Separable Graphs, Minors and the Reconstruction Conjecture

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Separable Graphs, Minors and the Reconstruction Conjecture

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Abstract

This doctoral thesis deals with the reconstruction conjecture in graph theory. This over 70 year old conjecture asks the question of how to uniquely determine a graph by its substructures. In this particular case, one has the isomorphism types of all induced subgraphs in which, with respect to the original graph, exactly one vertex and its adjacent edges are missing. The question now is about the uniqueness of the subgraphs of a graph, that is, whether there exists exactly one graph or at least two different graphs that contain the same isomorphism types as subgraphs in the given number. The conjecture itself reads as follows: "All simple, finite and undirected graphs on at least three vertices are reconstructible." Reconstructible means that the set of the isomorphism types of the induced subgraphs belong to exactly one graph and that there exists no other, different graph which contains the same subgraphs.

Due to the lack of a universal approach to the problem, we help ourselves with the following concept. We show that a class of graphs with a certain property is reconstructible, provided that the graphs of this class have this property. Over the last decades a wide range of classes of graphs have been proven to be reconstructible by using this principle. The hope is that one day we find enough reconstructible graph classes such that the union of these classes cover the set of all graphs and therefore will prove the correctness of the reconstruction conjecture. A second approach is to prove that certain invariants are reconstructible. This means that the value of an invariant is already determined by the induced subgraphs of the graph. In this respect we try to find a complete set of reconstructible invariants that uniquely determine a graph.

In this thesis, the author shows mainly two results. Regarding the first result, the author generalizes a result of Bondy about separable graphs. Bondy was able to show that separable graphs with no vertices of degree one are reconstructible. Furthermore, he was able to show that certain separable graphs with vertices of degree one, are reconstructible, too. The author extends and generalizes Bondy's findings, adds new insights, and thus increases the subclass of separable graphs with vertices of degree one that are reconstructible.

The second result aims at graph minors. The author shows that the fact whether one graph contains a certain other graph as a minor is often reconstructible. This depends on the structure of the minor and the order and size of the original graph. For that the graph and minor are distinguished by their connectivities. In addition, the author points out that many graph invariants can be defined by certain minors. As a consequence, it is shown that the Hadwiger number and the treewidth for certain graph classes are reconstructible.

The thesis concludes with a generalization of the reduction of Yang as well as the reduction of Ramachandran and Monikandan. The author shows in this regard that the problem of the reconstruction of self-complementary classes of graphs can be reduced to a smaller problem, thus simplifying potential reconstruction proofs of these classes.

Zusammenfassung

Die Doktorarbeit bezieht sich auf die Rekonstruktionsvermutung in der Graphentheorie. Diese über 70 Jahre alte Vermutung geht der Frage nach, wie man einen Graphen eindeutig bestimmen kann. In diesem Falle besitzt man die Isomorphietypen aller induzierten Teilgraphen, in denen bezüglich des Ursprungsgraphen genau ein Knoten und seine adjazenten Kanten fehlen. Die Frage ist nun nach der Eindeutigkeit der Teilgraphen eines Graphen, das heißt, ob es genau einen Graphen oder mindestens zwei verschieden Graphen gibt, die die Isomorphietypen als Teilgraphen in vorgegebener Anzahl besitzten. Die Vermutung selbst lautet: "Alle einfachen, endlichen und ungerichteten Graphen auf mindestens drei Knoten sind rekonstruierbar." Rekonstruierbar bedeutet in diesem Sinne, dass die Menge der Teilgraphen in dieser Form nur Teilgraphen von genau einem Graphen sind und das kein weiterer Graph existiert, der die gleichen Teilgraphen enthält.

Aufgrund des Fehlens eines universellen Lösungsansatzes zu der Fragestellung behilft man sich mit folgendem Konzept. Man zeigt das eine Menge oder Klasse von Graphen, die eine bestimmte Eigenschaft besitzen, unter Voraussetzung dieser Eigenschaft rekonstruierbar sind. Damit wurden über die letzten Jahrzehnte eine Menge von Klassen als rekonstruierbar bewiesen und man hofft, dass eines Tages eine ausreichende Anzahl an rekonstruierbaren Klassen gefunden wird, so dass diese die Menge aller Graphen überdeckt und damit die Rekonstruktionsvermutung beweist. Ein zweiter Ansatz ist zu beweisen, dass bestimmte Invarianten rekonstruierbar sind. Das heißt, dass der Wert einer Invariante bereits durch die Teilgraphen des Graphen festgelegt ist. Diesbezüglich versucht man eine vollständige Menge von rekonstruierbaren Invarianten zu finden, so dass diese einen Graphen eindeutig definieren.

In der Arbeit zeigt der Autor hauptsächlich zwei Ergebnisse. Bezüglich dem ersten Ergebnis geht der Autor auf ein Ergebnis von Bondy über separable Graphen ein. Bondy konnte zeigen, dass separable Graphen ohne Knoten vom Grad eins rekonstruierbar sind. Des weiteren konnte er zeigen, dass bestimmte separable Graphen, die Knoten vom Grad eins besitzen, auch rekonstruierbar sind. Der Autor erweitert und verallgemeinert die Ergenisse von Bondy, fügt neue Erkenntnisse hinzu und vergrößert so die Teilklasse der separablen Graphen mit Grad eins, die rekonstruierbar sind.

Das zweite Ergebnis bezieht sich auf Minoren in Graphen. Der Autor zeigt, das der Fall, ob ein Graph einen speziellen Minor enthält oder nicht enthält, oft rekonstruierbar ist. Dies geschieht in Abhängigkeit von der Gestalt des Minors und in Abhängigkeit der Ordnung und Größe des ursprünglichen Graphens. Dafür wird eine Unterscheidung der Minoren und des ursprünglichen Graphens bezüglich ihres Zusammenhangs angewandt. Darüber hinaus weist der Autor darauf hin, dass viele

Invarianten in der Graphentheorie über bestimmte Minoren definiert werden können. Solche Invarianten können mit Hilfe von "verbotenen Minorensätzen" beschrieben werden. Die Arbeit selbst zeigt, dass als Folge aus der Minorenbetrachtung die Hadwigerzahl und die Baumweite für bestimmte Graphenklassen, abhängig von ihrer Ordnung und Größe, rekonstruierbar sind.

Die Arbeit schließt mit einer Verallgemeinerung der Reduktionsbeweise von Yang beziehungsweise Ramachandran und Monikandan. Der Autor zeigt diesbezüglich auf, wie das Problem der Rekonstruierbarkeit von selbst komplementären Klassen auf einer kleineres Problem reduziert werden kann und vereinfacht damit potentielle Beweise dieser Klassen.

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Introduction

The reconstruction conjecture for graphs was first investigated by Ulam and his doctoral student Kelly in the early 1940s. First results were accomplished by Kelly in his doctoral thesis. Kelly also laid the foundation for some of the most used theorems within the theory of the reconstruction conjecture. Over the last 75 years many researchers tried to tackle the reconstruction conjecture. We had some major and minor leaps that brought us closer to solving the reconstruction conjecture on the whole. To this day however, we are missing the big breakthrough that solves the reconstruction conjecture. Therefore this doctoral thesis gets into the line of the many works of the past decades that tackled varied aspects of the reconstruction conjecture, yet failed to solve it on the whole.

What does reconstructing actually mean? Reconstructing any mathematical structure in our meaning is to solve a specific kind of puzzle. Yet while we know that the puzzle has a solution, we are not interested in the solution itself. To work out a solution of a given puzzle is merely a matter of time, algorithms and computing power. However the aim of reconstructing a structure is to show that the solution to the puzzle is unique. Hence there are not two or more different solutions to the same puzzle.

In our field of mathematics the structures are graphs. For our purpose we are given the isomorphism types of all subgraphs of a specific graph. Compared to the original graph those subgraphs are missing always exactly one vertex and their incident edges. Then the question to answer is to determine if there are more than two non-isomporphic graphs with the same isomorphism types as subgraphs or if there is only one graph unique up to isomorphism that contains all these isomorphism types as subgraphs.

This doctoral thesis deals with the reconstruction conjecture and edge-reconstruction conjecture for graphs. On the one hand we work on separable graphs regarding the reconstruction conjecture and on the other hand we work with minors of graphs regarding the edge-reconstruction conjecture.

The first chapter displays the basic definitions in graph theory used in this doctoral thesis. In addition, there are three main parts to this chapter. The first of these three main parts, section 1.2, displays almost all known results regarding the reconstruction conjecture. First the reconstruction conjecture is stated. Next we give the structural theorems and results that have been proven for the reconstruction conjecture. After that, we introduce in section 1.2.2 the first non-structural approach. We list the graph classes that are known to be reconstructible. This first approach tries to solve the reconstruction conjecture by solving the problem for specific graph classes in the hope of finding enough specific classes to eventually solve the reconstruction conjecture on the whole. Another approach is given in section 1.2.3. That states all known results regarding the reconstruction of graph invariants and parameters. The hope regarding this approach is to find a complete set of reconstructible invariants that uniquely define a graph. The second main part, section 1.3, is structured like the first main part. First, we state the edge-reconstruction conjecture. Next, we give the structural results regarding the edge-reconstrucion conjecture. After that all known edge-reconstructible graph classes are displayed and last, we give the edge-reconstructible graph invariants. The third and last main part of the first chapter, section 1.4, broadens our view of the topic and deals with theories and results settled around the reconstruction conjecture as well as other, different reconstruction conjectures related to the reconstruction conjecture for graphs.

The second chapter picks up an idea by Bondy. Bondy did show that separable graphs without vertices of degree 1 are reconstructible. He also did some work on the remaining separable graphs, in particular separable graphs with vertices of degree 1. He was able to show that, if the 1-connected parts of the graph including a vertex of degree 1 have a sepcific structure, then these graphs are reconstructible. We delve deeper into that idea, generalize it and are able to give a certain restriction on the 1-connected parts of the separable graph containing a vertex of degree 1. We also look at the automorphism group of parts of the separable graph to further strengthen our results.

The third chapter deals with minors in the reconstruction conjecture and the edge-reconstruction conjecture. We pose the question whether containing a specific minor or not is reconstructible or edge-reconstructible for certain graphs. The classes of graphs we investigate are based upon their connectness. We give a bound for 2-connected graphs and connected minors as well as we show that the problem is edge-reconstructible for a range of other graph classes. The chapter closes with an application of the results. In particular we were able to show that the Hadwiger number and the treewidth is edge-reconstructible for a wide range of cases. We also give a bound for the edge-reconstruction of the Hadwiger number and the treewidth based upon the ratio between the order of the graph and the order of the minor. Furthermore these results can easily be applied to various *exluded minor theorems* (a graph has a specific property if and only if it has no minors isomorphic to a specific set of graphs).

The last chapter states options for future research. It discusses an idea about reducing the problem of the reconstruction of self-complementary graph classes to subclasses of that class. Those subclasses offer more structure and therefore might make the original problem of reconstructing a specific self-complementary graph class easier. In particular we use Yangs reduction about *all graphs are reconstructible if and only if all 2-connected graphs are reconstructible* and generalize it to *all graphs in a self-complementary graph class are reconstructible if and only if all 2-connected graphs in that self-complementary* *graph class are reconstructible*. We proceed in the same manner with a theorem by Ramachandran and Monikandan and generalize it to self-complementary graph classes. In effect this will yield that *every graph in a self-complementary graph class is reconstructible if and only if all* 2*-connected graphs in that class with diameter* 2 *or diameter* 3 *and the complement holds diameter* 3 *are reconstructible*. This might make proving the reconstructability of perfect graphs or other self-complementary graph classes considerably easier.

1 Basic Definitions and the Reconstruction Conjecture

This chapter contains the basic definitions and an overview of the state of the art regarding the reconstruction and edge-reconstruction conjecture. Section 1.1 deals with most graph theoretical defintions that are needed for this doctoral thesis. Section 1.2 deals with the reconstruction conjecture while section 1.3 deals with the edge-reconstruction conjecture. In section 1.4 are a range of problems displayed that are related to the reconstruction conjecture and edge-reconstruction conjecture.

1.1 Basic Definitions

This subchapter deals with most definitions that are needed for our theoretical work regarding the reconstruction conjecture and edge-reconstruction conjecture. While the first few definitions and notations are crucial for every aspect of this doctoral thesis we encourage the reader to skip the remaining basic definitions and return to this chapter when a specific notation or definition is needed.

First we define the graph itself.

Definition 1.1. (graph, vertex, edge)
Let V and E be a pair of disjoint sets. Then $G = (V, E)$ is called a *graph* if $E \subseteq \{vw \mid v, w \in V\}$ holds. The elements in $V(G) := V$ are called *vertices* and form the *vertex set of* G. The elements of $E(G) := E$ are called *edges* and form the *edge set of* G. Furthermore for every edge $e = vw \in E(G)$ between two vertices $v, w \in V(G)$ holds $vw = wv$ and this edge is only listed once as either vw or wv.

Two vertices that are related or connected to each other are called adjacent. An edge attached to a vertex is called incident to that vertex.

Definition 1.2. (adjacent, incident)
Let G be a graph and $v, w \in V(G)$. If $vw \in E(G)$ holds, then the vertices v and w are called *adjacent* and the vertices v and w are called *incident* to the edge vw.

The size of the vertex set and edge set are given by the order and the size of a graph respectively.

Definition 1.3. (order, size)
Let G be a graph. The number of vertices in G is called the *order of* G and is denoted by $n(G) := |V(G)|$. The number of edges in G is called the *size of* G and is denoted by $m(G) := |E(G)|$.

The number of edges incident to a vertex is given by the next definition and is called the degree of that vertex.

Definition 1.4. (degree, minimal/maximal degree, end vertex, isolated vertex)
Let G be a graph and $v \in V(G)$. Then the number of edges incident to v is called the *degree of v in G* and is denoted by $d_G(v)$. The *minimal degree of* G is denoted by $\delta(G) := \min_{v \in V(G)} d_G(v)$ and the *maximal degree of* G is denoted by $\Delta(G) := \max_{v \in V(G)} d_G(v)$. A vertex of degree 1 is called an *end vertex* and a vertex of degree zero is called an *isolated vertex*.

The collection of the degrees of all vertices of a graph is called the degree sequence.

Definition 1.5. (degree sequence, neighbourhood degree sequence)
Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$. Then the *degree sequence of* G is a sequence of its vertex degrees, e.g. $(d_G(v_1), d_G(v_2), ..., d_G(v_n))$. The *neighbourhood degree sequence of* $v \in V(G)$ is a sequence of the degrees of the adjacent vertices of v.

The first theorem of this doctoral thesis is one of the most basic and well known theorems in graph theory. The so called Handshaking Lemma is frequently used in this thesis.

Theorem 1.6. (Handshaking Lemma)
Let G be a graph. Then

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|$$

holds.

Next we are going to define substructures of a graph that are graphs themselves.

Definition 1.7. (subgraph, induced subgraph, spanning subgraph)
Let G and F be two graphs. $F \subseteq G$ is called a *subgraph of* G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$ holds. In addition $F \subseteq G$ is called an *induced subgraph of* G if $V(F) \subseteq V(G)$ and $E(F) := \{vw \mid v, w \in V(F) \text{ and } vw \in E(G)\} \subseteq E(G)$. We denote the induced subgraph of G on the vertex set $V(F)$ with $G[V(F)]$. $F \subseteq G$ is called a *spanning subgraph* if F is a subgraph of G and $V(F) = V(G)$ holds.

One of the most important subgraphs is where each vertex of a subset of the vertex set is adjacent to all other vertices of that subset. Such a subgraph is called clique. If a graph is a clique itself, than it is called complete graph.

Definition 1.8. (complete graph, clique, clique number)
Let G be a graph with $E(G) = \{vw \mid v, w \in V(G)\}$. Then G is called a *complete graph on $n(G)$ vertices* and is denoted by $K_{n(G)}$. If G contains an induced subgraph G' which is complete, then G' is called a *clique* in G. We denote by $\omega(G)$ the order of the largest clique of G and call it the *clique number*.

The complement of a graph G contains all edges that G does not contain but no more edges than these. In other words if G is a subgraph of the complete graph $K_{n(G)}$ on the same vertex set, then the complement of G on the same vertex set contains all edges of $K_{n(G)}$ except the edges of the subgraph G.

Definition 1.9. (complement)
Let G and $\overline{G}$ be two graphs with the same vertex set. Then $\overline{G}$ is the *complement of* G if the following holds for all $v, w \in V(G)$: $vw \in E(G)$ holds if and only if $vw \notin E(\overline{G})$ holds.

The complement of a clique is called a stable set.

Definition 1.10. (stable set, stability number)
Let G be a graph. $V' \subseteq V(G)$ is called a *stable set* if $G[V']$ contains no edges. We denote by $\alpha(G)$ the order of the largest stable set of G and call it the *stability number*.

An independent or stable subset of the edges is called a matching.

Definition 1.11. (matching, perfect matching, 1-factor, matching number)
Let G be a graph. A *matching* is a subset $E' \subseteq E(G)$ in which no pair of edges of E' share a common vertex. A *perfect matching* is a matching of size $\frac{1}{2}|V(G)|$. A perfect matching is also known as 1-*factor*. The *matching number* $\nu(G)$ denotes the size of a largest matching in G.

We say that two graphs are structurally equal if they are isomorph to each other.

Definition 1.12. (graph isomorphism, isomorph)
Let G and H be two graphs. A *graph isomorphism* is a bijection

$$\varphi : V(G) \to V(H),$$

with

$$vw \in E(G) \Leftrightarrow \varphi(v)\varphi(w) \in E(H).$$

Two graphs G and H are called *isomorph* if there exists a graph isomorphism between G and H. We denote this with $G \cong H$.

A series of vertices that are connected by edges is called a walk or in specific cases a path.

Definition 1.13. (walk, path)
Let G be a graph. Let $P = v_1, v_2, v_3, ..., v_k, v_1$ be an ordered sequence of vertices $v_1, v_2, ..., v_k \in V(G)$, where every pair of consecutive vertices $v_i, v_{i+1} \in V(G)$ of that sequence is connected by an edge. Then P is called a *walk in G*. If furthermore there is no repitition of the vertices in that sequence, then C is called a *path*. The number of edges in a walk or path is called the *length* of the walk of path.

If one may trace a path from one vertex to another, then those two vertices are called connected. If this property holds for all pairs of vertices, then the graph itself is called connected.

Definition 1.14. (connected, disconnected, cut vertex, bridge)
A graph G is called *connected* if there is a path between v and w for all $v, w \in V(G)$. If there exist vertices v and w such that there is no path between them, then G is called *disconnected*. A vertex $v \in V(G)$ is called a *cut vertex* if G is connected and $G[V(G) \setminus \{v\}]$ is disconnected. An edge $e \in E(G)$ is called a *bridge* if G is connected and $(V(G), E(G) \setminus \{e\})$ is disconnected.

A separating set is a subset of the vertices that disconnects a connected graph. The measure of how good a graph is connected is given by the connectivity.

Definition 1.15. (separating set, connectivity, edge-connectivity)
Let G be a connected graph and $S \subsetneq V(G)$ a subset. Then S is called a *separating set* if $G[V(G) \setminus S]$ is disconnected. The *connectivity* $\kappa(G)$ of G is the size of a minimal separating set. If S is a minimal separating set with $|S| = k$, then G is called *k-connected*. Analoguesly we define $S_e \subsetneq E(G)$ as an *edge-separating set* if $(V(G), E(G) \setminus S_e)$ is disconnected. The *edge-connectivity* $\lambda(G)$ of G is the size of a minimal edge-separating set. If S_e is a minimal separating set with $|S_e| = k$, then G is called *k-edge-connected*.

A graph class of connected graphs that contains a cut vertex is called separable.

Definition 1.16. (separable graph)
A connected graph is called *separable* if it contains a cut vertex. In other words the graph is 1-connected but not 2-connected.

Bridges and maximal subgraphs that are 2-connected are called blocks.

Definition 1.17. (block)
Let G be a graph. A maximal, 2-connected subgraph of G and bridges of G with their end vertices are called *blocks*.

Next we will define the distance between two vertices in a graph. Upon that we may define a measure for the spread of a graph. We call this the diameter.

Definition 1.18. (distance, eccentricity, diameter)
Let G be a graph. The distance between two vertices $v, w \in V(G)$ is the length of a shortest path connecting the two vertices. We denote this number with $d(v, w)$. Then the *eccentricity* of a vertex v is defined as

$$e(v) = \max_{w \in V(G)} d(v, w)$$

with $v \in V(G)$. The *diameter* of a connected graph is the maximal eccentricity, hence

$$\text{diam}(G) = \max_{v \in V(G)} e(v).$$

An important theorem about the diameter of a graph is given by Harary and Robinson in [36].

Theorem 1.19. (Harary and Robinson, 1985)
Let G be a graph. If $\text{diam}(G) > 3$ holds, then $\text{diam}(\overline{G}) < 3$.

Substructures that form a closed path are called cycles and its special form where no edge is repeated are called circuits.

Definition 1.20. (cycle, circuit, length, girth, cycle graph)
Let G be a graph and $C = v_1, v_2, v_3, ..., v_k, v_1$ be an ordered sequence of vertices $v_1, v_2, ..., v_k \in V(G)$, where the first vertex matches the last vertex and every pair of consecutive vertices $v_i, v_{i+1} \in V(G)$ of that sequence as well as v_k, v_1 are connected by an edge. Then C is called a *cycle in* G. If there is no repetition of the vertices in the sequence, except the first and the last vertex, then C is called a *circuit*. The number of vertices in a cycle where the first and last vertex are only counted once, is called the *length* of the cycle or circuit. The length of a smallest circuit in G is called the *girth of* G. If G itself is a circuit, then G is called a *cycle graph*.

A disconnected graph without circuits is called a forest whereas a connected graph without circuits is called a tree.

Definition 1.21. (tree, forest)
A connected graph G that does not contain cycle graphs as subgraphs is called a *tree*. A *forest* is a disconnected graph that does not contain cycle graphs as subgraphs.

An eulerian graph contains a special cycle.

Definition 1.22. (eulerian graph)
A connected graph G is called *eulerian* if G contains a cycle C that visits all edges of G exactly once. That cycle is called a *eulerian cycle*.

Similar to the eulerian graph a hamiltonian graph contains a specific cycle. Yet this cycle is a circuit that contains all vertices exactly once.

Definition 1.23. (hamiltonian, hamiltonian graph)
A graph G is called *hamiltonian* if G contains a spanning subgraph which is a cycle graph. That circuit is called a *hamilton cycle*.

We may define circuits of length 3 as triangles and graphs that do not contain such subgraphs as triangle-free graph.

Definition 1.24. (triangle-free graph)
A graph G is called *triangle-free* if G does not contain an induced cycle or circuit of length 3.

Another graph class form the bipartite graphs. The vertex set of bipartite graphs may be partitioned into two disjoint sets such that the sets itself form stable sets.

Definition 1.25. (bipartite graph)
A graph G is called *bipartite* if there exist two non-empty sets $V_1, V_2 \subsetneq V(G)$ with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(G)$ as well as the induced subgraphs $G[V_1]$ and $G[V_2]$ contain no edges.

Chordal graphs do not contain induced subgraphs that are circuits of length 4 or larger.

Definition 1.26. (chordal graph, chord, split graph)
Let G be a graph. G is called *chordal* if every circuit with length at least 4 has a *chord* that is an edge between two non-consecutive vertices of the circuit. A *split graph* is a chordal graph which can be partitioned into two disjoint sets $V_1, V_2 \subseteq V(G)$ with $V_1 \cup V_2 = V(G)$ as well as V_1 is a clique and V_2 is an independent set.

The treewidth may be defined in several ways. We use the definition via the chordal extension. The treewidth states how close a graph is to a tree.

Definition 1.27. (chordal extension, treewidth)
Let G be a graph. A *chordal extension of a graph* G is a chordal graph H on the same vertex set as G with $G \subseteq H$. The *treewidth* of G is defined as the size of the largest clique minus one in a chordal extension of G with the smallest clique number. We denote the treewidth with $tw(G)$.

Next we are going to define a graph theoretical invariant called the chromatic polynomial and with it the chromatic number. The chromatic polynomial counts the number of colorings of a graph where a graph is colored such that the colors of adjacent vertices are different from each other. Such a coloring is called a proper (vertex) coloring.

Definition 1.28. (chromatic polynomial)
Let G be a graph. Then $P(G,x)$ counts the number of maps $h : V(G) \rightarrow \{1,2,...,x\}$ with $h(v) \neq h(w)$ for all $v,w \in V(G)$ with $vw \in E(G)$. $P(G,x)$ is called the *chromatic polynomial of* G.

Birkhoff has shown in 1912/13 [6] that the chromatic polynomial $P(G,x)$ is actually a polynomial.

The chromatic number is the least number of colors a graph may be colored with such that the colors of adjacent vertices are different from each other.

Definition 1.29. (chromatic number)
Let G be a graph. Then the least positive integer x such that $P(G,x) > 0$ holds, is called the *chromatic number of* G. The chromatic number is denoted by $\chi(G)$.

The class of perfect graphs are defined via a relation of all their subgraphs.

Definition 1.30. (perfect graph)
A graph G is called *perfect* if $\chi(G[V_1]) = \omega(G[V_1])$ holds for every subset $V_1 \subseteq V(G)$. That is the clique number equals the chromatic number for every induced subgraph of G.

An important property of perfect graphs is given by the weak perfect graph theorem by Lovász [48,50]. It states that the complement of a perfect graph is a perfect graph, too.

The line graph is a graph based on another graph. It takes the edges of the original graph as vertices and connects them if the respective edges shared a common vertex in the original graph.

Definition 1.31. (line graph)
Let G be a graph. Then $L(G) = (V,E)$ is called the *line graph of* G with

1. $V = E(G)$, that is each vertex of $L(G)$ represents an edge of G,
2. for $v,w \in V$ holds $vw \in E$ if and only if the corresponding edges of the vertices v and w share a common end vertex in G.

A minor is a substructure of a graph that may be accomlished by a series of different operations.

Definition 1.32. (minor, edge contraction)
A graph $H = (V', E')$ is a *minor* of a graph $G = (V, E)$ if H can be obtained from G by a series of vertex deletions, edge deletions, and *edge contractions*, where an edge contraction is the operation that replaces two adjacent vertices v, w by one that is adjacent to all vertices that were adjacent to v or w. For edge contractions, parallel edges will be ignored and only a single edge will be inserted. G having H as a minor is denoted by $H \preccurlyeq G$.

The last two structures are the vertex cover and the edge cover. The vertex cover of a graph is a subset of the vertices such that all edges of the graph are also incident to to vertices of that subset. Hence, all edges are covered by vertices.

Definition 1.33. (vertex cover, vertex cover number)
Let G be a graph. A subset $V' \subseteq V(G)$ is called a *vertex cover* if for all $e \in E(G)$ there exists a $v \in V'$ with v is an end vertex of e. The size of a minimal vertex cover, called the *vertex cover number*, is denoted by $\tau(G)$.

In comparison to the vertex cover the edge cover of a graph is a subset of the edges of the graph such that all vertices of the graph are incident to at least one edge of the edge cover. Hence, all vertices are covered by edges.

Definition 1.34. (edge cover, edge cover number)
Let G be a graph. A subset $E' \subseteq E(G)$ is called an *edge cover* if for all $v \in V(G)$ there exists an edge $e \in E'$ with v is an end vertex of e. The size of a minimal edge cover, called the *edge cover number*, is denoted by $\rho(G)$.

1.2 The Reconstruction Conjecture

There is a good visualisation for the problem of the reconstruction conjecture which was already given by Harary [32] and may come in handy. Imagine you are dealt a deck of cards. Each of the cards has printed on one subgraph where exactly one vertex is missing. The cards itself only hold the isomorphism types of the subgraphs which are the unlabelled subgraphs. Furthermore the same card may occur multiple times because the same isomorphism type of a subgraph may be derived from deleting different vertices. Now, the problem to be solved is not to find a graph that has all cards with the right multiplicity as subgraphs. This would be more or less a routine matter to find the original graph. The problem to be solved is to show that the graph that has all cards with the right multiplicity as subgraphs is unique and that there is no other different graph that holds the same subgraphs with the right multiplicity.

Having an idea what the reconstruction conjecture is all about, we give the exact mathematical formulations to fill this idea with life. We start by defining the deck and its contents, namely the cards. Figure 1 holds the visualisation of the definition of the vertex-deleted subgraphs, cards and the deck.

Definition 1.35. (vertex-deleted subgraph, card of G, deck of G)
Let G be a graph. Then the unlabelled *vertex-deleted subgraph* $G_v := G[V(G) \setminus \{v\}]$ for some $v \in V(G)$ is called a *card of* G. The multiset of all cards of G is called the *deck of* G and is denoted by $D(G)$. A card of G can occur multiple times in the deck of G.

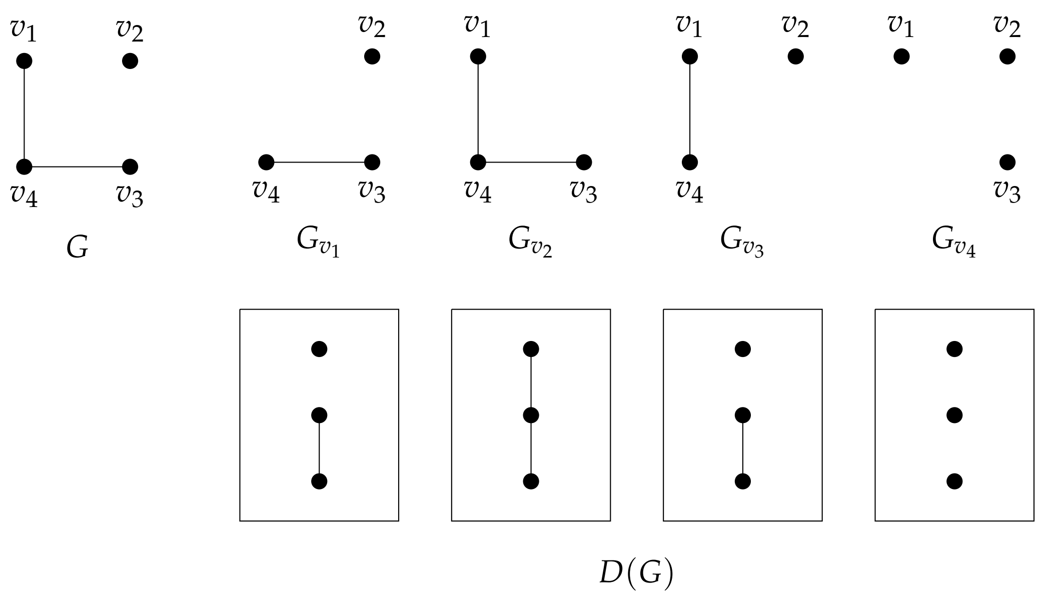


Figure 1: Graph G, the vertex-deleted subgraphs of G and the deck of G

With the cards from the deck we may reconstruct the original graph. Until the reconstruction conjecture is proven to be correct, we need to take into account that there is the possibility of different original graphs. We call these original graphs reconstructions.

Definition 1.36. (reconstruction of G)
Let G be a graph. Then the graph H is a *reconstruction of* G if $V(H) = V(G)$ and $H_v \cong G_v$ holds for all $v \in V(G)$.

Since there might be different, non-isomorphic reconstructions for the same deck, we define the word reconstructible for graphs with a unique reconstruction.

Definition 1.37. (reconstructible)
A graph G is called *reconstructible* if all reconstructions of G are isomorph to G. A graph invariant or graph parameter of G is called *reconstructible* if it has the same value for all reconstructions of G.

The reconstruction conjecture then states that every graph on at least 3 vertices is reconstructible.

Conjecture 1.38. (Reconstruction Conjecture)
All finite simple undirected graphs with at least three vertices are reconstructible.

The reconstruction conjecture itself is also known as *Ulam's Conjecture* because it was first considered by P. J. Kelly and S. M. Ulam in 1941, when Kelly wrote his doctoral thesis under Ulam. The reconstruction conjecture is proposed to work for all graphs with at least 3 vertices. The following example is the only known example with non-isomorphic reconstructions of simple graphs and it is based on graphs with 2 vertices.

Example 1.39.
Let $G = (\{v, w\}, \{vw\}) \cong K_2$ and $H = (\{v, w\}, \emptyset)$ be two graphs (see figure 2). Then $G_v \cong H_v$ and $G_w \cong H_w$. So H is a reconstruction of G because $D(G) = D(H)$ holds. In addition $G \ncong H$ and hence G and H are not reconstructible.

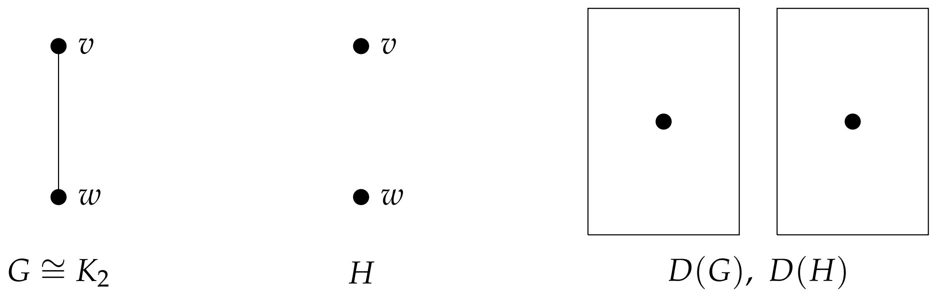


Figure 2: Two non isomorph graphs G and H with the same deck

1.2.1 Structural Properties

Over the last decades researchers accomplished to prove some fundamental structural theorems regarding the reconstruction conjecture. This subchapter will list structural approaches and structural aids for solving the reconstruction conjecture. One strong property of the reconstruction conjecture is that a graph is reconstructible if and only if its complement is reconstructible. This obersavtion is due to Kelly [39].

Theorem 1.40. (Kelly, 1957)
Let G be a graph. Then G is reconstructible if and only if its complement $\overline{G}$ is reconstructible.

Proof.
Let G be a graph and $\overline{G}$ its complement on the same vertex set. The deck $D(\overline{G})$ may be determined from the deck $D(G)$ by the identity $\overline{G_v} = \overline{G}_v$ which holds for all $v \in V(G)$. This is valid for both directions and therefore yields the claim. □

This basic property comes in handy to reconstruct classes of graphs, graph invariants and also for reductions of the reconstruction conjecture. Furthermore this property holds for the reconstruction conjecture itself but not for the later defined edge-reconstruction conjecture even though the edge-reconstruction conjecture is thought to be the easier problem of the two.

The next theorem is probably the most famous and most useful property regarding the reconstruction conjecture. It is obvious, that by having the deck of a graph G we also get all subgraphs of G on at most $n(G) - 1$ vertices. Kelly's Lemma goes further by counting all subgraphs with order at most $n(G) - 1$ and giving us the exact number of how often a specific isomorphism type of subgraph occurs in G. When Kelly published his paper "*A Congruence Theorem for Trees*" [39] he did show the first reconstructible class of graphs as well as the first structural property of the reconstruction conjecture.

Theorem 1.41. (Kelly's Lemma, 1957)
Let F and G be two graphs with $n(F) < n(G)$. Then the number $s(F, G)$ of subgraphs of G isomorphic to F is reconstructible.

Proof.
Each subgraph of G isomorphic to F occurs in exactly $n(G) - n(F)$ cards G_v. Summing over all cards of G yields

$$s(F,G) \cdot (n(G) - n(F)) = \sum_{v \in V(G)} s(F, G_v)$$

$$\Leftrightarrow \qquad s(F,G) = \sum_{v \in V(G)} \frac{s(F, G_v)}{n(G) - n(F)}.$$

The parameter $s(F, G_v)$ is reconstructible because the deck $D(G)$ is known. Furthermore $n(G) = |D(G)|$ holds and $n(F)$ is known. Hence, the right hand side of the equation is reconstructible and therefore the left hand side is reconstructible, too. □

As a corollary from Kelly's Lemma we are able to deduce that the size of a graph is reconstructible.

Corollary 1.42. (Kelly, 1957)
Let G be a graph. Then $m(G)$ is reconstructible.

Proof.
The chosen subgraph in Kelly's Lemma is $F = K_2$ that is the complete graph on two vertices. Thus Kelly's Lemma counts the number of edges. □

Tutte started counting subgraphs to reconstruct different structures and subgraphs. Some of those that he found in his paper "*All the King's Horses (A Guide to Reconstruction)*" [76] are given in the next theorem. This paper laid the foundation to a new method to reconstruct a whole range of graph invariants and sub-structures. It was later improved by Kocay into what we know today as Kocay's Lemma.

Theorem 1.43. (Tutte, 1979)
Let G be a graph and $\mathbf{F} = (F_1, F_2, ..., F_m)$ a sequence of graphs. Then

i) the number of disconnected spanning subgraphs of G with m components isomorphic to $F_1, F_2, ..., F_m$ is reconstructible,

ii) the number of separable spanning subgraphs of G with m blocks isomorphic to $F_1, F_2, ..., F_m$ is reconstructible,

iii) the number of nonseparable spanning subgraphs of G with a specified number of edges is reconstructible.

The proof of theorem 1.43 will be postponed after Kocay's Lemma. We will use the refined method by Kocay to show Tutte's result. In order to state and prove Kocay's Lemma we need to define a *cover*.

Definition 1.44. (cover, number of covers)
Let G be a graph and $\mathbf{F} := (F_1, F_2, ..., F_m)$ be a sequence of not necessarily distinct graphs. A *cover of G by F* is a sequence $(G_1, G_2, ..., G_m)$ of subgraphs of G such that $G_i \cong F_i$ for $1 \leq i \leq m$ and $\bigcup_{i=1}^{m} G_i = G$. The *number of covers of G by F* is denoted by $c(F, G)$.

Now, we are able to state Kocay's Lemma as given in Kocay's paper "*Some new methods in reconstruction theory*" [41]. This is the refined version of the method first stated by Tutte. He both gave a vertex and an edge version of this method. Remark that both Tutte's method as well as Kocay's Lemma are designed for spanning subgraphs. So this result neatly complements Kelly's Lemma since Kelly's Lemma works for subgraphs on at most $n(G) - 1$ vertices.

Theorem 1.45. (Kocay's Lemma, 1981)
Let G be a graph, $\mathbf{F} := (F_1, F_2, ..., F_m)$ be a sequence of graphs such that $n(F_i) < n(G)$ holds for all $i \in \{1, ..., m\}$ and $\mathbf{X}$ the set of all isomorphism types of graphs with order $n(G)$. Then

$$\sum_{X \in \mathbf{X}} c(\mathbf{F}, X) s(X, G)$$

is reconstructible.

Proof.
Let $\mathbf{X}'$ be the set of all isomorphism types of graphs. Then we have

$$\prod_{i=1}^{m} s(F_i, G) = \sum_{X \in \mathbf{X}'} c(\mathbf{F}, X) s(X, G),$$

where we count the sequence $(G_1, G_2, ..., G_m)$ of subgraphs of G such that $G_i \cong F_i$ holds for all $i \in \{1, ..., m\}$ in two different ways. $n(F_i) < n(G)$ holds by the definition of a cover. Hence, the left hand side is reconstructible by Kelly's Lemma. A part of the sum of the right hand side is reconstructible in itself. Consider the summands $c(\mathbf{F}, X)s(X, G)$. If $n(X) > n(G)$ holds, then $s(X, G) = 0$ follows. If $n(X) < n(G)$ holds, then $s(X, G)$ is reconstructible by Kelly's Lemma and therefore $c(\mathbf{F}, X)s(X, G)$ is reconstructible, too. Hence, we may reorder the previous sum into

$$\begin{aligned} \prod_{i=1}^{m} s(F_i, G) &= \sum_{X \in \mathbf{X}'} c(\mathbf{F}, X) s(X, G) \\ \Leftrightarrow \prod_{i=1}^{m} s(F_i, G) - \sum_{X \in \mathbf{X}' \setminus \mathbf{X}} c(\mathbf{F}, X) s(X, G) &= \sum_{X \in \mathbf{X}} c(\mathbf{F}, X) s(X, G). \end{aligned}$$

As stated all terms of the left hand side are reconstructible and therefore the right hand side is reconstructible, too. □

Now, with the proof of Kocay's Lemma we may prove Theorem 1.43. As stated before, Kocay refined the method that Tutte used to prove Theorem 1.43 and by doing so he made this theorem a corollary to Kocay's Lemma. For the original proof and method by Tutte refer to Tutte's "*All the King's Horses (A Guide to Reconstruction)*" [76].

Proof. (of theorem 1.43)

i) Without loss of generality assume $m \geq 2$ because the spanning subgraph is disconnected.

$$\sum_{i=1}^{m} n(F_i) = n(G)$$

holds by the same argument. Furthermore we may restrict the isomorphism types X to those that satisfy $c(\mathbf{F}, X) > 0$. Thus X is a disconnected graph with $n(X) = n(G)$ and components isomorphic to $F_1, F_2, ..., F_m$ and therefore $|\mathbf{X}| = 1$ holds. Hence, $c(\mathbf{F}, X)$ is reconstructible and by Kocay's Lemma the parameter $p = \sum_{X \in \mathbf{X}} c(\mathbf{F}, X) s(X, G)$ is reconstructible. So by Kocay's Lemma we have that

$$\begin{aligned} p &= \sum_{X \in \mathbf{X}} c(\mathbf{F}, X) s(X, G) \\ \Leftrightarrow \frac{p}{c(\mathbf{F}, X)} &= s(X, G) \end{aligned}$$

holds with X as described above. Thus the number of disconnected spanning subgraphs of G with components isomorphic to $F_1, F_2, ..., F_m$, namely $s(X, G)$, is reconstructible, too.

ii) Without loss of generality assume $m \geq 2$ because the spanning subgraph is separable with m blocks isomorphic to $F_1, F_2, ..., F_m$.

$$\sum_{i=1}^{m}(n(F_i) - 1) = n(G) - 1$$

holds by the same argument. Again we may restrict the isomorphism types X to those that satisfy $c(\mathbf{F}, X) > 0$. Thus $n(X) = n(G)$ holds and X is either disconnected or a separable graph with blocks isomorphic to $F_1, F_2, ..., F_m$. Denote the set of the first isomorphism type with $\mathbf{X}_d$. This case and the contribution to Kocay's Lemma is solvable with part $i)$ of this theorem. Denote the set of the second isomorphism type with $\mathbf{X}_s$. The contribution of this case to Kocay's Lemma is known, yet the summands need to be determined. The number of covers $c(\mathbf{F}, X)$ is in this case independent from the isomorphism type X and therefore some fixed number $p_s = c(\mathbf{F}, X)$ holds for all $X \in \mathbf{X}_s$. Hence, with $p := \sum_{X \in \mathbf{X}} c(\mathbf{F}, X)s(X, G)$ and Kocay's Lemma we deduce

$$p = \sum_{X \in \mathbf{X}_d \cup \mathbf{X}_s} c(\mathbf{F}, X)s(X, G)$$

$$\Leftrightarrow \frac{p - \sum_{X \in \mathbf{X}_d} c(\mathbf{F}, X)s(X, G)}{p_s} = \sum_{X \in \mathbf{X}_s} s(X, G)$$

As stated, the left hand side is reconstructible and thus the number of separable spanning subgraphs of G with blocks isomorphic to $F_1, F_2, ..., F_m$, namely $\sum_{X \in \mathbf{X}_s} s(X, G)$, is reconstructible, too.

iii) Set $F_i \cong K_2$ for all $i \in \{1, 2, ..., m\}$ in Kocay's Lemma. If the isomorphism type X is disconnected, then part $i)$ of this theorem counts the contribution of that isomorphism type to Kocay's Lemma. If X is separable, then part $ii)$ of this theorem counts the contribution to Kocay's Lemma. If $n(X) < n(G)$ holds, then $s(X, G)$ and the contribution to Kocay's Lemma is reconstructible by Kelly's Lemma. Hence, the contribution from non-separable spanning subgraphs of G with at most m edges to Kocay's Lemma is reconstructible. This holds true for all values of m. Finally the contribution to Kocay's Lemma of non-separable spanning graphs with exactly m edges is reconstructible by substracting the sum of up to $m - 1$ edges from the sum of up to m edges. Denote the set of isomorphism types X of non-separable spanning graphs with exactly m edges with $\mathbf{X}_m$. The number of covers is given by $c(\mathbf{F}, X) = m!$ for all $X \in \mathbf{X}_m$. Hence, with $p := \sum_{X \in \mathbf{X}} c(\mathbf{F}, X)s(X, G)$ and Kocay's Lemma we deduce

$$\begin{aligned} p &= \sum_{X \in \mathbf{X}} c(\mathbf{F}, X)s(X, G) \\ \Leftrightarrow\ p - \sum_{X \in \mathbf{X} \setminus \mathbf{X}_m} c(\mathbf{F}, X)s(X, G) &= \sum_{X \in \mathbf{X}_m} m! \cdot s(X, G) \\ \Leftrightarrow\ \frac{1}{m!}\left(p - \sum_{X \in \mathbf{X} \setminus \mathbf{X}_m} c(\mathbf{F}, X)s(X, G)\right) &= \sum_{X \in \mathbf{X}_m} s(X, G) \end{aligned}$$

As stated with the arguments above, the left hand side is reconstructible and thus the number of non-separable spanning subgraphs of G with exactly m edges, namely $\sum_{X \in \mathbf{X}_m} s(X, G)$, is reconstructible, too. □

Another strong result which concludes this section was given by Müller in his paper "*Probabilistic reconstruction from subgraphs*" [59]. He showed in a probabilistic way that almost all graphs are reconstructible, giving us a strong point that the reconstruction conjecture might be true.

Theorem 1.46. (Müller, 1976)
Almost all graphs are reconstructible.

1.2.2 Recognizable and Reconstructible Graph Classes

Besides proving the reconstruction conjecture by general and structural methods there are two widely used ways. The first is to show that all classes of graphs are reconstructible. Hence, you start by showing that some classes of graphs are reconstructible and hopefully are able to show this eventually for all classes of graphs. The second way is to show that graph invariants are reconstructible and that there exists a combination of different graph invariants that define a graph completely. Results for the second way and for the reconstruction of graph invariants are given in the next subchapter.

In this section we will present a wide range of results regarding the reconstruction of classes of graphs that have been accomplished since the reconstruction conjecture has first been proposed.

The first result that has been proven in respect to the reconstruction conjecture was given by Kelly. In his paper "*A Congruence Theorem for Trees*" [39] he has proven that trees are reconstructible. This is both the first class that was reconstructed as well as the first result in this field.

Theorem 1.47. (Kelly, 1957)
Trees are reconstructible.

The proof is based upon the fact that every tree has either a central vertex or a central edge and uses that the branches of a tree are countable. The method of counting substructures in general graphs is known as Kelly's Lemma and originates in his paper. For a shorter and more precise proof see [10].

In the same paper by Kelly it was shown that disconnected graphs are reconstructible.

Theorem 1.48. (Kelly, 1957)
Disconnected graphs are reconstructible.

In general the reconstruction of classes of graphs divides naturally into two steps. The two step method consists of showing that a graph is *recognizable* and *weakly reconstructible*.

Definition 1.49. (recognizable, weakly reconstructible)
Let $\mathbf{F}$ be a class of graphs and $G \in \mathbf{F}$. The class $\mathbf{F}$ is called *recognizable* if for every reconstruction H of G already $H \in \mathbf{F}$ holds. The class $\mathbf{F}$ is called *weakly reconstructible* if every reconstruction $H \in \mathbf{F}$ of G is isomorphic to G.

The term *recognizable* states that every reconstruction of a certain graph is in the same class as the graph itself. The second step consists of showing that the graph is *weakly reconstructible*. This means to show that the graph and its reconstruction are already isomorphic under the assumption that they both are in the same class. Hence, for recognizable classes of graphs weakly reconstructible is then equivalent to reconstructible.

To apply this method to an example we give the result by Kelly [39] and prove that regular graphs are reconstructible.

Definition 1.50. (regular, *k*-regular)
Let G be a graph and $k \in \mathbb{N}$. Then G is called *regular* if $d_G(v) = k$ holds for all $v \in V(G)$. If all vertices of G have degree k, then the graph is also called *k-regular*.

Theorem 1.51. (Kelly, 1957)
Regular graphs are reconstructible.

Proof.
recognizable: The degree sequence is reconstructible as a direct result of Kelly's Lemma. For example the degree of the missing vertex $v \in V$ of a card G_v is given by

$$d_G(v) = m(G) - m(G_v)$$

where $m(G)$ is reconstructible by Corollary 1.42. Hence, regular graphs are recognizable.
weakly reconstructible: Let G be a k-regular graph. Then every card G_v has exactly $n(G) - k - 1$ vertices of degree k and k vertices of degree $k - 1$. The unique way to reconstruct G from its deck $D(G)$ to yield a k-regular graph is to connect in any card G_v the missing vertex v of degree k to the exactly k vertices in G_v of degree $k - 1$. Alternatively, $n(G)$ and k already define a k-regular graph completely up to isomorphism.
Thus, regular graphs are reconstructible because they are both recognizable and weakly reconstructible. □

We may generalize the idea of the regular graph. A regular graph has only vertices of one degree which is either even or odd. We may skip the condition where there is only one degree allowed and generalize this to graphs of either only even or odd degree. We start by showing that graphs where every vertex is of even degree are reconstructible.

Theorem 1.52.
Graphs where every vertex has even degree are reconstructible.

Proof.
recognizable: The degree sequence of a graph is reconstructible and therefore graphs where every vertex has even degree are recognizable.
weakly reconstructible: Let G be a graph where every vertex has even degree. If G contains an isolated vertex v, then G is reconstructible because G_v contains all non trivial components. So we may assume that G contains no isolated vertices. Let $v \in V(G)$. Then G_v has exactly $d_G(v)$ vertices of odd degree. The unique way to reconstruct a graph where every vertex is of even degree from G_v is to connect the missing vertex v to the $d_G(v)$ vertices of odd degree.
Thus graphs where all vertices have even degree are reconstructible because they are both recognizable and weakly reconstructible. □

Corollary 1.53.
Connected graphs where every vertex has odd degree are reconstructible.

Proof.
recognizable: The degree sequence of a graph is reconstructible and therefore graphs where every vertex has odd degree are recognizable.
weakly reconstructible: Let G be a graph where every vertex has odd degree. By the Handshaking Lemma $n(G)$ is even because every graph has an even number of vertices of odd degree. Hence

$$d_{\overline{G}}(v) = n(G) - d_G(v) - 1$$

is even for all $v \in V(\overline{G})$. So $\overline{G}$ is reconstructible by Theorem 1.52. By Theorem 1.40 is a graph reconstructible if and only if its complement is reconstructible. Therefore, G is weakly reconstructible.
Thus graphs where all vertices have odd degree are reconstructible because they are both recognizable and weakly reconstructible. □

Hence, that regular graphs are reconstructible follows directly from Theorem 1.52 and its Corollary 1.53. As a direct result from Theorem 1.52, we have that eulerian graphs are reconstructible, too.

Corollary 1.54.
Eulerian graphs are reconstructible.

Next we give results for the reconstruction of certain types of blocks. Namely we look at critical blocks and line-critical blocks.

Definition 1.55. (critical block, line-critical block)
Let G be a graph. A block B is called a *critical block* if $B - \{v\}$ is separable for all $v \in V(B)$. The graph G is also called a *critical block* if $G = B$ holds. A block B' is called a *line-critical block* if for all edges $e \in E(B')$ the subgraph $B' - \{e\}$ is separable. A graph G is also called a *line-critical block* if $G = B'$ holds.

In 1979 Krishnamoorthy and Parthasarathy showed in their paper "*Reconstruction of Critical Blocks*" [43] that critical blocks are reconstructible.

Theorem 1.56. (Krishnamoorthy and Parthasarathy, 1979)
Critical blocks are reconstructible.

In the same year Fleischner proved in his paper "*The reconstruction of line-critical blocks*" [25] that line-critical blocks are reconstructible.

Theorem 1.57. (Fleischner, 1979)
Line-critical blocks are reconstructible.

Bondy discussed in his paper "*On Ulam's Conjecture for Separable Graphs*" [8] several subclasses of separable graphs. First he did show that separable graphs without vertices of degree 1 are reconstructible.

Theorem 1.58. (Bondy, 1969)
Separable graphs without end vertices are reconstructible.

Then he split the separable graph into different substructures, namely the trunk and the limbs.

Definition 1.59. (trunk, limb, root)
Let G be a graph. The *trunk* $T(G)$ of a graph G is the induced subgraph of G, remaining after successively removing all end vertices until none remain. A *limb* $L(G)$ of G is a nontrivial maximal connected subgraph of G having just one vertex in common with $T(G)$. This vertex is called the *root* of $L(G)$, denoted by $r(L(G))$.

He then did prove that every reconstruction of a separable graph has the same trunk and the same number and kind of limbs up to isomorphism.

Theorem 1.60. (Bondy, 1969)
Let G be a separable graph with end vertices. The trunk $T(G)$ and all limbs $L(G)$ are reconstructible.

Lastly he reconstructed some separable graphs with vertices of degree 1 where the trunk of that graph meets certain conditions.

Theorem 1.61. (Bondy, 1969)
Let G be a separable graph with end vertices. If the trunk $T(G)$ is isomorph to a complete graph, then G is reconstructible.

Another result on separable graphs is given by Ramachandran and Monikandan in their paper "*Graph reconstruction conjecture: Reductions using complement, connectivity and distance*" [67].

Theorem 1.62. (Ramachandran and Monikandan, 2009)
Separable graphs G with $\text{diam}(G) = 2$ are reconstructible.

The general case if separable graphs as a class are reconstructible is still unsolved.

Tutte did show in his paper "*All the King's Horses*" [76] that hamiltonian graphs are recognizable. This is actually a direct result of his counting technique that we discussed in the previous subchapter.

Theorem 1.63. (Tutte, 1979)
Hamiltonian graphs are recognizable.

Proof.
By Theorem 1.43 iii), the number of non-separable spanning subgraphs of G with a specified number of edges is reconstructible. Hence, set the numbers of edges to $n(G)$ and the theorem yields the number of hamilton cycles. So hamiltonian graphs are recognizable. □

We may give a specific subclass of hamiltonian graphs that is reconstructible. In order to do so, we need to show that 2-connected graphs that contain a path with 3 consecutive vertices of degree 2 are reconstructible.

Theorem 1.64.
2-connected graphs that contain a path with 3 consecutive vertices of degree 2 are reconstructible.

Proof.
recognizable: 2-connected graphs are recognizable as graphs with minimal degree at least 2 and not being disconnected or separable. Disconnected graphs are reconstructible by Theorem 1.48 and separable graphs without end vertices are reconstructible by Theorem 1.58. Furthermore, a 2-connected graph contains a path with 3 consecutive vertices of degree 2 if and only if there exists a card $G_v \in D(G)$ such that $d_G(v) = 2$ and G_v contains exactly two vertices of degree 1. Hence, 2-connected graphs with 3 consecutive vertices of degree 2 are recognizable.
weakly reconstructible: Let G be a 2-connected graph and P is a path of G, such that there are three consecutive vertices $u, v, w \in P$ with $d_G(u) = d_G(v) = d_G(w) = 2$. Let the sequence of these vertices be defined by $uv, vw \in E(G)$. Then G_v contains exactly two vertices of degree 1, namely u and w. Since G is 2-connected, there is exactly one possibility to add v to G_v to create a 2-connected graph. Hence add the vertex v of degree 2 adjacent to the two vertices u and w of degree one to reconstruct G. □

Now, we can use the class of 2-connected graphs that contain a path with 3 consecutive vertices of degree 2 to show that hamiltonian graphs, which exceed a specific order-size ratio, are reconstructible.

Theorem 1.65.
Let G be a hamiltonian graph with $m(G) \leq n(G) + \left\lfloor \frac{n(G)-1}{6} \right\rfloor$. Then G is reconstructible.

Proof.
Since G is hamiltonian, G has a circle of length $n(G)$. All other edges are chords of that circle. So, the hamiltonian circle has at most $\left\lfloor \frac{n(G)-1}{6} \right\rfloor$ chords. Consequently there are at least $n(G) - 2 \cdot \left\lfloor \frac{n(G)-1}{6} \right\rfloor$ vertices in G of degree 2 and at most $2 \cdot \left\lfloor \frac{n(G)-1}{6} \right\rfloor$ vertices of degree at least three. We partition the hamilton circle into subpaths. That means, partitioning the vertices of degree two evenly into connected paths of vertices of degree two, seperated by vertices of degree at least three will yield

$$\frac{n(G) - 2 \cdot \left\lfloor \frac{n(G)-1}{6} \right\rfloor}{2 \cdot \left\lfloor \frac{n(G)-1}{6} \right\rfloor} = \frac{n(G)}{2 \cdot \left\lfloor \frac{n(G)-1}{6} \right\rfloor} - 1$$
$$\geq \frac{n(G)}{2 \cdot \frac{n(G)-1}{6}} - 1$$
$$= \frac{6}{2} \cdot \frac{n(G)}{n(G) - 1} - 1$$
$$= 3 \cdot \frac{n(G)}{n(G) - 1} - 1$$
$$> 2$$

if G is a finite graph. Hence, there is at least one path with three consecutive vertices of degree two (pigeon hole principle). Furthermore, G is hamiltonian and hence 2-connected. So by Theorem 1.64, G is weakly reconstructible. The edge number is reconstructible and hamiltonian graphs are recognizable, thus G is reconstructible. □

Remark that in the last proof every graph that is too small (e.g. less than 10 vertices) is reconstructible by a computer study of McKay (see Theorem 1.103).

Now, we want to collect some results on the progress of the reconstruction conjecture on planar graphs. For this, we define planar graphs first:

Definition 1.66. (planar graphs, planar drawing, embedding, face)
A graph G is called *planar* if there exists an *embedding of* G into the plane, that is it can be drawn on the plane in such a way that edges only intersect at a common end vertex. Such a drawing is also called a *planar drawing*. A region of a planar drawing that is bounded by edges of the graph is called a *face*. A vertex belongs to a face if one of its incident edges is a bound of that face.

We state the results on some subclasses of planar graphs. We start with the subclass of maximal planar graphs.

Definition 1.67. (maximal planar graph)
A graph G is called a *maximal planar graph* if G is planar but adding any edge to G will result in a non-planar graph.

In 1978, Fiorini was able to show in his paper "*A theorem on planar graphs with an application to the reconstruction problem. I.*" [20] that maximal planar graphs with minimal degree at least 5 are reconstructible. In the same year Fiorini and Manvel improved that result and proved in their paper "*A theorem on planar graphs with an application to the reconstruction problem. II.*" [24] that maximal planar graphs with minimal degree at least 4 are reconstructible.

Theorem 1.68. (Fiorini and Manvel, 1978)
Maximal planar graphs G with $\delta(G) \geq 4$ are reconstructible.

The next two Theorems are an improvement of the results by Fiorini and Manvel and show by removing the minimal degree condition that maximal planar graphs in general are reconstructible. First, Fiorini and Lauri have shown in "*The reconstruction of maximal planar graphs. I. Recognition*" [21] that maximal planar graphs are recognizable. Hence leaving only the weak reconstruction to be done.

Theorem 1.69. (Fiorini and Lauri, 1981)
Maximal planar graphs are recognizable.

In the same journal Lauri published in the article "*The reconstruction of maximal planar graphs. II. Reconstruction*" [45] the second and final step to prove that maximal planar graphs are reconstructible. He did in fact show that maximal planar graphs are weakly reconstructible. Hence we get the following theorem:

Theorem 1.70. (Lauri, 1981)
Maximal planar graphs are reconstructible.

The next subclass of planar graphs is called outerplanar graphs.

Definition 1.71. (outerplanar graph)
Let G be a planar graph. Then G is called *outerplanar* if there exists a planar drawing of G such that all vertices belong to the outer face of the drawing.

Giles showed in his paper "*The reconstruction of outerplanar graphs*" [27] in 1974 that outerplanar graphs are reconstructible.

Theorem 1.72. (Giles, 1974)
Outerplanar graphs are reconstructible.

Bilinski, Kwon and Yu were able to show the next result for the whole class of planar graphs. In their paper "*On the reconstruction of planar graphs*" [5] they have shown that the whole class of planar graphs is recognizable. Hence the first step in the two step method of recognition and weak reconstruction for the entire class of planar graphs has been accomplished.

Theorem 1.73. (Bilinski, Kwon, Yu, 2007)
Planar graphs are recognizable.

In the same paper Bilinski, Kwon and Yu also reconstructed certain types of planar graphs. In particular they showed that 5-connected planar graphs with a special condition are reconstructible. For further information on that problem, see [5].

The class of P_4-reducible graphs was reconstructed by Thatte in his paper "*Some Results on the Reconstruction Problems. p-Claw-Free, Chordal, and P_4-Reducible graphs*" [75]. He uses the recursive construction of P_4-reducible graphs given by Jamison and Olariu to show the result.

Definition 1.74. (P_4-reducible graph)
Let G be a graph. G is called *P_4-reducible* if every vertex of G belongs to at most one path on 4 vertices as induced subgraph, namely a P_4.

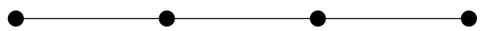


Figure 3: P_4

Theorem 1.75. (Thatte, 1995)
P_4-reducible graphs are reconstructible.

It is mathematical folklore that the class of bipartite graphs is recognizable.

Theorem 1.76.
Bipartite graphs are recognizable.

Monikandan and Balakumar presented the next two theorems in their paper "*Reconstruction of bipartite graphs and triangle-free graphs with connectivity two*" [57]. Their intend was to reduce the problem of the reconstruction conjecture to a smaller problem that excludes bipartite graphs. For this, see Theorem 1.102 and the section it is included in. In order to reduce the problem, they have shown that certain bipartite graphs of small diameter are reconstructible.

Theorem 1.77. (Monikandan and Balakumar, 2012)
All 2-connected, bipartite graphs G with $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

Theorem 1.78. (Monikandan and Balakumar, 2012)
Bipartite graphs G with $\text{diam}(G) = 2$ are reconstructible.

In the same paper Monikandan and Balakumar proved that 2 connected, triangle-free graphs of a certain low diameter are reconstructible.

Theorem 1.79. (Monikandan and Balakumar, 2012)
All 2-connected, triangle-free graphs G with $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

In 1983 von Rimscha showed in his paper "*Reconstructibility and perfect graphs*" [68] that perfect graphs as well as a range of subclasses of perfect graphs are recognizable. First we give the proof for perfect graphs.

Theorem 1.80. (von Rimscha, 1983)
Perfect graphs are recognizable.

Proof.
Let G be a graph. The clique number $\omega(G)$ of G is reconstructible by either Kelly's Lemma or if G is a complete graph as a regular graph by Theorem 1.51. The chromatic number $\chi(G)$ is reconstructible by Theorem 1.88. Hence, it is reconstructible if $\chi(G) = \omega(G)$ holds. It remains to be shown that $\chi(G') = \omega(G')$ holds for every induced subgraph of G. Since every reconstruction of G has the same deck as G and therefore the same induced subgraphs of G, we have $\chi(G') = \omega(G')$ for every induced subgraph $G' \subsetneq G$. So determining from its deck that the graph is perfect is recognizable. □

Several subclasses of perfect graphs were proven to be recognizable.

Theorem 1.81. (von Rimscha, 1983)
Chordal graphs, interval graphs, comparability graphs and split graphs are recognizable.

In addition von Rimscha was able to show that certain subclasses of interval graphs, comparability graphs and split graphs are reconstructible. For more details see [68]. One of the reconstructible subclasses of interval graphs are unit interval graphs.

Theorem 1.82. (von Rimscha, 1983)
Unit interval graphs are reconstructible.

In the next subchapter we are going to see that the chromatic polynomial is reconstructible. Hence, as a direct result we get that *chromatically unique graphs* are weakly reconstructible. Chromatically unique graphs are graphs which are defined by their chromatic polynomial up to isomorphism. That is if G and H are two graphs on the same vertex set, then $P(G,x) = P(H,x)$ already yields $G \cong H$.

Theorem 1.83.
Chromatically unique graphs are weakly reconstructible.

1.2.3 Reconstructible Graph Invariants

The second way to tackle the reconstruction conjecture is to find a combination of reconstructible graph invariants that determine a graph completely. There are a couple of basic graph invariants that are easy to proof. The *order* of the graph equals the cards in the deck. The *size* of a graph is reconstructible by Corollary 1.42.

Further reconstructible basic graph invariants are summarized in the following theorem:

Theorem 1.84.
Let G be a graph. The following graph invariants are reconstructible:

i) degree sequence of G

ii) neighbourhood degree sequence of the vertices of G

iii) clique number $\omega(G)$

iv) stability number $\alpha(G)$

v) matching number $\nu(G)$

vi) girth of G

vii) vertex cover number $\tau(G)$

viii) edge cover number $\rho(G)$

Proof.

i) The degree sequence of a graph is reconstructible by the identity

$$d_G(v) = m(G) - m(G_v)$$

for every vertex $v \in V(G)$. $m(G)$ is reconstructible by Corollary 1.42.

ii) The *neighbourhood degree sequence* states the degrees of the neighbours of a vertex. This can be determined by comparing the degree sequence of the graph with the degree sequence of the card.

iii) The clique number is reconstructible by Kelly's Lemma if the graph is not a complete graph. If the graph is complete, then $\omega(G)$ is reconstructible by Theorem 1.51 because regular graphs are reconstructible.

iv) Since a graph is reconstructible if and only if its complement is reconstructible we have that the stability number is reconstructible by the identity $\alpha(G) = \omega(\overline{G})$.

v) The matching number is reconstructible by Theorem 1.43 part i) if G contains a perfect matching. If the graph does not contain a perfect matching, then the largest matching is contained in one of the cards of the graph.

vi) The *girth* states the length of a smallest circuit in G. It is reconstructible if the smallest circle does not contain all vertices of the graph by Kelly's Lemma. If it does, then for example the number of hamilton circles is reconstructible by Theorem 1.43.

vii) The stability number of G and the order of a graph are reconstructible. Gallai [26] proved $\tau(G) = |V(G)| - \alpha(G)$ for all graphs G. Hence, the vertex covering number $\tau(G)$ is reconstructible.

viii) The order and the matching number of G are reconstructible. Gallai showed in [26] that $\rho(G) = |V(G)| - \nu(G)$ holds for all graphs G. Therefore, the edge cover number is reconstructible. □

There are more basic graph invariants which may be determined easily from the deck of G, but we want to move on to the more interesting graph invariants which include new techniques for solving the reconstruction conjecture.

Gupta, Mangal and Paliwal reported in their paper "*Some work towards the proof of the reconstruction conjecture*" [31] some progress on the diameter of a graph. They were able to show by solving a system of linear equations that the diameter of 2 is reconstructible and therefore, the associated graph class recognizable.

Theorem 1.85. (Gupta, Mangal and Paliwal, 2013)
Graphs of diameter 2 are recognizable.

By Theorem 1.40 we have that a graph is reconstructible if and only if its complement is reconstructible. Combining this with the fact that if a graph has diameter greater or equal to 3 then its complement has diameter less or equal to 3 we get the following theorem:

Theorem 1.86. (Gupta, Mangal and Paliwal, 2013)
Let G be a graph. If $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ holds, then G is recognizable.

Tutte published in his paper "*All the King's Horses (A Guide to Reconstruction)*" [76] a range of polynomials that may be reconstructed. As mentioned earlier on, Tutte was able to reconstruct a wide range of spanning subgraphs with his Theorem 1.43, which was later on generalized by Kocay. As a result, we get the chromatic polynomial.

Theorem 1.87. (Tutte, 1979)
The chromatic polynomial is reconstructible.

Proof.
Whitney proved in [77] that the chromatic polynomial might be determined as follows:

$$P(G,x) = \sum_{i=1}^{n(G)} b_i x^i$$

with

$$b_i = \sum_{X \in \mathbf{X}_i} (-1)^{m(X)} s(X,G)$$

where $\mathbf{X}_i$ is the set of all isomorphism types of graphs with order $n(G)$ and i components. The coefficients b_i may be reconstructed by Tutte's Theorem 1.43 part i) if $2 \leq i \leq n(G)$ holds because they are disconnected spanning subgraphs. If the spanning subgraph X is not disconnected, then $i = 1$ holds and the coefficient b_1 may be reconstructed by Theorem 1.43 parts ii) and iii). □

The chromatic number is determined by the chromatic polynomial and so Tutte reconstructed the chromatic number.

Corollary 1.88. (Tutte, 1979)
The chromatic number is reconstructible.

Proof.
The chromatic number is the least positive integer x such that $P(G,x) > 0$ holds. $P(G,x)$ is reconstructible by Theorem 1.87. □

As mentioned earlier on, we are looking for a range of different graph invariants that may define a graph completely. The chromatic polynomial in itself is not sufficient to define a graph up to isomorphism. For example, all trees on n vertices have the chromatic polynomial of $P(G,x) = x(x-1)^{n-1}$. Also, Figure 4 shows three graphs with the same chromatic polynomial.

However, there are graphs that are called *chromatically unique*. That is, if G and H are two graphs on the same vertex set, then $P(G,x) = P(H,x)$ already yields $G \cong H$. Hence, with the reconstruction of the chromatic polynomial we are able to show that chromatically unique graphs are weakly reconstructible as mentioned already in Theorem 1.83. Some classes that are chromatically unique are given for example in the paper "*Chromatically Unique Graphs*" [14] by Chao and Whitehead.

The dichromatic polynomial is a generalisation of the chromatic polynomial. It is also known as the Tutte polynomial.

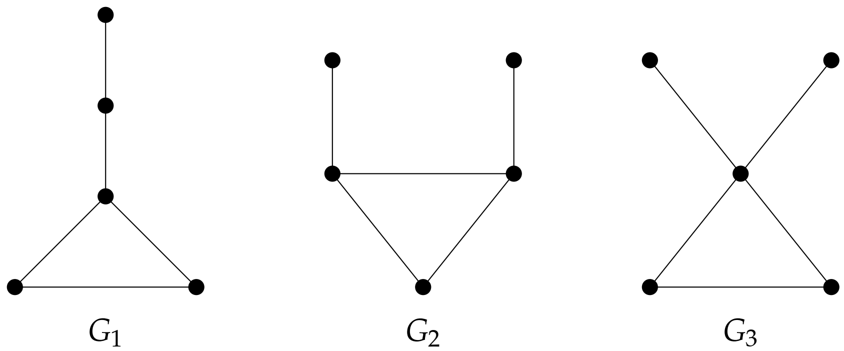


Figure 4: Three graphs with $P(G_i, x) = x(x-1)^3(x-2)$

Definition 1.89. (dichromatic polynomial)
The *dichromatic polynomial of a graph* G is defined as

$$Q(G,x,y) = \sum_{i=1}^{n(G)} \sum_{j=0}^{e(G)} c_{ij} \cdot x^i y^{i+j-n(G)}$$

with

$$c_{ij} = \sum_{X \in \mathbf{X}_{ij}} s(X,G)$$

where $\mathbf{X}_{ij}$ is the set of all isomorphism types of graphs with order $n(G)$, size j and i components.

The chromatic polynomial and the dichromatic polynomial are connected via the identity $P(G,x) = (-1)^{n(G)} Q(G,-x,-1)$. Tutte proved in his paper that this more general graph invariant is reconstructible. For the exact proof see [76].

Theorem 1.90. (Tutte, 1979)
The dichromatic polynomial is reconstructible.

Another graph invariant that Tutte was able to show is the characteristic polynomial.

Definition 1.91. (characteristic polynomial)
Let G be a graph. The *characteristic polynomial of* G is defined as

$$\chi(G,x) = \det(xI_{n(G)} - A),$$

where A is the adjacency matrix of G and $I_{n(G)}$ the identity matrix.

Theorem 1.92. (Tutte, 1979)
The characteristic polynomial is reconstructible.

Proof.
Sachs proved in [70] that the characteristic polynomial of a graph might be determined as follows:

$$\chi(G,x) = \sum_{i=0}^{n(G)} a_i x^{n(G)-i}$$

with

$$a_i = \sum_{X \in \mathbf{X}_i} (-1)^{c(X)} 2^{z(X)} s(X,G)$$

where $\mathbf{X}_i$ is the set of all isomorphism types of S-graphs with order i, $c(X)$ is the number of components of X and $z(X)$ is the number of circuits in X. An *S-graph* is a graph in which each component is either an edge or a circuit. Now, the coefficients a_i are reconstructible by Kelly's Lemma if $0 \leq i < n(G)$ holds. If $i = n(G)$ holds and X is disconnected, then $s(X,G)$ is reconstructible by Theorem 1.43 part i). If $i = n(G)$ holds and X is connected, then X is a Hamilton cycle. Hence in this case $s(X,G)$ is reconstructible by Theorem 1.43 part iii).
Therefore, the characteristic polynomial is reconstructible. □

Tutte generalized the characteristic polynomial to the idiosyncratic polynomial which he defined in his paper "*All the King's Horses (A Guide to Reconstruction)*" [76].

Definition 1.93. (idiosyncratic polynomial)
Let G be a graph. The *idiosyncratic polynomial of* G is defined as

$$\psi(G,x,y) = \det(xI_{n(G)} - B),$$

where $B = A + y(J - A)$, A is the adjacency matrix of G, $I_{n(G)}$ the $n(G) \times n(G)$ identity matrix and J the $n(G) \times n(G)$ matrix in which every entry is 1.

The more general graph invariant, the idiosyncratic polynomial is connected to the characteristic polynomial via the identity $\chi(G,x) = \psi(G,x,0)$. Tutte proved the reconstructibility of the idiosyncratic polynomial in [76].

Theorem 1.94. (Tutte, 1979)
The idiosyncratic polynomial is reconstructible.

In certain cases is the treewidth reconstructible. In particular we can show that the treewidth of disconnceted and separable graphs is reconstructible.

Theorem 1.95.
The treewidth of disconnected graphs and separable graphs is reconstructible.

Proof.
Disconnected graphs are reconstructible by Theorem 1.48. Hence, the treewidth is reconstructible, too.
Separable graphs without end vertices are reconstructible by Theorem 1.58. Hence its treewidth is reconstructible.
Now, assume that G is a separable graph with end vertices, that is G contains a vertex of degree 1. Consider the card G_v with $d_G(v) = 1$. Then $tw(G) = tw(G_v)$ holds, because both graphs contain the same minimal chordal extension. Adding the vertex v to G_v does not create a circle and therefore does not need to be taken into consideration. Hence the claim follows. □

As stated at the beginning of this subchapter there has no complete set of graph invariants be found that uniquely define a graph. A rather disappointing result was given by Izbicki in his paper "*Reguläre Graphen beliebigen Grades mit vorgegebenen Eigenschaften*" [38]. He did show how to construct an infinite number of regular graphs that have the same chromatic number, degree sequence and connectivity.

1.2.4 Reductions of the Reconstruction Conjecture

Besides the reconstruction of certain classes of graphs and graph invariants Yang tackled the problem of the reconstruction conjecture in a third way. In his paper "*The reconstruction conjecture is true if all* 2*-connected graphs are reconstructible*" [78] he showed that all graphs are reconstructible if a specific class of graphs is reconstructible. In effect he reduced the number of graphs that needs proof for the reconstruction conjecture. In particular he reduced the reconstruction conjecture of all graphs to the problem whether 2-connected graphs are reconstructible or not.

Theorem 1.96. (Yang, 1988)
Every connected graph is reconstructible if and only if every 2-connected graph is reconstructible.

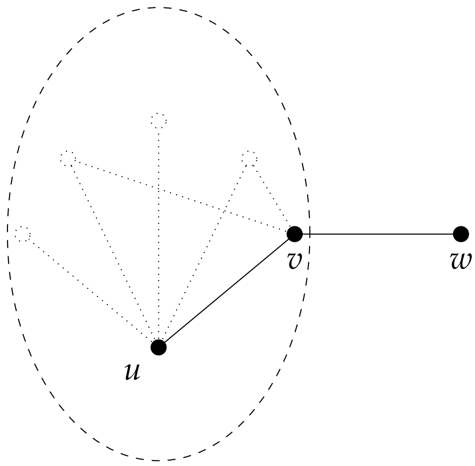
The proof itself makes heavy usage of Theorem 1.40 which states that a graph is reconstructible if and only if its complement is reconstructible. He shows that for a connected graph either the complement or the graph itself is 2-connected or the graph or its complement are already reconstructible. For that he uses a graph structure he calls a P-graph. First, we will define a P-graph and give his theorem about the reconstruction of certain types of P-graphs. Then, we will give Yang's proof for his reduction.

Definition 1.97. (P-graph)
Let G be a graph. G is called a *P-graph* if

i) G contains exactly two blocks where one of the blocks is an edge,

ii) there exists a vertex $u \in V(G)$ with $d_G(u) = n(G) - 2$ and u is not incident to the bridge of G,

holds.



block with vertex u satisfying $d_G(u) = n(G) - 2$

Figure 5: P-graph G

According to Yang [78] P-graphs are reconstructible if 2 connected graphs are reconstructible. In addition, he did show that certain P-graphs are already reconstructible.

Theorem 1.98. (Yang, 1988)
P-graphs are reconstructible if 2-connected graphs are reconstructible. In addition P-graphs satisfying any of the following conditions are reconstructible:

i) the P-graph contains no vertex of degree 2,

ii) there exists a vertex $u' \in V(G)$ with $d_G(u') = 2$ adjacent to a vertex of the bridge but not incident to the bridge itself.

Now, given that these types of P-graphs are reconstructible we state Yang's proof for Theorem 1.96.

Proof. (of theorem 1.96)
" $\Rightarrow$": Every 2-connected graph is also connected. Hence if connected graphs are reconstructible, then so are 2-connected graphs.
" $\Leftarrow$": Assume that all 2-connected graphs are reconstructible. Let G be a separable graph with $n(G) \geq 10$. Remark that graphs with fewer vertices are reconstructible by a computer study by McKay (see Theorem 1.103). We may further assume that

G has end vertices since separable graphs without end vertices are reconstructible by Theorem 1.58. Moreover we may assume that $\overline{G}$ contains a vertex $u \in V(G)$ of degree $d_G(u) = n(G) - 2$. Otherwise $\overline{G}$ contains no end vertices and is therefore reconstructible by either Theorem 1.58 or our hypothesis that 2-connected graphs are reconstructible. By Theorem 1.40 a graph is reconstructible if and only if its complement is reconstructible. Hence all restrictions we may achieve on a graph, we may also assume for its complement.
Assume that G contains at least two end vertices. Hence $\overline{G}$ contains at least two vertices of degree $n(G) - 2$. Then $\overline{G}$ meets one of three conditions. $\overline{G}$ is 2-connected and therefore reconstructible by our hypothesis. $\overline{G}$ is a P-graph and therefore reconstructible by Theorem 1.98. And last, $\overline{G}$ contains two non adjacent edges that are incident to end vertices. These edges are incident to different vertices of degree $n(G) - 2$. Let $v \in V(\overline{G})$ be an end vertex. Then the unique way to reconstruct $\overline{G}$ from $\overline{G}_v$ is to connect v to a vertex of degree $n(G) - 3$ in $\overline{G}_v$ that is not adjacent to at least one end vertex. All in all if G contains at least two end vertices, then the claim holds.
Now, assume that both G and $\overline{G}$ contain exactly one end vertex and exactly one vertex of degree $n(G) - 2$. Denote the vertex of degree 1 in G with $w \in V(G)$ and the vertex of degree $n(G) - 2$ in G with $u \in V(G)$. Then u is either a cut vertex or G is a P-graph and thus reconstructible by Theorem 1.98. Hence, assume u is a cut vertex. Let $v \in V(G)$ be the neighbour of w. Then u and v are the only cutvertices of G. Let B be the block containing both u and v. In the connected card G_s with at least one end vertex and with $s \in V(G) \setminus \{u, v, w\}$ the vertices u, v and w are identifiable. u as the only cut vertex of degree $n(G) - 3$, w as the only end vertex not adjacent to u and v as the neighbour of w. Hence, pick such a card G_s where u and v are contained in the same block and that block as large as possible. Hence, we know B and the location of u and v in it. From G_w we may reconstruct any other blocks and their connection to B at the vertex u. Thus, this case is also reconstructible. □

The condition that a graph is reconstructible if and only if its complement is reconstructible yields serveral other reductions of the reconstruction conjecture. In "*Some work towards the proof of the reconstruction conjecture*" [31] Gupta, Mangal and Paliwal used the theorem, that the complement of a graph of diameter greater than three has diameter less than three. Hence, they concluded the following reduction.

Theorem 1.99. (Gupta, Mangal and Paliwal, 2003)
The reconstruction conjecture is true if and only if all graphs G with $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are reconstructible.

Furthermore they showed in this paper that the class of graphs of diameter 2 and the class of graphs where both the graph and its complement have diameter 3 are recognizable. Hence missing for the proof of the reconstruction conjecture is the weak reconstruction of those two classes. They even refined it further by showing that the class of graphs where both the graph and its complement have diameter 2 and the class of graphs where a graph has diameter 2 and its complement has diameter greater than 2 are recognizable. They then split the class of graphs of diameter 2 into different subclasses to get a similar result.

Theorem 1.100. (Gupta, Mangal and Paliwal, 2003)
The reconstruction conjecture is true if and only if all graphs G with either

i) G is an edge-minimal graph of diameter 2,

ii) G is a non-edge minimal graph of diameter 2, or

iii) $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$

are reconstructible.

Gupta, Mangal and Paliwal also included the recognizability of these classes. Hence it remains to be shown that those three classes are weakly reconstructible to prove the reconstruction conjecture.

Ramachandran and Monikandan combined in their paper "*Graph reconstruction conjecture: Reductions using complement, connectivity and distance*" [67] Theorem 1.96 and Theorem 1.99 into the following stronger theorem.

Theorem 1.101. (Ramachandran and Monikandan, 2009)
The reconstruction conjecture is true if and only if all 2-connected graphs G with $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are reconstructible.

Monikandan and Balakumar combined the Theorems 1.96 and 1.99 with bipartite graphs by showing that 2-connected bipartite graphs of diameter 2 (see Theorem 1.78) and the class of graphs where a bipartite graph has diameter 3 and its complement has diameter 3 (see Theorem 1.77) are both reconstructible. Hence, the following theorem can be found in the paper "*Reconstruction of bipartite graphs and triangle-free graphs with connectivity two*" [57] where it is sufficient to reduce the problem to specific non-bipartite graphs.

Theorem 1.102. (Monikandan und Balakumar, 2012)
Every graph on at least 3 vertices is reconstructible if and only if all 2-connected graphs G containing an odd cycle with $\operatorname{diam}(G) = 2$ or $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 3$ are reconstructible.

Additional reductions can be found in this or other papers. For some other reductions refer to the papers [57] and [58].

1.2.5 Computer Studies

There are some computer studies carried out regarding the reconstruction conjecture. In 1977 Brendan D. McKay published the paper "*Computer reconstruction of small graphs*" [52] in which he started to prove the reconstruction conjecture for small graphs using computer techniques. His 1997 paper "*Small graphs are reconstructible*" [53] extended the classes of graphs he worked on even further and proved the reconstruction conjecture for graphs up to 11 vertices. Furthermore, he added special cases for classes of higher order. His initial attempt was to find a counterexample to the reconstruction conjecture, even though he states that a possible counterexample would probably not be small. Regardless it was a useful step. Showing that the reconstruction conjecture is true for small graphs came in handy for other researchers. To show that the reconstruction conjecture is true for a class, usually ends up in lots of case differentiations for small graphs. Here comes McKay's work in and we may reduce our efforts to graphs with order 12 or higher.

Theorem 1.103. (McKay, 1997)
The following classes of graphs are reconstructible:

i) graphs of order $4-11$,

ii) graphs of order 12 and maximum degree at most 5,

iii) triangle-free graphs of order $4-14$,

iv) square-free graphs of order $4-15$,

v) bipartite graphs of order $4-15$,

vi) bipartite graphs of order 16 and maximum degree at most 5.

The last Theorem 1.103 was actually proven for a stronger conjecture than the reconstruction conjecture. An *isomorphism-reduced deck* is a deck which contains only one isomorphism type of each card. Theorem 1.103 was proven for isomorphism-reduced decks. It is clear that an isomorphism-reduced deck can be derived from the deck of a graph.

The computer study itself is exhaustive. There are already one billion non isomorphic graphs on 11 vertices and the number increases exponentially with the order of the graph. Brendan D. McKay and Adolfo Piperno published the computer programms **nauty** (**n**o **aut**omorphisms, **y**es?) and **Traces** which come in handy to test graphs for isomorphisms. A user's guide may be found in [54, 55]. Furthermore, they provide a data base for all graphs up to a certain order and special classes of graphs of an even higher order.

1.3 The Edge-Reconstruction Conjecture

When talking about the edge-reconstruction conjecture we have the same picture in mind that we imagine for the reconstruction conjecture. Again we are dealt a set of cards which sum up the so called edge-deck. The difference is that in the cards are not vertices missing but always exactly one edge is missing. Hence, for each edge in a graph we are dealt a card and therefore the number of cards in the edge-deck corresponds to the number of edges of the graph.

Definition 1.104. (edge-deleted subgraph, card of G, edge-deck of G)
Let G be a graph. Then the unlabelled *edge-deleted subgraph* $G_e := (V(G), E(G) \setminus \{e\})$ for some $e \in E(G)$ is called a *card of* G. The multiset of all these cards of G is called the *edge-deck of* G and is denoted by $D_e(G)$. A card of G can occur mutliple times in the deck of G.

Figure 6 holds the visualisation of the definition of the edge-deleted subgraphs, cards and the edge-deck.

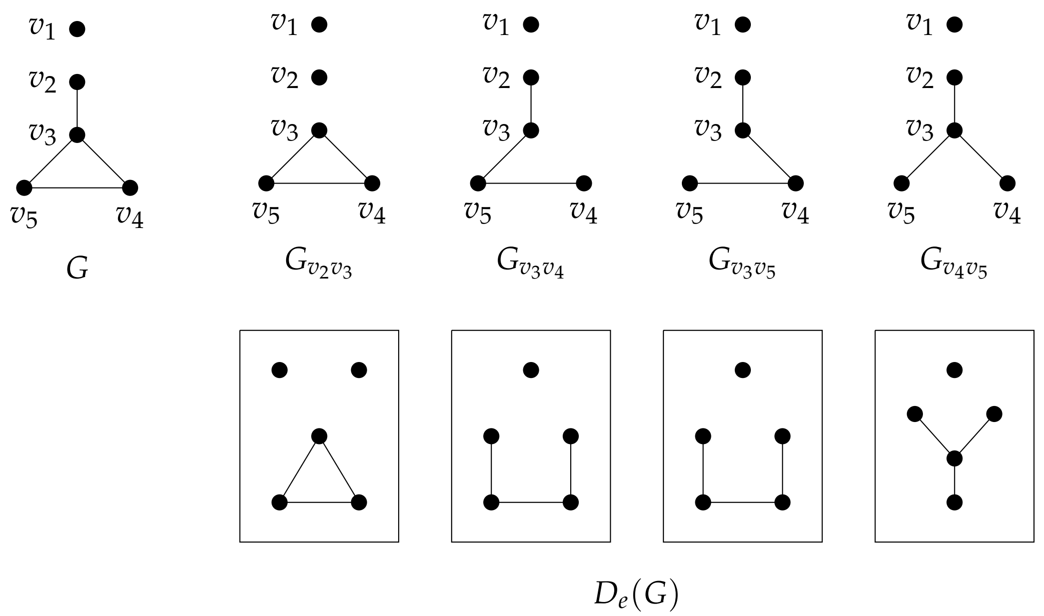


Figure 6: G, the edge-deleted subgraphs of G and the edge-deck of G

Analoguesly we define an edge-reconstruction as a graph that has the dealt cards with the right multiplicity as subgraphs.

Definition 1.105. (edge-reconstruction of G)
Let G and H be two graphs. Then the graph H is an *edge-reconstruction of* G if

$$D_e(H) \cong D_e(G)$$

holds, that is H and G have the same decks.

Hence, in figure 6 G is an edge-reconstruction of $D_e(G)$. If all reconstructions of a graph are isomorph to that graph then we will call it edge-reconstructible.

Definition 1.106. (edge-reconstructible)
A graph G is called *edge-reconstructible* if all edge-reconstructions of G are isomorph to G. A graph invariant or graph parameter of G is called *edge-reconstructible* if it has the same value for all edge-reconstructions of G.

The edge-reconstruction conjecture then states that every graph on at least 4 edges is reconstructible.

Conjecture 1.107. (Edge-Reconstruction Conjecture)
All finite simple undirected graphs with at least four edges are edge-reconstructible.

The following example is one known example with non-isomorphic reconstructions of simple graphs and it is based on graphs with 2 edges.

Example 1.108.
Let $G = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_3v_4\})$ and $H = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3\})$ be two graphs. Then H is an edge-reconstruction of G because $D_e(G) = D_e(H)$ holds. In addition $G \not\cong H$ holds and hence, G and H are not edge-reconstructible.

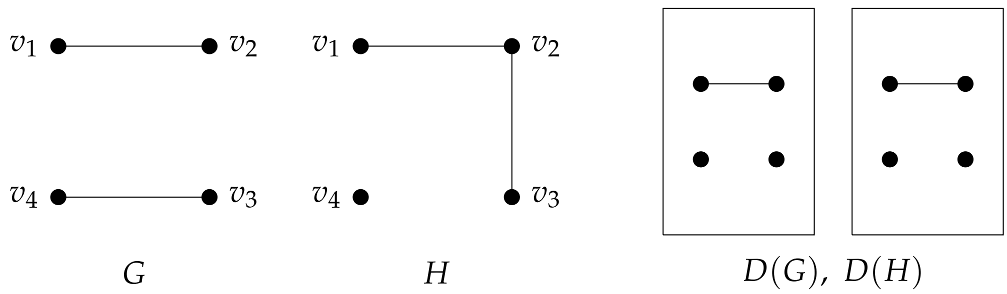


Figure 7: two non isomorphic graphs G and H with the same edge-deck

Another known example with non-isomorphic reconstructions is given in the following example. Now, the number of edges of the graphs is 3.

Example 1.109.
Let $G = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_1v_3, v_2v_3\})$ and $H = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_2v_4\})$ be two graphs. Then H is an edge-reconstruction of G because $D_e(G) = D_e(H)$ holds. In addition $G \not\cong H$ holds and hence, G and H are not edge-reconstructible.

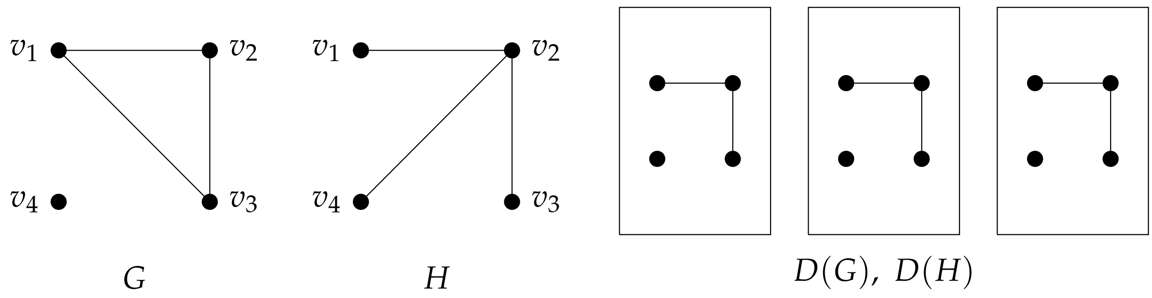


Figure 8: two non isomorphic graphs G and H with the same edge-deck

1.3.1 Structural Properties

Analoguesly to the reconstruction conjecture we give the structural Theorems of the edge-reconstruction conjecture. In structural terms the edge-reconstruction conjecture differs greatly from the reconstruction conjecture. We are no longer able to benefit from the Theorem that states that every graph is reconstructible if and only if its complement is reconstructible. On the other hand the edge-reconstruction conjecture is more open to algebraic methods.

We start this chapter with one of the most fundamental Theorems we have in both reconstruction conjectures: the counting of subgraphs. Analoguesly to Kelly's Lemma for the reconstruction conjecture there exists an edge version of that Lemma.

Theorem 1.110. (Kelly's Lemma, edge version)
Let F and G be two graphs with $m(F) < m(G)$. Then the number $s(F, G)$ of subgraphs of G isomorphic to F is edge-reconstructible.

As a result of Kelly's Lemma we get the following Corollary that may be found in [9]:

Corollary 1.111.
All graphs are edge-reconstructible if all graphs without isolated vertices are edge-reconstructible.

Proof.
Let G be a graph with isolated vertices. The degree sequence is reconstructible by Kelly's Lemma. Hence, the number of isolated vertices is known. Now, we may look at the edge-deck of G and create a subgraph $G' \subsetneq G$ such that G' contains all edges and vertices of G except the isolated vertices. We have $|D_e(G)| = |D_e(G')|$ and $G'_e \subsetneq G_e$ holds for all $e \in E(G)$. Remark that the edge-deck of this subgraph is edge-reconstructible. Now, assume that G' is edge-reconstructible. Then G is edge-reconstructible, too, because G equals the subgraph G' in addition with the isolated vertices. Both the number of isolated vertices and G' are edge-reconstructible. □

Basically we can identify the number of isolated vertices in a graph and then create a subgraph of the original graph without isolated vertices. If the subgraph is edge-reconstructible, then so is the original graph. Hence from now on, we restrict our focus to graphs without isolated vertices.

If a graph has no isolated vertices then its deck may be determined from its edge-deck. This usefull Theorem is due to Greenwell and can be found in his paper "*Reconstructing Graphs*" [29]. So more or less solving the reconstruction conjecture will also solve the edge-reconstruction conjecture.

Theorem 1.112. (Greenwell, 1971)
Let G be a graph with $m(G) \geq 4$ and $\delta(G) \geq 1$. If G is reconstructible, then G is edge-reconstructible.

After listing the result for the reconstruction conjecture in the previous subchapters Greenwell lets us use them for the edge-reconstruction conjecture as well. Hence, in the following chapters we will not list any classes of graphs or graph invariants that are already reconstructible. We will mostly list classes of graphs and graph invariants that are edge-reconstructible but are not known to be reconstructible yet.

Hemminger connects in his paper "*On reconstructing a graph*" [37] the reconstruction conjecture to the edge-reconstruction conjecture in a different way. By the definition of the line graph we are able to determine the deck of the line graph from the edge-deck of the graph itself because the missing edges in the cards of the edge-deck correspond to the missing vertices in the cards of the deck. Therefore, a graph is edge-reconstructible if and only if the line graph is reconstructible and we are able to determine the graph from its line graph.

Theorem 1.113. (Hemminger, 1969)
Let G be a graph. G is edge-reconstructible if and only if its line graph $L(G)$ is reconstructible and not K_3.

The next Theorem, Hoffman's Lemma, aims at finding an excludable configuration. Hoffman introduces the concept of a forced move and concludes that certain forced moves already imply that specific graphs are edge-reconstructible.

Definition 1.114. (forced edge, forced move)
Let G be a graph. An edge $e \in E(G)$ is called a *forced edge* if e is the only edge that can be added to G_e in order to yield an edge reconstruction of G. An ordered pair $e \rightarrow f$ is called a *forced move* if f is the only edge besides e that can be added to G_e in order to yield an edge reconstruction of G. A sequence $e_1 \rightarrow f_1,\ e_2 \rightarrow f_2, ...,\ e_k \rightarrow f_k$ of forced moves is called *conservative* if $\{e_1, e_2, ..., e_k\} = \{f_1, f_2, ..., f_k\}$ holds.

Now, Hoffman shows that if there exists a conservative, odd length sequence of forced moves in a specific graph, then this graph is edge-reconstructible. Hence, he shows that this conservative, odd length sequence is an excludable configuration which is not allowed to occur.

Theorem 1.115. (Hoffman's Lemma)
Let G be a graph and $e_1 \to f_1$, $e_2 \to f_2, ..., e_k \to f_k$ a conservative sequence of forced moves with $k \in \mathbb{N}$ odd. Then G is edge-reconstructible.

Proof.
Let H be an edge-reconstruction of G with $H \ncong G$. $e_1 \to f_1$ is a forced move. Then G and H are the only edge-reconstructions of G and $H \cong [V(G), E(G) \setminus \{e_1\} \cup \{f_1\}]$ holds. In general

$$H \cong [V(G), E(G) \setminus \{e_i\} \cup \{f_i\}]$$
$$G \cong [V(H), E(H) \setminus \{e_j\} \cup \{f_j\}]$$

holds with i odd and j even. Hence, $H \cong [V(G), E(G) \setminus \{e_k\} \cup \{f_k\}]$ holds with k odd. But also $G \cong [V(G), E(G) \setminus \{e_k\} \cup \{f_k\}]$ holds because the sequence of forced moves is conservative. So $G \cong H$ holds, a contradiction. □

Some result were able to be proven with the help of Hoffman's Lemma. For example he did show in cooperation with Myrvold and Ellingham that bidegreed graphs are edge-reconstructible. It is highly recommended to look at their paper "*Bidegreed graphs are edge reconstructible*" [63] for examples of forced moves and excludable configurations. For more information on Hoffman's Lemma see also [9].

The next structural property about the edge-reconstruction conjecture is given by Lovász. In his paper "*A Note on the Line Reconstruction Problem*" [49] he introduced a new concept of counting monomorphisms.

Definition 1.116. (monomorphisms, monomorphisms with defect)
Let G and H be two graphs. Let $\varphi : V(G) \to V(H)$ be $1-1$-mapping. Then φ is called a *monomorphism from G to H*. φ is called a *monomorphism from G to H with defect r* if there exist just r edges $vw \in E(G)$ such that $\varphi(v)\varphi(w) \notin E(H)$ holds. A monomorphism with defect 0 is a monomorphism in the usual sense. $G \to_0 H$ denotes the *set of all monomorphisms of G into H*. Also $G \to_r H$ denotes the *set of all monomorphisms with defect r of G into H*

He uses the inclusion-exclusion principle to count monomorphisms and show that beyond a certain size there will always be an isomorphism between every edge-reconstruction of a certain graph. The following method, introduced by Lovász was later refined by Müller and finally found its pinnacle in Nash-Williams' Lemma which is up to date the most powerful tool that we have in order to tackle the edge-reconstruction conjecture.

Theorem 1.117. (Lovász, 1972)
Let G be a graph of order $n(G)$ and size $m(G)$. If $m(G) > \frac{1}{2}\binom{n(G)}{2}$ holds, then G is edge-reconstructible.

Proof.
Let G and H be two graphs of order $n(G) = n(H)$ and size $m(G) = m(H)$ with $m(G) > \frac{1}{2}\binom{n(G)}{2}$. Furthermore $D_e(G) \cong D_e(H)$ holds that is H is a reconstruction of G.
By the sieve formula we have

$$|G \to_0 H| = \sum_{F \subseteq G} (-1)^{m(F)} |F \to_0 \overline{H}|$$

where F runs over all subgraphs of G with $V(F) = V(G)$ and $E(F) \subseteq E(G)$. With the inclusion-exclusion principle the right hand side maps no edges of G on edges of $\overline{H}$. Hence, all edges of G will be mapped on edges of H.
Now, we may apply the same for

$$|H \to_0 H| = \sum_{F \subseteq H} (-1)^{m(F)} |F \to_0 \overline{H}|.$$

H is a reconstruction of G and therefore has the same edge-deck as G. So

$$\sum_{F \subsetneq G} (-1)^{m(F)} |F \to_0 \overline{H}| = \sum_{F \subsetneq H} (-1)^{m(F)} |F \to_0 \overline{H}|$$

holds. Hence, the sum without the final summand are equal. Furthermore for the last summand holds

$$|G \to_0 \overline{H}| = |H \to_0 \overline{H}| = 0$$

because $m(G) > \frac{1}{2}\binom{n(G)}{2}$ holds. Hence, $|G \to_0 H| = |H \to_0 H|$ holds and with $|H \to_0 H| > 0$ yields the claim. □

An improved bound is given by Schmeichel in his paper "*A note on the edge reconstruction conjecture*" [71]. He copies the proof of Lovász with his idea about the inclusion-exclusion principle but replaces the argument $m(G) > \frac{1}{2}\binom{n(G)}{2}$ by a condition for the degree sequence.

Theorem 1.118. (Schmeichel, 1975)
Let G be a graph with degree sequence

$$(d_1 := d_G(v_1), d_2 := d_G(v_2), ..., d_{n(G)} := d_G(v_{n(G)}))$$

with $v_i \in V(G)$ for $i \in \{1, 2, ..., n(G)\}$ and

$$d_1 \leq d_2 \leq \ ... \ \leq d_{n(G)}.$$

If

$$d_i + d_{n(G)-i+1} \geq n(G)$$

holds for some i, then G is edge-reconstructible.

Proof.
Let G and H be two graphs of order n and size m with the above degree sequence. Furthermore $D_e(G) \cong D_e(H)$ holds that is H is an edge-reconstruction of G.
The proof follows almost exactly the proof of theorem 1.117 by Lovász up to the last argument. But we are not able to conclude that

$$|G \to_0 \overline{H}| = |H \to_0 \overline{H}| = 0$$

holds because we are missing the argument $m(G) > \frac{1}{2}\binom{n(G)}{2}$.
Hence, it remains to be shown, that $|G \to_0 \overline{H}| = |H \to_0 \overline{H}| = 0$ holds.
Consider the graph $\overline{H}$. Then the degree sequence of $\overline{H}$ is $d'_1 \leq d'_2 \leq \ ... \ \leq d'_p$ with $d'_i = (n(G) - 1) - d_{n(G)-i+1}$. Hence, with the condition $d_i + d_{n(G)-i+1} \geq n(G)$ we conclude that $d_i > d'_i$ holds for some i. Hence, there is no monomorphism that maps H onto $\overline{H}$ because the monomorphism maps at least one vertex of bigger degree on a vertex of smaller degree and therefore does not preserve the edges of H. So $|H \to_0 \overline{H}| = 0$ holds and therefore $|G \to_0 \overline{H}|$ holds, too.
Hence, the missing argument of $m(G) > \frac{1}{2}\binom{n(G)}{2}$ for Lovász's proof has been successfully replaced by the argument of the degree sequence from above and therefore concludes this proof. □

Müller refines Lovász's idea about the inclusion-exclusion principle by using not just monomorphisms but also monomorphisms with defect. So in his paper "*The Edge Reconstruction Hypothesis is True for Graphs with More than $n \cdot \log_2 n$ Edges*" [60] he successfully improves the ratio between edges to vertices from $m(G) > \frac{1}{2}\binom{n(G)}{2}$ to the stronger bound $m(G) > n(G) \cdot (\log_2 n(G) - 1)$. Remark that Müller actually did show that graphs with $2^{m(G)-1} > n(G)!$ are edge reconstructible yet he chose to state it for $m(G) > n(G) \cdot (\log_2 n(G) - 1)$. In the proof of Müller's Theorem the reader may also find the stronger bound of $2^{m(G)-1} > n(G)!$.

Theorem 1.119. (Müller, 1977)
Let G be a graph and $m(G) > n(G) \cdot (\log_2 n(G) - 1)$. Then G is edge-reconstructible.

Proof.
Let G be a graph with $m(G) > n(G) \cdot (\log_2 n(G) - 1)$. Let H be a graph of order $n(G)$ and size $m(G)$. Furthermore $D_e(G) \cong D_e(H)$ holds that is H is an edge-reconstruction of G.
This proof will be done by contradiction. Assume $G \ncong H$ that is $|G \rightarrow_0 H| = 0$. For $r = 0, 1, ..., m(G)$ we have by using the inclusion-exclusion principle

$$\begin{aligned}
|G \rightarrow_0 \overline{H}| &= |\emptyset \rightarrow_0 H| - \sum_{\substack{F \subseteq G \\ m(F)=1}} |F \rightarrow_0 H| + \sum_{\substack{F \subseteq G \\ m(F)=2}} |F \rightarrow_0 H| \; ... \\
&\quad ... \; + (-1)^{m(G)} |G \rightarrow_0 H| \\
&\vdots \\
|G \rightarrow_r \overline{H}| &= \sum_{\substack{F \subseteq G \\ m(F)=r}} |F \rightarrow_0 H| - \sum_{\substack{F \subseteq G \\ m(F)=r+1}} \binom{r+1}{r} |F \rightarrow_0 H| \; ... \\
&\quad ... \; + (-1)^{m(G)-r} \binom{m(G)}{r} |G \rightarrow_0 H| \\
&\vdots
\end{aligned}$$

where it is summed over all subgraphs F of G with $V(F) = V(G)$. Analoguesly we get

$$\begin{aligned}
|H \rightarrow_0 \overline{H}| &= |\emptyset \rightarrow_0 H| - \sum_{\substack{F \subseteq H \\ m(F)=1}} |F \rightarrow_0 H| + \sum_{\substack{F \subseteq H \\ m(F)=2}} |F \rightarrow_0 H| \; ... \\
&\quad ... \; + (-1)^{m(G)} |H \rightarrow_0 H| \\
&\vdots \\
|H \rightarrow_r \overline{H}| &= \sum_{\substack{F \subseteq H \\ m(F)=r}} |F \rightarrow_0 H| - \sum_{\substack{F \subseteq H \\ m(F)=r+1}} \binom{r+1}{r} |F \rightarrow_0 H| \; ... \\
&\quad ... \; + (-1)^{m(G)-r} \binom{m(G)}{r} |H \rightarrow_0 H| \\
&\vdots
\end{aligned}$$

All parts of the sums over the monomorphisms from subgraphs of G or H into H have the same value in the respected summands because $D_e(G) \cong D_e(H)$ holds and therefore contain the same subgraphs. The only difference is the last summand, namely

$$\begin{aligned}|H \to_0 \overline{H}| - |G \to_0 \overline{H}| &= (-1)^{m(G)}|H \to_0 H| \\ &\vdots \\ |H \to_r \overline{H}| - |G \to_r \overline{H}| &= (-1)^{m(G)-r}\binom{m(G)}{r}|H \to_0 H| \\ &\vdots\end{aligned}$$

Hence, the sum over the differences over all monomorphisms with certain defects yields

$$\sum_{r=0}^{m(G)} |H \to_r \overline{H}| - |G \to_r \overline{H}| = 2^{m(G)} \cdot |H \to_0 H|$$

So we have

$$\begin{aligned} 2^{m(G)} &\le 2^{m(G)}|H \to_0 H| \\ &= \sum_{r=0}^{m(G)} |H \to_r \overline{H}| - |G \to_r \overline{H}| \\ &\le 2 \cdot n(G)! \\ &< 2 \cdot \left(\frac{n(G)}{2}\right)^{n(G)} \\ \Leftrightarrow \qquad m(G) &< \log_2 \left(2 \cdot \left(\frac{n(G)}{2}\right)^{n(G)}\right) \\ &= \log_2 2 + n(G) \cdot \log_2 \left(\frac{n(G)}{2}\right) \\ &= 1 + n(G) \cdot (\log_2 n(G) - 1). \end{aligned}$$

This contradicts that $m(G) > n(G) \cdot (\log_2 n(G) - 1)$ holds. Remark that the Theorem is also shown for the condition $2^{m(G)-1} > n(G)!$ by the second last step of the first inequality. □

The method which was first introduced by Lovász found its pinnacle by Nash-Williams. Nash-Williams contributed the chapter *"The Reconstruction Problem"* to the book *"Selected Topics in Graph Theory"* [64] and in it he proved that a graph which may not be edge-reconstructible needs to meet a very restrictive condition.

Theorem 1.120. (Nash-Williams' Lemma, 1978)
Let G be a non edge-reconstructible, spanning subgraph of $K_{n(G)}$. Then for every subset E' of $E(G)$ such that $|E'| \equiv m(G)$ *modulo* 2 holds, there exists an automorphism φ of $K_{n(G)}$ such that $E(G \cap \varphi(G)) = E'$.

Up to date Nash-Williams' Lemma is still the most powerful structural Theorem for the edge-reconstruction conjecture that has been proven. After Nash-Williams published his Theorem many structural results were proven. First with Nash-Williams' Lemma at hand we can rewrite the proof of Theorem 1.119 by Müller and show that it is now a corollary to Nash-Williams' Lemma.

Proof. (of Theorem 1.119)
Assume that $2^{m(G)-1} > n(G)!$ holds with G is a spanning subgraph of $K_{n(G)}$. Then $m(G) > 0$ and hence $E(G)$ has $2^{m(G)-1}$ subsets E' such that $|E'| \equiv m(G)$ *modulo* 2 holds. However $K_{n(G)}$ has at most $n(G)!$ automorphisms on its $n(G)$ vertices. Since $n(G)! < 2^{m(G)-1}$ holds, it can not be true that, for every $E' \subseteq E(G)$ such that $|E'| \equiv m(G)$ *modulo* 2, there exists an automorphism φ of $K_{n(G)}$ such that $E(G \cap \pi(G)) = E'$ holds. So this yields a contradiction to Nash-Williams' Lemma and therefore G is edge-reconstructible. □

Due to Hoffman [63] we have a restrictive condition for non edge-reconstructible graphs to their average degree and minimal degree ratio.

Corollary 1.121. (Hoffman, 1987)
Let G be a non edge-reconstructible graph. Then $d \geq \delta(G) + 1 - \frac{1}{\delta(G)+1}$ holds, where d is the average degree of G.

Caunter and Nash-Williams [13] showed the following restrictive condition with regard to the ratio between average degree and maximal degree.

Corollary 1.122. (Caunter and Nash-Williams, 1982)
Let G be a connected, non edge-reconstructible graph. Then $2\log_2(2\Delta(G)) > d$ holds, where d is the average degree of G.

For a different and more algebraic approach to Nash-Williams' Lemma the reader is encouraged to look at Bondy's survey *"A Graph Reconstructor's Manual"* [9]. The following chapter contain many classes of graphs for which the proofs rely heavily on Nash-Williams' Lemma.

1.3.2 Edge-Reconstructible Graph Classes

We start this chapter by reminding the reader of Theorem 1.112 by Greenwell which more or less states that graphs that are reconstructible are also edge-reconstructible. Therefore, we do not necessarily give information on the edge-reconstruction of certain graphs which have already been shown to be reconstructible.

The first results are the known results for the class of planar graphs. While we are still not able to reconstruct the class of planar graphs for both the reconstruction conjecture and edge-reconstruction conjecture we have a wide range of results for specific subclasses of the class of planar graphs.

Fiorini was able to show in his paper "*On the edge-reconstruction of planar graphs*" [19] that maximal planar graphs with minimal degree 4 are edge-reconstructible. Remark that Theorem 1.70 by Lauri (1981) states that maximal planar graphs are reconstructible and therefore are edge-reconstructible, too. The result is given for additional information on how to reconstruct graphs from the edge-deck.

Theorem 1.123. (Fiorini, 1978)
Maximal planar graphs G with $\delta(G) \geq 4$ are edge-reconstructible.

In the same paper Fiorini proved that 4-connected planar graphs with minimal degree 5 are edge-reconstructible.

Theorem 1.124. (Fiorini, 1978)
4-connected planar graphs G with $\delta(G) \geq 5$ are edge-reconstructible.

In 1982 Fiorini and Lauri were able to show in their paper "*Edge-Reconstruction of 4-Connected Planar Graphs*" [22] that 4-connected planar graphs with minimal degree 4 are edge reconstructible. This is a relaxation of the degree condition given in the previous result and therefore improving the minimal degree from 5 to 4.

Theorem 1.125. (Fiorini and Lauri, 1982)
4-connected planar graphs with $\delta(G) \geq 4$ are edge-reconstructible.

The last result for planar graphs is of a more general nature and only holds restrictions to its minimal degree. Lauri was able to show in his paper "*Edge-reconstruction of planar graphs with minimum valency* 5" [44] that planar graphs with minimal degree of at least 5 are edge-reconstructible.

Theorem 1.126. (Lauri, 1979)
Planar graphs G with $\delta(G) \geq 5$ are edge-reconstructible.

Another result by Fiorini and Lauri is the edge-reconstruction of 3-connected graphs that triangulate a surface. Heavy usage is made of the topology of graphs that are embedded in the plain or the real projective plain. For more information see their paper "*On the edge-reconstruction of graphs which triangulate surfaces*" [23].

Theorem 1.127. (Fiorini and Lauri, 1982)
3-connected graphs that triangulate a surface are edge-reconstructible.

An application of Hoffman's Lemma and its theory of forced moves and excludable configurations is to show that bidegreed graphs are edge-reconstructible. A *bidegreed graph* is a graph that has only two different degrees in its degree sequence. Hence, all vertices are of either minimal or maximal degree. Ellingham, Hoffman and Myrvold [63] showed in 1987 that bidegreed graphs are edge-reconstructible.

Theorem 1.128. (Ellingham, Hoffman and Myrvold, 1987)
Bidegreed graphs are edge-reconstructible.

The actual problem to show that bidegreed graphs are reconstructible lies in the single subclass where all vertices are of either minimal degree $\delta(G)$ or of the maximal degree $\Delta(G) = \delta(G) + 1$. Hence, we have two consecutive degrees. If the minimal degree and maximal degree are not consecutive integers then there exists a card where the degrees of exactly two vertices do not belong to the degree sequence G. Hence, the graph is edge-reconstructible. The proof where minimal degree and maximal degree are consecutive integers is a long case by case study. As a generalization of the last Theorem Ellingham, Hoffman and Myrvold have shown the following:

Theorem 1.129. (Ellingham, Hoffman and Myrvold, 1987)
All graphs that do not have three consecutive integers in their degree sequence are edge-reconstructible.

The next step would be to show that graphs with three consecutive degrees are edge-reconstructible. Ellingham, Hoffman and Myrvold implied that the proof of tridegreed graphs might suggest an induction argument to prove the edge-reconstruction conjecture on the whole.

Pyber showed in his paper "*The edge reconstruction of Hamiltonian graphs*" [65] that hamiltonian graphs of a sufficiently high order are edge-reconstructible. He uses Nash-Williams' Lemma to show that if a hamiltonian graph is not edge-reconstructible, then that graph needs to have a high number of hamiltonian cycles. He is then able to prove, that for hamiltonian graphs of sufficiently high order this number of hamiltonian cycles can not be met and therefore yields a contradiction to Nash-Williams' Lemma. His method works for hamiltonian graphs of order 240 or higher.

Theorem 1.130. (Pyber, 1990)
Hamiltonian graphs of sufficiently high order are edge-reconstructible.

Analoguesly to the reconstruction conjecture we may give a bound on the order-size ratio of hamiltonian graphs in the edge-reconstruction conjecture. Due to the nature of the edge-reconstruction conjecture we are able to improve that bound. First we need the following general Theorem which is a again an improved version of Theorem 1.64. In the edge-reconstruction conjecture two adjacent vertices of degree 2 are sufficient to prove that a 2-connected graph is edge-recosntructible.

Theorem 1.131.
Let G be a 2-connected graph. If G contains two vertices $v, w \in V(G)$ with $vw \in E(G)$ and $d_G(v) = d_G(w) = 2$, then G is edge-reconstructible.

Proof.
2-connected graphs are recognizable. Let G be 2-connected. Let $v, w \in V(G)$ be two vertices with $vw \in E(G)$ and $d_G(v) = d_G(w) = 2$. For all vertices $u \in V(G)$ holds $d_G(u) \geq 2$. Consider G_{vw}. G_{vw} contains exactly 2 vertices of degree 1, namely v and w. Hence, G_{vw} with the edge vw that is $(V, E \setminus \{vw\} \cup \{vw\})$ is the only reconstruction of G. So G is edge-reconstructible. □

Now, we can use the class of 2-connected graphs that contain a path with two consecutive vertices of degree 2 to show that hamiltonian graphs which exceed a specific order-size ratio are edge-reconstructible. This improves the bound of Theorem 1.65. The bound for the reconstruction of hamiltonian graphs was $m(G) \leq n(G) + \left\lfloor \frac{n(G)-1}{6} \right\rfloor$.

Theorem 1.132.
Let G be a hamiltonian graph with $m(G) \leq n(G) + \left\lfloor \frac{n(G)-1}{4} \right\rfloor$. Then G is edge-reconstructible.

Proof.
Since G is hamiltonian, G has a circle of lenght $n(G)$. All other edges are chords of that circle. So the hamiltonian circle has at most $\left\lfloor \frac{n(G)-1}{4} \right\rfloor$ chords. So there are at least $n(G) - 2 \cdot \left\lfloor \frac{n(G)-1}{4} \right\rfloor$ vertices of G of degree 2 and at most $2 \cdot \left\lfloor \frac{n(G)-1}{4} \right\rfloor$ vertices of degree at least three. Partitioning the vertices of degree two evenly into connected paths of vertices of degree two, seperated by vertices of degree at least three will yield

$$
\begin{aligned}
\frac{n(G) - 2 \cdot \left\lfloor \frac{n(G)-1}{4} \right\rfloor}{2 \cdot \left\lfloor \frac{n(G)-1}{4} \right\rfloor} &= \frac{n(G)}{2 \cdot \left\lfloor \frac{n(G)-1}{4} \right\rfloor} - 1 \\
&\geq \frac{n(G)}{2 \cdot \frac{n(G)-1}{4}} - 1 \\
&= \frac{4}{2} \cdot \frac{n(G)}{n(G)-1} - 1 \\
&= 2 \cdot \frac{n(G)}{n(G)-1} - 1 \\
&> 1,
\end{aligned}
$$

if G is a finite graph. Hence, there is at least one path with two consecutive vertices of degree two (pidgeon hole principle). Furthermore G is hamiltonian and hence 2-connected. So by Theorem 1.131 G is weakly reconstructible. The size of G and the property hamiltonian are recognizable, so G is edge-reconstructible. □

Remark, that in the last proof, every graph that is too small (e.g. less than 10 vertices) is reconstructible by a computer study of McKay (see Theorem 1.103).

A long case by case study by Thatte in 1995 proved that p-claw-free graphs are edge-reconstructible.

Definition 1.133. (claw, claw-point, claw-free, p-claw-free)
A *claw* is a bipartite graph $K_{1,3} := (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_1v_3, v_1v_4\})$. The *claw-point* is the vertex of degree 3 in a claw. A *claw-free graph* is a graph that does not contain a claw as an induced subgraph. A *p-claw-free graph* is a graph that has no circuit of length p with a claw-point on it.

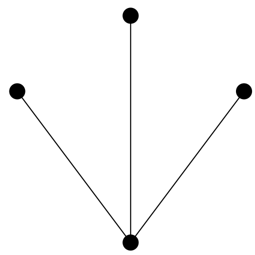


Figure 9: A claw - the bipartite graph $K_{1,3}$

Thatte was able to show in his paper "*Some Results on the Reconstruction Problems. p-Claw-Free, Chordal, and P_4-Reducible graphs*" [75] that p-claw-free graphs are edge reconstructible by applying Nash-Williams' Lemma and a range of excludable configurations.

Theorem 1.134. (Thatte, 1995)
p-claw-free graphs are edge-reconstructible.

The generalisation of p-claw-free graphs, namely the claw-free graphs was first proven by Ellingham, Pyber and Xingxing Yu in 1988. In their paper "*Claw-free graphs are edge reconstructible*" [18] Ellingham, Pyber and Xingxing Yu did prove that claw-free graphs are edge-reconstructible. The proof exploits Nash-Williams' Lemma. Remark that a claw-free graph is p-claw-free for every p. Thatte did prove that claw-free graphs are edge-reconstructible by showing that both p-claw-free graphs and chordal graphs are edge-reconstructible.

Theorem 1.135. (Ellingham, Pyber and Xingxing Yu, 1988 - Thatte, 1995)
Claw-free graphs are edge-reconstructible.

Line graphs are a special case of claw-free graphs. Therefore, Ellingham, Pyber and Xingxing Yu also showed that line-graphs are edge-reconstructible.

Corollary 1.136. (Ellingham, Pyber and Xingxing Yu, 1988)
Line-graphs are edge-reconstructible.

Next we are going to show that chordal graphs are edge-reconstructible. The proof is another example of the two step method in which you first show that a class of graphs is edge-recognizable and then weakly edge-reconstructible. For the proof we use a different verion of Nash-Williams' Lemma as given in [75]. It introduces the concept of an edge-replacing set.

Definition 1.137. (replacing edge, edge-replacing set)
Let G be an induced subgraph of $K_{n(G)}$ and $E' \subseteq E(G)$. Then $E'' \subseteq E(K_{n(G)})$ is called an *edge-replacing set of* E' if $E(G) \cap E'' = \emptyset$ and $D_e(G - E' + E'') = D_e(G)$ holds. A collection of all edge-replacing sets of E' is denoted by $rep(E')$. If $|E'| = 1$ holds, then we call E'' also a *replacing edge*.

Nash-Williams' Lemma states now that if the graph G is not edge-reconstructible, then for each subset of the edges of G there exists an edge-replacing set such that replacing those edges will yield either a reconstruction isomorphic to G or a reconstruction not isomorphic to G depending on the size of the edge-replacing set.

Theorem 1.138. (Nash-Williams' Lemma, 1978)
Let G and H be two graphs with $D_e(G) \cong D_e(H)$ but $G \ncong H$. Then there exists for all $E' \subseteq E(G)$ a set $E'' \subseteq rep(E')$ such that either $G - E' + E'' \cong G$ or $G - E' + E'' \cong H$ depending respectively upon wether $|E'|$ is even or odd.

With the help of Nash-Williams' Lemma Thatte was able to prove in his paper *"Some Results on the Reconstruction Problems. p-Claw-Free, Chordal, and P_4-Reducible graphs"* [75] in a very short way that chordal graphs are edge-reconstructible. He basicly looks at two cases, one of which are outerplanar graphs and the other is solvable by Nash-Williams' Lemma. Remark for the proof it is of no interest that the edge-replacing set has a specific size. Since we use mainly a proof by contradiction to Nash-Williams' Lemma it is sufficient to argue that there exists no edge-replacing set.

Theorem 1.139. (Thatte, 1995)
Chordal graphs are edge-reconstructible.

Proof.
edge-recognizable: Chordal graphs are recognizable by Theorem 1.81 from their deck. The deck is reconstructible from its edge-deck according to Theorem 1.112. Hence, chordal graphs are edge-recognizable.
weakly edge-reconstructible: Assume that chordal graphs are not edge-reconstructible. Let G be a chordal graph.
Assume that G contains a K_4 on the vertices v_1, v_2, v_3, v_4. Then replacing the edges

$E' = \{v_1v_2, v_3v_4\} \subseteq E(G)$ by $E'' \in rep(E(G))$ with Nash-Williams' Lemma yields a chordless cycle v_1, v_2, v_3, v_4, v_1 of length 4. A contradiction to G being chordal and edge-recognizable. Hence, G does not contain a K_4 as an induced subgraph.
Assume now that there exists an edge $v_1v_2 \in E(G)$ such that v_1v_2 is an edge of at least three K_3. Denote them with v_1, v_2, v_3, v_1, v_2, v_4 and v_1, v_2, v_5. Remark that there exists no edges v_3v_4, v_3v_5 and v_4v_5. Otherwise G would have a K_4 as an induced subgraph. Then there exists no replacing edge for v_1v_2 which is a contradiction to Nash-Williams' Lemma. Therefore, G does not contain an edge that is on at least three K_3. So G is outerplanar.
Outerplanar graphs are reconstructible by Theorem 1.72. Therefore outerplanar graphs are also edge-reconstructible by Theorem 1.112.
Hence, chordal graphs are edge-reconstructible because they are edge-recognizable and weakly edge-reconstructible. □

1.3.3 Edge-Reconstructible Graph Invariants

There is nothing to report on the matter of edge-reconstructible graph invariants that haven't already been proven to be reconstructible. Therefore, refer to the section about reconstructible graph invariants and use Theorem 1.112 to prove that these invariants are also edge-reconstructible graph invariants.

1.4 Reconstruction Numbers and related Reconstruction Conjectures

This subchapter deals with questions related to the reconstruction conjecture. The first part about reconstruction numbers is about the question of how many cards are needed to show that a graph is reconstructible. The second part deals with partial decks of the original deck of a graph and modified decks. The last part is about the reconstruction conjecture in directed graphs.

1.4.1 Reconstruction Numbers

In 1985 Harary and Plantholt introduced in their paper "*The Graph Reconstruction Number*" [35] the concept of reconstruction numbers. This concept is actually a stronger statement than the reconstruction conjecture itself. We start with their definition:

Definition 1.140. (reconstruction number)
Let G be a graph. The *reconstruction number of* G is the minimal k for which there are $\{v_1, v_2, ..., v_k\} \subseteq V(G)$ such that if H is a graph with $\{w_1, w_2, ..., w_k\} \subseteq V(H)$ and $G_{v_i} \cong H_{w_i}$ for all $i \in \{1, 2, ..., k\}$, then $H \cong G$ holds. Denote that number with $rn(G)$.

In other words, the reconstruction number of a graph G is the minimal number of cards required in order to uniquely identify the original graph G. Hence if a reconstruction number exists for a graph or a class of graphs then this graph or class of graphs is reconstructible. This is because just a partial subdeck of the original deck is required to show that the graph itself is reconstructible.

There are some classes of graphs for which the reconstruction number has been shown. For example Myrvold proved in her paper "*The Ally-Reconstruction Number of a Tree with Five or More Vertices Is Three*" [61] the reconstruction number for trees. In particular Myrvold showed the result for the *ally-reconstruction number* which may be distinguished from the *adversary-reconstruction number*. The difference between the ally-reconstruction number and the adversary-reconstruction number is simply a best-case and worst-case scenario. Hence we look for the minimal and maximal number of cards required to uniquely identify the original graph. Therefore the ally-reconstruction number is the same as the originally introduced reconstruction number by Harary and Plantholt. So for this thesis we continue the use of the term reconstruction number.

Theorem 1.141. (Myrvold, 1990)
Let G be a tree with $n(G) \geq 5$. Then $rn(G) = 3$.

For further information about reconstruction numbers the reader is referred to [2, 35, 56, 62]. Those will give a good overview and a good basis for getting into this field.

However there is one result for reconstruction numbers that holds particular interest. Harary and Plantholt conjectured in their paper "*The Graph Reconstruction Number*" [35] that in a probabilistic sense almost all graphs have reconstruction number equal to three. Bollobás picked up that idea in his paper "*Almost Every Graph has Reconstruction Number Three*" [7] and proved the conjecture by using random graphs.

Theorem 1.142. (Bollobás, 1990)
Almost every graph has reconstruction number three.

This is a stronger result than Theorem 1.46 given by Müller in 1976. From this point of view Müller is a corollary from Bollobás since having a reconstruction number for a class already implies that this class is reconstructible.

Interesting about Bollobás's result is that a computer study by McMullen and Radziszowski [56] showed that of the more than 12 million graphs on ten vertices only 12 graphs have reconstruction number greater than 3. That means finding a class with a higher reconstruction number than 3 might already be an interesting and difficult task. The authors themselves introduced some of those classes.

Moving on from the vertex induced reconstruction numbers there exists analogously to the edge-reconstruction conjecture the concept of the edge reconstruction number, both in its ally and adversary form. For a survey of open questions about the edge reconstruction number refer for example to [2] or one result is given, for example, in [3].

1.4.2 The Isomorphism-Reduced Deck and Related Decks

There are several reconstruction conjectures out there. Some work on different structures and some are weaker or stronger versions on the same structures as the reconstruction conjecture itself. When reconstructing graphs we have all unlabelled subgraphs as cards at our disposal. Yet a stronger version might not have all subgraphs available, but still needs to show that all reconstructions are isomorph to each other. For that there is for example the isomorphism-reduced deck and isomorphism-reduced edge-deck. Those decks contain only one card of each isomorphism type.

Definition 1.143. (isomorphism-reduced deck, isomorphism-reduced edge-deck)
Let G be a graph. $ID(G) \subseteq D(G)$ and $ID_e(G) \subseteq D_e(G)$ are maximal subsets of the deck of G or edge-deck of G such that each isomorphism type of a card of G is included exactly once. Then $ID(G)$ is called the *isomorphism-reduced deck of G* and $ID(G)$ is called the *isomorphism-reduced edge-deck of G*.

A stronger version of the reconstruction conjecture or edge-reconstruction conjecture would be if one would not have its deck or edge-deck at their disposal but only isomorphism-reduced deck or isomorphism-reduced edge-deck. The related conjecture would then be defined as:

Conjecture 1.144.
All finite simple undirected graphs with at least three vertices are reconstructible from their isomorphism-reduced deck.

There are some results that have been proven for this stronger conjecture. For example Theorem 1.103 by McKay was a computer study about the reconstruction conjecture. In fact he did show that all graphs with at least 11 vertices are not just reconstructible by their deck, but they are already reconstructible by their isomorphism-reduced deck. For more information about that computer study see [53].

Similary the stronger version of the edge-reconstuction conjecture would be:

Conjecture 1.145.
All finite simple undirected graphs with at least four edges are edge-reconstructible from their isomorphism-reduced edge-deck.

Other decks would be *shuffled decks*, *k-decks* or *vertex-switching decks*. For more information, the definitions and some results see the "*Graph Reconstructor's Manual*" [9] by Bondy. There is also the question about *legitimate decks*. The question for legitimate decks is, if to any given deck there exists at least one reconstruction or one at all. For more information see "*Graph Reconstruction - A Survey*" [10] by Bondy and Hemminger.

1.4.3 The Reconstruction Conjecture for Digraphs

A reconstruction conjecture related to our reconstruction conjecture but with a different set of structures is the reconstruction conjecture for digraphs. A digraph is a graph where each edge has a direction. Hence for two vertices $v, w \in V(G)$ would $vw \neq wv$ hold. The definition reads as follows:

Definition 1.146. (digraph, arc)
Let V and E be a pair of disjoint sets. Then $G = (V, E)$ is called a *digraph* if $E \subseteq \{vw \mid v, w \in V\}$ holds. The elements in $V(G) := V$ are called *vertices* and form the *vertex set of* G. The elements of $E(G) := E$ are called *arcs* and form the *edge set of* G. Furthermore for every arc $e = vw \in E(G)$ between two vertices $v, w \in V(G)$ holds $vw \neq wv$ where $vw \in E(G)$ indicates an arc from v to w and $wv \in E(G)$ indicates an arc from w to v.

The cards, decks, reconstructions and the term reconstructible for digraphs might be defined analogously to those of undirected graphs. Hence, a related conjecture might be proposed as followed:

Conjecture 1.147. (Reconstruction Conjecture for Digraphs)
All finite simple digraphs with at least three vertices are reconstructible.

Unfortunately this conjecture has been proven to be false. Beineke and Parker [4] found counterexamples to the reconstruction conjecture for digraphs on 5 and 6 vertices. Stockmeyer [72–74] improved that result by finding other small counterexamples and then discovered a general proof that an infinite number of *tournaments* (a digraph that has exactly one arc between each pair of vertices) are non-reconstructible. In fact he did show that there are non-reconstructible pairs of tournaments for every order $2^i + 2^j$ with i and j not both equal to zero. For this and more information about the reconstruction conjecture for digraphs see also [10].

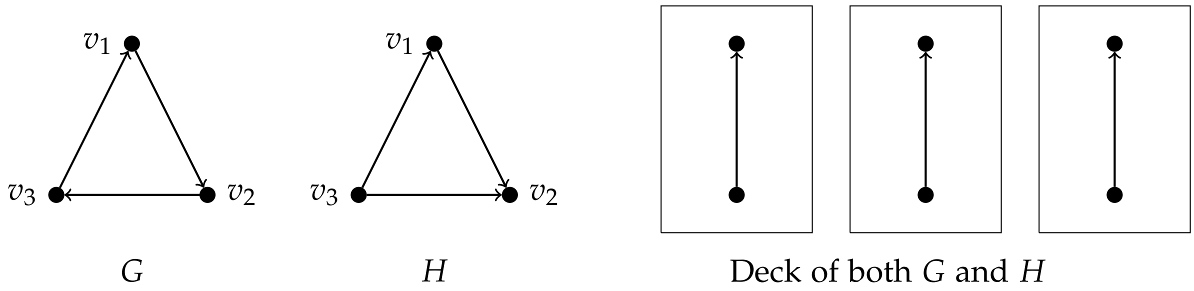


Figure 10: two non isomorphic tournaments G and H with the same deck

One example of non-reconstructible digraphs on 3 vertices are given in Figure 10.

However some positive progress has been made for the reconstruction conjecture for digraphs. There are some graph classes that are reconstructible despite the fact that the reconstruction conjecture for digraphs is false in general. One example is given in "*On the problem of reconstructing a tournament from subtournaments*" [34] by Harary and Palmer. They did prove that *nondiconnected tournaments* (in diconnected tournaments there are for every two vertices v, w oriented paths from v to w and from w to v) on at least 5 vertices are reconstructible.

Theorem 1.148. (Harary and Palmer, 1967)
Nondiconnected tournaments on at least 5 vertices are reconstructible.

2 Reconstructing Separable Graphs

In this chapter we want to pick up an idea by Bondy about the sizes of the limbs of separable graphs with end vertices and improve it by delving deeper into the structure of the subgraphs.

As stated earlier, Bondy did show in his paper *"On Ulam's Conjecture for Separable Graphs"* [8] that separable graphs without end vertices are reconstructible. That are seperable graphs with minimal degree greater or equal to 2.

Theorem 2.1. (Bondy, 1969)
Separable graphs without end vertices are reconstructible.

Bondy gave some results for separable graphs with end vertices, too. For that he split the separable graphs with end vertices into different substructures, namely the trunk and the limbs.

Definition 2.2. (trunk, limb, root)
Let G be a graph. The *trunk* $T(G)$ of a graph G is the induced subgraph of G, remaining after successively removing all end vertices until none remain.
A *limb* $L(G)$ of G is a nontrivial maximal connected subgraph of G having just one vertex in common with $T(G)$. This vertex is called the *root* of $L(G)$, denoted by $r(L(G))$.

He then did prove that every reconstruction of a separable graph has the same trunk and the same number and kind of limbs up to isomorphism.

Theorem 2.3. (Bondy, 1969)
Let G be a separable graph with end vertices. The trunk $T(G)$ and all limbs $L(G)$ are reconstructible.

Proof.
By definition the trunk of a graph is the induced subgraph of G, remaining after successively removing all end vertices until none remain. Hence, choose a card G_v with $d_G(v) = 1$. Now, $T(G_v) = T(G)$ holds. So the trunk is reconstructible.
The limbs are defined as the nontrivial maximal connected subgraphs of G having just one vertex in common with $T(G)$. Let $L_1(G), ..., L_k(G)$ be the limbs of G with $k \geq 1$.
$k = 1$ and $|L_1(G)| = 2$ holds if and only if $D(G)$ contains exactly one card G_v such that $G_v = T(G)$ holds. Hence, this kind of graph is recognizable and this limb is reconstructible.
$k = 1$ and $|L_1(G)| \geq 3$ holds if and only if $D(G)$ contains exactly one disconnected card G_v such that one component of G_v is isomorphic to $T(G)$ and the other one is an isolated vertex. Hence, this kind of graph is recognizable. The structure of the

limb itself is now reconstructible with Kelly's Lemma and the fact, that the trunk is reconstructible.
Now, let $k \geq 2$ hold. Consider the subdeck $SD(G) \subseteq D(G)$ that contains all cards G_v of G with $d_G(v) = 1$. Hence, it contains the cards where end vertices of limbs are missing. Suppose that a limb $L_i(G)$ has j end vertices. Then $L_i(G)$ is an induced subgraph in at least $|SD(G)| - j$ cards of $SD(G)$ where $L_i(G)$ does not share more than the root with the trunk. Ordering the limbs by size and the help of Kelly's Lemma allows us to count the number of limbs, their sizes and structures. So the number of limbs and the isomorphism class of the limbs are reconstructible. □

Let G be a separable graph with end vertices and G_v a card with v is a root of a limb of G. A natural question that arises is if we can complete the trunk of G and attach all missing limbs with that root and therefore reconstruct G. The answer to that question is somehow disappointing. For this problem we can define the following kind of vertices.

Definition 2.4. (similar, pseudo-similar)
Let G be a graph and $v, w \in V(G)$. The vertices v and w are called *similar* if there exists an automorphism, that sends one vertex onto the other. The vertices v and w are called *pseudo-similar* if $G - v \cong G - w$ holds.

Now, it is clear that if two vertices are similar, then both vertices are pseudo-similar, too. However, Harary and Palmer [33] did show that the converse is not true. The following example shows a graph with two pseudo-similar vertices v and w, which are not similar.

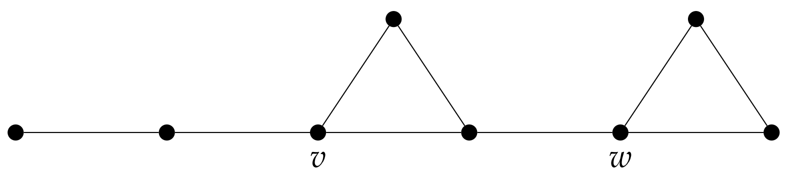


Figure 11: Counter example by Harary and Palmer

Harary and Palmer also presented in their paper "*On similar points of a graph*" [33] an infinite class of graphs for which there are pseudo-similar vertices that are not similar. Hence, this problem does not just arise for separable graphs with end vertices but it may arise in other classes of graphs as well.

Bondy did show that separable graphs with end vertices are reconstructible if the trunk of those graphs meets a very special condition. For example, we have seen in Theorem 1.61 that a separable graph with end vertices is reconstructible if the trunk is a complete graph. There are also other conditions for the trunk under which a separable graph with end vertices is reconstructible.

On the other hand, Bondy [8] gave also a condition for the limbs of a separable graph with end vertices. This is the condition that we improve.

Theorem 2.5. (Bondy, 1969)
Let G be a separable graph with end vertices and $L_1(G), ..., L_k(G)$ be the limbs of G. Let $L_i(G)$ with $i \in \{1, ..., k\}$ have at least 3 vertices. If there exists no limb $L_j(G)$, $j \in \{1, ..., k\} \setminus \{i\}$ such that

$$|V(L_j(G))| = |V(L_i(G))| - 1$$

holds, then G is reconstructible.

Proof.
Let $L_1(G), ..., L_k(G)$ be the limbs of G. Let $L_i(G)$, $1 \leq i \leq k$ be a limb on at least 3 vertices such that there exists no limb $L_j(G)$, $1 \leq j \leq k$ and $j \neq i$ such that $|V(L_j(G))| = |V(L_i(G))| - 1$ holds. Then there exists a card G_v with $d_G(v) = 1$ such that G_v contains all limbs except $L_i(G)$. Furthermore G_v contains a limb of size $|V(L_i(G))| - 1$. A limb of such size does not exist in G. Hence, this limb in G_v needs to be extended to $L_i(G)$ in order to yield a reconstruction of G. But then every reconstruction of G is isomorph to G and therefore G is reconstructible. □

Remark that there might be multiple ways to yield a limb $L_i(G)$ from its sublimb $L_i(G) \setminus \{v\}$. However, all those extended limbs with the fixed root $r(L_1(G))$ are isomorph to each other and hence, all reconstructions are isomorphic. Remark further that applying this theorem consequtively to the largest limb and then going to the smaller limbs yields for a non-reconstructible graph that it has limbs of all sizes up to the size of the largest limb.

Example 2.6.
Let G be a separable graph with end vertices. G contains two limbs $L_1(G)$ and $L_2(G)$ of sizes 5 and 2 respectively (see Figure 12). The graph is reconstructible by Theorem 2.5. If we picture the card G_v, then $T(G_v) = T(G)$ holds and $L_2(G)$ is both a limb in G and G_v. However, $L_1(G) \setminus \{v\}$ is not a limb in G. By Theorem 2.3, all limbs and the trunk of G are reconstructible. The only two ways to extend $L_1(G) \setminus \{v\}$ to $L_1(G)$ is to connect the missing vertex v in G_v to either v_1 or v_2. Hence, both ways yield a reconstruction of G from the card G_v. Furthermore those reconstructions are isomorphic and $G_1 \cong G_2$ holds with $G_1 := (V(G), E(G))$ and $G_2 := (V(G), (E(G) \setminus \{vv_1\}) \cup \{vv_2\})$.

The idea given is that a card G_v contains a subgraph, in our case a limb, which is not in G. Hence, this limb needs to be extended in order to yield a subgraph that is in G. In this case if the limbs are reconstructible, then a card with a limb too small gives us the unique sublimb that needs to be extended into the limb of the original graph. This

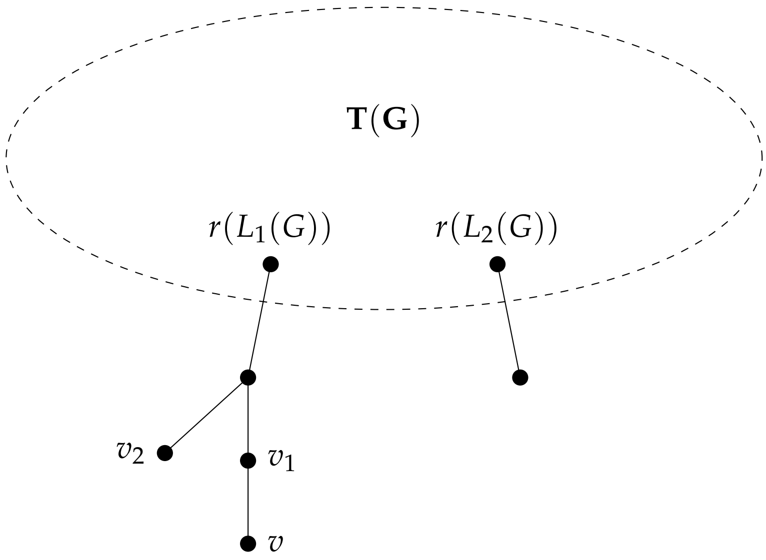


Figure 12: Separable graph with end vertices with limbs of sizes 2 to 5

idea may be refined. By generalizing this idea we are able to extend the subclasses of separable graphs with end vertices that are reconstructible.

Up to now we only looked at the size of the sublimbs. But we can also look deeper into the structure of the limbs. For our first case, there may be limbs that are exactly of order one less than the limb we are looking at. Yet they might be of a different isomorphism type thus making the original limb uniquely extendable into the original graph.

Theorem 2.7.
Let G be a separable graph with end vertices and limbs $L_1(G), ..., L_k(G)$. Let $L_1(G)$ be a limb of order at least 3 and $v \in V(L_1(G))$ be an end vertex of G. If there exists no limb $L_i(G)$, $i \in \{2, ..., k\}$ such that there is an isomorphism

$$\varphi : L_i(G) \to L_1(G) \setminus \{v\}$$

with

$$\varphi(r(L_i(G))) = r(L_1(G) \setminus \{v\}),$$

then G is reconstructible.

Proof.
Let $L_1(G), ..., L_k(G)$ be the limbs of G. Let $L_1(G)$ be a limb on at least 3 vertices and and $v \in V(L_1(G))$ be an end vertex of G. Assume that there exists no limb $L_i(G)$, $2 \leq i \leq k$ such that there is an isomorphism $\varphi : L_i(G) \to L_1(G)$ with $\varphi(r(L_i(G))) = r(L_1(G) \setminus \{v\})$. Then the card G_v is identifiable as a card, that contains a sublimb of $L_1(G)$ namely $L_1(G) \setminus \{v\}$ which is no limb of G. Furthermore, $T(G) \cong T(G_v)$

holds and G_v contains all limbs $L_2(G), ..., L_k(G)$. Hence, $L_1(G) \setminus \{v\}$ in G_v needs to be extended with v in order to yield a reconstruction of G. But then every reconstruction of G is isomorph to G and therefore G is reconstructible. □

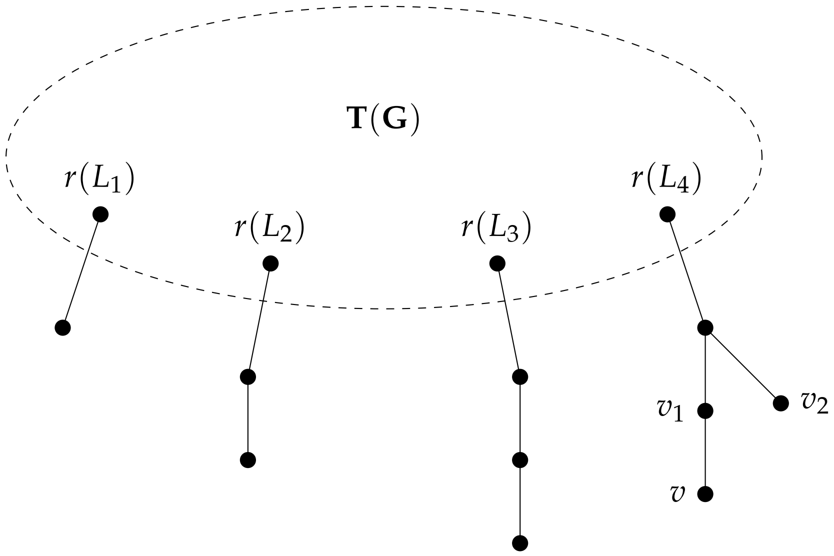


Figure 13: Separable graph with end vertices with limbs of sizes 1 to 4

Example 2.8.
Let G be a separable graph with end vertices. G contains four limbs $L_1 := L_1(G), ...,$ $L_4 := L_4(G)$ of sizes 1 to 4 (see Figure 13). We may not prove that G is reconstructible by Theorem 2.5 because for each limb there exists a limb of size exactly one less than the original one. However, G is reconstructible by Theorem 2.7. If we picture the card G_v, then $T(G_v) = T(G)$ holds and L_1, L_2 and L_3 are both limbs in G and G_v. However, $L_4 \setminus \{v\}$ is not a limb in G. By Theorem 2.3 all limbs and the trunk of G are reconstructible. The only two ways to extend $L_4 \setminus \{v\}$ to L_4 is to connect the missing vertex v in G_v to either v_1 or v_2. Hence, both ways yield a reconstruction of G from the card G_v. Furthermore those reconstructions are isomorphic and $G_1 \cong G_2$ holds with $G_1 := (V(G), E(G))$ and $G_2 := (V(G), (E(G) \setminus \{vv_1\}) \cup \{vv_2\})$.

In the last theorem we removed only end vertices from the limbs to yield a sublimb, which is not a limb of the original graph. We may generalize this to any removed vertex of a limb which is not a neighbour of the root or the root itself by applying Theorem 2.7 recursively on the sublimbs that the original limb requires in order to have a chance of not being reconstructible. Thus we get the following corollary.

Corollary 2.9.
Let G be a separable graph with end vertices and limbs $L_1(G), ..., L_k(G)$. Let $L_1(G)$ be a limb of order at least 3 and

$$v \in V(L_1(G)) \setminus \{r(L_1(G)) \cup N_G(r(L_1(G)))\}$$

that is a vertex $v \in L_1(G)$ which is neither the root nor its unique neighbour. Let $L \subseteq L_1(G) \setminus \{v\}$ be the connected component containing the root of $L_1(G)$ and set $r(L) := r(L_1(G))$. If there exists no limb $L_i(G)$, $i \in \{2,...,k\}$ such that there is an isomorphism

$$\varphi : L_i(G) \to L$$

with

$$\varphi(r(L_i(G))) = r(L),$$

then G is reconstructible.

Proof.
Let $L_1(G),...,L_k(G)$ be the limbs of G. Let $L_1(G)$ be a limb of order at least 3 and $v \in V(L_1(G)) \setminus \{r(L_1(G)) \cup N_G(r(L_1(G)))\}$ that is a vertex $v \in L_1(G)$ which is neither the root nor its unique neighbour. Let $L \subseteq L_1(G)$ be the connected component containing the root of $L_1(G)$ and set $r(L) := r(L_1(G))$. Assume that there exists no limb $L_i(G)$, $i \in \{2,...,k\}$ such that there is an isomorphism $\varphi : L_i(G) \to L$ such that $\varphi(r(L_i(G))) = r(L)$ holds.
We prove the corollary by induction over the order of the limb:
base case: Let $L_1(G)$ be a limb of order 3. Then v is an end vertex of G and Theorem 2.7 may be applied.
inductive step: Assume that the statement is true for all limbs of a fixed order $k \geq 3$. Let $L_1(G)$ be a limb of order $k+1$.
Assume that $v \in L_1(G)$ is an end vertex of G. If there exists no limb $L_i(G)$, $i \in \{2,...,k\}$ such that there is an isomorphism $\varphi : L_i(G) \to L_1(G) \setminus \{v\}$ with $\varphi(r(L_i(G))) = r(L_1(G) \setminus \{v\})$, then G is reconstructible by Theorem 2.7.
Now assume that $v \in L_1(G)$ is not an end vertex in G. Assume that Theorem 2.7 may not be applied on the end vertex $v_1 \in L_1(G)$. Then there exists a limb $L_2(G)$ and an isomorphism $\varphi : L_2(G) \to L_1(G) \setminus \{v_1\}$ such that $\varphi(r(L_2(G))) = r(L_1(G))$ holds. $L_2(G)$ is of order k. Furthermore $\varphi(v) = w$ holds for some $w \in L_2(G)$. Remark that w is neither the root nor the neighbour of a root in $L_2(G)$. Hence, by the induction hypothesis the statement is true for w and therefore true for v via the isomorphism φ. This concludes the proof. □

For Corollary 2.9 we may also present an alternative and direct proof which is analoguesly to the proof of Theorem 2.7.

Proof.
Let $L_1(G),...,L_k(G)$ be the limbs of G. Let $L_1(G)$ be a limb of order at least 3 and $v \in V(L_1(G)) \setminus \{r(L_1(G)) \cup N_G(L_1(G))\}$ that is a vertex v of limb $L_1(G)$ which is neither the root nor its unique neighbour. Let $L \subseteq L_1(G)$ be the connected component containing the root of $L_1(G)$ and set $r(L) := r(L_1(G))$. Disregard the other connected

components not containing the trunk when identifying the limbs. Assume that there exists no limb $L_i(G)$, $2 \leq i \leq k$ such that there is an isomorphism $\varphi : L_i(G) \to L$ with $\varphi(r(L_i(G))) = r(L)$. Then the card G_v is identifiable as a card, that contains a sublimb of $L_1(G)$ namely L which is no limb of G. Furthermore $T(G) \cong T(G_v)$ holds and G_v contains all limbs $L_2(G), ..., L_k(G)$. In addition it contains some trees as other components. Hence, L in G_v needs to be extended with v and the other tree components to $L_1(G)$ in order to yield a reconstruction of G. But then every reconstruction of G is isomorph to G and therefore G is reconstructible. □

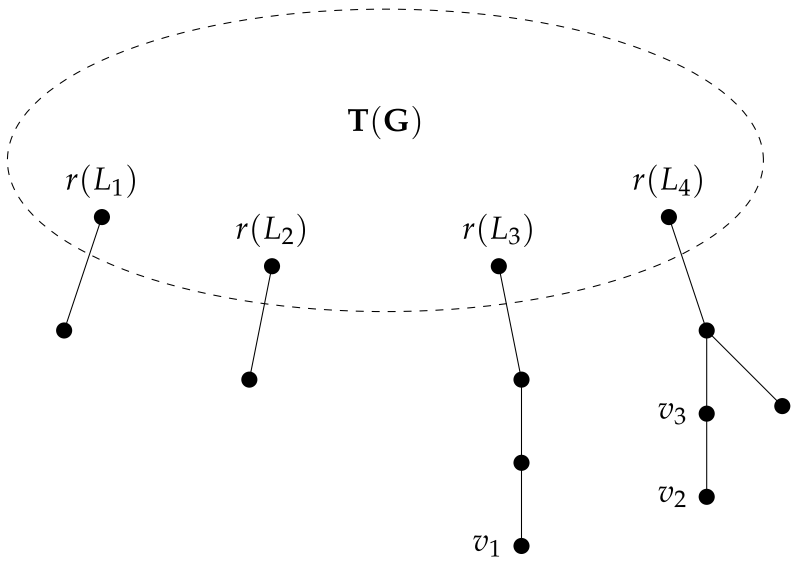


Figure 14: Separable graph with end vertices

Example 2.10.
Let G be a separable graph with end vertices. G contains four limbs $L_1 := L_1(G), ...,$ $L_4 := L_4(G)$ (see Figure 14). Now, we can prove in various ways that G is reconstructible. We may apply Theorems 2.5, 2.7 or Corollary 2.9 for the card G_{v_1}. We may apply Theorem 2.7 or Corollary 2.9 for the card G_{v_2}. And last we may apply Corollary 2.9 for the card G_{v_3}. Hence, we see that Corollary 2.9 is the most generalized one, which in turn is a direct result of Theorem 2.7. This actually means that if Corollary 2.9 may be applied for some card G_v for which Theorem 2.7 may not be applied, then there exists a different card $G_w \ncong G_v$ for which both Theorem 2.7 and Corollary 2.9 may be applied. In this case the missing limb that triggers both Theorem 2.7 and Corollary 2.9 is a limb isomorphic to a path of length 2. Yet the resulting cards are G_{v_1} and G_{v_3} respectively.

We can now combine the structures of the limb with results that we have for the graph itself. The limbs and their structures are known and we have established theorems to show in certain cases that the graph is reconstructible. Now, we add insights of the graphs itself to the previous theorems. In particular we look at the structure and isomorphism type of the trunk. For that we want to divide the trunk's vertices into orbits under its automorphism group.

Definition 2.11. (orbit)
Let G be a graph and $Aut(G)$ its automorphism group. Then for any vertex $v \in V(G)$ the *orbit of v under $Aut(G)$* is defined as

$$Aut(G) \cdot v = \{g \cdot v \mid g \in Aut(G)\}.$$

Corollary 2.9 states that the graph G needs to have for every limb $L_i(G)$ all possible sublimbs of $L_i(G)$ as limbs of the original graph in order to be non-reconstructible. Extending this result on the orbits of the trunk under the automorphism group of the trunk yields the following:

Theorem 2.12.
Let G be a separable graph with end vertices and limbs $L_1(G), ..., L_k(G)$. Let $L_1(G)$ be a limb of order at least 3 and

$$v \in V(L_1(G)) \setminus \{r(L_1(G)) \cup N_G(r(L_1(G)))\}$$

that is a vertex $v \in L_1(G)$ which is neither the root nor its unique neighbour. Let $L \subseteq L_1(G)$ be the connected component after removing v containing the root of $L_1(G)$ and set $r(L) := r(L_1(G))$. If there exists no limb $L_i(G)$, $i \in \{2, ..., k\}$ with

$$r(L_i(G)) \in Aut(T(G)) \cdot r(L)$$

such that there is an isomorphism

$$\varphi : L_i(G) \to L$$

with

$$\varphi(r(L_i(G))) = r(L),$$

then G is reconstructible.

Proof.
Let $L_1(G), ..., L_k(G)$ be the limbs of G. Let $L_1(G)$ be a limb of order at least 3 and $v \in V(L_1(G)) \setminus \{r(L_1(G)) \cup N_G(L_1(G))\}$ that is a vertex v of limb $L_1(G)$ which is neither the root nor its unique neighbour. Let $L \subseteq L_1(G)$ be the connected component containing the root of $L_1(G)$ and set $r(L) := r(L_1(G))$. Disregard the other connected components not containing the trunk when identifying the limbs. The trunk itself is reconstructible by Theorem 2.3. Therefore, the automorphism group of the trunk and

the orbits of the trunk under the automorphism group of the trunk are reconstructible. Assume that there exists no limb $L_i(G)$, $2 \leq i \leq k$ with $r(L_i(G)) \in Aut(T(G)) \cdot r(L)$ such that there is an isomorphism $\varphi : L_i(G) \to L$ with $\varphi(r(L_i(G))) = r(L)$. Then the card G_v is identifiable as a card, that contains a sublimb of $L_1(G)$ namely L which may be a limb of G but whose root $r(L)$ is in $Aut(T(G)) \cdot r(L)$. G itself on the other hand, does not contain any limb isomorph to L where the root of that limb is in $Aut(T(G)) \cdot r(L_1(G))$. Hence, L in G_v needs to be extended with v and the other tree components to $L_1(G)$ in order to yield a reconstruction of G. But then every reconstruction of G is isomorph to G and therefore G is reconstructible. □

The previous theorem states that for any limb all sublimbs of that limb need to be limbs of the graph itself where the roots are in the same orbit under the automorphism group of the trunk in order to be non-reconstructible. The following example visualizes Theorem 2.12.

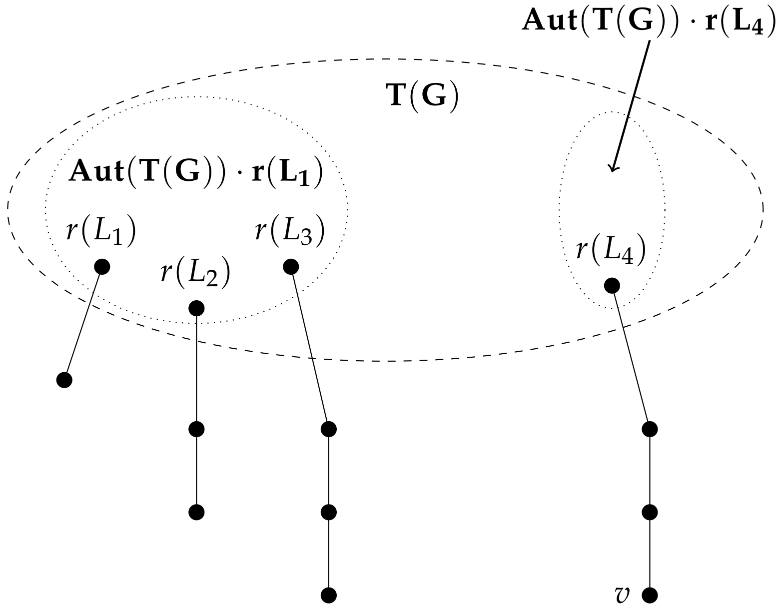


Figure 15: Separable graph with end vertices with limbs of sizes 1 to 4

Example 2.13.
Let G be a separable graph with end vertices. G contains four limbs $L_1 := L_1(G), ..., L_4 := L_4(G)$ (see Figure 15). The roots of the limbs L_1, L_2 and L_3 are in one orbit under the automorphism group of the trunk. The root of L_4 is not in the same orbit under the automorphism group as the roots of the other three limbs. Now, we are not able to prove that G is reconstructible by either Theorem 2.7 or Corollary 2.9. Theorem 2.12 on the other hand states that the card G_v contains two limbs of the isomorphism type of the limb L_2. The graph G only contains one of those. Furthermore, we are able to

distinguish those two limbs in G_v because the roots of the two limbs are in different orbits under the automorphism group of the trunk. Since we are able to reconstruct in which orbit the roots of the various isomorphism types of the limbs are in the original graph, the only option for a reconstruction of G is to add v to the limb of size three which root is not in the same orbit as the other three. Hence, G is reconstructible.

The remaining separable graphs with end vertices that we are not yet able to reconstruct have a precise and rather symmetric structure. So we may propose two problems that contain the entire separable graphs with end vertices that haven't been proven to be reconstructible yet. We divide the remaining graphs by the structure of its trunks and how the limbs are attached to the trunk. The first problem contains all graphs where the roots of all limbs are in one orbit under the automorphism group of the trunk.

Problem 2.14.
Let G be a separable graph with end vertices and Theorem 2.12 is not applicable. Show that G is reconstructible if all roots of the limbs of G are in one orbit under the automorphism group of the trunk.

A special case of the first problem was already proven by Bondy and also stated earlier on. When the trunk is isomorph to a complete graph, then all vertices of the trunk are in the same orbit under the automorphism group of the trunk. Furthermore the only problem to attach the limbs to the trunk is to find out, what isomorphism types of limbs share a common root. The following theorem is also a special case of the more general problem, where the automorphism group of the trunk acts transitive (i.e., there exists only one orbit under the group action) on the vertices of $G[V(T(G))]$. We restate Theorem 1.61:

Theorem 2.15. (Bondy, 1969)
Let G be a separable graph with end vertices. If the trunk $T(G)$ is isomorph to a complete graph, then G is reconstructible.

The second problem consists of the graphs where the limbs are attached to the trunk in at least two different orbits.

Problem 2.16.
Let G be a separable graph with end vertices and Theorem 2.12 is not applicable. Show that G is reconstructible if the trunk has at least two orbits under the automorphism group of the trunk that contain roots of limbs of G.

3 Reconstructing Minors

This chapter deals with the reconstruction and edge-reconstruction of minors contained in graphs. In particular we want to determine by the deck or the edge-deck of a graph G if G contains a certain minor H or not. Hence, we want to show when G having H as a minor is reconstructible or edge-reconstructible. The main idea is to show that minors of certain sizes need to be contained already in a card of a graph if the graph itself is of sufficient high order or of sufficient high size. We will give a general limit on what minors are recognizable in which graphs.

We start by defining a k-minor coloring. This coloring is the basis for the theory in this chapter. Plainly spoken if G contains a minor H on k vertices, then we color the graph G such that the vertices that are combined via edge contractions during the process of finding a minor must share the same color. The combined vertex has then the same color as the vertices it consists of. Furthermore, H is colored in k differnt colors. The proper definiton reads as follows:

Definition 3.1. ((edge-proper/proper) k-minor coloring)
Let G be a graph and H a minor of G on k vertices. A map $\varphi : V(G) \to \{1, 2, ..., k\}$: $v \mapsto \varphi(v)$ is called a *k-minor coloring* if the minor H may be obtained from G by a series of the following operations:

i) Deleting a vertex of G.

ii) Deleting an edge of G.

iii) Let v, w be two vertices with $\varphi(v) = \varphi(w) = i \in \{1, 2, ..., k\}$ and vw an edge. Contracting along the edge vw by replacing v and w by one vertex u that is adjacent to all vertices that were adjacent to v or w. Furthermore the contracted graph G^* inherits the colors from its original graph and u has the same color as v and w, that is

$$\varphi|_{G^*} : V(G^*) \to \{1, 2, ..., k\} : \begin{cases} x \mapsto \varphi(x) & , x \neq u \\ x \mapsto \varphi(v) & , x = u. \end{cases}$$

Furthermore, all vertices of H are of different color.
A k-minor coloring φ is called *proper* if there exists no subgraph $G' \subsetneq G$ with $V(G') \subsetneq V(G)$, such that G' has a k-minor coloring.
A k-minor coloring φ is called *edge-proper* if there exists no subgraph $G' \subsetneq G$, such that G' has a k-minor coloring.

Remark that **proper** means in particular that the operation $i)$ is not available. Remark further that **edge-proper** means in particular that the operations $i)$ and $ii)$ are not

available.
Remark that the k-minor coloring is **not** necessarily unique.

The proper and edge-proper k-minor coloring will be used to show when containing a minor is reconstructible or edge-reconstructible respectively. In particular, a minor H is both a minor of some card $G_v \in D(G)$ or $G_e \in D_e(G)$ and the graph G itself if G has no proper or no edge-proper k-minor coloring respectively. Proofs for that claim can be found in the next subchapters. We will then show, when such a k-minor coloring may or may not exist.

The first subchapter about minors in the reconstruction conjecture focuses on minors for the vertex version of the reconstruction conjecture. The next subchapter will be about minors for the edge-reconstruction conjecture.

3.1 Minors regarding the Reconstruction Conjecture

In this section we show for a few classes under which conditions it is reconstructible if such a graph contains a certain minor or not. In particular we show results for disconnected and separable graphs and give some ideas for 2-connected graphs. Yet the structure of the reconstruction conjecture limits our results, whereas the structure of the edge-reconstruction conjecture is more open to our ideas. The basic idea is to show when a minor of a graph is already given in a certain subgraph of that graph.

The first result is that a disconnected graph containing any minor is reconstructible. This is due to the fact that disconnected graphs are already reconstructible.

Theorem 3.2.
Let G be a disconnected graph and H a graph. Then G containing H as a minor is reconstructible.

Proof.
Disconnected graphs are reconstructible by Theorem 1.48. Hence, G containing any minor is reconstructible, too. □

The second result is that 2-connected minors in separable graphs are reconstructible. This is due to the fact that 2-connected minors need to be contained in a block of G. The result follows with the fact that every separable graph has at least 2 blocks or some vertices of degree one.

Theorem 3.3.
Let G be a separable graph and H a 2-connected graph. Then G containing H as a minor is reconstructible.

Proof.
Separable graphs without end vertices are reconstructible by Theorem 1.58. Hence, G containing any minor is reconstructible, too.
Now assume that G is a separable graph with end vertices, that is G contains a vertex of degree 1. If H is two connected, then H needs to be a minor of a block of G. Hence choose a vertex $v \in V(G)$ with $d_G(v) = 1$, then H is a minor of G_v. So a separable graph G containing a 2-connected minor is reconstructible. □

The case of a separable minor in a separable graph seems to be as difficult as the reconstruction conjecture itself.

For 2-connected graphs we give some ideas for the bounds on the order and size of the graph in comparison to its minor. Yet no general bound could be found because an arbitrary large graph could contain a very small minor without the minor occuring in any card. For the bound that we are going to give we use the proper k-minor coloring and give some properties for it. Remark that the results for a proper k-minor coloring are very crude. However, we will use these basic ideas of the proper k-minor coloring and refine them for the edge-proper k-minor coloring and are able to show in the next subchapter some interesting results with it.

The first structural result for proper k-minor colorings is that a k-minor coloring exists if and only if no cards contain that minor as a minor itself.

Theorem 3.4.
Let G be a graph and H a minor in G on k vertices. Then G has a proper k-minor coloring if and only if there exists no card $G_v \in D(G)$ such that G_v contains H as a minor.

Proof.
"$\Rightarrow$": Let $\varphi : V(G) \to \{1, 2, ..., k\} : v \mapsto \varphi(v)$ be a proper k-minor coloring. Assume that there exists a vertex $v \in V(G)$ such that H is a minor of G_v. Then G_v has a k-minor coloring with $G_v \subsetneq G$ and $V(G_v) \subsetneq V(G)$. This contradicts that φ is proper.
"$\Leftarrow$": Let H not be a minor in G_v for all $v \in V(G)$ but a minor of G. Then G has a k-minor coloring φ. Moreover, there exists no proper subgraph G' of G with $V(G') \subsetneq V(G)$ such that this subgraph contains H as a minor. Hence, the k-minor coloring φ is proper. □

Hence, as a result of this Theorem we are able to prove that if there is no k-minor coloring, then there exists a card such that this card contains the minor as a minor itself. Hence, it is reconstructible from the deck of the graph whether that graph contains this minor or not.

Corollary 3.5.
Let G be a graph and H a graph on k vertices. If there exists a k-minor coloring that is not proper, then G containing H as a minor is reconstructible.

Proof.
Let φ be a k-minor coloring that is not proper. Then by Theorem 3.4 there exists a card $G_v \in D(G)$ such G_v contains H as a minor. Since the deck of G is known, we find a card that contains H as a minor. Hence, G having H as a minor is reconstructible. □

An interesting result of the proper k-minor coloring is that all vertices of a single color class induce a connected subgraph.

Theorem 3.6.
Let G be a graph and H a minor of G on k vertices. Let G have a proper k-minor coloring and $V_i \subseteq V(G)$ contains all vertices of color $i \in \{1, 2, ..., k\}$. Then V_i induces a connected subgraph in G.

Proof.
Let φ denote the proper k-minor coloring in G. By operation iii) of the k-minor coloring definiton we may only contract edges where the end vertices have the same color. Furthermore since φ is proper, operation i) of the k-minor coloring definition is not available. In H all vertices are of different color. Therefore all vertices in V_i need to merge into a single vertex in H via edge contractions. Assume that V_i induces not a connected subgraph in G. Then there are vertices v and w in V_i such that v and w are in different components in the induced subgraph $G[V_i]$. Hence, there is no edge that connects the components by which one may contract. Therefore, v and w will be merged in different vertices of H. So H contains two vertices of the same color. A contradiction. □

With the result that all vertices of a single color class induce a connected subgraph of the graph, we are able to give some bounds on such a color class. The general idea is to limit the order and size of the original graph if you are looking for a specific subgraph. In particular, the next result is for 2-connected subset of a color class and limits its size.

Theorem 3.7.
Let G be a graph and H a minor of G on k vertices. Let G have a proper k-minor coloring and $V_i \subseteq V(G)$ contains all vertices of color $i \in \{1, 2, ..., k\}$. If $V_i' \subseteq V_i$ is maximal with V_i' induces a 2-connected subgraph of G, then $|V_i'| \leq d_H(v)$ holds where $v \in V(H)$ is of color i.

Proof.
Let G be a graph and H a minor of G on k vertices. Let G have a proper k-minor coloring and $V_i \subseteq V(G)$ contains all vertices of color $i \in \{1, 2, ..., k\}$. Furthermore $V_i' \subseteq V_i$ is maximal with V_i' induces a 2-connected subgraph of G.
Assume that $|V_i'| > d_H(v)$ holds where $v \in V(H)$ is of color i. First remark that the sequence in which to contract the edges within a single color class is insignificant.

Hence contract all edges within the color class i except the edges between vertices of V_i'. Now $v \in V(H)$ has $d_H(v)$ neighbours in the minor H. Hence the previously contracted graph needs at least $d_H(v)$ edges between the vertex set V_i' and the other color classes. Remark that for all color classes which are incident to v in H, there is at least one edge between a vertex of V_i' and a vertex of that specific color class in the previously contracted graph. Hence there exists a vertex $v' \in V_i'$ such that the same contracted graph minus the vertex v' still has all the necessary edges between $V_i' \setminus \{v'\}$ and the other color classes. This holds because there are $d_H(v)$ edges needed between $V_i' \setminus \{v'\}$ and the other color classes and we have $|V_i' \setminus \{v'\}| \geq d_H(v)$. Furthermore the induced subgraph on the vertices $V_i' \setminus \{v'\}$ is connected because the induced subgraph on V_i' is 2-connected. Hence all vertices in $V_i' \setminus \{v'\}$ and therefore all remaining vertices of color i can be merged to the single vertex $v \in V(H)$ via edge contractions. This is a contradiction to G having a proper k-minor coloring. Hence $|V_i'| \leq d_H(v)$ holds. □

The results for a proper k-minor coloring and the reconstruction conjecture are very limited. However, the idea to limit the order and size of a graph is more applicable for the edge-reconstruction conjecture than for the reconstruction conjecture itself. In the next subchapter we are going to show how this idea can truly shine.

3.2 Minors regarding the Edge-Reconstruction Conjecture

This section deals with reconstructing minors for the edge-reconstruction conjecture. Hence, we show when a graph containing a specific minor is edge-reconstructible. All in all, we look at disconnected, separable and 2-connected graphs. Since disconnected graphs are reconstructible, we have that containing a minor for disconnected graphs is edge-reconstructible. For separable graphs we show that in some cases they are always reconstructible. Finally, for 2-connected graphs we prove that there is a specific ratio between the order and size of the graph and the order and the size of the minor which tells us if a graph contains that minor. In addition, we are going to give some edge-reconstructible graph invariants that may be defined with the concept of minors.

As in the previous subchapter we are going to use the theory of the k-minor coloring. In this case we are going to use the edge-proper k-minor coloring. As seen before for the ratio between the graph and its minor we will determine if a minor must already be contained in a card of the edge-deck of the graph.

Hence, the first Theorem that we are going to prove is that a graph has no edge-proper k-minor coloring if and only if that minor is contained in some card of the graph.

Theorem 3.8.
Let G be a graph without isolated vertices and H a minor in G on k vertices. Then G has an edge-proper k-minor coloring if and only if there exists no card $G_e \in D_e(G)$ such that G_e contains H as a minor.

Proof.
" $\Rightarrow$": Let $\varphi : V(G) \rightarrow \{1, 2, ..., k\} : v \mapsto \varphi(v)$ be an edge-proper k-minor coloring. Assume that there exists an edge $e \in E(G)$ such that H is a minor of G_e. Then G_e has a k-minor coloring with $G_e \subsetneq G$. This contradicts that φ is edge-proper.
" $\Leftarrow$": Let H not be a minor in G_e for all $e \in E(G)$ but a minor of G. Then G has a k-minor coloring φ. For every G_v with $v \in V(G)$ holds $m(G_v) < m(G)$ because G has no isolated vertices. Hence there exists no proper subgraph of G such that this subgraph contains H as a minor. Hence the k-minor coloring φ is edge-proper. □

As a direct conclusion from Theorem 3.8, we have that if there exists no edge-proper k-minor coloring, then the question whether the graph contains that minor or not is edge-reconstructible.

Corollary 3.9.
Let G be a graph without isolated vertices and H a minor in G on k vertices. If there exists a k-minor coloring that is not edge-proper, then G containing H as a minor is edge-reconstructible.

Proof.
Let φ be a k-minor coloring that is not edge-proper. Then by Theorem 3.8 there exists a card $G_e \in D_e(G)$ such G_e contains H as a minor. Since the edge-deck of G is known we find a card that contains H as a minor. Hence G having H as a minor is edge-reconstructible. □

We can derive from the proper k-minor coloring of the previous subchapter a connection to the edge-proper k-minor coloring. In particular, an edge-proper k-minor coloring needs to be proper. Hence, we have for proper k-minor colorings are valid for edge-proper k-minor colorings, too.

Theorem 3.10.
Let G be a graph and H a minor of G on k vertices. Let φ be a k-minor coloring. If φ is edge-proper, then φ is proper, too.

Proof.
Assume, that φ is not proper. That is, there exist vertices $v_1, ..., v_t \in V$ such that $G' = G[V']$ with $V' = V \setminus \{v_1, ..., v_t\}$ has a k-minor coloring. Hence, $G' \subsetneq G$ holds and therefore φ is not edge-proper. □

We will continue with some basic properties for edge-proper k-minor colorings in order to delve deeper into the theory of edge-proper k-minor colorings. The first property that we are going to show is that the induced subgraph of one color class of an edge-proper k-minor coloring forms always a tree.

Theorem 3.11.
Let G be a graph and H a minor of G on k vertices. Let φ be an edge-proper k-minor coloring, then any induced subgraph by one color class of G is a tree.

Proof.
Let φ be a proper k-minor coloring. Let $V' \subseteq V(G)$ be the set of all vertices of a fixed color $i \in \{1, 2, ..., k\}$, that is $V' \subseteq V(G)$ is maximal with for all $v \in V'$ holds $\varphi(v) = i$. Assume that the spanning subgraph $G[V']$ on the vertices V' is not a tree. Then $G[V']$ is either disconnected or contains at least $|V'|$ edges. $G[V']$ needs to be connected. By Theorem 3.6 every color class of a proper k-minor coloring induces a connected subgraph of G. Theorem 3.10 states that the edge-proper k-minor coloring φ is also a proper k-minor coloring. Hence, $G[V']$ is connected. So the connected induced subgraph $G[V']$ contains at least $|V'|$ edges. Hence, there exists at least one edge, that we do not contract along. Denote this edge by e. Then φ is a k-minor coloring for $G^* = [V, E - \{e\}]$. This yields a contradiction because the k-minor coloring φ is edge-proper, that is, there is no graph $G^* \subsetneq G$, such that G^* has a k-minor coloring.□

Next, we are going to show for an edge-proper k-minor coloring that the vertex sets of two different color classes may be connected by at most one edge. In particular this edge exists if and only if the two vertices of these two colors are adjacent in the minor.

Theorem 3.12.
Let G be a graph and H a minor of G on k vertices. Let φ be an edge-proper k-minor coloring. Let $i, j \in \{1, 2, ..., k\}$ be two colors and $V_i \subseteq V(G)$ and $V_j \subseteq V(G)$ be all vertices of color i and j respectively. Then $v, w \in V(H)$ with $\varphi(v) = i$ and $\varphi(w) = j$ are adjacent if and only if there exists two vertices $v' \in V_i$ and $w' \in V_j$ such that $v'w' \in E(G)$ holds. In particular, $|\{v'w' \in E(G) \mid v' \in V_i \text{ and } w' \in V_j\}| \leq 1$ holds.

Proof.
$'' \Rightarrow''$: Assume that no vertex of color class i has a neighbour in color class j. φ is edge-proper. Therfore only edges will be contracted, where the end vertices of that edge share the same color. So all vertices of the color class i and only the vertices of color class i will merge into a single vertex in H. In addition, contracting edges will give the resulting vertex the union of neighbours of the end vertices of the edge. Yet, no vertex of color i has a neighbour of color j. Hence, the resulting vertex of the color class i is not adjacent to the vertices of color class j and therefore neither to the resulting vertex of the color class j.

" $\Leftarrow$": Let $V_i \subseteq V(G)$ and $V_j \subseteq V(G)$ be all vertices of color i and j respectively. Let $v' \in V_i$ and $w' \in V_j$ with $v'w' \in E(G)$. Then, the vertices of color class i will merge into a single vertex $v \in V(H)$ with $\varphi(v) = i$. Analogusly, V_j will merge into $w \in V(H)$ with $\varphi(w) = j$. Contracting edges will give the resulting vertex the union of neighbours of the end vertices of the edge. Since φ is edge-proper, no edges may be deleted. Hence, v is adjacent to w in H.
It remains to be shown that $|\{v'w' \in E(G) \mid v' \in V_i \text{ and } w' \in V_j\}| \leq 1$ holds. Assume that there are at least two edges between the color classes V_i and V_j. Denote those edges with $v_1'w_1' \in E(G)$ and $v_2'w_2 \in E(G)$ with $v_1', v_2' \in V_i$ and $w_1', w_2' \in V_j$. Let $w_1' \neq w_2'$ hold while v_1' and v_2' may be the same vertex. Consider the subgraph $G_1 := [V(G), E(G) \setminus \{v_2'w_2'\}]$. With the above results in this proof the resulting vertex of color class V_i is adjacent to the resulting vertex of color class V_j in the minor H_1 of G_1 because the edge $v_1'w_1' \in V(G_1)$ exists. All other edges in the resulting minor H_1 are unchanged compared to the minor H of G. Hence $H \cong H_1$ holds. This is a contradiction to φ being edge-proper because $G_1 \subsetneq G$ has an edge-proper k-minor coloring. Hence $|\{v'w' \in E(G) \mid v' \in V_i \text{ and } w' \in V_j\}| \leq 1$ holds. □

With these two properties we show when a graph containing a specific minor is edge-reconstructible. We start with the first case for disconnected graphs.

Theorem 3.13.
Let G be a disonnceted graph and H a graph on k vertices. Then G containing H as a minor is edge-reconstructible.

Proof.
Disconnected graphs are reconstructible by Theorem 1.48. Then they are also edge-reconstructible by Theorem 1.112 and Corollary 1.111. Hence, G containing H as a minor is edge-reconstructible. □

Next, we prove that for some separable graphs it is edge-reconstructible whether a graph contains a specific minor or not. In this case we have a separable graph with a 2-connected minor. We actually get the stronger result that this case is reconstructible. But with Theorem 1.112 and Corollary 1.111 it is also edge-reconstructible.

Theorem 3.14.
Let G be a separable graph and H a 2-connected graph. Then G containing H as a minor is reconstructible.

Proof.
If G is separable without end vertices, then G is reconstructible by Theorem 1.58. Therefore, G containing H as a minor is reconstructible.
Now, assume that G is separable with end vertices. H is 2-connected. Therefore H is a minor in a 2-connected subgraph of G. Let $v \in V(G)$ be a vertex of degree 1, that is an end vertex of G. Then H is a minor in the card G_v. Hence, G containing H as a minor is reconstructible. □

Analogously, we have the same argument for a separable graph and a disconnected minor.

Theorem 3.15.
Let G be a separable graph and H a disconnected graph. Then G containing H as a minor is edge-reconstructible.

Proof.
Let G be a separable graph and H a disconnected graph. Since H is disconnected there are at least two components of H. We restrict this proof to the case with two components of H. The proof for more components works analogously. Hence, there are two disjoint subgraphs of G that contain these two components of the minor H as minors. In G those two subgraphs are connected by at least one edge $e \in E(G)$ because G is separable. Hence, $G_e \in D_e(G)$ contains those two subgraphs as well. So G containing H as a minor is edge-reconstructible. □

The last case for separable graphs might be very difficult to prove. That case consists of a separable graph with a separable minor. This might be as difficult as edge-reconstructing separable graphs itself.

Now, we move on to the final case. After showing results for disconnected and separable graphs we prove a specific ratio for 2-connected graphs. That is the graph G is 2-connected. Analogously to Theorem 3.15 we get the following results for 2-connected graphs and a disconnected minor.

Theorem 3.16.
Let G be a 2-connected graph and H a disconnected graph. Then G containing H as a minor is edge-reconstructible.

Proof.
Let G be a 2-connected graph and H a disconnected minor in G. Since H is disconnected there are at least two components of H. We restrict this proof to the case with two components of H. The proof for more components works analogously. Hence, there are two disjoint subgraphs of G that contain these two components of the minor H as minors. In G those two subgraphs are connected by at least one edge $e \in E(G)$ because G is 2-connected. Hence, $G_e \in D_e(G)$ contains those two subgraphs as well. So G containing H as a minor is edge-reconstructible. □

The case for 2-connected graphs and a connected minor H is shown for a specific ratio between the order and size of the graph and the order and size of its minor. The basis for the limit on the order-size ratio for 2-connected graphs is given by the following edge-reconstructible class. Remark that we restate Theorem 1.131.

Theorem 3.17.
Let G be a 2-connected graph. If G contains two vertices $v, w \in V(G)$ with $vw \in E(G)$ and $d_G(v) = d_G(w) = 2$, then G is edge-reconstructible.

Hence, from now on we may restrict our attention to 2-connected graphs that do not contain two adjacent vertices of degree 2. In the end we want to use Theorem 3.8, which states that a graph contains a minor already in some cards if and only if it does not have an edge-proper k-minor coloring. Hence, we are going to show how small the original graph can get in comparison to the order and size of its minor and the original graph still contains the minor in some card of its edge-deck.

In order to achieve that, we first give a sharp bound for the number of vertices of a color class if that graph has an edge-proper k-minor coloring.

Theorem 3.18.
Let G be a 2-connected graph that does not contain two adjacent vertices of degree 2 and H a connected minor in G. Let φ be an edge-proper k-minor coloring and let $v \in V(H)$ with $d_H(v) \geq 2$ have color $i \in \{1, 2, ..., k\}$. Then

$$|\{w \in V(G) \mid \varphi(w) = i\}| \leq 3d_H(v) - 5$$

holds that is there are at most $3d_H(v) - 5$ vertices of color i in G.

Proof.
Let H be a minor of G on k vertices and φ be an edge-proper k-minor coloring. Denote the color classes with $V_1 \subseteq V(G), V_2 \subseteq V(G), ..., V_k \subseteq V(G)$ where V_i contains all vertices with color i. Remark that in order to have the maximal size of a color class, we may assume that V_i contains only vertices of degree at most 3. If V_i contains a vertex v of degree $d \geq 4$, then this vertex might be replaced by two adjacent vertices of degree 3 and $d - 1$, where the first vertex is adjacent to two neighbours of v and the other vertex is adjacent to the remaining neighbours of v. Hence, this yields a bigger color class and therefore a contradiction. So without loss of generalization we may assume that V_i contains only vertices of degree 2 and 3.
Let $v \in V(H)$ be of degree 2 and of color $i \in \{1, 2, ..., k\}$. Hence, there are exactly 2 edges from color class V_i into two different other color classes in G by Theorem 3.12. By Theorem 3.11 the color class V_i is a tree in G. Furthermore there are no two adjacent vertices of degree 2 in G. Hence, the color class V_i consists of exactly one vertex in G and therefore $|V_i| = \{w \in V(G) \mid \varphi(w) = i\}| = 1 = 3d_H(v) - 5$ holds.
Let $v \in V(H)$ be of degree 3 and of color $i \in \{1, 2, ..., k\}$. Again by Theorem 3.11 the color class V_i forms a tree as a subgraph of G. Furthermore there are exactly 3 edges from the color class V_i into three different other color classes in G by Theorem 3.12. There are no two adjacent vertices of degree 2 in G. Hence, the color class consists of one vertex of degree three. Then to this vertex we may attach at most three vertices of degree 2. Since we are not able to attach vertices of higher degree than three and we

are not able to have two vertices of degree two adjacent to each other, our color class V_i ends there. Hence, we have $|V_i| = |\{w \in V(G) \mid \varphi(w) = i\}| \leq 4 = 3d_H(v) - 5$.
Now, let $v \in V(H)$ be of degree $d \geq 4$ and of color $i \in \{1, 2, ..., k\}$. Let $G' \subseteq G$ be a graph such that G' fullfills all conditions as G does, except it may contain fewer vertices than G. All these vertices come from one color class V_i and there is a minor $H' \subseteq H$ of G' such that the difference between H' and H is exactly one edge which is incident to v. Identify the vertex v for H' with v'. Hence, $d_{H'}(v') = d - 1$ holds. Denote all vertices of color i in G' with V_i'. By induction we assume that the statement is true for some value $d - 1 \in \mathbb{N}$ which means it is true for color i in G'. Hence, we have for $d_{H'}(v') = d - 1$ and $|\{w \in V(G') \mid \varphi(w) = i\}| \leq 3d_{H'}(v') - 5 = 3(d_H(v) - 1) - 5$. We want to extend the color class V_i' to V_i by adding and replacing vertices and edges and therefore yielding G from G'.
By Theorem 3.11 the color classes V_i' and V_i form trees. In order to extend V_i' to V_i we may replace a vertex $w' \in V_i'$ with a tree that contains vertices of degree 2 and 3. The form of the tree depends on the degree w' and its placement within the color class V_i'. Assume the vertex w' has $d_{G'}(w') = 2$. Then either exactly one or both neighbours of w' are in V_i'. Assume that exactly one neighbour of w' is in V'. Then replace w' with a tree of order at most 4, which in turn is connected to vertices of G. Exactly one vertex of that tree has degree 3 in G in order to yield a connection to another, third color class. Furthermore that tree is connected to both neighbours of w'. Hence, the tree consists of exactly one vertex of degree 3 and at most three vertices of degree 2 which are then connected to the other color classes or the neighbour of w' in G'. Hence, we replace one vertex by up to four vertices. So

$$\begin{aligned}
|\{w \in V(G) \mid \varphi(w) = i\}| &\leq |\{w \in V(G') \mid \varphi(w) = i\}| - 1 + 4 \\
&\leq 3d_{H'}(v') - 5 + 3 \\
&= 3(d_H(v) - 1) - 5 + 3 \\
&= 3d_H(v) - 5
\end{aligned}$$

holds. Now assume that both neighbours of w' are in V_i'. Then replace w' with a tree of order at most 4, which in turn is connected to vertices of G. Exactly one vertex of that tree has degree 3 in G in order to yield a connection to another color class. Furthermore that tree is connected to both neighbours of w' in V_i'. Hence, the tree consists of exactly one vertex of degree 3 and at most three vertices of degree 2. Up to two vertices of degree 2 are connected to the neighbours of w' in V' and up to one vertex of degree 2 is connectected to another color class. So we replace one vertex by up to four vertices. Hence,

$$\begin{aligned}
|\{w \in V(G) \mid \varphi(w) = i\}| &\leq |\{w \in V(G') \mid \varphi(w) = i\}| - 1 + 4 \\
&\leq 3d_{H'}(v') - 5 + 3 \\
&= 3(d_H(v) - 1) - 5 + 3 \\
&= 3d_H(v) - 5
\end{aligned}$$

holds.
Last assume that w' has degree 3 in G. Then replace w' with a tree of order at most 4, which in turn is connected to vertices of G. exactly two vertices of this tree have degree 3. There might be a vertex of degree 2 adjacent to both of these vertices of degree 3. Then there might be another vertex of degree 2 adjacent to one of the vertices of degree 3 and adjacent to another color class. The remaining neighbours of the tree are the same neighbours that w' had in G'. So we replace one vertex by up to four vertices. Hence,

$$\begin{aligned} |\{w \in V(G) \mid \varphi(w) = i\}| &\leq |\{w \in V(G') \mid \varphi(w) = i\}| - 1 + 4 \\ &\leq 3d_{H'}(v') - 5 + 3 \\ &= 3(d_H(v) - 1) - 5 + 3 \\ &= 3d_H(v) - 5 \end{aligned}$$ □

holds. This concludes the proof.

Using the bound on the size of the color class we may sum over all color classes and get a restriction on the order of the original graph if that graph contains an edge-proper k-minor coloring.

Theorem 3.19.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. Furthermore G does not contain two adjacent vertices of degree 2. If there exists an edge-proper k-minor coloring, then

$$n(G) \leq m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)$$

holds, where $V' \subseteq V(H)$ is the set of all vertices of degree 3 or greater in the minor H.

Proof.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. Let $V'' = V(H) \setminus V'$ be the set of all vertices of degree 2 in H and $H[V']$ is the induced subgraph of H on the vertex set V'. Furthermore G does not contain two adjacent vertices of degree 2. Let φ be an edge-proper k-minor coloring. The requirements of Theorem 3.18 are then fullfilled for any vertex $v \in H$ with color $i \in \{1, 2, ..., k\}$ and any color class $V_i \subseteq V(G)$.
Let $v \in V(H)$. Then the size of its color class V_i is bounded by $3d_H(v) - 5$. Now we sum all color classes up.
Remark that if two vertices of degree 3 or more are adjacent to each other in H, then there might not be two vertices of degree 2 and of different color be adjacent to each other in G. The proof of Theorem 3.18 states that a maximal color class has only

vertices of degree 2 adjacent to other color classes. Hence, we need to substract one vertex of a pair of two color classes if they are adjacent in H and both have degree 3 or greater. This will add up to $m(H[V'])$.
Remark further that if a vertex of degree 2 of color i is adjacent to a vertex of degree 3 of color j in H, then the corresponding vertex of color i in G is adjacent to a vertex of degree 3 or greater of color j in G. By the same argument about the proof of Theorem 3.18 we need to substract one vertex for that connection. This will add up to $2|V''|$ because each vertex of degree 2 in H is adjacent to two vertices of degree 3 or greater in H. Otherwise if two vertices of degree 2 are adjacent in H then there would be two vertices of degree 2 adjacent in G. That would yield a contradiction to the requirements.
Last add one vertex in G for every vertex of degree 2 in H. This sums up to $|V''|$.
Now with $V(G) = V_1 \cup V_2 \cup ... \cup V_k$ we get the following:

$$\begin{aligned}
n(G) \leq & \sum_{v \in V'} (3d_H(v) - 5) - m(H[V']) - 2|V''| + |V''| \\
= & \sum_{v \in V'} (2d_H(v) - 5) + \sum_{v \in V'} d_H(v) - m(H[V']) - |V''| \\
= & \sum_{v \in V'} 2d_H(v) - 5|V'| + m(H) + m(H[V']) - m(H[V']) - |V''| \\
= & m(H) - n(H) - 4|V'| + 2 \sum_{v \in V'} d_H(v) \\
= & m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2).
\end{aligned}$$

□

Now inverting the last Theorem we get that if the number of vertices exceeds this ratio, then there exists a card that contains the minor of the original graph already as a minor itself. Hence, in this case the graph containing that minor is edge-reconstructible.

Theorem 3.20.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. If

$$n(G) > m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)$$

holds where $V' \subseteq V(H)$ is the set of all vertices of degree 3 or greater in H. Then G having H as a minor is edge-reconstructible.

Proof.
Let G be 2-connected and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. By Theorem 3.17 2-connected graphs G with two adjacent vertices of degree 2 are edge-reconstructible. Hence we may assume, that G does not contain

two adjacent vertices of degree 2. Now all requirements for Theorem 3.20 are satisfied. Hence, we conculde, that if

$$n(G) > m(H) - n(H) + 2\sum_{v \in V'} (d_H(v) - 2)$$

holds that G does not have an edge-proper k-minor coloring. Then by Theorem 3.9 G containing H as a minor is edge-reconstructible for those conditions. □

As an example and a conclusion of Theorem 3.20 we get the following sharp bound for the number of vertices of the original graph if the minor is a complete graph. The bound for a minor that is a complete subgraph is also an upper bound for all minors of the same order. Since a complete graph has the maximal number of edges on a certain vertex set this will also produce the biggest bound for all minors on the same number of vertices.

Corollary 3.21.
Let G be a 2-connected graph and the complete graph K_k is a minor of G with $k \geq 4$. If $n(G) > \frac{5}{2}k(k-3)$ holds, then G containing K_k as a minor is edge-reconstructible.

Proof.
It is sufficient to show that Theorem 3.20 holds. Hence, we show

$$n(G) > m(K_k) - n(K_k) + 2\sum_{v \in V'} (d_{K_k}(v) - 2)$$

holds, where $V' \subseteq V(K_k)$ is the set of all vertices of degree 3 or greater in K_k. Then $|V'| = k$ holds and $d_{K_k}(v) = k - 1$ holds for all $v \in V(K_k)$. Furthermore we have $m(K_k) = \frac{k(k-1)}{2}$. Hence, we get

$$\begin{aligned} n(G) &> m(K_k) - n(K_k) + 2\sum_{v \in V'} (d_{K_k}(v) - 2) \\ &= \frac{1}{2}k(k-1) - k + 2\sum_{v \in V'} (k - 1 - 2) \\ &= \frac{1}{2}k(k-1) - k + 2k(k-3) \\ &= \frac{5}{2}k(k-3). \end{aligned}$$

□

Using Theorem 3.18 again and summing over all color classes gives us a bound on the size of the graph, too, if the graph contains an edge-proper coloring.

Theorem 3.22.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. Furthermore G does not contain two adjacent vertices of degree 2. If there exists an edge-proper k-minor coloring, then

$$m(G) \leq 2m(H) - 2n(H) + 2\sum_{v \in V'} (d_H(v) - 2)$$

holds, where $V' \subseteq V(H)$ is the set of all vertices of degree 3 or greater in the minor H.

Proof.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. Let $V'' = V(H) \setminus V'$ be the set of all vertices of degree 2 in H and $H[V']$ is the induced subgraph of H on the vertex set V'. Furthermore G does not contain two adjacent vertices of degree 2. Let φ be an edge-proper k-minor coloring. The requirements of Theorem 3.18 are then fullfilled for any vertex $v \in H$ with color $i \in \{1, 2, ..., k\}$ and any color class $V_i \subseteq V(G)$.
Let $v \in V(H)$. Then the number of edges within its color classV_i is bounded by $m(H[V_i]) \leq 3d_H(v) - 6$ because it is a tree and the number of vertices is bounded by $3d_H(v) - 5$. Now we sum all color classes up and the edges in between them.
Remark that if two vertices of degree 3 or more are adjacent to each other in H, then there might not be two vertices of degree 2 and of different color be adjacent to each other in G. The proof of Theorem 3.18 states that a maximal color class has only vertices of degree 2 adjacent to other color classes. Hence, we need to substract one vertex of a pair of two color classes if they are adjacent in H and both have degree 3 or greater. This will add up to $m(H[V'])$.
We then add $m(H)$ that is the number of edges connecting different color classe in G. Remark further that if a vertex of degree 2 of color i is adjacent to a vertex of degree 3 of color j in H, then the corresponding vertex of color i in G is adjacent to a vertex of degree 3 or greater of color j in G. By the same argument about the proof of Theorem 3.18 we need to substract one edge for that connection. This will add up to $2|V''|$ because each vertex of degree 2 in H is adjacent to two vertices of degree 3 or greater in H. Otherwise if two vertices of degree 2 are adjacent in H then there would be two vertices of degree 2 adjacent in G. That would yield a contradiction to the requirements.
The sum yields with the identity $\sum\limits_{v \in V'} d_H(v) = m(H) + m(H[V'])$ the following:

$$\begin{aligned}
m(G) &\leq \sum_{v \in V'} (3d_H(v) - 6) + m(H) - m(H[V']) - 2|V''| \\
&= \sum_{v \in V'} (2d_H(v) - 6) + \sum_{v \in V'} d_H(v) + m(H) - m(H[V']) - 2|V''| \\
&= \sum_{v \in V'} 2d_H(v) - 6|V'| + m(H) + m(H[V']) + m(H) - m(H[V']) - 2|V''| \\
&= 2m(H) - 2n(H) - 4|V'| + \sum_{v \in V'} 2d_H(v) \\
&= 2m(H) - 2n(H) + 2 \sum_{v \in V'} (d_H(v) - 2).
\end{aligned}$$

□

Inverting the bound of the last Theorem we get that a graph containing a specific minor is edge-reconstructible if the number of edges exceeds the bound necessary for an edge-proper k-minor coloring. Hence, we get the following ratio for the number of edges of the original graph.

Theorem 3.23.
Let G be a 2-connected graph and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. If

$$m(G) > 2m(H) - 2n(H) + 2\sum_{v \in V'} (d_H(v) - 2)$$

holds where $V' \subseteq V(H)$ is the set of all vertices of degree 3 or greater in H. Then G having H as a minor is edge-reconstructible.

Proof.
Let G be 2-connected and H a connected minor of G on k vertices with $d_H(s) \geq 2$ for all $s \in V(H)$. By Theorem 3.17 2-connected graphs G with two adjacent vertices of degree 2 are edge-reconstructible. Hence we may assume, that G does not contain two adjacent vertices of degree 2. Now all requirements for Theorem 3.22 are satisfied. Hence we conclude, that if

$$m(G) > 2m(H) - 2n(H) + 2\sum_{v \in V'} (d_H(v) - 2)$$

holds that G does not have an edge-proper k-minor coloring. Then by Theorem 3.9 G containing H as a minor is edge-reconstructible for those conditions. □

As an example and a conclusion of Theorem 3.23 we get the following sharp bound for number of edges of the original graph if the minor is a complete graph.

Corollary 3.24.
Let G be a 2-connected graph and the complete graph K_k is a minor of G with $k \geq 4$. If $m(G) > 3k(k-3)$ holds, then G containing K_k as a minor is edge-reconstructible.

Proof.
It is sufficient to show that Theorem 3.23 holds. Hence, we show

$$m(G) > 2m(K_k) - 2n(K_k) + 2\sum_{v \in V'} (d_{K_k}(v) - 2)$$

holds, where $V' \subseteq V(K_k)$ is the set of all vertices of degree 3 or greater in K_k. Then $V' = V(K_k)$ holds and $d_{K_k}(v) = k-1$ holds for all $v \in V(K_k)$. Furthermore we have $m(K_k) = \frac{k(k-1)}{2}$. Hence, we get

$$\begin{aligned} m(G) &> 2m(K_k) - 2n(K_k) + 2\sum_{v \in V'} (d_{K_k}(v) - 2) \\ &= k(k-1) - 2k + 2\sum_{v \in V'} (k-1-2) \\ &= k(k-1) - 2k + 2k(k-3) \\ &= 3k(k-3). \end{aligned}$$

□

Summing up all the different cases we get the following:

Theorem 3.25.
Let G be a graph and H a minor. G containig H as a minor is edge-reconstructible if any of the following holds:

i) G is disconnected,

ii) G is separable and H is either disconnected or 2-connected,

iii) G is 2-connected and H is disconnected,

iv) G is 2-connected and H is connected and either

$$n(G) > m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)$$

or

$$m(G) > 2m(H) - 2n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)$$

holds, where $V' \subseteq V(H)$ is the set of all vertices of degree 3 or greater in H.

Proof.

i) The claim is given by Theorem 3.13.

ii) The claim is given by Theorem 3.15 and Theorem 3.14 respectively.

iii) The claim is given by Theorem 3.16.

iv) The claim is given by Theorem 3.20 and Theorem 3.23 respectively. □

3.2.1 Hadwiger number, Treewidth and the Edge-Reconstruction Conjecture

There are a range of invariants that might be defined via minors of graphs. Two examples are the Hadwiger number and the treewidth of a graph.

The Hadwiger number states the order of a largest clique that is a minor in a graph.

Definition 3.26. (Hadwiger number)
Let G be a graph. Let $k \in \mathbb{N}$ be maximal such that the complete graph K_k is a minor of G. Then $h(G) = k$ is called the *Hadwiger number* of G.

With the results for minors in the edge-reconstruction conjecture in the previous subchapter we are able to give a bound for the edge-recognition of the Hadwiger number. The complete graph is 2-connected and therefore either always edge-reconstructible as a minor or edge-reconstructible as a minor above a specific bound.

The Hadwiger number is reconstructible if the graph is either disconnected or separable.

Corollary 3.27.
Let G with $\delta(G) \geq 1$ be either a disconnected or a separable graph. Then the Hadwiger number $h(G)$ is edge-reconstructible.

Proof.
For a disconnected graph some component contains the largest clique as a minor. For a separable graph some block contains the largest clique as a minor. Both components and blocks are induced subgraphs in some cards of the deck of G. This yields the claim. □

If G is neither disconnected nor separable, then G is 2-connected. Hence, we may use the bounds from the previous subchapter. For the bound on the order of the graph we get:

Corollary 3.28.
2-connected graphs with Hadwiger number $h(G)$ are edge-recognizable if

$$n(G) > \frac{5}{2}(h(G)+1)(h(G)-2)$$

holds.

Proof.
Let G be 2-connected. G containing K_{k+1} and K_k as a minor is edge-reconstructible by Corollary 3.21 if $n(G) > \frac{5}{2}(k+1)(k-2) \geq \frac{5}{2}k(k-3)$ holds. □

Since we also have a bound on the number of edges of the original graph, we are able to show a bound for the Hadwiger number for the size of a graph.

Corollary 3.29.
2-connected graphs with Hadwiger number $h(G)$ are edge-recognizable if

$$m(G) > 3(h(G)+1)(h(G)-2)$$

holds.

Proof.
Let G be 2-connected. G containing K_{k+1} and K_k as a minor is edge-reconstructible by Corollary 3.24 if $m(G) > 3(k+1)(k-2) \geq 3k(k-3)$ holds. □

The other invariant is the treewidth. It is a prime example of an excluded minor Theorem. The treewidth for disconnected and separable graphs is edge-reconstructible since it is reconstructible.

Theorem 3.30.
Let G be a disconnected or separable graph. Then the treewidth of G is edge-reconstructible.

Proof.
Disconnected graphs are reconstructible by Theorem 1.48. Hence, with Theorem 1.112 by Greenwell and Corollary 1.111 disconnected graphs are edge-reconstructible, too. So the treewidth of disconnected graphs is edge-reconstructible.
By Theorem 1.95 the treewidth is reconstructible for separable graphs. Hence with Theorem 1.112 by Greenwell and Corollary 1.111, the treewidth has been proven to be edge-reconstructible. □

The remaining question is, whether the treewidth of a graph is edge-reconstructible or not if that graph is 2-connected. The treewidth may be defined via a finite set of forbidden minors. For that we define the forbidden minors as an obstruction set and give the forbidden graph characterisation.

Definition 3.31. (forbidden graph characterisation, obstruction set)
Let $\mathbf{X}$ be a set of graphs or a graph class. Then

$$Forb_{\preccurlyeq}(\mathbf{X}) := \{G \mid H \not\preccurlyeq G \ \forall \ H \in \mathbf{X}\}$$

is the class of all graphs that do not contain any graph of $\mathbf{X}$ as a minor. $Forb_{\preccurlyeq}(\mathbf{X})$ is called a *forbidden graph characterisation* by the minors isomorphic to the graphs in $\mathbf{X}$ and $\mathbf{X}$ is called an *obstruction set*.

Then as a result of Robertsons and Seymour's graph minor Theorem [69], we get a forbidden graph characterisation for graphs with treewidth at most k.

Theorem 3.32. (Robertson and Seymour, 1990)
Let $k \in \mathbb{N}$ and $\mathbf{G}$ be the graph class with for all $G \in \mathbf{G}$ holds $tw(G) \leq k$. Then $\mathbf{G} = Forb_{\preccurlyeq}(\mathbf{X})$ holds for a finite set $\mathbf{X}$ of graphs.

In other words, the class of all graphs with treewidth at most k may be defined with the help of an obstruction set. The property if 2-connected graphs G do or do not contain minors from a set $\mathbf{X}$, is edge-reconstructible if a certain order and size ratio is met. Hence, we get the following Theorem as a result of Theorem 3.25.

Theorem 3.33.
Let G be a graph and $\mathbf{G}_k := Forb_{\preccurlyeq}(\mathbf{X}_k)$ the class of all graphs with treewidth at most k. Then G having treewidth k is edge-reconstructible if either

$$n(G) > \max_{H \in \mathbf{X}_k} \{m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)\}$$

or

$$m(G) > \max_{H \in \mathbf{X}_k} \{2m(H) - 2n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)\}$$

holds, where $V' \subseteq V$ is the set of all vertices of degree 3 or greater in H and $\mathbf{X}_k$ is the obstruction set for $\mathbf{G}_k$.

Proof.
Let G be a graph with treewidth k and $\mathbf{G}_k$ the class of all graphs with treewidth at most k. Then by Theorem 3.32 both the class $\mathbf{G}_{k-1}$ and the class $\mathbf{G}_k$ have a forbidden graph characterisation.
Let $\mathbf{X}_k$ be the obstruction set of all graphs with treewidth at most k. In order to verify that G has treewidth exactly k, we need to show that G contains no H as a minor such that $H \in \mathbf{X}_k$ holds. Furthermore we need to show that G contains at least one graph of $\mathbf{X}_{k-1}$ as a minor in order to have a treewidth greater than $k-1$.
By the implication $H \preccurlyeq G$ holds $tw(H) \leq tw(G)$ (see for example [17, p. 290]), we have that the graphs in $\mathbf{X}_{k-1}$ are minors of subgraphs of the graphs in $\mathbf{X}_k$. Hence by Theorem 3.25, we can distinguish that G contains a minor of $\mathbf{X}_{k-1} \cup \mathbf{X}_k$ if one of the following holds:

$$n(G) > \max_{H \in \mathbf{X}_k} \{m(H) - n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)\}$$

or

$$m(G) > \max_{H \in \mathbf{X}_k} \{2m(H) - 2n(H) + 2 \sum_{v \in V'} (d_H(v) - 2)\}$$

where $V' \subseteq V$ is the set of all vertices of degree 3 or greater in H. □

4 Future Research

For future work there are a range of problems that might be of interest. There are yet a lot of graph classes that haven't been proven to be neither reconstructible nor edge-reconstructible. Furthermore, the reconstruction of some graph invariants is still open.

One specific class of graphs regarding the reconstruction conjecture comes to mind. In chapter 1.2.4 about the reductions of the reconstruction conjecture almost all reductions use the property of Theorem 1.40 that is a graph is reconstructible if and only if its complement is reconstructible. Hence, we are not just able to use the reductions to simplify the reconstruction conjecture for all graphs but to also simplify certain graph classes. In this case we are able to adapt the proofs such that they hold for self-complementary classes of graphs as well. In particular we can deduce the following Theorem from the reductions.

Theorem 4.1.
Let $\mathbf{G}$ be a graph class that is closed under the complement operation. That is for every $G \in \mathbf{G}$ holds $\overline{G} \in \mathbf{G}$. Then all graphs in $\mathbf{G}$ are reconstructible if and only if all 2-connected graphs $G \in \mathbf{G}$ with $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

The Theorem is a generalisation of Theorem 1.101 by Ramachandran and Monikandan from 2009. But instead of using all graphs, we use a self-complementary graph class $\mathbf{G}$. That is if $G \in \mathbf{G}$ holds, then $\overline{G} \in \mathbf{G}$ follows. Since the class of all graphs is self-complementary, the last Theorem gives a generalisation.
The proof of Theorem 4.1 can be easily done with the following ideas. Theorem 1.96 by Yang states that every connected graph is reconstructible if and only if every 2-connected graph is reconstructible. The proofs main argument is Theorem 1.40 that is the Theorem about the complement of a graph. Hence, the proof can be adapted to self-complementary graphs. The second part is about the argument of the diameter. It is widely known that if a graph has diameter greater or equal to three, then its complement has a diameter of three or less. Hence, this can be adapted to self-complementary graphs as well. These ideas outline the proof for the Theorem above.

However we will give the exact proofs below. Remark that the following proofs are identical to the original proofs except that they do not just work on all graphs but on self-complementary graph classes, too. Most of the times we only add a single line that states that we now work on self-complementary graph classes.

We begin with proving that P graphs in a self-complementary graph class are reconstructible if 2-connected graphs in that graph class are reconstructible. The original proof is given by Yang Yongzhi in [78]. Remark that the entire proof was already true for self-complementary graph classes. Therefore, the proof is almost copied in its entirety.

Theorem 4.2.
Let **G** be a graph class closed under the complement operation. P-graphs in **G** are reconstructible if 2-connected graphs in **G** are reconstructible.

Proof.
Let **G** be a graph class closed under the complement operation. Hence, for every $G \in \mathbf{G}$ holds $\overline{G} \in \mathbf{G}$. Furthermore, define the following vertices in the P-graph $G \in \mathbf{G}$. Let $u \in V(G)$ be a vertex adjacent to all vertices except an end vertex. Hence, $d_G(v) = n(G) - 2$ holds. Furthermore, let $v \in V(G)$ be incident to the bridge and adjacent to an end vertex. And last let $w \in V(G)$ be an end vertex of G different from u.
The prove uses induction on $n(G)$. Assume that $n(G) \geq 10$ holds. Remark that graphs with fewer vertices are reconstructible by a computer study by McKay (see Theorem 1.103). P-graphs with no vertices of degree 2 or with at least one vertex adjacent to v and of degree 2 are reconstructible by Theorem 1.98. Hence, we may assume that G does contain vertices of degree 2 of which none is adjacent to v. Let $V' \subseteq V(G) \setminus \{u\}$ be the set of vertices that are adjacent to some vertices of degree 2. Let $t \in V'$ and $s \in V(G)$ with $st \in E(G)$ and $d_G(s) = 2$. We distinguish four cases:

1.) There exists a $t \in V'$ with $d_G(t) = n(G) - 2$. In G_s the vertex s needs to be reattached to two vertices of degree $n(G) - 3$, neither of which can be v, to get a reconstruction H. No matter which two vertices of degree $n(G) - 3$ we choose, H will always be isomorph to G and therefore reconstructible.

2.) There exists a $t \in V'$ such that $d_G(t) = n(G) - 3$ and t is adjacent to at least two vertices of degree 2 in G. Consider G_s. We have the following degrees $d_G(u) = n(G) - 2$, $d_G(v) \leq n(G) - 3$, $d_G(w) = 1$ and $d_G(t) = n(G) - 3$. All other vertices can not be adjacent to the vertices of degree 2 that t is adjacent to and therefore have degree of at most $n(G) - 4$. Hence, u is identifiable in G_s as the only vertex of degree $n(G) - 3$ that is not adjacent to the vertex w of degree 1.
Since v is not adjacent to a vertex of degree 2, the only way to reconnect s to G_s and get a reconstruction H not isomorph to G is to connect it to u and a vertex t' of degree $n(G) - 4$ in G that is adjacent to one less vertex of degree 2 than t is. Since $d_G(t) = n(G) - 3$ holds, this can only happen if there are exactly three vertices of degree 2 in G; two adjacent to t and one adjacent to t'. Denote the vertex of degree 2 adjacent to t' with s'. Since t and t' are adjacent to all other vertices except w, the mapping that swaps t for t', s for s' and is the identity otherwise, is an isomorphism between G and H. Hence, this case is reconstructible.

3.) For any $t \in V'$, $d_G(t) \leq n(G) - 3$ and G has at least two vertices of degree 2. We consider three subcases.

3.1.) There exists a vertex $t \in V'$ such that t is adjacent to all vertices of degree 2 in G. Thus u and t are the only vertices adjacent to all 2-vertices in G_s. Hence, $H \cong G$ holds.
So from now on we may assume that 3.1.) is not the case that is there is no vertex besides u that is adjacent to all vertices of degree 2. Hence, all vertices except u have degree less or equal to $n(G) - 3$. Moreover if there exists a vertex with degree $n(G) - 3$, then that vertex is adjacent to all vertices except w and one of the vertices of degree 2. Hence, with case 2.) we can assume in this case there are exactly two vertices of degree 2 in G and that vertex is adjacent to exactly one of them.

3.2.) There exists a vertex $t \in V'$ with $d_G(t) = n(G) - 3$. In this case we can, by the last paragraph, let s' be the other vertex of degree 2 and t' and u are its neighbours. Hence, we may also assume that $d_G(t') \leq n(G) - 3$ holds.
So we get H from G_s by reconnecting s to a vertex $u' \in \{u, t'\}$ of degree $n(G) - 3$ and a vertex $t'' \in V(G_s) \setminus \{u, t'\}$ of degree $n(G) - 4$ that is adjacent to all vertices of G_s except s' and w. s' is identifiable as the only vertex of degree 2 in G_s and u and t' are its end vertices. If $d_G(t') \neq n(G) - 3$ holds, then we can identify u in G_s and if $d_G(d') = n(G) - 3$ holds, then it makes no difference whether we reconnect s to u or t'. Likewise if $t'' \neq t$ holds, then, other than s, t'' and t are adjacent to the same vertices. That is the mapping swaps u' for u, t'' for t, and the identity otherwise, is an isomorphism of G onto H.
The last subcase is where the two previous cases fail to hold.

3.3.) u is the only vertex adjacent to all vertices of degree 2 in G and for all $t \in T$ holds $d_G(t) \leq n(G) - 4$. First we note that there is no vertex $x \neq v$ with $d_G(x) = n(G) - 3$; for such a vertex x is different from u and so is not adjacent to at least one vertex of degree 2, nor to w, and so is adjacent to all other vertices - which includes another vertex of degree 2 contradicting our assumption in this case. Thus we can assume that $d_G(x) \leq n(G) - 4$ for all $x \in V(G) \setminus \{u, v\}$. Consider the case $d_G(v) = n(G) - 3$. To get H from G_u, u must be reconnected to all vertices except for the only end vertex that is adjacent to the unique vertex of degree $n(G) - 4$.
Thus we can assume that $d_G(x) \leq n(G) - 4$ for all vertices $x \in V(G) \setminus \{u\}$. Since $d_G(u) = n(G) - 2$, $f(u) = u$ and for all $x \neq u$, every isomorphism $\sigma_x : G_v \cong H_{f(x)}$ must have $\sigma_x(u) = u$. Moreover, for $x \neq u$ or w, w is identifiable in both G_x and $H_{f(x)}$ as the only end vertex not adjacent to u. So we must also have $\sigma_x(w) = w$. Let w be adjacent to q in H. Thus $f(v) = q$ and, for $x \neq u$ or w, we obviously have $\sigma_x(v) = q$. Now

let $G' = (V(G) \setminus \{w\}, E(G) \setminus \{uv\})$ and $H' = (V(H) \setminus \{w\}, E(H) \setminus \{uq\})$. Then $G'_v \cong H'_q$, $G'_u \cong H'_u$ and the observation show that, for $x \neq u$ or v, $\sigma_x | G'_x : G'_x \cong H'_{f(x)}$.

Thus f is an hypomorphism from G' to H'. Remark that a hypomorphism f from G' to H' in Yang Yongzhi's meaning is, that H' is a reconstruction of G' or the other way around. Hence, for a hypomorphism $f : V(G') \to V(H')$ in Yang Yongzhi's meaning would follow, that $G'_x \cong H'_{f(x)}$ holds for all $x \in V(G')$. We have three possibilities for G': it is 2-connected, it is separable without end vertices or it is separable with end vertices. The first case is covered by our hypthesis and the second case is covered by Theorem 1.58, but actually implies that G_w is separable and so does not occur. In the third case let y be the neighbour of v. Then y is the only cut vertex of G' because if u were also a cut vertex, then $G' + uv = G_w$ would be separable. This yields a contradiction. So G' is a P-graph on $n(G) - 1$ vertices and is reconstructible by our induction hypothesis.

We have shown that $G' \cong H'$. But $v \in V(G')$ and $q \in V(H')$ are identifiable as the only vertices not adjacent to the identifiable vertex u. Hence, under any isomorphism of G' onto H', v must be mapped to q and u to u. It follows that G is isomorphic to H. That completes case 3.).

4.) G has exactly one 2-vertex and for the only member t of V', $d_G(t) \leq n(G) - 3$ holds. Once again we have three subcases.

4.1.) First we have $d_G(v) \neq d_G(t)$. The only way to reconnect u to G_u to get $H \ncong G$ is to connect u to w instead of s, since s and w are the only vertices of degree 1 in G_u. But for that to happen, we must have $f(s) = w$ and the degrees of the neighbours of s in G are $\{d_G(u), d_G(t)\}$, which is different to the degrees of the neighbours of s in H that is $\{d_G(u), d_G(v)\} = \{d_V(u), d_H(v)\}$. This is a contradiction, since, if f is a hypomorphism from G to H the degrees of the neighbours need to stay the same for all vertices y and $f(y)$.

4.2.) We have $d_G(v) = d(t) < n(G) - 3$. So, in G_x, for any vertex $x \neq s$ or u, u is the only vertex of degree $n(G) - 3$ that joins all the vertices of degree 2, or that joins an end vertex if there are no vertices of degree 2 in G_x. The latter occurs when $x = t$ holds. And u is similarly identifiable in $H_{f(x)}$. So we have $\sigma_x(u) = u$ for any isomorphism $\sigma_x : G_x \cong H_{f(x)}$.

Now let $\sigma : G_s \cong H_{f(s)}$. Since $d_G(t) = d_G(v) \geq 3$, w is the only end vertex in G_s. Thus $\sigma(u) = u'$ is a vertex of degree $n(G) - 3$ and so is adjacent to all vertices of $H_{f(s)}$ except w. Hence, the mapping τ that swaps u and u' and is the identity otherwise is an automorphism of $H_{f(s)}$. Consequently $\tau\sigma : G_s \cong H_{f(s)}$ has $\tau\sigma(u) = u$. So we have, for all $x \neq u$, the existence of an isomorphism $\sigma_v : G_x \cong H_{f(x)}$ with $\sigma_x(u) = u$. Since w is the only end

vertex not joining u in G_x we must have $\sigma_x(w) = (w)$.
We now proceed as in case 3.3.) with the construction of G' and H', which are hypomorphic via f, and we again get that G' is either 2-connected or separable with v of degree 1. The latter happens when $d_G(v) = 3$. As before, it now follows that G is isomorphic to H.

4.3.) Last we have $d_G(t) = d_G(v) = n(G) - 3 \geq 7$. Suppose that we get $H \ncong G$ by reconnecting w to G_w at $q \neq v$; that is, $H = (V(G), E(G) \setminus \{vw\} \cup \{qw\})$. Now G_u has no automorphism that takes v to t (otherwise, G can be reconstructed by considering G_u), $H_{f(u)}$ has no automorphism that takes q to t (since $G \ncong H$), and G_w has no automorphism that takes v to q (since $G \ncong H$). Thus, by our degree conditions, there are unique vertices $t', v', q' \neq w, s$ that are not adjacent, respectively, to t, v, and q. Moreover, we can assume that t', v' and q' are all distinct, and that t, v, and q are pairwise adjacent: for $t' = v'$ implies there is an automorphism of G_u that takes v to t; $t' = q'$ implies that we have an automorphism of G_w that takes v to q; and assuming that some pair of t, v or q are not adjacent leads to a similar contradiction.
Thus $d_G(w) = 1$, $d_G(s) = 2$, v', q', and t' are adjacent to u and to two of v, q, and t, while all other vertices have degree at least four in G. We will use this to show that the set $\{s, w\}$ is identifiable in G_x for all $x \in V(G) \setminus \{s, w\}$. First note that s is the unique vertex of degree 2 in G_v for all $x \in V(G) \setminus \{q, s, t, u, v\}$, and that w is the unique end vertex in G_x for all $x \in V(G) \setminus \{t, u, v, w\}$. And s and w are the only end vertices in G_u. And in G_t, s and w are the only end vertices. In G_q, t' might be a vertex of degree 2 but it is of distance 2 from the unique end vertex w, while s is of distance 3 from w; v' might also be a vertex of degree 2 but it is indistinguishable from s, so it does not matter which we call s.
Finally, in G_v, w is the unique isolated vertex and the only two possible vertices of degree 2 other than s are q' and t'. But then q' and s are indistinguishable. And since v is adjacent to both q and t, q has degree less than t has in G_v; thus s can be distinguished from t' by its neighbours.
Now let G'' and H'' be constructed from G and H, respectively, by adding the edge sw. But then, since $\{s, w\}$ is identifiable for each $x \in V(G) \setminus \{s, w\}$, each isomorphism $\sigma_x : G_x \cong H_{f(x)}$ is also an isomorphism of G''_x onto H''_x. And since for $x \in \{s, w\}$, $f(x) \in \{s, w\}$, and $G''_x = G_x = H_{f(x)} = H''_{f(x)}$, f is also a hypomorphism of G'' onto H''. Thus $G'' \cong H''$ since G'' is 2-connected. But w is the only vertex of degree 2 in G'' and s is its only neighbour of degree 3, so we also have $G \cong H$. This contradiction completes this case and hence, completes the proof of this Theorem. □

Next we give the second proof by Yang Yongzhi [78]. Again remark that the proof did already hold for self-complementary graph classes. Hence, we just replace the class of all graphs by a self-complementary graph class and cite his proof.

Theorem 4.3.
Let **G** be a graph class closed under the complement operation. Every connected graph in **G** is reconstructible if and only if every 2-connected graph $G \in \mathbf{G}$ is reconstructible.

Proof.
Let **G** be a graph class closed under the complement operation. Hence, for every $G \in \mathbf{G}$ holds $\overline{G} \in \mathbf{G}$.
" $\Rightarrow$": Let $G \in \mathbf{G}$ be a 2-connected graph. Every 2-connected graph is also connected. Hence, if all connected graphs in **G** are reconstructible, then so is G.
" $\Leftarrow$": Assume that all 2-connected graphs in **G** are reconstructible. Let $G \in \mathbf{G}$ be a separable graph with $n(G) \geq 10$. Remark that graphs with fewer vertices are reconstructible by a computer study by McKay (see Theorem 1.103). We may further assume that G has end vertices since separable graphs without end vertices are reconstructible by Theorem 1.58. Moreover we may assume that $\overline{G} \in \mathbf{G}$ contains a vertex $u \in V(G)$ of degree $d_G(u) = n(G) - 2$. Otherwise $\overline{G} \in \mathbf{G}$ contains no end vertices and is therefore reconstructible by either Theorem 1.58 or our hypothesis that 2-connected graphs are reconstructible. By Theorem 1.40 a graph is reconstructible if and only if its complement is reconstructible. Furthermore, for every $G \in \mathbf{G}$ holds $\overline{G} \in \mathbf{G}$. Hence, all restrictions we may achieve on a graph, we may also assume for its complement. We distinguish two cases:
Assume that G contains at least two end vertices. Hence, $\overline{G}$ contains at least two vertices of degree $n(G) - 2$. Then $\overline{G}$ meets one of three conditions. $\overline{G} \in \mathbf{G}$ is 2-connected and therefore reconstructible by our hypothesis. $\overline{G} \in \mathbf{G}$ is a P-graph and therefore reconstructible by Theorem 4.2. And last $\overline{G} \in \mathbf{G}$ contains two non adjacent edges that are incident to end vertices. These edges are incident to different vertices of degree $n(G) - 2$. Let $v \in V(\overline{G})$ be an end vertex. Then the unique way to reconstruct $\overline{G}$ from $\overline{G}_v$ is to connect v to a vertex of degree $n(G) - 3$ in $\overline{G}_v$ that is not adjacent to at least one end vertex. Then we apply Theorem 1.40 which states that a graph is reconstructible if and only if its complement is reconstructible. All in all if G contains at least two end vertices then the claim holds.
Now assume that both $G \in \mathbf{G}$ and $\overline{G} \in \mathbf{G}$ contain exactly one end vertex and exactly one vertex of degree $n(G) - 2$. Denote the vertex of degree 1 in G with $w \in V(G)$ and the vertex of degree $n(G) - 2$ in G with $u \in V(G)$. Then u is either a cut vertex or $G \in \mathbf{G}$ is a P-graph and thus reconstructible by Theorem 4.2. Hence, assume u is a cut vertex. Let $v \in V(G)$ be the neighbour of w. Then u and v are the only cutvertices of G. Let B be the block containing both u and v. In the connected card G_s with at least one end vertex and with $s \in V(G) \setminus \{u, v, w\}$ the vertices u, v and w are identifiable. u as the only cut vertex of degree $n(G) - 3$, w as the only end vertex not adjacent to u and v as the neighbour of w. Hence, pick such a card G_s where u and v are contained in the same block and that block as large as possible. Hence, we know B and the

location of u and v in it. From G_w we may reconstruct any other blocks and their connection to B at the vertex u. Thus this case is also reconstructible. □

Last we modify a Theorem by Ramachandran and Monikandan given in [67] that states that all 2-connected graphs are reconstructible if and only if all 2-connected graphs G having a vertex $v \in V(G)$ lying on more than one induced P_4 such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible. Remark again that their main argument is based upon the complement and therefore holds for self-complementary graph classes as well.

Theorem 4.4.
Let $\mathbf{G}$ be a graph class closed under the complement operation. Then all 2-connected graphs in $\mathbf{G}$ are reconstructible if and only if all 2-connected graphs $G \in \mathbf{G}$ having a vertex $v \in V(G)$ lying on more than one induced P_4 such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

Proof.
" $\Rightarrow$": Every 2-connected graph with $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ is in particular a 2-connected graph, too.
" $\Leftarrow$": Let $G \in \mathbf{G}$ be a 2-connected graph. Remark that $\overline{G} \in \mathbf{G}$ holds. Assume that all vertices of G are on at most one induced P_4. Then G is a P_4-reducible graph and therefore reconstructible by Theorem 1.75. Hence, we may assume from now on that there exists at least one vertex $v \in V(G)$ that lies on more than one induced P_4. Remark that vertices that induce a P_4 in G also induce a P_4 in $\overline{G}$. Now we differentiate some cases:
First assume that $\text{diam}(G) = 2$ holds. Then G is reconstructible by assumption.
Next assume that $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ holds. Then G is also reconstructible by assumption.
Next assume that $\text{diam}(G) = 3$ and $\text{diam}(\overline{G}) \neq 3$ holds. By Theorem 1.19 the diameter of $\overline{G}$ needs to be less than 3. Hence, $\overline{G}$ is connected with $\text{diam}(\overline{G}) = 2$. If $\overline{G}$ is separable, then $\overline{G}$ is reconstructible by Theorem 1.62. If it is 2-connected, then $\overline{G}$ is reconstructible by assumption. Hence, since $\overline{G}$ is reconstructible, so is G by Theorem 1.40.
Last assume that $\text{diam}(G) > 3$ holds. Then $\text{diam}(\overline{G}) < 3$ holds by Theorem 1.19 and we apply the same reasons as in the previous case. Hence, this case is also reconstructible. □

With the modified proofs we may prove the generalisation of Ramachandran and Monikandan's main Theorem 1.101 given in [67]. That generalisation is Theorem 4.1.

Proof. (of Theorem 4.1)
The Theorem follows directly from Theorem 4.3, Theorem 4.4 and Theorem 1.75. □

One of the most widely known self-complenetary graph classes is the class of perfect graphs. Hence, we can deduce the following corollary from 4.1.

Corollary 4.5.
Let **G** be the class of perfect graphs. Then all perfect graphs are reconstructible if and only if all 2-connected graphs $G \in \mathbf{G}$ with $\mathrm{diam}(G) = 2$ or $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 3$ are reconstructible.

The last corollary reduces significantly the number of perfect graphs that need consideration for reconstruction. Hence, we propose the class of perfect graphs for reconstruction with the idea to reduce the number of perfect graphs that need consideration.

Problem 4.6.
Show that perfect graphs are reconstructible.

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References

[1] N. Alon, Y. Caro, I. Krasikov, and Y. Roditty. Combinatorial reconstruction problems. *Journal of Combinatorial Theory (B)*, 47:153–161, 1989.

[2] K.J. Asciak, M.A. Francalanza, J. Lauri, and W. Myrvold. A survey of some open questions in reconstruction numbers. *Ars Combinatoria*, 97:443–456, 2010.

[3] K.J. Asciak and J. Lauri. On the edge-reconstruction number of disconnected graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 63:87–100, 2011.

[4] L.W. Beineke and E.T. Parker. On nonreconstructable tournaments. *Journal of Combinatorial Theory*, 9:324–326, 1970.

[5] Mark Bilinski, Young Soo Kwon, and Xingxing Yu. On the reconstruction of planar graphs. *Journal of Combinatorial Theory (B)*, 97:5:745–756, 2007.

[6] George D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Annals of Mathematics. Second Series*, 14:42–46, 1912/13.

[7] Béla Bollobás. Almost Every Graph has Reconstruction Number Three. *Journal of Graph Theory*, 14:1:1–4, 1990.

[8] J.A. Bondy. On Ulam's Conjecture for Separable Graphs. *Pacific Journal of Mathematics*, 31:2:281–288, 1969.

[9] J.A. Bondy. A Graph Reconstructor's Manual. *Surveys in Combinatorics*, pages 221–252, 1991.

[10] J.A. Bondy and R.L. Hemminger. Graph Reconstruction - A Survey. *Journal of Graph Theory*, 1:227–268, 1977.

[11] Andrew Bowler, Paul Brown, and Trevor Fenner. Families of pairs of graphs with a large number of common cards. *Journal of Graph Theory*, 63:2:146–163, 2010.

[12] Andrew Bowler, Paul Brown, Trevor Fenner, and Wendy Myrvold. Recognizing connectedness from vertex-deleted subgraphs. *Journal of Graph Theory*, 67:4:285–299, 2011.

[13] J. Caunter and C. ST. J. A. Nash-Williams. Degree conditions for edge reconstruction. *manuscript*, 1982.

[14] Chong Yun Chao and Earl Glen Whitehead Jr. Chromatically unique graphs. *Discrete Mathematics*, 27:171–177, 1979.

[15] Gary Chartrand and Linda Lesniak. *Graphs & Digraphs*. Chapman & Hall/CRC, 4th edition, 2005.

[16] Dragoš Cvetković. On the reconstruction of the characteristic polynomial of a graph. *Discrete Mathematics*, 212:45–52, 2000.

[17] R. Diestel. *Graphentheorie*. Springer, 4th edition, 2010.

[18] M.N. Ellingham, L. Pyber, and Yu Xinxing. Claw-free graphs are edge reconstructible. *Journal of Graph Theory*, 12:445–451, 1988.

[19] S. Fiorini. On the edge-reconstruction of planar graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 83:31–35, 1978.

[20] S. Fiorini. A theorem on planar graphs with an application to the reconstruction problem. i. *The Quarterly Journal of Mathematics. Oxford. Second Series*, 29:353–361, 1978.

[21] S. Fiorini and J. Lauri. The reconstruction of maximal planar graphs. I. Recognition. *Journal of Combinatorial Theory (B)*, 30:188–195, 1981.

[22] S. Fiorini and J. Lauri. Edge-Reconstruction of 4-Connected Planar Graphs. *Journal of Graph Theory*, 6:33–42, 1982.

[23] S. Fiorini and J. Lauri. On the edge-reconstruction of graphs which triangulate surfaces. *The Quarterly Journal of Mathematics. Oxford. Second Series*, 33:191–214, 1982.

[24] S. Fiorini and B. Manvel. A theorem on planar graphs with an application to the reconstruction problem. II. *Journal of Combinatorics, Information & System Sciences*, 3:200–216, 1978.

[25] Herbert Fleischner. The reconstruction of line-critical blocks. *Ars Combinatoria*, 7:223–254, 1979.

[26] T. Gallai. Über extreme Punkt- und Kantenmengen. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica*, 2:133–138, 1959.

[27] William B. Giles. The reconstruction of outerplanar graphs. *Journal of Combinatorial Theory (B)*, 16:215–226, 1974.

[28] William B. Giles. Point deletions of outerplanar blocks. *Journal of Combinatorial Theory (B)*, 20:103–116, 1976.

[29] D.L. Greenwell. Reconstructing Graphs. *Proceedings of the American Mathematical Society*, 30:431–433, 1971.

[30] D.L. Greenwell and R.L. Hemminger. Reconstructing Graphs. *The Many Facets of Graph Theory*, pages 91–114, 1969.

[31] S.K. Gupta, Pankaj Mangal, and Vineet Paliwal. Some work towards the proof of the reconstruction conjecture. *Discrete Mathematics*, 272:291–296, 2003.

[32] Frank Harary. On the reconstruction of a graph from a collection of subgraphs. *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 47–52, 1964.

[33] Frank Harary and Ed Palmer. On similar points of a graph. *Journal of Mathematics and Mechanics*, 15:623–630, 1966.

[34] Frank Harary and Ed Palmer. On the problem of reconstructing a tournament from subtournaments. *Monatshefte für Mathematik*, 71:14–23, 1967.

[35] Frank Harary and Michael Plantholt. The Graph Reconstruction Number. *Journal of Graph Theory*, 9:451–454, 1985.

[36] Frank Harary and Robert W. Robinson. The diameter of a graph and its complement. *American Mathematical Monthly*, 92:3:211–212, 1985.

[37] R.L. Hemminger. On reconstructing a graph. *Proceedings of the American Mathematical Society*, 20:185–187, 1969.

[38] Herbert Izbicki. Reguläre Graphen beliebigen Grades mit vorgegebenen Eigenschaften. (German). *Monatshefte für Mathematik*, 64:15–21, 1960.

[39] Paul J. Kelly. A Congruence Theorem for Trees. *Pacific Journal of Mathematics*, 7:961–968, 1957.

[40] William L. Kocay. On reconstructing spanning subgraphs. *Ars Combinatoria*, 11:301–313, 1981.

[41] William L. Kocay. Some new methods in reconstruction theory. *Combinatorial mathematics*, IX:89–114, 1982.

[42] I. Krasikov and Y. Roditty. Some applications of the Nash-Williams lemma to the edge-reconstruction conjecture. *Graphs and Combinatorics*, 6:37–39, 1990.

[43] V. Krishnamoorthy and K.R. Parthasarathy. Reconstruction of Critical Blocks. *Journal of Mathematical and Physical Sciences*, 13:3:219–239, 1979.

[44] J. Lauri. Edge-reconstruction of planar graphs with minimum valency 5. *Journal of Graph Theory*, 3:269–286, 1979.

[45] J. Lauri. The reconstruction of maximal planar graphs. II. Reconstruction. *Journal of Combinatorial Theory (B)*, 30:196–214, 1981.

[46] J. Lauri. Graph reconstruction-some techniques and new problems. *Ars Combinatoria*, 24:35–61, 1987.

[47] Josef Lauri and Raffaele Scapellato. *Topics in Graph Automorphisms and Reconstruction*. London Mathematical Society Student Texts 54. Cambridge University Press, 1st edition, 2003.

[48] L. Lovász. A characterization of perfect graphs. *Journal of Combinatorical Theory (B)*, 13:95–98, 1972.

[49] L. Lovász. A Note on the Line Reconstruction Problem. *Journal of Combinatorical Theory (B)*, 13:309–310, 1972.

[50] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2:253–267, 1972.

[51] Alicia Maccari, Olga Rueda, and Vilma Viazzi. A survey on edge reconstruction of graphs. *Journal of Discrete Mathematical Sciences & Cryptography*, 5:1:1–11, 2002.

[52] Brendan D. McKay. Computer Reconstruction of small graphs. *Journal of Graph Theory*, 1:281–283, 1977.

[53] Brendan D. McKay. Small Graphs are Reconstructible. *Australasian Journal of Combinatorics*, 15:123–126, 1997.

[54] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, {II}. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014.

[55] Brendan D. McKay and Adolfo Piperno. nauty and Traces User's Guide (Version 2.6). *http://pallini.di.uniroma1.it/*, 2016.

[56] Brian McMullen and Stanisław P. Radziszowski. Graph reconstruction numbers. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 62:85–96, 2007.

[57] S. Monikandan and J. Balakumar. Reconstruction of bipartite graphs and triangle-free graphs with connectivity two. *The Australasian Journal of Combinatorics*, 53:141–150, 2012.

[58] S. Monikandan and J. Balakumar. A reduction of the graph reconstruction conjecture. *Discussiones Mathematicae. Graph Theory*, 34:3:529–537, 2014.

[59] Vladimír Müller. Probabilistic reconstruction from subgraphs. *Commentationes Mathematicae Universitatis Carolinae*, 17:709–719, 1976.

[60] Vladimír Müller. The Edge Reconstruction Hypothesis is True for Graphs with More than $n \cdot \log_2 n$ Edges. *Journal of Combinatorical Theory (B)*, 22:281–283, 1977.

[61] Wendy Myrvold. The Ally-Reconstruction Number of a Tree with Five or More Vertices Is Three. *Journal of Graph Theory*, 14:149–166, 1990.

[62] Wendy Myrvold. A report on the ally reconstruction problem. *Graph theory, combinatorics, and applications*, 2:949–956, 1991.

[63] W.J. Myrvold, M.N. Ellingham, and D.G. Hoffman. Bidegreed graphs are edge reconstructible. *Journal of Graph Theory*, 11:3:281–302, 1987.

[64] C. St. J. A. Nash-Williams. The Reconstruction Problem. *Selected Topics in Graph Theory*, pages 205–236, 1978. London: Academic Press.

[65] L. Pyber. The edge reconstruction of Hamiltonian graphs. *Journal of Graph Theory*, 14:2:173–179, 1990.

[66] S. Ramachandran. Graph Reconstruction - Some New Developments. *AKCE International Journal of Graphs and Combinatorics*, 1:51–61, 2004.

[67] S. Ramachandran and S. Monikandan. Graph reconstruction conjecture: Reductions using complement, connectivity and distance. *Bulletin of the Institute of Combinatorics and its Applications*, 56:103–108, 2009.

[68] Michael von Rimscha. Reconstructibility and perfect graphs. *Discrete Mathematics*, 47:283–291, 1983.

[69] N. Robertson and P. D. Seymour. Graph Minors I-XXIII. *Journal of Combinatorial Theory (B)*, 1983–2012.

[70] Horst Sachs. Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristischen Polynom (German). *Publicationes Mathematicae Debrecen*, 11:119–134, 1964.

[71] E.F. Schmeichel. A note on the edge reconstruction conjecture. *Bulletin of the Australian Mathematical Society*, 12:27–30, 1975.

[72] Paul K. Stockmeyer. The reconstruction conjecture for tournaments. *Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975)*, pages 561–566, 1975.

[73] Paul K. Stockmeyer. New counterexamples to the digraph reconstruction conjecture. *Notices of the American Mathematical Society*, 23:A654, 1976.

[74] Paul K. Stockmeyer. The falsity of the reconstruction conjecture for tournaments. *Journal of Graph Theory*, 1:1:19–25, 1977.

[75] Bhalchandra D. Thatte. Some Results on the Reconstruction Problems. p-Claw-Free, Chordal, and P_4-Reducible graphs. *Journal of Graph Theory*, 19:4:549–561, 1995.

[76] W.T. Tutte. All the King's Horses (A Guide to Reconstruction). *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pages 15–33, 1979.

[77] Hassler Whitney. A logical expansion in mathematics. *Bulletin of the American Mathematical Society*, 38:8:572–579, 1932.

[78] Yang Yongzhi. The reconstruction conjecture is true if all 2-connected graphs are reconstructible. *Journal of Graph Theory*, 12:2:237–243, 1988.