1)^{-1}(Q_x + Q_y)$ of `frac` as well as its newly computed right representation $(q^4 x^2 Q_x + x^2 Q_y + q^2 Q_y) \cdot (x^4 + (q^2 + 1)x^2 + q^2)^{-1}$. We can convince ourselves that the two representations are equal and that the right denominator of `result` is contained in S :

```
f * result[4] == g * result[3];
-> 1
isInS(result[4],2,v);
-> 1
```

The second example consists of the addition of two left fractions in various localizations of the second Weyl algebra $A_2 = \mathbb{Q}\langle x, y, \partial_x, \partial_y \mid F \rangle$, where the set of relations F is as in [Example 12.21](#):

```
LIB "olga.lib";
ring R = 0,(x,y,dx,dy),dp; // comm. polynomial ring
def W = Weyl(); setring W; // creates A_2 from R
poly g1 = x+3; poly g2 = x*y+y;
list L = g1,g2;
frac1 = [g1,dx,0,0]; frac2 = [g2,dy,0,0];
vector resm = addLeftFractions(frac1,frac2,0,L);
```

Here, `resm` contains the left fraction

$$(x^3y + 7x^2y + 15xy + 9y, \quad x^2y\partial_x + 4xy\partial_x + x^2\partial_y + 3y\partial_x + 6x\partial_y + 9\partial_y)$$

as an element of the monoidal localization of A_2 at $S = [x + 3, xy + y]$.

```
ideal p = y-3;
vector resg = addLeftFractions(frac1,frac2,1,p);
```

Now `resg` contains

$$(x^2y + 4xy + 3y, \quad xy\partial_x + y\partial_x + x\partial_y + 3\partial_y)$$

and belongs to the geometric localization of A_2 at $S = \mathbb{Q}[x, y] \setminus \langle y - 3 \rangle$.

```
intvec rat = 2,4;
frac1 = [y+3,dx,0,0]; frac2 = [dy-1,x,0,0];
vector resr = addLeftFractions(frac1,frac2,2,rat);
```

Lastly, `resr` is given by

$$(y\partial_y^2 - 2y\partial_y + 3\partial_y^2 + y - 4\partial_y + 1, \quad x^2y\partial_y + \partial_x\partial_y^2 - xy + 3x\partial_y - 2\partial_x\partial_y - x + \partial_x),$$

it is an element of the rational localization of A_2 at $S = \mathbb{Q}\langle y, \partial_y \mid \partial_y y = y\partial_y + 1 \rangle \setminus \{0\}$, which can be written as $\text{Quot}(\mathbb{Q}\langle y, \partial_y \mid \partial_y y = y\partial_y + 1 \rangle)\langle x, \partial_x \mid \partial_x x = x\partial_x + 1 \rangle$, a polynomial Weyl algebra in variables $\{x, \partial_x\}$ over the quotient field of a Weyl algebra in $\{y, \partial_y\}$.

The library `ncloc.lib`

Since `ncloc.lib` is mainly an object-oriented interface for the localization-related functionality of `olga.lib`, `ncloc.lib` depends on `olga.lib`.

Data structure

The library `ncloc.lib` introduces a new data type called `ncloc`, which serves as a wrapper for the localization data as required by `olga.lib`. Therefore, an object of type `ncloc` has two members:

`int locType`: value 0 for monoidal, 1 for geometric and 2 for rational localization.

`def locData`: see [Section 13.5](#) for details.

Procedures

`isDenom(poly p, ncloc loc)`: determines whether the polynomial `p` is contained in the left Ore set specified by `loc`.

Operations

`string(ncloc loc)`: returns a `string` representation of `loc`.

`print(ncloc loc)`: prints `string(loc)`.

`status(ncloc loc)`: reports on the status resp. validity of `loc` via `locStatus` from `olga.lib`.

`test(ncloc loc)`: tests if `loc` specifies valid localization data via `status`, throwing an error if this is not the case.

Infix operations

`ncloc == ncloc`: compares two objects of type `ncloc` to see whether they describe the same localization up to R -fixing isomorphism in the sense of [Proposition 7.18](#) as long as they have the same type.

`ncloc != ncloc`: returns the inverted result of `ncloc == ncloc`.

`ncloc = list/poly`: creates a monoidal localization from the given `poly` resp. `list` of polys.

`ncloc = ideal`: creates a geometric localization from the given `ideal`.

`ncloc = intvec`: creates a rational localization from the given `intvec`.

For more information on how localizations are created from given data see [Sections 13.5](#) and [13.5](#).

Examples

```
LIB "ncloc.lib";
ring R = QQ[x,d];
def W = Weyl(); setring W; // working in A_1
ncloc loc = x; print(loc);
-> monoidal localization generated by:
-> x
ncloc loc2 = x^2;
loc == loc2;
-> 1
status(loc);
-> [1]:
```

```

-> 1
-> [2]:
-> valid localization
ncloc locinvalid;
status(locinvalid);
-> [1]:
-> 0
-> [2]:
-> uninitialized or invalid localization: locData has to be defined
ncloc loc = x*(x+1); loc;
-> monoidal localization generated by:
-> x+1, x

```

The library `ncfrac.lib`

Since `ncfrac.lib` is mainly an interface for the fraction-related functionality of `olga.lib` and makes use of the library `ncloc.lib`, `ncfrac.lib` depends both on `olga.lib` and `ncloc.lib`.

Data structure

The library `ncfrac.lib` introduces a new data type called `ncfrac`, which represents a (non-commutative) fraction. An object `frac` of type `ncfrac` has five members:

`poly lnum`: the left numerator of `frac`.

`poly lden`: the left denominator of `frac`.

`poly rden`: the right denominator of `frac`.

`poly rnum`: the right numerator of `frac`.

`ncloc loc`: the localization to which the fraction `frac` belongs.

Procedures

`hasLeftDenom(ncfrac frac)`: checks whether `frac` has a left representation.

`hasRightDenom(ncfrac frac)`: checks whether `frac` has a right representation.

`isZeroNcfrac(ncfrac frac)`: determines whether `frac` represents the left fraction $(1, 0)$.

`isOneNcfrac(ncfrac frac)`: determines whether `frac` represents the left fraction $(1, 1)$.

`zeroNcfrac(ncloc loc)`: returns the object of type `ncfrac` representing the left fraction $(1, 0)$ in the localization given by `loc`.

`oneNcfrac(ncloc loc)`: returns the object of type `ncfrac` representing the left fraction $(1, 1)$ in the localization given by `loc`.

`ensureLeftNcfrac(ncfrac frac)`: computes a left representation of `frac` if it does not have one already.

`ensureRightNcfrac(ncfrac frac)`: computes a right representation of `frac` if it does not have one already.

`negateNcfrac(ncfrac frac)`: returns the additive inverse of `frac`.

`isInvertibleNcfrac(ncfrac frac)`: determines whether `frac` is multiplicatively invertible via `isInvertibleLeftFraction` from `olga.lib` (see there for specific behavior).

`invertNcfrac(ncfrac frac)`: computes the multiplicative inverse of `frac`. Returns the result of `zeroNcfrac(frac.loc)` if `isInvertibleNcfrac(frac)` returns 0.

Operations

`string(ncfrac frac)`: returns a `string` representation of `frac`.

`print(ncfrac frac)`: prints `string(frac)`.

`status(ncfrac frac)`: reports on the status resp. validity of `frac` via `fracStatus` from `olga.lib`.

`test(ncfrac frac)`: tests if `frac` specifies a valid fraction via `status`, throwing an error if this is not the case.

Infix operations

`ncfrac == ncfrac`: compares two objects of type `ncfrac`.

`ncfrac != ncfrac`: returns the inverted result of `ncfrac == ncfrac`.

`ncfrac + ncfrac`: adds two objects of type `ncfrac`.

`ncfrac - ncfrac`: subtracts two objects of type `ncfrac`.

`ncfrac * ncfrac`: multiplies two objects of type `ncfrac`.

`ncfrac / ncfrac`: divides two objects of type `ncfrac` (if the second argument is invertible according to `isInvertibleNcfrac`, throws error otherwise).

Note that all binary infix operations above first check whether both arguments are fractions in the same localization in the sense of `ncloc.lib` and will throw an error if this is not the case.

`ncfrac = int/number/poly`: creates an object of type `ncfrac` with

- left and right denominator equal to 1,
- left and right numerator according to the input,
- localization data describing the trivial monoidal localization at [1].

`ncfrac = vector`: creates an object of type `ncfrac` from a `vector` `v` with unspecified localization such that `lden, lnum, rnum, rden = v[1], v[2], v[3], v[4]` (note: without specifying a localization afterwards this results in an invalid fraction).

`ncfrac = list`: creates an object of type `ncfrac` from a `list` `L` as follows:

- tries to create an object of type `ncfrac` from `L[1]` as above,
- if `L[2]` is of type `ncloc`, sets the localization of the fraction to `L[2]`

Examples

```
LIB "ncfrac.lib";
ring R = 0,(x,y,Dx,Dy),dp; ring W = Weyl(); // working in A_2
poly g1 = x+3; poly g2 = x*y+y;
ncloc loc = list(g1,g2);
ncfrac frac1 = list([g1,Dx],loc); frac1;
-> left repr.: (x+3,Dx)
-> right repr.: (0,0)
ncfrac frac2 = list([g2,Dx],loc); frac2;
-> left repr.: (x*y+y,Dx)
-> right repr.: (0,0)
frac1 + frac2;
-> left repr.: (x^2*y+4*x*y+3*y,x*y*Dx+x*Dx+y*Dx+3*Dx)
-> right repr.: (0,0)
frac1 * frac2;
-> left repr.: (x^3*y^2+5*x^2*y^2+7*x*y^2+3*y^2,x*y*Dx^2+y*Dx^2-y*Dx)
-> right repr.: (0,0)
```

Chapter 14.

Closure algorithms

In this chapter we give algorithms to compute the local closure in several interesting special cases. Note that this topic is closely related to *desingularization* as described in [Remark 10.9](#), thus a lot of work has been done already ([\[Tsa00; Chy+10; CKS16; Zha16\]](#) and many others).

Legal note. The material of [Section 14.2](#) has been published before ([\[HL18b\]](#)); it is based on the author's master thesis ([\[Hof16\]](#)) and thus the author's original work.

14.1. Central saturation

If we generalize the classical concept of ideal quotients and saturation by principal ideals to non-commutative rings, we would once more have to distinguish between left and right versions. Here, we will only consider quotients with respect to central elements, so the distinction is unnecessary:

Definition 14.1. Let R be a ring, $q \in Z(R)$, $k \in \mathbb{N}$ and P a left R -submodule of R^k .

- The *(central) quotient* of P by q is

$$P : q := \{f \in R^k \mid qf \in P\} = \{f \in R^k \mid fq \in P\} = \text{LSat}_{\{q\}}(P).$$

- The *central saturation* of P by q is

$$P : q^\infty := \bigcup_{i \in \mathbb{N}_0} (P : q^i) = \{f \in R^k \mid \exists n \in \mathbb{N}_0 : q^n f \in P\} = \text{LSat}_{[q]}(P).$$

- The *(central) saturation index* of P by q is

$$\text{Satindex}(P, q) := \min(\{n \in \mathbb{N}_0 \mid (P : q^\infty) = (P : q^n)\} \cup \{\infty\}). \quad \diamond$$

Remark 14.2. Let R be a ring, $q \in Z(R)$, $k \in \mathbb{N}$ and P a left R -submodule of R^k . Then $P : q$ is a left R -submodule of R^k . Furthermore, if R is Noetherian, then $\text{Satindex}(P, q) < \infty$. $\diamond$

Computational Remark 14.3. In the situation of [Definition 14.1](#), consider the left R -module homomorphism $\phi : R^k \rightarrow R^k/P$, $f \mapsto fq + P$. We have

$$\ker(\phi) = \{f \in R^k \mid qf + P = \phi(f) = 0 + P\} = \{f \in R^k \mid qf \in P\} = P : q.$$

Thus, if we can compute kernels of left R -module homomorphisms, we can also compute central quotients. Furthermore, if we can decide equality of left R -modules, then we can also compute the central saturation iteratively, provided the saturation index is finite. $\diamond$

Lemma 14.4. *Let L be a left ideal in a ring R and $q \in Z(R)$. If $n := \text{Satindex}(L, q) < \infty$, then $L = {}_R\langle L, q^n \rangle \cap (L : q^n)$.*

Proof: Let $J := {}_R\langle L, q^n \rangle \cap (L : q^n)$. Since $L \subseteq {}_R\langle L, q^n \rangle$ and $L \subseteq (L : q^n)$ we clearly have $L \subseteq J$. On the other hand, let $a \in J$, then $q^n a \in L$ (since $a \in (L : q^n)$) and $a = b + r q^n$ for some $b \in L$ and $r \in R$ (since $a \in {}_R\langle L, q^n \rangle$). Now

$$q^{2n}r = q^n r q^n = q^n(a - b) = q^n a - q^n b \in L$$

shows that $r \in (L : q^{2n}) = (L : q^n)$, which implies $r q^n = q^n r \in L$, thus $a = b + r q^n \in L$. $\square$

Remark 14.5. Let R be a ring, $k \in \mathbb{N}$, P a left R -submodule of R^k , $q \in Z(R)$ and $S := [q]$. Then $P^S = \text{LSat}_S(P) = \text{LSat}_{[q]}(P) = P : q^\infty$. $\diamond$

14.2. The central closure algorithm

Convention 14.6. In this section let K be a field, $n, m, r \in \mathbb{N}$ and A a G -algebra over K generated by two sets of variables $x = \{x_1, \dots, x_n\}$ and $y = \{y_1, \dots, y_m\}$ such that x generates a sub- G -algebra B of A with $B \subseteq Z(A)$. Then $S := B \setminus \{0\}$ is a left Ore set in B as well as in A by [Proposition 5.31](#). Furthermore, let $\leq = (\leq_1, \leq_2)$ be an (n, m) -antiblock ordering satisfying the ordering condition for G -algebras on A and $\preceq := \leq^{\text{POT}}$. Finally, let $\varepsilon := \varepsilon_{S, A, A^r}$, $\rho := \rho_{S, A}$ and $\preceq_2 = \leq_2^{\text{POT}}$. $\diamond$

Example 14.7. A typical setting would be $A = K[s_1, \dots, s_n] \otimes_K A_{\frac{m}{2}}(K)$ over a field K , where the first set $\{s_1, \dots, s_n\}$ induces a commutative polynomial ring that commutes with the Weyl algebra in the second set. Algebras of this type are important in $\mathcal{D}$ -module theory. $\diamond$

From the setting alone we get the following observations:

- Since A is a domain, A^r is S -torsion-free. Therefore $\ker(\varepsilon) = t_S(A^r) = \{0\}$ implies that ε is injective.
- Since $B \subseteq Z(A)$ we can identify B with the commutative polynomial ring $K[x] = K[x_1, \dots, x_n]$. Then we have $S^{-1}B \cong K(x)$.
- We can view $S^{-1}A$ as a G -algebra in the variables $y_1, \dots, y_m$ over the field $K(x)$ with the relations inherited from A , thus the Gröbner basis theory of G -algebras applies.
- The monomials in $S^{-1}(A^r) \cong (S^{-1}A)^r$ are of the form $\varepsilon(y^\alpha e_i) = \rho(y^\alpha) \varepsilon(e_i)$. Let $(s, f) \in S^{-1}A^r$, then (s, f) and $(1, f) = \varepsilon(f)$ have the same leading exponent and the same leading monomial with respect to $\preceq_2$.

Lemma 14.8. *Let $f \in A^r \setminus \{0\}$ and $\text{le}_\preceq(f) = (i, (\alpha_1, \alpha_2))$, then $\text{le}_{\preceq_2}(\varepsilon(f)) = (i, \alpha_2)$.*

Proof: Let

$$f = \sum_{(j, (\beta, \gamma)) \in \mathbb{Z} \times \mathbb{N}_0^{n+m}} c_{(j, (\beta, \gamma))} x^\beta y^\gamma e_j = \sum_{(j, \gamma) \in \mathbb{Z} \times \mathbb{N}_0^m} \underbrace{\left(\sum_{\beta \in \mathbb{N}_0^n} c_{(j, (\beta, \gamma))} x^\beta \right)}_{=: \tilde{c}_{(j, \gamma)} \in K[x]} y^\gamma e_j \quad \text{with} \quad c_{(j, (\beta, \gamma))} \in K,$$

then $(j, (\beta, \gamma)) \preceq (i, (\alpha_1, \alpha_2)) = \text{le}_{\preceq}(f)$ whenever $c_{(j, (\beta, \gamma))} \neq 0$. Furthermore,

$$\varepsilon(f) = \varepsilon \left(\sum_{(j, \gamma) \in \mathcal{I} \times \mathbb{N}_0^m} \tilde{c}_{(j, \gamma)} y^\gamma e_j \right) = \sum_{(j, \gamma) \in \mathcal{I} \times \mathbb{N}_0^m} \varepsilon(\tilde{c}_{(j, \gamma)} y^\gamma e_j) = \sum_{(j, \gamma) \in \mathcal{I} \times \mathbb{N}_0^m} \rho(\tilde{c}_{(j, \gamma)}) \cdot \varepsilon(y^\gamma e_j)$$

implies that it suffices to show that $(j, \gamma) \preceq_2 (i, \alpha_2)$ whenever $\tilde{c}_{(j, \gamma)} \neq 0$. The last condition implies that there is some $\beta \in \mathbb{N}_0^n$ such that $c_{(j, (\beta, \gamma))} \neq 0$. Now $(j, (\beta, \gamma)) \preceq (i, (\alpha_1, \alpha_2)) = \text{le}_{\preceq}(f)$ implies $(j, \gamma) \preceq_2 (i, \alpha_2)$ by [Lemma 12.15](#), thus $\text{le}_{\preceq_2}(\varepsilon(f)) = (i, \alpha_2)$. $\square$

Proposition 14.9. *Let I be a left A -submodule of A^r and G a left Gröbner basis of I with respect to $\preceq$. Then $\varepsilon(G)$ is a left Gröbner basis of $J := S^{-1}I$ with respect to $\preceq_2 = \leq_2^{\text{POT}}$.*

Proof: Let $z \in J \setminus \{0\}$, then $z = (s, f)$ for some $s \in S$ and $f \in I$. Since G is a left Gröbner basis of I there exists $g \in G$ such that $\text{lm}_{\preceq}(g) \mid \text{lm}_{\preceq}(f)$. In terms of leading exponents, where $(i, (\alpha_1, \alpha_2)) = \text{le}_{\preceq}(g)$ and $(j, (\beta_1, \beta_2)) = \text{le}_{\preceq}(f)$, this means $i = j$ and $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$, in particular, we have $\alpha_2 \leq_2 \beta_2$. Since $\text{le}_{\preceq_2}(\varepsilon(g)) = (i, \alpha_2)$ and $\text{le}_{\preceq}((s, f)) = \text{le}_{\preceq_2}(\varepsilon(f)) = (j, \beta_2)$ by [Lemma 14.8](#), we have $\text{lm}_{\preceq_2}(\varepsilon(g)) \mid \text{lm}_{\preceq_2}(z)$. $\square$

Definition 14.10. Consider a polynomial $f \in K[x] \setminus K$. Since $K[x]$ is a unique factorization domain, f has a representation as a product of a unit and finitely many irreducible elements. The *square-free part* of f , denoted $\text{sfp}(f)$, is the product of all unique irreducible elements that occur in this factorization. $\diamond$

The following algorithm is a generalization of the algorithm EXTCONT in Table 8.8 of [\[BW93\]](#):

Algorithm 14.11 (CENTRALCLOSURE).

Input: A left A -submodule I of A^r .

Output: A Gröbner basis $G \subseteq A^r$ of I^S with respect to $\leq$.

```

1 begin
2    $H := \text{LEFTGRÖBNERBASIS}(I, \leq);$ 
3    $h := \text{sfp} \left( \prod_{g \in H} \text{lc}_{\preceq_2}(\varepsilon(g)) \right) \in K[x] \setminus \{0\};$ 
4    $k := \text{Satindex}(I, h);$ 
5    $G := \text{LEFTGRÖBNERBASIS}(I : h^k, \leq);$ 
6   return  $G$ ;
7 end
```

Remark 14.12. Consider the situation of [Algorithm 14.11](#).

- We have $I^S \stackrel{10.19}{=} (I^e)^c = \varepsilon^{-1}(I^e) \stackrel{10.18}{=} \varepsilon^{-1}(S^{-1}I)$ with respect to ε .
- For any $g \in H$ there exists $l \in \mathbb{N}$ such that $\text{lc}_{\preceq_2}(\varepsilon(g)) \mid h^l$ by construction of h . $\diamond$

Proposition 14.13. *Algorithm 14.11 terminates and is correct.*

Proof: The saturation index computation terminates since G -algebras are Noetherian, thus termination of the whole algorithm is ensured. To prove correctness we have to show that $I^S = I : h^\infty = I : h^k$.

First, let $f \in I : h^k$, then $h^k f \in I$ and $\varepsilon(f) = (1, f) = (h^k, h^k f) \in S^{-1}I$. Thus we have $f \in \varepsilon^{-1}(S^{-1}I) = I^S$, which implies $I : h^k \subseteq I^S$.

For the other inclusion, let $f \in I^S = \varepsilon^{-1}(S^{-1}I)$, then $\varepsilon(f) \in S^{-1}I$. Now $\varepsilon(H)$ is a left Gröbner basis of $S^{-1}I$ with respect to $\preceq_2$ by [Proposition 14.9](#). Furthermore, [Theorem 12.32](#) implies that $\text{LeftNF}(\varepsilon(f)|\varepsilon(H)) = 0$. We now prove $f \in I : h^k$ by an induction on the minimal number $N \in \mathbb{N}$ of steps necessary in [Algorithm 12.31](#) to reduce $\varepsilon(f)$ to zero:

Induction base: If $N = 0$, then $\varepsilon(f) = 0$. Since ε is injective we have $f = 0$, which trivially implies $f \in I : h^\infty$.

Induction hypothesis: Assume that for any $\tilde{f} \in I^S$, such that $\varepsilon(\tilde{f})$ can be reduced to zero in $N - 1$ steps by the left normal form algorithm with respect to $\varepsilon(H)$, we have $\tilde{f} \in I : h^k$.

Induction step: Let $f \in I^S$ such that the left normal form algorithm needs at least N steps to reduce $\varepsilon(f)$ to zero with respect to $\varepsilon(H)$. Then there exists $g \in H$ such that $\text{lm}_{\preceq_2}(\varepsilon(g)) \mid \text{lm}_{\preceq_2}(\varepsilon(f))$. Let $(i_f, \alpha) = \text{le}_{\preceq_2}(\varepsilon(f))$ and $(i_g, \beta) = \text{le}_{\preceq_2}(\varepsilon(g))$, then $i_g = i_f$ and

$$t := \varepsilon(f) - \frac{\text{lc}_{\preceq_2}(\varepsilon(f))}{\text{lc}_{\preceq_2}(\varepsilon(y^{\alpha-\beta}g))} \rho(y^{\alpha-\beta})\varepsilon(g) \in S^{-1}A^r$$

can be reduced to zero in $N - 1$ steps with respect to $\varepsilon(H)$. Since the relations between the variables in A have the form $y_j y_i = c_{i,j} y_i y_j + d_{i,j}$ for some $c_{i,j} \in K \setminus \{0\} = U(K)$ and $d_{i,j} \in A$ such that $\text{le}_{\preceq}(d_{i,j}) \prec \text{le}_{\preceq}(y_i y_j)$, we have

$$\text{lc}_{\preceq_2}(\varepsilon(y^{\alpha-\beta}g)) = u \cdot \text{lc}_{\preceq_2}(\varepsilon(g))$$

for some $u \in U(K)$, which is just the product of all $c_{i,j}$ that occur while bringing $\varepsilon(y^{\alpha-\beta}g)$ in standard monomial form, and thus

$$t = \varepsilon(f) - \frac{\text{lc}_{\preceq_2}(\varepsilon(f))}{u \text{lc}_{\preceq_2}(\varepsilon(g))} \rho(y^{\alpha-\beta})\varepsilon(g).$$

Since $\text{lc}_{\preceq_2}(\varepsilon(g))$ divides a power of h there exists $l \in \mathbb{N}$ such that

$$c := \frac{h^l}{u \text{lc}_{\preceq_2}(\varepsilon(g))} \in K[x] \setminus \{0\}$$

and therefore

$$\tilde{f} := h^l f - c \text{lc}_{\preceq_2}(\varepsilon(f)) y^{\alpha-\beta} g \in I^S,$$

since $f \in I^S$ by assumption and $g \in H \subseteq I \subseteq I^S$. Now

$$\begin{aligned} h^l t &= h^l \varepsilon(f) - \frac{h^l}{u \text{lc}_{\preceq_2}(\varepsilon(g))} \text{lc}_{\preceq_2}(\varepsilon(f)) \rho(y^{\alpha-\beta}) \varepsilon(g) \\ &= h^l \varepsilon(f) - c \text{lc}_{\preceq_2}(\varepsilon(f)) \rho(y^{\alpha-\beta}) \varepsilon(g) \\ &= \varepsilon(h^l f - c \text{lc}_{\preceq_2}(\varepsilon(f)) y^{\alpha-\beta} g) \\ &= \varepsilon(\tilde{f}), \end{aligned}$$

thus we can apply the induction hypothesis: we have $\tilde{f} \in I^S$ such that $\varepsilon(\tilde{f}) = h^l t$ can be reduced to zero in $N - 1$ steps with respect to $\varepsilon(H)$, since $h^l \in S$ is invertible in $S^{-1}A$ and thus does not change the reducibility of t . This gives us $\tilde{f} \in I : h^k$ or $h^k \tilde{f} \in I$. Now

$$h^{l+k} f = h^k \tilde{f} + h^{l+k} c \text{lc}_{\preceq_2}(\varepsilon(f)) y^{\alpha-\beta} g \in I$$

implies $f \in I : h^{l+k} = I : h^k$, which shows $I^S \subseteq I : h^k$. $\square$

14.3. The central Weyl closure algorithm

In this section, we give an algorithm to compute the $K[x, s] \setminus \{0\}$ -closure of a left ideal in the algebra $K[s_1, \dots, s_m] \otimes_K A_n(K)$, utilizing the central closure algorithm presented above in combination with the famous Weyl closure algorithm from [Tsa00].

Convention 14.14. In this section let K be a field, $n, m \in \mathbb{N}$ and A_n^K the n -th Weyl algebra over K in the variables $x = \{x_1, \dots, x_n\}$ and $\partial = \{\partial_1, \dots, \partial_n\}$. Furthermore, we consider the additional commutative indeterminates $s = \{s_1, \dots, s_m\}$ and $S := K[s] \setminus \{0\}$. Lastly, let $\rho := \rho_{S, A_n^K[s]}$. $\diamond$

Note that the extended ring $A_n^K[s] = A_n^K \otimes_K K[s]$ is no longer a Weyl algebra, but still a G -algebra. Localizing $A_n^K[s]$ at S yields $A_n^{K(s)}$, which is isomorphic to $A_n^{K(s)}$, the n -th Weyl algebra over the field $K(s)$. The structural homomorphism ρ is injective, since $A_n^K[s]$ is a domain.

Definition 14.15. Let I be a left ideal in $A_n^K[s]$. Define

$$G := I^{K[x, s] \setminus \{0\}} = \text{LSat}_{K[x, s] \setminus \{0\}}^{A_n^K[s]}(I) = \{r \in A_n^K[s] \mid \exists w \in K[x, s] \setminus \{0\} : wr \in I\}$$

and

$$H := \text{LSat}_{K(s)[x] \setminus \{0\}}^{A_n^{K(s)}}((I^S)^e) = \{r \in A_n^{K(s)} \mid \exists w \in K(s)[x] \setminus \{0\} : wr \in (I^S)^e\}. \quad \diamond$$

Lemma 14.16. We have $H = G^e$ with respect to ρ .

Proof: Let $r \in G$, then there exists $w \in K[x, s] \setminus \{0\}$ such that $wr \in I$. Now

$$\rho(w)\rho(r) = \rho(wr) \in \rho(I) \subseteq \rho(I^S) \subseteq (I^S)^e$$

and $\rho(w) \in \rho(K[x, s] \setminus \{0\}) \subseteq K(s)[x] \setminus \{0\}$, thus $\rho(r) \in H$ and therefore $\rho(G) \subseteq H$. Since H is a left ideal in $A_n^{K(s)}$ it contains G^e , the left ideal generated by $\rho(G)$, thus $G^e \subseteq H$.

Now let $(t, r) \in H$, where $t \in S$ and $r \in A_n^K[s]$, then there exists $(q, w) \in K(s)[x] \setminus \{0\}$, where $q \in S$ and $w \in K[x, s] \setminus \{0\}$, such that $(q, w) \cdot (t, r) \in (I^S)^e$. Since w and t commute we have $(q, w) \cdot (t, r) = (tq, wr)$. Now $(tq, wr) \in (I^S)^e$, therefore there exist $\tilde{t} \in S$ and $p \in I^S$ such that $(tq, wr) = (\tilde{t}, p)$. This implies the existence of $\bar{t} \in S$ and $\bar{r} \in A_n^K[s]$ such that $\bar{t}tq = \bar{r}\tilde{t}$ and $\bar{t}wr = \bar{r}p \in I^S$, thus there exist $\hat{t} \in S$ satisfying $\hat{t}\bar{t}wr \in I$. Since $\hat{t}\bar{t}w \in K[x, s] \setminus \{0\}$ we have $r \in G$, thus $(t, r) = (t, 1) \cdot (1, r) = (t, 1) \cdot \rho(r) \in G^e$ and therefore $H \subseteq G^e$. $\square$

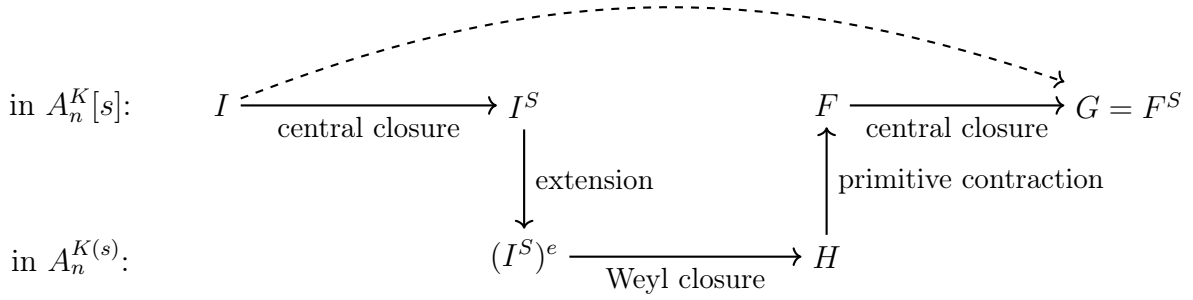
Algorithm 14.17 (CENTRALWEYLCLOSURE).

Input: A left ideal I of $A_n^K[s]$.

Output: The $K[x, s] \setminus \{0\}$ -closure of I .

```

1 begin
2   compute  $I^S$  via central closure in  $A_n^K[s]$ ;
3   extend  $I^S$  from  $A_n^K[s]$  to  $A_n^{K(s)}$ ;
4   compute the  $K(s)[x] \setminus \{0\}$ -closure  $H = \langle \rho(h_1), \dots, \rho(h_k) \rangle$  of  $(I^S)^e$  in  $A_n^{K(s)}$  via Weyl
   closure;
5   let  $F := \langle h_1, \dots, h_k \rangle \subseteq A_n^K[s]$ ;
6   compute  $F^S$  via central closure in  $A_n^K[s]$ ;
7   return  $F^S$ ;
8 end
```



Proof: Termination is obvious. Since $A_n^{K(s)}$ is Noetherian, the ideal H is finitely generated: let $H = \langle (a_1, h_1), \dots, (a_k, h_k) \rangle$ for some $a_i \in S$ and $h_i \in A_n^K[s]$. Since $(a_i, 1) \in U(A_n^{K(s)})$ for all i we have $H = \langle (1, h_1), \dots, (1, h_k) \rangle = \langle \rho(h_1), \dots, \rho(h_k) \rangle$. It remains to show that $G = F^S$. First, let $g \in G$, then $\rho(g) \in G^e = H$ by [Lemma 14.16](#), thus there exist $t_i \in S$ and $r_i \in A_n^K[s]$ such that

$$\rho(g) = \sum_{i=1}^k (t_i, r_i) \rho(h_i) = \sum_{i=1}^k (t_i, r_i h_i) = (t, \sum_{i=1}^k \tilde{r}_i r_i h_i)$$

for some common left denominator $t \in S$ and $\tilde{r}_i \in A_n^K[s]$. Now

$$\rho(tg) = \rho(t)\rho(g) = \rho(t)(t, \sum_{i=1}^k \tilde{r}_i r_i h_i) = \rho\left(t, \sum_{i=1}^k \tilde{r}_i r_i h_i\right).$$

Since ρ is injective we have $tg = \sum_{i=1}^k \tilde{r}_i r_i h_i \in F$, therefore $g \in F^S$ and thus $G \subseteq F^S$.

For the second inclusion let $y \in F^S$, then there exists $t \in S$ such that $ty \in F$. Now $\rho(ty) \in F^e \subseteq H$, since $F \subseteq H^c$ implies $F^e \subseteq (H^c)^e = H$. The ideal H is left $K(s)[x] \setminus \{0\}$ -saturated by construction, thus $\rho(t) \in K(s)[x] \setminus \{0\}$ implies $\rho(y) \in G^e$. Then $y \in (G^e)^c = G^S = G$ since G is left S -saturated, therefore $F^S \subseteq G$. $\square$

Remark 14.18. Let $f_i \in K[x]$ and consider the annihilator $\text{Ann}_{D[s]} f^s$ of the special function $f^s := f_1^{s_1} \cdots f_r^{s_r}$ in the algebra $D[s] = A_n^K[s]$. The *logarithmic annihilator* of f^s , denoted $\text{Ann}_{D[s]}^{(1)} f^s$, is generated by all elements $r \in \text{Ann}_{D[s]} f^s$ with $\text{tdeg}_\partial(r) = 1$. Since $\text{Ann}_{D[s]} f^s$ is left $K[x, s] \setminus \{0\}$ -saturated, we can compute $\text{Ann}_{D[s]} f^s$ by applying [Algorithm 14.17](#) to $\text{Ann}_{D[s]}^{(1)} f^s$. This at long last presents an alternative to the algorithm of Briançon and Maisonobe ([[BM02](#)]), which up until now has been the only way to compute $\text{Ann}_{D[s]} f^s$. We plan to investigate the viability of this approach in the future. $\diamond$

14.4. Theta-closure

Let L be a left ideal in the first complex Weyl algebra $A_1(\mathbb{C})$. How can we compute L^Θ , where $\Theta = [\{x\partial + z \mid z \in \mathbb{Z}\}]$?

As a first step, we know from [Lemma 8.25](#) that $\text{LSat}(\Theta) = \text{LSat}([x, \partial])$, thus $L^\Theta = L^{[x, \partial]}$ by [Lemma 6.3](#). To utilize the recipe described in [Proposition 6.13](#) we have to be able to compute the local closure at $[x]$ and $[\partial]$.

The first ingredient is Algorithm 2.2.3 in [[Tsa00](#)], which allows us to compute $L^{[f]}$ for a left ideal L in $A_n(\mathbb{C})$ such that A_n/L is holonomic and $f \in K[x]$ based on Algorithm 3 in [[OTW00](#)].

The condition on L is always satisfied for proper non-zero left ideals L in the case $n = 1$ by Chapter 10 in [Cou95].

With Lemma 6.5, we can compute the closure at $[\partial]$ by utilizing the closure at $[x]$ in combination with a suitable automorphism of the Weyl algebra, which is the so-called *Fourier automorphism* $\mathcal{F} : A_1(\mathbb{C}) \rightarrow A_1(\mathbb{C})$ induced by $x \mapsto -\partial$ and $\partial \mapsto x$.

Algorithm 14.19 (THETACLOSURE).

Input: A proper non-zero left ideal L of $A_1(\mathbb{C})$, $f \in \mathbb{C}[x]$.

Output: L^Θ , where $\Theta = [x\partial + \mathbb{Z}]$.

```

1 begin
2   compute  $L_x := L^{[x]}$ ;
3   set  $L_\partial := \{0\}$  and  $i := 1$ ;
4   while  $L_x \neq L_\partial$  do
5     if  $2 \mid i$  then
6       compute  $L_x := L_\partial^{[x]}$ ;
7     else
8       compute  $L_\partial := \mathcal{F}^{-1}(\mathcal{F}(L_x)^{[x]})$ ;
9     end
10    set  $i := i + 1$ ;
11  end
12  return  $L_x$ ;
13 end

```

Proof: By Lemma 8.25 we have $\text{LSat}(\Theta) = \text{LSat}([x, \partial])$, thus $L^\Theta = L^{[x, \partial]}$ by Lemma 6.3. By Lemma 6.5, $L_\partial = \mathcal{F}^{-1}(\mathcal{F}(L_x)^{[x]}) = \mathcal{F}^{-1}(\mathcal{F}(L_x)^{\mathcal{F}([\partial])}) = L_x^{[\partial]}$, thus for all $i \geq 2$ we have that L_x is left $[x]$ -saturated and L_∂ is left $[\partial]$ -saturated. By Lemma 6.12, if $L_x = L_\partial$, then $L_x = L_x^{[x, \partial]} = L_x^\Theta = L^\Theta$ by Proposition 6.13. Since $A_1(\mathbb{C})$ is (left) Noetherian, this will happen after finitely many steps. $\square$

Conclusion and future work

In this thesis, we introduced the notion of left saturation closure, a general object with innumerable ties to left Ore localization. This gave rise to the canonical form $\text{LSat}(S)$ for a left denominator set S in a ring R , which determines the isomorphism class of the localization $S^{-1}R$ under homomorphisms of localizations that fix the common base ring R . As a slight drawback, we noted that $\text{LSat}(S)$ itself is not a left denominator set in general. In fact, we showed that this being the case is equivalent to $S^{-1}R$ being a Dedekind-finite ring. In the future, it would be interesting to characterize the rings satisfying this property for all left denominator sets S , since in this setting the theory of left saturation closure reaches its full potential. So far, we know that such a ring must be Dedekind-finite itself, while being zero-insertive or semi-commutative is a sufficient condition. In particular, this holds for domains, where we showed that any saturated left Ore set is uniquely determined by the irreducible elements it contains.

Since the techniques involved are nearly identical, we also investigated Ore localizations of monoids and showed that most of the theory mentioned so far holds in this context as well.

After applying the results concerning $\text{LSat}(S)$ to the structure theory of Ore-localized modules, we identified the already known notion of local closure as another special case of left saturation closure and discussed the connection to local torsion, including the Weyl closure which is of great importance in D -module theory and its applications. In this light, a torsion-centric approach to local closure and desingularization might be something worth looking into.

Finally, we presented solutions to several algorithmic problems related to Ore localization in the setting of G -algebras, including basic arithmetic in the localization and local closure of important special cases. We plan to further improve and extend our implementation of basic arithmetic in Ore-localized G -algebras. In particular, we are open to suggestions for additional interesting types of localizations to include. Also, we plan to complete the implementation of our small contribution to the ever-growing collection of closure algorithms. In particular, we want to be able to treat left Ore sets in the Weyl algebras involving the Euler operators, since they are also of interest in other fields of scientific study.

Appendix A.

Proof of Proposition 5.20

Proposition 5.20. *Let S be a left denominator set in a ring R and M a left R -module.*

(a) $(S^{-1}M, +_M, (1, 0))$ is an Abelian group.

(b) For all $(s_1, r_1) \in S^{-1}R$ and all $(s_2, m_2), (s_3, m_3) \in S^{-1}M$ we have

$$(s_1, r_1) \cdot_M ((s_2, m_2) +_M (s_3, m_3)) = ((s_1, r_1) \cdot_M (s_2, m_2)) +_M ((s_1, r_1) \cdot_M (s_3, m_3)).$$

(c) For all $(s_1, r_1), (s_2, r_2) \in S^{-1}R$ and all $(s_3, m_3) \in S^{-1}M$ we have

$$((s_1, r_1) +_R (s_2, r_2)) \cdot_M (s_3, m_3) = ((s_1, r_1) \cdot_M (s_3, m_3)) +_M ((s_2, r_2) \cdot_M (s_3, m_3)).$$

Proof: (a) Let $(s, m), (s_1, m_1), (s'_1, m'_1), (s_2, m_2), (s'_2, m'_2), (s_3, m_3) \in S^{-1}M$.

Well-definedness: Let $(s_1, m_1) \sim (s'_1, m'_1)$ and $(s_2, m_2) \sim (s'_2, m'_2)$, then there exist $\hat{s}_1, \hat{s}_2 \in S$ and $\hat{r}_1, \hat{r}_2 \in R$ such that

$$\hat{s}_1 s_1 = \hat{r}_1 s'_1, \tag{S1}$$

$$\hat{s}_1 m_1 = \hat{r}_1 m'_1, \tag{S2}$$

$$\hat{s}_2 s_2 = \hat{r}_2 s'_2, \tag{S3}$$

$$\hat{s}_2 m_2 = \hat{r}_2 m'_2. \tag{S4}$$

We have

$$(s_1, m_1) +_M (s_2, m_2) = (\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2)$$

and

$$(s'_1, m'_1) +_M (s'_2, m'_2) = (\tilde{s}'s'_1, \tilde{s}'m'_1 + \tilde{r}'m'_2),$$

where $\tilde{s}, \tilde{s}' \in S$ and $\tilde{r}, \tilde{r}' \in R$ satisfy

$$\tilde{s}s_1 = \tilde{r}s_2, \tag{S5}$$

$$\tilde{s}'s'_1 = \tilde{r}'s'_2. \tag{S6}$$

We have to show $(\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2) = (\tilde{s}'s'_1, \tilde{s}'m'_1 + \tilde{r}'m'_2)$. By the left Ore condition there exist $s_a, s_b, s_c \in S$ and $r_a, r_b, r_c \in R$ such that

$$s_a \tilde{s}'s'_1 = r_a \tilde{s}s_1, \tag{S7}$$

$$s_b r_a \tilde{s} = r_b \hat{s}_1, \tag{S8}$$

$$s_c r_a \tilde{r} = r_c \hat{s}_2. \tag{S9}$$

By left reversibility,

$$s_b s_a \tilde{s}' s'_1 \stackrel{(S7)}{=} s_b r_a \tilde{s} s_1 \stackrel{(S8)}{=} r_b \hat{s}_1 s_1 \stackrel{(S1)}{=} r_b \hat{r}_1 s'_1$$

implies that there exists $s_d \in S$ such that

$$s_d s_b s_a \tilde{s}' = s_d r_b \hat{r}_1, \quad (S10)$$

while

$$s_c s_a \tilde{r}' s'_2 \stackrel{(S6)}{=} s_c s_a \tilde{s}' s'_1 \stackrel{(S7)}{=} s_c r_a \tilde{s} s_1 \stackrel{(S5)}{=} s_c r_a \tilde{r} s_2 \stackrel{(S9)}{=} r_c \hat{s}_2 s_2 \stackrel{(S3)}{=} r_c \hat{r}_2 s'_2$$

implies the existence of $s_e \in S$ such that

$$s_e s_c s_a \tilde{r}' = s_e r_c \hat{r}_2. \quad (S11)$$

By the left Ore condition there exist $s_f \in S$ and $r_f \in R$ such that

$$s_g := s_f s_d s_b = r_f s_e s_c \in S. \quad (S12)$$

Define $\mathring{s} := s_g s_a \in S$ and $\mathring{r} := s_g r_a \in R$, then

$$\mathring{s} \tilde{s}' s'_1 = s_g s_a \tilde{s}' s'_1 \stackrel{(S7)}{=} s_g r_a \tilde{s} s_1 = \mathring{r} \tilde{s} s_1$$

and

$$\begin{aligned} \mathring{s}(\tilde{s}' m'_1 + \tilde{r}' m'_2) &= s_g s_a \tilde{s}' m'_1 + s_g s_a \tilde{r}' m'_2 \\ &\stackrel{(S12)}{=} s_f s_d s_b s_a \tilde{s}' m'_1 + r_f s_e s_c s_a \tilde{r}' m'_2 \\ &\stackrel{(S10)}{=} s_f s_d r_b \hat{r}_1 m'_1 + r_f s_e s_c s_a \tilde{r}' m'_2 \\ &\stackrel{(S11)}{=} s_f s_d r_b \hat{r}_1 m'_1 + r_f s_e r_c \hat{r}_2 m'_2 \\ &\stackrel{(S2)}{=} s_f s_d r_b \hat{s}_1 m_1 + r_f s_e r_c \hat{r}_2 m'_2 \\ &\stackrel{(S4)}{=} s_f s_d r_b \hat{s}_1 m_1 + r_f s_e r_c \hat{s}_2 m_2 \\ &\stackrel{(S8)}{=} s_f s_d s_b r_a \tilde{s} m_1 + r_f s_e r_c \hat{s}_2 m_2 \\ &\stackrel{(S9)}{=} s_f s_d s_b r_a \tilde{s} m_1 + r_f s_e s_c r_a \tilde{r} m_2 \\ &\stackrel{(S12)}{=} s_g r_a \tilde{s} m_1 + s_g r_a \tilde{r} m_2 \\ &= \mathring{r}(\tilde{s} m_1 + \tilde{r} m_2) \end{aligned}$$

show that $(\tilde{s} s_1, \tilde{s} m_1 + \tilde{r} m_2) = (\tilde{s}' s'_1, \tilde{s}' m'_1 + \tilde{r}' m'_2)$, thus $+_M$ is well-defined.

Associativity: We have

$$\begin{aligned} ((s_1, m_1) +_M (s_2, m_2)) +_M (s_3, m_3) &= (s_a s_1, s_a m_1 + r_a m_2) +_M (s_3, m_3) \\ &= (s_b s_a s_1, s_b(s_a m_1 + r_a m_2) + r_b m_3) \end{aligned}$$

and

$$\begin{aligned} (s_1, m_1) +_M ((s_2, m_2) +_M (s_3, m_3)) &= (s_1, m_1) +_M (s_c s_2, s_c m_2 + r_c m_3) \\ &= (s_d s_1, s_d m_1 + r_d(s_c m_2 + r_c m_3)), \end{aligned}$$

where $s_a, s_b, s_c, s_d \in S$ and $r_a, r_b, r_c, r_d \in R$ satisfy

$$s_a s_1 = r_a s_2, \quad (\text{T1})$$

$$s_b s_a s_1 = r_b s_3, \quad (\text{T2})$$

$$s_c s_2 = r_c s_3, \quad (\text{T3})$$

$$s_d s_1 = r_d s_c s_2. \quad (\text{T4})$$

We have to show $(s_b s_a s_1, s_b(s_a m_1 + r_a m_2) + r_b m_3) = (s_d s_1, s_d m_1 + r_d(s_c m_2 + r_c m_3))$. By the left Ore condition there exist $s_e \in S$ and $r_e \in R$ such that

$$s_e s_d s_1 = r_e s_b s_a s_1. \quad (\text{T5})$$

By left reversibility, there exists $s_f \in S$ such that

$$s_f s_e s_d = s_f r_e s_b s_a. \quad (\text{T6})$$

Now

$$s_e r_d s_c s_2 \stackrel{(\text{T4})}{=} s_e s_d s_1 \stackrel{(\text{T5})}{=} r_e s_b s_a s_1 \stackrel{(\text{T1})}{=} r_e s_b r_a s_2$$

implies that there exists $s_g \in S$ such that

$$s_g s_e r_d s_c = s_g r_e s_b r_a, \quad (\text{T7})$$

while

$$s_e r_d r_c s_3 \stackrel{(\text{T3})}{=} s_e r_d s_c s_2 \stackrel{(\text{T4})}{=} s_e s_d s_1 \stackrel{(\text{T5})}{=} r_e s_b s_a s_1 \stackrel{(\text{T2})}{=} r_e r_b s_3$$

implies the existence of $s_h \in S$ such that

$$s_h s_e r_d r_c = s_h r_e r_b. \quad (\text{T8})$$

The left Ore condition gives us $s_i \in S$ and $r_i \in R$ such that

$$s_i s_f = r_i s_g \quad (\text{T9})$$

as well as $s_j \in S$ and $r_j \in R$ such that

$$s_j s_i s_f = r_j s_h, \quad (\text{T10})$$

which implies

$$s_k := r_j s_h \stackrel{(\text{T10})}{=} s_j s_i s_f \stackrel{(\text{T9})}{=} s_j r_i s_g \in S. \quad (\text{T11})$$

Define $\mathring{s} := s_k s_e \in S$ and $\mathring{r} := s_k r_e \in R$. Then

$$\mathring{s} s_d s_1 = s_k s_e s_d s_1 \stackrel{(\text{T5})}{=} s_k r_e s_b s_a s_1 = \mathring{r} s_b s_a s_1$$

and

$$\begin{aligned} \mathring{s}(s_d m_1 + r_d(s_c m_2 + r_c m_3)) &= s_k s_e s_d m_1 + s_k s_e r_d s_c m_2 + s_k s_e r_d r_c m_3 \\ &\stackrel{(\text{T11})}{=} s_j s_i s_f s_e s_d m_1 + s_j r_i s_g s_e r_d s_c m_2 + r_j s_h s_e r_d r_c m_3 \\ &\stackrel{(\text{T6})}{=} s_j s_i s_f r_e s_b s_a m_1 + s_j r_i s_g s_e r_d s_c m_2 + r_j s_h s_e r_d r_c m_3 \\ &\stackrel{(\text{T8})}{=} s_j s_i s_f r_e s_b s_a m_1 + s_j r_i s_g r_e s_b r_a m_2 + r_j s_h s_e r_d r_c m_3 \\ &\stackrel{(\text{T7})}{=} s_j s_i s_f r_e s_b s_a m_1 + s_j r_i s_g r_e s_b r_a m_2 + r_j s_h r_e r_b m_3 \\ &\stackrel{(\text{T11})}{=} s_k r_e s_b s_a m_1 + s_k r_e s_b r_a m_2 + s_k r_e r_b m_3 \\ &= \mathring{r}(s_b s_a m_1 + s_b r_a m_2 + r_b m_3) \\ &= \mathring{r}(s_b(s_a m_1 + r_a m_2) + r_b m_3) \end{aligned}$$

show that $(s_b s_a s_1, s_b(s_a m_1 + r_a m_2) + r_b m_3) = (s_d s_1, s_d m_1 + r_d(s_c m_2 + r_c m_3))$, thus $+_M$ is associative.

Commutativity: We have

$$(s_1, m_1) +_M (s_2, m_2) = (\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2) \quad \text{and} \quad (s_2, m_2) +_M (s_1, m_1) = (\hat{s}s_2, \hat{s}m_2 + \hat{r}m_1),$$

where $\tilde{s}, \hat{s} \in S$ and $\tilde{r}, \hat{r} \in R$ satisfy

$$\tilde{s}s_1 = \tilde{r}s_2, \tag{U1}$$

$$\hat{s}s_2 = \hat{r}s_1. \tag{U2}$$

We have to show $(\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2) = (\hat{s}s_2, \hat{s}m_2 + \hat{r}m_1)$. Since $\hat{s}s_2 \in R$ and $\tilde{s}s_1 \in S$, there exist $s_a \in S$ and $r_a \in R$ such that

$$s_a \hat{s}s_2 = r_a \tilde{s}s_1. \tag{U3}$$

By left reversibility,

$$s_a \hat{s}s_2 \stackrel{(U3)}{=} r_a \tilde{s}s_1 \stackrel{(U1)}{=} r_a \tilde{r}s_2$$

implies that there exists $s_b \in S$ such that

$$s_b s_a \hat{s} = s_b r_a \tilde{r}, \tag{U4}$$

while

$$s_a \hat{r}s_1 \stackrel{(U2)}{=} s_a \hat{s}s_2 \stackrel{(U3)}{=} r_a \tilde{s}s_1$$

implies the existence of $s_c \in S$ such that

$$s_c s_a \hat{r} = s_c r_a \tilde{s}. \tag{U5}$$

By the left Ore condition there exist $s_d \in S$ and $r_d \in R$ such that

$$s_e := r_d s_c = s_d s_b \in S. \tag{U6}$$

Define $\mathring{s} := s_e s_a \in S$ and $\mathring{r} := s_e r_a \in R$, then

$$\mathring{s}\hat{s}s_2 s_e s_a \hat{s}s_2 \stackrel{(U3)}{=} s_e r_a \tilde{s}s_1 = \mathring{r}\tilde{s}s_1$$

and

$$\begin{aligned} \mathring{s}(\hat{s}m_2 + \hat{r}m_1) &= s_e s_a(\hat{s}m_2 + \hat{r}m_1) = s_e s_a \hat{s}m_2 + s_e s_a \hat{r}m_1 \stackrel{(U6)}{=} s_d s_b s_a \hat{s}m_2 + r_d s_c s_a \hat{r}m_1 \\ &\stackrel{(U4)}{=} s_d s_b r_a \tilde{r}m_2 + r_d s_c s_a \hat{r}m_1 \stackrel{(U5)}{=} s_d s_b r_a \tilde{r}m_2 + r_d s_c r_a \tilde{s}m_1 \\ &\stackrel{(U6)}{=} s_e r_a \tilde{r}m_2 + s_e r_a \tilde{s}m_1 = s_e r_a(\tilde{r}m_2 + \tilde{s}m_1) = \mathring{r}(\tilde{s}m_1 + \tilde{r}m_2) \end{aligned}$$

show that $(\tilde{s}s_1, \tilde{s}m_1 + \tilde{r}m_2) = (\hat{s}s_2, \hat{s}m_2 + \hat{r}m_1)$, thus $+_M$ is commutative.

Neutral element: Let $\tilde{s} := 1 \in S$ and $\tilde{r} := s \in R$, then $\tilde{s}s = 1 \cdot s = s \cdot 1 = \tilde{r} \cdot 1$, thus

$$(s, m) +_M (1, 0) = (\tilde{s}s, \tilde{s}m + \tilde{r} \cdot 0) = (\tilde{s}s, \tilde{s}m) = (1 \cdot s, 1 \cdot m) = (s, m).$$

By commutativity we also have $(1, 0) +_M (s, m) = (s, m)$, therefore $(1, 0)$ is the neutral element with respect to $+_M$ in $S^{-1}M$.

Inverse element: Let $\tilde{s} := 1 \in S$ and $\tilde{r} := 1 \in R$, then $\tilde{s}s = 1 \cdot s = \tilde{r}s$, thus

$$\begin{aligned}(s, m) +_M (s, -m) &= (\tilde{s}s, \tilde{s}m + \tilde{r}(-m)) = (1 \cdot s, 1 \cdot m + 1 \cdot (-m)) \\ &= (s, m + (-m)) = (s, 0) = (1, 0).\end{aligned}$$

By commutativity we also have $(s, -m) +_M (s, m) = (1, 0)$, therefore $(s, -m)$ is the inverse of (s, m) with respect to $+_M$ in $S^{-1}M$. $\square$

(b) We have

$$\begin{aligned}(s_1, r_1) \cdot_M ((s_2, m_2) +_M (s_3, m_3)) &= (s_1, r_1) \cdot_M (s_a s_2, s_a m_2 + r_a m_3) \\ &= (s_b s_1, r_b (s_a m_2 + r_a m_3))\end{aligned}$$

as well as

$$\begin{aligned}((s_1, r_1) \cdot_M (s_2, m_2)) +_M ((s_1, r_1) \cdot_M (s_3, m_3)) &= (s_c s_1, r_c m_2) +_M (s_d s_1, r_d m_3) \\ &= (s_e s_c s_1, s_e r_c m_2 + r_e r_d m_3),\end{aligned}$$

where $s_a, s_b, s_c, s_d, s_e \in S$ and $r_a, r_b, r_c, r_d, r_e \in R$ satisfy

$$s_a s_2 = r_a s_3, \tag{V1}$$

$$s_b r_1 = r_b s_a s_2, \tag{V2}$$

$$s_c r_1 = r_c s_2, \tag{V3}$$

$$s_d r_1 = r_d s_3, \tag{V4}$$

$$s_e s_c s_1 = r_e s_d s_1. \tag{V5}$$

We have to show $(s_b s_1, r_b (s_a m_2 + r_a m_3)) = (s_e s_c s_1, s_e r_c m_2 + r_e r_d m_3)$. By the left Ore condition there exist $s_f \in S$ and $r_f \in R$ such that

$$s_f s_e s_c s_1 = r_f s_b s_1, \tag{V6}$$

thus there exists $s_g \in S$ such that

$$s_g s_f s_e s_c = s_g r_f s_b. \tag{V7}$$

Again by left reversibility,

$$s_g s_f s_e r_c s_2 \stackrel{(V3)}{=} s_g s_f s_e s_c r_1 \stackrel{(V7)}{=} s_g r_f s_b r_1 \stackrel{(V2)}{=} s_g r_f r_b s_a s_2$$

implies that there exists $s_h \in S$ such that

$$s_h s_g s_f s_e r_c = s_h s_g r_f r_b s_a, \tag{V8}$$

while

$$r_f s_b s_1 \stackrel{(V6)}{=} s_f s_e s_c s_1 \stackrel{(V5)}{=} s_f r_e s_d s_1$$

implies the existence of $s_i \in S$ such that

$$s_i r_f s_b = s_i s_f r_e s_d, \tag{V9}$$

lastly,

$$s_i s_f r_e r_d s_3 \stackrel{(V4)}{=} s_i s_f r_e s_d r_1 \stackrel{(V9)}{=} s_i r_f s_b r_1 \stackrel{(V2)}{=} s_i r_f r_b s_a s_2 \stackrel{(V1)}{=} s_i r_f r_b r_a s_3$$

gives us $s_j \in S$ such that

$$s_j s_i s_f r_e r_d = s_j s_i r_f r_b r_a. \quad (V10)$$

By the left Ore condition there exist $s_k \in S$ and $r_k \in R$ such that

$$s_l := s_k s_h s_g = r_k s_j s_i \in S. \quad (V11)$$

Define $\mathring{s} := s_l s_f \in S$ and $\mathring{r} := s_l r_f \in R$, then

$$\mathring{s} s_e s_c s_1 = s_l s_f s_e s_c s_1 \stackrel{(V6)}{=} s_l r_f s_b s_1 = \mathring{r} s_b s_1$$

and

$$\begin{aligned} \mathring{s}(s_e r_c m_2 + r_e r_d m_3) &= s_l s_f s_e r_c m_2 + s_l s_f r_e r_d m_3 \\ &\stackrel{(V11)}{=} s_k s_h s_g s_f s_e r_c m_2 + r_k s_j s_i s_f r_e r_d m_3 \\ &\stackrel{(V8)}{=} s_k s_h s_g r_f r_b s_a m_2 + r_k s_j s_i s_f r_e r_d m_3 \\ &\stackrel{(V10)}{=} s_k s_h s_g r_f r_b s_a m_2 + r_k s_j s_i r_f r_b r_a m_3 \\ &\stackrel{(V11)}{=} s_l r_f r_b s_a m_2 + s_l r_f r_b r_a m_3 \\ &= \mathring{r} r_b (s_a m_2 + r_a m_3) \end{aligned}$$

show that $(s_b s_1, r_b (s_a m_2 + r_a m_3)) = (s_e s_c s_1, s_e r_c m_2 + r_e r_d m_3)$.

(c) We have

$$((s_1, r_1) +_R (s_2, r_2)) \cdot_M (s_3, m_3) = (s_a s_1, s_a r_1 + r_a r_2) \cdot_M (s_3, m_3) = (s_b s_a s_1, r_b m_3)$$

as well as

$$\begin{aligned} ((s_1, r_1) \cdot_M (s_3, m_3)) +_M ((s_2, r_2) \cdot (s_3, m_3)) &= (s_c s_1, r_c m_3) +_M (s_d s_2, r_d m_3) \\ &= (s_e s_c s_1, s_e r_c m_3 + r_e r_d m_3) \\ &= (s_e s_c s_1, (s_e r_c + r_e r_d) m_3), \end{aligned}$$

where $s_a, s_b, s_c, s_d, s_e \in S$ and $r_a, r_b, r_c, r_d, r_e \in R$ satisfy

$$s_a s_1 = r_a s_2, \quad (W1)$$

$$s_b (s_a r_1 + r_a r_2) = r_b s_3, \quad (W2)$$

$$s_c r_1 = r_c s_3, \quad (W3)$$

$$s_d r_2 = r_d s_3, \quad (W4)$$

$$s_e s_c s_1 = r_e s_d s_2. \quad (W5)$$

We have to show $(s_b s_a s_1, r_b m_3) = (s_e s_c s_1, (s_e r_c + r_e r_d) m_3)$. By the left Ore condition there exist $s_f \in S$ and $r_f \in R$ such that

$$s_f s_e s_c s_1 = r_f s_b s_a s_1, \quad (W6)$$

thus there exists $s_g \in S$ such that

$$s_g s_f s_e s_c = s_g r_f s_b s_a. \quad (\text{W7})$$

Again by left reversibility,

$$s_g r_f s_b r_a s_2 \stackrel{(\text{W1})}{=} s_g r_f s_b s_a s_1 \stackrel{(\text{W7})}{=} s_g s_f s_e s_c s_1 \stackrel{(\text{W5})}{=} s_g s_f r_e s_d s_2$$

implies that there exists $s_h \in S$ such that

$$s_h s_g r_f s_b r_a = s_h s_g s_f r_e s_d, \quad (\text{W8})$$

while

$$\begin{aligned} s_h s_g r_f r_b s_3 &\stackrel{(\text{W2})}{=} s_h s_g r_f s_b (s_a r_1 + r_a r_2) \\ &= s_h s_g r_f s_b s_a r_1 + s_h s_g r_f s_b r_a r_2 \\ &\stackrel{(\text{W7})}{=} s_h s_g s_f s_e s_c r_1 + s_h s_g r_f s_b r_a r_2 \\ &\stackrel{(\text{W8})}{=} s_h s_g s_f s_e s_c r_1 + s_h s_g s_f r_e s_d r_2 \\ &\stackrel{(\text{W3})}{=} s_h s_g s_f s_e r_c s_3 + s_h s_g s_f r_e s_d r_2 \\ &\stackrel{(\text{W4})}{=} s_h s_g s_f s_e r_c s_3 + s_h s_g s_f r_e r_d s_3 \\ &= s_h s_g s_f (s_e r_c + r_e r_d) s_3 \end{aligned}$$

implies the existence of $s_i \in S$ such that

$$s_i s_h s_g r_f r_b = s_i s_h s_g s_f (s_e r_c + r_e r_d). \quad (\text{W9})$$

Define $\mathring{s} := s_i s_h s_g s_f \in S$ and $\mathring{r} := s_i s_h s_g r_f$, then

$$\mathring{s} s_e s_c s_1 = s_i s_h s_g s_f s_e s_c s_1 \stackrel{(\text{W6})}{=} s_i s_h s_g r_f s_b s_a s_1 = \mathring{r} s_b s_a s_1$$

and

$$\mathring{s} (s_e r_c + r_e r_d) m_3 = s_i s_h s_g s_f (s_e r_c + r_e r_d) m_3 \stackrel{(\text{W9})}{=} s_i s_h s_g r_f r_b m_3 = \mathring{r} r_b m_3$$

show that $(s_b s_a s_1, r_b m_3) = (s_e s_c s_1, (s_e r_c + r_e r_d) m_3)$.

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